One-step Heyting algebras and hypersequent calculi with the bounded proof property

Nick Bezhanishvili[1], Silvio Ghilardi[2], and Frederik Möllerström Lauridsen[1]

[1]ILLC, Universiteit van Amsterdam, The Netherlands
[2]Università degli Studi di Milano, Italy

Abstract

We introduce a new weakly analytic subformula property (the bounded proof property) of hypersequent calculi for intermediate logics. We define one-step Heyting algebras and establish semantic criteria characterizing the bounded proof property in terms of these algebraic structures. Finally, using these criteria, we provide a number of examples of calculi for intermediate logics with and without the bounded proof property.

Keywords: intermediate logics, hypersequent calculi, bounded proof property, finite model property, duality.

1 Introduction

Having a well-behaved proof system for a logic can help determine various desirable properties of this logic such as consistency, decidability, interpolation etc. Gentzen-style sequent calculi have for a long time played a pivotal role in proof theory [37] and proving admissibility of the cut-rule has been one of the main tools for establishing good proof theoretic properties of sequent calculi. However, for various non-classical logics finding a cut-free sequent calculus can be a difficult task, even when the logic in question has a very simple semantics. In fact, in many cases no such calculus seems to exist. In the 1980's Pottinger [34] and Avron [2] introduced hypersequent calculi for handling certain modal and relevance logics. Hypersequents are nothing more than finite (multi)sets of sequents but they give rise to simple cut-free calculi for many logics for which no ordinary cut-free calculus has been found. Since then cut-free hypersequent calculi for various modal and intermediate logics have been given [3, 17, 16, 29, 20, 32, 33]. However, establishing cut-elimination for Gentzen-style sequent or hypersequent calculi by syntactic means can be very cumbersome. Although the basic idea behind syntactic proofs of cut-elimination is simple, each individual calculus will need its own proof of cut-elimination and proofs obtained for one calculus do not necessarily transfer easily to other – even very similar – calculi. Recently some steps to ameliorate this situation have been taken. For example, [32, 33] provide general methods for obtaining cut-free calculi for larger classes of logics based on their semantics.

Semantic proofs of cut-elimination have been known since at least 1960 [36], but in recent years a general algebraic approach to proving cut-elimination for various substructural logics via McNeille completions has been developed [18, 19]. One of the attractive features of this approach is that it allows one to establish cut-elimination for large classes of logics in a uniform way. Moreover, [18, 19] also provide algebraic criteria determining when cut-free (hyper)sequent calculi for a given substructural logic can be obtained.[1] This algebraic approach suggests that algebraic semantics can be used to detect other desirable features of a proof system. It is this kind of *algebraic proof theory* that is the subject of the present paper. However, we will take a somewhat different approach to connecting algebra and proof theory than the one outlined above. In particular, we will be focusing on characterizing a proof-theoretic property weaker than – though in some ways similar to – cut-elimination.

The free algebra of a propositional logic encodes a lot of information about the logic. For instance it is well-known that the finitely generated free algebras constitute a powerful tool when it comes to

[1]However, these criteria only cover the lower levels ($\mathcal{N}_2$ and $\mathcal{P}_3$) of the substructural hierarchy of [18].

establishing meta-theoretical properties for various propositional logics such as interpolation, definability, admissiblity of rules etc. In [24] it was shown how to construct finitely generated free Heyting algebras as (chain) colimits of finite distributive lattices. In [25] a similar construction for for finitely generated free modal algebras was presented; showing how these algebras arise as colimits of finite Boolean algebras.[2] The intuition behind these constructions is that one builds the finitely generated free algebra in stages by freely adding the Heyting implication (or in the case of modal algebras the modal operator) step by step. Lately this construction has received renewed attention in [14, 9] (for Heyting algebras) and in [13, 26, 27, 12] (for modal algebras). Finally, in [22] sufficient criteria are given for this construction to succeed for finitely generated free algebras in an arbitrary variety.

It was realized in [10] that the so-called *modal one-step algebras* arising as consecutive pairs of algebras in the colimit construction of finitely generates free modal algebras can be used to characterize a certain weak analytic subformula property of proof systems for modal logics. This property – called *the bounded proof property* – holds in an axiom system Ax if for every finite set of formulas $\Gamma \cup \{\phi\}$ of modal depth[3] at most n such that Γ entails ϕ over Ax there exists a derivation in Ax witnessing this in which all the formulas have modal depth at most n. We write $\Gamma \vdash^n_{\mathrm{Ax}} \phi$ if this is the case. With this notation the bounded proof property may be expressed as

$$\Gamma \vdash_{\mathrm{Ax}} \phi \implies \Gamma \vdash^n_{\mathrm{Ax}} \phi.$$

Even though this is a fairly weak property it does, e.g., bound the search space when searching for proofs and thus it ensures decidability of logics with a finite axiomatization. Furthermore, having this property might serve as an indication of robustness of the axiom system in question. In this way it is like cut-elimination although in general it is much weaker.

In light of the original colimit construction of finitely generated free Heyting algebras it seems natural to ask if one can adapt the work of [10] to the setting of intuitionistic logic and its extensions. That is, we ask if it is possible to formulate the bounded proof property for intuitionistic logic and define a notion of one-step Heyting algebras which can characterize proof systems of intermediate logics with the bounded proof property.

In order to do this one first needs to choose a proof theoretic framework for which to ask this question. In this respect there are two remarks to be made. First of all as any use of *modus ponens* will evidently make the bounded proof property with respect to implications fail, we will have to consider proof systems different from natural deduction or Hilbert-style proof systems. Therefore, a Gentzen-style sequent calculus might be a better option. In these systems modus ponens is replaced with the cut-rule which for good systems can be eliminated or at least restricted to a well-behaved fragment of the logic in question. Secondly, as mentioned in the beginning of the introduction, ordinary sequent calculi are often ill-suited when it comes to giving well-behaved calculi for concrete intermediate logics, in that they generally do not admit cut-elimination. Therefore, keeping up with the recent trend in proof theory of non-classical logics, we base our approach on hypersequent calculi. This makes our results more general and more importantly allows us to consider more interesting examples of proof systems for intermediate logics. This approach is also in line with [11] where the results of [10] are generalised to the framework of multi-conclusion rule systems for modal logics.

We define a notion of one-step Heyting algebras and develop a theory of these algebras parallel to the theory of one-step modal algebras [10]. We show that just as in the modal case the bounded proof property for intuitionistic hypersequent calculi can be characterised algebraically using one-step Heyting algebras. We also develop a notion of intuitionistic one-step frames dual to that of one-step Heyting algebras. Finally, we test the obtained criterion of the bounded proof property on a number of examples of hypersequent calculi for intermediate logics. In particular, we show that every stable intermediate logic [4, 7] has the bounded proof property. This class contains all the logics in the class $\mathcal{P}_3$ of the substructural hierarchy of [18].

The paper is organized as follows. In Section 2 we recall hypersequent calculi for intermediate logics, and define when such calculi admit the bounded proof property. In Section 3 we introduce one-step Heyting algebras and one-step intuitionistic frames and in Section 4 we prove a semantic characterization

[2]The basic idea of constructing finitely generated free modal algebras in an incremental way is in some sense already present in [23] and [1]. Note that [1] is based on a talk given at the BCTCS in 1988.

[3]Recall that the *modal depth* of a formula ϕ is the maximal number of nestings of modalities occurring in ϕ.

of the bounded proof property in terms of these algebras and frames. Finally, Section 5 discusses a number of examples of intermediate logics with and without the bounded proof property.

2 Hypersequent calculi and universal classes of Heyting algebras

Let Prop be a set of propositional variables and let $Form(\mathsf{Prop})$ denote the set of formulas determined by the following grammar:

$$\phi ::= \bot \mid p \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi,$$

$p \in \mathsf{Prop}$.

We then define the *implicational degree* $d(\phi)$ of a formula ϕ by the following recursion: $d(\bot) = 0$ and $d(p) = 0$ for all $p \in \mathsf{Prop}$. Moreover,

$$d(\phi \wedge \psi) = d(\phi \vee \psi) = \max\{d(\phi), d(\psi)\} \quad \text{and} \quad d(\phi \to \psi) = \max\{d(\phi), d(\psi)\} + 1.$$

For $n \in \omega$ we let $Form_n(\mathsf{Prop})$ denote the subset of $Form(\mathsf{Prop})$ consisting of formulas of implicational degree at most n. A crucial fact is that if Prop is a finite then $Form_n(\mathsf{Prop})$ will be finite (up to provable equivalence) for all $n \in \omega$.

A *sequent* is a pair of finite (possible empty) multisets of formulas written as $\Gamma \Rightarrow \Delta$ and a *hypersequent* is a finite multiset of hypersequents written as

$$\Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n.$$

The sequents $\Gamma_k \Rightarrow \Delta_k$ are called the *components* of the hypersequent.

We will let lower case letters $s, s_0, s_1, \dots$ denote sequents while upper case letter $G, H, S, S_0, S_1, \dots$ will denote hypersequents. Note that the notion of implicational degree extends to sequents and hypersequents as follows:

$$d(\Gamma \Rightarrow \Delta) = \max\{d(\phi) \colon \phi \in \Gamma \cup \Delta\} \quad \text{and} \quad d(s_1 \mid \dots \mid s_n) = \max\{d(s_k) \colon k \leq n\}.$$

We say that a Heyting algebra $\mathfrak{A}$ *validates a sequent* $\Gamma \Rightarrow \Delta$ *under a valuation* v, written $(\mathfrak{A}, v) \vDash \Gamma \Rightarrow \Delta$, if $v(\bigwedge \Gamma) \leq v(\bigvee \Delta)$, and we say that $\mathfrak{A}$ *validates a hypersequent* $\Gamma_1 \Rightarrow \Delta_1 \mid \dots \mid \Gamma_n \Rightarrow \Delta_n$ *under a valuation* v, if $(\mathfrak{A}, v) \vDash \Gamma_k \Rightarrow \Delta_k$ for some $1 \leq k \leq n$. Finally, we say that $\mathfrak{A}$ *validates a sequent* (or *hypersequent*) if it validates it under all valuations.

2.1 Hypersequent proofs and hypersequent calculi

A *hypersequent rule* is a pair consisting of a finite set of hypersequent $\{S_1, \dots, S_n\}$, called the *premises*, and a single hypersequent S, called the *conclusion.* We write hypersequent rules as

$$\frac{S_1 \dots S_m}{S}\ (r)$$

Given a Heyting algebra $\mathfrak{A}$ and a hypersequent rule (r) we say that $\mathfrak{A}$ validates (r) if for each valuation v on $\mathfrak{A}$ we have that the conclusion S is valid under v if all the premisses S_j are valid under v.

Definition 2.1. Let $\{S, S_1, \dots, S_n\}$ be a set of hypersequents and let

$$\frac{S'_1 \dots S'_n}{S'}\ (r)$$

be a hypersequent rule. We say that a hypersequent S is *obtained from* $S_1, \dots, S_n$ *by an application of the rule* (r), if there exists a substitution σ and a hypersequent G such that S is of the kind $G \mid S'\sigma$ and S_i is of the kind $G \mid S'_i\sigma$ for $i \in \{1, \dots, n\}$.[4]

[4]Due to the presence of the external weakening rule (ew) (see Definition 2.2 below), this is the same as saying that S_i is of the kind $G_i \mid S'_i\sigma$ and that S is of the kind $G \mid S'\sigma$ for some $G \supseteq \bigcup_{i=1}^{n} G_i$.

In this way uniform substitution and external weakening are taken into account in the definition of rule application.

We here present the rules for the multi-succedent sequent hypersequent calculus for **IPC**.

Definition 2.2 ([20])**.** The calculus HJL' consists of the following rule schemas. Thus in the following ϕ and ψ range over formulas, Γ and Δ over finite multisets of formulas and G over hypersequents. This means that each rule schema represents infinitely many rules.

Axioms:

$$\frac{}{G \mid \phi \Rightarrow \phi}\,(\text{init}) \qquad \frac{}{G \mid \bot \Rightarrow}\,(l\bot)$$

External structural rules:

$$\frac{G \mid \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \Delta}\,(ec) \qquad \frac{G}{G \mid \Gamma \Rightarrow \Delta}\,(ew)$$

Internal structural rules:

$$\frac{G \mid \Gamma \Rightarrow \phi, \phi, \Delta}{G \mid \Gamma \Rightarrow \phi, \Delta}\,(ric) \qquad \frac{G \mid \Gamma, \phi, \phi \Rightarrow \Delta}{G \mid \Gamma, \phi \Rightarrow \Delta}\,(lic)$$

$$\frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma, \phi \Rightarrow \Delta}\,(liw) \qquad \frac{G \mid \Gamma \Rightarrow \Delta}{G \mid \Gamma \Rightarrow \phi, \Delta}\,(riw)$$

Logical rules:

$$\frac{G \mid \Gamma \Rightarrow \phi, \Delta \qquad G \mid \Gamma, \psi \Rightarrow \Delta}{G \mid \Gamma, \phi \to \psi \Rightarrow \Delta}\,(l \to) \qquad \frac{G \mid \Gamma, \phi \Rightarrow \psi}{G \mid \Gamma \Rightarrow \phi \to \psi}\,(r \to)$$

$$\frac{G \mid \Gamma, \phi, \psi \Rightarrow \Delta}{G \mid \Gamma, \phi \wedge \psi \Rightarrow \Delta}\,(l\wedge) \qquad \frac{G \mid \Gamma \Rightarrow \phi, \Delta \qquad G \mid \Gamma \Rightarrow \psi, \Delta}{G \mid \Gamma \Rightarrow \phi \wedge \psi, \Delta}\,(r\wedge)$$

$$\frac{G \mid \Gamma, \phi \Rightarrow \Delta \qquad G \mid \Gamma, \psi \Rightarrow \Delta}{G \mid \Gamma, \phi \vee \psi \Rightarrow \Delta}\,(l\vee) \qquad \frac{G \mid \Gamma \Rightarrow \phi, \psi, \Delta}{G \mid \Gamma \Rightarrow \phi \vee \psi, \Delta}\,(r\vee)$$

The cut rule:

$$\frac{G \mid \Gamma \Rightarrow \phi, \Delta \qquad G' \mid \phi, \Sigma \Rightarrow \Pi}{G \mid G' \mid \Gamma, \Sigma \Rightarrow \Pi, \Delta}\,(cut)$$

As we will only be interested in calculi for intermediate logics we shall understand by a *hypersequent calculus* any collection of hypersequent rules extending a hypersequent calculus for **IPC**, e.g., the multi-succedent calculus presented above. This means that rules such as external contraction and the cut-rule belong to every hypersequent calculus even though they may not be eliminable.

If $\mathscr{S} \cup \{S\}$ is a set of hypersequents and HC is a hypersequent calculus we say that S is *derivable* (or *provable*) *from* $\mathscr{S}$ *over* HC, written $\mathscr{S} \vdash_{\text{HC}} S$, if there exists a finite sequence of hypersequents $S_1, \ldots, S_n$ such that S_n is the hypersequent S and for all $1 \leq k < n$ either S_k belongs to $\mathscr{S}_k$ or S_k is obtained by applying a rule from HC to some subset of $\{S_1, \ldots, S_{k-1}\}$.

Note that it is not allowed to apply substitutions to hypersequents in $\mathscr{S}$. Thus $\vdash_{\text{HC}}$ denotes the global consequence relation, in the sense that the members of $\mathscr{S}$ will be taken as axioms, i.e. leaves in a derivation tree.

Definition 2.3. A hypersequent rule $(S_1, \ldots S_n)/S$ is *derivable* in a hypersequent calculi HC if

$$\{S_1, \ldots, S_n\} \vdash_{\text{HC}} S.$$

Two hypersequent calculi HC and HC$'$ will be *equivalent* if all the rules of HC are derivable in HC$'$ and vice versa.

Note that if HC and HC$'$ are equivalent then for all finite sets $\mathscr{S} \cup \{S\}$ of hypersequents we have that

$$\mathscr{S} \vdash_{\text{HC}} S \quad \text{iff} \quad \mathscr{S} \vdash_{\text{HC}'} S.$$

The next proposition will be used throughout the paper.

Proposition 2.4. *Every hypersequent calculus* HC *is equivalent to a hypersequent calculus consisting only of rules with single component hypersequents as premisses.*

Proof. For each hypersequent rule $(r) = (S_1, \ldots, S_n)/S$ of HC we let C_r be the set of functions selecting a component from each of the premisses S_k. Then we let HC′ be the set of rules given by

$$\frac{c(S_1) \ldots c(S_n)}{S} \, (r_c)$$

where (r) ranges over HC and c over C_r. That HC and HC′ are indeed equivalent is an easy consequence of having external contraction and weakening. More precisely, we have that

$$\frac{\dfrac{c(S_1)}{S_1}\,(ew) \qquad \ldots \qquad \dfrac{c(S_n)}{S_n}\,(ew)}{S}\,(r)$$

for any choice of $c \in \mathsf{C}_r$. From which it follows that the rules $(r_c)_{c \in \mathsf{C}_r}$ are derivable from (r).

For the converse implication let m_1 be the number of components of S_1, say $S_1 = s_{11} \mid \ldots \mid s_{1m_1}$. We show that (r) is equivalent to the set of rules

$$\frac{s_{1k} \; S_2 \ldots S_n}{S} \, (r_{1k})$$

The following derivation show that S is indeed derivable from $\{S_1, \ldots, S_n\}$ using the rules $(r_{1k})_{k=1}^{m_1}$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{S_1 \ldots S_n}{s_{12} \mid \ldots \mid s_{1m_1} \mid S}\,(r_{11}) \qquad S_2 \ldots S_n}{s_{13} \mid \ldots \mid s_{1m_1} \mid S \mid S}\,(r_{12})}{s_{13} \mid \ldots \mid s_{1m_1} \mid S}\,(ec)}{\vdots} \qquad}{}$$

$$\dfrac{\dfrac{s_{1m_1} \mid S \qquad\qquad S_2 \ldots S_n}{S \mid S}\,(r_{1m_1})}{S}\,(ec)$$

Given this the desired conclusion can be obtained by a straightforward inductive argument on the number of multi-component premisses of (r). □

In order to establish soundness and completeness of derivability of hypersequent rules with respect to Heyting algebras we will need the following facts

Lemma 2.5. *Let* $\mathscr{S} \cup \{S\}$ *be a set of hypersequents and let* s *be a sequent. Then for every hypersequent calculus* HC *we have that*

$$(\mathscr{S} \cup \{s\} \vdash_{\mathrm{HC}} S \quad \textit{and} \quad \mathscr{S} \vdash_{\mathrm{HC}} s \mid S) \quad \Longrightarrow \quad \mathscr{S} \vdash_{\mathrm{HC}} S.$$

Proof. Assuming that $\mathscr{S} \vdash_{\mathrm{HC}} s \mid S$ we see that for any hypersequent S' if $\mathscr{S} \cup \{s\} \vdash_{\mathrm{HC}} S'$, then, by induction on the length of a derivation witnessing this, we must have that $\mathscr{S} \vdash_{\mathrm{HC}} S' \mid S$. Therefore, if $\mathscr{S} \vdash_{\mathrm{HC}} s \mid S$ and $\mathscr{S} \cup \{s\} \vdash_{\mathrm{HC}} S$ we may conclude that $\mathscr{S} \vdash_{\mathrm{HC}} S \mid S$, whence by applying external contraction we obtain that $\mathscr{S} \vdash_{\mathrm{HC}} S$, as desired. □

We then introduce a variant of the well-known Lindenbaum-Tarski construction.

Proposition 2.6. *For every hypersequent calculus* HC *and every set of hypersequents* $\mathscr{S} \cup \{S\}$ *such that* $\mathscr{S} \nvdash_{\mathrm{HC}} S$ *there exists a Heyting algebra* $\mathfrak{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ *validating* HC *and a valuation on* $\mathfrak{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ *under which* $\mathfrak{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ *validates* $\mathscr{S}$ *but not* S.

Proof. Let Prop be the set of propositional letters occurring in $\mathscr{S} \cup \{S\}$ and let $\widetilde{\mathscr{S}}$ be a maximal set of hypersequents, based on $Form(\mathsf{Prop})$, extending $\mathscr{S}$ such that $\widetilde{\mathscr{S}} \nvdash_{\mathrm{HC}} S$. Assuming Zorn's Lemma such a set always exists. Then define an equivalence relation $\approx$ on the formula algebra $Form(\mathsf{Prop})$ by

$$\phi \approx \psi \iff \widetilde{\mathscr{S}} \vdash_{\mathrm{HC}} \Rightarrow \phi \leftrightarrow \psi.$$

Since HC extends a hypersequent calculus of **IPC** one may readily verify that $\mathfrak{LT}_{\mathrm{HC}}(\mathscr{S}, S) = Form(\mathsf{Prop})/{\approx}$ is a Heyting algebra.

We observe that by the maximality of $\widetilde{\mathscr{S}}$, Lemma 2.5 together with the assumption that $\widetilde{\mathscr{S}} \nvdash_{\mathrm{HC}} S$ yields that

$$\widetilde{\mathscr{S}} \vdash_{\mathrm{HC}} s_1 \mid \ldots \mid s_m \mid S \implies \widetilde{\mathscr{S}} \vdash_{\mathrm{HC}} s_i \text{ for some } 1 \leq i \leq m \;, \tag{1}$$

for all sequents $s_1, \ldots, s_m$. For suppose not, then in particular $\widetilde{\mathscr{S}} \cup \{s_1\} \vdash_{\mathrm{HC}} S$ by maximality of $\widetilde{\mathscr{S}}$. So by Lemma 2.5 we must have that $\widetilde{\mathscr{S}} \vdash_{\mathrm{HC}} s_2 \mid \ldots \mid s_m \mid S$. Thus after repeating this argument m times we obtain $\widetilde{\mathscr{S}} \vdash_{\mathrm{HC}} S$, in direct contradiction with the initial assumption.

Observe that from (1) and external weakening it follows that if $\widetilde{\mathscr{S}} \vdash_{\mathrm{HC}} s_1 \mid \ldots \mid s_m$ then $\widetilde{\mathscr{S}} \vdash_{\mathrm{HC}} s_i$ for some $1 \leq i \leq m$. From this it is easy to verify that $\mathfrak{LT}_{HC}(\mathscr{S}, S)$ validates all the rules of HC.

Finally, we claim that under the valuation determined by sending propositional variables to their respective equivalence classes of the equivalence relation $\approx$, the algebra $\mathfrak{LT}_{HC}(\mathscr{S}, S)$ validates all the hypersequents from $\mathscr{S}$ but does not validate the hypersequent S. This, however, is evident. ☐

Remark 2.7. One could initially be tempted to believe that the construction in the proof of Proposition 2.6 will yield free algebras for the universal class of Heyting algebras validating the calculus HC. However, this is not the case as universal classes of algebras do not necessarily have free algebras. To see why the construction fails to give free algebras note that $\phi \approx \psi$ does not imply that the corresponding terms are identified in all Heyting algebras validating HC, but only that they may consistently (relative to HC) be identified. Therefore, it is not well defined to map an equivalence class of formulas to the Heyting algebra term of a formula from the equivalence class.

Proposition 2.8 (Algebraic soundness and completeness)**.** *Let* HC *be a hypersequent calculus and let* (r) *be a hypersequent rule. Then the following are equivalent:*

1. *The rule* (r) *is derivable in* HC*;*
2. *All Heyting algebras validating* HC *also validates* (r)*.*

Proof. That item 1 implies item 2 follows from a straightforward induction on the length of derivations of rules. That item 2 implies item 1 is immediate from Proposition 2.6. ☐

2.2 Hypersequents calculi, multi-conclusion rules and universal classes of Heyting algebras

Given a hypersequent calculus HC we obtain an intermediate logics $\Lambda(\mathrm{HC}) := \{\phi \in Form \colon \vdash_{\mathrm{HC}} \Rightarrow \phi\}$. We say that a hypersequent calculus HC is a calculus for an intermediate logic L if $\Lambda(\mathrm{HC}) = L$. This means that derivability relations $\vdash_L$ and $\vdash_{\mathrm{HC}}$ coincides for sequents in the sense that

$$\vdash_L \bigwedge \Gamma \to \bigvee \Delta \quad \text{iff} \quad \vdash_{\mathrm{HC}} \Gamma \Rightarrow \Delta \tag{2}$$

obtains for all sequents $\Gamma \Rightarrow \Delta$. Note, however, that the corresponding version of (2) does not necessarily obtain for hypersequents. As our primary interest in hypersequents is to obtain analytic calculi for logics, this does not constitute a problem.

Given a hypersequent calculus HC the class $\mathcal{U}(\mathrm{HC})$ of Heyting algebras validating HC will evidently be a universal class. Conversely, given a universal class U of Heyting algebras, determined by a set of universal sentences Φ, we obtain a hypersequent calculus $\mathcal{HC}(U)$ by adding for each universal sentence $\sigma = \forall \underline{x}(\bigwedge_{k=1}^{m}(\phi_k(\underline{x}) = \top) \implies \bigvee_{l=1}^{m}(\psi_l(\underline{x}) = \top)) \in \Phi$ the rule schema

$$\frac{\Rightarrow \phi_1 \ldots \Rightarrow \phi_n}{\Rightarrow \psi_1 \mid \ldots \mid \Rightarrow \psi_m}\,(r_\sigma)$$

to a hypersequent calculus for **IPC**.

Using Proposition 2.8 it is easy to verify that $\mathcal{U}(\mathcal{HC}(U)) = U$ and that $\mathcal{HC}(\mathcal{U}(\mathrm{HC}))$ will be equivalent to HC. Thus we have a one-to-one correspondence between hypersequent calculi for intermediate logics (modulo equivalence) and universal classes of Heyting algebras.

Similarly we obtain a correspondence between multi-conclusion rules [31, 7] and hypersequent calculi. Given a multi-conclusion rule $(r) = (\phi_1, \ldots, \phi_n)/(\psi_1, \ldots, \psi_m)$ we obtain a hypersequent rule:

$$\frac{\Rightarrow \phi_1 \ldots \Rightarrow \phi_n}{\Rightarrow \psi_1 \mid \ldots \mid \Rightarrow \psi_m}\ (r_H)$$

Conversely, given a hypersequent rule with single component premises

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \ldots \Gamma_n \Rightarrow \Delta_n}{\Sigma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Sigma_m \Rightarrow \Pi_m}\ (r)$$

we obtain a multi-conclusion rule:

$$\frac{\bigwedge \Gamma_1 \to \bigvee \Delta_1, \ldots, \bigwedge \Gamma_n \to \bigvee \Delta_n}{\bigwedge \Sigma_1 \to \bigvee \Pi_1, \ldots, \bigwedge \Sigma_m \to \bigvee \Pi_m}\ (r_M)$$

Evidently a Heyting algebra validates a multi-conclusion rule (reps. hypersequent rule) iff it validates the corresponding hypersequent rule (reps. multi-conclusion rule). Since by Proposition 2.4 every hypersequent calculus is equivalent to one only consisting of rules with single component premisses this yields (modulo equivalence) a correspondence between multi-conclusion consequence relations and hypersequent calculi. Thus, for the proposes of axiomatizing intermediate logics hypersequent calculi and multi-conclusion consequence relations may be used interchangeably.

2.3 The bounded proof property

We say that a hypersequent calculus HC has the *bounded proof property* if whenever $\mathscr{S} \cup \{S\}$ is a set of hypersequents of implicational degree at most n such that $\mathscr{S} \vdash_{\mathrm{HC}} S$ then $\mathscr{S} \vdash^n_{\mathrm{HC}} S$, i.e., there exists a proof witnessing $\mathscr{S} \vdash_{\mathrm{HC}} S$ consisting only of hypersequents of degree at most n. The bounded proof property is thus a very weak form of analyticity. However, having this property will indicate some kind of robustness of the hypersequent calculus in question. For instance the subformula property will entail the bounded proof property. Therefore, if a hypersequent calculus enjoys cut-elimination it will also, under mild additional assumptions, have the subformula property and hence the bounded proof property. Finally, as in the modal case [10, 11], having the bounded proof property will ensure that the derivability relation $\vdash_{\mathrm{HC}}$ is decidable, given that HC consists of finitely many rules. This is due to the fact that for a given finite set of propositional variables Prop there are only finitely many non-equivalent formulas in Prop of implicational degree at most n.

The next proposition shows that the bounded proof property is completely determined by the degree 1 case.

Proposition 2.9. *A hypersequent calculus* HC *has the bounded proof property iff for each set $\mathscr{S} \cup \{S\}$ consisting of hypersequents of degree at most* 1*, we have*

$$\mathscr{S} \vdash_{\mathrm{HC}} S \quad \textit{iff} \quad \mathscr{S} \vdash^1_{\mathrm{HC}} S.$$

Proof. The left-to-right direction is evident.

For the converse implication let $\mathscr{S} \cup \{S\}$ be a set of hypersequents of degree at most n. We define a sequence of triples $(\mathscr{S}_i, S_i, \sigma_i)_{i=0}^{n-1}$ such that

i) $\mathscr{S}_i \cup \{S_i\}$ is a set of hypersequents of degree at most $n - i$ and σ_i is a substitution such that $d(\chi\sigma_i) \leq d(\chi) + 1$ for all formulas χ occurring in $\mathscr{S}_i \cup \{S_i\}$;

ii) $S_{i+1}\sigma_{i+1} = S_i$;

iii) $\mathscr{S}_{i+1}\sigma_{i+1}$ equals $\mathscr{S}_i$ union some set of sequents of the form $\chi \Rightarrow \chi$;

iv) $\mathscr{S}_{i+1} \vdash_{\mathrm{HC}} S_{i+1} \iff \mathscr{S}_i \vdash_{\mathrm{HC}} S_i$.

Let $\mathscr{S}_0$ be $\mathscr{S}$, S_0 be S and let σ_0 be the identity substitution. Now assume that the triple $(\mathscr{S}_i, S_i, \sigma_i)$ has been defined. Then for each subformula of the form $\phi \to \psi$ with $d(\phi) = d(\psi) = 0$ occurring in some formula of some sequent of some hypersequent in $\mathscr{S}_i \cup \{S_i\}$ introduce a fresh variable $p_{\phi\psi}$ and replace $\phi \to \psi$ with $p_{\phi\psi}$ everywhere. Let $\mathscr{S}'_i$ and S_{i+1} be the result of such replacements. Finally, let

$$\mathscr{S}_{i+1} = \mathscr{S}'_i \cup \{p_{\phi\psi} \Rightarrow \phi \to \psi,\ \phi \to \psi \Rightarrow p_{\phi\psi}\}_{\phi \to \psi}.$$

The substitution σ_{i+1} is then defined as $\sigma_{i+1}(p_{\phi\psi}) = \phi \to \psi$.

With this definition i)-iii) are easily seen to hold. For item iv) it suffices to observe that the derivability relation is structural, i.e., preserved by substitutions.

Now if $\mathscr{S} \vdash_{\mathrm{HC}} S$ then by construction we must have that $\mathscr{S}_{n-1} \vdash_{\mathrm{HC}} S_{n-1}$. Moreover, as per item i) the degree of $\mathscr{S}_{n-1} \cup \{S_{n-1}\}$ is at most 1, hence the initial hypothesis yields $\mathscr{S}_{n-1} \vdash^1_{HC} S_{n-1}$. This suffices to establish the proposition as soon as we observed that if $\mathscr{S}_{n-k} \vdash^k_{\mathrm{HC}} S_{n-k}$ then $\mathscr{S}_{n-(k+1)} \vdash^{k+1}_{\mathrm{HC}} S_{n-(k+1)}$. However, this is an immediate consequence of items ii) and iii) together with the fact that for any hypersequent S we have that $d(S\sigma_{i+1}) \leq d(S) + 1$. □

We say that a formula *occurs* in a hypersequent S if it is either a subformula on the right or the left hand side of the sequent arrow of some sequent belonging to S. We say that a hypersequent rule (r) is *reduced* if all the formulas occurring in (r) have implicational degree at most 1.

Proposition 2.10. *Any hypersequent rule is equivalent to a reduced hypersequent rule.*

Proof. Given a hypersequent rule $(r) = (S_1, \dots, S_m)/S_{m+1}$ of depth $n+1$ with $n \geq 1$ and an *occurrence* of a formula α of degree $n+1$ in (r) the main connective of which is $\to$, we produce an equivalent rule with one less occurrence of the formula α.

Let S_i be the hypersequent with the given occurrence of α and let $\Gamma \Rightarrow \Delta$ be the sequent in S_i with the given occurrence of α. As the formula α is of depth $n+1$ it must be of the form $\phi \to \psi$ with $\max\{d(\phi), d(\psi)\} = n$. We introduce a fresh variable p and replace the given occurrence of α in S_i with $p \to \psi$ or $\phi \to p$, depending on whether $d(\phi) = n$ or $d(\psi) = n$. If both $d(\phi)$ and $d(\psi) = n$ we introduce two fresh variables. Let S'_i be the hypersequent resulting from such a replacement. Evidently S'_i has one less occurrence of the formula α than S_i . Moreover if $i \leq m$ let S''_i be the hypersequent obtained by replacing the sequent $\Gamma \Rightarrow \Delta$ in S_i with the sequent $p \Rightarrow \psi$ or $\phi \Rightarrow p$ depending on whether we replace $\phi \to \psi$ with $\phi \to p$ or with $p \to \psi$ in S_i. If $i = m+1$ let S''_i be the hypersequent consisting of the single component hypersequent $p \Rightarrow \psi$ or $\phi \Rightarrow p$ again depending on whether we replace $\phi \to \psi$ with $\phi \to p$ or with $p \to \psi$ in S_i.

In this way we obtain a rule

$$\frac{S_1 \ \dots \ S_{i-1} \ S'_i \ S_{i+1} \ \dots \ S_m \ S''_i}{S_{m+1}} \ (r')$$

By Proposition 2.8 this rule must be equivalent to the rule (r).

Continuing this procedure for each occurrence of a formula of degree $n+1$ in (r) we obtain a rule (r_1) of depth n which is equivalent to (r). In this way we obtain a sequence $(r_{n+1}), (r_n) \dots, (r_1)$ of equivalent rules such that $(r_{n+1}) = (r)$ and $d(r_k) = k$, for all $1 \leq k \leq n+1$. □

Note as the above procedure abstracts away one occurrence of a formula of the form $\phi \to \psi$ at a time, and since we first abstract away outermost occurrences, it is always clear whether to replace the formula occurring negatively or positively in the formula $\phi \to \psi$.

In light of Proposition 2.10 we may without loss of generality assume that all hypersequent calculi are reduced, i.e., only consisting of reduced rules.

3 One-step Heyting algebras

Let bDL denote the category of bounded distributive lattices and bounded lattice homomorphisms. Then a well-known theorem by Birkhoff says that the category bDL_ω of finite bounded distributive lattice is dually equivalent to the category Pos_ω of finite posets and order-preserving maps. This dually is established via the downsets functor $\mathrm{Do}\colon \mathsf{Pos} \to \mathsf{bDL}$ and the functor $J\colon \mathsf{bDL} \to \mathsf{Pos}$ mapping a bounded distributive lattice D to the poset of completely join-irreducible elements of D. If $f\colon P \to P'$

is an order-preserving map between posets then $\mathrm{Do}(f)\colon \mathrm{Do}(P') \to \mathrm{Do}(P)$ is the preimage function $f^*(U) := f^{-1}(U)$. If $h\colon D \to D'$ is a homomorphism between finite bounded distributive then h has a left adjoint $h^\flat\colon D' \to D$ given by

$$h^\flat(a') := \bigwedge_{a' \leq h(a)} a.$$

We may therefore let $J(h)\colon J(D') \to J(D)$ be $h^\flat \restriction J(D')$.

Recall that any finite bounded distributive lattice D is in fact a Heyting algebra with Heyting implication

$$a \to b := \bigwedge \{c\colon a \wedge c \leq b\}$$

Therefore the category HA_ω of finite Heyting algebras and Heyting algebra homomorphisms will be a (non-full) subcategory of bDL_ω. Let $\mathsf{Pos}^{\mathrm{open}}$ denote the category of posets and open order-preserving maps, where a map between posets $f\colon P \to Q$ is open if

$$\forall a \in P \; \forall b \in Q \; (b \leq f(a) \implies \exists a' \in P \; (a' \leq a \; \& \; f(a') = b)).$$

Theorem 3.1 (Folklore)**.** *The dual equivalence of the categories bDL_ω and Pos_ω restricts to a dual equivalence between the categoires HA_ω and $\mathsf{Pos}_\omega^{open}$.*

We now introduce algebraic structures which may interpret the fragment of intuitionisitic logic consisting of formulas of implicational degree at most 1.

Definition 3.2. A *one-step Heyting algebra* is a triple (D_0, D_1, i) such that $i\colon D_0 \to D_1$ is a homomorphism between bounded distributive lattices with the property that for all $a, b \in D_0$ the Heyting implication $i(a) \to i(b)$ exists in D_1. We say that a one-step Heyting algebra (D_0, D_1, i) is *conservative* if $i\colon D_0 \to D_1$ is an embedding of bounded distributive lattices and D_1 is generated (as a bounded distributive lattice) by the set $\{i(a) \to i(b)\colon a, b \in D_0\}$. Finally, we say that (D_0, D_1, i) is *finite* if both D_0 and D_1 are finite.

Definition 3.3. A *one-step homomorphism* between two one-step Heyting algebras $\mathcal{H} = (D_0, D_1, i)$ and $\mathcal{H}' = (D_0', D_1', i')$ is a pair (g_0, g_1) of bounded lattice homomorphisms $g_0\colon D_0 \to D_0'$ and $g_1\colon D_1 \to D_1'$ making the diagram

$$\begin{array}{ccc} D_0 & \xrightarrow{g_0} & D_0' \\ \big\downarrow i & & \big\downarrow i' \\ D_1 & \xrightarrow{g_1} & D_1' \end{array}$$

commute, and such that for all $a, b \in D_0$

$$g_1(i(a) \to i(b)) = g_1(i(a)) \to g_1(i(b)).$$

A *one-step extension* of a one-step Heyting algebra $\mathcal{H}_0 := (D_0, D_1, i_0)$ is a one-step Heyting algebra $\mathcal{H}_1 := (D_1, D_2, i_1)$ such that $(i_0, i_1)\colon \mathcal{H}_0 \to \mathcal{H}_1$ is a one-step homomorphism with i_1 injective.

Note that if $\mathfrak{A}$ is Heyting algebra, then $\mathcal{H}_{\mathfrak{A}} = (\mathfrak{A}, \mathfrak{A}, \mathrm{Id})$ is a one-step Heyting algebra. Consequently we may, given a one-step Heyting algebra $\mathcal{H}$, speak of one-step homomorphism between $\mathfrak{A}$ and $\mathcal{H}$ by way of $\mathcal{H}_{\mathfrak{A}}$.

The above definitions determines a category OSHA of one-step Heyting algebras and one-step homomorphisms between them. This is a non-full subcategory of the arrow category $\mathsf{bDist}^{\to}$. We let OSHA_ω and $\mathsf{OSHA}_\omega^{\mathrm{cons}}$ denote the full subcategories of OSHA consisting of finite one-step Heyting algebras and finite conservative one-step Heyting algebras, respectively.

3.1 Duality

Since in the following we are only concerned with finite one-step Heyting algebras the duality is particularly well-behaved as there will be no need to introduce topology. We construct categories dually equivalent to the categories OSHA_ω and $\mathsf{OSHA}_\omega^{\mathrm{cons}}$. To this end we need the following well-known proposition.

Proposition 3.4. *Let $f\colon P \to Q$ and $g\colon Q \to R$ be order-preserving maps between finite posets. Then the following are equivalent:*

1. *The bounded lattice homomorphism $f^*\colon \mathrm{Do}(Q) \to \mathrm{Do}(P)$ preserves all Heyting implications of the form $g^*(U) \to g^*(V)$, for $U, V \in \mathrm{Do}(R)$;*

2. $\forall a \in P\ \forall b \in Q\ (b \leq f(a) \implies \exists a' \in P\ (a' \leq a\ \&\ g(f(a')) = g(b)))$

Proof. Straightforward. □

Definition 3.5 ([24])**.** Given order-preserving maps $f\colon P \to Q$ and $g\colon Q \to R$ satisfying one (and therefore both) of the conditions of Proposition 3.4 we say that f is *open relative to g* or simply that f is *g-open.*

Definition 3.6. An *intuitionistic one-step frame* is a triple (P_1, P_0, f) such that $f\colon P_1 \to P_0$ is an order-preserving map between posets. We say that an intuitionistic one-step frame $(P_1, P_0.f)$ is *conservative* if $f\colon P_1 \to P_0$ is a surjection satisfying

$$\forall a, b \in P_1\ (f[{\downarrow}a] \subseteq f[{\downarrow}b] \implies a \leq b).$$

Definition 3.7. A one-step map from an intuitionistic one-step frame $\mathcal{F}' = (P_1', P_0', f')$ to an intuitionistic one-step frame $\mathcal{F} = (P_1, P_0, f)$ is a pair (μ_1, μ_0) of order-preserving maps $\mu_1\colon P_1' \to P_1$ and $\mu_0\colon P_0' \to P_0$, with μ_1 is f-open, making the diagram

$$\begin{array}{ccc} P_1' & \xrightarrow{\mu_1} & P_1 \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ P_0' & \xrightarrow{\mu_0} & P_0 \end{array}$$

commute.

A *one-step extension* of an intuitionistic one-step frame $\mathcal{F}_0 = (P_1, P_0, f_0)$ is an intuitionistic one-step frame $\mathcal{F}_1 = (P_2, P_1, f_1)$ such that $(f_1, f_0)\colon \mathcal{F}_1 \to \mathcal{F}_0$ is a one-step map, with f_1 surjective.

It is easy to check that this yields a category IOSFrm of intuitionistic one-step frames and one-step maps. Moreover, the finite and the finite conservative intuitionistic one-step algebras form full subcategories IOSFrm_ω and $\mathsf{IOSFrm}_\omega^{\mathrm{cons}}$ of IOSFrm.

Note that if $\mathfrak{F}$ is an intuitionistic Kripke frame then $\mathcal{F}_{\mathfrak{F}} = (\mathfrak{F}, \mathfrak{F}, \mathrm{Id})$ will be an intuitionistic one-step frame. Consequently we may, given an intuitionistic one-step frame $\mathcal{F}$, speak of one-step homomorphism between $\mathfrak{F}$ and $\mathcal{F}$ by way of $\mathcal{F}_{\mathfrak{F}}$.

Proposition 3.8. *The categories OSHA_ω and IOSFrm_ω are dually equivalent. Moreover, this dual equivalence restricts to a dual equivalence between the categories $\mathsf{OSHA}_\omega^{cons}$ and $\mathsf{IOSFrm}_\omega^{cons}$.*

Proof. That the duality between bDL_ω and Pos_ω extends to a duality between the categories OSHA_ω and IOSFrm_ω is straightforward given Proposition 3.4.

To see that the dual equivalence between OSHA_ω and IOSFrm_ω restricts to a dual equivalence between $\mathsf{OSHA}_\omega^{\mathrm{cons}}$ and $\mathsf{IOSFrm}_\omega^{\mathrm{cons}}$ it suffices to note that under the isomorphism between the poset of bounded sublattices of $\mathrm{Do}(P)$ and the poset of compatible quasi-orders on P ([35, Thm. 3.7], [5, Thm. 6.15]) the sublattice generated by the set $\mathscr{U} \subseteq \mathrm{Do}(P)$ corresponds to the compatible quasi-order $\preceq_{\mathscr{U}}$ given by

$$a \preceq_{\mathscr{U}} b \quad \text{iff} \quad \forall U \in \mathscr{U}\ (b \in U \implies a \in U).$$

Thus $\mathscr{U} \subseteq \mathrm{Do}(P)$ generates $\mathrm{Do}(P)$ as a bounded distributive lattice iff the quasi-order $a \preceq_{\mathscr{U}} b$ coincides with the order on P.

From this it is easy to see that (P_1, P_0, f) is a conservative intuitionistic one-step frame if and only if $(\mathrm{Do}(P_0), \mathrm{Do}(P_1), f^*)$ is a conservative one-step Heyting algebra. □

3.2 One-step semantics

Given two disjoint finite sets Prop_0 and Prop_1 of propositional variables, a *valuation* on a one-step algebra $\mathcal{H} = (D_0, D_1, i)$ is a pair of functions $v = (v_0, v_1)$ such that $v_0 \colon \mathsf{Prop}_0 \to D_0$ and $v_1 \colon \mathsf{Prop}_1 \to D_1$.

Given a one-step algebra $\mathcal{H}$ together with a valuation $v = (v_0, v_1)$ for every formula $\phi(\vec{p}) \in Form_0(\mathsf{Prop}_0)$ we define an element $\phi^{\mathrm{v}_0} \in D_0$ as follows:

$$\bot^{\mathrm{v}_0} = \bot \quad \text{and} \quad \top^{\mathrm{v}_0} = \top \quad \text{and} \quad p_i^{\mathrm{v}_0} = v_0(p_i) \quad \text{for } p_i \in \vec{p},$$

and

$$(\phi_1 * \phi_2)^{\mathrm{v}_0} = \phi_1^{\mathrm{v}_0} * \phi_2^{\mathrm{v}_0}, \qquad * \in \{\wedge, \vee\}.$$

Moreover, for every formula $\psi(\vec{p}, \vec{q}) \in Form_1(\mathsf{Prop}_0 \cup \mathsf{Prop}_1)$, where the elements of $\vec{q} \subseteq \mathsf{Prop}_1$ do not have any occurrence in the scope of an implication, we define an element $\psi^{\mathrm{v}_1} \in D_1$ as follows:

$$\bot^{\mathrm{v}_1} = \bot \quad \text{and} \quad q^{\mathrm{v}_1} = v_1(q) \quad \text{and} \quad p^{\mathrm{v}_1} = i(v_0(p)) \quad \text{for } q \in \vec{q} \text{ and } p \in \vec{p},$$

and

$$(\psi_1 * \psi_2)^{\mathrm{v}_1} = \psi_1^{\mathrm{v}_1} * \psi_2^{\mathrm{v}_1} \qquad * \in \{\wedge, \vee\}.$$

Finally, for $\phi_1, \phi_2 \in Form_0(\mathsf{Prop}_0)$ we let,

$$(\phi_1 \to \phi_2)^{\mathrm{v}_1} = i(\phi_1^{\mathrm{v}_0}) \to i(\phi_2^{\mathrm{v}_0}).$$

By the definition of a one-step Heyting algebra the implications of the form $i(a) \to i(a)$ exist in D_1 and so the above is indeed well-defined.

Since, the function i preserves $\bot$ as well as the connectives $\wedge$ and $\vee$ it is easily seen that $i(\phi^{\mathrm{v}_0}) = \phi^{\mathrm{v}_1}$, for all $\phi \in Form_0(\mathsf{Prop}_0)$.

A valuation $v = (v_0, v_1)$ on a one-step algebra $\mathcal{H}$ is *suitable* for an expression (i.e. for a formula, sequent, or hypersequent) α of degree at most 1 iff the domain of v_0 includes all propositional variables having in α an occurrence located inside an implication; a *0-valuation* is an valuation $v = (v_0, v_1)$ where the domain of v_1 is empty (thus, a 0-evaluation is always suitable for every expression α).

We say that a one-step algebra $\mathcal{H}$ *validates* a sequent $\Gamma \Rightarrow \Delta$ of degree at most 1 *under a suitable valuation* $v = (v_0, v_1)$ if

$$\left(\bigwedge \Gamma\right)^{\mathrm{v}_1} \leq \left(\bigvee \Delta\right)^{\mathrm{v}_1},$$

with the convention that $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$.

A one-step algebra $\mathcal{H}$ validates a hypersequent $S = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ under a suitable valuation v if it validates at least one of the sequents $\Gamma_k \Rightarrow \Delta_k$ under v. We write $(\mathcal{H}, v) \models S$, if this is the case.

Finally, we say that a one-step algebra $\mathcal{H}$ validates a hypersequent S if it validates it under all possible suitable valuations v on $\mathcal{H}$, in which case we write $\mathcal{H} \models S$. Moreover, if $(r) = (S_1, \ldots, S_n)/S$ is a hypersequent rule of degree at most 1 we say that $\mathcal{H}$ validates (r) if for all valuations v on $\mathcal{H}$ we have that if $(\mathcal{H}, v) \models S_i$ for all $i \in \{1, \ldots, n\}$ then $(\mathcal{H}, v) \models S$.

We say that an intuitionistic one-step frame $\mathcal{F} = (P_1, P_0, f)$ validates a sequent, hypersequent or hypersequent rule if its dual one-step Heyting algebra $\mathcal{F}^* = (\mathrm{Do}(P_0), \mathrm{Do}(P_1), f^*)$ does. The notion of *0-validation* of a sequent, hypersequent or hypersequent rule is defined in the same way, by restricting to 0-valuations.

With these definitions we can then establish the soundness of the derivability relation with respect to the one-step semantics. There is a subtlety to take care of here, however: a propositional variable p not occurring in a set of hypersequents $\mathscr{S} \cup \{S\}$ under the scope of an implication may still occur inside an implication in a derivation witnessing $\mathscr{S} \vdash^1_{\mathrm{HC}} S$. Thus if p is in the domain of v_1 when we evaluate S in $\mathcal{H}$, it may happen that we cannot give a meaning to such derivation inside $\mathcal{H}$. This is why the correct semantics for the relation $\mathscr{S} \vdash^1_{\mathrm{HC}} S$ requires the restriction to 0-valuations for $\mathscr{S} \cup \{S\}$ (but not for the rules of HC, because the variables from the latter can be instantiated indifferently with formulas of degree 0 or 1).

Proposition 3.9. *Let $\mathcal{H}$ be a one-step algebra,* HC *a reduced hypersequent calculus, and $\mathscr{S} \cup \{S\}$ a set of hypersequents of degree at most* 1*. If $\mathscr{S} \vdash^1_{\mathrm{HC}} S$ and $\mathcal{H}$ validates HC, then $\mathcal{H}$ 0-validates $\mathscr{S}/S$.*

Proof. By induction on the length of a derivation witnessing $\mathscr{S} \vdash^1_{\mathrm{HC}} S$ (notice that we can assume that in such derivation only propositional variables occurring in $\mathscr{S}/S$ occur, because extra variables can be replaced by, say, $\top$). □

If $(g_0, g_1)\colon \mathcal{H} \to \mathcal{H}'$ is a one-step homomorphism such that both g_0 and g_1 are injective then we say that (g_0, g_1) is an embedding. The following lemma shows that the embeddings between one-step Heyting algebras preserve validity.

Lemma 3.10. *Let $(g_0, g_1)\colon \mathcal{H} \to \mathcal{H}'$ be an embedding of one-step algebras. If $v = (v_0, v_1)$ and $v' = (v'_0, v'_1)$ are valuations on $\mathcal{H}$ and $\mathcal{H}'$, respectively, such that $v'_0(p) = g_0(v_0(p))$ and $v'_1(q) = g_1(v_1(p))$, for all $p \in \mathsf{Prop}_0$ and $q \in \mathsf{Prop}_1$, then for any hypersequent rule (r) of degree at most 1 we have that $(\mathcal{H}, v)$ validates (r) iff $(\mathcal{H}', v')$ validates (r).*

Proof. It suffices to show that for all formulas $\phi, \psi \in Form_1(\mathsf{Prop}_0 \cup \mathsf{Prop}_1)$

$$\phi^{\mathrm{v}_1} \leq \psi^{\mathrm{v}_1} \iff \phi^{\mathrm{v}'_1} \leq \psi^{\mathrm{v}'_1} \tag{3}$$

Since (g_0, g_1) is a map of one-step algebras an easy inductive argument shows that the assumption $v'_0(p) = g(v_0(p))$ and $v'_1(q) = g_1(v_1(p))$ for all $p \in \mathsf{Prop}_0, q \in \mathsf{Prop}_1$ implies that $\phi^{\mathrm{v}'_1} = g_1(\phi^{\mathrm{v}_1})$ for all $\phi \in Form_1(\mathsf{Prop}_0 \cup \mathsf{Prop}_1)$. From this (3) readily follows as any injective lattice homomorphism will necessarily be both order-preserving and order-reflecting. □

In particular, we have that if $\mathcal{H}'$ is a one-step Heyting algebra validating HC and $\mathcal{H}$ embeds into $\mathcal{H}'$ then $\mathcal{H}$ validates HC as well.

We wish to establish an algebraic completeness result of $\vdash^1$ with respect to one-step Heyting algebras.

Proposition 3.11. *Let $\mathscr{S} \cup \{S\}$ be a finite set of hypersequents of implicational degree at most 1, and let* HC *be a (reduced) hypersequent calculus. If all one-step Heyting algebras validating* HC *0-validate the hypersequent rule $\mathscr{S}/S$ then $\mathscr{S} \vdash^1_{\mathrm{HC}} S$.*

Proof. Let $\mathscr{S} \cup \{S\}$ be a finite set of hypersequents of degree at most 1 such that $\mathscr{S} \nvdash^1_{\mathrm{HC}} S$. We then construct a one-step Heyting algebra algebra $\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ validating HC. Moreover, under some 0-valuation $\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ will validate $\mathscr{S}$ and refute S. This is completely similar to the construction found in the proof of Proposition 2.6. As before we let Prop be the set of propositional letters occurring in $\mathscr{S} \cup \{S\}$ and let $\widetilde{\mathscr{S}}$ be a maximal set of hypersequents, based on $Form(\mathsf{Prop})$, containing $\mathscr{S}$ such that $\widetilde{\mathscr{S}} \nvdash^1_{\mathrm{HC}} S$. We then have that if $s_1, \ldots, s_n$ are sequents of degree at most 1

$$\widetilde{\mathscr{S}} \vdash^1_{\mathrm{HC}} s_1 \mid \ldots \mid s_n \mid S \implies \widetilde{\mathscr{S}} \vdash^1_{\mathrm{HC}} s_i \text{ for some } i \leq n. \tag{4}$$

Letting D_k be the set of equivalence classes of formulas of degree at most k, for $k = 0, 1$, of the equivalence relation

$$\phi \approx \psi \text{ iff } \Rightarrow \phi \leftrightarrow \psi \in \widetilde{\mathscr{S}},$$

we obtain a (finite conservative) one-step Heyting algebra $\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S) := (D_0, D_1, i)$ where $i\colon D_0 \to D_1$ is the evident inclusion. From (4) we see that $\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ validates HC and moreover that under the valuation v on $\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ determined by sending propositional variables to the corresponding equivalence classes in D_0,[5] we have that $(\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S), v) \models \mathscr{S}$ but $(\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S), v) \not\models S$. □

Note that since there is only finitely many formulas of degree at most 1 when Prop is finite, the one-step algebra $\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ obtain in the proof of Proposition 3.11 is in fact a finite conservative one-step Heyting algebra.

[5]Notice that this is a 0-valuation.

4 Characterizing the bounded proof property

Given a finite conservative one-step Heyting algebra $\mathcal{H} = (D_0, D_1, i)$ we will define the *diagram* associated to $\mathcal{H}$. This construction is analogous to the diagrams of a finite conservative one-step modal algebra from [10]. In fact they are a two-sorted version of the diagrams know from model theory [15].

We introduce a set of propositional variables $\mathsf{Prop}_0^{\mathcal{H}} = \{p_a : a \in D_0\}$. Then by the conservativity of $\mathcal{H}$ it follows that for each $a \in D_1$ there exists a formula $\theta_a \in Form_1(\mathsf{P}_{\mathcal{H}})$ such that $\theta_b^{\mathsf{v}_1} = b$, where v is the *natural* 0-valuation on $\mathcal{H}$ given by $v_0(p_a) = a$. In particular, we have that $\theta_{i(a)} = p_a$ for all $a \in D_0$.

Now let

$$\begin{aligned}\mathscr{S}^0_{\mathcal{H}} \coloneqq &\{p_{a\wedge b} \Rightarrow p_a \wedge p_b,\ p_a \wedge p_b \Rightarrow p_{a\wedge b} : a, b \in D_0\}\\ &\cup \{p_{a\vee b} \Rightarrow p_a \vee p_b,\ p_a \vee p_b \Rightarrow p_{a\vee b} : a, b \in D_0\}\\ &\cup \{p_\bot \Rightarrow \bot,\ \bot \Rightarrow p_\bot\}, \cup\{p_\top \Rightarrow \top,\ \top \Rightarrow p_\top\},\end{aligned}$$

and

$$\begin{aligned}\mathscr{S}^1_{\mathcal{H}} \coloneqq &\{\theta_{a\wedge b} \Rightarrow \theta_a \wedge \theta_b,\ \theta_a \wedge \theta_b \Rightarrow \theta_{a\wedge b} : a, b \in D_1\}\\ &\cup \{\theta_{a\vee b} \Rightarrow \theta_a \vee \theta_b,\ \theta_a \vee \theta_b \Rightarrow \theta_{a\vee b} : a, b \in D_1\}\\ &\cup \{\theta_{i(a)\to i(b)} \Rightarrow \theta_{i(a)} \to \theta_{i(b)},\ \theta_{i(a)} \to \theta_{i(b)} \Rightarrow \theta_{i(a)\to i(b)} : a, b \in D_0\}\end{aligned}$$

We then define the *positive diagram of* $\mathcal{H}$ to be $\mathscr{S}_{\mathcal{H}} \coloneqq \mathscr{S}^0_{\mathcal{H}} \cup \mathscr{S}^1_{\mathcal{H}}$.

For each $a, b \in D_1$ we let s_{ab} be the sequent $\theta_a \Rightarrow \theta_b$ if $a \not\leq b$ and the empty sequent if $a \leq b$. We then define the *negative diagram of* $\mathcal{H}$ to be the hypersequent

$$S_{\mathcal{H}} \coloneqq \{s_{ab} : a, b \in D_1\}.$$

Definition 4.1. By the *diagram* of a finite conservative one-step Heyting algebra we will understand the hypersequent rule $\mathscr{S}_{\mathcal{H}}/S_{\mathcal{H}}$.

We say that a step algebra $\mathcal{H}'$ *refutes* a diagram $\mathscr{S}_{\mathcal{H}}/S_{\mathcal{H}}$ under a 0-valuation v if $(\mathcal{H}', v) \models \mathscr{S}_{\mathcal{H}}$ but $(\mathcal{H}', v) \not\models S_{\mathcal{H}}$.

The following proposition shows why we are interested in diagrams.

Proposition 4.2. *Let $\mathcal{H} = (D_0, D_1, i)$ and $\mathcal{H}' = (D'_0, D'_1, i')$ be one-step Heyting algebras with $\mathcal{H}$ finite and conservative. Then the following are equivalent:*

1. *There exists a one-step embedding from $\mathcal{H}$ into $\mathcal{H}'$;*
2. *There exists a 0-valuation v on $\mathcal{H}'$ such that $(\mathcal{H}', v')$ refutes the diagram of $\mathcal{H}$.*

Proof. First assume that there exists a one-step embedding $(g_0, g_1) : \mathcal{H} \to \mathcal{H}'$. We then define a 0-valuation $v' = (v'_0, v'_1)$ on $\mathcal{H}'$ by $v_0(p_a) = g_0(a)$. Then as $\mathcal{H}$ evidently refutes its own diagram under the natural valuation $v_0(p_a) = a$ it immediately follows from Lemma 3.10 that $(\mathcal{H}', v')$ refutes $\mathscr{S}_{\mathcal{H}}/S_{\mathcal{H}}$ as well.

Conversely if there exists a 0-valuation $v' = (v'_0, v'_1)$ on $\mathcal{H}'$ such that $(\mathcal{H}', v')$ refutes the diagram of $\mathcal{H}$, then we claim that defining $(g_0, g_1) : \mathcal{H} \to \mathcal{H}'$ by

$$g_0(a) = v'_0(p_a) \quad \text{and} \quad g_1(b) = \theta_b^{\mathsf{v}'_1},$$

yields an embedding of one-step algebras.

First of all since $\mathcal{H}$ is conservative the function g_1 is well-defined, and because i' is an injection and $(\mathcal{H}', v') \models \mathscr{S}^0_{\mathcal{H}}$ we see that g_0 must be a bounded lattice homomorphism. Since $(\mathcal{H}', v')$ also validates $\mathscr{S}^1_{\mathcal{H}}$ we see that g_1 is a bounded lattice homomorphism as well.

Now to see that $i' \circ g_0 = g_1 \circ i$ we simply observe that for all $a \in D_0$

$$i(g_0(a)) = i(v'_0(p_a)) = p_a^{\mathsf{v}'_1} = \theta_{i(a)}^{\mathsf{v}'_1} = g_1(i(a)).$$

From the assumption that $(\mathcal{H}', v')$ does not validate any of the sequents $\theta_a \Rightarrow \theta_b$ when $a \not\leq b$ it immediately follows that g_1 is an injection. So as i is an injection we must have that g_0, being the first component of the injection $g_1 \circ i$, is an injection as well.

Finally because $(\mathcal{H}', v')$ validates all sequents of the form $\theta_{i(a)\to i(b)} \Rightarrow \theta_{i(a)} \to \theta_{i(b)}$ and $\theta_{i(a)} \to \theta_{i(b)} \Rightarrow \theta_{i(a)\to i(b)}$ we have that

$$g_1(i(a) \to i(a)) = i'(g_0(a)) \to i'(g_0(b)),$$

and so we can conclude that (g_0, g_1) is indeed an embedding of one-step algebras. □

Definition 4.3. A class K of one-step Heyting algebras (or one-step intuitionistic one-step frames) has the extension property if all members of K have a one-step extensions also belonging to K.

Lemma 4.4. *Let* HC *be a hypersequent calculus and let* $Con_\omega^{Alg}(\mathrm{HC})$ *be the class of finite conservative one-step Heyting algebras validating* HC*. If every* $\mathcal{H} \in Con_\omega^{Alg}(\mathrm{HC})$ *embeds into some standard Heyting algebra validating* HC *then the class* $Con_\omega^{Alg}(\mathrm{HC})$ *has the extension property.*

Proof. Let $\mathcal{H} = (D_0, D_1, i)$ be a finite (conservative) one-step Heyting algebra and suppose that there exists an embedding $(g_0, g_1)\colon \mathcal{H} \to \mathfrak{A}$ into some Heyting algebra $\mathfrak{A}$ validating HC. Letting A be the bounded lattice reduct of $\mathfrak{A}$, we see that $\mathcal{H}'' = (D_1, A, g_1)$ is a one-step algebra validating HC and extending $\mathcal{H}$.

To obtain a finite conservative one-step Heyting algebra validating HC and extending $\mathcal{H}$ let D_2 be the bounded distributive sublattice of A generated by the set $\{g_1(a) \to g_1(b)\colon a, b \in D_1\}$. As the variety of bounded distributive lattices is locally finite D_2 is finite. Moreover, we have $g_1[D_1] \subseteq D_2$. Therefore, $\mathcal{H}' = (D_1, D_2, g_1)$ will be a finite conservative one-step algebra validating HC and extending $\mathcal{H}$. □

Theorem 4.5. *Let* HC *be a (reduced) hypersequent calculus. Then the following are equivalent:*

1. *The calculus* HC *has the bounded proof property;*
2. *The class of finite conservative one-step Heyting algebras validating* HC *has the extension property;*
3. *The class of finite conservative intuitionistic one-step frames validating* HC *has the extension property.*

Proof. That item 2 and 3 are equivalent is an immediate consequence of the dual equivalence between the categories $\mathsf{OSHA}_\omega^{\mathrm{cons}}$ and $\mathsf{IOSFrm}_\omega^{\mathrm{cons}}$.

To see that item 1 implies item 2 let $\mathcal{H}$ be a finite conservative one-step Heyting algebra validating HC. Since $\mathcal{H}$ refutes its own diagram $\mathscr{S}_\mathcal{H}/S_\mathcal{H}$ we obtain from Proposition 3.9 that $\mathscr{S}_\mathcal{H} \not\vdash^1_{\mathrm{HC}} S_\mathcal{H}$. Therefore, if HC enjoys the bounded proof property it follows that $\mathscr{S}_\mathcal{H} \not\vdash_{\mathrm{HC}} S_\mathcal{H}$. By algebraic completeness we must have a Heyting algebra $\mathfrak{A}$ validating HC and refuting $\mathscr{S}_\mathcal{H}/S_\mathcal{H}$. But then by Proposition 4.2 there exists embedding $(g_0, g_1)\colon \mathcal{H} \to \mathfrak{A}$ and so by Lemma 4.4 we may conclude that the class of finite conservative Heyting algebras validating HC has the extension property.

Finally, to see that item 2 implies item 1 let $\mathscr{S} \cup \{S\}$ be a finite set of hypersequents of implicational degree at most 1 such that $\mathscr{S} \not\vdash^1_{\mathrm{HC}} S$. By Proposition 2.9 it then suffices to show that $\mathscr{S} \not\vdash_{\mathrm{HC}} S$.

Let $\mathcal{H}_0 = (D_0, D_1, i_0)$ be the finite conservative one-step Heyting algebra $\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ constructed in the proof of Proposition 3.11. Moreover, let v^0 be a 0-valuation on $\mathcal{H}_0$ such that $(\mathcal{H}_0, v^0) \models \mathscr{S}$ but $(\mathcal{H}_0, v^0) \not\models S$. If the class of finite conservative one-step algebras validating HC has the extension property then we have a one-step extension in form of a finite conservative one-step Heyting algebra $\mathcal{H}_1 = (D_1, D_2, i_1)$ validating HC. Moreover, i_0, i_1 induce a 0-valuation v^1 on $\mathcal{H}_1$ under which $\mathscr{S}$ is valid but S it not. In this way we obtain a chain

$$D_0 \xhookrightarrow{i_0} D_1 \xhookrightarrow{i_1} \dots \xhookrightarrow{i_{n-1}} D_{n-1} \xhookrightarrow{i_n} D_n \xhookrightarrow{i_{n+1}} \dots$$

of Heyting algebras in the category bDL_ω, with the property that

$$i_{n+1}(i_n(a) \to_{n+1} i_n(b)) = i_{n+1}(i_n(a)) \to_{n+2} i_{n+1}(i_n(b)).$$

Consequently taking the colimit of the above diagram, in the category bDL_ω, we obtain a Heyting algebra $\mathfrak{A}$ with Heyting implication

$$[a] \to [b] := [i_{n,k+1}(a) \to_{k+2} i_{m,k+1}(b)], \quad k = \max\{n, m\},$$

for $a \in D_n$ and $b \in D_m$ and $i_{n,k}\colon D_n \to D_k$ the evident map for $n \leq k$.

It is then easy to see that $\mathfrak{A}$ must validate HC and moreover that the 0-valuations v^n on $\mathcal{H}_n$ induces a valuation v on $\mathfrak{A}$ which by the injectivity of the i_n's is such that $(\mathfrak{A}, v) \models \mathscr{S}$ and $(\mathfrak{A}, v) \not\models S$. We may therefore conclude that $\mathscr{S} \not\vdash_{\mathrm{HC}} S$. ☐

In concrete cases it is not so easy to work with one-step extensions of frames. However, assuming the finite model property we obtain a version of Theorem 4.5 which avoids the concept of one-step extensions altogether.

Definition 4.6. We say that a hypersequent calculus HC has the *(global) finite model property* if for each set $\mathscr{S} \cup \{S\}$ of hypersequents, $\mathscr{S} \not\vdash_{\mathrm{HC}} S$ iff there exists a finite Heyting algebra $\mathfrak{A}$ validating HC and a valuation v on $\mathfrak{A}$ such that $(\mathfrak{A}, v)$ validates all the hypersequents from $\mathscr{S}$ but not the hypersequent S.

Proposition 4.7. *A hypersequent calculus* HC *has the finite model property iff for each set* $\mathscr{S} \cup \{S\}$ *of hypersequents,* $\mathscr{S} \not\vdash_{\mathrm{HC}} S$ *iff there exists a finite intuitionistic Kripke frame* $\mathfrak{F}$ *validating* HC *and a valuation* v *on* $\mathfrak{F}$ *such that* $(\mathfrak{F}, v)$ *validates all the hypersequents from* $\mathscr{S}$ *but not the hypersequent* S*.*

Proof. Immediate by the duality between finite Heyting algebras and finite intuitionistic Kripke frames. ☐

Lemma 4.8. *Let* HC *be a hypersequent calculus. Then* HC *has the finite model property iff if for each set* $\mathscr{S} \cup \{S\}$ *of hypersequents of degree at most* 1*,* $\mathscr{S} \not\vdash_{\mathrm{HC}} S$ *iff there exists a finite Heyting algebra* $\mathfrak{A}$ *together with a valuation* v *such* $(\mathfrak{A}, v)$ *validates* HC *and all the hypersequents from* $\mathscr{S}$ *but not the hypersequent* S*.*

Proof. The statement follows from the fact that given $\mathscr{S}, S$ it is possible to produce $\mathscr{S}', S'$ having degree at most 1, such that for every Heyting algebra $\mathfrak{A}$ validating HC (finite or not) we have that $\mathfrak{A}$ validates $\mathscr{S}/S$ iff it validates $\mathscr{S}'/S'$ (thus, in particular, $\mathscr{S} \vdash_{\mathrm{HC}} S$ iff $\mathscr{S}' \vdash_{\mathrm{HC}} S'$ by Proposition 2.8). In order to build $\mathscr{S}', S'$ out of $\mathscr{S}, S$, we just need to abstract out implicative subformulas with fresh propositional variables (we have already applied this procedure e.g. in the proof of Propositions 2.9 and 2.10). ☐

Theorem 4.9. *Let* HC *be a (reduced) hypersequent calculus. Then the following are equivalent:*

1. *The calculus* HC *has the bounded proof property and the finite model property;*
2. *Each finite conservative one-step algebra* $\mathcal{H}$ *validating* HC *embeds into some finite Heyting algebra validating* HC*;*
3. *Each finite conservative intuitionistic one-step frame* $\mathcal{F}$ *validating* HC *is the relative open image of some finite intuitionistic Kripke frame validating* HC*.*

Proof. As in the proof of Theorem 4.5 it is immediate that item 2 and item 3 are equivalent.

To see that item 1 implies item 2 we observe that if $\mathcal{H}$ is a finite conservative one-step Heyting algebra validating HC then as $\mathcal{H}$ refutes its diagram $\mathscr{S}_{\mathcal{H}}/S_{\mathcal{H}}$ we must have that $\mathscr{S}_{\mathcal{H}} \not\vdash^1_{\mathrm{HC}} S_{\mathcal{H}}$ by Proposition 3.9. Consequently it follows from the assumption that HC has the bounded proof property that $\mathscr{S}_{\mathcal{H}} \not\vdash_{\mathrm{HC}} S_{\mathcal{H}}$ and therefore as HC has the finite model property we obtain a finite Heyting algebra $\mathfrak{A}$ which validates HC and refutes the diagram $\mathscr{S}_{\mathcal{H}}/S_{\mathcal{H}}$. By Proposition 4.2 it then follows that $\mathcal{H}$ embeds into $\mathfrak{A}$.

Conversely to see that item 2 implies item 1 we first note that by Theorem 4.5 item 2 implies that HC enjoys the bounded proof property. To see that it also enjoys the finite model property it suffices, by Lemma 4.8, to consider finite set of hypersequents $\mathscr{S} \cup \{S\}$ of degree at most 1. Given such a set $\mathscr{S} \cup \{S\}$ with the property that $\mathscr{S} \not\vdash_{\mathrm{HC}} S$ let $\mathcal{H}$ be the finite conservative one-step algebra $\mathcal{LT}_{\mathrm{HC}}(\mathscr{S}, S)$ as constructed in the proof of Proposition 3.11. Then by assumption we have a finite Heyting algebra $\mathfrak{A}$ validating HC such that $\mathcal{H}$ embeds into $\mathfrak{A}$, and this embedding induces a valuation on $\mathfrak{A}$ under which $\mathscr{S}$ is valid but S is not. ☐

5 Examples

In this section we provide a number of examples showing how to use the methods developed above to determine whether or not a given sequent or hypersequent calculus enjoys the bounded proof property.

We warn the reader that as we base the duality between one-step Heyting algebras and intuitionistic one-step frames on the downset functor Do: $\mathsf{Pos}_\omega^{open} \to \mathsf{HA}_\omega$ the partial order on Kripke frames may be the opposite of what the reader is familiar with.

It is possible to adapt the algorithmic correspondence theory for intuitionistic logic (see e.g., [21]) to the framework of one-step semantics for hypersequent rules. However, as the examples we will be considering here are rather simple we will derive the correspondence results we need manually.

Finally, we would like to mention the following result[6] due to Ciabattoni, Galatos and Terui:

Theorem 5.1 ([18])**.** *There is an effective procedure which given an axiom ϕ belonging to the level $\mathcal{P}_3$ of the substructural hierarchy produces a finite set of structural hypersequent rules $\mathcal{R}_\phi$ such that when added to the hypersequent version of* **LJ** *yields a hypersequent calculus for* $\mathbf{IPC} + \phi$ *enjoying cut-elimination and the subformula property.*

The hypersequent calculi obtained by this procedure evidently have the bounded proof property. Thus in order to obtain truly novel results of a positive nature using Theorem 4.5 and 4.9 it will be necessary to consider axioms at the level $\mathcal{N}_3$ of the substructrual hierarchy [18]. Since all intermediate logics are axiomatizable by canonical formulas [38] which belong to the level $\mathcal{N}_3$ over **IPC** the substructural hierarchy collapses at this level[7].

5.1 Calculi for LC

The intermediate logic **LC**, known as the Gödel-Dummett logic is obtained by adding the axiom $(p \to q) \vee (q \to p)$ to a Hilbert-style presentation of **IPC**. Using our methods we show that the sequent calculus obtained by adding the rule

$$\frac{}{\Rightarrow (\phi \to \psi) \vee (\psi \to \phi)}\ (r_{LC})$$

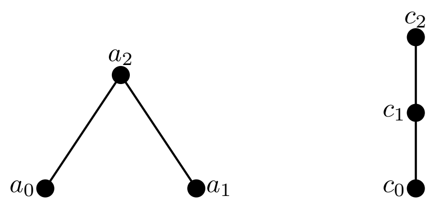
does not enjoy the bounded proof property.

Proposition 5.2. *A intuitionistic one-step frame (P_1, P_0, f) validates the rule (r_{LC}) iff*

$$\forall a, b, b' \in P_1\ (b \leq a\ \&\ b' \leq a \implies (f(b) \leq f(b')\ \text{or}\ f(b') \leq f(b))).$$

Proof. Straightforward. ☐

To see that adding the rule (r_{LC}) does not yield a sequent calculus with the bounded proof property, consider the one-step frame $\mathcal{F} = (P_1, P_0, f)$ presented as:



That is, P_1 is a 2-fork and P_0 is a 3-chain. The function f is the obvious map given by $a_i \mapsto c_i$ for $i \in \{0, 1, 2\}$. This is easily seen to be a finite conservative one-step frame validating the rule (r_{LC}). Now suppose towards a contradiction that $\mathcal{F}$ has a one-step extension, say $\mathcal{F}' = (P_2, P_1, g)$. As f is bijective it follows from the assumption that g is f-open that g must be an open map. Therefore, we must have $z_0, z_1, z_2 \in P_2$ with $z_0, z_1 \leq z_2$, such $g(z_i) = a_i$ for $i \in \{0, 1, 2\}$. But this shows that $\mathcal{F}'$ fails to validate the rule (r_{LC}) and consequently that $\mathcal{F}$ does not have any one-step extension validating (r_{LC}).

Thus, by Theorem 4.5, we see that the hypersequent calculus obtained by adding the rule (r_{LC}) does not have the bounded proof property.

However, we know from [3] that adding the so-called communication rule

[6] In fact it holds more generally for Full Lambek calculus with exchange $\mathbf{FL}_e$.

[7] Rather surprisingly this is also the case for $\mathbf{FL}_e$ as recently established in [30].

$$\frac{G \mid \Gamma_1, \Gamma_1' \Rightarrow \Pi \qquad G \mid \Gamma_2, \Gamma_2' \Rightarrow \Pi'}{G \mid \Gamma_1, \Gamma_2 \Rightarrow \Pi \mid \Gamma_1', \Gamma_2' \Rightarrow \Pi'}\ (com)$$

to the hypersequent version of **LJ** yields a hypersequent calculus for the logic **LC** which preserves cut-eliminability.

Since this rule is structural we see that whether or not an intuitionistic one-step frame (P_1, P_0, f) validates the rule (com) only depends on P_1. Consequently, it follows from Theorem 4.9 that the rule (com) enjoys the bounded proof property and the finite model property.

In fact, it is easy to see that an intuitionistic one-step frame (P_1, P_0, f) validates the rule (com) iff P_1 is a linear order.

5.2 Calculi for KC

Recall that the logic **KC** is obtained by adding the axiom $\neg p \vee \neg\neg p$ to **IPC**. It is well-known that this is the logic of (finite) directed frames.

Now consider the rule

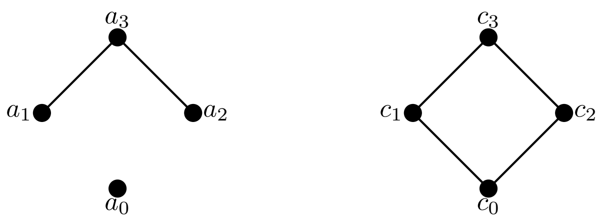
$$\frac{\phi \wedge \psi \Rightarrow \bot}{\Rightarrow \neg\phi \vee \neg\psi}\ (r_{KC})$$

Proposition 5.3. *A step frame (P_1, P_0, f) validates the rule (r_{KC}) iff*

$$\forall a, b_1, b_2 \in P_1\ (b_1 \leq a \ \&\ b_2 \leq a \implies \exists c \in P_0\ (c \leq f(b_1) \ \&\ c \leq f(b_2)).$$

Proof. Straightforward. □

Consider the one-step frame $\mathcal{F} = (P_1, P_0, f)$ presented as



with f given by $a_i \mapsto c_i$. Then $\mathcal{F}$ is a finite conservative one-step frame validating the rule (r_{KC}). If P_2 is a finite poset and $g\colon P_2 \to P_1$ is a f-open surjection, then as f is a bijection the f-openness condition on g implies that g will be an open surjection and therefore, that for $a \in f^{-1}(a_3)$ we have $b, b' \leq a$ such that $g(b) = a_1$ and $g(b') = a_2$. But as $\downarrow a_1$ and $\downarrow a_2$ are disjoint we see that (P_2, P_1, g) will not validate the rule (r_{KC}), and thus $\mathcal{F}$ does not have any one-step extension validating (r_{KC}).

By Theorem 4.5, it then immediately follows that the calculus obtained by adding the rule (r_{KC}) does not have the bounded proof property.

However, we know from [17] that adding the rule

$$\frac{G \mid \Gamma, \Gamma' \Rightarrow}{G \mid \Gamma \Rightarrow \mid \Gamma' \Rightarrow}\ (lq)$$

to a hypersequent version of **LJ** yields a hypersequent calculus for the logic **KC**, which enjoys cut-elimination.

It is easy to verify that a finite intuitionistic one-step frame (P_1, P_0, f) validates the rule (lq) iff P_1 satisfies

$$a_1, a_2 \in P_1\ \exists a \in P_1\ (a \leq a_1 \ \&\ a \leq a_2).$$

From this we obtain as an immediate consequence of Theorem 4.9 that the rule (lq) enjoys the bounded proof property and the finite model property.

5.3 Calculi for $\mathbf{BW}_n$

Consider the logic $\mathbf{BW}_n$ obtained by adding the axiom

$$\bigvee_{i=0}^{n}(p_i \to \bigvee_{j\neq i} p_j) \tag{$\mathbf{bw}_n$}$$

to **IPC**. It is well-known that a Kripke frame $\mathfrak{F} = (W, \leq)$ validates $\mathbf{bw}_n$ iff

$$\forall w, w_0, \ldots, w_n \ (w_0 \leq w \ \& \ \ldots \ \& \ w_n \leq w \implies \exists i, j \leq n \ (i \neq j \ \& \ w_i \leq w_j)) .$$

It follows that $\mathbf{BW}_n$ is the logic of frames $\mathfrak{F}$ such that every rooted subframe of $\mathfrak{F}$ does not contain any anti-chains of more that n nodes.

Proposition 5.4. *An intuitionistic one-step frame* (P_1, P_0, f) *validates the rule*

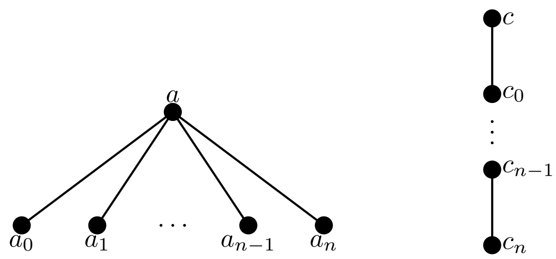
$$\frac{}{\Rightarrow \mathbf{bw}_n}\ (r_{bw_n})$$

iff

$$\forall a, a_0, \ldots, a_n \ (a_0 \leq a \ \& \ \ldots \ \& \ a_n \leq a \implies \exists i, j \leq n \ (i \neq j \ \& \ f(a_i) \leq f(a_j))) .$$

Proof. Straightforward. □

We show that adding the rule (r_{bw_n}) does not yield a calculus with the bounded proof property. To see this let $\mathcal{F} = (P_1, P_0, f)$ be the intuitionistic one-step frame presented as



with $f(a) = c$ and $f(a_i) = f(c_i)$. This is evidently a finite conservative one-step frame and by the above proposition $\mathcal{F}$ validates (r_{bw_n}). Now if $g\colon P_2 \to P_1$ is such that (P_2, P_1, g) is a finite conservative one-step frame we must have that g is open since f is an injection. Thus taking $b \in g^{-1}(a)$ since $a_i \leq g(b)$ for all $i \in \{0, \ldots, n\}$, we must have that there exists $b_i \leq b$, for $i \in \{0, \ldots, n\}$ such that $g(b_i) = a_i$. But then we have that $g(b_i) \not\leq g(b_j)$ when $i \neq j$ and so (P_2, P_1, f) does not validate (r_{bw_n}). We therefore conclude that the class of finite conservative one-step frames validating (r_{bw_n}) does not have the extension property and therefore by Theorem 4.5 adding the axiom (r_{bw_n}) does not yield a calculus with the bounded proof property.

Of course as $\mathbf{bw}_n$ belongs to level $\mathcal{P}_3$ this axiom may also be transformed into an equivalent structural hypersequent rule which preserves cut-eliminability when added to the hypersequent version of **LJ** see e.g. [16]. Once again, the bounded proof property for this structural rule easily follows from our results.

5.4 Calculi for stable logics

Recall [4, 7] that an intermediate logic L is *stable* if for all subdirectly irreducible Heyting algebras $\mathfrak{A}$ and $\mathfrak{B}$ such that $\mathfrak{A}$ is isomorphic to a bounded sublattice of $\mathfrak{B}$ we have that $\mathfrak{B} \models L$ implies that $\mathfrak{A} \models L$. Stable modal logics were defined in [6]. In [11, Thm. 5.3] it was proven that stable modal logics have multi-conclusion axiomatizations with the bounded proof property. We show that the same is the case for stable intermediate logics.

Definition 5.5. By the *stable canonical hypersequent rule* $\eta(\mathfrak{A})$ *associated to* a finite Heyting algebra $\mathfrak{A}$ we shall understand the hypersequent rule $\mathscr{S}/S$ with

$$\begin{aligned} \mathscr{S} := &\{p_{a\wedge b} \Rightarrow p_a;\ p_{a\wedge b} \Rightarrow p_b;\ p_a, p_b \Rightarrow p_{a\wedge b} \colon a, b \in A\} \cup \\ &\{p_{a\vee b} \Rightarrow p_a, p_b;\ p_a \Rightarrow p_{a\vee b};\ p_b \Rightarrow p_{a\vee b} \colon a, b \in A\} \cup \{p_\bot \Rightarrow\ ;\ \ \Rightarrow p_\top\} \end{aligned}$$

and S the hypersequent consisting of the sequents $p_a \Rightarrow p_b$ with a and b ranging over all $a, b \in A$ such that $a \not\leq b$.

Notice that the rule $\eta(\mathfrak{A})$ is obtain by applying invertible rules to the hypersequent rule $\rho(\mathfrak{A})_H$ obtained from the stable multi-conclusion rule $\rho(\mathfrak{A})$ as defined in [7, Def. 3.1]. Thus from the correspondence between multi-conclusion consequence relations and hypersequent calculi outlined in section 2.2 we obtain the following proposition as an immediate consequence of results of [7].

Proposition 5.6 ([7, Prop. 4.5, Thm. 5.3])**.** *An intermediate logic is stable if and only if it is axiomatized by stable canonical hypersequent rules.*

As hypersequent rules of the form $\eta(\mathfrak{A})$ do not contain any propositional variables having an occurence under the scope of an implication we obtain that a finite conservative one-step algebra (D_0, D_1, i) validates $\eta(\mathfrak{A})$ iff and only D_1 does. The following theorem is then an easy consequence of Theorem 4.9.

Proposition 5.7. *Let* K *be a class of finite Heyting algebras. Then the hypersequent calculus determined by the hypersequent rules* $(\eta(\mathfrak{A}))_{\mathfrak{A}\in\mathsf{K}}$ *has the bounded proof property and the finite model property.*

By [4, Thm. 6.13] this provides us with continuum many examples of intermediate logics with hypersequent calculi enjoying the bounded proof property and the finite model property. In particular, **LC**, **KC** and $\mathbf{BW}_n$ for each $n \in \omega$ discussed in the previous sections, are all stable logics.

Remark 5.8. We note that using the normal form representation given in [18] it is easy to see that each formula appearing at level $\mathcal{P}_3$ of the substructural hierarchy is provably equivalent (over **IPC**) to an ONNILLI-formula [8]. Consequently, all formulas in the class $\mathcal{P}_3$ axiomatize stable intermediate logics [8, Thm. 5].[8] We are not aware of any definite examples of a stable logic not axiomatized by $\mathcal{P}_3$-axioms. That is, it is an open question whether or not all stable logics are axiomatizable by $\mathcal{P}_3$-axioms. Furthermore, it is also unclear at the moment how the hypersequent rules obtained from $\mathcal{P}_3$-axiomatizations via the above construction compare with hypersequent rules obtained via Theorem 5.1.

6 Conclusion and future work

We have shown how to transfer the techniques and results of [10, 11] from the setting of modal logic to the setting of intermediate logics. That is, we have established semantic criteria determining when a given hypersequent calculi for an intermediate logic enjoys a certain weakly analytic subformula property; namely the bounded proof property. Analogous to the modal case these criteria are based on extension properties of structures interpreting the degree 1 fragment of the language of **IPC**. Furthermore, we have tested these criteria on a number of examples and shown how to obtain hypersequent calculi with the bounded proof property for a large class of semantically specified intermediate logics viz. stable logics.

The results obtained in this paper suggest that the methodology introduced in [10] is fairly modular and that it may successfully be applied to obtain similar results for other non-classical logics. For instance we expect that in the case of intermediate logics it would also be possible to characterize (hyper)sequent calculi for which the maximal number of $\vee$-nestings is bounded. Moreover, we find it worth investigating if similar results can be obtained for substructural logics. That is, given a connective $*$ and a substructural logic L such that the $*$-free reduct is locally tabular over L can extension properties of appropriate one-step structures characterising the bounded proof property with respect to $*$ of (hyper)sequent calculi for extension of L?

Showing that a given calculus has the bounded proof property and the finite model property via the semantic characterization of Theorem 4.9 looks an automatizable task: one applies some version of algorithmic correspondence theory and then looks for the appropriate pattern in order to trasform one-step frames into Kripke frames. Experience shows that such patterns are classifiable, so that we feel that the relevant metateory of these logics should effectively be handled with the help of a proof assistant.

Complexity issues are still to be investigated: although the mere invocation of bounded proof property yields heavy (usually non-optimal) complexity bounds, there is still the possibility that semantic

[8]Insidentaly, each formula appearing at level $\mathcal{P}_2$ is provably equivalent to a NNIL-formula and thus $\mathcal{P}_2$-formulas axiomatize subframe logics [8].

constructions employed in this paper could give useful search bounds for sufficient classes of 'one-step' countermodels.

Finally, we point out yet another open question: is it possible to find a class $\mathcal{Q}$ of formulas extending $\mathcal{P}_3$ and an effective procedure, similar to the one found in [18], yielding for each $\phi \in \mathcal{Q}$ a set of (logical) hypersequent rules $\mathcal{R}_\phi$ which determine a hypersequent calculus for $\mathbf{IPC} + \phi$ with the bounded proof property?

References

[1] S. Abramsky. A Cook's tour of the finitary non-well-founded sets. In S. N. Artëmov et al., editor, *We Will Show Them! Essays in Honour of Dov Gabbay, Volume One*, pages 1–18. College Publications, 2005.

[2] A. Avron. A constructive analysis of RM. *J. Symb. Log.*, 52(4):939–951, 1987.

[3] A. Avron. The method of hypersequents in the proof theory of propositional non-classical logics. In W. Hodges et al., editor, *Logic: From Foundations to Applications*, pages 1–32. Oxford University Press, 1996.

[4] G. Bezhanishvili and N. Bezhanishvili. Locally finite reducts of Heyting algebras and canonical formulas. *Notre Dame Journal of Formal Logic.* To appear. Available as Utrecht University Logic Group Preprint Series Report 2013-305.

[5] G. Bezhanishvili, N. Bezhanishvili, D. Gabelaia, and A. Kurz. Bitopological duality for distributive lattices and Heyting algebras. *Math. Structures Comput. Sci.*, 20(3):359–393, 2010.

[6] G. Bezhanishvili, N. Bezhanishvili, and R. Iemhof. Stable canonical rules. *J. Symb. Log.* To appear. Available as ILLC Prepublication Series Report PP-2015-08.

[7] G. Bezhanishvili, N. Bezhanishvili, and J. Ilin. Cofinal stable logics. 2015. Submitted. Available as ILLC Prepublication Series Report PP-2015-08.

[8] N. Bezhanishvili and D. de Jongh. Stable formulas in intuitionistic logic. *Notre Dame Journal of Formal Logic*, 2014. To appear. Available as ILLC Prepublication Series Report PP-2014-19.

[9] N. Bezhanishvili and M. Gehrke. Finitely generated free Heyting algebras via Birkhoff duality and coalgebra. *Logical Methods in Comput. Sci.*, 7(2), 2011.

[10] N. Bezhanishvili and S. Ghilardi. The bounded proof property via step algebras and step frames. *Ann. Pure Appl. Logic*, 165(12):1832–1863, 2014.

[11] N. Bezhanishvili and S. Ghilardi. Multiple-conclusion rules, hypersequents syntax and step frames. In Goré et al. [28], pages 54–73.

[12] N. Bezhanishvili, S. Ghilardi, and M. Jibladze. Free modal algebras revisited: the step-by-step method. In G. Bezhanishvili, editor, *Leo Esakia on Duality in Modal and Intuitionistic Logics*, Trends in Logic, pages 43–62, 2014.

[13] N. Bezhanishvili and A. Kurz. Free modal algebras: A coalgebraic perspective. In T. Mossakowski et al., editor, *Algebra and Coalgebra in Computer Science, 2th International Conference, CALCO 2007, Proceedings*, volume 4624 of *Lecture Notes in Computer Science*, pages 143–157. Springer, 2007.

[14] C. Butz. Finitely presented Heyting algebras. Technical report, BRIC Aarhus, 1998.

[15] C. C. Chang and H. J. Keisler. *Model theory.* Studies in logic and the foundations of mathematics. North-Holland publishing company New York, Amsterdam, Londres, 1973.

[16] A. Ciabattoni and M. Ferrari. Hypersequent calculi for some intermediate logics with bounded Kripke models. *J. Log. Comput.*, 11(2):283–294, 2001.

[17] A. Ciabattoni, D. M. Gabbay, and N. Olivetti. Cut-free proof systems for logics of weak excluded middle. *Soft Comput.*, 2(4):147–156, 1998.

[18] A. Ciabattoni, N. Galatos, and K. Terui. From axioms to analytic rules in nonclassical logics. In *Proc. 23th Annual IEEE Symposium on Logic in Computer Science, LICS 2008,*, pages 229–240. IEEE Computer Society, 2008.

[19] A. Ciabattoni, N. Galatos, and K. Terui. Algebraic proof theory for substructural logics: Cut-elimination and completions. *Ann. Pure Appl. Logic*, 163(3):266–290, 2012.

[20] A. Ciabattoni, P. Maffezioli, and L. Spendier. Hypersequent and labelled calculi for intermediate logics. In D. Galmiche et al., editor, *Automated Reasoning with Analytic Tableaux and Related Methods - 22th International Conference, TABLEAUX 2013. Proceedings*, volume 8123 of *Lecture Notes in Computer Science*, pages 81–96. Springer, 2013.

[21] W. Conradie, Y. Fomatati, A. Palmigiano, and S. Sourabh. Algorithmic correspondence for intuitionistic modal mu-calculus. *Theor. Comput. Sci.*, 564:30–62, 2015.

[22] D. Coumans and S. J. van Gool. On generalizing free algebras for a functor. *J. Log. Comput.*, 23(3):645–672, 2013.

[23] K. Fine. Normal forms in modal logic. *Notre Dame J. of Formal Logic*, (XVI):229–237, 1975.

[24] S. Ghilardi. Free Heyting algebras as bi-Heyting algebras. *C. R. Math. Rep. Acad. Sci. Canada*, (24):240–244, 1992.

[25] S. Ghilardi. An algebraic theory of normal forms. *Ann. Pure Appl. Logic*, 71(3):189–245, 1995.

[26] S. Ghilardi. Continuity, freeness, and filtrations. *J. of Appl. Non-Classical Logics*, 20(3):193–217, 2010.

[27] S. J. van Gool. Free algebras for Gödel-Löb provability logic. In Goré et al. [28], pages 217–233.

[28] R. Goré, B. P. Kooi, and A. Kurucz, editors. *Advances in Modal Logic 10.* College Publications, 2014.

[29] A. Indrzejczak. Cut-free hypersequent calculus for S4.3. *Bull. of the Section of Log.*, pages 89–104, 2012.

[30] E. Jeřábek. A note on the substructural hierarchy. *Math. Log. Quart.* To appear.

[31] E. Jeřábek. Canonical rules. *J. of Symb. L.*, 74(4):1171–1205, 2009.

[32] O. Lahav. From frame properties to hypersequent rules in modal logics. In *Proc. 28th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2013*, pages 408–417. IEEE Computer Society, 2013.

[33] O. Lahav and A. Avron. A unified semantic framework for fully structural propositional sequent systems. *ACM Trans. Comput. Log.*, 14(4):27, 2013.

[34] G. Pottinger. Uniform, cut-free formulations of T, S4, S5. *J. Symb. Log.*, 48:900, 1983.

[35] J. Schmid. Quasiorders and sublattices of distributive lattices. *Order*, 19(1):11–34, 2002.

[36] K. Schütte. Syntactical and semantical properties of simple type theory. *J. Symb. Log.*, 25:305–325, 1960.

[37] A. S. Troelstra and H. Schwichtenberg. *Basic Proof Theory.* Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, second edition, 2000.

[38] M. Zakharyaschev. Syntax and semantics of superintuitionistic logics. *Algebra and Logic*, 28(4):262–282, 1989.