Intuitionistic Master Modality

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Abstract

We present a cyclic sequent calculus for intuitionistic modal logic with the master modality. Formulas of the logic are evaluated over bi-relational Kripke models with three different frame conditions: functional frames, 'triangle' confluent frames, and arbitrary frames. It is shown that the calculus is sound and complete for all three classes of models. This, in particular, proves that intuitionistic modal logic with the master modality cannot distinguish between arbitrary models and functional models. Soundness is established by a standard argument while completeness is proven via a detour to non-wellfounded proofs, using a proof-search argument that draws on analyticity of the calculus. The framework is robust in the sense that it can be naturally adapted to account for various frame conditions, such as serial models, reflexive models or S4-models, as well as for a polymodal extension that can be interpreted as intuitionistic common knowledge.

Keywords: Modal logic, Intuitionistic logic, Sequent calculus, Cyclic proofs.

1 Introduction

Intuitionistic modal logic has a long history with contributions from various fields, ranging from proof theory and philosophical logic to type theory and programming language theory. The logics studied can be roughly divided into two camps: *intuitionistic modal logics*, aimed at capturing an intuitionistic meta-reading of possible world semantics [20], and *constructive modal logics*, built for modelling particular computational properties such as staged or contextual computation [11]. More recently, extensions of these logics with fixed point operators, referred to as Intuitionistic Fixed Point Modal Logics (IFPML), have gained increasing attention. Examples include intuitionistic linear-time

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temporal logic [7,2,3,4,1], intuitionistic common knowledge logic [10], and intuitionistic modal μ-calculus [17].

The mathematical theory underpinning IFPML is little explored compared to its classical counterpart. In the classical realm, games, automata and, more recently, cyclic proofs have shown to be particularly suitable for the study of fixed point modal logics [9,6,19]. In contrast to the more traditional finitary proof systems with induction rules, cyclic proof systems are often analytic, and therefore better suited for proof search. In the realm of IFPML, non-wellfounded and cyclic proof systems have so far only been developed for intuitionistic linear-time temporal logic [1,14].

This work is part of a larger programme to establish frameworks and techniques for studying IFPML ranging from intuitionistic versions of basic modal logics to intuitionistic modal μ-calculus. Here, we study the language $\mathcal{L}_{\mathrm{IM}}$ which extends the language of IPC with the basic modality $\Box$ and the master modality $\boxplus$. A formula $\boxplus\varphi$ is characterised as the greatest fixed point of the propositional function $p \mapsto \varphi \wedge \Box p$. Formulas are evaluated over bi-relational Kripke models $(W, \leq, R, V)$, where $\leq$ is the intuitionistic partial order and R the modal accessibility relation. The monotonicity property, that $w \leq v$ and $w \models \varphi$ implies $v \models \varphi$, can be built directly into the semantics, as we will initially do. An alternative approach is to impose frame conditions on $\leq$ and R, such as *triangle confluence*: if $w \leq v$ and vRu, then wRu.

For triangle confluent models, the truth conditions for the modalities reduce to the classical ones. We consider the class of all bi-relational models, the class of models with a functional modal relation, and the class of triangle confluent models, inducing the logics IM_{K}, IM_{f}, IM_{t}, respectively. The logic IM_{f} can be viewed as a weak version of intuitionistic linear-time temporal logic.

We introduce a cyclic proof system cIM, and establish soundness with respect to IM_{K} and completeness with respect to both IM_{f} and IM_{t}. This implies that the three logics are equivalent, thereby showing that our language cannot distinguish between arbitrary bi-relational models, functional models, and triangle models. While the result for arbitrary bi-relational models and triangle models was already known for $\mathcal{L}_{\mathrm{IM}}$ without the master modality (see e.g., [12]), the fact that $\mathcal{L}_{\mathrm{IM}}$ cannot distinguish functional models from arbitrary ones, is, to the best of our knowledge, a new result.

The calculus cIM is a natural modal extension of the standard multi-conclusion calculus for intuitionistic propositional logic (see e.g., [15]). To ensure soundness, the calculus uses a focus annotation that keeps track of good traces. As cIM is cut-free, it is analytic and hence suitable for effective proof search. A similar calculus for classical modal logic with the master modality over S5-frames is presented in [18]; that calculus differs from cIM in that it requires analytic cuts due to the S5 frame conditions.

Completeness of cIM proceeds via a detour into a non-wellfounded proof calculus specifically designed for proof-search. Inspired by the game-theoretic arguments in [16], we present a modular framework for proof-search as a two-player infinite game between Prover and Refuter, such that every unprovable sequent induces a countermodel of a particular form. The form of Prover's turn can be adapted as to obtain a particular frame condition. In this way, we obtain completeness of the non-wellfounded calculus for triangle and functional models. Completeness of cIM is obtained by showing that a non-wellfounded proof induces a cyclic proof. In addition, we also show completeness of a single-conclusion version of cIM.

Due to the modular approach, the calculus cIM and the proof methods are robust in the sense that they can easily be adapted to account for various frame conditions, such as serial frames, reflexive frames, and S4-frames. Furthermore, cIM can be adapted to a polymodal version of $\mathcal{L}_{\mathrm{IM}}$ to obtain an analytic calculus for the intuitionistic common knowledge logic considered in [10].

2 Syntax and semantics

The *language* of $\mathcal{L}_{\mathrm{IM}}$ consists of a countable set of atomic propositions Prop, logical connectives $\wedge, \vee, \to$ and modal operators $\Box$ and $\boxplus$. The operator $\boxplus$ is called the *master modality. Formulas* of $\mathcal{L}_{\mathrm{IM}}$ are given by the grammar:

$$\varphi ::= \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi \mid \boxplus\varphi$$

where $p \in \mathsf{Prop}$. Define $\top := \bot \to \bot$ and $\Box^k\varphi$ by $\Box^0\varphi := \varphi$ and $\Box^{k+1}\varphi := \Box\Box^k\varphi$. The set of formulas of $\mathcal{L}_{\mathrm{IM}}$ is denoted Fm. Greek letters $\varphi, \psi, \ldots$ etc., possibly with subscript, are meta-variables for formulas.

Definition 2.1 The *closure* of a formula φ is the smallest set $\mathsf{Cl}(\varphi)$ which contains φ, is closed under the subformula relation, and contains $\Box\boxplus\psi$ whenever $\boxplus\psi \in \mathsf{Cl}(\varphi)$.

The following lemma is proven by a simple induction on the structure of the formula φ.

Lemma 2.2 *For any formula φ, the closure $\mathsf{Cl}(\varphi)$ is finite.*

Formulas are evaluated in bi-relational (Kripke) models.

Definition 2.3 A *(bi-relational) model* is a tuple $M = (W, \leq, R, V)$ where

(i) $W \neq \emptyset$ is a set;

(ii) $(W, \leq)$ is a partial order;

(iii) $V \colon W \to \mathcal{P}(\mathsf{Prop})$ is monotone in $\leq$: if $w \leq v$ then $V(w) \subseteq V(v)$;

(iv) $R \subseteq W \times W$ is a binary relation.

Elements of W are called *worlds*, and given some world $w \in W$, we call the tuple (M, w) a *pointed model.* The function V is called a *valuation*, the relation $\leq$ is called the *intuitionistic order* and R is called the *modal accessibility relation.* If $w \leq v$ then v is called an *intuitionistic successor* of w, and if wRv then we call v a *modal successor* of w. A model is called *functional* if the modal accessibility relation R is functional, i.e., if wRv and wRu, then $v = u$.

For any binary relation S, we let S^* denote the reflexive and transitive closure of S. Given a model $M = (W, \leq, R, V)$, we let $\tilde{R}$ denote the composition

Fig. 1. Forth-down confluence (left) and triangle confluence (right). Dashed lines represent the relations each confluence condition stipulates the existence of.

$\leq; R$. Note that, since $\leq$ is reflexive, $w\tilde{R}^*v$ holds if and only if there exists a natural number n and worlds $u_0, \ldots, u_n$ such that $u_0 = w$, $u_n = v$ and for all $0 \leq i < n$ we have $u_i R u_{i+1}$ or $u_i \leq u_{i+1}$. The *truth relation* $\models$ is defined inductively by the following clauses, where $p \in \mathsf{Prop}$ and $w \in W$.

$M, w \not\models \bot$,		
$M, w \models p$	iff	$p \in V(w)$
$M, w \models \varphi \wedge \psi$	iff	$M, w \models \varphi$ and $M, w \models \psi$,
$M, w \models \varphi \vee \psi$	iff	$M, w \models \varphi$ or $M, w \models \psi$,
$M, w \models \varphi \to \psi$	iff	for all $v \geq w$ if $M, v \models \varphi$, then $M, v \models \psi$,
$M, w \models \Box\varphi$	iff	for all $v \in W$, if $w\tilde{R}v$ then $M, v \models \varphi$,
$M, w \models \boxplus\varphi$	iff	for all $v \in W$, if $w\tilde{R}^*v$ then $M, v \models \varphi$.

Validity and satisfiability over a class of models are defined as expected.

As remarked, monotonicity is built-in to the semantics:

Lemma 2.4 (Monotonicity of $\models$) *Let $\varphi \in \mathsf{Fm}$ and let $M = (W, \leq, R, V)$ be a model with $w, v \in W$. If $w \leq v$ and $M, w \models \varphi$, then $M, v \models \varphi$.*

2.1 Triangle models

We introduce a subclass of models in which the intuitionistic order and the modal accessibility relation satisfy a particular confluence property. For this subclass of models, the classical truth conditions for the modalities suffice to obtain the monotonicity lemma (cf. Lemma 2.4).

Definition 2.5 A *triangle model* is a model $M = (W, \leq, R, V)$ where $\leq$ and R are *triangle confluent*: if $w \leq v$ and vRu, then wRu (see Figure 1).

Given a triangle model $M = (W, \leq, R, V)$, a second truth relation $\models_t \subseteq W \times \mathsf{Fm}$ can be given which differs from $\models$ only in the modal clauses:

$M, w \models_t \Box\varphi$	iff	for all $v \in W$, if wRv thn $M, v \models_t \varphi$,
$M, w \models_t \boxplus\varphi$	iff	for all $v \in W$, if wR^*v then $M, v \models_t \varphi$.

Triangle confluence implies $R = \tilde{R}$, whence the next two lemmas obtain.

Lemma 2.6 (Monotonicity of $\models_t$) *Let $\varphi \in \mathsf{Fm}$ and let $M = (W, \leq, R, V)$ be a triangle model with $w, v \in W$. If $w \leq v$ and $M, w \models_t \varphi$, then $M, v \models_t \varphi$.*

Lemma 2.7 *Let $\varphi \in \mathsf{Fm}$ and let (M, w) be a pointed triangle model. Then $M, w \models_t \varphi$ if and only if $M, w \models \varphi$.*

Triangle confluence is a special case of *forth-down confluence*: if $w \leq v$ and vRu, then there exists $s \in W$ with wRs and $s \leq u$ (illustrated in Figure 1).

Forth-down confluence is sufficient for monotonicity. However, every model $M = (W, \leq, R, V)$, and so, in particular, every forth-down confluent model, induces a triangle model $M' = (W, \leq, (\leq; R), V)$ in a truth-preserving way. So the logic over forth-down models is identical to the logic over triangle models.

Denote by $\mathrm{IM_K}$, $\mathrm{IM_f}$ and $\mathrm{IM_t}$ the set of valid formulas over the class of bi-relational models, the class of functional models and the class of triangle models, respectively. By definition, $\mathrm{IM_K} \subseteq \mathrm{IM_f} \cap \mathrm{IM_t}$. In the next section we present a non-wellfounded and a cyclic calculus which are each sound and analytically complete for these logics. As a corollary, the three notions of validity coincide: $\mathrm{IM_K} = \mathrm{IM_f} = \mathrm{IM_t}$.

3 Proof systems

An *annotated formula* is a pair (φ, a) where φ is a formula and $a \in \{\mathsf{f}, \mathsf{u}\}$, where f designates that the formula is *in focus* and u that the formula is *unfocused.* Annotated formulas are usually written as φ^a. Finite sets of annotated formulas are denoted by $\Gamma, \Delta, \Sigma, \Pi$ and Ω with or without subscripts. For a set of annotated formulas Γ define

$$\Gamma^- = \{\varphi \mid \varphi^a \in \Gamma\} \text{ and } \Gamma^\mathsf{u} = \{\varphi^\mathsf{u} \mid \varphi^a \in \Gamma\}$$

A *sequent* is an ordered pair $\Gamma \Rightarrow \Delta$ where Γ and Δ are finite sets of annotated formulas, such that the following conditions hold.

(i) Every formula in Γ is unfocused.

(ii) At most one formula in Δ is in focus.

(iii) If a formula φ is in focus, then $\varphi = \boxplus\psi$ or $\varphi = \Box\boxplus\psi$ for some formula ψ.

We use σ to denote sequents and write Γ_σ and Δ_σ for the left and right side of σ respectively. The *interpretation* of σ is the formula $\sigma^I := \bigwedge \Gamma_\sigma^- \to \bigvee \Delta_\sigma^-$ where $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$. Note, annotations convey no semantic meaning. Given a pointed model (M, w) we write $M, w \models \sigma$ iff $M, w \models \sigma^I$. The *closure* of σ is the set $\mathsf{Cl}(\sigma) := \mathsf{Cl}(\Gamma_\sigma) \cup \mathsf{Cl}(\Delta_\sigma)$.

Our calculi employ multi-conclusion sequents, i.e., sequents $\Gamma \Rightarrow \Delta$ where Δ may contain more than one formula. This streamlines the proof-search argument for completeness as it allows writing the disjunction and left-implication rules in invertible form. But it is not an essential restriction; Section 5.4 demonstrates how a single-conclusion proof can be obtained from any multi-conclusion one.

Definition 3.1 The sequent calculus IM consists of the rules depicted in Table 1 for all values of $a \in \{\mathsf{u}, \mathsf{f}\}$.

The rules id and $\bot$ are called *axioms.* The rules u and f govern the focus annotations: the rule u takes a sequent with no formula in focus and puts one formula in focus. The rule f does the opposite: it takes a sequent with a formula in focus and changes its annotation to unfocused. The names of these rules are motivated by the fact that later we will usually read rules bottom-up. The rules $\boxplus\mathsf{L}$ and $\boxplus\mathsf{R}$ reflect the equivalence $\boxplus\varphi \leftrightarrow \varphi \wedge \Box\boxplus\varphi$.

$$\frac{}{\Gamma, \varphi^{\mathsf{u}} \Rightarrow \varphi^{a}, \Delta}\ \mathsf{id} \qquad \frac{}{\Gamma, \bot^{\mathsf{u}} \Rightarrow \Delta}\ \bot$$

$$\frac{\Gamma, \varphi^{\mathsf{u}}, \psi^{\mathsf{u}} \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi^{\mathsf{u}} \Rightarrow \Delta}\ \wedge\mathsf{L} \qquad \frac{\Gamma \Rightarrow \varphi^{\mathsf{u}}, \Delta \quad \Gamma \Rightarrow \psi^{\mathsf{u}}, \Delta}{\Gamma \Rightarrow \varphi \wedge \psi^{\mathsf{u}}, \Delta}\ \wedge\mathsf{R}$$

$$\frac{\Gamma, \varphi^{\mathsf{u}} \Rightarrow \Delta \quad \Gamma, \psi^{\mathsf{u}} \Rightarrow \Delta}{\Gamma, \varphi \vee \psi^{\mathsf{u}} \Rightarrow \Delta}\ \vee\mathsf{L} \qquad \frac{\Gamma \Rightarrow \varphi^{\mathsf{u}}, \psi^{\mathsf{u}}, \Delta}{\Gamma \Rightarrow \varphi \vee \psi^{\mathsf{u}}, \Delta}\ \vee\mathsf{R}$$

$$\frac{\Gamma, \varphi \to \psi^{\mathsf{u}} \Rightarrow \varphi^{\mathsf{u}}, \Delta \quad \Gamma, \psi^{\mathsf{u}} \Rightarrow \Delta}{\Gamma, \varphi \to \psi^{\mathsf{u}} \Rightarrow \Delta}\ {\to}\mathsf{L} \qquad \frac{\Gamma, \varphi^{\mathsf{u}} \Rightarrow \psi^{\mathsf{u}}}{\Gamma \Rightarrow \varphi \to \psi^{\mathsf{u}}, \Delta}\ {\to}\mathsf{R}$$

$$\frac{\Gamma, \varphi^{\mathsf{u}}, \Box\boxast\varphi^{\mathsf{u}} \Rightarrow \Delta}{\Gamma, \boxast\varphi^{\mathsf{u}} \Rightarrow \Delta}\ \boxast\mathsf{L} \qquad \frac{\Gamma \Rightarrow \varphi^{\mathsf{u}}, \Delta \quad \Gamma \Rightarrow \Box\boxast\varphi^{a}, \Delta}{\Gamma \Rightarrow \boxast\varphi^{a}, \Delta}\ \boxast\mathsf{R}$$

$$\frac{\Gamma \Rightarrow \Delta^{\mathsf{u}}}{\Gamma \Rightarrow \Delta}\ \mathsf{u} \qquad \frac{\Gamma \Rightarrow \varphi^{\mathsf{f}}, \Delta}{\Gamma \Rightarrow \varphi^{\mathsf{u}}, \Delta}\ \mathsf{f}$$

$$\frac{\Gamma \Rightarrow \varphi^{a}}{\Pi, \Box\Gamma \Rightarrow \Box\varphi^{a}, \Sigma}\ \Box$$

Table 1
The rules of the calculus IM

Note that the rules $\to\mathsf{R}$ and $\Box$ have single-conclusion premises and all other rules are invertible in the sense that the conclusion is valid if and only if all premises are. We therefore refer to $\Box$ and $\to\mathsf{R}$ as the *non-invertible* rules and the other rules as *invertible*. For each rule, the distinguished formula in the conclusion is called *principal* and the distinguished formula(s) in the premises are called its *residual(s)*. For example, for $\to\mathsf{L}$ the principal formula is $\varphi \to \psi^{\mathsf{u}}$ and its residuals are $\varphi \to \psi^{\mathsf{u}}$, φ^{u} and ψ^{u}. For the rule $\Box$, all formulas in the conclusion are principal and each formula in the premise is the residual of its corresponding principal formula (formulas in Σ and Π have no residuals). In every rule application, any formula that is neither principal nor residual is called a *side formula*.

The condition that sequents have at most one formula in focus imposes restrictions on rule applications, as is illustrated by the following lemma.

Lemma 3.2 *If in an instance of* $\boxast\mathsf{R}$ *the principal formula is in focus, then the left premise has no formula in focus.*

In the following we introduce a non-wellfounded and a cyclic proof system based on the rules of IM. We remark that the annotations are not vital for the non-wellfounded system; it is possible to define a sound and complete non-wellfounded proof system based on the rules of IM without annotations. [5]

[5] For such a system the global soundness condition on infinite branches is formulated in a different way than presented here using formula traces.

However, the annotations are required to guarantee soundness of the cyclic system.

3.1 Non-wellfounded calculus nIM

A *derivation* in nIM of a sequent σ is a finite or infinite tree whose nodes are labelled by sequents according to the rules of IM and whose root is labelled by σ. We read derivations 'upwards', so that the premise of a rule is considered to be a successor of the conclusion. Given a derivation π, a *path* through π is a finite or infinite sequence of nodes $\rho = \rho_0, \rho_1, \rho_2, \ldots$ of π such that for each index i the node ρ_{i+1} (if it exists) is a direct successor of ρ_i. A path is *maximal* if it ends in a leaf or is infinite. A maximal path starting at the root is also called a *branch*. We will often tacitly identify a node in a derivation with the sequent labelling it and thereby paths with sequences of sequents.

Definition 3.3 A nIM-*proof* of a sequent $\Gamma \Rightarrow \Delta$ is a derivation in nIM of $\Gamma \Rightarrow \Delta$, such that every leaf is labelled by an axiom and every infinite branch ρ has a *good* suffix ρ': every sequent in ρ' has a formula in focus and ρ' contains infinitely many applications of $\boxplus$R where the principal formula is in focus.

Lemma 3.4 *Every good suffix contains infinitely many applications of* $\Box$.

3.2 Cyclic calculus cIM

A *derivation* in cIM of a sequent σ is a nIM-derivation of σ that is finite.

Definition 3.5 A path ρ in a cIM-derivation is *successful* if the following hold.

(i) Every sequent in ρ has a formula in focus.

(ii) The path ρ passes through at least one instance of $\boxplus$R where the principal formula is in focus.

Given a cIM-derivation π, a pair of nodes (u, v) of π is called a *repetition* if there exists a path from u to v and both nodes are labelled by the same sequent. A repetition (u, v) is *successful* if the path from u to v is successful.

Definition 3.6 A cIM-*proof* of a sequent $\Gamma \Rightarrow \Delta$ is a derivation π in cIM, such that every leaf l of π is either labelled by an axiom or there exists a node $c(l)$ in π such that $(c(l), l)$ is a successful repetition.

We analogously define *single-conclusion* derivations and proofs in nIM and cIM, where instead of the multi-conclusion rules of IM we use their standard single-conclusion version, in which every sequent $\Gamma \Rightarrow \Delta$ satisfies $|\Delta| \leq 1$ (see the appendix for an explicit definition).

4 Soundness

This section establishes soundness of cIM with respect to bi-relational models. The proof closely follows the soundness proof of [18] and makes essential use of the focus annotations. Proofs in this section are deferred to the appendix.

Let σ be a sequent that has a formula in focus, i.e., Δ_σ contains a formula of the form $\Box^j \boxplus \varphi^{\mathsf{f}}$ for $j \in \{0, 1\}$. Denote by $\sigma(n)$ the sequent $\Gamma_\sigma \Rightarrow \Delta_\sigma, \Box^j \Box^n \varphi^{\mathsf{u}}$, i.e., the sequent expanding the right side of σ by formula $\Box^j \Box^n \varphi^{\mathsf{u}}$.

Lemma 4.1 *If σ has a formula in focus and is invalid, then there exists a natural number n such that $\sigma(n)$ is invalid.*

As a consequence, every invalid sequent σ with a formula in focus can be associated a measure:

$$\mu(\sigma) := \min\{n \in \omega \mid \sigma(n) \text{ is invalid}\}.$$

Lemma 4.2 *Suppose*

$$\frac{\sigma_1 \quad \cdots \quad \sigma_n}{\sigma}\ \mathsf{r}$$

is a rule instance of IM. *If σ is invalid, then there is an i such that σ_i is invalid. If both σ and σ_i have a formula in focus then, moreover,*

$$\mu(\sigma_i) \leq \mu(\sigma),$$

where the inequality is strict if $\mathsf{r} = \boxplus\mathsf{R}$ *and the principal formula is in focus.*

From these two lemmas, global soundness of cIM is easily established.

Theorem 4.3 *If there is a* cIM*-proof of a sequent σ, then σ is valid over the class of bi-relational models.*

The above result also implies that cIM is sound for the class of functional models and the class of triangle models. In addition, soundness of the single-conclusion version of cIM follows, as any single-conclusion proof induces a multi-conclusion proof via weakening.

5 Completeness

We now turn our attention to completeness of the cyclic calculus with respect to triangle and functional models. The argument proceeds in two steps. First, we set up a general framework for completeness via proof-search games, from which completeness of the ill-founded calculus nIM can be deduced. We then show how to transform an arbitrary nIM-proof into single-conclusion one, and lastly how to transform a (single-conclusion) nIM-proof into a (single-conclusion) cIM-proof.

5.1 Proof-search games

Each sequent σ will be associated a *proof-search tree* which will form the arena of a two-player game between *Prover*, whose winning strategies establish proofs of σ, and *Refuter*, whose winning strategies describe countermodels for σ. Completeness becomes a corollary of determinacy of the game.

A proof-search tree for σ is built by applying rules bottom-up to σ. The invertible rules are applied first until a *saturated* sequent is obtained.

Definition 5.1 A sequent $\Gamma \Rightarrow \Delta$ is *saturated* if the following hold.

(i) If $\varphi \wedge \psi^{\mathsf{u}} \in \Gamma$, then $\varphi^{\mathsf{u}} \in \Gamma$ and $\psi^{\mathsf{u}} \in \Gamma$.

(ii) If $\varphi \vee \psi^{\mathsf{u}} \in \Gamma$, then $\varphi^{\mathsf{u}} \in \Gamma$ or $\psi^{\mathsf{u}} \in \Gamma$.

(iii) If $\varphi \to \psi^{\mathsf{u}} \in \Gamma$, then $\varphi^{\mathsf{u}} \in \Delta$ or $\psi^{\mathsf{u}} \in \Gamma$.

(iv) If $\boxast\varphi^{\mathsf{u}} \in \Gamma$, then $\varphi^{\mathsf{u}} \in \Gamma$ and $\Box\boxast\varphi^{\mathsf{u}} \in \Gamma$.

(v) If $\varphi \wedge \psi^{\mathsf{u}} \in \Delta$, then $\varphi^a \in \Delta$ or $\psi^{\mathsf{u}} \in \Delta$.

(vi) If $\varphi \vee \psi^{\mathsf{u}} \in \Delta$, then $\varphi^a, \psi^a \in \Delta$.

(vii) If $\boxast\varphi^a \in \Delta$, then $\varphi^{\mathsf{u}} \in \Delta$ or $\Box\boxast\varphi^a \in \Delta$.

Given a sequent σ, a formula occurring in σ is said to be *saturated* if σ satisfies the corresponding clause above for that formula.

As we are working with set sequents, formulas can simultaneously function as principal and as side formulas. We call an application of a rule *preserving* if the principal formula(s) is also a side formula. For example, an application of $\boxast\mathsf{L}$ as depicted in Table 1 is preserving if $\boxast\varphi^{\mathsf{u}} \in \Gamma$.

The particular form of the proof-search tree depends on the kind of countermodel one wants to obtain from a refutation. In a general form suitable for our needs, proof-search trees have the following structure.

Definition 5.2 Fix some inference rule C and a sequent σ. A *proof-search tree (with choice rule* C*)* for σ is a finite or infinite tree T whose nodes are labelled by sequents according to C and the invertible logical rules of IM such that:

(i) The root is labelled by $\Gamma_\sigma \Rightarrow \Delta_\sigma$;

(ii) Every invertible rule is applied preservingly;

(iii) No invertible rule is applied to a sequent in which the principal formula is already saturated;

(iv) A node is a leaf if and only if it is labelled by an axiom or by a saturated sequent to which the C-rule cannot be applied;

(v) The C-rule is only applied to saturated sequents.

Each completeness proof we present will be relative to a suitable choice rule C.

Note that every sequent σ has a proof-search tree. Due to property (ii), every sequent can be saturated by finitely many invertible rule applications. By property (iii), we then obtain the following result.

Lemma 5.3 *Every infinite branch of a proof-search tree contains infinitely many applications of* C*.*

Given a proof-search tree T with choice rule C for a sequent σ, a game $G(T, \mathsf{C})$ can be defined between players 'Prover' and 'Refuter' where a play corresponds to a branch in T: reading upwards, invertible rules represent a choice of admissible moves for Refuter and the C-rule represents a choice of moves for Prover. Prover wins a play ρ if and only if ρ is finite and ends in an axiom, or ρ is infinite and has a good suffix. All other plays are won by Refuter. A winning strategy for Refuter then corresponds to a *refutation* of σ.

Definition 5.4 A *refutation* of a sequent σ is a subtree S of a proof-search tree T for σ satisfying the following properties.

(i) S contains the root of T.

(ii) No leaf is an axiom.

(iii) No infinite branch of S has a good suffix.

(iv) If S contains a node u that is labelled by the conclusion of a C-application, then S contains all direct successors of u in T.

(v) If S contains a node u that is labelled by the conclusion of any other rule than C, then S contains exactly one direct successor of u in T.

Whereas a winning strategy for Refuter corresponds to a refutation of σ, a winning strategy for Prover should correspond to a proof of σ; it must be checked that this is indeed the case for a particular choice for C.

It is routine to check that the set of winning plays in $G(T, \mathsf{C})$ for each player is Borel, and so it follows from Martin's determinacy theorem [13] that every sequent has a refutation or a proof. Thus, in order to prove completeness, the key result to obtain is that a refutation induces a countermodel. To show this, we will make use of the following 'canonical' model construction.

Definition 5.5 Let σ be a sequent and let S be a refutation of σ. The *canonical model* based on S is the model $M_S = (W, \leq, R, V)$ defined as follows.

(i) $W = S/\sim$, where $s \sim t$ iff there exists a path between s and t in which no instance of the C-rule occurs.

(ii) $\leq$ is the reflexive, transitive closure of the relation $\leq_0 \subseteq W \times W$ given by

$$w \leq_0 v \text{ iff there exist } s \in w \text{ and } t \in v \text{ such that } s \text{ is the conclusion and } t \text{ a left premise of the same } \mathsf{C}\text{-rule instance.}$$

(iii) $R \subseteq W \times W$ is such that

$$wRv \text{ iff there exist } s \in w \text{ and } t \in v \text{ such that } s \text{ is the conclusion and } t \text{ is a right premise of the same } \mathsf{C}\text{-rule instance.}$$

(iv) $V \colon w \mapsto \Gamma_w \cap \mathsf{Prop}$ where Γ_w is the left side of the sequent labelling the unique node in w that is the conclusion of a C-rule application.

5.2 Completeness of nIM with respect to triangle models

To show completeness of nIM with respect to triangle models, we consider the following choice rule.

$$\frac{\Pi, \Box\Gamma, \varphi_0^{\mathsf{u}} \Rightarrow \psi_0^{\mathsf{u}} \quad \cdots \quad \Pi, \Box\Gamma, \varphi_l^{\mathsf{u}} \Rightarrow \psi_l^{\mathsf{u}} \quad \Gamma \Rightarrow \chi_1^{b_0} \quad \cdots \quad \Gamma \Rightarrow \chi_m^{b_m}}{\Pi, \Box\Gamma \Rightarrow \{(\varphi_i \to \psi_i)^{\mathsf{u}}\}_{i=0}^{l}, \{\Box\chi_i^{a_i}\}_{i=0}^{m}, \Sigma} \; \mathsf{C_t}$$

where the annotations b_i are equal to f whenever the underlying formula χ_i is a $\boxplus$-formula, and equal to u otherwise. Moreover, we require that $\Pi \cup \Sigma$ contains no $\Box$-formulas and that Σ contains no $\to$-formulas. We call the premises of the form $\Gamma \Rightarrow \chi_i^{b_i}$ the *right premises* and the other premises the *left premises*.

The following lemma is a direct consequence of the definition of the winning conditions and the form of C_t.

Lemma 5.6 *If T is a proof-search tree for σ with choice rule $\mathsf{C_t}$, then a winning strategy for Prover in $G(T, \mathsf{C}_t)$ corresponds to a proof of σ.*

Proposition 5.7 *If a sequent has a refutation with choice rule* $\mathsf{C_t}$*, then it is falsified in a triangle model.*

Proof. Let T be a proof-search tree for σ and let S be a subtree of T that is a refutation of σ. Let $M_S = (W, \leq, R, V)$ be the canonical model based on S, and let $M = (W, \leq, (\leq; R), V)$ be the induced triangle model.

Let φ be a formula. By induction on the logical complexity of φ, we simultaneously prove that for any $w \in W$ we have (a) $M, w \models \varphi$ if $\varphi \in \Gamma_w^-$ and (b) $M, w \not\models \varphi$ if $\varphi \in \Delta_w^-$. The proof relies on the fact that the sequent $\Gamma_w \Rightarrow \Delta_w$ is saturated, since it is the conclusion of a $\mathsf{C_t}$-application. We only treat the connectives $\to$ and $\boxplus$. Recall that, for triangle models, we can simply use the classical truth conditions for the modalities.

The case of $\to$. (a). If $\varphi \to \psi \in \Gamma_w^-$ and $v \geq w$, then by definition of $\mathsf{C_t}$ and the fact that invertible rules are applied preservingly, we have $\varphi \to \psi \in \Gamma_v^-$. So by saturation and the induction hypothesis (IH), we have $M, v \models \psi$ or $M, v \not\models \varphi$, so we obtain $M, w \models \varphi \to \psi$. (b). If $\varphi \to \psi \in \Delta_w$, then by construction of $\leq$ there exists a $v \geq_0 w$ such that $\varphi \in \Gamma_v^-$ and $\psi \in \Delta_v^-$. So by the IH, we obtain $M, v \models \varphi$ and $M, v \not\models \psi$, so $M, w \not\models \varphi \to \psi$.

The case of $\boxplus$. (a). Let $\boxplus\varphi \in \Gamma_w^-$ and wR^*v. Saturation implies that $\Box\boxplus\varphi \in \Gamma_w^-$, so by definition of $\mathsf{C_t}$ and the fact that invertible rules are applied preservingly, we have $\Box\boxplus\varphi \in \Gamma_u^-$ for all $u \geq w$. This means that $\boxplus\varphi \in \Gamma_s^-$ if wRs. Iterating the argument, we find that $\boxplus\varphi \in \Gamma_v^-$. Saturation then gives $\varphi \in \Gamma_v^-$, so $M, v \models \varphi$ by the IH. (b). If $\boxplus\varphi \in \Delta_w^-$, then saturation implies $\varphi \in \Delta_w^-$ or $\Box\boxplus\varphi \in \Delta_w^-$. Suppose, for contradiction, that for all wR^*v we have $\varphi \notin \Delta_v^-$. Let $s \in w$ be the last node in w, i.e., s is the conclusion of a $\mathsf{C_t}$-application. Then we can define an infinite path ρ in S starting from s as follows: at each $\mathsf{C_t}$-application, we pick the right premise that has $\boxplus\varphi^{\mathsf{f}}$ as consequent. Note that saturation and the fact that no wR^*v satisfies $\varphi \in \Delta_v^-$ implies that this is always possible. The path ρ then forms a good suffix of the infinite branch of S in which it is contained, contradicting that S is a refutation. So there must be some wR^*v with $\varphi \in \Delta_v^-$, and thus $M, v \not\models \varphi$ by the IH. We conclude that $M, w \not\models \boxplus\varphi$.

We conclude that the root of M falsifies the sequent σ. $\Box$

Corollary 5.8 *The calculus* nIM *is complete for* $\mathrm{IM_t}$.

5.3 Completeness of nIM with respect to functional models

When constructing the proof-search tree for $\mathrm{IM_f}$, we have to ensure that the induced countermodel will be functional. This means that, when we reach a saturated sequent of the form $\Gamma \Rightarrow \Box\chi_1, \ldots, \Box\chi_m, \Delta$ we can only pick *one* χ_i that will be falsified in the (unique) modal successor. This problem can be solved by adding in extra intuitionistic successors, so that the remaining χ_i can be falsified at *their* modal successor. To keep track of which right $\Box$-formula has to be 'taken care of' at a particular step, the proof-search tree will be labelled by *indexed sequents* $\Gamma \Rightarrow_k \Delta$, that is, sequents equipped with a natural number k that we call the *index* of the sequent.

Definition 5.9 An *indexed proof-search tree* for a sequent σ consists of an enumeration $\chi_0, \chi_1, \dots, \chi_n$ of formulas in $\Box^{-1}\mathsf{Cl}(\sigma) := \{\chi \mid \Box\chi \in \mathsf{Cl}(\sigma)\}$ and a finite or infinite tree T whose nodes are labelled by indexed sequents such that:

(i) T is a proof-search tree for σ with choice rule [6]

$$\dfrac{\{\Pi, \Box\Gamma, \varphi_i^{\mathsf{u}} \Rightarrow_0 \psi_i^{\mathsf{u}}\}_{i=0}^{l} \quad \Gamma_\tau \Rightarrow_{(k+1)_m} \Delta_\tau \quad \Gamma \Rightarrow_0 \chi_{i_k}^{a}}{\Pi, \Box\Gamma \Rightarrow_k \{\varphi_i \to \psi_i^{\mathsf{u}}\}_{i=0}^{l}, \{\Box\chi_{i_j}^{a_i}\}_{j=0}^{m}, \Sigma}\ \mathsf{C_f}$$

where τ is the sequent labelling the conclusion, and a equals f if χ_{i_k} is a $\boxplus$-formula and equals u otherwise. We require that $i_0 < i_1 < \cdots < i_m$, $k < m$ and $(k+1)_m$ denotes $k+1$ modulo m. Moreover, $\Pi \cup \Sigma$ contains no $\Box$-formulas and Σ contains no $\to$-formulas. Note that the premise $\Gamma_\tau \Rightarrow_{(k+1)_m} \Delta_\tau$ differs from the conclusion only in the index. We call the rightmost premise the *right premise* and the others *left premises*.

(ii) Invertible rule applications leave the index of a sequent unchanged.

Lemma 5.10 *If T is a proof-search tree for σ with choice rule $\mathsf{C_f}$, then a winning strategy for Prover in $G(T, \mathsf{C_f})$ corresponds to a proof of σ.*

Proposition 5.11 *If a sequent has a (indexed) refutation with C-rule $\mathsf{C_f}$, then it has a functional countermodel.*

Proof. Let σ be a sequent and T be a proof-search tree based on some enumeration $\chi_1, \dots, \chi_n$ of $\Box^{-1}\mathsf{Cl}(\sigma)$. Let S be a subtree of T that is an indexed refutation of σ and let $M = (W, \leq, R, V)$ be the canonical model based on S. Note, R is functional, as every $\mathsf{C_f}$-rule application has only one right premise.

Let φ be a formula. By induction on the logical complexity of φ, we simultaneously prove that for any $w \in W$ we have (a) $M, w \models \varphi$ if $\varphi \in \Gamma_w^-$ and (b) $M, w \not\models \varphi$ if $\varphi \in \Delta_w^-$. We only treat the connective $\Box$.

(a). If $\Box\varphi \in \Gamma_w^-$ and $w \leq vRu$ then, by definition of the $\mathsf{C_f}$-rule and the fact that invertible rules are applied preservingly, $\varphi \in \Gamma_u^-$. The IH then implies $M, u \models \varphi$, so $M, w \models \Box\varphi$.

(b). Let $\Box\varphi \in \Delta_w^-$. As σ_w is the conclusion of the $\mathsf{C_f}$-rule, it must be of the form $\Pi, \Box\Gamma \Rightarrow_k \{\varphi_i \to \psi_i\}_{i=0}^{l}, \{\Box\chi_{i_j}\}_{j=0}^{m}, \Sigma$ with $\varphi = \chi_{i_p}$ for some p. Now, by construction of $\leq$ and the rule $\mathsf{C_f}$, it follows that there exists a $v \geq w$ such that σ_v is equal to $\Pi, \Box\Gamma \Rightarrow_p \{\varphi_i \to \psi_i\}_{i=0}^{l}, \{\Box\chi_{i_j}\}_{j=0}^{m}, \Sigma$. So, by construction of R, there exists a u with vRu and $\chi_{i_p} \in \Delta_u^-$. The IH then implies $M, u \not\models \varphi$, so $M, w \not\models \Box\varphi$.

We conclude that the root of M falsifies the sequent σ. $\Box$

Corollary 5.12 nIM *is complete for* $\mathrm{IM_f}$.

[6] Strictly speaking, we only defined proof-search trees for 'plain' sequents, that is, sequents without an index. However, if we extend the syntax by allowing formulas of the form k with $k \in \omega$, then we can simply define an indexed sequent $\Gamma \Rightarrow_k \Delta$ as the plain sequents $k, \Gamma \Rightarrow \Delta$. We prefer the former notation as it highlights the specific role of the index k.

5.4 Completeness of the cyclic calculus

With completeness of the ill-founded calculus to hand, we are ready to prove completeness of the cyclic calculus and its single-conclusion version. We first show that our use of multi-conclusion sequents is not a real restriction.

Lemma 5.13 *If a single-conclusion sequent σ has a* nIM*-proof, then it has a single-conclusion* nIM*-proof.*

Proof. Given any nIM-proof π, let its *trunk* π_{tr} be the derivation obtained from π by cutting off each branch after the lowest application of $\to\mathsf{R}$, $\Box$, $\bot$ or id. As π is a proof, note that π_{tr} must be finite. Working top-down, each sequent in π_{tr} will be replaced by a single-conclusion one. The first rule instances will be of type $\to\mathsf{R}$, $\Box$, $\bot$ or id, which are straightforward to treat; the premises are single-conclusion by definition, and if, for example, the conclusion is $\Pi, \Box\Gamma \Rightarrow \Box\varphi^a, \Delta$ with $\Box\varphi^a$ principal, replace it by $\Pi, \Box\Gamma \Rightarrow \Box\varphi^a$. Now consider a highest sequent $\Gamma \Rightarrow \Delta$ in π_{tr} with $|\Delta| > 1$. This will occur as the conclusion of a (possibly incorrect) rule instance

$$\frac{\sigma_1 \quad \cdots \quad \sigma_n}{\Gamma \Rightarrow \Delta}\ \mathsf{r}$$

with single-conclusion premises. Then, either (1) there exists a $\delta \in \Delta$ such that

$$\frac{\sigma_1 \quad \cdots \quad \sigma_n}{\Gamma \Rightarrow \delta}\ \mathsf{r}$$

is a correct instance of r, or (2) there is a premise σ_i such that $\Delta_{\sigma_i} \subseteq \Delta$. We treat $\mathsf{r} = {\to}\mathsf{L}$ as an exemplary case. If the conclusion is $\Gamma', \varphi \to \psi^{\mathsf{u}} \Rightarrow \Delta$ with $\varphi \to \psi^{\mathsf{u}}$ principal, then the premises are of the form $\Gamma', \varphi \to \psi^{\mathsf{u}} \Rightarrow \chi^a$ and $\Gamma', \psi^{\mathsf{u}} \Rightarrow \zeta^b$. If $\chi^a = \varphi^{\mathsf{u}}$, we pick $\delta = \zeta^b$. Otherwise, we must have $\chi^a \in \Delta$.

Property (1) means that $\Gamma \Rightarrow \Delta$ can be replaced by $\Gamma \Rightarrow \delta$, whereas (2) means that the node labelled by $\Gamma \Rightarrow \Delta$ can simply be deleted. Iterating this, we then obtain a single-conclusion derivation π_{tr}^{sc} such that replacing the trunk π_{tr} by π_{tr}^{sc} in π yields a nIM-proof π'. By construction, if π proves the sequent $\Gamma \Rightarrow \Delta$ then π' proves $\Gamma \Rightarrow \delta$ for some $\delta \in \Delta$.

Now let π be an nIM-proof of σ. Given a node s in π, we let $\uparrow s$ denote the nIM-proof induced by the upset of s in π. We define a sequence $(\pi_i)_{i<\omega}$ of finite, single-conclusion derivations as follows. Let π_0 be π_{tr}^{sc}, and given π_i, let π_{i+1} be the result of replacing each leaf s in π_i by the derivation $(\uparrow s)_{tr}^{sc}$. It is then easy to see that the limit π' of this construction gives a single-conclusion nIM-derivation of σ. Moreover, π' is a proof, as Lemma 3.4 and the fact that $\Box$ has a single-conclusion premise ensures that good suffixes are preserved. $\Box$

Theorem 5.14 *If a sequent σ has a (single-conclusion)* nIM*-proof, then it has a (single-conclusion)* cIM*-proof.*

Proof. Let π be a (single-conclusion) nIM-proof of σ. First note that π contains only finitely many sequents, as each such sequent only contains formulas in the finite set $\mathsf{Cl}(\sigma)$. Now let π' be the derivation obtained from π by cutting off each branch after the first successful repeat (if it exists). We prove that π' is finite. Suppose, for contradiction, that it is not. By König's lemma, π' then has an infinite branch ρ. Then ρ is also an infinite branch of the proof π, and thus it must have a good suffix ρ'. However, as ρ' contains only finitely many sequents, it follows that ρ' must contain a successful repeat. This contradicts that ρ is an infinite branch of π'. Thus π' is a (single-conclusion) cIM-proof. $\square$

From soundness of cIM (Theorem 4.3) and the two completeness results (Corollary 5.8 and 5.12) for nIM, we then obtain the following result.

Corollary 5.15 *The calculus* cIM *and its single-conclusion version are sound and complete for* $\mathrm{IM_K}$*,* $\mathrm{IM_t}$ *and* $\mathrm{IM_f}$*. In particular, we have* $\mathrm{IM_K} = \mathrm{IM_f} = \mathrm{IM_t}$.

6 Discussion

We close the paper by summarising some natural adaptions of the system cIM and possible interpretations of the language $\mathcal{L}_{\mathrm{IM}}$.

6.1 Intuitionistic temporal logic

The system cIM can be adapted to a sound and complete system cIM^s with respect to serial models and total functional models. [7] The modal rule $\square$ is replaced by the rule:

$$\frac{\Gamma \Rightarrow \Delta_0}{\Pi, \square\Gamma \Rightarrow \square\Delta, \Sigma}\ \square_s$$

where $\Delta_0 \subseteq \Delta$ and $|\Delta_0| \leq 1$. As in Section 5.3, completeness of the non-wellfounded calculus with respect to total functional frames is shown by proof-search on indexed sequents. The choice rule $\mathsf{C_f}$ is adapted so as to allow a right premise with an empty consequent in case the conclusion contains no $\square$-formula. As a result, each world in the induced canonical model necessarily has a modal successor, so the obtained countermodel will be total functional.

Completeness for total functional models induces an interpretation of the language $\mathcal{L}_{\mathrm{IM}}$ as an intuitionistic version of linear-time temporal logic (LTL). For each world w, the unique modal successor $R(w)$ may be interpreted as its temporal successor. The modal operator $\square$ is interpreted as the 'next' operator X and the master modality $\boxplus$ as the 'henceforth' operator. In contrast to classical LTL, the evaluation of a formula $\mathsf{X}\varphi$ at world w does not depend solely on $R(w)$, but also on $R(v)$ for all worlds $v \geq w$. As we have no confluence condition on $\leq$ and R, classical temporal tautologies such as $\mathsf{X}(\varphi \vee \psi) \to (\mathsf{X}\varphi \vee \mathsf{X}\psi)$ do not hold in this setting. The obtained temporal logic is therefore weaker than those considered in [4,1].

6.2 Intuitionistic common knowledge

Jäger and Marti introduce an intuitionistic version of common knowledge logic in [10] employing a polymodal extension of the language $\mathcal{L}_{\mathrm{IM}}$ with finitely many box operators $\square_0, \ldots, \square_n$. The formula $\square_i\varphi$ is read as *agent i knows φ* and $\boxplus\varphi$ as *φ is common knowledge*. This language is interpreted over triangle models with

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[7] We call a model $(W, \leq, R, V)$ *serial* if the relation R is serial and *total functional* if R is both serial and functional.

a modal relation for each $i \leq n$. Jäger and Marti present a finitary calculus for this logic based on an induction rule, which is complete for the class of triangle models and can be extended to complete calculi for reflexive models and S4-models. The proof of completeness, however, makes essential use of the cut rule, and a cut-elimination theorem is not given.

The calculus cIM can be adapted to the polymodal language by incorporating rules

$$\frac{\Gamma \Rightarrow \varphi}{\Pi, \Box_i \Gamma \Rightarrow \Box_i \varphi^a, \Delta}\ \Box_i$$

for each $i \leq n$, and appropriate modification of the rules for $\boxplus$. The resulting system cIM_p is easily shown to be sound and complete with respect to (polymodal) triangle models using the presented methods and appropriate adaption of the choice rule $\mathsf{C_t}$. Moreover, cIM_p (and, as it happens, cIM) can be extended to account for reflexive and for S4-models. For reflexive models we add for each i the rule $\Box_i^T$ below to cIM_p, and for S4-models we additionally replace the rules $\Box_i$ by $\Box_i^{S4}$:

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \Box_i \varphi \Rightarrow \Delta}\ \Box_i^T \qquad\qquad \frac{\Box_i \Gamma \Rightarrow \varphi}{\Pi, \Box_i \Gamma \Rightarrow \Box_i \varphi, \Sigma}\ \Box_i^{S4}$$

To establish completeness, the choice rule $\mathsf{C_t}$ needs only be adapted for S4-models, which is given by simply replacing Γ by $\Box_i \Gamma$ in the right premises of (the polymodal) $\mathsf{C_t}$. As cIM_p and its extensions are cut-free and analytic, they may be considered an improvement of Jäger and Marti's work. Whether cIM_p can be adapted to account for S5-models is unknown to us.

6.3 Future work

We have presented cyclic calculi for intuitionistic modal logic with $\Box$ and the master modality $\boxplus$. Two natural directions for further research are to extend the language by diamonds, or to allow for more fixed point operators. Concerning the former, note that $\Box$ and $\Diamond$ are not interdefinable in the intuitionistic setting. As a result, obtaining monotonicity in the presence of diamond operators requires other confluence conditions that are less robust than triangle confluence with respect to proof-search. It seems that more complex calculi are needed in this case, such as a nested or labelled calculi [20,8,5]. With respect to adding more fixed points, the current work seems to generalise more readily. A natural candidate in this regard is intuitionistic modal logic with $\Box$ and arbitrary least and fixed points. As triangle confluence still suffices in this general case, proof-search can be carried out in a similar fashion as done here.

Another open question is the complexity of the validity-checking problem for $\mathrm{IM_K}$. We conjecture that the validity problem has an EXPTIME upper bound and suspect that a similar approach as taken in [18] works: translate the calculus cIM into a parity game with a constant number of priorities. As such a game can be decided in polynomial time in the size of the arena (due to the fact that the number of priorities is constant [9]), and the size of the arena is exponential in the size of the formula (by analiticity of cIM), an exponential upper bound follows. Whether the lower bound is also exponential is unclear.

Appendix

A The single-conclusion calculus

$$\frac{}{\Gamma, \varphi^{\mathsf{u}} \Rightarrow \varphi^{a}}\ \mathsf{id} \qquad \frac{}{\Gamma, \bot^{\mathsf{u}} \Rightarrow \Delta}\ \bot$$

$$\frac{\Gamma, \varphi^{\mathsf{u}}, \psi^{\mathsf{u}} \Rightarrow \Delta}{\Gamma, \varphi \wedge \psi^{\mathsf{u}} \Rightarrow \Delta}\ \wedge\mathsf{L} \qquad \frac{\Gamma \Rightarrow \varphi^{\mathsf{u}} \quad \Gamma \Rightarrow \psi^{\mathsf{u}}}{\Gamma \Rightarrow \varphi \wedge \psi^{\mathsf{u}}}\ \wedge\mathsf{R}$$

$$\frac{\Gamma, \varphi^{\mathsf{u}} \Rightarrow \Delta \quad \Gamma, \psi^{\mathsf{u}} \Rightarrow \Delta}{\Gamma, \varphi \vee \psi^{\mathsf{u}} \Rightarrow \Delta}\ \vee\mathsf{L} \qquad \frac{\Gamma \Rightarrow \varphi_i^{\mathsf{u}}}{\Gamma \Rightarrow \varphi_0 \vee \varphi_1^{\mathsf{u}}}\ \vee_i\mathsf{R}$$

$$\frac{\Gamma, \varphi \to \psi^{\mathsf{u}} \Rightarrow \varphi^{\mathsf{u}} \quad \Gamma, \psi^{\mathsf{u}} \Rightarrow \Delta}{\Gamma, \varphi \to \psi^{\mathsf{u}} \Rightarrow \Delta}\ \to\mathsf{L} \qquad \frac{\Gamma, \varphi^{\mathsf{u}} \Rightarrow \psi^{\mathsf{u}}}{\Gamma \Rightarrow \varphi \to \psi^{\mathsf{u}}}\ \to\mathsf{R}$$

$$\frac{\Gamma, \varphi^{\mathsf{u}}, \Box\boxplus\varphi^{\mathsf{u}} \Rightarrow \Delta}{\Gamma, \boxplus\varphi^{\mathsf{u}} \Rightarrow \Delta}\ \boxplus\mathsf{L} \qquad \frac{\Gamma \Rightarrow \varphi^{\mathsf{u}} \quad \Gamma \Rightarrow \Box\boxplus\varphi^{a}}{\Gamma \Rightarrow \boxplus\varphi^{a}}\ \boxplus\mathsf{R}$$

$$\frac{\Gamma \Rightarrow \Delta^{\mathsf{u}}}{\Gamma \Rightarrow \Delta}\ \mathsf{u} \qquad \frac{\Gamma \Rightarrow \varphi^{\mathsf{f}}}{\Gamma \Rightarrow \varphi^{\mathsf{u}}}\ \mathsf{f}$$

$$\frac{\Gamma \Rightarrow \varphi^{a}}{\Pi, \Box\Gamma \Rightarrow \Box\varphi^{a}}\ \Box$$

Table A.1
The single-conclusion version of IM, where $|\Delta| \leq 1$.

B Soundness of the cyclic calculus

We present the omitted proofs from Section 4.

Lemma B.1 *If the conclusion of a rule instance* r *of* IM *is invalid, then there exists a premise of* r *that is invalid.*

Proof. Straightforward by inspection of the rules. $\Box$

Lemma B.2 *If* σ *has a formula in focus and is invalid, then there exists a natural number* n *such that* $\sigma(n)$ *is invalid.*

Proof. Let σ be an invalid sequent with a formula in focus. Then there exists a formula $\Box^j\boxplus\varphi^{\mathsf{f}} \in \Delta_\sigma$ for $j \in \{0, 1\}$, and a pointed model (M, w) with $M, w \not\models \sigma$. So in particular $M, w \not\models \Box^j\boxplus\varphi$. If $j = 0$, then there exists a world v with $w\tilde{R}^*v$ and $M, v \not\models \varphi$. Since $\leq$ is reflexive there are worlds $u_0, \ldots, u_{2n}$ such that $u_0 = w$, $u_{2n} = v$ and for all $0 \leq i < 2n$ holds that if i is even, then $u_i \leq u_{i+1}$ and if i i odd, then $u_i R u_{i+1}$. Therefore $w\tilde{R}^n v$, implying that $M, w \not\models \Box^n\varphi$. If $j = 1$, then there is a world v with $w\tilde{R}v$ and $M, v \not\models \boxplus\varphi$. By the previous case we have that $M, v \not\models \Box^n\varphi$ for some n. Hence $M, w \not\models \Box\Box^n\varphi$. Therefore $M, w \not\models \sigma(n)$ for some natural number n. $\Box$

Lemma B.3 *Suppose*

$$\frac{\sigma_1 \quad \cdots \quad \sigma_n}{\sigma}\ \mathsf{r}$$

is a rule instance of IM*. If σ is invalid, then there is an i such that σ_i is invalid. If both σ and σ_i have a formula in focus then, moreover,*

$$\mu(\sigma_i) \leq \mu(\sigma),$$

where the inequality is strict if $\mathsf{r} = \boxplus\mathsf{R}$ *and the principal formula is in focus.*

Proof. By Lemma B.1 it suffices to only consider the case where both the conclusion and at least one premise have a formula in focus. We first treat the case that the formula in focus is not principal. Then $\mathsf{r} \notin \{\to\mathsf{R}, \Box\}$, as this would contradict the existence of a premise with a focused formula. By inspection of the rules, note that then *every* premise must have a formula in focus, and so the following is a correct rule instance of r.

$$\frac{\sigma_1(\mu(\sigma)) \quad \cdots \quad \sigma_n(\mu(\sigma))}{\sigma(\mu(\sigma))}\ \mathsf{r}$$

By Lemma B.1, since $\sigma(\mu(\sigma))$ is invalid, there exists a premise $\sigma_i(\mu(\sigma))$ that is invalid. Hence σ_i is invalid and $\mu(\sigma_i) \leq \mu(\sigma)$.

Now suppose that the formula in focus is principal in r. Then $\mathsf{r} = \boxplus\mathsf{R}$ or $\mathsf{r} = \Box$. In the first case, σ is of the form $\Gamma \Rightarrow \boxplus\varphi^{\mathsf{f}}, \Delta$ with premises σ_1 and σ_2 given by $\Gamma \Rightarrow \varphi^{\mathsf{u}}, \Delta$ and $\Gamma \Rightarrow \Box\boxplus\varphi^{\mathsf{f}}, \Delta$, respectively. As there exists a pointed model (M, w) that falsifies $\sigma(\mu(\sigma))$, w has an intuitionistic successor v such that $M, v \models \Gamma$ and $M, v \not\models \boxplus\varphi \vee \Box^{\mu(\sigma)}\varphi \vee \bigvee \Delta^-$. If $\mu(\sigma) = 0$, then $M, v \not\models \varphi$, so (M, v) falsifies the left premise σ_1. By Lemma 3.2, σ_1 does not have a formula in focus, and so the statement of the lemma holds. If $\mu(\sigma) > 0$, then $M, v \not\models \Box\Box^{\mu(\sigma)-1}\varphi$. Hence (M, v) falsifies $\sigma_2(\mu(\sigma) - 1)$. So σ_2 is invalid and we have $\mu(\sigma_2) < \mu(\sigma)$.

In the second case, the conclusion σ is of the form $\Pi, \Box\Gamma \Rightarrow \Box\boxplus\varphi^{\mathsf{f}}, \Sigma$ and the single premise σ_1 of the form $\Gamma \Rightarrow \boxplus\varphi^{\mathsf{f}}$. Note that invalidity of $\sigma(\mu(\sigma))$ implies the invalidity of $\bigwedge \Gamma^- \to \Box^{\mu(\sigma)}\varphi$, which in turn implies invalidity of $\sigma_1(\mu(\sigma))$. So σ_1 is invalid and we have $\mu(\sigma_1) \leq \mu(\sigma)$. □

Theorem B.4 (Global soundness) *If there is a* cIM*-proof of a sequent σ, then σ is valid over the class of bi-relational models.*

Proof. Let π be a cIM-proof of σ and suppose for contradiction that σ is invalid. By repeatedly applying Lemma B.3 we obtain a path of invalid sequents

$$\rho = \sigma_1, \sigma_2 \ldots, \sigma_n$$

through π such that $\sigma = \sigma_1$ and σ_n is a leaf. As σ_n cannot be an axiom and π is a proof, there exists some σ_i such that (σ_i, σ_n) is a successful repetition. Then the path from σ_i to σ_n always has a formula in focus and passes through at least one instance of $\boxplus\mathsf{R}$ in which the formula in focus is principal. Hence, by construction, we have $\mu(\sigma_n) < \mu(\sigma_i)$, contradicting that $\sigma_n = \sigma_i$. □

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