Algebraic Monoidal Model Categories and Path Category Structures for Effective Kan Fibrations

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

In this document, we first develop a general framework to lift the pushout-product axioms in classical homotopy theory to the structured context of algebraic weak factorisation systems. The outcome is a notion of an *algebraic monoidal model category*. Within this framework, we are able to formulate and prove a structured analogue of Joyal-Tierney calculus, with the help of which we put an algebraic monoidal structure on *effective Kan fibrations*.

Effective Kan fibration is a new notion of fibration introduced in [63], in order to develop a constructive model of homotopy type theory on simplicial sets. In the second part of this document, we show there is a *path category* structure on the full subcategory of effective Kan fibrations over any object, using the algebraic monoidal structure constructed earlider. This brings us closer to showing the existence of a full model structure. Finally, we also identify a key semantic property, which we refer to as *Moore equivalence extension*. Based on the results in this document, we are able to show that if simplicial sets satisfy this property, then there exists a full algebraic monoidal model category structure for effective Kan fibrations.

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Contents

Chapter 1

Introduction

1.1 Background on Homotopy Type Theory

Martin-Löf type theory, henceforth MLTT, was initially formulated as a formal system to lay the foundation for constructive mathematics [45]. As its name indicates, it is a system of *types*, which we intuitively think of as collections of things. Each type can have *terms*, which we think of as elements belonging to that collection. The formal calculus of type theory is then a calculus about collections and their elements, within which we can express usual mathematical statements; see [33] or [58].

In contrast to set-theoretic foundation of mathematics, MLTT is not built upon some prior notion of *logic*. On the contrary, MLTT *subsumes* first-order intuitionistic logic by interpreting *propositions as (some) types*; see the exposition [70]. Intuitively, a proposition can be identified as some type, whose terms are regarded as its *proofs*. For various type theories, there is a precise sense in which terms of a type could be identified as proofs of a logical statement in natural deduction style. This correspondence is often called the *Curry-Howard isomorphism* [55]. As mentioned, another intuition for types is that they are certain *collections*, and terms of types will be *elements* of these collections. The crucial point is that MLTT provides a uniform calculus of propositions with their proofs, and mathematical objects with their elements.

A notable difference between the two approaches is the treatment of *equality*. In a set-theoretic foundation, if two sets X, Y are constructed, the equality proposition $X = Y$ will be an *external* statement in the underlying first-order logic. In particular, $X = Y$ does *not* belong to the domain of discourse of the set-theoretic foundation. In other words, $X = Y$ cannot be regarded as a set. Correspondingly, the deduction pricinples concerning equality statements are *not* specified by set theory itself, but are specified as structural rules in the underlying logic [24].

On the other hand, under the philosophy of propositions as types, equalities in MLTT are not treated as external predicates in the underlying logic, but as *types.* More precisely, given two terms of the same type $a, b : A$, there will be an *equality type* $a =_A b$, whose terms we now think of as proofs of the statement a is equal to b. This way, equality statements are *internalised* in MLTT. The deduction principles of equality, say reflexivity, symmetry, and transitivity, also *follows from* the calculus of equality types within MLTT.[1]

When first formulated by Per Martin-Löf, it wasn't clear how different the two approaches are. Related to this is the problem of *uniqueness of equality proofs* in MLTT: For any type A and terms $a, b : A$, given two proofs that a and b are equal, viz. given two terms $p, q : a =_A b$, is it the case that p, q themselves must be equal, in the sense that we can find an element $r : p =_{a=_A b} q$? Notice that this problem does not even make sense in the foundational approach based on first-order logic, since we are not able to state *within* the theory whether two proofs in this meta-theory are equal or not.

The intuition is that, if the principle of uniqueness of equality proofs holds, then our intuition for types in MLTT could be brought much closer to sets in the usual foundation of mathematics: We only need to worry about whether there exists a proof of $a =_A b$ and do not need to distinguish different proofs of equality, since any two such terms will be equal again, and thus will behave the same from the perspective of MLTT.

However, in the 1990s, Hofmann and Streicher established a *negative* answer to this in [31], by providing a *groupoid* model of MLTT. In this model, types are interpreted as groupoids. Closed terms of the type are interpreted as points in this groupoid, and identity proofs between closed terms are modelled as *paths* between the two points, i.e. morphisms in this groupoid. In particular, there could be distinct parallel arrows between two points in a groupoid, thus there could be multiple distinct proofs of a single equality type in this model.

This opened up the possibility of a *homotopical interpretation* of MLTT. The authors already conjectured in the *loc. cit.* that there could be a model of MLTT using higher groupoids, viz. homotopy types. Furthermore, foreshadowing the later development of homotopy type theory, they already realised that the groupoid model satisfies the socalled *universe extensionality* property, which is nothing more than the now famous univalence principle [69] restricted to 1-types.

The first full higher categorical model of equality types was constructed by Awodey and Warren in the paper [4], where they showed that Quillen model categories [46] can

[1] To be more precise, there is also a notion of external equality in MLTT, usual referred to as *definitional equality.* Given $a, b : A$, the statement that a, b are definitionally equal, denoted as $a \equiv b$, is also an external statement and is not represented as a type. However, $a \equiv b$ in general is strictly stronger than the fact that equality type $a =_A b$ has a term. The *mathematical equalities* we intend to formalise and reason about in type theory are still represented by the internal equality types.

be used to build models of equality types in MLTT. Later, [64] and [41] showed that the equality types in MLTT equip any type with the structure of a *weak ω-groupoid.* Around the same time, Voevodsky was independently developing a system called *homotopical λ-calculus*, and sketched an interpretation of this system in simplicial sets [68]. These works established an important link between type theory, homotopy theory, and higher category theory, where *homotopy type theory* was born. The above mentioned works also suggest that a more correct intuition about types in MLTT is that they represent certain *spaces*, rather than discrete collections of elements (cf. [54]).

Homotopy type theory, henceforth HoTT, is an extension of MLTT, with *higher inductive types* and the *univalence axiom* added. Higher inductive types allow one to perform internally in the type theory many constructions in classical homotopy theory, including cell complexes, homotopy colimits, truncations, etc.; see [42, 19].

The univalence principle mainly concerns the behaviour of equality type of *universes* in MLTT. In hindsight, if types are interpreted as spaces, or even as discrete sets, it should not be too surprising that the equality type for two such entities may possess non-trivial structures. Suppose MLTT is augmented with a universe $\mathcal{U}$, whose terms $A, B : \mathcal{U}$ we think of as types. In mathematical practice, the most useful understanding of the equality type $A =_{\mathcal{U}} B$ is not the strict equality in Frege's sense [22]. Rather, we view A, B as the same object when they are *isomorphic* or *equivalent* in some way. From this perspective, it seems natural that the type $A =_{\mathcal{U}} B$ may have multiple terms, since there can be more than one way two mathematical objects are equivalent. Univalence principle says exactly that the equality types $A =_{\mathcal{U}} B$ is characterised by the type of equivalences between A and B in a precise sense (cf. [69]).

From this perspective, the univalence axiom can also be viewed as an *internalisation* of the *invariance principle*: All properties definable in the type system are *invariant* under type *equivalence*; see [2]. Thus, HoTT provides a foundation for (higher) mathematics that is coherent with structuralism's philosophy, i.e. equivalent mathematical objects should never be distinguished.

1.2 The Problem of Computational Interpretation

As mentioned, MLTT was initially developped by Per Martin-Löf as a foundation for constructive mathematics. In particular, this system is itself computationally adequate, in the sense that it enjoys *normalisation*: Any well-typed term of MLTT has a normal form, and the normal form can be effectively computed from the term itself. Moreover, the normal forms of terms of base types like Nat or Bool are what we expect, i.e. natural numbers $\mathsf{s} \cdots \mathsf{s}0$ or booleans t, f. This property is often referred to as *canonicity* of type theory.

However, due to the fact that univalence is added as an *axiom* to MLTT in HoTT, the plain type system of HoTT looses this form of computational adequacy. Roughly speaking, an axiom in type theory is introduced as an additional term u of a special type. However, the mere existence of u does not tell one how to compute with it. If we write down a term of type Nat using u, when we compute this term we would get stuck at evaluating u and cannot proceed, thus canonicity will be violated.

A related problem is that the original model of HoTT based on the Kan-Quillen model structure on simplicial sets developped by Voevodsky, later presented in [37], is *not constructive*. As already described in [4], dependent types will be modelled as fibrations in a model category, and the fibrations in the Kan-Quillen model structure on simplicial sets are *Kan fibrations*.[2]. However, it is observed in [9] that one key semantic fact in building this model, that Kan fibrations are closed under pushforward along Kan fibrations, is not constructively provable. This property is equivalent to the socalled *Frobenius* property of Kan fibrations; see e.g. [65, 23, 63]. In the categorical semantics of type theory, pushforwards are used to model dependent products of types. Thus, this result means that if we model dependent types as Kan fibrations in the usual sense, then we cannot constructively prove that dependent types are closed under dependent products in this model.

Afterwards, the research on models of HoTT advanced quickly. At this point, we already know that all $(\infty, 1)$-toposes provide semantics for HoTT, thus HoTT can be used as an internal language of $(\infty, 1)$-toposes [53].[3] However, the proof of this result is again based on simplicial homotopy theory, and in particular relies on classical reasoning including using the axiom of choice.

One response to the problem of a computational understanding of HoTT is the so-called *cubical* approach. By working with cubical sets rather than simplicial sets, people have successfully shown the existence of a model structure, and built a model of HoTT. Furthermore, the semantic development of cubical sets has inspired the construction of a *cubical type theory*, within which univalence follows as a *theorem*, rather than being a postulate; see e.g. [7, 18, 1]. In particular, people have successfully shown that cubical type theory enjoys canonicity [20, 32] and normalisation [56].

However, these results are not fully satisfactory. For one thing, cubical type theory introduces new syntactic features such as interval types and face formulas extending the ordinary syntax of HoTT. This makes it harder to serve as an internal language for *all* ∞-toposes. In particular, at this point there are no corresponding results stating that all ∞-toposes can model cubical type theory. Furthermore, most of the cubical models constructed in the literature do not even represent the ∞-topos of ∞-groupoids classically;

[2] For an introduction to the Kan-Quillen model structure in simplicial homotopy theory, see e.g. [25].
[3] See [43] for the notion of $(\infty, 1)$-toposes.

for more details see [14].

Thus, we would still like to find a more general method of modelling HoTT constructively, which can be applied to simplicial sets at least. This computational understanding then has the potential for HoTT to benefit from the rich structure of homotopy theory and higher category theory.

One of the guiding principle of a constructive approach is to work with *structures* on maps, rather than treating them as *properties*. As shown in [23], one way to solve the mentioned failure of Kan fibrations being closed under pushforward constructively is to define a new notion of *uniform* Kan fibration, which coincides with the usual Kan fibrations *classically*. A uniform Kan fibration structure on a map assigns diagonal lifts against a class of lifting problems in a suitably compatible way. In particular, in the *loc. cit.* the authors are able to show constructively that uniform Kan fibrations satisfy the Frobenius property.

One key technical tool for carrying out this structured approach towards constructive model categories is the notion of an *algebraic weak factorisaion system* [27, 12, 11]. This notion "structuralises" the ordinary notion of weak factorisation systems, which are basic constituents of model structures. Based on this, there is a notion of an *algebraic model category* [47] that replaces the usual notion of a model category in the structured context.

To model the full system of HoTT, we also require that the algebraic model structure supports further features. For instance, to support *universes* in type theory, one needs to construct a *universal fibration* that classifies all small fibrations. The only known construction of such a universal fibration is due to Hofmann and Streicher [30], and it can only be applied when the fibration structure is *local*; see e.g. [63, Ch 2.]; an early definition also appeared in [53]. As mentioned, one can show constructively that uniform Kan fibrations satisfy Frobenius. Though the notion of uniform Kan fibration in [23] solves the problem raised in [9], this fibration structure is *not* local. A proof due to Sattler can be found in [63, App. D]. This suggests that uniform Kan fibrations cannot be used to build a full model of HoTT, either.

In this work, we tackle this problem based on the approach proposed in [63]. In particular, in *loc. cit.*, the authors describe a new structure of Kan fibrations, called *effective Kan fibrations*, and have shown constructively that effective Kan fibrations satisfy Frobenius. Another nice feature of this approach is that the notion of effective Kan fibrations does not depend on a particular presheaf model. Instead, it is defined *abstractly* on a variety of categories equipped with certain structures. The result that effective Kan fibrations satisfy Frobenius is proven in this abstract framework, which implies it can be applied to *any* situation where the axioms involved are satisfied.

The book [63] also contains a detailed discussion of the corresponding notion effective Kan fibration in simplicial sets. In particular, it proves that in simplicial sets, effective Kan fibration is a *local* notion of fibration structure. Furthermore, it is also classically correct, in the sense that any Kan fibration in the usual sense can be equipped with the structure of an effective Kan fibration if we use classical logic. Thus, this notion of an effective Kan fibration has the potential to serve as the basis for a constructive model of HoTT on simplicial sets.

Thus, the broader context and the underlying motivation behind this work is to approach the construction of an algebraic model structure using the notion of effective Kan fibration, so that it can be realised as a genuine constructive model of MLTT and HoTT. We will discuss the main contributions of this work in more detail in the next section.

1.3 Main Contributions

This document can be roughly divided in two parts. The first part lifts certain important techniques in classical homotopy theory to the structured context. This enriches the toolbox available for developing homotopy theory with algebraic weak factorisation systems. Building upon this, the second part of this document will show that the full subcategory of effective Kan fibrant objects over any object can be equipped with a path category structure. This will be an intermediate step which brings us closer to a full model structure. We will now discuss these two parts in more detail below.

1.3.1 Algebraic Monoidal Model Structures

One important fact about the classical Kan-Quillen model structure on simplicial sets is that it is *monoidal* w.r.t. Cartesian product. More concretely, this means that the *pushout-product* of two cofibrations is again a cofibration, and the pushout-product of cofibrations with trivial cofibrations are trivial fibrations. These are usually referred to as *pushout-product axioms* in the context of monoidal and enriched model categories; see [46] or [43].

From the works of Cisinski [16, 17], it becomes clear that the pushout-product axioms are extremely useful in the construction of a model structure. In particular, the mentioned pushout-product axioms help one to characterise weak equivalences between fibrant objects as certain homotopy equivalences. The latter is usually much easier to describe than the former. Thus, our first task is to study more closely how to express the pushout-product axioms structurally, and define an algebraic notion of monoidal model category.

Our approach is inspired by an unpublished note of Benno van den Berg and John Bourke [61], which uses *n-fold categories.* To motivate this choice, recall that in an algebraic weak factorisation system, the two classes of maps are in fact described by *double categories*; see Chapter 3. Thus, when expressing for instance that cofibrations are closed under pushout-product, we are in fact seeking a *functor*, which when given two maps equipped with cofibration structures, produces another cofibration structure on the pushout-product of the two underlying maps. This assignment should respect the double category structure on cofibrations, which means it should be functorial for both *horizontal morphisms* and *vertical compositions* between cofibrations.

However, the horizontal and vertical compatibility on *both* of the inputting cofibrations together makes the functoriality involved to be *higher dimensional*. Concretely, it means that we must have a proper framework that can describe how the resulting cofibration structure of the pushout-product functorially depends on the vertical and horizontal morphisms of both of the inputs. Also see the beginning of Chapter 4.

It turns out that the general theory of strict *n-fold categories* describes such a situation exactly. Concretely, n-fold categories are the n-fold iterations of the *internal category* construction over **Set**. In particular, 0-fold categories are simply sets; 1-fold categories are ordinary categories; and 2-fold categories are double categories. In Chapter 2, we will develop a presheaf model of general n-fold categories, for any $n \in \mathbb{N} \cup \{\infty\}$.

Before formulating a structured version of pushout-product axiom in Chapter 4, we will briefly recall the framework of algebraic weak factorisation systems in Chapter 3. We also take the chance to introduce the main examples we will be concerned with, viz. effective Kan fibration and other related notions defined in [63]. The majority of the contents in this chapter can either be found in the literature of algebraic weak factorisation systems [27, 12, 11], or in the book [63].

Chapter 4 contains one of the main results of this work. The unpublished note [61] has stated the possibility to realise the structured pushout-product axioms as morphisms in a certain *multicategory* of algebraic weak factorisation system. The construction of this multicategory is carefully given in Section 4.2. In fact, our version will be a modification of the proposed one in the *loc. cit.*, which makes it simultaneously more general and easier to work with.

The additional generality also allows us to further develop a structured formalism of *Joyal-Tierney calculus*, a result that expresses the fundamental duality between pushout-product and its dual notion, which is usually referred to as *pullback-exponential* in homotopy theory (cf. [36]). Practically, Joyal-Tierney calculus is a symbolic calculus that facilitates the computation of iterated lifting problems. Section 4.3 describes how the duality expressed in Joyal-Tierney calculus manifests itself in this structured context.

Based on this framework, the culmination of Chapter 4 is the notion of an *algebraic monoidal model category*. A version of this notion has also been considered by Riehl in [48]. However, as also felt by the authors of the unpublished note [61], the version given in the *loc. cit.* is too weak since the description of pushout-product axioms there does not involve functoriality of vertical compositions. In Section 4.4, we will provide our axiomatisation of this concept using *monoids* and *bimodules* in the multicategory constructed in Section 4.3. As a bonus, we will also show that the structured Joyal-Tierney calculus can be used to give equivalent formulations of such algebraic structures, either by considering pushout-products or pullback-exponentials.

As a major application, in Chapter 5 we will construct an algebraic monoidal structure for effective Kan fibrations and other related algebraic weak factorisation systems.

1.3.2 Path Category Structures on Effective Kan Fibrations

The notion of a path category is introduced in [66], and is a slight strenghening of the notion of a *category of fibrant objects* à la Brown [13]. In particular, if all objects in a model category are cofibrant, as in the Kan-Quillen model structure on simplicial sets, then the full subcategory of fibrant objects will be an example of a path category. Similar to the framework of category of fibrant objects, many familiar notions and results in homotopy theory can already be formulated in the context of path categories.

Moreover, path categories are more closely connected to the syntax of type theory. It is shown in [59] that the syntactic category of any type theory with *propositional equality types* can be equipped with the structure of a path category; on the other hand, any path category can serve as a model of type theory with propositional equality types.[4] It is also observed in [60] that univalence can also be formulated for path categories. Thus, the notion of a path category is a good intermediate step that bridges the syntactic side of type theory and the semantic side of model categories.

In the section part of this document, we will first show in Chapter 6 that the full subcategory of (effective) Kan fibrations over another object can be equipped with a path category structure. As mentioned, through the work of Cisinski, the monoidal structure developed in Chapter 5 will be extremely useful here. The homotopical techniques involved of proving this result is largely inspired Cisinski's work [17] and another unpublished note of Benno van den Berg and Eric Faber [62].

As also suggested in [62], to extend the above result to a full algebraic model structure for effective Kan fibrations, one possible approach is to follow the argument given in [52]. We carefully examine such an approach in Chapter 7. The main contribution there is that, in Section 7.2, we have identified a single key property, which we call *Moore equivalence extension*, such that if it holds for effective Kan fibrations, we will then be able to show the existence of a full algebraic monoidal model structure.

[4] Propositional equality types are weaker than the usual equality types appearing in HoTT, in that its computational rule does not hold definitionally, but only propositionally, i.e. is witnessed by another term in a higher equality type.

Chapter 2

Formal Theory of n-Fold Categories

As already discussed in Section 1.3, to properly develop the pushout-product axioms in the structured context, we need a language that can formulate the compositional structures of n-fold categories.[1]

Intuitively, an n-fold category contains a collection of objects with morphisms in n different dimensions. It also specifies higher dimensional morphisms forming squares, cubes, etc., among the arrows in different dimensions, with strict compositional operators that "past" these (higher) cubes together.

As mentioned in the Introduction, formally n-fold categories are specified as the n-fold iteration of the internal category construction, which is originally due to Ehresmann [21]. However, this inductive formulation makes it harder to use in practice.

Given the intuition above, in this chapter we develop the formal theory of n-fold categories via *presheaf models*, based on a version of cubical sets. We call them *D-cubical sets*, short for *dimensional cubical sets*, where we emphasis the cubical operations are book keeping tools for different dimensions. The name is to distinguish it from the various other cubical sets used in homotopy theory and cubical type theory, which are arguably more topologically minded.

In Section 2.1 we will introduce the underlying site $\mathbb{C}$ of D-cubical sets, which we call the *shape category of D-cubes*. In fact, there is a family of such categories, each corresponding to a selection of different dimensions. We will show that the (opposite) category of D-cubes form an elegant Reedy category. D-cubical sets are thus defined as *copresheaves* on $\mathbb{C}$, and can be viewed as a collection of higher dimensional cubes with specified faces and degeneracies.

Section 2.2 provides the definition of n-fold categories via D-cubical sets. The definition there makes it clear that n-fold categories are D-cubical sets equipped with further

[1] Besides explicitly mentioning, all n-fold categories in this document are assumed to be *strict*.

compositional data, which means we can glue cubes together with correct boundaries. It turns out that the presheaf description of n-fold categories given in this section is essentially the same as that given in [26, Ch. 6]. However, we furthermore provide the correctness proof of this presheaf model, in the sense that we show the category of n-fold categories thus defined is equivalent to the n-fold iteration of the internal category construction in **Set**.

Finally, at the end of this chapter, Section 2.3 will define a notion of *n-fold structures*, which is simply an $n+1$-fold category with one special dimension. This notion will be the basis of Chapter 4 where we define the structured version of pushout-product axioms and define algebraic monoidal model categories.

2.1 D-Cubical Sets

Let us define the category $\mathbb{C}$ of the shape category of D-cubes, short for the category of cube dimensions. Let $V = \{v_1, v_2, \cdots\}$ be a countably infinite set of variable names, which we refer to as *dimensions*.[2] We will also use $x, y, z, \cdots$ as *meta-variables* for these variable names.

Definition 2.1. The *shape category of D-cubes* $\mathbb{C}$ is defined as follows:

- Its objects are finite subsets of V;
- Morphisms $f : I \longrightarrow J$ in $\mathbb{C}$ are *functions* $f : I \longrightarrow J \cup \{0, 1\}$,[3] where for any $x \in I$ we have $f(x) \in \{x, 0, 1\}$. ◇

The composition of morphisms in $\mathbb{C}$ is given as follows: For any two maps $f : I \longrightarrow J$ and $g : J \longrightarrow K$, the composite map

$$gf : I \longrightarrow K$$

sends $x \in I$ to x, if $f(x) = x$ and $g(x) = x$. Otherwise, as long as $f(x) = i$ or $g(x) = i$, then $gf(x) = i$. The identity on I simply takes any $x \in I$ to itself. It is easy to see that the composition is associative and unitary.

For any $I \in \mathbb{C}$ and any $x \in I$, let us denote the subset $I - \{x\}$ as I^x. Then there are two canonical *face maps* $\delta_0^x, \delta_1^x : I \longrightarrow I^x$ in $\mathbb{C}$ as follows,

$$\delta_0^x(y) = \begin{cases} y & y \neq x \\ 0 & y = x \end{cases}$$

[2]For the constructively minded readers: Formally, V should be viewed as a copy of $\mathbb{N}$, so that it is also *decidable*, i.e. has decidable equality. This justifies the use of case distinctions in the following texts.

[3]We assume that the variable names in V signifying dimensions are *disjoint* from $\{0, 1\}$.

$$\delta_1^x(y) = \begin{cases} y & y \neq x \\ 1 & y = x \end{cases}$$

More generally, given a map $f : I \longrightarrow J$ in $\mathbb{C}$, we say it is *face map* on $x \in I$ if $f(x) = 0$ or $f(x) = 1$. There is also a canonical *degeneracy map* $\rho^x : I^x \longrightarrow I$,

$$\rho^x(y) = y.$$

Similarly, we say a map $f : I \longrightarrow J$ is a *degeneracy map* if $f(x) = x$ for all $x \in I$. The first observation is that these two classes of maps generate the entire shape category $\mathbb{C}$ of D-cubes:

Lemma 2.2. *Morphisms in $\mathbb{C}$ are generated by d_0, d_1, ρ under the following equations:*

- *For any $x \in I$,*
$$\delta_0^x \rho^x = 1, \quad \delta_1^x \rho^x = 1,$$
- *For any $x \neq y \in I$,*
$$\delta_0^y \rho^x = \rho^x \delta_0^y, \quad \delta_1^y \rho^x = \rho^x \delta_1^y.$$
- *For any $x \neq y \in I$,*
$$\delta_i^x \delta_j^y = \delta_j^y \delta_i^x, \quad \rho^x \rho^y = \rho^y \rho^x.$$
for any $i, j \in \{0, 1\}$.

Proof. It is easy to verify the correctness of the above equations. Furthermore, for any $f : I \longrightarrow J$ in $\mathbb{C}$, it has a unique factorisation as follows,

$$I \overset{\delta}{\twoheadrightarrow} I^f \overset{\rho}{\rightarrowtail} J$$

where I^f is the subset of I consisting of those elements x that $f(x) = x$. Here d can be realised as a composite of δ_i^x and similarly ρ can be realised as a composite of ρ^y. This concludes the proof. □

In the homotopy theory of diagrams in a model category, one useful notion is that of a *Reedy structures*:

Definition 2.3. A Reedy category $\mathcal{R}$ is a category equipped with two wide subcategories $\mathcal{R}_+, \mathcal{R}_-$, and a total ordering $|-| : \mathbf{Ob}(\mathcal{R}) \longrightarrow \mathbb{N}$ on objects called *degree*, such that:

- Every non-identity map in $\mathcal{R}_+$ ($\mathcal{R}_-$) increases (reduces) degrees strictly;

- Every map in $\mathscr{R}$ factors uniquely as a map in $\mathscr{R}_-$ followed by one in $\mathscr{R}_+$. ◊

For further references on the importance of Reedy structures, see [50, 49]. We can indeed put a *Reedy structure* on $\mathbb{C}$:

- For any $I \in \mathbb{C}$, its *degree* $|I|$ is defined as the *cardinality* of I.
- The two family of classes $\mathbb{C}_-, \mathbb{C}_+$ can be defined as composites of face maps and degeneracy maps, respectively.

We have also shown in Lemma 2.2 that any morphism $f : I \longrightarrow J$ in $\mathbb{C}$ can be uniquely factored as

$$I \xrightarrow{\delta} I^f \xrightarrow{\rho} J$$

where now $\delta \in \mathbb{C}_-$ and $\rho \in \mathbb{C}_+$. In fact, the Reedy structure on (the opposite of) $\mathbb{C}$ turns out to also be *elegant* (cf. [6]), which means that every cospan of maps in $\mathbb{C}_+$ has an absolute pullback in $\mathbb{C}_+$. However, since we will not use this result in the future, we do not include a proof here.

More generally, for any $I \in \mathbb{C}$, there is a full subcategory of $\mathbb{C}$, denoted as $\mathbb{C}_I$, whose objects are *subsets* of I. For $n \in \mathbb{N}$, we also denote V_n as the subset of V consisting of those elements $\{v_1, \cdots, v_n\}$, and we use $\mathbb{C}_n$ to denote $\mathbb{C}_{V_n}$. It is easy to see that the elegant Reedy structure on $\mathbb{C}$ is also inherited in these full subcategories.

From an external perspective, the shape category $\mathbb{C}_I$ only depends on the *degree* of I:

Lemma 2.4. *For any I, J, there is an isomorphism $\mathbb{C}_I \cong \mathbb{C}_J$ iff $|I| = |J|$.*

Proof. Evidently, any isomorphism $\mathbb{C}_I \cong \mathbb{C}_J$ is induced by an isomorphism $I \cong J$. □

This implies that in the formulation of n-fold categories in the next section, we only need to consider categories of the form $\mathbb{C}_n$. However, the categories $\mathbb{C}_I$ for an arbitrary finite set $I \subseteq V$ are still useful, as we will see.

A *D*-cubical set is nothing but a copresheaf on the shape category of D-cubes:

Definition 2.5. A *D-cubical set* is a *copresheaf* on $\mathbb{C}$. The category of D-cubical sets will be denoted as **cSet**. Similarly, the category of copresheaves on $\mathbb{C}_I$ will be denoted as $\mathbf{cSet}_I$, whose objects will be called *I-truncated* D-cubical sets, or simply an I-D-cubical set. ◊

As mentioned, we think of elements in V as dimensions. Intuitively, given any cubical set $X : \mathbb{C} \longrightarrow \mathbf{Set}$ and any $I \in \mathbb{C}$, we can view X_I as the set of *higher cubes* extending in dimensions in I, or simply *I-cubes*. For instance, $X_\varnothing$ is the set of *objects*, $X_{\{x\}}$ is the set of *arrows* along dimension x, and $X_{\{x,y\}}$ is the set of *squares* along dimension x, y, etc..

The action of the face maps and degeneracies maps in $\mathbb{C}$ gives the boundary and degeneracies of the cubes, respectively. For example, given any $x \in I$, the induced maps

$$\delta_0^x, \delta_1^x \,:\, X_I \longrightarrow X_{I^x}$$

takes any I-cube to the two one dimensional lower faces along dimension x. Similarly, the degeneracy

$$\rho^x \,:\, X_{I^x} \longrightarrow X_I$$

takes any I^x-cube to the degenerate I-cube, whose extension along dimension x is intuitively trivial. This way, the notion of D-cubical sets provides a convenient framework for defining n-fold categories, which is the task of the next section.

2.2 n-Fold Categories

To simultaneously work with n-fold categories for all $n \in \mathbb{N}$ and also possibly for $n = \infty$, from now on we assume n to be an element in the augmented natural numbers $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. We have defined the category $\mathbb{C}_n$ for $n \in \mathbb{N}$, and we take $\mathbb{C}_\infty$ to mean $\mathbb{C}$ itself. Similarly, we have the category of n-trucated D-cubical sets $\mathbf{cSet}_n$, where now $n \in \overline{\mathbb{N}}$, with $\mathbf{cSet}_\infty$ standing for $\mathbf{cSet}$.

As mentioned at the end of the previous section, for any n-D-cubical set

$$A \,:\, \mathbb{C}_n \longrightarrow \mathbf{Set},$$

we think of A_I as the set of higher cubes along dimensions in I. Furthermore, for any $I \in \mathbb{C}_n$ and $x \in I$, we have a reflexive pair

$$\begin{tikzcd} A_I \arrow[r, shift left=2, "\delta_0^x"] \arrow[r, shift right=2, "\delta_1^x"'] & A_{I^x} \arrow[l, "\rho"'] \end{tikzcd}$$

which takes faces and degeneracies of higher cubes. Given any higher cube $a \in A_I$, we also write

$$a \,:\, b \longrightarrow_x c$$

to mean that

$$\delta_0^x a = b, \quad \delta_1^x a = c.$$

An n-fold category is such an n-D-cubical set with additional *compositional data*, such that we can compose higher cubes with correct boundary conditions:

Definition 2.6. An *n-fold category* $\mathcal{A}$, for any $n \in \overline{\mathbb{N}}$, is an object in $\mathbf{cSet}_n$

$$A : \mathbb{C}_n \longrightarrow \mathbf{Set},$$

such that for any $I \in \mathbb{C}_n$ and $x \in I$ there is a map

$$\mu^x : A_I \times_{A_{I^x}} A_I \longrightarrow A_I,$$

where the domain is the pullback of δ_0^x and δ_1^x,

$$A_I \times_{A_{I^x}} A_I = A_I \;{}_{\delta_1^x}\!\times_{\delta_0^x} A_I.$$

Given $a, b \in A_I$, we say they are *composible along dimension* x, if

$$\delta_1^x a = \delta_0^x b.$$

For such a, b, we also write $b \circ_x a$ for $\mu^x(a, b)$. These composition operators are subject to the following requirements:

- *Unitary* along any dimension: For any $a : a_0 \longrightarrow_x a_1$ in A_I,

  $$a \circ_x \rho^x a_0 = a = \rho^x a_1 \circ_x a.$$

- *Associativity* along any dimension: For $a, b, c \in A_I$ composible along dimension x,

  $$c \circ_x (b \circ_x a) = (c \circ_x b) \circ_x a.$$

- *Naturality*: For any $f : I \longrightarrow J$ such that $fx = x$,

  $$\begin{tikzcd} A_I \times_{A_{I^x}} A_I \arrow[r, "\mu^x"] \arrow[d, "f\times f"'] & A_I \arrow[d, "f"] \\ A_J \times_{A_{J^x}} A_J \arrow[r, "\mu^x"'] & A_J \end{tikzcd}$$

  In equation, for any composible $a, b \in A_I$ along dimension x, we have

  $$f(b \circ_x a) = f(b) \circ_x f(a).$$

- *Interchange*: For any $x \neq y \in I$ and $a, b, c, d \in A_I$ such that a, c and b, d are composible along dimension x, and a, b, c, d are composible along dimension y,

  $$(d \circ_x b) \circ_y (c \circ_x a) = (d \circ_y c) \circ_x (b \circ_x a). \qquad \Diamond$$

The notion of an *n-fold functor* between n-fold categories now is straight forward:

Definition 2.7. An *n-fold functor* between two n-fold categories

$$F : \mathscr{A} \longrightarrow \mathscr{B}$$

is a natural transformation $F : A \longrightarrow B$ in $\mathbf{cSet}_n$ which preserves all the compositions: For any $I \in \mathbb{C}_n$ and $x \in I$, for any $a, b \in A_I$ composible along dimension x,

$$F(b \circ_x a) = F(b) \circ_x F(a). \qquad \diamondsuit$$

Notice that since in Definition 2.6, identities are modelled via degeneracies, naturality of F already ensures that it preserves identities, thus we only need to additional specify it preserves the composition operator.

We will write $\mathbf{Cat}^n$ for the category of n-fold categories. Under Definitions 2.6 and 2.7, it is not hard to verify directly that $\mathbf{Cat}^0 \cong \mathbf{Set}$ and $\mathbf{Cat}^1 \cong \mathbf{Cat}$. Going one dimension up, $\mathbf{Cat}^2$ is the category of *double categories*; see e.g. [21, 26].

As mentioned, for $n \in \mathbb{N}$, our notion of n-fold categories should be *correct*, in the sense that $\mathbf{Cat}^{n+1}$ should be the category of internal categories in $\mathbf{Cat}^n$. To prove this, the following auxiliary notion turns out to be useful:

Definition 2.8. An *I-restricted* n-fold category $\mathscr{A}$ is the same as an n-fold category, but with composition restricted to dimensions μ^x only for those $x \in I$. $\diamondsuit$

Similarly, when we talk about *m-restricted* n-fold categories for $m \le n$, we mean V_m-restricted n-fold categories. The category of I-restricted n-fold categories will be denoted as $\mathbf{Cat}^n_I$. By definition,

$$\mathbf{Cat}^n_\varnothing \cong \mathbf{cSet}_n, \quad \mathbf{Cat}^n_n \cong \mathbf{Cat}^n.$$

For any left exact category $\mathscr{C}$, viz. categories with finite limits, there is a notion of *internal categories*; see e.g. [34, Ch. B1]. Concretely, an internal category $\mathscr{A}$ in $\mathscr{C}$ is a diagram as follows,

$$A_1 \ {}_t\!\times_s\! A_1 \xrightarrow{\ \mu\ } A_1 \underset{t}{\overset{s}{\underset{\longleftarrow e \longrightarrow}{\rightrightarrows}}} A_0$$

satisfying the algebraic definition of being a category. In fact, there is a 2-functor

$$\mathbf{Cat}(-) : \mathbf{Lex} \longrightarrow \mathbf{Lex},$$

taking a left exact category to its category of *internal categories*. To show

$$\mathbf{Cat}^{n+1} \cong \mathbf{Cat}(\mathbf{Cat}^n),$$

we intend to prove the following more general fact that for any $x \in I$

$$\mathbf{Cat}^n_I \cong \mathbf{Cat}(\mathbf{Cat}^{n-1}_{I^x}),$$

We first observe the following fact on the level of sites:

Lemma 2.9. *For any $I \in \mathbb{C}$ and $x \in I$, we have*

$$\mathbb{C}_I \cong \mathbb{C}_{I^x} \times \mathbb{C}_1.$$

Proof. For convenience, we rename the objects and arrows in $\mathbb{C}_1$ as follows,

$$1 \underset{\partial_1}{\overset{\partial_0}{\underset{\longleftarrow e \longrightarrow}{\longrightarrow}}} 0$$

For any $J \in \mathbb{C}_I$, we identify J as the pair (J^x, i) where $J^x = J$ if $x \notin J$, and in that case $i = 0$; otherwise, J^x as previously defined is the set $J - \{x\}$, and in this case $i = 1$. This gives us a functor

$$\mathbb{C}_I \longrightarrow \mathbb{C}_{I^x} \times \mathbb{C}_1,$$

where on morphisms, we have for any $y \neq x \in I$,

$$d_i^y \longmapsto (d_i^y, 1), \quad \rho^y \longmapsto (\rho^y, 1),$$

while for x, we have

$$\delta_i^x \longmapsto (1, \partial_i), \quad \rho^x \longmapsto (1, e).$$

Verify that this gives us an isomorphism between categories is straightforward. □

The category $\mathbb{C}_1$ is the evidently *classifying category* of reflexive graphs, in the sense that if $\mathbf{Graph}_\mathrm{r}(-)$ denotes the 2-functor

$$\mathbf{Graph}_\mathrm{r}(-) : \mathbf{Cat} \longrightarrow \mathbf{Cat},$$

which takes the category of internal reflexive graphs in any category, then we have a natural isomorphism

$$\mathbf{Graph}_\mathrm{r}(\mathscr{C}) \cong [\mathbb{C}, \mathscr{C}],$$

for any category $\mathscr{C}$, where the latter is the functor category from $\mathbb{C}$ to $\mathscr{C}$.

The above result then firstly means we have the following:

Lemma 2.10. *For any $I \in \mathbb{C}$ and $x \in I$, we have*

$$\mathbf{cSet}_I \cong \mathbf{Graph}_\mathrm{r}(\mathbf{cSet}_{I^x}).$$

Proof. By Lemma 2.9, we have

$$\mathbf{cSet}_I \cong [\mathbb{C}_I, \mathbf{Set}] \cong [\mathbb{C}_{I^x} \times \mathbb{C}_1, \mathbf{Set}] \cong [\mathbb{C}_1, \mathbf{cSet}_{I^x}] \cong \mathbf{Graph_r}(\mathbf{cSet}_{I^x}).$$

The final isomorphism is due to the fact that $\mathbb{C}_1$ is exactly the classifying category for reflexive graphs. □

We can describe more clearly what is the constructed isomorphism between $\mathbf{cSet}_I$ and $\mathbf{Graph_r}(\mathbf{cSet}_{I^x})$. Given any reflexive graph in $\mathbf{cSet}_{I^x}$ as follows,

$$\begin{tikzcd} A_1 \arrow[r, shift left=2, "\partial_0"] \arrow[r, shift right=2, "\partial_1"'] & A_0 \arrow[l, "e" description] \end{tikzcd}$$

the associated I-D-cubical set A has components

$$A_J = \begin{cases} A_{0,J} & x \notin J \\ A_{1,J^x} & x \in J \end{cases}$$

It has evident face maps d_i^y for $y \in I^x$. For x itself, the face maps for any $J \ni x$ is given by

$$\delta_i^x = a \mapsto \partial_i a \;:\; A_J = A_{1,J^x} \longrightarrow A_{0,J^x} = A_{J^x},$$

for $i = 0, 1$; similarly, for $J^x \in \mathbb{C}_I$, the degeneracy map on x is given by

$$\rho^x = a \mapsto ea \;:\; A_{J^x} = A_{0,J^x} \longrightarrow A_{1,J^x} = A_J.$$

Naturality of ∂_0, ∂_1 and e makes sure that these face and degeneracy maps in the x-dimension interacts well with the other dimension, thus provides a well-defined I-D-cubical set.

With this set up, now we can show:

Proposition 2.11. *For any $n \in \mathbb{N}$, $I \in \mathbb{C}_n$ and $x \in I$, we have*

$$\mathbf{Cat}_I^n \cong \mathbf{Cat}(\mathbf{Cat}_{I^x}^{n-1}).$$

Proof. Consider an internal category in $\mathbf{Cat}_{I^x}^{n-1}$,

$$\begin{tikzcd} \mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_1 \arrow[r, "m"] & \mathcal{A}_1 \arrow[r, shift left=2, "\partial_0"] \arrow[r, shift right=2, "\partial_1"'] & \mathcal{A}_0 \arrow[l, "e" description] \end{tikzcd}$$

By the forgetful functor $\mathbf{Cat}_{I^x}^{n-1} \longrightarrow \mathbf{cSet}_{n-1}$, we get a reflexive graph in $\mathbf{cSet}_{n-1}$, which as we have shown corresponds to some $A \in \mathbf{cSet}_n$. We put a I-restricted n-fold category structure on A. Note that for any $y \neq x \in I$ and $J \not\ni x$ in $\mathbb{C}_I$, $A_J = A_{0J}$, thus we already have a multiplication coming from $\mathscr{A}_0$,

$$\mu^y = \mu_0^y : A_J \times_{A_{Jy}} A_J = A_{0J} \times_{A_{0,J^y}} A_{0J} \longrightarrow A_{0J} = A_J.$$

For those $J \ni x$ in $\mathbb{C}_I$, we also have a corresponding multiplication comming from $\mathscr{A}_1$,

$$\mu^y = \mu_1^y : A_J \times_{A_{Jy}} A_J = A_{1,J^x} \times_{A_{1,J^{x,y}}} A_{1,J^x} \longrightarrow A_{1,J^x} = A_J.$$

Finally, for the composition along dimension x, for any $J \ni x$ we use the multiplication $m : \mathscr{A}_1 \times_{\mathscr{A}_0} \mathscr{A}_1 \longrightarrow \mathscr{A}_1$ to construct the multiplication

$$\mu^x = m_{J^x} : A_J \times_{A_{Jx}} A_J = A_{1,J^x} \times_{A_0,J^x} A_{1,J^x} \longrightarrow A_{1,J^x} = A_J.$$

This newly defined composition along dimension x is evidently associative since m is; it is also unitary, because the degeneracy in dimension x in A by construction is given by e; finally, naturality of m ensures that μ^x satisfies interchange with other dimensions. □

Corollary 2.12. *For any $n \in \overline{\mathbb{N}}$ and $I \in \mathbb{C}_n$, $\mathbf{Cat}_I^n$ is complete and Cartesian closed.*

Proof. It is well-known that the category of internal categories in any complete and Cartesian closed category is again complete and Cartesian closed. □

Corollary 2.13. *For any $n \in \mathbb{N}$, we have*

$$\mathbf{Cat}^{n+1} \cong \mathbf{Cat}(\mathbf{Cat}^n).$$

In particular,

$$\mathbf{Cat}^n \cong \mathbf{Cat}^n(\mathbf{Set}).$$

Proof. This follows from Proposition 2.11, by realising

$$\mathbf{Cat}^{n+1} \cong \mathbf{Cat}^{n+1}_{n+1} \cong \mathbf{Cat}(\mathbf{Cat}^n_n) \cong \mathbf{Cat}(\mathbf{Cat}^n).$$

Unfolding this n times gives us $\mathbf{Cat}^n \cong \mathbf{Cat}^n(\mathbf{Set})$. □

2.3 n-Fold Structures

As mentioned in Chapter 1, our primary use for the theory of n-fold categories is to describe structure of morphisms on an ordinary category, and express their compositional properties. Thus, we would like to have a notion that treats n-fold categories more like higher dimensional structures on an ordinary category. This consideration leads us to define the following notion:

Definition 2.14. For any $n \in \overline{\mathbb{N}}$, an *n-dimensional structure* is an $n+1$-fold category $\mathscr{A}$, where one dimension is assumed to be *horizontal*, while the other n dimensions are assumed to be *vertical*. ◊

For convenience, when we think of n-dimensional structures, we always assume the horizontal dimension is to be labelled by a special variable name v_0, which is outside V, so that the above definition honestly applies also to the case when $n = \infty$.

In an n-dimensional structure, we always think of the horizontal dimension as the arrows in the underlying category, where this structure lives. More generally, given an n-dimensional structure $\mathscr{A}$, considered as an $n+1$-fold category on the variable set $\{v_0, \cdots, v_n\}$, for any $I \subseteq V_n = \{v_1, \cdots, v_n\}$ we may construct a *category* $\mathscr{A}_I$ as follows: Its objects are I-D-cubes in A_I, while a morphism from b to c in $\mathscr{A}_I$

$$a : b \longrightarrow_0 c$$

is a cube $a \in A_{I \cup \{v_0\}}$, with the two faces on 0-dimension given by x, y respectively. Composition is provided by the 0-dimension composition operator. This construction in fact gives us the structure of an $\mathbb{C}_n$-category

$$\mathscr{A}_- : \mathbb{C}_n \longrightarrow \mathbf{Cat},$$

where for any $f : I \longrightarrow J$ in $\mathbb{C}_n$ we have a functor

$$f : \mathscr{A}_I \longrightarrow \mathscr{A}_J$$

simply sending any $a \in A_I$ to $fa \in A_J$.

From this, we can define the *underlying category* $|\mathscr{A}|$ of an n-dimensional structure $\mathscr{A}$ as the evaluation on $\emptyset$, when viewed as an object in $[\mathbb{C}_n, \mathbf{Cat}]$. Concretely, objects of $|\mathscr{A}|$ are simply objects of $\mathscr{A}$, viz. $A_\emptyset$; morphisms of $|\mathscr{A}|$ are arrows in dimension 0.

To better describe this forgetful functor, we define a *2-category* of n-dimensional structures as a subcategory of $[\mathbb{C}_n, \mathbf{Cat}]$ which is *full on 2-cells*:

Definition 2.15. The *2-category of n-dimensional structures*, denoted as $\mathbf{Str}_n$, has objects as n-dimensional structures, 1-cells as $n+1$-fold functors, and 2-cells inherited from $[\mathbb{C}_n, \mathbf{Cat}]$. ◊

Concretely, a morphism between n-fold structures is simply an $n+1$-fold functor

$$F : \mathscr{A} \longrightarrow \mathscr{B}.$$

which in particular, induces a morphism in $[\mathbb{C}_n, \mathbf{Cat}]$,

$$F_- : \mathscr{A}_- \longrightarrow \mathscr{B}_-.$$

A 2-cell between two such morphisms F, G is a 2-cell in $[\mathbb{C}_n, \mathbf{Cat}]$, consisting of natural transformations for any $I \in \mathbb{C}_n$

$$\varphi_I : F_I \longrightarrow G_I,$$

which are compatible with actions in $\mathbb{C}_n$: For any $f : I \longrightarrow J$ in $\mathbb{C}_n$, we require the following diagram to commute,

$$\begin{tikzcd} \mathscr{A}_I \arrow[r, "f"] \arrow[d, bend right, "F_I"'] \arrow[d, bend left, "G_I"] \arrow[d, phantom, "= \varphi_I \Rightarrow"] & \mathscr{A}_J \arrow[d, bend right, "F_J"'] \arrow[d, bend left, "G_J"] \arrow[d, phantom, "= \varphi_J \Rightarrow"] \\ \mathscr{B}_I \arrow[r, "f"'] & \mathscr{B}_J \end{tikzcd}$$

The fact that we have realised the 2-category $\mathbf{Str}_n$ as a sub 2-category of $[\mathbb{C}_n, \mathbf{Cat}]$ gives us many ways to extract information from it. For instance, for any $I \subseteq V_n$, there is a corresponding composite as follows,

$$\mathbf{Str}_n \rightarrowtail [\mathbb{C}_n, \mathbf{Cat}] \xrightarrow{\mathrm{ev}_I} \mathbf{Cat}$$

where we denote this composite as

$$|-|_I : \mathbf{Str}_n \longrightarrow \mathbf{Cat}.$$

If $I = \emptyset$, we simply write $|-|$, and if $I = V_n$, we write $|-|_n$. This way, for any n-dimensional structure $\mathscr{A}$, it can be viewed as a structure on higher cubes on the underlying category $|\mathscr{A}|$, where this forgetful functor forgets all the vertical dimensions in $\mathscr{A}$. The n-forgetful functor $|\mathscr{A}|_n$ does not forget the structure on all the higher cubes, but it forgets all the composition structure on vertical directions.

In fact, the construction $\mathbf{Str}_-$ gives us a pseudo $\overline{\mathbb{N}}$-*graded monoid* of 2-categories; for the precise notion of a graded monoid w.r.t. another commutative monoid, see e.g. [38, Ch. 2].

Proposition 2.16. $\mathbf{Str}_-$ *gives us a pseudo* $\overline{\mathbb{N}}$*-graded monoid, such that there are 2-functors*

$$- \otimes - : \mathbf{Str}_n \times \mathbf{Str}_m \longrightarrow \mathbf{Str}_{n+m},$$

and a unit

$$1 : \mathbf{1} \longrightarrow \mathbf{Str}_0,$$

which is associative and unitary in the obvious sense.

Proof. Given any $\mathscr{A} \in \mathbf{Str}_n$ and $\mathscr{B} \in \mathbf{Str}_m$, for convenience let us assume that $\mathscr{B}$ is defined on the dimensions $\{v_0\} \cup V'_m$ where $V'_m = \{v_{n+1}, \cdots, v_{n+m}\}$.[4] Then the tensor product $\mathscr{A} \otimes \mathscr{B}$ as an $n+m$-fold category can be described as follows: For any $I \subseteq V_{n+m}$, we have

$$(\mathscr{A} \otimes \mathscr{B})_I := \mathscr{A}_{I \cap V_n} \times \mathscr{B}_{I \cap V'_m}.$$

In other words, an $I \cup J$-cube in $\mathscr{A} \otimes \mathscr{B}$ with $I \subseteq V_n$ and $J \subseteq V'_m$ simply consists of an I-cube in $\mathscr{A}$ and an J-cube in $\mathscr{B}$; similarly, a horizontal morphism between them is again a pair of horizontal morphisms in $\mathscr{A}$ and in $\mathscr{B}$. The composition operators of $\mathscr{A} \otimes \mathscr{B}$ are thus inherited from $\mathscr{A}$ and $\mathscr{B}$ in obvious ways.

The unit $1 \in \mathbf{Str}_0$ is simply the singleton set, which is evident that it is the unit. The pseudo associativity of $\otimes$ is inherited from that of the Cartesian product. □

According to Proposition 2.16, the underlying category of $\mathscr{A} \otimes \mathscr{B}$ is simply the Cartesian product of the underlying category of $\mathscr{A}$ and and that of $\mathscr{B}$,

$$|\mathscr{A} \otimes \mathscr{B}| \cong |\mathscr{A}| \times |\mathscr{B}|.$$

Thus, we think of the tensor product $\mathscr{A} \otimes \mathscr{B}$ as a way of combining an n-structure and an m-structure on two categories to an $n+m$-structure on their product. However, the first n dimensions and the last m dimensions have no interactions in this $n+m$-structure.

Since our aim is to use the notion of n-dimensional structures to describe structure of morphisms on an ordinary category, one important class of examples of higher dimensional structures will be the plain structure of commutative cubes in an ordinary category:

Example 2.17. For any $n \in \overline{\mathbb{N}}$ there is an associated n-dimensional structure, which we denote as $\mathrm{cube}_n(\mathscr{C})$. It suffices to note that for any $I \subseteq V_n$, the category $\mathrm{cube}_n(\mathscr{C})_I$ is defined as follows,

$$\mathrm{cube}_n(\mathscr{C})_I \cong [\mathbf{2}^I, \mathscr{C}],$$

where $\mathbf{2}^I$ is the functor category from the discrete set I to the classifying category of arrows $\mathbf{2} = \{0 < 1\}$. A functor $\mathbf{2}^I \longrightarrow \mathscr{C}$ is exactly a commutative higher cube in $\mathscr{C}$, and the horizontal morphisms between these are simply given by natural transformations. The composition operators of the higher cubes are evident. Concretely for low dimensions, $\mathrm{cube}_0(\mathscr{C})$ can be identified as $\mathscr{C}$ itself, while $\mathrm{cube}_1(\mathscr{C})$ is the *double category of arrows and commutative squares* in $\mathscr{C}$.

As mentioned, the tensor product of two higher dimensional structures can be viewed as a structure over the Cartesian product of the underlying categories. In the case of

[4] When $m = \infty$, we simply take V'_m to be $\{v_{n+1}, \cdots\}$; if both $n, m = \infty$, we simply take two disjoint infinite sets in V_n.

higher structure of commutative cubes for categories, given any categories $\mathscr{C}, \mathscr{D}$, the tensor product thus embeds in the following way,

$$\mathrm{cube}_n(\mathscr{C}) \otimes \mathrm{cube}_m(\mathscr{D}) \rightarrowtail \mathrm{cube}_{n+m}(\mathscr{C} \times \mathscr{D}).$$

The image of the tensor product in $\mathrm{cube}_{n+m}(\mathscr{C} \times \mathscr{D})$ consists of those $n+m$-dimensional cubes whose edges are *constant* on $\mathscr{D}$ along the first n dimensions, and *constant* on $\mathscr{C}$ along the last m dimensions. ◇

The content in this chapter will be used extensively in Chapter 4, where we give a structured treatment of the pushout-product axioms and define algebraic monoidal model categories. However, to be able to simultaneously apply these concepts to the primary examples we have in mind, in the next chapter we first briefly recall the framework of algebraic weak factorisation systems, and then discuss the example of effective Kan fibrations and various other related structures.

Chapter 3

Lifting Structures and Algebraic Weak Factorisation Systems

In this chapter we briefly recall the notion of *lifting structures* and *algebraic weak factorisation systems.* Most of the contents on these topics are not new, and can be found in [12, 27, 11], or the first part of the book [63]. However, to make this document self-contained, and to form a narrative that makes the various definitions seem more natural to the readers, we have included these contents here for sake of readability.

For the unstructured notion, a *weak factorisation system*, or *WFS*, on a category $\mathscr{C}$ consists of two classes of maps $(\mathbf{L}, \mathbf{R})$, satisfying the following conditions:

- Any map in $\mathbf{L}$ *lifts against* any map in $\mathbf{R}$;
- Any morphism in $\mathscr{C}$ *factors* as a morphism in $\mathbf{L}$ followed by a morphism in $\mathbf{R}$.
- $\mathbf{L}, \mathbf{R}$ are closed under *retracts.*

The three axioms can be viewed as the axiom of *lifting*, that of *factorisation*, and that of *retract*; see e.g. [49, 10] for more. In the structured context, the first two conditions are *algebraised*, while the third axiom is dropped.

Section 3.1 will introduce the algebraisation of the lifting axiom. In particular, we describe how the lifting condition is replaced by the algebraic notion of *lifting operators*, which in turn can be viewed as structures on maps in a category. In this section we will also introduce special classes of structures on a category $\mathscr{C}$, which we refer to as *left* and *right* structures. We will show that the lifting structures are always of these forms. The notions of left and right structures are fundamental to our discussion in the next chapter.

In Section 3.2 we will introduce the algebraisation of the factorisation axiom, and define the notion of an algebraic weak factorisation system. There we will also discuss the different roles played by the retract axiom in the structured and unstructured context.

Finally, Section 3.3 will list some of the important examples we intend to study in the remaining part of this document, including the notion of *effective Kan fibrations.* As mentioned in the Introduction, these examples are introduced and studied in the book [63], but we also include them here for completeness.

3.1 Lifting Structures

Let $\mathscr{C}$ be a category with finite limits and finite colimits. Recall from Section 2.2 that we use $\mathrm{cube}_1(\mathscr{C})$ to denote the *double category* of arrows of $\mathscr{C}$. A structure on arrows of $\mathscr{C}$ can thus be easily described as a double functor

$$\mathbb{L} \longrightarrow \mathrm{cube}_1(\mathscr{C}).$$

In particular, given any map $f : X \longrightarrow Y$ in $\mathscr{C}$, an $\mathbb{L}$-*structure* on f is a *vertical arrow* α in $\mathbb{L}$ over f. The fact that $\mathbb{L}$ is a double category over $\mathrm{cube}_1(\mathscr{C})$ signifies that there is a notion of *horizontal morphisms* of $\mathbb{L}$-structures over *squares* in $\mathscr{C}$, and that $\mathbb{L}$-structures can be vertically composed.

To better understand the situation, recall from Section 2.3 that for any n-structure, there is a notion of n-forgetful functor $|-|_n$. In this context, for the double category $\mathrm{cube}_1(\mathscr{C})$, its image under the 1-forgetful functor is the category of arrows in $\mathscr{C}$,

$$|\mathrm{cube}_1(\mathscr{C})|_1 = \mathscr{C}^{\rightarrow}.$$

Similarly, any double category $\mathbb{L}$ over $\mathrm{cube}_1(\mathscr{C})$, applying $|-|_1$ gives us an ordinary functor

$$\mathscr{L} \longrightarrow \mathscr{C}^{\rightarrow},$$

where $\mathscr{L}$ is the category of $\mathbb{L}$-structures on morphisms of $\mathscr{C}$ with *horizontal* maps. Thus, the difference between $\mathbb{L}$ and $\mathscr{L}$ is that the former also records how $\mathbb{L}$-structures can be vertically composed. For us, all of the examples considerred in this document will be *concrete* in the following sense:

Definition 3.1. We say $\mathbb{L} \longrightarrow \mathrm{cube}_1(\mathscr{C})$ is a *concrete* structure over $\mathscr{C}$, if its image under the 1-forgetful functor

$$\mathscr{L} \longrightarrow \mathscr{C}^{\rightarrow}$$

is *faithful.* ◊

By definition, the functor $\mathscr{L} \longrightarrow \mathscr{C}^{\rightarrow}$ being faithful simply means that it is a *property*, rather than additional data, that whether a square in $\mathscr{C}$ is a morphism between two $\mathbb{L}$-structures.

Besides being concrete, for the majority of structures arising in the context of homotopy theory, they satisfy additional closure properties w.r.t. pushouts or pullbacks. This leads us to define the following notions of left and right structures on a category:

Definition 3.2. A *left structure* on $\mathscr{C}$ is a concrete structure $\mathbb{L}$ over $\mathrm{cube}_1(\mathscr{C})$ that is furthermore *discretely opfibred*: For any vertical arrow α in $\mathbb{L}$ over f, and for any pushout $a : f \longrightarrow g$ in $\mathscr{C}$, there exists a *unique* horizontal morphism $a_* : \alpha \longrightarrow a_*\alpha$ in $\mathbb{L}$ over a, indicated as follows,

$$\begin{tikzcd} \alpha \arrow[r, dashed, "a_*"] \arrow[d, mapsto] & a_*\alpha \arrow[d, mapsto] \\ \bullet \arrow[r] \arrow[d, "f"'] & \bullet \arrow[d, "g"] \\ \bullet \arrow[r] \arrow[u, phantom, "\overset{a}{\Longrightarrow}\ \ulcorner" near end] & \bullet \end{tikzcd}$$

Similarly, a *right structure* on $\mathscr{C}$ is a concrete structure $\mathbb{R}$ over $\mathrm{cube}_1(\mathscr{C})$ that is *discretely fibred*, which means it has *unique* lifts against pullback squares in $\mathscr{C}$. ♢

Now if $\mathbb{L}$ is a left structure over $\mathscr{C}$, then the corresponding 1-category $\mathscr{L}$ can be described as a copresheaf

$$\mathscr{L} : \mathscr{C}^{\rightarrow}_{\mathrm{coCart}} \longrightarrow \mathbf{Set},$$

where $\mathscr{C}^{\rightarrow}_{\mathrm{coCart}}$ is the category of arrows in $\mathscr{C}$ with morphisms being pushout squares. For any morhpism f, $\mathscr{L}(f)$ will be the set of $\mathbb{L}$-structures over f. Functoriality of $\mathscr{L}$ is guaranteed by the uniqueness of lifts against pushout squares. Similarly, given a right structure $\mathbb{R}$ on $\mathscr{C}$, its underlying category $\mathscr{R}$ can be described as a presheaf

$$\mathscr{R} : (\mathscr{C}^{\rightarrow}_{\mathrm{Cart}})^{\mathrm{op}} \longrightarrow \mathbf{Set}.$$

The importance of the notion of left and right structures lies in the fact that the canonical structures arising from the algebraisation of *lifting* conditions will be of these forms. Let us first recall the notion of lifting condition in the usual unstructured context:

Definition 3.3. Given any maps $i : A \longrightarrow B$ and $f : X \longrightarrow Y$ in $\mathscr{C}$, we say i has the *left lifting property* against f, or equivalently f has the *right lifting property* against i, denoted as $i \pitchfork f$, if for any solid square as follows,

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r, "v"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

there exists a diagonal lift indicated as above. ♢

Given Definition 3.3, for any class of maps $\mathbf{L}$ on a category $\mathscr{C}$, one usually define the *right lifting class* $\mathbf{R}$ generated by $\mathbf{L}$ as follows,

$$\mathbf{R} = \{\, f \mid \forall i \in \mathbf{L}.\ i \pitchfork f \,\}.$$

We also say that $\mathbf{R}$ is *cofibrantly generated* by $\mathbf{L}$. Completely dually, given $\mathbf{R}$, there is also an associated *left lifting class* of $\mathbf{R}$, which is *fibrantly generated* by $\mathbf{R}$.

To algebraise this notion in the structured context, we not only care about the *proposition* that for any $i \in \mathbf{L}$, $i \pitchfork f$, but specific *lifting operators* that assigns f diagonal lifts to any lifting problem against a morphism $i \in \mathbf{L}$, preferrably in a certain compatible way. This leads to the following definition:

Definition 3.4. Let $\mathscr{L}$ be a category over the category of arrows $\mathscr{C}^{\rightarrow}$. The category $\mathscr{L}^{\pitchfork}$ of *right lifting structures* w.r.t. $\mathscr{L}$ is a category over $\mathscr{C}^{\rightarrow}$ as follows:

- An $\mathscr{L}^{\pitchfork}$-structure on a map $f : X \longrightarrow Y$ is a *right lifting operator* against $\mathscr{L}$, such that given any commutative diagram with an $\mathscr{L}$-structure α on i,

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "f"] \\
B \arrow[r, "v"'] \arrow[ur, dashed, "\varphi(\alpha)[u,v]" description] & Y
\end{tikzcd}$$

  φ assigns a diagonal lift $\varphi(\alpha)[u, v]$ indicated as above. If it is evident which square we are talking about, we also simply write $\varphi(\alpha)$. We require φ to be compatible with horizontal morphisms between $\mathscr{L}$-structures: For any morphism $\eta : \alpha' \longrightarrow \alpha$ of $\mathscr{L}$-structures over the square $a : i' \longrightarrow i$ in $\mathscr{C}$, we have

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "i'"'] & A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\
B' \arrow[r] \arrow[urr, dashed, "\varphi(\alpha')" description] & B \arrow[r] \arrow[ur, dashed, "\varphi(\alpha)" description] & Y
\end{tikzcd}$$

- A morphism $\eta : (f, \varphi) \longrightarrow (g, \psi)$ between two $\mathscr{L}^{\pitchfork}$-maps is a commutative square in $\mathscr{C}$, such that for any $\mathscr{L}$-map structure α on $i : A \longrightarrow B$, their corresponding lifts make the following diagram commute,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X \arrow[r] \arrow[d, "f"] & Z \arrow[d, "g"] \\
B \arrow[r] \arrow[ur, dashed, "\varphi(\alpha)" description] \arrow[urr, dashed, "\psi(\alpha)" description] & Y \arrow[r] & W
\end{tikzcd}$$

  The identity and composition of morphisms between $\mathscr{L}^{\pitchfork}$-structures is simply the identity and composition of squares in $\mathscr{C}$. ◊

Completely dually, given any category $\mathscr{R}$ over $\mathscr{C}^{\rightarrow}$, we can also define a category ${}^{\pitchfork}\mathscr{R}$ over $\mathscr{C}^{\rightarrow}$ of *left lifting structures* w.r.t. $\mathscr{R}$, whose objects are arrows in $\mathscr{C}$ equipped with left lifting operators against $\mathscr{R}$ that are compatible with horizontal morphisms in $\mathscr{R}$, and whose morphisms are squares in $\mathscr{C}$ compatible with the lifts.

Based on Definition 3.4, we can furthermore define the right and left lifting structures w.r.t. a *double category* over $\mathrm{cube}_1(\mathscr{C})$:

Definition 3.5. Let $\mathbb{L}$ be a double category over $\mathrm{cube}_1(\mathscr{C})$. The *double category* $\mathbb{L}^{\pitchfork}$ of *right lifting structures* over $\mathrm{cube}_1(\mathscr{C})$ is given as follows:

- An $\mathbb{L}^{\pitchfork}$-structure on a map $f : X \longrightarrow Y$ is similarly a *right lifting operator* φ against $\mathbb{L}$-structures, that besides being compatible with the horizontal morphisms in $\mathbb{L}$, we also require the following *vertical* compatibility: For $\mathbb{L}$-structures α on $i : A \longrightarrow B$ and β on $j : B \longrightarrow C$ with the following diagram in $\mathscr{C}$,

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[dd, "f"] \\ B \arrow[d, "j"'] \arrow[ur, dashed, "\varphi(\alpha)"] & \\ C \arrow[r, "v"'] \arrow[uur, dashed, "\varphi(\beta)"] & Y \end{tikzcd}$$

  we require that the map obtained by first lifing against α then lifting against β coincides with the lift against the vertical composite $\beta \bullet \alpha$ on ji. In equation form, this means
$$\varphi(\beta)[\varphi(\alpha)[u, vj], v] = \varphi(\beta \bullet \alpha)[u, v].$$

- A horizontal morphism $\eta : (f, \varphi) \longrightarrow (g, \psi)$ between two $\mathbb{L}^{\pitchfork}$-maps is similarly defined as a commutative square that are compatible with the lifting structures.

- There is also vertical composition on $\mathbb{L}^{\pitchfork}$-structures. Given φ on $f : X \longrightarrow Y$ and ψ on $g : Y \longrightarrow Z$, we define the vertical composition $\psi \bullet \varphi$ on gf as the following lifting operator,

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[dd, "i"'] & X \arrow[d, "f"] \\ & Y \arrow[d, "g"] \\ B \arrow[r, "v"'] \arrow[ur, dashed, "\psi(\alpha){[fu,v]}"] \arrow[uur, dashed, "\varphi(\alpha){[u,\psi(\alpha)]}"] & Z \end{tikzcd}$$

In other words, we have that

$$(\psi \bullet \varphi)(\alpha)[u, v] = \varphi(\alpha)[u, \psi(\alpha)[fu, v]]. \qquad \diamondsuit$$

It can be directly verified that the vertical composition of two $\mathbb{L}^{\pitchfork}$-structures is well-defined, thus $\mathbb{L}^{\pitchfork}$ is a well-defined double category over $\mathrm{cube}_1(\mathscr{C})$; see [11], also [63, Ch. 2.2]. Completely dually, for any double category $\mathbb{R}$ over $\mathrm{cube}_1(\mathscr{C})$, we can also define a double category ${}^{\pitchfork}\mathbb{R}$ of *left lifting structures* against $\mathbb{R}$.

Remark 3.6. Definition 3.4 and 3.5 have introduced an overloading of notations, i.e. the lifting structures for both 1-categories and double categories are denoted by the same operators $(-)^{\pitchfork}$ and ${}^{\pitchfork}(-)$, even though they are distinct constructions in separate cases. We believe though such a choice will not introduce confusion, since it will be clear from the contexts whether we apply them to a 1-category or a double category. $\diamondsuit$

Terminology 3.7. If $\mathbf{R} \cong \mathscr{L}^{\pitchfork}$ or $\mathbb{R} \cong \mathbb{L}^{\pitchfork}$, we say $\mathbf{R}, \mathbb{R}$ are *cofibrantly generated* by the category $\mathscr{L}$ or the double category $\mathbb{L}$, respectively. Dually, if $\mathbf{L} \cong {}^{\pitchfork}\mathscr{R}$ or $\mathbb{L} \cong {}^{\pitchfork}\mathbb{R}$, we say $\mathbf{L}, \mathbb{L}$ are *fibrantly generated* by the category $\mathscr{R}$ or the double category $\mathbb{R}$.[1] $\diamondsuit$

As mentioned, the right and left lifting structures against a double category are typical examples of left and right structures over a category:

Lemma 3.8. *For any double category $\mathbb{L}$ over $\mathrm{cube}_1(\mathscr{C})$, $\mathbb{L}^{\pitchfork}$ is a right structure on $\mathscr{C}$. Dually, for any double category $\mathbb{R}$ over $\mathrm{cube}_1(\mathscr{C})$, ${}^{\pitchfork}\mathbb{R}$ is a left structure on $\mathscr{C}$.*

Proof. From Definition 3.5, it is easy to see that $\mathbb{L}^{\pitchfork}$ is concrete over $\mathrm{cube}_1(\mathscr{C})$. Suppose g in $\mathscr{C}$ is equipped with a right lifting structure ψ against $\mathbb{L}$, and let f be a pullback of g. For any lifting problem of f against an $\mathbb{L}$-map α on i, we may define the lift φ on f as follows,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & Z \arrow[d, "g"] \\
B \arrow[r] \arrow[ur, dashed, "\varphi(\alpha)"] \arrow[urr, dashed, bend right, "\psi(\alpha)"'] & Y \arrow[r] & W
\end{tikzcd}$$

Here $\psi(\alpha)$ is induced by ψ, while $\varphi(\alpha)$ is induced by the universal property of the pullback. Since the induced lift $\varphi(\alpha)$ is unique, it is straight foward to verify that φ is a well-defined lifting structure on f. Dually, ${}^{\pitchfork}\mathbb{R}$ also forms a right structure on $\mathscr{C}$. $\square$

[1] Our choice of symbols here will be clear when we describe algebraic weak factorisation systems in Section 3.2.

In fact, the construction given in Definition 3.5 is *functorial*. Recall from Section 2.2 that we use $\mathbf{Cat}^2$ to denote the category of double categories. The operator $\mathbb{L} \mapsto \mathbb{L}^{\pitchfork}$ forms a functor

$$(-)^{\pitchfork} : (\mathbf{Cat}^2/\mathrm{cube}_1(\mathscr{C}))^{\mathrm{op}} \longrightarrow \mathbf{Cat}^2/\mathrm{cube}_1(\mathscr{C}).$$

Similarly, $\mathbb{R} \mapsto {}^{\pitchfork}\mathbb{R}$ can be viewed as a functor as follows,

$${}^{\pitchfork}(-) : \mathbf{Cat}^2/\mathrm{cube}_1(\mathscr{C}) \longrightarrow (\mathbf{Cat}^2/\mathrm{cube}_1(\mathscr{C}))^{\mathrm{op}}.$$

It is shown in [11] that these two functors are adjoint to each other:

Proposition 3.9. *There is an adjunction as follows,*

$$\begin{tikzcd}
(\mathbf{Cat}^2/\mathrm{cube}_1(\mathscr{C}))^{\mathrm{op}} \arrow[r, bend right, "(-)^{\pitchfork}"'] \arrow[r, phantom, "\perp"] & \mathbf{Cat}^2/\mathrm{cube}_1(\mathscr{C}) \arrow[l, bend right, "{}^{\pitchfork}(-)"']
\end{tikzcd}$$

Applying the 1-forgetful functor, there is a similar adjunction on the level of ordinary categories:

Corollary 3.10. *There is also an adjunction on the level of 1-categories,*

$$\begin{tikzcd}
(\mathbf{Cat}/\mathscr{C}^{\to})^{\mathrm{op}} \arrow[r, bend right, "(-)^{\pitchfork}"'] \arrow[r, phantom, "\perp"] & \mathbf{Cat}/\mathscr{C}^{\to} \arrow[l, bend right, "{}^{\pitchfork}(-)"']
\end{tikzcd}$$

3.2 Algebraic Weak Factorisation System

Recall from the start of this chapter that a WFS on $\mathscr{C}$ consists of two classes of maps $\mathbf{L}, \mathbf{R}$ satisfying the axioms of lifting, factorisation, and retracts. The previous section successfully algebraises the lifting axiom, now we proceed to algebraise the factorisation axiom.

What we want ultimately is a structured notion of lifting operators and factorisations, where the two interact nicely with each other. Most of the content of this section could be found in [12] or [11]. We include them here for the sake of readability.

To algebraise the notion of factorisation, we start with the following notion:

Definition 3.11. A *functorial factorisation* on $\mathscr{C}$ is a *section* of the composition functor

$$\mathscr{C}^{\to}{}_{\mathrm{dom}}\!\times_{\mathrm{cod}}\mathscr{C}^{\to} \longrightarrow \mathscr{C}^{\to}. \qquad \diamondsuit$$

Concretely, a functorial factorisation consists of three functors

$$L, R : \mathscr{C}^{\to} \longrightarrow \mathscr{C}^{\to}, \quad E : \mathscr{C}^{\to} \longrightarrow \mathscr{C},$$

such that for any $f : X \longrightarrow Y$, they produces a factorisation of f as follows,

$$X \xrightarrow{L_f} E_f \xrightarrow{R_f} Y$$

Functoriality of L, R, E implies that, given any square in $\mathscr{C}$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ U \arrow[r, "g"'] & V \end{tikzcd}$$

viewed as a morphism $(u, v) : f \longrightarrow g$ in $\mathscr{C}^{\to}$, the functorial factorisation produces a diagram as follows,

$$\begin{tikzcd} X \arrow[r, "L_f"] \arrow[d, "u"'] & E_f \arrow[r, "R_f"] \arrow[d, dashed, "E(u,v)"] & Y \arrow[d, "v"] \\ U \arrow[r, "L_g"'] & E_g \arrow[r, "R_g"'] & V \end{tikzcd}$$

Notice that due to the nature of functorial factorisation, the two functors L and R are automatically *copointed* $\epsilon : L \longrightarrow 1$ and *pointed* $\eta : 1 \longrightarrow R$, respectively. For any morphism f, the component of ϵ and η on f are give as follows,

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "L_f"'] & X \arrow[d, "f"] \\ E_f \arrow[r, "R_f"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "L_f"] \arrow[d, "f"'] & E_f \arrow[d, "R_f"] \\ Y \arrow[r, equal] & Y \end{tikzcd}$$

This way, we may look at the (co)algebras for the (co)pointed functors:

Terminology 3.12. We use (L, ϵ)-*coalgebras* to denote the coalgebras for the copointed functor (L, ϵ). The category of (L, ϵ)-coalgebras will be denoted as $\mathbf{L}$, and we also use $\mathbf{L}$-*structures* to refer to (L, ϵ)-coalgebras. Similarly, (R, η)-*algebras* are algebras for the pointed functor (R, η), and the category of (R, η)-algebras will be denoted as $\mathbf{R}$. ◊

One reason we are interested in (L, ϵ)-coalgebras and (R, η)-algebras is that, the data of these allows us to construct certain *lifting operators* between the two. For instance, by definition, an (L, ϵ)-coalgebra structure on $i : A \longrightarrow B$ and an (R, η)-algebra structure on $f : X \longrightarrow Y$ are exactly given by diagonal lifts as follows,

$$
\begin{tikzcd}
A \arrow[r, "L_i"] \arrow[d, "i"'] & E_i \arrow[d, "R_i"] \\
B \arrow[r, equal] \arrow[ur, dashed, "\alpha"] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "L_f"'] & X \arrow[d, "f"] \\
E_f \arrow[r, "R_f"'] \arrow[ur, dashed, "\varphi"] & Y
\end{tikzcd}
$$

In fact, the above two diagonal lifts are in a certain sense *universal* for functorial factorisation, since they can be used to construct solutions for *any* lifting problems against i and f:

Lemma 3.13. *Given a functorial factorisation (L, R, E) on $\mathscr{C}$, if $i \in \mathbf{L}$ and $f \in \mathbf{R}$, then for any lifting problem of i against f as follows,*

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "f"] \\
B \arrow[r, "v"'] & Y
\end{tikzcd}
$$

we can explicitly construct a diagonal lift, verifying that $i \pitchfork f$.

Proof. Using the functorial factorisation and the fact that i is an (L, ϵ)-coalgebra and f is an (R, η)-algebra, we construct the diagonal lift as follows,

$$
\begin{tikzcd}
 & & X \arrow[r, equal] \arrow[d, "L_f"] & X \arrow[d, "f"] \\
A \arrow[urr, "u"] \arrow[r, "L_i"] \arrow[d, "i"'] & E_i \arrow[r, "E(u,v)"] \arrow[d, "R_i"] & E_f \arrow[r, "R_f"] \arrow[ur, dashed, "\beta"] & Y \\
B \arrow[r, equal] \arrow[ur, dashed, "\alpha"] & B \arrow[urr, "v"'] & &
\end{tikzcd}
$$

In other words, the diagonal lift can be chosen as

$$\beta E(u, v)\alpha : B \longrightarrow X. \qquad \square$$

Thus, for a functorial factorisation (L, R, E) on $\mathscr{C}$, we do have that for any i equipped with an $\mathbf{L}$-structure and f with an $\mathbf{R}$-structure, $i \pitchfork f$ holds. In fact, we can say more on the level of structures:

Lemma 3.14. *Suppose* (L, R, E) *is a functorial factorisation on* $\mathscr{C}$*. The diagonal lift given in Lemma 3.13 can be promoted to functors over* $\mathscr{C}^{\to}$ *as follows,*

$$\mathbf{R} \to \mathbf{L}^{\pitchfork}, \quad \mathbf{L} \to \mathbf{R}^{\pitchfork}.$$

Proof. Straight forward verification. □

The pair $(\mathbf{L}, \mathbf{R})$ turns out to also satisfy the retract axiom, in the sense that if i is equipped with an $\mathbf{L}$-structure and j is a retract of i in the arrow category $\mathscr{C}^{\to}$, then j can also be equipped with an $\mathbf{L}$-structure.

To this end, the only difference between the two structures $(\mathbf{L}, \mathbf{R})$ on $\mathscr{C}$ with a genuine WFS is that, in the functorial factorisation for any morphism $f : X \longrightarrow Y$ as follows,

$$X \xrightarrow{L_f} E_f \xrightarrow{R_f} Y$$

the two maps L_f and R_f are not equipped with explicit (L, ϵ)-coalgebra and (R, η)-algebra structures.

One evident enrichment of the structure of a functorial factorisation on $\mathscr{C}$ that canonically equips L_f with an $\mathbf{L}$-structure and R_f with an $\mathbf{R}$-structure is when (L, ϵ) can be extended to a *comonad* and (R, η) can be extended to a *monad*. This is one of the reasons that motivates [27] to introduce the notion of a *natural weak factorisation system*:

Definition 3.15. A *natural weak factorisation system*, or *NWFS*, on $\mathscr{C}$ is a functorial factorisation (L, R, E) on $\mathscr{C}$ such that there are natural transformations $\delta : L \longrightarrow LL$ and $\mu : RR \longrightarrow R$ making (L, ϵ, δ) a comonad, and (R, η, μ) a monad. ◊

Similar to the case for the (co)algebra of the (co)pointed functors, we have the following terminologies:

Terminology 3.16. Given a NWFS (L, ϵ, δ) and (R, η, μ) on $\mathscr{C}$, we use *L-coalgebras* to denote the coalgebras for the *comonad* (L, ϵ, δ). The category of L-coalgebras will be denoted as $\mathscr{L}$. Similarly, *R-algebras* are algebras for the *monad* (R, η, μ), and the category of R-algebras will be denoted as $\mathscr{R}$. ◊

As an easy consequence, we have the following result:

Proposition 3.17. *For any NWFS* (L, ϵ, δ) *and* (R, η, μ) *on* $\mathscr{C}$*, the pair* $(\mathbf{L}, \mathbf{R})$ *consists of a WFS on* $\mathscr{C}$*.*

Proof. As mentioned, the only thing missing is for L_f and R_f to be equipped with an $\mathbf{L}$-structure and an $\mathbf{R}$-structure, respectively. This is solved for NWFS, since L_f has a cofree L-coalgebra structure, and R_f has a free R-algebra structure. □

Terminology 3.18. For any NWFS with comonad (L, ϵ, δ) and monad (R, η, μ), we will call the pair $(\mathbf{L}, \mathbf{R})$ the *underlying* WFS of this NWFS, justified by Proposition 3.17. $\diamond$

Notice that there are evident inclusion functors $\mathscr{L} \subseteq \mathbf{L}$ and $\mathscr{R} \subseteq \mathbf{R}$ over $\mathscr{C}^{\to}$, thus by Corollary 3.10 we have induced inclusions

$$\mathbf{L}^{\pitchfork} \to \mathscr{L}^{\pitchfork}, \quad {}^{\pitchfork}\mathbf{R} \to {}^{\pitchfork}\mathscr{R}.$$

The importance of the classes of (co)algebras lies in the following theorem:

Theorem 3.19. *For any NWFS (L, ϵ, δ) and (R, η, μ) on $\mathscr{C}$, if we compose the functors in Lemma 3.14 with the inclusions $\mathbf{L}^{\pitchfork} \subseteq \mathscr{L}^{\pitchfork}$ and ${}^{\pitchfork}\mathbf{R} \subseteq {}^{\pitchfork}\mathscr{R}$, we get* isomorphisms *of structures over $\mathscr{C}^{\to}$:*

$$\mathbf{R} \cong \mathscr{L}^{\pitchfork}, \quad \mathbf{L} \cong {}^{\pitchfork}\mathscr{R}.$$

Proof. See [12, Sec. 2.7]. $\square$

However, the problem now is that the two classes $\mathscr{L}$ and $\mathscr{R}$ are *not* closed under retracts anymore. It is indeed the case that any map equipped with an $\mathbf{L}$-structure can be written as a retract of a map equipped with an $\mathscr{L}$-structure, viz. the cofree L-coalgebra generated by L. Similarly for R. Thus, in the context of an NWFS, we may view the WFS $(\mathbf{L}, \mathbf{R})$ as the retract closure of the pair $(\mathscr{L}, \mathscr{R})$. However, if we retreat to the unstructured context, we do have the following result:

Corollary 3.20. *Given a NWFS (L, ϵ, δ) and (R, η, μ) on $\mathscr{C}$, the following are equivalent (similarly for the dual statements):*

- *For any i equipped with an $\mathbf{L}$-structure, $i \pitchfork f$;*
- *f can be equipped with an $\mathbf{L}^{\pitchfork}$-structure;*
- *f can be equipped with an $\mathscr{L}^{\pitchfork}$-structure;*
- *f can be equipped with an $\mathbf{R}$-structure;*

Proof. We have already shown the equivalence for the last three conditions. For the equivalence of the first conditions with the others, we only need to notice that to be an $\mathbf{R}$-map it suffices to have diagonal lift w.r.t. a *single* $\mathscr{L}$-map, viz. L_f. $\square$

Though the framework of natural weak factorisation systems allows us to algebraise the notion of left and right classes of maps in a weak factorisation system and obtain isomorphisms between structures as Theorem 3.19 states, there are still two defects we would like to improve.

One important point the current framework fails to explicate is that in a WFS, the left and right classes of maps are also *closed under composition*, which follows from the axioms of a WFS; see e.g. the corresponding sections in [29]. However, such composition operations are not algebraised in the structured categories of (co)algebras $\mathscr{L}$ or $\mathscr{R}$.

Secondly, the isomorphisms we get in Theorem 3.19 are $\mathscr{L}^{\pitchfork} \cong \mathbf{R}$ and ${}^{\pitchfork}\mathscr{R} \cong \mathbf{L}$, which is *asymmetric* since they relate the lifting structures of (co)algebras with (co)algberas of the (co)pointed functors.

It turns out that to account for composition of left and right classes of maps and to restore balance, a single solution suffices, i.e. to construct certain *double categories* of (co)algebras. This can be very elegantly achieved if we assume an additional *distributivity law* between the comonad L and the monad R in the framework of an algebraic weak factorisation system:

Definition 3.21. An *algebraic weak factorisation system*, or an *AWFS*, is a NWFS (L, ϵ, δ) and (R, η, μ) on $\mathscr{C}$ satisfying the following property: There is a canonical natural transformation $LR \longrightarrow RL$ such that for any f, its component is given by the following square

$$\begin{tikzcd} Ef \arrow[r, "\delta_f"] \arrow[d, "L_{R_f}"'] & EL_f \arrow[d, "R_{L_f}"] \\ ER_f \arrow[r, "\mu_f"'] & Ef \end{tikzcd}$$

We require this to form a *distributivity law* between the comonad L and the monad R. Equivalently, this is to require the following equation (cf. [63, Prop. 2.4]):

$$\delta_f \mu_f = \mu_{L_f} E(\delta_f, \mu_f) \delta_{R_f}. \qquad \diamond$$

It is shown in [12] that in an AWFS, we can define *vertical compositions* of L-coalgebras and R-algebras, so that they naturally form two *double categories*, which will be denoted as $\mathbb{L}, \mathbb{R}$, respectively. In particular, the underlying categories of $\mathbb{L}, \mathbb{R}$ will be $\mathscr{L}, \mathscr{R}$, respectively. From now on, we will use the pair $(\mathbb{L}, \mathbb{R})$ to denote an AWFS on a category $\mathscr{C}$. This solves our first problem, i.e. now we also have a fully algebraic description of the composition structures.

Furthermore, we have the following result, which is a *balanced* isomorphism between the structures of coalgebras and algebras:

Theorem 3.22. *For an AWFS* $(\mathbb{L}, \mathbb{R})$ *on* $\mathscr{C}$*, there are isomorphisms*

$$\mathbb{L}^{\pitchfork} \cong \mathbb{R}, \quad {}^{\pitchfork}\mathbb{R} \cong \mathbb{L},$$

which are transposes to each other under the adjunction described in Proposition 3.9.

Proof. See [12, Prop. 20]. $\square$

From Theorem 3.22 and under Lemma 3.8, it follows that the left class $\mathbb{L}$ in an AWFS is always a *left structure* on $\mathscr{C}$, because it is isomorphic to ${}^{\pitchfork}\mathbb{R}$ over $\mathrm{cube}_1(\mathscr{C})$. Similarly, the right class $\mathbb{R}$ will always be a *right structure* on $\mathscr{C}$.

There is also a notion of *morphisms* of AWFSs, which is important for the definition of algebraic model structures; see Section 4.23. However, a direct formulation of morphisms between AWFS is somewhat complicated. Here we follow [11, Prop. 9] and use the following result to equivalently characterise morphisms between AWFSs:

Proposition 3.23. *A morphism between two AWFSs* $(\mathbb{C}_0, \mathbb{F}_0) \longrightarrow (\mathbb{C}_1, \mathbb{F}_1)$ *is equivalently described as*

- *a double functor of the left structures* $\mathbb{C}_0 \longrightarrow \mathbb{C}_1$ *over* $\mathrm{cube}_1(\mathscr{C})$*;*
- *a double functor of the right structures* $\mathbb{F}_1 \longrightarrow \mathbb{F}_0$ *over* $\mathrm{cube}_1(\mathscr{C})$*.*

3.3 Examples of Algebraic Weak Factorisation Systems

In this section we briefly recall various AWFSs described in the book [63], which will be the main focus of the second part of this document. We work with an underlying category $\mathscr{C}$ with finite limits, finite colimits, and is locally Cartesian closed.

3.3.1 Cofibrations and Effective Trivial Fibrations

The notion of dominance was first introduced in [51] for toposes. A dominance Σ on $\mathscr{C}$ is a family of *monomorphisms* satisfying the following properties:

- Σ is closed under isomorphisms, composition, and pullbacks.
- There is a Σ-*classifier* $\top : 1 \longrightarrow \Sigma$, which is the terminal object of the category $\mathrm{Cart}(\Sigma)$, with objects as Σ-maps and morphisms as Cartesian squares between them.

For us, we will assume the additional property that

- Σ is closed under finite unions.

In concrete examples, e.g. in simplicial sets, the choice of a dominance structure will be a class of monomorphisms that supports some form of classical reasoning in a constructive context; see e.g. [63, Ch. 8].

Given a dominance Σ on $\mathscr{C}$, there exists an explicit construction of an AWFS $(\Sigma, \mathbb{T})$ on $\mathscr{C}$. For any map $f : X \longrightarrow Y$, we have the following factorisation

$$X \xrightarrow{L_f} E_f = \sum_{y:Y} \sum_{\sigma:\Sigma} X_y^\sigma \xrightarrow{R_f} Y$$

where E_f is Σ-partial map classifier into X over Y. For more detailed description of the (co)monad structures of this AWFS, we refer the readers to [12, Sec. 4.4] and [63, Ch. 3].

For us, the most important feature of this AWFS $(\Sigma, \mathbb{T})$ on $\mathscr{C}$ is that we have a very explicit description of the double category of coalgebras:

- Σ-coalgebra structure is *propositional*, i.e. a map can be equipped with a Σ-structure iff it *belongs to* Σ; see [63, Prop. 3.2].
- The vertical composition of Σ-maps are simply composition of maps in Σ. A horizontal morphism of Σ-maps is a *pullback square*; see [63, Lem. 3.1].
- Σ is closed under retracts; see [63, Lem. 4.2].

This completely determines the double category of Σ-coalgebras. Even nicer, since Σ is closed under retracts, the coalgebras for the comonad and the coalgebras for the copointed functor for this AWFS *coincide*.

This way, the right class for this AWFS is also completely determined as $\mathbb{T} \cong \Sigma^{\pitchfork}$ by Theorem 3.22. For the class $\mathbf{T}$ of algebras of the pointed functor, by Theorem 3.19 we also have $\mathbf{T} \cong \Sigma^{\pitchfork}$, where now Σ is viewed as the underlying *category* of Σ-maps. The underlying WFS for this AWFS is simply given by $(\Sigma, \mathbf{T})$.

From now on, we introduce the following terminology for this AWFS:

Terminology 3.24. We say a morphism is a *cofibration* iff it *belongs to* Σ. A morphism equipped with a $\mathbb{T}$-structure will be referred to as an *effective trivial fibration*. A morphism in $\mathbf{T}$ will simply be denoted as a *trivial fibration*. ◊

3.3.2 HDRs and Naïve Fibrations

Now assume $\mathscr{C}$ is also equipped with a *symmetric Moore structure*. Concretely, it consists of the following data:

- There is a *pullback-preserving* endo-functor M on $\mathscr{C}$, taking every object X to its *Moore path object* MX.

- There are natural transformations $r : 1 \longrightarrow M$, $s, t : M \longrightarrow 1$, $\mu : M\ {}_t{\times}_s M \longrightarrow M$, and $\tau : M \longrightarrow M$, making every object X in $\mathscr{C}$ an *internal* †*-category* with object of arrows MX.[2]

- There is a *connection* $\Gamma : M \longrightarrow MM$ making (M, s, Γ) a *comonad*,

- There is also a strength α making M a strong Cartesian comonad.[3]

These data are further required to satisfy some additional conditions, which for the convenience of the readers we have recorded in Appendi A. For more discussions on the relevance of these axioms, we refer the readers to [63, App. A.1].

Furthermore, it is shown in [63, Ch. 4] that such a Moore structure also induces an AWFS $(\mathbb{H}, \mathbb{N})$ on $\mathscr{C}$. For any map $f : X \longrightarrow Y$, the functorial factorisation is obtained as follows,

$$X \xrightarrow{\langle rf,1\rangle} MY\ {}_t{\times}_f X \xrightarrow{s\pi_{MY}} Y$$

Similar to the case of the AWFS from the dominance structure, in this case we can also completely characterise the left class $\mathbb{H}$, or its retract closure $\mathbf{H}$. Recall that in the symmetric Moore structure we have a comonad (M, s, Γ), and we use (M, s)-coalgebras and M-coalgebras to refer to the coalgebras for the copointed functor (M, s) and the comonad (M, Γ, s), respectively. We have the following general characterisations:

Proposition 3.25. *The categories* $\mathbf{H}$ *and* $\mathscr{H}$ *are equivalent over* $\mathrm{cod} : \mathscr{C}^{\rightarrow} \longrightarrow \mathscr{C}$ *to the category of* (M, s)*-coalgebras and* M*-coalgebras, respectively.*

Proof. See [63, Prop. 4.4]. □

An important consequence of Proposition 3.25 is that both the categories $\mathscr{H}$ and $\mathbf{H}$ now have pullbacks:

Corollary 3.26. *The categories* $\mathbf{H}$ *and* $\mathscr{H}$ *have pullbacks.*

Proof. This is due to the fact that M preserves pullbacks, thus the forgetful functor from the category of (M, s)-coalgebras and M-coalgebras to $\mathscr{C}$ creates pullbacks. □

---

[2] A †-category $\mathscr{E}$ is a category equipped with an involutive, identity-on-object functor $^\dagger : \mathscr{E}^{\mathrm{op}} \longrightarrow \mathscr{E}$; see e.g.[28, Ch. 2.3].

[3] For the notion of strength of monads, see e.g. [39].

It is also instructive to look at the relationship between **H** and Σ. It is observed in [63, Lem. 4.1] that for any $\partial : A \longrightarrow B$ in $\mathbb{H}$, or in fact in **H**, the following will be a *pullback*

$$\begin{tikzcd} A \arrow[r, "\partial"] \arrow[d, "\partial"'] & B \arrow[d, "H"] \\ B \arrow[r, "r"'] & MB \end{tikzcd}$$

From the assumption on the symmetric Moore structure, r is always cofibrant. It follows that ∂ is also a cofibration. It is also observed in [63, Prop. 4.5] that, under the assumption of r being a Cartesian natural transformation, horizontal morphisms in $\mathbb{H}$ and in **H** are always pullback squares. This implies the following result:

Proposition 3.27. *There is a morphism of AWFSs on $\mathscr{C}$ as follows*

$$(\mathbb{H}, \mathbb{N}) \longrightarrow (\Sigma, \mathbb{T}).$$

Proof. By Proposition 3.23, it suffices to give a double functor for the two left structures $\mathbb{H} \longrightarrow \Sigma$ over $\mathrm{cube}_1(\mathscr{C})$. This is evident, since the structure Σ is propositional. $\square$

We now introduce terminology for referring to maps in the left and right classes of this AWFS:

Terminology 3.28. Morphisms equipped with $\mathbb{H}$-structures are called *hyperdeformation retracts*, or *HDRs* in short. For maps in the retract closure **H**, we simply denote them as *Hdrs*. For the right class, we will call $\mathbb{N}$-maps *effective naïve fibrations*, and maps in **N** simply as *naïve fibrations*. $\Diamond$

One consequence of Proposition 3.27 is that there will also be a morphism of right structures $\mathbb{T} \longrightarrow \mathbb{N}$, which means every effective trivial fibration is also an effective naïve fibration. The central notion of *effective Kan fibration* is an intermediate structure between them, and will be the focus of next subsection.

3.3.3 Effective Kan Fibrations

As described in [63, Ch. 6], effective Kan fibrations are cofibrantly generated by *mould squares*. These are essentially Cartesian cofibrations between HDRs:

Definition 3.29. A *mould square* is a square as follows,

$$\begin{tikzcd} A \arrow[r, "\partial_a"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ C \arrow[r, "\partial_c"'] & D \end{tikzcd}$$

where i, j are cofibrations and ∂_a, ∂_c are HDRs, and the square is a *Cartesian* map of HDRs, which means it is a morphism of HDRs and both of the following squares are pullbacks,

$$\begin{tikzcd} A \arrow[r, "\partial_a"] \arrow[d, "i"'] & B \arrow[r, "\sigma_a"] \arrow[d, dashed, "j"] & A \arrow[d, "i"] \\ C \arrow[r, "\partial_c"'] & D \arrow[r, "\sigma_c"'] & C \end{tikzcd}$$

Here σ_a, σ_c are the retracts of ∂_a, ∂_c induced by the HDR structure (cf. [63, Ch 4.2]). $\diamond$

One important type of mould squares comes from squares spanned by a cofibration and an HDR:

Lemma 3.30. *Suppose $i : A \longrightarrow B$ is a cofibration and $\partial : C \longrightarrow D$ is an HDR, then the following square is a mould square,*

$$\begin{tikzcd} A \times C \arrow[r, "A\times\partial"] \arrow[d, "i\times C"'] & A \times D \arrow[d, "i\times D"] \\ B \times C \arrow[r, "B\times\partial"'] & B \times D \end{tikzcd}$$

Proof. From Corollary 3.26 we know that the product of two HDRs are again HDRs, thus the two horizontal arrows are HDRs. We also know that cofibrations are closed under pullbacks, thus the vertical maps are cofibrations. It is also easy to see that this square consists of a Cartesian morphism of HDRs. $\square$

The lifting problem associated to a mould square is formulated as follows. For any given map $f : X \longrightarrow Y$, a lifting problem against a mould square is indicated as the solid part of the diagram below,

$$\begin{tikzcd} A \arrow[r, "\partial_a"] \arrow[d, "i"'] & B \arrow[r, "u"] \arrow[d, dashed, "j"] & X \arrow[d, "f"] \\ C \arrow[r, "\partial_c"'] \arrow[urr, bend left, "d"] & D \arrow[r, "v"'] \arrow[ur, dashed, "\varphi"] & Y \end{tikzcd}$$

And we say this lifting problem has a solution if there exists a dashed arrow φ as above making everything commute.

In essence, we think of mould squares as a representation for its *cofibre arrow*, a notion we will be discussing in Chapter 4; see Definition 4.5. For the mould square above, the cofibre arrow of it is simply the universally induced arrow from the *pushout* of B and C

along A into D. It is easy to see that the above lifting problem is equivalent to the one below,

$$\begin{tikzcd}
B +_A C \arrow[r, "{[u,d]}"] \arrow[d, "{[j,\partial_c]}"'] & X \arrow[d, "f"] \\
D \arrow[r, "v"'] \arrow[ur, dashed, "\varphi"] & Y
\end{tikzcd}$$

which is the more familiar way of formulating lifting problems as discussed in Section 3.1.

In fact, one of the motivation for defining mould squares is to better formulate lifting structures w.r.t. the *pushout-products* of cofibrations and HDRs, which are cofibre arrows of those mould squares given in Lemma 3.30 (cf. [63, Ch. 6]). The advantage of using the notion of mould squares is that they have several natural composition structures that allow us to easily define compatibility conditions.

We start with a description of the category $\mathscr{M}$ of mould squares. From Corollary 3.26 we know that HDRs can be pulled back along morphisms of HDRs. As a direct consequence, it is easy to see that mould squares can also be pulled back along morphisms of HDRs, resulting cubes of the following form,

$$\begin{tikzcd}
 & A \arrow[rr, "\partial_a"] \arrow[dd, "i" near start] & & B \arrow[dd, "j"] \\
A' \arrow[ur] \arrow[rr, "\partial'_a" near start] \arrow[dd, "i'"'] & & B' \arrow[ur] \arrow[dd, "j'" near start] & \\
 & C \arrow[rr, "\partial_c" near start] & & D \\
C' \arrow[ur] \arrow[rr, "\partial'_c"'] & & D' \arrow[ur] &
\end{tikzcd}$$

where the top and bottom faces are morphisms of HDRs, and all the faces are Cartesian. To be more concrete, we can define a category $\mathscr{M}$ of mould squares, with objects being mould squares and morphisms being cubes as shown above. It can indeed be viewed as a category *over* $\mathscr{C}^{\rightarrow}$, associating each mould square to its cofibre arrow as follows,

$$\begin{tikzcd}
A \arrow[r, "\partial_a"] \arrow[d, "i"'] & B \arrow[d, "j"] & & B +_A C \arrow[d, "{[j,\partial_c]}"] \\
C \arrow[r, "\partial_c"'] & D & \mapsto & D
\end{tikzcd}$$

We thus define the notion of *Kan fibrations* as cofibrantly generated by this category:

Definition 3.31. A *Kan fibration* is a map f equipped with a right lifting structure w.r.t. the category $\mathscr{M}$. The category of Kan fibrations will be denoted as $\mathbf{F}$, with $\mathbf{F} \cong \mathscr{M}^{\pitchfork}$. $\diamondsuit$

Remark 3.32. In the classical homotopy theory for simplicial sets, the term "Kan fibration" denotes those maps that have the right lifting property against all horn inclusions (cf. [25]). Our notion of Kan fibration when specialised to simplicial sets is closely related to this definition, and indeed coincide assuming classical logic [63, Ch. 12]. $\diamondsuit$

We think of $\mathbf{F}$ as the retract closure of the effective structure that we are going to define in a moment. Notice that this view is compatible with Proposition 3.19, where we think of $\mathscr{M}$ as a category of generating effective trivial cofibrations.

For the effective structure, notice that mould squares can be composed in *two* directions, viz. horizontal and vertical, which are inherited from the vertical compositions of HDRs and cofibrations, respectively. It is also straight forward to see that the horizontal and vertical compositions of mould squares are again mould squares. An *effective* Kan fibration is thus a lifting structure against mould squares, with compatibility conditions in these three dimensions:

Definition 3.33. We say a Kan fibration $f : X \longrightarrow Y$ is an *effective* Kan fibration if its lifting structure against mould squares is furthermore compatible with horizontal and vertical compositions of mould squares. $\diamondsuit$

From Definition 3.5, it is not hard to specify what exactly are these compatibility conditions, and we refer the readers to [63, Sec. 6.1] for a more detailed description. The crucial thing for us is that the notion of effective Kan fibrations again organise themselves as a *right structure* $\mathbb{F}$ on $\mathscr{C}$. Based on $\mathbb{F}$, we can also define its corresponding left class:

Definition 3.34. We say a map is a *trivial cofibration* if it is equipped with an ${}^{\pitchfork}\mathscr{F}$-structure, where $\mathscr{F}$ is the underlying category of effective Kan fibrations. Similarly, we say a map is an *effective trivial cofibration* if it is equipped with an ${}^{\pitchfork}\mathbb{F}$-structure. $\diamondsuit$

The category of trivial cofibrations will be denoted as $\mathbf{C}$, with $\mathbf{C} \cong {}^{\pitchfork}\mathscr{F}$. The double category of effective trivial cofibrations will be denoted as $\mathbb{C}$, with $\mathbb{C} \cong {}^{\pitchfork}\mathbb{F}$. In particular, if a map arise as the cofibre arrow of a mould square, then by Definition 3.31 we know that it is an effective trivial cofibration.

Notice that, HDRs in fact embeds into mould squares, and mould squares embeds into Σ. From any HDR $\partial_a : A \longrightarrow B$, we may form a mould square as follows,

$$\begin{tikzcd}
\varnothing \arrow[r, equal] \arrow[d] & \varnothing \arrow[d] \\
A \arrow[r, "\partial_a"'] & B
\end{tikzcd}$$

A lifting solution against ∂_a is indeed equivalent to a lifting solution against this mould square. It is easy to see that the above construction extends to a functor $\mathscr{H} \longrightarrow \mathscr{M}$, which suggests that there will be a morphism $\mathbf{F} \longrightarrow \mathbf{N}$ under the adjunction in Corollary 3.10. Furthermore, the vertical compatibility condition of lifting against HDRs translates to the horizontal compatibility of lifts against mould squares, thus we similarly have a morphism of right structures $\mathbb{F} \longrightarrow \mathbb{N}$.

On the other hand, by the assumption on the dominance, cofibrations are closed under finite unions. Due to the fact that HDRs are also cofibrations established in Proposition 3.27, it follows that there is a functor $\mathscr{M} \longrightarrow \Sigma$ over $\mathscr{C}^{\rightarrow}$, which again induces a map $\mathbf{T} \longrightarrow \mathbf{F}$. Similarly we can show that this extends to right structures $\mathbb{T} \longrightarrow \mathbb{F}$. For more details see [63, Sec. 6.1].

Currently from an axiomatic perspective, we do not know whether there is a general method of constructing an AWFS on $(\mathbb{C}, \mathbb{F})$. However, at least concerning the primary application of this framework on *simplicial sets*, we know that the corresponding right class of effective Kan fibrations is cofibrantly generated by a small, in fact *countable*, double category [63, Ch. 12], [5]. Hence, it is hopeful to adapt the small object argument for double categorical cofibrant generation described in [12] to this case, and obtain a *constructive* argument for the existence of an AWFS. But this goal is quite orthogonal to the objective of this document, and we will not try to answer it here.

Chapter 4

Algebraic Monoidal Model Structures

This chapter contains the main contribution of the first part of this document, in which we provide a structured approach towards pushout-product axioms, and obtain a notion of an algebraic monoidal model category. Recall the definition of ordinary pushout-products of two maps $i : A \longrightarrow B$ and $m : X \longrightarrow Y$ in a category $\mathscr{C}$ with a monoidal product $\otimes$, which is given by the *cofibre arrow* of the following square,

$$\begin{tikzcd} A \otimes X \arrow[r, "A\otimes m"] \arrow[d, "i\otimes X"'] & A \otimes Y \arrow[d, "i\otimes Y"] \\ B \otimes X \arrow[r, "B\otimes m"'] & B \otimes Y \end{tikzcd}$$

where we recall from Section 3.3.3 that the cofibre arrow denotes the following uniquely induced map

$$B \otimes X +_{A\otimes X} A \otimes Y \longrightarrow B \otimes Y.$$

Now the pushout-product axioms in its most general form says that if i has structure $\mathbf{C}_0$, m has structure $\mathbf{C}_1$, then the induced cofibre arrow has structure $\mathbf{C}_2$, for some *left* classes of maps $\mathbf{C}_0, \mathbf{C}_1, \mathbf{C}_2$ in some WFSs.

From a structural perspective, after the discussion in Chapter 3 of AWFSs, these structures should all be understood as certain *left structures* over $\mathscr{C}$, with explicit notion of vertical compositions and horizontal morphisms. Furthermore, the pushout-product axioms should also be understood as certain *functorial operators*, that transform $\mathbb{C}_0$-maps and $\mathbb{C}_1$-maps to $\mathbb{C}_2$-structures on the cofibre arrow of the associated square.

While discussing mould squares and effective Kan fibrations in Section 3.3.3, we have noticed that for such cofibre arrows, the best way to formulate its functorial property is to stay on the level of squares. Intuitively, if we understand the $\mathbb{C}_2$-structure on the cofibre arrow as a structure on this *square*, then it makes sense to say whether this $\mathbb{C}_2$-structure depends functorially on the vertical and horizontal morphisms of $\mathbb{C}_0$ and $\mathbb{C}_1$-maps, since all of them induce evident horizontal and vertical maps of squares. This observation makes it clear that we would like to study structures on *higher* cubes, which is the main reason we have introduced the framework of n-fold categories in Chapter 2. Furthermore, it is evident that each edge of a square generated by pushout-products are *constant* on all but one dimensions, which suggests that the graded monoidal structure of n-structures we have defined in Section 2.3 will also be of use.

In Section 4.1 we will make this notion of structures on higher cubes as structures on their cofibre arrows precise. Based on this, in Section 4.2, we will introduce a **Grpd**-enriched *multicategory* of left structures, such that the pushout-product axioms can now be realised as certain *multimorphisms*. Intuitively, the multicategory structure arises because the pushout-product axiom takes two structures on *arrows* to a single structure on *squares*.

In Section 4.3, we describe the structured version of Joyal-Tierney calculus, which will be useful for the later part of this document. Finally, based on all the previous development within this chapter, Section 4.4 arives at a notion of algebraic monoidal model categories, and give several equivalent characterisations for it using the structured Joyal-Tierney calculus.

4.1 Cubes and Cofibre Arrows

To describe the abstract shape of higher cubes and their cofibre arrows, it turns out that the category Δ_+ is quite useful. Δ_+ can be seen as the skeletal category of finite linear orders, whose objects are finite ordinals $n \in \mathbb{N}$, and morphisms are *monotone* maps between them.

Δ_+ has a universal property: It is the *free monoidal category with a monoid.* The monoidal structure $\otimes$ on Δ_+ is the ordinal sum, i.e. $n \otimes m = n + m$. The reason we write $\otimes$ rather than $+$ for the ordinal sum is that this is *not* a braided monoidal structure. The monoid in Δ_+ is given by the diagram

$$0 \longrightarrow 1 \longleftarrow 2 = 1 \otimes 1$$

The maps are unique because 1 is the terminal object in Δ_+. It can be easily seen that all the maps in Δ_+ are generated by the above unit and multiplication maps, which are face and degeneracies, under the monoidal structure.

Now in **Cat** equipped with the Cartesian monoidal structure, there is a monoid $(\mathbf{2}, \wedge, 1)$ on the linear order $\mathbf{2} = \{0 < 1\}$ with multiplication being conjunction. This gives us a strict monoidal functor

$$B : \Delta_+ \longrightarrow \mathbf{Cat},$$

taking $n \in \Delta_+$ to 2^n. The action on morphisms are completely induced by the conjunction monoid structure on $\mathbf{2}$. Concretely, for the i-th face map $d_i : n \longrightarrow n+1$ the action

$$B(d_i) : \mathbf{2}^n \longrightarrow \mathbf{2}^{n+1}$$

is given by inserting the element 1 at the i-th place, if we view elements in $\mathbf{2}^n$ as sequences of length n. For the i-th degeneracy map $s_i : n+1 \longrightarrow n$

$$B(s_i) : \mathbf{2}^{n+1} \longrightarrow \mathbf{2}^n,$$

the action takes the conjunction of the i-th and $i+1$-th place.

Using the functor $B : \Delta_+ \longrightarrow \mathbf{Cat}$ we have constructed above, there is an associated *nerve* construction (cf. [44]):

$$N_B : \mathbf{Cat} \longrightarrow \mathbf{Cat}_{\Delta_+},$$

where $\mathbf{Cat}_{\Delta_+}$ is defined to be the functor category $[\Delta_+^{\mathrm{op}}, \mathbf{Cat}]$, and the nerve is given by

$$N_B(\mathscr{C}) := \mathbf{Cat}(B(-), \mathscr{C}).$$

In particular, for any $n \in \mathbb{N}$, the nerve $(N_B\mathscr{C})_n$ is the category of n-cubes in $\mathscr{C}$, which is the image of the n-fold structure $\mathrm{cube}_n(\mathscr{C})$ under the n-forgetful functor; see Section 2.3. However, the difference here is that this nerve $N_B(\mathscr{C})$ is not indexed by $\mathbb{C}$, but by Δ_+, and the action on morphisms between these cube categories is different. For concreteness, we look at examples at lower dimensions:

Example 4.1. By definition, for $n = 0, 1$ we have

$$(N_B\mathscr{C})_0 \cong \mathscr{C}, \quad (N_B\mathscr{C})_1 \cong \mathscr{C}^{\rightarrow},$$

where $\mathscr{C}^{\rightarrow}$ is the category of arrows in $\mathscr{C}$. The functor B takes the map $0 \longrightarrow 1$ in Δ_+ to the inclusion functor $\wedge_0 : \mathbf{1} \longrightarrow \mathbf{2}$ as follows,

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r, mapsto, dashed] & \bullet \end{tikzcd}$$

By precomposing with this functor, the action on the nerve

$$\wedge_0^* : \mathscr{C}^{\rightarrow} \cong (N_B\mathscr{C})_1 \longrightarrow (N_B\mathscr{C})_0 \cong \mathscr{C}$$

takes any morphism f in $\mathscr{C}$ to its codomain,

$$\wedge_0^* \cong \mathrm{cod} : \mathscr{C}^{\rightarrow} \longrightarrow \mathscr{C}. \qquad \diamondsuit$$

Example 4.2. By definition, for $n = 2$ we have

$$(N_B\mathscr{C})_2 \cong \mathscr{C}^{\square},$$

where $\mathscr{C}^{\square}$ is the category of commutative squares in $\mathscr{C}$. Notice that the functor B takes the multiplication $2 \longrightarrow 1$ in Δ_+ to the conjunction functor $\wedge \,:\, \mathbf{2}{\times}\mathbf{2} \longrightarrow \mathbf{2}$ in **Cat** represented by the following diagram,

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] \arrow[dr] \arrow[rr, dashed, mapsto, bend left] & \bullet \arrow[r, dashed, mapsto] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[urr, dashed, mapsto] & \bullet \arrow[r, dashed, mapsto] & \bullet
\end{tikzcd}$$

By precomposing with this functor, the action on the nerve

$$\wedge^* \,:\, \mathscr{C}^{\to} \cong (N_B\mathscr{C})_1 \longrightarrow (N_B\mathscr{C})_2 \cong \mathscr{C}^{\square}$$

takes any morphism f in $\mathscr{C}$ to the following square,

$$\begin{tikzcd}
\bullet \arrow[d, "f"'] & & \bullet \arrow[r, equal] \arrow[d, equal] & \bullet \arrow[d, "f"] \\
\bullet & \mapsto & \bullet \arrow[r, "f"'] & \bullet
\end{tikzcd}$$

As one can see, this does not send a morphism to the degenerate square, hence is different from the D-cubical acions described in Section 2.3. $\diamondsuit$

As mentioned, for any $f \,:\, n \longrightarrow m$ in Δ_+ the action on the nerve is given by precomposition with $B(f)$,

$$(N_B\mathscr{C})_m \cong \mathbf{Cat}(\mathbf{2}^m, \mathscr{C}) \xrightarrow{B(f)^*} \mathbf{Cat}(\mathbf{2}^n, \mathscr{C}) \cong (N_B\mathscr{C})_n.$$

If $\mathscr{C}$ has finite colimits, these precomposition functors will have *left adjoints*, given by *left Kan extensions* (cf. [44, 49]). This way, for finitely cocomplete $\mathscr{C}$ there is also an associated augmented (pseudo) cosimplicial object in **Cat**, again sending n to $(N_B\mathscr{C})_n$ while sending $f \,:\, n \longrightarrow m$ to the left Kan extension

$$B(f)_! \,:\, (N_B\mathscr{C})_n \longrightarrow (N_B\mathscr{C})_m.$$

Again, let us look at some examples in low dimensions:

Example 4.3. As shown in Example 4.1, the precomposition acts as follows,

$$\wedge_0^* \cong \mathrm{cod} \,:\, \mathscr{C}^{\to} \longrightarrow \mathscr{C},$$

where now its left Kan extension

$$\wedge_{0,!} : \mathscr{C} \longrightarrow \mathscr{C}^{\rightarrow}$$

takes any object X to the map $\varnothing \longrightarrow X$. $\Diamond$

Example 4.4. As shown in Example 4.2, on the multiplication map of the universal monoid in Δ_+ we have the following adjunction between $\mathscr{C}^{\square}$ and $\mathscr{C}^{\rightarrow}$

$$\wedge_! \dashv \wedge^*,$$

where now the left Kan extension $\wedge_!$ takes any square in $\mathscr{C}$ to its *cofibre arrow* (!):

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "u"] & & B +_A C \arrow[d, "{[u,v]}"] \\ C \arrow[r, "v"'] & D & \mapsto & D \end{tikzcd}$$

More generally, for the n-fold multiplication map $\wedge_n : n \longrightarrow 1$, the precomposition and left Kan extension adjunction gives us

$$\wedge_{n,!} : (N_B\mathscr{C})_n \rightleftarrows \mathscr{C}^{\rightarrow} : \wedge_n^*,$$

where the left adjoint forgets the terminal object in $\mathbf{2}^n$ and takes the colimit of the remaining diagram, and maps it to the uniquely induced map from the colimit to the original terminal object. $\Diamond$

Although already appeared multiple times, here we give the official definition of the cofibre arrow of a general higher cube:

Definition 4.5. For any higher cube in a finitely cocomplete category $\mathscr{C}$, viz. a functor $\mathbf{2}^n \longrightarrow \mathscr{C}$ in $(N_B\mathscr{C})_n$, we denote the image of it under $\wedge_{n,!}$ as its *cofibre arrow.* $\Diamond$

This way, we can easily specify the idea that for any structure of $\mathscr{L}$-maps on a category $\mathscr{C}$, an $\mathscr{L}$-structure on a higher cube can be defined as an $\mathscr{L}$-structure on its cofibre arrow:

Definition 4.6. Given a category $\mathscr{L}$ over $\mathscr{C}^{\rightarrow}$, the category of $\mathscr{L}$*-cofibre n-cubes*, denoted as $\mathrm{cof}_n(\mathscr{L})$, is defined as the following strict pullback

$$\begin{tikzcd} \mathrm{cof}_n(\mathscr{L}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{L} \arrow[d] \\ (N_B\mathscr{C})_n \arrow[r, "\wedge_{n,!}"'] & (N_B\mathscr{C})_1 \cong \mathscr{C}^{\rightarrow} \end{tikzcd}$$

$\Diamond$

By definition, an object in $\mathrm{cof}_n(\mathscr{L})$ is consists of an n-cube and an $\mathscr{L}$-structure on the cofibre arrow of this n-cube. Morphisms between them are simply horizontal morphisms of n-cubes, with an associated horizontal morphisms of $\mathscr{L}$-structures on their cofibre arrows.

However, currectly $\mathrm{cof}_n(\mathscr{L})$ is still an ordinary category. If $\mathscr{L}$ is the category of coalgebras coming from an AWFS $(\mathbb{L}, \mathbb{R})$, we would like to furthermore define full vertical compositions on higher cubes with compatible compositions on cofibre arrows to obtain an n-structure on $\mathscr{C}$, such that $\mathrm{cof}_n(\mathscr{L})$ becomes its image under the n-forgetful functor.

It turns out that this works more generally for all *discretely opfibred* structures over $\mathscr{C}$; see Definition 3.2.

Theorem 4.7. *If $\mathscr{L}$ is induced from a left structure $\mathbb{L}$ over $\mathrm{cube}_1(\mathscr{C})$ under the 1-forgetful functor, then the above category $\mathrm{cof}_n(\mathscr{L})$ of $\mathscr{L}$-cofibre n-cubes can be promoted to an n-fold structure $\mathrm{cof}_n(\mathbb{L})$ of $\mathscr{L}$-cofibre n-cubes which lives over $\mathrm{cube}_n(\mathscr{C})$,*

$$\mathrm{cof}_n(\mathbb{L}) \longrightarrow \mathrm{cube}_n(\mathscr{C}),$$

and whose image under $|-|_n$ is given by

$$\mathrm{cof}_n(\mathscr{L}) \longrightarrow (N_B\mathscr{C})_n.$$

Proof. To makes $\mathrm{cof}_n(\mathscr{L})$ into an n-fold structure over $\mathrm{cube}_n(\mathscr{C})$, we need to construct the vertical composition on each dimension. We explicitly construct the composition operator for the case of $n = 2$, which is in most cases what we actually need. The proof for higher dimensions is completely analogous.

Suppose now we have the following composible 2-cubes

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{a}{\Longrightarrow}"] & C \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{b}{\Longrightarrow}"] & E \arrow[d] \\
B \arrow[r] & D \arrow[r] & F
\end{tikzcd}$$

with $\mathbb{L}$-structures α on $B +_A C \longrightarrow D$ and β on $D +_C E \longrightarrow F$. We need to define an $\mathbb{L}$-structure on the pushout-product of the composed square (A, B, E, F). The crucial observation is that we have the following diagram,

$$\begin{tikzcd}
B +_A C \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{c}{\Longrightarrow}\ulcorner"] & B +_A E \arrow[d] & \\
D \arrow[r] & D +_C E \arrow[r] & F
\end{tikzcd}$$

with the left square a *pushout*. Let us use c to denote this pushout, thus by assumption $c_*\alpha$ will be an $\mathbb{L}$-structure on $B +_A E \longrightarrow D +_C E$, since $\mathbb{L}$ is discretely opfibred over $\mathscr{C}$. Then we can define the $\mathbb{L}$-structure on the total pushout-product $B +_A E \longrightarrow F$ to be $\beta \bullet c_*\alpha$, the vertical composition of the $\mathbb{L}$-structure on $B +_A E \longrightarrow D +_C E$ with $D +_C E \longrightarrow F$. The composition operator on the other dimension is completely similar.

The verification of unity, associativity and naturality are all straight forward, and we leave them for the readers. Here we explicitly verify the interchange law of the composition operators, which is less trivial. Suppose we have four composible $\mathscr{L}$-cofibre squares as follows,

$$\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{a}{\Longrightarrow}"] & C \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{b}{\Longrightarrow}"] & E \arrow[d] \\
B \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{u}{\Longrightarrow}"] & D \arrow[r] \arrow[d] \arrow[dr, phantom, "\overset{v}{\Longrightarrow}"] & F \arrow[d] \\
U \arrow[r] & V \arrow[r] & W
\end{tikzcd}$$

with $\mathbb{L}$-structures α, β, μ, ν on the cofibre arrows of a, b, u, v, respectively. The two ways of composing the $\mathbb{L}$-structures on the cofibre arrows, either by first composing to the right then downwards, or first downwards and then to the right, can be summarised in the following commuting diagram,

$$\begin{tikzcd}
B +_A C \arrow[rr] \arrow[dr] \arrow[dd] \arrow[rr, phantom, shift right=4, "\overset{c}{\Longrightarrow}"] & & B +_A E \arrow[dr] \arrow[dd] & & & & \\
& U +_A C \arrow[rr] \arrow[dd] \arrow[dl, phantom, shift left=4, "\overset{e}{\Longrightarrow}"] \arrow[r, phantom, shift right=6, "\overset{f}{\Longrightarrow}"] & & U +_A E \arrow[dd] \arrow[dl, phantom, shift left=4, "\overset{d}{\Longrightarrow}"] & & & \\
D \arrow[rr] \arrow[dr] & & D +_C E \arrow[rr] \arrow[dr] & & F \arrow[dr] \arrow[dl, phantom, shift right=2, "\overset{g}{\Longrightarrow}"] & & \\
& U +_B D \arrow[rr] \arrow[dr] \arrow[rr, phantom, shift right=5, "\overset{h}{\Longrightarrow}"] & & U +_B D +_C E \arrow[rr] \arrow[dr] \arrow[rr, phantom, shift right=5, "\overset{i}{\Longrightarrow}"] & & U +_B F \arrow[dr] & \\
& & V \arrow[rrr] & & & V +_C E \arrow[r] & V +_D F
\end{tikzcd}$$

Concretely, if we first compose to the right and then downwards, the $\mathbb{L}$-structure on the cofibre arrow of the total square is given by

$$\nu \bullet (ih)_*\mu \bullet (g \bullet d)_*(\beta \bullet c_*\alpha) = \nu \bullet i_*h_*\mu \bullet g_*\beta \bullet d_*c_*\alpha.$$

On the other hand, if we first compose downwards and the to the right, the $\mathbb{L}$-structure we get is given by

$$\nu \bullet (i \bullet g)_*\beta \bullet (h \bullet f)_* \bullet (\mu \bullet e_*\alpha) = \nu \bullet i_*g_*\beta \bullet h_*\mu \bullet f_*e_*\alpha.$$

Due to the commutativity of the diagram above and uniqueness of coCartesian lifts,

$$f_*e_*\alpha = d_*c_*\alpha,$$

and also

$$i_*h_*\mu \bullet g_*\beta = h_*\mu \bullet i_*g_*\beta.$$

Thus, the interchange law holds, and we have a genuine 2-structure on $\mathscr{C}$. □

Suppose $\mathscr{C}$ also has finite limits, then we can also talk about left structures on the opposite category $\mathscr{C}^{\mathrm{op}}$, which we think of as *right structures on* $\mathscr{C}$. In this case, the associated left Kan extension functor

$$(\mathscr{C}^{\mathrm{op}})^{\square} \longrightarrow (\mathscr{C}^{\mathrm{op}})^{\rightarrow}$$

when interpreted in $\mathscr{C}$, acts on squares in $\mathscr{C}$ as follows,

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "u"] \\ C \arrow[r, "v"'] & D \end{tikzcd} \quad \longmapsto \quad \begin{tikzcd} A \arrow[d, "\langle f,g \rangle"] \\ B \times_D C \end{tikzcd}$$

In other words, it takes a square to its *fibred arrow*. More generally, completely dual to Definition 4.5, the associated fibred arrow of a higher cube is the uniquely induced map from the initial vertex to the finite limit of the remaining diagram.

Given a right structure $\mathbb{R} \longrightarrow \mathrm{cube}_1(\mathscr{C})$, from Theorem 4.7 we also have an n-structure $\mathrm{fib}_n(\mathbb{R})$ of n-fibre $\mathbb{R}$-cubes, by formally viewing it as follows,

$$\mathrm{fib}_n(\mathbb{R})^{\mathrm{op}} \cong \mathrm{cof}_n(\mathbb{R}^{\mathrm{op}}).$$

Here we treat $\mathbb{R}^{\mathrm{op}}$ as a *left structure* on $\mathscr{C}^{\mathrm{op}}$. Such a construction allows us to also talk about right structures on $\mathscr{C}$ via left structures on $\mathscr{C}^{\mathrm{op}}$.

4.2 The Multicategory of Left Structures

Theorem 4.7 which associates an n-dimension structure of $\mathbb{L}$-cofibre n-cubes on $\mathscr{C}$ for any left structure $\mathbb{L}$ allows us to further define a **Grpd**-enriched *multicategory* $\mathbb{L}$**Str** of all left structures; for the notion of multicategories, see e.g. [40]. Here **Grpd** is the category of groupoids. But before that, we first define a **Grpd**-enriched multicategory $\mathbb{R}$**ex** of finitely cocomplete categories.

Definition 4.8. The **Grpd**-enriched multicategory $\mathbb{R}$**ex** consists of the following data:

- An object is a finitely cocomplete category;
- A morphism $F : (\mathscr{C}_1, \cdots, \mathscr{C}_n) \longrightarrow \mathscr{C}_0$ in $\mathbb{R}$**ex** is a functor
$$F : \mathscr{C}_1 \times \cdots \times \mathscr{C}_n \longrightarrow \mathscr{C}_0,$$
such that it is preserves finite colimits on each entry;[1]
- A 2-cell between two morphisms $\eta : F \cong G : (\mathscr{C}_1, \cdots, \mathscr{C}_n) \longrightarrow \mathscr{C}_0$ is simply a natural isomorphism between the underlying functors. ◊

In this case, the composition of multimorphisms and the **Grpd**-enrichment in $\mathbb{R}$**ex** are evidently inherited from the Cartesian product structure of the 2-category of finitely cocomplete categories. The **Grpd**-enriched multicategory $\mathbb{L}$**Str** will be a multicategory *over* $\mathbb{R}$**ex**. Concretely, it is defined as follows:

Definition 4.9. The **Grpd**-enriched multicategory of left structures $\mathbb{L}$**Str** consists of the following data:

- An object over a finitely cocomplete category $\mathscr{C}$ is a left structure $\mathbb{L}$ on $\mathscr{C}$;
- A morphism $(\mathbb{L}_1, \cdots, \mathbb{L}_n) \longrightarrow \mathbb{L}_0$ over a functor
$$F : \mathscr{C}_1 \times \cdots \times \mathscr{C}_n \longrightarrow \mathscr{C}_0,$$
is a lift of morphisms of n-structures as follows,
$$\begin{tikzcd}
\mathbb{L}_1 \otimes \cdots \otimes \mathbb{L}_n \arrow[r, "\hat{F}"] \arrow[d] & \mathrm{cof}_n(\mathbb{L}_0) \arrow[dd] \\
\mathrm{cube}_1(\mathscr{C}_1) \otimes \cdots \otimes \mathrm{cube}_1(\mathscr{C}_n) \arrow[d] & \\
\mathrm{cube}_n(\mathscr{C}_1 \times \cdots \times \mathscr{C}_n) \arrow[r, "\mathrm{cube}_n(F)"'] & \mathrm{cube}_n(\mathscr{C}_0)
\end{tikzcd}$$

[1]Notice that by definition, a morphism $() \longrightarrow \mathscr{C}$ is simply specifying an *arbitrary* object in $\mathscr{C}$, since the condition that it preserves finite colimits on each entry of the empty list is trivially satisfied.

Here the morphism

$$\mathrm{cube}_1(\mathscr{C}_1) \otimes \cdots \otimes \mathrm{cube}_1(\mathscr{C}_n) \longrightarrow \mathrm{cube}_n(\mathscr{C}_1 \times \cdots \times \mathscr{C}_n)$$

is described as in Example 2.17.

- A natural isomorphism $F \cong G$ in $\mathbb{R}\mathbf{ex}$ consists of a 2-cell in $\mathbb{L}\mathbf{Str}$ iff the $\mathbb{L}_0$-structures assigned by $\hat{F}$ and $\hat{G}$ coincide under this natural isomorphism. ◊

Let us discuss more closely what the definition of 2-cells in $\mathbb{L}\mathbf{Str}$ means. Notice that given any $F : \mathscr{C}_1 \times \cdots \times \mathscr{C}_n \longrightarrow \mathscr{C}_0$, the cofibre arrow construction induces a corresponding functor

$$\hat{F} : \mathscr{C}_1^{\rightarrow} \times \cdots \times \mathscr{C}_n^{\rightarrow} \longrightarrow \mathscr{C}_0,$$

where for any $\{f_i \in \mathscr{C}_i^{\rightarrow}\}_{1 \le i \le n}$, $\hat{F}(f_1, \cdots, f_n)$ is the cofibre arrow of the cube spanned by $(Ff_1, \cdots, Ff_n)$, which we call the *F-pushout-product* of $f_1, \cdots, f_n$. If we have a natural isomorphism $F \cong G$, it also induces one on the level of pushout-product. For this natural isomorphism to be a 2-cell in $\mathbb{L}\mathbf{Str}$, we require for any $\alpha_i \in \mathbb{L}_i$ over f_i, the two $\mathbb{L}_0$-structures $\hat{F}(\alpha_1, \cdots, \alpha_n)$ and $\hat{G}(\alpha_1, \cdots, \alpha_n)$ coincide under the isomorphism $\hat{F}(f_1, \cdots, f_n) \cong \hat{G}(f_1, \cdots, f_n)$. Notice that this makes sense because as discussed in Section 3.1, the underlying category $\mathscr{L}_0$ of $\mathbb{L}_0$ can be viewed as a copresheaf on $\mathscr{C}_0^{\rightarrow}$, and copresheaves preserve isomorphisms.

After we have shown that $\mathbb{L}\mathbf{Str}$ is a well-defined $\mathbf{Grpd}$-enriched multicategory, the forgetful functor $|-| : \mathbf{Str}_n \longrightarrow \mathbf{Cat}$ will also induce

$$|-| : \mathbb{L}\mathbf{Str} \longrightarrow \mathbb{R}\mathbf{ex}.$$

This definition on $\mathbb{L}\mathbf{Str}$ makes it *faithful* on 2-cells. On the other hand, we also have a multifunctor the other way around, taking any finitely cocomplete category $\mathscr{C}$ to its trivial structure $\mathrm{cube}_1(\mathscr{C})$,

$$\mathrm{cube}_1(-) : \mathbb{R}\mathbf{ex} \longrightarrow \mathbb{L}\mathbf{Str},$$

which exhibits $\mathbb{R}\mathbf{ex}$ as a retract of $\mathbb{L}\mathbf{Str}$. For any morphism in $\mathbb{R}\mathbf{ex}$

$$F : (\mathscr{C}_1, \cdots, \mathscr{C}_n) \longrightarrow \mathscr{C}_0,$$

the induced lifted morphism between n-structures

$$\hat{F} : \mathrm{cube}_1(\mathscr{C}_1) \otimes \cdots \otimes \mathrm{cube}_1(\mathscr{C}_n) \longrightarrow \mathrm{cof}_n(\mathrm{cube}_1(\mathscr{C}_0)),$$

simply takes a tuple of arrows $(f_1, \cdots, f_n)$ to their F-pushout-product $\hat{F}(f_1, \cdots, f_n)$.

Terminology 4.10. For left structures $\mathbb{L}_0, \cdots, \mathbb{L}_n$ on finitely cocomplete $\mathscr{C}_0, \cdots, \mathscr{C}_n$, we use $\mathbb{L}\mathbf{Str}(\mathbb{L}_1, \cdots, \mathbb{L}_n; \mathbb{L}_0)$ to denote the category of morphisms of type $(\mathbb{L}_1, \cdots, \mathbb{L}_n) \longrightarrow \mathbb{L}_0$ in $\mathbb{L}\mathbf{Str}$. Given any morphism $F : \mathscr{C}_1 \times \cdots \times \mathscr{C}_n \longrightarrow \mathscr{C}_0$ in $\mathbb{R}\mathbf{ex}$, we also use $\mathbb{L}\mathbf{Str}_F(\mathbb{L}_1, \cdots, \mathbb{L}_n; \mathbb{L}_0)$ to denote the fibre over F, viz. the *set* of morphisms in $\mathbb{L}\mathbf{Str}$ *over* F. ◊

To define the compositions in $\mathbb{L}\mathbf{Str}$, one useful observation is the following:

Lemma 4.11. *Given any morphism of n-structures*

$$\hat{F} : \mathbb{L}_1 \otimes \cdots \otimes \mathbb{L}_n \longrightarrow \mathbb{L}_0$$

over a functor in $\mathbb{R}\mathbf{ex}$*,*

$$F : \mathscr{C}_1 \times \cdots \times \mathscr{C}_n \longrightarrow \mathscr{C}_0,$$

$\hat{F}$ *preserves coCartesian lifts on each entry.*

Proof. Suppose we have an $\mathbb{L}_i$-structure α_i over f_i in $\mathscr{C}_i$ for all $1 \leq i \leq n$. Suppose we also have a coCartesian square $a : f_j \longrightarrow f_j'$ for some j. Let C, C' be the n-cube in $\mathscr{C}_1 \times \cdots \times \mathscr{C}_n$ spanned by $f_1, \cdots, f_j, \cdots, f_n$ and $f_1, \cdots, f_j', \cdots, f_n$, respectively. Here a can also be viewed as a coCartesian horizontal morphism $a : C \longrightarrow C'$. Now by definition, $\hat{F}(\alpha_1, \cdots, \alpha_j, \cdots, \alpha_n)$, $F(\alpha_1, \cdots, a_*\alpha_j, \cdots, \alpha_n)$ are $\mathbb{L}_0$-structures on the cofibre arrows g, g' of the cube $F(C), F(C')$, respectively. Since F preserves finite colimits on each entry, $F(a) : g \longrightarrow g'$ is a pushout. By uniqueness of coCartesian lifts for $\mathbb{L}_0$, we must have that

$$F(a)_*\hat{F}(\alpha_1, \cdots, \alpha_j, \cdots, \alpha_n) = \hat{F}(\alpha_1, \cdots, a_*\alpha_j, \alpha_n). \qquad \square$$

Using this observation, we can properly define the composition of multimorphisms in $\mathbb{L}\mathbf{Str}$, and establish that it is a well-defined multicategory:

Theorem 4.12. *There is a composition operator that makes the data in Definition 4.9 a well-defined* **Grpd***-enriched multicategory.*

Proof. The identity is easy to describe, since by construction we have

$$\mathrm{cof}_1(\mathbb{L}) = \mathbb{L},$$

thus the identity is simply represented by identity on $\mathbb{L}$. For the composition, the crucial observation is that, for an morphism over the functor F

$$(\mathbb{L}_1, \cdots, \mathbb{L}_n) \longrightarrow \mathbb{L}_0,$$

it also induces a morphim of r-structures for any $r = r_1 + \cdots + r_n$ as follows,

$$
\begin{tikzcd}
\mathrm{cof}_{r_1}(\mathbb{L}_1) \otimes \cdots \otimes \mathrm{cof}_{r_n}(\mathbb{L}_n) \arrow[r, "\hat{F}"] \arrow[d] & \mathrm{cof}_r(\mathbb{L}_0) \arrow[dd] \\
\mathrm{cube}_{r_1}(\mathscr{C}_1) \otimes \cdots \otimes \mathrm{cube}_{r_n}(\mathscr{C}_n) \arrow[d] & \\
\mathrm{cube}_r(\mathscr{C}_1 \times \cdots \times \mathscr{C}_n) \arrow[r, "\mathrm{cube}_r(F)"'] & \mathrm{cube}_r(\mathscr{C}_0)
\end{tikzcd}
$$

The r-morphism $\hat{F}$ acts as follows. Suppose we have an r_i-cube C_i in $\mathscr{C}_i$ with an $\mathbb{L}_i$-structure α_i on its cofibre arrow f_i, for all $1 \leq i \leq n$. Let C denote the corresponding *r-cube* on $\mathscr{C}_1 \times \cdots \times \mathscr{C}_0$ spanned by $C_1, \cdots, C_n$; similarly, let D denote the *n-cube* on $\mathscr{C}_1 \times \cdots \times \mathscr{C}_0$ spanned by $f_1, \cdots, f_n$. F preserving finite colimits on each entry implies that the cofibre arrow of the r-cube $F(C)$ *coincides* with the cofibre arrow of the n-cube $F(D)$. This way, we can simply use $\hat{F}(\alpha_1, \cdots, \alpha_n)$ to define the above functor on r-structures. To show that this operator preserves vertical composition on each dimension, we only need to notice that from Theorem 4.7 we know that the cofibre arrows of vertical composition of cubes are simply vertical compositions of coCartesian lifts of cofibre arrows of individual cubes. By definition the original morphism $\hat{F} : \mathbb{L}_1 \otimes \cdots \otimes \mathbb{L}_n \longrightarrow \mathbb{L}_0$ preserves vertical composition, and by Lemma 4.11 it also preserves coCartesian lifts. Thus, the generalised $\hat{F}$ also preserves vertical composition.

Now given any morphism

$$F : (\mathbb{L}_1, \cdots, \mathbb{L}_n) \longrightarrow \mathbb{L}_0$$

and any family of morphisms

$$G_i : (\mathbb{L}_{i,1}, \cdots, \mathbb{L}_{i,m_i}) \longrightarrow \mathbb{L}_i,$$

for all $1 \leq i \leq n$, their multicomposition

$$(G_1, \cdots, G_n) \circ F : (\mathbb{L}_{1,1}, \cdots, \mathbb{L}_{1,m_1}, \cdots, \mathbb{L}_{n,1}, \cdots, \mathbb{L}_{n,m_n}) \longrightarrow \mathbb{L}_0$$

is given by the following composition on higher structures,

$$\mathbb{L}_{1,1} \otimes \cdots \otimes \mathbb{L}_{n,m_n} \xrightarrow{\hat{G}_1 \otimes \cdots \otimes \hat{G}_n} \mathrm{cof}_{m_1}(\mathbb{L}_1) \otimes \cdots \otimes \mathrm{cof}_{m_n}(\mathbb{L}_n) \xrightarrow{\hat{F}} \mathrm{cof}_{m_1+\cdots+m_n}(\mathbb{L}_0).$$

where now $\hat{F}$ is induced by F as stated before. Verifying unity and associativity is straight forward, and the **Grpd**-enrichment is easily seen to be inherited from $\mathbb{R}$**ex**. ☐

Remark 4.13. Notice that the two multicategories $\mathbb{L}\mathbf{Str}$ and $\mathbb{R}\mathbf{ex}$ are in fact *symmetric*, since it is easy to observe that both the Cartesian product of categories and the monoidal product $\otimes$ on higher structures are symmetric. $\Diamond$

To see more concretely what will be a morphism in $\mathbb{L}\mathbf{Str}$, consider the case for $n = 2$. Suppose we have a certain monoidal product on $\mathscr{C}$ which preserves finite colimits on each entry

$$\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}.$$

For any $i : A \longrightarrow B, j : C \longrightarrow D \in \mathscr{C}$, the square in $\mathscr{C} \times \mathscr{C}$ spanned by i, j is mapped to the following square in $\mathscr{C}$,

$$\begin{tikzcd}
A \otimes C \arrow[r, "i \otimes C"] \arrow[d, "A \otimes j"'] & B \otimes C \arrow[d, "B \otimes j"] \\
A \otimes D \arrow[r, "i \otimes D"'] & B \otimes D
\end{tikzcd}$$

and the lift $\hat{\otimes}$ takes i, j to the cofibre arrow of this square, which is exactly the usually defined *pushout-product* of i, j as below

$$i \hat{\otimes} j : A \otimes D +_{A \otimes C} B \otimes C \longrightarrow B \otimes D.$$

Let $\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_2$ be left structures on $\mathscr{C}$. A morphism in $\mathbb{L}\mathbf{Str}$ over this monoidal product

$$\otimes : (\mathbb{L}_1, \mathbb{L}_2) \longrightarrow \mathbb{L}_0$$

now requires a morphism between 2-structures

$$\hat{\otimes} : \mathbb{L}_1 \otimes \mathbb{L}_2 \longrightarrow \mathrm{cof}_2(\mathbb{L}_0),$$

making the following diagram commute,

$$\begin{tikzcd}
\mathbb{L} \otimes \mathbb{M} \arrow[r, "\hat{\otimes}"] \arrow[d] & \mathrm{cof}_2(\mathbb{N}) \arrow[d] \\
\mathrm{cube}_2(\mathscr{C} \times \mathscr{C}) \arrow[r, "\mathrm{cube}_2(\otimes)"'] & \mathrm{cube}_2(\mathscr{C})
\end{tikzcd}$$

By our discussion previously, the functor $\hat{\otimes}$ takes any $\mathbb{L}_1$-structure α_1 on $i : A \longrightarrow B$ and any $\mathbb{L}_2$-structure α_2 on $j : C \longrightarrow D$, and assigns an $\mathbb{L}_0$-structure $\alpha_1 \hat{\otimes} \alpha_2$ on the cofibre arrow of the pushout-product $i \hat{\otimes} j$,

$$i \hat{\otimes} j : A \otimes D +_{A \otimes C} B \otimes C \longrightarrow B \otimes D.$$

This shows that the multicategory $\mathbb{L}\mathbf{Str}$ successfully describes the structure we want to capture: The existence of such a morphism in $\mathbb{L}\mathbf{Str}$ over $\otimes$ is exactly a structured way of expressing the fact the pushout-product of an $\mathbb{L}_1$-map with an $\mathbb{L}_2$-map gives us an $\mathbb{L}_0$-map.

The structural aspects means that, the morphism on higher structures

$$\hat{\otimes} : \mathbb{L}_1 \otimes \mathbb{L}_2 \longrightarrow \mathrm{cof}_2(\mathbb{L}_0)$$

now need to respect both horizontal morphisms and vertical compositions of $\mathbb{L}_1$-maps and $\mathbb{L}_2$-maps. For the horizontal direction, suppose we have a morphism $\eta : \alpha_1 \longrightarrow \alpha_1'$ of $\mathbb{L}_1$-maps over the square $a : i \longrightarrow i'$. Functoriality of $\hat{\otimes}$ for the horizontal map η then implies that we have a morphism of $\mathbb{L}_0$-structures

$$\eta \hat{\otimes} \alpha_2 : \alpha_1 \hat{\otimes} \alpha_2 \longrightarrow \alpha_1' \hat{\otimes} \alpha_2,$$

over the induced square $a \hat{\otimes} j : i \hat{\otimes} j \longrightarrow i' \hat{\otimes} j$ between the pushout-products. Similarly, this also applies to horizontal morphisms for the second entry.

The more subtle condition is for vertical compositions. Suppose we have another $\mathbb{L}_1$-structure α_1' on a map $k : B \longrightarrow E$ where we can vertically compose to get an $\mathbb{L}_1$-structure $\alpha_1' \bullet \alpha_1$ on ki. Then $\hat{\otimes}$ preserving this vertical composition means that the $\mathbb{L}_0$-structure $\alpha_1' \bullet \alpha_1 \hat{\otimes} \alpha_2$ on $ki \hat{\otimes} j$ should agree with the composition in $\mathrm{cof}_2(\mathbb{L}_0)$ we have defined in Theorem 4.7. Recall once again that the latter is obtained by the following composite,

$$\begin{tikzcd}
A \otimes D +_{A \otimes C} B \otimes C \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow a"] & A \otimes D +_{A \otimes C} E \otimes C \arrow[d] & \\
B \otimes D \arrow[r] & B \otimes D +_{B \otimes C} E \otimes C \arrow[r] & E \otimes D
\end{tikzcd}$$

Thus, preserving vertical composition of $\mathbb{L}_1$-maps means that we have

$$\alpha_1' \hat{\otimes} \alpha_2 \bullet a_*(\alpha_1 \hat{\otimes} \alpha_2) = (\alpha_1' \bullet \alpha_1) \hat{\otimes} \alpha_2.$$

Preserving vertical compositions of $\mathbb{L}_2$-maps is completely similar.

As a first example, we can already see quite easily that the structuralised pushout-product axiom now holds for the cofibration structure we have defined in Section 3.3.1:

Example 4.14. Recall that given any *dominance* Σ on a finitely complete, finitely co-complete and locally Cartesian closed category $\mathscr{C}$, there is an induced *propositional* left structure Σ on $\mathscr{C}$ consisting of *cofibrations*. In the structured context, the fact that cofibrations are closed under pushout-products is expressed by the existence of a morphism in $\mathbb{L}\mathbf{Str}$ as follows

$$(\Sigma, \Sigma) \longrightarrow \Sigma,$$

over the Cartesian product on $\mathscr{C}$. By definition, such a morphism sends a pair of cofibrations $i : A \longrightarrow B$ and $j : C \longrightarrow D$ to a cofibration structure on their pushout-product

$$i \hat{\otimes} j : A \times D +_{A \times C} B \times C \longrightarrow B \times D.$$

This is indeed also a cofibration, because by assumption maps in Σ maps are closed under pullback and unions. Since Σ is a propositional left structure, such an assignment automatically preserves horizontal and vertical compositions, which means it is a well-defined morphism. ◇

We will discuss more less trivial examples in Chapter 5 for the various other structures we have defined in Section 3.3, after we discuss the structured version of Joyal-Tierney calculus in the next section.

At the end of this section, we also discuss the relationship between the structured version of pushout-product axioms as morphisms in $\mathbb{L}\mathbf{Str}$ and the usual unstructured version. Recall from Section 3.2 that given any AWFS $(\mathbb{L}, \mathbb{R})$, its underlying WFS is given by $(\mathbf{L}, \mathbf{R})$. It is then easy to observe that the structured pushout-product axiom implies the usual version on the underlying WFSs:

Proposition 4.15. *Suppose we have a morphism in* $\mathbb{L}\mathbf{Str}$

$$F : (\mathbb{L}_1, \cdots, \mathbb{L}_n) \longrightarrow \mathbb{L}_0$$

If we use $\mathbf{L}_i$ *to denote the retract closure of* $\mathbb{L}_i$*-maps for any* $i \leq n$*, then we still have that the* F*-pushout-product of* $(f_i)_{1 \leq i \leq n}$ *with* $f_i \in \mathbf{L}_i$ *belongs to* $\mathbf{L}_0$*.*

Proof. Recall that the generalised F-pushout-product can be realised as a functor

$$\hat{F} : \mathscr{C}_1^{\rightarrow} \times \cdots \times \mathscr{C}_n^{\rightarrow} \longrightarrow \mathscr{C}_0^{\rightarrow}.$$

Now suppose we are given morphisms $f_i \in \mathbf{L}_i$ for all $1 \leq i \leq n$. By definition, we can find g_i such that there exists an $\mathbb{L}_i$-structure α_i on g_i, and f_i is a retract of g_i. By assumption $\hat{F}(\alpha_1, \cdots, \alpha_n)$ is an $\mathbb{L}_0$-structure on $\hat{F}(g_1, \cdots, g_n)$. By functoriality of $\hat{F}$, notice that $\hat{F}(f_1, \cdots, f_n)$ is again a retract of $\hat{F}(g_1, \cdots, g_n)$, which implies that the former belongs to $\mathbf{L}_0$. □

4.3 Structured Joyal-Tierney Calculus

As already mentioned in the Introduction, Joyal-Tierney calculus is one of the most basic techniques for homotopy theory. Dual to the notion of pushout-product, there is a notion of *pullback-exponentials.* The starting point of the Joyal-Tierney calculus is that given any monoidal product $\otimes$ on $\mathscr{C}$ with a *right closure* $[-,-]$, for any morphisms $i : A \longrightarrow B$, $j : C \longrightarrow D$, and $k : E \longrightarrow F$ in $\mathscr{C}$ we have

$$i \hat{\otimes} j \pitchfork k \iff i \pitchfork \exp(j, k),$$

where $\exp(j, k)$ is the pullback-exponential of j, k,

$$\exp(j, k) : [D, E] \longrightarrow [C, E] \times_{[C,F]} [D, F].$$

In other words, it is the *fibred arrow* of the following square in $\mathscr{C}$,

$$\begin{tikzcd}
{[D,E]} \arrow[r] \arrow[d] & {[C,E]} \arrow[d] \\
{[D,F]} \arrow[r] & {[C,F]}
\end{tikzcd}$$

From the discussion at the end of Section 4.1, this can also be seen as the *cofibre arrow* of the dual square in $\mathscr{C}^{\mathrm{op}}$.

Inspired by [15], in this section we will formulate a general version of Joyal-Tierney calculus for n-ary functors as certain action of a cyclic group on the multicategory $\mathbb{L}\mathbf{Str}$ of left structures. Given any $\mathscr{C}_0, \mathscr{C}_1, \cdots, \mathscr{C}_n$ in $\mathbb{R}\mathbf{ex}$, with $\mathscr{C}_0, \mathscr{C}_1$ also admitting finite limits. Consider any morphism in $\mathbb{R}\mathbf{ex}$ as follows,

$$F : \mathscr{C}_1 \times \cdots \times \mathscr{C}_n \longrightarrow \mathscr{C}_0.$$

We say it has a *right closure* if there exists another morphism in $\mathbb{R}\mathbf{ex}$

$$G : \mathscr{C}_2 \times \cdots \times \mathscr{C}_n \times \mathscr{C}_0^{\mathrm{op}} \longrightarrow \mathscr{C}_1^{\mathrm{op}},$$

such that for any $c_i \in \mathscr{C}_i$ for all $i \leq n$, there is an isomorphism

$$\mathscr{C}_0(F(c_1, \cdots, c_n), c_0) \cong \mathscr{C}_1(c_1, G(c_2, \cdots, c_n, c_0)),$$

natural in all entries. Notice that we need $\mathscr{C}_0, \mathscr{C}_1$ to have finite limits to recognise G as a valid morphism in $\mathbb{R}\mathbf{ex}$.

According to the discussion at the end of last section, they also induce two functors

$$\begin{aligned}
\hat{F} &: C_1^{\rightarrow} \times \cdots \times C_n^{\rightarrow} \longrightarrow C_0^{\rightarrow}, \\
\hat{G} &: C_2^{\rightarrow} \times \cdots \times C_n^{\rightarrow} \times (C_0^{\mathrm{op}})^{\rightarrow} \longrightarrow (C_1^{\mathrm{op}})^{\rightarrow}
\end{aligned}$$

As mentioned, $\hat{F}$ is the generalised F-pushout-product. More interestingly is what $\hat{G}$ does: As mentioned, cofibre arrows in $\mathscr{C}_1^{\mathrm{op}}$ can be viewed as fibred arrows in $\mathscr{C}_1$. Thus,

$\hat{G}$ assigns the fibred arrow of the cube spanned by $(f_2, \cdots, f_n, f_0)$, which can be viewed as generalised G-pullback-exponential. The upshot is that, if G is a right closure of F, then on the level of arrow categories we again have an adjunction

$$\mathscr{C}_0^{\to}(\hat{F}(f_1, \cdots, f_n), f_0) \cong \mathscr{C}_1^{\to}(f_1, \hat{G}(f_2, \cdots, f_n, f_0)).$$

This is usually called the *Leibniz adjunction*.

On the level of strucutres, recall from Section 3.1 that if $\mathscr{C}$ is finitely complete and finitely cocomplete, then for any left structure $\mathbb{L}$ over $\mathrm{cube}_1(\mathscr{C})$, there is an induced right lifting structure $\mathbb{L}^{\pitchfork}$ on $\mathscr{C}$. Thus equivalently, it can be viewed as a *left structure* on $\mathscr{C}^{\mathrm{op}}$, which will be denoted as $\mathbb{L}^{\circ}$, viz. $\mathbb{L}^{\circ} = (\mathbb{L}^{\pitchfork})^{\mathrm{op}}$.

The structured version of Joyal-Tierney calculus can now be stated as follows:

Theorem 4.16. *If a morphism F in* $\mathbb{R}\mathbf{ex}$

$$F : \mathscr{C}_1 \times \cdots \times \mathscr{C}_n \longrightarrow \mathscr{C}_0$$

has a right closure G, then for any left structures $\mathbb{L}_i$ on $\mathscr{C}_i$ for $i \leq n$, there will be a canonically induced function (recall Terminology 4.10)

$$\mathbb{L}\mathbf{Str}_F(\mathbb{L}_1, \cdots, \mathbb{L}_n; \mathbb{L}_0) \longrightarrow \mathbb{L}\mathbf{Str}_G(\mathbb{L}_2, \cdots, \mathbb{L}_n, \mathbb{L}_0^{\circ}; \mathbb{L}_1^{\circ}).$$

Proof. We provide a detailed proof in the case of $n = 2$, which covers all the applications we care about in practice. The proof with more parameters is completely similar.

We view a two variable functor as a tensor

$$- \otimes - : \mathscr{C}_1 \times \mathscr{C}_2 \longrightarrow \mathscr{C}_0,$$

which has a right closure

$$[-, -] : \mathscr{C}_2 \times \mathscr{C}_0^{\mathrm{op}} \longrightarrow \mathscr{C}_1^{\mathrm{op}}.$$

Suppose now we are given a morphism in $\mathbb{L}\mathbf{Str}$ over this product $\otimes$,

$$\otimes : (\mathbb{L}_1, \mathbb{L}_2) \longrightarrow \mathbb{L}_0.$$

We need to construct a new morphism

$$\exp_r : (\mathbb{L}_2, \mathbb{L}_0^{\circ}) \longrightarrow \mathbb{L}_1^{\circ}$$

over the right closure $[-, -]$. For any $\mathbb{L}_2$-structure β on $j : C \longrightarrow D$ and any right lifting structure φ against $\mathbb{L}_0$ on $f : X \longrightarrow Y$, we construct a right lifting structure $\hat{\exp}_r(\beta, \varphi)$ against $\mathbb{L}_1$ on their pullback-exponential

$$\hat{\exp}_r(j, f) : [D, X] \longrightarrow [C, X] \times_{[C,Y]} [D, Y].$$

Concretely, given any $\mathbb{L}_1$-structure α on $i : A \longrightarrow B$ and a square

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & {[D,X]} \arrow[d, "\hat{\exp}_r(j,f)"] \\
B \arrow[r] & {[C,X] \times_{[C,Y]} [D,Y]}
\end{tikzcd}$$

by the Leibniz adjunction we have a corresponding solid square

$$\begin{tikzcd}
A \otimes D +_{A \otimes C} B \otimes C \arrow[r] \arrow[d, "i \hat{\otimes} j"'] & X \arrow[d, "f"] \\
B \otimes D \arrow[r] \arrow[ur, dashed, "\varphi(\alpha \hat{\otimes} \beta)"] & Y
\end{tikzcd}$$

Now $\alpha \hat{\otimes} \beta$ by assumption is an $\mathbb{L}_0$-structure on the pushout-product $i \hat{\otimes} j$, thus the right lifting structure φ on f against $\mathbb{L}_0$ produces a lift as shown above. By transposing along the Leibniz adjunction again, we get a lift $\widetilde{\varphi(\alpha \hat{\otimes} \beta)}$ of $\exp_r(i, f)$ against i. In equation form,

$$\hat{\exp}_r(\beta, \varphi)(\alpha) = \widetilde{\varphi(\alpha \hat{\otimes} \beta)}.$$

We first need to show that the above construction of the lifting operator $\hat{\exp}_r(\beta, \varphi)$ is compatible with horizontal and vertical composition of $\mathbb{L}_1$-maps, thus is a well-defined $\mathbb{L}_1^{\circ}$-structure on f. For horizontal compatibility, suppose we are in the situation

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "i'"'] & A \arrow[r] \arrow[d, "i"'] & {[D,X]} \arrow[d, "\hat{\exp}_r(j,f)"] \\
B' \arrow[r] & B \arrow[r] & {[C,X] \times_{[C,Y]} [D,Y]}
\end{tikzcd}$$

where on the left is a square $a : i' \longrightarrow i$ underlying a horizontal morphism of $\mathbb{L}_1$-maps

$$\eta : \alpha' \longrightarrow \alpha.$$

Then for the dual diagram

$$\begin{tikzcd}
A' \otimes D +_{A' \otimes C} B' \otimes C \arrow[r] \arrow[d, "i' \hat{\otimes} j"'] & A \otimes D +_{A \otimes C} B \otimes C \arrow[r] \arrow[d, "i \hat{\otimes} j"'] & X \arrow[d, "f"] \\
B' \otimes D \arrow[r] \arrow[urr, dashed, "\varphi(\alpha' \hat{\otimes} \beta)"] & B \otimes D \arrow[r] \arrow[ur, dashed, "\varphi(\alpha \hat{\otimes} \beta)"] & Y
\end{tikzcd}$$

by functoriality of $\hat{\otimes}$ on the first entry, we also get a morphism of $\mathbb{L}_0$-maps

$$\eta\hat{\otimes}\beta \,:\, \alpha'\hat{\otimes}\beta \longrightarrow \alpha\hat{\otimes}\beta$$

over the induced horizontal square $a\hat{\otimes}j$. Hence, horizontal compatibility of the lifting structure φ on f makes sure the above two lifts are compatible, hence so are the two lifts in the dual diagram for $\hat{\exp}_r(j, f)$.

For vertical compatibility, suppose we have composible $\mathbb{L}_1$-maps $i_0 \,:\, A \longrightarrow B_0$ and $i_1 \,:\, B_0 \longrightarrow B_1$ with $\mathbb{L}_1$-structures α_0, α_1 on them, respectively. Consider the following lifting diagram,

$$\begin{tikzcd}
A \arrow[rr, "m"] \arrow[d, "i_0"'] & & {[D,X]} \arrow[dd, "\hat{\exp}_r(j,f)"] \\
B_0 \arrow[urr, dashed, "{\hat{\exp}_r(\beta,\varphi)(\alpha_0)[m,\langle u,v\rangle i_1]}"] \arrow[d, "i_1"'] & & \\
B_1 \arrow[uurr, dashed, "{\hat{\exp}_r(\beta,\varphi)(\alpha_1)[\hat{\exp}_r(\beta,\varphi)(\alpha_0),\langle u,v\rangle]}"] \arrow[rr, "{\langle u,v\rangle}"'] & & {[C,X] \times_{[C,Y]} [D,Y]}
\end{tikzcd}$$

Vertical compatibility requires us to show that

$$\hat{\exp}_r(\beta, \varphi)(\alpha_1)[\hat{\exp}_r(\beta, \varphi)(\alpha_0), \langle u, v\rangle] = \hat{\exp}_r(\beta, \varphi)(\alpha_1 \bullet \alpha_0)[m, \langle u, v\rangle].$$

To this end, we consider its dual diagram,

$$\begin{tikzcd}
A \otimes D +_{A\otimes C} B_0 \otimes C \arrow[r] \arrow[d, "i_0\hat{\otimes}j"'] & A \otimes D +_{A\otimes C} B_1 \otimes C \arrow[r, "{[\tilde{m},\tilde{v}]}"] \arrow[d] & X \arrow[dd, "f"] \\
B_0 \otimes D \arrow[rru, dashed, "\varphi(\alpha_0\hat{\otimes}\beta)"] \arrow[r] & B_0 \otimes D +_{B_0\otimes C} B_1 \otimes C \arrow[ru, dashed, "{[\varphi(\alpha_0\hat{\otimes}\beta),\tilde{v}]}"'] \arrow[d, "i_1\hat{\otimes}j"'] & \\
 & B_1 \otimes D \arrow[r, "\tilde{u}"'] \arrow[ruu, dashed, "\varphi(\alpha_1\hat{\otimes}\beta)"] & Y
\end{tikzcd}$$

The fact that $\hat{\otimes}$ preserves vertical composition for $\mathbb{L}_1$-maps implies that

$$(\alpha_1 \bullet \alpha_0)\hat{\otimes}\beta = (\alpha_1\hat{\otimes}\beta) \bullet a_*(\alpha_0\hat{\otimes}\beta),$$

where a is the pushout square on the top left. By horizontal compatibility of φ, the lift against $a_*(i_0\hat{\otimes}j)$, viz. the middle top arrow, is induced by pushout,

$$\varphi(a_*(\alpha_0\hat{\otimes}\beta)) = [\varphi(\alpha_0\hat{\otimes}\beta), \tilde{v}].$$

This way, we have that

$$\begin{aligned}
\varphi((\alpha_1 \bullet \alpha_0)\hat{\otimes}\beta)[[\widetilde{m}, \tilde{v}], \tilde{u}] &= \varphi((\alpha_1 \hat{\otimes} \beta) \bullet a_*(\alpha_0 \hat{\otimes} \beta))[[\widetilde{m}, \tilde{v}], \tilde{u}] \\
&= \varphi(\alpha_1 \hat{\otimes} \beta)[\varphi(a_*(\alpha_0 \hat{\otimes} \beta)), \tilde{u}] \\
&= \varphi(\alpha_1 \hat{\otimes} \beta)[[\varphi(\alpha_0 \hat{\otimes} \beta), \tilde{v}], \tilde{u}]
\end{aligned}$$

As mentioned, the first equality is by $\hat{\otimes}$ preserving vertical composition, and the second holds by vertical compatibility of φ. This concludes the proof that $\hat{\exp}_r(\beta, \varphi)$ is a well-defined right lifting structure on $\hat{\exp}_r(j, f)$.

Furthermore, we need to verify that $\hat{\exp}_r$ is a morphism between 2-structures, which means it preserves horizontal and vertical compositions of both entries. For the horizontal direction, suppose we have a horizontal morphism of $\mathbb{L}_2$-maps

$$\eta : \beta \longrightarrow \beta'$$

over the square $b : j \longrightarrow j'$. To show the induced square

$$\exp_r(j', b) : \exp_r(j', f) \longrightarrow \exp_r(j, f)$$

underlies a morphism between right lifting structures against $\mathbb{L}_1$, suppose we have a lifting problem against an $\mathbb{L}_1$-structure α on $i : A \longrightarrow B$ as follows,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & {[D', X]} \arrow[r] \arrow[d, "\hat{\exp}_r(j', f)"] & {[D, X]} \arrow[d, "\exp_r(j,f)"] \\
B \arrow[r] \arrow[ur, dashed, "\hat{\exp}_r(\beta', \varphi)(\alpha)"] \arrow[urr, dashed, "\hat{\exp}_r(\beta, \varphi)(\alpha)"'] & {[C', X] \times_{[C', Y]} [D', Y]} \arrow[r] & {[C, X] \times_{[C, Y]} [D, Y]}
\end{tikzcd}$$

By construction the lifting solutions are induced by

$$\begin{tikzcd}
A \otimes D +_{A \otimes C} B \otimes C \arrow[r] \arrow[d, "i \hat{\otimes} j"'] & A \otimes D' +_{A \otimes C'} B \otimes C' \arrow[r] \arrow[d, "i \hat{\otimes} j'"'] & X \arrow[d, "f"] \\
B \otimes D \arrow[r] \arrow[urr, dashed, "\varphi(\alpha \hat{\otimes} \beta)"] & B \otimes D' \arrow[r] \arrow[ur, dashed, "\varphi(\alpha \hat{\otimes} \beta')"] & Y
\end{tikzcd}$$

which by functoriality of $\hat{\otimes}$ in the second entry, the left square underlies a morphism $\alpha \hat{\otimes} \eta$ of $\mathbb{L}_0$-maps. Thus the two lifts are compatible due to horizontal compatibility of φ, which also implies that the two dual lifts $\hat{\exp}_r(\beta, \varphi)(\alpha)$ and $\hat{\exp}_r(\beta', \varphi)(\alpha)$ are also compatible.

Similarly, if we have a horizontal morphism of right lifting structures against $\mathbb{L}_0$,

$$(f, \varphi) \longrightarrow (f', \varphi'),$$

then again consider a lifting against an $\mathbb{L}_1$-map as follows,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & {[D,X]} \arrow[r] \arrow[d, "\hat{\exp}_r(j,f)"] & {[D,X']} \arrow[d, "\hat{\exp}_r(j,f')"] \\
B \arrow[r] \arrow[ur, dashed, "\hat{\exp}_r(\beta,\varphi)(\alpha)"] \arrow[urr, dashed, "\hat{\exp}_r(\beta,\varphi')(\alpha)"' near end] & {[C,X] \times_{[C,Y]} [D,Y]} \arrow[r] & {[C,X'] \times_{[C,Y']} [D,Y']}
\end{tikzcd}$$

The Leibniz adjunction produces

$$\begin{tikzcd}
A \otimes D +_{A \otimes C} B \otimes C \arrow[r] \arrow[d, "i \hat{\otimes} j"'] & X \arrow[r] \arrow[d, "f"] & X' \arrow[d, "f'"] \\
B \otimes D \arrow[r] \arrow[ur, dashed, "\varphi(\alpha \hat{\otimes} \beta)"] \arrow[urr, dashed, "\varphi'(\alpha \hat{\otimes} \beta)"'] & B \otimes D' \arrow[r] & Y
\end{tikzcd}$$

The fact that $f \longrightarrow f'$ is a morphism between the two corresopnding right lifting structures implies the above two lifts are compatible, thus so are the two dual lifts for the pullback-exponentials.

Now for vertical compatibility. Consider a vertical composition of $\mathbb{L}_2$-maps as follows

$$C \xrightarrow{j_0} D \xrightarrow{j_1} E$$

with $\mathbb{L}_2$-structures β_0, β_1 on them, respectively. Recall that the vertical composition of pullback exponentials is provided by the following diagram,

$$\begin{tikzcd}
{[E,X]} \arrow[r, "\hat{\exp}_r(j_1,f)"] & {[D,X] \times_{[D,Y]} [E,Y]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & {[D,X]} \arrow[d, "\hat{\exp}_r(j_0,f)"] \\
& {[C,X] \times_{[C,Y]} [E,Y]} \arrow[r] & {[C,X] \times_{[C,Y]} [D,Y]}
\end{tikzcd}$$

Now consider the following lifting problem against an $\mathbb{L}_1$-structure α on i,

$$\begin{tikzcd}
A \arrow[r, "m"] \arrow[dd, "i"'] & {[E,X]} \arrow[d, "\hat{\exp}_r(j_1,f)"] & \\
& {[D,X] \times_{[D,Y]} [E,Y]} \arrow[r] \arrow[d] & {[D,X]} \arrow[d, "\hat{\exp}_r(j_0,f)"] \\
B \arrow[r, "\langle u,v \rangle"'] \arrow[uur, dashed, bend left, "\hat{\exp}_r(\beta_1,\varphi)(\alpha)"] \arrow[ur, dashed, "\langle \hat{\exp}_r(\beta_0,\varphi)(\alpha), v \rangle"] \arrow[urr, dashed, "\hat{\exp}_r(\beta_0,\varphi)(\alpha)"' near end] & {[C,X] \times_{[C,Y]} [E,Y]} \arrow[r] & {[C,X] \times_{[C,Y]} [D,Y]}
\end{tikzcd}$$

The vertical compatibility requires us to show that

$$\hat{\exp}_r(\beta_1 \bullet \beta_0, \varphi)(\alpha) = \hat{\exp}_r(\beta_1, \varphi)(\alpha)[m, \langle \hat{\exp}_r(\beta_0, \varphi)(\alpha), v\rangle].$$

Now consider the dual diagram as follows,

$$\begin{tikzcd}
A \otimes D +_{A\otimes C} B \otimes C \arrow[r] \arrow[d, "i \hat{\otimes} j_0"'] & A \otimes E +_{A\otimes C} B \otimes C \arrow[r] \arrow[d] & X \arrow[dd, "f"] \\
B \otimes D \arrow[r] \arrow[urr, dashed, "\varphi(\alpha \hat{\otimes} \beta_0)"] & A \otimes E +_{A\otimes C} B \otimes D \arrow[ur, dashed, "{[\varphi(\alpha \hat{\otimes} \beta_0), \widetilde{m}]}"'] \arrow[d, "i \hat{\otimes} j_1"'] \arrow[ul, phantom, "\ulcorner", very near start] & \\
& B \otimes E \arrow[r, "\tilde{u}"'] \arrow[uur, dashed, bend right, "\varphi(\alpha \hat{\otimes} \beta_1)"] & Y
\end{tikzcd}$$

Notice that the left part of the diagram is the vertical composition of the two pushout-products. Denote the top left pushout to be a, then by the fact that $\hat{\otimes}$ preserves vertical comopsition of $\mathbb{L}_2$-maps, we have that

$$\alpha \hat{\otimes} (\beta_1 \bullet \beta_0) = (\alpha \hat{\otimes} \beta_1) \bullet a_*(\alpha \hat{\otimes} \beta_0).$$

Hence, we have the following computation,

$$\begin{aligned}
\varphi(\alpha \hat{\otimes} (\beta_1 \bullet \beta_0)) &= \varphi(\alpha \hat{\otimes} \beta_1 \bullet a_* \alpha \hat{\otimes} \beta_0) \\
&= \varphi(\alpha \hat{\otimes} \beta_1)[\varphi(a_* \alpha \hat{\otimes} \beta_0), \tilde{u}] \\
&= \varphi(\alpha \hat{\otimes} \beta_1)[[\varphi(\alpha \hat{\otimes} \beta_0), \widetilde{m}], \tilde{u}]
\end{aligned}$$

Again, the second holds by vertical compatibility of φ, and the third holds by the horizontal compatibility of φ. Hence, our construction $\hat{\exp}_r$ preserves vertical composition of $\mathbb{L}_2$-maps.

Now similarly, for the vertical composition of $\mathbb{L}_0^\circ$-structures, consider two right lifting structures

$$X \xrightarrow{(f,\varphi)} Y \xrightarrow{(g,\psi)} Z$$

The vertical composition of the two pullback-exponentials $\hat{\exp}_r(j, f)$ and $\hat{\exp}_r(j, g)$ is again obtained by first pullback $\hat{\exp}_r(j, g)$ and then vertically compose with $\hat{\exp}_r(j, f)$. For any lifting situation against an $\mathbb{L}_1$-map α on $i : A \longrightarrow B$,

$$\begin{tikzcd}
A \arrow[r, "m"] \arrow[dd, "i"'] & {[D, X]} \arrow[d, "\hat{\exp}_r(j,f)"] & \\
& {[C, X] \times_{[C,Z]} [D, Y]} \arrow[r] \arrow[d] & {[D, Y]} \arrow[d, "\hat{\exp}_r(j,g)"] \\
B \arrow[r, "\langle u,v \rangle"'] \arrow[uur, dashed, bend left, "\hat{\exp}_r(\beta,\psi)(\alpha)"] \arrow[ur, dashed, "\langle u{,}\hat{\exp}_r(\beta,\psi)(\alpha) \rangle"] \arrow[urr, dashed, "\hat{\exp}_r(\beta,\varphi)(\alpha)"] & {[C, X] \times_{[C,Z]} [D, Z]} \arrow[r] & {[C, Y] \times_{[C,Z]} [D, Z]}
\end{tikzcd}$$

consider its dual diagram as follows,

$$
\begin{tikzcd}
A \otimes D +_{A \otimes C} B \otimes C \arrow[r, "{[\widetilde{m}, \tilde{u}]}"] \arrow[dd, "i \hat{\otimes} j"'] & X \arrow[d, "f"] \\
& Y \arrow[d, "g"] \\
B \otimes D \arrow[r, "\tilde{v}"'] \arrow[uur, dashed, "\varphi(\alpha \hat{\otimes} \beta)"] \arrow[ur, dashed, "\psi(\alpha \hat{\otimes} \beta)"'] & Z
\end{tikzcd}
$$

This way, by how vertical composition of right lifting structures are defined, we have

$$(\psi \bullet \varphi)(\alpha \hat{\otimes} \beta) = \varphi(\alpha \hat{\otimes} \beta)[[\widetilde{m}, \tilde{u}], \psi(\alpha \hat{\otimes} \beta)],$$

which also implies the vertical compatibility of the dual diagram. Hence, $\hat{\exp}_r$ also preserves vertical composition of $\mathbb{L}_0^\circ$-maps. This way, $\hat{\exp}_r$ is a well-defined morphism between 2-structures. $\square$

Completely dually, we say an morphism F in $\mathbb{R}\mathbf{ex}$

$$F : \mathscr{C}_1 \times \cdots \times \mathscr{C}_n \longrightarrow \mathscr{C}_0$$

has a *left closure* if there is another morphism in $\mathbb{R}\mathbf{ex}$

$$H : \mathscr{C}_0^{\mathrm{op}} \times \mathscr{C}_1 \times \cdots \times \mathscr{C}_{n-1} \longrightarrow \mathscr{C}_n^{\mathrm{op}},$$

such that for any c_i in $\mathscr{C}_i$ for $i \leq n$, we have an isomorphism

$$\mathscr{C}_0(F(c_1, \cdots, c_n), c_0) \cong \mathscr{C}_n(c_n, H(c_0, \cdots, c_{n-1})).$$

Now we have implicitly assumed that $\mathscr{C}_0$ and $\mathscr{C}_n$ also have finite limits. By essentially the same proof as Theorem 4.16, we record the following result:

Theorem 4.17. *If F in $\mathbb{R}\mathbf{ex}$*

$$F : \mathscr{C}_1 \times \cdots \times \mathscr{C}_n \longrightarrow \mathscr{C}_0$$

has a left closure H, then for any left structures $\mathbb{L}_0, \cdots, \mathbb{L}_n$ over $\mathscr{C}_0, \cdots, \mathscr{C}_n$, we have a function

$$\mathbb{L}\mathbf{Str}_F(\mathbb{L}_1, \cdots, \mathbb{L}_n; \mathbb{L}_0) \longrightarrow \mathbb{L}\mathbf{Str}_H(\mathbb{L}_0^\circ, \mathbb{L}_1, \cdots, \mathbb{L}_{n-1}; \mathbb{L}_n^\circ).$$

With slightly stronger assumptions, combining Theorem 4.16 and 4.17 we can get a stronger result. Recall from Proposition 3.9 that the operation of taking the left and right lifting structures are adjoint to each other. This in particular means that for any left structure $\mathbb{L}$ over $\mathrm{cube}_1(\mathscr{C})$, there will be a canonical morphism of left structures

$$\mathbb{L} \longrightarrow (\mathbb{L}^\circ)^\circ,$$

over the *identity functor* on $\mathscr{C}$.

Definition 4.18. We say a left structure $\mathbb{L}$ on $\mathscr{C}$ is *complete* if the canonical morphism

$$\mathbb{L} \longrightarrow (\mathbb{L}^\circ)^\circ$$

is an isomorphism. Similarly we can define complete right structures. $\diamondsuit$

Corollary 4.19. *If $\mathbb{L}_1, \mathbb{L}_0$ are two complete left structures, then for any functor F with a right closure G, the map constructed in Theorem 4.16*

$$\mathbb{L}\mathbf{Str}_F(\mathbb{L}_1, \cdots, \mathbb{L}_n; \mathbb{L}_0) \cong \mathbb{L}\mathbf{Str}_G(\mathbb{L}_2, \cdots, \mathbb{L}_n, \mathbb{L}_0^\circ; \mathbb{L}_1^\circ)$$

will be an isomorphism.

Proof. We simply notice that if G is a right closure of F, then F will be a left closure of G: For any c_i in $\mathscr{C}_i$ for $i \leq n$, we have natural isomorphisms

$$\mathscr{C}_1^{\mathrm{op}}(G(c_2, \cdots, c_n, c_0), c_1) \cong \mathscr{C}_0^{\mathrm{op}}(c_0, F(c_1, \cdots, c_n)).$$

This way, by Theorem 4.17 we have another morphism

$$\mathbb{L}\mathbf{Str}_G(\mathbb{L}_2, \cdots, \mathbb{L}_n, \mathbb{L}_0^\circ; \mathbb{L}_1^\circ) \longrightarrow \mathbb{L}\mathbf{Str}_F((\mathbb{L}_1^\circ)^\circ, \cdots, \mathbb{L}_n; (\mathbb{L}_0^\circ)^\circ).$$

If both $\mathbb{L}_0$ and $\mathbb{L}_1$ are complete, then this is gives us a map

$$\mathbb{L}\mathbf{Str}_G(\mathbb{L}_2, \cdots, \mathbb{L}_n, \mathbb{L}_0^\circ; \mathbb{L}_1^\circ) \longrightarrow \mathbb{L}\mathbf{Str}_F(\mathbb{L}_1, \cdots, \mathbb{L}_n; \mathbb{L}_0).$$

We can directly verify this map is the inverse of the map given in Theorem 4.16. Again we provide the proof for the binary case, and the generalisation to multiple parameters is straight forward. Suppose we have a morphism

$$\otimes : (\mathbb{L}_1, \mathbb{L}_2) \longrightarrow \mathbb{L}_0,$$

over $\otimes : \mathscr{C}_1 \times \mathscr{C}_2 \longrightarrow \mathscr{C}_0$ with a right closure $[-, -]$, whose induced map under Joyal-Tierney calculus is given by

$$\exp_r : (\mathbb{L}_2, \mathbb{L}_0^\circ) \longrightarrow \mathbb{L}_1^\circ.$$

By viewing $\otimes$ as a left closure of $[-,-]$, this again induces a morphism

$$\boxtimes : ((\mathbb{L}_1^\circ)^\circ, \mathbb{L}_2) \longrightarrow (\mathbb{L}_0^\circ)^\circ.$$

Concretely, given any left lifting operator α against $\mathbb{L}_1^\pitchfork$ on $i : A \longrightarrow B$ and any $\mathbb{L}_2$-structure β on $j : C \longrightarrow D$, we get a left lifting operator $\alpha\hat{\boxtimes}\beta$ against $\mathbb{L}_0^\pitchfork$ on their pushout-product,

$$i\hat{\otimes}j : B \otimes C +_{A\otimes C} A \otimes D \longrightarrow B \otimes D.$$

By construction, for any $f : X \longrightarrow Y$ with an $\mathbb{L}_0^\pitchfork$-structure φ, the lifting $\alpha\hat{\boxtimes}\beta(\varphi)$

$$\begin{tikzcd}
B \otimes C +_{A\otimes C} A \otimes D \arrow[r] \arrow[d, "i\hat{\otimes}j"'] & X \arrow[d, "f"] \\
B \otimes D \arrow[r] \arrow[ur, dashed, "\alpha\hat{\boxtimes}\beta(\varphi)"] & Y
\end{tikzcd}$$

is induced by the transposed diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & {[D,X]} \arrow[d, "\hat{\exp}_r(j,f)"] \\
B \arrow[r] \arrow[ur, dashed, "\alpha(\hat{\exp}_r(\beta,\varphi))"] & {[C,X] \times_{[C,Y]} [D,Y]}
\end{tikzcd}$$

Now since $\mathbb{L}_1$ is complete, we can simultaneously view α as a genuine $\mathbb{L}_1$ structure, which means that the transpose of the above lift by construction is given as

$$\begin{tikzcd}
B \otimes C +_{A\otimes C} A \otimes D \arrow[r] \arrow[d, "i\hat{\otimes}j"'] & X \arrow[d, "f"] \\
B \otimes D \arrow[r] \arrow[ur, dashed, "\varphi(\alpha\hat{\otimes}\beta)"] & Y
\end{tikzcd}$$

It follows that the two lifts

$$\alpha\hat{\boxtimes}\beta(\varphi) = \varphi(\alpha\hat{\otimes}\beta).$$

Thus, under the completeness isomorphism of $\mathbb{L}_0$, we indeed have

$$\alpha\hat{\boxtimes}\beta = \alpha\hat{\otimes}\beta.$$

The other direction is similar. □

The above results corresponds to the usual application of Joyal-Tierney calculus: Suppose we have three WFSs $(\mathbf{C}_i, \mathbf{F}_i)$ for $i = 0, 1, 2$, and suppose we have shown that the pushout-product of $\mathbf{C}_1$ with $\mathbf{C}_2$ is a map in $\mathbf{C}_0$, then we can automatically conclude that the pullback-exponential of a $\mathbb{C}_2$-map with an $\mathbb{F}_0$-map gives us an $\mathbb{F}_1$-map.

At the end, let us describe a first easy application of the structured Joyal-Tierney calculus as follows. More serious examples and applications will be discussed in length in Chapter 5.

Example 4.20. As Example 4.4 shows, if we have a dominance structure Σ on $\mathscr{C}$, then we have a morphism in $\mathbb{L}\mathbf{Str}$

$$(\Sigma, \Sigma) \longrightarrow \Sigma$$

over the Cartesian product on $\mathscr{C}$. Furthermore, since Σ is a propositional structure, this morphism is *unique*. By Corollary 4.19 and Theorem 3.22 for AWFSs, this uniquely determines another morphism in $\mathbb{L}\mathbf{Str}$

$$(\Sigma, \mathbb{T}^{\mathrm{op}}) \longrightarrow \mathbb{T}^{\mathrm{op}},$$

now over the internal hom on $\mathscr{C}$. Notice that by Proposition 4.15, the existence of such a morphism implies the unstructured version, viz. the pullback-exponential of cofibrations w.r.t. trivial fibrations is also a trivial fibration, which in classical homotopy theory is indeed a consequence of the Joyal-Tierney calculus. However, the isomorphism stated in Corollary 4.19 provides much more information in the structured context. $\diamondsuit$

4.4 Algebraic Monoidal Model Categories

Based on previous developments in this chapter, in this section we will arive at a notion of an *algebraic monoidal model category*. Before giving the definition, we first describe what is a *monoidal model category* in the usual unstructured setting. We start by defining a model structure on a category:

Definition 4.21. A model structure on a category $\mathscr{C}$ consists of three classes of maps $(\mathbf{C}, \mathbf{F}, \mathbf{W})$ called *cofibrations*, *fibrations*, and *weak equivalences* respectively, such that

- $(\mathbf{C}, \mathbf{F} \cap \mathbf{W})$ and $(\mathbf{C} \cap \mathbf{W}, \mathbf{F})$ form two WFSs on $\mathscr{C}$;
- $\mathbf{W}$ contains all isomorphisms and has the *2-out-of-3 property*, i.e. given composible maps f, g, is two of the maps in f, g, fg belong to $\mathbf{W}$, then so is the third. $\diamondsuit$

For a model structure, maps in $\mathbf{C}, \mathbf{C} \cap \mathbf{W}$ are referred to as cofibrations and trivial cofibrations, respectively; similarly, maps in $\mathbf{F}, \mathbf{F} \cap \mathbf{W}$ are called fibrations and trivial fibrations. Now following [46], a model category $\mathscr{C}$ is simply a finite complete and finitely cocomplete category equipped with a model structure.[2] Also notice that it is a standard observation in homotopy theory that the model structure, if exists, is completely determined by the two WFSs $(\mathbf{C}, \mathbf{F} \cap \mathbf{W})$ and $(\mathbf{C} \cap \mathbf{W}, \mathbf{F})$; see e.g. [35]. Thus, in the future we may also refer to two WFSs on a category to form a model structure.

A model category is furthermore *monoidal*, if $\mathscr{C}$ has a monoidal product which interacts with the two WFSs in a compatible way:

Definition 4.22. A model structure $(\mathbf{C}, \mathbf{F}, \mathbf{W})$ on $\mathscr{C}$ is *monoidal* w.r.t. a monoidal structure $(I, \otimes)$, if the following conditions hold:

- *Unitary*: The monoidal unit is cofibrant, i.e. the unique map $\varnothing \longrightarrow I$ belongs to $\mathbf{C}$.[3]
- *Pushout-products*: Maps in $\mathbf{C}$ are closed under pushout-products; maps in $\mathbf{C} \cap \mathbf{W}$ are closed under pushout-products with maps in $\mathbf{C}$ on both sides. ◊

Notice that we can split Definition 4.22 as separate conditions on $\mathbf{C}$ and $\mathbf{C} \cap \mathbf{W}$: For $\mathbf{C}$, it requires that I is cofibrant and cofibrations being closed under pushout-products; For $\mathbf{C} \cap \mathbf{W}$, it further requires that maps in $\mathbf{C} \cap \mathbf{W}$ are closed under pushout-product w.r.t. maps in $\mathbf{C}$. In our structured approach, we will treat these two conditions separately.

We first define the notion of a *monoidal left structure*:

Definition 4.23. A *monoidal left structure* $\mathbb{L}$ on $\mathscr{C}$ consists of morphisms

$$\eta : () \longrightarrow \mathbb{L}, \quad \otimes : (\mathbb{L}, \mathbb{L}) \longrightarrow \mathbb{L},$$

making it a *pseudo monoid* in the **Grpd**-enriched multi-category $\mathbb{L}\mathbf{Str}$. ◊

We refer the readers to [57, 67] for the notion of pseudo monoids. Roughly speaking, they are objects with unit and multiplication maps equipped with explicit unitors and associators, satisfying similar equations as that of a monoidal category (cf. [44]). As mentioned in Remark 4.13, the multicategory $\mathbb{L}\mathbf{Str}$ is symmetric, thus there is a notion of pseudo *symmetric* monoids in $\mathbb{L}\mathbf{Str}$

To better understand Definition 4.23, we can also unfold the data of a monoidal left structure on $\mathscr{C}$. First notice that, as mentioned we have a forgetful multifunctor

$$|-| : \mathbb{L}\mathbf{Str} \longrightarrow \mathbb{R}\mathbf{ex}.$$

[2] In some modern references on homotopy theory, people may require the stronger axiom that the underlying category should admit *all* small limits and colimits.

[3] In some literature [43] one requires an even weaker condition, which says that there exists a cofibrant replacement I' of I, such that the induced map $I' \otimes X \longrightarrow X$ for any X is a weak equivalence.

It follows that a pseudo monoid in $\mathbb{L}\mathbf{Str}$ also induces one in $\mathbb{R}\mathbf{ex}$. This means exactly that if the followings are the underlying morphisms of η and $\otimes$ in $\mathbb{R}\mathbf{ex}$,

$$I : () \longrightarrow \mathscr{C}, \quad \otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C},$$

then $(\mathscr{C}, I, \otimes)$ will be a *monoidal category*, where the monoidal product preserves finite colimits on both sides. Based on this, we may describe more concretely what is required for a left structure $\mathbb{L}$ to be monoidal over it:

- η assigns an $\mathbb{L}$-structure on the map $\emptyset \longrightarrow I$, i.e. making I an $\mathbb{L}$-object, and $\otimes$ requires a functor over $\mathrm{cube}_2(\otimes)$ as below

  $$\hat{\otimes} : \mathbb{L} \otimes \mathbb{L} \longrightarrow \mathrm{cof}_2(\mathbb{L}),$$

  that for any $\mathbb{L}$-maps f, g, it assigns an $\mathbb{L}$-structure on the pushout-product $f \hat{\otimes} g$, which is compatible with the double categorical structures of $\mathbb{L}$.

- For the unitors on $\mathscr{C}$ to be unitors on $\mathbb{L}\mathbf{Str}$, by definition this means that given any $\mathbb{L}$-map $f : X \longrightarrow Y$, the pushout-product of the two maps assigned by $\hat{\otimes}$

  $$\begin{aligned} f \hat{\otimes} I &: \emptyset \otimes Y +_{\emptyset \otimes X} I \otimes X \cong X \longrightarrow I \otimes Y \cong Y, \\ I \hat{\otimes} f &: X \otimes I +_{X \otimes \emptyset} Y \otimes \emptyset \cong X \longrightarrow Y \otimes I \cong Y, \end{aligned}$$

  coincide with the $\mathbb{L}$-structure on f, modulo the unitors for I.[4]

- For the associator on $\mathscr{C}$ to be an associator in $\mathbb{L}\mathbf{Str}$, given three maps $f : X_0 \longrightarrow X_1$, $g : Y_0 \longrightarrow Y_1$ and $h : Z_0 \longrightarrow Z_1$ in $\mathscr{C}$, modulo the associators of the monoidal structure and obtain a cube spanned by f, g, h as follows,

$$\begin{tikzcd}
& X_0 \otimes Y_0 \otimes Z_1 \arrow[rr, "X_0\otimes g\otimes Z_1"] \arrow[dd, "f\otimes Y_0\otimes Z_1"' near start] & & X_0 \otimes Y_1 \otimes Z_1 \arrow[dd, "f\otimes Y_1\otimes Z_1"] \\
X_0 \otimes Y_0 \otimes Z_0 \arrow[ur, "X_0\otimes Y_0\otimes h"] \arrow[rr, "X_0\otimes g\otimes Z_0" near start] \arrow[dd, "f\otimes Y_0\otimes Z_0"'] & & X_0 \otimes Y_1 \otimes Z_0 \arrow[ur, "X_0\otimes Y_1\otimes h"] \arrow[dd, "f\otimes Y_1\otimes Z_0" near start] & \\
& X_1 \otimes Y_0 \otimes Z_1 \arrow[rr, "X_1\otimes g\otimes Z_1" near end] & & X_1 \otimes Y_1 \otimes Z_1 \\
X_1 \otimes Y_0 \otimes Z_0 \arrow[ur, "X_1\otimes Y_0\otimes h"] \arrow[rr, "X_1\otimes g\otimes Z_0"'] & & X_1 \otimes Y_1 \otimes Z_0 \arrow[ur, "X_1\otimes Y_1\otimes h"] &
\end{tikzcd}$$

[4]Notice that since $\otimes$ preserves finite colimits on each entry, we do have that $\emptyset \otimes X \cong \emptyset \cong X \otimes \emptyset$, thus the above isomorphisms do hold.

if we first assign $\mathbb{L}$-structures on $X_0 \otimes (g\hat{\otimes}h)$ and $X_1 \otimes (g\hat{\otimes}h)$, and then apply $\hat{\otimes}$ again to the following square[5]

$$\begin{tikzcd}
X_0 \otimes (Y_0 \otimes Z_1 +_{Y_0\otimes Z_0} Y_1 \otimes Z_0) \arrow[r, "X_0\otimes(g\hat{\otimes}h)"] \arrow[d] & X_0 \otimes Y_1 \otimes Z_1 \arrow[d, "f\otimes Y_1\otimes Z_0"] \\
X_1 \otimes (Y_0 \otimes Z_1 +_{Y_0\otimes Z_0} Y_1 \otimes Z_0) \arrow[r, "X_1\otimes(g\hat{\otimes}h)"'] & X_1 \otimes Y_1 \otimes Z_1
\end{tikzcd}$$

which results in an $\mathbb{L}$-structure on $f\hat{\otimes}g\hat{\otimes}h$, or if we first assign $\mathbb{L}$-structures on $(f\hat{\otimes}g) \otimes Z_0$ and $(f\hat{\otimes}g) \otimes Z_1$, and then apply $\otimes$ again to the following square

$$\begin{tikzcd}
(X_1 \otimes Y_0 +_{X_0\otimes Y_0} X_0 \otimes Y_1) \otimes Z_0 \arrow[r, "(f\hat{\otimes}g)\otimes Z_0"] \arrow[d] & X_1 \otimes Y_1 \otimes Z_0 \arrow[d, "f\otimes Y_1\otimes Z_0"] \\
(X_1 \otimes Y_0 +_{X_0\otimes Y_0} X_0 \otimes Y_1) \otimes Z_1 \arrow[r, "(f\hat{\otimes}g)\otimes Z_1"'] & X_1 \otimes Y_1 \otimes Z_1
\end{tikzcd}$$

which results in another $\mathbb{L}$-structure on $f\hat{\otimes}g\hat{\otimes}h$, the two structures would coincide, modulo the associators on $\otimes$.

For a *symmetric* pseudo monoid, we further require that for any $\mathbb{L}$-structures α, β on f, g, the associated $\mathbb{L}$-structures $\alpha\hat{\otimes}\beta$ and $\beta\hat{\otimes}\alpha$ coincide up to the canonical isomorphism $f\hat{\otimes}g \cong g\hat{\otimes}f$.

Again, for an easy example, the dominance structure on a category is always monoidal w.r.t. the Cartesian products:

Example 4.24. Suppose there is a dominance structure Σ on $\mathscr{C}$. From Example 4.14 we already know that there is a map $(\Sigma, \Sigma) \longrightarrow \Sigma$ over the Cartesian product on $\mathscr{C}$. The Cartesian product is indeed a monoidal structure. For the unit, recall from Section 3.3.1 that any object by assumption is cofibrant, thus there is a unique Σ-structure on the unit 1 of the Cartesian product. Finally, the existence of unitors, associators and symmetric braidings are trivial for Σ, because it is *propositional*, thus any diagram with codomain Σ commutes, if the underlying diagram in $\mathbb{R}$**ex** commutes. This also implies that the monoid Σ is also *symmetric*. $\Diamond$

The first condition of a monoidal model category on cofibrations thus is lifted to the data of a monoidal left structure. The second condition on trivial cofibrations, i.e. they

[5] Notice that the top horizontal arrow represents the cofibre arrow of the top face of the cube above; we can write it this way because by assumption $\otimes$ preserves finite colimits on each variable.

are closed under pushout-product over cofibrations, is naturally lifted to *actions* of this monoid. In other words, given a monoidal object in $\mathbb{L}\mathbf{Str}$, we can talk about a *module* over this monoid. Since the axiom requires $\mathbf{C} \cap \mathbf{W}$ to be closed under pushout-products of $\mathbf{C}$ on both sides, we in fact will consider a *bimodule* structure. However, for us, we do not consider arbitrary modules, since we want this action to lie over the monoidal product on $\mathscr{C}$. We call these the *monoidal bimodules*:

Definition 4.25. Given any monoidal left structure $(\mathbb{L}, \eta, \otimes)$ over the monoidal structure $(I, \otimes)$ on $\mathscr{C}$, a *monoidal bimodule* of $\mathbb{L}$ is a left structure $\mathbb{M}$ on $\mathscr{C}$ with a both a left and a right $\mathbb{L}$-action in $\mathbb{L}\mathbf{Str}$

$$\zeta : (\mathbb{L}, \mathbb{M}) \longrightarrow \mathbb{M}, \quad \xi : (\mathbb{M}, \mathbb{L}) \longrightarrow \mathbb{M},$$

both over the monoidal product $\otimes$ on $\mathscr{C}$, making it a pseudo bimodule of $\mathbb{L}$ in $\mathbb{L}\mathbf{Str}$. In particular, this means that they are pseudo modules respectively, such that the associators on $\otimes$ makes the following a 2-cell in $\mathbb{L}\mathbf{Str}$,

$$\begin{tikzcd} (\mathbb{L}, \mathbb{M}, \mathbb{L}) \arrow[r, "{(\zeta,1)}"] \arrow[d, "{(1,\xi)}"'] \arrow[dr, phantom, "\cong"] & (\mathbb{M}, \mathbb{L}) \arrow[d, "\xi"] \\ (\mathbb{L}, \mathbb{M}) \arrow[r, "\zeta"'] & \mathbb{M} \end{tikzcd}$$ ◇

Intuitively, the $\mathbb{L}$-actions on a monoidal bimodule $\mathbb{M}$ express structurally the fact that the pushout-product of an $\mathbb{M}$-map with an $\mathbb{L}$-map on both sides is again an $\mathbb{M}$-map. If the monoid $\mathbb{L}$ is symmetric, then the notion of a *left* and *right* module, and thus that of a *bimodule*, coincide.

At this point, we already have all the elements to define algebraic monoidal model categories. Recall from [47] we already have a notion of an *algebraic model structure*:

Definition 4.26. An *algebraic model structure* on a category $\mathscr{C}$ consists of two AWFSs $(\mathbb{C}_0, \mathbb{F}_0)$ and $(\mathbb{C}_1, \mathbb{F}_1)$, such that

- There is a morphism of AWFSs $(\mathbb{C}_1, \mathbb{F}_1) \longrightarrow (\mathbb{C}_0, \mathbb{F}_0)$ (cf. Proposition 3.23);
- The underlying WFSs $(\mathbf{C}_0, \mathbf{F}_0)$ and $(\mathbf{C}_1, \mathbf{F}_1)$ form a model structure on $\mathscr{C}$. ◇

Based on this, we may define what is furthermore an *algebraic monoidal model structure* on a monoidal category:

Definition 4.27. An *algebraic monoidal structure* on a monoidal category $(\mathscr{C}, I, \otimes)$ consists of two AWFSs $(\mathbb{C}_0, \mathbb{F}_0)$, $(\mathbb{C}_1, \mathbb{F}_1)$ and a morphism $(\mathbb{C}_1, \mathbb{F}_1) \longrightarrow (\mathbb{C}_0, \mathbb{F}_0)$, such that

- $\mathbb{C}_0$ is a monoidal left structure over $\otimes$;
- $\mathbb{C}_1$ is a monoidal bimodule on $\mathbb{C}_0$.
- $\mathbb{C}_1 \longrightarrow \mathbb{C}_0$ is a morphism of monoidal bimodules.[6]

We say this is an *algebraic monoidal model structure* if the two AWFSs also form an algebraic model structure on $\mathscr{C}$. $\diamondsuit$

Remark 4.28. As mentioned in the Introduction, Riehl in [48] also gives a notion of what she calls a *monoidal algebraic model structure*, again based on Definition 4.26. However, the monoidal structure defined there is weaker than our definition in two aspects. Firstly, the pushout-product axioms are realised as functor between the underlying 1-categories of the left classes of the AWFSs, thus only accounts for the functoriality of the horizontal morphisms. For us, the monoidal actions are formulated as morphisms in $\mathbb{L}$**Str**, thus we also account for the functoriality of vertical compositions. Secondly, the definition given in *loc. cit.* does not include the monoid and module axioms, either, which means there are no unity and associativity laws. $\diamondsuit$

An immediate question to ask is that what structures will be possessed by the corresponding *right* classes of an algebraic monoidal model structure on $\mathscr{C}$. Suppose the monoidal product $\otimes$ is *right closed*, with closure $[-,-]$. Then already from the structured Joyal-Tierney calculus we have described in Theorem 4.16, the algebraic monoidal model structure would furthermore imply the existence of the following morphisms in $\mathbb{L}$**Str**,

$$(\mathbb{C}_0, \mathbb{F}_0^{\mathrm{op}}) \longrightarrow \mathbb{F}_0^{\mathrm{op}}, \quad (\mathbb{C}_1, \mathbb{F}_1^{\mathrm{op}}) \longrightarrow \mathbb{F}_0^{\mathrm{op}}, \quad (\mathbb{C}_0, \mathbb{F}_1^{\mathrm{op}}) \longrightarrow \mathbb{F}_1^{\mathrm{op}}.$$

Thus, on the structured level we already have: (1) Pullback-exponentials of cofibrations against trivial fibrations will be trivial fibrations; (2) Pullback-exponentials of trivial cofibrations against fibrations will be trivial fibrations; (3) Pullback-exponentials of cofibrations agianst fibrations will be fibrations.

However, the monoidal structures on the left classes actually says more, since besides the morphisms of monoidal product and monoidal actions, we also have unity and associativity laws. To account for this on the side of right structures, we define a notion of an *exponential module*:

Definition 4.29. Suppose we have a monoidal left structure $(\mathbb{L}, \eta, \otimes)$ over a monoidal category $(\mathscr{C}, I, \otimes)$. If the monoidal product has a right closure $[-,-]$, then an *exponential left* $\mathbb{L}$*-module* is a right structure $\mathbb{R}$ on $\mathscr{C}$ with an action

$$\nu : (\mathbb{L}, \mathbb{R}^{\mathrm{op}}) \longrightarrow \mathbb{R}^{\mathrm{op}},$$

[6] Notice that any pseudo monoid is canonically a pseudo bimodule over itself.

over the right closure $[-,-]$, making $\mathbb{R}^{\mathrm{op}}$ a left $\mathbb{L}$-module in $\mathbb{L}\mathbf{Str}$. $\diamondsuit$

Again, if the pseudo monoid $\mathbb{L}$ is symmetric, then the two notions coincide. Let us unfold what the above definition amounts to. Suppose $\otimes$ is right closed, and there is a left $\mathbb{L}$-action on $\mathbb{R}^{\mathrm{op}}$ in $\mathbb{L}\mathbf{Str}$

$$\nu : (\mathbb{L}, \mathbb{R}^{\mathrm{op}}) \longrightarrow \mathbb{R}^{\mathrm{op}}.$$

It takes an $\mathbb{R}$-structure on $f : X \longrightarrow Y$ and an $\mathbb{L}$-structure on $i : A \longrightarrow B$, and assigns an $\mathbb{R}$-structure on the pullback exponential of the two morphism

$$\hat{\exp}_r(i, f) : [B, Y] \longrightarrow [A, Y] \times_{[A,X]} [B, X]$$

that is again compatible with the vertical composition on both sides. Notice that for any A, B, C in $\mathscr{C}$, we indeed have a canonical natural isomorphism

$$[A \otimes B, C] \cong [A, [B, C]],$$

which means that the following diagram commute in $\mathbb{R}\mathbf{ex}$,

$$\begin{tikzcd}
(\mathscr{C}, \mathscr{C}, \mathscr{C}^{\mathrm{op}}) \arrow[r, "{(1,[-,-])}"] \arrow[d, "{(\otimes,1)}"'] & (\mathscr{C}, \mathscr{C}^{\mathrm{op}}) \arrow[d, "{[-,-]}"] \\
(\mathscr{C}, \mathscr{C}^{\mathrm{op}}) \arrow[r, "{[-,-]}"'] & \mathscr{C}^{\mathrm{op}}
\end{tikzcd}$$

The associativity of the exponential left $\mathbb{L}$-module is thus over the above diagram in $\mathbb{R}\mathbf{ex}$. Based on the structured Joyal-Tierney calculus, the following result implies that we can equivalently using right structures to describe monoidal modules:

Lemma 4.30. *Given a monoidal left structure $\mathbb{L}$ over the monoidal category $(\mathscr{C}, I, \otimes)$ with a right closure $[-,-]$. For any complete left structure $\mathbb{M}$ (cf. Definition 4.18), given a right action over $\mathbb{L}$*

$$\xi : (\mathbb{M}, \mathbb{L}) \longrightarrow \mathbb{M},$$

it makes $\mathbb{M}$ a monoidal right $\mathbb{L}$-module iff the action

$$\nu : (\mathbb{L}, \mathbb{M}^{\circ}) \longrightarrow \mathbb{M}^{\circ}$$

induced by Theorem 4.16 makes $\mathbb{M}^{\pitchfork}$ an exponential left $\mathbb{L}$-module.

Proof. Suppose ξ makes $\mathbb{M}$ a monoidal $\mathbb{L}$-module. It means that for any $\mathbb{M}$-structure α on i, $\mathbb{L}$-structures β, γ on j, k, the two $\mathbb{M}$-structures $\hat{\xi}(\alpha, \beta \hat{\otimes} \gamma)$ and $\hat{\xi}(\hat{\xi}(\alpha, \beta), \gamma)$ coincide on $i \hat{\otimes} j \hat{\otimes} k$, modulo the associators.

Recall Theorem 4.16. Given any $\mathbb{L}$-structure β on j and $\mathbb{M}^{\pitchfork}$-structure φ on f, the $\mathbb{M}^{\pitchfork}$-structure $\hat{\nu}(\beta, \varphi)$ on $\hat{\exp}_r(j, f)$ lifts against any $\mathbb{M}$-structure α on i

$$\hat{\nu}(\beta, \varphi)(\alpha) = \varphi(\widetilde{\hat{\xi}(\alpha, \beta)}).$$

Now given $\mathbb{L}$-structures β, γ on j, k and $\mathbb{M}^{\pitchfork}$-structure φ on f, for any lifting problem against an $\mathbb{M}$-structure α on i, we have

$$\begin{aligned}\hat{\nu}(\beta, \hat{\nu}(\gamma, \varphi))(\alpha) &= \widetilde{\hat{\nu}(\gamma, \varphi)(\hat{\xi}}(\alpha, \beta)) = \varphi(\hat{\xi}(\widetilde{\hat{\xi}(\alpha, \beta)}, \gamma)) \\ &= \varphi(\widetilde{\hat{\xi}(\alpha, \beta \hat{\otimes} \gamma})) = \hat{\nu}(\beta \hat{\otimes} \gamma, \varphi)(\alpha)\end{aligned}$$

The first, second, and the last equalities are simply due to construction of ν; the third equality holds since ξ is a module action. This shows that ν makes $\mathbf{M}^{\pitchfork}$ an exponential left module. The other way arround is completely similar. □

Thus, the upshot of Lemma 4.30 is that when the monoidal product $\otimes$ is right closed, the notion of an exponential left module can be equivalently used to characterise monoidal right modules. To account for the other side, we assume the monoidal product also has a left closure $\langle -, - \rangle$. Notice that by the formulation in Section 4.3, the left closure will be a functor

$$\langle -, - \rangle : \mathscr{C}^{\mathrm{op}} \times \mathscr{C} \longrightarrow \mathscr{C}^{\mathrm{op}},$$

such that for any A, B, C in $\mathscr{C}$, we have

$$\mathscr{C}(A \otimes B, C) \cong \mathscr{C}(B, \langle C, A \rangle).$$

This notation slightly differs from the usual notation for exponentials, since we write $\langle C, A \rangle$ instead of $\langle A, C \rangle$. The reason we write it this way is due to the fact that, as mentioned, the Joyal-Tierney calculus given in Section 4.3 can essentially be viewed as describing a cyclic action on $\mathbb{L}$**Str**; see also [15]. As we will see, preserving the order between inputs will make the combinatorics involved correct.

Based on a left closure, one can similarly define the notion of an *exponential left* $\mathbb{L}$-*module*, which will be a right structure $\mathbb{R}$ with an action

$$\mu : (\mathbb{R}^{\mathrm{op}}, \mathbb{L}) \longrightarrow \mathbb{R}^{\mathrm{op}},$$

over $\langle -, - \rangle$. We again have the following result:

Lemma 4.31. *Given a monoidal left structure* $\mathbb{L}$ *over the monoidal category* $(\mathscr{C}, I, \otimes)$ *with a left closure* $\langle -, - \rangle$. *For any complete left structure* $\mathbb{M}$, *given a left action over* $\mathbb{L}$

$$\zeta : (\mathbb{L}, \mathbb{M}) \longrightarrow \mathbb{M},$$

it makes $\mathbb{M}$ *a monoidal left* $\mathbb{L}$*-module iff the action*

$$\mu : (\mathbb{M}^{\circ}, \mathbb{L}) \longrightarrow \mathbb{M}^{\circ}$$

induced by Theorem 4.17 makes $\mathbb{M}^{\pitchfork}$ *an exponential right* $\mathbb{L}$*-module.*

Proof. Completely similar to the proof of Lemma 4.30. □

Now to account for monoidal *bimodules*, we need to furthermore combine the actions of left and right exponential modules, and arrive at the notion of an *exponential bimodule*:

Definition 4.32. Let $\mathbb{L}$ be a monoidal left structure over $(\mathscr{C}, I, \otimes)$, which is both left and right closed. Now a right structure $\mathbb{R}$ on $\mathscr{C}$ is called an *exponential bimodule*, if it is equipped with an exponential left $\mathbb{L}$-module structure

$$\nu : (\mathbb{L}, \mathbb{R}^{\mathrm{op}}) \longrightarrow \mathbb{R}^{\mathrm{op}}$$

over the right closure $[-,-]$, and an exponential right $\mathbb{L}$-module structure

$$\mu : (\mathbb{R}^{\mathrm{op}}, \mathbb{L}) \longrightarrow \mathbb{R}^{\mathrm{op}}$$

over the left closure $\langle -,- \rangle$. Furthermore, they become a bimodule over $\mathbb{L}$ as follows,

$$\begin{tikzcd} (\mathbb{L}, \mathbb{R}^{\mathrm{op}}, \mathbb{L}) \arrow[r, "{(\nu,1)}"] \arrow[d, "{(1,\mu)}"'] \arrow[dr, phantom, "\cong"] & (\mathbb{R}^{\mathrm{op}}, \mathbb{L}) \arrow[d, "\mu"] \\ (\mathbb{L}, \mathbb{R}^{\mathrm{op}}) \arrow[r, "\nu"'] & \mathbb{R}^{\mathrm{op}} \end{tikzcd}$$ ◊

Again, notice that for any A, B, C in $\mathscr{C}$, we have a canonical natural isomorphism

$$\langle [A,B], C \rangle \cong [A, \langle B, C \rangle].$$

This can be easily shown by the following sequence of natural isomorphisms,

$$\begin{aligned} \mathscr{C}(D, \langle [A,B], C \rangle) &\cong \mathscr{C}(C \otimes D, [A,B]) \cong \mathscr{C}(C \otimes D \otimes A, B) \\ &\cong \mathscr{C}(D \otimes A, \langle B, C \rangle) \cong \mathscr{C}(D, [A, \langle B, C \rangle]) \end{aligned}$$

Hence, the above compatibility condition of the exponential bimodule is over the following diagram in $\mathbb{R}\mathbf{ex}$,

$$\begin{tikzcd} (\mathscr{C}, \mathscr{C}^{\mathrm{op}}, \mathscr{C}) \arrow[r, "{([-,-],1)}"] \arrow[d, "{(1,\langle -,- \rangle)}"'] & (\mathscr{C}^{\mathrm{op}}, \mathscr{C}) \arrow[d, "{\langle -,- \rangle}"] \\ (\mathscr{C}, \mathscr{C}^{\mathrm{op}}) \arrow[r, "{[-,-]}"'] & \mathscr{C}^{\mathrm{op}} \end{tikzcd}$$

If we do not write the left closure $\langle -,- \rangle$ the way above, the above isomorphism will necessarily involve certain commutators of the Cartesian products. The upshot is the following result:

Theorem 4.33. *Suppose $\mathbb{L}$ is a monoidal left structure over the monoidal category $(\mathscr{C}, I, \otimes)$. Suppose $\otimes$ has both a left and right closure. Then given any complete left structure $\mathbb{M}$ and a left and right monoidal module structure on $\mathbb{M}$,*

$$\zeta : (\mathbb{L}, \mathbb{M}) \longrightarrow \mathbb{M}, \quad \xi : (\mathbb{M}, \mathbb{L}) \longrightarrow \mathbb{M},$$

they form a monoidal bimodule iff the induced left and right exponential modules given by Lemma 4.30 and 4.31

$$\mu : (\mathbb{M}^{\circ}, \mathbb{L}) \longrightarrow \mathbb{M}^{\circ}, \quad \nu : (\mathbb{L}, \mathbb{M}^{\circ}) \longrightarrow \mathbb{M}^{\circ},$$

form an exponential bimodule on $\mathbb{M}^{\pitchfork}$.

Proof. Suppose $\mathbb{M}$ under ζ and ξ is a monoidal bimodule, which means that for any $\mathbb{L}$-structures α, β on i, k and $\mathbb{M}$-structure γ on j, we have

$$\hat{\xi}(\hat{\zeta}(\alpha, \gamma), \beta) = \hat{\zeta}(\alpha, \hat{\xi}(\gamma, \beta)),$$

over $i \hat{\otimes} j \hat{\otimes} k$, module the associator of $\otimes$. Now given $\mathbb{L}$-structures α, β on i, k and an $\mathbb{M}^{\pitchfork}$-structure φ on f, from the constructions in Theorem 4.16 and 4.17 we have that for any $\mathbb{M}$-structure γ on j,

$$\hat{\mu}(\varphi, \alpha)(\gamma) = \varphi(\overline{\hat{\zeta}(\alpha, \gamma)}), \quad \hat{\nu}(\beta, \varphi)(\gamma) = \varphi(\overline{\hat{\xi}(\gamma, \beta)}).$$

Thus, we have the following computation,

$$\begin{aligned} \hat{\mu}(\hat{\nu}(\beta, \varphi), \alpha)(\gamma) &= \hat{\nu}(\beta, \varphi)(\overline{\hat{\zeta}(\alpha, \gamma)}) = \varphi(\overline{\overline{\overline{\hat{\xi}(\hat{\zeta}(\alpha, \gamma), \beta)}}}) \\ &= \varphi(\overline{\overline{\hat{\zeta}(\alpha, \hat{\xi}(\gamma, \beta))}}) = \hat{\mu}(\varphi, \alpha)(\overline{\hat{\xi}(\gamma, \beta)}) \\ &= \hat{\nu}(\beta, \hat{\mu}(\varphi, \alpha))(\gamma) \end{aligned}$$

It thus follows that μ, ν do makes $\mathbb{M}^{\pitchfork}$ an exponential bimodule. The other way arround is completely similar, assuming $\mathbb{M}$ is complete. $\square$

The upshot of Theorem 4.33 is that if the monoidal structure for the underlying category is *biclosed*, then the second condition in Definition 4.27 of an algebraic monoidal structure can be equivalently described as $\mathbb{F}_1$ being an exponential bimodule of the monoid $\mathbb{C}_0$. This observation will be the basis of how we assign an algebraic monoidal structure for effective Kan fibrations in Chapter 5.

Chapter 5

Algebraic Monoidal Structure for Effective Kan Fibrations

In this chapter we describe an algebraic monoidal structure for effective Kan fibrations. We again work within a category $\mathscr{C}$ satisfying the same assumptions as in Section 3.3, i.e. $\mathscr{C}$ is finitely complete, finitely cocomplete, locally Cartesian closed, and equipped with a dominance and symmetric Moore structure.

Recall that in Example 4.24, we have seen that the left structure of cofibrations Σ indeed forms a *symmetric monoid*. Thus, provided the effective cofibrations and effective Kan fibrations described in Section 3.3.3 do form an AWFS $(\mathbb{C}, \mathbb{F})$, to construct an algebraic monoidal structure for these AWFSs as in Definition 4.27 we need to further equip $\mathbb{C}$ with a Σ-bimodule structure.

However, there is no explicit description of effective trivial cofibrations other than that it is the left lifting class for effective Kan fibrations. Thus, we use Theorem 4.33 and describe an *exponential bimodule* structure on effective Kan fibrations $\mathbb{F}$ w.r.t. Σ instead. Since Σ is symmetric, it actually suffices to describe a left or right exponential module structure. The upshot is that, if we do have an AWFS $(\mathbb{C}, \mathbb{F})$ for effective Kan fibrations, then the two AWFSs $(\Sigma, \mathbb{T})$ and $(\mathbb{C}, \mathbb{F})$ will be an algebraic monoidal structure on $\mathscr{C}$.

Also recall from Section 3.3 that we also have an auxiliary AWFS $(\mathbb{H}, \mathbb{N})$ of HDRs and effective naïve fibrations. In Section 3.3.3 we have seen that the underlying category $\mathscr{H}$ of HDRs embeds into the category $\mathscr{M}$ of mould squares, the latter of which cofibrantly generate the retract closure $\mathbf{F}$ of effective Kan fibrations. Thus, $\mathbb{H}$ should be viewed as a special class of effective trivial cofibrations. From Section 4.4 we know that if $(\Sigma, \mathbb{T})$ and $(\mathbb{C}, \mathbb{F})$ form an algebraic monoidal structure, then there would be a morphism in $\mathbb{L}\mathbf{Str}$

$$(\mathbb{C}, \mathbb{F}^{\mathrm{op}}) \longrightarrow \mathbb{T}.$$

over the exponential in $\mathscr{C}$. However, since we do not have an explicit description of $\mathbb{C}$, we instead construct a morphism

$$(\mathbb{H}, \mathbb{F}^{\mathrm{op}}) \longrightarrow \mathbb{T},$$

which is the main task of Section 5.1.

Section 5.2 proceed to construct a morphism

$$(\Sigma, \mathbb{F}^{\mathrm{op}}) \longrightarrow \mathbb{F}^{\mathrm{op}},$$

again over the exponential in $\mathscr{C}$. Section 5.3 will verify that this action does make $\mathbb{F}$ an exponential module over Σ, thus achieving our desired goal that $(\Sigma, \mathbb{T})$ and $(\mathbb{C}, \mathbb{F})$ has an algebraic monoidal structure, provided $(\mathbb{C}, \mathbb{F})$ forms an AWFS.

As mentioned in the Introduction, the pushout-product axioms are quite useful for homotopy theory. The various actions constructed in this chapter will also be applied in Chapter 6 when we describe path category structures for effective Kan fibrations.

5.1 Action of HDR on Effective Kan Fibration

In this section, we show that there exists a morphism in $\mathbb{L}\mathbf{Str}$

$$(\mathbb{H}, \mathbb{F}^{\mathrm{op}}) \longrightarrow \mathbb{T}^{\mathrm{op}},$$

over the exponential in $\mathscr{C}$. This is the structured version of the fact that pullback-exponential of HDRs against effective Kan fibrations gives us effective trivial fibrations. We first describe how to construct the effective trivial fibration structure on the pullback exponential:

Lemma 5.1. *If $\partial : C \longrightarrow D$ is an HDR and $f : X \longrightarrow Y$ is an effective Kan fibration, then their pullback-exponential*

$$\hat{\exp}(\partial, f) : [D, X] \longrightarrow [C, X] \times_{[C,Y]} [D, Y]$$

can be equipped with an effective trivial fibration structure.

Proof. Consider a lifting diagram where $i : A \longrightarrow B$ is a cofibration,

$$\begin{tikzcd} A \arrow[r, "m"] \arrow[d, "i"'] & {[D,X]} \arrow[d, "\hat{\exp}(\partial,f)"] \\ B \arrow[r, "\langle u,v \rangle"'] & {[C,X] \times_{[C,Y]} [D,Y]} \end{tikzcd}$$

We may write its dual diagram in the style of mould square as follows,

$$\begin{tikzcd}
A \times C \arrow[r, "A\times\partial"] \arrow[d, "i\times C"'] & A \times D \arrow[r, "\widetilde{m}"] \arrow[d, "i\times D"'] & X \arrow[d, "f"] \\
B \times C \arrow[r, "B\times\partial"] \arrow[urr, bend left, "\tilde{u}"] & B \times D \arrow[r, "\tilde{v}"'] \arrow[ur, dashed, "\varphi"] & Y
\end{tikzcd}$$

Now notice that the left square is indeed a mould square, thus a lift φ exists. Then the transpose $\tilde{\varphi}$ of φ provides the lift to the original diagram.

For horizontal compatibility, suppose we have a morphism between cofibrations, i.e. a pullback square, in a lifting problem as follows,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & A' \arrow[r] \arrow[d, "i'"] & {[D,X]} \arrow[d, "\widehat{\exp}(\partial,f)"] \\
B \arrow[r] \arrow[urr, dashed, "\tilde{\varphi}"] & B' \arrow[r] \arrow[ur, dashed, "\tilde{\varphi}'"] & {[C,X] \times_{[C,Y]} [D,Y]}
\end{tikzcd}$$

The two lifts are induced by the following dual diagram,

$$\begin{tikzcd}
A \times C \arrow[rr] \arrow[dr] \arrow[dd] & & A \times D \arrow[dr] \arrow[dd] & & \\
& A' \times C \arrow[rr] \arrow[dd] & & A' \times D \arrow[r] \arrow[dd] & X \arrow[dd, "f"] \\
B \times C \arrow[rr] \arrow[dr] & & B \times D \arrow[dr] \arrow[urr, dashed, "\varphi"] & & \\
& B' \times C \arrow[rr] \arrow[uurrr] & & A' \times D \arrow[r] \arrow[uur, dashed, "\varphi'"] & Y
\end{tikzcd}$$

The compatibility of φ and φ' follows from the perpendicular compatibility of f.

For the vertical compatibility, consider the following situation,

$$\begin{tikzcd}
A \arrow[r, "m"] \arrow[d, "i"'] & {[D,X]} \arrow[dd, "\widehat{\exp}(\partial,f)"] \\
B \arrow[d, "i'"'] \arrow[ur, dashed, "\tilde{\varphi}"] & \\
B' \arrow[r, "\langle u,v \rangle"'] \arrow[uur, dashed, "\tilde{\varphi}'"] & {[C,X] \times_{[C,Y]} [D,Y]}
\end{tikzcd}$$

Dually, the two lifts are induced by

$$
\begin{tikzcd}
A \times C \arrow[r] \arrow[d] & A \times D \arrow[r] \arrow[d] & X \arrow[dd] \\
B \times C \arrow[r] \arrow[d] & B \times D \arrow[d] \arrow[ur, dashed, "\varphi"] & \\
B' \times C \arrow[r] \arrow[urr, bend left=10] & B' \times D \arrow[r] \arrow[uur, dashed, "\varphi'"] & Y
\end{tikzcd}
$$

Thus the vertical compatibility of the two original lifts follows from the vertical compatibility of f. This implies that $\hat{\exp}(\partial, f)$ has a well-defined effective trivial fibration structure. $\square$

Now we verify that this construction produces the correct structural morphism:

Proposition 5.2. *Lemma 5.1 can be promoted to a morphism in* $\mathbb{L}\mathbf{Str}$

$$(\mathbb{H}, \mathbb{F}^{\mathrm{op}}) \longrightarrow \mathbb{T}^{\mathrm{op}}$$

over the exponential in $\mathscr{C}$.

Proof. We need to verify that the construction in Lemma 5.1 respects the horizontal and vertical compositions of $\mathbb{H}$, $\mathbb{F}$ and $\mathbb{T}$. For the horizontal functoriality of HDRs, suppose we have a morphism between HDRs,

$$a : \partial \longrightarrow \partial'.$$

We need to show that the induced square

$$
\begin{tikzcd}
{[D', X]} \arrow[r] \arrow[d, "\hat{\exp}(\partial', f)"'] & {[D, X]} \arrow[d, "\hat{\exp}(\partial, f)"] \\
{[C', X] \times_{[C', Y]} [D', Y]} \arrow[r] & {[C, X] \times_{[C, Y]} [D, Y]}
\end{tikzcd}
$$

consists of a morphism between effective trivial fibrations. To this end, given any cofibration $i : A \longrightarrow B$, consider the following lifting problem

$$
\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & {[D', X]} \arrow[r] \arrow[d, "\hat{\exp}(\partial', f)"] & {[D, X]} \arrow[d, "\hat{\exp}(\partial, f)"] \\
B \arrow[r] \arrow[ur, dashed, "\tilde{\varphi}'"] \arrow[urr, dashed, "\tilde{\varphi}"] & {[C', X] \times_{[C', Y]} [D', Y]} \arrow[r] & {[C, X] \times_{[C, Y]} [D, Y]}
\end{tikzcd}
$$

which is dual to the situation below

$$\begin{tikzcd}
A \times C \arrow[rr] \arrow[dr] \arrow[dd] & & A \times D \arrow[dr] \arrow[dd] & & \\
& A \times C' \arrow[rr] \arrow[dd] & & A \times D' \arrow[r] \arrow[dd] & X \arrow[dd, "f"] \\
B \times C \arrow[rr] \arrow[dr] & & B \times D \arrow[urr, dashed, "\varphi"] \arrow[dr] & & \\
& B \times C' \arrow[rr] \arrow[uurrr] & & B \times D' \arrow[r] \arrow[uur, dashed, "\varphi'"] & Y
\end{tikzcd}$$

Notice that the cube on the left is a morphism of mould squares. This follows from the fact that $i : A \longrightarrow B$ is a cofibration, and the morphism $a : \partial \longrightarrow \partial'$ between the two HDRs is a pullback square. Hence, the compatibility of $\tilde{\varphi}$ and $\tilde{\varphi}'$ follows from the perpendicular compatibility of f.

For the horizontal functoriality of effective Kan fibrations, suppose we have a morphism

$$b : f \longrightarrow g$$

between two effective Kan fibrations. Again, we need to verify that the following square

$$\begin{tikzcd}
{[D, X]} \arrow[r] \arrow[d, "\hat{\exp}(\partial, f)"'] & {[D, W]} \arrow[d, "\hat{\exp}(\partial, g)"] \\
{[C, X] \times_{[C,Y]} [D, Y]} \arrow[r] & {[C, W] \times_{[C,Z]} [D, Z]}
\end{tikzcd}$$

constitutes a morphism between effective trivial fibrations. To this end, again given a cofibration $i : A \longrightarrow B$, consider the following lifts,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & {[D, X]} \arrow[r] \arrow[d, "\hat{\exp}(\partial, f)"] \arrow[dl, dashed, "\tilde{\varphi}"'] & {[D, W]} \arrow[d, "\hat{\exp}(\partial, g)"] \\
B \arrow[r] \arrow[urr, dashed, "\tilde{\psi}"] & {[C, X] \times_{[C,Y]} [D, Y]} \arrow[r] & {[C, W] \times_{[C,Z]} [D, Z]}
\end{tikzcd}$$

The dual diagram gives us

$$\begin{tikzcd}
A \times C \arrow[r] \arrow[d] & A \times D \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "f"] & W \arrow[d, "g"] \\
B \times C \arrow[r] \arrow[urr] & B \times D \arrow[r] \arrow[ur, dashed, "\varphi"] \arrow[urr, dashed, "\psi"'] & Y \arrow[r] & Z
\end{tikzcd}$$

where now the compatibility of φ, ψ follows from the fact that $b : f \longrightarrow g$ is a morphism between effective Kan fibrations. Hence, the original two lifts are also compatible.

Finally, we need to verify vertical compatibility. Suppose now we have vertical composition of HDRs

$$C \xrightarrow{\partial_0} D \xrightarrow{\partial_1} E$$

we need to verify that the $\mathbb{T}$-structure we have constructed on $\hat{\exp}(\partial_1\partial_0, f)$ agrees with the one induced by the following diagram,

$$\begin{tikzcd}
{[E,X]} \arrow[r, "\hat{\exp}(\partial_1,f)"] & {[D,X] \times_{[D,Y]} [E,Y]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[D,X]} \arrow[d, "\hat{\exp}(\partial_0,f)"] \\
& {[C,X] \times_{[C,Y]} [E,Y]} \arrow[r] & {[C,X] \times_{[C,Y]} [D,Y]}
\end{tikzcd}$$

Consider then a lifting problem against a cofibration $i : A \longrightarrow B$ as follows,

$$\begin{tikzcd}
A \arrow[r] \arrow[dd] & {[E,X]} \arrow[d, "\hat{\exp}(\partial_1,f)"] & \\
& {[D,X] \times_{[D,Y]} [E,Y]} \arrow[r] \arrow[d] & {[D,X]} \arrow[d, "\hat{\exp}(\partial_0,f)"] \\
B \arrow[uur, dashed, "\tilde{\varphi}_1"] \arrow[ur, dashed, "\langle \tilde{\varphi}_0, v \rangle"] \arrow[urr, dashed, "\tilde{\varphi}_0"] \arrow[r, "\langle u,v \rangle"'] & {[C,X] \times_{[C,Y]} [E,Y]} \arrow[r] & {[C,X] \times_{[C,Y]} [D,Y]}
\end{tikzcd}$$

This then transposes to the following diagram,

$$\begin{tikzcd}
A \times C \arrow[r] \arrow[d] & A \times D \arrow[r] \arrow[d] & A \times E \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
B \times C \arrow[r] \arrow[urrr, bend left=10, "\tilde{u}"] & B \times D \arrow[r] \arrow[urr, dashed, "\varphi_0"] & B \times E \arrow[r, "v"'] \arrow[ur, dashed, "\varphi_1"] & Y
\end{tikzcd}$$

Now by horizontal compatibility of f, if we first lift against the left mould square to obtain φ_0 and then lift against right mould square to obtain φ_1, this coincides with the lift against the composite mould square. This implies our construction is compatible with the vertical composition of HDRs.

Now for the vertical composition of effective Kan fibrations, suppose we have another effective Kan fibration $g : Y \longrightarrow Z$. Again, we need to contemplate on the lifting problem for the following diagram,

$$\begin{tikzcd}
A \arrow[r] \arrow[dd, "i"'] & {[D,X]} \arrow[d, "\hat{\exp}(\partial,f)"] & \\
& {[C,X] \times_{[C,Y]} [D,Y]} \arrow[r] \arrow[d] & {[D,Y]} \arrow[d, "\hat{\exp}(\partial,g)"] \\
B \arrow[r, "\langle u,v \rangle"'] \arrow[uur, dashed, bend left, "\tilde{\varphi}"] \arrow[ur, dashed, "\langle u,\widetilde{\psi} \rangle"] \arrow[urr, dashed, "\widetilde{\psi}"] & {[C,X] \times_{[C,Z]} [D,Z]} \arrow[r] & {[C,Y] \times_{[C,Z]} [D,Z]}
\end{tikzcd}$$

Take its dual, we get

$$\begin{tikzcd}
A \times C \arrow[r] \arrow[dd] & A \times D \arrow[r] \arrow[dd] & X \arrow[d, "f"] \\
& & Y \arrow[d, "g"] \\
B \times C \arrow[r] \arrow[uurr, "u"] & B \times D \arrow[r, "v"'] \arrow[uur, dashed, "\varphi"] \arrow[ur, dashed, "\psi"] & Z
\end{tikzcd}$$

By the definition of vertical composition of effective Kan fibrations, the lift obtained by first lifting for g and then lifting for f coincides with the single lifting for the composite effective Kan fibration. This implies the vertical compatibility of vertical composition with effective Kan fibrations. □

5.2 Action of Cofibration on Effective Kan Fibration

In this section, we show the structured version of the statement that effective Kan fibrations are closed under pullback-exponential w.r.t. effective cofibrations. In other words, we will explicitly construct a morphism in $\mathbb{L}\mathbf{Str}$

$$(\Sigma, \mathbb{F}^{\mathrm{op}}) \longrightarrow \mathbb{F}^{\mathrm{op}},$$

again over the exponential in $\mathscr{C}$. As a first step, let us observe the following fact:

Lemma 5.3. *Mould squares are closed under pushout-products w.r.t. cofibrations, i.e. if the following is a mould square,*

$$\begin{tikzcd}
C \arrow[r, "\partial_c"] \arrow[d, "j"'] & D \arrow[d, "j'"] \\
C' \arrow[r, "\partial_{c'}"'] & D'
\end{tikzcd}$$

and $j : A \longrightarrow B$ is a cofibration, then the following is also a mould square,

$$\begin{tikzcd}
C \times B +_{C\times A} C' \times A \arrow[r, "{[\partial_c \times B, \partial_{c'} \times A]}"] \arrow[d, "j \hat{\otimes} i"'] & D \times B +_{D\times A} D' \times A \arrow[d, "j' \hat{\otimes} i"] \\
C' \times B \arrow[r, "\partial_{c'} \times B"'] & D' \times B
\end{tikzcd}$$

Proof. We know that cofibrations are closed under pushout-products, thus $j\hat{\otimes}i$ and $j'\hat{\otimes}i$ are also cofibrations. Also, we know from Corollary 3.26 that the category of HDRs has pullbacks, thus the two horizontal morphisms are also HDRs. □

Using this, we can explicitly construct an effective Kan fibration on the pullback-exponential of effective Kan fibrations w.r.t. cofibrations:

Lemma 5.4. *If $i : A \rightarrowtail B$ is a cofibration and $f : X \longrightarrow Y$ is an effective Kan fibration, then their pullback-exponential*

$$\hat{\exp}(i, f) : [B, X] \longrightarrow [A, X] \times_{[A,Y]} [B, Y]$$

is also an effective Kan fibration.

Proof. Consider any lifting problem against a mould square as follows,

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & {[B, X]} \arrow[d, "\hat{\exp}(i,f)"] \\
C' \arrow[r] \arrow[urr] & D' \arrow[r] & {[A, X] \times_{[A,X]} [B, Y]}
\end{tikzcd}$$

By the Leibniz adjunction, this transposes to a lifting problem as follows,

$$\begin{tikzcd}
C \times B +_{C\times A} C' \times A \arrow[r] \arrow[d] & D \times B +_{D\times A} D' \times A \arrow[r] \arrow[d] & X^f \arrow[d] \\
C' \times B \arrow[r] \arrow[urr] & D' \times B \arrow[r] \arrow[ur, dashed, "\varphi"] & Y
\end{tikzcd}$$

The left hand square is again a mould square by Lemma 5.3, hence the effective Kan fibration structure on f induces a lift φ as above, which transposes to a lift $\tilde{\varphi}$ to the original diagram. We need to verify that compatibilities for the constructed lifting operation against mould squares.

For the perpendicular direction, suppose we have a pullback cube of mould squares,

$$\begin{tikzcd}
& E \arrow[rr] \arrow[dl] \arrow[dd] & & F \arrow[dl] \arrow[dd] & \\
C \arrow[rr] \arrow[dd] & & D \arrow[rr] \arrow[dd] & & {[B,X]} \arrow[dd, "\hat{\exp}(i,f)"] \\
& E' \arrow[rr] \arrow[dl] & & F' \arrow[dl] \arrow[ur, dashed, "\widetilde{\psi}"] & \\
C' \arrow[rr] \arrow[uurrrr] & & D' \arrow[rr] \arrow[uurr, dashed, bend right, "\tilde{\varphi}"'] & & {[A,X] \times_{[A,Y]} [B,Y]}
\end{tikzcd}$$

the two lifts are induced by the following transposed diagram,

$$\begin{tikzcd}
& E \times B +_{E \times A} E' \times A \arrow[rr] \arrow[dl] \arrow[dd] & & F \times B +_{F \times A} F' \times A \arrow[dl] \arrow[dd] & \\
C \times B +_{C \times A} C' \times A \arrow[rr] \arrow[dd] & & D \times B +_{D \times A} D' \times A \arrow[rr] \arrow[dd] & & X \arrow[dd, "f"] \\
& E' \times B \arrow[rr] \arrow[dl] & & F' \times B \arrow[dl] \arrow[ur, dashed, "\psi"] & \\
C' \times B \arrow[rr] \arrow[uurrrr] & & D' \times B \arrow[rr] \arrow[uurr, dashed, bend right, "\varphi"'] & & Y
\end{tikzcd}$$

Notice that the left square is a again a pullback cube of mould squares, thus the compatibility of φ and ψ follows from the perpendicular compatibility of f. This implies that the two transposed lifts $\tilde{\varphi}$, $\widetilde{\psi}$ are also compatible.

Now for the horizontal direction, suppose we have a lifting diagram as follows,

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & {[B,X]} \arrow[d, "\hat{\exp}(i,f)"] \\
C' \arrow[r] \arrow[urrr] & D' \arrow[r] \arrow[urr, dashed, "\tilde{\varphi}"] & E' \arrow[r] \arrow[ur, dashed, "\widetilde{\psi}"] & {[A,X] \times_{[A,Y]} [B,Y]}
\end{tikzcd}$$

where the two lifts are induced by the following dual diagram,

$$\begin{tikzcd}
C \times B +_{C \times A} C' \times A \arrow[r] \arrow[d] & D \times B +_{D \times A} D' \times A \arrow[r] \arrow[d] & E \times B +_{E \times A} E' \times A \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
C' \times B \arrow[r] \arrow[urrr] & D' \times B \arrow[r] \arrow[urr, dashed, "\varphi"] & E' \times B \arrow[r] \arrow[ur, dashed, "\psi"] & Y
\end{tikzcd}$$

This way, the horizontal compatibility for $\hat{\exp}(i, f)$ follows from that of f.

Finally, for the vertical direction, suppose we are given a lifting situation as below,

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & D \arrow[r, "m"] \arrow[d, "j'"] & {[B,X]} \arrow[dd, "\hat{\exp}(i,f)"] \\
C' \arrow[r] \arrow[d] & D' \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[d, "k'"'] & \\
C'' \arrow[r] \arrow[uurr] & D'' \arrow[r, "\langle u,v \rangle"'] \arrow[uur, dashed, "\tilde{\varphi}'"] & {[B,X] \times_{[A,X]} [A,Y]}
\end{tikzcd}$$

By construction, the two step lifting comes from the following diagram,

$$\begin{tikzcd}
& C \times B +_{C \times A} C'' \times A \arrow[rr] \arrow[dd] & & D \times B +_{D \times A} D'' \times A \arrow[r, "{[\widetilde{m},\tilde{v}]}"] \arrow[dd] & X \arrow[dddd, "f"] \\
C \times B +_{C \times A} C' \times A \arrow[ur] \arrow[rr] \arrow[dd] & & D \times B +_{D \times A} D' \times A \arrow[ur] \arrow[dd] & & \\
& C' \times B +_{C' \times A} C'' \times A \arrow[rr] \arrow[dd] & & D' \times B +_{C' \times A} D'' \times A \arrow[dd] \arrow[uur, dashed, "{[\varphi,\tilde{v}]}"] & \\
C' \times B \arrow[ur] \arrow[rr] & & D' \times B \arrow[ur] \arrow[uuurr, dashed, "\varphi"] & & \\
& C'' \times B \arrow[rr] \arrow[uuuurrr] & & D'' \times B \arrow[r] \arrow[uuuur, dashed, "\varphi'"] & Y
\end{tikzcd}$$

Now notice that the mould square of the pushout-product of the vertically composed mould square w.r.t. i is the vertical composition of the two mould squares at the back in this diagram. Furthermore, the cube is both a pushout and a pullback for mould squares, thus the lift against the back of the cube is completely determined by the lift of the front of the cube. This way, the compatibility now follows from the perpendicular and vertical compatibility of f. □

Finally, we show that the above construction exhibits a morphism in $\mathbb{L}$**Str**:

Proposition 5.5. *Lemma 5.4 can be promoted to a morphism*

$$(\Sigma, \mathbb{F}^{\mathrm{op}}) \longrightarrow \mathbb{F}^{\mathrm{op}}$$

in $\mathbb{L}$**Str** *over the exponential in* $\mathscr{C}$.

Proof. We first need to verify the functoriality of the horizontal direction. Suppose we have a morphism between cofibrations, i.e. a pullback square

$$a \,:\, i \longrightarrow i',$$

we need to verify the following will be a morphism of effective Kan fibrations,

$$\begin{tikzcd}
{[B',X]} \arrow[r] \arrow[d, "\hat{\exp}(i',f)"'] & {[B,X]} \arrow[d, "\hat{\exp}(i,f)"] \\
{[A',X] \times_{[A',Y]} [B',Y]} \arrow[r] & {[A,X] \times_{[A,Y]} [B,Y]}
\end{tikzcd}$$

To this end, suppose we have a lifting problem against a mould square as follows,

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & {[B',X]} \arrow[r] \arrow[d, "\hat{\exp}(i',f)"'] & {[B,X]} \arrow[d, "\hat{\exp}(i,f)"] \\
C' \arrow[r] \arrow[urr, bend left] & D' \arrow[r] \arrow[ur, dashed, "\tilde{\varphi}'"] \arrow[urr, dashed, "\tilde{\varphi}"] & {[A',X] \times_{[A',Y]} [B',Y]} \arrow[r] & {[A,X] \times_{[A,Y]} [B,Y]}
\end{tikzcd}$$

By construction, the two lifts are induced by the following dual diagram,

$$\begin{tikzcd}
& C \times B +_{C\times A} C' \times A \arrow[rr] \arrow[dl] \arrow[dd] & & D \times B +_{D\times A} D' \times A \arrow[dl] \arrow[dd] & \\
C \times B' +_{C\times A'} C' \times A' \arrow[rr] \arrow[dd] & & D \times B' +_{D\times A'} D' \times A' \arrow[rr] \arrow[dd] & & X \arrow[dd, "f"] \\
& C' \times B \arrow[rr] \arrow[dl] & & D' \times B \arrow[dl] \arrow[ur, dashed, "\varphi"] & \\
C' \times B' \arrow[rr] \arrow[uurrrr] & & D' \times B' \arrow[rr] \arrow[uurr, dashed, bend right, "\varphi'"'] & & Y
\end{tikzcd}$$

Since $a \,:\, i \longrightarrow i'$ is a pullback, the left cube will be a pullback of mould squares. Thus the compatibility of φ and φ' follows from the perpenducilar compatibility of f.

Suppose now we have a morphism between effective Kan fibrations $b \,:\, f \longrightarrow g$. Similarly, we need to show that the following diagram will also be a morphism of effective Kan fibrations,

$$\begin{tikzcd}
{[B,X]} \arrow[r] \arrow[d, "\hat{\exp}(i,f)"'] & {[B,W]} \arrow[d, "\hat{\exp}(i,g)"] \\
{[A,X] \times_{[A,Y]} [B,Y]} \arrow[r] & {[A,W] \times_{[A,Z]} [B,Z]}
\end{tikzcd}$$

To this end, suppose we are given a lifting problem against a mould square,

$$
\begin{tikzcd}
C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & {[B,X]} \arrow[r] \arrow[d, "\mathrm{e\hat{x}p}(i,f)"] & {[B,W]} \arrow[d, "\mathrm{e\hat{x}p}(i,g)"] \\
C' \arrow[r] \arrow[urr, bend left] & D' \arrow[r] \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[urr, dashed, "\tilde{\psi}"] & {[A,X] \times_{[A,Y]} [B,Y]} \arrow[r] & {[A,W] \times_{[A,Z]} [B,Z]}
\end{tikzcd}
$$

The two lifts are induced by the following diagram,

$$
\begin{tikzcd}
C \times B +_{C \times A} C' \times A \arrow[r] \arrow[d] & D \times B +_{D \times A} D' \times A \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "f"] & W \arrow[d, "g"] \\
C' \times B \arrow[r] \arrow[urr] & D' \times B \arrow[r] \arrow[ur, dashed, "\varphi"] \arrow[urr, dashed, "\psi"] & Y \arrow[r] & Z
\end{tikzcd}
$$

now the compatibility follows from the fact that the square $f \longrightarrow g$ is a morphism between effective Kan fibrations.

To show the construction of M preserves the vertical composition of cofibrations, suppose we are given composible cofibrations $i : A \longrightarrow B$ and $j : B \longrightarrow C$. The vertical composition of the two pullback exponentials is induced by the following diagram,

$$
\begin{tikzcd}
{[C,X]} \arrow[r, "\mathrm{e\hat{x}p}(j,f)"] & {[B,X] \times_{[B,Y]} [C,Y]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[B,X]} \arrow[d, "\mathrm{e\hat{x}p}(i,f)"] \\
 & {[A,X] \times_{[A,Y]} [C,Y]} \arrow[r] & {[A,X] \times_{[A,Y]} [B,Y]}
\end{tikzcd}
$$

Now given a mould square, by the vertical composition of effective Kan fibrations, the lift is determined by two steps as follows,

$$
\begin{tikzcd}
D \arrow[r] \arrow[dd] & E \arrow[r, "w"] \arrow[dd] & {[C,X]} \arrow[d] & \\
 & & {[B,X] \times_{[B,Y]} [C,Y]} \arrow[r] \arrow[d] & {[B,X]} \arrow[d] \\
D' \arrow[r] \arrow[uurr, "d"] & E' \arrow[r, "\langle u,v \rangle"'] \arrow[uur, dashed, "m"] \arrow[ur, dashed, "\langle l,v \rangle"] \arrow[urr, dashed, "l"] & {[A,X] \times_{[A,Y]} [C,Y]} \arrow[r] & {[A,X] \times_{[A,Y]} [B,Y]}
\end{tikzcd}
$$

The first lift l is determined by the dual diagram

$$
\begin{tikzcd}
D \times B +_{D \times A} D' \times A \arrow[r] \arrow[d] & E \times B +_{E \times A} E' \times A \arrow[r, "{[\widetilde{w}|_B, \tilde{u}]}"] \arrow[d] & X \arrow[d, "f"] \\
D' \times B \arrow[r] \arrow[urr, "\tilde{d}|_B"] & E' \times B \arrow[r, "\tilde{v}|_B"'] \arrow[ur, dashed, "\tilde{l}"] & Y
\end{tikzcd}
$$

and the final lift m is determined by

$$
\begin{tikzcd}
D \times C \cup D' \times B \arrow[r] \arrow[d] & E \times C \cup E' \times B \arrow[r, "{[\widetilde{w}, \tilde{l}]}"] \arrow[d] & X \arrow[d, "f"] \\
D' \times C \arrow[r] \arrow[urr, "\tilde{d}" near start] & E' \times C \arrow[r, "\tilde{v}"'] \arrow[ur, dashed, "\widetilde{m}"] & Y
\end{tikzcd}
$$

On the other hand, if we directly think about the lift against the composite $\mathrm{e\hat{x}p}(ji, f)$, the dual diagram will look as follows,

$$
\begin{tikzcd}
& D \times B +_{D \times A} D' \times A \arrow[rr] \arrow[dl] \arrow[dd] & & E \times B +_{E \times A} E' \times A \arrow[dl] \arrow[dd] & \\
D \times C +_{D \times A} D' \times A \arrow[rr] \arrow[dd] & & E \times C +_{E \times A} E' \times A \arrow[rr, "{[\widetilde{w}, \tilde{u}]}"] \arrow[dd] & & X \arrow[ddd, "f"] \\
& D' \times B \arrow[rr] \arrow[dl] & & E' \times B \arrow[dl] \arrow[ur, dashed, "\tilde{l}"] & \\
D \times C +_{D \times B} D' \times B \arrow[rr] \arrow[d] & & E \times C +_{E \times B} E' \times B \arrow[d] \arrow[uurr, dashed, "{[\widetilde{w}, \tilde{l}]}"] & & \\
D' \times C \arrow[rr] \arrow[uuurrrr, "\tilde{d}" near start] & & E' \times C \arrow[rr, "\tilde{v}"'] \arrow[uuurr, dashed, "\widetilde{m}"] & & Y
\end{tikzcd}
$$

Notice that the front composite square is the dual lifting diagram we would like to think about. We have decomposed it into two composite mould square, where the top cube is both a pushout and pullback of mould squares. This way, by the vertical and perpendicular compatibility of f, it follows that the lifting of the composite front square indeed coincides with $\widetilde{m}$, thus this implies M is compatible with vertical composition of cofibrations.

Finally, for the vertical composition of effective Kan fibrations, suppose we are given two composible efective Kan fibrations $g : Y \longrightarrow Z$ and $f : X \longrightarrow Y$. The vertical composite of $\mathrm{e\hat{x}p}(i, g)$ with $\mathrm{e\hat{x}p}(i, f)$ is again given by the following composite of the $\mathbb{F}$-structure as follows,

$$
\begin{tikzcd}
{[B, X]} \arrow[r, "\mathrm{e\hat{x}p}(i, f)"] & {[A, X] \times_{[A, Y]} [B, Y]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & {[B, Y]} \arrow[d, "\mathrm{e\hat{x}p}(i, g)"] \\
& {[A, X] \times_{[A, Z]} [B, Z]} \arrow[r] & {[A, Y] \times_{[A, Z]} [B, Z]}
\end{tikzcd}
$$

Now given any lifting problem against a mould square, the lift of the above composite is given as follows,

$$\begin{tikzcd}
C \arrow[r] \arrow[dd] & D \arrow[r, "w"] \arrow[dd] & {[B,X]} \arrow[d] & \\
& & {[A,X] \times_{[A,Y]} [B,Y]} \arrow[r] \arrow[d] & {[B,Y]} \arrow[d] \\
C' \arrow[r] \arrow[uurr, "d"] & D' \arrow[r, "\langle u,v \rangle"'] \arrow[uur, dashed, "m"] \arrow[ur, dashed, "\langle l,v \rangle"] \arrow[urr, dashed, "l"] & {[A,X] \times_{[A,Z]} [B,Z]} \arrow[r] & {[A,Y] \times_{[A,Z]} [B,Z]}
\end{tikzcd}$$

If we transpose this using the Leibniz adjunction, we realise that it amounts to the following diagram,

$$\begin{tikzcd}
C \times B +_{C \times A} C' \times A \arrow[r] \arrow[dd] & D \times B +_{D \times A} D' \times A \arrow[r, "{[\widetilde{w},\tilde{v}]}"] \arrow[dd] & X \arrow[d, "g"] \\
& & Y \arrow[d, "f"] \\
C' \times B \arrow[r] \arrow[uurr, "\tilde{d}"] & D' \times B \arrow[r, "\tilde{u}"'] \arrow[uur, dashed, "\widetilde{m}"] \arrow[ur, dashed, "\tilde{l}"] & Z
\end{tikzcd}$$

By how vertical composition of effective Kan fibrations is defined, the exactly corresponds to the transpose of the effective Kan fibration on $\hat{\exp}(i, gf)$. $\square$

5.3 Effective Kan Fibration as an Exponential Module

Finally, we need to verify that our construction of the pullback-exponential of effective Kan fibrations w.r.t. cofibrations is also compatible with the monoid structure on cofibrations we have given in Example 4.24.

Proposition 5.6. *The morphism*

$$(\Sigma, \mathbb{F}^{\mathrm{op}}) \longrightarrow \mathbb{F}^{\mathrm{op}}$$

exhibits $\mathbb{F}$ *as an exponential module for the monoid* Σ *in* $\mathbb{L}\mathbf{Str}$.

Proof. Suppose we have two cofibrations $i : A \rightarrowtail B$ and $j : C \rightarrowtail D$ and an effective Kan fibration $f : Y \longrightarrow X$. If we first tensor together i, j producing the cofibration

$$i \hat{\otimes} j : A \times D \cup B \times C \rightarrowtail B \times D,$$

and then produces the pullback-exponential effective Kan fibration $\hat{\exp}(i \hat{\otimes} j, f)$, its lift against a mould square

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & F \arrow[r] \arrow[d] & {[B \times D, Y]} \arrow[d] \\
E' \arrow[r] \arrow[urr] & F' \arrow[r] & {[A \times D \cup B \times C, Y] \times_{[A \times D \cup B \times C, X]} [B \times D, X]}
\end{tikzcd}$$

is determined by the transpose indicated as below,

$$\begin{tikzcd}
E \times B \times D \cup E' \times A \times D \cup E' \times B \times C \arrow[r] \arrow[d] & F \times B \times D \cup F' \times A \times D \cup F' \times B \times C \arrow[r] \arrow[d] & Y \arrow[d] \\
E' \times B \times D \arrow[r] \arrow[urr] & F' \times B \times D \arrow[r] \arrow[ur, dashed] & X
\end{tikzcd}$$

On the other hand, if we first produce the effective Kan fibration $\hat{\exp}(j, f)$, and then apply the construction again to obtain the effective Kan fibration $\hat{\exp}(i, \hat{\exp}(j, f))$, according to the proof of Lemma 5.4, the lift of $\hat{\exp}(i, \hat{\exp}(j, f))$ against the same mould square first transports to the diagram below,

$$\begin{tikzcd}
E \times B \cup E' \times A \arrow[r] \arrow[d] & F \times B \cup F' \times A \arrow[r] \arrow[d] & {[D, Y]} \arrow[d] \\
E' \times B \arrow[r] & F' \times B \arrow[r] & {[C, Y] \times_{[C,X]} [D, X]}
\end{tikzcd}$$

and then transports again, which amounts to the same diagram as above. This way, the two way of constructing the effective Kan fibration structure coincide along the isomorphism $\hat{\exp}(i \hat{\otimes} j, f) \cong \hat{\exp}(i, \hat{\exp}(j, f))$, which exhibits $\mathbb{F}$ as a monoidal module over the cofibration left structure Σ. $\square$

Theorem 5.7. *If* $(\mathbb{C}, \mathbb{F})$ *forms an AWFS on* $\mathscr{C}$*, then the two AWFSs* $(\Sigma, \mathbb{T})$ *and* $(\mathbb{C}, \mathbb{F})$ *is an algebraic monoidal structure on* $\mathscr{C}$*.*

Proof. Combining Example 4.24, Proposition 5.6, and Theorem 4.33. The fact that the morphism $(\mathbb{C}, \mathbb{F}) \longrightarrow (\Sigma, \mathbb{T})$ is a bimodule morphism is again trivial, due to the fact that Σ is a propositional left structure. $\square$

Chapter 6

Path Category Structures for Effective Kan Fibrations

As mentioned in the Introduction, our ultimate goal is to construct an algebraic monoidal model category using the notion of effective Kan fibrations, which can be applied to simplicial sets. In the previous chapter, we have established in Theorem 5.7 that, provided the pair of left and right structures $(\mathbb{C}, \mathbb{F})$ form an AWFS, the two AWFSs $(\Sigma, \mathbb{T})$ and $(\mathbb{C}, \mathbb{F})$ has an algebraic monoidal structure on $\mathscr{C}$.

Given Definition 4.27, the other main property we would want is that $(\Sigma, \mathbb{T})$ and $(\mathbb{C}, \mathbb{F})$ will form an algebraic model structure. Recall from Definition 4.26 that they form an algebraic model structure iff there is a morphism $(\mathbb{C}, \mathbb{F}) \longrightarrow (\Sigma, \mathbb{T})$ of AWFSs and the underlying WFSs of them form a model structure. Given Proposition 3.23, to give a morphism between the two AWFSs, it suffices to give a morphism $\mathbb{T} \longrightarrow \mathbb{F}$, which we do have by the discussion in Section 3.3.3. Thus, the two results missing are:

- $(\mathbb{C}, \mathbb{F})$ form an AWFS;
- The two underlying WFSs $(\Sigma, \mathbf{T})$ and $(\mathbf{C}, \mathbf{F})$ form a model structure.

As commented at the end of Section 3.3.3, the two goals seem to be quite independent from each other, and the remaining part of this document mainly concerns the second problem. One consequence is that, from now on our focus will be the *retract closures* of the various left and right structures we have introduced so far on $\mathscr{C}$. Let us comment from the very beginning that, though now we work with these retract closures, the results we have obtained in Chapter 5 are still applicable here by Proposition 4.15. For instance, the morphism $(\mathbb{H}, \mathbb{F}^{\mathrm{op}}) \longrightarrow \mathbb{T}^{\mathrm{op}}$ in $\mathbb{L}\mathbf{Str}$ we have constructed in Proposition 5.2 also implies that the pullback-exponential of an Hdr against a Kan fibration gives us a trivial fibration.[1] In

[1] Recall from Section 3.3.2 that we call retract closures of HDRs as Hdrs.

the remaining part of this document, for simplicity we will no longer explicitly refer to Proposition 4.15, and trust the readers to implicitly understand this point.

In this chapter, we will establish an intermediate step towards the construction of a full model structure. We will show that, given *any* object A in $\mathscr{C}$, the full subcategory of *Kan fibrations over* A, which will be denoted as $\mathscr{C}_{\mathbf{F}}(A)$, has a *path category structure*. Furthermore, these path category structures will be *stable* under reindexing functors induced by any morphism $f : B \longrightarrow A$ in $\mathscr{C}$. One nice thing about this intermediate step is that, unlike a full model structure, the result here does *not* depend on the existence of an AWFS $(\mathbb{C}, \mathbb{F})$ or even a WFS $(\mathbf{C}, \mathbf{F})$.

As mentioned in the Introduction, the notion of a path category introduced in [66] is a slight strengthening of the notion of a category of fibrant objects à la Brown [13]. It is a framework that can formalise many homotopy theoretic notions and results and has a close connection with the syntax of type theory. More precisely, path categories can serve as models for type theories with propositional equality types, and the syntactic category of any type theory with propositional equality type has a canonical path category structure [59]. It is also observed in [60] that we can formulate the notion of *homotopy n-types* and *univalent fibrations* in the framework of path categories, thus making it a nice intermediate environment to investigate the relationship between syntax of type theory and semantics of model structures.

In particular, for any model category where all objects are cofibrant, it will indeed be the case that the full subcategory of fibrations over an arbitrary object has a path category structure, which is stable under reindexing. Thus, the result in this chapter is a necessary consequence of a full model structure for effective Kan fibrations, and in fact helps us achieving this (cf. Chapter 7).

Concretely, a path category is a category equipped with two class of maps called *fibrations* and *weak equivalences*, which satisfy certain axioms. We will review the notion of a path category in Section 6.3. For the category $\mathscr{C}_{\mathbf{F}}(A)$, fibrations will of course be *Kan fibrations*. We will realise weak equivalences as *homotopy equivalences* defined w.r.t. a cylinder object. This will be the topic of Section 6.1.

Section 6.2 establishes how Kan fibrations and homotopy equivalences interact in $\mathscr{C}_{\mathbf{F}}(A)$. In particular, we will show that a map in $\mathscr{C}_{\mathbf{F}}(A)$ is a trivial fibration, i.e. it belongs to $\mathbf{T}$, iff it is a Kan fibration and a relative homotopy equivalence. Using this, in Section 6.3 we put a path category structure on $\mathscr{C}_{\mathbf{F}}(A)$, and show it is stable under reindexing.

6.1 Cylinder Object and Homotopy Equivalence

As mentioned earlier, the weak equivalences in $\mathscr{C}_{\mathbf{F}}(A)$ will be realised as *homotopy equivalences*. To obtain a good notion of homotopy, we consider *effective cylinders*:

Definition 6.1. An *effective cylinder* in $\mathscr{C}$ is an *internal semi-lattice* $(\mathbf{I}, \vee, \partial_0, \partial_1)$, satisfying the following additional properties:

- Each end-point inclusion is an *HDR* $\partial_i : 1 \longrightarrow \mathbf{I}$ for $i = 0, 1$;
- The coproduct of two end-points is a *cofibration* $[\partial_0, \partial_1] : 1 + 1 \longrightarrow \mathbf{I}$. ◊

From now on, we will assume our category $\mathscr{C}$ is equipped with a chosen effective cylinder $\mathbf{I}$ in the sense of Definition 6.1. The disjunction operator $\vee$ on $\mathbf{I}$ will sometimes also be referred to as a *connection* (cf. [8]).

The existence of $\mathbf{I}$ also naturally induces an effective cylinder in each slice category $\mathscr{C}/A$ for any $A \in \mathscr{C}$, by taking the image of $\mathbf{I}$ under the pullback functor A^*. We use $\mathbf{I}_A$ to denote this corresponding cylinder in $\mathscr{C}/A$, whose underlying object is simply the projection $\mathbf{I} \times A \longrightarrow A$.

Definition 6.2. Given two maps $f, g : X \longrightarrow Y$ over an arbitrary object A in $\mathscr{C}$, a *homotopy H relative to A* between them is a map in $\mathscr{C}/A$

$$H : \mathbf{I}_A \times_A X \longrightarrow Y,$$

which is equivalently a map $H : \mathbf{I} \times X \longrightarrow Y$ over A, such that

$$H\partial_0 = f, \quad H\partial_1 = g.$$

In this case, we denote $H : f \sim_A g$. We also say that f, g are *homotopic relative to A*, denoted as $f \sim_A g$, if there exists such a homotopy between them. ◊

One important fact about homotopy is that it will be a *congruence* on maps between fibrant objects:

Lemma 6.3. *The relative homotopy relation $\sim_A$ is a* congruence *on $\mathscr{C}_{\mathbf{F}}(A)$.*

Proof. Suppose $X \longrightarrow A$ and $Y \longrightarrow A$ are Kan fibrations. We first show that the relation $\sim_A$ will be an equivalence relation on the hom set $\mathscr{C}/A(X, Y)$. Reflexivity is trivial, since for any $f : X \longrightarrow Y$ we have have a homotopy $R_f : f \sim f$ defined by

$$R_f = f \circ \pi_X : \mathbf{I} \times X \longrightarrow Y.$$

For symmetricity, suppose we have a homotopy $H : f \sim_A g$. By assumption of the cylinder object in Definition 6.1, $m = [\partial_0, \partial_1] : Y + Y \longrightarrow \mathbf{I} \times Y$ will be a cofibration, because cofibrations are closed under pullbacks. We also know from assumption that

$\partial_0 : 1 \longrightarrow \mathbf{I}$ is an HDR. Thus from Lemma 3.30, the following pushout-product will be the cofibre arrow of a mould square

$$m \hat{\otimes} \partial_0 : \sqsubset \times Y = (\mathbf{I} \times \{0\} \cup \{0\} \times \mathbf{I} \cup \{1\} \times \mathbf{I}) \times Y \longrightarrow \mathbf{I} \times \mathbf{I} \times Y.$$

This way, the following diagram has a diagonal lift,

$$\begin{tikzcd} \sqsubset \times Y \arrow[r, "R_f \cup H \cup R_f"] \arrow[d, "m\hat{\otimes}\partial_0"'] & X \arrow[d] \\ \mathbf{I} \times \mathbf{I} \times Y \arrow[r] \arrow[ur, dashed, "K"] & A \end{tikzcd}$$

We then look at the homotopy $L = K(\mathbf{I} \times \partial_1)$. From the diagram above, we have that

$$L\partial_0 = H\partial_1 = g, \quad L\partial_1 = R_f \partial_1 = f.$$

Hence, $L : g \sim_A f$, thus the homotopy relation is symmetric.

Similarly for transitivity, suppose we have $H : f \sim_A g$ and $K : g \sim_A h$ with $f, g, h : Y \longrightarrow X$. Consider the lifting

$$\begin{tikzcd} \sqsubset \times Y \arrow[r, "H \cup R_f \cup K"] \arrow[d, "m\hat{\otimes}\partial_0"'] & X \arrow[d] \\ \mathbf{I} \times \mathbf{I} \times Y \arrow[r] \arrow[ur, dashed, "L"] & A \end{tikzcd}$$

This way, defining $M = L(\mathbf{I} \times \partial_1)$, again we have $M : f \sim_A h$. Hence, $\sim_A$ is an equivalence relation. Homotopy relation being closed under compositions is standard. □

Based on the notion of a homotopy, we can define the notion of homotopy equivalence, which as mentioned will be the class of weak equivalences in the path category structure we are going to construct:

Definition 6.4. A map $f : X \longrightarrow Y$ over A is a *homotopy equivalence* if there is $g : Y \longrightarrow X$ over A which is inverse to f up to homotopy $H : gf \sim_A 1_X$ and $K : fg \sim_A 1_Y$. It is a *strong* homotopy equivalence if the two homotopies can be chosen coherently as follows,

$$\begin{tikzcd} \mathbf{I} \times X \arrow[r, "H"] \arrow[d, "\mathbf{I}\times f"'] & X \arrow[d, "f"] \\ \mathbf{I} \times Y \arrow[r, "K"'] & Y \end{tikzcd}$$

For such f, we way it is a *strong deformation retract* if H can be chosen as π_X; similarly, f is a *strong codeformation retract* if K can be chosen to be π_Y. ◊

Remark 6.5. Notice that the notion of a strong codeformation retract is *absolute*, in the sense that if $f : X \longrightarrow Y$ is a strong codeformation retract in $\mathscr{C}$, then when viewed as a morphism in any slice category $\mathscr{C}/A$, it will still be a strong codeformation retract. This holds because if f lives over A, then the homotopy H must also live over A. ◊

One immediate consequence of homotopy relation being a congruence is that homotopy equivalences are closed under composition and satisfy 2-out-of-6:

Corollary 6.6. *Homotopy equivalences in $\mathscr{C}_{\mathbf{F}}(A)$ are closed under compositions.*

Proof. Suppose f, g are composible homotopy equivalences in $\mathscr{C}_{\mathbf{F}}(A)$ with homotopy inverses u, v, respectively. Then the homotopy inverse of gf can be chosen as uv, and by Lemma 6.3 that homotopy equivalence is a congruence, we have

$$uvgf \sim_A uf \sim_A 1, \quad gfuv \sim_A gv \sim_A 1.$$

This implies that uv is the homotopy inverse of gf, thus homotopy equivalences are closd under compositions. □

Corollary 6.7. *Homotopy equivalences in $\mathscr{C}_{\mathbf{F}}(A)$ are closed under homotopies.*

Proof. Suppose f is a homotopy equivalence and $f \sim_A g$. Suppose u is the homotopy inverse of f, and again by Lemma 6.3, we have

$$gu \sim_A fu \sim_A 1, \quad ug \sim_A uf \sim_A 1.$$

Hence, u is also a homotopy inverse of g. □

Corollary 6.8. *Homotopy equivalences in $\mathscr{C}_{\mathbf{F}}(A)$ satisfy 2-out-of-6.*

Proof. Suppose we have f, g, h in $\mathscr{C}_{\mathbf{F}}(A)$, where both gf and hg are homotopy equivalences. Explicitly, we have u, v that

$$ugf \sim_A 1, \quad gfu \sim_A 1, \quad vhg \sim_A 1, \quad hgv \sim_A 1.$$

We first show that hgf is a homotopy equivalence with inverse ugv:

$$ugvhgf \sim_A ugf \sim_A 1, \quad hgfugv \sim_A hgv \sim_A 1.$$

Secondly, we know that

$$f \sim vhgf,$$

and both hgf and v are homotopy equivalences. By Corollary 6.6, so is $vhgf$, and by Corollary 6.7, f will also be a homotopy equivalence. Similarly we can show h is also a homotopy equivalence. □

6.2 Trivial Fibrations and Homotopy Equivalences

Our main goal in this section is to show that, in $\mathscr{C}_{\mathbf{F}}(A)$, a map is a trivial fibration $\mathbf{T}$ iff it is a Kan fibration and a relative homotopy equivalence. We start with the left to right direction. Firstly, we observe that trivial fibrations have sections:

Lemma 6.9. *If $f : X \to Y$ is a trivial fibration, then it has a section.*

Proof. Recall from Section 3.3.1 that any object Y in $\mathscr{C}$ will be cofibrant, in the sense that the map $0 \to Y$ is a cofibration. Thus, f being a trivial fibration implies that the following lifting problem has a solution (cf. Corollary 3.20)

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, equal] \arrow[ur, dashed, "s"] & Y \end{tikzcd}$$

□

Lemma 6.10. *If $f : X \to Y$ is a trivial fibration, then it is a strong codeformation retract.*

Proof. From Lemma 6.9 we already know that f has a section s. Now consider the following lifting problem,

$$\begin{tikzcd} X + X \arrow[r, "{[sf,1]}"] \arrow[d, "{[\partial_0,\partial_1]}"'] & X \arrow[d, "f"] \\ \mathbf{I} \times X \arrow[r, "f\pi_X"'] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

By our assumption for the effective cylinder object, the left vertical map is a cofibration, thus the lift H exists. It follows that $H : sf \sim 1_Y$. We observe that $f\pi_Y = \pi_X(\mathbf{I} \times f)$, which implies that the following diagram commute,

$$\begin{tikzcd} \mathbf{I} \times Y \arrow[r, "H"] \arrow[d, "\mathbf{I}\times f"'] & Y \arrow[d, "f"] \\ \mathbf{I} \times X \arrow[r, "\pi_X"'] & X \end{tikzcd}$$

□

By Remark 6.5, Lemma 6.10 implies that any trivial fibration in $\mathscr{C}_{\mathbf{F}}(A)$ will be a relative homotopy equivalence, because the notion of strong codeformation retract is absolute.

On the other hand, we also want to show the converse for Kan fibrations. In other words, we intend to show that if a Kan fibration in $\mathscr{C}_{\mathbf{F}}(A)$ is also a relative homotopy equivalence, then it is also a trivial fibration. To this end, the following characterisation of strong homotopy equivalences will be useful:

Lemma 6.11. *For any map $f : X \longrightarrow Y$, let $\sigma_f : \hat{\exp}(\partial_0, f) \longrightarrow f$ be the following square*

$$\begin{tikzcd} {[\mathbf{I}, X]} \arrow[r, "\mathrm{ev}_1"] \arrow[d, "\hat{\exp}(\partial_0,f)"'] & X \arrow[d, "f"] \\ X \times_Y [\mathbf{I}, Y] \arrow[r, "\mathrm{ev}_1 \circ \pi_{[\mathbf{I},Y]}"'] & Y \end{tikzcd}$$

where $\mathrm{ev}_1 = [\partial_1, X]$ *is the exponential of* $\partial_1 : 1 \longrightarrow \mathbf{I}$ *against* X*, and similarly for* Y*. Then* f *is a strong homotopy equivalence iff* σ_f *has a section.*

Proof. By definition, a section of σ_f is a diagram as follows,

$$\begin{tikzcd} X \arrow[r, "K"] \arrow[d, "f"'] & {[\mathbf{I}, X]} \arrow[r, "\mathrm{ev}_1"] \arrow[d, "\hat{\exp}(\partial_0,f)"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\langle s,H \rangle"'] & X \times_Y [\mathbf{I}, Y] \arrow[r, "\mathrm{ev}_1 \circ \pi_{[\mathbf{I},Y]}"'] & Y \end{tikzcd}$$

Let $\widetilde{K}, \widetilde{H}$ be the transpose of K, H, respectively. After inspection, the data of the above retract is equivalent to a map $s : Y \longrightarrow X$ and a pair of homotopies

$$\widetilde{K} : sf \sim 1, \quad \widetilde{H} : fs \sim 1,$$

such that we have

$$fK = H(\mathbf{I} \times f).$$

This exactly says that f is a strong homotopy equivalence with homotopy inverse s. □

Using this characterisation, we have:

Proposition 6.12. *If a Kan fibration is also a strong homotopy equivalence, then it is a trivial fibration.*

Proof. Suppose the Kan fibration f is a strong homotopy equivalence. By Lemma 6.11 it is a retract of $\hat{\exp}(\partial_0, f)$. Since ∂_0 is an HDR and f is a Kan fibration, by Proposition 5.2, $\hat{\exp}(\partial_0, f)$ will be a trivial fibration. Now f being a retract of $\hat{\exp}(\partial_0, f)$ also implies that f will be a trivial fibration as well. □

Given this, our remaining goal is to show that for any Kan fibration in $\mathscr{C}_{\mathbf{F}}(A)$, if it is a relative homotopy equivalence, then it is also a strong homotopy equivalence. We start with the following observation:

Lemma 6.13. *Let $f : Y \longrightarrow X$ be a Kan fibration over A with a section $g : X \longrightarrow Y$ up to homotopy $K : fg \sim_A 1_X$. Then we can find an actual section s of f such that $g \sim_A s$.*

Proof. Since f is a Kan fibration and $\partial_0 : X \longrightarrow \mathbf{I} \times X$ is an HDR, the following diagram has a diagonal lift

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "\partial_0"'] & Y \arrow[d, "f"] \\ \mathbf{I} \times X \arrow[r, "K"'] \arrow[ur, dashed, "L"] & X \end{tikzcd}$$

We then may define s to be $L\partial_1$. The commutativity above implies that s will be a section of f. Furthermore, since K is a homotopy over A, the above diagram also lies over A, which implies $L : g \sim_A s$. □

Now if we start with a section s of a Kan fibration f such that s is the relative homotopical inverse of f, then f will actually be a strong codeformation retract:

Lemma 6.14. *If $f : Y \longrightarrow X$ over A is a Kan fibration with a section s and a homotopy $H : sf \sim_A 1_Y$, then f is a strong codeformation retract.*

Proof. We need to modify the homotopy H so that it commutes with the projection map. Consider the following diagram over A,

$$\begin{tikzcd} \sqsubset \times Y \arrow[r, "H \cup sfH \cup \pi_Y"] \arrow[d, "m \hat{\otimes} \partial_0"'] & Y \arrow[d, "f"] \\ \mathbf{I} \times \mathbf{I} \times Y \arrow[r, "fH(\vee \times 1_Y)"'] \arrow[ur, dashed, "K"] & X \end{tikzcd}$$

Here $\vee : \mathbf{I} \times \mathbf{I} \longrightarrow \mathbf{I}$ is the connection on the cylinder $\mathbf{I}$ we have assumed to exists in Definition 6.1. Notice that since $fs = 1$, the outer square commutes because we have $fsfH = fH$. Thus, the diagonal lift K exists, and we may look at the homotopy $M = K(\mathbf{I} \times \partial_1) : sf \sim_A 1_Y$. The following diagram commutes,

$$\begin{tikzcd} \mathbf{I} \times Y \arrow[r, "M"] \arrow[d, "\mathbf{I} \times f"'] & Y \arrow[d, "f"] \\ \mathbf{I} \times X \arrow[r, "\pi_X"'] & X \end{tikzcd}$$

because by the above factorisation we have

$$fM = fK(\mathbf{I} \times \partial_1) = fH(\vee \times 1_Y)(\mathbf{I} \times \partial_1) = \pi_X(\mathbf{I} \times f).$$

Hence, f is a strong codeformation retract. □

We can combine the above two results as follows:

Lemma 6.15. *In $\mathscr{C}_{\mathbf{F}}(A)$, if $f : Y \longrightarrow X$ is a Kan fibration and a homotopy equivalence relative to A, then it is a strong codeformation retract.*

Proof. Suppose f is a homotopy equivalence relative to A, then it has a section g up to homotopy over A. From Lemma 6.13, it has an actual section s with a homotopy $g \sim_A s$. Now since the homotopy relation is a congruence by Lemma 6.3, $fs \sim_A fg \sim_A 1$ as well. Now we apply Lemma 6.14, and it follows that f is a strong codeformation retract. □

Proposition 6.16. *In $\mathscr{C}_{\mathbf{F}}(A)$, a map is a trivial fibration iff it is a Kan fibration and a homotopy equivalence relative to A.*

Proof. Combine Lemma 6.10, Lemma 6.15 with Proposition 6.12. □

6.3 A Path Category Structure for Kan Fibrant Objects

In this section we construct a path category structure on $\mathscr{C}_{\mathbf{F}}(A)$, whcih is stable under reindexing. Recall from [66] that a *path category* is a category $\mathscr{C}$ equipped with two classes of maps, called *fibrations* and *weak equivalences*. If a map is both a fibration and a weak equivalence, we call it an *acyclic fibration*. For any object, we say it is (acyclic) fibrant if the unique map to the terminal object is a(n acyclic) fibration.

The two classes of maps in a path category are subject to the following conditions:

1. Fibrations are closed under composition.

2. Pullback of a fibration along any map exists and is again a fibration.

3. Pullback of an acyclic fibration is again an acyclic fibration.

4. Weak equivalences satisfy 2-out-of-6.

5. Isomorphisms are acyclic fibrations and every acyclic fibration has a section.

6. Any object has a path object.

7. Every object is fibrant.

For us, fibrations in $\mathscr{C}_{\mathbf{F}}(A)$ will simply be Kan fibrations, and weak equivalences will be homotopy equivalences relative to A. With such choices, we already know that all the conditions but 6 are satisfied. 1 and 2 follow from the fact that Kan fibrations, being the retract closure of a right structure $\mathbb{F}$ on $\mathscr{C}$, are closed under pullback and composition. 3 and 5 follows from our characterisation in the previous section. In particular, by Proposition 6.16, acyclic fibrations in $\mathscr{C}_{\mathbf{F}}(A)$ are exactly trivial fibrations, thus they will be closed under pullback, and any trivial fibration has a section by Lemma 6.9. 4 holds by Corollary 6.8. 7 holds by our construction of the category $\mathscr{C}_{\mathbf{F}}(A)$.

Hence, the remaining axiom to check is that any object in $\mathscr{C}_{\mathbf{F}}(A)$ has a *path object*:

Definition 6.17. Suppose a category $\mathscr{C}$ has two classes of maps called fibrations and weak equivalences. For any object X in $\mathscr{C}$, a *path object* on X is a factorisation of the diagonal

$$X \longrightarrow PX \xrightarrow{\langle d^0, d^1 \rangle} X \times X$$

such that $X \longrightarrow PX$ is a weak equivalence, and $\langle d^0, d^1 \rangle$ is a fibration. ◊

For $\mathscr{C}_{\mathbf{F}}(A)$, we will choose the path object to be the relative function space $[\mathbf{I}_A, X]_A$, which is the exponential of $\mathbf{I}_A$ and X in the slice $\mathscr{C}/A$. The factorisation is easy to define,

$$X \xrightarrow{c_X} [\mathbf{I}_A, X]_A \xrightarrow{\langle \mathrm{ev}_0, \mathrm{ev}_1 \rangle} X \times_A X$$

where c_X is the transpose of the projection $\pi_X : \mathbf{I}_A \times_A X \longrightarrow X$. We first show that c_X is a homotopy equivalence relative to A:

Lemma 6.18. *For any X in $\mathscr{C}_{\mathbf{F}}(A)$, the constant map*

$$c_X : X \longrightarrow [\mathbf{I}_A, X]_A,$$

is a homotopy equivalence relative to A.

Proof. We may take its homotopy inverse to be $\mathrm{ev}_1 : [\mathbf{I}_A, X]_A \longrightarrow X$. Notice that c_X is a retract of ev_1, thus we only need to construct the homotopy for the other way. Using the semi-lattice structure of the effective cylinder $\mathbf{I}$, the following is a map over A,

$$K = \mathrm{ev}(\vee_A \times 1) : \mathbf{I}_A \times_A \mathbf{I}_A \times_A [\mathbf{I}_A, X]_A \longrightarrow X.$$

In the internal logic, given any $i, j : \mathbf{I}_A$ and any $p : [\mathbf{I}_A, X]_A$, we have

$$K(i, j, p) = p(i \vee_A j).$$

In particular, if $i = 0$, then $K_0(j, p) = p(j)$, while if $i = 1$, then $K_1(j, p) = p(1)$. It then follows that the transpose $\widetilde{K} : \mathbf{I}_A \times_A [\mathbf{I}_A, X]_A \longrightarrow [\mathbf{I}_A, X]_A$ given by

$$\widetilde{K}(i, p) = \lambda j. K(i, j, p)$$

can be viewed as a homotopy $\widetilde{K} : 1 \sim_A c_X \mathrm{ev}_1$. □

Now the remaining thing to show is that the pairing of the evaluation maps

$$\langle \mathrm{ev}_0, \mathrm{ev}_1 \rangle : [\mathbf{I}_A, X]_A \longrightarrow X \times_A X$$

is a Kan fibration.

Lemma 6.19. *For any object X in $\mathscr{C}_{\mathrm{F}}(A)$,*

$$\langle \mathrm{ev}_0, \mathrm{ev}_1 \rangle : [\mathbf{I}_A, X]_A \longrightarrow X \times_A X$$

is a Kan fibration, where both $\mathrm{ev}_0, \mathrm{ev}_1$ *are trivial fibrations.*

Proof. By assumption, $[\partial_0, \partial_1] : 1 + 1 \longrightarrow \mathbf{I}$ is a cofibration. Since $x : X \longrightarrow A$ by assumption is a Kan fibration, by Proposition 5.5 the following pullback exponential will also be a Kan fibration,

$$\hat{\exp}([\partial_0, \partial_1], x) = \langle [\mathbf{I}, x], \langle \mathrm{ev}_0, \mathrm{ev}_1 \rangle \rangle : [\mathbf{I}, X] \longrightarrow [\mathbf{I}, A] \times_{A \times A} (X \times X).$$

Now notice that the following is a pullback square,

$$\begin{tikzcd}
{[\mathbf{I}_A, X]_A} \arrow[r] \arrow[d, "\langle \mathrm{ev}_0, \mathrm{ev}_1 \rangle"'] & {[\mathbf{I}, X]} \arrow[d, "\langle [\mathbf{I}, x], \langle \mathrm{ev}_0, \mathrm{ev}_1 \rangle \rangle"] \\
X \times_A X \arrow[r, "\langle \widehat{x\pi_X}, 1 \rangle"'] & {[\mathbf{I}, A] \times_{A \times A} (X \times X)}
\end{tikzcd}$$

which implies that the map

$$\langle \mathrm{ev}_0, \mathrm{ev}_1 \rangle : [\mathbf{I}_A, X]_A \longrightarrow X \times_A X$$

will in fact be a Kan fibration.

Similarly, by Proposition 5.2, since $\partial_i : 1 \longrightarrow \mathbf{I}$ is an HDR for $i = 0, 1$ and $X \longrightarrow A$ is a Kan fibration, the pullback-exponential

$$\hat{\exp}(\partial_i, x) = \langle [\mathbf{I}, x], \mathrm{ev}_i \rangle : [\mathbf{I}, X] \longrightarrow [\mathbf{I}, A] \times_A X$$

will be a trivial fibration. Now again we have a pullback square as below,

$$\begin{tikzcd}
{[\mathbf{I}_A, X]_A} \arrow[r] \arrow[d, "\mathrm{ev}_i"'] & {[\mathbf{I}, X]} \arrow[d, "\langle [\mathbf{I}, x], \mathrm{ev}_i \rangle"] \\
X \arrow[r, "\langle \widehat{x\pi_X}, 1 \rangle"'] & {[\mathbf{I}, A] \times_A X}
\end{tikzcd}$$

Hence, $\mathrm{ev}_i : [\mathbf{I}_A, X]_A \longrightarrow X$ is an effective trivial fibration for any $i = 0, 1$. $\square$

Corollary 6.20. *For any object X in $\mathscr{C}_{\mathbf{F}}(A)$, the factorisation*

$$X \xrightarrow{c_X} [\mathbf{I}_A, X]_A \xrightarrow{\langle \mathrm{ev}_0, \mathrm{ev}_1 \rangle} X \times_A X$$

exhibits $[\mathbf{I}_A, X]_A$ as the path object of X in $\mathscr{C}_{\mathbf{F}}(A)$.

Proof. Combine Lemma 6.18 and Lemma 6.19. □

To this end, we have verified that Kan fibrations and homotopy equivalences relative to A on $\mathscr{C}_{\mathbf{F}}(A)$ satisfy all the axioms of being a path category. We record this as the following theorem:

Theorem 6.21. *For any object A, $\mathscr{C}_{\mathbf{F}}(A)$ has a path category structure with fibrations being Kan fibrations and weak equivalences being homotopy equivalences relative to A. Furthermore, for any morphism $f : A \longrightarrow B$ in $\mathscr{C}$, the corresponding reindexing functor*

$$f^* : \mathscr{C}_{\mathbf{F}}(B) \longrightarrow \mathscr{C}_{\mathbf{F}}(A)$$

preserves fibrations and weak equivalences.

Proof. As mentioned, we have verified that Kan fibrations and homotopy equivalences relative to A satisfy all the axioms for a path category structure. Now for the pullback functor $f^* : \mathscr{C}_{\mathbf{F}}(B) \longrightarrow \mathscr{C}_{\mathbf{F}}(A)$ induced by any morphism $f : A \longrightarrow B$, evidently it preserves fibrations, since Kan fibrations are closed under pullback. It also preserves weak equivalences, because the effective cylinder object we have chosen are stable under pullback $f^*(\mathbf{I}_B) \cong \mathbf{I}_A$. □

Remark 6.22. In this chapter we have chosen the fibration structure to be Kan fibrations, rather than *effective* Kan fibrations. Our motivation for such a choice has been clearly stated at the beginning of this chapter. However, using the techniques developed in this chapter, one can show that the category $\mathscr{C}_{\mathbb{F}}(A)$ of *effective* Kan fibrations over an arbitrary object A also has a path category structure, with fibrations being effective Kan fibrations and weak equivalences again being homotopy equivalences relative to A. The only caveat is that, Proposition 6.16 no longer holds. In general, a trivial fibration will *not* be an effective Kan fibration. On the other hand, it is also unclear how to eqiup an *effective* trivial fibration structure on an effective Kan fibration which is also a relative homotopy equivalence. ◊

Chapter 7

Towards an Algebraic Monoidal Model Structure

In this chapter, we continue our journey from Chapter 6 to construct a model structure on the full category $\mathscr{C}$. Our strategy of constructing a model structure on $\mathscr{C}$ follows [52]. Based on the approach in *loc. cit.*, in Section 7.1 we will identify the two key properties we need to show $\mathscr{C}$ is an algebraic monoidal model category: (1) $(\mathbb{C}, \mathbb{F})$ forms an AWFS, as mentioned; and (2) Kan fibrations have the socalled *extension property* along trivial cofibrations.

As mentioned at the beginning of Chapter 6, this document will not treat (1). Section 7.2 will look more closely at the extension property stated in (2), and show that it crucially depends on a notion of *Moore equivalence extension* which will be introduced there. The final section thus has singled out a single property central to future investigations on constructive model structure based on effective Kan fibrations.

7.1 Towards a Full Algebraic Model Structure

In this section we explore the possibilities of extending the model structure we have constructed in the previous section on fibrant objects to a full model structure on the total category. In fact, what we want is an algebraic monoidal model structure on $\mathscr{C}$.

As mentioned multiple times, the immediate obstacle we face is that we do not have a general method of constructing an AWFS $(\mathbb{C}, \mathbb{F})$ for effective Kan fibrations on $\mathscr{C}$. For now, we simply take the following working assumption:

Assumption 7.1. *The effective trivial cofibrations and effective Kan fibrations form an AWFS* $(\mathbb{C}, \mathbb{F})$ *on* $\mathscr{C}$.

As already discussed at the beginning of Chapter 6, under Assumption 7.1, the remaining obstacle is to show that the two underlying WFSs $(\Sigma, \mathbf{T})$ and $(\mathbf{C}, \mathbf{F})$ form a model structure on $\mathscr{C}$. In this section, we follow the strategy in [52], and we intend to identify the key property for this to hold.

Notice that if the two WFSs $(\Sigma, \mathbf{T})$ and $(\mathbf{C}, \mathbf{F})$ do form a model structure, then the class of weak equivalences will be determined by them. One possible definition of weak equivalences from two WFSs is as follows:

Definition 7.2. A morphism f in $\mathscr{C}$ is a *weak equivalence*, i.e. $f \in \mathbf{W}$, iff f can be factored as a trivial cofibration followed by a trivial fibration. ◊

The crucial consequence of Definition 7.2 is that it makes the following result true:

Lemma 7.3. $\mathbf{W} \cap \Sigma = \mathbf{C}$*, and* $\mathbf{W} \cap \mathbf{F} = \mathbf{T}$*.*

Proof. See e.g. [52, Lem. 2.1]. □

Remark 7.4. One can also verify that the above weak equivalences will be a sound extension of the path category structure we have constructed in Section 6.3, in the sense that weak equivalences between Kan fibrant objects are precisely homotopy equivalences. The crucial fact is that a cofibration between Kan fibrant objects is a trivial cofibration iff it is also a homotopy equivalence, which can be shown via a dual argument as in Section 6.2. For space limitations, we leave out the argument here. ◊

By Definition 4.21, it remains to show that weak equivalences satisfy *2-out-of-3*. The paper [52] has identified a family of sufficient conditions to ensure this:

Proposition 7.5. *Given two WFSs* $(\Sigma, \mathbf{T})$ *and* $(\mathbf{C}, \mathbf{F})$ *on a category* $\mathscr{C}$*. If the following conditions hold, then they form a model structure on* $\mathscr{C}$ *with weak equivalences as in Definition 7.2:*

- $\mathscr{C}$ *equipped with* $(\Sigma, \mathbf{T})$ *and* $(\mathbf{C}, \mathbf{F})$ *has the span property.*
- *Trivial fibrations satisfy 2-out-of-3 relative to Kan fibrations.*
- *Trivial fibrations and Kan fibrations extend along trivial cofibrations.*
- $(\mathbf{C}, \mathbf{F})$ *has the Frobenius property.*

Proof. See Theorem 2.8 of [52]. □

We will define the notions mentioned above below, and try to prove them for the two WFSs $(\Sigma, \mathbf{T})$ and $(\mathbf{C}, \mathbf{F})$ we have on $\mathscr{C}$. We start with the socalled *span property*:

Definition 7.6 (Span Property)**.** Two WFSs $(\mathbf{C}_0, \mathbf{F}_0)$ and $(\mathbf{C}_1, \mathbf{F}_1)$ on $\mathscr{C}$ has the *span property* if for any $\mathbf{C}_1$-maps i, j and $\mathbf{F}_1$-map f, if $j = fi$, then f also belongs to $\mathbf{F}_0$. ♢

For us, the span property does hold:

Lemma 7.7. *$\mathscr{C}$ equipped with $(\Sigma, \mathbf{T})$ and $(\mathbf{C}, \mathbf{F})$ has the span property.*

Proof. Suppose we have a diagram as below

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "j"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

where $i, j \in \mathbf{C}$ and $f \in \mathbf{F}$. By Lemma 6.11 and Proposition 5.2, to show $f \in \mathbf{T}$, it suffices to show that f is a strong homotopy equivalence. First notice that f has a section s as indicated as follows,

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"'] & X \arrow[d, "f"] \\ Y \arrow[r, equal] \arrow[ur, dashed, "s"] & Y \end{tikzcd}$$

Then consider the following lifting problem,

$$\begin{tikzcd} A \arrow[r, "\widetilde{i\pi_A}"] \arrow[d, "i"'] & {[\mathbf{I}, X]} \arrow[d, "\hat{\exp}([\partial_0,\partial_1],f)"] \\ X \arrow[r, "\langle\langle sf,1\rangle,\widetilde{f\pi_X}\rangle"'] \arrow[ur, dashed, "H"] & (X \times X) \times_{Y\times Y} [\mathbf{I}, Y] \end{tikzcd}$$

By Proposition 5.5, the right vertical map will again be a Kan fibration. Thus, the diagonal lift H exists, whose transpose $\widetilde{H} : \mathbf{I} \times X \longrightarrow X$ by construction makes f a strong codeformation retract. □

We move on to discuss the second property stated in Proposition 7.5:

Definition 7.8. When equipped with two WFSs $(\mathbf{C}_0, \mathbf{F}_0)$ and $(\mathbf{C}_1, \mathbf{F}_1)$, we say $\mathbf{F}_0$ satisfies *2-out-of-3 relative to* $\mathbf{F}_1$, if for any $\mathbf{F}_1$-maps p, q, r with $r = qp$, if two of them belong to $\mathbf{F}_0$, then so is the third. ♢

Again, we can show that the above property indeed holds for our model on $\mathscr{C}$:

Lemma 7.9. *Trivial fibrations in $\mathscr{C}$ satisfies 2-out-of-3 relative to Kan fibrations in $\mathscr{C}$.*

Proof. The same proof of [52, Lem. 4.5] can be applied here. $\square$

The extension property is defined as follows:

Definition 7.10 (Extension)**.** A class of maps **R** extends along a class of maps **L**, if whenever we are given a solid diagram as follows,

$$\begin{tikzcd} \bullet \arrow[r, dashed] \arrow[d, "\in\mathbf{R}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \bullet \arrow[d, dashed, "\in\mathbf{R}"] \\ \bullet \arrow[r, "\in\mathbf{L}"'] & \bullet \end{tikzcd}$$

it can be completed in the above way as a pullback. $\diamond$

For trivial fibrations, we indeed can show they extend along trivial cofibrations:

Lemma 7.11. *Trivial fibrations extend along trivial cofibrations in $\mathscr{C}$.*

Proof. Consider a trivial cofibration i and a trivial fibration f,

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "i_* f"] \\ A \arrow[r, "i"'] & B \end{tikzcd}$$

Here $i_* f$ is the pushforward of f along i. Since i is a monomorphism, the above square will be a pullback. Now since cofibrations are closed under pullbacks, it follows that trivial fibrations will be closed under pushforward along any morphism. This means that $i_* f$ will also be a trivial fibration. $\square$

The Frobenius property is defined as follows:

Definition 7.12 (Frobenius)**.** We say a WFS $(\mathbf{L}, \mathbf{R})$ has the Frobenius property if **L** is stable under pullback along maps in **R**. $\diamond$

As mentioned in the Introduction, one of the nice properties of the notion of effective Kan fibrations is that, we can show constructively effective Kan fibrations are closed under pushforward along effective Kan fibrations [63, Ch. 7]. Under Assumption 7.1, this implies that effective trivial cofibrations will be closed under pullback along effective Kan fibrations, i.e. the AWFS $(\mathbb{C}, \mathbb{F})$ has the Frobenius property.

To show the underlying WFS $(\mathbf{C}, \mathbf{F})$ has the Frobenius property, we observe that generally for any AWFS $(\mathbb{L}, \mathbb{R})$, if it satisfies Frobenius, then so does its underlying WFS. We start with the following observation:

Lemma 7.13. *If an AWFS* $(\mathbb{L}, \mathbb{R})$ *on* $\mathscr{C}$ *has the Frobenius property, i.e.* $\mathbb{L}$*-maps is closed under pullback along* $\mathbb{R}$*-maps, then* $\mathbf{L}$*-maps is also closed under pullback along* $\mathbb{R}$*-maps.*

Proof. Consider the following diagram,

$$\begin{tikzcd}
 & D \arrow[rr, "h"] \arrow[dd, "k"] \arrow[dl, bend left, "t_1"] \arrow[dr, phantom, "\lrcorner"] & & B \arrow[dd, "j"] \arrow[dl, bend left, "r_1"] \\
C \arrow[ur, bend left, "u_1"] \arrow[rr, "q"] \arrow[dd, "l"] \arrow[dr, phantom, "\lrcorner"] & & A \arrow[ur, bend left, "s_1"] \arrow[dd, "i"] & \\
 & W \arrow[rr, "g"] \arrow[dl, bend left, "t_0"] \arrow[dr, phantom, "\lrcorner"] & & Z \arrow[dl, bend left, "r_0"] \\
X \arrow[ur, bend left, "u_0"] \arrow[rr, "f"'] & & Y \arrow[ur, bend left, "s_0"] &
\end{tikzcd}$$

where f in $\mathbb{R}$ and i in $\mathbf{L}$, where i arises as a retract of the $\mathbb{L}$-map j. In the above diagram, g is the pullback of f along r_0, thus is again in $\mathbb{R}$. Now by Frobenius of the AWFS $(\mathbb{L}, \mathbb{R})$, the pullback k of j along g will be in $\mathbb{L}$. This makes l a retract of k, thus l will be in $\mathbf{L}$. □

Using this, we can show the following result:

Lemma 7.14. *If an AWFS* $(\mathbb{L}, \mathbb{R})$ *has the Frobenius property, then so does its underlying WFS* $(\mathbf{L}, \mathbf{R})$.

Proof. Consider a pullback diagram with $A \longrightarrow B$ belongs to $\mathbf{L}$ and $E \longrightarrow B$ belongs to $\mathbf{R}$ as shown below. Now since we have an AWFS $(\mathbb{L}, \mathbb{R})$, $E \twoheadrightarrow B$ will be a retract of the free R-algebra

$$\begin{tikzcd}
 & C \arrow[rr, "g"] \arrow[dd, "j"] \arrow[dl, bend left, "s"] \arrow[dr, phantom, "\lrcorner"] & & A \arrow[dd, "i"] \arrow[dl, equal] \\
D \arrow[ur, bend left, "p"] \arrow[rr, "h"] \arrow[dd, "k"] \arrow[dr, phantom, "\lrcorner"] & & A \arrow[dd, tail, "i"] & \\
 & E \arrow[rr, "f"] \arrow[dl, bend left, "L"] & & B \arrow[dl, equal] \\
F \arrow[ur, bend left, "r"] \arrow[rr, "R"] & & B &
\end{tikzcd}$$

Now since R is an $\mathbb{R}$-map, by Lemma 7.13 $D \longrightarrow F$ belongs to $\mathbf{L}$. Since $fr = R$, we have that $D \longrightarrow F$ is also the pullback of $A \longrightarrow B$, hence on the left square p is also a pullback of r. Now by the universal property of pullback, we have a uniquely induced map

$$s = \langle Lj, 1 \rangle : C \longrightarrow D,$$

which is well-defined because $rLj = j$. Now this implies that j is also a retract of k, thus j also belongs to $\mathbf{L}$. □

Corollary 7.15. *In $\mathscr{C}$, the WFS* $(\mathbf{C}, \mathbf{F})$ *has the Frobenius property.*

Proof. This follows from Lemma 7.14 and the fact that effective Kan fibrations are closed under pushforward along effective Kan fibrations as shown in [63, Ch. 7]. □

Now combine all the results above, the only missing hypothesis from Proposition 7.5 is the extension property of Kan fibrations:

Proposition 7.16. *Under Assumption 7.1, if Kan fibrations extend along trivial cofibrations, then* $(\Sigma, \mathbb{T})$ *and* $(\mathbb{C}, \mathbb{F})$ *form an algebraic monoidal model structure on* $\mathscr{C}$.

Proof. Combining Proposition 7.5, Lemma 7.7, Lemma 7.9, Lemma 7.11, and Corollary 7.15, if Kan fibrations extend along trivial cofibrations, then $(\Sigma, \mathbf{T})$ and $(\mathbf{C}, \mathbf{F})$ will be a model structure on $\mathscr{C}$. By Definition 4.26, the two AWFSs $(\Sigma, \mathbb{T})$ and $(\mathbb{C}, \mathbb{F})$ do form an algebraic model structure, since as mentioned we have a morphism between the AWFSs $(\mathbb{C}, \mathbb{F}) \longrightarrow (\Sigma, \mathbb{T})$. Then Theorem 5.7 implies that we further have an algebraic monoidal model category. □

Hence, the crucial property is to show that Kan fibrations extend along trivial cofibrations. We will provide a more detailed analysis of this problem in the next section.

7.2 Extension Property of Effective Kan Fibrations

In this section we propose a strategy to show Kan fibrations extend along trivial cofibrations. As in [52], whether this holds crucially depends on a certain *equivalence extension property*. In *loc. cit.*, the equivalence relation involved is *homotopy equivalence*. This is related to how the fibration structure is defined there. For Kan fibrations as defined in Section 3.3.3, it turns out the equivalence relation involved will be *Moore equivalence* in a suitable sense. The argument in this section will necessarily involve more deeply the symmetric Moore structure on $\mathscr{C}$, which we again refer the readers to Appendix A.

As a first observation, we note that the problem can be reduced to that of showing Kan fibrations extension along *generating* trivial cofibrations:

Lemma 7.17. *The class of cofibrations that Kan fibrations extend along is closed under coproducts, pushouts, transfinite compositions and retracts.*

Proof. See [52, Lem. 7.5]. □

Thus for us, we need to show that Kan fibrations extend along the cofibre arrow of any mould square (cf. Section 3.3.3). More concretely, suppose we have a mould square

$$\begin{tikzcd} A \arrow[r, "\partial_a"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ C \arrow[r, "\partial_c"'] & D \end{tikzcd}$$

and if we have a Kan fibration over $B +_A C$, we should be able to construct a Kan fibration over D whose pullback along $B +_A C \longrightarrow D$ is X. We can also reformulate this question more natively on mould squares: Given any mould square as above, with a Kan fibration X_B over B and a Kan fibration X_C over C such that they pullback to the same Kan fibration X_A over A, then there exists another Kan fibration $X \longrightarrow D$ which pullbacks to X_B, X_C along j, ∂_c, respectively.

Based on this reformulation, our first task in this section is to turn this property as a property of *extension of Hdrs*. One starting point is the observation that Kan fibrations do extend along *HDRs*. In fact more generally, any right class of maps extend along *any* map with a retract:

Lemma 7.18. *If* $\mathbf{R}$ *is stable under pullback, then they extend along any split mono.*

Proof. Suppose we have a split mono $i : A \longrightarrow B$ with retract $j : B \longrightarrow A$. Now suppose we have an $\mathbf{R}$-map $f : X \longrightarrow A$. Consider the following diagram,

$$\begin{tikzcd} X \arrow[r, "\langle if,1\rangle"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \times_A X \arrow[r, "l"] \arrow[d, "g"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ A \arrow[r, "i"'] & B \arrow[r, "j"'] & A \end{tikzcd}$$

where g is the pullback of f along j. Since both the outer square and the right square are pullbacks, so is the left square. Now since $\mathbf{R}$-maps are closed under pullbacks, g will also be an $\mathbf{R}$-map. This way, the left square exhibits g as the extension of f along i. □

In particular, Lemma 7.18 implies that Kan fibrations extend along Hdrs, since Hdrs have explicit retracts. This suggests that when given an extension problem against a mould square as stated before, we can first extend X_C along the HDR $\partial_c : C \longrightarrow D$ and obtain a Kan fibration Y over D, such that $X_C \longrightarrow Y$ is an Hdr. The caveat is that when we pullback Y along j, j^*Y, the result may *not* be isomorphic to X_B. But they will always be *Moore equivalent* in a suitable sense, as we will show below.

Notice that if we have an Hdr $\partial : A \longrightarrow B$ and a Kan fibration X over B, there will be an Hdr structure on X as well, making the morphism $X \longrightarrow B$ a morphism of Hdrs:

Lemma 7.19. *If $\partial_a : A \longrightarrow B$ is an Hdr and $f : X \longrightarrow B$ is a Kan fibration, then in the following pullback square,*

$$\begin{tikzcd} X_A \arrow[r, "\partial_x"] \arrow[d, "f_A"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ A \arrow[r, "\partial_a"'] & B \end{tikzcd}$$

∂_x is also an Hdr, and the square is a morphism of Hdrs.

Proof. This follows from the Frobenius construction given in [63, Prop. 5.2]. The *loc. cit.* shows that if f is an *effective* Kan fibration and ∂_a is an HDR, then ∂_x is also an HDR and the morphism will be a morphism of HDRs. It is easy to see that the same argument also applies to the case where f is simply a Kan fibration and ∂_a is an Hdr. □

Now if Y is obtained by extending a Kan fibration X over A along the Hdr $\partial_a : A \longrightarrow B$ as in Lemma 7.18, the induced morphism of Hdrs as in Lemma 7.19 will furthermore be *Cartesian*, because by construction Y is the pullback of X along σ_a.

The upshot is that, given an Hdr $\partial_a : A \longrightarrow B$, if we have a Kan fibration $f : X \longrightarrow B$, and let X_A be the Kan fibration over A obtained by pulling back X along ∂_a, the Kan fibration X and the extension Y of X_A along ∂_a as in Lemma 7.18 will be *Moore equivalent over B*, in a suitable sense. Recall from Proposition 3.25 the category of Hdrs is equivalent to the category of (M, s)-coalgebras, thus we will identify them freely.

Definition 7.20. Given an Hdr B and Kan fibrations X, Y over B, we say a morphism $f : X \longrightarrow Y$ over B is a *Moore equivalence over B*, if firstly it is a morphism of Hdrs, and there is another morphism $g : Y \longrightarrow X$, such that the induced Hdr structures on X, Y

$$H_X : X \longrightarrow MX, \quad H_Y : Y \longrightarrow MY,$$

serve as Moore homotopies $H_X : 1 \sim gf$ and $H_Y : 1 \sim fg$, respectively.[1] ◊

Remark 7.21. It turns out that Moore equivalences interact nicely with Kan fibrations and trivial fibrations. Given a Kan fibration $f : X \longrightarrow Y$ between Kan fibrant objects, by definition it is a Moore equivalence iff the following diagram is a retract,

$$\begin{tikzcd} X \arrow[r, "H_X"] \arrow[d, "f"'] & MX \arrow[r, "s"] \arrow[d, "\langle t, Mf \rangle"] & X \arrow[d, "f"] \\ Y \arrow[r, "\langle g, H_Y \rangle"'] & X \times_Y MY \arrow[r, "s \circ \pi_{MY}"'] & Y \end{tikzcd}$$

[1] We do not assume g to also live over B. In fact, from our definition, it cannot be.

By [63, Cor. 11.1], f is a Kan fibration in simplicial sets iff $\langle t, Mf \rangle : MX \longrightarrow X \times_Y MY$ is a trivial fibration. Thus, if f is a Moore equivalence, then it is also a trivial fibration. $\diamond$

One immediate consequence of two Kan fibrations X, Y being Moore equivalent over an Hdr B is that, their pullbacks along $\partial_a : A \longrightarrow B$ do coincide:

Lemma 7.22. *Suppose we have two Kan fibrations X, Y over an Hdr $\partial_a : A \longrightarrow B$. If we have a Moore equivalence $f : X \longrightarrow Y$ over B, then we have $f_A : X_A \cong Y_A$ over A, where f_A is the pullback of f along ∂_a.*

Proof. In the internal logic, we know that

$$X_A = \{\, x : X \mid H_X(x) = r_x \,\}, \quad Y_A = \{\, y : Y \mid H_Y(y) = r_y \,\},$$

since the following are pullback squares (cf. [63, Lem. 4.1]),

$$\begin{tikzcd} X_A \arrow[r, "\partial_x"] \arrow[d, "\partial_x"'] & X \arrow[d, "H_X"] \\ X \arrow[r, "r"'] & MX \end{tikzcd} \qquad \begin{tikzcd} Y_A \arrow[r, "\partial_y"] \arrow[d, "\partial_y"'] & Y \arrow[d, "H_Y"] \\ Y \arrow[r, "r"'] & MY \end{tikzcd}$$

We can thus directly construct an inverse $g_A : Y_A \longrightarrow X_A$ as follows: For any $y : Y_A$,

$$g_A(y) = \sigma_x g \partial_y y.$$

Now given any $x : X_A$ we have

$$g_A f_A(x) = \sigma_x g \partial_y f_A(x) = \sigma_x g f \partial_x x.$$

Since we know that $H_X : 1 \sim gf$ and that $H_X(\partial_x x) = r(\partial_x x)$, it follows that

$$gf\partial_x x = \partial_x x,$$

thus we do have $g_A f_A(x) = \sigma_x \partial_x x = x$.

On the other hand, given any $y : Y_A$, we have

$$f_A g_A(y) = f_A \sigma_x g \partial_y y = \sigma_y f g \partial_y y.$$

Again, since $H_Y : 1 \sim fg$ and $H_Y(\partial_y y) = r(\partial_y y)$, it follows that

$$f_A g_A(y) = \sigma_y \partial_y y = y.$$

This concludes the proof. $\square$

As mentioned, one typical example of a Moore equivalence over an Hdr comes from the following situation:

Lemma 7.23. *Suppose we have an Hdr $\partial_a : A \longrightarrow B$ and a Kan fibration $f : X \longrightarrow B$. Let X_A be the pullback of X along ∂_a. Let Y be a Kan fibration over B obtained via extending X_A along ∂_a as in Lemma 7.18, then the following comparision map*

$$\langle f, \sigma_x \rangle : X \longrightarrow Y = B \times_A X_A$$

with be a Moore equivalence between X, Y over B.

Proof. It is easy to see that the comparison map is a morphism of Hdrs. Recall the Hdr structure on Y

$$H_Y : Y = B \times_A X_A \longrightarrow MY \cong MB \times_{MA} MX_A$$

simply takes any $(b, x) : Y = B \times_A X_A$ to the following path,

$$H_Y(b, x) = (H_B(b), \alpha(x, |H_B(b)|)),$$

Now given any $x : X$, by construction we have that

$$\begin{aligned} M\langle f, \sigma_x \rangle H_X(x) &= \langle Mf H_X(x), M\sigma_x H_X(x) \rangle = \langle H_B(fx), \alpha(\sigma_x x, |H_X(x)|) \rangle \\ &= \langle H_B(fx), \alpha(\sigma_x x, |H_B(fx)|) \rangle = H_Y \langle fx, \sigma_x x \rangle \end{aligned}$$

The second and third equalities hold due to the fact that $f : X \longrightarrow B$ is a morphism of Hdrs. The inverse map is easy to define as follows,

$$\partial_x \pi_{X_A} : Y = B \times_A X_A \longrightarrow X.$$

To show that they form a Moore equivalence between X and Y over B, we need to further verify that indeed we have

$$H_X : 1 \sim \partial_x \pi_{X_A} \circ \langle f, \sigma_x \rangle, \quad H_Y : 1 \sim \langle f, \sigma_x \rangle \circ \partial_x \pi_{X_A}.$$

On one hand, given any $x : X$, by definition we have

$$\partial_x \pi_{X_A} \circ \langle f, \sigma_x \rangle(x) = \partial_x \sigma_x(x),$$

thus we do have $H_X : 1 \sim \partial_x \pi_{X_A} \circ \langle f, \sigma_x \rangle$. On the other hand, for any $(b, x) : B \times_A X_A = Y$,

$$\langle f, \sigma_x \rangle \circ \partial_x \pi_{X_A}(b, x) = \langle f \partial_x x, \sigma_x \partial_x x \rangle = \langle \partial_a \sigma_a b, x \rangle.$$

The final equality is due to the fact that by assumption, $f_A x = \sigma_a b$. Thus, we indeed have

$$H_Y(b, x) : (b, x) \sim (\partial_a \sigma_a b, x).$$

This implies that $H_Y : 1 \sim \langle f, \sigma_x \rangle \circ \partial_x \pi_{X_A}$. This concludes the proof. $\square$

Now consider a mould square as follows,

$$\begin{tikzcd}
A \arrow[r, "\partial_a"] \arrow[d, "i"'] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[r, "\sigma_a"] \arrow[d, dashed, "j"] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "i"] \\
C \arrow[r, "\partial_c"'] & D \arrow[r, "\sigma_c"'] & C
\end{tikzcd}$$

Suppose we have Kan fibrations $f_C : X_C \longrightarrow C$ and $f_B : X_B \longrightarrow B$ which are pulled back to the same map $f_A : X_A \longrightarrow A$. Let the extension of X_C along $\partial_c : C \longrightarrow D$ as constructed in Lemma 7.18 be $g : Y \longrightarrow D$, and let $g_B : Y_B \longrightarrow B$ be the pullback of Y along j. Notice that the following is a pullback,

$$\begin{tikzcd}
Y_B \arrow[r] \arrow[d, "g_B"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X_A \arrow[d, "f_A"] \\
B \arrow[r, "\sigma_a"'] & A
\end{tikzcd}$$

This is because we have

$$\sigma_a^* X_A \cong \sigma_a^* i^* X_C \cong j^* \sigma_c^* X_C \cong j^* Y \cong Y_B.$$

Thus, by Lemma 7.23, Y_B and X_B will be Moore equivalent over A. Given such a situation, it is natural to define the following *Moore equivalence extension property*:

Definition 7.24. We say Kan fibrations satisfy the *Moore equivalence extension property* if the following holds: Given any *Cartesian cofibrant* morphism of Hdr $B \longrightarrow D$, in other words a mould square, and given any Kan fibrations X, Y which are Moore equivalent over B

$$\begin{tikzcd}
X_B \arrow[rr, dashed] \arrow[dr, dashed, "\sim"] \arrow[dd] & & X \arrow[dr, dashed, "\sim"] \arrow[dd, dashed] & \\
& Y_B \arrow[rr] \arrow[dl] & & Y \arrow[dl] \\
B \arrow[rr] & & D &
\end{tikzcd}$$

then for any Kan fibration Y over D whose pullback along $B \longrightarrow D$ is Y_B, we can extend X_B along $B \longrightarrow D$ to a Kan fibration X over D as well as shown above, such that X and Y are Moore equivalent over D. $\diamond$

It is easy to see that if the Moore equivalence extension property holds for Kan fibrations, then Kan fibrations do extend along trivial cofibrations:

Proposition 7.25. *If Kan fibrations in $\mathscr{C}$ have the Moore equivalence extension property, then Kan fibrations extend along trivial cofibrations.*

Proof. As mentioned, by Lemma 7.17 we only need to show that Kan fibrations extend along cofibre arrows of mould squares. Suppose we have a mould square as follows,

$$\begin{tikzcd} A \arrow[r, "\partial_a"] \arrow[d, "i"'] & B \arrow[d, "j"] \\ C \arrow[r, "\partial_c"'] & D \end{tikzcd}$$

with Kan fibrations X_C over C and X_B over B which are pulled back to the same Kan fibration X_A over A. We need to find a Kan fibration X over D which coincide with X_B and X_C when restricted along B, C, respectively.

To this end, let us first extend X_C along the HDR $C \longrightarrow D$ as in Lemma 7.18 and get a Kan fibration Y over D. By the discussion before Definition 7.24, the restriction of Y to Y_B over B can be equivalently seen as the extension of X_A along ∂_a as indicated by Lemma 7.18. Now Lemma 7.23 implies that Y_B and X_B will be Moore equivalent over B. By the Moore equivalence extension property, we can find a Kan fibration X over D which is Moore equivalent to Y over D, and whose restriction on B coincide with X_B. By Lemma 7.22, X and Y have the same restriction along $C \longrightarrow D$, which also implies that X restricted to C will be X_C. Thus, X will be our desired extension. $\square$

At this point, we have reduced the problem of constructing an algebraic monoidal model structure on $\mathscr{C}$ to the problem of showing Kan fibrations have the Moore equivalence extension property (modulo the problem of constructing an AWFS $(\mathbb{C}, \mathbb{F})$, of course). We comment at the end that, from the perspective of building a model of HoTT, a version of equivalence extension property is closely related to the *fibrancy of the universe*; see e.g. [3]. This means that trying to show the Moore equivalence extension property for Kan fibrations, either abstractly in an axiomatic setting, or concretely in the specific example of simplicial sets, should be the focus of future investigations.

Chapter 8

Conclusion and Future Work

In this document, we have defined and used the framework of n-fold categories to construct a **Grpd**-enriched multicategory $\mathbb{L}$**Str** of left structures, whose morphisms represent generalised pushout-product axioms in the structured context. We have also lifted the important technique of Joyal-Tierney calculus in homotopy theory to the structured context, as certain cyclic action on morphisms in $\mathbb{L}$**Str**. This makes $\mathbb{L}$**Str** a convenient environment for formulating and investigating general relationships between different structures of morphisms.

The first main result of this document is an axiomatisation of an *algebraic monoidal model category*, which improves the existing notion stated in the literature [48]. Furthermore, we have shown that the primary example of our concern, viz. the effective Kan fibrations and effective trivial fibrations introduced in [63], can be equipped with an algebraic monoidal structure, using the structured Joyal-Tierney calculus.

Based on these results, we are able to construct a stable family of path category structures on categories of Kan fibrations over arbitrary objects. We have also carefully examined how the strategy of showing a full model category structure given in [52] can be applied for effective Kan fibrations. We have identified a key property called *Moore equivalence extension*, such that if it holds, then we can show the existence of an algebraic monoidal model structure for effective Kan fibrations. Of course, this relies on the assumption that effective Kan fibration is part of an AWFS.

As mentioned earlier, it seems the problem of constructing an AWFS for effective Kan fibrations, at least in the concrete example of simplicial sets, is solvable. Thus, the focus for future work is to first investigate whether the Moore equivalence extension property holds, especially for simplicial sets. Upon successfully obtaining such a full algebraic monoidal model category, we can proceed to construct a full model of HoTT, working towards a constructive interpretation of the univalence axiom.

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Appendix A

Symmetric Moore Structure

In this section we discuss the axioms of a *symmetric Moore structure* on a category $\mathscr{C}$. We assume $\mathscr{C}$ is a category having finite limits and finite colimits, and is locally Cartesian closed.

A.1 Internal †-Category Structure

A symmetric Moore structure on $\mathscr{C}$ first consists of a pullback-preserving endo-functor M, taking each object X in $\mathscr{C}$ to its *Moore path object* MX. Given $f : X \longrightarrow Y$, the map $Mf : MX \longrightarrow MY$ maps a path in X along f to a path in Y.

For MX to be deserving the name of a path object, we require it to equip naturally for each X in $\mathscr{C}$ a structure of an *internal* †*-category*,[1]

$$\begin{tikzcd} MX \,{}_t\!\times_s MX \arrow[r, "\mu"] & MX \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & MX \arrow[l, "r" description] \\ & MX \arrow[u, "\tau"] & \end{tikzcd}$$

where all the maps r, s, t, τ, μ are natural. The †-structure τ, or the symmetry, can simply be viewed as providing an *internal functor* from the category (X, MX, r, s, t, μ) to its *opposite*. For more detailed discussion on †-categories, see e.g. [28, Ch. 2.3].

We further require the following additional conditions for these structures:

- We impose that μ, r should be *Cartesian* natural transformations.

[1]For basic notions of internal categories, see e.g. [34, Ch. B2].

- When $\mathscr{C}$ is also equipped with a *dominance* Σ, we require that all components of r also lie in Σ.

The two conditions ensures that there is a *morphism of AWFSs* from the one induced by the symmetric More structure (see Section 3.3.2) to the one induced by the dominance structure (see Section 3.3.1).

Notationwise, for any $p : MX$, we use $p : x_0 \longrightarrow x_1$ to indicate $sp = x_0$ and $tp = x_1$. Given another $q : x_1 \longrightarrow x_2$, we simply use the juxtaposition pq to denote their composite $\mu(p, q)$. In the second path space MMX, there will be *two* composition structures. If two paths between paths $a : p \longrightarrow q$ and $b : q \longrightarrow u$ agree on the boundary, they can be composed as ab via $\mu(a, b)$. Diagramatically, we denote it as the following *horizontal composition*:

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \arrow[r] & \bullet \\
\bullet \arrow[r] \arrow[u, "p"] \arrow[ur, phantom, "\Rightarrow a"] & \bullet \arrow[r] \arrow[u, "q"] \arrow[ur, phantom, "\Rightarrow b"] & \bullet \arrow[u, "u"']
\end{tikzcd}$$

On the other hand, there is also a *vertical composition* provided by $M\mu$ as follows: If two a, a' they agree on the vertical boundary, viz. $Mt(a) = Ms(a')$, as indicated by the diagram below,

$$\begin{tikzcd}
\bullet \arrow[r] & \bullet \\
\bullet \arrow[r] \arrow[u, "p'"] \arrow[ur, phantom, "\Rightarrow a'"] & \bullet \arrow[u, "q'"'] \\
\bullet \arrow[r] \arrow[u, "p"] \arrow[ur, phantom, "\Rightarrow a"] & \bullet \arrow[u, "q"']
\end{tikzcd}$$

then we may apply $M\mu$ via the natural isomorphism

$$\nu : MMX \;{}_{Mt}{\times}_{Ms} MMX \cong M(MX \;{}_{t}{\times}_{s} MX),$$

which exists since M preserves pullback. We denote the vertical composition of a and a' as $a \bullet a'$, and concretely it is given by

$$a \bullet a' = M\mu.\nu(a, a').$$

A.2 Path Length and Constant Path

We think of $M1$ as the object of *path length*. For any X, the unique map $X \longrightarrow 1$ induces a function $MX \longrightarrow M1$, which we think of as taking a path in X to its associate path length.

For this reason, we write this map as $|-|$. Cartesianness of r implies that the following diagram is a pullback,

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[d] & MX \arrow[d, "|-|"] \\ 1 \arrow[r, "r"'] & M1 \end{tikzcd}$$

which means that a path in MX is trivial iff its length is trivial.

Besides trivial paths, for any path length we would also want to construct a *constant path* of this length. This is recorded in the structure of a *strength* (cf. [39]), which is a natural family of maps

$$\alpha_X : X \times M1 \longrightarrow MX,$$

taking a point x in X and a path length to the associated constant path on x of the *same length*. In particular, this means we have

$$\begin{tikzcd} X \times M1 \arrow[r, "\alpha"] \arrow[dr, "\pi_{M1}"'] & MX \arrow[d, "|-|"] \\ & M1 \end{tikzcd}$$

or in other words, for any $x : X$ and $\gamma : M1$ we have

$$|\alpha(x, \gamma)| = \gamma.$$

Furthermore, to make sure the image of α is indeed the constant path on a point, and that the path length composes in the right way, we require the following equations to hold,

$$s\alpha(x, \gamma) = t\alpha(x, \gamma) = x, \quad \tau\alpha(x, \gamma) = \alpha(x, \gamma), \quad \alpha(x, r) = r_x, \quad \alpha(x, \gamma)\alpha(x, \sigma) = \alpha(x, \gamma\sigma).$$

In the usual definition of a strength of a functor, we usually require a natural family of maps of the following type,

$$\alpha_{X,Y} : X \times MY \longrightarrow M(X \times Y).$$

However, since M preserves pullback, these are constructible from the α given above. Notice that we have

$$M(X \times Y) \cong MX \times_{M1} MY.$$

This means that we can define

$$\alpha_{X,Y}(x, p) = (\alpha(x, |p|), p).$$

It is easy to verify that these induced maps $\alpha_{X,Y}$ indeed form a strength in the usual sense, and also works well with the internal †-category structure. For the details, we refer the readers to [65].

A.3 Connection

Furthermore, to obtain a good theory of fibration from the path functor M, we assume there is a *contraction operator*, which is a map $\Gamma : M \longrightarrow MM$. Intuitively, we think of Γ as taking a path p and producing a higher path Γp that contracts p along itself, or in diagramatic form as follows,

$$\begin{tikzcd}
x_1 \arrow[dr, equal] & \\
x_0 \arrow[u, "p"] \arrow[r, "p"'] \arrow[ur, phantom, "\Rightarrow\Gamma p"] & x_1
\end{tikzcd}$$

To describe this algebraically, we impose the following conditions. Firstly, w.r.t. s, we should have that for any $p : MX$,

$$s\Gamma p = p = Ms(\Gamma p),$$

which means Γ satisfies the two counit laws w.r.t. s. These two equalities represent the fact that the source of the contraction Γp is p, and that if we project under M using s, we obtain p again. Simiarly, w.r.t. t, we require for any $p : MX$,

$$t\Gamma p = r_{tp}, \quad Mt(\Gamma p) = \alpha(tp, |p|).$$

These two equations encode that the target of the contraction Γp is the trivial path r, and the contraction is constant on the target.

Besides the above conditions, we furthermore require that (M, Γ, s) forms a *comonad*. The additional coassociativity law means that for any $p : MX$,

$$\Gamma\Gamma p = M\Gamma.\Gamma p.$$

Diagramatically, this means that the two ways of constructing the following higher contraction,

$$\begin{tikzcd}
x_1 \arrow[r, equal] \arrow[dr, equal] & x_1 \arrow[d, equal] \\
x_0 \arrow[u, "p"] \arrow[r, "p"'] \arrow[ur, dashed, "p"] & x_1
\end{tikzcd}$$

one by using Γ on the 2-dimensional path Γp, and the other by mapping Γp along $M\Gamma$, are equivalent. Similarly, we also impose that the contraction of a constant path itself is induced by the strength of the contraction of path lengths,

$$\Gamma\alpha(p, \gamma) = M\alpha(\alpha(x, \Gamma\gamma)).$$

Finally, we also require the connection Γ to be compatible with the internal †-category structure. For the identities, we require that the contraction applied to trivial paths results in trivial paths as wel,

$$\Gamma r = r_r.$$

For the composition μ, we impose the following *distributivity law*: For any $p : x_0 \longrightarrow x_1$ and $q : x_1 \longrightarrow x_2$ in MX,

$$\Gamma(pq) = (\Gamma p \bullet \alpha(q, |p|))\Gamma q.$$

In diagramatic form, it says that the following two ways of constructing the contraction on the composite path pq are equivalent,

$$
\begin{tikzcd}
x_2 \arrow[ddrr] & & \\
x_1 \arrow[u, "q"] & & \\
x_0 \arrow[u, "p"] \arrow[r, "p"'] \arrow[ur, phantom, "\Rightarrow\Gamma(pq)"] & x_1 \arrow[r, "q"'] & x_2
\end{tikzcd}
=
\begin{tikzcd}
x_2 \arrow[dr] & & \\
x_1 \arrow[u, "q"] \arrow[dr] \arrow[r, phantom, "\Rightarrow\alpha(q,|p|)"] & x_2 \arrow[dr] & \\
x_0 \arrow[u, "p"] \arrow[r, "p"'] \arrow[ur, phantom, "\Rightarrow\Gamma p"] & x_1 \arrow[u, "q"] \arrow[r, "q"'] \arrow[ur, phantom, "\Rightarrow\Gamma q"] & x_2
\end{tikzcd}
$$

Finally, for the symmetry τ, notice that we can define a new contraction operator

$$\Gamma^* = \tau M.M\tau.\Gamma.\tau,$$

but now w.r.t. to the dual Moore structure $(M, t, s, r, \mu^{\mathrm{op}})$. We require that (M, Γ^*, t) is also a strong comonad.