THE NOBLE ART OF LINEAR DECORATING

HAROLD SCHELLINX

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ACADEMISCH PROEFSCHRIFT

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Preface

Linear logic (Girard,1987) sprouts from the remarkable observation that a certain semantical decomposition of intuitionistic type constructors corresponds on a syntactical level to the banning of structural rules of weakening and contraction from the formulation of intuitionistic logic as a sequent calculus, followed by their resurrection in modalized form. It then is a small, but important, step to apply this latter, purely formal, manipulation to sequent calculi also for classical logic, and marvel at the consequences.

Soon following its introduction, linear logic became the topic of a quickly growing number of research- and survey-papers, and inspired workers in proof theory, category theory, complexity theory, theoretical and not-so-theoretical computer science, all eager to explore the possible, impossible, the more, as well as the less, probable, implications and applications. As a result, in much less than a decade, the field has become so extensive, that, in the present context, we will not even try to give a comprehensive overview.

A lot of the excitement originated from the fact that an inference system which marks the multiple use of premisses is, somehow, 'resource-conscious', thus establishing an obvious intuitive link with the practice of programming. More high expectations sprang from the motivation and 'explanation' of linear connectives in terms of different ways to process information, which suggested rather immediate applications, for example to both theory and practice of parallel computing.

What we would like to underline, however, is a different, in a sense more ideological, and maybe less glamorous, aspect: the conviction that, though, undoubtedly, 'logics' are interesting to logicians, it is 'logic' they should be after. And, that it is not so much a (logical) theorem, as the way in which it can be established, that is of importance: if there is a 'heart of logic' to be unveiled, it will lie in its proofs. It is in the bearing of these credos that we think linear logic finds its deepest appeal.

The optic of this thesis, then, is a fairly modest one: linear sequent calculus appears as a refinement of the known calculi for both intuitionistic and classical logic. It therefore can be considered a tool to investigate, as if through a microscope, behaviour and properties of intuitionistic and classical sequent derivations.

It is on a such proof theoretical study that we will embark.

Ars-en-Ré
August, 1993

Table of Contents

Guide

Take some derivation π in sequent calculus for classical or intuitionistic logic. Can we transform it into a sequent derivation in linear logic in a way that essentially preserves its structure, i.e. can we define a *linear decoration* of the original proof? And if 'yes', then is there an *optimal* way to do this?

The bulk of the considerations in this thesis originated in attempts to give some kind of an answer to this, natural, question. It describes our work on the subject, major parts of it done in close cooperation with Vincent Danos and Jean-Baptiste Joinet, under the agreeable lee of the Université Paris VII's 'Équipe de Logique', between spring 1992 and fall 1993.

Our main concern will be with *mappings*: from formulas to linear formulas, from classical and intuitionistic proofs to linear proofs, from classical (intuitionistic) proofs to classical (intuitionistic) proofs, and from linear proofs to linear proofs, mappings that in most cases will preserve, at least, what we call the *skeleton* of the original, in the case 'linear to linear' moreover its *dynamics* (i.e. behaviour under cut elimination).

FROM FORMULAS TO LINEAR FORMULAS

We define (chapter 2) *modal translations* of formulas into linear formulas, which are obtained by replacing each connective by one of its linear analogues, and prefixing each subformula by a *modality*, i.e. a (possibly empty) string of linear exponentials. Girard's original embedding of intuitionistic into linear logic is an example.

When turning our attention to *classical* logic, we find there exists no *unique* minimal choice for a modal translation. Instead we are confronted with *two* embeddings (Q and T), corresponding to distinct, one might say *dual*, linear decompositions of classical implication.

FROM CLASSICAL AND INTUITIONISTIC PROOFS TO LINEAR PROOFS

Though the minimal modal embeddings of chapter 2 in general do not automatically extend to derivations, there are (less economical) modal translations whose inductive application to sequent proofs defines a structure-preserving mapping of intuitionistic and classical to linear derivations. The existence of such *inductive decoration strategies* (chapter 3) provides a positive answer to the first part of our original question. Moreover we show that in the classical case there exist essentially *two* distinct modal translations whose inductive application to a proof preserves the skeleton. We call them q (which is closely related to the embedding Q of chapter 2) and t (related to the T-embedding). For certain fragments, most notably the one containing the rules for implication, universal first order quantification and universal second order (propositional) quantification, this 'linear decorating' of a classical proof is *deterministic*, and unambiguously defines a procedure for eliminating its cuts as a 'reflection' of the procedure for its linear image. It is e.g. immediate from the strong normalization theorem for linear logic that these reductions for (the given fragment of) classical sequent calculus enjoy strong normalization.

As it is essentially the *absence* of exponentials that provides us with information on the dynamics of a proof (section 4.4), we would like also to find *optimal* decorations, which, intuitively, should be obtainable by tracing the effects of occurrences of structural rules throughout a given proof. This is carried out in detail for intuitionistic implicational logic in chapter 4. In the case of derivations in classical sequent calculus, though, such a procedure cannot be defined unequivocally, and optimality results can only be relative to prior choices, e.g. by optimizing the results of the inductive application of the modal translations of chapter 3.

FROM CLASSICAL PROOFS TO CLASSICAL PROOFS, AND
FROM INTUITIONISTIC PROOFS TO INTUITIONISTIC PROOFS

The economic, non-decorating, embeddings of chapter 2 suggest restrictions on the form of rules, both in intuitionistic and classical sequent calculus. These restrictions can be built into alternative formulations of these calculi, that remain complete for provability, and for which the economic embeddings become decorating. We thus obtain the calculus **ILU** for intuitionistic logic, corresponding to Girard's embedding of intuitionistic into linear logic, and the calculi **LKT** and **LKQ** for classical logic, corresponding to the T-, resp. the Q-embedding of classical into linear logic. We characterize these calculi (chapter 3) as proper fragments of linear logic: they inherit linear logic's computational properties.

When inductively applying the economic embeddings to **IL**-, resp. **CL**-derivations, at several points we have to apply cuts, in order for the conclusion to remain within the scope of the embedding. We refer to these cuts as *correction cuts.* They are of a specific form: in all cases one cuts with the linear decoration of a derivation of an identity $A \Rightarrow A$. We show (chapter 6) that elimination of these cuts realizes the restrictions imposed by the embeddings on the form of the rules, whence we refer to them as *'constrictive morphisms'*: after elimination of the correction cuts from the derivation obtained by inductively applying Girard's translation to an **IL**-derivation, the skeleton of the result is an **ILU**-proof. Similarly when applying the Q- or T-translation to a **CL**-derivation π, elimination of the corresponding correction cuts transforms π into an **LKQ**-, resp. **LKT**-proof.

FROM LINEAR PROOFS TO LINEAR PROOFS

In chapter 5 we construct the *exponential graph* of a linear proof π, an artefact that displays the interdependencies of exponentials. Using this graph we characterize exponentials that are superfluous. Removing ('stripping') them determines a lattice of linear derivations with top π and as bottom a unique normal form $\pi^{\triangleright}$, having essentially the same set of reductions. When applying this removal to the linear derivation obtained by inductively applying a decorating modal translation to an *intuitionistic* derivation π, the result obtained is essentially equivalent to the linear derivation $\partial(\pi)$ constructed in chapter 4 by tracing the effects of occurrences of structural rules throughout π, whence $\partial(\pi)$ is shown to be the optimal linear decoration of π.

We similarly may apply stripping to the result of the inductive application of decorating modal translations to *classical* derivations, in which case we obtain optimal decorations relative to the chosen initial decoration (i.e. relative to the choice of a normalization protocol for the original proof).

'Stripping' in fact is a basic example of *dilatation* (chapter 8) of linear derivations, i.e. the replacement of exponentiated formulas by non-exponentiated ones in such a way that the reductions of the original can be simulated by reductions of the image. We show that a (fully expanded) linear derivation π is, in some strong sense, dilatable if and only if its exponential graph is acyclic.

1

Linear logic

Almost sixty years ago, in 1935, Gerhard Gentzen introduced the calculus of sequents for classical logic and, as a corollary to his celebrated *cut elimination theorem* (often referred to as 'the Hauptsatz'), showed that the Hilbertean ideal of 'purity of methods', because of Gödel's incompleteness theorem known to be in general unattainable in formalized mathematics, can be fulfilled for theorems of the predicate calculus: if ϕ is a valid first order formula, then ϕ is derivable in the sequent calculus using formulas occurring in the set of ϕ's *sub*formulas only. This is known as the 'subformula property' for first order logic.

The inference rules of the sequent calculus on the one hand correspond very closely to the semantical definition of classical truth[1], but also have a strong *operational* quality: they 'explain' the logical connectives by showing us *how* they are used. However, though the semantical interpretation of the connectives is quite insensitive to the precise formulation of the rules, this can hardly be said of the operational interpretation (which, in a way, is nothing *but* precisely this formulation!).

Gentzen originally conceived of a sequent $\Gamma \Rightarrow \Delta$ as consisting in finite *lists* of formulas $\Gamma \equiv G_1, \ldots, G_n$ and $\Delta \equiv D_1, \ldots, D_m$, and corresponding to the formula $(G_1 \wedge \ldots \wedge G_n) \rightarrow (D_1 \vee \ldots \vee D_m)$. Given the logical equivalence with $(G_{\sigma(1)} \wedge \ldots \wedge G_{\sigma(n)}) \rightarrow (D_{\tau(1)} \vee \ldots \vee D_{\tau(m)})$ for permutations σ of $\{1, \ldots, n\}$ and τ of $\{1, \ldots, m\}$, other authors

[1] This is particularly clear in the formulation of sequent derivations as 'semantical tableaux', due to Beth(1955). It was observed by Prawitz(1975) that in a certain sense Gentzen's sequent calculus can be seen as *the* natural system for generating logical truths.

interpreted Γ, Δ as *multisets* (i.e. lists modulo the order of the entries), thus rendering obsolete the structural rule of *exchange*. From there we might go even further and use the equivalence of a formula with both its n-fold conjunction and disjunction, for any $n \geq 1$, to interpret Γ and Δ as *sets* of formulas, thus rendering the notion of *contraction* superfluous.

For each of these interpretations of Γ, Δ one constructs Gentzen calculi that are complete with respect to provability and enjoy cut elimination. The specific form of the inference rules, however, will differ. Historically, they come in two main guises: the additive, and the multiplicative one (see appendix c). Though both can be made sense of in either interpretation, the additive versions of the 'two-premiss' logical rules (i.e. L$\rightarrow$, L$\vee$ and R$\wedge$) are the obvious ones to use when thinking of the contexts as *sets* while their multiplicative variants appear more naturally when thinking of lists or *multisets*. In the multiset-interpretation, the structural rules of weakening and contraction enable us to switch back and forth between the two.

The first step towards linear logic is discarding the use of these structural rules from sequent derivations in which the contexts are interpreted as either multisets in the commutative, or lists in the non-commutative case.[2] Consequently it will no longer be possible to derive the additive rules from the multiplicative ones (as we lack weakening), nor vice versa to get the multiplicative rules from the additive ones (for want of contraction), and we are confronted with genuine alternatives. One might like to choose the one, or the other. Or, as does linear logic, one might opt for both. However, in this last case, in order to keep eliminability of cut, it is necessary to clearly distinguish the additive from the multiplicative occurrences (see e.g. Schellinx(1991)). It is thus that we obtain the 'linear' *splitting* of the classical connectives and constants, which is summarized in figure 1.1. The provability relations between the linear constants are indicated in figure 1.2.

The multiplicative conjunction is called 'tensor' or 'times', the additive one 'and'; the additive conjunction is pronounced 'plus', while $⅋$ is known as 'par'. The multiplicative implication is referred to as 'linear implication'. (The additive implication usually is disregarded, as it

[2]We will consider here only *commutative* linear logic.

$$\left.\begin{array}{l}\text{multiplicative}\\ \text{additive}\end{array}\right\} \to \left\{\begin{array}{c}\multimap\\ \rightsquigarrow\end{array}\right. \quad \wedge\left\{\begin{array}{c}\otimes\\ \&\end{array}\right. \quad \vee\left\{\begin{array}{c}\parr\\ \oplus\end{array}\right. \quad \top\left\{\begin{array}{c}\mathbf{1}\\ \top\end{array}\right. \quad \bot\left\{\begin{array}{c}\bot\\ \mathbf{0}\end{array}\right.$$

Figure 1.1: The splitting of connectives and constants

lacks some of the very basic properties one would like logical 'arrows' to have (note that $A \rightsquigarrow A$ is not derivable). Therefore it goes without a name.)

The identity axiom and the cut rule together with the axioms and rules for the above constants and connectives form the propositional part of linear logic, also known as **MALL** (for 'Multiplicative Additive Linear Logic').

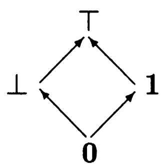


Figure 1.2: The linear constants

What is usually called just plainly *linear logic* arises from a reintroduction of structural rules, be it for a set of marked formulas: structural manipulation to the left of the entailment sign is allowed only for those formulas that start with the symbol "!", which in the literature appears under the names 'storage', 'of course', 'bang', 'shriek' as well as the more prosaic 'exclamation mark', while structural manipulation to the right is limited to formulas starting with a "?", referred to as 'costorage', 'why not' or simply 'question mark'. Our construction ends by adding the rules L?, R?, L! and R!, in order to allow the introduction of (the *exponentials*) "!" and "?" (other than by means of weakening or an instance of an axiom) in the course of a derivation.

Extending the system thus obtained with (the usual) rules for the first order quantifiers gives 'first order Classical Linear Logic' (**CLL**, appendix b; a sequent $\Gamma \Rightarrow \Delta$ in this calculus consists in finite multisets $\Gamma \equiv G_1, \ldots, G_n$ and $\Delta \equiv D_1, \ldots, D_m$, and corresponds to the formula

$(G_1 \otimes \ldots \otimes G_n) \multimap (D_1 \parr \ldots \parr D_m))$.

Via the involutive *linear negation*, $(\cdot)^\perp$, which is defined by $A^\perp := A \multimap \perp$ (or, equivalently, $A \leadsto \mathbf{0}$; but *not* $A \multimap \mathbf{0}$, which acts like an intuitionistic linear negation), linear logic exhibits wonderful symmetries, witnessed e.g. by the 'De Morgan'-dualities, and the fact that the additive connectives and constants are definable in terms of $\leadsto$ and $\mathbf{0}$, the multiplicatives in terms of $\multimap$ and $\perp$, just like their non-linear peers:

$$\begin{array}{rcl} A \parr B & := & (A \multimap \perp) \multimap B \\ A \otimes B & := & (A \multimap (B \multimap \perp)) \multimap \perp \\ \mathbf{1} & := & \perp \multimap \perp \\ A \oplus B & := & (A \leadsto \mathbf{0}) \leadsto B \\ A \& B & := & (A \leadsto (B \leadsto \mathbf{0})) \leadsto \mathbf{0} \\ \top & := & \mathbf{0} \leadsto \mathbf{0} \end{array}$$

The multiplicatives constitute the core of linear sequent calculus (recall that the entailment sign naturally corresponds to linear implication, the comma on the left to multiplicative conjunction, that on the right to multiplicative disjunction). The additives are linked to this multiplicative core by the exponentials, the leading characters in what follows:[3]

$$\begin{array}{rcl} !A \otimes !B & \Longleftrightarrow & !(A \,\&\, B) \\ ?A \parr ?B & \Longleftrightarrow & ?(A \oplus B) \\ !A \multimap ?B & \Longleftrightarrow & ?(A \leadsto B) \end{array}$$

A (possibly empty) sequence of exponentials is called a *modality*. Observe that for all modalities μ both $!A \Rightarrow \mu A$ and $\mu A \Rightarrow ?A$ are derivable: starting from an axiom $A \Rightarrow A$ and an application of !L, we can without restriction apply R? and R! to obtain μA in the first, and by the unrestricted possibility of using L! and L? starting from an

[3]Note the analogy with $2^a \cdot 2^b = 2^{a+b}$, which accounts for the name 'exponentials'.

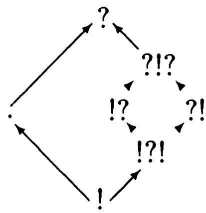


Figure 1.3: The lattice of linear modalities

axiom followed by an application of R? in the second case. Otherwise said, in the partial ordering on modalities induced by linear derivability ($\mu \preceq \nu$ iff **CLL** $\vdash \mu A \Rightarrow \nu A$ for any A), "!" is minimal, "?" is maximal.

If we consider the equivalence relation induced by this ordering, we find *seven* equivalence classes: calling $\cdot, ?, !, !?, ?!, !?!$ and $?!?$ (where "$\cdot$" stands for the void modality) *basic* modalities, one easily shows (e.g. using the *idempotency* of these basic modalities) that for any modality μ there is a *unique* basic μ_0 such that **CLL** $\vdash \mu A \Longleftrightarrow \mu_0 A$ for all A. So, modulo provable linear equivalence, there are precisely seven modalities in linear logic. Basic modalities are related as in figure 1.3, where an arrow from μ to ν indicates that $\mu \prec \nu$ (see Joinet(1993)).

As an easy corollary we then find that *all* modalities are idempotent: **CLL** $\vdash \mu A \Longleftrightarrow \mu\mu A$ for all μ, A.

1 Theme

The following is a crucial technical result.[4]

1.1. THEOREM. (Cut elimination) *A sequent* $\Gamma \Rightarrow \Delta$ *is derivable in* **CLL** *if and only if it is derivable without the use of cut.* ⊠

Stating it this way, however, is but telling the story less than half. Elimination of cuts from proofs means explicitating their content by

[4]Cf. the notes at the end of this chapter, page 27.

exhibiting normal forms. In this sense a cut-elimination *procedure* corresponds to a computation mechanism. It is the procedure which, potentially, provides a computational interpretation of a sequent calculus.

Thus Gentzen's Hauptsatz and its proof tell us that in theory we can eliminate the cuts from **CL**-proofs. In practice, however, it seems that we do not really know *how*. The 'classical' examples are proofs of a sequent $\Gamma, \Gamma' \Rightarrow \Delta, \Delta'$ obtained by a cut between proofs of sequents $\Gamma \Rightarrow A, \Delta$ and $\Gamma', A \Rightarrow \Delta'$, which both end with an application of a structural rule, e.g. as follows.

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Gamma \Rightarrow A, A, \Delta\end{matrix}}{\Gamma \Rightarrow A, \Delta} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\\ \Gamma', A, A \Rightarrow \Delta'\end{matrix}}{\Gamma', A \Rightarrow \Delta'}}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

In order to eliminate the cut, we have to duplicate either π_1 or π_2, but clearly need a coin or a similar oracle to tell us *which*.[5] Deciding for the one or the other will in general have non-trivial consequences, as it is bound to lead us to unreconciliably distinct normal forms. Moreover, as was observed by Lafont (see e.g. Girard et al.(1988)), if we consider a cut between derivations of $\Gamma \Rightarrow A, \Delta$ and $\Gamma, A \Rightarrow \Delta$ obtained by weakenening from *distinct* proofs of $\Gamma \Rightarrow \Delta$, it is immediate that an equivalence $\sim$ between proofs satisfying that $\pi \sim \pi'$ if π normalizes to π' is doomed to be degenerated, in the sense that we are forced to declare *all* proofs of a sequent $\Gamma \Rightarrow \Delta$ equivalent. This in turn prohibits any reasonable semantics of *proofs* (cf. Girard(1991)).

The example suggests that we should blame the structural rules. And indeed, this is confirmed by the intuitionistic calculus **IL**, where the use of structural rules is restricted to one, the left hand, side of sequents. Also in linear logic the problem is resolved through the asymmetric handling of structural rules: there simply can not be a cut between a derivation with conclusion $\Gamma \Rightarrow ?A, \Delta$ and one with conclusion

[5]The problem in fact is not due to a specific formulation of the calculus. It can not be resolved by, for example, chosing an additive rather than a multiplicative formulation of rules and treating contraction *implicitly*.

$\Gamma', !A \Rightarrow \Delta'$. In both cases it is the existence of a non trivial semantics of proofs (a *denotational* semantics) and the semantical soundness of the cut elimination procedure (i.e. if π reduces to π', then the interpretation of π is equal to that of π') which guarantees that remaining traces of non-determinism can be deemed more or less innocent.

Girard(1987a) introduces the systems **PN1**, of *proofnets* for propositional/exponential linear logic, and **PN2**, of proofnets for propositional/exponential linear logic extended with second order propositional quantification, and proves strong normalization as well as semantical soundness with respect to the interpretation in coherence spaces. A purely combinatorial proof of strong normalization for the multiplicative/exponential fragment of **PN1** can be found in Joinet(1993). For that fragment, as well as for its second-order extension, moreover the Church-Rosser property holds, cf. Danos(1990). Proofnets abstract from inessential distinctions due to the sequentiality of sequent derivations, and it is there that cut elimination in linear logic finds its most powerful expression.

So apparently we found a formal system that combines the (constructive) characteristics typical to intuitionistic logic (including a semantics of proofs) with the symmetries (including an involutive negation) of classical logic.

But what does it *mean*?

Well, linear implication $\multimap$ can be seen as representing a causal form of entailment. Under this reading $a \multimap b$ will represent an *action* that uses an object of type a in order to obtain an object of type b. Pushing this idea somewhat further, one might characterize the multiplicative fragment as '*a calculus of actions closed under composition*', and interpret the *additive* rules in terms of '*sharing of resources (premisses)*'. We'll be the first, though, to agree with all those objecting that this 'interpretation' is far from convincing. At the moment we will and can not do much better.

Linear logic is very much a 'proof theorist's' logic, and the appropriate answer when asked for the meaning of an 'expression' seems to be: exhibit its proof(s). And these *do* have a rather obvious interpretation.

If we *collapse* a linear derivation π, i.e. replace all symbols $\otimes, \&$ by $\wedge$, all symbols $⅋, \oplus$ by $\vee$, and each of $\rightsquigarrow, \multimap$ by $\rightarrow$, and moreover erase the modalities, delete resulting repetitions of sequents, then what we get is a derivation in classical, maybe even intuitionistic, logic, which we will refer to as π's *skeleton* $\mathsf{sk}(\pi)$.

Notwithstanding its triviality, the above observation is essential. It suggests the use of linear logic as a proof theoretical tool, as we may interpret derivations in linear sequent calculus as *annotated* classical or intuitionistic proofs.

Moreover, it has a converse.

1.2. THEOREM. *The skeleton* $\mathsf{sk}(\pi)$ *of a linear derivation* π *is a derivation in classical or intuitionistic sequent calculus; conversely each classical or intuitionistic sequent derivation occurs as the skeleton of a linear derivation.*

PROOF: The first part of the claim is obvious; the second part is an immediate corollary to the existence of skeleton-preserving translations of both classical and intuitionistic sequent calculus *proofs* into linear logic. These will be the subject of chapters 3 and 4. ⊠

Observe that this is a strong argument in favour of linear logic's claim to being a *refinement* of intuitionistic and classical logic. We may for example use it to obtain cut elimination for both classical and intuitionistic sequent calculus as a corollary to the linear cut elimination theorem.

1.3. THEOREM. *A linear derivation fixes a normalization protocol for its skeleton.*

PROOF: Let π be an intuitionistic or classical derivation and suppose τ is a linear derivation such that $\mathsf{sk}(\tau) = \pi$. Let $\overline{\mu}$ be a terminating reduction sequence leading from τ to a cut free proof τ'. The reflection of $\overline{\mu}$, obtained using sk, determines a reduction sequence leading from π to a cut free derivation π', see figure 1.4. ⊠

As the elimination procedure for (multiplicative) **CLL**, contrary to that for **CL**, is essentially deterministic, this means that a linear derivation π will force a *choice* among several possible **CL**-normalizations of its skeleton. Of course, given some classical or intuitionistic derivation π, the set of linear derivations τ such that $\mathsf{sk}(\tau) = \pi$ will always be infinite, and distinct elements of this set in general will correspond to

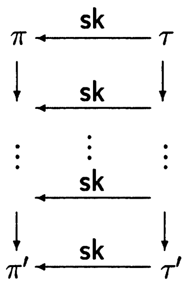


Figure 1.4: Normalization of π

different ways to normalize π. We will encounter a concrete example of this phenomenon later on, in section 4.4.

In view of the above, one might suspect that conversely it will be possible to obtain *any* possible normalization sequence for a classical derivation by a suitable choice of a linear derivation τ such that $\mathsf{sk}(\tau) = \pi$. We refer to this as *Joinet's conjecture.* Note that a proof of this claim would depend crucially on what one takes to be the collection of *all* normalization sequences starting from a given CL-derivation. Experience shows that indeed it does hold for large collections of such normalizations. Reductions that make use of Gentzen's highly symmetric 'cross-cuts' procedure, however, appear to be extremely resistant to simulation by means of linear decorations.

2 Weakening and contraction

Due to the absence of the structural rules of weakening and contraction in linear logic we can, given a derivable sequent $\Gamma \Rightarrow \Delta$ and a linear formula A, in general not conclude that either of the sequents $\Gamma, A \Rightarrow \Delta$ or $\Gamma \Rightarrow A, \Delta$ is derivable; similarly, from derivability of a sequent $\Gamma, A, A \Rightarrow \Delta$ or $\Gamma \Rightarrow A, A, \Delta$ we can conclude neither the derivability of $\Gamma, A \Rightarrow \Delta$, nor that of $\Gamma \Rightarrow A, \Delta$.

2.1. DEFINITION. We introduce the following sets of linear formulas:

$$\begin{array}{rcl} \mathcal{P} & := & \{A \mid \vdash A \Longleftrightarrow !A\} \\ \mathcal{N} & := & \{A \mid \vdash A \Longleftrightarrow ?A\} \\ \mathcal{W}_l & := & \{A \mid \forall\Gamma,\Delta : \text{if } \vdash \Gamma \Rightarrow \Delta \text{ then } \vdash \Gamma, A \Rightarrow \Delta\} \\ \mathcal{W}_r & := & \{A \mid \forall\Gamma,\Delta : \text{if } \vdash \Gamma \Rightarrow \Delta \text{ then } \vdash \Gamma \Rightarrow A, \Delta\} \\ \mathcal{C}_l & := & \{A \mid \forall\Gamma,\Delta : \text{if } \vdash \Gamma, A, A \Rightarrow \Delta \text{ then } \vdash \Gamma, A \Rightarrow \Delta\} \\ \mathcal{C}_r & := & \{A \mid \forall\Gamma,\Delta : \text{if } \vdash \Gamma \Rightarrow A, A, \Delta \text{ then } \vdash \Gamma \Rightarrow A, \Delta\}. \end{array} \quad \boxtimes$$

(Derivability here is understood to mean derivability in **CLL**. If we want to consider any of the above sets relative to a different fragment of linear logic, we indicate this by means of a superscript. E.g., $\mathcal{W}_l^{\mathbf{ILL}} := \{A \mid \forall\Gamma,\Delta : \text{if } \mathbf{ILL} \vdash \Gamma \Rightarrow \Delta \text{ then } \mathbf{ILL} \vdash \Gamma, A \Rightarrow \Delta\}$.)

$\mathcal{P}$, the set of formulas of *positive polarity*, contains $\mathbf{0}, \mathbf{1}$ and is closed under $!, \otimes, \oplus, \exists$. Dually, $\mathcal{N}$, the set of formulas of *negative polarity*, contains $\bot, \top$ and is closed under $?, \invamp, \&, \forall$.[6] For any pair of formulas $P \in \mathcal{P}, N \in \mathcal{N}$ we have that $\nvdash N \Rightarrow P$, as a cut free derivation of a sequent of the form $?X \Rightarrow !Y$ is impossible. It follows that $\mathcal{P} \cap \mathcal{N} = \emptyset$.

Note that $\mathcal{P} \subseteq \mathcal{W}_l \cap \mathcal{C}_l$, and $\mathcal{N} \subseteq \mathcal{W}_r \cap \mathcal{C}_r$: both weakening and contraction on the left are admissible for positive formulas, while structural manipulations on the right are admissible for negative formulas. It also is straightforward to show the following.

2.2. LEMMA. *Let A be a linear formula. Then:*

$$\begin{array}{rcl} A \in \mathcal{W}_l & \text{iff} & \vdash A \Rightarrow \mathbf{1} \\ A \in \mathcal{W}_r & \text{iff} & \vdash \bot \Rightarrow A \\ A \in \mathcal{C}_l & \text{iff} & \vdash A \Rightarrow A \otimes A \\ A \in \mathcal{C}_r & \text{iff} & \vdash A \invamp A \Rightarrow A. \end{array} \quad \boxtimes$$

Thus we can characterize linear formulas having full structural permission to the left or to the right.

---

[6]The terminology (*positive, negative (polarity)*) is as in Girard(1993), and is not to be confused with the standard notion of positive, negative occurrence of (sub)formula.

2.3. PROPOSITION. *For all linear formulas A:*

$$A \in \mathcal{W}_l \cap \mathcal{C}_l \quad \text{iff} \quad \vdash A \Longleftrightarrow \mathbf{1} \,\&\, (A \otimes A)$$
$$A \in \mathcal{W}_r \cap \mathcal{C}_r \quad \text{iff} \quad \vdash A \Longleftrightarrow \bot \oplus (A \parr A). \qquad \boxtimes$$

As $\nvdash \bot \Rightarrow \mathbf{1}$, we have $\mathcal{W}_l \cap \mathcal{W}_r = \emptyset$. Note that the intersection of $\mathcal{C}_r$ and $\mathcal{C}_l$, however, is non-empty. In fact $\mathcal{C}_r \cap \mathcal{C}_l$ is precisely the union of the anti-theorems in $\mathcal{C}_l$ and the theorems in $\mathcal{C}_r$. So for example $(p \multimap p) \parr ?(p \multimap p) \in \mathcal{C}_r \cap \mathcal{C}_l$.

$\mathcal{W}_l \cap \mathcal{C}_l$, like $\mathcal{P}$, is closed under $\otimes, \oplus, \exists$, and $\mathcal{W}_r \cap \mathcal{C}_r$, like $\mathcal{N}$, is closed under $\parr, \&, \forall$. Using the following lemma (and its obvious dual) the reader will have no difficulties in showing that in fact all formulas in $\mathcal{W}_l \cap \mathcal{C}_l$ are linearly equivalent to a formula of positive polarity or to a formula with main connective $\otimes, \oplus$ or $\exists$, and all formulas in $\mathcal{W}_r \cap \mathcal{C}_r$ to a formula of negative polarity or to a formula with main connective $\parr, \&$ or $\forall$.

2.4. LEMMA. *Let $\circ \in \{\&, \parr\}$ and suppose $\nvdash A_1 \circ A_2, \nvdash \forall x\, A$. Then*
(i) $A_1 \circ A_2 \in \mathcal{W}_l \cap \mathcal{C}_l$ iff $\exists i(A_i \in \mathcal{W}_l \cap \mathcal{C}_l$ and $\vdash A_i \Longleftrightarrow A_1 \circ A_2)$
(ii) $\forall x\, A \in \mathcal{W}_l \cap \mathcal{C}_l$ iff $\exists t(A[t/x] \in \mathcal{W}_l \cap \mathcal{C}_l$ and $\vdash A[t/x] \Longleftrightarrow \forall x\, A)$. $\boxtimes$

Let us call a linear formula *modality-free* if it does not contain any occurrence of the symbols !, ?. The next proposition characterizes the *modality-free* formulas in $\mathcal{W}_l \cap \mathcal{C}_l$ and $\mathcal{W}_r \cap \mathcal{C}_r$.

2.5. PROPOSITION. *Suppose A is modality-free. Then $A \in \mathcal{W}_l \cap \mathcal{C}_l$ iff $\vdash A \Longleftrightarrow \mathbf{0}$ or $\vdash A \Longleftrightarrow \mathbf{1}$. Dually, $A \in \mathcal{W}_r \cap \mathcal{C}_r$ iff $\vdash A \Longleftrightarrow \bot$ or $\vdash A \Longleftrightarrow \top$.*

PROOF: We will only prove the first claim, as the second one follows by duality. The right-to-left direction is clear. So let A be a modality-free linear first order formula, and suppose $A \in \mathcal{W}_l \cap \mathcal{C}_l$. We define A's complexity, $\|A\|$, inductively: put $\|A\| = 1$ for atomic A, $\|A \circ B\| = \|A\| + \|B\| + 1$ for $\circ \in \{\multimap, \otimes, \parr, \&, \oplus\}$, and $\|\mathcal{Q}x.A\| = \|A\| + 1$ for $\mathcal{Q} \in \{\forall, \exists\}$. Extend $\| \cdot \|$ to finite multisets in the obvious way, i.e. $\|\Gamma\| = \sum_{G \in \Gamma} \|G\|$.

Let $A \in \mathcal{W}_l \cap \mathcal{C}_l$. Then $\vdash A \Rightarrow A \otimes A$ and $A \Rightarrow \mathbf{1}$; hence, if $\vdash A$, then $\vdash A \iff \mathbf{1}$. So let's moreover suppose that $\not\vdash A$. We show by induction on $\|\Gamma \cup \Delta\|$ that whenever $\vdash \Gamma \Rightarrow \Delta, A \otimes A$, also $\vdash \Gamma \Rightarrow \Delta, \mathbf{0}$. It follows that in particular $\vdash A \Rightarrow \mathbf{0}$, and we are done.

Say we proved the claim for all Γ, Δ with $\|\Gamma \cup \Delta\| < n$. Let us then show it holds as well for derivable sequents $\Gamma \Rightarrow \Delta, A \otimes A$ with $\|\Gamma \cup \Delta\| = n$, by considering the possible cut free derivations in the non-exponential fragment of **CLL**:

- if the derivation is an axiom, observe that w.l.o.g. we may assume that our derivations have only atomic versions of the identity axioms. Therefore our derivation is an instance of R$\top$ or L$\mathbf{0}$. But then so is $\Gamma \Rightarrow \Delta, \mathbf{0}$;

- in all cases where the derivation ends with a rule in which $A \otimes A$ is not the main formula, we can apply the induction hypothesis to the premiss(es);

- in case $A \otimes A$ is main formula, we have in the premisses derivable sequents $\Gamma_i \Rightarrow \Delta_i, A$ with $\|\Gamma_i \cup \Delta_i\| < n (i = 1, 2)$. (Note that here we use that A is not a theorem.) As $\vdash A \Rightarrow A \otimes A$ we get $\vdash \Gamma_i \Rightarrow \Delta_i, A \otimes A$ (by an application of cut). We apply the induction hypothesis to these derivable sequents and finish by a cut with $\mathbf{0} \otimes \mathbf{0} \Rightarrow \mathbf{0}$. ⊠

Proposition 2.5 tells us that indeed the banning of 'full structural permissions' (weakening *plus* contraction) in the non-exponential fragment of **CLL** has been successful: both left- (resp. right-)weakening and contraction are admissible for a modality-free formula if and only if that formula is linearly equivalent to either the constant $\mathbf{0}$ (resp. $\perp$) or the constant $\mathbf{1}$ (resp. $\top$). (Observe that on the other hand left-weakening is admissible for all formulas of the form $\mathbf{1} \,\&\, A$, left-contraction is admissible for all linear theorems, etcetera.)

In view of the above, and the fact that in **CLL** full structural permission is restricted to modalized formulas, it is tempting to claim that $\mathcal{P} = \mathcal{W}_l \cap \mathcal{C}_l$, and, dually, $\mathcal{N} = \mathcal{W}_r \cap \mathcal{C}_r$. Though it is hard to imagine the shape of a possible counter-example, proper evidence for this conjecture is lacking.

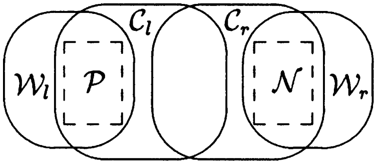


Figure 1.5: Section 2, summary

3 Deconstructing intuitionistic logic

Linear logic found its origin in a *semantical* decomposition of *intuitionistic* connectives, or, to be precise, in a semantical decomposition of *type constructors* corresponding (in the sense of the 'Curry-Howard-de Bruijn isomorphism' (Howard(1980))) to these connectives: the constructions of *sum-* and *function-*types appear as *compound* operations. E.g. in (the category $\mathcal{COH}$ of) coherence spaces the interpretation of the sum-type $A \vee B$ is of the form $!A^* \oplus !B^*$, where $\oplus$ is the direct sum, and $! : \mathcal{COH} \to \mathcal{COH}$ maps a coherence space X^* to a coherence space $!X^*$ with $|\,!X^*| = \{a \in X^* \mid a \text{ finite}\}$. Similarly $(A \to B)^*$ becomes $!A^* \multimap B^*$ (see Girard et al.(1988) for more details).

So, historically, linear logic emerged as a refinement of *intuitionistic* logic. This origin survives in what is known as *intuitionistic* linear logic or **ILL**. In retrospect, we obtain **ILL** by imposing the 'usual' intuitionistic restriction on **CLL** (let's say minus the rules for our weird additive arrow $\leadsto$): we allow as succedents only multisets that contain precisely one formula.

Doing so, the right rules for *par* and the exponential "?" are no longer applicable. Keeping only a *left* rule for '$\parr$' obviously is not very interesting. Also, we would no longer be able to expand identity axioms of the form $A \parr B \Rightarrow A \parr B$. As to the exponential '?', this would never be involved in a structural rule: we took away its 'raison d'être', whence it becomes *superfluous* (cf. chapter 5).

Therefore we 'loose' this connective and exponential. Moreover, we drop the neutral element for *par*, the constant $\bot$. Consequently **ILL** is precisely the fragment containing the identity axiom, cut, and axioms, rules for $\top, \mathbf{0}, \multimap, \otimes, \&, \oplus, \forall, \exists, !$, all restricted to singleton succedents.

Because of (essentially) the properties of the constant $\mathbf{0}$, provability in **ILL** is a more limited notion than provability in the corresponding fragment of **CLL**.

3.1. PROPOSITION. *Fragments of* **CLL** *in the language of* **ILL** *are conservative over* **ILL** *if and only if they do not include the constant* $\mathbf{0}$, *or do not include linear implication* $\multimap$.

PROOF. From right to left, let us suppose the fragment does not include the constant $\mathbf{0}$. Let π be a cut free derivation of $\Gamma \Rightarrow A$. If there is in π a sequent with multiple succedents then there is an instance of $\mathrm{L}\multimap$ in which the right premiss has an empty succedent set. We can then follow upwards a branch in the proof tree consisting solely of sequents with empty succedent set. Such a branch has to end in an instance of an axiom, but that is impossible in a fragment without $\mathbf{0}$.

Suppose the fragment does not include linear implication $\multimap$, and again let π be a cut free derivation of $\Gamma \Rightarrow A$. It is now straightforward by induction on the length of cut free derivations that *all* sequents in π have precisely one succedent. Therefore, in both cases, π is in fact an **ILL**-derivation.

From left to right, consider the following derivation in $\{\mathbf{0}, \multimap\}$:

$$\dfrac{\dfrac{C \Rightarrow C \quad \dfrac{\dfrac{\mathbf{0} \Rightarrow D, B}{\Rightarrow \mathbf{0} \multimap D, B} \quad A \Rightarrow A}{(\mathbf{0} \multimap D) \multimap A \Rightarrow B, A}}{\dfrac{C, C \multimap ((\mathbf{0} \multimap D) \multimap A) \Rightarrow B, A}{C \multimap ((\mathbf{0} \multimap D) \multimap A) \Rightarrow C \multimap B, A}} \quad \mathbf{0} \Rightarrow}{C \multimap ((\mathbf{0} \multimap D) \multimap A), (C \multimap B) \multimap \mathbf{0} \Rightarrow A}$$

One easily checks that the final sequent is not cut free derivable in **ILL**. Therefore it is not derivable in **ILL**. ⊠

The semantical decomposition of the intuitionistic type constructors in coherence spaces in turn gives rise to the following embedding of intuitionistic into linear logic:

3.2. DEFINITION. (*Girard's translation*) Define a mapping $(\cdot)^*$ of formulas to *linear* formulas as follows:

for atomic p let $p^* := p$; then put

$$\begin{array}{rcl} \bot^* & := & \mathbf{0} \\ (A \wedge B)^* & := & A^* \& B^* \\ (A \vee B)^* & := & !A^* \oplus !B^* \\ (A \to B)^* & := & !A^* \multimap B^* \\ (\forall x A)^* & := & \forall x A^* \\ (\exists x A)^* & := & \exists x !A^*. \end{array} \qquad \boxtimes$$

We will refer to the fragment of intuitionistic linear logic corresponding to this embedding (i.e. the fragment $\{\mathbf{0}, \&, \oplus, \multimap, \forall, \exists, !\}$) as *deconstructed intuitionistic logic* (**DIL**).

3.3. PROPOSITION. **IL** $\vdash \Gamma \Rightarrow A$ *if and only if* **DIL** $\vdash !\Gamma^* \Rightarrow A^*$.

PROOF: The left-to-right direction (*correctness*) is obtained by induction on the length of derivations in the sequent calculus for **IL**, where we take (as is suggested by the choice of $\&$ and $\oplus$ in the definition of the embedding) the *additive* formulation of the rules for $\wedge, \vee$. In the proof one uses derivability of

$$\begin{array}{rcl} !(A \& B) & \Rightarrow & !A \& !B \\ !(!A \multimap B) & \Rightarrow & !A \multimap !B \\ !\forall x A & \Rightarrow & \forall x !A. \end{array}$$

The right-to-left direction (*faithfulness*) is obtained, simply by observing that the skeleton of a **DIL**-derivation of $!\Gamma^* \Rightarrow A^*$ is an **IL**-derivation of $\Gamma \Rightarrow A$. $\boxtimes$

DIL has the following property.

3.4. PROPOSITION. $\mathcal{P}^{\mathbf{DIL}} = \mathcal{W}_l^{\mathbf{DIL}} \cap \mathcal{C}_l^{\mathbf{DIL}}$.

PROOF: First observe that $C \in \mathcal{W}_l^{\mathbf{ILL}} \cap \mathcal{C}_l^{\mathbf{ILL}}$ if and only if **ILL** $\vdash C \Longleftrightarrow \mathbf{1} \& (C \otimes C)$ if and only if **ILL** $\vdash C \Rightarrow !(D \multimap D) \& (C \multimap (C \multimap D)) \multimap D$, for all **ILL**-formulas D.

Let C be an **ILL**-formula, B a **DIL**-formula (i.e. B is tensor-free), such that **ILL** $\not\vdash C$, **ILL** $\vdash B \Rightarrow C \otimes C$, and moreover **ILL** $\vdash C, C \Rightarrow C$. One shows by a straightforward induction on the complexity of B that **ILL** $\vdash B \Rightarrow !C$.

Now take $A \in \mathcal{W}_l^{\mathbf{DIL}} \cap \mathcal{C}_l^{\mathbf{DIL}}$. If A is a theorem, then **DIL** $\vdash \mathbf{1} \Longleftrightarrow A$. So we suppose that **DIL** $\not\vdash A$. Because $A \in \mathcal{W}_l^{\mathbf{DIL}} \cap \mathcal{C}_l^{\mathbf{DIL}}$ we have that **DIL** $\vdash A \Rightarrow !(p \multimap p) \,\&\, (A \multimap (A \multimap p)) \multimap p$ for some atomic proposition p (not occurring in A). But then in **ILL** we can, for all D, derive $A \Rightarrow !(D \multimap D) \,\&\, (A \multimap (A \multimap D)) \multimap D$. So $A \in \mathcal{W}_l^{\mathbf{ILL}} \cap \mathcal{C}_l^{\mathbf{ILL}}$, and therefore, by the observation above, **ILL** $\vdash A \Rightarrow !A$. By the subformula property for **ILL**-derivations we find **DIL** $\vdash A \Rightarrow !A$. ⊠

In fact something even stronger holds: if $A \in \mathcal{C}_l^{\mathbf{DIL}}$ and A is not a **DIL**-theorem, then $A \in \mathcal{P}^{\mathbf{DIL}}$. Note that this does not hold in **ILL**: e.g. $p \otimes !p \in \mathcal{C}_l^{\mathbf{ILL}}$, and **ILL** $\not\vdash p \otimes !p$; but clearly $p \otimes !p \notin \mathcal{P}^{\mathbf{ILL}}$.

It might be useful to observe that Girard's translation provides a *minimal* modal translation[7] of intuitionistic logic into intuitionistic linear logic, in the sense that any mapping $(\cdot)^<$ obtained from it by deleting one or more of the exponentials occurring in definition 3.2, will no longer be correct: e.g., if we remove the exponential for the implication, we can no longer derive the translation of $p \to (p \to p)$. (It might amuse the reader to look for examples forcing the exponentials in the translation of disjunction and existential quantifier.) On the other hand, all mappings $(\cdot)^>$ obtained by *adding* exponentials, will be correct translations: again a sequent $\Gamma \Rightarrow A$ will be derivable in intuitionistic logic if and only if **DIL** proves $!\Gamma^> \Rightarrow A^>$ (one easily shows that for all mappings $(\cdot)^>$ and for all formulas A: **DIL** $\vdash !A^* \Longleftrightarrow !A^>$).

One final remark: Girard's embedding of intuitionistic logic maps the intuitionistic conjunction to the linear *'and'* ($\&$). It therefore fits most naturally the formulation of **IL** having additively formulated rules for conjunction. However, it is easy to define a translation that maps intuitionistic conjunction to tensor. This translation, say $(\cdot)^{*'}$, suggests itself automatically if one considers **IL** with multiplicatively formulated

[7]The concept *modal translation* will be given a precise meaning in the introduction to the next chapter.

rules for $\wedge$, and tries to prove soundness by induction on the length of derivations. It maps $A \wedge B$ to $!A^{*'} \otimes !B^{*'}$.

Notes

- The terminology that we use with respect to (derivations in) sequent calculi is more or less standard. The reader who is new to this subject will in most cases find the explanation of unfamiliar terms in appendix a.

- Some of the observations in this chapter, notably proposition 3.1, are taken from Schellinx(1991).

For an inspired discussion of Hilbert's program and the ideal of 'purity of methods' see Girard(1987b).

A thorough technical introduction to syntax and semantics of linear logic is given in Troelstra's "Lectures on Linear Logic". Those who prefer to draw directly from the source should consult Girard(1987a), as well as the intriguing "Towards a Geometry of Interaction" (Girard(1989)) for some of the earlier motivations and ideology.

- The cut elimination theorem 1.1 is essentially part of Girard(1987a), though a first detailed verification for the formulation of **CLL** as a two-sided sequent calculus was carried out early 1989 by Roorda (to be found in Roorda(1991), also in Troelstra(1992)). A proof of cut elimination for the one-sided version of **CLL** is given as Appendix A of Lincoln et al.(1992).

Throughout this thesis we will always eliminate cuts from sequent derivations as in the corresponding proof nets. For those familiar with proof nets and their reductions (Girard(1987a), Danos(1990)), the procedure to follow will be evident. Otherwise any of the above references can be taken as a guide. However, it might be useful to keep in mind that in case the cutformula is exponentiated while not being main formula, neither in the last rule of the left, nor in the last rule of the right premiss of the cut, we will by convention eliminate it by permuting upwards in the (unique) premiss where the occurrences of the cutformula can not possibly have been subjected to structural rules, and is introduced either in an axiom, or by a box rule: we 'look for the box'.

- An excellent account of proofs of the cut elimination theorems for **CL** and **IL**, the subtle and somewhat problematic relation between cut elimination in the intuitionistic sequent calculus and normalization in a corresponding term calculus, as well as a useful overview of several possible variations in the formulation of the calculi, can be found in Gallier(1993).

- The system **ILL** of intuitionistic linear logic defined in section 3 is the one originally presented in Girard and Lafont(1987). A conservative extension is obtained by adding the axiom and (the intuitionistically restricted version of) the rule for $\bot$ and/or the intuitionistic rules for negation. The resulting system is called **ILZ** in Troelstra(1992). It is shown in Došen(1992c) that the non-exponential fragment of **CLL** formulated in the restricted language $\{\otimes, \oplus, (\cdot)^{\perp}, \exists, \mathbf{1}, \bot, \top, \mathbf{0}\}$ is included in **ILZ**.

- Some authors consider a system of intuitionistic linear logic in which the rule R! is replaced by the ternary rule

$$\frac{A \Rightarrow B \quad A \Rightarrow \mathbf{1} \quad A \Rightarrow A \otimes A}{A \Rightarrow\, !B},$$

which originally appeared disguised as a term-calculus rule in Lafont(1988). Let us call this variant on the (intuitionistic) linear sequent calculus $\mathbf{ILL}^l$. It might be possible to prove cut elimination for this calculus, but observe that cut free derivations will in general not have the subformula property. Also note that axioms of the form $!A \Rightarrow\, !A$ can not be expanded.

One easily shows that all sequents provable in **ILL** are provable in $\mathbf{ILL}^l$. The converse, however, will hold if and only if $\mathcal{P}^{\mathrm{IIL}} = \mathcal{W}_l^{\mathrm{IIL}} \cap \mathcal{C}_l^{\mathrm{IIL}}$.

- Another, but less well known, restriction that, when applied to the sequent calculus for classical logic, results in a calculus sound and complete for intuitionistic logic, is limitation to singleton succedents *only* for the rules R$\rightarrow$ (and R$\forall$ in the first order case). We call the resulting system multi-succedent **IL**, or $\mathbf{IL}^{>}$ (appendix e). Though $\mathbf{IL}^{>}$ is closed under cut, the system lacks a 'decent' normalization *procedure*. On the other hand, e.g. the restriction on R$\rightarrow$ appears naturally under an operational reading of implication, determining so-called *deductive* (as opposed to *semantic*) tableaux (see Beth(1969)). We obtain intuitionistic logic because Peirce's law $(((A \rightarrow B) \rightarrow A) \rightarrow A)$ is no longer derivable.

The formal system obtained by imposing the equivalent restriction on **CLL** would have both $\oplus$ *and* $\parr$, unlike **ILL**. However, as we argued in Schellinx(1991), the obvious candidate for such a system of multisuccedent intuitionistic linear logic fails to be closed under cut. Inspired by observations in the category theoretical semantics of (intuitionistic) linear logic, the challenge of this negative result was taken up by Hyland and de Paiva(199x), who propose a more refined formulation (using a term assignment system providing additional information) that *does* satisfy cut elimination.

2

Embeddings

As the considerations in the previous chapter indicate, linear logic might be considered a 'symmetric completion' of a decomposition of intuitionistic logic. Moreover, because a **CLL**-derivation is (possibly among other things) the linear annotation of a derivation in **CL**, it seems reasonable to assume that we will be able also to recover *classical* logic.

In fact there are many possible embeddings, both of intuitionistic and of classical logic, into linear logic. In the early seventies Grishin defined an embedding of classical logic into a system of classical logic without contraction, in fact into what in our terminology would be called the non-exponential fragment of **CLL** extended with weakening. It was observed by Ono(1990) that one easily modifies this embedding to suit the system without weakening. And the fact that decidability of provability in **MALL** as well as in intuitionistic propositional logic is PSPACE-complete led Lincoln et al.(1993) to the construction of an embedding of intuitionistic implicational logic into the multiplicative additive fragment of linear logic, i.e. *not* using the exponential "!".

Here we will turn however our attention more specifically to those translations that *do* use the exponentials. Eventually, in the chapters to come, these will allow us not only to faithfully embed theorems, but also their proofs. We will speak of modal translations. Girard's embedding is a typical example.

DEFINITION. By a(n) *(inductive) modal translation* we mean a mapping $(\cdot)^{\checkmark}$ of formulas to linear formulas satisfying $p^{\checkmark} = \mu_0 p$ for atomic

p, and

$$
\begin{array}{rcl}
\bot^{\checkmark} & = & \mu_1 C, \text{with either } C \equiv \mathbf{0} \text{ or } C \equiv \bot \\
\top^{\checkmark} & = & \mu_2 C, \text{with either } C \equiv \top \text{ or } C \equiv \mathbf{1} \\
(A \to B)^{\checkmark} & = & \mu_3(\mu_4 A^{\checkmark} \circ \mu_5 B^{\checkmark}), \text{with } \circ \in \{\multimap, \leadsto\} \\
(A \wedge B)^{\checkmark} & = & \mu_6(\mu_7 A^{\checkmark} \circ \mu_8 B^{\checkmark}), \text{with } \circ \in \{\otimes, \&\} \\
(A \vee B)^{\checkmark} & = & \mu_9(\mu_{10} A^{\checkmark} \circ \mu_{11} B^{\checkmark}), \text{with } \circ \in \{\parr, \oplus\} \\
(\forall x A)^{\checkmark} & = & \mu_{12}(\forall x.\mu_{13} A^{\checkmark}) \\
(\exists x A)^{\checkmark} & = & \mu_{14}(\exists x.\mu_{15} A^{\checkmark}),
\end{array}
$$

for modalities $\mu_i (0 \leq i \leq 15)$. ⊠

Further on we are going to consider second-order extensions, in which case one needs the additional clauses

$$
\begin{array}{rcl}
(\forall X A)^{\checkmark} & = & \mu_{16}(\forall X.\mu_{17} A^{\checkmark}) \\
(\exists X A)^{\checkmark} & = & \mu_{18}(\exists X.\mu_{19} A^{\checkmark}),
\end{array}
$$

for modalities $\mu_i (16 \leq i \leq 19)$.

A modal translation $(\cdot)^{\checkmark}$ into second-order linear logic is said to be *compatible with substitution* just in case, for all formulas A, B, it holds that $(A[B/X])^{\checkmark}$ and $A^{\checkmark}[B^{\checkmark}/X]$ are identical.

LEMMA. *A modal translation is compatible with substitution if and only if it is the identity on atomic formulas (i.e. if and only if μ_0 is the empty modality).* ⊠

1 Embedding IL into CLL

Girard's translation (definition 1.3.2) of course defines not just an embedding of **IL** into intuitionistic, but a fortiori also into classical, linear logic, and one might wonder whether proposition 1.3.3 continues to hold if we replace **DIL** by **CLL**. Well, clearly the embedding will remain *correct*. But what about *faithfulness*? Given proposition 1.3.1, this no longer is obvious. We will show, by analyzing the properties of possible **CLL**-derivations of sequents $!\Gamma^* \Rightarrow A^*$, that it nevertheless is true.

First of all, given the fact that, at least for the moment, we are merely interested in *provability*, by eliminability of cut (or, more precisely, the *subformula property* that it entails) we may restrict our attention to derivations in the fragment $\{\mathbf{0}, \&, \oplus, \multimap, \forall, \exists, !\}$. We inductively define a measure ρ on cut free derivations π in this fragment as follows:

1. If π is an instance of an axiom, then $\rho(\pi) = 0$;
2. If π' is obtained from π by means of a unary rule other than $\mathrm{R}\multimap, \mathrm{R}\forall$, then $\rho(\pi') = \rho(\pi) + 1$;
3. If π is obtained from π_1, π_2 by means of one of the rules $\mathrm{L}\oplus, \mathrm{L} \multimap$ then $\rho(\pi') = \rho(\pi_1) + \rho(\pi_2) + 1$;
4. If π' is obtained from π by means of one of the rules $\mathrm{R}\multimap, \mathrm{R}\forall$, then $\rho(\pi') = \rho(\pi)$;
5. If π' is obtained from π_1, π_2 by means of the rule $\mathrm{R}\&$, then $\rho(\pi') = \rho(\pi_1) + \rho(\pi_2)$.

(Note that $\rho(\pi)$ just counts the number of applications of rules in π different from $\mathrm{R}\forall, \mathrm{R}\multimap, \mathrm{R}\&$.)

Now let $\vdash_n$ denote *"cut free derivable from atomic instances of axioms $p \Rightarrow p$ with $\rho(\pi) \leq n$."*. Then the following is easily checked by induction on the length of derivations.

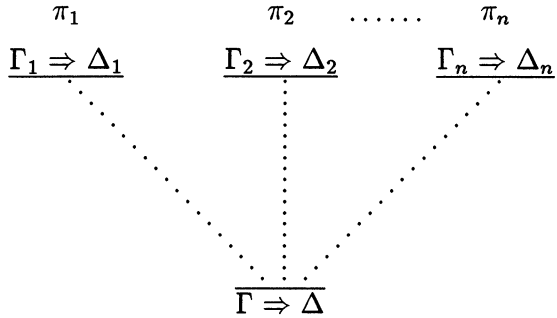
1.1. LEMMA. (Inversion Lemma)[1] *We have:*

$$\vdash_n \Gamma \Rightarrow A \multimap B, \Delta \quad \text{iff} \quad \vdash_n \Gamma, A \Rightarrow B, \Delta$$
$$\vdash_n \Gamma \Rightarrow A \,\&\, B, \Delta \quad \text{iff} \quad \vdash_n \Gamma \Rightarrow A, \Delta \text{ and } \vdash_n \Gamma \Rightarrow B, \Delta$$
$$\vdash_n \Gamma \Rightarrow \forall x A, \Delta \quad \text{iff} \quad \vdash_n \Gamma \Rightarrow A, \Delta$$

(in the last case of course with the usual precautions regarding variables.) ⊠

Lemma 1.1 tells us that we may assume that a cut free derivation π of a sequent $\Gamma \Rightarrow \Delta$ in our fragment ends with a (possibly empty) series of applications of $\mathrm{R}\multimap$, $\mathrm{R}\&$, $\mathrm{R}\forall$, starting from a collection of derivations π_i of sequents $\Gamma_i \Rightarrow \Delta_i$, where each formula in Δ_i is atomic or is of one of the forms $A \oplus B, \exists x A$ or $!A$. Moreover $\rho(\pi_i) \leq \rho(\pi)$:

[1] In fact this is but part of a general *inversion lemma* for **CLL**-derivations, see e.g. Troelstra(1992)



Let us call a translated intuitionistic formula C^* *primitive* if either it is atomic, or has one of the forms $!A^* \oplus !B^*$ or $\exists x!A^*$.Then the following hold:

1.2. LEMMA. *Suppose there is a derivation of*

(a) $!\Gamma^*, \Pi^* \Rightarrow !\Lambda^*$ *or*

(b) $!\Gamma^*, \Pi^* \Rightarrow !\Lambda^*, B^*$ *, with* B^* *primitive.*

Then we may assume the derivation to be such that all sequents with more than one succedent have one of the forms (i) or (ii) :

(i) $!\Sigma^*, \Delta^* \Rightarrow !\Theta^*, A^*$, *with* $|\Theta| \geq 1$ *and* A^* *primitive;*

(ii) $!\Sigma^*, \Delta^* \Rightarrow !\Theta^*$, *with* $|\Theta| \geq 2$.

PROOF: By induction on $\rho(\pi)$, for cut free derivations π of (a), (b).

If $\rho(\pi) = 0$, then the cut free derivation π of (a) or (b) is necessarily an axiom and there is nothing to prove.

A sequent of the form (a) can be derived by means of a right rule in the given fragment only if that rule is R! and moreover $\Pi = \emptyset$, $|\Lambda| = 1$:

$$\frac{!\Gamma^* \Rightarrow D^*}{!\Gamma^* \Rightarrow !D^*}.$$

Because of (the remarks following) lemma 1.1 we may assume that $!\Gamma^* \Rightarrow D^*$ is obtained solely through applications of R$\multimap$, R&, R$\forall$ starting from derivations π_i of sequents $!\Gamma_i^* \Rightarrow D_i^*$, with D_i^* -primitive. To these derivations we may apply the induction hypothesis for (b).

A sequent of the form (b) can be derived by means of a right rule only if that rule is either R$\oplus$ or R$\exists$. In all these cases we can apply the induction hypothesis for (a) to the premiss of the rule.

Finally, if (a) or (b) was obtained by applying a left rule (including W!, C!) the result follows directly by induction hypothesis. ⊠

1.3. LEMMA. *If the sequent* $!\Gamma^* \Rightarrow A^*$ *is derivable in* **CLL**, *we can assume the derivation to be cut free and such that all applications of* R⊸, R∀ *only use sequents with precisely one succedent.*

PROOF: In view of (the remark following) lemma 1.1 we may assume that we have obtained $!\Gamma^* \Rightarrow A^*$ by a number of applications of R⊸, R&, R∀ starting from a collection of sequents $!\Gamma_i^* \Rightarrow A_i^*$ with A_i^* primitive. By lemma 1.2 we know that also we may assume the derivations of the sequents $!\Gamma_i^* \Rightarrow A_i^*$ to be such that all occurrences of sequents with more than one succedent have either the form (i) or (ii). Would there be, in any one of these derivations, an application of R⊸, R∀ involving a sequent having more than one succedent, then we obtain a sequent of the form (i) or (ii) as a conclusion in an application of R⊸, R∀. Obviously this is not possible. ⊠

Now we are ready to prove that a formula A is an intuitionistic theorem if and only A^* is a linear theorem.

1.4. THEOREM. *Girard's translation faithfully embeds intuitionistic logic into* classical *linear logic.*

PROOF: If a sequent $!\Gamma^* \Rightarrow A^*$ is derivable in **CLL**, we know by lemma 1.3 that there is a cut free derivation π in which all applications of R⊸, R∀ involve only sequents with precisely *one* succedent. Therefore the skeleton of π is an $\mathbf{IL}^{>}$-derivation, so $\Gamma \Rightarrow A$ is intuitionistically provable. ⊠

2 Modal logic

Modal statements are about what *might* and what *must* be the case. Their study is the subject of *modal logic*, which in its simplest guise is ordinary classical logic extended with rules and axioms for the unary modality □ of 'necessity'

A sequent calculus formulation of what is known as the modal logic **S4** is obtained by adding to the sequent calculus for classical first-order logic the following two introduction rules for $\Box$:

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta} \qquad \frac{\Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}$$

(By defining a dual operator (the modality $\Diamond$ of 'possibility'), by $\Diamond B := \neg\Box\neg B$, we get the four modal introduction rules of appendix c (page 180)).

The similarity between the calculus thus obtained and classical linear logic is obvious[2], and in fact the only *formal* difference between (this formulation of) **S4** and **CLL** is the restriction of the use of structural rules to *modalized formulas* in the latter one. As a consequence, the additive and multiplicative formulations of the logical rules remain interderivable in **S4**. Therefore distinctions like $\otimes/\&$ and $⅋/\oplus$ are not apparent at the level of provability.

Let us (ab)use the formal similarity and, ignoring historical priorities, write "!" for the **S4**-modality $\Box$.

In the early thirties Gödel in a short note ("Eine Interpretation des intuitionistischen Aussagenkalküls", 1933f in Gödel(1986)) observes that (propositional) intuitionistic logic can be interpreted *"by means of the notions of the ordinary propositional calculus and the (informal) notion 'p is provable' "*. He defines the following translation:

for atomic p (including $\bot$) let $p^{\#} := p$; then put

$$\begin{aligned}(A \wedge B)^{\#} &:= A^{\#} \wedge B^{\#}\\ (A \vee B)^{\#} &:= !A^{\#} \vee !B^{\#}\\ (A \rightarrow B)^{\#} &:= !A^{\#} \rightarrow !B^{\#}\end{aligned}$$

Gödel states that this interpretation is correct, and conjectures faithfulness. The first *proof* of this fact seems to be due to McKinsey and Tarski(1948).

There are quite a few variants of (the extension to the first order calculus) of this embedding of intuitionistic logic into **S4**. Let us define *Gödel's translation* to be the following (which seems to be more or less standard):

[2] Observe that one also finds the same 'modality lattice', see e.g. Chellas(1990).

for atomic $p \neq \bot$ let $p^\circ := !p$; then put

$$\begin{array}{rcl} \bot^\circ & := & \bot \\ (A \wedge B)^\circ & := & A^\circ \wedge B^\circ \\ (A \vee B)^\circ & := & A^\circ \vee B^\circ \\ (A \rightarrow B)^\circ & := & !(A^\circ \rightarrow B^\circ) \\ (\forall x A)^\circ & := & !\forall x A^\circ \\ (\exists x A)^\circ & := & \exists x A^\circ. \end{array}$$

The following theorem is a 'classic'.

2.1. THEOREM. *Gödel's translation is correct and faithful, in the sense that* IL $\vdash \Gamma \Rightarrow A$ *iff* S4 $\vdash \Gamma^\circ \Rightarrow A^\circ$.

PROOF: This can be shown by means of elegant semantical arguments. E.g. for the propositional case one uses completeness of **IL** with respect to the class of all Heyting algebras, completeness of **S4** with respect to the class of Boolean algebras with a modality !, and the following facts:

- the lattice $!\mathcal{B} := \{!b \mid b \in \mathcal{B}\}$ is a Heyting algebra;

- for every Heyting algebra $\mathcal{H}$ there exists a Boolean algebra $\mathcal{B}$ with modality ! such that $\mathcal{H} = !\mathcal{B}$.

These form the core of the proof in McKinsey and Tarski(1948), which is easily extended to include first order quantifiers (see Rasiowa and Sikorski(1963)). ⊠

Observe that Girard's translation (definition 1.3.2), though similar to Gödel's, is not quite the same. But we could of course try its equivalent for S4:

for atomic p (including $\bot$) let $p^* := p$; then put

$$\begin{array}{rcl} (A \wedge B)^* & := & A^* \wedge B^* \\ (A \vee B)^* & := & !A^* \vee !B^* \\ (A \rightarrow B)^* & := & !A^* \rightarrow B^* \\ (\forall x A)^* & := & \forall x A^* \\ (\exists x A)^* & := & \exists x !A^*. \end{array}$$

On the level of *proofs* there are non-trivial distinctions between these translations (cf. section 3.1), but with respect to mere provability nothing is gained or lost: the $(\cdot)^{\#}$-, $(\cdot)^{\circ}$-, $(\cdot)^{*}$-translations are equivalent, in the sense that, by a simple induction on the complexity of **IL**- formulas, one shows that S4 proves $A^{\circ} \Longleftrightarrow\, !A^{\#}$ and $A^{\circ} \Longleftrightarrow\, !A^{*}$. Consequently also the $(\cdot)^{*}$-embedding of intuitionistic logic into S4 is correct and faithful:

$$\mathbf{IL} \vdash \Gamma \Rightarrow A \quad \text{iff} \quad \mathbf{S4} \vdash\, !\Gamma^{*} \Rightarrow A^{*}.$$

But the fragment $\{\mathbf{0}, \&, \oplus, \multimap, \forall, \exists, !\}$ of linear logic considered in section 1 is a proper fragment of S4 (to be precise, of the formulation with multiplicative rules for implication and additive rules for conjunction and disjunction): only structural rules for non-modalized formulas are missing. The above thus gives an indirect, alternative proof of theorem 1.4 and we find

$$\mathbf{CLL} \vdash\, !\Gamma^{*} \Rightarrow A^{*} \quad \text{iff} \quad \mathbf{IL} \vdash \Gamma \Rightarrow A \quad \text{iff} \quad \mathbf{S4} \vdash\, !\Gamma^{*} \Rightarrow A^{*}.$$

Hence we established the following fact: *with respect to the provability of (e.g. $(\cdot)^{*}$-)translated intuitionistic formulas restricting the use of structural rules in* S4 *to the left and to ('!'-)exponentiated formulas only, is conservative.*

3 Embedding system $\mathcal{F}$

Gödels translation and the proof of its sound- and faithfulness have been extended to e.g. first-order arithmetic, type theory and set theory (see e.g. Goodman(1984), Flagg(1985), Scedrov(1985)).

Flagg and Friedman(1986) give a syntactic formulation of the semantical argument sketched in the proof of theorem 2.1, which provides a uniform method to obtain these 'conservative extension'-results. We will here sketch their method for the extension of propositional intuitionistic logic by quantification over propositions ($\mathbf{I}^2_p\mathbf{L}$, which via a Curry-Howard-de Bruijn isomorphism corresponds to Girard's system $\mathcal{F}$ of variable types (Girard(1971))).

Gödels translation now becomes:

$$\begin{array}{rcl} (\forall pA)^\circ & := & !\forall pA^\circ \\ (\exists pA)^\circ & := & \exists pA^\circ . \end{array}$$

One easily verifies that the resulting embedding of $\mathbf{I}^2_p\mathbf{L}$ into the extension of (the propositional fragment of) $\mathbf{S4}$ with quantification over propositions ($\mathbf{S}^2_p\mathbf{4}$) is correct: if $\mathbf{I}^2_p\mathbf{L} \vdash \Gamma \Rightarrow A$, then $\mathbf{S}^2_p\mathbf{4} \vdash \Gamma^\circ \Rightarrow A^\circ$. (In the proof one uses $\mathbf{S}^2_p\mathbf{4} \vdash (A[T/p])^\circ \Longleftrightarrow A^\circ[T^\circ/p]$.)

In order to show *faithfulness*, we adapt the interpretation given by Flagg and Friedman(1986) of $\mathbf{S4}$ in $\mathbf{IL}$ to second order quantified formulas, in the obvious way:

3.1. DEFINITION. Write $\neg_E A$ for $A \to E$. Let Γ be a finite set of $\mathbf{I}^2_p\mathbf{L}$-formulas, and let $E \in \Gamma$. Then, for each $\mathbf{S}^2_p\mathbf{4}$-formula A we define an $\mathbf{I}^2_p\mathbf{L}$-formula $A^{(E)}_\Gamma$ as follows: for atomic A let $A^{(E)}_\Gamma := \neg_E\neg_E A$; then put

$$\begin{array}{rcl} (A \wedge B)^{(E)}_\Gamma & := & A^{(E)}_\Gamma \wedge B^{(E)}_\Gamma \\ (A \vee B)^{(E)}_\Gamma & := & \neg_E\neg_E(A^{(E)}_\Gamma \vee B^{(E)}_\Gamma) \\ (A \to B)^{(E)}_\Gamma & := & A^{(E)}_\Gamma \to B^{(E)}_\Gamma \\ (\forall pA)^{(E)}_\Gamma & := & \forall pA^{(E)}_\Gamma \\ (\exists pA)^{(E)}_\Gamma & := & \neg_E\neg_E\exists pA^{(E)}_\Gamma \\ (!A)^{(E)}_\Gamma & := & \neg_E\neg_E \bigwedge_{C\in\Gamma} A^{(C)}_\Gamma . \end{array} \qquad \boxtimes$$

One shows that, for all provable $\mathbf{S}^2_p\mathbf{4}$-formulas B, we have that $\mathbf{I}^2_p\mathbf{L} \vdash B^{(E)}_\Gamma$, for any set Γ of $\mathbf{I}^2_p\mathbf{L}$-formulas, and any $E \in \Gamma$. Then define inductively for $\mathbf{I}^2_p\mathbf{L}$-formulas F the set $\mathsf{Sub}(F)$ by

$$\begin{array}{rcl} \mathsf{Sub}(p) & := & \{p\}, \text{for atomic } p; \\ \mathsf{Sub}(A \circ B) & := & \mathsf{Sub}(A) \cup \mathsf{Sub}(B) \cup \{A \circ B\}, \text{for binary connectives } \circ; \\ \mathsf{Sub}(Qp.A) & := & \mathsf{Sub}(A) \cup \{Qp.A\}, \text{for quantifiers Q.} \end{array}$$

By induction on the complexity of A we find the following.

3.2. LEMMA. *Let A be an $\mathbf{I}^2_p\mathbf{L}$-formula, and suppose $\Delta \supseteq \mathsf{Sub}(A)$. Then $\mathbf{I}^2_p\mathbf{L} \vdash A \Longleftrightarrow \bigwedge_{C\in\Delta} A^{\circ(C)}_{\Delta}$.* ⊠

3.3. THEOREM. *For all $\mathbf{I}^2_p\mathbf{L}$-formulas A: $\mathbf{I}^2_p\mathbf{L} \vdash A$ iff $\mathbf{S}^2_p\mathbf{4} \vdash A^\circ$.*

PROOF: Suppose $\mathbf{S}^2_p\mathbf{4} \vdash A^\circ$. Let $\Gamma = \mathsf{Sub}(A)$. Then $\mathbf{I}^2_p\mathbf{L} \vdash \bigwedge_{C\in\Delta} A^{\circ(C)}_{\Delta}$. So $\mathbf{I}^2_p\mathbf{L} \vdash A$ by the lemma. ⊠

This of course extends directly to sequents:

$$\mathbf{I}^2_p\mathbf{L} \vdash \Gamma \Rightarrow A \text{ iff } \mathbf{S}^2_p\mathbf{4} \vdash \Gamma^\circ \Rightarrow A^\circ.$$

We get what we might call Girard's variant of this embedding by adding to the propositional clauses in the definition of the $(\cdot)^*$-translation of the previous section:

$$\begin{aligned} (\forall pA)^* &:= \forall pA^* \\ (\exists pA)^* &:= \exists p!A^*. \end{aligned}$$

But $\mathbf{S}^2_p\mathbf{4}$ proves $A^\circ \Longleftrightarrow !A^*$, so we also find:

$$\mathbf{I}^2_p\mathbf{L} \vdash \Gamma \Rightarrow A \text{ iff } \mathbf{S}^2_p\mathbf{4} \vdash !\Gamma^* \Rightarrow A^*.$$

As the propositional/exponential fragment of linear logic extended with quantification over propositions ($\mathbf{C}^2_p\mathbf{LL}$) can be seen as a fragment of $\mathbf{S}^2_p\mathbf{4}$, we obtain faithfulness of the linear analogue of this embedding.

3.4. COROLLARY. *Girard's translation is a correct and faithful embedding of second order propositional intuitionistic into linear logic:*

$$\mathbf{I}^2_p\mathbf{L} \vdash \Gamma \Rightarrow A \text{ iff } \mathbf{C}^2_p\mathbf{LL} \vdash !\Gamma^* \Rightarrow A^*. \quad ⊠$$

It is more or less obvious that adding first order quantifiers will be unproblematic. We refrain from verifying the details.

4 Decomposing classical logic

Like Girard's embedding of intuitionistic into linear logic, which, as we saw, is based upon a linear decomposition of intuitionistic connectives (most notably of intuitionistic implication $A \to B$ as $!A \multimap B$), we can construct modal embeddings of *classical* into linear logic. These then in turn provide us with linear decompositions of the *classical* connectives. Unlike in the intuitionistic case, however, there is no *unique* optimal choice.

One might be tempted to try and interpret classical implication $A \to B$ as $!A \multimap ?B$. However, the modal translation thus obtained fails to define a correct embedding of classical (even of *intuitionistic*) logic: if we put $p^\dagger := p$ for atomic p, and $(A \to B)^\dagger := !A^\dagger \multimap ?B^\dagger$, there are intuitionistic theorems ϕ whose translation $\phi^\dagger$ is not provable in linear logic. An example of such a ϕ is $((a \to b) \to a) \to ((a \to b) \to b)$. (One shows[3] that **CLL** $\not\vdash !(!(!a \multimap ?b) \multimap ?a), !(!a \multimap ?b) \Rightarrow ?b$, from which one obtains that **CLL** $\not\vdash \phi'$, using the fact that $\psi^\dagger \in \mathcal{N}$ (i.e. $\psi^\dagger \iff ?\psi^\dagger$) for all non-atomic ψ.

Further reflection leads us to *two* candidates for an embedding of classical logic. One, the T-*translation*, is based upon a linear decomposition of $A \to B$ as $!?A \multimap ?B$, the other, the Q-*translation*, interprets $A \to B$ as $!A \multimap ?!B$.[4]

They are defined inductively by the following clauses.[5]

In case of the Q-translation we put $p^{\mathrm{Q}} := p$ for atoms p (including the constants $\top, \bot$) and, according to whether we want to use the additive or multiplicative version of the logical connectives, take the corresponding clause of the Q-part of table 2.1.

In case of the T-translation, again put $p^{\mathrm{T}} := p$ for atoms p (including the constants $\top, \bot$) and, according to whether one wants to use the additive or multiplicative version of the logical connectives, take the corresponding clause of the T-part of table 2.1.

[3]This makes an interesting exercise in linear non-derivability.

[4]Here "T" stands for *tête*, "Q" for *queue*. See section 4 of the next chapter.

[5]We will from now on mostly concentrate on the second order propositional fragments of our logics. The reader who wishes to do so, will easily find the clauses necessary to include also first order quantification.

Q	multiplicative	additive
$(A \to B)^Q$	$!A^Q \multimap ?!B^Q$	$?!A^Q \leadsto !B^Q$
$(A \wedge B)^Q$	$!A^Q \otimes !B^Q$	$?!A^Q \& ?!B^Q$
$(A \vee B)^Q$	$?!A^Q \parr ?!B^Q$	$!A^Q \oplus !B^Q$
$(\forall p)^Q$	$\forall p ?!A^Q$	$\forall p ?!A^Q$
$(\exists p)^Q$	$\exists p !A^Q$	$\exists p !A^Q$

T	multiplicative	additive
$(A \to B)^T$	$!?A^T \multimap ?B^T$	$?A^T \leadsto !?B^T$
$(A \wedge B)^T$	$!?A^T \otimes !?B^T$	$?A^T \& ?B^T$
$(A \vee B)^T$	$?A^T \parr ?B^T$	$!?A^T \oplus !?B^T$
$(\forall p)^T$	$\forall p ?A^T$	$\forall p ?A^T$
$(\exists p)^T$	$\exists p !?A^T$	$\exists p !?A^T$

Table 2.1: The Q- and the T-translation

4.1. THEOREM. *Both the* T- *and* Q-*mapping are sound and faithful embeddings of classical into linear logic:*

$$\mathbf{CLL} \vdash !\Gamma^Q \Rightarrow ?!\Delta^Q \quad \text{iff} \quad \mathbf{CL} \vdash \Gamma \Rightarrow \Delta \quad \text{iff} \quad \mathbf{CLL} \vdash !?\Gamma^T \Rightarrow ?\Delta^T.$$

PROOF: As the skeleton of any linear proof of a modal translation of a formula A is a correct proof in classical logic, the *faithfulness* of both mappings is trivial.

In order to prove *correctness* one shows by induction on the length of C_p^2L-derivations (with multiplicatively formulated logical rules in case of the multiplicative, and additively formulated rules in case of the additive version) that derivability of a sequent $\Gamma \Rightarrow \Delta$ implies derivability in C_p^2LL of (1) $!\Gamma^Q \Rightarrow ?!\Delta^Q$ and (2) $!?\Gamma^T \Rightarrow ?\Delta^T$. One uses in both cases the fact that our mappings are compatible with substitution. Moreover, for (1) we use derivability of

$$!(!A \multimap ?!B) \quad \Rightarrow \quad ?!A \multimap ?!B$$

$$\begin{aligned} ?!A \rightsquigarrow ?!B &\Rightarrow ?!(?!A \rightsquigarrow !B) \\ !?!A \otimes ?!B &\Rightarrow ?!(!A \otimes !B) \\ ?!A_i &\Rightarrow ?!(!A_1 \oplus !A_2) \\ \exists p ?!A &\Rightarrow ?!\exists p !A, \end{aligned}$$

for (2) that of

$$\begin{aligned} !?(!?A \multimap ?B) &\Rightarrow !?A \multimap !?B \\ !?A \rightsquigarrow !?B &\Rightarrow ?(?A \rightsquigarrow !?B) \\ !?(?A \parr ?B) &\Rightarrow !?A \parr ?!?B \\ !?(?A_1 \& ?A_2) &\Rightarrow !?A_i \\ !?\forall p ?A &\Rightarrow \forall p !?A. \qquad \boxtimes \end{aligned}$$

Consequently a formula A is a classical theorem if and only if $?!A^{\mathrm{Q}}$ and $?A^{\mathrm{T}}$ are linear theorems.

When working out the details of the proof of theorem 4.1 the reader will find that instead of the derivable sequents

$$\begin{aligned} !?!A \otimes ?!B &\Rightarrow ?!(!A \otimes !B) \\ !?(?A \parr ?B) &\Rightarrow !?A \parr ?!?B, \end{aligned}$$

(s)he could as well use derivability of

$$\begin{aligned} ?!A \otimes !?!B &\Rightarrow ?!(!A \otimes !B) \\ !?(?A \parr ?B) &\Rightarrow ?!?A \parr !?B, \end{aligned}$$

or, said otherwise, there *two* ways to derive $!?!A \otimes !?!B \Rightarrow ?!(!A \otimes !B)$, and *two* ways to derive $!?(?A \parr ?B) \Rightarrow ?!?A \parr ?!?B$ (cut free, that is). We encounter, within a procedure that otherwise is completely unambiguous, *precisely* at these points the necessity to choose. We will come back to this (important!) fact, notably in chapters 3 and 6.

Observe that in case of the *additive* definition of the Q-conjunction there still exists a quite drastic optimization: due to derivability of $!(A \& B) \Rightarrow !A \& !B$ and $?!(?!A \& ?!B) \Rightarrow ?!(A \& B)$ we might in fact take

$(A \wedge B)^{\mathrm{Q}} := A^{\mathrm{Q}} \& B^{\mathrm{Q}}$. Dually, for the additive definition of the T-disjunction we might opt for $(A \vee B)^{\mathrm{T}} := A^{\mathrm{T}} \oplus B^{\mathrm{T}}$. We will try to motivate our choice later, cf. sections 3.5, 6.2.

Notes

- The first section appeared, in slightly different form, in Schellinx(1991). Section 4, like much of the material in the chapters to come, is based upon joint work with Vincent Danos and Jean-Baptiste Joinet.

- Interesting facts on the history of Gödel's and related modal translations can be found in Troelstra's "Introductory note to 1933f" in Volume I of Gödel's Collected Works (Gödel(1986)) and in several of Kosta Došen's papers, e.g. Došen(1992a). The observation in section 2 that faithfulness of the well-known embedding of intuitionistic logic into **S4** implies faithfulness of Girard's embedding of intuitionistic logic into **CLL** is more or less implicit in Došen's 'Modal translations in substructural logics' (Došen(1992b), proposition 14).

- Intuitionistic negation $A \to \bot$ under Girard's embedding becomes $!A^* \multimap \mathbf{0}$, which is not provably equivalent to $!A^*$'s linear negation $(!A^*)^\perp$ (i.e. $!A^* \multimap \bot$. Would one prefer to map intuitionistic to linear negation, one might use a translation mapping atoms p to $p \oplus \bot$, and $\bot$ to $\bot$ (Prijatelj and Schellinx(1991)). Note however that this is not a 'modal translation' in our, strict, sense.

Classical negation $A \to \bot$ is mapped by the Q-translation to $!A^{\mathrm{Q}} \multimap ?!\bot$, by the T-translation to $!?A^{\mathrm{T}} \multimap ?\bot$. As, for all modalities μ, we have that $\bot \Longleftrightarrow ?\mu\bot$, these are linearly equivalent to $(!A^{\mathrm{Q}})^\perp$, respectively $(!?A^{\mathrm{T}})^\perp$.

- The embeddings of classical into linear logic of section 4 are, a fortiori, embeddings of classical logic into **S4**. Given the observations of section 3 it will not come as too big a surprise that these are closely related to modal embeddings of classical logic into **S4** as previously studied by Fitting(1970) and Czermak(1974).

3

Decorations

We recall our main theme: the *skeleton* (the object obtained by deleting all exponentials, replacing the linear connectives by their non-linear counterparts and eliminating possible repetitions of sequents) of whatever derivation in linear logic is a derivation in a sequent calculus for classical (or intuitionistic) logic. So we may interpret a linear derivation as a 'dressed up' classical or intuitionistic proof. In the present chapter we will verify that the taking of the skeleton of a linear proof indeed has a converse: given a classical or intuitionistic derivation π, one can always 'dress π up' in such a way that the result is a linear proof $\delta(\pi)$, with π as its skeleton.

DEFINITION. A *decoration* of a (classical, intuitionistic) derivation π is a linear derivation $\delta(\pi)$ such that $\mathsf{sk}(\delta(\pi)) = \pi$; by a decoration strategy for a given (sequent)calculus we mean a uniform procedure (algorithm) that outputs a decoration for any given derivation in the calculus. ⊠

Just as for derivations, we will also refer to the obvious classical/intuitionistic formula that underlies a linear formula A, as its *skeleton* $\mathsf{sk}(A)$. Conversely we will speak of *decorations* of formulas: $\delta(A)$ is a *linear decoration* of A if it is obtained by replacing classical/intuitionistic connectives by linear ones, and by prefixing subformulas of A with strings of !, ?. So e.g. both $!?!!A \otimes B$ and $A \,\&\, ?B$ are linear decorations of $A \wedge B$; note that a modal translation $(\cdot)^{\checkmark}$ will always satisfy $\mathsf{sk}(A^{\checkmark}) = A$.

Also for *linear* formulas A we will refer to a formula $\delta(A)$ that is obtained by prefixing subformulas of A with strings of !, ? as a *decoration* of A. Obviously the number of distinct decorations of a given formula is infinite. Though, by the observations made at the end of the introduction to chapter 1, modulo provable linear equivalence this number will be finite, the manifold of literally distinct decorations does play a role as soon as we are no longer merely interested in mere provability, but also in the dynamics of proofs, cf. section 1.1, section 4.4.

1 Plethoric translations I

In order to produce linear decorations of intuitionistic and classical proofs, a possible approach is to try to transform a given derivation π into a linear derivation $\delta(\pi)$, by inductively applying a modal translation to the sequents occurring in π.

1.1. DEFINITION. Let a modal translation $(\cdot)^{\checkmark}$ and modalities μ, ν be given. We say that the triple $\langle(\cdot)^{\checkmark}, \mu, \nu\rangle$ determines an *inductive decoration strategy* for a sequent calculus $\mathcal{S}$ if

1/ for all $\mathcal{S}$-axioms $\Gamma \Rightarrow \Delta$ it holds that $\mu\Gamma^{\checkmark} \Rightarrow \nu\Delta^{\checkmark}$ is a **CLL**-axiom or obtainable from such solely by means of zero or more applications of exponential contextual and/or dereliction rules;

2/ for all $\mathcal{S}$-rules with conclusion $\Gamma \Rightarrow \Delta$ and premiss(es) $\Gamma_i \Rightarrow \Delta_i$ we can derive $\mu\Gamma^{\checkmark} \Rightarrow \nu\Delta^{\checkmark}$ in linear logic from $\mu\Gamma_i^{\checkmark} \Rightarrow \nu\Delta_i^{\checkmark}$ by one application of the corresponding **CLL**-rule preceded and/or followed by zero or more applications of exponential contextual and/or dereliction rules. ⊠

Obviously, by definition, if $\langle(\cdot)^{\checkmark}, \mu, \nu\rangle$ is an inductive decoration strategy for a calculus $\mathcal{S}$, then, given an $\mathcal{S}$-derivation π of a sequent $\Gamma \Rightarrow \Delta$, we can inductively apply the translation $(\cdot)^{\checkmark}$ to π and derive $\mu\Gamma^{\checkmark} \Rightarrow \nu\Delta^{\checkmark}$ by means of a linear derivation $\pi^{\checkmark}$ which is a decoration of the original one.

As the proofs of theorems 1.3.3 and 2.4.1 show, neither Girard's translation of intuitionistic logic, nor the Q- and T-translation of classical logic define proper translations of sequent-calculus *derivations*, in the sense that none of $\langle(\cdot)^*, !, \cdot\rangle$, $\langle(\cdot)^{\mathrm{Q}}, !, ?!\rangle$, $\langle(\cdot)^{\mathrm{T}}, !?, ?\rangle$ define an inductive decoration strategy for **IL**, **CL**. Using these translations, the inductive transformation of proofs in general will introduce cuts at several points. Consequently the skeleton of the linear derivation obtained will not be the intuitionistic or classical derivation we started from. In fact this is true in a *strong* sense: also after elimination of the cuts introduced (the '*correction cuts*') what we obtain is likely to be a derivation having a skeleton that is different from the original one. An example in the intuitionistic case is provided by the following derivation:

$$\frac{C \Rightarrow C \qquad \dfrac{A \Rightarrow A}{B, A \Rightarrow A}}{C, C \to B, A \Rightarrow A}$$

Applying Girard's translation and introducing the necessary 'correction cut' for the implication-left rule, we get

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{!C \Rightarrow !C \qquad B \Rightarrow B}{!C, !C \multimap B \Rightarrow B}}{!C, !(!C \multimap B) \Rightarrow B}}{!C, !(!C \multimap B) \Rightarrow !B}}{!(!C \multimap B) \Rightarrow !C \multimap !B} \qquad \dfrac{\dfrac{\dfrac{C \Rightarrow C}{!C \Rightarrow C}}{!C \Rightarrow !C} \qquad \dfrac{\dfrac{A \Rightarrow A}{!A \Rightarrow A}}{!B, !A \Rightarrow A}}{!C, !C \multimap !B, !A \Rightarrow A}}{!C, !(!C \multimap B), !A \Rightarrow A}$$

which reduces to

$$\frac{\dfrac{\dfrac{A \Rightarrow A}{!A \Rightarrow A}}{!C, !A \Rightarrow A}}{!C, !(!C \multimap B), !A \Rightarrow A}$$

Though the translation does *not* give us a decoration-strategy for IL-derivations, the example shows that after having eliminated the correction cut we obtain as skeleton an IL-derivation of the same sequent for which the inductive application of Girard's translation *does* result in a decoration. In fact this is not by accident, but (modulo precise formulation) a general property. (See section 6.3.)

The embeddings of chapter 2 fail to be inductive decoration strategies because of their economy in the use of exponentials. When one is willing to 'bang' less economically, it is mostly not too difficult, given any of the standard sequent calculi for classical or intuitionistic logic, to define a decoration via a sound and faithful translation into linear logic.

Let us consider second order intuitionistic propositional logic $\mathbf{I}^2_p\mathbf{L}$, with additively formulated rules for conjunction and disjunction. Define a mapping $(\cdot)^\circledast$ of formulas to *linear* formulas by:

for p atomic let $p^\circledast := p$; then put

$$\begin{array}{rcl} \bot^\circledast & := & \mathbf{0} \\ (A \wedge B)^\circledast & := & !A^\circledast \& \, !B^\circledast \\ (A \vee B)^\circledast & := & !A^\circledast \oplus !B^\circledast \\ (A \to B)^\circledast & := & !A^\circledast \multimap !B^\circledast \\ (\forall p A)^\circledast & := & \forall p !A^\circledast \\ (\exists p A)^\circledast & := & \exists p !A^\circledast. \end{array}$$

Then $\Gamma \Rightarrow A$ is derivable in $\mathbf{I}^2_p\mathbf{L}$ if and only if $!\Gamma^\circledast \Rightarrow A^\circledast$ is linearly derivable (observe e.g. that $\vdash !A^\circledast \Longleftrightarrow !A^*$). Moreover one easily checks that $\langle (\cdot)^\circledast, !, \cdot \rangle$ is an inductive decoration strategy for the chosen calculus[1]: we replace all sequents $\Gamma \Rightarrow A$ in a derivation π by $!\Gamma^\circledast \Rightarrow A^\circledast$, and apply R! just before each rule in which the succedent formula is active, e.g. before R$\to$ and before applications of L$\to$ and cut (in the left premiss).

We call the resulting linear derivation $\pi^\circledast$ the *f(ull)-decoration* of π, which, by construction has the *down-property*: each main formula in an application of R! is active in the rule below.

In general the number of shrieks thus introduced is somewhat overabundant, to put it mildly: the *potential* use of structural rules is anticipated by 'banging' each and every (sub)formula occurring in the antecedents of $\pi^\circledast$-sequents; but, on the other hand, it is the possibility

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[1] $(\cdot)^\circledast$ is the translation $(\cdot)^\checkmark$ that naturally suggests itself when we inductively transfer $\mathbf{IL}$-derivations to linear logic, replacing sequents $\Gamma \Rightarrow A$ by $!\Gamma^\checkmark \Rightarrow A^\checkmark$ and *not* using correction cuts: e.g. the rule R$\to$ forces that $!A^\checkmark \multimap B^\checkmark$, and the rule L$\to$ that $A^\checkmark \multimap !B^\checkmark$ is a subdecoration of $(A \to B)^\checkmark$. Whence $(A \to B)^\checkmark := !A^\checkmark \multimap !B^\checkmark$ is the 'natural' candidate.

of this 'modal anticipation' of non-linearity that enables our defining uniform translations inducing decoration strategies. In fact, the $(\cdot)^{\circledast}$-translation is the most economical decorating uniform translation of *intuitionistic* logic that is compatible with substitution ($(A[X/p])^{\circledast} \equiv A^{\circledast}[X^{\circledast}/p]$, for all A, X): mappings obtained by dropping exponentials in $(\cdot)^{\circledast}$'s definition will no longer be decorating. If we add exponentials, we loose the substitution-property and/or introduce multiple shrieks.

2 ILU and (sub)Girardian decorations

Related to the observation that Girard's translation is not an inductive decoration-strategy is the fact that *no* strategy can lead to decorations of **IL**-derivations that are always *subGirardian*, i.e. do not shriek (sub)formulas that are not banged in the $(\cdot)^*$-translation. If we consider again the example of the previous section, then clearly each decoration will contain the following *minimal* decoration:

$$\dfrac{C \Rightarrow C \qquad \dfrac{A \Rightarrow A}{!B, A \Rightarrow A}}{C, C \multimap !B, A \Rightarrow A}$$

The exclamation mark appearing in front of B is forced by the use of the structural rule of weakening. Deleting it results in a non-linear derivation. The 'root of all evil'[2] apparently is that intuitionistic sequent calculus allows applications of the rules L$\rightarrow$, L$\forall_2$, L$\wedge$ in case the active formula in the (right) premiss has been subjected to structural manipulation. We here find once more[3] illustrated how linear logic reveals the possibility of restrictions on derivations in its underlying calculi, restrictions leading to subsets of the collection of these proofs that nevertheless are complete. Namely, the *correctness* of the $(\cdot)^*$-translation shows that we can do without applications of those rules in the aforementioned cases: the collection of derivations that do *not* use them is complete for intuitionistic logic. Indeed, this is an immediate

[2]See Heesterbeek et al.(1992)

[3]A first example is provided by our observation concluding section 2 of the previous chapter.

Identity axiom:

$$A; \Rightarrow A$$

Logical rules:

$$\text{L}\!\to \frac{;\Gamma \Rightarrow A \qquad B;\Gamma' \Rightarrow C}{A \to B;\Gamma,\Gamma' \Rightarrow C} \qquad \text{R}\!\to \frac{\Pi;\Gamma,A \Rightarrow B}{\Pi;\Gamma \Rightarrow A \to B}$$

Rules for the second order quantifier (q not free in Γ, Π):

$$\text{L}\forall_2 \frac{A[X/p];\Gamma \Rightarrow B}{\forall p\, A;\Gamma \Rightarrow B} \qquad \text{R}\forall_2 \frac{\Pi;\Gamma \Rightarrow A[q/p]}{\Pi;\Gamma \Rightarrow \forall p\, A}$$

Structural rules:

$$\text{LW} \frac{\Pi;\Gamma \Rightarrow A}{\Pi;\Gamma,B \Rightarrow A} \qquad \text{LC} \frac{\Pi;\Gamma,B,B \Rightarrow A}{\Pi;\Gamma,B \Rightarrow A} \qquad \text{D} \frac{B;\Gamma \Rightarrow A}{;B,\Gamma \Rightarrow A}$$

Table 3.1: **ILU**, the cut-free fragment.

corollary to the subformula property and the fact that the skeleton of a cut free intuitionistic linear derivation of $!\Gamma^* \Rightarrow A^*$ is an **IL**-derivation.

This suggests a formulation of intuitionistic sequent calculus in which the use of these rules on such, non-linear, formulas is forbidden, and which as a consequence we should expect to allow (sub)Girardian decorations.

Such a formulation can be found by a rather straightforward abstraction of the structure of linear derivations of sequents of the form $!\Gamma^* \Rightarrow A^*$ (let us restrict ourselves to the formulation of the fragment with implication and universal second order propositional quantification, i.e. the fragment corresponding to Girard's system $\mathcal{F}$) [4] (table 3.1).

In a sequent $\Pi;\Gamma \Rightarrow A$ the symbol Π denotes a multiset containing *at most* one (the *head-*)formula whose occurrence in a sequent is distinguished by means of the ";". In the linear interpretation it corresponds

[4] Observe that the instances of rules that we will get rid of have no obvious equivalent in the natural deduction formulation of $\mathbf{I}^2_p\mathbf{L}$. Therefore this modified sequent calculus will be closer to natural deduction and the simply typed λ-calculus than the standard formulation. (The reader will find that the 'obvious' way to interpret a natural deduction derivation in sequent calculus is as a derivation in our modified calculus!)

to a formula that is not (yet) shrieked. The structural rule D is the equivalent of L!, the linear dereliction rule.

Included we find the (cut free) *neutral* fragment of intuitionistic implicational logic as it appears in Girard's system of Unified Logic (**LU**, Girard(1993)). We therefore refer to the above calculus as **ILU**.

We will show (theorem 2.3) that $\Pi; \Gamma \Rightarrow A$ is derivable in **ILU** if and only if $\Pi^*, !\Gamma^* \Rightarrow A^*$ is derivable in the $\{!, \multimap, \forall_2^*\}$-fragment of linear logic, where $\forall_2^*$ indicates abstraction limited to formulas of the form X^*. Moreover, by construction, $\langle (\cdot)^*, \cdot, !, \cdot \rangle$ determines an inductive decoration strategy (in the sense of definition 1.1, adapted to **ILU**-sequents in the obvious[5] way) for **ILU**-derivations π, resulting in linear derivations π^* (the *g(irard)-decorations*) with the down-property.

Cuts between g-decorated **ILU**-derivations come in two distinct forms.

(1) If π^* is a g-decorated **ILU**-derivation with conclusion $\Pi^*, !\Gamma^* \Rightarrow A^*$ and τ^* is a g-decorated **ILU**-derivation with conclusion $A^*, !\Delta^* \Rightarrow B^*$, we can form

$$\dfrac{\begin{matrix}\pi^* \\ \vdots \\ \Pi^*, !\Gamma^* \Rightarrow A^*\end{matrix} \qquad \begin{matrix}\tau^* \\ \vdots \\ A^*, !\Delta^* \Rightarrow B^*\end{matrix}}{\Pi^*, !\Gamma^*, !\Delta^* \Rightarrow B^*}$$

(2) If π^* is a g-decorated **ILU**-derivation with conclusion $!\Gamma^* \Rightarrow A^*$ and τ^* is a g-decorated **ILU**-derivation with conclusion $\Pi^*, !A^*, !\Delta^* \Rightarrow B^*$, we can form

$$\dfrac{\dfrac{\begin{matrix}\pi^* \\ \vdots \\ !\Gamma^* \Rightarrow A^*\end{matrix}}{!\Gamma^* \Rightarrow !A^*} \qquad \begin{matrix}\tau^* \\ \vdots \\ \Pi^*, !\Delta^*, (!A^*)^n \Rightarrow B^*\end{matrix}}{\Pi^*, !\Gamma^*, !\Delta^* \Rightarrow B^*}$$

[5]See the remark in the notes at the end of this chapter, page 77.

We then can apply elementary reduction steps of the cut elimination procedure for linear logic in such a way that the resulting reduct is either a g-decorated **ILU**-derivation, or contains only linear cuts of type (1), (2).

Strong normalization for these reductions in linear logic implies that the collection of g-decorated **ILU**-derivations is closed under linear cuts of the above form.

Hence as a corollary we get that **ILU** is closed under the

Cut rules:

$$\text{head}\quad \frac{\Pi;\Gamma_1 \Rightarrow A \qquad A;\Gamma_2 \Rightarrow B}{\Pi;\Gamma_1,\Gamma_2 \Rightarrow B}$$

$$\text{mid}\quad \frac{;\Gamma_1 \Rightarrow A \quad \Pi;A,\Gamma_2 \Rightarrow C}{\Pi;\Gamma_1,\Gamma_2 \Rightarrow C}$$

By the above observations we obtain an intrinsically definable, deterministic cut elimination procedure σ for **ILU**-derivations π, which corresponds to the linear cut elimination procedure σ_{LL} applied to π's g-decoration π^*. Observe however, that the correspondence between the two procedures is *not* step-by-step. Due to the fact that in **ILU** the exponential rule R! is 'invisible', in certain cases an elementary reduction step in **ILU** will correspond to *two* consecutive steps in the linear equivalent. Conversely, an elementary reduction step applied to π^* will always correspond to either an empty step (an instance of the 'repetition rule') in **ILU**, or an elementary **ILU**-reduction step, as illustrated in figure 3.1. We will see **ILU**-reductions in detail in section 6.2.

We express the fact that the g-decorations π^* of **ILU**-derivations π simulate the reductions of π, by saying that they are *strong* decorations with respect to σ.

2.1. DEFINITION. Let **L** be a sequent calculus and σ a procedure for cut elimination in **L**. A decoration strategy δ for **L** is said to be *strong* (with respect to σ) if and only if any elementary normalisation step in σ, transforming a derivation π in **L** into π', can be simulated by one

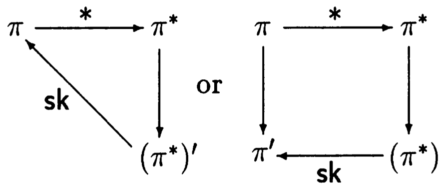
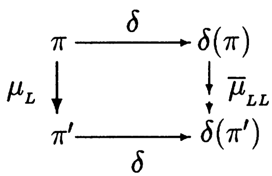


Figure 3.1: "...an empty, or an elementary step.."

or more elementary steps in the standard procedure for linear sequent calculus, leading from $\delta(\pi)$ to $\delta(\pi')$. ⊠

In other words, δ is a strong decoration strategy if for all **L**-derivations π and for all elementary reduction steps μ in σ there exists an **LL**-reduction $\overline{\mu}$ such that the following diagram commutes:



Note that for whatever decoration strategy δ for **L** there exists a normalization procedure, say σ_l, such that δ is strong with respect to σ_l: it suffices to define σ_l as the reflection of cut elimination in linear logic (cf. section 1.1). But often it will not be possible to formulate the corresponding procedure σ_l *directly*, and independently of $\delta(\pi)$, i.e. σ_l is not *intrinsically* definable. This will for example be the case whenever the decoration strategy in question is non-deterministic. Cf. section 3.

Sound- and completeness of **ILU** with respect to provability in (the corresponding fragment of) intuitionistic logic should be more or less evident from the correctness of Girard's translation of intuitionistic into linear logic. It is in any case an immediate corollary to our forthcoming theorem 2.3. Still, let us first give here an instructive direct argument using closure of **ILU** under cut:

2.2. PROPOSITION. *If* **ILU** $\vdash \Pi;\Gamma \Rightarrow A$, *then* $\mathbf{I}_p^2\mathbf{L} \vdash \Pi,\Gamma \Rightarrow A$. *Conversely, if* $\mathbf{I}_p^2\mathbf{L} \vdash \Gamma \Rightarrow A$, *then* **ILU** $\vdash;\Gamma \Rightarrow A$.

PROOF: Note that only the second claim is worth of our attention. Here we proceed by induction on the length of $\mathbf{I}_p^2\mathbf{L}$-derivations. We encounter a problem in case the last rule applied has been L$\rightarrow$ or L$\forall_2$. In the first case, by inductive hypothesis we have **ILU**-derivations of $;\Gamma \Rightarrow A$ and $;\Delta, B \Rightarrow C$, and we would like to get a derivation of $;\Gamma,\Delta,A \rightarrow B \Rightarrow C$. As B is not a head-formula, we cannot use the **ILU**-rule directly. However, it is easy to derive $;A, A \rightarrow B \Rightarrow B$ in **ILU**. So using mid-cuts we construct

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ ;\Gamma \Rightarrow A\end{matrix} \quad \begin{matrix}\vdots\\ ;A, A \rightarrow B \Rightarrow B\end{matrix}}{;\Gamma, A \rightarrow B \Rightarrow B} \qquad \begin{matrix}\vdots\\ ;\Delta, B \Rightarrow C\end{matrix}}{;\Gamma,\Delta, A \rightarrow B \Rightarrow C}$$

and we are done by closure of **ILU** under cut. In case of L$\forall_2$ we use derivability in **ILU** of $;\forall pA \Rightarrow A$. ⊠

The following theorem shows that a $\{!, \multimap, \forall_2\}$-derivation in which each formula occurring is a subformula of a formula of the form A^* in fact is an **ILU**-derivation.

2.3. THEOREM. *If* π *is a derivation in* $\{!, \multimap, \forall_2^*\}$ *of* $\Pi^*, !\Gamma^* \Rightarrow !\Sigma^*, \Delta^*$ *in which all cutformulas are of the form* A^* *or* $!A^*$, *and all identity axioms of the form* $A^* \Rightarrow A^*$, *then* $\mathsf{sk}(\pi)$ *is an ILU-derivation of* $\Pi;\Gamma \Rightarrow \Sigma \cup \Delta$.

PROOF: By induction on the length of π, proving simultaneously that always $|\Pi \cup \Sigma| \leq 1$ and $|\Sigma \cup \Delta| = 1$.

For the basis of the induction we consider the axioms $A^* \Rightarrow A^*$, which obviously satisfy these conditions, and whose skeletons are the **ILU**-axioms $A; \Rightarrow A$. For the induction step let us consider just some typical cases.

- If π ends by an application of L$\multimap$, the situation will be as follows:

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Pi_1^*, !\Gamma_1^* \Rightarrow !A^*, !\Sigma_1^*, \Delta_1^*\end{matrix} \quad \begin{matrix}\pi_2\\ \vdots\\ B^*, \Pi_2^*, !\Gamma_2^* \Rightarrow !\Sigma_2^*, \Delta_2^*\end{matrix}}{!A^* \multimap B^*, !\Gamma_1^*, !\Gamma_2^* \Rightarrow !\Sigma_1^*, !\Sigma_2^*, \Delta_1^*, \Delta_2^*}$$

Applying the induction hypothesis to π_i, we find that $\Pi_i = \Sigma_i = \Delta_1 = \emptyset$ and $|\Delta_2| = 1$, from which the desired conclusions follow.

- In case π ends by an application of L! we have

$$\dfrac{\begin{matrix}\pi' \\ \vdots \\ \Pi^*, A^*, !\Gamma^* \Rightarrow !\Sigma^*, \Delta^*\end{matrix}}{\Pi^*, !A^*, !\Gamma^* \Rightarrow !\Sigma^*, \Delta^*}$$

By induction hypothesis $\Pi = \Sigma = \emptyset$ and $|\Delta| = 1$. So $\mathsf{sk}(\pi')$ derives $A; \Gamma \Rightarrow \Delta$. We can continue in **ILU** by an application of D, giving $; A, \Gamma \Rightarrow \Delta$.

- In case of an application of R! the premiss has to be of the form $!\Gamma^* \Rightarrow A^*$, and the conclusion is $!\Gamma^* \Rightarrow !A^*$. By induction hypothesis we have an **ILU**-derivation of $; \Gamma \Rightarrow A$, and we continue by an application of the repetition rule (i.e., don't do a thing). ⊠

There is a subtle point to notice here: we do *not* claim that if π is a derivation, as in the theorem, of a sequent $\Pi^*, !\Gamma^* \Rightarrow !\Sigma^*, \Delta^*$, then $(\mathsf{sk}(\pi))^* = \pi$. In general this will not be the case.

3 Plethoric translations II

Let us try to define a uniform translation $(\cdot)^{\checkmark}$ of classical logic that, like the $(\cdot)^{\circledast}$-translation for intuitionistic logic, can be extended to an inductive decoration strategy for **CL**. In order to do so, we have to interpret sequents $\Gamma \Rightarrow \Delta$ as $\mu\Gamma^{\checkmark} \Rightarrow \nu\Delta^{\checkmark}$, where μ, ν are modalities. Then observe that in order to satisfy condition 2 of definition 1.1

1. in case of the structural rules we need that $\mu \equiv !\mu'$ and $\nu \equiv ?\nu'$, for modalities μ', ν';
2. in case of an application of cut, we need to be able to 'unify' the decorations $\mu A^{\checkmark}$ and $\nu A^{\checkmark}$ of the cut formula by some series of applications of dereliction- and/or promotion-rules. Clearly this

can be done if and only if either μ is a suffix of ν or ν is a suffix of μ.[6]

We will call a pair of modalities (μ, ν) satisfying these two conditions *adequate.*

3.1. THEOREM. *Let (μ, ν) be a pair of modalities. There exists a modal translation $(\cdot)^{\checkmark}$ such that $\langle(\cdot)^{\checkmark}, \mu, \nu\rangle$ is an inductive decoration strategy for* **CL** *if and only if (μ, ν) is adequate.*

PROOF: That adequacy is a necessary condition has already been shown. It is also sufficient: given an adequate pair (μ, ν) define $p^{\circledcirc} := p$ for p atomic; then take $\bot^{\circledcirc} := \mathbf{0}$, $\top^{\circledcirc} := \top$, and

	multiplicative	additive
$(A \to B)^{\circledcirc}$	$max(\mu,\nu)A^{\circledcirc} \multimap max(\mu,\nu)B^{\circledcirc}$	$max(\mu,\nu)A^{\circledcirc} \rightsquigarrow max(\mu,\nu)B^{\circledcirc}$
$(A \wedge B)^{\circledcirc}$	$\begin{cases} !\nu A^{\circledcirc} \otimes !\nu B^{\circledcirc}, & \text{if } \nu > \mu \\ \mu A^{\circledcirc} \otimes \mu B^{\circledcirc}, & \text{otherwise} \end{cases}$	$max(\mu,\nu)A^{\circledcirc} \,\&\, max(\mu,\nu)B^{\circledcirc}$
$(A \vee B)^{\circledcirc}$	$\begin{cases} ?\mu A^{\circledcirc} \parr ?\mu B^{\circledcirc}, & \text{if } \mu > \nu \\ \nu A^{\circledcirc} \parr \nu B^{\circledcirc}, & \text{otherwise} \end{cases}$	$max(\mu,\nu)A^{\circledcirc} \oplus max(\mu,\nu)B^{\circledcirc}$
$(\forall pA)^{\circledcirc}$	$\forall p.\, max(\mu,\nu)A^{\circledcirc}$	$\forall p.\, max(\mu,\nu)A^{\circledcirc}$
$(\exists pA)^{\circledcirc}$	$\exists p.\, max(\mu,\nu)A^{\circledcirc}$	$\exists p.\, max(\mu,\nu)A^{\circledcirc}$

It is straightforward to verify that $\langle(\cdot)^{\circledcirc}, \mu, \nu\rangle$ is an inductive decoration strategy for **CL**. In particular **CL** $\vdash \Gamma \Rightarrow \Delta$ if and only if **CLL** $\vdash \mu\Gamma^{\circledcirc} \Rightarrow \nu\Delta^{\circledcirc}$. ⊠

Note that we obtain a similar result for **IL** if we take μ, ν to be modalities consisting of 0 or more occurrences of '!' (and of course drop the condition that ν be of the for $?\nu'$ from the definition of adequacy).

We will separately state the modal translations corresponding (as in the proof of theorem 3.1) to the *two*, simplest possible, adequate pairs, namely (!, ?!) and (!?, ?), to which we refer as the q-, respectively the

[6] We will write $\mu < \nu$ in case the modality μ is a *proper* suffix of ν. E.g. !!?! <?!!!?!. If $\mu < \nu$ or $\nu < \mu$, then $max(\mu, \nu)$ indicates the largest of these two modalities with respect to this relation.

q	multiplicative	additive
$(A \rightarrow B)^{\mathsf{q}}$	$?!A^{\mathsf{q}} \multimap ?!B^{\mathsf{q}}$	$?!A^{\mathsf{q}} \leadsto ?!B^{\mathsf{q}}$
$(A \wedge B)^{\mathsf{q}}$	$!?!A^{\mathsf{q}} \otimes !?!B^{\mathsf{q}}$	$?!A^{\mathsf{q}} \& ?!B^{\mathsf{q}}$
$(A \vee B)^{\mathsf{q}}$	$?!A^{\mathsf{q}} \parr ?!B^{\mathsf{q}}$	$?!A^{\mathsf{q}} \oplus ?!B^{\mathsf{q}}$
$(\forall p)^{\mathsf{q}}$	$\forall p ?!A^{\mathsf{q}}$	$\forall p ?!A^{\mathsf{q}}$
$(\exists p)^{\mathsf{q}}$	$\exists p ?!A^{\mathsf{q}}$	$\exists p ?!A^{\mathsf{q}}$

t	multiplicative	additive
$(A \rightarrow B)^{\mathsf{t}}$	$!?A^{\mathsf{t}} \multimap !?B^{\mathsf{t}}$	$!?A^{\mathsf{t}} \leadsto !?B^{\mathsf{t}}$
$(A \wedge B)^{\mathsf{t}}$	$!?A^{\mathsf{t}} \otimes !?B^{\mathsf{t}}$	$!?A^{\mathsf{t}} \& !?B^{\mathsf{t}}$
$(A \vee B)^{\mathsf{t}}$	$?!?A^{\mathsf{t}} \parr ?!?B^{\mathsf{t}}$	$!?A^{\mathsf{t}} \oplus !?B^{\mathsf{t}}$
$(\forall p)^{\mathsf{t}}$	$\forall p !?A^{\mathsf{t}}$	$\forall p !?A^{\mathsf{t}}$
$(\exists p)^{\mathsf{t}}$	$\exists p !?A^{\mathsf{t}}$	$\exists p !?A^{\mathsf{t}}$

Table 3.2: The q- and the t-translation

t-translation. (As we will see in section 5.5, q and t are, in a way, the *unique* inductive decoration strategies for CL.)

Recall the modal translations Q, T of classical logic, introduced in section 4 of the previous chapter. The q-translation suggests itself when one tries inductively to transform CL-derivations into linear derivations, replacing sequents $\Gamma \Rightarrow \Delta$ by $!\Gamma^{\mathrm{Q}} \Rightarrow ?!\Delta^{\mathrm{Q}}$, the t-translation when replacing sequents $\Gamma \Rightarrow \Delta$ by $!?\Gamma^{\mathrm{T}} \Rightarrow ?\Delta^{\mathrm{T}}$, *without* using the correction cuts of (the proof of) theorem 2.4.1. The translation $(\cdot)^{©}$ in theorem 3.1 is completely determined by the fact that sequents $\Gamma \Rightarrow \Delta$ are to be interpreted as $\mu\Gamma^{©} \Rightarrow \nu\Delta^{©}$, cf. the footnote on page 47.

So the q-translation is related to the Q-translation, and the t-translation to the T-translation, as is the $(\cdot)^{\circledast}$-translation to Girard's embedding $(\cdot)^{*}$ in the intuitionistic case.

The q-translation is given by:

put $p^{\mathsf{q}} := p$ for atoms p; next take $\bot^{\mathsf{q}} := \mathbf{0}$, $\top^{\mathsf{q}} := \top$ and, according to whether we use the logical rules in their additive or multiplicative formulation, take the corresponding clause of the q-part of table 3.2.

The t-translation is given by:

put $p^{\mathsf{t}} := p$ for atoms p; again take $\bot^{\mathsf{t}} := \mathbf{0}$, $\top^{\mathsf{t}} := \top$ and, depending on the use of either additive or multiplicative versions of the logical connectives, take the corresponding clause of the t-part of table 3.2.

The inductive decoration strategies determined by q and t are *almost* free of ambiguity: in all but two cases there is only one possible way to continue. E.g., there is only *one* way to derive $!A^{\mathsf{q}} \Rightarrow ?!A^{\mathsf{q}}$ from $A^{\mathsf{q}} \Rightarrow A^{\mathsf{q}}$ by means of exponential contextual and dereliction rules; and given derivations of $!?\Gamma_1^{\mathsf{t}} \Rightarrow ?A^{\mathsf{t}}, ?\Delta_1^{\mathsf{t}}$ and $!?B^{\mathsf{t}}, !?\Gamma_2^{\mathsf{t}} \Rightarrow ?\Delta_2^{\mathsf{t}}$ there is but *one* way to derive $!?\Gamma_1^{\mathsf{t}}, !?\Gamma_2^{\mathsf{t}}, !?A^{\mathsf{t}} \multimap !?B^{\mathsf{t}} \Rightarrow ?\Delta_1^{\mathsf{t}}, ?\Delta_2^{\mathsf{t}}$ in accordance with definition 1.1.

The exceptions are the case of (multiplicative) L$\wedge$ for q, and (dually) that of (multiplicative) R$\vee$ for t. To take one of the two as an example, if we start from a derivation π of $!?\Gamma^{\mathsf{t}} \Rightarrow ?A^{\mathsf{t}}, ?B^{\mathsf{t}}, ?\Delta^{\mathsf{t}}$ we can continue in *two* different ways:

$$\cfrac{\cfrac{\cfrac{\cfrac{!?\Gamma^{\mathsf{t}} \Rightarrow ?A^{\mathsf{t}},\ ?B^{\mathsf{t}},\ ?\Delta^{\mathsf{t}}}{!?\Gamma^{\mathsf{t}} \Rightarrow !?A^{\mathsf{t}},\ ?B^{\mathsf{t}},\ ?\Delta^{\mathsf{t}}}}{!?\Gamma^{\mathsf{t}} \Rightarrow ?!?A^{\mathsf{t}},\ ?B^{\mathsf{t}},\ ?\Delta^{\mathsf{t}}}}{!?\Gamma^{\mathsf{t}} \Rightarrow ?!?A^{\mathsf{t}},\ !?B^{\mathsf{t}},\ ?\Delta^{\mathsf{t}}}}{!?\Gamma^{\mathsf{t}} \Rightarrow ?!?A^{\mathsf{t}},\ ?!?B^{\mathsf{t}},\ ?\Delta^{\mathsf{t}}} \quad \text{or, symmetrically,} \quad \cfrac{\cfrac{\cfrac{\cfrac{!?\Gamma^{\mathsf{t}} \Rightarrow ?A^{\mathsf{t}},\ ?B^{\mathsf{t}},\ ?\Delta^{\mathsf{t}}}{!?\Gamma^{\mathsf{t}} \Rightarrow ?A^{\mathsf{t}},\ !?B^{\mathsf{t}},\ ?\Delta^{\mathsf{t}}}}{!?\Gamma^{\mathsf{t}} \Rightarrow ?A^{\mathsf{t}},\ ?!?B^{\mathsf{t}},\ ?\Delta^{\mathsf{t}}}}{!?\Gamma^{\mathsf{t}} \Rightarrow !?A^{\mathsf{t}},\ ?!?B^{\mathsf{t}},\ ?\Delta^{\mathsf{t}}}}{!?\Gamma^{\mathsf{t}} \Rightarrow ?!?A^{\mathsf{t}},\ ?!?B^{\mathsf{t}},\ ?\Delta^{\mathsf{t}}}$$

Neither of these has an obvious reason to be preferable above the other, and if the CL-derivation π contains n instances of R$\vee$, we have to choose among 2^n distinct candidates for π^{t}. It is at this level, as soon as modalities of length > 2 are involved, that we encounter non-determinism in mapping a classical derivation to the subcollection of the collection of its linear decorations containing only decorations that are based upon a fixed modal translation. One should compare this to similar observations in Girard(1991). Cf. also sections 2.4 and 6.2.

Observe that the triple modalities in e.g. the t-decoration, will play a role *only* in the reductions of logical cuts. The two different decorations correspond to the *two* ways in which, in a CL-derivation π, we can reduce a cut of the form

$$
\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\\ \Gamma \Rightarrow A, B, \Delta\end{matrix}}{\Gamma \Rightarrow A \vee B, \Delta} \qquad \dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Gamma_1, A \Rightarrow \Delta_1\end{matrix} \quad \begin{matrix}\pi_2\\ \vdots\\ \Gamma_2, B \Rightarrow \Delta_2\end{matrix}}{\Gamma_1, \Gamma_2, A \vee B \Rightarrow \Delta_1, \Delta_2}}{\Gamma, \Gamma_1, \Gamma_2 \Rightarrow \Delta, \Delta_1, \Delta_2}
$$

Applying the first variant of the decoration (π^{t}_I) corresponds to first apply a cut to π and π_1, and next to the result and π_2 (let us note this as $\frac{\pi_1}{\pi_2}$; the second variant (π^{t}_{II}) will correspond to $\frac{\pi_2}{\pi_1}$. Choosing the one, rather than the other, will in general lead to *distinct* results. But in CL the two cuts commute: $\frac{\pi_1}{\pi_2}$ reduces to $\frac{\pi_2}{\pi_1}$ and vice versa. In the linear decorations, however, they *don't* commute: π^{t}_I reduces to $\frac{\pi^{\mathsf{t}}_1}{\pi^{\mathsf{t}}_2}$, and π^{t}_{II} to $\frac{\pi^{\mathsf{t}}_2}{\pi^{\mathsf{t}}_1}$. But $\frac{\pi^{\mathsf{t}}_1}{\pi^{\mathsf{t}}_2}$ does *not* reduce to $\frac{\pi^{\mathsf{t}}_2}{\pi^{\mathsf{t}}_1}$, neither $\frac{\pi^{\mathsf{t}}_2}{\pi^{\mathsf{t}}_1}$ to $\frac{\pi^{\mathsf{t}}_1}{\pi^{\mathsf{t}}_2}$.

This indicates, maybe not too surprisingly, that we still need some *additional* information. (The decorations suggest that a possible way out is the use of *directed* connectives. We could e.g. distinguish between $A \overset{\curvearrowleft}{\vee} B$, $A \overset{\curvearrowright}{\vee} B$, and translate

$$
\begin{aligned}
(A \overset{\curvearrowright}{\vee} B)^{\mathsf{t}} &:= \; !?A^{\mathsf{t}} \otimes ?!?B^{\mathsf{t}}\\
(A \overset{\curvearrowleft}{\vee} B)^{\mathsf{t}} &:= \; ?!?A^{\mathsf{t}} \otimes !?B^{\mathsf{t}}.
\end{aligned}
$$

For obvious[7] reasons here we will here not follow this train of thought any further.)

Note, however, that for some fragments, most notably $\{\rightarrow, \forall\}$, both the q- and the t-decoration are entirely deterministic. As a result each of them determines, unequivocally, a procedure to eliminate cuts from the original proof as the 'reflection' of linear cut elimination applied to the decoration. (The t- and q-decorations are *strong* with respect to these procedures.)

[7] Time, space, aesthetics.

For example, in the case of t, the left premiss, π_1, of a cut is, in the decoration, a *box*:

$$\dfrac{\dfrac{\boxed{\begin{array}{c}\pi_1^{\mathbf{t}}\\ \vdots \\ !?\Gamma_1^{\mathbf{t}} \Rightarrow ?\Delta_1^{\mathbf{t}}, ?A^{\mathbf{t}}\end{array}}}{!?\Gamma_1^{\mathbf{t}} \Rightarrow ?\Delta_1^{\mathbf{t}}, !?A^{\mathbf{t}}} \qquad \begin{array}{c}\pi_2^{\mathbf{t}}\\ \vdots \\ !?A^{\mathbf{t}}, !?\Gamma_2^{\mathbf{t}} \Rightarrow ?\Delta_2^{\mathbf{t}}\end{array}}{!?\Gamma_1^{\mathbf{t}}, !?\Gamma_2^{\mathbf{t}} \Rightarrow ?\Delta_1^{\mathbf{t}}, ?\Delta_2^{\mathbf{t}}}$$

So we start by permuting the cut upwards in the right premiss π_2, meanwhile duplicating and erasing π_1, until we reach the introduction of the cutformula by (in the decoration: a dereliction immediately preceded by *promotion* (box!) and) a logical rule or an axiom. In case of an axiom we reduce; otherwise we found the following configuration

$$\dfrac{\begin{array}{c}\pi_1^{\mathbf{t}}\\ \vdots \\ !?\Gamma_1^{\mathbf{t}} \Rightarrow ?\Delta_1^{\mathbf{t}}, ?A^{\mathbf{t}}\end{array} \qquad \dfrac{\boxed{\begin{array}{c}(\pi_2^{\mathbf{t}})'\\ \vdots \\ ?A^{\mathbf{t}}, !?\Sigma^{\mathbf{t}} \Rightarrow ?\Pi^{\mathbf{t}}\end{array}}}{!?A^{\mathbf{t}}, !?\Sigma^{\mathbf{t}} \Rightarrow ?\Pi^{\mathbf{t}}}}{!?\Gamma_1^{\mathbf{t}}, !?\Sigma^{\mathbf{t}} \Rightarrow ?\Delta_1^{\mathbf{t}}, ?\Pi^{\mathbf{t}}}$$

and permute the the box $(\pi_2^{\mathbf{t}})'$ upwards in (the copy of) π_1, again maybe duplicating and/or erasing, until we reach the introduction of $(?)A^{(\mathbf{t})}$ by an axiom or (a dereliction immediately preceded by) a logical rule. Now reduce in both cases, either the axiom cut, or the resulting logical (linear) cut.

It follows from the strong normalization theorem for linear logic, that this procedure of cut elimination for the fragment of classical sequent calculus under consideration enjoys strong normalization. What is more, we have *confluence* modulo permutations allowed in CLL: the t-decorations of the resulting normal forms will all correspond to one and the same proofnet.

4 LKT and LKQ

As we have seen in section 2, the fact that Girard's translation of intuitionistic logic does not give rise to an inductive decoration strategy for **IL**-derivations shows us that some of the rules in the calculus are more liberal than necessary. Moreover, by having a close look at the structure of linear derivations of g-decorated sequents, we were able to formulate a sequent calculus that incorporates precisely the restrictions suggested by Girard's translation.

In the case of classical sequent calculus we can proceed in a similar way, and construct calculi that incorporate restrictions suggested by the Q- and T-translations (section 2.4). Observe that formulations will differ according to whether one choses the additive or multiplicative formulation of a given logical rule. The principle of finding appropriate rules however is in all cases the same, and the reader might try her or his hand at finding Q-rules for e.g. multiplicative conjunction etcetera. Complete formulations of a Q-calculus with multiplicative implication and conjunction, additive disjunction and of a T-calculus with multiplicative implication and disjunction, additive conjunction, are given in Joinet(1993). As for **ILU** here we will restrict ourselves to the formulation of fragments with implication and universal second order propositional quantification. (Note however that e.g. the extension of these fragments to include the *first order* universal quantifier is completely straightforward, and all results stated in what follows hold for these extensions as well. In proofs and definitions the case of the first order quantifier is treated completely analogous to that of the second order one.)

We will base the rules of a sequent calculus derived from the Q-translation upon the structure of linear derivations of sequents of the form $!\Gamma^{Q} \Rightarrow ?!\Delta^{Q}, !A^{Q}$. We start by observing the following.

4.1. LEMMA. *If in* $\{!, ?, \multimap, \forall_2^{Q}\}$ *we can derive a sequent of the form* $\Gamma_1^{Q}, !\Gamma_2^{Q}, ?!\Gamma_3^{Q} \Rightarrow ?!\Delta_3^{Q}, !\Delta_2^{Q}, \Delta_1^{Q}$ *(eventually using cuts on formulas of type* $A^{Q}, !A^{Q}$ *or* $?!A^{Q}$*), then* $|\Gamma_1 \cup \Gamma_3 \cup \Delta_2| \leq 1$.

PROOF: By induction on the length of derivations. ⊠

Identity axiom:

$$!A^{\mathrm{Q}} \Rightarrow !A^{\mathrm{Q}}$$

Logical rules:

$$\text{L}\multimap\ \frac{!\Gamma \Rightarrow ?!\Delta, !A \qquad ?!B, !\Gamma' \Rightarrow ?!\Delta'}{!(!A \multimap ?!B), !\Gamma, !\Gamma' \Rightarrow ?!\Delta, ?!\Delta'} \qquad \text{R}\multimap\ \frac{!\Gamma, !A \Rightarrow ?!B, ?!\Delta}{!\Gamma \Rightarrow ?!\Delta, !(!A \multimap ?!B)}$$

Rules for the second order quantifier (q not free in Γ, Δ):

$$\text{L}\forall_2\ \frac{?!A[X^{\mathrm{Q}}/p], !\Gamma \Rightarrow ?!\Delta}{!\forall p\, ?!A, !\Gamma \Rightarrow ?!\Delta} \qquad \text{R}\forall_2\ \frac{!\Gamma \Rightarrow ?!\Delta, ?!A[q/p]}{!\Gamma \Rightarrow ?!\Delta, !\forall p\, ?!A}$$

Exponential structural rules:

$$\text{W!}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, !A^{\mathrm{Q}} \Rightarrow \Delta} \qquad \text{W?}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow ?!A^{\mathrm{Q}}, \Delta} \qquad \text{C!}\ \frac{\Gamma, !A, !A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta} \qquad \text{C?}\ \frac{\Gamma \Rightarrow ?A, ?A, \Delta}{\Gamma \Rightarrow ?A, \Delta}$$

Exponential contextual and dereliction rule:

$$\text{L?}\ \frac{!\Gamma, !A \Rightarrow ?!\Delta}{!\Gamma, ?!A \Rightarrow ?!\Delta} \qquad \text{D}\ \frac{!\Gamma \Rightarrow ?!\Delta, !A}{!\Gamma \Rightarrow ?!\Delta, ?!A}$$

Table 3.3: Auxiliary calculus **LLQ**

In order to be able to continue in a given derivation with the main formula after the introduction of its main connective $\multimap$, or the quantifier $\forall_2$, the Q-translation obliges us to bang, or bang and question, that formula: we have to apply an exponential contextual rule. Now, applying R$\multimap$ to a sequent $!\Gamma^{\mathrm{Q}}, !A^{\mathrm{Q}} \Rightarrow ?!\Delta^{\mathrm{Q}}, ?!B^{\mathrm{Q}}, \Sigma^{\mathrm{Q}}$ (‡), in order to derive $!\Gamma^{\mathrm{Q}} \Rightarrow ?!\Delta^{\mathrm{Q}}, !A^{\mathrm{Q}} \multimap ?!B^{\mathrm{Q}}, \Sigma^{\mathrm{Q}}$, we are somewhat stuck in case $\Sigma \neq \emptyset$, as the only rules that can be applied afterwards to give us sequents of the right form are R$\multimap$ and R$\forall_2$, until $\Delta = \emptyset$, and our derivation necessarily ends. But the occurrence of sequents like (‡) is not at all excluded (cf. lemma 4.1). In order to avoid them, we will apply R! immediately following R$\multimap$ or R$\forall_2$, forcing $\Sigma = \emptyset$. This is incorporated in the auxiliary calculus **LLQ** of table 3.3.

Obviously **LLQ** is a fragment of $\{?, !, \multimap, \forall_2^{\mathrm{Q}}\}$, as there all of its rules are derivable. The specific formulation is in each case motivated either by contextual constraints, or the cardinality properties expressed in lemma 4.1. It is immediate that all formulas occurring in a derivation in **LLQ** are of one of the forms $!A^{\mathrm{Q}}$ or $?!A^{\mathrm{Q}}$, and at most one formula of the form $!A^{\mathrm{Q}}$ can occur in the succedent of an **LLQ**-sequent. Also one can show the following using an argument similar to that for g-decorated **ILU**-derivations in section 2 (page 50).

4.2. LEMMA. **LLQ** *is closed under cut.* ⊠

Consequently **LLQ** is a proper fragment of linear logic, and we extract the sequent calculus **LKQ** (table 3.4) directly from this linear fragment. It has sequents $\Gamma \Rightarrow \Delta; \Pi$, where, as in **ILU**, the symbol Π denotes a multi-set containing *at most* one (the *'queue'* or *tail-*)formula whose occurrence in the succedent of a sequent is distinguished by means of the symbol ";". In the linear interpretation it corresponds to a formula that has not (yet) been questioned.

The relation with **LLQ** is evident.

4.3. LEMMA. **LLQ** $\vdash !\Gamma^{\mathrm{Q}} \Rightarrow\ ?!\Delta^{\mathrm{Q}}, !\Pi^{\mathrm{Q}}$ iff **LKQ** $\vdash \Gamma \Rightarrow \Delta; \Pi$.

PROOF: By construction the skeleton of an **LLQ**-derivation of a sequent $!\Gamma^{\mathrm{Q}} \Rightarrow\ ?!\Delta^{\mathrm{Q}}, !\Pi^{\mathrm{Q}}$ is an **LKQ**-derivation of $\Gamma \Rightarrow \Delta; \Pi$. Conversely, $\langle(\cdot)^{\mathrm{Q}}, !, ?!, !\rangle$ determines an inductive decoration strategy for **LKQ**-derivations π, which gives rise to linear derivations π^{Q} that in fact are **LLQ**-derivations: applications of L$\multimap$ always are followed by applications of L! on the main formula, etcetera. ⊠

As for **ILU** we find an intrinsically definable cut elimination procedure for **LKQ**-derivations π that corresponds to the linear cut elimination procedure applied to π's Q-decoration π^{Q}, i.e. $\langle(\cdot)^{\mathrm{Q}}, !, ?!, !\rangle$ is a *strong* decoration strategy with respect to this procedure. Hence **LKQ** inherits the computational properties of **LL**.

In order to show that derivability of $\Gamma \Rightarrow \Delta; \Pi$ in **LKQ** corresponds to linear derivability of $!\Gamma^{\mathrm{Q}} \Rightarrow ?!\Delta^{\mathrm{Q}}, \Pi^{\mathrm{Q}}$ we need one more lemma, showing that in fact **LLQ** is complete for derivability of sequents of the form $!\Gamma^{\mathrm{Q}} \Rightarrow ?!\Delta^{\mathrm{Q}}, !\Pi^{\mathrm{Q}}$.

4.4. LEMMA. *A sequent* $!\Gamma^{\mathrm{Q}} \Rightarrow ?!\Delta^{\mathrm{Q}}, !\Pi^{\mathrm{Q}}$ *is derivable in* $\{?, !, \multimap, \forall_2^{\mathrm{Q}}\}$ *if and only if it is derivable in* **LLQ**.

Identity axiom:

$$A \Rightarrow\, ; A$$

Logical rules:

$$\text{L}\!\to \frac{\Gamma \Rightarrow \Delta; A \qquad B, \Gamma' \Rightarrow \Delta';}{\Gamma, \Gamma', A \to B \Rightarrow \Delta, \Delta';} \qquad \text{R}\!\to \frac{\Gamma, A \Rightarrow \Delta, B;}{\Gamma \Rightarrow \Delta; A \to B}$$

Rules for the second order quantifier (q not free in Γ, Δ):

$$\text{L}\forall_2 \frac{\Gamma, A[X/p] \Rightarrow \Delta;}{\Gamma, \forall p\, A \Rightarrow \Delta;} \qquad \text{R}\forall_2 \frac{\Gamma \Rightarrow \Delta, A[q/p];}{\Gamma \Rightarrow \Delta; \forall p\, A}$$

Structural rules:

$$\text{D} \frac{\Gamma \Rightarrow \Delta; A}{\Gamma \Rightarrow \Delta, A;}$$

$$\text{LW} \frac{\Gamma \Rightarrow \Delta; \Pi}{\Gamma, A \Rightarrow \Delta; \Pi} \qquad \text{RW} \frac{\Gamma \Rightarrow \Delta; \Pi}{\Gamma \Rightarrow A, \Delta; \Pi}$$

$$\text{LC} \frac{\Gamma, A, A \Rightarrow \Delta; \Pi}{\Gamma, A \Rightarrow \Delta; \Pi} \qquad \text{RC} \frac{\Gamma \Rightarrow A, A, \Delta; \Pi}{\Gamma \Rightarrow A, \Delta; \Pi}$$

Cut rules:

$$\text{tail} \frac{\Gamma \Rightarrow \Delta; A \qquad A, \Gamma' \Rightarrow \Delta'; \Pi}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'; \Pi} \qquad \text{mid} \frac{\Gamma \Rightarrow \Delta, A; \Pi \qquad A, \Gamma' \Rightarrow \Delta';}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'; \Pi}$$

Table 3.4: The calculus **LKQ**

PROOF: Of course we only consider the left-to-right implication, and proceed by induction on the length of cut free derivations in the given fragment, with only atomic instances of the identity axiom. We need a variation on the inversion lemma of section 2.1: if in our fragment under the given conditions we have a derivation of length n of a sequent $\Gamma \Rightarrow A \multimap B, \Delta$ or $\Gamma \Rightarrow \forall pA, \Delta$, then there are derivations of length $\leq n$ of the sequents $\Gamma, A \Rightarrow B, \Delta$ and $\Gamma \Rightarrow A, \Delta$ (taking the usual care with respect to naming variables).

If the last rule applied was a structural rule, we are done by induction hypothesis.

If our derivation ends with an application of R!, then the premiss of this rule is the conclusion of a derivation of a sequent $!\Gamma^{Q} \Rightarrow ?!\Delta^{Q}, A^{Q}$.

$$
\begin{array}{c}
\pi' \\
\vdots \\
\dfrac{!B^{\mathrm{Q}}, !\Gamma_2^{\mathrm{Q}} \Rightarrow ?!\Delta_2^{\mathrm{Q}}}{?!B^{\mathrm{Q}}, !\Gamma_2^{\mathrm{Q}} \Rightarrow ?!\Delta_2^{\mathrm{Q}}} \\
\vdots \\
\text{structural rules} \\
\vdots \\
\dfrac{?!B^{\mathrm{Q}}, !\Gamma_1^{\mathrm{Q}} \Rightarrow ?!\Delta_1^{\mathrm{Q}}}{\forall p\, ?!B^{\mathrm{Q}}, !\Gamma_1^{\mathrm{Q}} \Rightarrow ?!\Delta_1^{\mathrm{Q}}} \\
\vdots \\
\text{structural rules} \\
\vdots \\
\dfrac{\forall p\, ?!B^{\mathrm{Q}}, !\Gamma_2^{\mathrm{Q}} \Rightarrow ?!\Delta_2^{\mathrm{Q}}}{!\forall p\, ?!B^{\mathrm{Q}}, !\Gamma_2^{\mathrm{Q}} \Rightarrow ?!\Delta_2^{\mathrm{Q}}}
\end{array}
\qquad \text{or} \qquad
\begin{array}{c}
\begin{array}{cc} \pi_1 & \pi_2 \\ \vdots & \vdots \end{array} \\
\dfrac{!\Gamma_{11}^{\mathrm{Q}} \Rightarrow ?!\Delta_{11}^{\mathrm{Q}}, !B^{\mathrm{Q}} \qquad ?!C^{\mathrm{Q}}, !\Gamma_{12}^{\mathrm{Q}} \Rightarrow ?!\Delta_{12}^{\mathrm{Q}}}{!B^{\mathrm{Q}} \multimap ?!C^{\mathrm{Q}}, !\Gamma_1^{\mathrm{Q}} \Rightarrow ?!\Delta_1^{\mathrm{Q}}} \\
\vdots \\
\text{structural rules} \\
\vdots \\
\dfrac{!B^{\mathrm{Q}} \multimap ?!C^{\mathrm{Q}}, !\Gamma^{\mathrm{Q}} \Rightarrow ?!\Delta^{\mathrm{Q}}}{!(!B^{\mathrm{Q}} \multimap ?!C^{\mathrm{Q}}), !\Gamma^{\mathrm{Q}} \Rightarrow ?!\Delta^{\mathrm{Q}}}
\end{array}
$$

Figure 3.2: "...in one of two possible ways..."

Now use the inversion lemma and apply the induction hypothesis.

If the last rule applied was R? we know by lemma 4.1 that the premiss of this rule is the conclusion of a derivation of $!\Gamma^{\mathrm{Q}} \Rightarrow ?!\Delta^{\mathrm{Q}}, !A^{\mathrm{Q}}$, to which we may apply the induction hypothesis.

Last but not least we need to consider the case that the final rule is L!. Once more using lemma 4.1 we know that the premiss in that case necessarily is the conclusion of a derivation of a sequent $A^{\mathrm{Q}}, !\Gamma^{\mathrm{Q}} \Rightarrow ?!\Delta^{\mathrm{Q}}$. Also, by the same lemma, this conclusion can be separated from an application of a logical rule having A^{Q} as main formula only by a certain number of applications of structural rules. Thus we find that our derivation ends in one of two possible ways, as depicted in figure 3.2.

We reach the desired conclusion by induction hypothesis and an easy permutation of rules. ⊠

4.5. PROPOSITION. $!\Gamma^{\mathrm{Q}} \Rightarrow ?!\Delta^{\mathrm{Q}}, \Pi^{\mathrm{Q}}$ *is derivable in* $\{?, !, \multimap, \forall_2^{\mathrm{Q}}\}$ *if and only if* $\Gamma \Rightarrow \Delta; \Pi$ *is derivable in* **LKQ**.

PROOF: As $!\Gamma^{\mathrm{Q}} \Rightarrow ?!\Delta^{\mathrm{Q}}, \Pi^{\mathrm{Q}}$ is derivable if and only if $!\Gamma^{\mathrm{Q}} \Rightarrow ?!\Delta^{\mathrm{Q}}, !\Pi^{\mathrm{Q}}$ is derivable this is immediate by lemmas 4.3 and 4.4. ⊠

In fact also $\langle (\cdot)^{\mathrm{Q}}, !, ?!, \cdot \rangle$ determines an inductive decoration strategy for **LKQ**-derivations π, resulting in linear derivations $\pi^{\mathrm{Q}'}$. It is again not difficult to verify that, in the same sense as for **ILU**, **LLQ**, the collection of Q′-decorated **LKQ**-derivations is closed under linear cut, which gives us an alternative intrinsically definable cut elimination procedure $\sigma^{\mathrm{Q}'}$ for **LKQ**-derivations. In general application of $\sigma^{\mathrm{Q}'}$ will result in normal forms that are *different* from those obtained using procedure σ^{Q}. (We can in fact indicate precisely in what respect σ^{Q} differs from $\sigma^{\mathrm{Q}'}$. Further on we will encounter a similar dichotomy in the case of the T-translation, for which we will describe the difference in detail.)

LKQ *lacks* the stronger property characterizing **ILU** (theorem 2.3). The skeleton of a linear derivation of the Q-translation of a sequent derivable in **LKQ** will not necessarily be an **LKQ**-derivation. The reader will easily find an example.[8]

This 'strong conservativity' however *does* hold for the calculus **LKT** (table 3.5) based upon the T-translation, which is the classical analogue of **ILU**. (Note that when we delete all occurrences of '?' in the T-translation, what we get is Girard's translation $(\cdot)^*$; for Q what we find is $(\cdot)^{\circledast}$.) Here we find sequents $\Pi; \Gamma \Rightarrow \Delta$, with Π containing at most one distinguished formula, the '*tête*' or *head*-formula. As in **ILU**, it corresponds to a formula that has not yet been subjected to non-linear manipulations on the left. Moreover, **ILU** is obtainable from **LKT** by the usual intuitionistic restriction of the succedents of sequents to singletons.

Included we find the *negative* fragment of classical implicational logic as it appears in **LU** (Girard(1993)).

As the next theorem, the 'classical version' of theorem 2.3, shows, derivability of $\Pi; \Gamma \Rightarrow \Delta$ in **LKT** corresponds in a strong sense to linear derivability of $\Pi^{\mathrm{T}}, !?\Gamma^{\mathrm{T}} \Rightarrow ?\Delta^{\mathrm{T}}$ in the $\{!, ?, \multimap, \forall_2^{\mathrm{T}}\}$-fragment of linear logic, where $\forall_2^{\mathrm{T}}$ indicates abstraction restricted to formulas of the form X^{T}.

[8]Hint: try the Q-translation $\Rightarrow ?!(!p \multimap ?!q), ?!p$ of $\Rightarrow p \to q, p;$.

Identity axiom:

$$A;\ \Rightarrow A$$

Logical rules:

$$\text{L}{\to}\ \frac{;\Gamma \Rightarrow \Delta, A \qquad B;\Gamma' \Rightarrow \Delta'}{A \to B;\Gamma,\Gamma' \Rightarrow \Delta,\Delta'} \qquad \text{R}{\to}\ \frac{\Pi;\Gamma, A \Rightarrow \Delta, B}{\Pi;\Gamma \Rightarrow \Delta, A \to B}$$

Rules for the second order quantifier (q not free in Π, Γ, Δ):

$$\text{L}\forall_2\ \frac{A[X/p];\Gamma \Rightarrow \Delta}{\forall p\, A;\Gamma \Rightarrow \Delta} \qquad \text{R}\forall_2\ \frac{\Pi;\Gamma \Rightarrow \Delta, A[q/p]}{\Pi;\Gamma \Rightarrow \Delta, \forall p\, A}$$

Structural rules:

$$\text{D}\ \frac{A;\Gamma \Rightarrow \Delta}{;A,\Gamma \Rightarrow \Delta}$$

$$\text{LW}\ \frac{\Pi;\Gamma \Rightarrow \Delta}{\Pi;\Gamma, A \Rightarrow \Delta} \qquad \text{RW}\ \frac{\Pi;\Gamma \Rightarrow \Delta}{\Pi;\Gamma \Rightarrow A, \Delta}$$

$$\text{LC}\ \frac{\Pi;\Gamma, A, A \Rightarrow \Delta}{\Pi;\Gamma, A \Rightarrow \Delta} \qquad \text{RC}\ \frac{\Pi;\Gamma \Rightarrow A, A, \Delta}{\Pi;\Gamma \Rightarrow A, \Delta}$$

Cut rules:

$$\text{head}\ \frac{\Pi;\Gamma \Rightarrow \Delta, A \qquad A;\Gamma' \Rightarrow \Delta'}{\Pi;\Gamma,\Gamma' \Rightarrow \Delta,\Delta'} \qquad \text{mid}\ \frac{;\Gamma \Rightarrow \Delta, A \qquad \Pi; A,\Gamma' \Rightarrow \Delta'}{\Pi;\Gamma,\Gamma' \Rightarrow \Delta,\Delta'}$$

Table 3.5: The calculus **LKT**

4.6. THEOREM. *If π is a derivation in $\{!, ?, \multimap, \forall_2^{\mathrm{T}}\}$ of a sequent*

$$\Gamma_1^{\mathrm{T}}, ?\Gamma_2^{\mathrm{T}}, !?\Gamma_3^{\mathrm{T}} \Rightarrow !?\Delta_3^{\mathrm{T}}, ?\Delta_2^{\mathrm{T}}, \Delta_1^{\mathrm{T}}$$

and all occurrences of cutformulas are of the form $A^{\mathrm{T}}, ?A^{\mathrm{T}}$ or $!?A^{\mathrm{T}}$, all axioms of the form $A^{\mathrm{T}} \Rightarrow A^{\mathrm{T}}$, then $\mathsf{sk}(\pi)$ is an **LKT***-derivation of*

$$\Gamma_1 \cup \Gamma_2; \Gamma_3 \Rightarrow \Delta_3, \Delta_2, \Delta_1.$$

PROOF: Similar to that of theorem 2.3. One proceeds by induction on the length of π, now showing simultaneously that the sequents considered always satisfy $|\Gamma_1 \cup \Gamma_2 \cup \Delta_3| \leq 1$. ⊠

As in the case for **LKQ** we find *two* inductive decoration strategies[9] for **LKT**-derivations π:

(1) We have the T-decoration $\langle (\cdot)^{\mathrm{T}}, ?, !?, ? \rangle$, resulting in linear derivations π^{T} and

(2) there is the T′-decoration $\langle (\cdot)^{\mathrm{T}}, \cdot, !?, ? \rangle$, resulting in linear derivations $\pi^{\mathrm{T}'}$.

Consequently, also for **LKT** we find *two* procedures (σ^{T} and $\sigma^{\mathrm{T}'}$) for eliminating cuts from an **LKT**-derivation π, corresponding to the reflection in **LKT** of linear cut elimination in π^{T}, respectively in $\pi^{\mathrm{T}'}$.

Let us have a closer look at the difference. Consider a mid-cut in an **LKT**-derivation π.

$$\dfrac{\begin{matrix}\pi_1 \\ \vdots \\ ;\Gamma_1 \Rightarrow \Delta_1, A\end{matrix} \quad \begin{matrix}\pi_2 \\ \vdots \\ \Pi; A, \Gamma_2 \Rightarrow \Delta_2\end{matrix}}{\Pi; \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

The resulting cut in π^{T} is of the following form:

$$\dfrac{\dfrac{\boxed{\begin{matrix}\pi_1^{\mathrm{T}} \\ \vdots \\ !?\Gamma_1^{\mathrm{T}} \Rightarrow ?\Delta_1^{\mathrm{T}}, ?A^{\mathrm{T}}\end{matrix}}}{!?\Gamma_1^{\mathrm{T}} \Rightarrow ?\Delta_1^{\mathrm{T}}, !?A^{\mathrm{T}}} \quad \begin{matrix}\pi_2^{\mathrm{T}} \\ \vdots \\ ?\Pi^{\mathrm{T}}, !?A^{\mathrm{T}}, !?\Gamma_2^{\mathrm{T}} \Rightarrow ?\Delta_2^{\mathrm{T}}\end{matrix}}{?\Pi^{\mathrm{T}}, !?\Gamma_1^{\mathrm{T}}, !?\Gamma_2^{\mathrm{T}} \Rightarrow ?\Delta_1^{\mathrm{T}}, ?\Delta_2^{\mathrm{T}}}$$

In $\pi^{\mathrm{T}'}$ we find:

[9] Observe that both are free from ambiguities: starting from axioms $A^{\mathrm{T}} \Rightarrow A^{\mathrm{T}}$, respectively $?A^{\mathrm{T}} \Rightarrow ?A^{\mathrm{T}}$ the decorations π^{T} and $\pi^{\mathrm{T}'}$ are uniquely determined. A similar remark applies to the decorations π^{Q} and $\pi^{\mathrm{Q}'}$ of an **LKQ**-derivation π.

$$\dfrac{\dfrac{\boxed{\begin{array}{c}\pi_1^{\mathrm{T}'}\\ \vdots\\ !?\Gamma_1^{\mathrm{T}} \Rightarrow ?\Delta_1^{\mathrm{T}}, ?A^{\mathrm{T}}\end{array}}}{!?\Gamma_1^{\mathrm{T}} \Rightarrow ?\Delta_1^{\mathrm{T}}, !?A^{\mathrm{T}}} \qquad \begin{array}{c}\pi_2^{\mathrm{T}'}\\ \vdots\\ \Pi^{\mathrm{T}}, !?A^{\mathrm{T}}, !?\Gamma_2^{\mathrm{T}} \Rightarrow ?\Delta_2^{\mathrm{T}}\end{array}}{\Pi^{\mathrm{T}}, !?\Gamma_1^{\mathrm{T}}, !?\Gamma_2^{\mathrm{T}} \Rightarrow ?\Delta_1^{\mathrm{T}}, ?\Delta_2^{\mathrm{T}}}$$

Hence, in both cases, in its linear interpretation the left premiss of an **LKT**-mid-cut is a *box*. Eliminating the cut means permuting the box upwards, meanwhile possibly duplicating and/or erasing it, while searching for the introductions by means of dereliction of the exclamation mark prefixing $?A^{\mathrm{T}}$. Having found (one of) these, the mid-cut becomes a head-cut:

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Pi; \Gamma_1 \Rightarrow \Delta_1, A\end{array} \qquad \begin{array}{c}\pi_2\\ \vdots\\ A; \Gamma_2 \Rightarrow \Delta_2\end{array}}{\Pi; \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

which in π^{T} is of the form

$$\dfrac{\begin{array}{c}\pi_1^{\mathrm{T}}\\ \vdots\\ ?\Pi^{\mathrm{T}}, !?\Gamma_1^{\mathrm{T}} \Rightarrow ?\Delta_1^{\mathrm{T}}, ?A^{\mathrm{T}}\end{array} \qquad \begin{array}{c}\pi_2^{\mathrm{T}}\\ \vdots\\ ?A^{\mathrm{T}}, !?\Gamma_2^{\mathrm{T}} \Rightarrow ?\Delta_2^{\mathrm{T}}\end{array}}{?\Pi^{\mathrm{T}}, !?\Gamma_1^{\mathrm{T}}, !?\Gamma_2^{\mathrm{T}} \Rightarrow ?\Delta_1^{\mathrm{T}}, ?\Delta_2^{\mathrm{T}}}$$

while in $\pi^{\mathrm{T}'}$ we find

$$\dfrac{\begin{array}{c}\pi_1^{\mathrm{T}'}\\ \vdots\\ \Pi^{\mathrm{T}}, !?\Gamma_1^{\mathrm{T}} \Rightarrow ?\Delta_1^{\mathrm{T}}, ?A^{\mathrm{T}}\end{array} \qquad \dfrac{\boxed{\begin{array}{c}\pi_2^{\mathrm{T}'}\\ \vdots\\ A^{\mathrm{T}}, !?\Gamma_2^{\mathrm{T}} \Rightarrow ?\Delta_2^{\mathrm{T}}\end{array}}}{?A^{\mathrm{T}}, !?\Gamma_2^{\mathrm{T}} \Rightarrow ?\Delta_2^{\mathrm{T}}}}{\Pi^{\mathrm{T}}, !?\Gamma_1^{\mathrm{T}}, !?\Gamma_2^{\mathrm{T}} \Rightarrow ?\Delta_1^{\mathrm{T}}, ?\Delta_2^{\mathrm{T}}}$$

Only in $\pi^{T'}$ the right premiss is necessarily a box. It is this difference which accounts precisely for the distinction between σ^{T} and $\sigma^{T'}$: when eliminating a *head-cut* in π^{T}, we are going to 'look for the box', i.e. we permute π_1^{T} upwards in π_2^{T}, until we reach the introduction of the question mark prefixing A^{T}. This is either an axiom (in which case we are done), or a promotion-rule, immediately preceded by a logical rule introducing the main connective of A^{T}. In this last case we 'found the box', which we then permute upwards in π_1^{T}, meanwhile possibly duplicating and/or erasing it, while searching for the introductions by means of dereliction of the question mark prefixing A^{T}.

On the other hand, eliminating a head-cut in $\pi^{T'}$ means permuting the box $\pi_2^{T'}$ upwards in $\pi_1^{T'}$, meanwhile possibly duplicating and/or erasing it, while searching for the introductions by means of dereliction of the question mark prefixing A^{T}. Having found (one of) these, it is necessarily preceded directly by a logical rule introducing A^{T}'s main connective, or an axiom. In the latter case we are done; in the former one we permute the corresponding subderivation upwards in $\pi_2^{T'}$ until we reach an axiom or the logical rule introducing A^{T}'s main connective.

So the difference between σ^{T} and $\sigma^{T'}$ is essentially in the size of the subderivations being erased and/or duplicated during elimination of head-cuts, this size being *maximal* in $\sigma^{T'}$, and *minimal* in σ^{T}.

5 Constructive classical logic

Recently there has been a lot of interest in the extraction of a computational meaning from proofs in classical logic, more specifically, in the extension of the *programming-with-proofs* paradigm, as known from intuitionistic[10], to *classical* (second order) logic (see e.g. Parigot(1993a)). As is well known, by a result dating back to Kreisel (see Friedman(1978), Leivant(1985)), the interest can not lie in the access to new representable functions, as all functions representable, in, say, classical second order predicate calculus, are already so in the *intuitionistic* second order system. No, the main goal is new *algorithms*: one will have access to other *proofs* of theorems of the form $\forall x(N(x) \to N(f(x)))$.

[10]Independently due to Jean-Louis Krivine and Daniel Leivant. See e.g. Krivine and Parigot(1990).

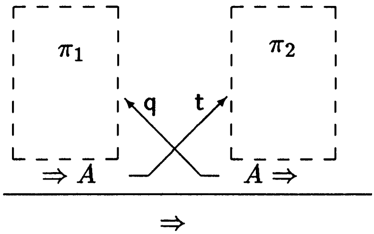
A sine qua non for this aim to be realizable, is a proof system for (a suitable fragment of) classical logic, having formulas to represent integers, lists, etcetera ('datatypes'), and with a 'reasonable' normalization or cut elimination procedure: it should be strongly normalizing and confluent (at least with respect to proofs of these datatypes).

As we saw in the previous section, **LKT** inherits the computational properties of linear logic. It is a proof system for classical logic that corresponds to a proper fragment of $\mathbf{CLL}_2$, and e.g. strong normalization of both σ^{T} and $\sigma^{\mathrm{T}'}$ is an immediate corollary to the strong normalization theorem for reductions in linear logic. A similar remark of course can be made with respect to **LKQ**. Moreover, confluence holds in both cases (that is (at least) for the representation of the derivations as proofnets). Hence both systems present themselves as potential participants in the 'programming-with-classical-logic'-quest. But they are by no means the only ones.

By the results of section 3, also e.g. the $\{\rightarrow, \forall, \forall_2\}$-fragment of **CL** in its usual formulation satisfies the criteria of 'strong normalizability' and (with a little care, cf. the end of section 3.3) 'confluence', provided we choose the adequate reduction. Either that defined as the reflection of the t-decoration ($\mathbf{LK}^{\mathsf{t}}$), or that defined by means of the q-decoration ($\mathbf{LK}^{\mathsf{q}}$) will do. And this is not yet the end of the story. Our technique of 'decorating' and 'pulling back' the linear elimination procedure to the skeleton of the proof provides us with a whole range of 'constructivizations of classical, through linear, logic'. One might for example define 'mixed' decorations of a proof. First arbitrarily assign 'types' t, q to occurrences of formulas in the proof, with the only restriction that the assignment should respect 'identity-classes'[11]. The multisets in a sequent $\Gamma \Rightarrow \Delta$ then, both in the succedent and the antecedent, are partitioned in formulas of type q and of type t, so the sequent has the form $\Gamma_{\mathsf{q}}, \Gamma_{\mathsf{t}} \Rightarrow \Delta_{\mathsf{t}}, \Delta_{\mathsf{q}}$. In the mixed decoration this will become $!\Gamma_{\mathsf{q}}^{\phi}, !?\Gamma_{\mathsf{t}}^{\phi} \Rightarrow ?\Delta_{\mathsf{t}}^{\phi}, ?!\Delta_{\mathsf{q}}^{\phi}$, where $(\cdot)^{\phi}$ now is a translation that will *depend on the types* of the components of the formula to be translated. If, given such a translation, the resulting inductive decoration strategy for the typed proof is *deterministic*, we again find a strongly normalizing and

[11]I.e. corresponding occurrences of the same formula get the same type. This notion will be introduced more formally in section 5.1.

'confluent' reduction as a reflection of the linear elimination procedure of the decorated proof. This reduction now will depend on the *type* of the cutformula.



If the cutformula is of type t, then the *left* premiss is a box (in the decoration); is it of type q, then the *right* premiss is a box. We now eliminate the box as in $\mathbf{LK}^{\mathsf{t}}$ in the first, as in $\mathbf{LK}^{\mathsf{q}}$ in the second case.

But what is the translation $(\cdot)^\Phi$? As always, it will be the identity on atoms, and if *both* components of a binary formula are of type t(q) we simply take the t(q)-translation. So we need only determine $(\cdot)^\Phi$ in case the components are of distinct type. Let us, as an example, look at $A \to B$ in case the rules are multiplicatively formulated.

- If A is of type t, and B of type q, then, inductively applying the $(\cdot)^\Phi$-translation we find by induction hypothesis in the premiss of the unary rule $!?A^\Phi \Rightarrow ?!B^\Phi$, in the premisses of the binary rule $\Rightarrow ?A^\Phi$ and $!B^\Phi \Rightarrow$. This suggests taking $(A \to B)^\Phi := !?A^\Phi \multimap ?!B^\Phi$, which indeed is easily shown to be correct: for the unary rule, we apply $R \multimap$, for the binary rule we are done by a promotion in the right and a promotion in the left premiss, followed by an application of $L \multimap$.

- If conversely A is of type q, and B of type t, then, inductively applying $(\cdot)^\Phi$, we find by induction hypothesis in the premiss of the unary rule $!A^\Phi \Rightarrow ?B^\Phi$, in the premisses of the binary rule $\Rightarrow ?!A^\Phi$ and $!?B^\Phi \Rightarrow$. Here we are forced to use *triple* modalities, and we stumble once more upon the nondeterminism we encountered earlier in the q-translation of the *conjunction*, and in the t-translation of the *disjunction*, and which (cf. section 3) is closely related to the *order* in which the two cuts, resulting from a logical cut where the main connective of the cutformula is the 'problematic' one, are performed. The diffluence in the case of $A_{\mathsf{q}} \to B_{\mathsf{t}}$ is illustrated in figure 3.3. Each of the two possible triple

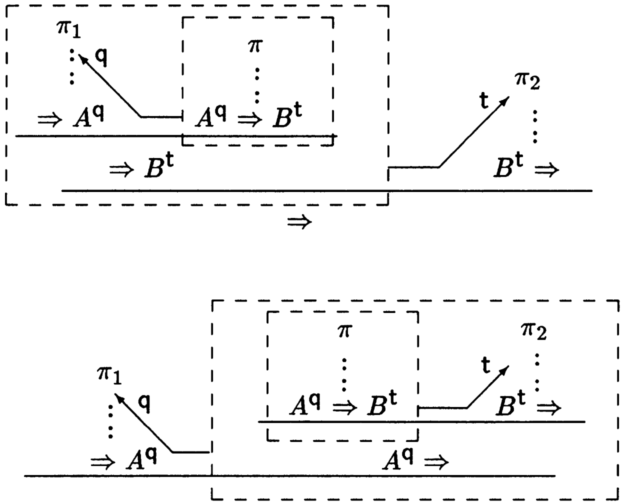


Figure 3.3: The q/t diffluence

decorations will lead to exactly *one* of these configurations. The reader will find that in the other three cases (t/t, q/q and t/q), where the decoration is deterministic, the order is irrelevant.

We leave the determination of the mixed translation for the other two binary connectives to the reader, and merely observe that for these we do not encounter any non-determinism.

Combining these observations with those of section 3, we find the following table:

A	B	$\to$	$\wedge$	$\vee$
t	t			↯
q	t	↯		
t	q			
q	q		↯	

A "↯" indicates that when decorating derivations that use *multi-*

plicatively formulated rules for these connectives, we need triple modalities, whence the decoration of all instances of the *unary* rule in question will be non-deterministic. It is here that the fundamental resistance of cut elimination in the classical sequent calculus to a computational interpretation resides. But it is also immediate that there are *several* ways out.

- A dubious one is to *go additive*: recall that the additive fragment of linear logic is far less well behaved than the multiplicative one, as confluence fails (though we do have a *semantical* invariance).
- One can opt for *oriented* connectives, as indicated in section 3. We leave this possibility open to further inquiries.
- We can ask for *linearity* of the active formulas in the (binary) rule dual to the problematic unary one, which will suppress diffluences like those of figure 3.3. This, as a matter of fact, is the solution in (the extension to the other connectives of) **LKT** and **LKQ** (see Joinet(1993)). The embeddings Q and T of section 2.4 tell us that imposing this restriction on proofs is conservative over provability in classical logic!
- Also we can take the 'additive way out' in a more subtle manner, by asking additivity *only* for the problematic unary rules.[12]

This diagnosis is not new, and neither are the cures. (The simple analysis and localization of the problem in terms of determinism or non-determinism of the linear decoration of classical sequent derivations, however, is (we think) surprising!). Similar observations can be found e.g. in Parigot(1991), who responded by constructing his system of 'Free Deduction' (**FD**), a fully symmetric system with an 'internal' notion of cut (as in natural deduction) in which, besides the identity axiom and (im- or explicitly) the usual structural rules, one has only *elimination* rules. Here are the **FD**-rules for implication:

[12]The attentive reader will object that in view of our interpretation in **CLL** this is problematic, as a cut between an additively and a multiplicatively translated formula will not be correct. She is right. It is however possible to use a *multiplicative* translation that enables us to *simulate* the reduction we are after. For lack of time and space we can not go into details, and have to leave these to forthcoming reports on this work.

$$\frac{\Gamma, A \to B \Rightarrow \Delta \quad \Pi, A \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad \frac{\Gamma, A \to B \Rightarrow \Delta \quad \Pi \Rightarrow B, \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$\frac{\Gamma \Rightarrow A \to B, \Delta \quad \Pi \Rightarrow A, \Sigma \quad \Pi', B \Rightarrow \Sigma'}{\Gamma, \Pi, \Pi' \Rightarrow, \Delta, \Sigma, \Sigma'}$$

Note that the first two are rules of an *additive* kind, while the third is in *multiplicative* formulation. In this system Parigot(1991) defines notions of *logical* and *structural* cut and their elimination, which in the logical case depends on a choice of *typing* (either, in his terminology, 'input *left*' (**l**) or 'input *right*' (**r**)) of the formulas involved. He shows that for every choice of 'inputs' (arbitrary, but, as above, respecting identity-classes) one obtains a deterministic computation mechanism satisfying confluence and strong normalization.

As he observes, one might replace the two additive rules given above by *one* multiplicative one. Then, however, non-determinism re-enters the ring.

There is a simple way to interpret an $\mathbf{FD}^{lr}$-proof (i.e. a 'typed' **FD**-proof): read q for **r**, and t for **l**, and transform each **FD**-rule by the corresponding instance of the cut-rule. E.g., using an additive ('irrelevant') implication right-rule, we replace

$$\frac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma, (A_l \to B_r)_l \Rightarrow \Delta\end{array} \quad \begin{array}{c}\pi_2\\ \vdots\\ \Pi, A_l \Rightarrow \Sigma\end{array}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

by

$$\frac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma, (A_{\mathsf{t}} \to B_{\mathsf{q}})_{\mathsf{t}} \Rightarrow \Delta\end{array} \quad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \Pi, A_{\mathsf{t}} \Rightarrow \Sigma\end{array}}{\Pi \Rightarrow (A_{\mathsf{t}} \to B_{\mathsf{q}})_{\mathsf{t}}, \Sigma}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

The result is, of course, an $\mathbf{LK}^{tq}$-derivation. What is more, we can interpret the notion of **FD**-cut in this interpretation, and simulate **FD**-reductions by t/q-reductions. So given the fact that we can in turn simulate these t/q-reductions in linear logic, we find as a corollary that **FD**-reductions are strongly normalizing.

A particular subsystem of **FD** with all inputs fixed to the left (and, as a matter of fact, with *multiplicative* rules for implication), is that of *classical natural deduction* (**CND**, see Parigot(1992)), a simple extension of intuitionistic natural deduction allowing the occurrence of multiple conclusions. Such a system is not new by its shape. The novelty of Parigot's approach is the 'Curry-Howard-De Bruijn correspondence' that he obtains in this classical case via an extension of lambda calculus, his so-called *$\lambda\mu$-calculus.*

CND corresponds to **LKT**, and the **CND**-reductions can be simulated in **LKT** by the reduction procedure σ^{T}. This is worked out in detail in Danos et al.(1993b). Again, as a corollary we find strong normalization for **CND**-reductions, and thus can provide an alternative for the argument in Parigot(1993b).

This discussion of constructivizations of classical logic would not be complete without mentioning Girard's *'Logique Classique'* (**LC**, Girard(1991)), which combines several of the features discussed above. As in **FD**, we find a 'typing' of formulas: in **LC** one distinguishes between formulas of *positive* and those of *negative polarity* (cf. section 1.2). However, here the typing is not an *arbitrary* assignment of signs to identity classes in a given proof. Instead it is built into the system.

Each atomic formula occurs in two types, and the polarity of a compound formula is determined by the polarity of its components as in the following *polarity-table.*

A	B	$\wedge$	$\vee$	$\to$
+	+	+	+	−
−	+	+	−	+
+	−	+	−	−
−	−	−	−	−

The remarkable thing about this table is that it is *unique*, in the sense that it can be shown to be, among all possibilities, the one that allows for an optimal number of isomorphisms in Girard's denotational semantics for **LC**.

A second feature of **LC** is the combination of additive and multiplicative formulations of rules. And thirdly, at several places the *linearity* restraint is introduced for a distinguished formula.

LC-derivations can be decorated in a straightforward way. One interprets positive formulas P as $!P^{\wr}$, negative formulas N as $!N^{\wr}$, by putting $p := !p^{\wr}$ for positive atoms, $n := !n^{\wr}$ for negative ones, and using the following *translation-table.*[13]

A	B	$\wedge$	$\vee$	$\rightarrow$
P	Q	$!P^{\wr}\otimes!Q^{\wr}$	$!P^{\wr}\oplus!Q^{\wr}$	$!P^{\wr}\multimap ?!Q^{\wr}$
N	Q	$!?N^{\wr}\otimes!Q^{\wr}$	$?N^{\wr}\parr ?!Q^{\wr}$	$?N^{\wr}\leadsto !Q^{\wr}$
P	M	$!P^{\wr}\otimes!?M^{\wr}$	$?!P^{\wr}\parr ?M^{\wr}$	$!P^{\wr}\multimap ?M^{\wr}$
N	M	$?N^{\wr}\&?M^{\wr}$	$?N^{\wr}\parr ?M^{\wr}$	$!?N^{\wr}\multimap ?M^{\wr}$

For quantified formulas, polarities and translations are as follows.

A	$\forall xA$		$\exists xA$	
P	$-$	$\forall x?!P^{\wr}$	$+$	$\exists x!P^{\wr}$
N	$-$	$\forall x?N^{\wr}$	$+$	$\exists x!?N^{\wr}$

Note that indeed, whenever a formula F has a certain sign accordaing to the polarity-table, then $F^{\wr}$ will be positive or negative in the sense of definition 1.2.1. Observe also that, when reconsidering the Q- and T-embedding of section 2.4, all entries of table 2.2.1 have a well-defined polarity which is independent of its constituents. E.g., when taken multiplicatively, $(A \rightarrow B)^{\mathrm{Q}}$ will *always* be negative; and, when taken additively, $(A \rightarrow B)^{\mathrm{T}}$ will *always* be positive. This property is lost when taking the even more economic translation $A^{\mathrm{Q}}\&B^{\mathrm{Q}}$ for the Q-conjunction, $A^{\mathrm{T}} \oplus B^{\mathrm{T}}$ for the T-disjunction.

[13] Girard(1991) formulates **LC** as a one-sided calculus. Our table in fact refers to a two-sided version, e.g. like the corresponding fragment of **LU** (Girard(1993)).

The Q-embedding and **LKQ** are clearly related to the +/+-part of **LC**, while the T-embedding and **LKT** relate to the −/−-part. These observations however are not yet sufficient to 'explain' **LC**.

We hope to come back to this question in later work.

Notes

- The material in this chapter is based upon corresponding material in Danos et al.(1993a), Danos et al.(1993b) and Danos et al.(1993c).

- The idea of sequents with compartments distinguishing linear and non-linear behaviour of formulas is due to Girard, who introduced it in his calculi **LC** (Girard(1991)) and **LU** (Girard(1993)). In theory of course there is no limit at all to the number of areas in a sequent that one might distinguish. It would e.g. be possible to formalize precisely the behaviour of linear derivations of sequents of the form $\Gamma_1^{\mathsf{T}}, ?\Gamma_2^{\mathsf{T}}, !?\Gamma_3^{\mathsf{T}} \Rightarrow !?\Delta_3^{\mathsf{T}}, ?\Delta_2^{\mathsf{T}}, \Delta_1^{\mathsf{T}}$ by means of a calculus containing sequents of the form $\Gamma_1; \Gamma_2; \Gamma_3 \Rightarrow \Delta_3; \Delta_2; \Delta_1$ where the three compartments at each side of the entailment sign contain formulas modalized by one of the modalities $\cdot, ?$ and $!?$. The notion of inductive decoration strategy for such calculi should be generalized accordingly: for a calculus built from sequents

$$\Gamma_1; \Gamma_2; \ldots; \Gamma_m \Rightarrow \Delta_1; \Delta_2; \ldots; \Delta_n,$$

a modal translation $(\cdot)^{\checkmark}$ and modalities $\mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n$ the $n+m+1$-tuple $\langle (\cdot)^{\checkmark}, \mu_1, \ldots, \mu_m, \nu_1, \ldots, \nu_n \rangle$ defines such a strategy for the calculus in question if the conditions of definition 1.1 are fulfilled when replacing sequents $\Gamma_1; \Gamma_2; \ldots; \Gamma_n \Rightarrow \Delta_1; \Delta_2; \ldots; \Delta_n$ by

$$\mu_1\Gamma_1; \mu_2\Gamma_2; \ldots; \mu_m\Gamma_m \Rightarrow \nu_1\Delta_1; \nu_2\Delta_2; \ldots; \nu_n\Delta_n.$$

- Note that not containing applications of L$\rightarrow$, L$\forall_2$ on non-linear formulas, though obviously *necessary*, is not a *sufficient* condition for an intuitionistic derivation in order to be (interpretable as) an **ILU**-derivation. Satisfying that restraint limits moreover the order in which rules can be applied to derive a certain sequent. Consider e.g. the following example:

$$\dfrac{A \Rightarrow A \qquad \dfrac{B \Rightarrow B \quad C \Rightarrow C}{B, B \rightarrow C \Rightarrow C}}{A, B \rightarrow C, A \rightarrow B \Rightarrow C}$$

- The 'strong conservativity' results of section 4 do not automatically extend to versions of the calculi with other connectives. For example, theorem 4.6 does no longer hold if we include $\otimes$.

4

Decorating intuitionistic derivations

There's another option that suggests itself when trying to transform a given intuitionistic or classical derivation into a linear one: start from a structural *source* (i.e. a weakened or contracted formula), trace it through the proof, and prefix its successive occurrences with a shriek (if the structural rule was to the left) or with a question mark (if it was to the right of the entailment sign). In the present chapter we will study this process in detail in a basic case, that of derivations in intuitionistic implicational logic (or, losely spoken, terms in the simply typed lambda calculus).

It is reasonable to expect that such a procedure, when properly applied, will result in decorations which in some sense are *minimal*, as opposed to those obtained by means of the inductive applications of plethoric modal translations described in the previous chapter.

Notice that for cut free proofs our task will be almost trivial: any source occurring will lead us straight to the conclusion, and the only small problem that we encounter resides in the possibility that we might pass a contraction 'on our way down'. If our derivation contains cuts, however, we may also enter a cut formula, which forces us to 'travel up' again in the subderivation determined by the other premiss. In that case, the formula we are tracing will emerge in this subderivation as succedent of some sequent. Banging it means introducing an instance of the *promotion*-rule, or of an *axiom* in which the succedent formula is shrieked. In both cases the linear derivation we are constructing can become *correct* only if all the formulas in the antecedent are banged as well. These formulas then in turn will act as sources, and we have to trace all of them, and shriek them throughout the proof. Thus each structural source will cause a 'cascade of shrieks' to cover the original derivation.

1 The lower decoration strategy I

It is clear how to proceed, though the formal description is somewhat tedious. In Joinet(1993) a notion of *path* is defined, somewhat similar to the *identity-classes* that we will encounter in the next chapter. Throughout this chapter we will assume that all identity axioms are atomic (we speak of 'fully expanded' derivations), and define for formulas F occurring in a derivation π in the calculus the *track* $T_\pi(F)$ of F in π, being (almost) a subtree of π with nodes labelled by a symbol $\sigma(F)$ denoting (among other things) the sign of F (i.e. $\sigma(F) =$ "+", "$\star$"or"$\triangledown$" if F occurs *positively*, $\sigma(F) =$ " $-$ " if F occurs *negatively* in the sequent $\Gamma \Rightarrow \Delta$ (i.e. in the formula $\bigwedge F \rightarrow \bigvee \Delta$)), or either $\circ$ or $\bullet$ (in case a (sub)formula "disappears" in a cut). Using the terminology of Regnier(1992), we will call a formula in π *terminal* if it either is a cutformula in π or a formula in the conclusion. Then, to be precise, we define $T_\pi(F)$ for (sub)formulas F of terminal formulas in π; in case F is a cutformula we put $T_\pi(F) := T_{\pi'}(F)$, where π' denotes the subderivation of π having the cut on F as a last rule. The occurrence of F in the terminal formula is called the *root* of $T_\pi(F)$.

Let us then state a formal inductive definition of $T_\pi(F)$ for the (multiplicative) sequent calculus formulation of intuitionistic implicational logic (appendix **d**).

1. if π is an axiom $F \Rightarrow F$, then, for the *left* occurrence of F, $T_\pi(F)$ is the one-node tree labelled by " $-$ "; for the right occurrence labelled by "$\triangledown$";
2. if π has been obtained from π' by means of rule R$\rightarrow$, then for all subformulas F of $\Gamma \cup \{A\} \cup \{B\}$, *except* B, we obtain $T_\pi(F)$ by adding a new node labelled by the appropriate sign of occurrence; we obtain $T_\pi(B)$ by adding a new node $\star$; $T_\pi(A \rightarrow B)$ is the one-node tree, labelled " $+$ ";
3. if π has been obtained from π_1 and π_2 by means of rule L$\rightarrow$, then for all subformulas F of $\Gamma_1 \cup \{A\}$, *except* A, we obtain $T_\pi(F)$ from $T_{\pi_1}(F)$ by adding a new node labelled by the appropriate sign of occurrence

(i.e. "+" or "−"); we obtain $T_\pi(A)$ by adding a new node labelled "$\star$"; for all subformulas F of $\Gamma_2 \cup \{B\} \cup \{C\}$ we obtain $T_\pi(F)$ from $T_{\pi_2}(F)$ by adding a new node labelled by the appropriate sign of occurrence, and $T_\pi(A \to B)$ is the one-node tree labelled by " − " (so a node labelled "$\star$" in $T_\pi(F)$ indicates that it is the immediate successor of a node corresponding to the *lowest* occurrence in π of F as a *succedent* (i.e. in a sequent $\Gamma \Rightarrow F$));

4. if π has been obtained from π' by means of a weakening with formula A, then for all subformulas F of A, $T_\pi(F)$ is the one-node tree labelled by the appropriate sign of occurrence; for all other F we get $T_\pi(F)$ from $T_{\pi'}(F)$ by adding a node labelled by the appropriate sign of occurrence;
5. if π has been obtained from π' by means of exchange (see the remark in appendix a), then we get $T_\pi(F)$ from $T_{\pi'}(F)$ by adding a node labelled by the appropriate sign of occurrence;
6. if π has been obtained from π' by means of a contraction on a formulas A, then for all subformulas A of F we obtain $T_\pi(F)$ by joining the trees $T_{\pi'}(F^{(1)})$ and $T_{\pi'}(F^{(2)})$ (where $F^{(1)}$ and $F^{(2)}$ denote the two distinct occurrences of F) to a node labelled by the appropriate sign of occurrence; for all other F we get $T_\pi(F)$ from $T_{\pi'}(F)$ by adding a new node, again labelled by the appropriate sign of occurrence;
7. if π has been obtained from π_1 and π_2 by means of application of cut on a formula A, then for all subformulas F of Γ_1 we obtain $T_\pi(F)$ from $T_{\pi_1}(F)$ by adding a new node labelled by the appropriate sign of occurrence, for all subformulas F of $\Gamma_2 \cup \{B\}$ we obtain $T_\pi(F)$ from $T_{\pi_2}(F)$ by adding a new node, again labelled by the appropriate sign of occurrence; for proper subformulas F of A, $T_\pi(F)$ is the tree obtained by joining $T_{\pi_1}(F)$ and $T_{\pi_2}(F)$ through a bottom-node labelled $\circ$. $T_\pi(A)$ is obtained by joining $T_{\pi_1}(A)$ and $T_{\pi_2}(A)$ through a bottom-node labelled " $\bullet$ " (so a node labelled " $\bullet$ " in $T_\pi(F)$ indicates (that F is a cutformula and) that it is the immediate successor of a node corresponding to the *lowest* occurrence in π of F as a succedent).

The idea is easily grasped by looking at some examples (see below).

1.1. PROPOSITION. *For each terminal formula F, $T_\pi(F)$ is either a finite tree with all nodes labelled $\neq$ " $\circ$ " (" $\bullet$ "), or a finite tree with bottom-node "$\circ$" ("$\bullet$"). Moreover, if all nodes are labelled $\neq$ "$\circ$" ("$\bullet$"), then either all nodes are labelled "+" ("$\star$", "$\triangledown$"), or all nodes are labelled*

"$-$". If the bottom-node is labelled "$\circ$" ("$\bullet$"), then the labels in the subtrees defined by the two predecessors of "$\circ$" ("$\bullet$") are all equal to "$-$" in one of the two, all unequal to "$-$" in the other. ⊠

Clearly *each occurrence* of a (sub)formula F in a given derivation π corresponds to a unique node i in $\mathcal{T}_\pi(F)$.

We define for a *negative* occurrence of a formula $N(i)$ the $N(i)$-*decoration* of $\mathcal{T}_\pi(N)$ as the labelled tree obtained from $\mathcal{T}_\pi(N)$ by the following instructions:

[1] replace the label at node i, as well as the labels "$-$" at all successors of i, by "$-!$";

[2] if the bottom-node is *not* labelled "$\circ$" or ("$\bullet$"), then we are done;

[3] if it is "$\bullet$" then change the label of its positive predecessor by "$+!$", and we are done;

[4] if it is "$\circ$", then for all branches of the *'positive'* subtree:

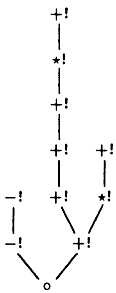
(a) if the branch contains a starred node, take the starred node's predecessor and add "!" to its label, as well as to the 'non-zero'-labels of all its successive successor-nodes;

(b) if the branch does *not* contain a starred node, then add "!" to the 'non-zero'-labels of all its nodes.

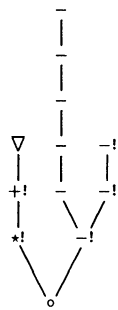
E.g. the decoration induced by the occurrence of AA (being an abbreviation of $A \to A$) introduced by weakening, respectively the decoration induced by the negative occurrence of A in the right-most axiom in the derivation

$$\dfrac{\dfrac{\dfrac{A\Rightarrow A}{A,AA\Rightarrow A}}{A\Rightarrow (AA)A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\Rightarrow A}{A,A\Rightarrow A}}{A\Rightarrow AA}\quad A\Rightarrow A}{A,(AA)A\Rightarrow A}}{(AA)A,A\Rightarrow A}}{(AA)A\Rightarrow AA}\qquad A\Rightarrow A}{(AA)A,(AA)A\Rightarrow A}}{(AA)A\Rightarrow A}}{A\Rightarrow A}$$

are



and



An $N(i)$-decoration induces in the obvious way a distribution of shrieks in the original derivation π: we 'bang' the occurrences of N that correspond to nodes of the $N(i)$-decoration whose label contain a "!". (We will refer to the result as a 'pre-decoration' of π.)

So, for the AA- and A-decoration given above we find the pre-decorations

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, !(AA) \Rightarrow A}}{A \Rightarrow !(AA)A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, A \Rightarrow A}}{A \Rightarrow !(AA)} \quad A \Rightarrow A}{A, !(AA)A \Rightarrow A}}{!(AA)A, A \Rightarrow A}}{!(AA)A \Rightarrow !(AA)} \quad A \Rightarrow A}{!(AA)A, !(AA)A \Rightarrow A}}{!(AA)A \Rightarrow A}}{A \Rightarrow A}$$

and

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, AA \Rightarrow !A}}{A \Rightarrow (AA)!A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, A \Rightarrow A}}{A \Rightarrow AA} \quad A \Rightarrow A}{A, (AA)A \Rightarrow A}}{(AA)A, A \Rightarrow A}}{(AA)A \Rightarrow AA} \quad !A \Rightarrow A}{(AA)A, (AA)!A \Rightarrow A}}{(AA)!A \Rightarrow A}}{A \Rightarrow A}$$

So this is what we should do: given a derivation we look at the collection of instances of weakening and contraction that have been used; we then start 'banging' the main formula, say $N(i)$, in the conclusion of such a rule ($N(i)$ is a '*primary source*'), and trace the formula through the derivation. This is done by means of the $N(i)$-decoration defined above. (Note that we made *one* choice: we stop putting shrieks as soon as we reach a *lowest* occurrence of the formula as *succedent* in a sequent. We will therefore speak of the *'lower decoration strategy'*; clearly there are other possibilities: one might stop only at the *highest* occurrences, or anywhere in between.)

The last shriek we put might be the conclusion of a sequent having $G_1, \ldots, G_n$ as premisses. In order to obtain a derivation that is correct in linear logic we have no choice but to bang these; consequently $G_1, \ldots, G_n$ will be '*secondary sources*', and the process continues until there is nothing left to be done.

To put this formally, starting with $N(i)$ we define a finite tree of decorations as follows:

[1] top node is the $N(i)$-decoration;

[2] let a node α be given, i.e. some $M(j)$-decoration;

(a) if α has a bottom-node labeled $-!$, then α has no successors;

(b) if α has bottom-node labeled $\circ$ or $\bullet$, then look at the highest nodes labeled ! in the positive subtree. If these are $\triangledown!$ or predecessors of $\star!$- nodes, then they correspond to sequents $\Gamma_1 \Rightarrow M(j), \ldots, \Gamma_n \Rightarrow M(j)$ in π. The successors of α then are the $G_m(i)$-decorations, for all $G_m(i) \in \Gamma_1 \cup \ldots \cup \Gamma_n$ that so far have not yet appeared in the tree.

(Observe that finiteness is clear, as there are but finitely many formulas in a given derivation.)

We thus obtain a finite tree of N-decorations for each primary source in the derivation. The *linearly decorated derivation* then is the original derivation with shrieks added in accordance with the superposition of all the corresponding pre-decorations.

It will suffice to look carefully at the following example to convince one-self that this is a completely self-evident process[1], though admittedly somewhat cumbersome to describe formally.

[1] As to the complexity of the procedure, this is easily seen to be linear in the *size* of (i.e. the number of subformulas appearing in) the original derivation.

1.2. EXAMPLE.

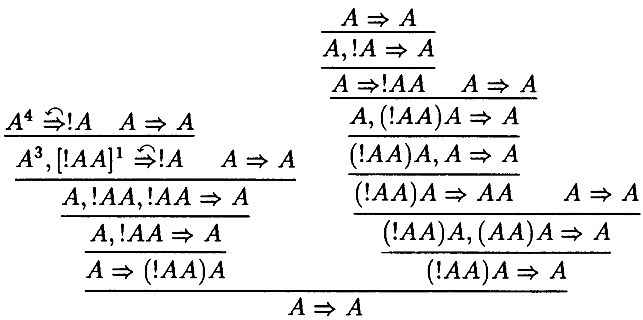
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad A \Rightarrow A}{A, AA \Rightarrow A} \quad A \Rightarrow A}{A, AA, AA \Rightarrow A}}{A, AA \Rightarrow A}}{A \Rightarrow (AA)A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, A \Rightarrow A}}{A \Rightarrow AA} \quad A \Rightarrow A}{A, (AA)A \Rightarrow A}}{(AA)A, A \Rightarrow A}}{(AA)A \Rightarrow AA} \quad A \Rightarrow A}{(AA)A, (AA)A \Rightarrow A}}{(AA)A \Rightarrow A}}{A \Rightarrow A}$$

In this derivation of $A \Rightarrow A$ there are *three* primary sources, namely the contracted occurrence of $(AA)A$, the contracted occurrence of AA in the left subtree and the weakened occurrence of A in the right subtree.

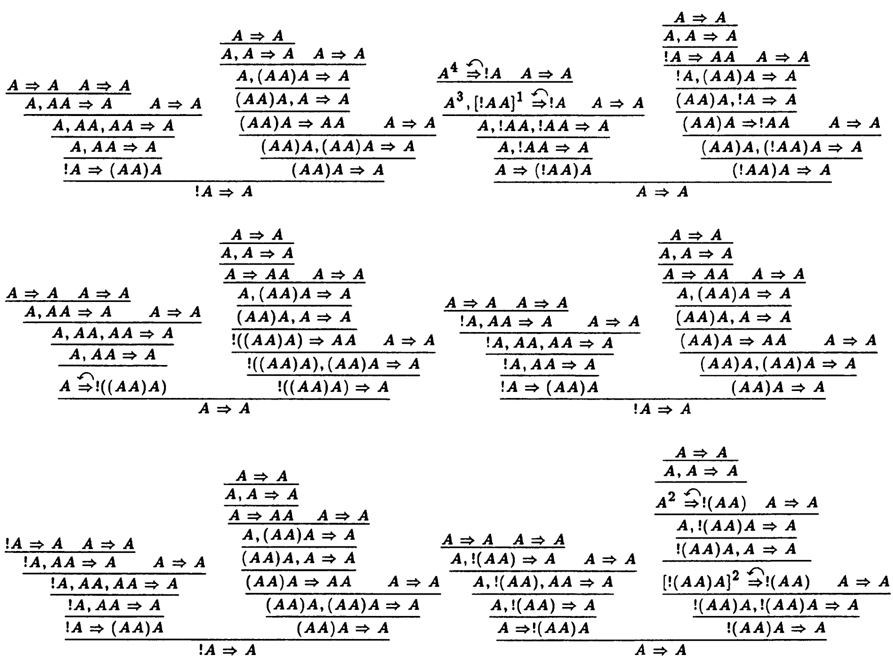
The induced pre-decorations are the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad A \Rightarrow A}{A, AA \Rightarrow A} \quad A \Rightarrow A}{A, AA, AA \Rightarrow A}}{A, AA \Rightarrow A}}{A^1 \overset{\frown}{\Rightarrow} !((AA)A)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, A \Rightarrow A}}{A \Rightarrow AA} \quad A \Rightarrow A}{A, (AA)A \Rightarrow A}}{(AA)A, A \Rightarrow A}}{(AA)A \Rightarrow AA} \quad A \Rightarrow A}{(AA)A, (AA)A \Rightarrow A}}{!((AA)A) \Rightarrow A}}{A \Rightarrow A}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad A \Rightarrow A}{A, AA \Rightarrow A} \quad A \Rightarrow A}{A, AA, AA \Rightarrow A}}{A, !(AA) \Rightarrow A}}{A \Rightarrow !(AA)A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, A \Rightarrow A}}{A^2 \overset{\frown}{\Rightarrow} !(AA)} \quad A \Rightarrow A}{A, !(AA)A \Rightarrow A}}{!(AA)A, A \Rightarrow A}}{[!(AA)A]^1 \overset{\frown}{\Rightarrow} !(AA)} \quad A \Rightarrow A}{!(AA)A, !(AA)A \Rightarrow A}}{!(AA)A \Rightarrow A}}{A \Rightarrow A}$$



We used $\overset{\frown}{\Rightarrow}$ to indicate where we get secondary sources, and marked those by means of superscripts: there are *six* of them, giving rise to the respectively the following pre-decorations:



As there are now no *new* secondary sources that have to be considered, our job ends by superposing all the pre-decorations obtained, thus giving us *the* decoration of our derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{!A\Rightarrow !A \quad A\Rightarrow A}{!A,!(!AA)\Rightarrow !A} \quad A\Rightarrow A}{!A,!(!AA),!AA\Rightarrow A}}{!A,!(!AA)\Rightarrow A}}{!A\Rightarrow !(!(!AA)A)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\Rightarrow A}{A,!A\Rightarrow A}}{!A\Rightarrow !(!AA)} \quad A\Rightarrow A}{!A,!(!AA)A\Rightarrow A}}{!(!AA)A,!A\Rightarrow A}}{!(!(!AA)A)\Rightarrow !(!AA)} \quad A\Rightarrow A}{!(!(!AA)A),!(!AA)A\Rightarrow A}}{!(!(!AA)A)\Rightarrow A}}{!A\Rightarrow A}$$

But then, did we, by applying the decoration strategy, in the end obtain a *correct* derivation in linear logic?

There are some deviations. E.g. we encounter instances of

$$\dfrac{\Gamma, C\Rightarrow B}{!\Gamma\Rightarrow !(CB)}$$

These, however, can be interpreted as being abbreviations for

$$\begin{array}{c}\dfrac{\Gamma, C\Rightarrow B}{\Gamma\Rightarrow CB} \\ \hline \vdots \\ \dfrac{!\Gamma\Rightarrow CB}{!\Gamma\Rightarrow !(CB)}\end{array}$$

More serious seems that in the decorated derivation we get contractions on formulas that, though having the same skeleton, are not identical as linear formulas. This is illustrated by the example.

In general, in a decorated derivation we will encounter instances of contraction of the form

$$(\dagger) \qquad \dfrac{\Gamma, \delta_1(A), \delta_2(A)\Rightarrow B}{\Gamma, \delta_3(A)\Rightarrow B},$$

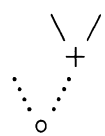
where $\delta_i(A)$ denotes a decoration of a formula A and all $\delta_i(A)$ are not necessarily identical.

We can however observe the following.

1.3. LEMMA. *If* (†) *is an instance of a contraction rule in a decorated derivation* π, *then* δ_1 *and* δ_2 *differ at most in the decoration of* positive *subformulas. Moreover* $\delta_3(A) = (\delta_1 \cup \delta_2)(A)$ *or* $\delta_3 = !(\delta_1 \cup \delta_2)(A)$ *(where* $\cup$ *stands for superposition of decorations).*

PROOF: It is sufficient to show that the decoration induced by each node of each finite tree belonging to the primary sources in π has the property.

Now for the decoration induced by A we have $\delta_1(A) = \delta_2(A) = A$ and $\delta_3(A) = !A$. Otherwise we have a decoration induced by a subformula of A occurring as a primary or secondary source somewhere else in π. If it is a *negative* subformula N, the source is in a subtree of π ending in a premiss of an instance of cut on a formula having A as a subformula, the other premiss being the conclusion of a subderivation of π containing (†). The instance of contraction corresponds to a splitting in the *positive* subtree of the track of N, and in fact we can take $\mathcal{T}_\pi(N)$ to be of the form



as neither the bottom-node of the splitting, nor any of its successors can be labelled "$\star$". But then, by definition of N-decoration *all* three nodes of the splitting will be shrieked. So the induced decoration satisfies $\delta_1(A) = \delta_2(A) = \delta_3(A)$.

If the subformula is positive, the source is in the subderivation of π ending with (†). Then only one of both occurrences of A in the premiss of (†) is decorated, but obviously it is a positive subformula that is shrieked. ⊠

If a linear formula A^+ has been obtained from a linear formula A by prefixing "!" to some number of *positive* subformulas of A, we call A^+ a *positive !-decoration* of A (note that A itself is not necessarily without modalities). One easily shows that $A^+ \Rightarrow A$ always is derivable in linear logic[2], and has a canonical derivation which is a decoration of the canonical derivation of the axiom $A \Rightarrow A$ (see also section 6.1).

So the instances (†) of contraction in our decorated derivation have the property that $\delta_3(A)$ is a positive !-decoration of both $\delta_1(A)$ and $\delta_2(A)$.

Let ϕ^+ be any positive !-decoration of ϕ, ϕ^-any negative !-decoration. By τ we denote the canonical derivation of $\phi^+ \Rightarrow \phi$ or $\phi \Rightarrow \phi^-$. Now consider a linear derivation π of a sequent $\Gamma, \phi \Rightarrow B$, or of a sequent $\Gamma \Rightarrow \phi$. We can use π and τ to construct a linear derivation θ as follows:

$$\dfrac{\begin{matrix}\tau \\ \vdots \\ \phi^+ \Rightarrow \phi\end{matrix} \quad \begin{matrix}\pi \\ \vdots \\ \Gamma, \phi \Rightarrow B\end{matrix}}{\Gamma, \phi^+ \Rightarrow B} \qquad \text{or} \qquad \dfrac{\begin{matrix}\pi \\ \vdots \\ \Gamma \Rightarrow \phi\end{matrix} \quad \begin{matrix}\tau \\ \vdots \\ \phi \Rightarrow \phi^-\end{matrix}}{\Gamma \Rightarrow \phi^-}$$

We then have the following property.

1.4. PROPOSITION. *We can eliminate the cut from θ in such a way that for the resulting reduct θ' it holds that* $\mathsf{sk}(\theta') = \mathsf{sk}(\pi)$.

PROOF: By induction on the complexity of ϕ: if $\phi \equiv p$ for some atom p, then $\phi^- = \phi$, so $\pi = \theta'$; ϕ^+ either is p or $!p$, so $\pi = \theta'$ or we obtain θ' from π by a linear dereliction (L!), which does not change the skeleton.

Let $\phi \equiv\, !\psi$. Then $\phi^+ =\, !\psi^+$, and $\phi^- =\, !\psi^-$. For the first case, we will prove the slightly stronger claim that in a derivation θ of the form

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\tau \\ \vdots \\ \psi^+ \Rightarrow \psi\end{matrix}}{!\psi^+ \Rightarrow \psi}}{!\psi^+ \Rightarrow !\psi} \quad \begin{matrix}\pi \\ \vdots \\ \Gamma, (!\psi)^n \Rightarrow B\end{matrix}}{\Gamma, (!\psi^+)^n \Rightarrow B}$$

(where $(\phi)^n$ stands for $n \geq 1$ occurrences of the formula ϕ) we can eliminate the (derivable) rule of "semi-multicut" in such a way that

[2]One might also look this up in the chapter on elementary syntactical results in Troelstra(1992).

for the reduct θ' we have that $\mathsf{sk}(\theta') = \mathsf{sk}(\pi)$. For this we proceed by induction on the length of π. We need the stronger hypothesis for the case where the last rule applied is a contraction on $!\psi$, which in the present form is trivially handled by inductive hypothesis, as are most other cases. The crucial one is the one where the last rule of π is a dereliction on ψ. Then θ has the form

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\tau\\ \vdots\\ \psi^+ \Rightarrow \psi\end{array}}{!\psi^+ \Rightarrow \psi}}{!\psi^+ \Rightarrow !\psi} \qquad \dfrac{\begin{array}{c}\pi\\ \vdots\\ \Gamma, (!\psi)^{n-1}, \psi \Rightarrow B\end{array}}{\Gamma, (!\psi)^n \Rightarrow B}}{\Gamma, (!\psi^+)^n \Rightarrow B}$$

If $n > 1$ we transform this into

$$\dfrac{\dfrac{\begin{array}{c}\tau\\ \vdots\\ \psi^+ \Rightarrow \psi\end{array} \qquad \dfrac{\begin{array}{c}\tau\\ \vdots\\ !\psi^+ \Rightarrow !\psi\end{array} \quad \begin{array}{c}\pi\\ \vdots\\ \Gamma, (!\psi)^{n-1}, \psi \Rightarrow B\end{array}}{\Gamma, (!\psi^+)^{n-1}, \psi \Rightarrow B}}{\Gamma, (!\psi^+)^{n-1}, \psi^+ \Rightarrow B}}{\Gamma, (!\psi^+)^n \Rightarrow B}$$

and we get the result by our inductive hypotheses.

For $\phi^- = !\psi^-$ we have θ of the form

$$\dfrac{\begin{array}{c}\pi\\ \vdots\\ \Gamma \Rightarrow !\psi\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\tau\\ \vdots\\ \psi \Rightarrow \psi^-\end{array}}{!\psi \Rightarrow \psi^-}}{!\psi \Rightarrow !\psi^-}}{\Gamma \Rightarrow !\psi^-}$$

and we apply induction on the length of π.

Finally we consider the case that $\phi \equiv \psi \multimap \chi$. Then $\phi^+ = !(\psi^- \multimap \chi^+)$ or $(\psi^- \multimap \chi^+)$ and $\phi^- = (\psi^+ \multimap \chi^-)$. Without loss of generality we may assume that $\phi^+ = \psi^- \multimap \chi^+$, and θ has the form

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\tau_1\\ \vdots\\ \psi \Rightarrow \psi^-\end{array} \quad \begin{array}{c}\tau_2\\ \vdots\\ \chi^+ \Rightarrow \chi\end{array}}{\psi^- \multimap \chi^+, \psi \Rightarrow \chi}}{\psi^- \multimap \chi^+ \Rightarrow \psi \multimap \chi} \quad \begin{array}{c}\pi\\ \vdots\\ \Gamma, \psi \multimap \chi \Rightarrow B\end{array}}{\Gamma, \psi^- \multimap \chi^+ \Rightarrow B}$$

We proceed rather straightforwardly by induction on the length of π. Similarly for ϕ^-. ⊠

By the above proposition (which, by the way, is an instance of a more general property, see section 6.1) we can replace any formula ϕ in the antecedents of sequents in a derivation in intuitionistic linear implicational sequent calculus by any of its positive !-decorations ϕ^+:

$$\frac{\Gamma, \phi \Rightarrow C}{\Gamma, \phi^+ \Rightarrow C}$$

is a derivable rule in a *strong* sense, i.e. when we justify it by means of a cut with $\phi^+ \Rightarrow \phi$ then we can eliminate that cut and obtain a derivation π^+ of $\Gamma, \phi^+ \Rightarrow C$ such that $\mathsf{sk}(\pi^+) = \mathsf{sk}(\pi)$. So elimination merely 'injects' a number of applications of exponential rules to 'adjust' the derivation.

(Note that this is in sharp contrast with the observations in section 3.1: also

$$\frac{\Gamma, !\phi \multimap !\psi \Rightarrow C}{\Gamma, !(!\phi \multimap \psi) \Rightarrow C}$$

is a derivable rule, but in this case elimination of the justifying cut may very well change the skeleton.)

We can apply this to our decorated intuitionistic derivations: (†) not only is derivable in linear logic, it is completely 'harmless', as, if we would wish so, we could apply cuts with $\delta_3(A) \Rightarrow \delta_i(A)$, eliminate these, and obtain a linear derivation in which all contractions are literally correct, with the same skeleton as our decoration.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{!A \Rightarrow !A \quad A \Rightarrow A}{!A, !(!AA) \Rightarrow !A} \quad A \Rightarrow A}{!A, !(!AA), !AA \Rightarrow A}}{!A, !(!AA), !(!AA) \Rightarrow A}}{!A, !(!AA) \Rightarrow A}}{!A \Rightarrow !(!(!AA)A)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{A, !A \Rightarrow A}}{!A \Rightarrow !(!AA)} \quad A \Rightarrow A}{!A, !(!AA)A \Rightarrow A}}{!(!AA)A, !A \Rightarrow A}}{!(!(!AA)A) \Rightarrow !(!AA)} \quad A \Rightarrow A}{!(!(!AA)A), !(!AA)A \Rightarrow A}}{!(!(!AA)A), !(!(!AA)A) \Rightarrow A}}{!(!(!AA)A) \Rightarrow A}}{!A \Rightarrow A}$$

Figure 4.1: $\partial(\pi)$

1.5. DEFINITION. Let π be a derivation in intuitionistic implicational logic. We denote by $\partial(\pi)$ the linear derivation obtained after correction of the instances of contraction in the result of the lower decoration-strategy applied to π. ⊠

So for the derivation π of example 1.2 we find $\partial(\pi)$ as in figure 4.1. Clearly $\partial(\pi)$ is a *minimal* decoration, in the sense that each shriek occurring in it has at least one structural justification. As, by construction, it moreover has the down-property, also the following is immediate:

1.6. PROPOSITION. *$\partial(\pi)$ is a subdecoration of the f-decoration $\pi^{\circledast}$ of π, i.e. if an occurrence A of a formula is shrieked in $\partial(\pi)$, then so is the corresponding occurrence in $\pi^{\circledast}$.* ⊠

The converse, of course, in general will not hold.

2 Decorating ILU

If we forget the semicolon, an **ILU**-derivation is just an 'ordinary' derivation in intuitionistic logic, and we can apply the lower decoration-strategy defined in the previous section. Recall that by definition this strategy stops shrieking a formula F at its *lowest* appearance in a sequent $\Gamma \Rightarrow F$. For **ILU**-derivations this gives rise to the following property.

2.1. PROPOSITION. *Let F be a negative non-head formula in an* **ILU**-*derivation π. Application of the lower strategy implies that the F-decoration never induces the shrieking of* head-*formulas.*

PROOF: If $\mathcal{T}_\pi(F)$ has bottom-node "−", this is trivial. Therefore, let $\mathcal{T}_\pi(F)$ have bottom-node $\circ$ or $\bullet$. We need to consider two cases, distinguishing between whether the cut at hand is *mid* or *head.*

Let us start with an instance of a mid-cut:

$$\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ ;\Gamma_1 \Rightarrow A\end{array} \quad \begin{array}{c}\pi_2\\ \vdots\\ \Pi;\Gamma_2, A \Rightarrow B\end{array}}{\Pi;\Gamma_1,\Gamma_2 \Rightarrow B}$$

F is a subformula of the cutformula A. If F 'originates' in π_1 (i.e. is *negative in* π_1), then A necessarily contains a subformula $F \rightarrow X$ which is introduced in π_1 by a *right* implication rule. So if $F \rightarrow X$ is introduced in π_2 by a logical rule, then it is by a *left* implication rule. Therefore lowest appearances of F in π_2 *to the right* of the entailment sign occur only in sequents without head-formula, and application of the lower strategy can not induce shrieking of head-formulas.

If F 'originates' in π_2 (i.e. is *negative* in π_2), then either $F \equiv A$ or there is a subformula $F \rightarrow X$ of A introduced in π_2 by a *right* implication rule. In the first case the lowest appearance of F in π_1 to the right of the entailment-sign is in the left premiss of the cut, which has no head-formula. In the second case we reason as before. Therefore also in this case application of the lower strategy can not induce shrieking of head-formulas.

In case of a head-cut

$$\dfrac{\begin{array}{cc}\pi_1 & \pi_2 \\ \vdots & \vdots \\ \Pi;\Gamma_1 \Rightarrow A & A;\Gamma_2 \Rightarrow B\end{array}}{\Pi;\Gamma_1,\Gamma_2 \Rightarrow B}$$

the argument is similar: just notice that when F originates in π_2 it will now not be possible for F to be identical to A. ⊠

As $\partial(\pi)$ has the down-property we get the following.

2.2. COROLLARY. *Let π be an* **ILU**-*derivation. Then $\partial(\pi)$ is a subdecoration of the g-decoration π^* of π (which in turn is a subdecoration of the f-decoration $\pi^{\circledast}$ of π, considered as an* **IL**-*proof).* ⊠

3 Decoration and normalization

When normalizing derivations in sequent calculus (i.e. eliminating the cuts) we come across counterparts of the structural rules of weakening and contraction: because in the derivation by means of an application of weakening a formula may suddenly appear as if out of thin air, there are elementary reduction steps in which complete subderivations are *erased*; and similarly, because in the derivation distinct occurrences of a formula can be contracted into a single one, we get elementary reduction steps in which subderivations are *duplicated.*

For any derivation in intuitionistic implicational logic we showed how to trace the linear consequences of occurrences of weakening and contraction in order to obtain a minimally decorated linear equivalent. Each 'shriek' occurring in such a decorated derivation has *at least one* structural cause. In this section we will look into the converse, namely whether a minimal decoration $\partial(\pi)$ of a derivation π can provide us with information as to the behaviour of π under reduction.[3]

Whereas weakening and contraction are about *formulas*, the notions of erasure and duplication are about (sub)*derivations.* However,

[3]Throughout this section we assume our derivations to be in **ILU**, but the reader will observe that our claims and arguments apply in fact to derivations in (the implicational fragment of intuitionistic) linear logic.

there are obvious candidates for the title of '*marked subderivation*' in a decorated derivation, namely those subderivations that end with an application of R!. As before, in analogy with the similar notion in proofnets, we speak of *boxes*.

3.1. DEFINITION. A subderivation π_1 of a derivation π (written as $\pi_1 \prec \pi$) is said to be *boxed* (or is said to be a *box*) in π iff it is a subderivation of a sequent $\Gamma \Rightarrow A$ that is *externally* decorated in $\partial(\pi)$ (i.e. all formulas in $\Gamma \cup \{A\}$ start with a bang). A subderivation $\pi_{\mathrm{s}} \prec \pi$ is said to be a *pseudo-box* (or *source*) in π if its last rule is an instance of weakening or contraction. (Note that a pseudo-box might be a box.) ⊠

With each box π_{i} we can uniquely associate a terminal formula in π. If this is the cutformula in an instance c_i of the cut rule in π, we call c_i is the *reflecting cut* for π_{i}. As an example, let us decorate (an **ILU**-version of) the derivation of page 82:

$$\dfrac{\dfrac{\dfrac{\dfrac{A;\Rightarrow A}{A;!(!AA) \overset{\mathrm{b}}{\Rightarrow} A}}{;A,!(!AA) \Rightarrow A}}{;!A \overset{3}{\Rightarrow} !(!(!AA)A)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A;\Rightarrow A}{A;!A \overset{\mathrm{a}}{\Rightarrow} A}}{A;\Rightarrow !AA}}{;!A \overset{1}{\Rightarrow} !(!AA)} \quad A;\Rightarrow A}{!(!AA)A;!A \Rightarrow A}}{!(!AA)A;\Rightarrow !AA}}{;!(!(!AA)A) \overset{2}{\Rightarrow} !(!AA)} \quad A;\Rightarrow A}{!(!AA)A;!(!(!AA)A) \Rightarrow A}}{;!(!AA)A,!(!(!AA)A) \Rightarrow A}}{;!(!(!AA)A) \overset{\mathrm{c}}{\Rightarrow} A}}{;!A \Rightarrow A}$$

We see that we have three boxes π_1, π_2 and π_3 (indicated above by 1, 2 and 3) and three sources $\pi_{\mathrm{a}}, \pi_{\mathrm{b}}$ and π_{c} (indicated by a, b and c) corresponding to the three primary sources in π: weakening of A, weakening of AA and contraction on $(AA)A$. (None of the pseudo-boxes is in fact a box.)

If we take the tree of decorations associated with a primary source N in π, then each (initial segment of a) branch of this tree determines a *chain* Σ consisting in a source followed by zero (a *trivial* chain) or more boxes in π. In the example we find the trivial chain π_a (the weakening on A has no consequences), the chain $\pi_b \to \pi_1$, the chain $\pi_b \to \pi_2 \to \pi_3$ and the chain $\pi_c \to \pi_3$.

3.2. DEFINITION. A box π_i in π is said to be *contained* in a box π_j (and π_j is said to *contain* π_i) if $\pi_i \prec \pi_j$. A chain $\Sigma \equiv \pi_s \to \pi_2 \to \ldots \to \pi_n$ is called *linear* if $\pi_i \prec \pi_j$ for no $i \neq j$. We say that $\pi_i \to \pi_{i+1}$ is a *linear link* in Σ if π_i, π_{i+1} neither are contained in, nor contain, other boxes from Σ. ⊠

In our example the two-element chains are linear (as are *all* two-element chains), but the three-element chain is not (as $\pi_b \prec \pi_3$).

Observe that a chain defines, in the obvious way, a *path* through the derivation π.[4]

3.3. DEFINITION. A non-trivial chain $\Sigma \equiv \pi_s \to \pi_1 \to \ldots \to \pi_n$ is called *strong* if all its reflecting cuts are distinct. It is called *adequate* if its induced path passes an instance of contraction always either by a side formula or by the *same* active formula. ⊠

Clearly, all two-element chains are strong. They are the *only* strong chains in our example. It is not true that all linear chains are strong.

Also it is quite obvious that strong chains always are adequate, while the converse is false. We will see in what follows that adequate chains (and therefore also strong chains) always are linear.

If we apply an elementary reduction step (of the procedure of cut elimination) to a derivation π and it turns out that in performing this step $\pi' \prec \pi$ is erased or duplicated, then π' is (a subderivation of) a box in π. This is clear, as then π either has the form

$$\dfrac{\begin{matrix}\pi_1\\ \vdots\\ ;\Gamma \Rightarrow A\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ \Pi;\Delta \Rightarrow B\end{matrix}}{\Pi;\Delta, A \Rightarrow B}}{\Pi;\Gamma,\Delta \Rightarrow B} \qquad \text{or} \qquad \dfrac{\begin{matrix}\pi_1\\ \vdots\\ ;\Gamma \Rightarrow A\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ \Pi;\Delta, A, A \Rightarrow B\end{matrix}}{\Pi;\Delta, A \Rightarrow B}}{\Pi;\Gamma,\Delta \Rightarrow B}$$

[4]The concepts of 'chain' and 'path' are similar to that of 'trace', which plays an important role in the study of proofnets, cf. Regnier(1992).

(with π_1 boxed) and reduces to respectively

$$\dfrac{\dfrac{\vdots}{\Pi;\Delta\Rightarrow B}}{\dfrac{\vdots}{\Pi;\Gamma,\Delta\Rightarrow B}} \qquad\text{and}\qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{;\Gamma\Rightarrow A}\quad \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\end{matrix}}{;\Gamma\Rightarrow A}\quad \dfrac{\vdots}{\Pi;\Delta,A,A\Rightarrow B}}{\Pi;\Gamma,\Delta,A\Rightarrow B}}{\Pi;\Gamma,\Gamma,\Delta\Rightarrow B}}{\dfrac{\vdots}{\Pi;\Gamma,\Delta\Rightarrow B}}$$

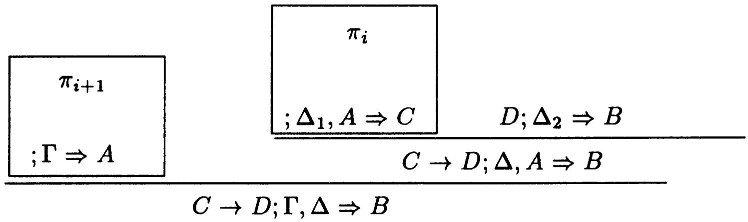
If $\pi_i \prec \pi$ is boxed and moreover element of an *adequate chain* Σ in π, then we can show the converse, i.e. there exists a series of reductions (a *reduction strategy*) starting from π, that eventually either will erase or duplicate (a copy of) π_i. An important step towards a proof of this is the following

3.4. PROPOSITION. *Let Σ be an adequate chain in π starting from a source s (so $\Sigma \equiv \pi_s \to \pi_1 \to \pi_2 \to \ldots \to \pi_n$ for some $n \geq 1$). Suppose $\pi_i \to \pi_{i+1}$ is a linear link in Σ. Then we can eliminate the reflecting cut c_i and decorate the reduct π' in such a way that we obtain an adequate chain $\Sigma' \equiv \pi_s \to \ldots \to \pi_{i-1} \to \pi^\star \to \pi_{i+2} \to \ldots \to \pi_n$. If $\pi_i \equiv \pi_s$, then π_1 will be erased or duplicated, depending on whether the source is a weakening or a contraction.*

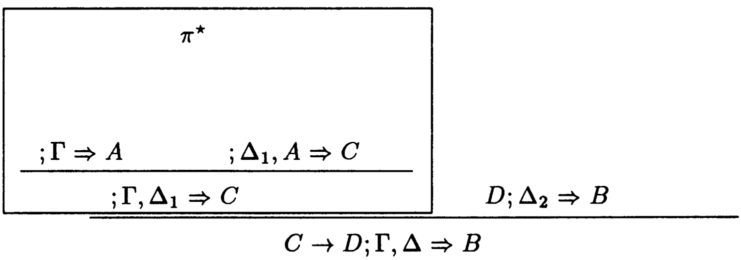
PROOF: To establish these claims one performs a long induction, considering all the possible configurations in which c_i can appear in π as in a proof of cut elimination. We will skip most of the details, and merely consider some important cases, indicating where the assumptions of linearity and adequacy are used.

- If $\pi_i \equiv \pi_s$, as basic cases we encounter precisely the two instances of erasure/duplication given above: the adequate chain $\pi_s \to \pi_1 \to \ldots \to \pi_n$ becomes the adequate chain $\pi_{s'} \to \pi_2 \to \ldots \to \pi_n$, where s′ is a weakening or contraction (of/on a formula in Γ), depending on whether s is a weakening or a contraction. (By linearity no boxes in Σ are contained in π_1.)

- 1. Consider the following situation:

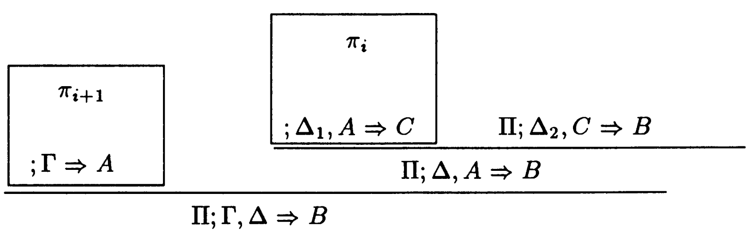


We reduce the reflecting cut as follows:



so π_i and π_{i+1} are 'merged', resulting in *one* box $\pi^\star$.

2. Similarly, performing the reduction, we get a 'merge' in the following situation:

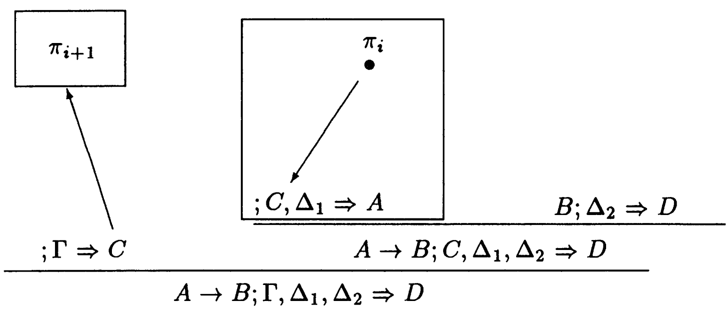


as one may readily check.

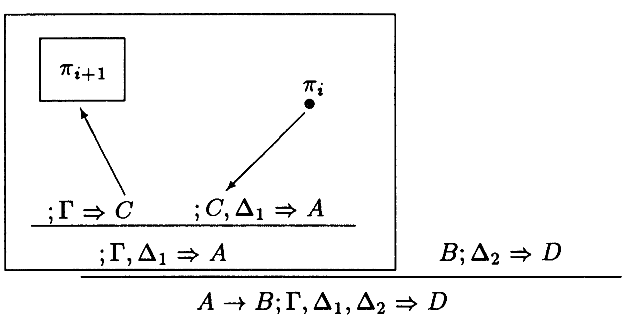
- In all other cases one can perform the reduction without effecting the adequate chain. However, one has to be careful, and both linearity of the link and adequacy of the chain are used.

  In case of a duplication of a box in the chain due to a reduction of a cut on a contracted formula, we find, due to adequacy, a copy of the box via which the chain can pass uneffected, and still adequate. But there are other 'traps': let us consider two important cases in detail.

1. Consider the situation:

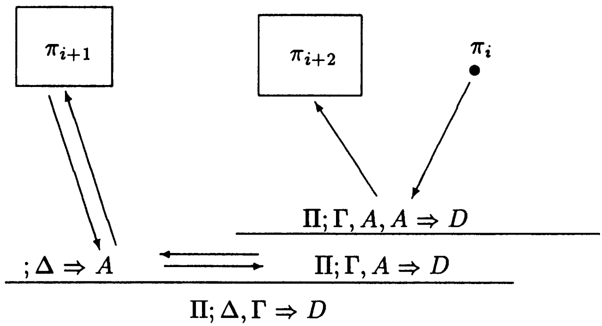


If the link is not linear, it might be the case that the subderivation of π with conclusion $;C, \Delta_1 \Rightarrow A$ is a box π_j in Σ, while at the same time we have no choice but to reduce the reflecting cut of π_i as follows:

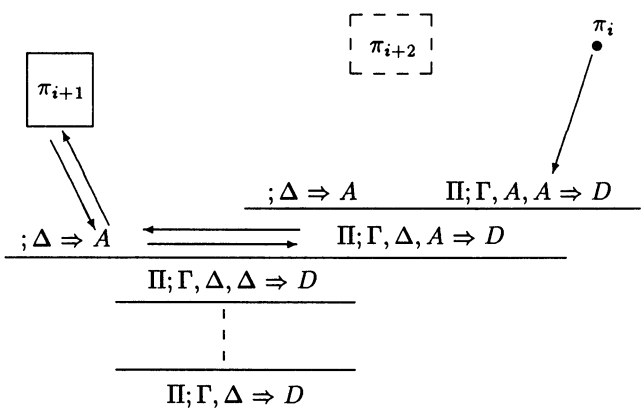


Decoration now gives us a box $\pi_{j'}$, with conclusion $; \Gamma, \Delta_1 \Rightarrow A$. Clearly $\pi_j \not\equiv \pi_{j'}$. Our assumption avoids this difficulty.

2. Another possible configuration is:



If Σ is not adequate, then we might have $c_i \equiv c_{i+1}$ and an induced path γ that passes from the source via the *right* occurrence of A, but then continues from π_{i+1} to π_{i+2} via the *left* occurrence of A. After reduction, however, we get a derivation π' in which we can only pass via the right occurrence of A, and π_2 simply ceases to be a box:



⊠

3.5. LEMMA. *Let* $\ldots \to \pi_{\mathtt{i}} \to \pi_{\mathtt{i+1}} \to \pi_{\mathtt{i+2}} \to \ldots$ *be a subchain of three (consecutive) boxes in an adequate chain. Then neither of the three contains nor is contained in any of the others.*

PROOF: This is evident if $c_{i+1} \not\equiv c_i$. So let $c_{i+1} \equiv c_i$. Suppose $\pi_{\mathtt{i}}$ has conclusion $;\Gamma, B \Rightarrow A$ (with B as secondary source), while $\pi_{\mathtt{i+1}}$ has conclusion $;\Delta, C \Rightarrow B$ (with C as secondary source). Let $\phi(B)$ denote a formula ϕ having B as a subformula. Then by adequacy, for some ϕ, $C \multimap \phi(B)$ has to be a subformula of the cutformula, and C emerges in the derivation of the other premiss of the reflecting cut as the succedent of the conclusion of a subderivation which is the left premiss of an application of L$\to$ with C and $\phi(B)$ as main formulas; $\pi_{\mathtt{i}}$ then has to be contained in the subderivation of the right premiss. ⊠

3.6. PROPOSITION. *Adequate chains are linear.*

PROOF: Suppose $\Sigma \equiv \pi_{\mathtt{s}} \to \pi_1 \to \pi_2 \to \ldots \to \pi_{\mathtt{n}}$ is an adequate *non-linear* chain. Then there is a smallest k such that $\pi_{\mathtt{k}}$ is contained in or contains a box $\pi_{\mathtt{i}}$ for some $i < k$. Consider

$$\Sigma^- \equiv \pi_{\mathtt{s}} \to \ldots \to \pi_{\mathtt{i}} \to \ldots \to \pi_{\mathtt{k}}.$$

By lemma 3.5 we have that $k - i > 2$. Therefore there is a linear link between $\pi_{\mathtt{i}}$ and $\pi_{\mathtt{k}}$. By proposition 3.4 we can eliminate its reflecting cut, and obtain an adequate chain with one box less between $\pi_{\mathtt{i}}$ and $\pi_{\mathtt{k}}$. Iterating this eventually will contradict lemma 3.5. ⊠

We now have found sufficiently many properties of decorated **ILU**-derivations to prove the following

3.7. THEOREM. *Let* Σ *be an adequate chain in* π *starting from a source* s, *so* $\Sigma \equiv \pi_{\mathtt{s}} \to \pi_1 \to \pi_2 \to \ldots \to \pi_{\mathtt{n}}$ *for some* $n \geq 1$. *If the source is a* weakening, *then there is a reduction strategy* σ *that will erase each of* $\pi_1, \pi_2, \ldots, \pi_{\mathtt{n}}$ *(precisely in that order); if the source is a* contraction *then there is a* σ *that will duplicate each of* $\pi_1, \ldots, \pi_{\mathtt{n}}$ *(precisely in that order).*

PROOF: Let c_i be the reflecting cut for π_i. The strategy will consist in eliminating $c_1, \ldots, c_n$ precisely in that order. All the work in fact has been done in the proofs of propositions 3.6 and 3.4. ⊠

In the example on page 95, if we apply an elementary reduction step to the **ILU**-skeleton and then decorate the reduct π', the result is:

$$
\dfrac{
\dfrac{\dfrac{\dfrac{A; \Rightarrow A}{A; !(!AA) \overset{\mathfrak{b}'}{\Rightarrow} A}}{; A, !(!AA) \Rightarrow A}}{; !A \overset{3}{\Rightarrow} !(!(!AA)A)}
\qquad
\dfrac{
\dfrac{\dfrac{\dfrac{A; \Rightarrow A}{A; !(!AA) \overset{\mathfrak{b}}{\Rightarrow} A}}{; A, !(!AA) \Rightarrow A}}{; A \Rightarrow !(!AA)A}
\quad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A; \Rightarrow A}{A; !A \overset{\mathfrak{a}}{\Rightarrow} A}}{A; \Rightarrow !AA}}{!A \overset{1}{\Rightarrow} !(!AA)} \quad A \Rightarrow A}{!(!AA)A; !A \Rightarrow A}}{!(!AA)A; \Rightarrow !AA}}{; !(!(!AA)A) \overset{2}{\Rightarrow} !(!AA)} \quad A; \Rightarrow A}{!(!AA)A; !(!(!AA)A) \Rightarrow A}
}{; A, !(!(!AA)A) \Rightarrow A}
}{\dfrac{; !A, A \Rightarrow A}{; !A \Rightarrow A}}
$$

The duplication of π_3 in fact duplicates the pseudo-box $\pi_{\mathfrak{b}}$, and we see that our original chain $\pi_{\mathfrak{b}} \to \pi_1$ can be found in the reduct as chain $\pi_{\mathfrak{b}'} \to \pi_1$. The remaining non-trivial chain, $\pi_{\mathfrak{b}} \to \pi_2 \to \pi_3$, is still there. Note that all non-trivial chains in π' are *strong*, so by 3.7 we have, for each of the boxes in π' (and a fortiori for each of the boxes in π) a reduction strategy that eventually will erase or duplicate it. However, this is not a general property: *the problematic situations indicated in our sketch of the proof of proposition 3.4 can be built into concrete examples of derivations with boxed subderivations, that, whatever strategy of reduction applied, neither are erased nor duplicated, but e.g. simply are 'un-boxed'.*

So we have to conclude that even when 'logically necessary', exponentials can be 'computationally superfluous', and (maybe not too unexpectedly) our (static) logical linearity analysis can provide us with but an *approximation* of the dynamics of a given proof.

This is due to (the identification of formulas in) *contractions*: the problematic situations do not occur in contraction-free derivations, i.e. in *affine* implicational logic. As trivially in contraction-free derivations all chains are adequate we find the following as a corollary to 3.7.

3.8. PROPOSITION. *Let* π *be a contraction-free derivation, and suppose* Σ *is a chain* $\pi_{\mathrm{s}} \to \pi_{\mathrm{1}} \to \ldots \to \pi_{\mathrm{n}}$, *with* $n \geq 1$. *Then there is a reduction strategy that will erase each of* $\pi_{\mathrm{1}}, \ldots, \pi_{\mathrm{n}}$ *(precisely in that order).* ⊠

4 Decorating classical derivations

Given the facts (i) that a linear derivation fixes a normalization protocol for its skeleton, and (ii) that the normalization procedure for a CL-derivation is essentially non-deterministic (cf. section 1 of chapter 1), when trying to apply the techniques introduced earlier in this chapter to *classical* derivations, one should expect to founder on the necessity to choose between non-equivalent possibilities.

It is most instructive to illustrate this by means of an example, a derivation that is an instance of the problematic situation discussed on page 16:

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad A \Rightarrow A}{A \vee A \Rightarrow A, A}}{A \vee A \Rightarrow A} \qquad \dfrac{\dfrac{A \Rightarrow A \quad A \Rightarrow A}{A, A \Rightarrow A \wedge A}}{A \Rightarrow A \wedge A}}{A \vee A \Rightarrow A \wedge A}$$

When searching for a minimal decoration by tracing the effects of the two structural rules throughout the derivation, the choice we have to make is that between prefixing the cutformula with the modality '?!' and prefixing it with '!?'. Thus one finds decorations depicted in figure 4.2.

Whereas the original derivation does not give us any clue whatsoever on the elimination procedure to follow, in the linear decorations the protocol is fixed: in the first case we have to duplicate the *right* subderivation, while in the second case we are going to duplicate the *left* one. Note that the choice is in fact that between a q- and a t-type decoration of the cutformula.

The situation changes when we consider not a CL-derivation, but one in **LKT** or **LKQ**. The corresponding uniform decoration strategies

$$\dfrac{\dfrac{\dfrac{\dfrac{!A \Rightarrow !A}{!A \Rightarrow ?!A} \quad \dfrac{!A \Rightarrow !A}{!A \Rightarrow ?!A}}{!A \parr !A \Rightarrow ?!A, ?!A}}{!A \parr !A \Rightarrow ?!A} \qquad \dfrac{\dfrac{\dfrac{\dfrac{!A \Rightarrow A \quad !A \Rightarrow A}{!A, !A \Rightarrow A \otimes A}}{!A \Rightarrow A \otimes A}}{!A \Rightarrow ?(A \otimes A)}}{?!A \Rightarrow ?(A \otimes A)}}{!A \parr !A \Rightarrow ?(A \otimes A)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow ?A \quad A \Rightarrow ?A}{A \parr A \Rightarrow ?A, ?A}}{A \parr A \Rightarrow ?A}}{!(A \parr A) \Rightarrow ?A}}{!(A \parr A) \Rightarrow !?A} \qquad \dfrac{\dfrac{\dfrac{?A \Rightarrow ?A}{!?A \Rightarrow ?A} \quad \dfrac{?A \Rightarrow ?A}{!?A \Rightarrow ?A}}{!?A, !?A \Rightarrow ?A \otimes ?A}}{!?A \Rightarrow ?A \otimes ?A}}{!(A \parr A) \Rightarrow ?A \otimes ?A}$$

Figure 4.2: Two minimal decorations

limit the collection of possible reductions beforehand, and the problem of constructing, as in the case of **ILU** described in section 2, a minimal decoration under the proviso that the result will be a subdecoration of the uniform one, allows an unambiguous solution using techniques similar to those that apply in the intuitionistic case. A full description for the case of **LKT** can be found in chapter 6 of Joinet(1993).

Returning to our example, it's immediately clear that this derivation allows for many different possible reduction sequences, among them even *infinite* ones. Correspondingly, there exist many distinct decoration, e.g. as in figure 4.3.

We leave it to the reader to find the corresponding normal form.

4.1. REMARK. This last decoration, and many more like them, are obtained, one might say, 'by hand': one chooses a decoration of the cutformula, and completes the proof in linear logic by 'working ones way up'. The number of essentially distinct decorations thus obtained is far larger than that of those within reach of the 'inductive decoration strategies' of the previous chapter, as we will see in section 5.5.

$$
\begin{prooftree}
\AxiomC{$A \Rightarrow ?A$}
\AxiomC{$A \Rightarrow ?A$}
\BinaryInfC{$A \parr A \Rightarrow ?A, ?A$}
\UnaryInfC{$!(A \parr A) \Rightarrow ?A, ?A$}
\UnaryInfC{$!(A \parr A) \Rightarrow ?A, !?A$}
\UnaryInfC{$!(A \parr A) \Rightarrow ?A, ?!?A$}
\UnaryInfC{$!(A \parr A) \Rightarrow !?A, ?!?A$}
\UnaryInfC{$!(A \parr A) \Rightarrow ?!?A, ?!?A$}
\UnaryInfC{$!(A \parr A) \Rightarrow ?!?A$}
\AxiomC{$?A \Rightarrow ?A$}
\UnaryInfC{$!?A \Rightarrow ?A$}
\AxiomC{$?A \Rightarrow ?A$}
\UnaryInfC{$!?A \Rightarrow ?A$}
\BinaryInfC{$!?A, !?A \Rightarrow ?A \otimes ?A$}
\UnaryInfC{$!?A, !?A \Rightarrow !(?A \otimes ?A)$}
\UnaryInfC{$!?A \Rightarrow !(?A \otimes ?A)$}
\UnaryInfC{$?!?A \Rightarrow !(?A \otimes ?A)$}
\BinaryInfC{$!(A \parr A) \Rightarrow !(?A \otimes ?A)$}
\end{prooftree}
$$

Figure 4.3: A less minimal decoration

Notes

- The material in this chapter, with the exception of section 4, is taken from Danos et al.(1993c).

- The lambda-term corresponding to the derivation in example 1.2 is $(F)A$, where $F \equiv \lambda f.(f)\lambda x.(f)\lambda y.x$ is the *right* and $A \equiv \lambda r.(r)(r)n$ is the *left* premiss. Variations on this derivation occur in examples at several places in this and other chapters of this thesis. As a matter of fact, the term F originally is a counterexample of D.P. Kierstead in the order 3 - case to an attempt of Kleene to construct a theory of higher order computability using oracles. See Kleene(1980).

5

The exponential graph

In the previous chapter we saw how, by tracing the effects of occurrences of structural rules, one may associate with any derivation in intuitionistic (implicational) logic a derivation in linear logic that has the same skeleton, and which moreover seems to be optimal.

The proofs obtained are subdecorations of those one gets using the modal translation $(\cdot)^{\circledast}$ (in case of **IL**), $(\cdot)^*$ (in case of **ILU**), and in this chapter we will make the relation between the decoration $\pi^{\circledast}$ (resp. π^*) and the decoration $\partial(\pi)$ obtained by applying the lower decoration strategy to an **IL**(**U**)-derivation π more precise.

Recall that, by the results of the previous chapter, it is essentially the *absence* of modalities that gives us information on the dynamics of a proof: if a subderivation is not 'boxed' we can be sure that at no point during normalization it will endure non-linear manipulations. We therefore have an interest, given a linear derivation π and an occurrence of an exponential, to find out *why* it occurs at that given spot.

We will find that each exponential "!", "?" that is not, directly or indirectly, caused by an instance of a structural rule is *superfluous* : we can remove it and obtain a proof that (1) is still correct, and (2) has the same dynamics as the original one.

Our object of study is the full system of second-order classical linear logic. To be precise, we are going to consider derivations in the two-sided sequent calculus of appendix b, extended with the rules for second-order quantifiers ($\mathbf{CLL}_2$).

1 Identity classes

Besides the two constraints originating in the modal condition imposed on formulas in the structural and contextual exponential rules of the linear sequent calculus (in the sequel we will refer to the *structural* and the *contextual* constraint), there is, as in any sequent calculus, another, fairly obvious, one: in writing down rules and derivations we implicitly demand the identity of some of the (sub)formulas occurring in the sequents appearing in it. In the sequel we will refer to the *identity* constraint. E.g. the occurrences of the contextual formulas $!\Gamma$ and $?\Delta$ in the premiss and conclusion of a promotion rule are occurrences of *identical* formulas. This implicit identity relation is made explicit in the following

1.1. DEFINITION. We call occurrences of (sub)formulas in a proof *identified* whenever they are the corresponding occurrences of the same[1] (sub)formula in

- the two formulas in an axiom;
- the cut formulas in a cut;
- the abstracted formulas in a second-order rule;
- an active formula and the corresponding subformula of the main formula in a logical or exponential rule (in the case of $L\forall_2$ and $R\exists_2$ rules a strict subformula of an abstracted occurrence, has no "correspondent" in the conclusion sequent of the rule);
- the up and down occurrences of passive or side-active formulas in a rule (this includes the implicit contextual contraction in additive binary rules). ⊠

Let us denote by "$\sim$" the reflexive, symmetric and transitive closure of the identification relation. Note once more that all elements of a $\sim$-equivalence class (or: *identity* class) are occurrences of the *same* (sub)formula (up to substitution).[2] In the sequel we will only deal with classes containing at least one formula whose main connective is an *exponential*. We denote the set of such classes in a proof π by $\mathcal{E}(\pi)$.

[1] Exactly the same or, in case of quantifier rules, the same up to substitution.

[2] In general the converse does not hold, of course.

1.2. EXAMPLE. Let N be some linear theorem. Then the following is a CLL_2-derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{N \Rightarrow N}{!N \Rightarrow N}}{!N \Rightarrow !N}}{\Rightarrow !N \multimap !N}}{\Rightarrow !(!N \multimap !N)} \quad \dfrac{\begin{matrix}\vdots\\ \Rightarrow !N\end{matrix} \quad !N \Rightarrow !N}{!N \multimap !N \Rightarrow !N}}{!(!N \multimap !N) \multimap (!N \multimap !N) \Rightarrow !N}}{\forall X\, !(X \multimap X) \multimap (X \multimap X) \Rightarrow !N}$$

The reader will easily verify that all occurrences of $!N$ are in the same identity-class, and $!(!N \multimap !N) \sim !(X \multimap X)$.

If in the last step of the derivation we would have abstracted not on $!N$, but on $!N \multimap !N$ (resulting in $\forall X\, !X \multimap X \Rightarrow !N$ as the final sequent), the left and right occurrences of $!N$ in $!N \multimap !N$ had *not* been in the same identity-class.

2 Stripping derivations

Let E be a subset of $\mathcal{E}(\pi)$. The *domain* of E is the union of the classes it contains. By a *strip* we mean the operation of simultaneous deletion in π of all external exponentials in the domain of E. The resulting pseudo-proof (which need in general not be a proof) is denoted by $\pi - E$. The corresponding instance of a rule r in $\pi - E$ is written as $r - E$.

Each formula B in π 're-appears' in $\pi - E$, though maybe slightly modified. To be precise it is modified if and only if some formula $!A$ or $?A$ in one of the classes in E is a subformula of B in π. If we want to specify the changes we will write $B - E$, though mostly we will continue to denote this, possibly modified, formula by B.

Take some box rule r in π, with its main occurrence in some class e and a side-active occurrence in some class e': we say that e *binds* e' (*via* r) and write this as $e \curvearrowright_1 e'$. The transitive closure of the relation $\curvearrowright_1$ will also be called *binding*, and is denoted by $\curvearrowright$. (Par abus de langage we will sometimes write $s \curvearrowright_1 s'$ and $s \curvearrowright s'$ also for proper *sub*sets of classes.)

This defines a directed graph, the *exponential graph* $\mathcal{G}(\pi)$ of π, with as vertices the classes in $\mathcal{E}(\pi)$, and an arrow from e to e' if and only if $e \curvearrowright_1 e'$. If an occurrence of an element of a class e is main formula in a structural rule in π, then we label the corresponding vertex of the exponential graph by "w" (for 'weakening'), "c" (for 'contraction'), or "w+c" (for 'weakening & contraction'), according to the kind of structural rule in which elements of e occur as main formula.

2.1. DEFINITION. A set $E \subset \mathcal{E}(\pi)$ is called *saturated* (or said to satisfy the *saturation condition*), in case for all $e' \in E$, if $e \curvearrowright e'$ for some $e \in \mathcal{E}(\pi)$, then also $e \in E$. If no class in E is labeled then we will say that E verifies the *no sources condition.* If E satisfies both the saturation and the no sources condition, we say that it is *not relevantly exponentiated* (abbreviated by *nre*) in π. A *redex* is any non-empty set E that is *nre* and *minimal*, i.e. no proper subset of E is nre. ⊠

The exponentials prefixing elements of the classes in an nre set E of π are, one might say, *superfluous*:

2.2. PROPOSITION. (Stripping preserves correctness) *Let π be a proof, r a rule in π, and E nre in π; $r - E$ is still a correct rule, and hence $\pi - E$ is a proof. More precisely, either $r - E$ and r are instances of the same rule, or $r - E$ is a repetition rule.*

PROOF: First observe that, whatever rule r, because only classes are stripped, all identity constraints are obviously still satisfied by $r - E$. Now, if r is a box rule, by the saturation condition, the (eventual) contextual constraint for $r - E$ will also be satisfied. And finally, if r is a structural rule, by the no sources condition, so is the structural constraint for $r - E$. (Clearly $r - E$ is a repetition rule only when r introduces an exponential that is stripped, i.e. when r is an exponential rule whose main formula is in the domain of E.) ⊠

2.3. REMARK. We will in the sequel adopt the convention that all occurrences of the repetition rule in $\pi - E$ are eliminated. So possible repetitions of sequents are identified.

2.4. LEMMA. *If E_1, E_2 are nre, then so are $E_1 \cap E_2, E_1 \cup E_2$.* ⊠

So $\mathcal{E}(\pi)$ contains a *largest* nre subset, which we denote by $E_{\mathsf{max}}(\pi)$. It is the largest saturated subset of $\mathcal{E}(\pi)$ that contains no labeled vertices.

2.5. LEMMA. *Let E be nre. Then $\mathcal{E}(\pi - E) = \mathcal{E}(\pi)\backslash E$, and the exponential graph of $\pi - E$ is a* full *subgraph of that of π.*

PROOF: For the first claim, observe that any class not in E remains a class in $\pi - E$, while all classes in $\pi - E$ are classes in π. For the second claim, note that for e', e in $\mathcal{E}(\pi)\backslash E$ we have that $e' \frown_1 e$ in $\pi - E$ if and only if $e' \frown_1 e$ in π. ⊠

2.6. LEMMA. *If E, E' are nre in π, and E' is a subset of E, then $E\backslash E'$ is nre in $\pi - E'$.*

PROOF: As no class in E is labeled, the same holds for $E\backslash E'$. As E is nre in π and $E' \subseteq E$, the only possible elements of $\mathcal{E}(\pi)$ that bind elements of $E\backslash E'$ are in E'. So $E\backslash E'$ is saturated in $\mathcal{E}(\pi - E')$. ⊠

2.7. LEMMA. *Suppose E_1 is nre in π. Then E_2 is nre in $\pi - E_1$ if and only if $E_1 \cup E_2$ is nre in π.*

PROOF: ($\Rightarrow$) As E_1 and E_2 are nre, none of their elements is labeled. Let $e' \in \mathcal{E}(\pi)$ bind an element of E_1. Then e' in E_1 by saturation. If it binds an element of E_2, and it is not an element of E_1, then $e' \in \mathcal{E}(\pi - E_1)$, so $e' \in E_2$, by saturation of E_2.

($\Leftarrow$) By lemma 2.6. ⊠

2.8. PROPOSITION. *Let R_1, R_2 be distinct redexes in π. Then R_2 is a redex in $\pi - R_1$.*

PROOF: Observe that, by lemma 2.4, $R_1 \cap R_2 = \emptyset$, from which the claim easily follows, using lemma 2.5. ⊠

2.9. COROLLARY. *Let R_1, R_2 be distinct redexes. Then $(\pi - R_1) - R_2$ is a correct linear derivation which is equal to $(\pi - R_2) - R_1$.* ⊠

Now define a reduction $\rhd$ on linear derivations by $\pi \rhd \pi - R$, for R a redex in π. Given some derivation π, clearly the number of potential redexes in π is finite. So all $\rhd$-reduction-sequences are finite, ending in a $\rhd$-normal form. As by the above $\rhd$ is locally (1-1) confluent, in fact for each π we obtain a *unique* $\rhd$-normal form, which we will denote by $\pi^{\triangleright}$. Thus $\triangleright$ defines a complete lattice of linear derivations with top π, bottom $\pi^{\triangleright}$, and $\pi_i \triangleright \pi_j$ if and only if there is a (possibly empty) $\triangleright$-reduction-sequence leading from π_i to π_j. We will refer to the lattice obtained as the "$\triangleright$-*lattice of* π".

2.10. LEMMA. *If E is nre in π, then $\pi \triangleright \pi - E$ and $E_{\mathsf{max}}(\pi - E) = E_{\mathsf{max}}(\pi)\backslash E$.*

PROOF: The first claim is shown by induction on the size of E, the second claim using lemma's 2.6, 2.7. ⊠

2.11. THEOREM. *$\pi^{\triangleright} = \pi - E_{\mathsf{max}}(\pi)$; the exponential graph of $\pi^{\triangleright}$ is precisely the union of all directed paths in the graph of π starting from a labeled vertex.*

PROOF: By lemma 2.10, $\pi \triangleright \pi - E_{\mathsf{max}}(\pi)$, so $\pi - E_{\mathsf{max}}(\pi) \triangleright \pi^{\triangleright}$. But as $E_{\mathsf{max}}(\pi - E_{\mathsf{max}}(\pi)) = \emptyset$ in fact $\pi - E_{\mathsf{max}}(\pi) = \pi^{\triangleright}$. The second claim is immediate by 2.5 and the fact that we obtain the exponential graph of $\pi^{\triangleright}$ by removing all saturated subgraphs of the graph of π that do *not* contain a labeled vertex. ⊠

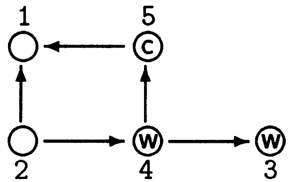
Consequently we have shown:

> *a class e remains in $\pi^{\triangleright}$ if and only if the corresponding class in π has a structural cause.*

2.12. EXAMPLE. In figure 5.1 we give an example of a derivation π and its exponential graph. We use indexed exponentials to distinguish the !-identity-classes. The places where binding occurs are indicated by $\stackrel{\frown}{\Rightarrow}$. $\pi^{\triangleright}$ is obtained by deletion of all exponentials $!_2$.

3 The 'mono-stable' fragment of CLL_2

Let us call derivations π in linear logic 'mono' if the only modalities prefixing the skeleton of each formula appearing in π are among '!', '?', '!?' and '?!'. Observe that the collection of all first-order 'mono'-derivations is closed under cut-elimination. To get the same property in the second-order case, abstraction on externally modalized formulas



$$A \Rightarrow A$$
$$A, \, !_4(!_3A \multimap A) \Rightarrow A$$
$$!_1A, \, !_4(!_3A \multimap A) \Rightarrow A$$
$$!_1A, \, !_4(!_3A \multimap A) \overset{\curvearrowleft}{\Rightarrow} \, !_2A$$
$$!_1A \Rightarrow \, !_4(!_3A \multimap A) \multimap \, !_2A$$
$$!_1A \overset{\curvearrowleft}{\Rightarrow} \, !_5(!_4(!_3A \multimap A) \multimap \, !_2A)$$

$$A \Rightarrow A$$
$$A, \, !_3A \Rightarrow A$$
$$A \Rightarrow \, !_3A \multimap A$$
$$!_3A \Rightarrow \, !_3A \multimap A$$
$$!_3A \overset{\curvearrowleft}{\Rightarrow} \, !_4(!_3A \multimap A) \qquad \frac{A \Rightarrow A}{!_2A \Rightarrow A}$$
$$!_3A, \, !_4(!_3A \multimap A) \multimap \, !_2A \Rightarrow A$$
$$!_4(!_3A \multimap A) \multimap \, !_2A \Rightarrow \, !_3A \multimap A$$
$$!_5(!_4(!_3A \multimap A) \multimap \, !_2A) \Rightarrow \, !_3A \multimap A$$
$$!_5(!_4(!_3A \multimap A) \multimap \, !_2A) \overset{\curvearrowleft}{\Rightarrow} \, !_4(!_3A \multimap A) \qquad \frac{A \Rightarrow A}{!_2A \Rightarrow A}$$
$$!_5(!_4(!_3A \multimap A) \multimap \, !_2A), \, !_4(!_3A \multimap A) \multimap \, !_2A \Rightarrow A$$
$$!_5(!_4(!_3A \multimap A) \multimap \, !_2A), \, !_5(!_4(!_3A \multimap A) \multimap \, !_2A) \Rightarrow A$$
$$!_5(!_4(!_3A \multimap A) \multimap \, !_2A) \Rightarrow A$$

$$!_1A \Rightarrow A$$

Figure 5.1: A derivation and its exponential graph

should be prohibited. This defines a *proper* fragment of second-order linear logic: the *mono-stable* fragment. For 'mono'-derivations we are able to strengthen proposition 2.2, in the sense that we now also have the converse:

3.1. THEOREM. *Let π be 'mono', and $E \subseteq \mathcal{E}(\pi)$. Then $\pi - E$ is a correct linear derivation if* and only *if E is nre.*

PROOF: ($\Rightarrow$) If E is *not* nre and π is 'mono', then the strip defined by E will result in a derivation $\pi - E$ in which there is either an application of a structural rule on a non-exponentiated (not properly exponentiated) formula, or an application of an exponential contextual rule where the context contains (a) non-exponentiated (not properly exponentiated) formula(s). So $\pi - E$ can not possibly be correct. ⊠

In general we can not be sure of the left-to-right direction: 'good' exponentials may be hidden (more or less directly) behind the 'stripped' ones, e.g. in case we strip in '!!' or '!??????!'.

Theorem 3.1 tells us that the minimum $\pi^{\triangleright}$ of the $\triangleright$-lattice of a 'mono'-derivation π is a minimum in a very strong sense: for *no* $E \subseteq \mathcal{E}(\pi^{\triangleright})$ the strip defined by E can possibly result in a derivation that is linearly correct.

This does *not* mean that for a 'mono'-derivation π, it is impossible to remove any more exponentials in $\pi^{\triangleright}$: what can't be done is remove one or more entire *classes*, but one still has the possibility to lower as much as possible the L! and R? rules that are left, in order to introduce them just before they are needed. If we apply this lowering of dereliction rules to $\pi^{\triangleright}$ we obtain derivations $(\pi^{\triangleright})'$. Clearly all of them have the same exponential graph. As a matter of fact, they will be identified in their proofnet representation and hence are denotationally equal: they have the same interpretation in coherence space. In other words, the difference between them is negligible.

4 Strips and normalization

Let c be a cut rule in a proof π. We will denote by $[c]$ the particular kind of *elementary normalization* or *reduction step* to be performed in order to eliminate the cut.

If a derivation π' can be obtained by the consecutive application of zero or more elementary reduction steps, starting from a derivation π, we say that π *reduces* to π' (notation: $\pi \twoheadrightarrow \pi'$).

Observe that the nature of $[c]$ is completely determined by:

- the rules r_g and r_d surmounting c (in the left, respectively the right premiss);

- the status in r_g and r_d (main, passive, side-active) of the cutformula.

Accordingly we distinguish four kinds of elementary normalization steps: permutation steps, logical steps, structural steps, axiom steps.

We recall steps that in the sequel ask for a non trivial treatment, namely those where r_g or r_d is a box rule whose main formula is the cutformula; also we display the configuration where r_g and r_d are second-order rules introducing the cutformula. (For each we will show only one among the possible cases.)

- If the cutformula is side-active in an exponential contextual rule surmounting c, we denote the associated reduction step by $[cc]$ ('commutative cut') being of the following form:

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ !\Gamma_1 \Rightarrow ?\Delta_1, A\end{matrix}}{!\Gamma_1 \Rightarrow ?\Delta_1, !A} \quad \dfrac{\begin{matrix}\pi_2\\ \vdots\\ !A, !\Gamma_2 \Rightarrow B, ?\Delta_2\end{matrix}}{!A, !\Gamma_2 \Rightarrow !B, ?\Delta_2}}{!\Gamma_1, !\Gamma_2 \Rightarrow !B, ?\Delta_1, ?\Delta_2} \quad \overset{[cc]}{\leadsto} \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ !\Gamma_1 \Rightarrow ?\Delta_1, A\end{matrix}}{!\Gamma_1 \Rightarrow ?\Delta_1, !A} \quad \begin{matrix}\pi_2\\ \vdots\\ !A, !\Gamma_2 \Rightarrow B, ?\Delta_2\end{matrix}}{!\Gamma_1, !\Gamma_2 \Rightarrow B, ?\Delta_1, ?\Delta_2}}{!\Gamma_1, !\Gamma_2 \Rightarrow !B, ?\Delta_1, ?\Delta_2}$$

- If the cutformula is main in a dereliction rule, we denote the associated reduction step by $[de]$ which is of the following form:

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ !\Gamma_1 \Rightarrow ?\Delta_1, A\end{matrix}}{!\Gamma_1 \Rightarrow ?\Delta_1, !A} \quad \dfrac{\begin{matrix}\pi_2\\ \vdots\\ A, \Gamma_2 \Rightarrow \Delta_2\end{matrix}}{!A, \Gamma_2 \Rightarrow \Delta_2}}{!\Gamma_1, \Gamma_2 \Rightarrow ?\Delta_1, \Delta_2} \quad \overset{[de]}{\leadsto} \quad \dfrac{\begin{matrix}\pi_1\\ \vdots\\ !\Gamma_1 \Rightarrow ?\Delta_1, A\end{matrix} \quad \begin{matrix}\pi_2\\ \vdots\\ A, \Gamma_2 \Rightarrow \Delta_2\end{matrix}}{!\Gamma_1, \Gamma_2 \Rightarrow ?\Delta_1, \Delta_2}$$

- If the cutformula is main in an instance of a contraction rule, we denote the associated reduction step by $[co]$ which is of the following form:

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ !\Gamma_1 \Rightarrow ?\Delta_1, A\end{array}}{!\Gamma_1 \Rightarrow ?\Delta_1, !A} \quad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ !A, !A, \Gamma_2 \Rightarrow \Delta_2\end{array}}{!A, \Gamma_2 \Rightarrow \Delta_2}}{!\Gamma_1, \Gamma_2 \Rightarrow ?\Delta_1, \Delta_2} \quad \overset{[co]}{\rightsquigarrow} \quad \dfrac{\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ !\Gamma_1 \Rightarrow ?\Delta_1, A\end{array}}{!\Gamma_1 \Rightarrow ?\Delta_1, !A} \quad \dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ !\Gamma_1 \Rightarrow ?\Delta_1, A\end{array}}{!\Gamma_1 \Rightarrow ?\Delta_1, !A} \quad \begin{array}{c}\pi_2\\ \vdots\\ !A, !A, \Gamma_2 \Rightarrow \Delta_2\end{array}}{!A, !\Gamma_1, \Gamma_2 \Rightarrow ?\Delta_1, \Delta_2}}{!\Gamma_1, !\Gamma_1, \Gamma_2 \Rightarrow ?\Delta_1, ?\Delta_1, \Delta_2}\\ \vdots\end{array}}{!\Gamma_1, \Gamma_2 \Rightarrow ?\Delta_1, \Delta_2}$$

- If the cutformula is main in an instance of a weakening rule, we denote the associated reduction step by $[w]$ which is of the following form:

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ !\Gamma_{\Rightarrow} ?\Delta_1, A\end{array}}{!\Gamma_1 \Rightarrow ?\Delta_1, !A} \quad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ \Gamma_2 \Rightarrow \Delta_2\end{array}}{!A, \Gamma_2 \Rightarrow \Delta_2}}{!\Gamma_1, \Gamma_2 \Rightarrow ?\Delta_1, \Delta_2} \quad \overset{[w]}{\rightsquigarrow} \quad \dfrac{\begin{array}{c}\underline{\begin{array}{c}\pi_2\\ \vdots\\ \Gamma_2 \Rightarrow \Delta_2\end{array}}\\ \vdots\end{array}}{!\Gamma_1, \Gamma_2 \Rightarrow ?\Delta_1, \Delta_2}$$

- If the cutformulas are main in $\forall_2$-rules, we denote the associated reduction step by $[\forall_2]$, which is of the following form:

$$\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ \Gamma_1 \Rightarrow \Delta_1, A[X]\end{array}}{\Gamma_1 \Rightarrow \Delta_1, \forall X A[X]} \quad \dfrac{\begin{array}{c}\pi_2\\ \vdots\\ A[T/X], \Gamma_2 \Rightarrow \Delta_2\end{array}}{\forall X A[X], \Gamma_2 \Rightarrow \Delta_2}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \quad \overset{[\forall_2]}{\rightsquigarrow} \quad \dfrac{\begin{array}{c}\pi_1[T/X]\\ \vdots\\ \Gamma_1 \Rightarrow \Delta_1, A[T/X]\end{array} \quad \begin{array}{c}\pi_2\\ \vdots\\ A[T/X], \Gamma_2 \Rightarrow \Delta_2\end{array}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Let μ be an elementary normalization step. Any occurrence of a (sub)formula F (resp. any instance of a rule r) in $\mu(\pi)$ comes, in the obvious way, from a unique occurrence of a (sub)formula (resp. a rule) in π. Let us denote by $\mu_\star$ this lifting application.

4.1. LEMMA. (Lifting of classes) *For any elementary normalization step μ in a proof π, $\mu_\star$ respects classes. I.e., if F, G are occurrences of subformulas in $\mu(\pi)$ and $F \sim_{\mu(\pi)} G$, then $\mu_\star(F) \sim_\pi \mu_\star(G)$.* ⊠

Hence each class e in $\mu(\pi)$ is mapped by $\mu_\star$ to a class e' of π (so $\mu_\star(e) \subset e'$). Note however, that this mapping is neither one-to-one, nor onto, in general.

4.2. LEMMA. (Lifting of binding) *For any elementary normalization step μ in a proof π, $\mu_\star$ respects binding. I.e., if e, e' are classes in $\mu(\pi)$ and $e \curvearrowright e'$ in $\mu(\pi)$, then $\mu_\star(e) \curvearrowright \mu_\star(e')$.*

PROOF: (Recall that $\curvearrowright$ is but the transitive closure of $\curvearrowright_1$.) Suppose μ is [*cc*], and $e \curvearrowright_1 e'$ via the box rule permuted by μ with the cut rule. Either $\mu_\star(e) \curvearrowright_1 \mu_\star(e')$ or there is in π a class e'' (namely the class of the cutformulas) such that $\mu_\star(e) \curvearrowright_1 e''$ and $e'' \curvearrowright_1 \mu_\star(e')$. In all other cases $\mu_\star(e) \curvearrowright_1 \mu_\star(e')$ (in particular, note that for $T \equiv !T'$ or $?T'$ in $[\forall_2]$, there will be no binding involving T in $\pi_1[T/X]$). ⊠

Let E be a set of classes in a proof π, and suppose μ is an elementary normalization step of π. Let us denote by $\mu(E)$ the set of classes in $\mu(\pi)$ mapped by $\mu_\star$ to a class in E. This makes sense, precisely because $\mu_\star$ respects classes.

4.3. LEMMA. *Let π be a proof, μ an elementary normalization step in π, and E a subset of $\mathcal{E}(\pi)$. If E is saturated, then so is $\mu(E)$.*

PROOF: Take a class e in $\mu(E)$ such that $e' \curvearrowright e$ for some class e' in $\mathcal{E}(\mu(\pi))$. By lemma 4.2, $\mu_\star(e') \curvearrowright \mu_\star(e)$. Because $e \in \mu(E)$, by definition $\mu_\star(e)$ is contained in a class of E. Hence, by saturation of E, the same holds for $\mu_\star(e')$, and, again by definition, $e' \in \mu(E)$. ⊠

4.4. LEMMA. *Let π be a proof, μ an elementary normalization step in π, and E a subset of $\mathcal{E}(\pi)$. If E is nre, then so is $\mu(E)$.*

PROOF: By lemma 4.3, $\mu(E)$ is saturated. Now suppose there is a labeled class e in $\mu(E)$. If an occurrence of a formula F is main in a contraction (resp. weakening) rule in $\mu(\pi)$, observe that either this already is the case for $\mu_\star(F)$ in π, or μ is [*co*] (resp. [*w*]), and $\mu_\star(F)$ is side-active in the box rule to be duplicated (resp. erased). So either $\mu_\star(e)$ is also labeled, or there is in $\mathcal{E}(\pi)$ a class e' labeled such that $e' \curvearrowright_1 e$, contradicting the hypothesis that E satisfies the no sources condition. ⊠

4.5. DEFINITION. Let π be a proof, E an nre set of classes in π, μ an elementary normalization step performable in π. The equivalent of μ in $\pi - E$, denoted by $\hat{\mu}$, is defined as follows:

- $\hat{\mu} = [id]$ (the empty operation) if μ is either $[de]$ with active formulas in the domain of E, or a permutation step where the cut is permuted upwards from the conclusion to the premiss of an exponential rule with main formula in the domain of E.[3]
- $\hat{\mu} = \mu$ in all other cases. ⊠

Let r be a rule in a proof π, and μ an elementary normalization step of π. We denote by $\mu(r)$ the set of instances of rules in $\mu(\pi)$ mapped by $\mu_\star$ to r.

4.6. THEOREM. (Stripping preserves normalization). *Let μ be an elementary normalization step in a proof π, and E nre in π. Then μ can be applied to π if and only if $\hat{\mu}$ can be applied to $\pi - E$, and $\hat{\mu}(\pi - E) = \mu(\pi) - \mu(E)$.* ⊠

At this point a detailed proof of theorem 4.6 would consist in a case-by-case inspection of all possible appearances of an instance of the cut rule in π. We will however encounter an alternative argument in chapter 7.

4.7. COROLLARY. *Let π be a proof, E nre in π, and μ an elementary normalization step in π. Then $\mu(\pi) \rhd \hat{\mu}(\pi - E)$.*

PROOF: By theorem 4.6 and lemma 2.10 ⊠

Note that $F \subseteq \mathcal{E}(\mu(\pi))$ might very well be nre, while $\mu_\star(F) \subseteq \mathcal{E}(\pi)$ is not (i.e., the converse of lemma 4.4 does *not* hold).

A typical example is the class of a main formula in a box rule to be duplicated by $[co]$, which might become nre after duplication.

4.8. THEOREM. *Let $\mu_k \ldots \mu_1$ be a reduction-sequence in π. Then*

$$\hat{\mu}_k \ldots \hat{\mu}_1(\pi^{\rhd}) \rhd (\mu_k \ldots \mu_1(\pi))^{\rhd}.$$

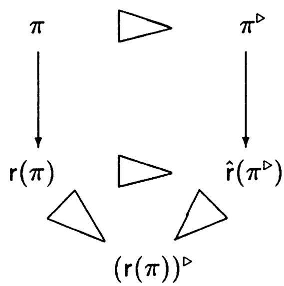
[3] Note that this clause includes $[cc]$!

PROOF: By iteration of corollary 4.7 we find $\mu_k \dots \mu_1(\pi) \rhd \hat{\mu}_k \dots \hat{\mu}_1(\pi^{\rhd})$. We conclude that $\hat{\mu}_k \dots \hat{\mu}_1(\pi^{\rhd})$ is in the $\rhd$-lattice of $\mu_k \dots \mu_1(\pi)$, where $(\mu_k \dots \mu_1(\pi))^{\rhd}$ is the bottom-element. ⊠

We established an important property of the $\rhd$-lattice of a derivation π that intuitively can be expressed as follows:

derivations in the $\rhd$-lattice of π have, essentially, the same set of reductions.

Writing r to denote a reduction sequence in π, the content of the above can be visualised in the following diagram:



We observed that if π is *not* 'mono', it may be possible to strip sets of classes in π that are not nre, and still get a derivation that is linearly correct. However, the result of such a strip is likely to have a behaviour under reduction quite different from that of π, and in general 4.6 will no longer hold.

5 Plethoric translations III

In this section we will apply stripping to linear decorations of **CL**-derivations π obtained by the inductive application of the plethoric translations defined in section 3.3. We show that, whatever adequate pair of modalities one chooses, the result in fact always is essentially the same as either π^{q} or π^{t}.

5.1. THEOREM. *$\langle(\cdot)^{\mathsf{q}}, !, ?!\rangle$ and $\langle(\cdot)^{\mathsf{t}}, !?, !\rangle$ are, essentially, the unique inductive decoration strategies for* **CL**.

PROOF: Let π be a **CL**-derivation, and suppose we have obtained $\pi^{\circledcirc}$ by inductively decorating π via $\langle(\cdot)^{\circledcirc}, \mu, \nu\rangle$, as in theorem 3.3.1. As (μ, ν) is *adequate*, either (1) $\mu \equiv !\alpha?\beta$ and $\nu \equiv ?\beta$, or (2) $\mu \equiv !\beta$ and $\nu \equiv ?\alpha!\beta$, for modalities α, β.

We will show that in $\pi^{\circledcirc}$ exponentials in the classes induced by α and β are always superfluous, and can be stripped. In case (1) we will find $(\pi^{\circledcirc})^{\triangleright} \equiv \pi^{\mathsf{t}}$, in case (2) $(\pi^{\circledcirc})^{\triangleright} \equiv \pi^{\mathsf{q}}$. The claim of the theorem then is immediate by the results of the previous sections.

We proceed by induction on the structure of π, and limit ourselves to case (2), that of (1) being completely similar. Let us just treat some of the steps.

- If π is an axiom $A \Rightarrow A$, then $\pi^{\circledcirc}$ is

$$\dfrac{!\beta A^{\circledcirc} \Rightarrow !\beta A^{\circledcirc}}{\begin{array}{c}\vdots\\ !\beta A^{\circledcirc} \Rightarrow ?\alpha!\beta A^{\circledcirc}\end{array}}$$

Clearly the exponentials α, β are superfluous and can be stripped. Moreover, we can replace $A^{\circledcirc}$ by A^{q}, thus obtaining π^{q}.

- Suppose π ends with an application of R$\rightarrow$. Then $\pi^{\circledcirc}$ is

$$\begin{array}{c}(\pi')^{\circledcirc}\\ \vdots\\ \underline{!\beta\Gamma^{\circledcirc}, !\beta A^{\circledcirc} \Rightarrow ?\alpha!\beta B^{\circledcirc}, ?\alpha!\beta\Delta^{\circledcirc}}\\ \vdots\\ \underline{!\beta\Gamma^{\circledcirc}, ?\alpha!\beta A^{\circledcirc} \Rightarrow ?\alpha!\beta B^{\circledcirc}, ?\alpha!\beta\Delta^{\circledcirc}}\\ \underline{!\beta\Gamma^{\circledcirc} \Rightarrow ?\alpha!\beta A^{\circledcirc} \multimap ?\alpha!\beta B^{\circledcirc}, ?\alpha!\beta\Delta^{\circledcirc}}\\ \vdots\\ !\beta\Gamma^{\circledcirc} \Rightarrow ?\alpha!\beta(?\alpha!\beta A^{\circledcirc} \multimap ?\alpha!\beta B^{\circledcirc}), ?\alpha!\beta\Delta^{\circledcirc}\end{array}$$

By induction hypothesis we can strip $(\pi')^{\circledcirc}$, and we obtain

$$\begin{array}{c}
(\pi')^{\mathsf{q}} \\
\vdots \\
!\Gamma^{\mathsf{q}}, !A^{\mathsf{q}} \Rightarrow ?!B^{\mathsf{q}}, ?!\Delta^{\mathsf{q}} \\
\hline
\vdots \\
!\Gamma^{\mathsf{q}}, ?\alpha!A^{\mathsf{q}} \Rightarrow ?!B^{\mathsf{q}}, ?!\Delta^{\mathsf{q}} \\
\hline
!\Gamma^{\mathsf{q}} \Rightarrow ?\alpha!A^{\mathsf{q}} \multimap ?!B^{\mathsf{q}}, ?!\Delta^{\mathsf{q}} \\
\hline
\vdots \\
!\Gamma^{\mathsf{q}} \Rightarrow ?\alpha!\beta(?\alpha!A^{\mathsf{q}} \multimap ?!B^{\mathsf{q}}), ?!\Delta^{\mathsf{q}}
\end{array}$$

The exponentials in α, β here all induce new classes, that are minimal in the exponential graph, and contain no sources. Therefore they can be stripped, and we get π^{q}.

- In case π ends with an application of L$\wedge$ we have e.g.

$$\begin{array}{c}
(\pi')^{\circledcirc} \\
\vdots \\
!\beta\Gamma^{\circledcirc}, !\beta A^{\circledcirc}, !\beta B^{\circledcirc} \Rightarrow ?\alpha!\beta\Delta^{\circledcirc} \\
\hline
\vdots \\
!\beta\Gamma^{\circledcirc}, !?\alpha!\beta A^{\circledcirc}, !\beta B^{\circledcirc} \Rightarrow ?\alpha!\beta\Delta^{\circledcirc} \\
\hline
\vdots \\
\hline
!\beta\Gamma^{\circledcirc}, !?\alpha!\beta A^{\circledcirc}, !?\alpha!\beta B^{\circledcirc} \Rightarrow ?\alpha!\beta\Delta^{\circledcirc} \\
\hline
!\beta\Gamma^{\circledcirc}, !?\alpha!\beta A^{\circledcirc} \otimes !?\alpha!\beta B^{\circledcirc} \Rightarrow ?\alpha!\beta\Delta^{\circledcirc} \\
\vdots \\
\hline
!\beta\Gamma^{\circledcirc}, !\beta(!?\alpha!\beta A^{\circledcirc} \otimes !?\alpha!\beta B^{\circledcirc}) \Rightarrow ?\alpha!\beta\Delta^{\circledcirc}
\end{array}$$

By induction hypothesis we can strip $(\pi')^{\circledcirc}$, to find

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\pi')^{\mathfrak{q}}}{\vdots}}{!\Gamma^{\mathfrak{q}}, !A^{\mathfrak{q}}, !B^{\mathfrak{q}} \Rightarrow ?!\Delta^{\mathfrak{q}}}}{\vdots}}{!\Gamma^{\mathfrak{q}}, !?\alpha!A^{\mathfrak{q}}, !B^{\mathfrak{q}} \Rightarrow ?!\Delta^{\mathfrak{q}}}}{\vdots}}{\dfrac{!\Gamma^{\mathfrak{q}}, !?\alpha!A^{\mathfrak{q}}, !?\alpha!B^{\mathfrak{q}} \Rightarrow ?!\Delta^{\mathfrak{q}}}{!\Gamma^{\mathfrak{q}}, !?\alpha!A^{\mathfrak{q}} \otimes !?\alpha!B^{\mathfrak{q}} \Rightarrow ?!\Delta^{\mathfrak{q}}}}}{\vdots}}{!\Gamma^{\mathfrak{q}}, !\beta(!?\alpha!A^{\mathfrak{q}} \otimes !?\alpha!B^{\mathfrak{q}}) \Rightarrow ?!\Delta^{\mathfrak{q}}}
$$

As in the previous case, the exponentials in α, β all induce new classes that are minimal and contain no source. Stripping them results in $\pi^{\mathfrak{q}}$. (Of course, here we might also strip the other exponentials introduced; but that's beside the point.)

- In case π ends with an application of LC, we have

$$
\dfrac{\dfrac{\dfrac{(\pi')^{\circledcirc}}{\vdots}}{!\beta\Gamma^{\circledcirc}, !\beta A^{\circledcirc}, !\beta A^{\circledcirc} \Rightarrow ?\alpha!\beta\Delta^{\circledcirc}}}{!\beta\Gamma^{\circledcirc}, !\beta A^{\circledcirc} \Rightarrow ?\alpha!\beta\Delta^{\circledcirc}}
$$

The result now is of course immediate by induction hypothesis.

- Let us finally consider the case that π ends with an application of cut:

$$
\dfrac{\dfrac{\dfrac{\pi_1^{\circledcirc}}{\vdots}}{!\beta\Gamma_1^{\circledcirc} \Rightarrow ?\alpha!\beta\Delta_1^{\circledcirc}, ?\alpha!\beta A^{\circledcirc}} \quad \dfrac{\dfrac{\dfrac{\dfrac{\pi_2^{\circledcirc}}{\vdots}}{!\beta A^{\circledcirc}, !\beta\Gamma_1^{\circledcirc} \Rightarrow ?\alpha!\beta\Delta_2^{\circledcirc}}}{\vdots}}{?\alpha!\beta A^{\circledcirc}, !\beta\Gamma_1^{\circledcirc} \Rightarrow ?\alpha!\beta\Delta_2^{\circledcirc}}}{!\beta\Gamma_1^{\circledcirc}, !\beta\Gamma_2^{\circledcirc} \Rightarrow ?\alpha!\beta\Delta_1^{\circledcirc}, ?\alpha!\beta\Delta_2^{\circledcirc}}
$$

The result in fact is immediate by the induction hypotheses for $\pi_1^{\circledcirc}$ and $\pi_2^{\circledcirc}$. ⊠

6 The lower decoration strategy II

Let us continue to apply the results on stripping and take another look at the linear derivation $\partial(\pi)$ obtained from a derivation π in intuitionistic implicational logic by applying the lower decoration strategy. Because of the assumption that π is fully expanded (i.e. all instances of the identity axiom are atomic), the exponential classes in π are precisely the tracks[4] of exponentiated formulas.

6.1. LEMMA. *Let $!A$ be a subformula of a terminal formula in $\partial(\pi)$. Then there is a directed path γ in $\mathcal{G}(\partial(\pi))$ from a labeled vertex to the vertex $T_\pi(!A)$.*

PROOF: This is immediate from the construction of $\partial(\pi)$. Formally we proceed by induction on the length of a branch in a finite tree of decorations starting from a primary bang-source: if $T_\pi(!A)$ contains that source, our claim evidently holds; otherwise we have a sequent $!\Gamma, !A \Rightarrow !B$ being the conclusion of an instance of R! in $\partial(\pi)$. By induction hypothesis there is a directed path in $\mathcal{G}(\partial(\pi))$ from a labeled vertex to $T_\pi(!B)$. But as $T_\pi(!B)$ binds $T_\pi(!A)$, there is a directed path from the labeled vertex to $T_\pi(!A)$. ⊠

As $\partial(\pi)$ is a subdecoration of $\pi^{\circledast}$, we can, in the obvious way consider $\mathcal{G}(\partial(\pi))$ as a subgraph of $\mathcal{G}(\pi^{\circledast})$. More so:

6.2. LEMMA. *$\mathcal{G}(\partial(\pi))$ is a* full *subgraph of $\mathcal{G}(\pi^{\circledast})$: if $T_\pi(!A), T_\pi(!B)$ are exponential classes in $\partial(\pi)$, and there is an arrow between the corresponding vertices in $\mathcal{G}(\pi^{\circledast})$, then that arrow exists also in $\mathcal{G}(\partial(\pi))$.*

PROOF: The arrow is there because of the conclusion of an instance of R! in $\pi^{\circledast}$. As both $\pi^{\circledast}$ and $\partial(\pi)$ have the down-property, the corresponding sequent in $\partial(\pi)$ is also the conclusion of R!, q.e.d. ⊠

Let us, for a vertex labeled "s", write $\vec{\mathsf{cx}}(\mathsf{s})$ to denote the union of all directed paths in $\mathcal{G}(\pi)$ starting from that vertex. We thus obtain:

[4] Cf. the notion of 'track' introduced in the previous chapter.

6.3. THEOREM. $\mathcal{G}(\partial(\pi)) = \bigcup_{s \in \mathcal{G}(\pi^{\circledast})} \vec{cx}(s)$.

PROOF: By lemma's 6.1 and 6.2. ⊠

Now $\bigcup_{s \in \mathcal{G}(\pi^{\circledast})} \vec{cx}(s)$ is precisely $\mathcal{G}((\pi^{\circledast})^{\triangleright})$ (theorem 2.11), so, by the results of the previous sections, $\partial(\pi)$ is an optimal linearization of π, which is essentially equal to $(\pi^{\circledast})^{\triangleright}$, though in general not identical to it, the difference being that in general applications of L! in $\partial(\pi)$ are 'postponed'. (Cf. the remark at the end of section 3.)

We can make analogous observations regarding the derivation $\partial(\pi)$ obtained by applying the lower decoration strategy to an **ILU**-derivation π (section 4.2): now $\mathcal{G}(\partial(\pi))$ is a (full) subgraph of $\mathcal{G}(\pi^*)$, as well as of $\mathcal{G}(\pi^{\circledast})$, and we find the following.

6.4. THEOREM. *Let π be an* **ILU***-derivation. Then*

$$\bigcup_{s \in \mathcal{G}(\pi^*)} \vec{cx}(s) = \mathcal{G}(\partial(\pi)) = \bigcup_{s \in \mathcal{G}(\pi^{\circledast})} \vec{cx}(s). \quad ⊠$$

Again we find that $\bigcup_{s \in \mathcal{G}(\pi^*)} \vec{cx}(s)$ is precisely $\mathcal{G}((\pi^*)^{\triangleright})$.

The minimality of the linearizations thus obtained are *optimal*, as both the f- and the g-decoration are what we might call 'mono' decorations.

6.5. DEFINITION. Let δ be a decoration strategy for calculus **L**. We will call δ a 'mono' decoration if $\delta(\pi)$ is a 'mono'-derivation for any proof π in **L**. ⊠

Therefore, by theorem 3.1 and the remarks at the end of section 3, we may consider $\partial(\pi)$ as *the* optimal linearization of an **IL(U)**-derivation π. In case of an **ILU**-derivation, recall that derivations in **ILU** are, in a way, derivations in linear logic: the g-decoration is *strong* (definition 3.2.1). We can apply the following.

6.6. THEOREM. *If δ is a strong 'mono' decoration strategy for calculus* **L**, *then $\delta(\pi)^{\triangleright}$ is an optimal linear version of π which simulates π's behaviour under reduction.*

PROOF: Optimality follows from the fact that δ is 'mono'. By the results of section 4 we know that $\delta(\pi)^{\triangleright}$ has essentially the same set of reductions as $\delta(\pi)$. As δ is strong, it simulates the reduction of π. ⊠

We have of course analogous results for decorations of derivations in sequent calculi for (fragments of) classical logic. E.g. the linear derivation $\partial(\pi)$ obtained by applying the decoration strategy of chapter 6 of Joinet(1993) to an **LKT**-derivation π will be an optimal linearization of π that is essentially equal to $(\pi^{\mathrm{T}})^{\triangleright}$. As the T-decoration is both 'mono' and strong, also in this case we may speak of *the* optimal linearization, and apply theorem 6.6.

As we mentioned before, (the implicational fragment of) **ILU** corresponds to the neutral fragment of intuitionistic implicational logic in Girard's system of unified logic **LU** (Girard(1993)), and (fragments of) **LKT** correspond to the negative fragment of Girard's classical logic **LC** (Girard(1991)). Indeed our methods are not limited to merely these fragments. If in an **LU**-derivation π we decorate negative atoms N as $?N$, positive atoms P as $!P$, and follow the linear definitions of the classical and intuitionistic connectives, what we get is a strong 'mono' decoration (cf. section 3.5). As a result theorem 6.6 will apply to *all* of non-linear **LU**, including **LC**.

Notes

- The first four sections and parts of section 5 of this chapter appeared, in condensed and slightly different form, as sections 2, 3 and 4 of Danos et al.(1993d). Other parts of section 5 are from Danos et al.(1993c).

- Observe the similarity between the notion of 'identity class' and that of 'logical flowgraph' as introduced for classical logic in Buss(1991). An identity class in fact is an undirected flowgraph. A similar concept of course is implicit in proofnets.

6

Constrictive morphisms

Recall the proofs of the *correctness* of Girard's embedding $(\cdot)^*$ of **IL** into **CLL** (proposition 1.3.3), of the Q- and T-embedding of **CL** into **CLL** (theorem 2.4.1). In each of these cases we inductively apply the translation to a given sequent calculus proof. At some points, however, applying the necessary logical rule to the translated sequents of the premiss does not result in the 'right' translation of the main formula, but to a formula that appears to have the (external) shape of its f-translation (in the intuitionistic case), or its q-, respectively t-translation (in the classical case).

We continue by using a *correction cut*: we cut with a small derivation of a sequent that brings the deviating main formula 'back into track'. The example one finds on page 46 is typical.

Correction cuts are derivations of a special kind: they are decorations of derivations of identities $A \Rightarrow A$. What is more, in all cases mentioned above, they are nothing but decorations of expansions of instances of an identity axiom (in the aforementioned example, a decoration of an expansion of $!C \multimap B \Rightarrow !C \multimap B$). In the present chapter we will look in some detail at linear derivations of this type, and ask what happens were we to *eliminate* these cuts. In fact, what we will show is that elimination of g-correction cuts transforms an **IL**-derivation into an **ILU**-derivation. Similarly, eliminating T-correction cuts maps **CL** to **LKT**, and eliminating correction cuts for the Q-translation will map a **CL**-proof to **LKQ**.

In this sense the economic, non-decorating, embeddings of intuitionistic and classical into linear logic find a natural interpretation as *transformations of derivations.*

1 Decorating the identity

In the obvious inductive way we define the *full expansion* π_A of the axiom $A \Rightarrow A$:

if A is an atom p, then π_p is just $p \Rightarrow p$. For non-atomic A we have e.g. $\pi_{C \multimap D}, \pi_{\forall X.A}, \pi_{?A}$:

$$\dfrac{\dfrac{\begin{matrix}\pi_C \\ \vdots \\ C \Rightarrow C\end{matrix} \quad \begin{matrix}\pi_D \\ \vdots \\ D \Rightarrow D\end{matrix}}{C \multimap D, C \Rightarrow D}}{C \multimap D \Rightarrow C \multimap D} \qquad \dfrac{\dfrac{\begin{matrix}\pi_{A[Y/X]} \\ \vdots \\ A[Y/X] \Rightarrow A[Y/X]\end{matrix}}{\forall X.A \Rightarrow A[Y/X]}}{\forall X.A \Rightarrow \forall X.A} \qquad \dfrac{\dfrac{\begin{matrix}\pi_A \\ \vdots \\ A \Rightarrow A\end{matrix}}{A \Rightarrow ?A}}{?A \Rightarrow ?A}$$

All other cases are similar.

Let π' be a derivation of $A \Rightarrow A$, and suppose $P_1 \Rightarrow P_1, \ldots, P_n \Rightarrow P_n$ are the identity axioms occurring in π'. We say that π' is an *expansion* of the axiom $A \Rightarrow A$ iff the proof obtained by replacing each axiom $P_i \Rightarrow P_i$ in π' by its full expansion π_{P_i} is precisely the full expansion π_A of $A \Rightarrow A$.

1.1. DEFINITION. Let π' be an expansion of $A \Rightarrow A$, and let some derivation π of $\Gamma, A \Rightarrow \Delta$ or $\Gamma \Rightarrow A, \Delta$ be given. We say that π' is *not further expanded* with respect to A *than* π (notation: $\pi' \preceq_A \pi$) if for each occurrence in π of an identity axiom for a formula in the $\sim$-class of an occurrence of a subformula F in A, the corresponding occurrence of F in π' stems from[1] an identity axiom for a formula G such that $F \sim F'$ for some subformula F' of G. ⊠

In the following example the left premiss of the cut (an expansion of $\forall Z(X \multimap Z) \Rightarrow \forall Z(X \multimap Z)$) is not further expanded (with respect to $\forall Z(X \multimap Z)$) than the right premiss:

$$\dfrac{\dfrac{\dfrac{X \multimap Y \Rightarrow X \multimap Y}{\forall Z(X \multimap Z) \Rightarrow X \multimap Y}}{\forall Z(X \multimap Z) \Rightarrow \forall Z(X \multimap Z)} \quad \dfrac{\dfrac{X \Rightarrow X \quad C \multimap C \Rightarrow C \multimap C}{X, X \multimap (C \multimap C) \Rightarrow C \multimap C}}{X, \forall Z(X \multimap Z) \Rightarrow C \multimap C}}{X, \forall Z(X \multimap Z) \Rightarrow C \multimap C}$$

[1]We will not bother to make this notion more precise.

($C \multimap C$ corresponds to the subformula Z of $\forall Z(X \multimap Z)$ which in the left premiss stems from the axiom for $X \multimap Y$, and $Z \sim Y$.)

Note that if $\pi_A \preceq_A \pi$, then $\pi' \preceq_A \pi$ for *all* expansions π' of $A \Rightarrow A$. We have $\pi_A \preceq_A \pi$ e.g. in case all instances of identity axioms in π are *atomic*.

1.2. LEMMA. *Let π be a derivation of $\Gamma, A \Rightarrow \Delta$ or $\Gamma \Rightarrow A, \Delta$; let π' be an expansion of $A \Rightarrow A$, and suppose $\pi' \preceq_A \pi$. Let $\tilde{\pi}$ the derivation obtained by a cut between π and π'. Then there is a reduction from $\tilde{\pi}$ to π.* ⊠

In case $\pi' \not\preceq_A \pi$, $\tilde{\pi}$ will reduce to an η-expansion of π, i.e. the result is π with one or more identity-axioms replaced by expansions.

1.3. REMARK. The property expressed by this lemma, the *existence* of a reduction, is usually somewhat stronger in the case of linear logic than it is in case of intuitionistic or classical sequent calculus. To illustrate this consider the following example.

$$
\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{A \to B, A \Rightarrow B}}{A \to B \Rightarrow A \to B} \qquad \dfrac{\dfrac{\begin{array}{c}\pi_1 \\ \vdots \\ \Gamma_1 \Rightarrow \Delta_1\end{array}}{\Gamma_1 \Rightarrow A, \Delta_1} \quad \begin{array}{c}\pi_2 \\ \vdots \\ \Gamma_2, B \Rightarrow \Delta_2\end{array}}{\Gamma_1, \Gamma_2, A \to B \Rightarrow \Delta_1, \Delta_2}}{\Gamma_1, \Gamma_2, A \to B \Rightarrow \Delta_1, \Delta_2}
$$

We reduce to

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1 \\ \vdots \\ \Gamma_1 \Rightarrow \Delta_1\end{array}}{\Gamma_1 \Rightarrow A, \Delta_1} \quad \dfrac{A \Rightarrow A \quad B \Rightarrow B}{A \to B, A \Rightarrow B}}{\Gamma_1, A \to B \Rightarrow \Delta_1, B} \qquad \begin{array}{c}\pi_2 \\ \vdots \\ \Gamma_2, B \Rightarrow \Delta_2\end{array}}{\Gamma_1, \Gamma_2, A \to B \Rightarrow \Delta_1, \Delta_2}
$$

and then might continue to permute the derivation of $\Gamma_1 \Rightarrow A, \Delta_1$ up to the identity axiom $A \Rightarrow A$; but also we could directly reduce the cut on A, and obtain

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Gamma_1 \Rightarrow \Delta_1\end{matrix}}{\Gamma_1, A \to B \Rightarrow \Delta_1}}{\Gamma_1, A \to B \Rightarrow B, \Delta_1} \quad \begin{matrix}\pi_2\\ \vdots\\ \Gamma_2, B \Rightarrow \Delta_2\end{matrix}}{\Gamma_1, \Gamma_2, A \to B \Rightarrow \Delta_1, \Delta_2}$$

Obviously it is the first alternative that is in accordance with the claim of the lemma. Note that in the linear sequent calculus we have no choice and *only* the first reduction is possible in a comparable situation, like

$$\dfrac{\dfrac{\dfrac{?A \Rightarrow ?A \quad ?B \Rightarrow ?B}{?A \multimap B, ?A \Rightarrow ?B}}{?A \multimap ?B \Rightarrow ?A \multimap ?B} \quad \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Gamma_1 \Rightarrow \Delta_1\end{matrix}}{\Gamma_1 \Rightarrow ?A, \Delta_1} \quad \begin{matrix}\pi_2\\ \vdots\\ \Gamma_2, ?B \Rightarrow \Delta_2\end{matrix}}{\Gamma_1, \Gamma_2, ?A \multimap ?B \Rightarrow \Delta_1, \Delta_2}}{\Gamma_1, \Gamma_2, ?A \multimap ?B \Rightarrow \Delta_1, \Delta_2}$$

However we can *force* the second reduction by cutting with a *decoration* of an expansion of $A \multimap ?B \Rightarrow A \multimap ?B$, as the reader should verify:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad ?B \Rightarrow ?B}{A \multimap ?B, A \Rightarrow ?B}}{!(A \multimap ?B), A \Rightarrow ?B}}{!(A \multimap ?B), ?A \Rightarrow ?B}}{!(A \multimap ?B) \Rightarrow ?A \multimap ?B} \quad \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \Gamma_1 \Rightarrow \Delta_1\end{matrix}}{\Gamma_1 \Rightarrow ?A, \Delta_1} \quad \begin{matrix}\pi_2\\ \vdots\\ \Gamma_2, ?B \Rightarrow \Delta_2\end{matrix}}{\Gamma_1, \Gamma_2, ?A \multimap ?B \Rightarrow \Delta_1, \Delta_2}}{\Gamma_1, \Gamma_2, !(A \multimap ?B) \Rightarrow \Delta_1, \Delta_2}$$

In fact, this observation might be judged the heart of coming sections' matter. ⊠

Let π' be an expansion of $A \Rightarrow A$, and suppose $e_1, \ldots, e_n$ are precisely the exponentiated subformulas of A. Write E_i for the set of elements of $\mathcal{E}(\pi')$ induced by the left and right occurrence of e_i in the conclusion (note that E_i is either a singleton, or has two elements). By a simple induction on the complexity of A we then find that $\pi' - E_i$ is an expansion of $A - E_i \Rightarrow A - E_i$. As a corollary we obtain the following

1.4. LEMMA. *Let $\delta(A)$ be a decoration of a formula A, and let π' be an expansion of $\delta(A) \Rightarrow \delta(A)$. Then π' is a decoration of an expansion of $A \Rightarrow A$.*

PROOF: Let $\{e_1, \dots, e_k\}$ be the exponentiated subformulas of $\delta(A)$ that are not exponentiated subformulas of A. Iterating the observation above we find that $\pi' - E_1 - \dots - E_k$ is an expansion of $\delta(A) - E_1 - \dots - E_k$, which is just A. ⊠

1.5. DEFINITION. Let $\langle\pi'\rangle$ be a decoration of an expansion of $A \Rightarrow A$. (I.e., $\langle\pi'\rangle$ derives $\delta(A) \Rightarrow \delta'(A)$, where $\delta(A), \delta'(A)$ are decorations of A.) We say that $\langle\pi'\rangle$ is an *invisible morphism* for a given derivation π of $\Gamma, \delta'(A) \Rightarrow \Delta$ or $\Gamma \Rightarrow \delta(A), \Delta$ if we can eliminate the final cut from the derivation $\tilde{\pi}$, being

$$\dfrac{\begin{matrix}\langle\pi'\rangle \\ \vdots \\ \delta(A) \Rightarrow \delta'(A)\end{matrix} \quad \begin{matrix}\pi \\ \vdots \\ \Gamma, \delta'(A) \Rightarrow \Delta\end{matrix}}{\Gamma, \delta(A) \Rightarrow \Delta} \qquad \text{resp.} \qquad \dfrac{\begin{matrix}\pi \\ \vdots \\ \Gamma \Rightarrow \delta(A), \Delta\end{matrix} \quad \begin{matrix}\langle\pi'\rangle \\ \vdots \\ \delta(A) \Rightarrow \delta'(A)\end{matrix}}{\Gamma \Rightarrow \delta'(A), \Delta}$$

in such a way that for the resulting reduct $\mathsf{r}(\tilde{\pi})$ it holds that $\mathsf{sk}(\mathsf{r}(\tilde{\pi})) = \mathsf{sk}(\pi)$. In case $\mathsf{sk}(\mathsf{r}(\tilde{\pi}))$ is always necessarily different from $\mathsf{sk}(\pi)$ we call $\langle\pi'\rangle$ *constrictive* for π. ⊠

As suggested by our terminology, we consider a decoration $\langle\pi'\rangle$ of an expansion π' of an identity axiom $A \Rightarrow A$ as a *morphism* that can be applied to a given derivation π, and that (via cut-elimination) maps π to a derivation $\mathsf{r}(\pi)$. The morphism is said to be *invisible* if its application to the linear derivation π does not affect the underlying classical (or intuitionistic) skeleton.

The decoration of the expansion of $A \multimap ?B \Rightarrow A \multimap ?B$ at the end of remark 1.3 is *not* invisible for the derivation to which it is applied, and therefore is an example of a constrictive morphism.

We can use the 'exponential removal'-theory of chapter 5 to identify invisible morphisms. The idea is to try to 'strip' $\langle\pi'\rangle$ down to an expansion of an identity axiom, which, under the condition of lemma 1.2, is trivially invisible.

By definition of decoration, for $\langle\pi'\rangle$ on its own, this of course can always be done. When $\langle\pi'\rangle$ is applied to a derivation π, this may no longer be possible. The following proposition, however, gives a sufficient condition for invisibility.

1.6. PROPOSITION. *Let $\tilde{\pi}$ be as in definition 1.5, and suppose that in $\tilde{\pi}$ the set E of elements of $\mathcal{E}(\tilde{\pi})$ determined by $\mathcal{E}(\langle\pi'\rangle)\backslash\mathcal{E}(\pi')$ is nre. If $\langle\pi'\rangle - E \preceq_{A-E} \pi - E$, then $\langle\pi'\rangle$ is invisible for π.*

PROOF: Suppose the exponential classes $e_1, \ldots, e_n$ of π' coincide in $\tilde{\pi}$ with elements of $\mathcal{E}(\langle\pi'\rangle)\backslash\mathcal{E}(\pi')$. Let $A' := A - E_1 - \ldots - E_n$ (so $A' \equiv A - E$). By lemma 1.4, π' is a decoration of an expansion π'' of $A' \Rightarrow A'$, and $\langle\pi'\rangle - E \equiv \pi''$.

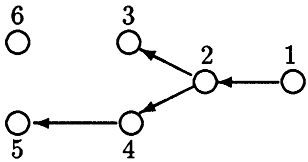
Now $\tilde{\pi} - E$ is the derivation obtained by a cut between π'' and $\pi - E$. As, by hypothesis, $\pi'' \preceq_{A'} \pi - E$, by lemma 1.2 $\tilde{\pi} - E \twoheadrightarrow \pi - E$. And as E is nre, by theorem 5.4.6, $\tilde{\pi}$ reduces to a derivation $\mathsf{r}(\tilde{\pi})$ such that $\mathsf{r}(\tilde{\pi}) \rhd \pi - E$. So $\mathsf{sk}(\mathsf{r}(\tilde{\pi})) = \mathsf{sk}(\pi - E) = \mathsf{sk}(\pi)$. ⊠

Note that in fact this proposition gives 'invisibility' in an even stronger sense, as, by the results of chapter 5, $\mathsf{r}(\tilde{\pi})$ will also have the same behaviour under reduction as π.

As an application let us use the above to give another proof of (a generalization of) proposition 1.4.

In analogy to the terminology used in section 4.1 we call a linear formula ϕ^+ that has been obtained from a formula ϕ by prefixing "!" to a number of *positive*, "?" to a number of *negative* subformulas, a *positive decoration* (of ϕ).[2] *Negative decoration* is defined in the obvious dual way. Clearly $\phi^+ \Rightarrow \phi$ and $\phi \Rightarrow \phi^-$ are always derivable, with derivations $\langle\pi_\phi\rangle$ that are decorations of the full expansion π_ϕ of $\phi \Rightarrow \phi$. In fact, one obtains $\langle\pi_\phi\rangle$ by merely adding a certain number of instances

[2] To be precise: ϕ is a positive decoration of ϕ; if ϕ^+ is a positive decoration of ϕ and ψ is a positive(negative) subformula of ϕ^+, then $\phi^+[!\psi/\psi](\phi^+[?\psi/\psi])$ is a positive decoration of ϕ.



$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{!_4A \Rightarrow !_4A \quad B \Rightarrow B}{!_4A, !_4A \multimap B \Rightarrow B}}{!_4A, !_3(!_4A \multimap B) \Rightarrow B}}{!_4A, !_3(!_4A \multimap B) \Rightarrow !_2B}}{!_3(!_4A \multimap B) \Rightarrow !_4A \multimap !_2B}
\quad
\dfrac{\dfrac{\dfrac{A \Rightarrow A}{!_5A \Rightarrow A}}{!_5A \Rightarrow !_4A} \quad \dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow B}{!_2B \Rightarrow B}}{!_2B \Rightarrow !_1B} \quad C \Rightarrow C}{!_1B \multimap C, !_2B \Rightarrow C}}{!_6(!_1B \multimap C), !_2B \Rightarrow C}}{!_4A \multimap !_2B, !_5A, !_6(!_1B \multimap C) \Rightarrow C}
}{!_3(!_4A \multimap B), !_5A, !_6(!_1B \multimap C) \Rightarrow C}
$$

Figure 6.1: The decoration of the expansion of $!A \multimap B \Rightarrow !A \multimap B$ is not invisible; the set consisting of the exponential classes 2 and 3 is not nre.

of dereliction to π_ϕ. Now let π be a derivation of a sequent $\Gamma, \phi \Rightarrow \Delta$ or $\Gamma \Rightarrow \phi, \Delta$, using only *atomic* instances of the identity axiom.

It is immediate that the set E of exponential classes determined by $\mathcal{E}(\langle\pi_\phi\rangle)\backslash\mathcal{E}(\pi_\phi)$ is nre. And, as $\phi - E \equiv \phi$, we have that $\langle\pi_\phi\rangle - E = \pi_\phi \preceq_\phi \pi = \pi - E$. So, by proposition 1.6, $\langle\pi_\phi\rangle$ is invisible for π. (Proposition 1.4 claimed this in the special case of intuitionistic linear implicational logic only.)

With a little more effort we are able to draw the same conclusion in case π derives $\Gamma, \phi^- \Rightarrow \Delta$ or $\Gamma \Rightarrow \Delta, \phi^+$ (again using only *atomic* instances of the identity axiom). Indeed, if e is an exponential class in π determined by an exponential in $\phi^+\backslash\phi$ or $\phi^-\backslash\phi$, then e contains no source: the exponential determining e occurs *positively* (in case it is "!"), *negatively* (in case it is "?"), in the conclusion; were e to contain a source, it should coincide with the class induced by an exponential occurring with the opposite sign somewhere in π and as π has only atomic instances of the identity-axiom, this cannot be the case; moreover e is always *minimal* in $\mathcal{G}(\pi)$, so $\{e\}$ is saturated. From these facts it is readily seen that once more the conditions of proposition 1.6 are satisfied.

2 Correction cuts

As observed in the introduction to this chapter, the essential examples of decorated axiom-expansions are the correction cuts introduced in order to prove correctness of economic non-decorating embeddings of intuitionistic and classical, into linear logic. Let us here list some of these morphisms, and briefly indicate why in general they will be *constrictive.*

In proving the correctness of Girard's translation by induction on the length of **IL**-derivations, in the case of L→ one cuts with

$$\dfrac{\dfrac{\dfrac{\dfrac{!A^* \Rightarrow !A^* \quad B^* \Rightarrow B^*}{!A^* \multimap B^*, !A^* \Rightarrow B^*}}{!(!A^* \multimap B^*), !A^* \Rightarrow B^*}}{!(!A^* \multimap B^*), !A^* \Rightarrow !B^*}}{!(!A^* \multimap B^*) \Rightarrow !A^* \multimap !B^*}$$

This decoration of (an expansion of) $!A^* \multimap B^* \Rightarrow !A^* \multimap B^*$ in general will be constrictive: if B in the original derivation has been subjected to structural manipulation, the 'nre'-condition of proposition 1.6 can not be fulfilled.

Similar observations hold for the Q- and T-implication, being decorations of respectively $!A^{\mathrm{Q}} \multimap ?!B^{\mathrm{Q}} \Rightarrow !A^{\mathrm{Q}} \multimap ?!B^{\mathrm{Q}}$ and $!?A^{\mathrm{T}} \multimap ?B^{\mathrm{T}} \Rightarrow !?A^{\mathrm{T}} \multimap ?B^{\mathrm{T}}$:

$$\begin{array}{c}
!A^{\mathrm{Q}} \Rightarrow\, !A^{\mathrm{Q}} \quad ?!B^{\mathrm{Q}} \Rightarrow\, ?!B^{\mathrm{Q}} \\ \hline
!A^{\mathrm{Q}} \multimap ?!B^{\mathrm{Q}}, !A^{\mathrm{Q}} \Rightarrow\, ?!B^{\mathrm{Q}} \\ \hline
!(!A^{\mathrm{Q}} \multimap ?!B^{\mathrm{Q}}), !A^{\mathrm{Q}} \Rightarrow\, ?!B^{\mathrm{Q}} \\ \hline
!(!A^{\mathrm{Q}} \multimap ?!B^{\mathrm{Q}}), ?!A^{\mathrm{Q}} \Rightarrow\, ?!B^{\mathrm{Q}} \\ \hline
!(!A^{\mathrm{Q}} \multimap ?!B^{\mathrm{Q}}) \Rightarrow\, ?!A^{\mathrm{Q}} \multimap ?!B^{\mathrm{Q}}
\end{array}
\qquad
\begin{array}{c}
!?A^{\mathrm{T}} \Rightarrow\, !?A^{\mathrm{T}} \quad ?B^{\mathrm{T}} \Rightarrow\, ?B^{\mathrm{T}} \\ \hline
!?A^{\mathrm{T}} \multimap ?B^{\mathrm{T}}, !?A^{\mathrm{T}} \Rightarrow\, ?B^{\mathrm{T}} \\ \hline
?(!?A^{\mathrm{T}} \multimap ?B^{\mathrm{T}}), !?A^{\mathrm{T}} \Rightarrow\, ?B^{\mathrm{T}} \\ \hline
!?(!?A^{\mathrm{T}} \multimap ?B^{\mathrm{T}}), !?A^{\mathrm{T}} \Rightarrow\, ?B^{\mathrm{T}} \\ \hline
!?(!?A^{\mathrm{T}} \multimap ?B^{\mathrm{T}}), !?A^{\mathrm{T}} \Rightarrow\, !?B^{\mathrm{T}} \\ \hline
!?(!?A^{\mathrm{T}} \multimap ?B^{\mathrm{T}}) \Rightarrow\, !?A^{\mathrm{T}} \multimap !?B^{\mathrm{T}}
\end{array}$$

Note that as decorations of the axiom expansions, the above derivations are uniquely determined. The same holds true for the correction cuts involving quantifiers, as we leave for the reader to verify.

This however is no longer true in case of the (multiplicative) Q-conjunction, neither in that of the (multiplicative) T-disjunction. As already mentioned in section 2.4 we are confronted, for each of these, with *two* symmetric possibilities to decorate in the required way an expansion of $!A^{\mathrm{Q}} \otimes !B^{\mathrm{Q}} \Rightarrow\, !A^{\mathrm{Q}} \otimes !B^{\mathrm{Q}}$ and $?A^{\mathrm{T}} \otimes ?B^{\mathrm{T}} \Rightarrow\, ?A^{\mathrm{T}} \otimes ?B^{\mathrm{T}}$.

To take the multiplicative T-disjunction as an example, we might take either

$$\begin{array}{c}
?A^{\mathrm{T}} \Rightarrow\, ?A^{\mathrm{T}} \quad ?B^{\mathrm{T}} \Rightarrow\, ?B^{\mathrm{T}} \\ \hline
?A^{\mathrm{T}} \parr ?B^{\mathrm{T}} \Rightarrow\, ?A^{\mathrm{T}},\ ?B^{\mathrm{T}} \\ \hline
?(?A^{\mathrm{T}} \parr ?B^{\mathrm{T}}) \Rightarrow\, ?A^{\mathrm{T}},\ ?B^{\mathrm{T}} \\ \hline
!?(?A^{\mathrm{T}} \parr ?B^{\mathrm{T}}) \Rightarrow\, ?A^{\mathrm{T}},\ ?B^{\mathrm{T}} \\ \hline
!?(?A^{\mathrm{T}} \parr ?B^{\mathrm{T}}) \Rightarrow\, !?A^{\mathrm{T}},\ ?B^{\mathrm{T}} \\ \hline
!?(?A^{\mathrm{T}} \parr ?B^{\mathrm{T}}) \Rightarrow\, ?!?A^{\mathrm{T}},\ ?B^{\mathrm{T}} \\ \hline
!?(?A^{\mathrm{T}} \parr ?B^{\mathrm{T}}) \Rightarrow\, ?!?A^{\mathrm{T}},\ !?B^{\mathrm{T}} \\ \hline
!?(?A^{\mathrm{T}} \parr ?B^{\mathrm{T}}) \Rightarrow\, ?!?A^{\mathrm{T}} \parr !?B^{\mathrm{T}}
\end{array}
\qquad \text{or} \qquad
\begin{array}{c}
?A^{\mathrm{T}} \Rightarrow\, ?A^{\mathrm{T}} \quad ?B^{\mathrm{T}} \Rightarrow\, ?B^{\mathrm{T}} \\ \hline
?A^{\mathrm{T}} \parr ?B^{\mathrm{T}} \Rightarrow\, ?A^{\mathrm{T}},\ ?B^{\mathrm{T}} \\ \hline
?(?A^{\mathrm{T}} \parr ?B^{\mathrm{T}}) \Rightarrow\, ?A^{\mathrm{T}},\ ?B^{\mathrm{T}} \\ \hline
!?(?A^{\mathrm{T}} \parr ?B^{\mathrm{T}}) \Rightarrow\, ?A^{\mathrm{T}},\ ?B^{\mathrm{T}} \\ \hline
!?(?A^{\mathrm{T}} \parr ?B^{\mathrm{T}}) \Rightarrow\, ?A^{\mathrm{T}},\ !?B^{\mathrm{T}} \\ \hline
!?(?A^{\mathrm{T}} \parr ?B^{\mathrm{T}}) \Rightarrow\, ?A^{\mathrm{T}},\ ?!?B^{\mathrm{T}} \\ \hline
!?(?A^{\mathrm{T}} \parr ?B^{\mathrm{T}}) \Rightarrow\, !?A^{\mathrm{T}},\ ?!?B^{\mathrm{T}} \\ \hline
!?(?A^{\mathrm{T}} \parr ?B^{\mathrm{T}}) \Rightarrow\, !?A^{\mathrm{T}} \parr ?!?B^{\mathrm{T}}
\end{array}$$

To see the effect of this choice, let us suppose that in the original derivation both formula A and formula B were introduced (to simplify matters say immediately before the application of L$\vee$) by weakening. The correction cut 'tells' us to replace these weakenings by a weakening directly on $A \vee B$. In order to realize this replacement we have to chose one of the premisses of the L$\vee$-rule and continue from there. Using the left rather than the right, or the right rather than the left correction cut, in fact is equivalent to making that choice. (Cf. section 3.3.)

These derivations of decorations of identities provide us with valuable information on the possibility of re-arranging the 'management' of structural rules in (the skeleton of) a linear derivation. They indicate e.g. that the application of structural rules to a formula $A \circ B$ (where $\circ$ is some binary connective) in deriving a sequent $\Gamma \Rightarrow \Delta$ can be replaced by structural rules on A and/or B separately, and vice versa. In a more general sense this holds of course not only for decorations of expansions of an axiom $A \Rightarrow A$, but also of other types of derivations of $A \Rightarrow A$.[3] Consider the following. We observed at the end of section 2.4 that in case of the Q-translation we left unused the possibility of a drastic optimization in the additive translation of the conjunction, as we might have chosen $(A \wedge B)^{\mathrm{Q}} := A^{\mathrm{Q}} \& B^{\mathrm{Q}}$ instead of $(A \wedge B)^{\mathrm{Q}} := ?!A^{\mathrm{Q}} \& ?!B^{\mathrm{Q}}$, due to derivability of $!(A\&B) \Rightarrow\, !A\&\,!B$ and $?!(?!A\&?!B) \Rightarrow\, ?!(A\&B)$. The first of these is derivable by means of a decoration of an expansion of $A\&B \Rightarrow A\&B$, as the reader will easily verify.

The second, on the other hand, is *not* the decoration of an expansion of the identity. It is derived e.g. as follows:

$$\begin{array}{c}
\dfrac{\dfrac{\dfrac{A \Rightarrow A}{!A \Rightarrow A}}{!A, !B \Rightarrow A} \qquad \dfrac{\dfrac{B \Rightarrow B}{!B \Rightarrow B}}{!A, !B \Rightarrow B}}{!A, !B \Rightarrow A\&B} \\ \hline
!A, !B \Rightarrow\, !(A\&B) \\ \hline
!A, !B \Rightarrow\, ?!(A\&B) \\ \hline
!A, ?!B \Rightarrow\, ?!(A\&B) \\ \hline
!A, ?!A\&?!B \Rightarrow\, ?!(A\&B) \\ \hline
!A, !(?!A\&?!B) \Rightarrow\, ?!(A\&B) \\ \hline
?!A, !(?!A\&?!B) \Rightarrow\, ?!(A\&B) \\ \hline
?!A\&?!B, !(?!A\&?!B) \Rightarrow\, ?!(A\&B) \\ \hline
!(?!A\&?!B), !(?!A\&?!B) \Rightarrow\, ?!(A\&B) \\ \hline
!(?!A\&?!B) \Rightarrow\, ?!(A\&B) \\ \hline
?!(?!A\&?!B) \Rightarrow\, ?!(A\&B)
\end{array}$$

[3]In this broader sense for example also the standard derivations of $!A \otimes !B \Rightarrow\, !(A\&B)$ and of $!(A\&B) \Rightarrow\, !A \otimes !B$ can be seen as 'constrictive morphisms'.

3 Mapping IL to ILU ...

Take some derivation π of a sequent $\Gamma \Rightarrow A$ in the $\{\rightarrow, \forall_2\}$-fragment of **IL**, and apply inductively Girard's translation $(\cdot)^*$. Identity axioms $A \Rightarrow A$ become

$$\frac{A^* \Rightarrow A^*}{!A^* \Rightarrow A^*},$$

and when encountering a right rule, one continues by applying the corresponding rule in linear logic. Something more interesting happens as soon as we stumble upon an application of L$\rightarrow$ or L$\forall_2$.

$$\frac{\Gamma_1 \Rightarrow A \quad \Gamma_2, B \Rightarrow C}{\Gamma_1, \Gamma_2, A \rightarrow B \Rightarrow C} \qquad \frac{\Gamma, A[T/p] \Rightarrow B}{\Gamma, \forall p.A \Rightarrow B}$$

It is at this point that, in order to ensure that the conclusion is the $(\cdot)^*$-translation of the conclusion in the original derivation, we have to introduce a correction cut. In case of L$\rightarrow$ we obtain the configuration

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{!A^* \Rightarrow !A^* \quad B^* \Rightarrow B^*}{!A^*, !A^* \multimap B^* \Rightarrow B^*}}{!A^*, !(!A^* \multimap B^*) \Rightarrow B^*}}{!A^*, !(!A^* \multimap B^*) \Rightarrow !B^*}}{!(!A^* \multimap B^*) \Rightarrow !A^* \multimap !B^*} \quad \dfrac{\dfrac{\begin{matrix}\pi_1^* \\ \vdots \\ !\Gamma_1^* \Rightarrow A^*\end{matrix}}{!\Gamma_1^* \Rightarrow !A^*} \quad \begin{matrix}\pi_2^* \\ \vdots \\ !\Gamma_2^*, !B^* \Rightarrow C^*\end{matrix}}{!\Gamma_1^*, !\Gamma_2^*, !A^* \multimap !B^* \Rightarrow C^*}}{!\Gamma_1^*, !\Gamma_2^*, !(!A^* \multimap B^*) \Rightarrow C^*}$$

which we will denote by θ_1. For L$\forall_2$ we get θ_2, being

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A^*[T^*/p] \Rightarrow A^*[T^*/p]}{\forall p.A^* \Rightarrow A^*[T^*/p]}}{!\forall p.A^* \Rightarrow A^*[T^*/p]}}{!\forall p.A^* \Rightarrow !A^*[T^*/p]}}{!\forall p.A^* \Rightarrow \forall p.!A^*} \quad \dfrac{\begin{matrix}(\pi')^* \\ \vdots \\ !\Gamma^*, !A^*[T^*/p] \Rightarrow B^*\end{matrix}}{!\Gamma^*, \forall p.!A^* \Rightarrow B^*}}{!\forall p.A^*, !\Gamma^* \Rightarrow B^*}$$

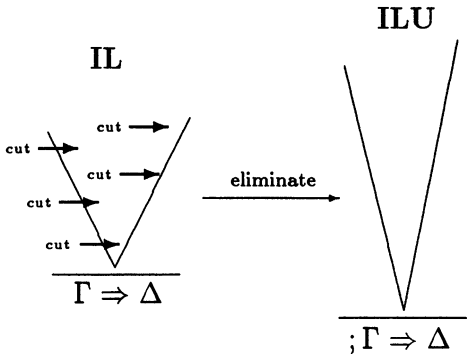


Figure 6.2: From **IL** to **ILU**

Thus we continue, introducing the appropriate cut at each occurrence of a left logical rule. The result is a linear derivation, say π^{G}, of the sequent $!\Gamma^* \Rightarrow A^*$.

We will show the following.

3.1. THEOREM. *If we eliminate the correction cuts introduced in* π^{G} *we obtain a linear derivation* κ*, the skeleton of which is an* **ILU***-derivation of* $;\Gamma \Rightarrow A$*. Moreover, we may assume that* $\kappa = (\mathsf{sk}(\kappa))^*$*.*

PROOF: The first half of the claim of course is an immediate corollary to theorem 3.2.3: after elimination of the correction cuts we have a derivation of $!\Gamma^* \Rightarrow A^*$ in which all identity axioms are of the form $A^* \Rightarrow A^*$, and all remaining cutformulas are of the form A^* or $!A^*$, so its skeleton then is an **ILU**-derivation of $;\Gamma \Rightarrow A$.

For the second half we need an additional argument (which will inspire an occasional lengthy aside).

Let us first reflect briefly upon the meaning of the claim: if our original **IL**-derivation is cut-free, then after elimination of all correction cuts we end up with a cut-free **ILU**-derivation. So application of the constrictive morphisms *realizes* the restrictions on sequent derivations demanded by the correctness of Girard's translation, and e.g. all shrieks immediately following a linear implication are rendered superfluous (cf.

section 3.2, the note on page 78, and the example of figure 6.1).

Consider some highest occurrence of a correction cut in π^{G}, say for an application of L$\to$. Obviously the derivations π_i^* of the premisses of the rule are g-decorations of **ILU**-derivations π_i, and reducing the correction cut we obtain the configuration θ_1', being

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\begin{matrix}\pi_1^*\\ \vdots\\ !\Gamma^* \Rightarrow A^*\end{matrix}}{!\Gamma^* \Rightarrow !A^*} \quad B^* \Rightarrow B^*}{!\Gamma^*, !A^* \multimap B^* \Rightarrow B^*}}{!\Gamma^*, !(!A^* \multimap B^*) \Rightarrow B^*}}{!\Gamma^*, !(!A^* \multimap B^*) \Rightarrow !B^*} \quad \begin{matrix}\pi_2^*\\ \vdots\\ \Pi^*, !\Delta^*, !B^* \Rightarrow C^*\end{matrix}}{\Pi^*, !\Gamma^*, !\Delta^*, !(!A^* \multimap B^*) \Rightarrow C^*}
$$

In case this highest occurrence is for an application of L$\forall_2$, we find θ_2', which is of the form

$$
\cfrac{\cfrac{\cfrac{\cfrac{A^*[T^*/p] \Rightarrow A^*[T^*/p]}{\forall p.A^* \Rightarrow A^*[T^*/p]}}{!\forall p.A^* \Rightarrow A^*[T^*/p]}}{!\forall p.A^* \Rightarrow !A^*[T^*/p]} \quad \begin{matrix}\pi^*\\ \vdots\\ \Pi^*, !\Gamma^*, !A^*[T^*/p] \Rightarrow B^*\end{matrix}}{\Pi^*, !\forall p.A^*, !\Gamma^* \Rightarrow B^*}
$$

Observe that already at this point the skeletons of the derivations under consideration are in **ILU**, as obviously

$$
\cfrac{\cfrac{\cfrac{\begin{matrix}\mathsf{sk}(\pi_1^*)\\ \vdots\\ ;\Gamma \Rightarrow A\end{matrix} \quad B; \Rightarrow B}{A \to B; \Gamma \Rightarrow B}}{; A \to B, \Gamma \Rightarrow B} \quad \begin{matrix}\mathsf{sk}(\pi_2^*)\\ \vdots\\ \Pi; \Delta, B \Rightarrow C\end{matrix}}{\Pi; A \to B, \Gamma, \Delta \Rightarrow C}
$$

is an **ILU**-derivation (because both $\mathsf{sk}(\pi_1^*)$ and $\mathsf{sk}(\pi_2^*)$ are).

Similarly

$$\dfrac{\dfrac{\dfrac{A[T/p]; \Rightarrow A[T/p]}{\forall p.A; \Rightarrow A[T/p]}}{; \forall p.A \Rightarrow A[T/p]} \quad \begin{array}{c}\mathsf{sk}(\pi^*)\\ \vdots \\ \Pi;\Gamma, A[T/p] \Rightarrow B\end{array}}{\Pi;\Gamma,\forall p.A \Rightarrow B}$$

is an **ILU**-derivation (because $\mathsf{sk}(\pi^*)$ is).

The first of these tells us that the rule

$$\frac{;\Gamma \Rightarrow A \qquad \Pi;\Gamma', B \Rightarrow C}{\Pi; A \to B, \Gamma, \Gamma' \Rightarrow C}$$

is *admissible*, yes, *derivable* in **ILU**. The second shows the same for the rule

$$\frac{\Pi;\Gamma, A \Rightarrow B}{\Pi;\Gamma,\forall p.A \Rightarrow B}.$$

(One should compare this to the proof of proposition 3.2.2!)

So the second claim of the theorem clearly holds, as we can eliminate the correction cuts as if eliminating the corresponding cuts in **ILU**! Let us do it this once in some detail, thus exhibiting the relation between cut elimination in **ILU**, and the elimination of cuts in its linear equivalent (section 3.2). To be precise, what we will do is the following.

Given the g-decoration π_1^* of an ILU-proof π_1 with conclusion $;\Gamma \Rightarrow A$ and the g-decoration π_2^* of an ILU-proof π_2 with conclusion $\Pi;\Delta,B^n \Rightarrow C$ (where B^n denotes $n \geq 1$ occurrences of the formula B), we construct, using the derivable rule of *multicut*, the derivation $(\theta_1^n)'$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1^*\\ \vdots \\ !\Gamma^* \Rightarrow A^*\end{array}}{!\Gamma^* \Rightarrow !A^*} \quad B^* \Rightarrow B^*}{!\Gamma^*, !A^* \multimap B^* \Rightarrow B^*}}{!\Gamma^*, !(!A^* \multimap B^*) \Rightarrow B^*}}{!\Gamma^*, !(!A^* \multimap B^*) \Rightarrow !B^*} \quad \begin{array}{c}\pi_2^*\\ \vdots \\ \Pi^*, !\Delta^*, (!B^*)^n \Rightarrow C^*\end{array}}{\Pi^*, !\Gamma^*, !\Delta^*, !(!A^* \multimap B^*) \Rightarrow C^*}$$

Similarly, given the g-decoration π^* of an **ILU**-proof π with conclusion $\Pi; \Gamma, (A[T/p])^n \Rightarrow B$ we construct the linear derivation $(\theta_2^n)'$:

$$\dfrac{\dfrac{\dfrac{\dfrac{A^*[T^*/p] \Rightarrow A^*[T^*/p]}{\forall p.A^* \Rightarrow A^*[T^*/p]}}{!\forall p.A^* \Rightarrow A^*[T^*/p]}}{!\forall p.A^* \Rightarrow !A^*[T^*/p]} \quad \begin{array}{c} \pi^* \\ \vdots \\ \Pi^*, !\Gamma^*, (!A^*[T^*/p])^n \Rightarrow B^* \end{array}}{\Pi^*, !\forall p.A^*, !\Gamma^* \Rightarrow B^*}$$

We then show that given a linear derivation of the form $(\theta_i^n)'$ we obtain, by reducing *only* the shown instance of the multicut and its descendants, a reduct $\mathsf{r}(\theta_i^n)'$ which is the g-decoration of an **ILU**-derivation κ with conclusion $\Pi; \Gamma, \Delta, A \to B \Rightarrow C$ in case $i = 1$, and conclusion $\Pi; \Gamma, \forall p.A \Rightarrow B$ in case $i = 2$

We only treat the case of $(\theta_1^n)'$, as that of $(\theta_2^n)'$ is completely similar, and proceed by induction on the length of π_2^*.

If the last rule applied in π_2^* is C!, R$\forall_2$ or R$\multimap$ we permute the left branch of the derivation upwards and get the result by induction hypothesis, in the last two cases followed by the corresponding **ILU**-rule.

If the last rule is a dereliction L! on a formula in Δ^*, then $\Pi = \emptyset$ (as π_2 is an ILU-derivation). We permute and get the result by induction hypothesis and L!, which corresponds to D in the ILU-derivation. Similarly, in case of L$\forall_2$ we permute and are done by induction hypothesis and L$\forall_2$.

If the last rule is a dereliction on a B^*, then again $\Pi = \emptyset$. The situation now is the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c} \pi_1^* \\ \vdots \\ !\Gamma^* \Rightarrow A^* \end{array}}{!\Gamma^* \Rightarrow !A^*} \quad B^* \Rightarrow B^*}{!\Gamma^*, !A^* \multimap B^* \Rightarrow B^*}}{!\Gamma^*, !(!A^* \multimap B^*) \Rightarrow B^*}}{!\Gamma^*, !(!A^* \multimap B^*) \Rightarrow !B^*} \quad \dfrac{\begin{array}{c} (\pi_2^*)' \\ \vdots \\ !\Delta^*, (!B^*)^{n-1}, B^* \Rightarrow C^* \end{array}}{!\Delta^*, (!B^*)^n \Rightarrow C^*}}{!\Gamma^*, !\Delta^*, !(!A^* \multimap B^*) \Rightarrow C^*}$$

If $n > 1$ we transform this into

$$
\dfrac{
  \dfrac{\begin{array}{c}\pi_1^* \\ \vdots \\ !\Gamma^* \Rightarrow A^*\end{array}}{!\Gamma^* \Rightarrow !A^*}
  \qquad
  \dfrac{
    \dfrac{\begin{array}{c}\vdots\end{array}}{
      \begin{array}{c}
      \dfrac{\dfrac{\begin{array}{c}\pi_1^* \\ \vdots \\ !\Gamma^* \Rightarrow A^*\end{array}}{!\Gamma^* \Rightarrow !A^*} \quad B^* \Rightarrow B^*}{\vdots} \\
      !\Gamma^*, !(!A^* \multimap B^*) \Rightarrow !B^*
      \end{array}}
    \quad
    \begin{array}{c}(\pi_2^*)' \\ \vdots \\ !\Delta^*, (!B^*)^{n-1}, B^* \Rightarrow C^*\end{array}
  }{B^*, !\Gamma^*, !\Delta^*, !(!A^* \multimap B^*) \Rightarrow C^*}
}{
  \dfrac{
    \dfrac{!A^* \multimap B^*, !(!A^* \multimap B^*), !\Gamma^*, !\Gamma^*, !\Delta^*}{!(!A^* \multimap B^*), !(!A^* \multimap B^*), !\Gamma^*, !\Gamma^*, !\Delta^*}
  }{
    \dfrac{\vdots}{!(!A^* \multimap B^*), !\Gamma^*, !\Delta^* \Rightarrow C^*}
  }
}
$$

and the result follows by induction hypothesis, the left rule for implication and contraction in ILU.

If $n = 1$ we get directly

$$
\dfrac{
  \dfrac{\begin{array}{c}\pi_1^* \\ \vdots \\ !\Gamma^* \Rightarrow A^*\end{array}}{!\Gamma^* \Rightarrow !A^*}
  \quad
  \begin{array}{c}(\pi_2^*)' \\ \vdots \\ B^*, !\Delta^* \Rightarrow C^*\end{array}
}{!A^* \multimap B^*, !\Gamma^*, !\Delta^* \Rightarrow C^*}
$$

which is the decoration of an ILU-derivation because π_1^* and π_2^* are.

If the last rule applied was L$\multimap$, we know (because π_2^* is the decoration of an ILU-derivation) that $(\theta_1^n)'$ has the following form:

$$
\dfrac{
  \begin{array}{c}\vdots \\ !\Gamma^*, !(!A^* \multimap B^*) \Rightarrow !B^*\end{array}
  \quad
  \dfrac{
    \dfrac{\begin{array}{c}\pi_{21}^* \\ \vdots \\ !\Delta_1^*, (!B^*)^{n_1} \Rightarrow E^*\end{array}}{!\Delta_1^*, (!B^*)^{n_1} \Rightarrow !E^*}
    \quad
    \begin{array}{c}\pi_{22}^* \\ \vdots \\ F^*, !\Delta_2^*, (!B^*)^{n_2} \Rightarrow C^*\end{array}
  }{!E^* \multimap F^*, !\Delta^*, (!B^*)^n \Rightarrow C^*}
}{!E^* \multimap F^*, !\Gamma^*, !\Delta^*, !(!A^* \multimap B^*) \Rightarrow C^*}
$$

If neither n_1 nor n_2 equals 0 we transform the derivation into

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\vdots}{!\Gamma^*, !(!A^* \multimap B^*) \Rightarrow !B^*} \quad \dfrac{\pi^*_{21} \atop \vdots}{!\Delta_1^*, (!B^*)^{n_1} \Rightarrow E^*}}{!\Gamma^*, !\Delta_1^*, !(!A^* \multimap B^*) \Rightarrow E^*}}{!\Gamma^*, !\Delta_1^*, !(!A^* \multimap B^*) \Rightarrow !E^*}
\quad
\dfrac{\dfrac{\vdots}{!\Gamma^*, !(!A^* \multimap B^*) \Rightarrow !B^*} \quad \dfrac{\pi^*_{22} \atop \vdots}{F^*, !\Delta_2^*, (!B^*)^{n_2} \Rightarrow C^*}}{F^*, !\Gamma^*, !\Delta_2^*, !(!A^* \multimap B^*) \Rightarrow C^*}
}{
\dfrac{!E^* \multimap F^*, !\Gamma^*, !\Gamma^*, !\Delta^*, !(!A^* \multimap B^*), !(!A^* \multimap B^*) \Rightarrow C^*}{\dfrac{\vdots}{!E^* \multimap F^*, !\Gamma^*, !\Delta^*, !(!A^* \multimap B^*) \Rightarrow C^*}}
}
$$

which gives us the desired result by induction hypothesis. If either n_1 or n_2 equals 0 we are done even more directly.

If the last rule applied has been a weakening on a formula in Δ^* or on an occurrence of B^* while $n > 1$, we are done by a permutation upwards and our induction hypothesis. Otherwise the situation is

$$
\dfrac{\dfrac{\vdots}{!\Gamma^*, !(!A^* \multimap B^*) \Rightarrow !B^*} \quad \dfrac{\dfrac{(\pi_2^*)' \atop \vdots}{\Pi^*, !\Delta^* \Rightarrow C^*}}{\Pi^*, !\Delta^*, !B^* \Rightarrow C^{*}\text{''}}}{\Pi^*, !\Gamma^*, !\Delta^*, !(!A^* \multimap B^*) \Rightarrow C^*}
$$

and we eliminate the cut by transforming to

$$
\dfrac{\dfrac{(\pi_2^*)' \atop \vdots}{\Pi^*, !\Delta^* \Rightarrow C^*}}{\Pi^*, !\Gamma^*, !\Delta^*, !(!A^* \multimap B^*) \Rightarrow C^*}
$$

which gives the desired result.

Finally we have to deal with the case that the last applied rule in π_2^* is a cut-rule. As π_2 is an ILU-derivation we have two possible situations, depending on whether the underlying cut is *head* or *mid.* Both cases are similar and are handled more or less as for L$\multimap$. We will show the case of a mid-cut. Then $(\theta_1^n)'$ has the form

$$\dfrac{\begin{matrix}\vdots\\ !\Gamma^*, !(!A^* \multimap B^*) \Rightarrow !B^*\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\pi^*_{21}\\ \vdots\\ !\Delta_1^*, (!B^*)^{n_1} \Rightarrow D^*\end{matrix}}{!\Delta_1^*, (!B^*)^{n_1} \Rightarrow !D^*} \quad \begin{matrix}\pi^*_{22}\\ \vdots\\ \Pi^*, !D^*, !\Delta_2^*, (!B^*)^{n_2} \Rightarrow C^*\end{matrix}}{\Pi^*, !\Delta^*, (!B^*)^{n} \Rightarrow C^*}}{\Pi^*, !\Gamma^*, !\Delta^*, !(!A^* \multimap B^*) \Rightarrow C^*}$$

which for n_1 and n_2 both not equal to 0 is transformed into

$$\dfrac{\begin{matrix}\dfrac{\dfrac{\begin{matrix}\vdots\\ !\Gamma^*, !(!A^* \multimap B^*) \Rightarrow !B^*\end{matrix} \quad \begin{matrix}\pi^*_{21}\\ \vdots\\ !\Delta_1^*, (!B^*)^{n_1} \Rightarrow D^*\end{matrix}}{!\Gamma^*, !\Delta_1^*, !(!A^* \multimap B^*) \Rightarrow D^*}}{!\Gamma^*, !\Delta_1^*, !(!A^* \multimap B^*) \Rightarrow !D^*} \quad \dfrac{\begin{matrix}\vdots\\ !\Gamma^*, !(!A^* \multimap B^*) \Rightarrow !B^*\end{matrix} \quad \begin{matrix}\pi^*_{22}\\ \vdots\\ \Pi^*, !D^*, !\Delta_2^*, (!B^*)^{n_2} \Rightarrow\end{matrix}}{\Pi^*, !D^*, !\Gamma^*, !\Delta_2^*, !(!A^* \multimap B^*) \Rightarrow C^*}\end{matrix}}{\dfrac{\Pi^*, !\Gamma^*, !\Gamma^*, !\Delta^*, !(!A^* \multimap B^*), !(!A^* \multimap B^*) \Rightarrow C^*}{\dfrac{\vdots}{\Pi^*, !\Gamma^*, !\Delta^*, !(!A^* \multimap B^*) \Rightarrow C^*}}}$$

Again, the case for n_1 or n_2 equal to 0 is even simpler to handle.

It now is routine to finish the proof. ⊠

4 ... and CL to LKT, LKQ

It will not come as too big a surprise that we can apply the procedure described in the previous section also to *classical* logic, by replacing Girard's translation either by T of Q, and using the corresponding correction cuts.

The pattern underlying this use of a non-decorating embedding into linear logic as a transformation of derivations in the original calculus is depicted in figure 6.3.

If the sequent calculus we start from is **IL**, the mapping χ sends a derivation π to the linear derivation $\chi(\pi)$ obtained by inductively applying $(\cdot)^*$ and adding correction cuts in order to stay within the collection of $(\cdot)^*$-translated formulas. When starting from **CL**, the mapping χ sends π to the linear proof $\chi(\pi)$ obtained by a similar use of

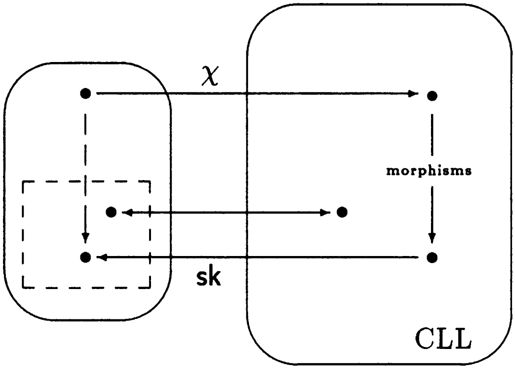


Figure 6.3: Proof transformation via constrictive morphisms

the embedding $(\cdot)^{\mathrm{T}}$ or $(\cdot)^{\mathrm{Q}}$.[4] One then eliminates the correction cuts. In the intuitionistic case the skeleton of the reduct is an **ILU**-derivation, in the classical case we find an **LKQ**-, resp. an **LKT**-proof.

4.1. PROPOSITION. *If, in the transformation above, the derivation we start from is (interpretable as) an* **ILU**/**LKT**(**Q**)*-derivation, then* $\mathsf{sk} \circ \mathtt{morphisms} \circ \chi = \mathsf{id}$.

PROOF: Indeed, if this is the case, then the exponentials that we *remove* by means of correction cuts, in fact are superfluous, whence the claim follows by a straightforward induction on the number of these cuts, using proposition 1.6: the correction cuts are invisible for these derivations. ⊠

The first half of the equivalent of theorem 3.1 for **LKT** of course is immediate from theorem 3.4.6, and the second half follows by an argument identical to the one given for **ILU**, i.e. one eliminates the cuts as if in **LKT**. Note that, lacking a characterization similar to that of

[4] In case of the extension of **LKQ** with multiplicative conjunction rules, or that of **LKT** with multiplicative disjunction rules, χ is not uniquely determined. Cf. section 2, and section 3.4.

theorem 3.4.6 for **LKQ**, the proof of the **LKQ**-analogue of theorem 3.1 will boil down to directly establishing the second claim, by eliminating the cuts as if in **LKQ**.

We feel that the main point has already been amply illustrated in the previous section, and therefore will skip the proofs for the classical case. (Annexe B of Joinet(1993) contains a detailed proof of the cut free derivability of 'mid'-logical rules in **LKQ**, which is (essentially) equivalent to the analogue of theorem 3.1 for **LKQ** (cf. the derivable **ILU**-rules of page 138, and the leftmost vertical arrow in figure 6.3.))

7

MULTICOLOUR linear logic

It is well known (and easy to see) that the logical connectives are determined by their introduction rules in the sequent calculus, in the following sense: were we to introduce a connective $*$ with the same rules as e.g. the tensor $\otimes$, then in the thus extended calculus we can derive $A * B \Longleftrightarrow A \otimes B$, for all formulas A, B. The status of the exponentials, on the other hand, is quite different: if we introduce a unary connective ¡, with the *same* introduction rules as for example the shriek '!', then neither $!A \Rightarrow ¡A$, nor $¡A \Rightarrow !A$ will be derivable. This of course is due to the contextual constraint imposed in the R!- and L?-rule.

The difference can be expressed in another way. Take some linear derivation π, and let $\mathcal{T}(\pi)$ be the collection of all identity classes of formulas whose main connective is a tensor. We now use a 'fresh' tensor for each element of $\mathcal{T}(\pi)$, or, otherwise said, give distinct *colours* to the main connective of different elements of $\mathcal{T}(\pi)$. The coloured version of π will be a correct derivation in the extension of linear logic with a pair of tensor introduction rules for (at least) all the colours that have been used. However, if we take a different colour for the exponentials in distinct elements of the collection of exponential classes $\mathcal{E}(\pi)$, the coloured version of π in general will *not* be a correct derivation in the corresponding extension of **CLL**. In fact, the coloured derivation will be correct only *modulo* some relation between the coloured exponentials, telling us, for the contextual rules, in the company of which other colours a given exponential can be introduced in a promotion rule. Indeed, the colouring of π will correspond precisely to the use of a different colour for each vertex in the exponential graph of π, and the relation we are looking will be none other than the binding relation.[1]

These observations suggest that we put things the other way round, and define extensions of linear logic with a set of distinct exponentials, the interdependencies of which are given beforehand, by some graph.

1 R-CLL

Extend the language of non-exponential linear logic with a set of indexed exponentials $\underset{a}{!}, \underset{a}{?}$, and let R be some binary relation on the set of indices. Then define the sequent calculus R-**CLL** by adding to the non-exponential fragment of **CLL**:

1. for all indexed exponentials $\underset{a}{!}, \underset{a}{?}$ the usual dereliction rules, i.e.

$$\text{L!}\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \underset{a}{!}A \Rightarrow \Delta} \qquad \text{R?}\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \underset{a}{?}A, \Delta}\ ;$$

2. structural permissions restricted to specified sets of indices $\mathcal{W}_{\mathsf{R}}$ and $\mathcal{C}_{\mathsf{R}}$: if $a \in \mathcal{W}_{\mathsf{R}}(\mathcal{C}_{\mathsf{R}})$, then weakening (contraction) on the left is allowed for a formula $\underset{a}{!}A$ and weakening (contraction) on the right is allowed for a formula $\underset{a}{?}A$;
3. a rule for the introduction of indexed shrieks on the right, namely

$$\text{R!}\ \frac{\underset{x_1}{!}G_1, \ldots, \underset{x_n}{!}G_n \Rightarrow A, \underset{y_1}{?}D_1, \ldots, \underset{y_m}{?}D_m}{\underset{x_1}{!}G_1, \ldots, \underset{x_n}{!}G_n \Rightarrow \underset{z}{!}A, \underset{y_1}{?}D_1, \ldots, \underset{y_m}{?}D_m}\ ,$$

provided that $z\mathsf{R}x_i, z\mathsf{R}y_j$ for all x_i, y_j, and, under the same condition, a rule for the introduction of indexed whynots on the left:

$$\text{L?}\ \frac{\underset{x_1}{!}G_1, \ldots, \underset{x_n}{!}G_n, A \Rightarrow \underset{y_1}{?}D_1, \ldots, \underset{y_m}{?}D_m}{\underset{x_1}{!}G_1, \ldots, \underset{x_n}{!}G_n, \underset{z}{?}A \Rightarrow \underset{y_1}{?}D_1, \ldots, \underset{y_m}{?}D_m}\ .$$

[1] In fact we already used 'colouring' of exponentials (by means of indices) as a notational device in the previous two chapters, in examples related to the exponential graph of derivations.

Note that we do not ask anything special of the binary relation R. However, of course some basic properties of the calculus will depend upon properties of the relation. It is, for example, easy to verify the following.

1.1. PROPOSITION. R *is reflexive if and only if for each index a all identity axioms* $\underset{a}{!}A \Rightarrow \underset{a}{!}A$ *and* $\underset{a}{?}A \Rightarrow \underset{a}{?}A$ *have non-trivial expansions in* R-CLL. ⊠

It is an obvious, but nevertheless important, observation that if we forget about all the indices, what we find is nothing more nor less than a derivation in CLL. And forgetting about the indices is essentially what we do when eliminating cuts in R-CLL-derivations. But of course then we have to assure ourselves that all the usual elementary reduction steps remain correct when performed in R-CLL.

1.2. PROPOSITION. R-CLL *allows cut elimination if and only if*

(1) R *is transitive, and*

(2) $\mathcal{W}_{\mathsf{R}}$ *and* $\mathcal{C}_{\mathsf{R}}$ *are upwardly closed.*[2]

PROOF: From left to right observe that (1) implies correctness of [*cc*], (2) correctness of [*w*] and [*co*].

For the other implication, suppose aRb and bRc. Then the following is a derivation in R-CLL:

$$\dfrac{\dfrac{\dfrac{p \Rightarrow p}{\underset{c}{!}p \Rightarrow p}}{\underset{c}{!}p \Rightarrow \underset{b}{!}p} \qquad \dfrac{\dfrac{p \Rightarrow p}{\underset{b}{!}p \Rightarrow p}}{\underset{b}{!}p \Rightarrow \underset{a}{!}p}}{\underset{c}{!}p \Rightarrow \underset{a}{!}p}$$

But obviously there is no cut free proof of $\underset{c}{!}p \Rightarrow \underset{a}{!}p$ if $(a, c) \notin R$.

Similarly we obtain contradicitions if $i \in \mathcal{W}_{\mathsf{R}}, iRj$ and $j \notin \mathcal{W}_{\mathsf{R}}$ or $i \in \mathcal{C}_{\mathsf{R}}, iRj$ and $j \notin \mathcal{C}_{\mathsf{R}}$, e.g. using the following derivations:

[2]I.e., if $i \in \mathcal{W}_{\mathsf{R}}(\mathcal{C}_{\mathsf{R}})$ and $i\mathsf{R}j$, then $j \in \mathcal{W}_{\mathsf{R}}(\mathcal{C}_{\mathsf{R}})$

$$
\dfrac{\dfrac{\dfrac{p \Rightarrow p}{!_j p \Rightarrow p}}{!_j p \Rightarrow !_i p} \qquad \dfrac{q \Rightarrow q}{!_i p, q \Rightarrow q}}{!_j p, q \Rightarrow q}
\qquad\qquad
\dfrac{!_j p \Rightarrow !_i p \qquad \dfrac{\dfrac{!_i p \Rightarrow p \quad !_i p \Rightarrow p}{!_i p, !_i p \Rightarrow p \otimes p}}{!_i p \Rightarrow p \otimes p}}{!_j p \Rightarrow p \otimes p}
$$

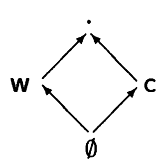
⊠

We get the standard calculus by taking for R the reflexive relation **1** on a one element index set. So **CLL** = **1-CLL**. This reflects the fact that in **CLL** all exponentiated formulas obtain *full* structural permissions.

The characterization in chapter 5 of superfluous exponentials by means of the exponential graph of a linear derivation π boils down to the identification of those modalized formulas in the proof for which there is purely *logical* evidence that at no point (during normalization) they will use the 'talents bestowed upon them'. In view of the above, we can reformulate their *removal* as the interpretation of the proof as a **2-CLL**-proof, where **2** denotes the reflexive closure of the relation $\emptyset \to \cdot$, distinguishing the usual expontials $!, ?$ from the 'no-permission' exponentials $!_\emptyset, ?_\emptyset$, which correspond to the superfluous exponentials in π. Stripping a derivation then means defining an optimal two-colouring, i.e. replacing as many exponentials as possible by 'no permission' ones.

Observe that an $\emptyset$-exponentiated formula during reduction will never be cutformula in a $[w]$ or $[co]$ reduction-step; moreover a derivation π and its interpretation $\mathbf{2}(\pi)$ have *exactly* the same set of reductions, which proves theorem 5.4.6.

It is an obvious subsequent step to consider **4**, being the reflexive transitive closure of the relation



In **4-CLL** we distinguish *four* types of exponentials corresponding to the four possible kinds of structural permission that occur in a linear logic proof: no permission, weakening only, contraction only, and *both* weakening and contraction.

It is easy to define an optimal four-colouring of a given linear derivation π by propagating source-labels ('colours') in $\mathcal{G}(\pi)$: a vertex e will become w-coloured if and only if there is a directed path from a vertex labeled w to e, it becomes c-coloured if and only if there is a directed path from a vertex labeled c to e, and $(\mathsf{w}+\mathsf{c})$-coloured precisely in case there are directed paths both from a vertex labeled w and from a vertex labeled c to e. E.g. an optimal four-colouring of the example on page 112 is obtained by replacing $\underset{2}{!}$ by $\underset{\emptyset}{!}$, $\underset{1}{!}$ and $\underset{5}{!}$ by the usual 'full permission' shriek, and both $\underset{4}{!}$ and $\underset{5}{!}$ by $\underset{w}{!}$.

2 A prelude to dilatation

Recall that in linear logic we can simulate weakening using the multiplicative constants and the additive connectives, as weakening to the *left* is derivable for all formulas of the form $\mathbf{1}\&A$, weakening to the *right* for all formulas of the form $\bot \oplus A$.

This indicates the possible existence of a natural extension of the stripping of superfluous exponentials: once the no-permission exponentials have been removed, we can continue and try, given a **4-CLL** interpretation of π, to replace all occurrences of $\underset{w}{!}A$ by $\mathbf{1}\&A$, of $\underset{w}{?}A$ by $\bot \oplus A$. In order for this replacement to result in a correct linear derivation, of course we have to do a bit more than mere replacing. We define inductively a re-writing of π as $\pi^{\triangleleft}$, in which all occurrences of $\underset{w}{!}\cdot$ have been replaced by $\mathbf{1}\&\cdot$. It consists in replacing all occurrences of rules for $\underset{w}{!}, \underset{w}{?}$ by small chunks of derivations. Thus we rewrite e.g.

$$
\dfrac{\begin{matrix}\pi\\ \vdots\\ \Gamma \Rightarrow \Delta\end{matrix}}{\Gamma, \underset{w}{!}A \Rightarrow \Delta}
\quad \text{as} \quad
\dfrac{\dfrac{\begin{matrix}\pi^{\triangleleft}\\ \vdots\\ \Gamma^{\triangleleft} \Rightarrow \Delta^{\triangleleft}\end{matrix}}{\Gamma^{\triangleleft}, \mathbf{1} \Rightarrow \Delta^{\triangleleft}}}{\Gamma^{\triangleleft}, \mathbf{1}\&A^{\triangleleft} \Rightarrow \Delta^{\triangleleft}}
$$

For the promotion rule, observe that *all* side-active formulas will be prefixed either by $\underset{w}{!}, \underset{w}{?}$, or $!, ?$. We then replace e.g.

$$\dfrac{\begin{matrix}\pi \\ \vdots \\ !\Gamma_1, \underset{w}{!}\Gamma_2, A \Rightarrow ?\Delta_1, \underset{w}{?}\Delta_2\end{matrix}}{!\Gamma_1, \underset{w}{!}\Gamma_2, \underset{w}{?}A \Rightarrow ?\Delta_1, \underset{w}{?}\Delta_2}$$

by

$$\dfrac{\begin{matrix}\underline{\bot \Rightarrow} \\ \vdots \\ !\Gamma_1^{\triangleleft}, 1\&\Gamma_2^{\triangleleft}, \bot \Rightarrow ?\Delta_1^{\triangleleft}, \bot \oplus \Delta_2^{\triangleleft}\end{matrix} \qquad \begin{matrix}\pi^{\triangleleft} \\ \vdots \\ !\Gamma_1^{\triangleleft}, 1\&\Gamma_2^{\triangleleft}, A^{\triangleleft} \Rightarrow ?\Delta_1^{\triangleleft}, \bot \oplus \Delta_2^{\triangleleft}\end{matrix}}{!\Gamma_1^{\triangleleft}, 1\&\Gamma_2^{\triangleleft}, \bot \oplus A^{\triangleleft} \Rightarrow ?\Delta_1^{\triangleleft}, \bot \oplus \Delta_2^{\triangleleft}}$$

Hence we proved the following.

2.1. PROPOSITION. *Given a linear derivation π, we can transform π into a linear derivation $\pi^{\diamond}$ $(= (\pi^{\triangleright})^{\triangleleft})$, in which all superfluous exponentials have been removed, all* w*-exponentials are replaced by* $1\&\cdot$. *Exponentials that remain in $\pi^{\diamond}$ are caused by contractions.* ⊠

At first sight the transformation $\pi \mapsto \pi^{\triangleleft}$ seems to be of a far more drastic nature than that of stripping. This impression, however, upon closer analysis turns out to be deceptive. We can once more turn things the other way round and *define* the modality $\underset{w}{!}$ as $1\&\cdot$. Similarly we can *define* the modality $\underset{\emptyset}{!}\cdot$ by putting $\underset{\emptyset}{!}A := A$. The rules for $\underset{\emptyset}{!}A$ in **2**- or **4-CLL** then are nothing but *repetition* rules, and the rules for $\underset{w}{!}A$ in **4-CLL** (by the above observations) are *derivable* in **CLL**. What is more, it is not difficult to verify that the exponential reduction steps for $\underset{w}{!}$ in a (**4-CLL**-)proof π can be simulated by reductions of $\pi^{\diamond}$ (cf. theorem 8.1.4).

Thus stripping and w-replacement in fact are transformations of a quite similar nature. They are the basic instances of what we will call *dilatation*, the replacement in a linear derivation π of exponentiated formulas by modality-free *approximations*, while preserving the essential (dynamic) properties of π.

The results of chapter 5 and the observations above show that dilatation is *always* possible when one considers superfluous or w-exponentials. One of course expects such a replacement to be, in general, impossible as soon as the exponentials under consideration also have contraction permission.

Notes

- This chapter is a modified and extended version of section 5 of Danos et al.(1993d). The possibility to interpret a linear proof via its exponential graph as a derivation in linear logic extended with a collection of exponentials ordered by the binding relation, was pointed out to us by Jean-Yves Girard, who also observed the potential interest of considering linear derivations whose exponential graph is acyclic, and the link with problems related to bounded systems of linear logic. These observations form the basis for the next and final chapter.

8

Dilatations

In this last chapter we define *dilatations*, which, like the *strippings* of chapter 5 and the transformation $\pi \mapsto \pi^{\triangleleft}$ of section 7.2, map $\mathbf{CLL}_{(2)}$-derivations to sequent derivations in the exponential-free fragment of linear logic, conserving both the (linear) skeleton *and* the dynamics.

The guiding intuition is that a shrieked formula can be thought of as a (potentially) infinite *tensor*, a questioned one as a (potentially) infinite *par*. This idea finds its earliest expression in the *approximation* theorem (Girard(1987a)), showing that, at least in (fully expanded) *cut free* first order linear derivations, we can always replace the exponentials by n-ary tensors and n-ary pars. It is also this approximation that is the driving force behind the system **BLL** of bounded linear logic studied in Girard et al.(1992).

Indeed, *dilatations* map linear sequent derivations to derivations in a system of bounded linear logic, and we will see that the *existence* of such a map can be related to the presence or absence of cycles in the exponential graph.

1 Dilated linear logic

Let $\overset{n}{\otimes}(\mathbf{1}\&A)$ be the n-fold tensor $\underbrace{(\mathbf{1}\&A)\otimes(\mathbf{1}\&A)\otimes\ldots\otimes(\mathbf{1}\&A)}_{n}$, and

$\overset{n}{⅋}(\bot\oplus A)$ the n-fold par $\underbrace{(\bot\oplus A)⅋(\bot\oplus A)⅋\ldots⅋(\bot\oplus A)}_{n}$.

1.1. DEFINITION. For all $n \in \mathbb{N}$ we define *n-dilated* or *pseudo exponentials*, written as $[n]$ and $\langle n\rangle$, by

$$[n]A = \begin{cases} A, & \text{if } n = 0 \\ \overset{n}{\otimes}(\mathbf{1}\&A), & \text{otherwise} \end{cases}$$

$$\langle n\rangle A = \begin{cases} A, & \text{if } n = 0 \\ \overset{n}{⅋}(\bot \oplus A), & \text{otherwise} \end{cases}$$

⊠

Pseudo exponentials can be seen as approximating exponentials, in the sense that $!A \Rightarrow [n]A$ and $\langle n\rangle A \Rightarrow ?A$ are derivable, for all n. Given a linear derivation π, we say that an exponential class in π is *dilatable*, if we can replace each occurrence of one of its elements $!A$ ($?A$) in π by a dilated exponential $[n]A$ ($\langle n\rangle A$)[1], and keep a correct linear derivation, in which certain instances of exponential rules are replaced by instances of (derivable) rules for dilated exponentials. It is easy to determine these rules, say, in order to simplify matters somewhat, within the one-sided sequent calculus for linear logic. They are given in table 8.1

The rules are such that, when we replace each $\langle n\rangle$ by '?', and each $[m]$ by '!' the result is correct as a **CLL**-derivation. Moreover they verify the following.

1.2. PROPOSITION. (Dilatation preserves correctness) *The rules of table 8.1 for dilated exponentials are derivable in linear logic.* ⊠

We will refer to the calculus that combines these rules with the usual exponential rules of linear logic as **DCLL**. The calculus that uses only *pure* pseudo exponential rules (i.e. without the occurrences of '?' in the contextual rules) will be called **DLL**, for *dilated linear logic.* Observe that **DLL** is but a small variation on the 'primitive' system of *bounded* linear logic which initiated the work described in Girard et al.(1992).

We call the **CLL**-derivation underlying a **DLL**- or **DCLL**-derivation π as its *linear skeleton* $\mathsf{lsk}(\pi)$. This leads naturally to the notion of being *dilatable.*

---

[1]Note that n is not supposed to be necessarily equal for all elements in the class!

Exponential dereliction rule:

$$(\mathrm{D'})\ \frac{\Gamma, A}{\Gamma, \langle n\rangle A}$$

Exponential structural rules:

$$(\mathrm{W'})\ \frac{\Gamma}{\Gamma, \langle n\rangle A}\ n \neq 0 \qquad (\mathrm{C'})\ \frac{\Gamma, \langle m\rangle A, \langle n\rangle A}{\Gamma, \langle m+n\rangle A}\ n, m \neq 0$$

Exponential contextual rules:

$$(\ddagger)\quad \frac{?\Gamma, \langle m_1\rangle G_1, \langle m_2\rangle G_2, \ldots, \langle m_j\rangle G_j, A}{?\Gamma, \langle m_1\rangle G_1, \langle m_2\rangle G_2, \ldots, \langle m_j\rangle G_j, [0]A}$$

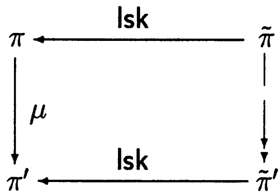
$$(\dagger)\quad \frac{?\Gamma, \langle m_1\rangle G_1, \langle m_2\rangle G_2, \ldots, \langle m_j\rangle G_j, A}{?\Gamma, \langle x \cdot m_1\rangle G_1, \langle x \cdot m_2\rangle G_2, \ldots, \langle x \cdot m_j\rangle G_j, [x]A}\ x, m_i \neq 0$$

Table 8.1: Rules for dilated exponentials

1.3. DEFINITION. A **CLL**-derivation π is *dilatable* if there exists a **DLL**-derivation π' such that $\mathsf{lsk}(\pi') = \pi$. Given an exponential class e in π, we call the pseudo exponentials $[n]$, $\langle m\rangle$ that have been assigned to elements $!A$, $?A$ of e their *dilation coefficients*, which are said to be *positive* in case of a $[\cdot]$, and *negative* in case of a $\langle\cdot\rangle$. The dilation coefficients corresponding to introduction (by means of dereliction, weakening, in a unary additive rule or in an axiom) of elements $?A$ in $\mathsf{lsk}(\pi)$ are called *initial.* ⊠

Note that the superfluous exponentials of chapter 5 here appear as exponential classes that can be uniformly '0'-dilated, the w-exponentials of the previous chapter correspond to those classes that permit a uniform '1'-dilatation. As we already observed in these special cases, an important point is the following.

1.4. THEOREM. (Dilatation preserves normalization) *A* **DLL**-*proof* $\tilde{\pi}$ *simulates the reductions of its linear skeleton* π*, i.e. for each elementary reduction step* μ*, leading from* π *to* π' *there exists a reduction of* $\tilde{\pi}$ *such that the following diagram commutes:*



PROOF: Consider e.g. a cut between $[n+m]A$ which has just been introduced by means of a pseudo promotion, and $\langle n+m\rangle A^{\perp}$ which comes from pseudo contracting $\langle n\rangle A^{\perp}$ and $\langle m\rangle A^{\perp}$:

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \langle k_i\rangle G_i, A\end{matrix}}{\langle (n+m)k_i\rangle G_i, [n+m]A} \qquad \dfrac{\begin{matrix}\pi_2\\ \vdots\\ \langle n\rangle A^{\perp}, \langle m\rangle A^{\perp}, \Gamma\end{matrix}}{\langle n+m\rangle A^{\perp}, \Gamma}}{\langle (n+m)k_i\rangle G_i, \Gamma}$$

We simulate the $[co]$-reduction step of **CLL** by transforming the proof into

$$\dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \langle k_i\rangle G_i, A\end{matrix}}{\langle mk_i\rangle G_i, [m]A} \qquad \dfrac{\dfrac{\begin{matrix}\pi_1\\ \vdots\\ \langle k_i\rangle G_i, A\end{matrix}}{\langle nk_i\rangle G_i, [n]A} \quad \begin{matrix}\pi_2\\ \vdots\\ \langle n\rangle A^{\perp}, \langle m\rangle A^{\perp}, \Gamma\end{matrix}}{\langle nk_i\rangle G_i, \langle m\rangle A^{\perp}, \Gamma}}{\langle (n+m)k_i\rangle G_i, \Gamma}$$

All other simulations are similar. ⊠

It will be clear that one cannot expect each and every **CLL**-derivation to be dilatable. This would make usual linear logic equivalent to *bounded* linear logic, which evidently is not the case. Indeed it is easy to give an example of a derivation that is *not* dilatable. Consider the following.

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A^{\perp}, A}{?_1 A^{\perp}, A}}{?_1 A^{\perp}, !_2 A}}{?_1 A^{\perp}, ?_1 A^{\perp}, !_2 A}}{?_1 A^{\perp}, !_2 A}}{?_1 A^{\perp} \parr !_2 A}}{!_3(?_1 A^{\perp} \parr !_2 A)}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A^{\perp}, A}{?_4 A^{\perp}, A}}{?_4 A^{\perp}, !_1 A} \quad \dfrac{\dfrac{A^{\perp}, A}{?_2 A^{\perp}, A}}{?_2 A^{\perp}, !_1 A}}{?_4 A^{\perp}, !_1 A \otimes ?_2 A^{\perp}, !_1 A} \quad \dfrac{A^{\perp}, A}{?_2 A^{\perp}, A}}{?_4 A^{\perp}, !_1 A \otimes ?_2 A^{\perp}, !_1 A \otimes ?_2 A^{\perp}, A}}{?_4 A^{\perp}, ?_3(!_1 A \otimes ?_2 A^{\perp}), !_1 A \otimes ?_2 A^{\perp}, A}}{?_4 A^{\perp}, ?_3(!_1 A \otimes ?_2 A^{\perp}), ?_3(!_1 A \otimes ?_2 A^{\perp}), A}}{?_4 A^{\perp}, ?_3(!_1 A \otimes ?_2 A^{\perp}), A}
}{?_4 A^{\perp}, A}
$$

Let y_1, y_2 be the initial dilation coefficients for $?_1$, respectively corresponding to the introduction by dereliction and by weakening. Let x be the dilation coefficient of $!_2$. Then the coefficients corresponding to $?_1, !_1$ in the cutformulas will be $xy_1 + y_2$, those corresponding to $?_2, !_2$ will be x. Suppose the left most initial dilation coefficient for $?_2$ to be z. Due to the introduction by promotion of $!_1 A$ with $?_2 A$ side-active, we will have to satisfy $x = z(xy_1 + y_2)$. The reader will readily verify that the possible (trivial) solutions to this equation are excluded as dilation coefficients.

In a **DLL**-derivation the negative dilation coefficients depend polynomially on the positive ones, and deciding whether a given linear proof π is dilatable boils down to solving a set of equations between polynomials. Solvability of the equations in turn seems to be closely related to the presence or absence of *cycles* from the exponential graph of π. (Observe that in the example above we have a cycle $1 \to 2 \to 1$.) However, this appears to be too coarse a criterion: the absence of cycles from its exponential graph is neither necessary, nor sufficient for a derivation to be dilatable.

Indeed, the derivation in the example above remains non-dilatable if we 'cut off' the three expansions of the identity axiom $!A, ?A^{\perp}$, which collapses the three exponential classes 1,2 and 4. The resulting exponential graph however is acyclic.[2]

And, in figure 8.1 one finds an example of a derivation π that is dilatable, though its exponential graph contains cycles.

If we consider the example more closely, an easy calculation shows that most of the initial dilation coefficients are fixed. Indeed, let us suppose the initial dilation coefficient for class 2 corresponding to the occurrence (the right most one) of $?_2A^\perp$ introduced by weakening to be x_1, and the coefficient corresponding to the occurrence introduced by dereliction x_2. Say the left most intitial dilation coefficient for class 3 is x_3. Then, if the dilation coefficient for $!_5$ is k, due to the highest introduction of this shriek by promotion, the coefficient for the terminal occurrence of $?_2$ becomes kx_1. Hence the terminal occurrence of $!_2$ will have kx_1 as dilation coefficient. Due to the right most introduction of $!_2$ by promotion, the coefficient of the terminal occurrence of $?_3$ becomes kx_1x_3, which in turn is the dilation coefficient of the terminal occurrence of $!_3$. So, when introducing $!_3$ by promotion in the derivation of the right premiss of the cut, we have to satisfy $kx_1 = kx_1x_3x_2$, which forces $x_2 = x_3 = 1$.

In fact, augmenting these (or most of the other) initial values, 'activates' the cycle and the derivation 'explodes'.

This motivates the following.

1.5. DEFINITION. An exponential class e in a **DCLL**-derivation π is said to be 1^+-dilatable if there exists a **DCLL**-derivation π' such that $\mathsf{lsk}(\pi) = \mathsf{lsk}(\pi')$ and all (or, equivalently, all *initial*) dilation coefficients for elements of e in π' are bigger than 1.

A **CLL**-derivation π then is called 1^+-dilatable if there exists a dilatation in which all coefficients are bigger than 1. ⊠

As figure 8.1 shows, there exist dilatable derivations that are not 1^+-dilatable.

[2] Though one might say that it contains a 'hidden cycle', as the logical flowgraph (in the sense of Buss(1991)) of the collapsed class is cyclic.

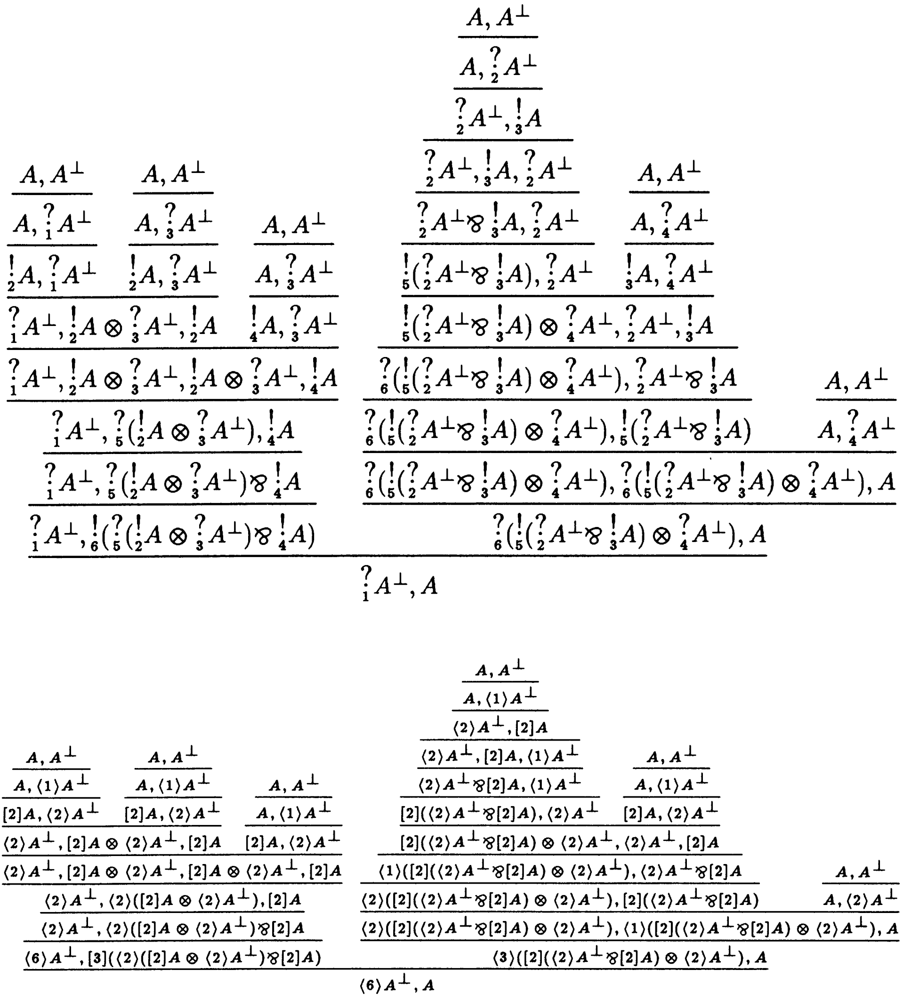
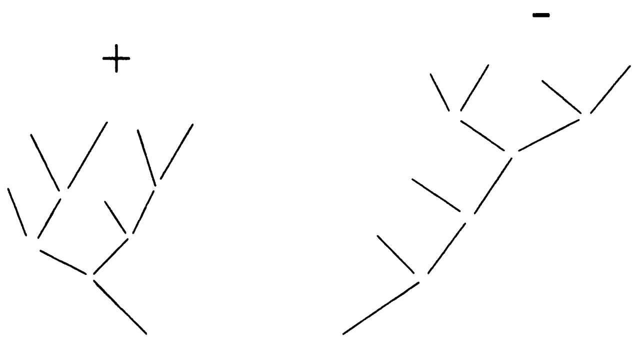


Figure 8.1: A dilatable, but not 1^+-dilatable, derivation



$$\frac{\Gamma, F[!A] \qquad F^{\perp}[?A^{\perp}], \Delta}{\Gamma, \Delta}\ \text{cut}$$

Figure 8.2: Track of an exponential class

2 A characterization of 1^+-dilatable derivations

If we limit ourselves to *fully expanded* CLL-derivations, i.e. derivations having only atomic instances of the identity axiom, the information provided by the exponential graph becomes far more decisive. In fact, as we will show, for this class of derivations acyclicity of the exponential graph is equivalent to 1^+-dilatability. The crucial step is provided by the following.

2.1. PROPOSITION. *Let π be a fully expanded* DCLL*-derivation, and e an exponential class that is* minimal *in $\mathcal{G}(\pi)$. Then e is (1^+-)dilatable.*

PROOF: Due to the minimality of e, no element $!A, ?A^{\perp}$ of the class is side-active in a 'real' promotion rule. Therefore we can construct a correct DCLL-derivation by only dilating positive ($!A$) and negative ($?A^{\perp}$) occurrences of elements of e. Moreover, as π is fully expanded, e in fact is a *tree* (namely, the *track* of $!A, ?A^{\perp}$, similar to the tracks defined in chapter 4). It will in general have the shape pictured in figure 8.2.

All leaves in the positive subtree correspond to instances of weakening (with a superformula of $!A$), a unary additive rule, or 'real' promotion; all leaves in the negative subtree correspond to instances of weakening, dereliction or the unary additive rule. We can calculate a value for n on the negative subtree, by successively calculating dilation coefficients for the binary nodes, which correspond to instances of contraction or the binary additive rule.

We start from one of the leaves corresponding to some highest binary node, with dilation coefficient $\langle x_1 \rangle$. Going downwards, in order to remain correct as a **DCLL**-derivation, we eventually have to change the coefficient when passing an instance of a (pseudo-)promotion rule for a 1^+-dilated pseudo-shriek. If we pass k instances of such rules, say with coefficients $[n_1], [n_2], \ldots, [n_k]$, we arrive at the binary node with coefficient $\langle n_1 \cdot n_2 \cdot \ldots \cdot n_k \cdot x_1 \rangle$. We repeat this procedure for the other corresponding leave, resulting in coefficient, say, $\langle m_1 \cdot m_2 \cdot \ldots \cdot m_{k'} \cdot x_2 \rangle$. If the binary rule then is an additive logical rule or a contraction on a superformula of $?A^\perp$, the coefficients have to be *equal* in order for the derivation so far to be correct. I.e. x_1 and x_2 should satisfy

$$n_1 \cdot n_2 \cdot \ldots \cdot n_k \cdot x_1 = m_1 \cdot m_2 \cdot \ldots \cdot m_{k'} \cdot x_2.$$

If the binary rule is a contraction on $?A^\perp$, we take (writing a_1 for the product of the n_i, a_2 for the product of the m_j) the sum $\langle a_1 \cdot x_1 + a_2 \cdot x_2 \rangle$ as coefficient, and continue towards the next binary node, in fact towards the first occurrence of a node corresponding to a contraction on a superformula or an additive rule. There we will have to satisfy an equation

$$\sum_{i=1}^{k_1} a_i \cdot x_i = \sum_{j=1}^{k_2} b_j \cdot y_j.$$

Continuing in this way, eventually the dilation coefficient for the terminal occurrence will be any strictly positive solution $(x_i, y_i, \ldots, z_i)$ of

$$\sum_{i=1}^{k_1} a_i \cdot x_i = \sum_{i=1}^{k_2} b_i \cdot y_i = \ldots = \sum_{i=1}^{k_n} c_i \cdot z_i,$$

where the $x_i, y_i, \ldots, z_i$ are (pairwise distinct) variables representing the dilation coefficients at the leaves, and $n - 1$ is the number of binary nodes in the negative subtree corresponding to contractions on superformulas and binary additive rules. Obviously there is a solution where all initial coefficients are strictly bigger than 1. ⊠

2.2. COROLLARY. *Suppose π is a fully expanded* DCLL*-derivation. If $\mathcal{G}(\pi)$ is acyclic, then π is $(1^+ -)$dilatable.*

PROOF: By induction on the number of vertices of $\mathcal{G}(\pi)$. Clearly, if π contains no real exponentials, then it is dilatable. Otherwise, as $\mathcal{G}(\pi)$ is acyclic, we can take a minimal exponential class e. By proposition 2.1 e is $(1^+$-)dilatable, and dilatation gives us a DCLL-proof π', whose exponential graph is precisely that of π, minus the vertex e. Therefore π' is dilatable by induction hypothesis. ⊠

2.3. LEMMA. *Suppose π is a fully expanded* CLL*-derivation. Then the binding relation on $\mathcal{E}(\pi)$ is irreflexive, i.e. $\mathcal{G}(\pi)$ contains no autocycles.*

PROOF: Suppose there is an auto-cycle. Then there is an instance of the promotion rule having the form

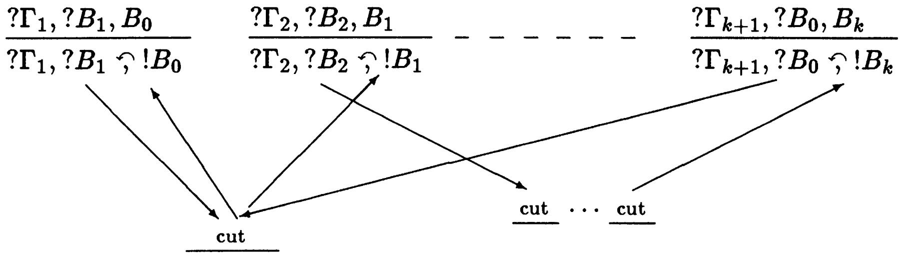
$$\frac{?\Gamma,\ ?A^{\perp}, A}{?\Gamma,\ ?A^{\perp},\ !A,}$$

where $!A$ and $?A^{\perp}$ are in the same identity class. But that is absurd in a fully expanded derivation, as identity classes are trees, and positive and negative elements of the class occur in distinct subtrees. ⊠

Now we are ready to prove:

2.4. THEOREM. *Let π be a fully expanded* CLL*-derivation. Then π is 1^+-dilatable if and only if $\mathcal{G}(\pi)$ is acyclic.*

PROOF: Suppose π is 1^+-dilatable. Because of lemma 2.3 a cycle in the exponential graph would necessarily be induced by a series of bindings of the form



Now consider a 1^+-dilatation of π. Writing n_i^+, n_i^- for the positive, respectively the negative dilation coefficient of class i in the shown instances of promotion, m_i^+, m_i^- for the values of these coefficients in the cut-formulas, we need to satisfy

$$n_0^+ < n_1^- \le m_1^- = n_1^+ < n_2^- \le \dots \le m_k^- = n_k^+ < n_0^- \le m_0^- = n_0^+.$$

But that is impossible.

The converse is immediate from corollary 2.2, as obviously the collection of fully expanded **DCLL**-derivation properly includes the collection of fully expanded **CLL**-derivations. ⊠

The approximation theorem of Girard(1987a) for fully expanded normal (first order) **CLL**-derivations of course is a corollary to theorem 2.4.

2.5. COROLLARY. (Approximation theorem) *If π is a fully expanded cut free* **CLL***-derivation, then π is dilatable.*

PROOF: If π is fully expanded and cut free, then (lemma 2.3) $\mathcal{G}(\pi)$ is acyclic, hence, by theorem 2.4, π is dilatable. ⊠

2.6. REMARK. Like the non-identity axiom and like the cut-rule, the second order rule of *extraction* ($\exists_2$) has the power to 'identify' positive and negative occurrences of a formula in a proof. It acts, one might say, as an n-ary version of the cut-rule, where n depends on the number of identifications being made. Consequently, in the above, we can treat it just like the cut-rule, and theorem 2.4 continues to hold for fully expanded second order derivations.

However, the approximation theorem *fails* in the second order case, as it is no longer true that the exponential graph of the full expansion of a normal derivation is acyclic. It will in general also fail to hold as soon as we consider derivations that are *not* fully expanded. We leave it to the reader to construct appropriate examples.

Hence in all case in which elements of an exponential class appear *both* negatively and positively in a proof π (due to axioms, cuts or extractions), dilatability of π can not be guaranteed.

3 Acyclic linear logic

Linear derivations with an exponential graph that is acyclic, correspond to derivations in a 'multicolour' system R-**CLL** where R is transitive and non-reflexive, i.e. an *order*. If $\mathcal{U}$ is a *universal* order[3], then any derivation with an acyclic exponential graph can be interpreted as a derivation in $\mathcal{U}$-**CLL**. By the results of the previous section, we know that a fully expanded linear derivation π will be (interpretable as) a $\mathcal{U}$-**CLL**-derivation if and only if it is 1^+-dilatable.

A small variation on $\mathcal{U}$-**CLL** is what we call **ALL** (for *acyclic linear logic*), obtained by adding to the non-exponential part of (the one-sided version of) the sequent calculus for linear logic the rules of table 8.2.

Each **ALL** derivation π determines, by the relations forced between the indices occurring in it, an order R_π. We say that R_π *realizes* $\mathsf{skl}(\pi)$ *in* **ALL**.

Of course, again, when forgetting about the indices in an **ALL**-derivation π, what we find is a **CLL**-derivation, to which, as in **DLL**, we will refer as π's linear skeleton $\mathsf{lsk}(\pi)$.

In fact **ALL** is obtained as an abstraction from **DLL**: we replace the concrete order induced by the dilation coefficients by an abstract one. It is, however, not equivalent to **DLL**. There exist (normal) **ALL**-derivations π such that $\mathsf{lsk}(\pi)$ is not dilatable. And, if we restrict ourselves once again to fully expanded derivations, we have, due to the strictness of the order, the following.

[3] I.e. $\mathcal{U}$ is a countable order into which any finite order X can be embedded, and having the property that for each such embedding ϕ of X, and for each finite extension Y of X, there exists an embedding ψ of Y whose restriction to X equals ϕ.

$$\text{weakening}\quad \frac{\Gamma}{\Gamma, \underset{i}{\overset{?}{\vdots}} A} \qquad \text{dereliction}\quad \frac{\Gamma, A}{\Gamma, \underset{i}{\overset{?}{\vdots}} A}$$

$$\text{contraction}\quad \frac{\Gamma, \underset{i_1}{\overset{?}{\vdots}} A, \underset{i_2}{\overset{?}{\vdots}} A}{\Gamma, \underset{i}{\overset{?}{\vdots}} A}\ i_1, i_2 < i$$

$$\text{promotion}\quad \frac{\underset{i_1}{\overset{?}{\vdots}} G_1, \ldots, \underset{i_n}{\overset{?}{\vdots}} G_n, A}{\underset{i'_1}{\overset{?}{\vdots}} G_1, \ldots, \underset{i'_n}{\overset{?}{\vdots}} G_n, \underset{j}{\overset{!}{\vdots}} A}\ j, i_k < i'_k$$

Table 8.2: Acyclic linear logic: the exponential rules.

3.1. PROPOSITION. *Let π be fully expanded. Then π is interpretable as an* **ALL**-*derivation if and only if it is* 1^+-*dilatable.*

PROOF: Suppose $\tilde{\pi}$ is a 1^+-dilatation of π. Take the dilation coefficients occurring in $\tilde{\pi}$ as indices. Then the set of indices together with the usual 'smaller than' order on the integers realizes π as an **ALL**-derivation.

Conversely, as in the proof of theorem 2.4, if π is realizable as an **ALL**-derivation, then $\mathcal{G}(\pi)$ contains no 'real' cycles. As π is fully expanded, moreover it contains no auto-cycles (lemma 2.3). Hence $\mathcal{G}(\pi)$ is acyclic, and π is 1^+-dilatable by theorem 2.4. ⊠

The interest of a system like **ALL** is that it might be the key to a system of bounded linear logic without explicit reference to resource polynomials, thus removing an obvious weakness from the result of Girard et al.(1992). As the above indicates, **ALL** as it stands nevertheless is too coarse for this purpose.

References

BETH, E. W. (1955). Semantic entailment and formal derivability. *Mededelingen der Koninklijke Nederlandse Akademie van Wetenschappen, afdeling Letterkunde, Nieuwe Reeks*, 18(13):309–342.

BETH, E. W. (1969). *Moderne Logica.* Van Gorcum & Comp. N.V., Assen.

BUSS, S. R. (1991). The undecidability of k-provability. *Annals of Pure and Applied Logic*, 53:75–102.

CHELLAS, B. F. (1990). *Modal Logic. An introduction.* Cambridge University Press.

CZERMAK, J. (1974). Embeddings of classical logic in S4. *Studia Logica*, 34(1):87–100.

DANOS, V. (1990). *La Logique Linéaire appliquée à l'étude de divers processus de normalisation (principalement du λ-calcul).* PhD thesis, Université Paris VII.

DANOS, V., JOINET, J.-B., AND SCHELLINX, H. (1993a). LKQ and LKT: Sequent calculi for second order logic based upon dual linear decompositions of classical implication. Manuscript.

DANOS, V., JOINET, J.-B., AND SCHELLINX, H. (1993b). LKT and $\lambda\mu$-calculus. Manuscript.

DANOS, V., JOINET, J.-B., AND SCHELLINX, H. (1993c). On the linear decoration of intuitionistic derivations. Prépublication 41, Équipe de Logique Mathématique, Université Paris VII.

DANOS, V., JOINET, J.-B., AND SCHELLINX, H. (1993d). The structure of exponentials: uncovering the dynamics of linear logic proofs. In Gottlob, G., Leitsch, A., and Mundici, D., editors, *Computational Logic and*

Proof Theory, pages 159–171. Springer Verlag. Lecture Notes in Computer Science 713, Proceedings of the Third Kurt Gödel Colloquium, Brno, Czech Republic, August 1993.

DOŠEN, K. (1992a). The first axiomatization of relevant logic. *Journal of Philosophical Logic*, 21:337–354.

DOŠEN, K. (1992b). Modal translations in substructural logics. *Journal of Philosophical Logic*, 21:283–336.

DOŠEN, K. (1992c). Nonmodal classical linear predicate logic is a fragment of intuitionistic linear logic. *Theoretical Computer Science*, 102:207–214.

FITTING, M. (1970). An embedding of classical logic in S4. *Journal of Symbolic Logic*, 35(4):529–534.

FLAGG, R. C. (1985). Epistemic set theory is a conservative extension of intuitionistic set theory. *Journal of Symbolic Logic*, 50(4):895–902.

FLAGG, R. C. AND FRIEDMAN, H. (1986). Epistemic and intuitionistic formal systems. *Annals of Pure and Applied Logic*, 32:53–60.

FRIEDMAN, H. (1978). Classically and intuitionistically provably recursive functions. In Müller, G. and Scott, D., editors, *Higher Set Theory*, pages 21–27. Springer Verlag. Lecture Notes in Mathematics 669.

GALLIER, J. (1993). Constructive logics. Part I: A tutorial on proof systems and typed λ-calculi. *Theoretical Computer Science*, 110:249–339.

GENTZEN, G. (1935). Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, 39:176–210, 405–431.

GIRARD, J.-Y. (1971). Une extension de l'interpretation fonctionelle de Gödel à l'analyse et son application à l'élimination des coupures dans l'analyse et la théorie des types. In Fenstad, J. E., editor, *Proceedings of the Second Scandinavian Logic Colloquium*, pages 63–92. North-Holland. Studies in Logic and the Foundations of Mathematics 63.

GIRARD, J.-Y. (1987a). Linear logic. *Theoretical Computer Science*, 50:1–102.

GIRARD, J.-Y. (1987b). *Proof Theory and Logical Complexity. Volume I.* Bibliopolis, Napoli. Studies in Proof Theory 1.

GIRARD, J.-Y. (1989). Towards a geometry of interaction. In Gray, J. W. and Scedrov, A., editors, *Categories in Computer Science and Logic*, pages 69–108. American Mathematical Society. Proceedings of the AMS-IMS-SIAM Joint Summer Research Conference, June 14-20, 1987, Boulder, Colorado; Contemporary Mathematics Volume 92.

GIRARD, J.-Y. (1991). A new constructive logic: classical logic. *Mathematical Structures in Computer Science*, 1(3):255–296.

GIRARD, J.-Y. (1993). On the unity of logic. *Annals of Pure and Applied Logic*, 59:201–217.

GIRARD, J.-Y. AND LAFONT, Y. (1987). Linear logic and lazy computation. In Ehrig, H., Kowalski, R., Levi, G., and Montanari, U., editors, *TAPSOFT '87; Volume 2*, pages 52–66. Springer Verlag.

GIRARD, J.-Y., LAFONT, Y., AND TAYLOR, P. (1988). *Proofs and Types.* Cambridge Tracts in Theoretical Computer Science 7. Cambridge University Press.

GIRARD, J.-Y., SCEDROV, A., AND SCOTT, P. J. (1992). Bounded linear logic: A modular approach to polynomial time computability. *Theoretical Computer Science*, 97:1–66.

GÖDEL, K. (1986). *Collected Works. Volume I: Publications 1929-1936.* Oxford University Press. Edited by Feferman, S., Dawson Jr., J.W., Kleene, S.C., Moore, G.H., Solovay, R.M. and van Heijenoort, J.

GOODMAN, N. D. (1984). Epistemic arithmetic is a conservative extension of intuitionistic arithmetic. *Journal of Symbolic Logic*, 49:192–203.

HEESTERBEEK, VAN NEERVEN, AND SCHELLINX (1992). *Das Fegefeuer Theorem (De Purgatorio). Eschatologische Axiomatik zum transorbitalen Sündenmanagement.* Litzelstetter Libellen, Ziemlich Neue Folge, Nr.2. Libelle Verlag, Bottighofen am Bodensee[4].

HOWARD, W. A. (1980). The formulae-as-types notion of construction. In Seldin, J. P. and Hindley, J. R., editors, *To H.B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, pages 479 – 490. Academic Press.

[4]ISBN 3-909081-55-X

HYLAND, M. AND DE PAIVA, V. (199x). Full intuitionistic linear logic. *Annals of Pure and Applied Logic*. To appear.

JOINET, J.-B. (1993). *Etude de la normalisation du calcul des séquents classique à travers la logique linéaire*. PhD thesis, Université Paris VII.

KLEENE, S. C. (1980). Recursive functionals and quantifiers of finite type revisited II. In Barwise, J., Keisler, H. J., and Kunen, K., editors, *The Kleene Symposium*, pages 1–20. North-Holland. Studies in Logic and the Foundations of Mathematics 101.

KRIVINE, J.-L. AND PARIGOT, M. (1990). Programming with proofs. *Journal of Information Processing and Cybernetics EIK*, 26(4):149–167.

LAFONT, Y. (1988). The linear abstract machine. *Theoretical Computer Science*, 59:157–180. (Some corrections in volume 62(1988), pp. 327-328.).

LEIVANT, D. (1985). Syntactic translations and provably recursive functions. *Journal of Symbolic Logic*, 50(3):682–688.

LINCOLN, P., MITCHELL, J., SCEDROV, A., AND SHANKAR, N. (1992). Decision problems for propositional linear logic. *Annals of Pure and Applied Logic*, 56:239–311.

LINCOLN, P., SCEDROV, A., AND SHANKAR, N. (1993). Linearizing intuitionistic implication. *Annals of Pure and Applied Logic*, 60:151–177.

MCKINSEY, J. AND TARSKI, A. (1948). Some theorems about the sentential calculi of Lewis and Heyting. *Journal of Symbolic Logic*, 13:1–15.

ONO, H. (1990). Structural rules and a logical hierarchy. In Petkov, P. P., editor, *Mathematical Logic*, pages 95–104, New York, London. Plenum Press. Proceedings of the Summer School and Conference on Mathematical Logic, Chaika, Bulgaria, 1988.

PARIGOT, M. (1991). Free Deduction: an analysis of computation in classical logic. In Voronkov, A., editor, *Russian Conference on Logic Programming*, pages 361–380. Springer Verlag. LNAI 592.

PARIGOT, M. (1992). $\lambda\mu$-Calculus: an algorithmic interpretation of classical natural deduction. In Voronkov, A., editor, *Logic Programming and Automated Reasoning*, pages 190–201. Springer Verlag. LNAI 624.

PARIGOT, M. (1993a). Classical proofs as programs. In Gottlob, G., Leitsch, A., and Mundici, D., editors, *Computational Logic and Proof Theory*, pages 263–276. Springer Verlag. LNCS 713.

PARIGOT, M. (1993b). Strong normalization for second order classical natural deduction. In *Logic in Computer Science*, pages 39–46. IEEE Computer Society Press. Proceedings of the Eighth Annual Symposium on Logic in Computer Science, Montréal, June 19-23, 1992.

PRAWITZ, D. (1975). Comments on Gentzen-type procedures and the classical notion of truth. In Dold, A. and Eckmann, B., editors, *Proof Theory Symposion, Kiel 1974*, pages 290–319. Springer Verlag. Lecture Notes in Mathematics 500.

PRIJATELJ, A. AND SCHELLINX, H. (1991). A note on embeddings of intuitionistic logic into classical linear logic. Manuscript.

RASIOWA, H. AND SIKORSKI, R. (1963). *The mathematics of metamathematics.* Pańafter Państowe Wydawnictwo Naukowe, Warszawa. Polska Akademia Nauk, Monografie Matematyczne, Tom 41.

REGNIER, L. (1992). *λ-Calcul et réseaux.* PhD thesis, Université Paris VII.

ROORDA, D. (1991). *Resource Logics: Proof-theoretical Investigations.* PhD thesis, University of Amsterdam.

SCEDROV, A. (1985). Extending Gödel's modal interpretation to type theory and set theory. In Shapiro, S., editor, *Intensional mathematics*, pages 81–120. North-Holland. Studies in Logic and the Foundations of Mathematics 113.

SCHELLINX, H. (1991). Some syntactical observations on linear logic. *Journal of Logic and Computation*, 1(4):537–559.

TROELSTRA, A. S. (1992). *Lectures on Linear Logic.* CSLI Lecture Notes 29. Center for the Study of Language and Information, Stanford.

a

Terminological conventions

Let us summarize and explain some of the terminology that we frequently use with respect to our main objects of study, *derivations in sequent calculi.*

Sequents are denoted by $\Gamma \Rightarrow \Delta$, where $\Rightarrow$ is the *entailment sign* of the calculus, and Γ, Δ are finite *multisets* of formulas, i.e. sets with multiplicities, or (equivalently) lists modulo the order in which the entries are given. In particular *exchange* mostly is implicit: $\{A, A, C, D\}$ and $\{C, A, D, A\}$ are identical as multisets. (In examples we sometimes explicitly indicate the use of exchange because this makes it easier to follow a specific occurrence of a formula in a derivation without having to add indices. It is used however merely as a *notational device.*) In a sequent $\Gamma \Rightarrow \Delta$ we call Γ the *antecedent*, Δ is the *succedent.* We denote the number of elements in a multiset Γ by $|\Gamma|$. E.g., $|\{C, A, D, A\}| = 4$.

If φ indicates an operation that maps formulas to formulas then, if $\Gamma = \{G_1, \ldots, G_n\}$, we write $\varphi\Gamma$ for the multiset $\{\varphi G_1, \ldots, \varphi G_n\}$. So $!\Gamma$ stands for $\{!G_1, \ldots, !G_n\}$, $\Gamma^\star$ for $\{G_1^\star, \ldots, G_n^\star\}$, etcetera.

Derivability of a sequent $\Gamma \Rightarrow \Delta$ in the sequent calculus $\mathbf{C}$ is written as $\mathbf{C} \vdash \Gamma \Rightarrow \Delta$. We write $\vdash A \Longleftrightarrow B$ as an abbreviation for '*both* $A \Rightarrow B$ *and* $B \Rightarrow A$ *are derivable*'. (In case $\mathbf{C}$ is (some fragment of) linear logic we will say that A and B are *'linearly equivalent'*.)

In diagrams we will use

$$\begin{array}{c} \pi \\ \vdots \\ \Gamma \Rightarrow \Delta \end{array}$$

as abbreviation for a derivation π with $\Gamma \Rightarrow \Delta$ as conclusion.

We will speak of L-formulas, being formulas that contain only connectives for which there are rules in **L**. An L-formula B is called an **L**-*theorem* iff $\mathbf{L} \vdash \; \Rightarrow B$. So a linear, intuitionistic, classical theorem will mean a formula derivable in linear, intuitionistic, classical logic. We usually write $\mathbf{L} \vdash B$ instead of $\mathbf{L} \vdash \; \Rightarrow B$. An L-formula A such that $\mathbf{L} \vdash A \Rightarrow$ is said to be an *anti-theorem.*

To a *fragment* of a sequent calculus **L** we will often refer by listing the connectives under consideration. E.g. the fragment $\{!, \otimes, \&, \forall\}$ of **CLL** is the fragment containing the identity axioms and *all* the rules for $!, \otimes, \&$ and $\forall$.

The following conventions are used in distinguishing between the occurrences of formulas in a given rule, e.g. L$\multimap$ in **CLL** (appendix b):

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad B, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \multimap B \Rightarrow \Delta_1, \Delta_2}$$

The formula $A \multimap B$ is called the *main* formula of the rule with main connective $\multimap$; the occurrences A and B in the premisses will be referred to as the *active* formulas; all other occurrences are said to be *passive*,and we distinguish in the obvious way between an *up* and a *down* occurrence of a given passive formula. The multisets Γ_i, Δ_i are referred to as the *context.*

In the case of second-order rules, e.g. L$\forall_2$ and R$\forall_2$

$$\frac{A[T], \Gamma \Rightarrow \Delta}{\forall X A[X], \Gamma \Rightarrow \Delta} \quad \text{and} \quad \frac{\Gamma \Rightarrow \Delta, A[Y]}{\Gamma \Rightarrow \Delta, \forall X A[X]}$$

the active occurrences are $A[T], A[Y]$. We refer to T, Y as the *abstracted* formulas.

We may encounter derivations that contain *repetitions* of sequents. We will in such cases sometimes speak of an application of the *repetition* rule, where all occurrences of formulas are said to be *passive.*

The rules for the exponentials have our special interest. We recall the *exponential contextual* or *promotion rules* L? and R! (in analogy with the proofnet formulation of linear logic also referred to as the *box* rules):

$$\frac{!\Gamma, C \Rightarrow ?\Delta}{!\Gamma, ?C \Rightarrow ?\Delta} \quad \text{and} \quad \frac{!\Gamma \Rightarrow C, ?\Delta}{!\Gamma \Rightarrow !C, ?\Delta}.$$

Observe that in these rules the formulas in the context *do* play an important role, in the sense that applicability of the rules depends crucially on their being 'exponentiated'. We therefore call them *side-active.*

An occurrence of a formula A in some sequent σ is called *linear* if no element of A's identity class[1], restricted to the subderivation π' with conclusion σ, is main formula in a structural rule in π'.

[1]See section 1 of chapter 5.

b

Classical linear logic (CLL)

Identity axiom and cut rule:

$$(\mathrm{Ax})\quad A \Rightarrow A \qquad\qquad (\mathrm{cut})\ \frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Rules and axioms for the constants:

$$(\mathbf{L1})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, \mathbf{1} \Rightarrow \Delta} \qquad\qquad (\mathbf{R1})\quad \Rightarrow \mathbf{1}$$

$$(\text{no } \mathrm{L}\top) \qquad\qquad (\mathrm{R}\top)\quad \Gamma \Rightarrow \top, \Delta$$

$$(\mathrm{L}\bot)\quad \bot \Rightarrow \qquad\qquad (\mathrm{R}\bot)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \bot}$$

$$(\mathbf{L0})\quad \Gamma, \mathbf{0} \Rightarrow \Delta \qquad\qquad (\text{no } \mathbf{R0})$$

Multiplicative logical rules:

$$(\mathrm{L}\multimap)\ \frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad B, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \multimap B \Rightarrow \Delta_1, \Delta_2} \qquad\qquad (\mathrm{R}\multimap)\ \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \multimap B, \Delta}$$

$$(\mathrm{L}\invamp)\ \frac{\Gamma_1, A \Rightarrow \Delta_1 \qquad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \invamp B \Rightarrow \Delta_1, \Delta_2} \qquad\qquad (\mathrm{R}\invamp)\ \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \invamp B, \Delta}$$

$$(\mathrm{R}\otimes)\ \frac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad \Gamma_2 \Rightarrow B, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow A \otimes B, \Delta_1, \Delta_2} \qquad\qquad (\mathrm{L}\otimes)\ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \otimes B \Rightarrow \Delta}$$

Additive logical rules:

$$(\mathrm{R}\leadsto)\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow A \leadsto B, \Delta} \qquad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \leadsto B, \Delta} \qquad (\mathrm{L}\leadsto)\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{\Gamma, A \leadsto B \Rightarrow \Delta}$$

$$(\mathrm{R}\oplus)\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow A \oplus B, \Delta} \qquad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \oplus B, \Delta} \qquad (\mathrm{L}\oplus)\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \oplus B \Rightarrow \Delta}$$

$$(\mathrm{L}\&)\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A \,\&\, B \Rightarrow \Delta} \qquad \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \,\&\, B \Rightarrow \Delta} \qquad (\mathrm{R}\&)\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \,\&\, B, \Delta}$$

Rules for the first-order quantifiers (y not free in Γ, Δ):

$$(\mathrm{L}\forall)\ \frac{\Gamma, A[t/x] \Rightarrow \Delta}{\Gamma, \forall x\, A \Rightarrow \Delta} \qquad (\mathrm{R}\forall)\ \frac{\Gamma \Rightarrow A[y/x], \Delta}{\Gamma \Rightarrow \forall x\, A, \Delta}$$

$$(\mathrm{L}\exists)\ \frac{\Gamma, A[y/x] \Rightarrow \Delta}{\Gamma, \exists x\, A \Rightarrow \Delta} \qquad (\mathrm{R}\exists)\ \frac{\Gamma \Rightarrow A[t/x], \Delta}{\Gamma \Rightarrow \exists x\, A, \Delta}$$

Exponential structural rules:

$$(\mathrm{W}!)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta} \qquad (\mathrm{W}?)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow ?A, \Delta}$$

$$(\mathrm{C}!)\ \frac{\Gamma, !A, !A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta} \qquad (\mathrm{C}?)\ \frac{\Gamma \Rightarrow ?A, ?A, \Delta}{\Gamma \Rightarrow ?A, \Delta}$$

Exponential contextual rules:

$$(\mathrm{L}?)\ \frac{!\Gamma, A \Rightarrow ?\Delta}{!\Gamma, ?A \Rightarrow ?\Delta} \qquad (\mathrm{R}!)\ \frac{!\Gamma \Rightarrow A, ?\Delta}{!\Gamma \Rightarrow !A, ?\Delta}$$

Exponential dereliction rules:

$$(\mathrm{R}?)\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow ?A, \Delta} \qquad (\mathrm{L}!)\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta}$$

The second-order calculus CLL_2 is obtained by adding:

Rules for the second-order quantifiers (Y not free in Γ, Δ):

$$(\mathrm{L}\forall_2)\ \frac{\Gamma, A[T/X] \Rightarrow \Delta}{\Gamma, \forall X\, A \Rightarrow \Delta} \qquad (\mathrm{R}\forall_2)\ \frac{\Gamma \Rightarrow \Delta, A[Y/X]}{\Gamma \Rightarrow \Delta, \forall X\, A}$$

$$(\mathrm{L}\exists_2)\ \frac{\Gamma, A[Y/X] \Rightarrow \Delta}{\Gamma, \exists X\, A \Rightarrow \Delta} \qquad (\mathrm{R}\exists_2)\ \frac{\Gamma \Rightarrow \Delta, A[T/X]}{\Gamma \Rightarrow \Delta, \exists X\, A}$$

Linear negation is defined by $A^\perp := A \multimap \perp$, and the following are provable:

$$
\begin{array}{rcl}
A & \Longleftrightarrow & (A^\perp)^\perp \\
\mathbf{1} & \Longleftrightarrow & \perp^\perp \\
\top & \Longleftrightarrow & \mathbf{0}^\perp \\
A \otimes B & \Longleftrightarrow & (A^\perp \parr B^\perp)^\perp \\
A \oplus B & \Longleftrightarrow & (A^\perp \,\&\, B^\perp)^\perp \\
\exists x A & \Longleftrightarrow & (\forall x A^\perp)^\perp \\
\exists X A & \Longleftrightarrow & (\forall X A^\perp)^\perp \\
!A & \Longleftrightarrow & (?A^\perp)^\perp
\end{array}
$$

The rules and axioms for $\mathbf{1}, \mathbf{0}, \otimes, \oplus, \exists, \exists_2, !$ are 'De Morgan'-derivable from those for $\perp, \top, \parr, \&, \forall, \forall_2, ?$.

This duality is fully exploited in the *one-sided* version of the calculus, which is formulated as follows:

Identity axiom and cut rule:

$$
(\mathrm{Ax})\quad A, A^\perp \qquad\qquad (\mathrm{cut})\ \frac{\Gamma, A \qquad \Delta, A^\perp}{\Gamma, \Delta}
$$

Rules and axioms for the constants:

$$
(\mathbf{1})\quad \mathbf{1} \qquad\qquad (\top)\quad \Gamma, \top
$$

$$
(\perp)\ \frac{\Gamma}{\Gamma, \perp} \qquad\qquad (\text{no rules for } \mathbf{0})
$$

Multiplicative logical rules:

$$
(\parr)\ \frac{\Gamma, A, B}{\Gamma, A \parr B} \qquad\qquad (\otimes)\ \frac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta, A \otimes B}
$$

Additive logical rules:

$$
(\oplus)\ \frac{\Gamma, A}{\Gamma, A \oplus B} \qquad \frac{\Gamma, B}{\Gamma, A \oplus B} \qquad\qquad (\&)\ \frac{\Gamma, A \qquad \Gamma, B}{\Gamma, A \,\&\, B}
$$

Rules for the first-order quantifiers (y not free in Γ):

$$(\forall)\ \frac{\Gamma, A[y/x]}{\Gamma, \forall x\, A} \qquad (\exists)\ \frac{\Gamma, A[t/x]}{\Gamma, \exists x\, A}$$

Rules for the second-order quantifiers (Y not free in Γ):

$$(\forall_2)\ \frac{\Gamma, A[Y/X]}{\Gamma, \forall X\, A} \qquad (\exists_2)\ \frac{\Gamma, A[T/X]}{\Gamma, \exists X\, A}$$

Exponential structural rules:

$$(\mathrm{W})\ \frac{\Gamma}{\Gamma, ?A} \qquad (\mathrm{C})\ \frac{\Gamma, ?A, ?A}{\Gamma, ?A}$$

Exponential contextual rule:

$$\frac{?\Gamma, A}{?\Gamma, !A}$$

Exponential dereliction rule:

$$\frac{\Gamma, A}{\Gamma, ?A}$$

C

Classical logic (CL)

Identity axiom and cut rule:

$$(\mathrm{Ax})\quad A \Rightarrow A \qquad\qquad (\mathrm{cut})\ \frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Axioms for the constants:

$$(\mathrm{f})\quad \Gamma, \bot \Rightarrow \Delta \qquad\qquad (\mathrm{t})\quad \Gamma \Rightarrow \top, \Delta$$

Multiplicative logical rules:

$$(\mathrm{L}{\to})\ \frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad B, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \to B \Rightarrow \Delta_1, \Delta_2} \qquad\qquad (\mathrm{R}{\to})\ \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta}$$

$$(\mathrm{L}\vee)\ \frac{\Gamma_1, A \Rightarrow \Delta_1 \qquad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \vee B \Rightarrow \Delta_1 \Delta_2} \qquad\qquad (\mathrm{R}\vee)\ \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta}$$

$$(\mathrm{R}\wedge)\ \frac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad \Gamma_2 \Rightarrow B, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow A \wedge B, \Delta_1, \Delta_2} \qquad\qquad (\mathrm{L}\wedge)\ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta}$$

Additive logical rules:

$$(\mathrm{R}{\to})\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow A \to B, \Delta} \quad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \to B, \Delta} \qquad\qquad (\mathrm{L}{\to})\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta}$$

$$(\mathrm{R}\vee)\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow A \vee B, \Delta} \quad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \vee B, \Delta} \qquad\qquad (\mathrm{L}\vee)\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee B \Rightarrow \Delta}$$

$$(\mathrm{L}\wedge)\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \quad \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \wedge B \Rightarrow \Delta} \qquad\qquad (\mathrm{R}\wedge)\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge B, \Delta}$$

Rules for the first-order quantifiers (y not free in Γ, Δ):

$$(\mathrm{L}\forall)\ \frac{\Gamma, A[t/x] \Rightarrow \Delta}{\Gamma, \forall x\, A \Rightarrow \Delta} \qquad (\mathrm{R}\forall)\ \frac{\Gamma \Rightarrow A[y/x], \Delta}{\Gamma \Rightarrow \forall x\, A, \Delta}$$

$$(\mathrm{L}\exists)\ \frac{\Gamma, A[y/x] \Rightarrow \Delta}{\Gamma, \exists x\, A \Rightarrow \Delta} \qquad (\mathrm{R}\exists)\ \frac{\Gamma \Rightarrow A[t/x], \Delta}{\Gamma \Rightarrow \exists x\, A, \Delta}$$

Structural rules:

$$(\mathrm{LW})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad (\mathrm{RW})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$(\mathrm{LC})\ \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad (\mathrm{RC})\ \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}$$

We get the second-order calculus $\mathbf{CL}_2$ by adding

Rules for the second-order quantifiers (Y not free in Γ, Δ):

$$(\mathrm{L}\forall_2)\ \frac{\Gamma, A[T/X] \Rightarrow \Delta}{\Gamma, \forall X\, A \Rightarrow \Delta} \qquad (\mathrm{R}\forall_2)\ \frac{\Gamma \Rightarrow \Delta, A[Y/X]}{\Gamma \Rightarrow \Delta, \forall X\, A}$$

$$(\mathrm{L}\exists_2)\ \frac{\Gamma, A[Y/X] \Rightarrow \Delta}{\Gamma, \exists X\, A \Rightarrow \Delta} \qquad (\mathrm{R}\exists_2)\ \frac{\Gamma \Rightarrow \Delta, A[T/X]}{\Gamma \Rightarrow \Delta, \exists X\, A}$$

Negation is defined by $\neg A := A \to \bot$.

The multiplicative and additive versions of the logical rules are interderivable (using the structural rules).

The modal logic **S4** is obtained by adding to (the first order fragment of **CL**) the introduction-rules

Modal contextual rules:

$$\mathrm{L}\Diamond\ \frac{\Box\Gamma, A \Rightarrow \Diamond\Delta}{\Box\Gamma, \Diamond A \Rightarrow \Diamond\Delta} \qquad \mathrm{R}\Box\ \frac{\Box\Gamma \Rightarrow A, \Diamond\Delta}{\Box\Gamma \Rightarrow \Box A, \Diamond\Delta}$$

Modal dereliction rules:

$$\mathrm{R}\Diamond\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow \Diamond A, \Delta} \qquad \mathrm{L}\Box\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}$$

d

Intuitionistic logic (IL)

Identity axiom and cut rule:

$$(\mathrm{Ax})\quad A \Rightarrow A \qquad\qquad (\mathrm{cut})\ \frac{\Gamma_1 \Rightarrow A \qquad A, \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow B}$$

Falsum axiom:

$$(\mathrm{f})\quad \Gamma, \bot \Rightarrow A$$

Multiplicative logical rules:

$$(\mathrm{L}{\rightarrow})\ \frac{\Gamma_1 \Rightarrow A \qquad B, \Gamma_2 \Rightarrow C}{\Gamma_1, \Gamma_2, A \rightarrow B \Rightarrow C} \qquad (\mathrm{R}{\rightarrow})\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B}$$

$$(\mathrm{R}\wedge)\ \frac{\Gamma_1 \Rightarrow A \qquad \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A \wedge B} \qquad (\mathrm{L}\wedge)\ \frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}$$

Additive logical rules:

$$(\mathrm{R}{\rightarrow})\ \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \qquad (\mathrm{L}{\rightarrow})\ \frac{\Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow C}{\Gamma, A \rightarrow B \Rightarrow C}$$

$$(\mathrm{R}\vee)\ \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \qquad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B} \qquad (\mathrm{L}\vee)\ \frac{\Gamma, A \Rightarrow C \qquad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C}$$

$$(\mathrm{L}\wedge)\ \frac{\Gamma, A \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \qquad \frac{\Gamma, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C} \qquad (\mathrm{R}\wedge)\ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

Rules for the first-order quantifiers (y not free in Γ, C*):*

$$(\mathrm{L}\forall)\ \frac{\Gamma, A[t/x] \Rightarrow C}{\Gamma, \forall x\, A \Rightarrow C} \qquad (\mathrm{R}\forall)\ \frac{\Gamma \Rightarrow A[y/x]}{\Gamma \Rightarrow \forall x\, A}$$

$$(\mathrm{L}\exists)\ \frac{\Gamma, A[y/x] \Rightarrow C}{\Gamma, \exists x\, A \Rightarrow C} \qquad (\mathrm{R}\exists)\ \frac{\Gamma \Rightarrow A[t/x]}{\Gamma \Rightarrow \exists x\, A}$$

Structural rules:

$$(\mathrm{W})\ \frac{\Gamma \Rightarrow C}{\Gamma, A \Rightarrow C} \qquad (\mathrm{C})\ \frac{\Gamma, A, A \Rightarrow C}{\Gamma, A \Rightarrow C}$$

We obtain the second-order system $\mathbf{IL}_2$ by adding:

Rules for the second-order quantifiers (Y not free in Γ, C*):*

$$(\mathrm{L}\forall_2)\ \frac{\Gamma, A[T/X] \Rightarrow C}{\Gamma, \forall X\, A \Rightarrow C} \qquad (\mathrm{R}\forall_2)\ \frac{\Gamma \Rightarrow A[Y/X]}{\Gamma \Rightarrow \forall X\, A}$$

$$(\mathrm{L}\exists_2)\ \frac{\Gamma, A[Y/X] \Rightarrow C}{\Gamma, \exists X\, A \Rightarrow C} \qquad (\mathrm{R}\exists_2)\ \frac{\Gamma \Rightarrow A[T/X]}{\Gamma \Rightarrow \exists X\, A}$$

Negation is defined by $\neg A := A \to \bot$.

The multiplicative and additive versions of the logical rules are interderivable (using the structural rules).

REMARK. Note that we do not have 'natural' multiplicative versions of the rules for $\vee$. The best we can come up with is the 'hybrid' set of rules

$$(\mathrm{R}\vee)\ \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B} \quad \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B} \qquad (\mathrm{L}\vee)\ \frac{\Gamma, A \Rightarrow C \qquad \Gamma', B \Rightarrow C}{\Gamma, \Gamma', A \vee B \Rightarrow C}$$

REMARK. One can also formalize intuitionistic logic as a sequent calculus $\mathbf{IL}'$, having sequents $\Gamma \Rightarrow \Delta$ with $\mid \Delta \mid \leq 1$, and a rule RW of the form

$$\frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}.$$

One easily verifies:

$$\begin{array}{lll} \mathbf{IL}' \vdash \Gamma \Rightarrow & \text{iff} & \mathbf{IL} \vdash \Gamma \Rightarrow \bot \\ \mathbf{IL}' \vdash \Gamma \Rightarrow A & \text{iff} & \mathbf{IL} \vdash \Gamma \Rightarrow A. \end{array}$$

Taxonomical divertissement

If we skip one or more of the structural rules, we can 'compose' fragments of intuitionistic logic by chosing either an additive or a multiplicative version of a rule for a connective. In general different choices will give rise to different, proper, fragments of **IL**.

To illustrate this (without any claim of exhaustiveness), let us construct some fragments of intuitionistic *implicational* logic (**IIL**).

Each of the fragments will contain (i) the identity axiom Ax and (ii) the rule R→. We will vary the presence of the structural rules (W,C) and the nature of the rule L→ (additive (a) or multiplicative (m)).

We then find the following possibilities:

(1) a (≡ a + C);

(2) m (*linear* implicational logic);

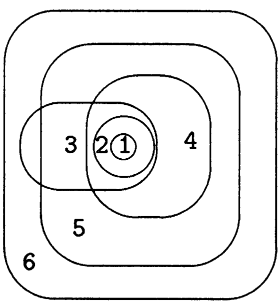
(3) m + C (*relevant* implicational logic);

(4) m + W (*affine* implicational logic);

(5) a + W;

(6) **IIL**.

The following figure indicates the relations of proper containment that hold between these fragments:



For a proof one uses e.g. the following sequent (writing XY for $X \to Y$):

$$BA, A(((BC)D)D), (((BC)D)D)C, (((BC)D)D)C \Rightarrow C,$$

which is *linearly* derivable (i.e. in m). The sequent that is obtained by contracting the two occurrences of $(((BC)D)D)C$, however, can not be derived in a + W.

e

Multi-succedent IL ($IL^{>}$)

Identity axiom and cut rule:

$$\text{(Ax)}\quad A \Rightarrow A \qquad\qquad \text{(cut)}\ \frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Falsum axiom:

$$\text{(f)}\quad \Gamma, \bot \Rightarrow \Delta$$

Multiplicative logical rules:

$$(\mathrm{L}\!\to)\ \frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad B, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \to B \Rightarrow \Delta_1, \Delta_2} \qquad (\mathrm{R}\!\to)\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

$$(\mathrm{L}\lor)\ \frac{\Gamma_1, A \Rightarrow \Delta_1 \qquad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \lor B \Rightarrow \Delta_1, \Delta_2} \qquad (\mathrm{R}\lor)\ \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \lor B, \Delta}$$

$$(\mathrm{R}\land)\ \frac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad \Gamma_2 \Rightarrow B, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow A \land B, \Delta_1, \Delta_2} \qquad (\mathrm{L}\land)\ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \land B \Rightarrow \Delta}$$

Additive logical rules:

$$(\mathrm{R}\!\to)\ \frac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B} \qquad \frac{\Gamma, A \Rightarrow}{\Gamma \Rightarrow A \to B} \qquad (\mathrm{L}\!\to)\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{\Gamma, A \to B \Rightarrow \Delta}$$

$$(\mathrm{R}\lor)\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow A \lor B, \Delta} \qquad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \lor B, \Delta} \qquad (\mathrm{L}\lor)\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \lor B \Rightarrow \Delta}$$

$$(\mathrm{R}\land)\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \land B, \Delta} \qquad (\mathrm{L}\land)\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A \land B \Rightarrow \Delta} \qquad \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \land B \Rightarrow \Delta}$$

Rules for the first-order quantifiers (y not free in Γ, Δ):

$$(L\forall)\ \frac{\Gamma, A[t/x] \Rightarrow \Delta}{\Gamma, \forall x\, A \Rightarrow \Delta} \qquad (R\forall)\ \frac{\Gamma \Rightarrow A[y/x]}{\Gamma \Rightarrow \forall x\, A}$$

$$(L\exists)\ \frac{\Gamma, A[y/x] \Rightarrow \Delta}{\Gamma, \exists x\, A \Rightarrow \Delta} \qquad (R\exists)\ \frac{\Gamma \Rightarrow A[t/x], \Delta}{\Gamma \Rightarrow \exists x\, A, \Delta}$$

Structural rules:

$$(LW)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad (RW)\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow A, \Delta}$$

$$(LC)\ \frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta} \qquad (RC)\ \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}$$

Negation is defined by $\neg A := A \to \bot$.

The multiplicative and additive versions of the logical rules are interderivable (using the structural rules).

REMARK. With respect to provability this sequent calculus is equivalent to the calculus **IL** of appendix d, in the sense that $\mathbf{IL}^> \vdash \Gamma \Rightarrow \Delta$ if and only if $\mathbf{IL} \vdash \Gamma \Rightarrow \bigvee \Delta$, where $\bigvee \Delta$ denotes the disjunction of *all* formulas in Δ (which by convention is $\bot$ if $\Delta = \emptyset$). See also Schellinx(1991)

The equivalent second-order calculus is obtained by adding:

Rules for the second-order quantifiers (Y not free in Γ, Δ):

$$(L\forall_2)\ \frac{\Gamma, A[T/X] \Rightarrow \Delta}{\Gamma, \forall X\, A \Rightarrow \Delta} \qquad (R\forall_2)\ \frac{\Gamma \Rightarrow A[Y/X]}{\Gamma \Rightarrow \forall X\, A}$$

$$(L\exists_2)\ \frac{\Gamma, A[Y/X] \Rightarrow \Delta}{\Gamma, \exists X\, A \Rightarrow \Delta} \qquad (R\exists_2)\ \frac{\Gamma \Rightarrow \Delta, A[T/X]}{\Gamma \Rightarrow \Delta, \exists X\, A}$$

Index

Miscellaneous

Sequent Calculi

Translations and Embeddings

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Beknopte weergave

Lineaire logica (Girard, 1987) is een verfijning van de formulering van klassieke logica als sequenten calculus (Gentzen, 1935). De 'ingreep' is eenvoudig: in de 'klassieke' formulering mag een formule, eenmaal als hypothese in een afleiding aanwezig, een in principe onbeperkt aantal keren als zodanig opgevoerd worden; bovendien kan iedere willekeurige formule als hypothese geïntroduceerd worden, ook als er nooit daadwerkelijk gebruik van wordt gemaakt. Verder geldt het een aantal keren afleiden van een bepaald theorema als equivalent met dat precies één keer doen, en mag aan een eenmaal bereikte conclusie X naar willekeur een conclusie Y (in de geest van 'X óf Y') toegevoegd worden.

In de 'lineaire' formulering wordt aan dat soort praktijken paal en perk gesteld. Contractie (meervoudig gebruik) van en verzwakking ('we konden er ook zonder') met een hypothese kan enkel onder de voorwaarde dat de betreffende formule, zeg A, met een uitroepteken ($!A$) gemerkt wordt: *"Natuurlijk, A!"*. Contractie (het samenrapen) van (identieke) conclusies of verzwakking met een willekeurige extra slotsom, mag enkel als we de betreffende kandidaat, zeg B, van een vraagteken ($?B$) voorzien: *"Waarom niet B?"*.[1] Dit blijkt een ingreep met vérstrekkende gevolgen.

Het schrappen van ongelimiteerde contractie en verzwakking als regels in de sequenten calculus leidt in de eerste plaats tot een *splitsing* van de bekende logische connectieven in elk twee varianten, een *additieve* en een *multiplicatieve*[2]. Door het opnieuw toevoegen van de

[1] De symbolen !, ? worden de 'exponenten' of 'modaliteiten' genoemd.

[2] We gebruiken conjunctie bijvoorbeeld in de omgangstaal in de *multiplicatieve* zin, als we implicatie ('als ... dan') hanteren om een actie aan te duiden die middels *gebruik* van het antecedent tot het succedent leidt: "*Als* ik tienduizend gulden heb,

structurele regels, maar nu enkel voor gemodaliseerde ('gemerkte') formules, is bovendien de expressieve kracht van het systeem niet minder dan dat van intuitionistische of klassieke logica. De belangrijkste consequentie echter is dat de resulterende logica, met name in haar 'bewijsnet'-formulering, in hoge mate *constructieve* eigenschappen bezit: zo is de snede-eliminatie procedure *sterk normalizerend* (reducties (i.e. *berekeningen*) zijn altijd eindig) en, voor belangrijke fragmenten, *confluent* (i.e. het resultaat is uniek). Bovendien is er een niet-triviale semantiek (coherentie-ruimten) voor *bewijzen*, invariant onder reductie.

Als we de lineaire typering 'vergeten', dan is een afleiding in lineaire logica niets anders dan een afleiding in *klassieke*, of misschien zelfs wel in *intuïtionistische* logica. We laten in dit proefschrift o.a. zien dat omgekeerd *elk* klassiek, en *elk* intuïtionistisch, bewijs voorkomt als 'skelet' van (in principe oneindig veel) lineaire bewijzen. Dit vormt het uitgangspunt voor ons werk, dat tweeledig van aard is: we bekijken afleidingen in de lineaire sequenten calculus *an sich*, én gebruiken lineaire logica als een bewijstheoretisch instrument voor het bestuderen van intuïtionistische en klassieke bewijzen.[3]

In hoofdstuk 5 introduceren we de *exponenten graaf* van een afleiding in klassieke (tweede orde) lineaire logica, een artefact hetwelk de onderlinge relatie tussen exponenten in een bewijs weergeeft. Met behulp van die graaf karakteriseren we exponenten die geen (directe of indirecte) *structurele* oorzaak hebben. Deze kunnen verwijderd worden, en het resultaat is een afleiding die (1) nog steeds correct is, en (2) dezelfde *dynamiek* (i.e. gedrag onder reductie) heeft als het origineel. Onder bepaalde voorwaarden is de zo verkregen afleiding boven-

---

dan koop ik tweedehands een zwarte BMW ", én "*Als* ik tienduizend gulden heb, *dan* koop ik een eerste druk van *De Avonden*". Samenstelling van beide acties door middel van de 'gewone' conjunctie (die 'idempotent' is, dus o.a. *contractie* toelaat) maakt mij niet alleen de bink, maar, en voor hetzelfde geld, 'n *'De Avonden'* rijker. We weten natuurlijk beter. Oók in de lineaire logica, welke soms tot, vaak op soortgelijk flauwe voorbeelden gebaseerde, ongerechtvaardigd hoge verwachtingen met betrekking tot toepasbaarheid in 'praktische' aangelegenheden aanleiding blijkt te geven.

[3] "... denn nicht das brechen des Strahls, sondern der Strahl selbst, wodurch die Wahrheit uns berührt, ist das Erkennen ..." (G.W.F. HEGEL. *Phänomenologie des Geistes.* Bamberg und Würzburg, 1807)

dien de *minimaal gemodaliseerde* met deze eigenschappen. Ook stelt de exponenten graaf ons in staat díe afleidingen in lineaire logica te karakteriseren welke *dilateerbaar* zijn, dat wil zeggen, waarin we *alle* gemodaliseerde formules kunnen vervangen door niet-modale formules. Wéér (dit is ons 'sine qua non') *zonder* structuur en dynamiek van het origineel essentieel te veranderen. Het belangrijkste resultaat hier is dat een volledig geëxpandeerde lineaire afleiding dilateerbaar is dan en slechts dan als haar exponenten graaf acyclisch is (hoofdstuk 8).

We bestuderen de intuïtionistische en klassieke sequenten calculus door middel van het inductief toepassen van modale vertalingen (hoofdstukken 2 en 3). We laten zien dat er in het 'klassieke' geval in essentie *twee* modale vertalingen zijn welke de structuur van de oorspronkelijke afleiding ongewijzigd laten. Voor bepaalde belangrijke fragmenten, bijvoorbeeld dat bestaande uit de regels voor implicatie, universele eerste orde quantificatie en universele tweede orde (propositionele) quantificatie, is dit procédé van 'lineair decoreren' volledig *deterministisch* (i.e. de wijze van modalizeren is eenduidig bepaald). Een gevolg is dat deze decoraties ondubbelzinnig een normalizatie procedure voor het betreffende fragment definiëren, zijnde de reflectie van de lineaire procedure toegepast op de decoratie. Het is daarom direct duidelijk dat deze procedure sterk normalizerend is.

In hoofdstuk 6 introduceren we het begrip *constrictief morfisme*, met behulp waarvan we de genoemde modale vertalingen kunnen optimalizeren. Dit leidt tot welgedefiniëerde restricties op regels van de sequenten calculus, die volledigheid met betrekking tot bewijsbaarheid behouden. Bovendien kunnen deze restricties in een gegeven afleiding gerealiseerd worden door 'toepassing van het morfisme', i.e. door eliminatie van de snede die het introduceert. Zo krijgen we 'alternatieve' sequentencalculi voor intuitïonistische en klassieke logica voor welke de optimale modale vertalingen decorerend zijn. Ze heten **ILU**, **LKT** en **LKQ**.

Stellingen

I,

II,

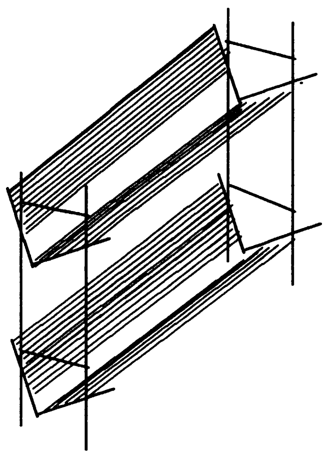
III,

⋮

IV.

Het gebouw van de Faculteit Wiskunde aan de Plantage Muidergracht huisvestte vroeger een farmaceutisch laboratorium. Laten we daarom spreken van "de Brouwerij".

V.



Stelling voor zesenzestig (66) kompakt plaatjes. Eigen ontwerp.

VI.

Onze wiskunde is uniek dan en alleen dan als onze wereld volledig is.

VII.

$$\sharp\left\{k \in \mathbb{N} \;\middle|\; \frac{m}{m+1}\cdot n \leq k < \frac{m+1}{m+2}\cdot n\right\} = \left\lfloor \frac{mod(n, m+2) + \left\lfloor \frac{n}{m+2} \right\rfloor}{m+1} \right\rfloor$$

H.A.J.M. SCHELLINX & J.A.P. HEESTERBEEK, *On sums of remainders and almost perfect numbers.* In: TW in Beeld, Mathematisch Centrum, Amsterdam 1988; propositie 3a.

VIII.

Een afvalemmer voorzien van een deksel met enkele fikse gaten, één welker aan het oog onttrokken door een trapachtige constructie leidend tot een middels een metalen veertje in onbelaste toestand gesloten luikje, teneinde rondscharrelend ongedierte, dat bij het verkennen van de ónbedekte gaten "Ha, daar trap ik niet in!" dient te piepen, te verleiden de treden te bestijgen om bij aankomst op het beweeglijke platje in de diepte te duikelen, getuigt eerder van literaire kwaliteiten dan van een bijzonder inzicht in de psychologie van de huismuis.

JOCHEM HARTZ, *"Een hoogst effectieve muizeval".* Persoonlijke mededeling.

IX.

Ieder machtig object in een zwak cartesisch gesloten categorie definieert een λ-algebra. En *elke* λ-algebra kan men via zo'n object verkrijgen.

RAYMOND HOOFMAN & HAROLD SCHELLINX, *Models of the untyped λ-calculus in semi cartesian closed categories.* ILLC Prepublication Series for Mathematical Logic and Foundations ML-93-05, Amsterdam 1993.

X.

$t\uparrow - \dagger > 0$

HEESTERBEEK, VAN NEERVEN & SCHELLINX, Das Fegefeuer-Theorem (De Purgatorio). Verlag Die Libelle, Bottighofen am Bodensee 1992; theorema 6.6.4.

XI.

Muziek is de wereld zoals ik die hoorde.

XIII.

Zijn $R \geq 0$ en $r \geq 1$ natuurlijke getallen. Definieer, voor $n \in \mathbb{N}$:

$$[R, n]_r = \begin{cases} n - R, & \text{als } n < R \\ mod(n - R, r), & \text{anders.} \end{cases}$$

Als we $R + [R, \cdot]_r$ samenstellen met de gebruikelijke optelling, dan maakt dit $\{0, 1, \ldots, R + r - 1\}$ tot een commutatieve monoïde met eenheid 0. Deze komt mooi van pas bij het modelleren van geknotte structurele regels.

HORI, ONO & SCHELLINX, *Extending intuitionistic linear logic with knotted structural rules.* Manuscript, Hiroshima 1993.

XIV.

Irreflexiviteit van de ouderschapsrelatie tussen individuen blijkt niet noodzakelijk voor een rigoreuze definitie van het begrip 'soort'. Nog is alle hoop op verzoening van moderne biologie met christelijke dogmatiek dus niet vervlogen.

D.J. KORNET & J.A.J. METZ, *Internodons as Equivalence Classes in the Genealogical Network: Building-Blocks for a Rigorous Species Concept.* Manuscript, Leiden 1993.

XV.

Uw logica is niet de mijne.

XVI.

$$\frac{\textbf{ILU}}{\lambda_2} \sim \frac{\textbf{LKT}}{\lambda\mu}$$

XVII.

Uitbreiding van de standaard quantumlogische formalismen met relativistische tijdsoperatoren blijft voorlopig een vrome wens.

HAROLD SCHELLINX, *Measurement and the logic of relativistic time.* Manuscript, Paris 1994.

XVIII.

Een aanzienlijk deel van het werk in dit proefschrift beschreven is van wezenlijk semantische aard.

XIX.

Het Plotkin-Scott graafmodel $\mathcal{P}\omega$ kent enkel het triviale automorfisme. Men kan evenwel voor een ieder willekeurig direct product $\mathcal{G} \equiv \prod_{i=1}^{\infty} G_i$, waarin elke factor isomorf is, hetzij met een symmetrische groep S_n, hetzij met $S(\mathbb{N})$, een één-éénduidige afbeelding $p_{\mathcal{G}}$ van $\mathbb{N} \times \mathbb{N}$ naar $\mathbb{N}$ aangeven, zodanig dat $\mathcal{G}$ juist de groep der automorfismen van het door $p_{\mathcal{G}}$ geïnduceerde graafmodel is.

HAROLD SCHELLINX, *Isomorphisms and nonisomorphisms of graph models.* Journal of Symbolic Logic, 56-1, 227-249.

" ... Puebla de formas la pared ... "

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