An Exploration of Contraction Free Arithmetic

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

This thesis is a proof-theoretic study of Contraction Free Arithmetic (CFA). We introduce CFA as a first order arithmetical theory based on a logic (GQC) which is the multiplicative fragment of LK without the structural rules of contraction. Our investigation starts by showing some basic properties of GQC, before proceeding to establish various properties of CFA. One such interesting property of CFA is that the (omitted) additive connectives become definable in it. A key characteristic of CFA is that it has the Induction Rule, for all formulas, in place of the Induction Schema. We justify this by showing that the presence of the induction schema would reintroduce contraction rules, and subsequently, the arithmetic would collapse to PA. We proceed to establish that induction schema restricted to Δ_0 formulas hold for CFA and utilize this to show, among other things, that any Π_2 sentence provable in the arithmetical theory $\mathsf{I}\Delta_0$ is also provable in CFA. This thesis culminates in the study of the computational strength of CFA via provably recursive functions of CFA. We establish that the class of provably recursive functions within CFA precisely coincides with the class of primitive recursive functions. Consequently, the essential result of the present work is that CFA not only expands the class of provably recursive functions beyond those of $\mathsf{I}\Delta_0$, but also establishes its distinctiveness from PA.

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$\otimes$	Multiplicative Conjunction
$\wedge$	Additive Conjunction
$\oplus$	Multiplicative Disjunction
$\vee$	Additive Disjunction
$\rightarrow$	Multiplicative Conditional
$\bot$	Falsum
$\top$	Truth
$\emptyset$	Empty Set
$\mathcal{C}(A)$	Complexity of formula A
LK	Classical Logic
PA	Peano Arithmetic
HA	Heyting Arithmetic
GPC	Grishin Propositional Calculus
GQC	Grishin Quantified Calculus
$\mathsf{I}\Delta_0$	First Order Bounded Arithmetic
CFA	Contraction Free Arithmetic
$\#$	Natural Sum of Ordinals

1 Introduction

This thesis studies arithmetic from a proof-theoretical perspective within the framework of an underlying logic which does not enjoy contraction. The logic falls under the class of logics called substructural logic. We begin by a brief discussion of what substructural logics are.

1.1 Substructural Logics

Gerhard Gentzen introduced both Natural Deduction and Sequent Calculi in his revolutionary doctoral dissertation titled *Untersuchungen über das logische SchlieSSen* (translated into English as Investigations into Logical Deduction) Gentzen [1935]. It is fitting to acknowledge that substructural logics trace their origins back to this thesis, as Gentzen demonstrated in logic there are rules of inference independent of any logical constants. To understand this concept, let us examine his system of sequent calculus for Classical logic, denoted as LK:

Definition 1.1. *(Variant of* LK*) The calculus* LK *for classical logic has the following postulates.*

Initial Sequents

$$A \Rightarrow A$$

Structural Rules

$$\text{(Cut)}\ \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$\text{(WL)}\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{(WR)}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

$$\text{(CL)}\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{(CR)}\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

Inference Rules

$$(\neg \mathrm{L})\ \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \qquad (\neg \mathrm{R})\ \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}$$

$$(\wedge \mathrm{L})\ \frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad (\wedge \mathrm{R})\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$(\vee \mathrm{L})\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad (\vee \mathrm{R})\ \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B}$$

$$(\rightarrow \mathrm{L})\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Pi \Rightarrow \Sigma}{A \rightarrow B, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad (\rightarrow \mathrm{R})\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}$$

$$(\exists R)\ \frac{\Gamma \Rightarrow \Delta, A(t)}{\Gamma \Rightarrow \Delta, \exists x A(x)} \qquad (\exists L)\ \frac{A(\mathbf{a}), \Gamma \Rightarrow \Delta}{\exists x A(x), \Gamma \Rightarrow \Delta}$$

$$(\forall R)\ \frac{\Gamma \Rightarrow \Delta, A(\mathbf{a})}{\Gamma \Rightarrow \Delta, \forall x A(x)} \qquad (\forall L)\ \frac{A(t), \Gamma \Rightarrow \Delta}{\forall x A(x), \Gamma \Rightarrow \Delta}$$

Where **a** (referred to as an *eigenvariable*) is a free variable that does not occur freely in $\Gamma \cup \Delta \cup \forall x A(x)$, t represents any term, while Γ, Δ, and so forth, denote multisets of formulas.

Structural rules (weakening, contraction or cut) are, as seen above, do not involve any logical constant and instead operate directly on sequents. The term "Substructural logics" is due to Schröder-Heister and Doen, who write, in the introduction of their edited collection Substructural Logics: Our proposal is to call logics that can be obtained in this manner, by restricting structural rules, substructural logics.(Schroeder-Heister and Došen [1993], p.6). Although, substructural logics are most naturally formalized via sequent calculus, it is also possible to characterize them via axiomatic systems. In the next chapter, we will do so in the particular case of a contraction free logic.

Remark 1.2. *Definition 1.1 differs from Gentzen's original formulation in that it considers Γ, Δ, and so forth, as multi-sets of formulas rather than sequences. As a result, we exclude the structural rule of exchange, which allows for the permutation of formulas within the sequences.*

A priori one might assume the same rules for logical constants $(\neg, \rightarrow, \wedge, \vee)$ in the logic whose structural rules we restrict, but it turns out that a weaker structural context may make classically equivalent constants split into nonequivalent constants. To illustrate this point, let us consider the following alternative rules for disjunction and conjunction:

$$(\wedge L')\ \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad (\wedge R')\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Pi \Rightarrow \Sigma, B}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, A \wedge B}$$

$$(\vee L')\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Pi \Rightarrow \Sigma}{A \vee B, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad (\vee R')\ \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}$$

In the presence of the structural rules of weakening and contraction, these alternate rules of disjunction and conjunction become derivable in LK. Conversely, it is possible to derive the original rules of disjunction and conjunction (as in definition 1.1) within the calculus where the alternative rules serve as primitives. For instance, let's see the equivalence between $\wedge L$ and $\wedge L'$:

- $\wedge L$ is derivable given $\wedge L'$ and the rest of LK (without $\wedge L$ of course):

$$(\wedge L')\ \dfrac{(WL)\ \dfrac{A, \Gamma \Rightarrow \Delta}{A, B, \Gamma \Rightarrow \Delta}}{A \wedge B, \Gamma \Rightarrow \Delta}$$

- $\wedge L'$ is derivable given $\wedge L$ and the rest of LK:

$$(CL)\ \dfrac{(\wedge L)\ \dfrac{(\wedge L)\ \dfrac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, B, \Gamma \Rightarrow \Delta}}{A \wedge B, A \wedge B, \Gamma \Rightarrow \Delta}}{A \wedge B, \Gamma \Rightarrow \Delta}$$

In general, the inter-derivability can be summarised as follows:

$$\begin{aligned}
&\wedge L' \xrightarrow{\text{WL}} \wedge L \xrightarrow{\text{CL}} \wedge L' \\
&\wedge R' \xrightarrow{\text{CL+CR}} \wedge R \xrightarrow{\text{WL+WR}} \wedge R' \\
&\vee L' \xrightarrow{\text{CL+CR}} \vee L \xrightarrow{\text{WL+WR}} \vee L' \\
&\vee R' \xrightarrow{\text{WR}} \vee R \xrightarrow{\text{CR}} \vee R'
\end{aligned}$$

Hence, during the study of substructural logics one has to be sensitive about the choice of inference rules as well.

As can be observed, the proofs of equivalence between the logical constants rest essentially on the presence of weakening and contraction. Without them, previously interchangeable rules can define distinct connectives. We refer to the connectives defined using the former rules as *additive*, denoted by the symbols $\wedge$ and $\vee$. In the literature, they are also known as "extensional" and "lattice-theoretical". The connectives defined using the latter rules are termed *multiplicative*, and following Paoli [2002], we use the symbols $\otimes$ and $\oplus$ for conjunction and disjunction, respectively (it is worth noting that in Girard's Linear Logic [Girard, 1987], the multiplicative conjunction and disjunction are denoted by $\otimes$ and $\parr$, respectively). Moreover, the additive rules are occasionally characterized as *context-dependent*, as they can only be applied when the "contexts" (side-formulas) of the premises are identical. In contrast, rules of the latter type are described as context-free, as they do not need to abide by such a restriction.

Remark 1.3. *Other interesting behaviour is observed when different combinations of rules are considered. For instance, in the calculus* LK*, if we omit CL and CR, and specify that conjunction is defined by $\wedge L'$, $\wedge R$, and disjunction is defined by $\vee L$, $\vee R'$, we then regain both contraction rules (see Paoli [2002], p.13). Consequently, we are ultimately left with unique disjunctive and conjunctive connectives.*

Numerous substructural logics have been thoroughly investigated to date, with one of the most significant being Intuitionistic logic which, as identified by Gentzen, can be formalized as the sequent calculus LJ. The classical sequent calculus LK yields LJ with the structural restriction that there be at most one formula in the succedent. Other explored substructural logics include Relevant logics (which reject weakening), BCK logic (which rejects contraction), Linear logic (which rejects both weakening and contraction), and the Lambek calculus (which rejects not only weakening and contraction but also exchange).

1.2 Contraction and Paradoxes

In this section, we delve into the structural rule of contraction and its role in certain paradoxes. In fact, in a way, the need to address paradoxes prompted the study of contraction-less logics.

The structural rule of contraction is given by:

$$\boxed{(\mathrm{CL})\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad\qquad (\mathrm{CR})\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}}$$

It is captured by the following in the principle (see Proposition 2.19), called the *law of absorption*:

$$\boxed{(A \to (A \to B)) \to (A \to B)}$$

Let us see how the seemingly innocuous contraction rule crops up in self-referential paradoxes. First, we consider a truth theoretic paradox ,and then we examine how it comes up in Naive Set Theory as the ubiquitous Russell's paradox.

Liar Paradox

The Liar Paradox generally refers to the class of truth-theoretic paradoxes in which an argument is given to show how reasoning about a Liar sentence leads to contradiction. The classic example of such a liar sentence is:

L := "This sentence is not true"

The common informal argument goes as follows: if L is true, then what it asserts is true. However, if what L asserts is true, then L itself is false. On the other hand, if L is false, then what it asserts must be false, but that would mean L is true. Consequently, we end up with a contradiction.

Let us try to concentrate on the essential features of the argument and formalize it (albeit in a rather loose manner). To begin with, the Liar paradox requires a language containing a truth predicate satisfying Tarski's T-Schema.

- Truth Predicate: $Tr(\overline{A}) \leftrightarrow A$

  Then, we require that the compound predicate $\neg Tr$ has a *fixed-point*, which we call the Liar sentence:

- Liar Sentence: $L \vdash \neg Tr(\overline{L})$ and $\neg Tr(\overline{L}) \vdash L$

  Furthermore, we require some basic logical laws, which are given below. As we have already seen how contraction can be implicit in additive connectives (in particular, $\vee$L and $\wedge$R), we will formulate the laws in terms of multiplicative connectives.

- Logical Laws:

  1. Excluded Middle: $\vdash A \oplus \neg A$
  2. Explosion: $\neg A, A \vdash B$
  3. Multiplicative Disjunction principle($\oplus L$): If $A \vdash C$ and $B \vdash C$ then $A \oplus B \vdash C, C$
  4. Multiplicative Adjunction($\otimes R$): If $A \vdash B$ and $A \vdash C$ then $A, A \vdash B \otimes C$

Given a system that satisfies these basic logical and linguistic properties, let us see how contraction plays a role in generating the liar paradox. Consider the following proof tree:

$$\text{(CL)}\,\dfrac{(\otimes R)\,\dfrac{\text{(Cut)}\,\dfrac{Tr(\overline{L}) \Rightarrow L \qquad L \Rightarrow \neg Tr(\overline{L})}{Tr(\overline{L}) \Rightarrow \neg Tr(\overline{L})} \qquad Tr(\overline{L}) \Rightarrow Tr(\overline{L})}{Tr(\overline{L}), Tr(\overline{L}) \Rightarrow \neg Tr(\overline{L}) \otimes Tr(\overline{L})}}{Tr(\overline{L}) \Rightarrow \neg Tr(\overline{L}) \otimes Tr(\overline{L})}$$

Similarly, we can get

$$\neg Tr(\overline{L}) \Rightarrow \neg Tr(\overline{L}) \otimes Tr(\overline{L})$$

Using Disjunction principle,

$$Tr(\overline{L}) \oplus \neg Tr(\overline{L}) \Rightarrow \neg Tr(\overline{L}) \otimes Tr(\overline{L}), \neg Tr(\overline{L}) \otimes Tr(\overline{L})$$

Finally using LEM, cut and CR, we get

$$\Rightarrow \neg Tr(\overline{L}) \otimes Tr(\overline{L})$$

Hence, we see how contraction plays a crucial role in deriving the contradiction. For more details, we refer the reader to Heck [2012].

Russell's Paradox

Perhaps no other paradox has stimulated the development of logic as much as Russell's paradox. In essence, it states that in *Naive set theory*, one can define a set C as the set containing all sets that are not members of themselves, i.e., $C := \{x : x \notin x\}$. It can be concluded about C that $C \in C \Leftrightarrow C \notin C$, which would then lead to a contraction. Curry [1942] showed how a contradiction can be reached using only contraction and cut:

Consider an arbitrary theory containing the unrestricted comprehension schema. Let A be any sentence in the language of the theory. Let $C := \{x : (x \in x) \rightarrow A\}$. Then we have the following axiomatic proof.

1.$C \in C \rightarrow (C \in C \rightarrow A)$	Comprehension
2.$(C \in C \rightarrow A) \rightarrow C \in C$	Comprehension
3.$(C \in C \rightarrow (C \in C \rightarrow A)) \rightarrow (C \in C \rightarrow A)$	Law of absorption
4.$C \in C \rightarrow A$	1,3 Modus Ponens
5.$C \in C$	2,4 Modus Ponens
6.A	4,5 Modus Ponens

Hence, we see that since A was arbitrary, the presence of unrestricted comprehension, together with the laws of absorption and modus ponens, is sufficient to render the theory trivial. Subsequently, in Grišin [1974], Grishin demonstrated via cut-elimination that unrestricted comprehension can be consistently added to a contraction-free predicate logic, thus showing that contraction is necessary for the paradox.

The usual way out of Russell's paradox was to stick to the underlying Classical Logic [Mares and Paoli, 2014] and weaken the non-logical axioms of Naive set theory, i.e, the axiom of comprehension (every predicate has an extension) and the axiom of extensionality (sets with the same members are identical). Since the principles of extensionality and comprehension seemed to encode all there is to say about the notion of set, this was unsatisfying to some. Starting from late 1950s, Thoralf Skolem suggested the opposite way out, he proposed the adoption of ukasiewicz's infinite-valued logic [Skolem, 1963]. It is known that contraction does not hold unconditionally in Lukasiewicz's logics (Paoli [2002], chapters 2,4). However, a result by Grišin [1982] showed that on adding extensionality to plain contraction-free logic gives back contraction. Hence, the resulting system on adding comprehension becomes trivial. Thus, Lukasiewicz's infinite-valued logic, which is stronger than plain contraction-free logic, would be too strong to do naive set theory.

1.3 Contraction Free Logic

The first logician to study the contraction-less fragment of classical logic seems to be V.N. Grišhin [Grišin, 1974]. Grišhin formulated[1] the logic BCK by rejecting contraction from Gentzen's calculus LK. His reason to work with such a logic was to evade set-theoretical paradoxes while keeping unrestricted comprehension (see Schroeder-Heister and Došen [1993], p.15). In Grišin [1974], he showed via cut-elimination that unrestricted comprehension can be consistently added to his BCK predicate logic and also concluded that BCK predicate logic is decidable (Wang [1962], chapter IX, p. 228). In subsequent papers, Grishin studied algebraic models of BCK logic, an analogue of Herbrand's Theorem, and the effect of adding both unrestricted comprehension and extensionality to BCK logic [Grišin, 1976, 1979, 1982, 1983, 1985]. The effect is that we regain classical logic and the paradoxes. Later on, Ono and Komori studied BCK logics semantically, especially through Kripke-Style models [Ono and Komori, 1985]. In Ketonen and Weyhrauch [1984], Ketonen and Weyhrauch studied the purely multiplicative fragment, via sequent calculus, of BCK, which they called *Direct Predicate Logic.* They give a decision procedure for this fragment and also show that the process is exponentially bounded. In fact, the logic Definition 2.23 we work with is essentially this one, but we name the sequent formulation as $\mathsf{GQC}^{\mathsf{G}}$ and its corresponding Hilbert style formulation as $\mathsf{GQC}^{\mathsf{H}}$. Finally, Greg Restall, in his doctoral thesis, considered various systems of contraction-less logics[2] and did not only a proof theoretic but also model theoretic study of his logics [Restall, 1994]. He further considered various applications of his logics, such as on information flow, linguistics, and also on paradoxes.

In literature, logics which reject contraction are also known as **Affine** logics. Affine logics are closely associated with Linear logic, which differs from Affine logic primarily by the lack of weakening. The term "Affine" was introduced by Jean-Yves Girard, alluding to affine transformations in vector spaces.

[1] However, logicians had previously considered systems such as the purely implication fragment of BCK. For example, see Tarski [1956].

[2] Of the various systems Restall considered in his dissertation, the multiplicative fragment of the calculus **CK** is closest to the logic we consider. The difference being that the additive fragment of CK includes the distributivity of additive connectives.

1.4 Arithmetic in Contraction-free Logic

Martin Löb showed in Löb [1955] that any provability predicate satisfying the minimal Hilbert-Bernays condition cannot satisfy $\vdash Pr(\overline{\alpha}) \to \alpha$ unless α is a theorem of the arithmetic. The proof of this assertion relied crucially on an application of contraction. Greg Restall (see Restall [1992] and Restall [1994]) noticed that contraction free arithmetic could potentially be used to tackle this limitation in formal arithmetic. In Restall [1992], he considered ukasiewicz's infinite-valued logic, which neither has excluded middle nor Contraction, as a basis for arithmetic and showed that this arithmetic collapses into Peano Arithmetic. In Restall [1994](Chp 11), Restall studied various contraction free arithmetics and established some basic properties showing that a lot of arithmetic can be done even in the absence of contraction. He further provided an overview of how arithmetization of syntax can be executed and subsequently showed how Gödel's first theorem would go through in his arithmetics[3].

In [Beklemishev and Shamkanov, 2016], Lev Beklemishev and Daniyar Shamkanov studied the second incompleteness theorem in an abstract setting. Their goal was to carve out the basic conditions that a consequence relation would need to satisfy in order for the second incompleteness theorem to go through. Inspired by Grišhin's result on consistency of contraction-free logic augmented with the unrestricted comprehension principle, they wanted to see if contraction is a requirement for the second incompleteness theorem. However, Shamkanov figured out that a weaker form of contraction suffices for the second incompleteness theorem, in particular, left contraction for Σ_1 (note that the provability predicate is typically Σ_1) formulas suffice. This discovery led to the inception of contraction-free arithmetic, wherein left contraction for Σ_1 formulas would be upheld (refer to Proposition 3.43). However, the findings were not documented, and thus, serving as an impetus for my thesis. It is relevant to note another interesting finding in the same paper. They also demonstrated a propositional contraction free logical system which satisfies Löb's conditions for a $\Box$ operator and in which Gödelian fixed point exists (i.e., a sentence p such that $p \leftrightarrow \Box\neg p$), but the second incompleteness theorem fails , i.e, $\Rightarrow \neg\Box\bot$ is provable.

1.5 Overview of this thesis

Inspired by the discussions above, we were motivated to investigate the consequence it would have on Arithmetic when we work with an underlying logic which is contraction-free. In such a setting then, we would be able to precisely understand the effects of removing contraction. We consider the *purely multiplicative fragment* of the calculus which results when we remove contraction from Gentzen calculus LK. We base our arithmetic, which we name Contraction-Free

[3] We note that contraction free arithmetics being a subsystem of Peano Arithmetic would inevitably have undecidable sentences.

Arithmetic, on this logic. The reason for choosing such an underlying logic is twofold. Firstly and most importantly, as we shall demonstrate in section 3.2, the additive fragment becomes definable is our arithmetic. Secondly, from our discussion in section 1.1 and 1.2, it can be understood that contraction is, in a way, implicit in the additive connectives and hence, working with the multiplicative fragment will give us a clearer idea as to the number of times a formula has been used in the premise of a deduction. Our objective is to investigate the deductive strength of our arithmetic by studying its computational content. In particular, we aim to classify the class of provably recursive functions in this arithmetic.

Chapter 2 introduces our contraction-free logic in both sequent calculus and Hilbert Style settings. After discussing some basic properties of this logic, we proceed to show that the cut-rule is eliminable in our system. Then, we discuss some consequences of cut-elimination and demonstrate a weak version of Herbrand's Theorem. Finally, we discuss possible notions of deduction from hypotheses and prove a deduction theorem for them.

Chapter 3 introduces contraction free arithmetic. We demonstrate several fundamental properties of our arithmetic and proceed to showcase the definability of additive connectives in our language. In the last section, we provide a justification for our selection of the Induction Rule over the Induction Schema in our axiomatization.The central result of this chapter is the derivability of Δ_0-induction schema in CFA. As a consequence of that, we establish that any Π_2 formula which is provable in I_0 is also provable in CFA under a suitable formula translation. Moreover, further consequences of Δ_0-induction schema are discussed.

Chapter 4 is dedicated to the study of the computational strength of CFA through provably recursive functions. We start by showing that the primitive recursive functions are provably recursive in CFA. In the last section, which may be considered the key achievement of this thesis, we demonstrate that the class of provably recursive functions coincides with the class of primitive recursive functions. We achieve our result by demonstrating the admissibility of the cut rule in an infinitary calculus capable of embedding CFA.

Chapter 5 delves into the conclusions that can be drawn from our work and it also provides a list of interesting avenues for future research.

Appendix A mainly consists of some relevant definition which were not included in the text and also mentions some well-known theorems.

Appendix B mainly consists of some proofs which were left out from the text to maintain a smoother flow.

2 Contraction Free Logic

In this chapter, we present an introduction to Contraction Free Logic, which we name Grišhin Logic in honor of the Russian logician V.N. Grišhin. To start with we introduce our propositional calculus, followed by the introduction of our predicate calculus. After demonstrating some elementary properties of our logic, we proceed to prove the cut-elimination theorem for first-order Grišhin Logic. In the concluding section, we define two consequence relations in our logic to capture deduction from hypotheses and prove some of their properties.

2.1 Grišhin Calculus

In this section, we initiate our basic proof-theoretic study of contraction free logic. We study the purely multiplicative fragment of first-order predicate logic in which we reject contraction. As a consequence, intuitively it entails that in our logic, although the sequent $A, A \to B \Rightarrow B$ will be provable, the sequent $A, (A \to B) \otimes (A \to C) \Rightarrow (B \otimes C)$ will not be provable. This is because the antecedent A needs to be used at most once in a proof of the former sequent but necessarily twice in a proof of the latter sequent.

2.1.1 Propositional Calculus

In this subsection, which is is heavily inspired by Paoli [2002], we begin by presenting our propositional logic through sequent calculus. After introducing some preliminary notions, we go on to show which classical theorems are no longer valid to get a better feel of our system. Following this, we present our axiomatic Hilbert calculus and conclude the subsection by showing its equivalence with the other formalization.

We abbreviate Grišhin Propositional Logic as GPC. Let us start by introducing our language.

Definition 2.1. ($\mathfrak{L}_{\mathsf{GPC}}$) *The propositional language of* GPC *consists of a denumerable set of propositional variables and a given number of connectives drawn from the set* $\mathfrak{L}_{\mathsf{GPC}} = \{\to, \bot\}$. *As usual we shall use* $p, q, \ldots$ *as metavariables* *for propositional variables. Formulas are constructed as usual and we will use $A, B, \ldots$ metavariables for generic formulas.*

Multisets are aggregates where the ordering of elements do not matter (unlike for sequences) but their multiplicity is taken into account (unlike for sets). Note that multisets can be rigorously defined (See, e.g., Troelstra [1992], p.2).

Definition 2.2. *(**Sequents**) The basic expressions of the calculus are inferences of the form $\Gamma \Rightarrow \Delta$ (read: Δ "is derivable from" Γ). Where Γ and Δ are finite, possibly empty, multisets of formulas of our language.*

We now present our basic sequent calculus, which we name $\mathsf{GPC}^{\mathsf{G}}$. Here, the superscript G stands for Gentzen, after Gerhard Gentzen who first introduced the formal system of sequent calculus in Gentzen [1935].

Definition 2.3. ($\mathsf{GPC}^{\mathsf{G}}$) *The sequent calculus* GPC*, defined over* $\mathfrak{L}_{\mathsf{GPC}}$*, has the following postulates in sequent calculus.*

Initial Sequents

$$A \Rightarrow A$$

$$\bot \Rightarrow$$

Structural Rules

$$(\mathrm{Cut})\ \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

$$(\mathrm{WL})\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad\qquad (\mathrm{WR})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$$

Inference Rules

$$(\rightarrow\mathrm{L})\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Pi \Rightarrow \Sigma}{A \rightarrow B, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad\qquad (\rightarrow\mathrm{R})\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}$$

Remark 2.4. *By taking Γ, Δ in a sequent to be multisets instead of sequences we ensure that* $\mathsf{GPC}^{\mathsf{G}}$ *covertly contains the structural rule of exchange. Hence, we are allowed to perform arbitrary permutations both in the antecedent and in the succedent of a sequent.*

Definition 2.5. *(Principal, auxiliary and side formulas) In all these rules, the formula occurrences in $\Gamma, \Pi, \Delta, \Sigma$ are called side formulas; the formula occurrence in the conclusion which is not a side formula is called principal, and the formula occurrences in the premises which are not side formulas are called auxiliary.*

We now define formally, in the standard way, the central object of study in proof theory : proofs.

Definition 2.6. ***(Proof)*** *A proof in* $\mathsf{GPC}^{\mathsf{G}}$ *is a finite labelled tree whose nodes are labelled by sequents, in such a way that leaves are labelled by axioms and each sequent at a node is obtained from sequents at immediate predecessor(s) node(s) according to one of the rules of* $\mathsf{GPC}^{\mathsf{G}}$*. We shall denote proofs by means of metavariables* $D, D', \ldots$*. If* D *is a proof, a sub-tree* D' *of* D *which is itself a proof is called a sub-proof of* D*.*
A sequent S *is provable in* $\mathsf{GPC}^{\mathsf{G}}$ *(which is denoted as* $\vdash_{\mathsf{GPC}^{\mathsf{G}}} S$*) iff it labels the root of some proof in* $\mathsf{GPC}^{\mathsf{G}}$ *(i.e., iff it is the end-sequent of such a proof).*

We will often use lower case Greek alphabets, in particular π, to denote a formal proof. Informally, we might also sometimes refer to a proof of a sequent as "derivation" of a sequent. Next, we define the notion[1] of "Height of a proof", which serves to be a crucial anchor point when we reason about formal proofs.

Definition 2.7. *(Height of proof) The height* $h(\pi)$ *of a proof* π *is defined inductively as follows:*

1. *Base Clause: If* π *consists of only one node, i.e, a node labelled by an initial sequent then* $h(\pi) = 1$*.*

2. *Inductive Clause: If the last inference in* π *is a single premise rule, then* $h(\pi) = h(\pi') + 1$ *where* π' *is the sub-proof obtained from* π *by removing the root. Else if the last inference in* π *is a double premise rule, then* $h(\pi) = max(h(\pi'), l(\pi'')) + 1$ *where* π' *and* π'' *are the sub-proofs obtained from* π *by removing the root.*

We introduce the following well-known abbreviations in our language.

Definition 2.8. *In the sequel, we frequently employ the following abbreviations:*

- $\neg A$ *abbreviates* $A \to \bot$
- $\top$ *abbreviates* $\neg\bot$
- $A \oplus B$ *abbreviates* $\neg A \to B$
- $A \otimes B$ *abbreviates* $\neg(\neg A \oplus \neg B)$
- $\Gamma \Leftrightarrow \Delta$ *as an abbreviation for* $\Gamma \Rightarrow \Delta$ *and* $\Delta \Rightarrow \Gamma$*.*

The following proposition justifies our starting with a minimal set of logical constants (i.e, $\{\to, \bot\}$) and introducing other ones as abbreviations.

[1] In literature, they are also know as depth or length of a proof.

Proposition 2.9. *The following rules are derivable in* $\mathsf{GPC}^{\mathsf{G}}$.

$$(\top R)\ \Gamma \Rightarrow \Delta, \top$$

$$(\neg\mathrm{L})\ \frac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta} \qquad (\neg\mathrm{R})\ \frac{A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg A}$$

$$(\oplus\mathrm{L})\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Pi \Rightarrow \Sigma}{A \oplus B, \Gamma, \Pi \Rightarrow \Delta, \Sigma} \qquad (\oplus\mathrm{R})\ \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \oplus B}$$

$$(\otimes\mathrm{L})\ \frac{A, B, \Gamma \Rightarrow \Delta}{A \otimes B, \Gamma \Rightarrow \Delta} \qquad (\otimes\mathrm{R})\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Pi \Rightarrow \Sigma, B}{\Gamma, \Pi \Rightarrow \Delta, \Sigma, A \otimes B}$$

We now present some useful theorems of $\mathsf{GPC}^{\mathsf{G}}$, without proofs as they are essentially the same as those in LK.

Proposition 2.10. *The following sequents are provable in* $\mathsf{GPC}^{\mathsf{G}}$.

(i) $\Rightarrow A \to A$;

(ii) $A \to (B \to C) \Rightarrow B \to (A \to C)$;

(iii) $A \to B, B \to C \Rightarrow A \to C$;

(iv) $A \otimes (B \otimes C) \Leftrightarrow (A \otimes B) \otimes C$;

(v) $A \oplus (B \oplus C) \Leftrightarrow (A \oplus B) \oplus C$;

(vi) $A \oplus B \Leftrightarrow B \oplus A$;

(vi)′ $A \otimes B \Leftrightarrow B \otimes A$;

(vii) $A, B \Rightarrow A \otimes B$;

(viii) $A \to (B \to C) \Leftrightarrow (A \otimes B) \to C$;

(ix) $A \Leftrightarrow \neg\neg A$;

(x) $A \to \neg B \Rightarrow B \to \neg A$;

(xi) $A \to B \Leftrightarrow \neg A \oplus B$;

(xiv) $\neg\top \Leftrightarrow \bot$;

(xv) $A \otimes \top \Leftrightarrow A$

(xvi) $A \oplus \bot \Leftrightarrow A$.

The next few theorems are provable only because of the presence of weakening:

$(xvii)$ $A \Rightarrow B \to A$

$(xviii)$ $A \Rightarrow \neg A \to B$

(xix) $A \otimes A \Rightarrow A$ *and* $A \Rightarrow A \oplus A$

Definition 2.11. *(Formula-translation of a sequent). If* $A_1, .., A_n \Rightarrow B_1, .., B_m$ *is a sequent, its formula translation is defined as follows:*

$$t(A_1, .., A_n \Rightarrow B_1, .., B_m) := \begin{cases} A_1 \otimes \ldots \otimes A_n \to B_1 \oplus \ldots B_m & (n, m > 0) \\ \neg(A_1 \otimes \ldots \otimes A_n) & (n > 0, m = 0) \\ B_1 \oplus \ldots B_m & (n = 0, m > 0) \\ \bot & (n = 0, m = 0) \end{cases}$$

The above definition will come into play when we show the equivalence (see Proposition 2.17) between $\mathsf{GPC}^{\mathsf{G}}$ and its corresponding Hilbert style system. Now, we present the following proposition (without proof), which sheds light on the rationale behind defining formula translation as we did.

Proposition 2.12.

$$\vdash_{\mathsf{GPC}^{\mathsf{G}}} A_1, .., A_n \Rightarrow B_1, .., B_m \textit{ iff } \vdash_{\mathsf{GPC}^{\mathsf{G}}} \;\Rightarrow t(A_1, .., A_n \Rightarrow B_1, .., B_m)$$

To get a better understanding of the limitations of $\mathsf{GPC}^{\mathsf{G}}$ here, we present[2] some formulas which are theorems of classical propositional logic but fail in $\mathsf{GPC}^{\mathsf{G}}$.

Proposition 2.13. *(**Non-Theorems** of* $\mathsf{GPC}^{\mathsf{G}}$*) The following sequents, which are classically provable (i.e, in* LK*), are not provable in* $\mathsf{GPC}^{\mathsf{G}}$*:*

1. $p \to (p \to r) \Rightarrow p \to r$
2. $p \to (q \to r) \Rightarrow (p \to q) \to (p \to r)$
3. $\Rightarrow ((p \to r) \to p) \to p$
4. $\neg p \to p \Rightarrow p$

Proof. Consider the following truth table:

$\to$	0	$1/2$	1
0	1	1	1
$1/2$	$1/2$	1	1
1	0	$1/2$	1

[2] The proposition is based on Proposition 2.13 of Paoli [2002].

Given that $\bot$ is always evaluated to 0. By inducting on the height of the proof, one can demonstrate that if $\Gamma \Rightarrow \Delta$ is provable in $\mathsf{GPC}^{\mathsf{G}}$, then its formula translation $t(\Gamma \Rightarrow \Delta)$ evaluates to 1 (using the provided truth table), under any assignment of values $0, 1/2, 1$ to variables occurring in the sequent.

Now for the proposition:
In all parts : Assign p to $1/2$, $q$to $1/2$ and r to 0. In all these cases $t(\Gamma \Rightarrow \Delta) = 1/2$. Hence, we can conclude that they are not theorems of $\mathsf{GPC}^{\mathsf{G}}$.

□

We now present the corresponding Hilbert style calculus for $\mathsf{GPC}^{\mathsf{G}}$. Subsequently, we show (Theorem 2.17) that the sequent calculus and the Hilbert style calculus are equivalent.

Definition 2.14. ($\mathsf{GPC}^{\mathsf{H}}$) *The axiomatic calculus*[3] $\mathsf{GPC}^{\mathsf{H}}$*, defined over* $\mathfrak{L}_{\mathsf{GPC}}$*, has the following postulates.*

Axioms

(F1) $A \to A$

(F2) $(A \to B) \to ((B \to C) \to (A \to C))$

(F3) $(A \to (B \to C)) \to (B \to (A \to C))$

(F4) $((A \to \bot) \to \bot) \to A$

(F5) $(A \to (B \to \bot)) \to (B \to (A \to \bot))$

(F6) $A \to (B \to A)$

Modus Ponens rule

$$(\text{MP})\ \frac{\text{A} \qquad A \to B}{\text{B}}$$

[3] The subscript 'H', stands for Hilbert

Definition 2.15. *(Proof in $\mathsf{GPC^H}$) A proof in $\mathsf{GPC^H}$ is a sequence $B_1, \dots, B_n$ of well-formed formulas (wffs) such that, for each i, B_i is either an axiom of $\mathsf{GPC^H}$ or a direct consequence of some of the preceding wffs in the sequence by virtue of modus ponens. A formula B is provable in $\mathsf{GPC^H}$ (denoted as $\vdash_{\mathsf{GPC^H}} B$) if there exists a proof $B_1, \dots, B_n$ where $B = B_n$. Moreover, we refer to "n" as the length of proof of B.*

In $\mathsf{GPC^H}$, as well, we introduce the same abbreviations as in Definition 2.8. In addition, we use $A \leftrightarrow B$ as an abbreviation for $(A \to B) \otimes (B \to A)$.

Proposition 2.16. *The following formulas are provable in $\mathsf{GPC^H}$.*

(F7) $(A \to B) \leftrightarrow (\neg B \to \neg A)$

(F8) $A \otimes B \leftrightarrow \neg(\neg A \oplus \neg B)$

(F9) $A \oplus B \leftrightarrow (\neg A \to B)$

(F10) $A \oplus B \leftrightarrow B \oplus A$

We are now in a position to demonstrate that $\mathsf{GPC^G}$ and $\mathsf{GPC^H}$, despite their entirely distinct presentations, are indeed equivalent in a specific sense.

Proposition 2.17. *(Equivalence)*

$$\vdash_{\mathsf{GPC^G}} \Gamma \Rightarrow \Delta \text{ \textit{iff} } \vdash_{\mathsf{GPC^H}} t(\Gamma \Rightarrow \Delta)$$

Proof.
Right to Left direction.
It is sufficient to show that if $\vdash_{\mathsf{GPC^H}} A$ then $\vdash_{\mathsf{GPC^G}} \Rightarrow A$. As then, if $\vdash_{\mathsf{GPC^H}} t(\Gamma \Rightarrow \Delta)$, we will also have $\vdash_{\mathsf{GPC^G}} \Rightarrow t(\Gamma \Rightarrow \Delta)$. Subsequently, by Proposition 2.12, we will have that $\vdash_{\mathsf{GPC^G}} \Gamma \Rightarrow \Delta$.

In order to show that sufficient condition holds good we proceed by induction on the length of the proof of A in $\mathsf{GPC^H}$.
First, we check for any axiom B of $\mathsf{GPC^H}$, that $\vdash_{\mathsf{GPC^G}} \Rightarrow A$. But Proposition 2.10, gives us what is required.
Then, we need to ensure that the rules of $\mathsf{GPC^H}$ preserve such a property. I.e., we show in $\mathsf{GPC^G}$, that if $\Rightarrow A$ and $\Rightarrow A \to B$ then $\Rightarrow B$.

Consider the following proof tree:

$$\text{(Cut)}\;\dfrac{\text{(Cut)}\;\dfrac{(\to\text{L})\;\dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A \to B, A \Rightarrow B} \qquad \begin{matrix}\vdots \\ \Rightarrow A \to B\end{matrix}}{A \Rightarrow B} \qquad \begin{matrix}\vdots \\ \Rightarrow A\end{matrix}}{\Rightarrow B}$$

Left to Right direction.

We argue by induction on the structure of the proof of $\Gamma \Rightarrow \Delta$ in $\mathsf{GPC}^{\mathsf{G}}$.

Base Case. $\Gamma \Rightarrow \Delta$ is an initial sequent, i.e, $A \Rightarrow A$ or $\bot \Rightarrow$. In that case $t(A \Rightarrow A) = A \to A$ and $t(\bot \Rightarrow) = \bot \to \bot$, which are nothing but instances of axiom F1 . Hence, $\vdash_{\mathsf{GPC}^{\mathsf{H}}} t(\Gamma \Rightarrow \Delta)$
Inductive Case. Suppose $\Gamma \Rightarrow \Delta$ has been derived by applying a rule of $\mathsf{GPC}^{\mathsf{G}}$. We need to approach the problem by dividing it into cases based on the rule that was used to infer $\Gamma \Rightarrow \Delta$. We will demonstrate the case of weakening (WL and WR). The rest can be handled similarly.
Say $\Gamma = A_1, \dots, A_n$ and $\Delta = B_1, \dots B_m$. Let $\gamma := A_1 \otimes .. \otimes A_n$ and $\delta := B_1 \oplus .. \oplus B_m$. Then, observe that $t(\Gamma \Rightarrow \Delta) = \gamma \to \delta$.

- *WL*. Suppose $\vdash_{\mathsf{GPC}^{\mathsf{H}}} t(\Gamma \Rightarrow \Delta)$, i.e., $\vdash_{\mathsf{GPC}^{\mathsf{H}}} \gamma \to \delta$. Then we have to show that $\vdash_{\mathsf{GPC}^{\mathsf{H}}} t(A, \Gamma \Rightarrow \Delta)$ or, $\vdash_{\mathsf{GPC}^{\mathsf{H}}} A \otimes \gamma \to \delta$.

  Proof:

  1. $\gamma \to \delta$
  2. $\neg\gamma \to (A \to \neg\gamma)$ (**F6**)
  3. $(A \to \neg\gamma) \to (\neg A \oplus \neg\gamma)$ (Abbr)
  4. $\neg\gamma \to (\neg A \oplus \neg\gamma)$ (2,3, F2 MP)
  5. $\neg(\neg A \oplus \neg\gamma) \to \neg\neg\gamma$ (4,F7 MP)
  6. $A \otimes \gamma \to \gamma$ (F4,F8,F2 MP)
  7. $A \otimes \gamma \to \delta$ (1,6, F2 MP)

- *WR* : Say $\vdash_{\mathsf{GPC}^{\mathsf{H}}} t(\Gamma \Rightarrow \Delta)$, i.e., $\vdash_{\mathsf{GPC}^{\mathsf{H}}} \gamma \to \delta$. Then we have to show that $\vdash_{\mathsf{GPC}^{\mathsf{H}}} t(\Gamma \Rightarrow \Delta, A)$ or, $\vdash_{\mathsf{GPC}^{\mathsf{H}}} \gamma \to \delta \oplus A$.

  Proof:

  1. $\gamma \to \delta$
  2. $\delta \to (\neg A \to \delta)$ (**F6**)
  3. $(\neg A \to \delta) \to \delta \oplus A$ (F9, F10, F2 MP)
  4. $\delta \to \delta \oplus A$ (2,3, F2 MP)

5. $\gamma \to \delta \oplus A$ (1,4, F2 MP)

Then, by induction on the structure of proof of $\Gamma \Rightarrow \Delta$, we will have that $\mathsf{GPC}^{\mathsf{H}} \vdash t(\Gamma \Rightarrow \Delta)$

□

Remark 2.18. *(Weakening) In the inductive cases of WR and WL in the preceding proof, observe how the axiom (F6) played a crucial role to capture the weakening rule. In fact, (F6) is the axiom that exactly corresponds to the weakening rule of sequent calculus. To be precise, we have that* $\mathsf{GPC}^{\mathsf{G}} - WR - WL$ *is equivalent to* $\mathsf{GPC}^{\mathsf{H}} - (A \to (A \to B)) \to (A \to B)$. *Henceforth, we will occasionally refer to* $A \to (B \to A)$ *as 'weakening'.*

As noted in Section [1.2], similar to weakening, we have a formula that corresponds to the contraction rule in sequent calculus. This formula is $(A \to (A \to B)) \to (A \to B)$, and it is sometimes referred to as the Law of Absorption (see Paoli [2002]). The following proposition will make things more precise.

Proposition 2.19. *(Contraction)*

$$\mathsf{GPC}^{\mathsf{G}} + CR + CL \textit{ is equivalent to } \mathsf{GPC}^{\mathsf{H}} + (A \to (A \to B)) \to (A \to B)$$

2.1.2 Predicate Calculus

In this subsection, we introduce Grišhin Quantified Calculus[4] (GQC), as the first-order extension of GPC. Following this, we will present some important theorems of GQC that will be required later in our exploration. We begin by providing the language for GQC.

Definition 2.20. *The language* $\mathfrak{L}_{\mathsf{GQC}}$ *of first-order Grišhin logic consists of* $\mathfrak{L}_{\mathsf{GPC}} \cup \{\forall\}$ *together with denumerable supply of variables, n-place predicate symbols for all $n \in \mathbb{N}$, symbols for n-ary functions for all $n \in \mathbb{N}$ and symbols for* 0*-ary functions (also called constants).*

We will now introduce several standard concepts essential for constructing our formal system.

Definition 2.21. *(Terms and formulas)*
The set of ***terms*** *are defined inductively as follows:*

- *Any variable is a term.*
- *Any constant symbol is a term.*
- *If $t_1, \ldots, t_n$ are terms then $f(t_1, \ldots, t_n)$ is a term, where f is an n-ary function symbol.*

[4] We note that (GQC) is essentially the same as Direct Predicate Calculus introduced by Ketonen and Weyhrauch [1984].

- *Nothing else is a term.*

*The set of **formulas** are defined inductively as follows:*

- *Atomic Formulas:* $\bot$ *is a formula ; If* $t_1, \ldots, t_n$ *are terms, then* $P(t_1, \ldots, t_n)$ *is a formula where* P *is an n-ary predicate symbol.*
- *If* A *and* B *are formulas, then* $A \to B$ *is a formula.*
- *If* A *is a formula and* x *is a variable, then* $\forall x A(x)$ *is also a formula.*
- *Nothing else is a formula.*

If a formula has no occurrences of $\forall x$, for any variable x, then it is called **quantifier-free**.

Definition 2.22. *(Free and bound variables) We define inductively, for each formula, what it means for* x *to occur **free** in* A.

- *For atomic* A, x *occurs free in* A *iff* x *occurs in* A.
- x *occurs free in* $A \to B$ *iff* x *occurs free in* A *or in* B.
- x *occurs free in* $\forall z A$ *if* x *occurs free in* A *and* $z \neq x$.

If an occurrence of x *is not free in* A*, then that occurrence of* x *is said to be **bound**.*

We use the following *metavariables* for certain syntactic categories (though locally different conventions might be introduced) : $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ for bound variables; $\mathbf{a}, \mathbf{b}, \ldots$ for free variables; $\mathbf{f}, \mathbf{g}, \mathbf{h}$ for arbitrary function symbols; $\mathbf{c}, \mathbf{d}$ for constants; $\mathbf{t}, \mathbf{s}, \mathbf{r}$ for terms.

Now we are in a position to introduce our predicate calculus.

Definition 2.23. ($\mathsf{GQC}^{\mathsf{G}}$) *The first-order Grišhin logic, denoted as* $\mathsf{GQC}^{\mathsf{G}}$ *and axiomatized in sequent calculus, is defined as an extension of the propositional system* $\mathsf{GPC}^{\mathsf{G}}$*. That is,* $\mathsf{GQC}^{\mathsf{G}}$ *contains all inference and structural rules of* $\mathsf{GPC}^{\mathsf{G}}$ *together with:*

Initial Sequents

$$A \Rightarrow A$$

where A is atomic

$$\bot \Rightarrow$$

Inference Rules

$$(\forall R)\ \dfrac{\Gamma \Rightarrow \Delta, A(\mathbf{a})}{\Gamma \Rightarrow \Delta, \forall x A(x)} \qquad\qquad (\forall L)\ \dfrac{A(t), \Gamma \Rightarrow \Delta}{\forall x A(x), \Gamma \Rightarrow \Delta}$$

Where **a** (called *eigenvariable*) is a free variable and does not occur freely in $\Gamma \cup \Delta \cup \{\forall x A(x)\}$, and t is any term.

The concepts of Auxiliary, Side, Principal formulas, Proof, Proof Height are adapted from Definitions 2.5, 2.6 and 2.7.

Proposition 2.24. $(\mathsf{GQC}^{\mathsf{G}})$ *Given any formula A, $\vdash_{\mathsf{GQC}^{\mathsf{G}}} A \Rightarrow A$.*

Proof. It is proved by induction on the structure of A. □

Contrary to standard practice, we constructed our logical system using only universal quantifiers. This choice simplifies the process of inductive reasoning on the structure of formal proofs, as it reduces the number of cases we need to consider. Furthermore, our selection does not significantly impact our study, as evidenced by the following proposition.

Definition 2.25. *(Abbreviation) $\exists x A(x)$ is the abbreviation for $\neg\forall x \neg A(x)$*

Proposition 2.26. $(\mathsf{GQC}^{\mathsf{G}})$ *The following rules can be derived:*

$$(\exists R)\ \dfrac{\Gamma \Rightarrow \Delta, A(t)}{\Gamma \Rightarrow \Delta, \exists x A(x)} \qquad\qquad (\exists L)\ \dfrac{A(\mathbf{a}), \Gamma \Rightarrow \Delta}{\exists x A(x), \Gamma \Rightarrow \Delta}$$

Where **a** (called *eigenvariable*) is a free variable and does not occur freely in $\Gamma \cup \Delta \cup \{\forall x A(x)\}$. And t is any term.

Remark 2.27. *Though we might be a bit informal sometimes, formally, the height of a proof depends only on $\rightarrow$ and $\forall$ rules, and not on the derived rules.*

We now introduce the corresponding Hilbert-style calculus.

Definition 2.28. ($\mathsf{GQC}^{\mathsf{H}}$) *The first order Grišhin logic, denoted as* $\mathsf{GQC}^{\mathsf{H}}$ *and axiomatized in Hilbert system, is defined as an extension of the* $\mathsf{GPC}^{\mathsf{H}}$ *together with :*

Axioms

(P1) $\forall x B(x) \to B(t/x)$
where B admits t for x (i.e., no free variable of t becomes bound in $B(t/x)$).

(P2) $\forall x(B \to C) \to (B \to \forall x C)$
where x is not free in B.

Rule

$$\text{(Generalization)}\ \frac{\text{B}}{\forall x B}$$

Exactly like in the case of propositional logic, the sequent and Hilbert style formulation of Grišhin logic is equivalent.

Proposition 2.29. *(Equivalence)*

$$\vdash_{\mathsf{GQC}^{\mathsf{G}}} \Gamma \Rightarrow \Delta \textit{ iff } \vdash_{\mathsf{GQC}^{\mathsf{H}}} t(\Gamma \Rightarrow \Delta)$$

Proof. Proof is analogous to Proposition 2.17. □

In light of the discussion in Chapter 1 regarding the collapse of multiplicative and additive connectives to a unique connective in the presence of contraction-without worrying too much about the languagewe, roughly, observe the following relation:

$$\boxed{\mathsf{LK} \equiv \mathsf{GQC}^{\mathsf{G}} + CR + CL}$$

Next we give a sufficient and necessary condition for contraction to hold for a formula.

Proposition 2.30. $\vdash_{\mathsf{GQC}^{\mathsf{G}}} A \Rightarrow A \otimes A$ *and* $\vdash_{\mathsf{GQC}^{\mathsf{G}}} A \oplus A \Rightarrow A$ *iff left and right contraction holds for A.*

Proof.
Right to Left direction. This is easy to see. We focus on the other direction.
Left to Right direction. The following proof trees shows how we can derive the rule of contraction for A.
Left Contraction:

$$(\text{Cut})\ \dfrac{A \Rightarrow A \otimes A \qquad \dfrac{A, A, \Gamma \Rightarrow \Delta}{A \otimes A, \Gamma \Rightarrow \Delta}}{A, \Gamma \Rightarrow \Delta}$$

Right Contraction:

$$(\text{Cut})\ \dfrac{\dfrac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A \oplus A} \qquad A \oplus A \Rightarrow A}{\Gamma \Rightarrow \Delta, A}$$

□

We conclude our brief introduction to Grišhin logic by presenting some theorems of $\mathsf{GQC}^{\mathsf{G}}$ that will prove useful for our subsequent study.

Proposition 2.31. ($\mathsf{GQC}^{\mathsf{G}}$) *Predicate theorems of Grišhin logic:*

1. $\vdash_{\mathsf{GQC}^{\mathsf{G}}} \exists x(B(x) \otimes C(x)) \Rightarrow \exists x B(x) \otimes \exists x C(x)$
2. $\vdash_{\mathsf{GQC}^{\mathsf{G}}} \forall x A(x) \otimes \forall x B(x) \Rightarrow \forall x(A(x) \otimes B(x))$
3. $\vdash_{\mathsf{GQC}^{\mathsf{G}}} \forall x(P \to Q) \Rightarrow \exists x P \to \exists x Q$
4. $\vdash_{\mathsf{GQC}^{\mathsf{G}}} \exists x(P \to Q) \Rightarrow \forall x P \to \exists x Q$
5. $\vdash_{\mathsf{GQC}^{\mathsf{G}}} \forall x(A(x) \oplus B(x)) \Rightarrow \exists x A(x) \oplus \forall x B(x)$
6. $\vdash_{\mathsf{GQC}^{\mathsf{G}}} \forall v P \otimes \exists v Q \Rightarrow \exists x(P(x) \otimes Q(x))$
7. $\vdash_{\mathsf{GQC}^{\mathsf{G}}} \forall x \exists y A(x,y) \otimes \forall x \exists z B(x,z) \Rightarrow \forall x \exists y \exists z(A(x,y) \otimes B(y,z))$

Proof. 1. Consider the following proof tree.

$$(\exists L)\ \dfrac{(\otimes L)\ \dfrac{(\otimes R)\ \dfrac{(\exists R)\ \dfrac{B(x) \Rightarrow B(x)}{B(x) \Rightarrow \exists x B(x)} \qquad (\exists R)\ \dfrac{C(x) \Rightarrow C(x)}{C(x) \Rightarrow \exists x C(x)}}{B(x), C(x) \Rightarrow \exists x B(x) \otimes \exists x C(x)}}{B(x) \otimes C(x) \Rightarrow \exists x B(x) \otimes \exists x C(x)}}{\exists x(B(x) \otimes C(x)) \Rightarrow \exists x B(x) \otimes \exists x C(x)}$$

2. Consider the following proof tree.

$$(\forall R)\,\dfrac{(\otimes R, \otimes L)\,\dfrac{(\forall L)\,\dfrac{A(x) \Rightarrow A(x)}{\forall x A(x) \Rightarrow A(x)} \qquad (\forall L)\,\dfrac{A(x) \Rightarrow A(x)}{\forall x B(x) \Rightarrow B(x)}}{\forall x A(x) \otimes \forall x B(x) \Rightarrow A(x) \otimes B(x)}}{\forall x A(x) \otimes \forall x B(x) \Rightarrow \forall x (A(x) \otimes B(x))}$$

3. Consider the following proof tree.

$$(\exists L)\,\dfrac{(\exists R)\,\dfrac{(\forall L)\,\dfrac{\begin{matrix}\vdots\\ P(x) \to Q(x), P(x) \Rightarrow Q(x)\end{matrix}}{\forall x (P(x) \to Q(x)), P(x) \Rightarrow Q(x)}}{\forall x (P(x) \to Q(x)), P(x) \Rightarrow \exists x Q(x)}}{\forall x (P(x) \to Q(x)), \exists x P(x) \Rightarrow \exists x Q(x)}$$

4. Consider the following proof tree.

$$(\exists L)\,\dfrac{(\forall L)\,\dfrac{(\exists R)\,\dfrac{\begin{matrix}\vdots\\ P(x) \to Q(x), P(x) \Rightarrow Q(x)\end{matrix}}{P(x) \to Q(x), P(x) \Rightarrow \exists x Q(x)}}{P(x) \to Q(x), \forall x P(x) \Rightarrow \exists x Q(x)}}{\exists x (P(x) \to Q(x)), \forall x P(x) \Rightarrow \exists x Q(x)}$$

5. Proposition 2.10 (xi) gives us $A \oplus B \Leftrightarrow \neg A \to B$. Now consider the following proof tree.

$$(cut)\,\dfrac{\begin{matrix}\vdots\\ \forall x (A(x) \oplus B(x)) \Rightarrow \forall x (\neg A(x) \to B(x))\end{matrix} \qquad \begin{matrix}\vdots\\ \forall x (\neg A(x) \to B(x)) \Rightarrow \forall x \neg A(x) \to \forall x B(x)\end{matrix}}{(abbr)\,\dfrac{(abbr)\,\dfrac{\forall x (A(x) \oplus B(x)) \Rightarrow \forall x \neg A(x) \to \forall x B(x)}{\forall x (A(x) \oplus B(x)) \Rightarrow \neg \forall x \neg A(x) \oplus \forall x B(x)}}{\forall x (A(x) \oplus B(x)) \Rightarrow \exists x A(x) \oplus \forall x B(x)}}$$

Moreover, as a corollary, we have that: $\vdash_{\mathsf{GQC^G}} \forall x B(x) \oplus C \Leftrightarrow \forall x (B(x) \oplus C)$ where x does not occur in C.

6. It can be easily established that $P \to Q \Leftrightarrow \neg Q \to \neg P$. Given this, it is easy to observe that $\forall v P \otimes \exists v Q \Rightarrow \exists x (P(x) \otimes Q(x))$ follows from part 5. Moreover, as a corollary, we have that: $\exists x B(x) \oplus C \Leftrightarrow \exists x (B(x) \oplus C)$

7. The proof of this is a bit long. Hence, we provide an outline of steps, from which a proof tree can be directly constructed using multiple applications of the cut rule. Each of the following sequents is provable in $\mathsf{GQC}^{\mathsf{G}}$:

   (a) $\forall x\exists yA(x,y) \otimes \forall x\exists zB(x,z) \Rightarrow \exists yA(x,y) \otimes \forall x\exists zB(x,z)$

   (b) $\exists yA(x,y) \otimes \forall x\exists zB(x,z) \Rightarrow \exists y(A(x,y) \otimes \forall x\exists zB(x,z))$

   (c) $\exists y(A(x,y) \otimes \forall x\exists zB(x,z)) \Rightarrow \exists y(A(x,y) \otimes \exists zB(y,z))$

   (d) $\exists y(A(x,y) \otimes \exists zB(y,z)) \Rightarrow \exists y\exists z(A(x,y) \otimes \exists zB(y,z))$

   (e) $\exists y\exists z(A(x,y) \otimes \exists zB(y,z)) \Rightarrow \forall x\exists y\exists z(A(x,y) \otimes \exists zB(y,z))$

   Therefore, using cut multiple times we can construct a proof tree whose root is labelled by $\forall x\exists yA(x,y)\otimes\forall x\exists zB(x,z) \Rightarrow \forall x\exists y\exists z(A(x,y)\otimes\exists zB(y,z))$.

□

2.2 Cut-Elimination in $\mathsf{GQC}^{\mathsf{G}}$

In this section, we establish that the cut-rule can be eliminated from $\mathsf{GQC}^{\mathsf{G}}$. Cut elimination for a sequent calculus states that every sequent which has a derivation also has a proof that does not make use of the cut rule. Intuitively, an essential property of a proof which does not use cut is that it is not "roundabout", i.e., no concepts enter into the proof beyond those necessary for reaching the final result (see Gentzen [1935]). As such proofs without cut possess the subformula property (See Proposition 2.44) and this insight was used by Gentzen to prove that LK is consistent[5] (i.e., it cannot derive contradictions). He also used the technique of cut-elimination to prove the consistency of PA[6]. Subsequently, the study of the cut rule has become one of the corner stones of proof theory. For a comprehensive introduction to cut-elimination we refer the reader to Troelstra and Schwichtenberg [2000].

Cut-elimination for contraction-free predicate logic, particularly in BCK logic, was initially established by Grišhin in Grišin [1974] (in Russian). Subsequently, others have examined cut-elimination in various contraction-free logics. Notably, Ketonen and Weyhrauch [1984] noted the validity of cut-elimination for our logic, en route to proving the logic's decidability. Indeed, in the absence of contraction, cut-elimination proves to be simpler. The following proof we present is our own and is adapted from Mancosu et al. [2021] and Paoli [2002].

We start by introducing some preliminary notions. To capture the number of connectives occurring of in formula, we define the complexity of a formula as follows.

[5] As a cut-free proof cannot derive the empty sequent, i.e., $\emptyset \Rightarrow \emptyset$.

[6] However, it's worth noting that cut cannot be eliminated from PA because of the presence of the Induction Schema. Nonetheless, Gentzen found an ingenious way to overcome this obstacle for his consistency proof.

Definition 2.32. *(Complexity of a formula) The complexity $\mathcal{C}(A)$ of a formula A is defined inductively as follows:*

- *Base Clause: If A is atomic or $A = \bot$ then $\mathcal{C}(A)$=0*
- *Inductive Clause:*
  - *If A is of the form $\forall x B(x)$ then $\mathcal{C}(A)$=1+$\mathcal{C}(B)$*
  - *If A is of the form $B \to D$ then $\mathcal{C}(A)$=$\mathcal{C}(B) + \mathcal{C}(D) + 1$*

Cut-elimination heavily relies on the fact that certain proofs can be transformed into proofs of the same end-sequent but in which a certain free variable is replaced by some other term. The following lemma on variable replacement will be useful towards this end.

Proposition 2.33. *(Variable Replacement) Suppose $\pi(a)$ is a proof, t is a term not containing any eigenvariables of $\pi(a)$, and **a** is a free variable that is not used as an eigenvariable of an inference in $\pi(a)$. Then $\pi(t)$, which results from $\pi(a)$ by replacing every occurrence of **a** by **t**, is a correct proof.*

Proof. We show this by induction on the height of the proof $\pi(a)$ with end-sequent $\Gamma \Rightarrow \Delta$.
Base Case. Height is 1, meaning that $\Gamma \Rightarrow \Delta$ is an initial sequent. Then it is either $A \Rightarrow A$ or $\bot \Rightarrow$. Moreover, since $A[a/t] \Rightarrow A[a/t]$ is also an initial sequent, we have that $\pi(t)$ is also a correct proof.
Inductive Case. Say $\pi(a)$ is a proof with proof height n(> 1). We analyse by breaking up into cases based on the last rule that has been applied. For instance, if the last rule applied is **WL**, then $\pi(a)$ takes the following form:

$$\text{(WL)}\ \frac{\begin{array}{c}\vdots\pi_1(a)\\ \Gamma'(a) \Rightarrow \Delta(a)\end{array}}{A(a), \Gamma'(a) \Rightarrow \Delta(a)}$$

where $\Gamma'(a) = A(a), \Gamma'(a)$.

Now, since the proof $\pi_1(a)$ of $\Gamma'(a) \Rightarrow \Delta(a)$ is of height $n - 1(< n)$ and by assumption, a is a free variable that is not used as an eigenvariable of any inference in $\pi_1(a)$ and t does not contain an eigenvariable of $\pi_1(a)$. So the induction hypothesis applies to $\pi_1(a)$. Hence, we have that $\pi_1(t)$ is a correct proof. Then the following proof, i.e, $\pi(t)$ is also correct.

$$\text{(WL)}\ \frac{\begin{array}{c}\vdots\pi_1(t)\\ \Gamma'(t) \Rightarrow \Delta(t)\end{array}}{A(t), \Gamma'(t) \Rightarrow \Delta(t)}$$

Cases other than $\forall R$ (which involve eigenvariable condition) can be worked out similarly. Here, we show how to deal with $\forall R$.
Now say the last rule that has been applied is $\forall R$. Then $\pi(a)$ has the following form:

$$\begin{array}{c} \vdots \\ (\forall R)\ \dfrac{\Gamma(a) \Rightarrow \Delta'(a), A(a,b)}{\Gamma(a) \Rightarrow \Delta'(a), \forall x A(a,x)} \end{array}$$

where $\Delta(a) = \Delta'(a), \forall x A(a,x)$

First, we note that the eigenvariable $\mathbf{b}$ is different from variable $\mathbf{a}$ as we assumed a not to be used as an eigenvariable in $\pi_1(a)$ (in fact, the necessity of the assumption is brought out in this step).
Say $\pi_1(a)$ is the proof of $\Gamma'(a) \Rightarrow \Delta(a), A(a,b)$. Since $\pi_1(a)$ is of height $n-1(<n)$, inductive hypothesis applies to it and thus $\pi_1(t)$ is a proof of $\Gamma'(t) \Rightarrow \Delta(t), A(t,b)$. Furthermore, as $\Gamma(a) \Rightarrow \Delta'(a)$ and the term t does not contain b, the eigenvariable condition is also satisfied and the following, i.e $\pi(t)$, proof is also correct

$$\begin{array}{c} \vdots \\ (\forall R)\ \dfrac{\Gamma(t) \Rightarrow \Delta'(t), A(t,b)}{\Gamma(t) \Rightarrow \Delta'(t), \forall x A(t,x)} \end{array}$$

Then, by strong induction on the height of the proof, we have our required result.

□

Corollary 2.34. *Suppose a proof π ends in $\forall R$ with eigenvariable $\boldsymbol{a}$, contains no other $\forall R$ inference with eigenvariable $\boldsymbol{a}$, and $\boldsymbol{b}$ is a variable not occurring in π. Then the result of replacing $\boldsymbol{a}$ by $\boldsymbol{b}$ throughout π, is a proof of the same end-sequent.*

Proof. We consider the case when π ends in $\forall R$. Then π has the following form:

$$\begin{array}{c} \pi_1(a)\vdots \\ (\forall R)\ \dfrac{\Gamma \Rightarrow \Delta, A(a)}{\Gamma \Rightarrow \Delta, \forall x A(x)} \end{array}$$

Call $\pi_1(a)$ the proof of the sequent $\Gamma \Rightarrow \Delta, A(a)$. As per our assumptions we have that $\mathbf{b}$ does not occur in $\pi_1(a)$ and $\mathbf{a}$ is not an eigenvariable in $\pi_1(a)$. Also, since $\mathbf{a}$ was the eigenvariable in $\pi(a)$, it does not occur in Γ or Δ. Thus, we can apply Proposition 2.33 to get the proof $\Gamma \Rightarrow \Delta, A(b)$. Furthermore, since b does not occur in Γ or Δ, the eigenvariable condition is satisfied and we have the following proof-

$$\dfrac{\begin{array}{c}\pi_1(b)\vdots \\ \Gamma \Rightarrow \Delta, A(b)\end{array}}{\Gamma \Rightarrow \Delta, \forall x A(x)}\,(\forall R)$$

as required.

□

Definition 2.35. *(Regular proof) A proof in* GQC$^{\mathsf{G}}$ *is* ***regular*** *if every eigenvariable is the eigenvariable of a single* $\forall R$ *inference and occurs only above (perhaps more than once) that one inference.*

The notion of regular proofs becomes crucial when we want to transform proofs with $\forall$ rules. Regular proofs help ensure that the eigenvariable condition remains satisfied when we transform proofs.The next proposition demonstrates, through the use of the Proposition on variable replacement, that any proof can be transformed into a suitable regular proof.

Proposition 2.36. *Every proof* π *can be transformed into a regular proof* π' *of the same end-sequent by replacing eigenvariables only.*

Proof. We proceed by induction on the number of applications of the rule $\forall R$ in π *in which* the eigenvariable used occurs above it **and** as an eigenvariable of another inference *or* in some place *other than* above it.
Base Case: If n is 0, then the proof is regular by definition.

Inductive Case: Say $n > 0$ then choose the $\forall R$ inference with its conclusion sequent having the least proof height (i.e, the top-most such node in the proof π) such that its eigenvariable is used in another inference *or* occurs in some place *other than* above it. Say the eigenvariable of this inference is **a**.
Consider the sub-proof π_1 ending with this inference : It is a proof that ends in $\forall R$ and contains no other eigenvariable inferences with eigenvariable **a**. Hence, we apply Corollary 2.34 to replace the sub-proof π_1 by π_1' in which we replace **a** everywhere in π_1 by some free variable **b** not in π. In the resulting proof the eigenvariables of other inferences have not been changed and so the number of $\forall R$ inferences in the resultant proof in which the eigenvariable used occurs above it **and** as an eigenvariable of another inference *or* in some place *other than* above it is less than **n**. Hence, induction hypothesis applies. Thus, by mathematical induction we have our required result.

□

In the literature, there are various techniques for eliminating cuts from a proof system (see Troelstra and Schwichtenberg [2000]). The strategy we employ here is to remove the top-most cut from a proof, ensuring that the sub-proof containing the top-most cut has no other occurrences of the cut-rule. Then, we replace the transformed cut-free sub-proof back into the proof and repeat the process (see Theorem 2.43). In order to make things precise, we require the following definitions.

Definition 2.37. *(Cut-proofs and Cut-free proofs) A proof π in* GQC$^{\mathsf{G}}$ *is called a cut-proof iff it contains just one application of Cut, whose conclusion is the end-sequent(root) of the proof; it is called a cut-free proof iff it contains no application of cut.*

Definition 2.38. *(Reducible proofs) A cut-proof π which can be transformed into a proof with the same end-sequent in which cut has been not applied is said to be reducible.*

We now embark on our journey towards cut-elimination. The following lemma, both straightforward and crucial, will give us a taste of what lies ahead.

Lemma 2.39. *Suppose π is a cut-proof such that one of the premises of cut-rule is an initial sequent. Then π is reducible.*

Proof. Without loss of generality say the left premise of the cut rule is an initial sequent (which means it cannot be $\bot \Rightarrow$), then π has the following form:

$$\text{(Cut)}\ \frac{A \Rightarrow A \qquad \begin{array}{c}\pi_1 \vdots \\ A, \Pi \Rightarrow \Sigma\end{array}}{A, \Pi \Rightarrow \Sigma}$$

Then the cut-free proof π_1 with the end-sequent $A, \Pi \Rightarrow \Sigma$ is our required cut-free proof. Thus, π is reducible.

□

Our strategy would be to inductively remove cuts from a proof. To achieve this, we will require suitable notions upon which we can base our induction. This is the objective of the following definition.

Definition 2.40. *(Rank of a sequent in a cut-proof) Let π be a cut-proof whose final inference is:*

$$\textit{(Cut)}\ \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

The rank of the sequent S in π is denoted by $r(S)$ and is so defined:

- *If S is not $\Gamma, \Pi \Rightarrow \Delta, \Sigma$ then $r(S)$ is the height of the sub-proof π' ending with S;*
- $r(\Gamma, \Pi \Rightarrow \Delta, \Sigma) = r(\Gamma \Rightarrow \Delta, A) + r(A, \Pi \Rightarrow \Sigma)$

Moreover, we define the cut-rank of the proof (abbreviated as $r(\pi)$) to be $r(\Gamma, \Pi \Rightarrow \Delta, \Sigma)$.

Note that the cut-rank of a proof is at least 2.
As an abuse of notation, given a cut-proof π, by $\mathcal{C}(\pi)$ we mean the complexity[7] of the cut-formula in π. Now, we introduce the following notion on which we would induct on in Lemma 2.42.

Definition 2.41. *A cut-proof π_1 is said to be less complex than a cut-proof π_2 iff either $\mathcal{C}(\pi_1) < \mathcal{C}(\pi_2)$ or, $\mathcal{C}(\pi_1) = \mathcal{C}(\pi_2)$ and $r(\pi_1) < r(\pi_2)$.*

We are now in a position to show how, by inductively arguing on complexity of proofs, we can transform any (regular) cut-proof into a cut-free proof.

Lemma 2.42. *(Main Lemma) Any regular cut-proof is reducible.*

Proof. The proof of the Main Lemma is by double induction on the rank and complexity of a cut-proof (i.e. how *complex* a proof is). With more weight given to the complexity.

1. Base Case: Every regular proof π such that $\mathcal{C}(\pi)$=0 and $r(\pi) = 2$ is reducible:

   As $r(\pi) = 2$ then both premises of the cut rule is an initial sequent. We thus apply Lemma 2.39 to get the required result.

2. Inductive Case: Suppose every regular proof π' which is less complex than π (i.e, $\mathcal{C}(\pi') < \mathcal{C}(\pi)$ or $d(\pi') = d(\pi)$ and $r(\pi') < r(\pi)$) is reducible, then we show that π is reducible. We break it up into two parts:

   [A] $\mathcal{C}(\pi) > 0$ and $r(\pi) = 2$. Similarly to the base case, π is reducible.
   [B] $r(\pi) > 2$. In this part, the argument is again divided into two cases:

   (i) The rank of the right premises of the cut rule is 1. This case follows directly from Lemma 2.39.

   (ii) The rank of right premise of the cut rule is > 1. Furthermore, if the rank of the left premise of the cut rule is 1 then we can again apply Lemma 2.39, hence we only need to work out the case where the left rank > 1.

   Then π is of the following form:

$$\text{(Cut)}\ \frac{\overset{\pi_1 \vdots}{\Gamma \Rightarrow \Delta, A} \qquad \overset{\pi_2 \vdots}{A, \Pi \Rightarrow \Sigma}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

[7] For instance, in Definition 2.40, it would mean $\mathcal{C}(A)$.

where $r(\Gamma \Rightarrow \Delta, A) > 1$ and $r(A, \Pi \Rightarrow \Sigma) > 1$

We divide it into cases based on the rule whose conclusion is the right premise $A, \Pi \Rightarrow \Sigma$.

(a) **WL**

- Consider the case when the principal formula of WL is the cut formula A. Then proof has the following form:

$$(\mathrm{Cut})\,\dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow \Delta, A\end{matrix} \qquad (\mathrm{WL})\,\dfrac{\begin{matrix}\vdots\\ \Pi \Rightarrow \Sigma\end{matrix}}{A, \Pi \Rightarrow \Sigma}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

  Then we have the following cut-free proof of $\Gamma, \Pi \Rightarrow \Delta, \Sigma$:

$$(\mathrm{WL{,}WR})\,\dfrac{\begin{matrix}\vdots\\ \Pi \Rightarrow \Sigma\end{matrix}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

- Now consider the case when the principal formula of WL is not the cut formula A. Then proof has the following form:

$$(\mathrm{Cut})\,\dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow \Delta, A\end{matrix} \qquad (\mathrm{WL})\,\dfrac{\begin{matrix}\vdots\\ A, \Pi' \Rightarrow \Sigma\end{matrix}}{A, C, \Pi' \Rightarrow \Sigma}}{\Gamma, C, \Pi' \Rightarrow \Delta, \Sigma}$$

  Which can be transformed into the following proof:

$$(\mathrm{WL})\,\dfrac{(\mathrm{Cut})\,\dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow \Delta, A\end{matrix} \qquad \begin{matrix}\vdots\\ A, \Pi' \Rightarrow \Sigma\end{matrix}}{\Gamma, \Pi' \Rightarrow \Delta, \Sigma}}{\Gamma, C, \Pi' \Rightarrow \Delta, \Sigma}$$

  Now the sub-proof whose end-sequent is $\Gamma, \Pi' \Rightarrow \Delta, \Sigma$ (i.e. the conclusion of the cut-rule) is a cut-proof and has cut-rank one lower than π. Thus, this sub-proof is less complex than π. Subsequently, we apply the induction hypothesis to transform it into a cut-free proof. Hence, π is reducible.

(b) **WR**

The proof will have the following form:

$$\text{(Cut)}\ \dfrac{\begin{matrix}\vdots\\ \Gamma\Rightarrow\Delta,A\end{matrix}\qquad \text{(WR)}\ \dfrac{\begin{matrix}\vdots\\ A,\Pi\Rightarrow\Sigma'\end{matrix}}{A,\Pi\Rightarrow\Sigma',C}}{\Gamma,\Pi\Rightarrow\Delta,\Sigma',C}$$

Which can be transformed into the following proof:

$$\text{(WR)}\ \dfrac{\text{(Cut)}\ \dfrac{\begin{matrix}\vdots\\ \Gamma\Rightarrow\Delta,A\end{matrix}\qquad \begin{matrix}\vdots\\ A,\Pi\Rightarrow\Sigma'\end{matrix}}{\Gamma,\Pi\Rightarrow\Delta,\Sigma'}}{\Gamma,\Pi\Rightarrow\Delta,\Sigma',C}$$

Now since the sub-proof has cut-rank one lower than π, we apply the induction hypothesis to transform it into a cut-free proof. Thus, π is reducible.

(c) $\forall$**L**

- Consider the case when the principal formula of $\forall L$ is the cut formula $\forall xA(x)$. Then proof has the following form:

$$\text{(Cut)}\ \dfrac{\begin{matrix}\vdots\\ \Gamma\Rightarrow\Delta,\forall xA(x)\end{matrix}\qquad (\forall L)\ \dfrac{\begin{matrix}\vdots\\ A(t),\Pi\Rightarrow\Sigma\end{matrix}}{\forall xA(x),\Pi\Rightarrow\Sigma}}{\Gamma,\Pi\Rightarrow\Delta,\Sigma}$$

  Now since $r(\Gamma \Rightarrow \Delta, \forall xA) > 1$ we can further distinguish cases based on the last inference rule applied to get the left premise $\Gamma \Rightarrow \Delta, \forall xA(x)$. The only interesting case to consider is when $\forall xA(x)$ is the principal formula of the rule. The only possibilities of that is the $\forall R$ rule and the WR rule.

  If the rule is **WR** then we have the following cut-free proof:

$$\text{(WL,WR)}\ \dfrac{\begin{matrix}\vdots\\ \Gamma\Rightarrow\Delta\end{matrix}}{\Gamma,\Pi\Rightarrow\Delta,\Sigma}$$

If the rule is $\forall\mathbf{R}$ then π has the following form:

$$(\text{Cut})\,\dfrac{(\forall R)\,\dfrac{\overset{\pi_1(b)\vdots}{\Gamma \Rightarrow \Delta, A(b)}}{\Gamma \Rightarrow \Delta, \forall x A(x)} \qquad (\forall L)\,\dfrac{\overset{\vdots}{A(t), \Pi \Rightarrow \Sigma}}{\forall x A(x), \Pi \Rightarrow \Sigma}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

Note that since π is a regular proof, the free variable b is not an eigenvariable of the proof $\pi_1(b)$ and the term t does not contain any eigenvariables of π and hence, also of $\pi_1(b)$. Thus by Proposition 2.33 we have that $\pi_1(t)$ is a proof with end-sequent $\Gamma \Rightarrow \Delta, A(t)$. Then we have the following proof of $\Gamma, \Pi \Rightarrow \Delta, \Sigma$ which has cut-rank less than $r(\pi)$:

$$(\text{Cut})\,\dfrac{\overset{\pi_1(t)\vdots}{\Gamma \Rightarrow \Delta, A(t)} \qquad \overset{\vdots}{A(t), \Pi \Rightarrow \Sigma}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

Thus, we can apply the induction hypothesis to transform it into a cut-free proof. Hence, π is reducible.

- Now consider the case when the principal formula of $\forall L$ is not the cut formula A. Then proof has the following form:

$$(\text{Cut})\,\dfrac{\overset{\vdots}{\Gamma \Rightarrow \Delta, A} \qquad (\forall L)\,\dfrac{\overset{\vdots}{A, B(t), \Pi' \Rightarrow \Sigma}}{\forall x B(x), \Pi' \Rightarrow \Sigma}}{\Gamma, \forall x B(x), \Pi' \Rightarrow \Delta, \Sigma}$$

This can be transformed into the following proof:

$$(\forall L)\,\dfrac{(\text{Cut})\,\dfrac{\overset{\vdots}{\Gamma \Rightarrow \Delta, A} \qquad \overset{\vdots}{A, B(t), \Pi' \Rightarrow \Sigma}}{\Gamma, B(t), \Pi' \Rightarrow \Delta, \Sigma}}{\Gamma, \forall x B(x), \Pi' \Rightarrow \Delta, \Sigma}$$

Now since the sub-proof has cut-rank one lower than π, we apply the induction hypothesis to transform it into a cut-free proof. Thus, π is reducible.

(d) $\forall$**R**.

The proof has the following form:

$$\text{(Cut)}\ \dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow \Delta, A\end{matrix} \qquad (\forall R)\ \dfrac{\begin{matrix}\vdots\\ A, \Pi \Rightarrow \Sigma', B(b)\end{matrix}}{A, \Pi \Rightarrow \Sigma', \forall x B(x)}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma', \forall x B(x)}$$

Since π is a *regular* proof by assumption, the eigenvariable b does not not occur in Γ or Δ. Hence, π can be transformed into the following proof :

$$(\forall R)\ \dfrac{\text{(Cut)}\ \dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow \Delta, A\end{matrix} \qquad \begin{matrix}\vdots\\ A, \Pi \Rightarrow \Sigma', B(b)\end{matrix}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma', B(b)}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma', \forall x B(x)}$$

Now since its sub-proof has cut-rank one lower than π, we apply the induction hypothesis to transform it into a cut-free proof. Thus, π is reducible.

(e) $\rightarrow$ **R**

This case is dealt analogously to the WR case.

(f) $\rightarrow$ **L**

There are two possibilities:

- Consider the case when the principal formula of the $\rightarrow L$ rule is not the cut formula. Then π has the following form:

$$\text{(Cut)}\ \dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow \Delta, A\end{matrix} \qquad (\rightarrow L)\ \dfrac{\begin{matrix}\vdots\\ A, \Pi_1 \Rightarrow \Sigma_1, C\end{matrix} \qquad \begin{matrix}\vdots\\ D, \Pi_2 \Rightarrow \Sigma_2\end{matrix}}{A, C \rightarrow D, \Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2}}{C \rightarrow D, \Gamma, \Pi_1, \Pi_2 \Rightarrow \Delta, \Sigma_1, \Sigma_2}$$

  This can be transformed into the following proof :

$$(\rightarrow L)\ \dfrac{\text{(Cut)}\ \dfrac{\begin{matrix}\vdots\\ \Gamma \Rightarrow \Delta, A\end{matrix} \qquad \begin{matrix}\vdots\\ A, \Pi_1 \Rightarrow \Sigma_1, C\end{matrix}}{\Gamma, \Pi_1 \Rightarrow \Delta, \Sigma_1, C} \qquad \begin{matrix}\vdots\\ D, \Pi_2 \Rightarrow \Sigma_2\end{matrix}}{C \rightarrow D, \Gamma, \Pi_1, \Pi_2 \Rightarrow \Delta, \Sigma_1, \Sigma_2}$$

Now since the sub-proof has cut-rank one lower than π, we apply the induction hypothesis to transform it into a cut-free proof[8]. Thus, π is reducible.

- Now consider the important case when the principal formula of the $\to L$ rule is the cut formula $C \to D$. Then π has the following form:

$$(\text{Cut})\,\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, C \to D\end{array} \qquad (\to L)\,\dfrac{\begin{array}{c}\vdots\\ \Pi_1 \Rightarrow \Sigma_1, C\end{array} \qquad \begin{array}{c}\vdots\\ D, \Pi_2 \Rightarrow \Sigma_2\end{array}}{C \to D, \Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2}}{\Gamma, \Pi_1, \Pi_2 \Rightarrow \Delta, \Sigma_1, \Sigma_2}$$

Now since $r(\Gamma \Rightarrow \Delta, C \to D) > 1$ we can further distinguish cases based on the last inference rule applied to get the left premise $\Gamma \Rightarrow \Delta, C \to D$. The only interesting case to consider is when $C \to D$ is the principal formula of the rule. The only possibilities of that is the $\to R$ rule and the WR rule. If the rule is **WR** then we have the following cut-free proof:

$$(\text{WL,WR})\,\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta\end{array}}{\Gamma, \Pi_1, \Pi_2 \Rightarrow \Delta, \Sigma_1, \Sigma_2}$$

If the rule is $\to$ **R** then π has the following form:

$$\begin{array}{r}(\to R)\\(\text{Cut})\end{array}\,\dfrac{\dfrac{\begin{array}{c}\vdots\\ C, \Gamma \Rightarrow \Delta, D\end{array}}{\Gamma \Rightarrow \Delta, C \to D} \qquad (\to L)\,\dfrac{\begin{array}{c}\vdots\\ \Pi_1 \Rightarrow \Sigma_1, C\end{array} \qquad \begin{array}{c}\vdots\\ D, \Pi_2 \Rightarrow \Sigma_2\end{array}}{C \to D, \Pi_1, \Pi_2 \Rightarrow \Sigma_1, \Sigma_2}}{\Gamma, \Pi_1, \Pi_2 \Rightarrow \Delta, \Sigma_1, \Sigma_2}$$

This can be transformed into the following proof in which both cuts are of **complexity** lower than $\mathcal{C}(\pi)$.

$$(\text{Cut})\,\dfrac{(\text{Cut})\,\dfrac{\begin{array}{c}\vdots\\ C, \Gamma \Rightarrow \Delta, D\end{array} \qquad \begin{array}{c}\vdots\\ \Pi_1 \Rightarrow \Sigma_1, C\end{array}}{\Gamma, \Pi_1 \Rightarrow \Delta, \Sigma_1, D} \qquad \begin{array}{c}\vdots\\ D, \Pi_2 \Rightarrow \Sigma_2\end{array}}{\Gamma, \Pi_1, \Pi_2 \Rightarrow \Delta, \Sigma_1, \Sigma_2}$$

Now, we take that sub-proof whose end-sequent is the conclusion of the *upper cut*. Since this sub-proof (which is a cut-proof) is less complex than π, we apply the induction hypothesis to transform it into a cut-free proof. Then, we apply the induction hypothesis again on the thus obtained proof to get a cut-free proof of $\Gamma, \Pi_1, \Pi_2 \Rightarrow \Delta, \Sigma_1, \Sigma_2$. Hence, π is reducible.

[8] Note that the other possibility, when we instead have the sequent $D, A, \Pi_2 \Rightarrow \Sigma_2$ in π, can also be dealt with similarly.

Since this exhausts all possible rules, we have our proof for [B].

Thus, from [A] and [B], we have our inductive case. Finally, by double induction, we have that any regular proof is reducible.

□

Theorem 2.43. *(Cut-Elimination in $\mathsf{GQC}^{\mathsf{G}}$) The cut-rule is eliminable from $\mathsf{GQC}^{\mathsf{G}}$. That is, given any proof in $\mathsf{GQC}^{\mathsf{G}}$ it is possible to construct another proof with the same end-sequent but that does not contain any cut.*

Proof. Take any arbitrary proof in $\mathsf{GQC}^{\mathsf{G}}$ and using Proposition 2.36 modify the proof to ensure that it is regular, call it π_0.
Now we take the left-most and upper most application of cut in π_0, and let $\Pi \Rightarrow \Sigma$ be its conclusion. Then the sub-proof π_0' of π_0 is a cut-proof. We apply Lemma 2.42 to π_0' to get a cut-free proof of $\Pi \Rightarrow \Sigma$. Replace π_0' in π_0 with this cut-free proof. Call this new proof π_1. Note that π_1 has the same end-sequent as π_0 and has one less application of cut. By repeating this procedure as many times as there are applications of cut in π_0, we get the required transformation. □

We conclude this section with a brief remark on how cut-elimination differs in the presence of contraction.

Remark 2.44. *(Difference from classical cut-elimination)*
The presence of contraction in logic significantly complicates cut-elimination (c.f. Lemma 2.42). For instance, look at the following proof:

$$\textit{(Cut)}\,\dfrac{\begin{matrix}\vdots\\ \Pi \Rightarrow \Sigma, A\end{matrix} \qquad \textit{(CL)}\,\dfrac{\begin{matrix}\vdots\\ \Gamma, A, A \Rightarrow \Delta\end{matrix}}{\Gamma, A \Rightarrow \Delta}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

If we attempt to push the cut upwards then we would like to have a transformation as follows

$$\textit{(CR,CL)}\,\dfrac{\textit{(Cut)}\,\dfrac{\begin{matrix}\vdots\\ \Pi \Rightarrow \Sigma, A\end{matrix} \qquad \textit{(Cut)}\,\dfrac{\begin{matrix}\vdots\\ \Pi \Rightarrow \Sigma, A\end{matrix} \qquad \begin{matrix}\vdots\\ \Gamma, A, A \Rightarrow \Delta\end{matrix}}{\Pi, \Gamma, A \Rightarrow \Delta}}{\Pi, \Pi, \Gamma \Rightarrow \Delta, \Sigma, \Sigma}}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}$$

But then, we see that this would not give a proof which is less complex. As the complexity of the lowest cut need not be less. To solve this problem Gentzen introduced his Mix rule which is a derivable generalization of the cut rule.

$$\textit{(Mix on A)}\ \frac{\Gamma \Rightarrow \Delta \qquad \Pi \Rightarrow \Sigma}{\Gamma, \Pi^* \Rightarrow \Delta^*, \Sigma}$$

Where both Δ and Π contain at least one occurrence of A, while Δ^* and Π^* contain none. Gentzen showed that the systems LK and $LK - cut + mix$ prove the same sequents and by eliminating mix from $LK - cut + mix$ he achieved his Hauptsatz. Furthermore, we note that the definition of rank of a cut-proof (c.f. Definition 2.40) would need to be more involved in the presence of contraction.

Consequences of Cut-elimination

In this subsection, we briefly discuss some direct consequences of the cut-elimination theorem. Let's start with the following definition.

Definition 2.45. *(Sub-formula) Given a formula A we define the set, $Sub(A)$, of sub-formulas of A inductively as follows:*

- *Base Clause: $Sub(A) = \{A\}$ if A is atomic.*
- *Inductive Clause:*
  If $A = C \to B$ then $Sub(C \to B) = \{C \to B\} \cup Sub(C) \cup Sub(B)$;
  If $A = \forall x B(x)$ then $Sub(\forall x B(x)) = \{\forall x B(x)\} \bigcup_{t \in terms} Sub(B(x/t))$

A proof π is said to have *sub-formula property* if any formula occurring in π is a sub-formula of the formulas in the sequent at the root of π.
The following is perhaps one of the most important consequences of the cut-elimination theorem.

Proposition 2.46. *(Sub-formula property) Given any sequent $\Gamma \Rightarrow \Delta$, if we have that $\vdash_{\mathsf{GQC}^{\mathsf{G}}} \Gamma \Rightarrow \Delta$ then there is a proof π of $\Gamma \Rightarrow \Delta$ in $\mathsf{GQC}^{\mathsf{G}}$ which has the sub-formula property.*

Corollary 2.47. *(**Non-Theorems** of $\mathsf{GQC}^{\mathsf{G}}$) The following sequents, which are classically provable (i.e, in LK), are not provable in $\mathsf{GPC}^{\mathsf{G}}$:*

- $\forall x(A(x) \otimes B(x)) \Rightarrow \forall x A(x) \otimes \forall x B(x)$.
- $\exists x B(x) \oplus \exists x C(x) \Rightarrow \exists x(B(x) \oplus C(x))$
- $\Rightarrow \exists y \forall x(P(y) \to P(x))$

Remark 2.48. *From Corollary 2.47, it becomes evident that not all formulas within GQC can be reduced to an equivalent prenex form, which stands in contrast to classical logic.*

On the other hand, the following sequents are provable in $\mathsf{GPC}^{\mathsf{G}}$:

- $\forall x(A(x) \otimes B(x)) \otimes \forall x(A(x) \otimes B(x)) \Rightarrow \forall x A(x) \otimes \forall x B(x)$.
- $\exists x B(x) \oplus \exists x C(x) \Rightarrow \exists x(B(x) \oplus C(x)) \oplus \exists x(B(x) \oplus C(x))$
- $\exists y \forall x(P(y) \to P(x)) \oplus \exists y \forall x(P(y) \to P(x))$

Next we present, an intriguing corollary- a (weak) Herbrand theorem for GQC. We also give a proof this time, to give a flavour of how such arguments looks like.

Corollary 2.49. *Given any formula $A(x)$, suppose that $\vdash_{\mathsf{GQC^G}} \Rightarrow \exists x A(x)$ then there is a term t, such that, $\vdash_{\mathsf{GQC^G}} \Rightarrow A(t)$.*

Proof. If $\vdash_{\mathsf{GQC^G}} \Rightarrow \exists x A(x)$, then by the cut-elimination theorem there is a cut-free proof π of $\Rightarrow \forall x(A(x) \to \bot) \to \bot$ in $\mathsf{GQC^G}$. Since this proof has the sub-formula property, the only possibility for the last applied rule in π is $\to R$. Then π must have the following form:

$$(\to R)\ \dfrac{\begin{matrix}\vdots \\ \forall x(A(x) \to \bot) \Rightarrow \bot\end{matrix}}{\Rightarrow \forall x(A(x) \to \bot) \to \bot}$$

Hence, there is a cut-free proof of $\forall x(A(x) \to \bot) \Rightarrow \bot$ in $\mathsf{GQC^G}$. Now, there are two valid pathways through which $\forall x(A(x) \to \bot) \Rightarrow \bot$ could be inferred: either via WR or $\forall L$ (since $\Rightarrow \bot$ cannot be proved in $\mathsf{GQC^G}$, WL is not a valid pathway).

Let's take the case of $\forall L$. Then π must be of the following form:

$$(\to R)\ \dfrac{(\forall L)\ \dfrac{\begin{matrix}\vdots \\ A(t) \to \bot \Rightarrow \bot\end{matrix}}{\forall x(A(x) \to \bot) \Rightarrow \bot}}{\Rightarrow \forall x(A(x) \to \bot) \to \bot}$$

Hence, there is a cut-free proof of $A(t) \to \bot \Rightarrow \bot$ in $\mathsf{GQC^G}$. Again, there are two valid pathways through which $A(t) \to \bot \Rightarrow \bot$ could be inferred: either via WR or $\to L$. Let's take the case of $\to L$. Then π must be of the following form:

$$(\to R)\ \dfrac{(\forall L)\ \dfrac{(\to L)\ \dfrac{\begin{matrix}\vdots \\ \Rightarrow A(t)\end{matrix} \qquad \begin{matrix}\vdots \\ \bot \Rightarrow \bot\end{matrix}}{A(t) \to \bot \Rightarrow \bot}}{\forall x(A(x) \to \bot) \Rightarrow \bot}}{\Rightarrow \forall x(A(x) \to \bot) \to \bot}$$

Thus, we have that for some term t, $\vdash_{\mathsf{GQC^G}} \Rightarrow A(t)$.

Furthermore, reasoning similarly in the cases we have left out, the same conclusion can be reached, i.e., for some term t, $\vdash_{\mathsf{GQC^G}} \Rightarrow A(t)$. Hence, we will have what we require.

□

Remark 2.50. *In classical logic,* LK*, the above proposition famously fails. Instead,* LK *offers a weaker result (known as Herbrand's Theorem): if we have* $\vdash_{\mathsf{LK}} \Rightarrow \exists x A(x)$*, then there exists a finite sequence of terms* $(t_i)_{i=1}^n$ *such that* $\vdash \mathsf{LK} \Rightarrow \bigvee_{i=1}^n A(t_i)$*. This phenomenon arises precisely due to the presence of CR in* LK*.*

We conclude this section by mentioning an important observation made by Ketonen and Weyhrauch in Ketonen and Weyhrauch [1984]: they extended the scope of Corollary 2.48, establishing a suitable Herbrand's theorem applicable to any prenex formula. Additionally, they demonstrated that this theorem no longer hold for formulas that are not in prenex form.

2.3 Deduction from hypotheses

In this section, we briefly discuss two notions of deduction from hypotheses-External and Internal consequence[9]. This is particularly interesting in case of substructural logics as the distinction between two notions of consequence breaks down in case of Classical logic and in fact, this can be a potential source for logical paradoxes (see Mares and Paoli [2014]). Thus, the study of consequence is richer in case of contraction free logics. The sequent style formulations of consequence has been adapted from Avron [1988].

External Consequence

We begin with the notion of consequence which is the standard way of capturing when a formula can be deduced from a *set* of formulae (i.e. hypothesis). We formulate it in Hilbert style axiomatization and then define an equivalent notion in sequent calculus.

Definition 2.51. *(Weak derivability from assumptions,* $\mathsf{GQC^H}$*)* B *is weakly derivable from a* ***set*** Γ *(abbreviated as* $\Gamma \vdash^e B$*) if there is a sequence* $B_1, \ldots, B_n$ *of well-formed formulas (wffs) such that,* $B_n = B$ *and for each* i*,* B_i *is either an axiom of* GQC^H*, or is in* Γ*, or is a direct consequence of some of the preceding wffs in the sequence by virtue of modus ponens or by generalization.*

It is easy to observe that if $\vdash_{GQC^H} A$ then $\vdash^e A$.

Definition 2.52. *(External Consequence,* $\mathsf{GQC^G}$*) We say that* B *is an external consequence of a* ***set*** Γ *if and only if* $\Rightarrow B$ *is provable in* $\mathsf{GQC^G}$ *by adding, as initial sequents,* $\Rightarrow A$ *where* $A \in \Gamma$*.*

The next proposition demonstrates the equivalence between the notion of weak derivability and external consequence.

Proposition 2.53. *Given a set of formulae* Γ *and a formula* B*,* $\Gamma \vdash^e B$ *iff* B *is an external consequence of* Γ*.*

[9] The propositional counterparts of notions presented in this section are by now well-established (see Mares and Paoli [2014]).

Proof.

Left to Right direction

Suppose $\Gamma \vdash^e B$ and say $A_1, \ldots, A_n = B$ is a derivation of B. We proceed by induction on the length of the sequence.

Base case $n = 1$. Then $A_1 = B$, meaning B is either an axiom of $\mathsf{GQC}^{\mathsf{H}}$ or $B \in \Gamma$. Using Proposition 2.29, in the former case, $\Rightarrow B$ is provable in $\mathsf{GQC}^{\mathsf{G}}$. In the latter case since $B \in \Gamma$, $\Rightarrow B$ is added as an initial sequent and thus is provable. Hence, in either case, B is an external consequence of Γ.

Inductive case. Now assume the induction hypothesis holds for proofs of length up to n, and consider the following derivation of B: $A_1, \ldots, A_{n+1} = B$.

The important cases to address are when A_{n+1} is a result of either modus ponens (MP) or generalization (Gen).

MP We have that $A_n = A_i \to A_{n+1}$ for some $i < n$. We have that $\Gamma \vdash^e A_i$ and $\Gamma \vdash^e A_n$, both having deductions of length less than or equal to n. Thus, we apply the induction hypothesis and construct the following proof in $\mathsf{GQC}^{\mathsf{G}}$ with assumptions from Γ.

$$\dfrac{\begin{matrix}\vdots\\ \Rightarrow A_i\end{matrix} \qquad (\text{cut})\,\dfrac{\begin{matrix}\vdots\\ \Rightarrow A_i \to A_{n+1}\end{matrix} \qquad \dfrac{A_i \Rightarrow A_i \qquad A_{n+1} \Rightarrow A_{n+1}}{A_i \to A_{n+1}, A_i \Rightarrow A_{n+1}}}{A_i \Rightarrow A_{n+1}}}{\Rightarrow A_{n+1}}$$

Therefore, we have a proof of $\Rightarrow A_{n+1}$ in $\mathsf{GQC}^{\mathsf{G}}$ with assumptions from Γ.

Gen We have that $A_{n+1} = \forall x A_n$ as an application of generalization. Then we have $\Gamma \vdash^e A_n$. We apply induction hypothesis and construct the following proof tree in $\mathsf{GQC}^{\mathsf{G}}$.

$$(\forall R)\,\dfrac{\begin{matrix}\vdots\\ \Rightarrow A_n\end{matrix}}{\Rightarrow \forall x A_n}$$

Therefore, we have a proof of $\Rightarrow A_{n+1}$ in $\mathsf{GQC}^{\mathsf{G}}$ with assumptions from Γ.

By induction on the length of deduction we are done.

Right to Left direction

We establish a more general result: if the sequent $\Pi \Rightarrow \Delta$ is provable in $\mathsf{GQC}^{\mathsf{G}}$ by adding initial sequents $\Rightarrow A$ where $A \in \Gamma$, then $\Gamma \vdash^e t(\Pi \Rightarrow \Delta)$ (refer to definition 2.11). We proceed by induction on the structure of proofs in $\mathsf{GQC}^{\mathsf{G}}$.

Base Case. $\Pi \Rightarrow \Delta$ is an initial sequent. Then we have the following possibilities for $\Pi \Rightarrow \Delta$: $B \Rightarrow B$ or, $\bot \Rightarrow$ or, $\Rightarrow A$ where $A \in \Gamma$. The first two cases are dealt exactly in the same way as in Proposition 2.29 to show that $\vdash_{GQC^H} t(\Pi \Rightarrow \Delta)$ and subsequently, $\Gamma \vdash^e t(\Pi \Rightarrow \Delta)$. Regarding the last one, we note that since $A \in \Gamma$, it is derivable from Γ. Hence, we have $\Gamma \vdash^e t(\Pi \Rightarrow \Delta)$.

Inductive Case. Then $\Pi \Rightarrow \Delta$ has been arrived at by an application of a rule of $\mathsf{GQC^G}$ with additional initial sequents from Γ. This is dealt exactly as in the inductive case of left to right direction of Theorem 2.29.
By induction on the structure of proofs in $\mathsf{GQC^G}$ we are done. □

One of the key ways to get a better understanding of consequence is to formulate and understand a suitable Deduction Theorem for it. A Deduction theorem not only simplifies proofs but also provides a better understanding of how implication works. In classical logic, we can demonstrate that if A is closed (i.e., A contains no free variables), then $\Gamma, A \vdash B$ if and only if $\Gamma \vdash A \to B$. This is possible because classical logic, equipped with contraction, doesn't require tracking the number of times an assumption (such as A in this case) has been used. We do not have that liberty in our system and thus, we need an alternative deduction theorem. This is the content of the next proposition.
We let A^n be an abbreviation for $A \otimes \ldots \otimes A$ (n times) and $A^0 := \top$. Then we get the following deduction theorem.

Proposition 2.54. (*Deduction Theorem*^{e}) *$\Gamma, A \vdash^e B$ iff $\Gamma \vdash^e A^n \to B$, for some $n \geq 1$, where A is a closed formula.*

Proof.
Right to Left direction This is easy to see, so we focus on the converse.
Left to Right direction
Say $\Gamma, A \vdash^e B$ and thus say $A_1, \ldots, A_n = B$ is a deduction. We prove the proposition by induction on the length of the sequence.
Base Case. $n = 1$. Then $A_1 = B$, i.e, B is an axiom of GQC^H or, $B = A$ or, $B \in \Gamma$. In all these possibilities using the GQC^H axioms $A \to A$ and $B \to (A \to B)$ we will have that $\Gamma \vdash^e A \to B$.

Now assume the induction hypothesis for deductions of length up to n and say we have the following derivation of B: $A_1, \ldots, A_{n+1} = B$.

Inductive Case. The important cases we need to deal with are when A_{n+1} is a result of MP or Gen.

MP We have that $A_n = A_i \to A_{n+1}$ for some $i < n$. Now, from $\Gamma, A \vdash^e A_i$ and $\Gamma, A \vdash^e A_n$, by applying the induction hypothesis, we obtain, for some positive integers p and q, $\Gamma \vdash^e A^p \to A_i$ and $\Gamma \vdash^e A^q \to (A_i \to A_{n+1})$. Now utilizing the axiom $P \to (Q \to R) \to (Q \to (P \to R))$ and modus ponens, we deduce $\Gamma \vdash^e A_i \to (A^q \to A_{n+1})$. Furthermore, using the axiom $(P \to R) \to ((R \to Q) \to (P \to Q))$ and $\Gamma \vdash^e A^p \to A_i$, modus ponens yields $\Gamma \vdash^e A^p \to (A^q \to A_{n+1})$. Or equivalently, $\Gamma \vdash^e A^p \otimes A^q \to A_{n+1}$, or simply $\Gamma \vdash^e A^{p+q} \to A_{n+1}$, as required.

Gen We have that $A_{n+1} = \forall x A_n$ as a result of generalization. Consequently, we have $\Gamma, A \vdash^e A_n$. By applying the induction hypothesis, we obtain, for some positive integer p, $\Gamma \vdash^e A^p \to A_n$. Since A is closed, so is A^p. Therefore, employing the rule of generalization and the axiom $\forall x(A^p \to A_n) \to A^p \to \forall x A_n$, we deduce $\Gamma \vdash^e A^p \to \forall x A_n$ using MP.

By induction on the length of deductions we have our required result.

□

Remark 2.55. *We note that in classical logic we have $p \to (q \to r) \to ((p \to q) \to (p \to q))$ as a theorem. Due to which, in the inductive case of MP, we could reason as follows: From $\Gamma \vdash A \to A_i$ and $\Gamma \vdash A \to (A_i \to A_{n+1})$, we can conclude $\Gamma \vdash A \to A_{n+1}$. This is why in classical logic we have $\Gamma, A \vdash B$ iff $\Gamma \vdash A \to B$.*

Internal Consequence

In order to get a notion of derivability where we have the original deduction theorem, we need a tighter notion. This is the content of this part.

Definition 2.56. *(Internal Consequence, $\mathsf{GQC}^{\mathsf{G}}$) We say that B is an internal consequence of a **multiset** Γ if and only if $\Gamma \Rightarrow B$ is provable in $\mathsf{GQC}^{\mathsf{G}}$.*

We adapt the following definition of derivability from Troelstra Troelstra [1992].

Definition 2.57. *(Strong derivability from assumptions, $\mathsf{GQC}^{\mathsf{H}}$) A strong derivation is a labeled tree D with leaves labeled by expressions of the form $A \vdash A$, where A is a formula of the language, or of the form $\vdash B$, where B is an axiom of $\mathsf{GQC}^{\mathsf{H}}$. Concerning the labels of the other nodes of D, they are obtained by the following rules (where Γ, Δ are finite **multisets** of formulae):*

$$(\to E)\,\frac{\Gamma \vdash A \qquad \Delta \vdash A \to B}{\Gamma, \Delta \vdash B} \qquad\qquad (Gen)\,\frac{\vdash A}{\vdash \forall x A}$$

*We shall say A is **strongly** derivable from Γ (abbreviated as $\Gamma \vdash^i B$) if there is a derivation whose root is labelled by $\Gamma \vdash A$.*

It is easy to observe that if $\vdash_{GQC^H} A$ then $\vdash^i A$.
The next proposition shows that the notion of strong derivability and internal consequence coincide.

Proposition 2.58. *Given a multi-set of formulae Γ and a formula B,*

$$\Gamma \vdash^i B \textit{ iff } \vdash_{GQC^G} \Gamma \Rightarrow B.$$

Proof. Right to Left
Suppose $\Gamma = A_1, \ldots, A_n$, let $\gamma := A_1 \otimes \ldots A_n$.
Since $\vdash_{GQC^G} \Gamma \Rightarrow B$ we have from Proposition 2.29 that $\vdash_{GQC^H} \gamma \to B$. Thus, we have a strong derivation whose root is labelled with $\vdash \gamma \to B$. Now using the $\mathsf{GQC^H}$ theorem $(P \otimes Q \to R) \to (P \to (Q \to R))$ and $(\to E)$ we can construct a strong derivation of $\vdash A_1 \to (A_2 \otimes \ldots A_n \to B)$. Now consider the following derivation:

$$(\to E)\ \dfrac{\begin{array}{c}\vdots\\ \vdash A_1 \to (A_2 \otimes \ldots \otimes A_n \to B)\end{array} \qquad A_1 \vdash A_1}{A_1 \vdash (A_2 \otimes \ldots \otimes A_n \to B)}$$

We can repeat this process n times to eventually get a derivation whose root is labelled by $\Gamma \vdash B$. Thus, $\Gamma \vdash^i B$.

Left to Right
We show this by induction on the structure of strong derivations. If the derivation only consists of a leaf then the root is labelled either by $A \vdash A$ where A is any formula or, $\vdash B$ where B is an axiom of $\mathsf{GQC^H}$. But $A \Rightarrow A$ is an initial sequent of $\mathsf{GQC^G}$ and $\Rightarrow B$ from Proposition 2.29. Thus, we will have $\vdash_{GQC^G} \Gamma \Rightarrow B$.
Now, say the node labelled by $\Gamma \vdash B$ is obtained by one of the rules. We need to show the following:

- Preservation under $\to E$:
  Say $\Gamma \Rightarrow A$ and $\Delta \Rightarrow A \to B$ is provable in $\mathsf{GQC^G}$. We need to show that $\Gamma, \Delta \Rightarrow B$. Consider the following proof tree in $\mathsf{GQC^G}$:

$$(\mathrm{Cut})\ \dfrac{(\mathrm{Cut})\ \dfrac{(\to\mathrm{L})\ \dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A \to B, A \Rightarrow B} \qquad \begin{array}{c}\vdots\\ \Delta \Rightarrow A \to B\end{array}}{\Delta, A \Rightarrow B} \qquad \begin{array}{c}\vdots\\ \Gamma \Rightarrow A\end{array}}{\Gamma, \Delta \Rightarrow B}$$

  Hence, $\Gamma, \Delta \Rightarrow B$ is also provable in $\mathsf{GQC^G}$.

- Preservation under Gen(eralization).
  Say $\Rightarrow B(x)$ is provable in $\mathsf{GQC^G}$. As the eigenvariable condition is clearly satisfied, we apply the $\forall R$ rule in $\mathsf{GQC^G}$ to get that $\Rightarrow \forall x B(x)$ is provable.

Thus, by induction on the structure of stroong derivations, we get that if $\Gamma \vdash^i A$ then $\vdash_{GQC^G} \Gamma \Rightarrow B$.

□

Corollary 2.59. (*Deduction Theorem*^{i}) *If* $\Gamma, A \vdash^i B$ *then* $\Gamma \vdash^i A \to B$.

Corollary 2.60. *(Generalization) If* $\Gamma \vdash^i B(x)$ *and* x *is not free in any formula in* Γ*, then* $\Gamma \vdash^i \forall x B(x)$.

We conclude this section with some remarks on the relationship between internal and external consequence.

Remark 2.61.

- *It is easy to observe that by repeated applications of cut, if* B *is an internal consequence of (a multiset)* Γ *then* B *is also an external consequence of* Γ *(taken as a set).*
- *On the other hand the converse is not true:* $C \otimes C$ *is an external consequence of* $\{C\}$ *but not an internal consequence. Consider the following proof tree in* $\mathsf{GQC}^{\mathsf{G}}$ *where* $\Rightarrow C$ *has been added as an initial sequent.*

$$(\otimes R)\ \frac{C \Rightarrow C \qquad \Rightarrow C}{C \Rightarrow C \otimes C}$$

- *Classical logic cannot distinguish between internal and external consequence. In the presence of weakening and contraction, an external consequence is also an internal consequence.*

3 | Contraction Free Arithmetic

In this chapter, we introduce Contraction Free Arithmetic (CFA), utilizing first-order Grišhin logic as its underlying framework. We demonstrate several fundamental properties of our arithmetic and proceed to showcase the definability of additive connectives in our language. In section [3.3], we provide a justification for our selection of the Induction Rule over the Induction Schema in our axiomatization. In the same section, we establish Δ_0-Induction within our system. As a consequence of that, we show that any Π_2 formula which is provable in $\mathsf{I}\Delta_0$ is also provable in CFA under a suitable formula translation. We conclude the chapter by showing that left contraction for Σ_1 formulas hold in CFA.

3.1 The Formal System of CFA

The language of *Contraction-Free Arithmetic* (CFA) is a first-order vocabulary, given by,

$$L_{\mathsf{CFA}} := \{0, S, +, \cdot, =\}$$

where **0** is a constant symbol; **S** is an unary function symbol; **(+),** $(\cdot)$ are binary function symbols ; $=$ is a binary predicate symbol.

Definition 3.1. ($\mathsf{CFA}^{\mathsf{H}}$) *We define the* ***Hilbert-style*** *proof system for* CFA *over the logic* $\mathsf{GQC}^{\mathsf{H}}$ *in the language* L_{CFA} *as follows:*

Axioms for Equality

1. $\forall x(x = x)$.
2. $\forall \vec{x} \forall \vec{y}((x_1 = y_1 \otimes \ldots \otimes x_n = y_n) \to f(\vec{x}) = f(\vec{y}))$,
   where f is any function symbol.
3. $\forall \vec{x} \forall \vec{y}((x_1 = y_1 \otimes \ldots \otimes x_n = y_n) \to (R(\vec{x}) \to R(\vec{y})))$,
   where R is any atomic formula.

Axioms for Arithmetic

1. $\forall x \neg (S(x) = 0)$
2. $\forall x \forall y (S(x) = S(y) \to x = y)$
3. $\forall x (x + 0 = x)$
4. $\forall x \forall y (x + S(y) = S(x + y))$
5. $\forall x (x \cdot 0 = 0)$
6. $\forall x \forall y (x \cdot S(y)) = x \cdot y + x$

Rule of Induction

$$(\mathrm{IR})\ \frac{\phi(0) \qquad \forall y(\phi(y) \to \phi(S(y)))}{\forall x(\phi(x))}$$

Definition 3.2. *(**Proof**) A proof in* CFA *is a sequence $B_1, \ldots, B_n$ of well-formed formulas (wffs) such that, for each i, B_i is either an axiom of* CFA *or a direct consequence of some of the preceding wffs in the sequence by virtue of one of the rules of inference of* CFA*. A formula B is provable in* CFA *(abbreviated as $\mathsf{CFA}^{\mathsf{H}} \vdash B$) if there exists a proof $B_1, \ldots, B_n$ where $B = B_n$.*

Remark 3.3. *In the presence of contraction, such as in LK, the rule of Induction and the Induction schema are equivalent. However, in its absence, while the Induction schema entails the rule, the converse does not hold. We refer to Section 3.3 for our justification for choosing the Induction rule.*

We now give an sequent-style formulation of CFA.

Definition 3.4. ($\mathsf{CFA}^{\mathsf{G}}$) *We define the **Sequent-style** proof system for* CFA *over the logic* $\mathsf{GQC}^{\mathsf{G}}$ *in the language L_{CFA} as follows*[1]*:*

- **Initial sequents for Equality**
    1. $\Rightarrow t = t$
    2. $t_1 = s_1, \ldots, t_n = s_n \Rightarrow f(t_1, \ldots, t_n) = f(s_1, \ldots, s_n)$.
       Where f is any function symbol.
    3. $t_1 = s_1, \ldots, t_n = s_n, R(t_1, \ldots, t_n) \Rightarrow R(s_1, \ldots, s_n)$.
       Where R is any atomic formula.
- **Initial sequents for Arithmetic**
    1. $\Rightarrow \neg(S(s) = 0)$
    2. $S(s) = S(t) \Rightarrow s = t$

[1] G is after Gentzen

3. $\Rightarrow t + 0 = t$
4. $\Rightarrow s + S(t) = S(s + t)$
5. $\Rightarrow s \cdot 0 = 0$
6. $\Rightarrow s \cdot S(t) = s \cdot t + s$

- **Rule of Induction**[2] **(IR)**

$$\text{(IR)}\ \frac{\Rightarrow \phi(0) \qquad \phi(x) \Rightarrow \phi(S(x))}{\Rightarrow \phi(t)}$$

Definition 3.5. *(**Proof**) A proof in* CFA *is a finite labelled tree whose nodes are labelled by sequents, in such a way that leaves are labelled by initial sequents of* CFA *and each sequent at a node is obtained from sequents at immediate predecessor(s) node(s) according to one of the rules of* CFA*. We shall denote proofs by means of metavariables $D, D', \ldots$. If D is a proof, a subtree D' of D which is itself a proof is called a subproof of D.*
A sequent $\Gamma \Rightarrow \Delta$ is provable in CFA *($CFA \vdash_{GQC^G} \Gamma \Rightarrow \Delta$) iff it labels the root of some proof in* CFA *(i.e. iff it is the end-sequent of such a proof).*

Proposition 3.6. $\mathsf{CFA}^{\mathsf{G}} \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathsf{CFA}^{\mathsf{H}} \vdash t(\Gamma \Rightarrow \Delta)$.

Given the preceding proposition, henceforth, we will often just write $\mathsf{CFA} \vdash$ for $\mathsf{CFA}^{\mathsf{G}} \vdash$ or $\mathsf{CFA}^{\mathsf{H}} \vdash$, if the context allows us to do so.

Remark 3.7. *The usual properties of addition and multiplication like commutativity or distributivity can be readily established using the Induction rule, akin to standard proofs utilizing the induction schema.*

The *standard model* of CFA is a model with universe $\mathbb{N} = \{0, 1, \ldots\}$ such that all symbols have the usual interpretation. Further we say for any $n \in \mathbb{N}$ and $\overline{n}$ is the abbreviation for the term $S(S(..S(0)..))$ (i.e. n occurrences of S) in the language of CFA. We call the term $\overline{n}$ the nth numeral.

Definition 3.8. *For abbreviation, we let $y < x$ stand for $\exists z(y + S(z) = x)$ and $y \leq x$ for $y < x \oplus y = x$.*

As usual, for any term t and variable x, $\exists x \leq t\ \phi(x)$ and $\forall x \leq t\ \phi(x)$ is an abbreviation for $\exists x(x \leq t \otimes \phi)$ and $\forall x(x \leq t \to \phi)$ respectively[3]. Furthermore, we call the quantifiers occurring as $(\exists x \leq t)$ or $(\forall x \leq t)$ *bounded*.

Definition 3.9. *(Arithmetic hierarchy) A formula is called a bounded formula if it contains only bounded quantifiers. The set of bounded formulas is denoted by Δ_0. For $n \geq 0$, the classes Σ_n and Π_n of first order formulas are inductively defined by:*

[2] Note that its an implicit assumption that x does not occur in $\phi(0)$, otherwise the rule will not be sound w.r.t. $\mathbb{N}$

[3] By convention, x and t are distinct.

1. $\Sigma_0 = \Pi_0 = \Delta_0$
2. Σ_{n+1} *is the set of formulas of the form* $(\exists x)A$ *where* $A \in \Pi_n$.
3. Π_{n+1} *is the set of formulas of the form* $(\forall x)A$ *where* $A \in \Sigma_n$.

Remark 3.10.
Any formula ϕ *that is equivalent to a* Σ_n *formula is* informally *also referred to as a* Σ_n *formula. Similarly, for* Π_n *formulas. Henceforth, we will work with this informal usage.*
At the same time, it's important to note that unlike classical logic, where any formula is equivalent to some formula in the arithmetical hierarchy, CFA *presents a different scenario. Here, the arithmetical hierarchy does not exhaust all formulas in the language of* CFA. *This occurs because there are formulas in the language of* CFA *that are not equivalent to any prenex formula.*

Proposition 3.11. (CFA) *Given a formula* $\phi(x, y)$ *with* x, y *as free variables, the following sentences are provably equivalent:*

1. $\exists x \exists y \phi(x, y)$
2. $\exists w(\exists x \leq w)(\exists y \leq w)\phi(x, y)$*, where* w *does not occur in* ϕ.

Proof. We reason within CFA formalized in Hilbert style.
From (1) to (2) : For any a, b we have, $\phi(a, b) \to (a \leq (a + b) \otimes b \leq (a + b) \otimes \phi(a, b))$. From which, by $\exists-$introduction we have, $\phi(a, b) \to \exists x \leq (a + b) \exists y \leq (a + b)\phi(x, y)$. Applying $\exists-$introduction again, gives us $\phi(a, b) \to \exists w(\exists x \leq w)(\exists y \leq w)\phi(x, y)$. Now, $\exists$ elimination gives us, $\exists x \exists y \phi(x, y) \to \exists w(\exists x \leq w)(\exists y \leq w)\phi(x, y)$ as needed.
From (2) to (1) : Since w does not occur in ϕ, unpacking the abbreviations involved in the bounded quantifiers is enough to show that $\exists w(\exists x \leq w)(\exists y \leq w)\phi(x, y) \to \exists x \exists y \phi(x, y)$ □

Corollary 3.12. *Any formula* starting *with* $n > 1$ *unbounded existential quantifiers is provably equivalent to a formula starting with just a single unbounded quantifier.*

3.2 Additive Connectives

In this section, we demonstrate that, in a sense, the additive connectives become definable in CFA. We establish this by demonstrating the stronger result that the system of contraction-free arithmetic together with additive connectives is equivalent to CFA (Theorem 3.18).
For our purposes, we introduce an extension CFA′ of CFA in the language L' and in the present section, we mainly reason in sequent style.

Definition 3.13. (CFA′) *Let* $L' := L_{CFA} \cup \{\wedge, \vee\}$. CFA′ *is the calculus defined over* CFA *in the language* L' *with the following additional rules of inference.*

$$(\wedge L)\ \frac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad (\wedge R)\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$(\vee L)\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad (\vee R)\ \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B}$$

We will demonstrate that these newly introduced connectives can be expressed in terms of multiplicative connectives alone within CFA'. First, we present some basic preliminary lemmas.

Lemma 3.14. (CFA)

(a) $\top \to B \Leftrightarrow B$

(b) $\bot \to B \Leftrightarrow \top$

(c) $0 = 0 \Leftrightarrow \top$

(d) If a is a numeral different from 0 then, $a = 0 \Leftrightarrow \bot$.

(e) $\top \to B \otimes \bot \to C \Rightarrow B$

(f) $B \wedge C \Rightarrow \bot \to B \otimes \top \to C$

Proposition 3.15. (CFA') *Let B, C be any formula where x does not occur free, then :*

1. $B \wedge C \Leftrightarrow \forall x(x = 0 \to B \otimes \neg(x = 0) \to C)$

2. $B \vee C \Leftrightarrow \exists x(x = 0 \to B \otimes \neg(x = 0) \to C)$

Proof.
1. "$\Leftarrow$"
With the help of Lemma 3.14, we have the following proof tree[4].

$$(\wedge\text{R})\ \frac{(\forall\text{L})\ \dfrac{\vdots \atop 0 = 0 \to B \otimes 0 \neq 0 \to C \Rightarrow B}{\forall x(x = 0 \to B \otimes x \neq 0 \to C) \Rightarrow B} \qquad (\forall\text{L})\ \dfrac{\vdots \atop S(0) = 0 \to B \otimes S(0) \neq 0 \to C \Rightarrow C}{\forall x(x = 0 \to B \otimes x \neq 0 \to C) \Rightarrow C}}{\forall x(x = 0 \to B \otimes x \neq 0 \to C) \Rightarrow B \wedge C}$$

"$\Rightarrow$"

Let $\phi(x) := B \wedge C \to (x = 0 \to B \otimes \neg(x = 0) \to C)$. We proceed with Induction Rule on the formula ϕ.
Basis: Since $\Rightarrow B \wedge C \to B$, it is easy to check that $\phi(0)$ is provable.

[4] Actually, the proofs are only a proof sketch from which formal proof trees can be directly constructed.

Inductive Step: To show that $\phi(x) \Rightarrow \phi(S(x))$ is provable, consider the following proof tree :

$$\text{(WL)}\ \dfrac{(\to R)\ \dfrac{\begin{array}{c}\vdots\\ B \wedge C \Rightarrow (S(x) = 0 \to B) \otimes (S(x) \neq 0 \to C)\end{array}}{\Rightarrow B \wedge C \to ((S(x) = 0 \to B) \otimes (S(x) \neq 0 \to C))}}{B \wedge C \to ((x = 0 \to B) \otimes (x \neq 0 \to C)) \Rightarrow B \wedge C \to ((S(x) = 0 \to B) \otimes (S(x) \neq 0 \to C))}$$

Hence, we have that $\phi(x) \Rightarrow \phi(S(x))$ is provable.

Thus, by applying the Induction rule in CFA' we get $\Rightarrow \forall x \phi(x)$. Moreover, as x is not free in $B \wedge C$, we get

$$B \wedge C \Rightarrow \forall x((x = 0 \to B) \otimes (x \neq 0 \to C)).$$

2. The proof is similar to part 1. We refer to Appendix B.

□

Returning to our arithmetic CFA, we note that a similar phenomenon occurs: $\forall x(x = 0 \to A \otimes \neg(x = 0) \to B)$ and $\exists x(x = 0 \to A \otimes \neg(x = 0) \to B)$ behave exactly like $A \wedge B$ and $A \vee B$ respectively. That is to say,

Proposition 3.16. (CFA) *The following rules are derivable :*

1.

$$\dfrac{A, \Gamma \Rightarrow \Delta}{\forall x(x = 0 \to A \otimes \neg(x = 0) \to B), \Gamma \Rightarrow \Delta}$$

$$\dfrac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, \forall x(x = 0 \to A \otimes \neg(x = 0) \to B)}$$

2.

$$\dfrac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{\exists x(x = 0 \to A \otimes \neg(x = 0) \to B), \Gamma \Rightarrow \Delta}$$

$$\dfrac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, \exists x(x = 0 \to A \otimes \neg(x = 0) \to B)}$$

Proof. The proofs of these derivable rules have exactly the same flavor as those in the previous proposition. □

With Propositions 3.15 and 3.16 at hand, we are now almost ready to show the equivalence. In order to formally capture what we mean by the systems CFA' and CFA being equivalent, we first introduce the notion of formula translation.

Definition 3.17. *(Formula translation) Let ϕ be any formula in the language of CFA'. The formula translation ϕ^* of ϕ, in the language of CFA is defined inductively on the structure of ϕ.*

- *If ϕ is atomic then $\phi^* := \phi$.*
- *If $\phi = A \to B$ then $\phi^* := A^* \to B^*$; if $\phi = \forall x(B(x))$ then $\phi^* := \forall x(B(x)^*)$.*
- *If $\phi = A \wedge B$ then $\phi^* := \forall x((x = 0 \to A^*) \otimes (x \neq 0 \to B^*))$, where x does not occur in ϕ.*
- *If $\phi = A \vee B$ then $\phi^* := \exists x((x = 0 \to A^*) \otimes (x \neq 0 \to B^*))$, where x does not occur in ϕ.*

We naturally extend the formula translation to multiset of formulae. If Γ is a multiset, then Γ^* is the multiset that contains only the translations of formulas in Γ.

Theorem 3.18. *(Equivalence) $\mathsf{CFA}' \vdash\ \Rightarrow \phi$ iff $\mathsf{CFA} \vdash\ \Rightarrow \phi^*$, where ϕ^* is the formula translation of ϕ into CFA.*

Proof. We break the problem into the following claims.

Claim (i) $\mathsf{CFA}' \vdash \phi^* \Rightarrow \phi$.
We proceed with induction on the structure of ϕ. The basis is straightforward, following directly from the definition of formula translation. For the inductive steps, we will rely on both the definition of formula translation and Proposition 3.15. For instance, we work out the case where $\phi = A \wedge B$: By our induction hypothesis, we have $\mathsf{CFA}' \vdash A^* \Rightarrow A$ and $\mathsf{CFA}' \vdash B^* \Rightarrow B$. Consequently, we can derive the following sequent within CFA':
$\forall x(x = 0 \to A^* \otimes \neg(x = 0) \to B^*) \Rightarrow \forall x(x = 0 \to A \otimes \neg(x = 0) \to B)$.
Now, utilizing Proposition 3.15 and the cut rule, we obtain:

$$\mathsf{CFA}' \vdash \forall x(x = 0 \to A^* \otimes \neg(x = 0) \to B^*) \Rightarrow A \wedge B$$

i.e. $\mathsf{CFA}' \vdash \phi^* \Rightarrow \phi$ as required.

Claim (ii) If a sequent is provable in CFA' then its translated sequent is provable in CFA, i.e. if $\mathsf{CFA}' \vdash \Gamma \Rightarrow \Delta$ then $\mathsf{CFA}' \vdash \Gamma^* \Rightarrow \Delta^*$.

We proceed by induction on the height of the proof of $\vdash \Gamma \Rightarrow \Delta$ in CFA'.
Base case: If the height of the proof is $\mathbf{0}$, then $\Gamma \Rightarrow \Delta$ is an initial sequent of CFA'. However, the initial sequents of CFA' are exactly the initial sequents of CFA. Furthermore, $\Gamma^* = \Gamma$ and $\Delta^* = \Delta$. Thus, $CFA \vdash \Gamma^* \Rightarrow \Delta^*$ as required.

Inductive Case: Suppose $\Gamma \Rightarrow \Delta$ has a proof of height $\mathbf{n}$ (> 0) in CFA'. We proceed by considering cases based on the last rule applied.

- Case $\to R$.
  Suppose $A \to B$ is the principal formula. Then in CFA' we have,

  $$\begin{array}{c}\vdots\\ (\to R)\,\dfrac{A, \Gamma \Rightarrow \Delta', B}{\Gamma \Rightarrow A \to B, \Delta'}\end{array}$$

  where $\Delta = A \to B, \Delta'$.
  Now $A, \Gamma \Rightarrow \Delta', B$ has a proof of height $n - 1$ in CFA'. By induction hypothesis, we have, $\mathsf{CFA} \vdash A^*, \Gamma^* \Rightarrow \Delta'^*, B^*$. Then we obtain the following proof in CFA:

  $$\begin{array}{c}\vdots\\ (\to R)\,\dfrac{A^*, \Gamma^* \Rightarrow \Delta'^*, B^*}{\Gamma^* \Rightarrow A^* \to B^*, \Delta'^*}\end{array}$$

  Hence, $\mathsf{CFA} \vdash \Gamma^* \Rightarrow A^* \to B^*, \Delta'^*$. According to our definition of formula translation $(A \to B)^* = A^* \to B^*$. Therefore, $CFA \vdash \Gamma^* \Rightarrow (A \to B)^*, \Delta'^*$.

- Cases $\to L, \forall R, \forall L$, IR.
  Note that since $(A \to B)^* = A^* \to B^*$ and $(\forall x A(x))^* = \forall x(A(x)^*)$. The same strategy as in the previous case will apply.

- Case $\wedge L$.
  Suppose $A \wedge B$ is the principal formula. Then in CFA' we have

  $$\begin{array}{c}\vdots\\ (\wedge L)\,\dfrac{A, \Gamma' \Rightarrow \Delta}{A \wedge B, \Gamma' \Rightarrow \Delta}\end{array}$$

  Where $\Gamma = A \wedge B, \Gamma'$
  Now $A, \Gamma' \Rightarrow \Delta$ has a proof of height $n-1$ in CFA'. By induction hypothesis we have, $CFA \vdash A^*, \Gamma'^* \Rightarrow \Delta^*$. Then by Proposition 3.16,

  $$\mathsf{CFA} \vdash \forall x(x = 0 \to A^* \otimes \neg(x = 0) \to B^*), \Gamma^* \Rightarrow \Delta^*$$

  i.e.

  $$\mathsf{CFA} \vdash (A \wedge B)^*, \Gamma^* \Rightarrow \Delta^*$$

  as required.

- Case $\wedge R$.
  Suppose $A \wedge B$ is the principal formula, then in CFA' we have:

  $$(\wedge\text{R})\ \frac{\Gamma \Rightarrow \Delta', A \qquad \Gamma \Rightarrow \Delta', B}{\Gamma \Rightarrow \Delta', A \wedge B}$$

  where $\Delta = \Delta', A \wedge B$.
  Now $\Gamma \Rightarrow \Delta', A$ and $\Gamma \Rightarrow \Delta', B$ have a proof of height $< \mathbf{n}$ in CFA'. By induction hypothesis we have, $CFA \vdash \Gamma'^* \Rightarrow \Delta'^*, A^*$ and $CFA \vdash \Gamma'^* \Rightarrow \Delta'^*, B^*$. Then by Proposition 3.16,

  $$\mathsf{CFA} \vdash \Gamma^* \Rightarrow \Delta^*, \forall x(x = 0 \to A^* \otimes \neg(x = 0) \to B^*)$$

  i.e.

  $$\mathsf{CFA} \vdash \Gamma^* \Rightarrow \Delta^*, (A \wedge B)^*$$

  as required.

- $\vee R, \vee L$. They are analogous to the two preceding cases.

As these cases exhaust all possible inference rules, we conclude that $CFA \vdash \Gamma^* \Rightarrow \Delta^*$. Thus, by strong induction on the height of proof we have our required result.

Now, given that CFA' extends CFA, it directly follows from Claim **(i)** and **(ii)** that $\mathsf{CFA}' \vdash\Rightarrow \phi$ if and only if $\mathsf{CFA} \vdash\Rightarrow \phi^*$.

□

Remark 3.19. *Theorem [3.18] explains why adding additive connectives to* CFA *will not influence our arithmetic in any way. Furthermore, it allows us to introduce $\wedge, \vee$ in our language (as abbreviations) freely so that we can exploit them to ease our treatment of the subject. Henceforth, $A \wedge B$ and $A \vee B$ will be treated as abbreviations for $\forall x(x = 0 \to A \otimes \neg(x = 0) \to B)$ and $\exists x(x = 0 \to A \otimes \neg(x = 0) \to B)$ respectively.*

Remark 3.20. *Given the previous remark, we are now in a position to utilize theorems about non-contractive logics found in the literature. We will be particularly interested in the sequent-style system LA and its equivalent Hilbert system HA from Paoli [2002]. Essentially, LA is a propositional logic without contraction, containing the usual rules of connectives, including additive ones. In fact, all theorems of LA will be valid in* CFA.

In the remainder of this section, we prove some useful results in CFA. To begin with, we show that if a formula A in the language ofCFAsatisfies excluded middle with additive connective , i.e. $A \vee \neg A$, then left and right contraction holds for A.

Proposition 3.21. (CFA) *Suppose we have* $\mathsf{CFA} \vdash\ \Rightarrow A \vee \neg A$ *then,* *1.* $\mathsf{CFA} \vdash A \Rightarrow A \otimes A$

2. $\mathsf{CFA} \vdash A \oplus A \Rightarrow A$

3. Left and Right contraction is admissible for A.

Proof.

1. Let C be an abbreviation for $A \to (A \otimes A)$. Then following sequents are provable in CFA[5]: (a) $A \to C, \neg A \to C \Rightarrow A \lor \neg A \to C$; (b) $\Rightarrow A \to C$; (c) $\Rightarrow \neg A \to C$.

   Using them we construct the following proof tree[6]:

$$(\mathrm{Cut})\,\dfrac{(\mathrm{Cut})\,\dfrac{\begin{array}{c}\vdots\\ A \to C, \neg A \to C \Rightarrow A \lor \neg A \to C\end{array}\qquad\begin{array}{c}\vdots\\ \Rightarrow A \to C\end{array}}{\neg A \to C \Rightarrow A \lor \neg A \to C}\qquad\begin{array}{c}\vdots\\ \Rightarrow \neg A \to C\end{array}}{\Rightarrow A \lor \neg A \to C}$$

   Since $\Rightarrow A \lor \neg A \to C$, from Proposition 2.12, we have that $A \lor \neg A \Rightarrow C$.

   Now consider the following proof tree.

$$(\mathrm{Cut})\,\dfrac{\begin{array}{c}\vdots\\ A \lor \neg A \Rightarrow C\end{array}\qquad\begin{array}{c}\vdots\\ \Rightarrow A \lor \neg A\end{array}}{\Rightarrow C}$$

   Thus, we have $CFA \vdash A \Rightarrow A \otimes A$.

2. Proposition 2.10.(ix) provides us with $A \Leftrightarrow \neg\neg A$, from which it follows that $A \lor \neg A \Leftrightarrow \neg A \lor \neg\neg A$. Hence, based on our assumption, we have $\neg A \lor \neg\neg A$. Then, utilizing part 1, we deduce $\Rightarrow \neg A \to \neg A \otimes \neg A$.
   Now consider the following proof tree:

$$(\mathrm{Cut})\,\dfrac{\begin{array}{c}\vdots\\ \Rightarrow \neg A \to \neg A \otimes \neg A\end{array}\qquad\begin{array}{c}\vdots\\ \neg A \to \neg A \otimes \neg A \Rightarrow \neg(\neg A \otimes \neg A) \to \neg\neg A\end{array}}{\Rightarrow \neg(\neg A \otimes \neg A) \to \neg\neg A}$$

   Since $\Rightarrow \neg(\neg A \otimes \neg A) \to \neg\neg A$, from Proposition 2.12 we have that $\neg(\neg A \otimes \neg A) \Rightarrow A$.

   Now consider the following proof tree.

[5] For details, the reader can refer to Proposition 2.13 in Paoli [2002].

[6] This is actually only a proof sketch, from which a proof tree can be directly constructed

$$\text{(Cut)}\ \frac{\begin{array}{c}\vdots\\ \neg(\neg A\otimes\neg A)\Rightarrow A\end{array}\qquad \begin{array}{c}\vdots\\ A\oplus A\Rightarrow\neg(\neg A\otimes\neg A)\end{array}}{A\oplus A\Rightarrow A}$$

Thus, we have $CFA \vdash A \oplus A \Rightarrow A$.

3. Immediate from Proposition 2.30.

□

Now, we will provide some basic theorems in CFA that will be necessary for future reference.

Proposition 3.22. *(CFA)* [7]

1. $A \otimes (B \vee C) \Leftrightarrow (A \otimes B) \vee (A \otimes C)$
2. $(B \wedge \neg C) \vee (\neg B \wedge C) \vee (\neg B \wedge \neg C) \Rightarrow \neg(B \wedge C)$
3. $(B \otimes \neg C) \vee (\neg B \otimes C) \vee (\neg B \otimes \neg C) \Rightarrow \neg(B \otimes C)$
4. $y < x \Rightarrow y < S(x)$
5. *Double Induction.*

$$\textit{(DoubleInd)}\ \frac{\Rightarrow A(0,0)\quad \Rightarrow \forall x A(0,x)\quad \Rightarrow \forall x A(x,0)\qquad \Rightarrow \forall x\forall y(A(x,y)\to A(S(x),S(y)))}{\Rightarrow \forall x\forall y A(x,y)}$$

We end this section by highlighting that while the additive connectives become expressible in CFA, certain properties that hold for them in **LK** may not necessarily hold in CFA in the absence of contraction. For instance, the following are usually not the case in CFA,

- $A \to (B \to C) \Rightarrow A \wedge B \to C$; this requires left contraction for $A \wedge C$.
- $A \to B \Rightarrow \neg A \vee B$; this requires right contraction for $\neg A \vee B$.
- $A \wedge (A \to B) \Rightarrow B$; this requires left contraction for $A \wedge (A \to B)$.
- $A \wedge (B \vee C) \Rightarrow (A \wedge B) \vee (A \wedge C)$
- $(A \vee B) \wedge (A \vee C) \Rightarrow A \vee (B \wedge C)$

[7] We direct the reader to Appendix B for proofs of certain parts of this proposition.

3.3 Induction

In this section, we discuss Induction inside CFA. The study of Induction is perhaps one of the most interesting aspects of arithmetic. In fact, a significant portion of research in arithmetic consists of studying various systems of arithmetic by restricting induction schema to specific levels arithmetical hierarchy. In our axiomatization of CFA, we were careful to incorporate the Induction Rule instead of the Induction schema. Near the end of this section (Proposition 3.35), we justify our choice. The primary result of this section is the derivation of the induction schema for bounded formulas in CFA (Theorem 3.32). Subsequently, we show (Theorem 3.38) that any Π_2 formula provable in the arithmetical theory $\mathsf{I}\Delta_0$ become provable in CFA under a natural formula translation.

In order to establish that Δ_0 induction schema is provable in CFA, we have to go through some preliminary lemmas which involve establishing some basic arithmetical results in CFA. Our initial focus is to show that atomic formulas allow left and right contraction.

Definition 3.23. *(Abbreviation) We abbreviate $A \leftrightarrow B$ to mean $(A \to B) \otimes (B \to A)$.*

Lemma 3.24. (CFA)

1. $\mathsf{CFA} \vdash\ \Rightarrow \forall x(x = 0 \vee \neg(x = 0))$
2. $\mathsf{CFA} \vdash\ \Rightarrow \forall y \forall z \exists x(y = z \leftrightarrow x = 0)$

Proof.

1. Consider the following proof tree:

$$(\mathrm{IR})\ \dfrac{\Rightarrow 0 = 0 \vee \neg(0 = 0) \qquad (\mathrm{WL})\ \dfrac{\vee R\ \dfrac{\Rightarrow \neg(S(x) = 0)}{\Rightarrow (S(x) = 0) \vee \neg(S(x) = 0)}}{x = 0 \vee \neg(x = 0) \Rightarrow (S(x) = 0) \vee \neg(S(x) = 0)}}{\Rightarrow \forall x(x = 0 \vee \neg(x = 0))}$$

2. Let $A(y, z) := \exists x(y = z \leftrightarrow x = 0)$. We demonstrate the lemma by employing double induction on A. To achieve this, we establish the following:

   - $\Rightarrow \exists x(0 = 0 \leftrightarrow x = 0)$

$$(\exists R)\ \dfrac{\Rightarrow (0 = 0 \leftrightarrow 0 = 0)}{\Rightarrow \exists x(0 = 0 \leftrightarrow x = 0)}$$

- $\Rightarrow \forall y \exists x(y = 0 \leftrightarrow x = 0)$.

  Let a be any free variable and consider the following proof tree.

$$(\forall R)\,\dfrac{(\exists R)\,\dfrac{\Rightarrow (a = 0 \leftrightarrow a = 0)}{\Rightarrow \exists x(a = 0 \leftrightarrow x = 0)}}{\Rightarrow \forall y \exists x(y = 0 \leftrightarrow x = 0)}$$

- $\Rightarrow \forall z \exists x(0 = z \leftrightarrow x = 0)$.
  Let a be any free variable and consider the following proof tree.

$$(\forall R)\,\dfrac{(\exists R)\,\dfrac{\begin{array}{c}\vdots \\ \Rightarrow (0 = a \leftrightarrow a = 0)\end{array}}{\Rightarrow \exists x(0 = a \leftrightarrow x = 0)}}{\Rightarrow \forall z \exists x(0 = z \leftrightarrow x = 0)}$$

- $\forall y \forall z(\exists x(y = z \leftrightarrow x = 0) \rightarrow \exists x(S(y) = S(z) \leftrightarrow x = 0))$.
  Observe that $a = b \Leftrightarrow S(a) = S(b)$ follows from the initial sequents of arithmetic and equality. Now consider the following proof tree.

$$(\forall R)\,\dfrac{(\rightarrow R)\,\dfrac{\begin{array}{c}(a = b \leftrightarrow x = 0) \Rightarrow (S(a) = S(b) \leftrightarrow x = 0) \\ \vdots \\ \exists x(a = b \leftrightarrow x = 0) \Rightarrow \exists x(S(a) = S(b) \leftrightarrow x = 0)\end{array}}{\Rightarrow \exists x(a = b \leftrightarrow x = 0) \rightarrow \exists x(S(a) = S(b) \leftrightarrow x = 0)}}{\Rightarrow \forall y \forall z(\exists x(y = z \leftrightarrow x = 0) \rightarrow \exists x(S(y) = S(z) \leftrightarrow x = 0))}$$

Thus, we have established the necessary premise for double induction on A. Therefore, through double induction, we obtain $\Rightarrow \forall y \forall z \exists x(y = z \leftrightarrow x = 0)$.

□

Lemma 3.25. (CFA) *If A is an atomic formula, then* CFA $\vdash A \vee \neg A$ *and subsequently, left and right contraction holds for A.*

Proof. As A is an atomic formula, A is of the form $t_1 = t_2$. Without loss of generality assume that x is does not occur in $t_1 = t_2$. We reason within CFA formalized in Hilbert style. From Lemma 3.24.2 it follows that $\vdash \exists x((x = 0) \vee \neg(x = 0) \rightarrow (t_1 = t_2) \vee \neg(t_1 = t_2))$, and from Lemma 3.24.1 we have $\vdash \forall x(x = 0 \vee \neg(x = 0))$. Furthermore, from Proposition 2.31.4, if x is not free in Y, we can derive $\exists x(X \rightarrow Y) \rightarrow (\forall x X \rightarrow Y)$. Combining these results, we obtain CFA $\vdash (t_1 = t_2) \vee \neg(t_1 = t_2)$. Finally, Proposition 3.21 gives us that left and right contraction holds for A. □

As a corollary of the preceding lemma, we demonstrate a very important property of equality: terms that are equal can be substituted for each other in any formula.

Corollary 3.26. *(Properties of equality in* CFA)

1. *Equality is symmetric.*
2. *Equality is transitive.*
3. *For any formula A, $t_1 = s_1, \ldots, t_n = s_n, A(t_1, \ldots, t_n) \Rightarrow A(s_1, \ldots, s_n)$*

Proof.

- For proofs of equality being symmetric and transitive, we refer the reader to Appendix B.
- We show it for $n = 1$, the general proof follows analogously. We proceed by induction on the structure of A.
  Base Case: A is atomic, then it follows from initial sequents of equality.
  Inductive Case:
  When $A(y) := B(y) \to C(y)$. First, observe that Lemma 3.25 guarantees contraction for atomic formulas of the form $t_i = s_i$. Now consider the following proof tree, which is constructed using the induction hypothesis.

$$
(\mathrm{CL})\,\dfrac{(\mathrm{Cut})\,\dfrac{(\to R)\,\dfrac{(\to L)\,\dfrac{\begin{matrix}\vdots\\ s_1 = t_1, B(s_1) \Rightarrow B(t_1)\end{matrix}\qquad \begin{matrix}\vdots\\ t_1 = s_1, C(t_1) \Rightarrow C(s_1)\end{matrix}}{s_1 = t_1, t_1 = s_1, B(t_1) \to C(t_1), B(s_1) \Rightarrow C(s_1)}}{s_1 = t_1, t_1 = s_1, B(t_1) \to C(t_1) \Rightarrow B(s_1) \to C(s_1)}\qquad \begin{matrix}\vdots\\ t_1 = s_1 \Rightarrow s_1 = t_1\end{matrix}}{t_1 = s_1, t_1 = s_1, B(t_1) \to C(t_1) \Rightarrow B(s_1) \to C(s_1)}}{t_1 = s_1, B(t_1) \to C(t_1) \Rightarrow B(s_1) \to C(s_1)}
$$

  When $A(y) := \forall x B(x, y)$. Consider the following proof tree, which is constructed using the induction hypothesis.

$$
(\forall R)\,\dfrac{(\forall L)\,\dfrac{\begin{matrix}\vdots\\ t_1 = s_1, B(x, t_1) \Rightarrow B(x, s_1)\end{matrix}}{t_1 = s_1, \forall x B(x, t_1) \Rightarrow B(x, s_1)}}{t_1 = s_1, \forall x B(x, t_1) \Rightarrow \forall x B(x, s_1)}
$$

□

Another important corollary, which will be key in our proofs later on, is that the less-than predicate (which is a priori Σ_1) is in fact equivalent to a bounded formula, thus it can be assumed to be bounded.

Corollary 3.27. (CFA) *$y < x$, i.e. $\exists z(y + S(z) = x)$, is equivalent to the bounded formula $\exists z < x(y + S(z) = x)$.*

Proof. It is easy to see that $\exists z < x(y + S(z) = x) \to \exists z(y + S(z) = x)$. We prove the other direction.
Axioms of arithmetic and equality would give us $(y + S(z) = x) \to z < x$. Combining it with $(y + S(z) = x) \to (y + S(z) = x)$, we get

$$(y + S(z) = x) \otimes (y + S(z) = x) \to (z < x) \otimes (y + S(z) = x)$$

Since we have *contraction* for atomic formulas, we have that $(y + S(z) = x) \to (y + S(z) = x) \otimes (y + S(z) = x)$. Thus we get,

$$(y + S(z) = x) \to (z < x) \otimes (y + S(z) = x)$$

Now $\exists$ introduction would give us,

$$(y + S(z) = x) \to \exists z((z < x) \otimes (y + S(z) = x))$$

Finally, from $\exists$ elimination we have,

$$\exists z(y + S(z) = x) \to \exists z((z < x) \otimes (y + S(z) = x))$$

or,

$$\exists z(y + S(z) = x) \to \exists z < x(y + S(z) = x).$$

□

We can now demonstrate that contraction holds for Δ_0.

Theorem 3.28. (CFA) *For any Δ_0 formula A, we have that* CFA $\vdash A \vee \neg A$

Proof. We proceed by induction on the structure of A.

Base Case: This is Lemma 3.25.

Inductive Step: We break it up into the following cases[8] and reason in CFA formalized in Hilbert-style:

- $A = \neg B$. By the induction hypothesis, we have $B \vee \neg B$. Combining this with $B \leftrightarrow \neg\neg B$, we derive $\neg\neg B \vee \neg B$. Lastly, applying the commutativity of $\vee$, we obtain $\neg B \vee \neg\neg B$.

[8] As $A \to B \Leftrightarrow \neg(A \otimes \neg B)$ and $\neg A \leftrightarrow A \to \bot$, it is sufficient to work out the cases when $A = \neg B$ and $A = B \otimes C$

- $A = B \otimes C$. By the induction hypothesis, we have $B \vee \neg B$ and $C \vee \neg C$. Combining them, we obtain $(B \vee \neg B) \otimes (C \vee \neg C)$. Now, applying the distributivity (Proposition 3.22.1) of $\otimes$ over $\vee$, we arrive at $(B \otimes C) \vee ((B \otimes \neg C) \vee (\neg B \otimes C) \vee (\neg B \otimes \neg C))$. Finally, from Proposition 3.22.3, we conclude $(B \otimes C) \vee \neg(B \otimes C)$.

- $A = \forall y < zB(y) = \forall y(y < z \to B(y))$.

  Let $\phi(z) = \forall y(y < z \to B(y)) \vee \neg\forall y(y < z \to B(y))$. We will proceed by applying CFA's induction Rule on ϕ and establish that for any z, $\phi(z)$ holds in CFA.

  (1) To show $\phi(0)$.
  For any y, we can easily derive $y < 0 \to B(y)$ from the axioms of arithmetic. Generalization would then give us $\forall y(y < 0 \to B(y))$. Since for any formula A, B we have $A \to A \vee B$, we can finally derive $\forall y(y < 0 \to B(y)) \vee \neg\forall y(y < 0 \to B(y))$. Thus, we have proved $\phi(0)$ in CFA.

  (2) To show $\forall x(\phi(x) \to \phi(S(x)))$.
  Our strategy is to first show that $\forall y(y < x \to B(y)) \to \phi(S(x))$ and then show $\neg\forall y(y < x \to B(y)) \to \phi(S(x))$. Since for any formulas A, B and C we have: If $A \to B$ and $C \to B$, then $(A \vee C) \to B$. Thus, using this we can derive $\phi(x) \to \phi(S(x))$. Finally, generalization would give us $\forall x(\phi(x) \to \phi(S(x)))$.

  To show $\forall y(y < x \to B(y)) \to \phi(S(x))$. It suffices to demonstrate that $B(x) \otimes \forall y(y < x \to B(y)) \to \forall y(y < S(x) \to B(y))$ and $\neg B(x) \otimes \forall y(y < x \to B(y)) \to \neg\forall y(y < S(x) \to B(y))$. As from these, we can derive $(B(x) \vee \neg B(x)) \otimes \forall y(y < x \to B(y)) \to \neg\forall y(y < S(x) \to B(y))$. Then by induction hypothesis, we obtain $B(x) \vee \neg B(x)$, and subsequently, we can conclude $\forall y(y < x \to B(y)) \to \neg\forall y(y < S(x) \to B(y))$ as needed.

  Note that from induction hypothesis, we have $B(x) \to B(x) \otimes B(x)$ and $B(x) \oplus B(x) \to B(x)$. **...(i)**

  For $B(x) \otimes \forall y(y < x \to B(y)) \to \forall y(y < S(x) \to B(y))$. **...(ii)**
  Properties of equality gives us $B(x) \to (y = x \to B(y))$. Combining that with $(y < x \to B(y))$, we get $(B(x) \otimes (y < x \to B(y))) \to ((y = x \to B(y))) \otimes (y < x \to B(y)))$. In our logic, for any formulae L_1, L_2, M_1, M_2, we have $(L_1 \to M_1) \otimes (L_2 \to M_2) \to (L_1 \oplus L_2 \to M_1 \oplus M_2)$. Thus we get, $(y < x \to B(y)) \otimes B(x) \to (y < x \oplus y = x \to B(y) \oplus B(y))$. Now from *(i)*, we deduce, $(y < x \to B(y)) \otimes B(x) \to (y \leq x \to B(y))$. Generalization and properties of universal quantifier gives us $\forall y((y < x \to B(y)) \otimes B(x)) \to \forall y(y \leq x \to B(y))$. Since[9] y does not occur freely in

[9] This assumption is safe to make, as otherwise, we can simply replace y in the formulation

$B(x)$, we get $\forall y(y < x \to B(y)) \otimes B(x) \to \forall y(y \leq x \to B(y))$. Finally[10], as $y < S(x) \to y \leq x$, we get $\forall y(y < x \to B(y)) \otimes B(x) \to \forall y(y < S(x) \to B(y))$., as required.

For $\neg B(x) \otimes \forall y(y < x \to B(y)) \to \neg\forall y(y < S(x) \to B(y))$. **...(iii)**
Axioms of universal quantifier gives us $\forall y(y = x \to B(y)) \to (x = x \to B(x))$. Since $x = x$ follows from the axioms of arithmetic, we get $\forall y(y = x \to B(y)) \to B(x)$. Then contra-position would give us, $\neg B(x) \to \neg\forall y(y = x \to B(y))$. From which we get $\neg B(x) \to \neg\forall y(y < S(x) \to B(y))$. Finally, weakening gives us $\forall y(y < x \to B(y)) \otimes \neg B(x) \to \neg\forall y(y < S(x) \to B(y))$.
Thus, from **(ii)** and **(iii)** we get $\forall y(y < x \to B(y)) \to \phi(S(x))$. **(R1)**

Now, we show that $\neg\forall y(y < x \to B(y)) \to \phi(S(x))$.
From Proposition 3.22.4 we have for any variable y, that $y < x \to y < S(x)$ holds. Using axiom (F2) of our logic, we get $(y < S(x) \to B(y)) \to (y < x \to B(y))$. Generalization and properties of universal quantifier gives us, $\forall y(y < S(x) \to B(y)) \to \forall y(y < x \to B(y))$. Furthermore, contra-position gives us $\neg\forall y(y < x \to B(y)) \to \neg\forall y(y < S(x) \to B(y))$. Finally as for any formula A, B we have $A \to A \vee B$, we can conclude that $\neg\forall y(y < x \to B(y)) \to \phi(S(x))$. **(R2)**

Therefore, from R1 and R2 we have $\forall x(\phi(x) \to \phi(S(x))$. Now using CFA's Induction Rule on this and $\phi(0)$ we have our required result.

□

Corollary 3.29. *If A is a Δ_0 formula, then left and right contraction holds for A.*

Remark 3.30. *Say we have that C, B are Δ_0 formulas. Then it can be shown that left and right contraction holds for $C \wedge B$ and $C \vee B$ (See Appendix B). Subsequently, we will have that $C \wedge B \Leftrightarrow C \otimes B$ and $C \oplus B \Leftrightarrow C \oplus B$.*

We need an additional lemma, important for our proof of Δ_0 induction, which hinges on bounded formulas having contraction.

Lemma 3.31. (CFA) *If $A(x)$ is a bounded formula then,*

1. $(\forall x \leq yA(x)) \otimes A(y+1) \leftrightarrow \forall x \leq y+1A(x)$.
2. $\forall yA(y) \leftrightarrow \forall y(\forall x < yA(x)) \leftrightarrow \forall y(\forall x \leq yA(x))$.

Proof. Since A is bounded, from Theorem 3.28 we get that $A(x) \to A(x) \otimes A(x)$ and $A(x) \oplus A(x) \to A(x)$.

of A with any variable that does not occur in B.

[10] Note that although $y < S(x) \to y \leq x$, we can not yet prove that $y \leq x \to y < S(x)$. To establish the latter we would need contraction for the Δ_0 formula $y < S(x)$, which we will have once we have proven the theorem.

1. *(Right to Left)*
   Clearly we have that
   $$(\forall x \leq y+1 A(x)) \otimes (\forall x \leq y+1 A(x)) \to \forall x \leq y A(x) \otimes A(y+1).$$
   Now since $\forall x \leq y+1 A(x)$ is also a Δ_0 formula we have,
   $$(\forall x \leq y+1 A(x)) \to \forall x \leq y A(x) \otimes A(y+1).$$
   (Left to Right)
   From $A(y+1) \to (x = y+1 \to A(x))$ and $(x \leq y \to A(x)) \to (x \leq y \to A(x))$ we get,
   $$(x \leq y \to A(x) \otimes A(y+1)) \to ((x \leq y \to A(x)) \otimes (x = y+1 \to A(x)))$$
   We can further derive,
   $$((x \leq y \to A(x)) \otimes A(y+1)) \to (x \leq y \oplus x = y+1 \to A(x) \oplus A(x))$$
   Since we have $A(x) \oplus A(x) \to A(x)$ and $x \leq y \leftrightarrow x < y+1$[11], we get
   $$(x \leq y \to A(x)) \otimes A(y+1) \to (x \leq y+1 \to A(x))$$
   Then generalization and properties of universal quantifiers would yield,
   $$\forall x \leq y A(x) \otimes A(y+1) \to \forall x \leq y+1 A(x).$$
2. $\forall y A(y) \leftrightarrow \forall y(\forall x < y A(x))$ is straightforward. For $\forall y(\forall x < y A(x)) \leftrightarrow \forall y(\forall x \leq y A(x))$, we just need to note that $x \leq y \to x < y+1$, is a consequence of $x < y+1$ being a Δ_0 formula.

□

Theorem 3.32. (Δ_0-*Induction Schema*) *The Induction schema for bounded formulas is derivable in* CFA*, i.e, if $A(x)$ is any Δ_0 formula then*
$$\mathsf{CFA} \vdash A(0) \otimes \forall x(A(x) \to A(S(x))) \to \forall x A(x).$$

Proof. We have to show that given any Δ_0 formula A, $A(0) \otimes \forall x(A(x) \to A(S(x))) \to \forall x A(x)$ holds in CFA.
We reason within CFA formalized in Hilbert style. For our purposes we define $P(y) := A(0) \otimes \forall x < y(A(x) \to A(S(x))) \to \forall x \leq y A(x)$, which is a Δ_0 formula. We will show that $P(0)$ and $\forall y(P(y) \to P(S(y))$ holds. Then we can apply induction rule on P to derive $\forall y P(y)$. From which we get,
$$A(0) \otimes \forall y(\forall x < y(A(x) \to A(S(x)))) \to \forall y(\forall x \leq y A(x))$$

[11] As mentioned earlier, $x \leq y \leftrightarrow x < y+1$ is a consequence of having contraction for Δ_0 formulas

Finally, using Lemma 3.31.2 we will get,

$$A(0) \otimes \forall x(A(x) \to A(S(x))) \to \forall x A(x).$$

We proceed by showing the premises required for the induction rule.

1. $P(0)$. This is direct.

2. $\forall y(P(y) \Rightarrow P(y+1))$.
   Let $\gamma := P(y) \otimes A(0) \otimes \forall x < y+1(A(x) \to A(S(x)))$. We show that

   $$\gamma \otimes \gamma \otimes \forall x < y+1(A(x) \to A(S(x))) \to \forall x \leq y A(x) \otimes A(y+1).$$

   Clearly, we have that $\gamma \to \forall x \leq y A(x)$. **(i)**
   Moreover, from $\forall x \leq y A(x) \to A(y)$ and $A(y) \otimes \forall x < y+1(A(x) \to A(S(x))) \to A(y+1)$ we can derive $\forall x \leq y A(x) \otimes \forall x < y+1(A(x) \to A(S(x))) \to A(y+1)$. Finally, from (i) we will get $\gamma \otimes \forall x < y+1(A(x) \to A(S(x))) \to A(y+1)$. **(ii)**

   Thus, combining (i) and (ii), gives us

   $$\gamma \otimes \gamma \otimes \forall x < y+1(A(x) \to A(S(x))) \to \forall x \leq y A(x) \otimes A(y+1).$$

   Now as P, A and $\forall x < y+1(A(x) \to A(S(x)))$ are all Δ_0 formulas, we use Corollary 3.29 to derive,

   $$P(y) \otimes A(0) \otimes \forall x < y+1(A(x) \to A(S(x)) \to \forall x \leq y A(x) \otimes A(y+1).$$

   Finally, we use Lemma 3.31.1 to conclude,

   $$P(y) \otimes A(0) \otimes \forall x < y+1(A(x) \to A(S(x)) \to \forall x \leq y+1 A(x)$$

   or,

   $$P(y) \to P(y+1)$$

   or,

   $$\forall y(P(y) \to P(y+1)).$$

□

Remark 3.33. *The proof of the above theorem indicates that if we had contraction for all formulas in the language of* CFA*, we would be able to derive the Induction Schema.*

Corollary 3.34. *If $A(x)$ is any Δ_0 formula then*

$$\mathsf{CFA} \vdash \forall x(A(0) \otimes \forall y < x(A(x) \to A(S(x))) \to A(x)).$$

We are now in a position to appreciate why we chose to have the Induction Rule over Induction Schema in our formulation of CFA. This is the content of our next proposition.

Proposition 3.35. *Call the Arithmetic which results from replacing the Induction Rule with Induction Schema in* CFA *as* **T**. *Then* **T** *is equivalent to classical Peano Arithmetic* (PA).

Proof. In order to prove the proposition we claim that it is sufficient to show that contraction holds for all formulas in **T** - The presence of contraction in **T** would entail that the additive and multiplicative connectives would collapse into unique connectives (See section 1.1). Thus, we will be in a situation where **T** and PA are based on the same arithmetical language. Now, in order to prove the proposition we need to show that the theory of **T** and the theory of PA are the same.
T is based on an underlying logic which is a fragment of Classical logic LK and has the same non-logical axioms as PA, thus it is clearly a sub-theory of PA. What remains to be shown is that PA is also a sub-theory of **T**. But in the presence of contraction in **T**, all reasoning that we can do within PA can also be done within **T**. Hence, we would have that PA is a sub-theory of **T**.

We start by noting that in **T**, for any formula B in its language, $B \to B \otimes B$ is provable. From which we will also be able to conclude $\neg B \to \neg B \otimes \neg B$ and hence $B \oplus B \to B$ will also hold. Therefore, we will have that contraction holds for B.
Let B be any formula in the language of **T**. Assume that x does not occur in B and define $A(x) := (x = 0 \to \top) \otimes (x = 1 \to B) \otimes (x = 2 \to B \otimes B)$.

We reason within **T**.
First observe that $A(0) \leftrightarrow \top$, $A(1) \leftrightarrow B$ and $A(2) \leftrightarrow B \otimes B$. We also have $A(0) \otimes \forall x(A(x) \to A(S(x))) \to \forall x A(x)$. Consequently,

$$A(0) \otimes \forall x(A(x) \to A(S(x))) \to A(2)$$

is provable in **T**. Hence, in order to demonstrate $B \to B \otimes B$, it suffices to show that $B \to A(0) \otimes \forall x(A(x) \to A(S(x)))$ and $A(2) \to B \otimes B$ are provable.

- $A(2) \to B \otimes B$. This is obvious since $A(2) \leftrightarrow B \otimes B$.
- $B \to A(0) \otimes \forall x(A(x) \to A(S(x)))$.
  Define $\phi(x) := B \to (A(x) \to A(S(x)))$. We will use Induction Schema for ϕ to show $\forall x \phi(x)$. Then, since x is not free in B, we will have $B \to \forall x(A(x) \to A(S(x)))$. Moreover, as $A(0) \leftrightarrow \top$, we will have $B \to A(0) \otimes \forall x(A(x) \to A(S(x)))$, as required.

  To show $\phi(0)$.
  $\phi(0) = B \to (\top \to B)$. From axiom (F6), $\phi(0)$ holds.

To show $\forall x(\phi(x) \to \phi(S(x)))$.
For any formulas P, Q we have $P \to ((P \to Q) \to Q)$ in our logic. Also, axioms of arithmetic gives us $S^2(x) = 2 \to S(x) = 1$. Together they yield,

$$S^2(x) = 2 \to ((S(x) = 1 \to B) \to B).$$

From which we get,

$$(B \otimes S^2(x) = 2 \otimes (S(x) = 1 \to B)) \to B \otimes B$$

or,

$$B \otimes (S(x) = 1 \to B) \to (S^2(x) = 2 \to B \otimes B)$$

...(E1)

Axioms of arithmetic ensure that $\top \leftrightarrow (S^2(x) = 0 \to \top)$ and $\top \leftrightarrow (S^2(x) = 1 \to B)$. Thus, from (E1) we have,

$$B \otimes (S(x) = 1 \to B) \to A(S^2(x)).$$

Further, from weakening (i.e. $P \to (Q \to P)$) we get,

$$B \otimes (S(x) = 0 \to \top) \otimes (S(x) = 1 \to B) \otimes (S(x) = 2 \to B \otimes B) \to A(S^2(x)).$$

or,

$$B \otimes A(x) \to A(S^2(x)).$$

or,

$$B \to (A(x) \to A(S^2(x)))$$

or,

$$\phi(S(x))$$

Another usage of weakening gives us,

$$\phi(x) \to \phi(S(x))$$

Finally, generalization gives us,

$$\forall x(\phi(x) \to \phi(S(x))).$$

Now applying the induction schema for ϕ we have $\forall x \phi(x)$.

Therefore, we conclude that that $B \to B \otimes B$ is provable in **T**.

□

We can summarize this roughly as: $\mathsf{PA} \equiv \mathsf{CFA} - IR + IA$. Thus, we can see how taking induction Schema would make the study of contraction-free arithmetic redundant.

Remark 3.36. *In the sequent-calculus formulation of* CFA *(Definition [3.4]) we took care to exclude side formulas from the Induction Rule. We did so as having side formulas would imply the Induction Schema. To illustrate this point, consider the following proof tree:*

$$\textit{(IR with Side Formulas)}\ \frac{A(0) \Rightarrow A(0) \qquad \forall x(A(x) \to A(S(x)) \Rightarrow \forall x(A(x) \to A(S(x))}{A(0), \forall x(A(x) \to A(S(x)) \Rightarrow \forall x A(x)}$$

Thus, the sequent $A(0), \forall x(A(x) \to A(S(x)) \Rightarrow \forall x A(x)$ *would have been provable.*

Now we present an important corollary of Theorem 3.32. Namely, that any Π_2 formula provable in first-order bounded arithmetic (i.e., $\mathsf{I}\Delta_0$) is, in a way, also provable in CFA. But first in order to make things precise, we introduce a formula translation from the language of $\mathsf{I}\Delta_0$ to the language of CFA.

Definition 3.37. *(Formula translation) Let* ϕ *be any formula in the language of* I_0. *The formula translation* ϕ^T *in the language of* CFA *is defined inductively on the structure of* ϕ*:*

- *If* ϕ *is atomic then* $\phi^T := \phi$.
- *If* $\phi = A \wedge B$ *then* $\phi^T := A^T \otimes B^T$
- *If* $\phi = A \to B$ *then* $\phi^T := A^T \to B^T$
- *If* $\phi = \neg A$ *then* $\phi^T := \neg A^T$.
- *If* $\phi = \forall x(B(x))$ *then* $\phi^T := \forall x(B(x)^T)$ *or, if* $\phi = \exists x(B(x))$ *then* $\phi^T := \exists x(B(x)^T)$

Note that formula translation can be naturally extended to multi-set Γ *of formulae.*

The formula translation essentially replaces all additive connectives in a formula of $\mathsf{I}\Delta_0$ with the corresponding multiplicative connectives. Now, to establish that any Π_2 provable sentence in $\mathsf{I}\Delta_0$ is also provable in CFA, we first demonstrate that any bounded formula provable in $\mathsf{I}\Delta_0$ is also provable in CFA. This step is perhaps the most involved one in our proof. The proof relies crucially on two factors: Firstly, on an application of cut-elimination theorem for LK; secondly, the availability of contraction for bounded formulas in CFA.

Theorem 3.38. *If for a bounded formula* ϕ*, we have that* $\mathsf{I}\Delta_0 \vdash \phi$ *then* $\mathsf{CFA} \vdash \phi^T$.

Proof. We have $\mathsf{I}\Delta_0 \vdash \phi$. From which we get that for some finite multi-set Ω containing instances of axioms of $\mathsf{I}\Delta_0$, $\mathsf{LK} \vdash \Omega \Rightarrow \phi$. Moreover, there is a cut-free proof (say) π of $\Omega \Rightarrow \phi$ in LK. Thus, by the sub-formula property, any formula in

π is a sub-formula of $\Omega \cup \{\phi\}$. Which means any formula in π is either bounded or, a sub-formula of instances of the Δ_0 induction schema. Now an instance[12] of the Δ_0 induction schema is of the form

$$\forall x(A(0) \wedge \forall y < x(A(x) \to A(S(x))) \to A(x)).$$

This means that any unbounded formula in π is of the form: **..(SFs)**

- $\forall x(B(0) \wedge \forall y < x(B(x) \to B(S(x))) \to B(x))$.

where B is a bounded formula.

Now let S' denote the result of deleting from a sequent S, all sub-formulas of Ω which are not bounded.

Claim: Given any sequent S in π, the corresponding sequent $(S')^T$ is provable in CFA.

We proceed by induction on the structure of π.

Base Case. Let S be a leaf of π, i.e., S is an initial sequent of LK. But since initial sequents of LK and $\mathsf{GQC}^{\mathsf{G}}$ are the same, we have what we require.

Inductive Case. We break it up into cases based on the rule that has been applied to get S.

(Weakening) We deal with WL, WR is analogous. Suppose the proof of $S = A, \Gamma \Rightarrow \Delta$ in LK has the following form:

$$(WL)\ \dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta\end{array}}{A, \Gamma \Rightarrow \Delta}$$

By induction hypothesis, $\mathsf{CFA} \vdash (\Gamma')^T \Rightarrow (\Delta')^T$. Now, if A is not bounded, then $((A, \Gamma \Rightarrow \Delta)')^T$ is nothing but $((\Gamma \Rightarrow \Delta)')^T$. Hence, $(S')^T$ is provable in CFA.

On the other hand, if A is bounded then consider the following proof tree in CFA:

$$(WL)\ \dfrac{\begin{array}{c}\vdots\\ (\Gamma')^T \Rightarrow (\Delta')^T\end{array}}{A^T, (\Gamma')^T \Rightarrow (\Delta')^T}$$

Thus, $(S')^T$ is provable in CFA.

(Contraction) We deal with CL, CR is analogous. Suppose the proof of $S = A, \Gamma \Rightarrow \Delta$ in LK has the following form:

[12] Usually, Δ_0 induction schema is taken to be $A(0) \wedge \forall x(A(x) \to A(S(x))) \to \forall x A(x)$, but this has more quantifier alternations which is not suitable for our proof. Thus, we assume that $\mathsf{I}\Delta_0$ is axiomatized by the given formulation of Δ_0 induction.

$$\dfrac{\vdots}{} \qquad (CL)\,\dfrac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}$$

By induction hypothesis, $\mathsf{CFA} \vdash ((A, A, \Gamma)')^T \Rightarrow (\Delta')^T$. Now, if A is not bounded, then $((A, A, \Gamma)')^T \Rightarrow (\Delta')^T$ is the same as $((A, \Gamma)')^T \Rightarrow (\Delta')^T$. Hence, $(S')^T$ is provable in CFA.
On the other hand, if A is bounded then so is A^T. Which means, from Corollary 3.29, we have contraction for A. Now consider the following proof tree in CFA:

$$\begin{array}{c}\vdots\end{array} \qquad (CL)\,\dfrac{(A')^T, (A')^T, (\Gamma')^T \Rightarrow (\Delta')^T}{(A')^T, (\Gamma')^T \Rightarrow (\Delta')^T}$$

Thus, $(S')^T$ is provable in CFA.

(Implication)

$(\rightarrow R)$. Suppose the proof of $S = \Gamma \Rightarrow \Delta, A \rightarrow B$ in LK has the following form:

$$\begin{array}{c}\vdots\end{array} \qquad (\rightarrow R)\,\dfrac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}$$

By induction hypothesis, $\mathsf{CFA} \vdash ((A, \Gamma)')^T \Rightarrow ((\Delta, B)')^T$. Now, if A and B are both bounded [13] or both unbounded, it is straightforward that $\mathsf{CFA} \vdash (S')^T$.
On the other hand if A is bounded and B is unbounded, then $A \rightarrow B$ is an unbounded formula. But then, it should be a sub-formula of some instance of Δ_0 induction axiom as per **(SFs)**. Which is not possible. We arrive at a similar contradiction when A is unbounded and B is bounded.

$(\rightarrow L)$ Suppose the proof of $S = A \rightarrow B, \Gamma, \Pi \Rightarrow \Delta, \Sigma$ in LK has the following form:

$$(\rightarrow\mathrm{L})\,\dfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, A\end{array} \qquad \begin{array}{c}\vdots\\ B, \Pi \Rightarrow \Sigma\end{array}}{A \rightarrow B, \Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

[13] Note that $(A \rightarrow B)^T = A^T \rightarrow B^T$

By induction hypothesis, $\mathsf{CFA} \vdash ((\Gamma)')^T \Rightarrow ((\Delta, A)')^T$ and $\mathsf{CFA} \vdash ((B, \Pi)')^T \Rightarrow ((\Sigma)')^T$. Now, if A and B are both bounded then, it is straightforward to see that $\mathsf{CFA} \vdash (S')^T$.

Now suppose one of them is unbounded, without loss of generality say it is A. Then $A \to B$ is also unbounded. Moreover, $((\Gamma)')^T \Rightarrow ((\Delta, A)')^T$ is nothing but $((\Gamma)')^T \Rightarrow ((\Delta)')^T$, which is thus provable in CFA. Now consider the following proof tree in CFA:

$$\text{(WL,WR)}\ \frac{\begin{array}{c}\vdots\\ ((\Gamma)')^T \Rightarrow ((\Delta)')^T\end{array}}{((\Gamma)')^T, ((\Pi)')^T \Rightarrow ((\Delta)')^T, ((\Sigma)')^T}$$

Hence, $\mathsf{CFA} \vdash (S')^T$.

(Conjunction)

$(\wedge R)$ Suppose the proof of $S = \Gamma \Rightarrow \Delta, A \wedge B$ in LK has the following form:

$$(\wedge\text{R})\ \frac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, A\end{array} \qquad \begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, B\end{array}}{\Gamma \Rightarrow \Delta, A \wedge B}$$

By induction hypothesis, $\mathsf{CFA} \vdash ((\Gamma)')^T \Rightarrow ((\Delta, A)')^T$ and $\mathsf{CFA} \vdash ((\Gamma)')^T \Rightarrow ((\Delta, B)')^T$. If both A and B are bounded, then it is straightforward to see that $\mathsf{CFA} \vdash (S')^T$. Now suppose that at least one of A and B is unbounded. Without loss of generality say it is A, then $((\Gamma)')^T \Rightarrow ((\Delta, A)')^T$ is nothing but $((\Gamma)')^T \Rightarrow ((\Delta)')^T$. Moreover, as $A \wedge B$ is also bounded, $((\Gamma)')^T \Rightarrow ((\Delta, A \wedge B)')^T$ is nothing but $(\Gamma')^T \Rightarrow (\Delta')^T$. Thus, $\mathsf{CFA} \vdash (S')^T$.

$(\wedge L)$ Suppose the proof of $S = A \wedge B, \Gamma \Rightarrow \Delta$ in LK has the following form:

$$(\wedge L)\ \frac{\begin{array}{c}\vdots\\ A, \Gamma \Rightarrow \Delta\end{array}}{A \wedge B, \Gamma \Rightarrow \Delta}$$

By induction hypothesis, $\mathsf{CFA} \vdash ((A, \Gamma)')^T \Rightarrow ((\Delta)')^T$. Now, say A and B are both bounded then consider the following proof tree in CFA:

$$\cfrac{\cfrac{\begin{array}{c}\vdots\\ A^T, (\Gamma')^T \Rightarrow (\Delta')^T\end{array}}{A^T, B^T, (\Gamma')^T \Rightarrow (\Delta')^T}\,(\text{WL})}{A^T \otimes B^T, \Gamma \Rightarrow \Delta}\,(\otimes L)$$

Thus[14], we have that $\mathsf{CFA} \vdash (S')^T$. Now, if A is unbounded then $A \wedge B$ will also be unbounded. Moreover, $((A, \Gamma)')^T \Rightarrow (\Delta')^T$ will be nothing but $(\Gamma')^T \Rightarrow (\Delta')^T$. Hence, we will also have that $\mathsf{CFA} \vdash (S')^T$.

Finally, let's consider the scenario where A is bounded, while B is unbounded. In such a case, $A \wedge B$ would also be unbounded. According to (SFs), this means that $A \wedge B$ has to be an instance of the given Δ_0 induction schema. However, considering the form of $A \wedge B$, this is not possible. Hence, the case when A is bounded and B is unbounded, is impossible.

(Disjunction)

The proof for disjunction rules are similar to conjunction rules.

(Negation)

The proof for negation rules are rather straightforward.

(Universal Quantifier)

($\forall$**R**) Suppose the proof of $S = \Gamma \Rightarrow \Delta, \forall x A(x)$ in LK has the following form:

$$\cfrac{\begin{array}{c}\vdots\\ \Gamma \Rightarrow \Delta, A(a)\end{array}}{\Gamma \Rightarrow \Delta, \forall x A(x)}\,(\forall R)$$

We will argue to demonstrate that this case is an impossibility. Suppose, $\Gamma \Rightarrow \Delta, \forall x A(x)$ is a sequent in π. Since the end-sequent of π is $\Omega \Rightarrow \phi$ where ϕ is a bounded formula, at some point in the course of the proof π, $\forall x A(x)$ being unbounded needs to be moved to the antecedent. This is only possible through the $\rightarrow L$ rule. But then for some formula B, $\forall x A(x) \rightarrow B$ will be the principle formula of that $(\rightarrow L)$ rule. But an unbounded formula of the form $\forall x A(x) \rightarrow B$ cannot be a sub-formula of $\Omega \cup \{\phi\}$ according to SFs. Thus, this case is not possible.

($\forall$**L**) Suppose the proof of $S = \forall x A(x), \Gamma \Rightarrow \Delta$ in LK has the following form:

[14] Note that $(A \wedge B)^T = A^T \otimes B^T$

$$(\forall L)\ \dfrac{\begin{array}{c}\vdots\\ A(t), \Gamma \Rightarrow \Delta\end{array}}{\forall x A(x), \Gamma \Rightarrow \Delta}$$

By induction hypothesis, $\mathsf{CFA} \vdash A(t)^T, (\Gamma')^T \Rightarrow (\Delta')^T$. Moreover, since $\forall x A(x)$ is an unbounded formula in π, it has to be an instance of the induction schema. Given Corollary 3.34, we have that $\mathsf{CFA} \vdash \Rightarrow A(t)^T$. Now consider the following proof in CFA:

$$(\text{Cut})\ \dfrac{\begin{array}{c}\vdots\\ A(t)^T, (\Gamma')^T \Rightarrow (\Delta')^T\end{array} \qquad \begin{array}{c}\vdots\\ \Rightarrow A(t)^T\end{array}}{(\Gamma')^T \Rightarrow (\Delta')^T}$$

Thus, we have that $\mathsf{CFA} \vdash (S')^T$.

Conclusion: From the claim we get that since $\Omega \Rightarrow \phi$ is also a sequent in π, $(\Omega')^T \Rightarrow (\phi)^T$ is provable in CFA. Since the formula translation of any axiom of $\mathsf{I}\Delta_0$ is derivable in CFA, using cut we get that $\Rightarrow \phi^T$ is provable in CFA.

□

Corollary 3.39. *If $\phi(x)$ is a bounded formula such that $\mathsf{I}\Delta_0 \vdash \forall x \phi(x)$, then we have that $\mathsf{CFA} \vdash \forall x \phi(x)^T$.*

Proof. The proof is a direct application of Theorem 3.38 and then, the induction rule in CFA. □

Corollary 3.40. *If $\phi(x, y)$ is a bounded formula with no additional free variables, and we have that $\mathsf{I}\Delta_0 \vdash \forall x \exists y \phi(x, y)$, then $\mathsf{CFA} \vdash \forall x \exists y \phi(x)^T$.*

Proof. Suppose, $\mathsf{I}\Delta_0 \vdash \forall x \exists y \phi(x, y)$. Then, according to Parikh's Theorem (see Parikh [1971]), there exists a term $t(x)$ such that $\mathsf{I}\Delta_0 \vdash \forall x \exists y < t(x) \phi(x, y)$. Since $\exists y < t(x)\phi(x, y)$ is a bounded formula, the preceding corollary implies that $\mathsf{CFA} \vdash \forall x \exists y < t(x)\phi(x, y)$. Moreover, Proposition 2.31.1 asserts that $\mathsf{CFA} \vdash \forall x \exists y < t(x)\phi(x, y) \rightarrow \forall x \exists y \phi(x, y)$. Therefore, we have $\mathsf{CFA} \vdash \forall x \exists y \phi(x, y)$, as required. □

Thus, any Π_2 formula provable in $\mathsf{I}\Delta_0$ is also provable in CFA under a natural formula translation.

Corollary 3.41. CFA *is Σ_1 complete with respect to $\mathbb{N}$.*

Proof. It follows from Corollary 3.40 and Σ_1 completeness of $\mathsf{I}\Delta_0$. □

We conclude this section by presenting another consequence of having contraction for Δ_0 formulas.

Proposition 3.42. (CFA) *For any Σ_1 formula A, $\mathsf{CFA} \vdash A \Rightarrow A \otimes A$, i.e, left contraction is derivable for A.*

Proof. Let $A = \exists x B(x)$ where $B(x)$ is Δ_0. Since $B(x)$ is Δ_0, we have that $B(x) \Rightarrow B(x) \otimes B(x)$. We reason in CFA formalized in sequent style. Consider the following proof tree:

$$(\exists L)\ \dfrac{(\text{cut})\ \dfrac{(\exists R)\ \dfrac{\begin{matrix}\vdots \\ B(x) \Rightarrow B(x) \otimes B(x)\end{matrix}}{B(x) \Rightarrow \exists x(B(x) \otimes B(x))} \qquad \begin{matrix}\vdots \\ \exists x(B(x) \otimes B(x)) \Rightarrow \exists x B(x) \otimes \exists x B(x)\end{matrix}}{B(x) \Rightarrow \exists x B(x) \otimes \exists x B(x)}}{\exists x B(x) \Rightarrow \exists x B(x) \otimes \exists x B(x)}$$

Thus, we conclude that for any Σ_1 formula A, $\mathsf{CFA} \vdash A \Rightarrow A \otimes A$.

□

Remark 3.43. *The above proposition holds crucial significance if we aim to derive Gödel's second incompleteness theorem for* CFA*. We refer the reader to the Chapter 5 (Further work) for a brief discussion.*

4 Provably Recursive Functions

In this chapter, we delve into a classic problem (Buss [1998]) in proof theory: the classification of provably recursive functions. Firstly, we establish that primitive recursive functions are provably total. Subsequently, we go on to show that in fact, any provably recursive function of CFA is primitive recursive in contrast to classical PA.

4.1 Provably Recursive Functions

What exactly are provably recursive functions, and why are we interested to study them? Put simply, provably recursive functions are computable functions whose behavior can be formally established within a system of arithmetic. Characterizing this class of functions serves as a measure of the deductive strength of that system of arithmetic. Without further ado lets define[1]them formally.

Definition 4.1. *Let T be a subtheory of PA and $f : \mathbb{N}^k \to \mathbb{N}$. The function is provably recursive iff there is a Σ_1 formula $A(x_1, .., x_k, y)$ such that*

1. *$A(\vec{n}, m)$ holds in $\mathbb{N}$ iff $f(\vec{n}) = m$ for all naturals $\vec{n}$, m.*
2. *$\mathsf{T} \vdash \forall\vec{x}\exists! y A(\vec{x}, y)$*[2]

Then, we say that A represents f.

Intuition: If f is a provably recursive function of T, then the theory T should prove that some Turing machine M which computes f, halts on all appropriate inputs. This is because, $A(x, y)$ can be taken to be a Σ_1-formula expressing there is a w which codes a halting M-computation with input x and output y.

Remark 4.2. *Some direct consequences of Definition [4.1]*

[1] For definitions of primitive recursive functions and recursive functions, we refer to the Appendix A.

[2] $\exists! y A(y)$ is an abbreviation for $\exists x A(x) \otimes \forall y \forall z((A(y) \otimes A(z)) \to y = z)$

- *From Σ_1 completeness ($\Rightarrow$) and Σ_1 soundness ($\Leftarrow$) of T, we get that for all naturals $\vec{n}$ and m, $f(\vec{n}) = m \Leftrightarrow \mathsf{T} \vdash A(\vec{n}, m)$.*

- *A classical result (c.f. Odifreddi [1992]) says that the recursively enumerable (r.e.) sets are exactly those subsets of $\mathbb{N}^k$ which are definable by a Σ_1 formula in $\mathbb{N}$. Thus, in a theory T which is sound with respect to $\mathbb{N}$, any function which is provably recursive in T will have a r.e. graph due to which the function will be (total) recursive[3]. Hence, provably recursive functions of T can be thought of as those recursive functions whose totality can be proven in T.*

Remark 4.3. *Further interesting points concerning provably recursive functions of an arithmetical theory T which is Σ_1 sound and r.e.*

- *A fundamental property of partial recursive functions is their ability to be indexed effectively. This means that a one-to-one correspondence can be established between non-negative integers and instructions for computing partial recursive functions. Consequently, given an index, one can effectively generate the corresponding instructions for computing the partial recursive function, and conversely, given a set of instructions, one can effectively determine its index. In contrast, the class of (total) recursive functions cannot be indexed in this manner; this theorem stands as one of the fundamental principles (See Odifreddi [1992]) of recursive function theory. In light of this, a natural question arises: can we delineate a subclass of (total) recursive functions which can be indexed effectively? Indeed, the class of provably recursive functions of a recursively axiomatizable theory constitutes such a* ***proper*** *subclass (See Fischer [1967]).*

- *Since the class of provably recursive functions of a theory can be effectively indexed, it would admit computable diagonalization (consider $D(n) := f_n(n) + 1$). Thus, the diagonal function is a concrete example of a total recursive function which cannot be a Provably recursive function.*

- *Say we have a recursive function f, which is not a provably recursive function of a theory T. Then the statement that "function f is total" is an example of a natural combinatorial statement which is true but unprovable in T.*

Thus, we see that the examination of provably recursive functions within an arithmetical theory can offer valuable insights into its nature. It not only gives a measure of its computational power; it also serves to delimit its mathematical power in providing natural examples of true mathematical statements it cannot prove. The *classification* of provably recursive functions of classical Peano Arithmetic was first studied by Kreisel (Kreisel [1952]) with roots going back to Ackermann (Ackermann [1940]). Kreisel showed that, roughly speaking, the class of provably recursive functions of PA is exactly the class of functions that

[3] If f is a total function then f is recursive *iff* the graph of f is recursively enumerable.

is definable by recursions over standard well-orderings of the natural numbers with order types less than ϵ_0. In Kreisel [1958], he also showed that the same class of functions can be proved to be total in the intuitionistic arithmetic of Heyting. Let us now embark on the study of the provably recursive functions of CFA.

4.2 Primitive Recursive Functions

In this section, we show that primitive recursive functions are provably total in CFA. We begin with a brief word about coding of sequences in CFA.

Remark 4.4. *In order to reason about primitive recursion inside* CFA*, it is crucial to formally talk about finite sequences inside* CFA*. Consequently, it becomes essential to devise a suitable encoding for sequences (see Appendix A for definitions) and formulate its associated properties such as the n-th term of a sequence. Thanks to Theorem 3.38, we would be able to replicate coding of finite sequences as in Bounded Arithmetic* ($\mathsf{I}\Delta_0$) *without difficulty. We refer the interested reader to Chapter 5 of Pudlák and Hájek [1993].*[4]

In order to prove that primitive recursive functions are provably recursive in CFA, we adapt the strategy used in arithmetical theories like $\mathsf{EA}+\Sigma_1\text{-}IR$ which is the theory we get when extending EA[5] with the Induction rule over Σ_1 formulas (cf. Beklemishev [1997]). The central point of difference is in proving that the class of provably recursive functions is closed under primitive recursion. Firstly, we need to formulate the formula representing the function used to define primitive recursion in a slightly different way, so as to ensure that the formula is indeed Σ_1. Secondly, we employ the Induction Rule instead of the standard Σ_1 induction schema, as seen in theories like PA, to prove the totality of the function thus defined.

Theorem 4.5. (CFA) *Any primitive recursive function f, is a provably recursive function of* CFA.

Proof. We prove this by induction on the generation of the primitive recursive functions. Thus, we first need to show that the basic functions are provably recursive. Subsequently, we show that the provably recursive nature of functions is preserved under compositions and primitive recursion. We proceed in the sequent calculus[6].

[4] In particular, see Definition 3.5 and Definition 3.26, in Chapter 5 of the book.

[5] Which is a theory formulated in the language of PA where induction schema is restricted to bounded formulas and additionally, it has an axiom stating 2^x is total.

[6] Sequent calculus proofs in arithmetic often tend to be lengthy, as they require multiple applications of cut. Therefore, typically only proof sketches are provided, from which a proof tree can be easily constructed.

The Basic functions
They are $\lambda x.0$, $\lambda x.x+1$ and $\lambda x_1,\dots,x_n.x_i$. These functions are easily seen to provably recursive. For instance, consider the case of $\lambda x.x+1$. Take $A(x,y) := y = S(x)$.

- For *totality* consider the following proof tree:

$$\dfrac{\dfrac{\Rightarrow S(x)=S(x)}{\Rightarrow \exists y(y=S(x))}\,(\exists R)}{\Rightarrow \forall x\exists y(y=S(x))}\,(\forall R)$$

- For *functionality* consider the following proof tree:

$$\dfrac{\dfrac{\dfrac{y=S(x), z=S(x)\Rightarrow y=z}{(y=S(x)\otimes z=S(x))\Rightarrow y=z}\,(\otimes L)}{\Rightarrow (y=S(x)\otimes z=S(x))\rightarrow y=z}\,(\rightarrow R)}{\Rightarrow \forall y\forall z(y=S(x)\otimes z=S(x))\rightarrow y=z}\,(\forall R)$$

Composition
Say $F(x) = G(H_1(x),...H_m(x))$. Suppose by induction hypothesis G and H_i are provably recursive and are represented by $\psi,\chi_1,..,\chi_m$ respectively. Then we show that F is represented by the formula-

$$\phi(x,z) := \exists z_1,..\exists z_m(\chi_1(x,z_1)\otimes ..\otimes \chi_m(x,z_m)\otimes \psi(z_1,..,z_m,z))$$

Since χ_i and ψ are all Σ_1, we will be able to pull out all the existential quantifiers outside and then from Proposition 3.11, we can deduce that ϕ is also Σ_1. The fact that ϕ defines the graph of F is straightforward, so we focus on proving that $\Rightarrow \forall x\exists! y\phi(x,y)$.

- *Totality:* $\Rightarrow \forall x\exists z\phi(x,z)$

  We have that for all $i\leq m$, $\Rightarrow \forall x\exists z_i\chi_i(x,z_i)$ and $\Rightarrow \forall\vec{x}\exists z\psi(\vec{x},z)$. Applying Proposition 2.31.7, which states that $\Rightarrow \forall x\exists yA(x,y)\otimes\forall x\exists zB(x,z)\rightarrow \forall x\exists y\exists z(A(x,y)\otimes B(y,z))$, we obtain: $\Rightarrow \forall x\exists z\exists\vec{z_i}(\bigotimes_{i=1}^m \chi_i(x,z_i)\otimes\psi(z_1,..,z_m,z))^7$, which is nothing but $\Rightarrow \forall x\exists z\phi(x,z)$.

- *Functionality:* $\Rightarrow \phi(x,y_1)\otimes\phi(x,y_2)\rightarrow y_1=y_2$

  Proposition 2.31.6, gives us $P\otimes\exists xQ(x) \Rightarrow \exists y(P\otimes Q(y))$, thus from $\phi(x,y_1)\otimes\phi(x,y_2)$ we obtain:

$$\exists\vec{z_i}(\chi_1(x,z_1)..\otimes\chi_1(x,z_m)\otimes\psi(\vec{z_i},y_1))\otimes\exists\vec{k_i}(\chi_1(x,k_1)..\otimes\chi_1(x,k_m)\otimes\psi(\vec{k_i},y_2))$$
$$\Rightarrow \exists\vec{z_i}\exists\vec{k_i}(\chi_1(x,z_1)\otimes\chi_1(x,k_1)\cdots\otimes\chi_m(x,z_m)\otimes\chi_m(x,k_m)\otimes\psi(\vec{z_i},y_1)\otimes\psi(\vec{k_i},y_2))$$

[7] $\bigotimes_{i=1}^m A_i$ is an abbreviation for $A_1\otimes\ldots\otimes A_m$

$$\ldots(\mathbf{i})$$

Now using the functionality of H_i, i.e, $\Rightarrow \chi_i(x, z_i) \otimes \chi_i(x, k_i) \to z_i = k_i$ we get-

$$\begin{aligned}&\exists \vec{z_i} \exists \vec{k_i}(\chi_1(x, z_1) \otimes \chi_1(x, k_1) \cdots \otimes \chi_m(x, z_m) \otimes \chi_m(x, k_m)\\&\otimes \psi(\vec{z_i}, y_1) \otimes \psi(\vec{k_i}, y_2)) \Rightarrow \exists \vec{z_i} \exists \vec{k_i}(\bigotimes_{i=1}^{m}(z_i = k_i) \otimes \psi(\vec{z_i}, y_1) \otimes \psi(\vec{k_i}, y_2))\end{aligned}$$

$$\ldots(\mathbf{ii})$$

Now the properties of $(=)$ gives us,

$$\exists \vec{z_i} \exists \vec{k_i}(\bigotimes_{i=1}^{m}(z_i = k_i) \otimes \psi(\vec{z_i}, y_1) \otimes \psi(\vec{k_i}, y_2)) \Rightarrow \exists \vec{z_i} \exists \vec{k_i}(\psi(\vec{k_i}, y_1) \otimes \psi(\vec{k_i}, y_2))$$

$$\ldots(\mathbf{iii})$$

Finally, we use the functionality of G, i.e, $\Rightarrow \psi(\vec{k_i}, y_1) \otimes \psi(\vec{k_i}, y_2) \to y_1 = y_2$ to get -

$$\exists \vec{z_i} \exists \vec{k_i}(\psi(\vec{k_i}, y_1) \otimes \psi(\vec{k_i}, y_2)) \Rightarrow y_1 = y_2$$

$$\ldots(\mathbf{iv})$$

Therefore, three applications of cut on (i),(ii),(iii), and (iv) would yield, $\Rightarrow \phi(x, y_1) \otimes \phi(x, y_2) \to y_1 = y_2$.

Primitive recursion

Suppose $g(n, x)$ is defined by

$$\begin{aligned}&g(0, x) = e(x)\\&g(n + 1, x) = h(g(n, x), n, x)\end{aligned}$$

Where e, h are provably recursive and represented by $E(x, y)$ and $H(z, n, x, y) = \exists v H_0(v, z, n, x, y)$ with H_0 being a bounded formula. Then we show that g is represented by the following Σ_1 formula[8]-

$$g(n, x) = y \leftrightarrow \phi(n, x, y) := \exists s \exists v \in Seq(E(x, (s)_0) \otimes \forall i < n H_0((v)_i, (s)_i, i, x, (s)_{i+1}) \otimes (s)_n = y)$$

$$\ldots(\mathbf{E1})$$

It is not difficult to see that ϕ indeed defines the graph of g. We focus on proving

[8] Which is an abbreviation for $\phi(n, x, y) := \exists s \exists v(Seq(s) \otimes Seq(v) \otimes E(x, (s)_0) \otimes \forall i < n H_0((v)_i, (s)_i, i, x, (s)_{i+1}) \otimes (s)_n = y)$

the totality and functionality of g in CFA.

Totality. It is shown using Induction Rule on $\psi(i) := \exists y(g(i,x) = y)$.
Base Case To show $\Rightarrow \exists y(g(0,x) = y)$, we use the totality of e in CFA.
Steps towards a proof:

1. Totality of e and that $\langle y\rangle$ is a sequence of length 1 gives us,
$\Rightarrow \exists y(E(x,y)) \otimes \forall y(\langle y\rangle)_0 = y \otimes Seq(\langle y\rangle))$

2. From Proposition 2.31.6 we have,
$\exists y(E(x,y)) \otimes \forall y((\langle y\rangle)_0 = y \otimes Seq(\langle y\rangle)) \Rightarrow \exists y(E(x,y) \otimes (\langle y\rangle)_0 = y \otimes Seq(\langle y\rangle))$

3. $\exists R$ rule on $\langle y\rangle$ gives us,
$\exists y(E(x,y) \otimes (\langle y\rangle)_0 = y \otimes Seq(\langle y\rangle)) \Rightarrow \exists y \exists s(E(x,y) \otimes (s)_0 = y \otimes Seq(s))$

4. Rules of equality and commutativity of $\otimes$ gives us,
$\exists y \exists s(E(x,y) \otimes (s)_0 = y \otimes Seq(s)) \Rightarrow \exists y \exists s(Seq(s) \otimes E(x,(s)_0) \otimes (s)_0 = y)$

5. Finally, from the definition of ϕ we have,
$\exists y \exists s(Seq(s) \otimes E(x,(s)_0) \otimes (s)_0 = y) \Rightarrow \exists y(g(0,x) = y)$

Therefore, multiple applications of (Cut) would yield $\Rightarrow \psi(0)$.

Inductive Case To show $\psi(n) \Rightarrow \psi(n+1)$

The informal argument proceeds as follows: Let's assume that $g(n,x) = y$, thus yielding two sequences, denoted as s and v, of length $n+1$ and n respectively, satisfying (E1). Our objective is to construct suitable sequences of length $n+2$ and $n+1$. Given that the function h is provably total, we can get a z such that $h(y,n,x) = z$. Consequently, there exists a w such that $H_0(w,y,n,x,z)$ holds true. Then append the element z to the end of sequence s, and w to the end of sequence v, thereby obtaining the required sequences.

Let's proceed to give a formal proof. Steps towards a proof:
Take $s'' = \langle (s)_0, .., (s)_n, y\rangle$ and $v'' = \langle (v)_0, .., (v)_{n-1}, v'\rangle$.

1. *Since*
$(\forall i < nH_0((v)_i, (s)_i, i, x, (s)_{i+1})) \otimes (s)_n = y_1, H_0(v', y_1, n, x, y) \Rightarrow$
$(\forall i < n+1H_0((v'')_i, (s'')_i, i, x, (s'')_{i+1}))$

   we have,

   $Seq(s) \otimes Seq(v) \otimes E(x,(s)_0) \otimes (\forall i < nH_0((v)_i, (s)_i, i, x, (s)_{i+1})) \otimes (s)_n = y_1, H_0(v', y_1, n, x, y) \Rightarrow$
$Seq(s'') \otimes Seq(v'') \otimes E(x,(s'')_0) \otimes (\forall i < n+1H_0((v'')_i, (s'')_i, i, x, (s'')_{i+1})) \otimes (s'')_{n+1} = y$

2. *Using* $(\exists R)$ *on* y, s'', v'' *we get,*

$E(x,(s'')_0) \otimes (\forall i < n+1 H_0((v'')_i,(s'')_i,i,x,(s'')_{i+1})) \otimes (s'')_{n+1} = y \Rightarrow$ $\exists y \exists s'' \exists v'' \in Seq(E(x,(s'')_0) \otimes (\forall i < n+1 H_0((v'')_i,(s'')_i,i,x,(s)_{i+1})) \otimes (s'')_{n+1} = y)$

3. *As the succedent is nothing but $\exists y g(n+1,x) = y)$, applying (cut) on 1 and 2 we get,*

   $Seq(s) \otimes Seq(v) \otimes E(x,(s)_0) \otimes (\forall i < nH_0((v)_i,(s)_i,i,(s)_{i+1})) \otimes (s)_n = y_1, H_0(v',y_1,n,x,y) \Rightarrow \exists y g(n+1,x) = y)$

4. *Using $(\exists L)$ rule on s, v and v' we get,*

   $\exists s \exists v \in \mathrm{Seq}(E(x,(s)_0) \otimes (\forall i < nH_0((v)_i,(s)_i,i,(s)_{i+1})) \otimes (s)_n = y_1), \exists y \exists v'(H_0(v',y_1,n,x,y)) \Rightarrow \exists y g(n+1,x) = y)$

5. *Using the totality of h, i.e., $\Rightarrow \exists y \exists v' H_0(v',y_1,n,x,y)$ by cut we get,*

   $\exists s \exists v \in \mathrm{Seq}(E(x,(s)_0) \otimes (\forall i < nH_0((v)_i,(s)_i,i,x,(s)_{i+1})) \otimes (s)_n = y_1) \Rightarrow \exists y g(n+1,x) = y)$

6. *Furthermore using $(\exists L)$ on y_1 we get,*

   $\exists y \exists s \exists v \in \mathrm{Seq}(E(x,(s)_0) \otimes (\forall i < nH_0((v)_i,(s)_i,i,x,(s)_{i+1})) \otimes (s)_n = y) \Rightarrow \exists y g(n+1,x) = y)$

Thus, we get that $\exists y(g(n,x) = y) \Rightarrow \exists y(g(n+1,x) = y)$ or, $\psi(n) \Rightarrow \psi(n+1)$.

Now, the Induction Rule of CFA yields, $\Rightarrow \forall n \exists y g(n,x) = y$. An application of the $\forall x R$ rule would give us, $\Rightarrow \forall x \forall n \exists y g(n,x) = y$, thus establishing the totality of g in CFA.
Functionality of g. To show $\Rightarrow \phi(n,x,y_1) \otimes \phi(n,x,y_2) \to y_1 = y_2$

Say $\phi(n,x,y) = \exists s \exists v R(n,s,v,x,y)$. Suppose we have $R(n,s_1,v_1,x,y_1) \otimes R(n,s_2,v_2,x,y_2)$, then our aim is to establish $\forall i \leq n((s_1)_i = (s_2)_i)$. Once we establish that $\forall i \leq n((s_1)_i = (s_2)_i)$, it follows that $(s_1)_n = (s_2)_n$, hence $y_1 = y_2$.
Now, in order to establish $\Rightarrow R(n,s_1,v_1,x,y_1) \otimes R(n,s_2,v_2,x,y_2) \to (\forall i \leq n((s_1)_i = (s_2)_i))$, we apply Induction Rule on $\psi(i) := R(n,s_1,v_1,x,y_1) \otimes R(n,s_2,v_2,x,y_2) \to (i \leq n \to (s_1)_i = (s_2)_i)$. The basis $(\Rightarrow \psi(0))$ and the induction step $(\psi(x) \Rightarrow \psi(x+1))$ would follow directly from the functionality of e and h. Thus, we will get that $R(n,s_1,v_1,x,y_1) \otimes R(n,s_2,v_2,x,y_2) \Rightarrow y_1 = y_2$. Since s_j, v_j are free-variables, using $(\exists L)$ we get $\Rightarrow \phi(n,x,y_1) \otimes \phi(n,x,y_2) \to y_1 = y_2$, as required

□

Remark 4.6. *Usually g is represented by the following Σ_1 formula:*

$$g(n,x) = y \leftrightarrow \phi(n,x,y) := \exists s \in Seq(E(x,(s)_0) \otimes \forall i < nH((s)_i,i,(s)_{i+1}) \otimes (s)_n = y)$$

But in the absence of Σ_1 collection principle[9]*, this might not be equivalent to a Σ_1 formula in our arithmetic. Hence, we define it in as shown above.*

[9] The Σ_1 collection refers to the following axiom schema for Σ_1 formulas : $\forall u[(\forall x \leq u)\exists y \psi(x,y) \to \exists v(\forall x \leq u)(\exists y \leq v)\psi(x,y)]$

We conclude this section by emphasizing that merely containing Δ_0 induction is not sufficient for arithmetical theories to establish that primitive recursive functions are provably recursive within it. For instance, in EA, provably recursive functions are limited to elementary functions[10] and the elementary functions form a proper subset of primitive recursive functions (consider, super-exponentiation function).

4.3 Classifying Provably Recursive Functions

In this section, we address the classification problem of provably recursive functions in CFA. We demonstrate that the class of provably recursive functions precisely corresponds to the class of *primitive recursive functions*. Our approach is inspired by Weiermann (c.f. Weiermann [2006]), who classified the provably recursive functions of classical PA using certain operators that control the witness information in derivations.

4.3.1 CFA^∞

We will operate within a semi-formal infinitary Gentzen calculus denoted by CFA^∞, the design of which is based on the truth definition for the standard model $\mathbb{N}$. The defining feature of this system is an infinitary ω rule replacing the $\forall R$ rule. We show that the symmetric nature of the ω rule makes the cut-rule admissible in CFA^∞ and this will help us derive the induction rule for this system. We start by formally defining the system.
We introduce CFA^∞ in the language of CFA. The *terms* of CFA^∞ are closed terms of CFA and its *formulas* are the closed formulas of CFA. For each closed term t in the language we have a standard interpretation $val(t)$ called the value of t. In particular, for $n \in \mathbb{N}$ we have that $val(\overline{n}) = n$.

Definition 4.7. *The infinitary proof system* CFA^∞, *where* Γ, Δ, Π *and* Σ *are finite multisets of closed formulas, is defined by the following postulates:*

Initial Sequents

$$\Gamma \Rightarrow \Delta$$

Where, Γ *contains a (under the standard interpretation) false atomic formula or* Δ *contains a true atomic formula.*

Inference Rules:

$$(\rightarrow L)\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Pi \Rightarrow \Sigma}{A \rightarrow B, \Gamma, \Pi \Rightarrow \Delta, \Sigma}$$

[10] The class of elementary functions is the least set of functions containing the basic functions and closed under composition and bounded minimization operators.

$$(\to R)\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

$$\omega R\ \frac{\Gamma \Rightarrow A(0), \Delta \ldots \Gamma \Rightarrow A(n), \Delta \ldots (n \in \mathbb{N})}{\Gamma \Rightarrow \forall x A(x), \Delta}$$

$$(\forall L)\ \frac{A(k), \Gamma \Rightarrow \Delta \qquad (k \in \mathbb{N})}{\forall x A(x), \Gamma \Rightarrow \Delta}$$

Definition 4.8. *A* CFA^∞ *derivation* D *is a well-founded tree*[11] *of sequents being locally correct w.r.t the above axioms and rules, or:*

1. *The sequents at the top nodes of* D *are initial sequents,*
2. *Every other sequent is obtained from the sequent(s) immediately above it by one of the rules.*

Abbreviation: $\vdash^\alpha \Gamma \Rightarrow \Delta$ abbreviates that there exists a CFA^∞ derivation with end-sequent $\Gamma \Rightarrow \Delta$ with (ordinal) height $\leq \alpha$.

Remark 4.9. *In* CFA^∞*, height is defined conventionally (as described in Definition 2.7), with the only distinction being that proof height increases by +1, following ordinal addition. Additionally, the presence of the* ω *Rule necessitates the utilization of well-orders (thus, the incorporation of ordinals) in general, as the well-ordering of* $\mathbb{N}$ *alone would not be adequate.*

Remark 4.10. *Due to the absence of the* Cut-Rule *in* CFA^∞*, it possesses the subformula property. Although later, we will demonstrate that* Cut *is in fact derivable in* CFA^∞*.*

The fact that the design of this calculus is modeled after the truth definition for the standard model, becomes evident in the following proposition.

Proposition 4.11. (CFA^∞) *If for some* α *and* $\Gamma \Rightarrow \Delta$*,* $\vdash^\alpha \Gamma \Rightarrow \Delta$ *then there is a* $\phi \in \Gamma$ *s.t.* $\mathbb{N} \nvDash \phi$ *or, there is a* $\psi \in \Delta$ *s.t.* $\mathbb{N} \vDash \psi$*.*

Proof. Proof by transfinite induction on the height of the proof. □

Corollary 4.12. *If for some* α *and* ϕ*,* $\vdash^\alpha \Rightarrow \phi$ *then* $\mathbb{N} \vDash \phi$*.*

Another significant feature of this calculus will be the incorporation of operator control via functions $F : \mathbb{N} \to \mathbb{N}$ which will control the computational content of the derivations. This method of operator controlled derivations was first introduced by Buchholz in Buchholz [1992] for ordinal analysis of strong impredicative theories. In our context, operators are weakly increasing and inflationary (i.e, $x \leq f(x)$) functions F. The underlying concept behind their use

[11] A well-founded tree is essentially a tree where starting from any node in the tree and following the directed edges, one will eventually reach a leaf node in finite number of steps.

is quite straight forward, for instance, in the case of an existential introduction $A(\bar{k}) \vdash \exists x A(x)$, one keeps track of the existential witness by demanding that $k \leq F(0)$. This requirement ensures that if we have a analytic (cut-free) derivation of $F \vdash \exists x A(x)$ then there is a $k \leq F(0)$ s.t. $\mathbb{N} \vDash A[k]$. The relation $F \vdash$ will be so set up that this crucial information on witnesses is conserved during derivations.

Definition 4.13. *(Skolem Operator/ Control Functions) A skolem operator is a function $F : \mathbb{N} \to \mathbb{N}$, such that if $m \leq n$ then $F(m) \leq F(n)$ (i.e, weakly monotone) and $m \leq F(m)$ (i.e, inflationary).*
Some notions related to skolem operators:

- *Given a skolem operator F we define $F[m_1, .., m_n](x) := F(max(m_1, \ldots, m_n, x))$.*
- *We say $F \leq G$ iff for all x, $F(x) \leq G(x)$*

Now we define the $F \vdash^\alpha$ relation in a manner that guarantees if $F \vdash^\alpha \Gamma \Rightarrow \Delta$, then $\vdash^\alpha \Gamma \Rightarrow \Delta$ holds within CFA^∞.

Definition 4.14. *We define the $F \vdash^\alpha$ relation inductively, $F \vdash^\alpha \Gamma \Rightarrow \Delta$ holds iff*

- *Base Clause: (Ax) Γ contains a false atomic formula or Δ contains a true atomic formula.*
- *Inductive Clause:*

($\to L$) *If there exists $\alpha_1, \alpha_0 < \alpha$ such that $F \vdash^{\alpha_0} \Gamma_1 \Rightarrow \Delta_1, \phi$ and $F \vdash^{\alpha_1} \Gamma_2, \psi \Rightarrow \Delta_2$ for some Γ_i, Δ_i s.t. $\Gamma_1, \Gamma_2 = \Gamma \backslash \{\phi \to \psi\}$[12] and $\Delta_1, \Delta_2 = \Delta$.* ***Or,***

($\to R$) *If there exists $\alpha_0 < \alpha$ such that $F \vdash^{\alpha_0} \Gamma, \phi \Rightarrow \Delta \backslash \{\phi \to \psi\}, \psi$ and $\Delta_1, \Delta_2 = \Delta$.* ***Or,***

($\forall L$) *If there exists $\alpha_0 < \alpha$ such that $F \vdash^{\alpha_0} \Gamma \backslash \{\forall x \phi(x)\}, \phi(\bar{k}) \Rightarrow \Delta$ for some $k \leq F(0)$.* ***Or,***

(ωR) *If for all i there exists $\alpha_i < \alpha$ such that $F[i] \vdash^{\alpha_i} \Gamma \Rightarrow \phi(\bar{i}), \Delta'$.*

Lemma 4.15. (CFA^∞) *Given an ordinal α,*

1. *(**Weakening**) If $F \vdash^\alpha \Gamma \Rightarrow \Delta$ and $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$, $\alpha \leq \alpha'$, $F \leq G$ then $G \vdash^{\alpha'} \Gamma' \Rightarrow \Delta'$*
2. *If $F \vdash^\alpha \Gamma, \phi \Rightarrow \Delta$ and ϕ is a true atomic formula then $F \vdash^\alpha \Gamma \Rightarrow \Delta$*
3. *If $F \vdash^\alpha \Gamma \Rightarrow \phi, \Delta$ and ϕ is a false atomic formula then $F \vdash^\alpha \Gamma \Rightarrow \Delta$.* ***(Inversions)***
4. *If $F \vdash^\alpha \Gamma \Rightarrow \phi \to \psi, \Delta$ then $F \vdash^\alpha \Gamma, \phi \Rightarrow \psi, \Delta$.*

[12] Here, $\{\phi \to \psi\}$ is treated as a multi-set. Also, as an example of multi-set subtraction, if $\Gamma_1 = \{A, A, B\}$ and $\Gamma_2 = \{A\}$ then $\Gamma_1 \backslash \Gamma_2 = \{A, B\}$

5. *If $F \vdash^{\alpha} \Gamma \Rightarrow \Delta, \forall x\phi(x)$ then for all i, $F[i] \vdash^{\alpha} \Gamma \Rightarrow \Delta, \phi(\bar{i})$*

Proof.

(1) We proceed by transfinite induction on α, i.e, the height of the derivation.
Base Case: Say $F \vdash^{0} \Gamma \Rightarrow \Delta$. Then $\Gamma \Rightarrow \Delta$ is an initial sequent and so is $\Gamma' \Rightarrow \Delta'$. Since $F \leq G$, we have $G \vdash^{\alpha'} \Gamma' \Rightarrow \Delta'$.
Inductive Case: $\Gamma \Rightarrow \Delta$ is arrived at by an application of a rule of the calculus. We only deal with the case of $\forall$ rules, the rest are straight forward.

- (ωR) Then we have that for all i, $F[i] \vdash^{\alpha_i} \Gamma \Rightarrow \phi(\bar{i}), \Delta \backslash \{\forall x\phi(x)\}$ and for all i, $\alpha_i < \alpha$. Since $F[i] \leq G[i]$ by induction hypothesis we have that $G[i] \vdash^{\alpha_i} \Gamma' \Rightarrow \phi(\bar{i}), \Delta' \backslash \{\forall x\phi(x)\}$ for all i, $\alpha_i < \alpha$. Now since $\alpha_i < \alpha \leq \alpha'$ by ωR rule we have that $G \vdash^{\alpha'} \Gamma' \Rightarrow \Delta'$.

- $(\forall L)$ Then we have for some $k \leq F(0)$ and $\alpha_0 < \alpha$, $F \vdash^{\alpha_0} \Gamma \backslash \{\forall x\phi(x)\}, \phi(\bar{k}) \Rightarrow \Delta$. By induction hypothesis we have that $G \vdash^{\alpha_0} \Gamma' \backslash \{\forall x\phi(x)\}, \phi(\bar{k}) \Rightarrow \Delta'$. As $\alpha_0 < \alpha \leq \alpha'$ and $k \leq F(0) \leq G(0)$ by $\forall L$ rule, $G \vdash^{\alpha'} \Gamma' \Rightarrow \Delta'$.

Thus, by transfinite induction we have our proof.

(5) We proceed by transfinite induction on the height α.
Base Case: Then $\Gamma \Rightarrow \Delta$ is an initial sequent. Since $F \leq F[i]$, an application of Weakening yields $F[i] \vdash^{\alpha} \Gamma \Rightarrow \Delta, \phi(\bar{i})$.
Inductive Case: $F \vdash^{\alpha} \Gamma \Rightarrow \Delta$ is reached by applying a rule of the calculus. We will focus on the case of the ω rule; for the other cases, a direct application of the Induction Hypothesis leads to the conclusion.

- Then we have that for all i, there exists $\alpha_i < \alpha$ such that $F[i] \vdash^{\alpha_i} \Gamma \Rightarrow \Delta', \phi(\bar{i})$. But then $F[i] \vdash^{\alpha} \Gamma \Rightarrow \Delta', \phi(\bar{i})$ as required.

Thus, by transfinite induction we conclude our proof.

((2),(3),(4)) can similarly be proven using transfinite induction on α □

Remark 4.16. *Note that as a direct corollary to Weakening, we obtain that if $F \vdash^{\alpha} \Gamma \Rightarrow \Delta$ and $F \leq G$, then $G \vdash^{\alpha} \Gamma \Rightarrow \Delta$.*

With the help of the previous lemma, the following proposition illustrates how the Skolem operators keep track of existential witnesses during derivations in CFA^{∞}.

Proposition 4.17 (Witness control)**.**

1. *If $F \vdash^{\alpha} \Rightarrow \exists x\phi(x)$ then there is a $k \leq F(0)$ s.t. $\mathbb{N} \models \phi(k)$*

2. *If $F \vdash^{\alpha} \Rightarrow \forall x \exists y\phi(x, y)$ where ϕ then for all m there is $n \leq F(m)$ s.t. $\mathbb{N} \models \phi[m, n]$*

Proof.

1. Suppose $F \vdash^{\alpha} \Rightarrow \exists x \phi(x)$, which means $F \vdash^{\alpha} \Rightarrow \forall x(\phi(x) \to \bot) \to \bot$. By lemma 4.15.4, we have $F \vdash^{\alpha} \forall x(\phi(x) \to \bot) \Rightarrow \bot$. Then, by Lemma 4.15.2, we obtain $F \vdash^{\alpha} \forall x(\phi(x) \to \bot) \Rightarrow$. Thus, the last rule applied to its derivation must be $\forall L$, hence $F \vdash^{\alpha_0} (\phi(\overline{k}) \to \bot) \Rightarrow$ for some $\mathbf{k} \leq F(0)$ and $\alpha_0 < \alpha$. Finally, applying Proposition 4.11, we conclude $\mathbb{N} \vDash \neg(\phi(\overline{k}) \to \bot)$ or, $\mathbb{N} \vDash \phi(\overline{k})$, as required.

2. Suppose $F \vdash^{\alpha} \Rightarrow \forall x \exists y \phi(x, y)$. Then, by Lemma 4.15.5, we infer that for any m, $F[m] \vdash^{\alpha} \Rightarrow \exists y \phi(\overline{m}, y)$. Consequently, according to part 1, there exists an $n \leq F[m](0) = F(m)$ such that $\mathbb{N} \vDash \phi[\overline{m}, \overline{n}]$.

□

Remark 4.18. *Proposition 4.17 plays a central role in our proof of classification and provides insight into the subsequent steps. By establishing Proposition 4.17, we realize that any provably recursive function in* CFA *(assuming we can embed* CFA *within* CFA^{∞}*) will be bounded by a Skolem operator. Consequently, if we can further characterize these Skolem operators, we can draw additional conclusions about the provably recursive functions in* CFA *(see Proposition 4.35 and Lemma 4.36).*

4.3.2 Admissibility of Cut in CFA^{∞}

In this subsection, we demonstrate the admissibility of the Cut-Rule in CFA^{∞}. This is essential if we aim to replicate the reasoning conducted in CFA within CFA^{∞}. Importantly, we capture the effect on skolem operators during an application of the (derived)cut-rule.

Proposition 4.19. *If F and G are primitive recursive skolem operators*[13]*, then so are the following:*

- *$F[k]$ for any $k \in \mathbb{N}$*
- *$F \circ G$*

In order to capture the effect on control operators during an application of cut, we define an operation on the control operators as follows.

Definition 4.20. *Given skolem operators F and G and $n \in \mathbb{N}^{+}$, we define $A^{n}(F, G)$ recursively (where, A stands for 'Admissible'):*

- *$A(F, G) = A^{1}(F, G) = F \circ G$*
- *$A^{n+1}(F, G) = A(A^{n}(F, G), A^{n}(F, G))$*

Remark 4.21. *The rationale behind defining such an operation on Skolem operators will become clear during the proof of the Cut-admissibility Theorem (Theorem 4.28). The primary idea is that, given Skolem operators F and G we have $F, G \leq F \circ G$ and importantly, if $k \leq G(0)$ then $F[k] \leq F \circ G$.*

[13] I.e, Skolem operators which are primitive recursive functions.

Proposition 4.22. *If F and G are primitive recursive skolem operators then for all $n, m \in \mathbb{N}^+$:*

1. *$A^n(F, G)$ is a primitive recursive skolem operator.*
2. *$A^n(F, G) = F \circ G \ldots F \circ G$ where $F \circ G$ is repeated 2^{n-1} times.*
3. *$A^m(A^n(F, G), A^n(F, G)) = A^{n+m}(F, G)$*
4. *If $n \leq m$ then $F, G \leq A^n(F, G) \leq A^m(F, G)$*
5. *If $F_1 \leq F_2$ and $G_1 \leq G_2$ then $A^n(F_1, G_1) \leq A^n(F_2, G_2)$.*
6. *If $k \leq G(0)$, then $A^n(F[k], G) = A^n(F, G)$*
7. *For all i, $A^n(F, G[i]) = A^n(F, G)[i]$*
8. *For all i, $A^n(F[i], G[i]) = A^n(F, G)[i]$*

To keep track of the height of derivations, we define the following operation on ordinals.

Definition 4.23. *(Natural Sum) Let $\alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_k}$ and $\beta = \omega^{\beta_1} + \ldots + \omega^{\beta_k}$ be in CNF. Let $\gamma_1, \ldots \gamma_{k+l}$ be $\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_{k+l}$ sorted in non-decreasing order. Then*

- *The natural sum of α and β, $\alpha \# \beta := \omega^{\gamma_1} + \ldots + \omega^{\gamma_k}$.*

Proposition 4.24. *(Properties of natural sum)*

1. *Natural sum is commutative and associative.*
2. *If $\alpha < \beta$ then $\alpha \# \gamma < \beta \# \gamma$.*

Remark 4.25. *These properties are not valid for the usual ordinal sum. It is for this reason that when working with height of proofs in CFA^∞ we instead work with natural sum .*

Lemma 4.26. *If we have $F \vdash^\alpha \Gamma \Rightarrow \Delta, B$ and $G \vdash^\beta B, \Pi \Rightarrow \Sigma$, with B being an atomic formula, then it follows that $A(F, G) \vdash^{max(\alpha,\beta)} \Gamma, \Pi \Rightarrow \Delta, \Sigma$.*

Proof.
Since B is closed, we have two possibilities: either $\mathbb{N} \vDash B$ or $\mathbb{N} \nvDash B$. Consequently, we consider two cases:

- If $B \in True_0$[14]: Applying Lemma 4.15.2 to $G \vdash^\beta B, \Pi \Rightarrow \Sigma$ yields $G \vdash^\beta \Pi \Rightarrow \Sigma$. Then, by applying weakening, we obtain $A(F, G) \vdash^{max(\alpha,\beta)} \Gamma, \Pi \Rightarrow \Delta, \Sigma$.

[14] Where, $True_0$ is the set of true atomic formula (Of course, in the language of CFA^∞) and $False_0$ is the set of false atomic formula

- If $B \in False_0$: Utilizing Lemma 4.15.3 on $F \vdash^{\alpha} \Gamma_1 \Rightarrow \Delta_1, B$ gives $F \vdash^{\alpha} \Gamma_1 \Rightarrow \Delta_1$. By applying weakening again, we arrive at $A(F,G) \vdash^{max(\alpha,\beta)} \Gamma, \Pi \Rightarrow \Delta, \Sigma$.

Therefore, we conclude $A(F,G) \vdash^{max(\alpha,\beta)} \Gamma, \Pi \Rightarrow \Delta, \Sigma$ as required.

□

Lemma 4.27. *Suppose $F \vdash^{\alpha} \Gamma \Rightarrow \Delta, B$ and $G \vdash^{\beta} B, \Pi \Rightarrow \Sigma$, where either $\alpha = 0$ or $\beta = 0$ (i.e., one of them is an initial sequent). Then $A(F,G) \vdash^{max(\alpha,\beta)} \Gamma, \Pi \Rightarrow \Delta, \Sigma$.*

Proof. Without loss of generality, let's assume $\alpha = 0$. Then we have two possibilities based on whether B is a true atomic formula or not:

- If $B \in True_0$: We apply Lemma 4.15.2 to $G \vdash^{\beta} B, \Pi \Rightarrow \Sigma$ and obtain $G \vdash^{\beta} \Pi \Rightarrow \Sigma$. Now, by applying weakening , we get $A(F,G) \vdash^{max(\alpha,\beta)} \Gamma, \Pi \Rightarrow \Delta, \Sigma$.
- If $B \notin True_0$: Then $\Gamma \Rightarrow \Delta$ is an initial sequent. Hence, $F \vdash^{0} \Gamma \Rightarrow \Delta$. Now, by applying weakening, we obtain $A(F,G) \vdash^{max(\alpha,\beta)} \Gamma, \Pi \Rightarrow \Delta, \Sigma$.

Thus, we have $A(F,G) \vdash^{max(\alpha,\beta)} \Gamma, \Pi \Rightarrow \Delta, \Sigma$ as required. □

With the preceding Lemmas at hand, we proceed to prove the admissibility of cut. Recall (see Definition 2.32) that $\mathcal{C}(B)$ denotes the complexity of a formula B.

Theorem 4.28. *(Admissibility of Cut,* CFA^{∞}*) If $F \vdash^{\alpha} \Gamma_1 \Rightarrow \Delta_1, B$ and $G \vdash^{\beta} B, \Gamma_2 \Rightarrow \Delta_2$ then $A^{\mathcal{C}(A)+1}(F,G) \vdash^{\gamma} \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ for some $\gamma > \alpha, \beta$*

Proof. We begin with a primary induction on the complexity of B (i.e. $\mathcal{C}(B)$), accompanied by a subsidiary transfinite induction on $\alpha\#\beta$.
Base Case: When $\mathcal{C}(B) = 0$, meaning B is an atomic formula. This is Lemma 4.26.
Inductive Case: Assume $\mathcal{C}(B) = n$ and that for all formulas of degree less than n, the formula can be eliminated.
We proceed by a *subsidiary induction* on $\alpha\#\beta$,

- Base case. Let's consider $\alpha\#\beta = 0$. We can then apply Lemma 4.27 to obtain our desired result.
- Inductive Case.

  By virtue of lemma 4.27, we assume that both $\alpha, \beta > 0$. Then we divide the problem into cases based on the rule whose conclusion is $G \vdash^{\beta} B, \Gamma_2 \Rightarrow \Delta_2$.

1. $\forall$**L**

   In this case, further two possibilities arise.

   Firstly, let's consider the case when the principal formula of the $\forall L$ rule is the cut formula $B = \forall x D(x)$.

   Then for some $\beta_0 < \beta$ and $k \leq G(0)$, we have $G \vdash^{\beta_0} D(\overline{k}), \Gamma_2 \Rightarrow \Delta_2$, i.e,

$$\begin{array}{c} \vdots \\ (\forall L)\ \dfrac{G \vdash^{\beta_0} D(\overline{k}), \Gamma_2 \Rightarrow \Delta_2}{G \vdash^{\beta} \forall x D(x), \Gamma_2 \Rightarrow \Delta_2} \end{array}$$

   According to Lemma 4.15.5, we have $F[k] \vdash^{\alpha} \Gamma_1 \Rightarrow \Delta_1, D(\overline{k})$. Since $\alpha \# \beta_0 < \alpha \# \beta$, by applying the induction hypothesis on $F[k] \vdash^{\alpha} \Gamma_1 \Rightarrow \Delta_1, D(\overline{k})$ and $G \vdash^{\beta_0} D(\overline{k}), \Gamma_2 \Rightarrow \Delta_2$ we obtain $A^{\mathcal{C}(D(x))+1}(F[k], G) \vdash^{\gamma} \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ for some $\gamma > \alpha, \beta_0$. By applying Proposition 4.22.6, we obtain $A^{\mathcal{C}(D(x))+1}(F, G) \vdash^{\gamma} \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$. Therefore, $A^{\mathcal{C}(B)+1}(F, G) \vdash^{\gamma} \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$.

   Secondly, let's consider the case when the principal formula of $\forall L$ is not the cut formula B. Suppose $\Gamma_2 = \forall x D(x), \Gamma_2'$

   Then for some $\beta_0 < \beta$ and $k \leq G(0)$, we have $G \vdash^{\beta_0} B, D(\overline{k}), \Gamma_2' \Rightarrow \Delta_2$, i.e,

$$\begin{array}{c} \vdots \\ (\forall L)\ \dfrac{G \vdash^{\beta_0} B, D(\overline{k}), \Gamma_2' \Rightarrow \Delta_2}{G \vdash^{\beta} B, \forall x D(x), \Gamma_2 \Rightarrow \Delta_2} \end{array}$$

   Since $\alpha \# \beta_0 < \alpha \# \beta$, by applying the induction hypothesis on $F \vdash^{\alpha} \Gamma_1 \Rightarrow \Delta_1, B$ and $G \vdash^{\beta_0} B, D(\overline{k}), \Gamma_2' \Rightarrow \Delta_2$, we have for some $\gamma_0 > \alpha, \beta_0$ that $A^{\mathcal{C}(B)+1}(F, G) \vdash^{\gamma_0} \Gamma_1, D(\overline{k}), \Gamma_2' \Rightarrow \Delta_1, \Delta_2$. According to Proposition 4.22.4, $k \leq G(0) \leq A^{\mathcal{C}(B)+1}(F, G)(0)$, thus by applying the $\forall L$ rule on $B(k)$, we obtain $A^{\mathcal{C}(B)+1}(F, G) \vdash^{\gamma} \Gamma_1, \forall x B(x), \Gamma_2' \Rightarrow \Delta_1, \Delta_2$ where $\gamma_0 < \gamma$.

2. ω**R**

   Say $\Delta_2 = \Delta_2', \forall x D(x)$. Then we have the following:

$$(\omega R)\ \dfrac{\begin{array}{c}\vdots \\ G[0] \vdash^{\beta_0} B, \Gamma_2 \Rightarrow \Delta_2', D(0)\end{array} \qquad \begin{array}{c}\vdots \\ G[1] \vdash^{\beta_1} B, \Gamma_2 \Rightarrow \Delta_2', D(\overline{1}) \ldots\end{array}}{G \vdash^{\beta} B, \Gamma_2 \Rightarrow \Delta_2', \forall x D(x)}$$

   where $\beta_i < \beta$.

   Then, for every i, by applying the Induction Hypothesis on $G[i] \vdash^{\beta_i} B, \Gamma_2 \Rightarrow \Delta_2', D(\bar{i})$ and $F \vdash^{\alpha} \Gamma_1 \Rightarrow \Delta_1, B$ we obtain $A^{\mathcal{C}(B)+1}(F, G[i]) \vdash^{\gamma_i} \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2, D(\bar{i})$ where $\gamma_i > \alpha, \beta_i$. From Proposition 4.9.7, we deduce $A^{\mathcal{C}(B)+1}(F,G)[i] \vdash^{\gamma_i} \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2', D(\bar{i})$. Now, by applying the ωR rule, we conclude that $A^{\mathcal{C}(B)+1}(F,G) \vdash^{\gamma} \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2', \forall x D(x)$, where for all i, $\gamma_i < \gamma$.

3. $\to$ **R**

   Suppose we have the following:

$$\begin{array}{c} \vdots \\ (\to R)\ \dfrac{G \vdash^{\beta_0} B, \Gamma_2, C \Rightarrow \Delta_2', D}{G \vdash^{\beta} B, \Gamma_2 \Rightarrow \Delta_2} \end{array}$$

   Where, $\Delta_2 = \Delta_2', C \to D$.

   Applying the Induction Hypothesis to $G \vdash^{\beta_0} B, \Gamma_2, C \Rightarrow \Delta_2', D$ and $F \vdash^{\alpha} \Gamma_1 \Rightarrow \Delta_1, B$ we obtain $A^{\mathcal{C}(B)+1}(F,G) \vdash^{\gamma_0} \Gamma_1, \Gamma_2, C \Rightarrow \Delta_1, \Delta_2', D$ for some $\gamma_0 > \alpha, \beta_0$. An application of $\to R$ rule yields $A^{\mathcal{C}(B)+1}(F,G) \vdash^{\gamma} \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2', C \to D$ where $\gamma_0 < \gamma$.

4. $\to$ **L**

   In this case as well, further two possibilities arise.

   Firstly, the case when the principal formula of the $\to L$ rule is not the cut formula: It's a straightforward application of the (subsidiary) induction hypothesis.

   Secondly, now consider the important case when the principal formula of the $\to L$ rule is the cut formula $B = C \to D$.

   Say we have the following:

$$(\to L)\ \dfrac{\begin{array}{c}\vdots \\ G \vdash^{\beta_0} \Gamma_{21} \Rightarrow \Delta_{21}, C\end{array} \qquad \begin{array}{c}\vdots \\ G \vdash^{\beta_0} D, \Gamma_{22} \Rightarrow \Delta_{22}\end{array}}{G \vdash^{\beta} C \to D, \Gamma_2 \Rightarrow \Delta_2}$$

   Applying inversion (Lemma 4.15.4) to $F \vdash^{\alpha} \Gamma_1 \Rightarrow \Delta_1, C \to D$ yields $F \vdash^{\alpha} \Gamma_1, C \Rightarrow \Delta_1, D$. Since $\alpha \# \beta_0 < \alpha \# \beta$, applying the (subsidiary) induction hypothesis to this and $G \vdash^{\beta_0} \Gamma_{21} \Rightarrow \Delta_{21}, C$ yields $A^{\mathcal{C}(C)+1}(F,G) \vdash^{\gamma_0} \Gamma_1, \Gamma_{21} \Rightarrow \Delta_1, \Delta_{21}, D$ for some $\gamma_0 > \alpha, \beta_0$.

   Given that $\mathcal{C}(D) < \mathcal{C}(C \to D) = n$, by the *primary* induction hypothesis on $A^{\mathcal{C}(C)+1}(F,G) \vdash^{\gamma_0} \Gamma_1, \Gamma_{21} \Rightarrow \Delta_1, \Delta_{21}, D$ and $G \vdash^{\beta_0} D, \Gamma_{22} \Rightarrow \Delta_{22}$, we obtain $A^{\mathcal{C}(D)+1}(G, A^{\mathcal{C}(C)+1}(F,G)) \vdash^{\gamma} \Gamma_1, \Gamma_{21}, \Gamma_{22} \Rightarrow \Delta_1, \Delta_{21}, \Delta_{22}$ for some $\gamma > \gamma_0, \beta_0$.

   By Lemma 4.15.5 and weakening, we further deduce

$$A^{\mathcal{C}(D)+1}(A^{\mathcal{C}(C)+1}(F,G), A^{\mathcal{C}(C)+1}(F,G)) \vdash^{\gamma} \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$$

   Again, since $\mathcal{C}(C) + 1 + \mathcal{C}(D) + 1 = \mathcal{C}(C \to D) + 1$ from Lemma 4.15.3, we obtain $A^{\mathcal{C}(C \to D)+1}(F,G) \vdash_0^{\gamma} \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$, as required.

Thus, by double induction we have our theorem.

□

Remark 4.29.

- *For the above proof, the absence of contraction from* CFA^∞ *was crucial. In the corresponding infinitary calculus of* PA *which would need to have contraction, for very similar reasons as in Remark 2.44, the argument presented in Theorem 2.28 would not go through.*

4.3.3 Embedding CFA in CFA^∞

With Cut at our disposal, we can now proceed to embed CFA within CFA^∞. We start by establishing some preliminary lemmas.

Proposition 4.30. *Let $k \in \mathbb{N}$ be a constant and F be a primitive recursive Skolem operator. Then there exists a primitive recursive Skolem operator G such that for all $i \in \mathbb{N}^+$ we have $A^{i\cdot k+i}(F,F)[i] \leq G[i]$.*

Proof. Define a function F' from F using primitive recursion as follows:

$$F'(x,0) = F(x)$$
$$F'(x,n+1) = F(F'(x,n))$$

Now, define $G(x) := F'(x, 2^{x\cdot k+x})$. Clearly, G is itself primitive recursive.
Claim: For any $i \in \mathbb{N}$, $A^{i\cdot k+i}(F,F)[i] \leq G[i]$.
Without loss of generality, assume $i \leq x$. As F is a Skolem operator, we have:

$$A^{i\cdot k+i}(F,F)[i](x) = F^{2^{i\cdot k+i}}(x) \leq F^{2^{x\cdot k+x}+1}(x) = F'(x, 2^{x\cdot k+x}) = G(x).$$

□

Lemma 4.31. *Every bounded formula in* CFA *defines a primitive recursive predicate, i.e., its characteristic function is primitive recursive.*

Proof. We proceed by induction on the structure of a bounded formula in CFA. Since primitive recursive predicates are easily seen to be closed under propositional connectives and bounded quantification, we only need to consider the base case, i.e., when the formula is atomic.
Let the formula be $t(x_1,\ldots,x_n) = s(x_1,\ldots,x_n)$. Then let the predicate defined by the formula be $X(m_1,\ldots,m_n) = 1$ if and only if $\mathbb{N} \vDash t(m_1,\ldots,m_n) = s(m_1,\ldots,m_n)$. Now, since t and s are built up from primitive recursive functions such as addition $(+)$, multiplication $(.)$, and successor (S), which are primitive recursive functions, $val(t(x_1,\ldots,x_n))$ and $val(s(x_1,\ldots,x_n))$ are also primitive recursive (since every polynomial is primitive recursive). Furthermore, since equality $(=)$ is also a primitive recursive predicate, we conclude that X is also a primitive recursive predicate. □

Lemma 4.32. *Given a term $t(x_1,\ldots,x_n)$ in the language of* CFA^∞*, the function $f : \mathbb{N}^n \to \mathbb{N}$ defined by $f(x_1,\ldots,x_n) = val(t(x_1,\ldots,x_n))$ is a weakly monotone primitive recursive function, i.e., for any $\vec{n},\vec{m} \in \mathbb{N}$ where for all i, $n_i \leq m_i$, we have $f(\vec{n}) \leq f(\vec{m})$.*

Proof. We proceed by induction on the complexity of the build-up of the term t. Note that the functions S, $+$, and $\times$ are all monotone. □

Lemma 4.33. (CFA^∞) *Suppose* $F \vdash^\alpha \Gamma, \phi(t) \Rightarrow \Delta$, *and if* $val(t) = n$ *where* $n \in \mathbb{N}$, *then* $F \vdash^\alpha \Gamma, \phi(\overline{n}) \Rightarrow \Delta$

Proof. Transfinite induction on α □

We make some preliminary comments about the proof strategy. In order to embed CFA within CFA^∞-

- Firstly, we need to demonstrate that the initial sequents of CFA are derivable in CFA^∞, with derivations bounded by appropriate primitive recursive Skolem operators. (Refer to Proposition 4.35 and Lemma 4.36 to know the necessity of primitive recursion in this context.)
- Then we can argue inductively on the proof of a derivation in CFA.
- The critical step in this proof would be to show that the primitive recursive nature of Skolem operators are preserved during Cut rule and the Induction Rule. These are the junctures where we rely on applying the theorem on Cut-admissibility. Moreover, the crucial scenario of the Induction rule, where repeated application of the cut-rule is necessary to emulate induction within CFA^∞, is solved with the aid of Propositions 4.15 and 4.30.

Theorem 4.34. *(Embedding) If* $\mathsf{CFA} \vdash \Gamma(\vec{a}) \Rightarrow \Delta(\vec{a})$ *then there exists a primitive recursive operator* F *and an ordinal* γ *s.t for all* $\vec{m}$, $F[m_1, \ldots, m_n] \vdash^\gamma \Gamma(\vec{m}) \Rightarrow \Delta(\vec{m})$.

Proof. We proceed by induction on the proof of $\Gamma(\vec{a}) \Rightarrow \Delta(\vec{a})$ in CFA.
Base Case: We have two possibilities-

- $\Gamma(\vec{a}) \Rightarrow \Delta(\vec{a})$ is an initial sequent of GQC^G. We take $F(x) = x$. Now, if $\bot = \Gamma(\vec{a})$ we are done. Otherwise say it is $A(\vec{x}) \Rightarrow A(\vec{x})$ where A is atomic. Then for any $\vec{m}$ either $\mathbb{N} \vDash A(\vec{m})$ or $\mathbb{N} \nvDash A(\vec{m})$. In either case we have that $A(\vec{m}) \Rightarrow A(\vec{m})$ is an initial sequent of CFA^∞. Consequently, we obtain $F[\vec{m}] \vdash^0 \Gamma(\vec{m}) \Rightarrow \Delta(\vec{m})$.
- $\Gamma(\vec{a}) \Rightarrow \Delta(\vec{a})$ is an initial sequent of the arithmetic.
  We consider $\mathsf{CFA} \vdash\Rightarrow \neg(S(t(\vec{a})) = 0)$ (the remainder can be argued for analogously). Now given any $\vec{m} \in \mathbb{N}$, $\mathbb{N} \nvDash S(t(\vec{\overline{m}})) = 0$. Thus, taking $F(x) = x$ we have that $F[\vec{m}] \vdash^0 S(t(\vec{\overline{m}})) = 0 \Rightarrow \bot$. Applying the $\to R$ rule, we obtain $F[\vec{m}] \vdash^1 \Rightarrow (S(t(\vec{\overline{m}}))) = 0 \to \bot)$ as needed.

Inductive Case: $\Gamma(\vec{a}) \Rightarrow \Delta(\vec{a})$ is arrived at by an application of a rule of the calculus CFA.

- $(\to L), (\to R)$: Follows directly from Induction Hypothesis.

- $(WL), (WR)$: Follows from lemma [4.6.1] and the Induction Hypothesis.
- (Cut) : Say we have the following application of cut in CFA

  $$\text{(Cut)}\ \frac{\Gamma_1(\vec{a}) \Rightarrow \Delta_1(\vec{a}), A(\vec{a}) \qquad A(\vec{a}), \Gamma_2(\vec{a}) \Rightarrow \Delta_2(\vec{a})}{\Gamma_1(\vec{a}), \Gamma_2(\vec{a}) \Rightarrow \Delta_1(\vec{a}), \Delta_2(\vec{a})}$$

  We apply Induction Hypothesis to get $F[\vec{m}] \vdash^{\gamma_1} \Gamma_1(\vec{m}) \Rightarrow \Delta_1(\vec{m}), A(\vec{m})$ and $G[\vec{m}] \vdash^{\gamma_2} A(\vec{m}), \Gamma_2(\vec{m}) \Rightarrow \Delta_2(\vec{m})$. Then, according to Theorem 4.28, we have $A^{C(A)+1}(F[\vec{m}], G[\vec{m}]) \vdash^{\gamma} \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$. From Proposition 4.15.8, we have $A^{C(A)+1}(F, G)[\vec{m}] \vdash^{\gamma} \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$. Additionally, from Proposition 4.15.1, we know that $A^{C(A)+1}(F, G)$ is primitive recursive.
- $(\forall L)$ Suppose $\mathsf{CFA} \vdash \Gamma(\vec{a}), \phi(t(\vec{a})) \Rightarrow \Delta(\vec{a})$. Then by the Induction hypothesis we have in CFA^{∞} that there is a primitive recursive function F such that for all $\vec{m} \in \mathbb{N}$, $F[m_1, ..., m_n] \vdash^{\gamma} \Gamma(\vec{m}), \phi(t(\vec{m})) \Rightarrow \Delta(\vec{m})$. Now by Lemma 4.33 we have that $F[m_1, ..., m_n] \vdash^{\gamma} \Gamma(\vec{m}), \phi(\bar{k}) \Rightarrow \Delta(\vec{m})$, where $k = val(t(\vec{m}))$.

  We define $F'(x) := F(x) + val(t(x, \dots, x))$. Finally, from lemma 4.32 we have that $k = val(t(\vec{m})) \leq t(max(\vec{m}), \dots, max(\vec{m})) \leq F'[\vec{m}](0)$. Then by $\forall L$ rule we will get that $F'[m_1, ..., m_n] \vdash^{\gamma \# 1} \Gamma(\vec{m}), \forall x \phi(x) \Rightarrow \Delta(\vec{m})$.
- $(\forall R)$: Say $\mathsf{CFA} \vdash \Gamma(\vec{a}) \Rightarrow \Delta(\vec{a}), \phi(\vec{a}, x)$ where x is not among $\vec{a}$. Then by Induction hypothesis we have in CFA^{∞} that there is a primitive recursive operator F_1 such that for all $\vec{m}, i \in \mathbb{N}$, $F_1[m_1, ..., m_n, i] \vdash^{\gamma_i} \Gamma(\vec{m}) \Rightarrow \Delta(\vec{m}), \phi(\vec{m}, i)$. Since $F_1[m_1, ..., m_n, i] = F_1[m_1, ..., m_n][i]$ an application of the ωR rule yields $F[m_1, ..., m_n] \vdash_0^{\gamma} \Gamma(\vec{m}) \Rightarrow \Delta(\vec{m}), \forall x \phi(\vec{m}, x)$, where $\gamma_i < \gamma$.
- Inductive Rule: Say $\mathsf{CFA} \vdash \Rightarrow \phi(\vec{a}, 0)$ and $\mathsf{CFA} \vdash \phi(\vec{a}, x) \Rightarrow \phi(\vec{a}, S(x))$. By induction hypothesis we have in CFA^{∞} that for some primitive recursive operators F_1 and F_2, for all $\vec{m}, i \in \mathbb{N}$, $F_1[m_1, ..., m_n] \vdash^{\gamma_1} \Rightarrow \phi(\vec{m}, 0)$ and $F_2[m_1, ..., m_n, i] \vdash^{\gamma_2} \phi(\vec{m}, \bar{i}) \Rightarrow \phi(\vec{m}, S(\bar{i}))$.

  Define $F = F_1 + F_2$. Then we will also have that $F[m_1, ..., m_n] \vdash^{\gamma_1} \Rightarrow \phi(\vec{m}, 0)$ and $F[m_1, ..., m_n, i] \vdash^{\gamma_2} \phi(\vec{m}, \bar{i}) \Rightarrow \phi(\vec{m}, S(\bar{i}))$.

  Now applying the Cut-admissibility theorem repeatedly we get for every[15] $i > 0$,

  $$A^{i.\mathcal{C}(\phi)+i}(F, F)[\vec{m}][i] \vdash^{\beta_i} \Rightarrow \phi(\vec{m}, \bar{i})$$

  Now, according to Proposition 4.30, we can construct a primitive recursive Skolem operator G such that for all positive i, $A^{i.\mathcal{C}(\phi)+i}(F, F)[i] \leq G[i]$.

[15] For instance, when $i = 1$, a direct application of Theorem [4.13] gives us $A^{\mathcal{C}(\phi)+1}(F, F)[\vec{m}][0] \vdash^{\beta_1} \Rightarrow \phi(\vec{m}, 1)$. Thus, weakening gives us $A^{\mathcal{C}(\phi)+1}(F, F)[\vec{m}][1] \vdash^{\beta_1} \Rightarrow \phi(\vec{m}, 1)$ as needed. An induction argument can easily generalize the result.

Consequently, we also have $A^{i.\mathcal{C}(\phi)+i}(F,F)[\vec{m}][i] \leq G[\vec{m}][i]$. Therefore, by weakening, we obtain for all i, $G[\vec{m}][i] \vdash^{\beta_i} \Rightarrow \phi(\vec{m},\bar{i})$.[16]

Finally, an application of the ωR rule would yield $G[\vec{m}] \vdash^{\beta} \Rightarrow \forall x\phi(\vec{m},x)$ where $\beta_i < \beta$. Since G is a primitive recursive skolem operator, we have what we needed.

□

We are finally ready to classify the provably recursive functions. However, before proceeding, we require a couple of additional results concerning primitive recursion.

Proposition 4.35. *Let F be a function satisfying the following conditions*

1. The graph of F is primitive recursive.
2. F is bounded above by some primitive recursive function.

Then F is itself a primitive recursive function.

Proof. Let X_F denote the primitive recursive characteristic function of the graph of F, defined as follows:

$$\begin{aligned} X_F(x,y) &= 1 && \text{if } F(x)=y \\ X_F(x,y) &= 0 && \text{otherwise} \end{aligned}$$

Given that F is bounded above by a primitive recursive function (say) G, and since bounded search is primitive recursive, we can define a function $F'(x)$ as follows:

$$F'(x) = \mu y \leq G(x)(X_F(x,y)=1)$$

This function $F'(x)$ is also primitive recursive. Moreover, note that $F'(x)$ essentially represents $F(x)$ for all x. Therefore, F is indeed primitive recursive. □

Lemma 4.36. (CFA) *If f is a provably recursive function in* CFA*, then it has a primitive recursive graph.*

Proof. Let f be a provably recursive function represented by the formula $\exists z\phi(z,x,y)$, where ϕ is a bounded formula. Subsequently, we have $\mathsf{CFA} \vdash \forall x\exists y\exists z\phi(z,x,y)$. By the embedding theorem (Theorem 4.34), there exists a primitive recursive Skolem operator G such that $G \vdash^{\alpha} \Rightarrow \forall x\exists y\exists z\phi(z,x,y)$.

Following the argument in Proposition 4.17, we will have that for any $m \in \mathbb{N}$ there exists $n_0 \leq G[m](0) = G(m)$ and $\alpha_1 < \alpha_0$, such that $G[m] \vdash^{\alpha_1} \Rightarrow \exists z\phi(z,\overline{m},\overline{n_0})$. Further, by the same argument, there exists $n_1 \leq G(m)$ and $\alpha_2 < \alpha_1$, such that $G[m] \vdash^{\alpha_2} \Rightarrow \phi(\overline{n_1},\overline{m},\overline{n_0})$.

Thus, from Corollary 4.12, for all m, there exist $n_0, n_1 \leq G(m)$ such that $\mathbb{N} \vDash \phi(\overline{n_1},\overline{m},\overline{n_0})$. **. . . (i)**

[16] Note that when $i=0$, the way we define G in Proposition 4.30 ensures that $F \leq G$ and $F[\vec{m}] \leq A^{m.\mathcal{C}(\phi)+m}(F,F)[\vec{m}] \leq G[\vec{m}][0]$.

Let X_ϕ be the characteristic function of the primitive recursive predicate defined by the bounded formula ϕ (Existence of which is guaranteed by Lemma 4.31). Then, $f(m) = n$
iff $\mathbb{N} \vDash \exists z\phi(z, \overline{m}, \overline{n})$
iff (from **(i)**) for some $z \leq G(m)$, $\mathbb{N} \vDash \phi(\overline{z}, \overline{m}, \overline{n})$
iff for some $z \leq G(m)$, $X_\phi(z, m, n) = 1$.
Thus, we define the characteristic function X_f of f as follows:

$$X_f(m, n) = 1 \text{ if and only if } \exists z \leq G(m)(X_\phi(z, m, n) = 1)$$

Then, X_f is a primitive recursion function. It is because bounded quantification, G and X_ϕ are all primitive recursive. □

Theorem 4.37. (CFA) *f is a provably recursive function of* CFA *if and only if f is a primitive recursive function.*

Proof.
Right to Left direction. This follows from Theorem 4.5.
Left to Right direction. Let f be a provably recursive function of CFA, represented by a Σ_1 formula $\phi(x, y)$ with x, y as the only free variables in ϕ.
We have $\mathsf{CFA} \vdash \forall x \exists y \phi(x, y)$. By the embedding theorem, in CFA^∞, we have $G \vdash^\gamma \forall x \exists y \phi(x, y)$, for some primitive recursive Skolem operator G. An application of Proposition 4.17 yields that for any m, there exists $n \leq G(m)$ such that $\mathbb{N} \vDash \phi(\overline{m}, \overline{n})$. Thus, for any m, we have $f(m) \leq G(m)$, indicating that f is bounded by a primitive recursive function G.
Furthermore, from Lemma 4.36, we know that f has a primitive recursive graph. Therefore, according to Proposition 4.35, f is a primitive recursive function. □

Remark 4.38.

- *While the proof of the theorem focuses on provably recursive functions of a single variable, we note that through subsequent generalizations of Proposition 4.35, Lemma 4.36, and Proposition 4.17 (which are straightforward), the proof can be directly extended to n-ary provably recursive functions.*

- *The theorem conclusively shows that* CFA *is distinct from* PA *because the class of provably recursive functions of* PA *contains (total) recursive functions which are not primitive recursive. For instance, the Ackermann function*[17]*, is provably recursive in* PA *but is not a primitive recursive function.*

[17] For the definition, refer to Appendix A.

5 Conclusion and Further Work

Conclusion

In this thesis, we conducted a proof-theoretic study of Contraction Free Arithmetic (CFA). We introduced a contraction free predicate logic (GQC), which is essentially the multiplicative fragment of LK without contraction rules. After a brief study of the logic, we defined the arithmetic over this logic. In Section 3.2, we established that the additive connective (which were omitted) in fact become derivable in CFA. Moreover, in our axiomatization of CFA, we were careful to select the induction rule over the induction axiom, and we justified our claim in Proposition 3.35, demonstrating that contraction-free arithmetic would collapse to PA in the presence of the induction schema. Thus, it becomes plausible to claim that our choice of the underlying logic and our particular axiomatization provide the most natural framework to study the consequences of removing the logical rule of contraction from arithmetic.

One of the key results (Theorem 3.32) established in this thesis is that the induction schema for bounded formulas is derivable in CFA, which gives a lot of impetus to our study. As a consequence of the presence of Δ_0 induction schema, we showed (Corollary 3.40) that any Π_2 formula provable in the arithmetical theory $\mathsf{I}\Delta_0$ is also provable in CFA under a natural formula translation. This result makes the study of CFA much smoother, as it allows us to freely make use of various known properties of $\mathsf{I}\Delta_0$.

This thesis culminated in the classification of provably recursive functions of CFA, which serves to be an important measure of the deductive strength of an arithmetical theory. In Section 4.2, we showed that primitive recursive functions are provably recursive in CFA, thereby establishing that CFA surpasses the capabilities of $\mathsf{I}\Delta_0$, which lacks the ability to verify the totality of all primitive recursive functions. While, in Section 4.3, we established that any provably recursive function in CFA is, in fact, primitive recursive. For our proof, we embedded (theorem 4.34) CFA in an infinitary calculus CFA^∞ in which derivations are controlled by certain operators (called skolem operators). The pivotal step (Theorem 4.28) in our proof was demonstrating the admissibility of the cut-rule in CFA^∞, wherein we observed that the behavior of the Skolem operators remains favorable throughout cut-admissibility. We used these operators to bound the provably recursive functions of CFA and thus, help us show that any provable recursive function is primitive recursive (see Proposition 4.35 and Lemma 4.36). Given that PA has provably recursive functions which are not primitive recursive, we conclude that CFA constitutes a distinct arithmetical theory from PA.

Further Work

Our work represents just the initial phase in the exploration of *Contraction Free Arithmetic*. In this concluding subsection, we note some questions that we would be interested to investigate next.

- An interesting direction of study would be to investigate the constructive nature of CFA. It is inspired by certain properties it shares with HA. Notably, in the underlying logic of CFA, akin to intuitionistic logic, the provability of $\Rightarrow \exists x A(x)$ implies the provability of $\Rightarrow A(t)$ for some term t. Furthermore, similar to CFA, the (additive) excluded middle for bounded formulas holds in HA. Given the similarities, we would like to investigate, in particular, whether disjunction property and numerical existence property holds in CFA: A theory T is said to have the disjunction property, if for any sentences A and B we have that $\mathsf{T} \vdash A \vee B$, then we have that either $\mathsf{T} \vdash A$ or $\mathsf{T} \vdash B$ holds; a theory T is said to have the numerical existence property, if for any formula $A(x)$ with no free variable other than x, is we have that $\mathsf{T} \vdash A \vee B$ then we have that either $\mathsf{T} \vdash \exists x A(x)$ then there is a numeral n such that $\mathsf{T} \vdash A(\overline{n})$ holds[1].

- A natural progression for future research involves determining its position relative to other arithmetic systems. In particular, we are interested in CFA's relation with systems of arithmetic which have the same class of provably recursive functions (i.e. primitive recursive functions). For instance, we are interested in CFA's relation with PRA and $\mathsf{I}\Sigma_1$. To begin with, we would be interested to see whether Π_2 formulas provable in PRA is also provable in CFA. Moreover, it is a well known result by Parson's that, $\mathsf{I}\Sigma_1$ is Π_2-conservative over PRA. Then, this would also imply that Π_2 formulas provable in $\mathsf{I}\Sigma_1$ is also provable in CFA.

- Furthermore, we wish to rigorously establish Gödel's Second Incompleteness Theorem for CFA, which was not pursued due to time constraints. In Section 3.3, we established that any Π_2 sentence which is provable in $\mathsf{I}\Delta_0$ would also be provable in CFA under a suitable formula translation. Given that, akin to the case of $\mathsf{I}\Delta_0$, we would be able to execute arithmetization of syntax in CFA. Moreover, we also have that CFA is Σ_1

[1] Recall that, in CFA, $A \vee B \leftrightarrow \exists x((x = 0 \rightarrow A) \otimes (x \neq 0 \rightarrow B)$. Hence, if CFA has the numerical existence property then it will also have the disjunction property.

complete. Furthermore, even though we have the induction rule instead of the induction schema, we will be able to define a suitable Σ_1 provability predicate for CFA. In the same section, we also showed that our CFA enjoys left contraction for Σ_1 formulas. Building upon the insights from the work done in Beklemishev and Shamkanov [2016], it appears feasible to verify that CFA satisfies the constraints outlined in that paper for the second incompleteness theorem. However, careful work needs to be done to establish a fixed point lemma. Moreover, we need to ensure that our modified provability predicate satisfies the Hilbert-Bernays conditions.

- Closely related to the study of second incompleteness theorem in CFA, is the study of its Provability logic. The provability logic (say P) of an arithmetical theory T is a propositional modal logic which captures the notion of provability with its $\Box$ operator. To make things a bit more precise, we define the notion of $T-interpretation$ $f_T(\phi)$ of a modal formula ϕ which assigns to each propositional atom of modal logic a sentence of arithmetic, and it also satisfies :

  - $f_T(\bot) = \bot$;
  - $f_T(A \to B) = f_T(A) \to f_T(B)$;
  - $f_T(\Box A) = Prov_T(\overline{f_T(A)})$. Where $Prov_T$ is the provability predicate for T.

  Further, we have the notion of arithmetical soundness and completeness:

  - Arithmetical Soundness. Given any modal formula A and T− interpretation f_T, if P $\vdash A$ then T $\vdash f_T(A)$.
  - Arithmetical completeness. Given any modal formula A, if T $\vdash f_T(A)$ for any T− interpretation f_T, then P $\vdash A$.

  Then, we say that P is the provability logic of an arithmetical theory T iff P is arithmetically sound and complete with respect to T. The landmark result in provability is R. Solovay's arithmetical completeness theorem of 1976, where he showed that the modal logic GL is arithmetically complete with respect to PA. Provability logic has played an important and vibrant role in the study of the foundations of mathematics and mathematical logic at large. For a thorough introduction, we direct the reader to Artemov and Beklemishev [2004]. The provability logics of various *classical* subsystems of PA have been investigated, while, on the other hand, provability logic of arithmetical theories based on non-classical logic has been under-investigated. For instance, the provability logic of HA was established (although it is not yet peer-reviewed) only recently in 2022 (see Mojtahedi [2022]). There is an important result (see de Jongh et al. [1991]), in our context, in the field of provability logics for (*classical*) sub-systems of PA which says that GL is arithmetically sound and complete with respect to an arithmetical theory T if the following conditions hold[2]:

1. T proves induction schema for Δ_0 formulas, and T proves the totality EXP[3].
2. T is a sound arithmetical theory.

Considering that CFA establishes the provability of the Δ_0 induction schema and that EXP is a primitive recursive function (thus, its totality is provable in CFA), there arises a compelling possibility that GL serves as the provability logic of CFA. However, it is important to carefully consider the implications of removing contraction on this result. Hence, exploring the provability logic of CFA could be an interesting research direction.

- The next direction of research would be an investigation of the *reflection principles* of CFA. Reflection principles, in the context of a recursively enumerable theory T, are formal schemata that assert the soundness of T, meaning they state that 'every sentence provable in T is true'. More precisely, if $Prov_T(x)$ denotes a Σ_1 provability predicate for T, then the (uniform) reflection principle for T is the schema:

$$\forall x(Prov_T(\overline{A(x)}) \to A(x))$$

for all formulas $A(x)$. The schema is denoted by $RFN(\mathsf{T})$. Partial reflection principles are derived by restricting the formula A to range over specific sub-classes of T-formulas. Commonly, these classes are chosen from the arithmetical hierarchy, such as Σ_n or Π_n. The resulting partial reflection principles are labeled as $RFN_{\Sigma_n}(\mathsf{T})$ and $RFN_{\Pi_n}(\mathsf{T})$, respectively. The following classical facts are well known :

1. $RFN_{\Sigma_n}(\mathsf{T})$ is equivalent to $RFN_{\Pi_{n+1}}(\mathsf{T})$ over Elementary arithmetic(EA) for $n \geq 1$. Moreover, $RFN_{\Pi_1}(\mathsf{T})$ is equivalent to $Con(\mathsf{T})$, the consistency assertion for T (c.f. Smorynski [1977]).
2. $\mathsf{PA} \equiv \mathsf{EA} + RFN(\mathsf{EA})$ or, in other words, an alternative axiomatization of PA over EA can be obtained by replacing the induction schema with the full (uniform) reflection schema for EA.
3. Over EA, the induction schema for Σ_n formulas is equivalent to $RFN_{\Sigma_{n-1}}(\mathsf{T})$ (c.f.Leivant [1983]).

In Beklemishev [1997], Lev Beklemishev showed that fragments of arithmetic axiomatized by various forms of induction rules can also be axiomatized in terms of reflection principles. For instance, consider the following

[2] It is yet unknown whether condition 1 gives a lower bound on the scope of provability logic.

[3] EXP is the the Π_1 formula expressing that for all x, its power 2^x exists

progression of formal systems over EA:

$$\begin{aligned}
&\mathsf{EA_0} = \mathsf{EA}\\
&\mathsf{EA_{n+1}} = \mathsf{EA_n} + \{RFN_{\Sigma_n}(\mathsf{EA_n})\}\\
&\mathsf{EA}_\omega = \bigcup_{n\in\omega} \mathsf{EA_n}
\end{aligned}$$

Then, $\mathsf{EA}_\omega \equiv \mathsf{EA} + \Sigma_n$-Induction Rule. In order words, the closure of EA under the induction rule for Σ_n is equivalent to the ω iterated Σ_n reflection principle.

Considering the close link between induction rules and reflection schemas in arithmetical theories with a classical base logic, and noting that CFA is axiomatized with induction rules, it becomes intriguing to explore reflection principles for contraction-free arithmetic. Specifically, we aim to investigate whether a similar relationship between contraction-free arithmetics, axiomatized by restricted classes of induction rules, and reflection principles can be established.

We conclude this thesis with the remark that further exploration of contraction-free arithmetic will require a deeper comprehension of the classes of formulas within this arithmetic. Since not all formulas are equivalent to some prenex formula, we will encounter classes of formulas that are more diverse than those provided by the arithmetical hierarchy.

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A | Appendix A

Appendix A mainly consists of some relevant definition which were not included in the text and also mentions some well-known theorems.

Definition A.1. *(Prenex formulas) A prenex formula is a formula of the form:*

$$Q_1 x_1 \dots Q_n x_n B$$

where Q_i is either $\forall$ or $\exists$ and the formula B is quantifier-free.

Remark A.2. *A theorem of classical logic states that given any formula, we can find another formula which is in prenex form and is equivalent to the original formula. On the other hand, this does not hold in intuitionistic logic.*

Definition A.3. *(Primitive Recursion) A function $f : \mathbb{N}^k \to \mathbb{N}$ is said to defined from g and h by primitive recursion if :*

$$\begin{aligned} &f(\vec{x}, 0) = g(\vec{x}) \\ &f(\vec{x}, n+1) = h(f(\vec{x}, n), \vec{x}, n) \end{aligned}$$

where $\vec{x} = x_1, \dots, x_m$. We also allow $m = 0$, in which case g is interpreted as a constant and $f(n+1)$ is defined in terms of the previous value $f(n)$.

Definition A.4. *(Primitive Recursive Functions, Gödel) The class of primitive recursive functions is the smallest class of functions :*

1. *Containing the initial functions.*

$$\begin{aligned} &0^m(\vec{x}) = 0 && \textit{(m-ary constant function zero)} \\ &S(n) = n + 1 && \textit{(Succesor Function)} \\ &P_i^m(\vec{x}) = x_i && \textit{(m-ary Projection Function)} \end{aligned}$$

2. *Closed under composition,i.e, the schema that defines*

$$f(\vec{x}) = h(g_1(\vec{x}), \dots, g_m(\vec{x}))$$

from given functions g_i and h.

3. *Closed under primitive recursion.*

Definition A.5. *(Partial Recursive Functions, Kleene) The class of recursive functions is the smallest class of functions :*

1. *Containing the initial functions.*
2. *Closed under composition.*
3. *Closed under primitive recursion.*
4. *Closed under μ-recursion, i.e, the schema that defines*

   $$\phi(\vec{x}) = \mu y[(\forall z \leq y \psi(\vec{x}, z) \downarrow) \otimes \psi(\vec{x}, y) = 0]$$

   from given function ψ

A (general) recursive (also called computable) function is a function which happens to be $Total$. Hence, any primitive recursive function is a recursive function.

Definition A.6. *(Recursive Enumerable set) A set $A \subseteq \mathbb{N}^k$ is said to be recursively enumerable if there is a recursive function f s.t. $A := rng(f)$.*

Definition A.7. *(Ackermann function) Ackermann function, A, is a function from $\mathbb{N}^2$ to $\mathbb{N}$ and is defined recursively in the following way:*

$$\begin{aligned} &A(0, n) = n + 1 \\ &A(m + 1, 0) = A(m, 1) \\ &A(m + 1, n + 1) = A(m, A(m + 1, n)) \end{aligned}$$

Remark A.8. *In recursive function theory, the Ackermann function, is a rather classic example of a total recursive function which is not primitive recursive. Moreover, the Ackermann is a provably recursive function of* PA.

Definition A.9. *A* ***coding of finite sequences*** *consists of a primitive recursive set $Seq \subset \mathbb{N}$ and primitive recursive functions:*

1. *Unary function lh; called the length of a sequence s.*
2. *Binary function $memb$; $memb(s, i)$ is the i th member of s.*
3. *Binary function $prolong$; $prolong(s, k)$ is the result of juxtaposing k with s.*

such that the following holds for each s, $s' \in Seq$:

- *$lh(s) \leq s$ and, for each $i < lh(x)$, $memb(s, i) < s$*
- *there exists an empty sequence ε with $lh(\varepsilon) = 0$*
- *for each $k \in \mathbb{N}$, if $s' = prolong(s, k)$ then $lh(s') = lh(s) + 1$, for $i < lh(s)$ we have $memb(s, i) = memb(s', i)$ and for $i = lh(s)$ we have $memb(s', i) = k$.*

- *if $lh(s) \leq lh(s')$ and, for each $i < lh(s)$, $memb(s,i) \leq memb(s',i)$ then $s \leq s'$*
- *$\mathbb{N} - Seq$ is infinite*

Remark A.10. *We will informally write $(s)_i$ to mean the same number as $memb(s,i)$.*

Theorem A.11. *There is a Δ_0 coding of finite sequences*[1]*.*

Theorem A.12. *Some useful primitive recursive functions concerning sequences.*

1. For each $n \geq 1$, there is an n-ary primitive recursive function associating with each $k_0, .., k_{n-1} \in \mathbb{N}$ the n-tuple $\langle k_0, .., k_{n-1}\rangle$, i.e. the sequence s of length n such that, for each $i < n$, $(s)_i = k_i$.
2. Concatenation: For $s, t \in Seq$, $s \frown t$ denotes the concatenation of s, t.

Definition A.13. (Σ_1 *completeness of a theory* T) *An arithmetical Theory T is said to be Σ_1 complete if for any Σ_1 formula that holds true in $\mathbb{N}$ is provable in T.*

Definition A.14. (Σ_1 *soundness of a theory* T) *An arithmetical Theory T is said to be Σ_1 sound if any Σ_1 formula provable in T is true in $\mathbb{N}$.*

Definition A.15. *(Language L_0) L_0 is a first-order vocabulary order vocabulary, given by,*

$$L_0 := \{0, S, +, \cdot, =, <\}$$

*where **0** is a constant symbol; **S** is an unary function symbol; **(+),** $(\cdot)$ are binary function symbols ; **=,<** are binary predicate symbols.*

Definition A.16. *(Parikh [1971])* ($\mathsf{I}\Delta_0$) $\mathsf{I}\Delta_0$ *is the theory in the Language L_0 over classical logic with the following axioms:*

1. $\neg(S(x) = 0)$
2. $(S(x) = S(y) \to x = y)$
3. $\neg(x = 0) \to \exists y(x = S(y))$
4. $x + 0 = x$
5. $x + S(y) = S(x + y)$
6. $x \cdot 0 = 0$
7. $x \cdot S(y)) = x \cdot y + x$
8. $x < y \leftrightarrow \exists z(x + S(z) = y)$

[1] See Chapter 5.3.(f) of Pudlák and Hájek [1993]

Together with the induction schema for bounded formulas. I.e. given any bounded formula $\phi(x)$:

$$\phi(0) \wedge \forall x(\phi(x) \to \phi(S(x)) \to \forall x \phi(x)$$

Remark A.17. *Instead of the bounded induction schema taken in Definition A.16, we can equivalently axiomatize $\mathsf{I}\Delta_0$ with the following form of bounded induction schema:*

$$\forall x(A(0) \wedge \forall y < x(A(y) \to A(S(y)) \to A(x))$$

Remark A.18. $\mathsf{I}\Delta_0$ *is also an* $\mathsf{I}\Sigma_0$ *in literature. For more details on* $\mathsf{I}\Delta_0$ *consult Pudlák and Hájek [1993].*

Proposition A.19. ($\mathsf{I}\Delta_0$) *Given a formula ϕ in the language of* $\mathsf{I}\Delta_0$*, if we have $I\Delta_0 \vdash \phi$ then there exists a finite set Δ of formulas which contains instances of the axioms of* $\mathsf{I}\Delta_0$ *such that there exists a cut-free proof of the sequent $\Delta \Rightarrow \phi$ in* LK.

Definition A.20. (PRA), PRA *is an arithmetical theory axiomatized over* $\mathsf{I}\Delta_0$ *in a language that includes a function symbol for every primitive recursive function, along with their defining equations.*

Definition A.21. ($\mathsf{I}\Sigma_1$) $\mathsf{I}\Sigma_1$ *is the arithmetical theory axiomatized over* $\mathsf{I}\Delta_0$ *which contains in addition the induction schema for Σ_1 formulas.*

Definition A.22. (GL) *Gödel-Löb logic* (GL) *is a modal logic which results from adding Löb's axiom to modal logic **K**. Where Löb's axiom is the following principle:*

$$\Box(\Box p \to p) \to \Box p.$$

Definition A.23. *(Conservative Extension) A theory* T′ *is a* conservative extension *of theory* T*, if all theorems of* T *are theorems of* T′ *and additionally, all theorems of* T′ *in the language of* T *are also theorems of* T.

B | Appendix B

Appendix B mainly consists of some proofs which were left out from the text to maintain a smoother flow.

Proposition B.1. *(CFA) If C, D are bounded formulas then it can be shown that left and right contraction holds for $B \wedge C$ and $B \vee C$.*

Proof. As C, D are bounded formulas we have from Theorem 3.28 that $\Rightarrow C \vee \neg C$ and $\Rightarrow B \vee \neg B$. Now consider the following proof tree:

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Rightarrow C \vee \neg C$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$C \vee \neg C, B \Rightarrow (B \wedge C) \vee \neg C$}
\LeftLabel{(Cut)}
\BinaryInfC{$B \Rightarrow (B \wedge C) \vee \neg C$}
\LeftLabel{$(\vee R)$}
\UnaryInfC{$B \Rightarrow (B \wedge C) \vee \neg B \vee \neg C$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\neg B \Rightarrow (B \wedge C) \vee \neg B \vee \neg C$}
\LeftLabel{$(\vee L)$}
\BinaryInfC{$B \vee \neg B \Rightarrow (B \wedge C) \vee \neg B \vee \neg C$}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\Rightarrow B \vee \neg B$}
\LeftLabel{(Cut)}
\BinaryInfC{$\Rightarrow (B \wedge C) \vee \neg B \vee \neg C$}
\UnaryInfC{$\Rightarrow (B \wedge C) \vee \neg (B \wedge C)$}
\end{prooftree}
$$

Thus, we will have that left and right contraction holds for $B \wedge C$. Similarly, it can be established for $B \vee C$.

□

Proposition B.2. *(CFA)*

1. *Double Induction:*

$$
\textit{(DoubleInd)}\ \frac{\Rightarrow A(0,0) \quad \Rightarrow \forall y A(0,y) \quad \Rightarrow \forall x A(x,0) \qquad \Rightarrow \forall x \forall y (A(x,y) \rightarrow A(S(x), S(y)))}{\Rightarrow \forall x \forall y A(x,y)}
$$

2. $y < x \Rightarrow y < S(x)$

Proof. 1. Consider the following (partial) proof tree.

$$\text{(IndRule)}\ \dfrac{\dfrac{\begin{array}{c}\vdots\\ \Rightarrow A(S(x),0)\end{array}}{\Rightarrow A(x,0)\to A(S(x),0)} \qquad \begin{array}{c}\vdots\\ \Rightarrow A(x,y)\to A(S(x),S(y))\end{array}}{\Rightarrow \forall y(A(x,y)\to A(S(x),S(y)))}$$

$$\vdots$$

$$\text{(IndRule)}\ \dfrac{\Rightarrow \forall yA(x,y)\to\forall yA(S(x),S(y))) \qquad\qquad \Rightarrow\forall yA(0,y)}{\forall x\forall yA(x,y)}$$

2. Consider the following proof tree.

$$\dfrac{(\exists L)\ \dfrac{(\exists R)\ \dfrac{\dfrac{\begin{array}{c}\vdots\\ y+S(c)=x\Rightarrow S(y+S(c))=S(x)\end{array} \qquad \begin{array}{c}\vdots\\ \Rightarrow y+S(S(c))=S(y+S(c))\end{array}}{y+S(c)=x\Rightarrow y+S(S(c))=S(x)}}{y+S(c)=x\Rightarrow\exists z(y+S(z)=S(x))}}{\exists c(y+S(c)=x)\Rightarrow\exists z(y+S(z)=S(x))}}{y<x\Rightarrow y<S(x)}$$

□

Proposition B.3.

1. *Equality is symmetric.*
2. *Equality is transitive.*

Proof.

- Define a predicate $P(x) := x = s$. Now, consider the following proof tree.

$$\text{(cut)}\ \dfrac{\dfrac{\begin{array}{c}\vdots\\ s=t,P(s)\Rightarrow P(t)\end{array}}{s=t,s=s\Rightarrow t=s} \qquad \Rightarrow s=s}{s=t\Rightarrow t=s}$$

- Define a predicate $P(x) := x = r$. Now, using part 1, we construct the following proof tree.

$$\text{(cut)}\ \dfrac{\begin{array}{c}\vdots\\ t=s,P(t)\Rightarrow P(s)\end{array} \qquad \begin{array}{c}\vdots\\ s=t\Rightarrow t=s\end{array}}{s=t,P(t)\Rightarrow P(s)}$$

□

Proposition B.4. (CFA$'$) *Let B, C be any formula where x does not occur free, then : $B \lor C \Leftrightarrow \exists x(x = 0 \to B \otimes \neg(x = 0) \to C)$*

Proof. "$\Rightarrow$"
With the help of Lemma 3.14, we construct the following proof tree:

$$(\lor \mathrm{L})\ \dfrac{(\exists \mathrm{R})\ \dfrac{\begin{matrix}\vdots\\ B \Rightarrow 0 = 0 \to B \otimes 0 \neq 0 \to C\end{matrix}}{B \Rightarrow \exists x(x = 0 \to B \otimes x \neq 0 \to C)} \qquad (\exists \mathrm{R})\ \dfrac{\begin{matrix}\vdots\\ C \Rightarrow S(0) = 0 \to B \otimes S(0) \neq 0 \to C\end{matrix}}{C \Rightarrow \exists x(x = 0 \to B \otimes x \neq 0 \to C)}}{B \lor C \Rightarrow \exists x(x = 0 \to B \otimes x \neq 0 \to C)}$$

"$\Leftarrow$"

Now let $\phi(x) := ((x = 0 \to B) \otimes (x \neq 0 \to C)) \to B \lor C$, we proceed with Induction rule on ϕ:

Basis.

$$(\lor R)\ \dfrac{\begin{matrix}\vdots\\ (0 = 0 \to B) \otimes (0 \neq 0 \to C) \Rightarrow B\end{matrix}}{(0 = 0 \to B) \otimes (0 \neq 0 \to C) \Rightarrow B \lor C}$$

Hence, we have $\Rightarrow \phi(0)$
Inductive step.

$$(\to R, WL)\ \dfrac{\begin{matrix}\vdots\\ (S(x) = 0 \to B) \otimes (S(x) \neq 0 \to C) \Rightarrow B \lor C\end{matrix}}{(x = 0 \to B) \otimes (x \neq 0 \to C) \to B \lor C \Rightarrow (S(x) = 0 \to B) \otimes (S(x) \neq 0 \to C) \to B \lor C}$$

Hence, we have $\phi(x) \Rightarrow \phi(S(x))$

Thus, applying Induction rule in CFA$'$, we get that $\Rightarrow \forall x \phi(x)$
Further, as x is not free in $B \lor C$ and we have that $\Rightarrow \forall x(P \to Q) \to \exists x P \to Q$, we obtain $\exists x((x = 0 \to B) \otimes (x \neq 0 \to C)) \to B \lor C$, as required. □

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I conclude by acknowledging the influence of the poet Rabindranath Tagore (after all, it's blasphemous for a Bengali not to do so) in the development of my conscience, and by quoting [1] him on something I came to understand gradually–

> "A mind all logic is like a knife all blade. It makes the hand bleed that uses it
>
> If you shut your doors to all errors, truth will be shut out"

[1] Stray Birds, Rabindranath Tagore, 1916