The temporal Heyting calculus

An algebraic, topological, and frame-theoretic analysis

MSc Thesis (*Afstudeerscriptie*)

written by

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Abstract

This thesis studies Esakia's *temporal Heyting calculus* **tHC**, a temporal intuitionistic modal logic, by employing algebraic, topological, and frame-theoretic methods. First, we develop a general theory of temporal Heyting algebras, showing **tHC** to be sound and complete with respect to the variety and studying an important class of filters called ⧫*-filters* as well as an important class of elements called ⧫*-compatible* elements. Next, we develop a general theory of temporal Esakia spaces, studying an important class of subsets called *archival* subsets and using this notion to define two notions of « reachability » on our spaces : one order-topological and the other purely frame-theoretic. Next, we establish and develop an Esakia duality between the categories of temporal Heyting algebras and temporal Esakia spaces, including a full contravariant equivalence as well as a congruence/filter/closed-upset correspondence. Next, we use our duality theory to study the relational models of **tHC**, establishing relational soundness and completeness, developing a method of filtration on our models, and establishing the relational finite model property. Finally, we apply several of our duality and relational results to prove several facts about the variety of temporal Heyting algebras, including the algebraic finite model property. We conclude by proving the main results of the thesis : a lattice-theoretic and order-topological characterisation of both simple and subdirectly-irreducible temporal Heyting algebras (in both the general and finite case) as well as a final relational completeness result that combines finiteness and a type of « rootedness » defined in terms of the above-mentioned notion of « reachability ». We also include, in an appendix, a brief description of an algebraic symbolic model checker called `thacheck` that was authored for the variety of temporal Heyting algebras.

Keywords : intuitionistic modal logic, temporal logic, temporal Heyting algebras, Esakia duality

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Thesis

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Chapter 1

Introduction

This thesis studies the *temporal Heyting calculus*, a temporal intuitionistic modal logic that was defined by Esakia in 2006 in [24]. Until now, this logic has never been the subject of an extensive treatment, but it has been investigated and presented-on at conferences by Jibladze [30] and Alshibaia [1]. The temporal Heyting calculus is the temporal version of the *modalised Heyting calculus*, which has been been studied in [24, 38, 39, 29, 30]. It can be seen as an application of the tradition of « temporalising » *classical* modal logics [48] to the intuitionistic setting.

We study this logic in three ways : algebraically, order-topologically, and frame-theoretically. In the first case, we utilise algebraic semantics for intuitionistic logics to establish soundness and completeness with respect to the variety of *temporal Heyting algebras*. This allows us to study the temporal Heyting calculus via *algebraic* methods. In the second case, we establish an Esakia duality between the class of temporal Heyting algebras and a class of ordered topological spaces called *temporal Esakia spaces*. This allows us to study the variety of temporal Heyting algebras, and, thereby, the temporal Heyting calculus, via *topological* methods. In the final case, we utilise relational semantics to establish soundness and completeness with respect to the class of *temporal transits*. This allows us to study the temporal Heyting calculus via *frame-theoretic* methods.

We provide the following chapter-by-chapter outline.

In Chapter 2, we establish the preliminary definitions and facts necessary to understand the theory developed in the thesis. This includes the basics of intuitionistic modal logic, order theory, universal algebra, lattice theory, ordered topological spaces, and Esakia duality. We provide references for each of these fields and attempt to present the material in a way that is extensive enough to keep the thesis self-contained but minimal enough to never include unnecessary theory, all the while maintaining consistency with modern literature on the topics.

In Chapter 3, we develop a general theory of temporal Heyting algebras. We first establish algebraic soundness and completeness for the temporal Heyting calculus. We then study the class of filters corresponding to congruences and, subsequently, the class of elements corresponding to both of these in the finite case.

In Chapter 4, we develop a general theory of temporal Esakia spaces. We study a class of subsets called *archival* subsets which will allow us to define and study two notions of « reachability » on temporal Esakia spaces : one order-topological and the other purely frame-theoretic. These notions of « reachability » are analogous to the well-known *specialisation ordering* as well as the notion of « topo-reachability » in [46]. We then show that the two notions coincide in the finite case.

In Chapter 5, we establish and study a duality theory between the category of temporal Heyting algebras and temporal Esakia spaces. This is an extension of Esakia duality between the non-modal versions of these categories. We establish an extension of the correspondence between congruences, filters, and closed upsets that is present in Esakia duality. We also characterise the injective and surjective temporal Heyting algebra homomorphisms in the dual category.

In Chapter 6, we develop a theory of relational models of the temporal Heyting calculus. We use the duality established in Chapter 5 to prove relational soundness and completeness with respect to the class of temporal transits. We then develop a method of filtration on our models and use it to prove the finite model property for relational models.

In Chapter 7, we apply the theory developed in Chapter 3 through Chapter 6 to prove several theorems about the variety of temporal Heyting algebras. This includes using the relational finite model property to establish the finite model property for algebraic models and subsequently proving a stronger algebraic soundness and completeness result. We then prove the main results of the thesis : a lattice-theoretic and order-topological characterisation of both simple and subdirectly-irreducible temporal Heyting algebras (in both the general and finite case), as well as a relational completeness result combining finiteness and a type of « rootedness » defined in terms of the above-mentioned notion of « reachability ». We distinguish the general and finite cases for the above-mentioned characterisations because they are given in different terms (filter-wise vs. element-wise and order-topological vs. frame-theoretic). Our final completeness result is a culmination of all of the theory developed in this thesis and provides us with a simple class of frames, limited in both size and shape, for the temporal Heyting calculus.

In Chapter 8, we briefly summarise the path taken to our main results and subsequently outline several ideas for future work on temporal Heyting calculus and the related algebraic, topological, and frame-theoretic classes.

In Appendix A, we briefly describe a symbolic model checker that was written for temporal Heyting algebras. We give a brief tour of the package and provide examples of how it can be used.

Chapter 2

Preliminaries

In this chapter, we present all the preliminary definitions, concepts, and results necessary for the reader to understand the theory developed in Chapter 3 through Chapter 6 as well as the main results when they are presented in Chapter 7. Great effort has been put forth to make this text as self-contained as possible, but it is assumed that the reader is familiar with basic set theory and mathematical notions. Furthermore, readers will definitely benefit from having studied some basic modal logic, order theory, lattice theory, universal algebra, and topology. We include general references for each of these fields in their respective sections. The use of theory proper to the field of category theory has been intentionally kept to a minimum, but some category-theoretic notions had to be included to fully appreciate the main duality results, which are inherently categorical. As a general reference for category theory, we recommend [41].

Before beginning, we establish some basic set-theoretic and categorical definitions.

Notation 2.0.1 : Given a set X and a subsets $S, T \subseteq X$, we let $S - T$ denote set-theoretic difference, i.e. $\{x \in S \mid x \notin T\}$, and we let $-S$ denote set-theoretic complement , i.e. $X - S$.

Given a set X, we let the following denote the *diagonal relation on* X and the *all relation on* X respectively.

$$\Delta_X := \{\langle x, y\rangle \in X \times X \mid x = y\} \qquad \nabla_X := X \times X$$

When the set X is clear from context, we simply write Δ and ∇.

Given a category $\mathbf{C}$ and a $\mathbf{C}$-object $\mathbb{X}$, we denote the identity morphism on $\mathbb{X}$ by $\mathsf{id}_{\mathbb{X}}$. When working with sets, this is the function $\mathbb{X} \to \mathbb{X}$ defined by the rule $x \mapsto x$.

Definition 2.0.1 (Isomorphism) **:** Given a category $\mathbf{C}$ and a $\mathbf{C}$-morphism $f : \mathbb{X} \to \mathbb{Y}$, we call f a $\mathbf{C}$-*isomorphism* if there exists some other $\mathbf{C}$-morphism $f^{-1} : \mathbb{Y} \to \mathbb{X}$ such that $f^{-1} \circ f = \mathsf{id}_{\mathbb{X}}$ and $f \circ f^{-1} = \mathsf{id}_{\mathbb{Y}}$. When working with sets, all **C**-isomorphisms are bijections.

2.1 Intuitionistic modal logic

This section introduces several intuitionistic modal logics. As a general reference for intuitionistic logic, we recommend [6] and [18]. As a general reference for classical modal, we recommend [10], [18], and [9].

We start by defining some basic logical notation.

Notation 2.1.1 : Let Prop denote a set of propositional variables, usually denoted by the letters $p, q, \ldots$.

Given formulas φ, χ and a variable $p \in \mathsf{Prop}$, let $\varphi[\chi/p]$ denote the formula that results from uniformly substituting all instances of p in φ for χ.

Here we inductively define the languages to be used in the current text, superscripting them with i to indicate « intuitionistic » and subscripting them with m or t, to indicate « modal » or « temporal » respectively.

Definition 2.1.1 :

$$\begin{aligned}
\mathcal{L}^{\mathsf{i}} &:= p \in \mathsf{Prop} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \bot \mid \top \\
\mathcal{L}^{\mathsf{i}}_{\mathsf{m}} &:= \varphi \in \mathcal{L}^{\mathsf{i}} \mid \Box\varphi \\
\mathcal{L}^{\mathsf{i}}_{\mathsf{t}} &:= \varphi \in \mathcal{L}^{\mathsf{i}}_{\mathsf{m}} \mid \blacklozenge\varphi
\end{aligned}$$

Here we define some *rules of inference* for our logics.

Definition 2.1.2 : Given formulas $\varphi, \chi \in \mathcal{L}^{\mathsf{i}}$ and $p \in \mathsf{Prop}$, we define the following rules of inference.

$$(\mathsf{MP})\ \frac{\varphi \quad \varphi \to \chi}{\chi}$$

$$(\mathsf{US})\ \frac{\varphi}{\varphi[\chi/p]}$$

$$(\mathsf{PD})\ \frac{\varphi \to \chi}{\blacklozenge\varphi \to \blacklozenge\chi}$$

Given a logic **L** and one of the above-defined rules (R), say that **L** *is closed under* (R) if whenever the upper formulas of (R) are present in **L**, the lower formula of (R) is also present in **L**.

Now we define the Heyting calculus via the Hilbert-style axiomatisation given in [6, §3].

Definition 2.1.3 (Heyting calculus - **HC**) **:** The *Heyting calculus*[1], denoted by **HC**, is the smallest subset of $\mathcal{L}^i$ that contains the following axioms and is closed under (MP) and (US).

(HC.1) $p \to (q \to p)$

(HC.2) $p \to (q \to r) \to ((p \to q) \to (p \to r))$

(HC.3) $p \wedge q \to p$

(HC.4) $p \wedge q \to q$

(HC.5) $p \to (q \to (p \wedge q))$

(HC.6) $p \to p \vee q$

(HC.7) $q \to p \vee q$

(HC.8) $(p \to r) \to ((q \to r) \to ((p \to q) \to r))$

(HC.9) $\bot \to p$

Any logic that contains **HC** is called *superintuitionistic*.

The Heyting calculus was introduced by Arend Heyting to formalise what he and his doctoral advisor L. E. J. Brouwer considered to be epistemologically-sound reasoning in accordance with their philosophy of *Intuitionism*. For more information on the motivations and history of the Heyting calculus and Intuitionism, see [6, §2] and [36].

We now introduce the concept of an *intuitionistic modal logic*. Typically, when one begins studying logic, one sees *either* modal *or* (non-classical) logics —indeed, these concepts are treated as disjoint in well-known texts such as [18]—. In this sense, the modalised Heyting calculus is somewhat novel in that it is *both* modal *and* non-classical. When working in a classical context, there is a very standard way of doing basic modal logic. This largely comes down to the fact that the negation present in classical logic links the modalities $\Box$ and $\Diamond$ in a very obvious and intuitive way. However, when one does not have access to classical negation, (as we do not, working within **HC**,) it is a non-trivial question as to how one should do modal logic, that is, how it should be axiomatised, what its semantics should look like, and which of its properties should be conservative when the law of excluded middle is added, placing us in a classical setting again.

[1] In other literature, this logic is sometimes known as the *intuitionistic propositional calculus* and denoted by **IPC** [6].

Several intuitionistic modal logics (including first-order varieties) have been proposed and studied through the years. The modalised Heyting calculus represents one of these efforts, with several motivations and justifications which can be referenced in [24, §2].

Definition 2.1.4 (Modalised Heyting calculus - **mHC**) **:** The *modalised Heyting calculus*, denoted by **mHC**, is the smallest subset of $\mathcal{L}^{\mathsf{i}}_{\mathsf{m}}$ that contains **HC**, the following axioms, and is closed under (MP) and (US).

(mHC.1) $\Box(p \to q) \to (\Box p \to \Box q)$

(mHC.2) $p \to \Box p$

(mHC.3) $\Box p \to (q \vee q \to p)$

The conference slides [30] are an excellent resource on this logic, providing a great deal of intuition.

One should note that **mHC** differs in expressivity from classical modal logics as it has $\Box$, but not $\Diamond$. While there have been attempts to add *both* modalities to the Heyting calculus [26, 21, 19, 12], this presents a very non-trivial task as there is no classical negation to link the modalities. For this reason, logicians find themselves imposing extra axioms to regain as much of classical $\Box/\Diamond$ duality as possible, often making trade-offs along the way.

One of the central goals of modal logic is to study what relational structure can be defined solely through the validity of formulas (in the context of some semantics). When we augment our language, and, subsequently, our semantics, we achieve greater expressive power to impose this structure. This can be done in a variety of creative ways, such as including « global » modalities [10, Example 2.4], « next » modalities [33], etc., but a very common and natural method of augmenting expressivity is through the addition of *temporal* modalities. In the same way that relational semantics for the traditional $\Box$ and $\Diamond$ are defined in terms of reachability through a *forward*-looking relation, relational semantics for temporal modalities are defined in terms of reachability through a *backward*-looking relation which is simply the inverse of the forward-looking one. In the non-temporal setting, while we have enough expressivity to say « p is true in all of the worlds that I can see » with the formula $\Box p$, we do *not* have enough expressivity to say « A world where p is true can see *me* ». If we take our forward-looking relation to represent the flow of time, what we cannot express is that « At one point in the past, p was true ». This temporal reading of modal logic has been covered thoroughly in [48].

We now define the temporal Heyting calculus, which is the subject of the current text. This logic seeks to add the above-described temporal expressivity to a non-classical modal setting. Now in a classical modal setting, when we make a logic temporal, we leverage our two forward-looking modalities $\Box$ and $\Diamond$ to axiomatise our backward-looking modalities $\blacksquare$ and $\blacklozenge$, however, as mentioned above, in the setting of **mHC**, we have no $\Diamond$ in our language and no classical negation to define it. For this reason, we are only able to leverage our modality $\Box$ to axiomatise the behaviour of a backward-looking modality $\blacklozenge$, leading us to the following definition for our temporal Heyting calculus.

Definition 2.1.5 (Temporal Heyting calculus - $\mathbf{tHC}$) **:** The *temporal Heyting calculus*, denoted by $\mathbf{tHC}$, is the smallest subset of $\mathcal{L}_t^i$ that contains $\mathbf{mHC}$, the following axioms, and is closed under (MP), (US), and (PD).

(tHC.1) $\blacklozenge(p \vee q) \to (\blacklozenge p \vee \blacklozenge q)$

(tHC.2) $\blacklozenge\bot \to \bot$

(tHC.3) $p \to \Box\blacklozenge p$

(tHC.4) $\blacklozenge\Box p \to p$

The logic $\mathbf{tHC}$ was first-defined in [24], but has never been presented in-depth in any published source. Some investigation was carried out by Jibladze [30] and Alshibaia [1] and presented at conferences (TACL 2011 and ToLo IV respectively), but no proofs were ever presented and only the slides and abstracts remain. In particular, [1] posits several facts that will be proven to be true in the current text as well as one which needed to be slightly modified (the dual characterisation of subdirectly-irreducible temporal Heyting algebras).

Having defined all necessary logics, we now introduce some notation useful for establishing connections between logics and classes of structures once we have defined algebraic and relational semantics for our logics.

Notation 2.1.2 : Given a logic $\mathbf{L}$ and a class of structures $\mathbf{K}$ with some semantics $\models$,

$$\begin{aligned} \mathbf{K} \models \mathbf{L} &\quad :\Longleftrightarrow \quad \varphi \in \mathbf{L} \text{ implies } (\forall \mathbb{X} \in \mathbf{K})(\mathbb{X} \models \varphi) \\ \mathbf{L} \vdash \mathbf{K} &\quad :\Longleftrightarrow \quad (\forall \mathbb{X} \in \mathbf{K})(\mathbb{X} \models \varphi) \text{ implies } \varphi \in \mathbf{L} \\ \mathbf{L} \dashv\vdash \mathbf{K} &\quad :\Longleftrightarrow \quad \mathbf{L} \vdash \mathbf{K} \text{ and } \mathbf{K} \models \mathbf{L} \end{aligned}$$

The first notational convention captures the notion of « soundness », the second captures the notion of « completeness », and the third captures the combination of the first two.

Given a class of structures $\mathbf{K}$, we denote the class of *finite* structures in $\mathbf{K}$ by $\mathbf{K}_{\mathsf{fin}}$.

We include a note on the rule of inference (US).

Remark 2.1.1 : It is well-known, often-cited [16, p. 197], and seldom-proven that in this context, any formula that can be derived using (US) can be derived by performing (US) *only* on the axioms of the logic. (The proof for this is simple, but tedious, involving an induction on the maximal number of non-(US) rules that are applied before (US) appears in a deduction.) For this reason, when completing soundness proofs, we need only check that (US) preserves validity on the axioms, which will always be obvious from the proofs of the validity of the axioms themselves. In this sense, the rule of inference (US) is not entirely necessary in our context : we could have removed (US) entirely and opted for the corresponding *schemata* to each axiom (i.e. $\varphi \to \Box\varphi$ instead of $p \to \Box p$) and all proofs would proceed identically. However, we decided to include (US) for the sake of conformity with the definitions of $\mathbf{HC}$ and $\mathbf{mHC}$ given in [6] and [24] respectively and simply include the current remark.

Finally, we define the finite model property.

Definition 2.1.6 (Finite model property (FMP)) **:** Given a logic $\mathbf{L}$ and a class of structures $\mathbf{K}$ such that $\mathbf{L} \dashv\vdash \mathbf{K}$, we say that $\mathbf{L}$ has the *finite model property* (FMP) if $\mathbf{L} \dashv\vdash \mathbf{K}_{\mathsf{fin}}$.

Establishing the FMP is of crucial importance when studying a logic as its presence implies that to decide whether a formula is present in a logic, we only need check its validity on *finite* models. One can imagine that if we are working with an actual model-checking algorithm (as will be presented in Appendix A), this means the difference between a routine that is guaranteed to terminate and one that could potentially run forever. For this reason, the FMP is tightly linked to questions of decidability [10, §6.2]. Furthermore, even outside the realm of computation, it is often much simpler to work on finite structures. To give an example, when we begin working with relational semantics on topological spaces in §2.6, we will see (Fact 2.6.2) that in the finite case, the topological aspects of our structures trivialise, allowing us to concern ourselves with only their relational structure.

2.2 Frames

Here we turn to the field of *order theory*, which studies sets with binary relations. Several well-known properties of these relations, such as transitivity, reflexivity, antisymmetry, etc., can be combined in different ways to define different classes of structures called *frames*. This section defines several such classes which turn out to be closely linked to the logics defined in §2.1. As a general reference for order theory, we recommend [20].

We begin by establishing some notation when working with binary relations.

Notation 2.2.1 : Given a set X, a binary relation $R \subseteq X^2$, a subset $S \subseteq X$, and a point $x \in X$,

$$\begin{aligned} R[S] &:= \{y \in X \mid (\exists x \in S)(x\, R\, y)\} \\ R[x] &:= R[\{x\}] \\ R^{-1}[S] &:= \{w \in X \mid (\exists x \in S)(w\, R\, x)\}. \end{aligned}$$

Given more points $x_1, x_2, x_3 \in X$, the notation $x_1\, R\, x_2\, R\, x_3$ is taken to mean $x_1\, R\, x_2$

and $x_2 \, R \, x_3$. The same goes for longer chains.

Given another binary relation $R' \subseteq X^2$ and another point $z \in X$, we define the *composition of R and R'*, denoted by $R; R'$, as follows.

$$x \, (R; R') \, z \quad :\Longleftrightarrow \quad (\exists y \in X)(x \, R \, y \, R' \, z)$$

Definition 2.2.1 (Frame) **:** A *frame* is tuple $\langle X, R_1, \dots, R_k \rangle$ where X is a set and $R_i \subseteq X^2$. Given a tuple $\mathbb{X} := \langle X, R_1, \dots, R_k, \Xi \rangle$ such that Ξ is *not* a binary relation, we call $\langle X, R_1, \dots, R_k \rangle$ the *frame of* $\mathbb{X}$. This is the frame that results from « forgetting » the additional structure on $\mathbb{X}$. (In the case of relational models (§2.3), Ξ will be a relational valuation ; in the case of ordered topological spaces (§2.6), Ξ will be a topology.)

Frames will be denoted by the symbols $\mathbb{X}$ and $\mathbb{Y}$.

We now define several frame-theoretic properties and, subsequently, several classes of frames.

Definition 2.2.2 : Given a a set X and a binary relation $R \subseteq X^2$, we define the following properties of R.

(Transitivity) $(\forall x, y, z \in X)(x \, R \, y \, R \, z \Rightarrow x \, R \, z)$

(Reflexivity) $(\forall x \in X)(x \, R \, x)$

(Antisymmetry) $(\forall x, y \in X)(x \, R \, y \, R \, x \Rightarrow x = y)$

(Symmetry) $(\forall x, y \in X)(x \, R \, y \Rightarrow y \, R \, x)$

Relations satisfying transitivity are called *transitive*. Relations satisfying reflexivity are called *reflexive* while relations failing to satisfy reflexivity are called *irreflexive*. Relations satisfying antisymmetry are called *antisymmetric*. Relations satisfying symmetry are called *symmetric*.

Definition 2.2.3 (Quasi-ordered set) **:** A *quasi-ordered set* (or *qoset*) is a frame $\langle X, R \rangle$ satisfying the following conditions.

(QOS.1) R is transitive

(QOS.2) R is reflexive

The class of quasi-ordered sets is denoted by **QOS**.

Definition 2.2.4 (Transit) **:** A *transit* is a frame $\langle X, R \rangle$ satisfying the following conditions.

(Tran.1) R is transitive

(Tran.2) R is antisymmetric

The class of transits is denoted by $\mathbf{Tran}$.

The terminology « transit » was introduced by [24], though a slightly different, but equivalent, definition was given.

Definition 2.2.5 (Temporal transit) **:** A *temporal transit* is a frame $\langle X, R^{\triangleleft}, R^{\triangleright} \rangle$ satisfying the following conditions for all $x, y \in X$.

(tTran.1) $\langle X, R^{\triangleright} \rangle$ is a transit

(tTran.2) $x \; R^{\triangleright} \; y$ if and only if $y \; R^{\triangleleft} \; x$

The class of temporal transits is denoted by $\mathbf{tTran}$.

Beginning in Definition 2.2.6, we sometimes define a category as opposed to simply a class of structures. This is only done when it is necessary, for the results of the current text, to have a more definitive understanding of the morphisms associated with a particular class of structures.

Definition 2.2.6 ($\mathbf{POS}$ - Partially-ordered set) **:**

We define the category $\mathbf{POS}$.

A $\mathbf{POS}$-object, called a *partially-ordered set* (or *poset*), is a frame $\langle X, \leqslant \rangle$ satisfying the following conditions.

(POS.o.1) $\leqslant$ is transitive

(POS.o.2) $\leqslant$ is reflexive

(POS.o.3) $\leqslant$ is antisymmetric

A $\mathbf{POS}$-morphism, called a *monotone map*, is a function $f : \mathbb{X} \to \mathbb{Y}$ satisfying the following conditions for all $x_1, x_2 \in X$.

(POS.m.1) $x_1 \leqslant x_2$ implies $fx_1 \leqslant fx_2$

$$\begin{array}{ccc} x_2 & \xrightarrow{\quad f \quad} & fx_2 \\ \uparrow \leqslant & & \uparrow \leqslant \\ x_1 & \xrightarrow{\quad f \quad} & fx_1 \end{array}$$

We now state several definitions and facts related to posets.

Notation 2.2.2 : Given $\langle X, \leqslant \rangle \in \mathbf{POS}$ and $S \subseteq X$,

$$\uparrow S := \{y \in X \mid (\exists x \in S)(x \leqslant y)\} \qquad \downarrow S := \{w \in X \mid (\exists x \in S)(w \leqslant x)\}.$$

(These are equivalent to $\leqslant[S]$ and $\geqslant[S]$ respectively, but the above notation is standard for posets.)

Given $x, y \in X$,

$$[x, y] := \uparrow x \cap \downarrow y \qquad [x, y) := [x, y] - \{y\} \qquad (x, y] := [x, y] - \{x\}.$$

Definition 2.2.7 (Upset/Downset) **:** Given $\mathbb{X} := \langle X, \leqslant \rangle \in \mathbf{POS}$ and $S \subseteq X$, we call S an *upset* if for all $x, y \in X$,

$$x \in S \text{ and } x \leqslant y \text{ implies } y \in S.$$

We call S a *downset* if for all $w, x \in X$,

$$x \in S \text{ and } w \leqslant x \text{ implies } w \in S.$$

We denote the sets of upsets and downsets of $\mathbb{X}$ by $\mathsf{Up}(\mathbb{X})$ and $\mathsf{Down}(\mathbb{X})$ respectively.

Fact 2.2.1 : Given $S \subseteq \mathbb{X} \in \mathbf{POS}$

$$S \in \mathsf{Up}(\mathbb{X}) \iff -S \in \mathsf{Down}(\mathbb{X}).$$

Finally, we define notation and facts relating transits and posets.

Definition 2.2.8 (Reflexivisation/Irreflexivisation) **:** Given a set X and a binary relation $R \subseteq X^2$, the *reflexivisation of* R, denoted by $\dot{R}$, is defined as follows for all $x, y \in X$.

$$x \, \dot{R} \, y \quad :\iff \quad x \, R \, y \text{ or } x = y$$

The *irreflexivisation of* R, denoted by $\mathring{R}$, is defined as follows for all $x, y \in X$.

$$x \mathring{R} y \quad :\Longleftrightarrow \quad x R y \text{ and } x \neq y$$

Fact 2.2.2 : Given $\langle X, R\rangle \in \mathbf{Tran}$, we have $\langle X, \dot{R}\rangle \in \mathbf{POS}$.

Notation 2.2.3 : Given $\mathbb{X} := \langle X, R\rangle \in \mathbf{Tran}$,

$$\mathsf{Refl}(\mathbb{X}) := \{x \in X \mid x R x\}.$$

Due to Fact 2.2.2, there is always an implicit posetal structure on any transit. For this reason, we make use of the notation $\leqslant := \dot{R}$ and $< := \mathring{R}$. This means that we always have $< \subseteq R \subseteq \leqslant$.

Fact 2.2.3 : Given $\langle X, R\rangle \in \mathbf{Tran}$ and $x, y \in X$, we have $R = \leqslant; R; \leqslant$, i.e.

$$x R y \quad \Longleftrightarrow \quad (\exists x', y')(x \leqslant x' R y' \leqslant y).$$

Proof : ($\Rightarrow$) Given $x R y$, we have $x \leqslant x R y \leqslant y$. ($\Leftarrow$) Given $x \leqslant x' R y' \leqslant y$, equiv. $x \dot{R} x' R y' \dot{R} y$, we distinguish four cases : (1) $x = x'$ and $y' = y$, (2) $x R x'$ and $y' = y$, (3) $x = x'$ and $y' R y$, and (4) $x R x'$ and $y' R y$. (1) Here we have $x = x' R y' = y$, so we have $x R y$. (2) Here we have $x R x' R y' = y$, so, by (Tran.1), we have $x R y' = y$, so we have $x R y$. (3) Here we have $x = x' R y' R y$, so, by (Tran.1), we have $x = x' R y$, so we have $x R y$. (4) Here we have $x R x' R y' R y$, so, by (Tran.1), we have $x R y$. ⊠

2.3 Relational models

Here we define the relational models of our logics. These are frames paired with maps called *relational valuations* that assign atomic facts (i.e. propositional variables), to the elements of the structure where they are taken to be « true ». In the case of classical logics, these valuations are completely unrestricted, but in the non-classical case, they must satisfy the so-called *heredity* or *persistence* [6, §3.3] condition. We follow [19, §2.2] and say, equivalently, that they must map into the upsets of the structure.

Definition 2.3.1 ((Intuitionistic) Relational valuation) **:** Given a frame $\langle X, R_1, \ldots, R_k\rangle$, a *relational valuation on* X is a function $\nu : \mathsf{Prop} \to \wp X$.

Given $\mathbb{X} \in \mathbf{POS}$ and a relational valuation ν on $\mathbb{X}$, we call ν *intuitionistic* if $\nu : \mathsf{Prop} \to \mathsf{Up}(\mathbb{X})$.

Definition 2.3.2 (Relational model) **:** A *relational model* (or *Kripke model*) is a tuple $\langle X, R_1, \ldots, R_k, \nu\rangle$ such that $\langle X, R_1, \ldots, R_k\rangle$ is a frame and ν is a relational valuation on X.

Relational models will be denoted by the symbol $\mathbb{M}$.

We now define the relational models corresponding to the logic **HC**.

Definition 2.3.3 (Intuitionistic Kripke model) **:** An *intuitionistic Kripke model* is a relational model $\langle X, \leqslant, \nu\rangle$ satisfying the following conditions.

(IKM.1) $\langle X, \leqslant\rangle$ is a poset

(IKM.2) ν is an intuitionistic relational valuation on X

The class of intuitionistic Kripke models is denoted by **IKM**.

Next, we define the relational models corresponding to the logic **mHC**.

Definition 2.3.4 (Frontal intuitionistic Kripke model) **:** An *frontal intuitionistic Kripke model* is a relational model $\langle X, R, \nu\rangle$ satisfying the following conditions.

(fIKM.1) $\langle X, R\rangle$ is a transit

(fIKM.2) $\langle X, \dot{R}, \nu\rangle$ is an intuitionistic Kripke model

The class of frontal intuitionistic Kripke models is denoted by **fIKM**.

Finally, we define the relational models that will be shown to correspond to the logic **tHC**.

Definition 2.3.5 (Temporal intuitionistic Kripke model) **:** An *temporal intuitionistic Kripke model* is a relational model $\langle X, R^{\triangleleft}, R^{\triangleright}, \nu\rangle$ satisfying the following conditions.

(tIKM.1) $\langle X, R^{\triangleleft}, R^{\triangleright}\rangle$ is temporal transit.

(tIKM.2) $\langle X, R^{\triangleright}, \nu\rangle$ is a frontal intuitionistic Kripke model

The class of temporal intuitionistic Kripke models is denoted by **tIKM**.

Given these relational models, we now define relational semantics on **tIKM**.

Definition 2.3.6 (Relational semantics) **:** We let $\mathbb{M} := \langle X, R^{\triangleleft}, R^{\triangleright}, \nu\rangle$ be a temporal intuitionistic Kripke model. We also let $x \in X$ and $p \in \mathsf{Prop}$ and $\varphi, \chi \in \mathcal{L}_{\mathsf{t}}^{\mathsf{i}}$.

$$
\begin{array}{lll}
\langle \mathbb{M}, x \rangle \models p & :\iff & x \in \nu p \\
\langle \mathbb{M}, x \rangle \models \top & :\iff & \text{True} \\
\langle \mathbb{M}, x \rangle \models \bot & :\iff & \text{False} \\
\langle \mathbb{M}, x \rangle \models \varphi \wedge \chi & :\iff & \langle \mathbb{M}, x \rangle \models \varphi \text{ and } \langle \mathbb{M}, x \rangle \models \chi \\
\langle \mathbb{M}, x \rangle \models \varphi \vee \chi & :\iff & \langle \mathbb{M}, x \rangle \models \varphi \text{ or } \langle \mathbb{M}, x \rangle \models \chi \\
\langle \mathbb{M}, x \rangle \models \varphi \to \chi & :\iff & (\forall y \in X)(x \leqslant y \text{ and } \langle \mathbb{M}, y \rangle \models \varphi \text{ implies } \langle \mathbb{M}, y \rangle \models \chi) \\
\langle \mathbb{M}, x \rangle \models \blacklozenge \varphi & :\iff & (\exists w \in X)(x \, R^{\triangleleft} \, w \text{ and } \langle \mathbb{M}, w \rangle \models \varphi) \\
\langle \mathbb{M}, x \rangle \models \Box \varphi & :\iff & (\forall y \in X)(x \, R^{\triangleright} \, y \text{ implies } \langle \mathbb{M}, y \rangle \models \varphi)
\end{array}
$$

(Recall that in the case of $\mathbf{tTran}$, we have $\leqslant \; := \dot{R}^{\triangleright}$.) In all these cases, when the model $\mathbb{M}$ is clear from context, we simply write $x \models \varphi$ instead of $\langle \mathbb{M}, x \rangle \models \varphi$.

Note that, given these semantics, we can augment a relational valuation $\nu : \mathsf{Prop} \to \wp X$ to a relational valuation $\nu : \mathcal{L}_{\mathsf{t}}^{\mathsf{i}} \to \wp X$ by defining

$$\nu\varphi := \{x \in X \mid \langle \mathbb{M}, x \rangle \models \varphi\}.$$

If ν intuitionistic, it can be easily checked that $\nu : \mathcal{L}_{\mathsf{t}}^{\mathsf{i}} \to \mathsf{Up}(\mathbb{M})$ (i.e. that $\nu\varphi \in \mathsf{Up}(\mathbb{M})$ for all $\varphi \in \mathcal{L}_{\mathsf{t}}^{\mathsf{i}}$).

Next, we extend our semantics to the model level for $\mathbb{M} \in \mathbf{tIKM}$.

$$\mathbb{M} \models \varphi \quad :\iff \quad (\forall x \in \mathbb{M})(\langle \mathbb{M}, x \rangle \models \varphi)$$

Finally, we extend our semantics to the frame level for $\mathbb{X} := \langle X, R^{\triangleleft}, R^{\triangleright} \rangle \in \mathbf{tTran}$.

$$\mathbb{X} \models \varphi \quad :\iff \quad \text{For all intuitionistic relational valuations } \nu, \text{ we have } \langle X, R^{\triangleleft}, R^{\triangleright}, \nu \rangle \models \varphi.$$

Given this semantics, we state some relevant well-known relational soundness and completeness results.

Fact 2.3.1 : The following soundness and completeness results are well-known.

$$\mathbf{HC} \dashv\vdash \mathbf{POS} \qquad \mathbf{mHC} \dashv\vdash \mathbf{Tran}$$

The former of these results was established by Kripke in [31] while the latter was stated by Esakia in [24].

We conclude this section with the following remark addressing a potential point of confusion for readers familiar with basic results in classical modal logic.

Remark 2.3.1 : Readers may recall that the logic **K4** [10, p. 36] is sound and complete with respect to the class of transitive frames [10, Theorem 4.27]. Now our class **Tran** is a strict subclass of the class of transitive frames (as **Tran** stipulates antisymmetry (Tran.2)), but readers may be familiar with the idea that antisymmetry is essentially a *modally-insignificant* frame-theoretic property : classical modal formulas *cannot* distinguish those frames that are antisymmetric from those that are not. (See the first few paragraphs of [10, §4.5] for an enlightening discussion on this fact.) For this reason, the reader may wonder if **K4** is also complete with respect to **Tran**. Readers familiar with the well-known method of *unravelling* [10, Definition 4.51] may guess that it can be used on the canonical modal of **K4** to make the canonical model antisymmetric as was done for the logic **S4** (which contains **K4**) in [10, Theorem 4.54]. The answer to this question is *yes* : we *can* prove that **K4** is complete with respect to **Tran** in just this way. This, however, does *not* imply that **mHC** and **K4** are equivalent. The key insight here is that in the setting of **K4**, our semantics is different. At a point level, the semantics for $\to$ is Boolean in the classical case and relational in the intuitionistic case. At a frame level in the classical case, relational valuations are not required to be intuitionistic and are defined directly in terms of the relation on the frame. At a frame level in the intuitionistic case, relational valuations *are* required to be intuitionistic and are defined in terms of the *reflexivisation* of the relation on the frame ($\dot{R}^{\triangleright}$). So we are working with a distinct semantics both point-wise and frame-wise, allowing us to have soundness and completeness results for two different, non-equivalent logics with respect to the same class of frames. This ambiguity could be resolved by skipping frame-level semantics altogether and only stating soundness and completeness results with respect to classes of *models* (i.e. **mHC** $\dashv\vdash$ **fIKM**), which would make it very clear which valuations are being considered, but logicians have long stated soundness and completeness results with respect to classes of *frames* and this text will not be the one to make them change their ways.

2.4 Universal algebra

Here we introduce several basic concepts in universal algebra. Since all of the algebras relevant for the current text are *lattices* (§2.5), our treatment of universal algebra is sometimes less than completely general, often taking advantage of methods of simplifying concepts that are available when restricting our study to these structures. We mention this so the reader is not puzzled when referencing other definitions of concepts like algebraic semantics or subdirect-irreducibility. As a general reference for universal algebra, we recommend [13]. Indeed, most proofs involving universal algebra in the current section rely on facts proven in [13].

Definition 2.4.1 (Type) **:** A *type* is an element of $\mathbb{N}^n$ for some $n \in \mathbb{N}$ (eg. $\langle 2, 3, 1, 0, 2\rangle$).

Definition 2.4.2 (Algebra) **:** An *algebra* is a tuple $\mathbb{A} := \langle A, f_1, \ldots, f_k \rangle$ such that A is a set and $f_i : A^n \to A$ for some $n \in \mathbb{N}$. Given a type $\langle n_1, \ldots, n_k \rangle \in \mathbb{N}^k$, we say that $\mathbb{A}$ *is of type* $\langle n_1, \ldots, n_k \rangle$ if f_i is of arity n_i. (We typically arrange the f_i's so that the type of $\mathbb{A}$ is in descending order.) We call two algebras of the same type *similar*.

Algebras will be denoted by the symbols $\mathbb{A}, \mathbb{B}, \mathbb{C}$ and $\mathbb{D}$.

In some settings, algebras are also allowed to have infinitely-many operations of infinite arity, but we follow [13, Definition 1.3], only considering finitely many operations of finite arity.

Definition 2.4.3 (Homomorphism) **:** Given two similar algebras $\langle A, f_1, \ldots, f_k \rangle$ and $\langle B, g_1, \ldots, g_k \rangle$, we call a function $h : A \to B$ a *homomorphism* if for all $f_i : A^n \to A$ and all $a_1, \ldots, a_n \in A^n$, we have

$$hf_i(a_1, \ldots, a_n) = g_i(ha_1, \ldots, ha_n).$$

We now define several notions that will allow us to relate logic to algebras.

Notation 2.4.1 : Given a class of similar algebras $\mathbf{K}$, we denote the $\mathbf{K}$-algebra freely generated by Prop by $\mathsf{Term}(\mathbf{K})$. (See [13, Definition 10.4] for details.) This can be thought of as the completely syntactic $\mathbf{K}$-algebra with as little structure as possible, satisfying only the constraints of the class $\mathbf{K}$. Because this algebra has such minimal structure, it can be homomorphically mapped into any $\mathbf{K}$-algebra [27, Lemma 1.1].

Definition 2.4.4 (Equation) **:** Given a class of similar algebras $\mathbf{K}$, a $\mathbf{K}$-*equation* is a string of the form $\varphi \approx \chi$ where $\varphi, \chi \in \mathsf{Term}(\mathbf{K})$. When $\mathbf{K}$ is clear from context, we simply call $\varphi \approx \chi$ an *equation*.

Definition 2.4.5 (Equational algebraic semantics) **:** Given a class of similar algebras $\mathbf{K}$, an algebra $\mathbb{A} \in \mathbf{K}$, and a $\mathbf{K}$-equation $\varphi \approx \chi$, we define the following semantics.

$$\mathbb{A} \models \varphi \approx \chi \quad :\Longleftrightarrow \quad \text{For all } \mathbf{K}\text{-morphisms } \nu : \mathsf{Term}(\mathbf{K}) \to \mathbb{A}, \text{ we have } \nu\varphi = \nu\chi.$$

Such $\mathbf{K}$-morphisms ν are called *algebraic valuations on* $\mathbb{A}$.

It should be noted that « algebraic semantics » can refer to a much more general notion (see [27, §3.2]). Fortunately, the algebras that we work with in the the current text allow for the use of the equational definition given above.

Definition 2.4.6 (Congruence) **:** Given an algebra $\langle A, f_1, \ldots, f_k \rangle$ and an equivalence relation $\theta \subseteq A^2$, we call θ a *congruence* if for all $f_i : A^n \to A$ and all $a_1, \ldots, a_n, b_1, \ldots, b_n \in A^n$,

$$a_1 \,\theta\, b_1 \text{ and } \ldots \text{ and } a_n \,\theta\, b_n \implies f_i(a_1, \ldots, a_n) \,\theta\, f_i(b_1, \ldots, b_n).$$

Given that an algebra $\mathbb{A}$ can often be considered to be of several different types (by considering its reducts or obvious extensions), we denote the collection of $\mathbf{K}$-congruences on $\mathbb{A}$ by $\mathsf{Cong}^{\kappa}(\mathbb{A})$ (where $\mathbf{K}$ is assumed to be a class of similar algebras).

The relations $\Delta_{\mathbb{A}}$ and $\nabla_{\mathbb{A}}$ are congruences on all algebras $\mathbb{A}$ and are the smallest and largest congruences respectively.

We use the notion of a « congruence » to define several types of algebras.

Definition 2.4.7 (Subdirectly-irreducible algebra) **:** Given a class of similar algebras $\mathbf{K}$ and $\mathbb{A} \in \mathbf{K}$, we call $\mathbb{A}$ *subdirectly-irreducible* if $\mathsf{Cong}^{\kappa}(\mathbb{A})$ has a second-least element. Given a class of similar algebras $\mathbf{K}$, we denote the class of subdirectly-irreducible elements of $\mathbf{K}$ by $\mathbf{K}_{\mathsf{si}}$ and the class of *finite* subdirectly-irreducible elements of $\mathbf{K}$ by $\mathbf{K}_{\mathsf{fsi}}$.

There is a more categorical definition of subdirect-irreducibility which is given in terms of subdirect products and embeddings [13, Definition 8.3], but since the two definitions are well-known to be equivalent [13, Theorem 8.4], the above-definition was given as it is more relevant for the work to be done in the current text.

Definition 2.4.8 (Simple algebra) **:** Given a class of similar algebras $\mathbf{K}$ and $\mathbb{A} \in \mathbf{K}$, we call $\mathbb{A}$ *simple* if $\mathsf{Cong}^{\kappa}(\mathbb{A}) = \{\Delta, \nabla\}$.

Note that all simple algebras are subdirectly-irreducible as ∇ serves as the second-least congruence.

Definition 2.4.9 (Semisimple) **:** We call a class of similar algebras $\mathbf{K}$ *semisimple* if every subdirectly-irreducible algebra in $\mathbf{K}$ is simple. Given the remark immediately above, this is equivalent to the condition that the class of subdirectly-irreducible algebras and the class of simple algebras are identical.

All of the algebras to be defined in the current text will be defined using only equations. This need not be the case for classes of algebras in general, but when it *is* the case, we have a special kind of class called a *variety*.

Definition 2.4.10 (Variety) **:** A *variety* is a class of similar algebras $\mathbf{V}$ for which there exists a set of $\mathbf{V}$-equations Γ such that the following holds for all algebras $\mathbb{A}$.

$$\mathbb{A} \in \mathbf{V} \iff (\forall \varphi \approx \chi \in \Gamma)(\mathbb{A} \models \varphi \approx \chi)$$

Given a class of similar algebras $\mathbf{K}$, we denote that smallest variety containing $\mathbf{K}$ by $\mathsf{Var}(\mathbf{K})$.

(This definition is technically non-standard, but is well-known to be equivalent to the standard definition [13, Definition 9.3].)

Here we define some well-known class-operators and state some well-known facts about varieties.

Definition 2.4.11 (H, S, P) **:** Given a class of similar algebras $\mathbf{K}$, we define the following.

$$\begin{aligned} \mathsf{H}(\mathbf{K}) &:= \{\mathbb{A} \mid \mathbb{A} \text{ is the homomorphic image of some algebra in } \mathbf{K}\} \\ \mathsf{S}(\mathbf{K}) &:= \{\mathbb{A} \mid \mathbb{A} \text{ is a subalgebra of some algebra in } \mathbf{K}\} \\ \mathsf{P}(\mathbf{K}) &:= \{\mathbb{A} \mid \mathbb{A} \text{ is the product of some set of algebras in } \mathbf{K}\} \end{aligned}$$

Readers unfamiliar with any of these terms should reference [13, Ch. II], though, in reality, they will play a very small role for our purposes.

We now state two foundational theorems in the theory of varieties that relate them to the class-operators defined above as well as the notion of « subdirect-irreducibility ».

Fact 2.4.1 (Birkhoff's Theorem) **:** Given a variety $\mathbf{V}$, we have $\mathbf{V} = \mathsf{HSP}(\mathbf{V}_{\mathsf{si}})$.

Proof : See [13, Theorem 8.6]. ⊠

Fact 2.4.2 (Tarski's Theorem) **:** Given a class of similar algebras $\mathbf{K}$, we have $\mathsf{Var}(\mathbf{K}) = \mathsf{HSP}(\mathbf{K})$.

Proof : See [13, Theorem 9.5]. ⊠

Finally, we relate the class-operators to equation-satisfaction.

Fact 2.4.3 : Given a class of similar algebras $\mathbf{K}$ and a $\mathbf{K}$-equation $\varphi \approx \chi$,

$$\begin{aligned} \mathbf{K} \models \varphi \approx \chi \quad &\Longleftrightarrow \quad \mathsf{H}(\mathbf{K}) \models \varphi \approx \chi \\ \mathbf{K} \models \varphi \approx \chi \quad &\Longleftrightarrow \quad \mathsf{S}(\mathbf{K}) \models \varphi \approx \chi \\ \mathbf{K} \models \varphi \approx \chi \quad &\Longleftrightarrow \quad \mathsf{P}(\mathbf{K}) \models \varphi \approx \chi. \end{aligned}$$

Proof : See [13, Lemma 11.3]. ⊠

Fact 2.4.4 : Given a variety $\mathbf{V}$, an algebra $\mathbb{A} \in \mathbf{V}$, and a $\mathbf{V}$-equation $\varphi \approx \chi$,

$$\mathbf{V}_{\mathsf{si}} \models \varphi \approx \chi \quad \Longrightarrow \quad \mathbf{V} \models \varphi \approx \chi.$$

Proof : Arguing via the contrapositive, we assume $\mathbf{V} \not\models \varphi \approx \chi$, implying there exists some $\mathbb{A} \in \mathbf{V}$ such that $\mathbb{A} \not\models \varphi \approx \chi$. Since $\mathbb{A} \in \mathbf{V}$, Fact 2.4.1 implies that $\mathbb{A} \in \mathsf{HSP}(\mathbf{V}_{\mathsf{si}})$, implying that $\mathbb{A}$ is the homomorphic image of some algebra $\mathbb{B} \in \mathsf{SP}(\mathbf{V}_{\mathsf{si}})$, that $\mathbb{B}$ is a subalgebra of some algebra $\mathbb{C} \in \mathsf{P}(\mathbf{V}_{\mathsf{si}})$, and that $\mathbb{C}$ is the product of some set $\{\mathbb{D}_i\}_{i \in I} \subseteq \mathbf{V}_{\mathsf{si}}$. By Fact 2.4.3, $\mathbb{A} \not\models \varphi \approx \chi$ implies $\mathbb{B} \not\models \varphi \approx \chi$, implying $\mathbb{C} \not\models \varphi \approx \chi$, implying there exists some $\mathbb{D}_k$ such that $\mathbb{D}_k \not\models \varphi \approx \chi$. We let $\mathbb{A}' := \mathbb{D}_k$ and observe that since $\mathbb{A}' \in \mathbf{V}_{\mathsf{si}}$, we have $\mathbf{V}_{\mathsf{si}} \not\models \varphi \approx \chi$. ⊠

2.5 Lattices

We now shift to definitions and facts from lattice theory, which studies algebras with operations $\wedge$ and $\vee$ (called *meet* and *join* respectively) that mirror the logical connectives « and » and « or ». As a general reference for lattice theory, we recommend [20] and [13].

Definition 2.5.1 (**BDL** - Bounded distributive lattice) **:**

We define the category **BDL**.

A **BDL**-object, called a *bounded distributive lattice*[2], is an algebra $\langle A, \wedge, \vee, 0, 1\rangle$ of type $\langle 2, 2, 0, 0\rangle$ satisfying the following conditions for all $a, b, c \in A$.

(BDL.o.1) $a \wedge b = b \wedge a$

(BDL.o.2) $a \vee b = b \vee a$

(BDL.o.3) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

(BDL.o.4) $a \vee (b \vee c) = (a \vee b) \vee c$

(BDL.o.5) $a \wedge a = a$

(BDL.o.6) $a \vee a = a$

(BDL.o.7) $a \wedge (a \vee b) = a$

(BDL.o.8) $a \vee (a \wedge b) = a$

(BDL.o.9) $a \wedge 0 = 0$

(BDL.o.10) $a \vee 1 = 1$

[2] In other literature, **BDL**-objects are sometimes known as simply *distributive lattices*.

(BDL.o.11) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(BDL.o.12) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

A **BDL**-morphism is a homomorphism over the type of **BDL**-objects.

We also define some notation on bounded distributive lattices.

Notation 2.5.1 : Given $\mathbb{A} \in \mathbf{BDL}$ and a *finite* subset $\{a_i\}_{i=1}^n \subseteq \mathbb{A}$,

$$\bigwedge \{a_i\}_{i=1}^n := \bigwedge_{i=1}^n a_i := a_1 \wedge \ldots \wedge a_n \qquad \bigvee \{a_i\}_{i=1}^n := \bigvee_{i=1}^n a_i := a_1 \vee \ldots \vee a_n.$$

Bounded distributive lattices have been studied extensively and are the subject of what is known as *Priestley duality*, a generalisation of Stone's celebrated *Stone duality* [43]. They also form the basis of an algebraic semantics for many substructural logics such as $\mathbf{wK}_\sigma$ in [17]. As a general reference for bounded distributive lattices, we recommend [28].

We now state a particular case of our equational algebraic semantics which we will refer to as our *algebraic semantics*. Note that this is only available to us when working with a subvariety of **BDL**.

Definition 2.5.2 (Algebraic semantics) **:** Given a variety $\mathbf{V} \subseteq \mathbf{BDL}$, an algebra $\mathbb{A} \in \mathbf{V}$, and a formula $\varphi \in \mathsf{Term}(\mathbf{V})$, we define the following shorthand.

$$\mathbb{A} \models \varphi \quad :\Longleftrightarrow \quad \mathbb{A} \models \varphi \approx \top$$

Note that, since $\nu\top = 1$ for all algebraic morphisms on $\mathbb{A}$, this is equivalent to the following definition.

$$\mathbb{A} \models \varphi \quad :\Longleftrightarrow \quad \text{For all algebraic morphisms } \nu \text{ on } \mathbb{A}\text{, we have } \nu\varphi = 1.$$

Here we state several facts relating bounded distributive lattices to posets.

Fact 2.5.1 : Given $\mathbb{A} \in \mathbf{BDL}$, one can always define a partial order $\leqslant$ on $\mathbb{A}$. Given $a, b \in \mathbb{A}$,

$$a \leqslant b \quad :\Longleftrightarrow \quad a \wedge b = a \quad \Longleftrightarrow \quad a \vee b = b.$$

Proof : See [13, p. 8]. ⊠

The above fact is crucial to note because several of the algebraic structures to be defined in the current section are defined in terms of $\leqslant$ as opposed to $=$. Fact 2.5.1 ensures that such constraints are equivalent to equations, keeping us within the realm of educationally-defined classes, i.e. varieties.

Fact 2.5.2 : Given a $\mathbf{BDL}$-morphism $h : \mathbb{A} \to \mathbb{B}$ and $a_1, a_2 \in \mathbb{A}$,

$$a_1 \leqslant a_2 \quad \Longrightarrow \quad ha_1 \leqslant ha_2.$$

Given an *injective* $\mathbf{BDL}$-morphism $h : \mathbb{A} \to \mathbb{B}$ and $a_1, a_2 \in \mathbb{A}$,

$$a_1 \leqslant a_2 \quad \Longleftrightarrow \quad ha_1 \leqslant ha_2.$$

Now we define and state several facts about two important classes of subsets on bounded distributive lattices called *filters* and *ideals*.

Definition 2.5.3 ((Prime) Filter/Ideal) **:** Given $F \subseteq \mathbb{A} \in \mathbf{BDL}$, we call F a *filter* if the following condition holds for all $a, b \in \mathbb{A}$.

- $1 \in F$
- $a, b \in F$ implies $a \wedge b \in F$
- $F \in \mathsf{Up}(\langle A, \leqslant \rangle)$

Given $I \subseteq \mathbb{A} \in \mathbf{BDL}$, we call F an *ideal* if the following condition holds for all $a, b \in \mathbb{A}$.

- $0 \in I$
- $a, b \in I$ implies $a \vee b \in I$
- $I \in \mathsf{Down}(\langle A, \leqslant \rangle)$

The sets of filters and ideals on $\mathbb{A}$ are denoted by $\mathsf{Filt}(\mathbb{A})$ and $\mathsf{Ideal}(\mathbb{A})$ respectively.

Given $\{a_i\}_{i \in I} \subseteq \mathbb{A}$, the *filter generated by* $\{a_i\}_{i \in I}$ is defined as follows.

$$[\{a_i\}_{i \in I}) := \uparrow \left\{ \bigwedge_{j \in J} a_j \mid J \subseteq I \text{ and } J \text{ finite} \right\}$$

Similarly, the *ideal generated by* $\{a_i\}_{i \in I}$ is defined as follows.

$$(\{a_i\}_{i \in I}] := \downarrow \left\{ \bigvee_{j \in J} a_j \mid J \subseteq I \text{ and } J \text{ finite} \right\}$$

Given $F \in \mathsf{Filt}(\mathbb{A})$, we call F *prime* if $F \neq \mathbb{A}$ (equiv. $0 \notin F$) and the following condition is met for all $a, b \in \mathbb{A}$.

$$a \vee b \in F \quad \Longrightarrow \quad a \in F \text{ or } b \in F$$

Given $I \in \mathsf{Ideal}(\mathbb{A})$, we call I *prime* if $I \neq \mathbb{A}$ (equiv. $1 \notin I$) and the following condition is met for all $a, b \in \mathbb{A}$.

$$a \wedge b \in I \quad \Longrightarrow \quad a \in I \text{ or } b \in I$$

The sets of prime filters and prime ideals on $\mathbb{A}$ are denoted by $\mathsf{PrFilt}(\mathbb{A})$ and $\mathsf{PrIdeal}(\mathbb{A})$ respectively.

Prime filters will be denoted by the letters v, w, x, y, z.

Fact 2.5.3 : Given $x \subseteq \mathbb{A} \in \mathbf{BDL}$

$$x \in \mathsf{PrFilt}(\mathbb{A}) \quad \Longleftrightarrow \quad -x \in \mathsf{PrIdeal}(\mathbb{A}).$$

Fact 2.5.4 (Prime Filter Theorem (PFT)) **:** Given $\mathbb{A} \in \mathbf{BDL}$ and $F \in \mathsf{Filt}(\mathbb{A})$ and $I \in \mathsf{Ideal}(\mathbb{A})$ such that $F \cap I = \varnothing$, there exists some $x \in \mathsf{PrFilt}(\mathbb{A})$ such that $F \subseteq x$ and $x \cap I = \varnothing$.

Proof : See [28, Theorem 3.10]. ⊠

We now define and state several facts about the algebraic structures corresponding to the logic **HC**.

Definition 2.5.4 (**HA** - Heyting algebra) **:**

We define the category **HA**.

An **HA**-object, called a *Heyting algebra*[3], is an algebra $\langle A, \wedge, \vee, \rightarrow, 0, 1\rangle$ of type $\langle 2, 2, 2, 0, 0\rangle$ satisfying the following conditions for all $a, b, c \in A$.

(HA.o.1) $\langle A, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice

(HA.o.2) $a \rightarrow a = 1$

(HA.o.3) $(a \rightarrow b) \wedge b = b$

(HA.o.4) $a \wedge (a \rightarrow b) = a \wedge b$

(HA.o.5) $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$

(HA.o.6) $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$

An **HA**-morphism is a homomorphism over the type of **HA**-objects.

[3] In other literature, **HA**-objects are sometimes known as *pseudo-Boolean algebras*.

Notation 2.5.2 : Given $a, b \in \mathbb{A} \in \mathbf{HA}$,

$$a \leftrightarrow b := (a \to b) \wedge (b \to a).$$

Remark 2.5.1 : $\mathbf{HA}$-objects can be equivalently defined as algebras $\langle A, \wedge, \vee, \to, 0, 1\rangle$ such that $\langle A, \wedge, \vee, 0, 1\rangle \in \mathbf{BDL}$ and for all $a, b, c \in A$,

$$a \wedge b \leqslant c \iff a \leqslant b \to c.$$

In particular, this implies

$$a \leqslant b \iff 1 \wedge a \leqslant b \iff 1 \leqslant a \to b \iff 1 = a \to b.$$

On Heyting algebras, there is a well-known correspondence between the congruences of an algebra and its filters. This allows us to study congruences, which are of great importance from a universal algebra perspective, via the lattice structure on the algebra. This correspondence is often present on subclasses of Heyting algebras as well, sometimes tweaking the *type* of filter.

Definition 2.5.5 : Given $\mathbb{A} \in \mathbf{HA}$, we define the following maps.

$$\begin{aligned} \circ_{\bullet} : \mathsf{Cong}^{\mathbf{HA}}(\mathbb{A}) &\longrightarrow \wp\,\mathbb{A} \\ \theta &\longmapsto [1]_\theta \end{aligned} \qquad \begin{aligned} \circ^{\bullet} : \mathsf{Filt}(\mathbb{A}) &\longrightarrow \wp\,\mathbb{A} \\ F &\longmapsto \left\{\langle a, b\rangle \in \mathbb{A}^2 \mid a \leftrightarrow b \in F\right\} \end{aligned}$$

The above-defined maps turn out to $\mathbf{POS}$-isomorphically map between congruences and filters on a given algebra.

Fact 2.5.5 : Given $\mathbb{A} \in \mathbf{HA}$,

$$\langle \mathsf{Cong}^{\mathbf{HA}}(\mathbb{A}), \subseteq\rangle \cong^{\mathbf{POS}} \langle \mathsf{Filt}(\mathbb{A}), \subseteq\rangle.$$

Proof : It is well-known that that $\circ_{\bullet}$ and $\circ^{\bullet}$ are monotone inverses of each other [23, Proposition 2.4.9]. ⊠

We now define and state several facts about the algebraic structures corresponding to the logic $\mathbf{mHC}$.

Definition 2.5.6 ($\mathbf{fHA}$ - Frontal Heyting algebra) **:**

We define the category $\mathbf{fHA}$.

An $\mathbf{fHA}$-object, called a *frontal Heyting algebra*[4], is an algebra $\langle A, \wedge, \vee, \to, \square, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ satisfying the following conditions for all $a, b \in A$.

(fHA.o.1) $\langle A, \wedge, \vee, \rightarrow, 0, 1\rangle$ is a Heyting algebra

(fHA.o.2) $\Box(a \wedge b) = \Box a \wedge \Box b$

(fHA.o.3) $a \leqslant \Box a$

(fHA.o.4) $\Box a \leqslant b \vee (b \rightarrow a)$

An **fHA**-morphism[5] is a homomorphism over the type of **fHA**-objects.

Frontal Heyting algebras have been discussed in [24] and studied in greater depth in [15].

Fact 2.5.6 : Given $a, b \in \mathbb{A} \in$ **fHA**,

$$a \leqslant b \implies \Box a \leqslant \Box b.$$

Proof : Given $a \leqslant b$, we have $a \wedge b = a$, implying $\Box(a \wedge b) = \Box a$, implying $\Box a \wedge \Box b = \Box a$, implying $\Box a \leqslant \Box b$. ⊠

Fact 2.5.7 : Given $\mathbb{A} \in$ **fHA**,

$$\langle \mathsf{Cong}^{\mathbf{fHA}}(\mathbb{A}), \subseteq\rangle \cong^{\mathbf{POS}} \langle \mathsf{Filt}(\mathbb{A}), \subseteq\rangle.$$

Proof : Well-definedness of $\circ_{\bullet}$ and $\circ^{\bullet}$ and be easily checked. The former follows from Fact 2.5.5 and the latter follows easily, relying on the following inequalities.

$$\begin{array}{ccccccc} & & a \rightarrow b & \leqslant & \Box(a \rightarrow b) & \leqslant & \Box a \rightarrow \Box b \\ & \rotatebox{45}{$\leqslant$} & & & & & \\ a \leftrightarrow b & & & & & & \\ & \rotatebox{-45}{$\leqslant$} & & & & & \\ & & b \rightarrow a & \leqslant & \Box(b \rightarrow a) & \leqslant & \Box b \rightarrow \Box a \end{array}$$

⊠

We now define and state several facts about the algebraic structures that will be shown to correspond to the logic **tHC**.

Definition 2.5.7 (**tHA** - Temporal Heyting algebra) **:**

We define the category **tHA**.

[4] In other literature, **fHA**-objects are sometimes known as **mHC**-*algebras* [38].

[5] In other literature, **fHA**-morphisms are sometimes known as *frontal Heyting morphisms* [15].

A $\mathbf{tHA}$-object, called a *temporal Heyting algebra*, is an algebra $\langle A, \wedge, \vee, \rightarrow, \blacklozenge, \Box, 0, 1\rangle$ of type $\langle 2, 2, 1, 1, 0, 0\rangle$ satisfying the following conditions for all $a, b \in A$.

(tHA.o.1) $\langle A, \wedge, \vee, \rightarrow, \Box, 0, 1\rangle$ is a frontal Heyting algebra

(tHA.o.2) $\blacklozenge 0 = 0$

(tHA.o.3) $\blacklozenge(a \vee b) = \blacklozenge a \vee \blacklozenge b$

(tHA.o.4) $a \leqslant \Box\blacklozenge a$

(tHA.o.5) $\blacklozenge\Box a \leqslant a$

A $\mathbf{tHA}$-morphism is a homomorphism over the type of $\mathbf{tHA}$-objects.

Fact 2.5.8 : Given $a, b \in \mathbb{A} \in \mathbf{tHA}$,

$$a \leqslant b \quad \Longrightarrow \quad \blacklozenge a \leqslant \blacklozenge b.$$

Proof : Given $a \leqslant b$, we have $a \vee b = b$, implying $\blacklozenge(a \vee b) = \blacklozenge b$, implying $\blacklozenge a \vee \blacklozenge b = \blacklozenge b$, implying $\blacklozenge a \leqslant \blacklozenge b$. ⊠

Fact 2.5.9 : Given $a, b \in \mathbb{A} \in \mathbf{tHA}$,

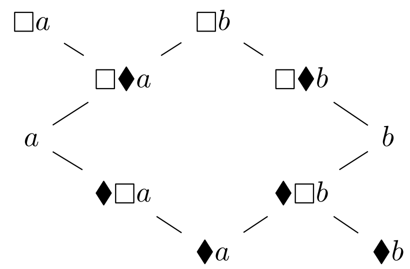
$$\blacklozenge a \leqslant b \quad \Longleftrightarrow \quad a \leqslant \Box b.$$

Proof : ($\Rightarrow$) $\blacklozenge a \leqslant b$ implies $\Box\blacklozenge a \leqslant \Box b$, implying, since $a \leqslant \Box\blacklozenge a$, that $a \leqslant \Box b$. ($\Leftarrow$) $a \leqslant \Box b$ implies $\blacklozenge a \leqslant \blacklozenge\Box b$, implying, since $\blacklozenge\Box b \leqslant b$, that $\blacklozenge a \leqslant b$. ⊠

Given Fact 2.5.9, we say that on $\mathbf{tHA}$, the operation $\blacklozenge$ is the *left adjoint to* $\Box$. This relationship is the main *algebraic* justification for studying $\mathbf{tHC}$ as the addition of a left-adjoint to $\Box$ on $\mathbf{fHA}$ corresponds to the temporal extension of $\mathbf{mHC}$. The structure gained when adding an adjoint to a modal operator as well as the naturality and tradition of this practice has been covered thoroughly in [35], in which the variety $\mathbf{tHA}$ makes an appearance as Example 3.3.

The following remark is included to provide readers with intuition with respect to adjoints to modal operations.

Remark 2.5.2 : Given $a, b \in \mathbb{A} \in \mathbf{tHA}$ such that $a \leqslant \Box b$ (equiv. $\blacklozenge a \leqslant b$), we have the following.



We now present a pair of facts about prime filters on temporal Heyting algebras.

Fact 2.5.10 : Given $\mathbb{A} \in \mathbf{tHA}$ and $x \in \mathsf{PrFilt}(\mathbb{A})$, we have $\Box^{-1}[x] \in \mathsf{Filt}(\mathbb{A})$.

Proof : First, note $\Box 1 = 1 \in x$, so $1 \in \Box^{-1}[x]$. Now, given $a, b \in \Box^{-1}[x]$, we have $\Box a, \Box b \in x$, implying $\Box(a \wedge b) = \Box a \wedge \Box b \in x$, finally implying $a \wedge b \in \Box^{-1}[x]$. Finally, given $a \in \Box^{-1}[x]$ and $a \leqslant b$, we have $\Box a \in x$ and, by Fact 2.5.6, $\Box a \leqslant \Box b$, implying $\Box b \in x$, finally implying $b \in \Box^{-1}[x]$. ⊠

Fact 2.5.11 : Given $\mathbb{A} \in \mathbf{tHA}$ and $x \in \mathsf{PrFilt}(\mathbb{A})$, we have $\blacklozenge^{-1}[-x] \in \mathsf{Ideal}(\mathbb{A})$.

Proof : First, note that, by Fact 2.5.3, $-x \in \mathsf{PrIdeal}(\mathbb{A})$, so $\blacklozenge 0 = 0 \in -x$, implying $0 \in \blacklozenge^{-1}[-x]$. Now, given $a, b \in \blacklozenge^{-1}[-x]$, we have $\blacklozenge a, \blacklozenge b \notin x$, implying $\blacklozenge(a \vee b) = \blacklozenge a \vee \blacklozenge b \notin x$, finally implying $a \vee b \notin \blacklozenge^{-1}[-x]$. Finally, given $b \in \blacklozenge^{-1}[-x]$ and $a \leqslant b$, we have $\blacklozenge b \in -x$ and, by Fact 2.5.8, $\blacklozenge a \leqslant \blacklozenge b$, implying $\blacklozenge a \in -x$, finally implying $x \in \blacklozenge^{-1}[-x]$. ⊠

To conclude, we state some relevant well-known algebraic soundness and completeness results.

Fact 2.5.12 : The following algebraic soundness and completeness results are well-known.

$$\mathbf{HC} \dashv\vdash \mathbf{HA} \qquad \mathbf{mHC} \dashv\vdash \mathbf{fHA}$$

For an in-depth look at the former result, the reader should reference [6, §4.3]. The latter was stated by Esakia in [24].

2.6 Ordered topological spaces

Since we are in a non-classical setting, all of the topological spaces considered in the current text will be *ordered*, that is, they will be equipped with at least one binary relation whose interaction with the topology is governed by some constraints. As a general reference for ordered topological spaces, we recommend [28, §2.3].

In this text, it is assumed that the reader is familiar with basic topological notions such as bases and subbases, common separation axioms, and compactness. As a general reference for topology, we recommend [22].

Definition 2.6.1 (Ordered topological space) **:** An *ordered topological space* is a tuple $\langle X, R_1, \ldots, R_k, \Omega\rangle$ such that $\langle X, R_1, \ldots, R_k\rangle$ is a frame and $\langle X, \Omega\rangle$ is a topological space.

Ordered topological space will be denoted by the symbols $\mathbb{X}$ and $\mathbb{Y}$.

Notation 2.6.1 : Given a topological space $\mathbb{X} := \langle X, \Omega\rangle$ and $S \subseteq X$,

$$\begin{aligned} \overline{\Omega} &:= \text{the set of closed sets in } \mathbb{X} &= \{C \subseteq X \mid -C \in \Omega\} \\ \mathsf{Clop}(\mathbb{X}) &:= \text{the set of clopen sets in } \mathbb{X} &= \Omega \cap \overline{\Omega}. \end{aligned}$$

Given an ordered topological space $\mathbb{X} := \langle X, \leqslant, \Omega\rangle$ such that $\langle X, \leqslant\rangle \in \mathbf{POS}$,

$$\begin{aligned} \mathsf{OpUp}(\mathbb{X}) &:= \Omega \cap \mathsf{Up}(\langle X, \leqslant\rangle) & \mathsf{ClUp}(\mathbb{X}) &:= \overline{\Omega} \cap \mathsf{Up}(\langle X, \leqslant\rangle) \\ \mathsf{OpDown}(\mathbb{X}) &:= \Omega \cap \mathsf{Down}(\langle X, \leqslant\rangle) & \mathsf{ClDown}(\mathbb{X}) &:= \overline{\Omega} \cap \mathsf{Down}(\langle X, \leqslant\rangle) \end{aligned}$$

$$\begin{aligned} \mathsf{ClopUp}(\mathbb{X}) &:= \mathsf{Clop}(\langle X, \Omega\rangle) \cap \mathsf{Up}(\langle X, \leqslant\rangle) \\ \mathsf{ClopDown}(\mathbb{X}) &:= \mathsf{Clop}(\langle X, \Omega\rangle) \cap \mathsf{Down}(\langle X, \leqslant\rangle). \end{aligned}$$

Here we define the category of spaces dual to **HA**.

Definition 2.6.2 (**ES** - Esakia space) **:**

We define the category **ES**.

An **ES**-object, called an *Esakia space*[6], is an ordered topological space $\langle X, \leqslant, \Omega\rangle$ satisfying the following conditions for all $x, y \in \mathbb{X}$ and all $K \subseteq \mathbb{X}$.

(ES.o.1) $\langle X, \leqslant\rangle$ is a poset

(ES.o.2) $\langle X, \Omega\rangle$ is a compact topological space

(ES.o.3) $x \not\leqslant y$ implies $\exists K \in \mathsf{ClopUp}(\mathbb{X})$ such that $x \in K \not\ni y$

(ES.o.4) $K \in \mathsf{Clop}(\mathbb{X})$ implies $\downarrow K \in \mathsf{Clop}(\mathbb{X})$

The condition (ES.o.3) is often known as the *Priestley Separation Axiom* (PSA).

[6]In other literature, **ES**-objects are sometimes known as *Heyting spaces* [15] or *strict hybrids* [23].

An **ES**-morphism, called a *Esakia morphism*[7], is a function $f : \mathbb{X} \to \mathbb{Y}$ satisfying the following conditions for all $x_1 \in \mathbb{X}$ and $y \in \mathbb{Y}$.

(ES.m.1) $f : \langle X, \leqslant \rangle \to \langle Y, \leqslant \rangle$ is a monotone map

(ES.m.2) $f : \langle X, \Omega \rangle \to \langle Y, \Omega \rangle$ is a continuous map

(ES.m.3) $fx_1 \leqslant y$ implies $\exists x_2 \in X$ such that $x_1 \leqslant x_2$ and $fx_2 = y$.

$$\begin{array}{ccc} x_2 & \xrightarrow{\quad f \quad} & y \\ \uparrow \leqslant & & \uparrow \leqslant \\ x_1 & \xrightarrow{\quad f \quad} & fx_1 \end{array}$$

Here we state some well-known facts about Esakia spaces that will be utilised to develop our duality theory for **tHA**.

Fact 2.6.1 : The following are well-known and stated without proof.

- All Esakia spaces are Hausdorff [22, p. 37].
- Given $x \in \mathbb{X} \in$ **ES**, we have $\uparrow x \in \mathsf{ClUp}(\mathbb{X})$.
- All Esakia morphisms are closed [22, p. 31] .

Fact 2.6.2 : Given $\mathbb{X} := \langle X, \Omega \rangle \in$ **ES**, the following are equivalent.

1. $\mathbb{X}$ is finite
2. $\Omega = \wp X$ (i.e. all sets are open)
3. $\mathsf{Clop}(\mathbb{X}) = \wp X$ (i.e. all sets are clopen)

Proof : This follows from Fact 2.6.1 and [20, p. 278]. ⊠

We now define the category of spaces dual to **fHA**.

Definition 2.6.3 (**fES** - Frontal Esakia space) **:**

We define the category **fES**.

An **fES**-object, called a *frontal Esakia space*[8], is an ordered topological space $\langle X, R, \Omega \rangle$ satisfying the following conditions for all $x \in \mathbb{X}$ and all $K \subseteq \mathbb{X}$.

[7] In other literature, **ES**-morphisms are sometimes known as *continuous p-morphisms* [28] or *continuous strongly isotone maps* [23].

[8] In other literature, **fES**-objects are sometimes known as *Rf-Heyting spaces* [15].

(fES.o.1) $\langle X, R\rangle$ is a transit

(fES.o.2) $\langle X, \dot{R}, \Omega\rangle$ is an Esakia space

(fES.o.3) $K \in \mathsf{ClopUp}(\mathbb{X})$ implies $-R^{-1}[-K] \in \mathsf{ClopUp}(\mathbb{X})$

(fES.o.4) $R[x] \in \overline{\Omega}$

An **fES**-morphism, called a *frontal Esakia morphism*, is a function $f : \mathbb{X} \to \mathbb{Y}$ satisfying the following conditions for all $x_1, x_2 \in X$ and $y \in Y$.

(fES.m.1) $f : \langle X, \leqslant\rangle \to \langle Y, \leqslant\rangle$ is an Esakia morphism

(fES.m.2) $x_1 \, R \, x_2$ implies $fx_1 \, R \, fx_2$

$$\begin{array}{ccc} x_2 & \xrightarrow{\ f\ } & fx_2 \\ \uparrow{\scriptstyle R} & & \uparrow{\scriptstyle R} \\ x_1 & \xrightarrow{\ f\ } & fx_1 \end{array}$$

(fES.m.3) $fx_1 \, R \, y$ implies $\exists x_3 \in X$ such that $x_1 \, R \, x_3$ and $fx_3 = y$

$$\begin{array}{ccc} x_3 & \xrightarrow{\ f\ } & y \\ \uparrow{\scriptstyle R} & & \uparrow{\scriptstyle R} \\ x_1 & \xrightarrow{\ f\ } & fx_1 \end{array}$$

Finally, we define the category of spaces that will be shown to be dual to **tHA**.

Definition 2.6.4 (**tES** - Temporal Esakia space) **:**

We define the category **tES**.

A **tES**-object, called a *temporal Esakia space*, is an ordered topological space $\langle X, R^{\lhd}, R^{\rhd}, \Omega\rangle$ satisfying the following conditions for all $x \in \mathbb{X}$ and all $K \subseteq \mathbb{X}$.

(tES.o.1) $\langle X, R^{\lhd}, R^{\rhd}\rangle$ is a temporal transit

(tES.o.2) $\langle X, R^{\rhd}, \Omega\rangle$ is a frontal Esakia space

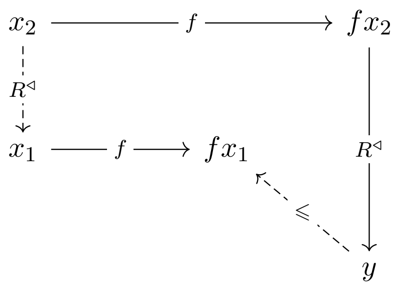
(tES.o.3) $K \in \mathsf{ClopUp}(\mathbb{X})$ implies $R^{\rhd}[K] \in \mathsf{ClopUp}(\mathbb{X})$

(tES.o.4) $R^{\lhd}[x] \in \overline{\Omega}$

An **tES**-morphism, called a *temporal Esakia morphism*, is a function $f : \mathbb{X} \to \mathbb{Y}$ satisfying the following conditions for all $x_2 \in X$ and $y \in Y$.

(tES.m.1) f is a frontal Esakia morphism

(tES.m.2) $fx_2 \, R^{\triangleleft} \, y$ implies $\exists x_1 \in X$ such that $x_2 \, R^{\triangleleft} \, x_1$ and $y \leqslant fx_1$



The reader should take note of condition (tES.m.2). In the classical case, (duality for *classical* temporal logic [47, §8.1],) this condition would be a mirror image of the conditions for the forward-looking relation (fES.m.2) and (fES.m.3). But in our non-classical setting, we have this weaker condition. This is due to the fact that when we turn to studying duality for **tES**, the morphisms in the dual category are not strong enough to fulfil the classical condition.

To conclude the section, we mention that, given a temporal transit, we can always build a temporal Esakia space in a trivial way.

Fact 2.6.3 : Given $\langle X, R^{\triangleleft}, R^{\triangleright} \rangle \in \mathbf{tTran}$, we have $\langle X, R^{\triangleleft}, R^{\triangleright}, \wp X \rangle \in \mathbf{tES}$.

2.7 Esakia duality

Here we define the basics of Esakia duality, that is, the duality between the categories $\mathbf{HA} \leftrightarrows \mathbf{ES}$, as well as how this has been extended to the categories $\mathbf{fHA} \leftrightarrows \mathbf{fES}$.

Definition 2.7.1 : Given $\mathbb{A} \in \mathbf{HA}$ and $\mathbb{X} \in \mathbf{ES}$, we define the following maps.

$$\begin{aligned} \pi_{\mathbb{A}} : \mathbb{A} &\longrightarrow \wp(\mathsf{PrFilt}(\mathbb{A})) \\ a &\longmapsto \{x \in \mathsf{PrFilt}(\mathbb{A}) \mid a \in x\} \end{aligned} \qquad \begin{aligned} \gamma_{\mathbb{X}} : \mathbb{X} &\longrightarrow \wp(\mathsf{ClopUp}(\mathbb{X})) \\ x &\longmapsto \{K \in \mathsf{ClopUp}(\mathbb{X}) \mid x \in K\} \end{aligned}$$

The map $\pi_{\mathbb{A}}$[9] is known as the *spectral map of* $\mathbb{A}$. When the algebra $\mathbb{A}$ and space $\mathbb{X}$ are clear from context, we simply write π and γ.

We construct the functors that will witness the contravariant equivalence [41, §1.5] between the categories $\mathbf{HA} \leftrightarrows \mathbf{ES}$ and $\mathbf{fHA} \leftrightarrows \mathbf{fES}$.

[9] In other literature, the symbol $\widehat{a}$ is sometimes used instead of πa [28].

Construction 2.7.1 : We define the contravariant functor $◌_* : \mathbf{HA} \to \mathbf{ES}$ as follows. On objects, $\mathbb{A}$ maps to $\langle \mathsf{PrFilt}(\mathbb{A}), \subseteq, \Omega_{\mathbb{A}} \rangle$ where $\Omega_{\mathbb{A}}$ is the topology generated [22, §1.2] by the subbasis $\pi[\mathbb{A}] \cup -[\pi[\mathbb{A}]]$, i.e.

$$\{\pi a \mid a \in \mathbb{A}\} \cup \{-\pi a \mid a \in \mathbb{A}\}.$$

On morphisms, $h : \mathbb{A} \to \mathbb{B}$ maps to $h^{-1}[◌] : \mathbb{B}_* \to \mathbb{A}_*$.

We also define the contravariant functor $◌^* : \mathbf{ES} \to \mathbf{HA}$ as follows. On objects, $\mathbb{X}$ maps to $\langle \mathsf{ClopUp}(\mathbb{X}), \cap, \cup, \to, \varnothing, X \rangle$ where, given $K_1, K_2 \in \mathsf{ClopUp}(\mathbb{X})$,

$$K_1 \to K_2 := -{\downarrow}(K_1 \cap -K_2).$$

On morphisms, $f : \mathbb{X} \to \mathbb{Y}$ maps to $f^{-1}[◌] : \mathbb{Y}^* \to \mathbb{X}^*$.

The functors $◌_*$ and $◌^*$ can be extended to $\mathbf{fHA} \rightleftarrows \mathbf{fES}$ as follows.

For $\mathbf{fHA} \to \mathbf{fES}$, the object $\langle A, \wedge, \vee, \to, \Box, 0, 1 \rangle$ maps to $\langle X, R, \Omega \rangle$ where $\langle X, \subseteq, \Omega \rangle := \langle A, \wedge, \vee, \to, 0, 1 \rangle_*$ and for all $x, y \in X$,

$$x \, R \, y \quad :\Longleftrightarrow \quad (\forall a \in A)(\Box a \in x \Rightarrow a \in y).$$

It can be checked that $\dot{R} = \subseteq$ and $\mathring{R} = \subsetneq$ [15, p. 208].

For $\mathbf{fES} \to \mathbf{fHA}$, the object $\langle X, R, \Omega \rangle$ maps to $\langle A, \wedge, \vee, \to, \Box, 0, 1 \rangle$ where $\langle A, \wedge, \vee, \to, 0, 1 \rangle := \langle X, \dot{R}^{\triangleright}, \Omega \rangle^*$ and for all $K \in A$,

$$\Box K := -R^{-1}[-K] = \{x \in X \mid R[x] \subseteq K\}.$$

For convenience, we follow [46] and also define the function $◌_+$ which maps an algebra $\mathbb{A}$ to the frame of $\mathbb{A}_*$. This maps $\mathbf{HA} \to \mathbf{POS}$ and $\mathbf{fHA} \to \mathbf{Tran}$.

Fact 2.7.1 : The functors $◌_*$ and $◌^*$ and their appropriate extensions are pseudo-inverse between the categories $\mathbf{HA} \rightleftarrows \mathbf{ES}$ and $\mathbf{fHA} \rightleftarrows \mathbf{fES}$. These facts were established in [25] and [15] respectively. This establishes a $\mathbf{HA}/\mathbf{fHA}$-isomorphism between an algebra $\mathbb{A} \in \mathbf{HA}/\mathbf{fHA}$ and $\mathsf{ClopUp}(\mathbb{A}_*)$ as well as an $\mathbf{ES}/\mathbf{fES}$-isomorphism between a space $\mathbb{X} \in \mathbf{ES}/\mathbf{fES}$ and $\mathsf{PrFilt}(\mathbb{X}^*)$.

$$(\mathbf{HA}/\mathbf{tHA}) \qquad \begin{array}{ccc} \mathbb{A} & \overset{\pi_{\mathbb{A}}}{\rightarrowtail\!\!\twoheadrightarrow} & \mathbb{A}^*_* \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h^*_*} \\ \mathbb{B} & \overset{\pi_{\mathbb{B}}}{\rightarrowtail\!\!\twoheadrightarrow} & \mathbb{B}^*_* \end{array} \qquad \begin{array}{ccc} \mathbb{X} & \overset{\gamma_{\mathbb{X}}}{\rightarrowtail\!\!\twoheadrightarrow} & \mathbb{X}^*_* \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^*_*} \\ \mathbb{Y} & \overset{\gamma_{\mathbb{Y}}}{\rightarrowtail\!\!\twoheadrightarrow} & \mathbb{Y}^*_* \end{array} \qquad (\mathbf{ES}/\mathbf{fES})$$

Having established the duality between the categories $\mathbf{HA} \leftrightarrows \mathbf{ES}$, we present a well-known correspondence between filters on Heyting algebras and closed upsets on Esakia spaces.

Fact 2.7.2 : Given $\mathbb{A} \in \mathbf{HA}$ and $\mathbb{X} \in \mathbf{ES}$, the following maps are $\mathbf{POS}$-isomorphisms.

$$\begin{aligned} \bigcap \pi[\circ] : \langle \mathsf{Filt}(\mathbb{A}), \subseteq \rangle &\longrightarrow \langle \mathsf{ClUp}(\mathbb{A}_*), \supseteq \rangle \\ F &\longmapsto \{x \in \mathbb{A}_* \mid F \subseteq x\} \\ \bigcap \gamma[\circ] : \langle \mathsf{ClUp}(\mathbb{X}), \subseteq \rangle &\longrightarrow \langle \mathsf{Filt}(\mathbb{X}^*), \supseteq \rangle \\ C &\longmapsto \{K \in \mathbb{X}^* \mid C \subseteq K\} \end{aligned}$$

Proof : It can be checked that $\bigcap : \mathsf{ClUp}(\mathbb{A}_*) \to \mathsf{Filt}(\mathbb{A})$ and $\bigcap : \mathsf{Filt}(\mathbb{X}^*) \to \mathsf{ClUp}(\mathbb{X})$ are monotone inverses of $\bigcap \pi[\circ]$ and $\bigcap \gamma[\circ]$ respectively (keeping in mind that the subset relation is flipped in some cases). ⊠

Corollary 2.7.1 : Given $\mathbb{A} \in \mathbf{HA}$ and $\mathbb{X} \in \mathbf{ES}$,

$$\langle \mathsf{Filt}(\mathbb{A}), \subseteq \rangle \cong^{\text{POS}} \langle \mathsf{ClUp}(\mathbb{A}_*), \supseteq \rangle \qquad \langle \mathsf{ClUp}(\mathbb{X}), \subseteq \rangle \cong^{\text{POS}} \langle \mathsf{Filt}(\mathbb{X}^*), \supseteq \rangle$$

A similar correspondence is present between ideals on Heyting algebras and open downsets on Esakia spaces.

Fact 2.7.3 : Given $\mathbb{A} \in \mathbf{HA}$ and $\mathbb{X} \in \mathbf{ES}$, the following maps are $\mathbf{POS}$-isomorphisms.

$$\begin{aligned} \bigcup \pi[\circ] : \langle \mathsf{Ideal}(\mathbb{A}), \subseteq \rangle &\longrightarrow \langle \mathsf{OpUp}(\mathbb{A}_*), \subseteq \rangle \\ I &\longmapsto \{x \in \mathbb{A}_* \mid x \cap I \neq \varnothing\} \\ \bigcup \gamma[\circ] : \langle \mathsf{OpUp}(\mathbb{X}), \subseteq \rangle &\longrightarrow \langle \mathsf{Ideal}(\mathbb{X}^*), \subseteq \rangle \\ O &\longmapsto \{K \in \mathbb{X}^* \mid K \cap O \neq \varnothing\} \end{aligned}$$

Finally, we present a correspondence between the congruences on Heyting algebras and closed upsets on Esakia spaces.

Corollary 2.7.2 : Given $\mathbb{A} \in \mathbf{HA}$ and $\mathbb{X} \in \mathbf{ES}$,

$$\langle \mathsf{Cong}^{\text{HA}}(\mathbb{A}), \subseteq \rangle \cong^{\text{POS}} \langle \mathsf{ClUp}(\mathbb{A}_*), \supseteq \rangle \qquad \langle \mathsf{ClUp}(\mathbb{X}), \subseteq \rangle \cong^{\text{POS}} \langle \mathsf{Cong}^{\text{HA}}(\mathbb{X}^*), \supseteq \rangle.$$

Proof : This follows directly from Theorem 3.2.1 and Theorem 5.2.1. ⊠

Chapter conclusion

Having established all of the necessary preliminaries for the current text, we return to universal algebra and lattice theory to study the structure of temporal Heyting algebras.

Chapter 3

Temporal Heyting algebras

In this chapter we study the structure of temporal Heyting algebras. We provide the following section-by-section outline.

§3.1 We prove algebraic soundness and completeness for **tHC** and **tHA**.

§3.2 We define a class of filters on **tHA** and subsequently establish a correspondence between this class and **tHA**-congruences.

§3.3 We define a class of elements on **tHA** and subsequently establish a correspondence between this class, ⧫-filters, and **tHA**-congruences in the *finite* case.

3.1 Soundness and completeness

Here we prove soundness and completeness for algebraic models of **tHC**. Due to the intuitionistic base of our logic, this proof is largely routine, so only the novel steps are provided.

Theorem 3.1.1 : tHC $\dashv\vdash$ **tHA**.

Proof : ($\vdash$) This can be shown via a standard argument using the Lindenbaum-Tarski algebra for **tHA** [27, §2][6, §4.3]. (In fact, this method is so formulaic that Font outlines what he calls « the Lindenbaum-Tarski process » in [27, p. 74].) Arguing via the contrapositive, we take some $\varphi \notin \mathbf{tHC}$. We then define a congruence $\equiv \subseteq \mathsf{Term}(\mathbf{tHA})^2$ by

$$\chi \equiv \psi \quad :\Longleftrightarrow \quad \chi \leftrightarrow \psi \in \mathbf{tHC}.$$

We then let $\mathbb{A} := \mathsf{Term}(\mathbf{tHA})/\equiv$ and show that $\mathbb{A} \in \mathbf{tHA}$ and $\mathbb{A} \not\models \varphi$, implying that $\mathbf{tHA} \not\models \varphi$. Since this argument is so standard, we prove only the key fact that $\equiv$ is a congruence over ⧫. Given $\chi \equiv \psi$, we have $\chi \leftrightarrow \psi \in \mathbf{tHC}$, implying $\chi \to \psi \in \mathbf{tHC}$

and $\psi \to \chi \in \mathbf{tHC}$. By (PD), we have $\blacklozenge\chi \to \blacklozenge\psi \in \mathbf{tHC}$ and $\blacklozenge\psi \to \blacklozenge\chi \in \mathbf{tHC}$, implying that $\blacklozenge\chi \leftrightarrow \blacklozenge\psi \in \mathbf{tHC}$, finally implying that $\blacklozenge\chi \equiv \blacklozenge\psi$.

($\dashv$) Given Fact 2.5.12, all that needs to be checked is the validity of (tHC.1) through (tHC.4) on **tHA** and that (MP), (US), and (PD) preserve validity on **tHA**. Since the axioms, (MP), and (US) are quite trivial to check, we give only a proof for (PD). We assume $\mathbf{tHA} \models \varphi \to \chi$ and show $\mathbf{tHA} \models \blacklozenge\varphi \to \blacklozenge\chi$. Given $\mathbb{A} \in \mathbf{tHA}$ and an algebraic valuation ν on $\mathbb{A}$, we have $\mathbb{A} \models \varphi \to \chi$, implying $1 = \nu(\varphi \to \chi) = \nu\varphi \to \nu\chi$, implying, by Fact 2.5.1, that $\nu\varphi \leqslant \nu\chi$. By Fact 2.5.8, we have $\blacklozenge\nu\varphi \leqslant \blacklozenge\nu\chi$, implying

$$1 = \blacklozenge\nu\varphi \to \blacklozenge\nu\chi = \nu\blacklozenge\varphi \to \nu\blacklozenge\chi = \nu(\blacklozenge\varphi \to \blacklozenge\chi),$$

finally implying that $\mathbb{A} \models \blacklozenge\varphi \to \blacklozenge\chi$. Since $\mathbb{A}$ was taken arbitrarily in **tHA**, we can conclude that $\mathbf{tHA} \models \blacklozenge\varphi \to \blacklozenge\chi$. ⊠

Having established Theorem 3.1.1, we can leverage Fact 2.4.4 to make an even stronger statement.

Theorem 3.1.2 : $\mathbf{tHC} \dashv\vdash \mathbf{tHA}_{\mathsf{si}}$.

Proof : ($\vdash$) Given $\mathbf{tHA}_{\mathsf{si}} \models \varphi$, we have $\mathbf{tHA}_{\mathsf{si}} \models \varphi \approx \top$, implying, by Fact 2.4.4, that $\mathbf{tHA} \models \varphi \approx \top$, finally implying that $\mathbf{tHA} \models \varphi$. Since $\mathbf{tHA} \models \varphi$, Theorem 3.1.1 implies that $\varphi \in \mathbf{tHC}$. ($\dashv$) Since $\mathbf{tHA}_{\mathsf{si}} \subseteq \mathbf{tHA}$, Theorem 3.1.1 implies that $\mathbf{tHA}_{\mathsf{si}} \models \mathbf{tHC}$. ⊠

With these algebraic soundness and completeness results established, we continue in our investigation of the variety **tHA**, with the knowledge that several of the algebraic notions covered will have direct logical analogues in **tHC**.

3.2 Congruences and ⧫-filters

Here we establish a correspondence between congruences and a class of filters on temporal Heyting algebras called *⧫-filters*. This is analogous to the correspondence established for Heyting algebras in Fact 2.5.5. Such a class of filters is of interest both logically and algebraically. Logically, these filters correspond to subsets of the logic that are closed under all rules of inference, i.e. a *deductively-closed theory* of the logic in some sense. Algebraically, such a correspondence allows us to use lattice theory to study congruences, which are central in studying some of the universal-algebraic properties of the variety in question, such as conditions for simplicity and subdirect-irreducibility.

Now we define our special class of filters on temporal Heyting algebras.

Definition 3.2.1 ($\blacklozenge$-filter) **:** Given $\mathbb{A} \in \mathbf{tHA}$ and $F \in \mathsf{Filt}(\mathbb{A})$, we call F a $\blacklozenge$-*filter* if for all $a, b \in \mathbb{A}$,

$$a \to b \in F \quad \Longrightarrow \quad \blacklozenge a \to \blacklozenge b \in F.$$

We denote the set of $\blacklozenge$-filters on $\mathbb{A}$ by $\blacklozenge\mathsf{Filt}(\mathbb{A})$.

In the following two lemmas, we establish the well-definedness of our operations $\circ_{\bullet}$ and $\circ^{\bullet}$ respectively.

Lemma 3.2.1 : Given $\mathbb{A} \in \mathbf{tHA}$ and $\theta \in \mathsf{Cong}^{\mathrm{tHA}}(\mathbb{A})$, we have $\theta_{\bullet} \in \blacklozenge\mathsf{Filt}(\mathbb{A})$.

Proof : Since $\theta \in \mathsf{Cong}^{\mathrm{tHA}}(\mathbb{A})$, we have $\theta \in \mathsf{Cong}^{\mathrm{fHA}}(\mathbb{A})$ implying, by Fact 2.5.7, that $\theta_{\bullet} \in \mathsf{Filt}(\mathbb{A})$, so it only remains to show that $\theta_{\bullet}$ satisfies the additional condition to be a $\blacklozenge$-filter. Given $a \to b \in \theta_{\bullet}$, we have $a \to b\,\theta\,1$, implying $a \wedge b = a \wedge (a \to b)\,\theta\,a \wedge 1 = a$. Now since $a \wedge b\,\theta\,a$, we have $\blacklozenge(a \wedge b)\,\theta\,\blacklozenge a$, implying $\blacklozenge a\,\theta\,\blacklozenge(a \wedge b)$. Also, since $a \wedge b\,\theta\,a$, we have $b = (a \wedge b) \vee b\,\theta\,a \vee b$, implying $\blacklozenge b\,\theta\,\blacklozenge(a \vee b)$. Note that since $a \wedge b \leqslant a \vee b$, we have $\blacklozenge(a \wedge b) \leqslant \blacklozenge(a \vee b)$, implying $\blacklozenge(a \wedge b) \to \blacklozenge(a \vee b) = 1$. Finally, since $\blacklozenge a\,\theta\,\blacklozenge(a \wedge b)$ and $\blacklozenge b\,\theta\,\blacklozenge(a \vee b)$, we have

$$\blacklozenge a \to \blacklozenge b\,\theta\,\blacklozenge(a \wedge b) \to \blacklozenge(a \vee b) = 1,$$

implying $\blacklozenge a \to \blacklozenge b \in \theta_{\bullet}$ as desired. ⊠

Lemma 3.2.2 : Given $\mathbb{A} \in \mathbf{tHA}$ and $F \in \blacklozenge\mathsf{Filt}(\mathbb{A})$, we have $F^{\bullet} \in \mathsf{Cong}^{\mathrm{tHA}}(\mathbb{A})$.

Proof : Since $F \in \blacklozenge\mathsf{Filt}(\mathbb{A})$, we have $F \in \mathsf{Filt}(\mathbb{A})$, implying, by Lemma 2.5.7, that $F^{\bullet} \in \mathsf{Cong}^{\mathrm{fHA}}(\mathbb{A})$, so it only remains to show that $F^{\bullet}$ is a congruence over $\blacklozenge$, i.e. that $(\forall a, b \in \mathbb{A})(a\,F^{\bullet}\,b \Rightarrow \blacklozenge a\,F^{\bullet}\,\blacklozenge b)$. Given $a\,F^{\bullet}\,b$, we have $a \leftrightarrow b \in F$, implying $a \to b, b \to a \in F$, implying $\blacklozenge a \to \blacklozenge b, \blacklozenge b \to \blacklozenge a \in F$, implying $\blacklozenge a \leftrightarrow \blacklozenge b \in F$, finally implying $\blacklozenge a\,F^{\bullet}\,\blacklozenge b$. ⊠

We combine these lemmas to state the following theorem.

Theorem 3.2.1 : Given $\mathbb{A} \in \mathbf{tHA}$,

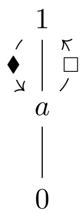
$$\langle \mathsf{Cong}^{\mathrm{tHA}}(\mathbb{A}), \subseteq \rangle \cong^{\mathrm{POS}} \langle \blacklozenge\mathsf{Filt}(\mathbb{A}), \subseteq \rangle.$$

Proof : This follows from Lemma 3.2.1, Lemma 3.2.2, and the fact that $\circ^{\bullet}$ and $\circ_{\bullet}$ are monotone inverses of each other ([23, Proposition 2.4.9]). ⊠

We now leverage Theorem 3.2.1 to prove a proposition about the variety **tHA**.

Proposition 3.2.1 : The variety **tHA** is *not* semisimple.

Proof : We give an example of an $\mathbb{A} \in \mathbf{tHA}$ with a ⧫-filter F such that ⧫Filt$(\mathbb{A}) = \{\{1\}, F, \mathbb{A}\}$. In light of Theorem 3.2.1, this implies that $\mathsf{Cong}^{\mathbf{tHA}}(\mathbb{A}) = \{\{1\}^\bullet, F^\bullet, \mathbb{A}^\bullet\}$, implying that $\mathbb{A}$ is subdirectly-irreducible (as witnessed by $F^\bullet$), but not simple.



Letting $F := {\uparrow}a$, it can be easily checked that $F \in$ ⧫Filt$(\mathbb{A})$ and ⧫Filt$(\mathbb{A}) = \{\{1\}, F, \mathbb{A}\}$. ⊠

3.3 ⧫-compatible elements

Here we study a particular class of elements called ⧫-*compatible* on a given temporal Heyting algebra. These elements are interesting for two main reasons. The first is that, in the finite case, they will be shown to correspond to ⧫-filters and, thereby, congruences, giving us the ability to characterise simple and subdirectly-irreducible algebras element-wise, as is done using the notion of an *opremum* on Heyting algebras [23, Theorem 2.4.11]. The second reason is that they simulate « compatibility », a notion that is of interest from a universal-algebraic point of view. This latter point will be discussed in greater detail in Remark 3.3.1.

Definition 3.3.1 (⧫-compatible) **:** Given $a \in \mathbb{A} \in \mathbf{tHA}$, we call a ⧫-*compatible* if for all $b \in \mathbb{A}$,

$$a \wedge \blacklozenge b \leqslant \blacklozenge(a \wedge b).$$

We denote the set of ⧫-compatible elements of $\mathbb{A}$ by ⧫Com$(\mathbb{A})$.

We now include a fact about temporal Heyting algebras and a subsequent remark on the notion of « compatibility ».

Fact 3.3.1 : Given $a, b \in \mathbb{A} \in \mathbf{tHA}$,

$$a \wedge \blacklozenge b = a \wedge \blacklozenge(a \wedge b) \iff a \wedge \blacklozenge b \leqslant \blacklozenge(a \wedge b).$$

Proof : ($\Rightarrow$) By weakening, we have $a \wedge \blacklozenge b \leqslant a \wedge \blacklozenge(a \wedge b)$, implying that $a \wedge \blacklozenge b \leqslant \blacklozenge(a \wedge b)$.
($\Leftarrow$) Conversely, given $a \wedge \blacklozenge b \leqslant \blacklozenge(a \wedge b)$, we have

$$a \wedge \blacklozenge b = a \wedge a \wedge \blacklozenge b \leqslant a \wedge \blacklozenge(a \wedge b).$$

Also, since $a \wedge b \leqslant b$, Fact 2.5.8 implies $\blacklozenge(a \wedge b) \leqslant \blacklozenge b$, implying $a \wedge \blacklozenge(a \wedge b) \leqslant a \wedge \blacklozenge b$. Combining these facts, we get $a \wedge \blacklozenge b = a \wedge \blacklozenge(a \wedge b)$. ⊠

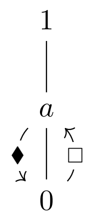
Remark 3.3.1 : The terminology introduced in Definition 3.3.1 comes from [14] where, given $\mathbb{A} \in \mathbf{HA}$, we call a function $f : \mathbb{A}^n \to \mathbb{A}$ *compatible* if for all $a, b_1, \ldots, b_n \in \mathbb{A}$,

$$a \wedge f(b_1, \ldots, b_n) = a \wedge f(a \wedge b_1, \ldots, a \wedge b_n).$$

In the case of $\blacklozenge$, Fact 3.3.1 shows that this is equivalent to the condition

$$a \wedge \blacklozenge b \leqslant \blacklozenge(a \wedge b).$$

In general, $\blacklozenge$ is *not* a compatible function. This can be seen in the following temporal Heyting algebra.



Here we have

$$a \wedge \blacklozenge 1 = a \wedge 1 = a \not\leqslant 0 = \blacklozenge a = \blacklozenge(a \wedge 1).$$

However, though $\blacklozenge$ is not compatible in *general*, there are elements such as 0 and 1 (Proposition 3.3.4) that make $\blacklozenge$ *behave* like a compatible function when fixed in the place of a. It is for this reason that these elements have been referred to as $\blacklozenge$-compatible elements in the current text.

We now prove several propositions about the lattice-theoretic and universal-algebraic structures that correspond to $\blacklozenge$-compatible elements in the *finite* case.

Proposition 3.3.1 : Given $a \in \mathbb{A} \in \mathbf{tHA}$,

$$a \in \blacklozenge\mathsf{Com}(\mathbb{A}) \iff {\uparrow} a \in \blacklozenge\mathsf{Filt}(\mathbb{A}).$$

Proof : ($\Rightarrow$) Clearly, ${\uparrow} a \in \mathsf{Filt}(\mathbb{A})$, so we show only the additional condition. Given $b \to c \in {\uparrow} a$, we have $a \leqslant b \to c$, implying $a \wedge b \leqslant c$, implying $\blacklozenge(a \wedge b) \leqslant \blacklozenge c$. Since $a \in \blacklozenge\mathsf{Com}(\mathbb{A})$, we have $a \wedge \blacklozenge b \leqslant \blacklozenge(a \wedge b)$, so $a \wedge \blacklozenge b \leqslant \blacklozenge c$, implying $a \leqslant \blacklozenge b \to \blacklozenge c$, finally implying $\blacklozenge b \to \blacklozenge c \in {\uparrow} a$. ($\Leftarrow$) Arguing via the contrapositive, we assume $a \notin \blacklozenge\mathsf{Com}(\mathbb{A})$, implying there is some $b \in \mathbb{A}$ such that $a \wedge \blacklozenge b \not\leqslant \blacklozenge(a \wedge b)$, implying that $a \not\leqslant \blacklozenge b \to \blacklozenge(a \wedge b)$. Now since $a \wedge b \leqslant a \wedge b$, we have $a \leqslant b \to (a \wedge b)$. So we have

$$b \to (a \wedge b) \in {\uparrow} a \not\ni \blacklozenge b \to \blacklozenge(a \wedge b),$$

implying that $\uparrow a \notin \blacklozenge\mathsf{Filt}(\mathbb{A})$. ☒

Proposition 3.3.2 : Given $\mathbb{A} \in \mathbf{tHA}_{\mathsf{fin}}$,

$$\langle \blacklozenge\mathsf{Filt}(\mathbb{A}), \subseteq \rangle \cong^{\text{POS}} \langle \blacklozenge\mathsf{Com}(\mathbb{A}), \geqslant \rangle.$$

Proof : In light of Proposition 3.3.1, it can be easily checked that $\bigwedge : \blacklozenge\mathsf{Filt}(\mathbb{A}) \to \blacklozenge\mathsf{Com}(\mathbb{A})$ and $\uparrow : \blacklozenge\mathsf{Com}(\mathbb{A}) \to \blacklozenge\mathsf{Filt}(\mathbb{A})$. With this established, one need only check that basic facts that these maps are order-reversing and inverses of one another, that is, $\uparrow \bigwedge F = F$ and $\bigwedge \uparrow a = a$. ☒

Proposition 3.3.3 : Given $\mathbb{A} \in \mathbf{tHA}_{\mathsf{fin}}$,

$$\langle \mathsf{Cong}^{\text{tHA}}(\mathbb{A}), \subseteq \rangle \cong^{\text{POS}} \langle \blacklozenge\mathsf{Com}(\mathbb{A}), \geqslant \rangle.$$

Proof : This follows from Theorem 3.2.1 and Proposition 3.3.2. ☒

Proposition 3.3.4 : Given $\mathbb{A} \in \mathbf{tHA}$, the set $\blacklozenge\mathsf{Com}(\mathbb{A})$ forms a sub-**BDL** of $\mathbb{A}$.

Proof : Let $a, b \in \blacklozenge\mathsf{Com}$ and $c \in \mathbb{A}$. We check that $\blacklozenge\mathsf{Com}(\mathbb{A})$ is closed under the operations $\{0, 1, \wedge, \vee\}$.

(0) $0 \wedge \blacklozenge c = 0 \leqslant \blacklozenge(0 \wedge c)$.

(1) $1 \wedge \blacklozenge c = \blacklozenge c = \blacklozenge(1 \wedge c)$.

($\wedge$) First observe that, since $b \in \blacklozenge\mathsf{Com}(\mathbb{A})$, we have $b \wedge \blacklozenge c \leqslant \blacklozenge(b \wedge c)$. This implies that $a \wedge b \wedge \blacklozenge c \leqslant a \wedge \blacklozenge(b \wedge c)$. Since $a \in \blacklozenge\mathsf{Com}(\mathbb{A})$, we have $a \wedge \blacklozenge(b \wedge c) \leqslant \blacklozenge(a \wedge b \wedge c)$. So we have $(a \wedge b) \wedge \blacklozenge c \leqslant \blacklozenge((a \wedge b) \wedge c)$ as desired.

($\vee$) First observe that, since $a, b \in \blacklozenge\mathsf{Com}(\mathbb{A})$, we have $a \wedge \blacklozenge c \leqslant \blacklozenge(a \wedge c)$ and $b \wedge \blacklozenge c \leqslant \blacklozenge(b \wedge c)$. This implies that

$$(a \wedge \blacklozenge c) \vee (b \wedge \blacklozenge c) \leqslant \blacklozenge(a \wedge c) \vee \blacklozenge(b \wedge c),$$

implying $(a \vee b) \wedge \blacklozenge c \leqslant \blacklozenge((a \wedge c) \vee (b \wedge c))$, finally implying $(a \vee b) \wedge \blacklozenge c \leqslant \blacklozenge((a \vee b) \wedge c)$ as desired. ☒

Remark 3.3.2 : For the reader who has studied *closure algebras* [23, Definition 2.2.1] (also called *interior algebras* [11]), we remark that ♦-compatible elements are highly analogous to *open elements* [23, Definition 2.2.3]. Open elements also form a sub-**BDL** of their algebras and also correspond to *open filters* (also called *skeletal filters* or *modal filters*) [23, Proposition 2.4.14] in the finite case. Because of this, they also correspond to *congruences* in the finite case. The analogy between these classes of elements suggests that there may be more to say about the structure of ⧫-compatible elements. This idea will be expanded upon in Chapter 8 as potential future work.

Finally, we define a notion that will be of use in §7.3.

Definition 3.3.2 (⧫-opremum) **:** Given $a \in \mathbb{A} \in \mathbf{tHA}$, we call a a ⧫-*opremum* if a is the second-greatest element of $\langle ⧫\mathsf{Com}(\mathbb{A}), \leqslant \rangle$. Since Proposition 3.3.4 shows that $1 \in ⧫\mathsf{Com}(\mathbb{A})$, this is equivalent to a satisfying the following conditions.

- $a \in ⧫\mathsf{Com}(\mathbb{A})$
- $a \neq 1$
- $(\forall b \in \mathbb{A})(b \in ⧫\mathsf{Com}(\mathbb{A})$ and $b \neq 1$ implies $b \leqslant a)$

Chapter conclusion

Having gained a better understanding of the structure of temporal Heyting algebras, we continue on to study the structure of temporal Esakia spaces.

Chapter 4

Temporal Esakia spaces

In this chapter we study the structure of temporal Esakia spaces. We provide the following section-by-section outline.

- §4.1 We define and study a class of subsets on **tES** relevant to duality theory.
- §4.2 We define and study the notion of « topo-reachability » on temporal Esakia spaces.
- §4.3 We define and study the notion of « Z-reachability » on *finite* temporal Esakia spaces.
- §4.4 We prove the coincidence of « topo-reachability » and « Z-reachability » in the *finite* case.

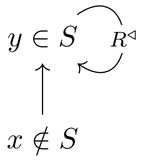
4.1 Archival subsets

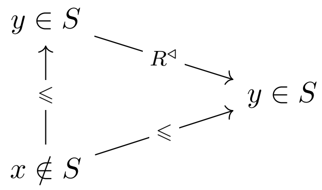
Here we define a class of subsets on temporal Esakia spaces analogous to the *hereditary* subsets of [42], which considered modal algebras and modal Stone spaces. The terminology « archival » was chosen to draw attention to the fact that they are defined in terms of the past-viewing relation $R^{\triangleleft}$. They will similarly play a role in characterising the subdirectly-irreducible algebras of **tHA** once duality is established.

Consider the following situation. We have a point $x \in \mathbb{X} \in \mathbf{tES}$ that can « see » into a subset $S \subseteq \mathbb{X}$ from outside of the subset.

$$\begin{array}{c} y \in S \\ \uparrow \\ x \notin S \end{array}$$

Now if $\mathbb{X}$ were $R^{\triangleright}$-reflexive (equiv. $R^{\triangleleft}$-reflexive), we would know that x can « see » something in S that y can see through the $R^{\triangleleft}$-relation, namely y itself.





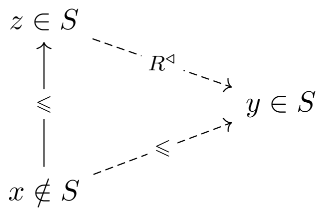
However, due to the sometimes-irreflexive nature of temporal Esakia spaces, this is not always the case. We could be in the irreflexive situation described above in which ${\uparrow}x \cap R^{\triangleleft}[y] \cap S = \varnothing$.

Here we define the class of *archival* subsets: subsets which are guaranteed to avoid the situation described above.

Definition 4.1.1 (Archival) **:** Given $S \subseteq \mathbb{X} \in \mathbf{tES}$, we call S *archival* if for all $x, z \in \mathbb{X}$,

$$x \notin S \ni z \text{ and } x \leqslant z \quad \Longrightarrow \quad {\uparrow}x \cap R^{\triangleleft}[z] \cap S \neq \varnothing.$$

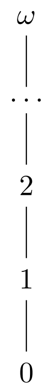
This is depicted as follows.



We denote the set of archival subsets of $\mathbb{X}$ by $\mathsf{Arc}(\mathbb{X})$, the set of archival *upsets* of $\mathbb{X}$ by $\mathsf{ArcUp}(\mathbb{X})$, and the set of *closed* archival upsets of $\mathbb{X}$ by $\mathsf{ClArcUp}(\mathbb{X})$.

The reader might have noticed that given the situation described in Definition 4.1.1, we now find ourselves, once again, in the situation $x \leqslant y$ and $x \notin S \ni y$, implying that the closure principle will apply again and again, *ad infinitum*. If $\mathbb{X}$ is infinite, then it is entirely possible that we always have a new, distinct witness as, exemplified in the following example.

Example 4.1.1 : Consider the algebra $\mathbb{A} \in \mathbf{HA}$ (depicted on the left) and its dual $\mathbb{A}_* \in \mathbf{ES}$ (depicted on the right).

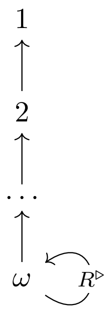


(In the dual space, a is taken to mean $\uparrow a$ for all $a \in \mathbb{A}$. This can be done because all prime filters on $\mathbb{A}$ are principal.) Here we have

$$\Omega = \{O \subseteq \mathbb{A}_* \mid \omega \notin O \text{ and } O \text{ finite}\} \cup \{O \subseteq \mathbb{A}_* \mid \omega \in O \text{ and } O \text{ cofinite}\}.$$

By Fact 2.7.1, we also have $\mathsf{ClopUp}(\mathbb{A}_*) = \pi[\mathbb{A}] = \{\uparrow x \mid x \in \mathbb{A}_*\}$.

Now if we define the relation $R^{\triangleright} := < \cup \{\langle \omega, \omega \rangle\}$ and define $R^{\triangleleft}$ to be the inverse of $R^{\triangleright}$, we get the following.



It can be checked, via several simple calculations, that this is a temporal Esakia space.

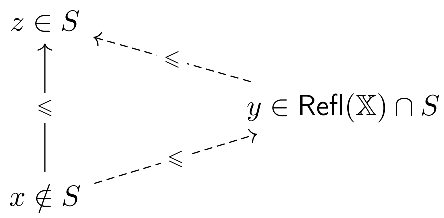
Finally, we consider the subset $\mathbb{N}$, which we claim to be archival. Observe the fact that $\omega \leqslant n$ and $\omega \notin \mathbb{N} \ni n$ for all n. However, we always have $\omega \leqslant n + 1 \; R^{\triangleright} \; n$ with $n + 1 \in \mathbb{N}$, implying that $\uparrow\omega \cap R^{\triangleleft}[n] \cap \mathbb{N} \neq \varnothing$. So we can see that $\mathbb{N}$ is an archival subset with an infinitely descending chain of distinct irreflexive witnesses.

However, if our space is finite, there can be no such infinite chain. For this reason, if a subset S is to be archival, we must always descend until we encounter a reflexive point. This intuition is formalised in the following lemma.

Proposition 4.1.1 : Given $S \subseteq \mathbb{X} \in \mathbf{tES}_{\mathsf{fin}}$, we have $S \in \mathsf{Arc}(\mathbb{X})$ if and only if the following holds for all $x, y \in \mathbb{X}$,

$$x \leqslant z \text{ and } x \notin S \ni z \quad \Longrightarrow \quad {\uparrow}x \cap {\downarrow}z \cap \mathsf{Refl}(\mathbb{X}) \cap S \neq \varnothing.$$

This is depicted as follows.



Proof : ($\Rightarrow$) We let ${\uparrow}x \cap R^{\lhd}z \cap S \neq \varnothing$ and show that ${\uparrow}x \cap {\downarrow}z \cap \mathsf{Refl}(\mathbb{X}) \cap S \neq \varnothing$. Since $\mathbb{X}$ is finite, every subset is well-founded, so consider some *minimal* $y \in {\uparrow}x \cap R^{\lhd}[z] \cap S$. Now since $y \in {\uparrow}x \cap R^{\lhd}[z]$, we have $x \leqslant y$ and $z\, R^{\lhd}\, y$, implying $y\, R^{\rhd}\, z$, further implying $x \leqslant y \leqslant z$, so we know that $y \in {\uparrow}x \cap {\downarrow}z \cap S$. So we need only show that $y \in \mathsf{Refl}(\mathbb{X})$ for y to witness the truth of the desired statement. Since we have $x \leqslant y$ and $x \notin S \ni y$ and $S \in \mathsf{Arc}(\mathbb{X})$, there exists some $w \in {\uparrow}x \cap R^{\lhd}[y] \cap S$. Now since $w \in R^{\lhd}[y]$, we have $w R^{\rhd} y$, implying $w \leqslant y$. But since y was assumed to be minimal, it must be the case that $y = w\, R^{\rhd}\, y$, implying that $y \in \mathsf{Refl}(\mathbb{X})$ as desired. ($\Leftarrow$) We let ${\uparrow}x \cap {\downarrow}z \cap \mathsf{Refl}(\mathbb{X}) \cap S \neq \varnothing$ and show that ${\uparrow}x \cap R^{\lhd}[z] \cap S \neq \varnothing$. Let $x \leqslant y \leqslant z$ such that $y \in \mathsf{Refl}(\mathbb{X}) \cap S$. Since $y \leqslant y\, R^{\rhd}\, y \leqslant z$, Fact 2.2.3 implies that $y\, R^{\rhd}\, z$, implying that $y \in {\uparrow}x \cap R^{\lhd}[z] \cap S$. ⊠

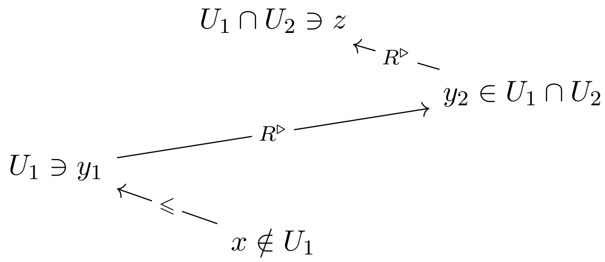
Having defined archival subsets and characterised them in the finite case, we conclude by stating an important lemma to be used in the following section.

Lemma 4.1.1 : Given $\mathbb{X} \in \mathbf{tES}$,

$$\{U_i\}_{i=1}^{n} \subseteq \mathsf{ArcUp}(\mathbb{X}) \quad \Longrightarrow \quad \bigcap_{i=1}^{n} U_i \in \mathsf{ArcUp}(\mathbb{X})$$

Proof : Arguing via induction on n, the base case is trivial and the inductive case reduces to a binary case. So given $U_1, U_2 \in \mathsf{ArcUp}(\mathbb{X})$, we show $U_1 \cap U_2 \in \mathsf{ArcUp}(\mathbb{X})$. Obviously, $U_1 \cap U_2 \in \mathsf{Up}(\mathbb{X})$, so we show that $U_1 \cap U_2$ is archival. Consider some $x, z \in \mathbb{X}$ such that $x \leqslant z$ and $x \notin U_1 \cap U_2 \ni z$. Without loss of generality, we can assume $x \notin U_1$, implying, since U_1 is archival, that there exists some $y_1 \in U_1$ such that $x \leqslant y_1\, R^{\rhd}\, z$. Now if $y_1 \in U_2$ then we are done (because $y_1 \in {\uparrow}x \cap R^{\lhd}[z] \cap U_1 \cap U_2$), so we assume $y_1 \notin U_2$, implying $y_1 \leqslant z$ and $y_1 \notin U_2 \ni z$, which implies, since U_2 is archival, that there exists some $y_2 \in U_2$ such that $y_1 \leqslant y_2\, R^{\rhd}\, z$. Now since $y_1 \leqslant y_2$ and

$U_1 \in \mathsf{Up}(\mathbb{X})$, we have $y_2 \in U_1$.



So $y_2 \in U_1 \cap U_2$ and $x \leqslant y_1 \leqslant y_2 \; R^{\triangleright} \; z$, implying $y_2 \in \cap{\uparrow}x \cap R^{\triangleleft}[z] \cap U_1 \cap U_2$. ⊠

We now continue on to study two notions of « reachability » on temporal Esakia spaces that will be very closely tied to the notion of « archival subsets ».

4.2 Topo-reachability

Here we define a notion of « reachability » on a temporal Esakia space in terms of its order-topological structure. For this reason, it is referred to as *topo*-reachability. This terminology comes from [46], which considered Boolean algebras with operators (BAOs) (which correspond to classical modal logics) and showed that, having established a Stone duality with a variety of BAOs $\mathbf{V}$, one can characterise the simple and subdirectly-irreducible elements of $\mathbf{V}$ via the notion of « topo-reachability » on their dual spaces. Subsequently, [8] generalised this work to provide these characterisations in the setting of « distributive modal algebras ». We will see in Theorems 7.2.1 and 7.3.1 that our notion of « topo-reachability » allows to accomplish the same thing for $\mathbf{tHA}$.

We first define an function $\widehat{\circ} : \wp\mathbb{X} \to \wp\mathbb{X}$ that will be used to define « topo-reachability ».

Definition 4.2.1 : Given $S \subseteq \mathbb{X} \in \mathbf{tES}$,

$$\widehat{S} := \bigcap \{C \in \mathsf{ClArcUp}(\mathbb{X}) \mid S \subseteq C\}.$$

Given $x \in \mathbb{X}$, we write $\widehat{x}$ instead of $\widehat{\{x\}}$.

We now establish an important lemma about the function $\widehat{\circ}$.

Lemma 4.2.1 : Given $S \subseteq \mathbb{X} \in \mathbf{tES}$, the set $\widehat{S}$ is the smallest closed archival upset containing S, i.e.

$$S \subseteq \widehat{S} \in \mathsf{ClArcUp}(\mathbb{X}) \quad \text{and} \quad (\forall C \in \wp\mathbb{X})(S \subseteq C \in \mathsf{ClArcUp}(\mathbb{X}) \Rightarrow \widehat{S} \subseteq C).$$

Proof : Clearly, if we can show that $S \subseteq \widehat{S} \in \mathsf{ClArcUp}(\mathbb{X})$, it will the smallest such set, so we need only show this fact. However, it should be clear that $S \subseteq \widehat{S}$, and since $\widehat{S}$ is the intersection of closed upsets, it is itself a closed upset, so it only remains to show that $\widehat{S}$ is archival. Assume, toward a contradiction, that $\widehat{S}$ is *not* archival, implying there exists some $x, z \in \mathbb{X}$ such that $x \leqslant z$ and $x \notin \widehat{S} \ni z$ and $\uparrow x \cap R^{\lhd}[z] \cap \widehat{S} = \varnothing$. Recall that this means

$$\uparrow x \cap R^{\lhd}[z] \cap \bigcap \{C \in \mathsf{ClArcUp} \mid S \subseteq C\} = \varnothing.$$

Now, since $\mathbb{X}$ is compact and $\uparrow x$ and $R^{\lhd}[z]$ are closed (by Fact 2.6.1 and (tES.o.4) respectively), there is a finite subcollection $\{C_i\}_{i=1}^n \subseteq \{C \in \mathsf{ClArcUp} \mid S \subseteq C\}$ such that $\uparrow x \cap R^{\lhd}[z] \cap C_\star = \varnothing$ where $C_\star := \bigcap_{i=1}^n C_i$. Now since $\{C_i\}_{i=1}^n \subseteq \overline{\Omega}$, we know $C_\star \in \overline{\Omega}$. And since $\{C_i\}_{i=1}^n \subseteq \mathsf{ArcUp}(\mathbb{X})$, Lemma 4.1.1 implies that $C_\star \in \mathsf{ArcUp}(\mathbb{X})$, so we can conclude that $C_\star \in \mathsf{ClArcUp}(\mathbb{X})$. Now since $x \in \uparrow x \cap R^{\lhd}[z]$, we know $x \notin C_\star$, and since $z \in \widehat{S}$ and $C_\star \in \{C \in \mathsf{ClArcUp} \mid S \subseteq C\}$, we know $z \in C_\star$. But then $x \leqslant z$ and $x \notin C_\star \ni z$ and $\uparrow x \cap R^{\lhd}[z] \cap C_\star = \varnothing$, contradicting the fact that $C_\star$ is archival. ☒

We now define our notion of « topo-reachability ».

Definition 4.2.2 (Topo-reachability) **:** Given $x, y \in \mathbb{X} \in \mathbf{tES}$, we say that y *is topo-reachable from* x and write $x \trianglelefteq y$, if $y \in \widehat{x}$.

Given Lemma 4.2.1, this implies that if y is topo-reachable from x if and only if y is in every closed archival upset containing x. This can be see as a type of *specialisation ordering* [28, p. 37] where, given some topological space $\langle X, \Omega \rangle$ with points $x, y \in X$, we say

$$x \prec y \quad :\Longleftrightarrow \quad (\forall C \in \overline{\Omega})(x \in C \Rightarrow y \in C) \quad \Longleftrightarrow \quad y \in \overline{x}$$

(though in [28], an order-inverse definition is given in terms of open sets). Compare this with our situation:

$$x \trianglelefteq y \quad :\Longleftrightarrow \quad (\forall C \in \mathsf{ClArcUp}(\mathbb{A}))(x \in C \Rightarrow y \in C) \quad \Longleftrightarrow \quad y \in \widehat{x}.$$

This analogy goes even further as the closed sets of Stone spaces correspond to the congruences of Boolean algebras and we will see, symmetrically, that the closed archival upsets of temporal Esakia spaces correspond to the congruences of temporal Heyting algebras (Theorem 5.2.2).

Here we state a fact for the reader to gain some intuition on topo-reachability.

Fact 4.2.1 : Given $\mathbb{X} \in \mathbf{tES}$, we have $\langle \mathbb{X}, \trianglelefteq \rangle \in \mathbf{QOS}$.

Proof : (QOS.1) Given $x \lhd y \lhd z$, we have $z \in \widehat{y}$ and $y \in \widehat{x}$. By Lemma 4.2.1, this implies that $\widehat{y} \subseteq \widehat{x}$, implying that $z \in \widehat{x}$, implying that $x \lhd z$. (QOS.2) This is obvious as $x \in \widehat{x}$, implying $x \lhd x$. ⊠

We now define two order-topological notions in terms of « topo-reachability » .

Definition 4.2.3 (Topo-root) **:** Given $x \in \mathbb{X} \in \mathbf{tES}$, we call x a *topo-root* if every point in $\mathbb{X}$ is topo-reachable from x. This is equivalent to the condition $(\forall y \in \mathbb{X})(x \lhd y)$ and the condition $\widehat{x} = \mathbb{X}$. If there exists such an x in $\mathbb{X}$, we call $\mathbb{X}$ *topo-rooted*. We denote the set of topo-roots of $\mathbb{X}$ by $\mathsf{ToRo}(\mathbb{X})$.

Definition 4.2.4 (Topo-connected) **:** Given $\mathbb{X} \in \mathbf{tES}$, we call $\mathbb{X}$ *topo-connected* if every point in $\mathbb{X}$ is a topo-root, i.e. $\mathsf{ToRo}(\mathbb{X}) = \mathbb{X}$.

Finally, we prove a lemma that will be critical in proving Theorem 7.3.1, a main result of the current text.

Lemma 4.2.2 : Given $\mathbb{X} \in \mathbf{tES}$, we have $-\mathsf{ToRo}(\mathbb{X}) \in \mathsf{ArcUp}(\mathbb{X})$.

Proof : We show that (1) $-\mathsf{ToRo}(\mathbb{X}) \in \mathsf{Up}(\mathbb{X})$ and (2) $-\mathsf{ToRo}(\mathbb{X})$ is archival.

(1) To show that $-\mathsf{ToRo}(\mathbb{X}) \in \mathsf{Up}(\mathbb{X})$, Fact 2.2.1 implies that it suffices to show that $\mathsf{ToRo}(\mathbb{X}) \in \mathsf{Down}(\mathbb{X})$. We take $w, x \in \mathbb{X}$ such that $w \leqslant x$ and $x \in \mathsf{ToRo}(\mathbb{X})$. Now since $x \in \mathsf{ToRo}(\mathbb{X})$, we have $\widehat{x} = \mathbb{X}$. Since $\widehat{w} \in \mathsf{Up}(\mathbb{X})$ and $w \leqslant x$, we have $x \in \widehat{w}$, implying, by Lemma 4.2.1, that $\widehat{x} \subseteq \widehat{w}$. So we can conclude that $\mathbb{X} = \widehat{x} \subseteq \widehat{w}$, implying that $\mathbb{X} = \widehat{w}$ and, therefore, $w \in \mathsf{ToRo}(\mathbb{X})$.

(2) Now to see that $-\mathsf{ToRo}(\mathbb{X})$ is archival, we take some $x, z \in \mathbb{X}$ such that $x \leqslant z$ and $x \notin -\mathsf{ToRo}(\mathbb{X}) \ni z$, implying $\widehat{x} = \mathbb{X} \supsetneq \widehat{z}$. Now since $\widehat{z} \subsetneq \mathbb{X}$, we know that $x \notin \widehat{z}$ (because $\mathbb{X}$ is the *smallest* closed archival upset containing x). So we have $x \leqslant z$ and $x \notin \widehat{z} \ni z$, implying that there exists some $y \in \widehat{z}$ such that $x \leqslant y \; R^{\rhd} \; z$. Now since $y \in \widehat{z}$, Lemma 4.2.1 implies that $\widehat{y} \subseteq \widehat{z} \subsetneq \mathbb{X}$, implying that $\widehat{y} \subsetneq \mathbb{X}$, finally implying that $y \notin \mathsf{ToRo}(\mathbb{X})$. So we have found our $y \in {\uparrow}x \cap R^{\lhd}[z] \cap -\mathsf{ToRo}(\mathbb{X})$ as desired. ⊠

4.3 Z-reachability

Here we define a completely frame-theoretic alternative notion of « reachability » on *finite* temporal Esakia spaces. Given Fact 2.6.1, we know that all finite temporal Esakia spaces are discrete, implying that their topological structure is trivial. This means that throughout this section the reader can essentially imagine that we are simply studying finite temporal transits.

Definition 4.3.1 : Given $w, x \in \mathbb{X} \in \mathbf{tES}_{\mathsf{fin}}$ we define the following binary relation $B \subseteq \mathbb{X}^2$.

$$x\ B\ w \quad :\iff \quad w \leqslant x \text{ and } (w, x] \cap \mathsf{Refl}(\mathbb{X}) = \varnothing.$$

The relation B formalises the idea of moving « backward » until encountering either a reflexive or minimal point. Note that $x\ B\ x$ because $x \leqslant x$ and $(x, x] = \varnothing$.

We also define the following relations for all $n \in \mathbb{N}$.

$$\begin{aligned} Z_0 &:= \Delta_{\mathbb{X}} \\ Z_{n+1} &:= Z_n; B; \leqslant \\ Z &:= \bigcup_{m \in \mathbb{N}} Z_m \end{aligned}$$

The relation Z formalises the idea of « zig-zagging » down via B and back up via $\leqslant$ a finite number of times. (This is reminiscent of the zig-zagging relation that can be used to characterise simple and subdirectly-irreducible *bi-Heyting algebras* [7, Corollary 3.4].)

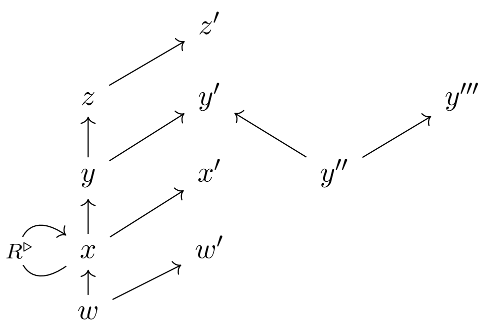
Note that, given $S \subseteq \mathbb{X}$, we have $S \subseteq Z[S]$ because given $x \in S$, we have $x\ Z_0\ x$, implying $x \in Z[S]$. We also have $\uparrow S \subseteq Z[S]$ because given $x \in S$ and $x \leqslant y$, we have $x\ B\ x \leqslant y$, implying $x\ Z_1\ y$, implying $y \in Z[S]$.

We now introduce some terminology defined in terms of our relation Z.

Definition 4.3.2 (Z-reachability) **:** Given $x, y \in \mathbb{X} \in \mathbf{tES}_{\mathsf{fin}}$, we say y *is Z-reachable from* x if $x\ Z\ y$.

The following example is included to provide readers with more intuition on Z-reachability.

Example 4.3.1 : Observe the following $\mathbb{X} \in \mathbf{tES}_{\mathsf{fin}}$.

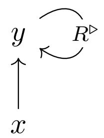


$$\begin{aligned} Z_0[z] &= \{z\} \\ B[Z_0[z]] &= \{x, y, z\} \\ Z_1[z] &= \{x, y, z, x', y', z'\} \\ B[Z_1[z]] &= \{x, y, z, x', y', z', y''\} \\ Z_2[z] &= \{x, y, z, x', y', z', y'', y'''\} \\ &\dots \\ Z[z] &= \{x, y, z, x', y', z', y'', y'''\} \end{aligned}$$

Here, everything is Z-reachable from z except for w and w'. This is because B allows us to descend from z to x, but does not allow us to descend to w because $x \in (w, z] \cap \mathsf{Refl}(\mathbb{X})$. Once we have descended as far as possible, we take upsets and repeat the process again.

We note that Z-reachability is *not* symmetric in general, as exemplified in Example 4.3.2.

Example 4.3.2 : Consider the following finite temporal Esakia space with two points x and y.



Here we have $x \; B \; x \leqslant y$, so we have $x \; Z \; y$. However, since $y \in (x, y] \cap \mathsf{Refl}(\mathbb{X})$, we do *not* have $y \; B \; x$, so we do *not* have $y \; Z \; x$.

We now define two more frame-theoretic notions in terms of « Z-reachability », mirroring Definitions 4.2.3 and 4.2.4.

Definition 4.3.3 (Z-root) **:** Given $x \in \mathbb{X} \in \mathbf{tES}_{\mathsf{fin}}$, we call x a *Z-root* if every point in $\mathbb{X}$ is Z-reachable from x. This is equivalent to the condition $(\forall y \in \mathbb{X})(x \; Z \; y)$ and the condition $Z[x] = \mathbb{X}$. If there exists such an x in $\mathbb{X}$, we call $\mathbb{X}$ *Z-rooted*.

Definition 4.3.4 (Z-connected) **:** Given $\mathbb{X} \in \mathbf{tES}_{\mathsf{fin}}$, we call $\mathbb{X}$ *Z-connected* if every point in $\mathbb{X}$ is a Z-root.

Now we prove three lemmas relating Z-reachability to archival subsets.

Lemma 4.3.1 : Given $w, x \in \mathbb{X} \in \mathbf{tES}_{\mathsf{fin}}$ and $S \in \mathsf{Arc}(\mathbb{X})$,

$$x \in S \text{ and } x \; B \; w \quad \Longrightarrow \quad w \in S.$$

Proof : This will be proven via induction on the length of the longest chain in $(w, x]$, denoted by n. This chain must be finite because $\mathbb{X}$ is assumed to be finite. $(n = 0)$ This means that $w = x$, implying, since $x \in S$, that $w \in S$. $(n = m + 1)$ This means that we have a chain of unique elements $\{y_i\}_{i=1}^{m+1} \subseteq (w, x]$ such that

$$w < y_1 < \cdots < y_m < y_{m+1} = x.$$

Now since $(w, x] \cap \mathsf{Refl}(\mathbb{X}) = \varnothing$, we have $(y_1, x] \cap \mathsf{Refl}(\mathbb{X}) = \varnothing$. Combining this with the fact that $y_1 \leqslant x$, and that the length of the longest chain in $(y_1, x]$ is $\leqslant m$, we can apply our induction hypothesis, implying that $y_1 \in S$. Now assume, toward a contradiction, that w is *not* in S. This implies that $w \leqslant y_1$ and $w \notin S \ni y_1$, implying, since $S \in \mathsf{Arc}(\mathbb{X})$, that there exists some $y \in S$ such that $w \leqslant y \; R^{\triangleright} \; y_1$. Now we cannot have $y = y_1$ because then $y_1 \; R^{\triangleright} \; y_1$, contradicting $y_1 \notin \mathsf{Refl}(\mathbb{X})$. We also cannot have $w = y$ because $w \notin S \ni y$. So it must be the case that $w < y < y_1$. But observe that this contradicts the fact that $\{y_i\}_{i=1}^{m+1}$ was taken to be the longest chain in $(w, x]$. ☒

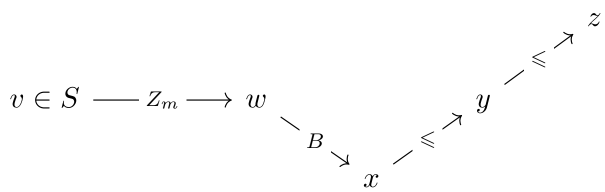
Lemma 4.3.2 : Given $x, y \in \mathbb{X} \in \mathbf{tES}_{\mathsf{fin}}$ and $U \in \mathsf{ArcUp}(\mathbb{X})$,

$$x \in U \text{ and } x \; Z \; y \quad \Longrightarrow \quad y \in U.$$

Proof : Since $x \; Z \; y$, we have $x \; Z_n \; y$ for some $n \in \mathbb{N}$. We proceed via induction on n. $(n = 0)$ Given $x \, Z_0 \, y$, we have $x = y$, implying that $y \in U$. $(n = m+1)$ Given $x \, Z_{m+1} \, y$, we have $x \; Z_m \; v \; B \; w \leqslant y$ for some $v, w \in \mathbb{X}$. By induction hypothesis, we have $v \in U$. Since $v \; B \; w$ and $v \in U \in \mathsf{ArcUp}(\mathbb{X})$, we know, by Lemma 4.3.1, that $w \in U$. And since $w \leqslant y$ and $w \in U \in \mathsf{Up}(\mathbb{X})$, we can conclude that $y \in U$. ☒

Lemma 4.3.3 : Given $S \subseteq \mathbb{X} \in \mathbf{tES}_{\mathsf{fin}}$, we have $Z[S] \in \mathsf{ArcUp}(\mathbb{X})$.

Proof : To see that $Z[S] \in \mathsf{Up}(\mathbb{X})$, take some $y \in Z[S]$ such that $y \leqslant z$. Now since $y \in Z[S]$, we have $v \in S$ such that $v \; Z_n \; y$ for some $n \in \mathbb{N}$. If $n = 0$, then $v \, Z_0 \, y \, B \, y \leqslant z$, implying that $v \, Z_1 \, z$, further implying that $z \in Z[S]$, so we consider the inductive case where $n = m+1$ for some $m \in \mathbb{N}$. This implies that $v \, Z_m \, w \, B \, x \leqslant y$ for some $w, x \in \mathbb{X}$. But, since $y \leqslant z$, we have $v \; Z_m \; w \; B \; x \leqslant z$, implying that $v \; Z_{m+1} \; z$, finally implying that $z \in Z[S]$.



Now, to see that $Z[S]$ is archival, we let $x \leqslant z$ and $x \notin Z[S] \ni z$. Recall Proposition 4.1.1 implies that it suffices to show that $\uparrow x \cap \downarrow z \cap \mathsf{Refl}(\mathbb{X}) \cap Z[S] \neq \varnothing$. Now since $x \notin Z[S]$, it cannot be the case that $z \; B \; x$ as we would then have $z \; B \; x \leqslant x$, implying $x \in Z[S]$. Since $x \leqslant z$ and $z \not B \, x$, it must be the case that $(x, z] \cap \mathsf{Refl}(\mathbb{X}) \neq \varnothing$. Since $\mathbb{X}$ is finite, let y be some *maximal* point in $(x, z] \cap \mathsf{Refl}(\mathbb{X})$. This implies that $y \leqslant z$ and $(y, z] \cap \mathsf{Refl}(\mathbb{X}) = \varnothing$, implying that $z \; B \; y$, implying, by Lemma 4.3.1, that $y \in Z[S]$. So we have found our point $y \in \uparrow x \cap \downarrow z \cap \mathsf{Refl}(\mathbb{X}) \cap Z[S]$. ☒

4.4 Coincidence of notions of reachability

In this section, we establish that the order-topological notion of « topo-reachability » and the frame-theoretic notion of « Z-reachability » coincide in the finite case. This implies that in the finite case, all the universal-algebraic theory that will be expressible in terms of « topo-reachability » will also be expressible using only the frame-theoretic notion of Z-reachability.

Theorem 4.4.1 : Given $x, y \in \mathbb{X} \in \mathbf{tES}_{\mathsf{fin}}$,

$$y \text{ is topo-reachable from } x \iff y \text{ is } Z\text{-reachable from } x$$

i.e.

$$x \trianglelefteq y \iff x\, Z\, y.$$

Proof : ($\Rightarrow$) Since y is topo-reachable from x, we have $y \in \widehat{x}$. Now, by Lemma 4.3.3, we know that $Z[x] \in \mathsf{ArcUp}(\mathbb{X})$. Since $\mathbb{X}$ is finite, Fact 2.6.2 implies that $\mathbb{X}$ is discrete, so $Z[x] \in \overline{\Omega}$, implying $Z[x] \in \mathsf{ClArcUp}(\mathbb{X})$. Finally, since $\widehat{x}$ is the *smallest* closed archival upset containing x (Lemma 4.2.1), we have $\widehat{x} \subseteq Z[x]$, implying $y \in Z[x]$, finally implying that y is Z-reachable from x. ($\Leftarrow$) Since y is Z-reachable from x, we have $x\, Z\, y$. By Lemma 4.3.2, $x \in \widehat{x} \in \mathsf{ArcUp}(\mathbb{X})$ and $x\, Z\, y$ imply $y \in \widehat{x}$, so we can conclude that y is topo-reachable from x. ⊠

We now establish the coincidence of the notion of a « topo-root » and a « Z-root » as well as the notion of « topo-connectedness » and « Z-connectedness ».

Corollary 4.4.1 : Given $x \in \mathbb{X} \in \mathbf{tES}_{\mathsf{fin}}$,

$$x \text{ is a topo-root} \iff x \text{ is a } Z\text{-root.}$$

Proof : x is a topo-root if and only if $(\forall y \in \mathbb{X})(y$ is topo-reachable from $x)$. By Theorem 4.4.1, this is the case if and only if $(\forall y \in \mathbb{X})(y$ is Z-reachable from $x)$, which is the case if and only if x is a Z-root. ⊠

Corollary 4.4.2 : Given $\mathbb{X} \in \mathbf{tES}_{\mathsf{fin}}$,

$$\mathbb{X} \text{ is a topo-connected} \iff \mathbb{X} \text{ is a } Z\text{-connected.}$$

Proof : $\mathbb{X}$ is topo-connected if and only if $(\forall x \in \mathbb{X})(x$ is a topo-root$)$. By Corollary 4.4.1, this is the case if and only if $(\forall x \in \mathbb{X})(x$ is a Z-root$)$, which is the case if and only if $\mathbb{X}$ is Z-connected. ⊠

Finally, we observe that Example 4.3.2 applies to $\trianglelefteq$ and Fact 4.2.1 applies to Z.

Corollary 4.4.3 : Given $\mathbb{X} \in \mathbf{tES}$,

- $\langle \mathbb{X}, \trianglelefteq \rangle$ is *not* symmetric in general.

- If $\mathbb{X}$ is finite, then $\langle \mathbb{X}, Z \rangle \in \mathbf{QOS}$.

Chapter conclusion

Having gained a better understanding of the structure of temporal Esakia spaces, we apply the theory developed in Chapter 3 as well as the current chapter to develop a duality theory between the categories $\mathbf{tHA}$ and $\mathbf{tES}$.

Chapter 5

Duality for tHA $\leftrightarrows$ tES

In this chapter, we establish and study a duality theory between the categories **tHA** and **tES**. Establishing such a duality in the context of superintuitionistic logics has proven to be extremely valuable as, given an algebraic soundness and completeness theorem, it allows us to study the associated logic (in this case **tHC**) from both algebraic and order-topological perspectives. Important examples of such dualities include the already-mentioned **HA** $\leftrightarrows$ **ES** [25] and **fHA** $\leftrightarrows$ **fES** [15] as well as **BDL** with Priestley spaces [40], Boolean algebras with Stone spaces [43], modal algebras with modal spaces [4], and tense algebras and tense spaces [45]. We provide the following section-by-section outline.

§5.1 We establish a full contravariant equivalence between the categories **tHA** $\leftrightarrows$ **tES**.

§5.2 We establish a correspondence between $\blacklozenge$-filters on **tHA** and closed archival upsets on **tES**. We connect this to previous results about congruences in §3.2.

§5.3 We give a dual characterisation of injective and surjective **tHA**-morphisms.

5.1 Contravariant equivalence

We begin by augmenting our functors $\circ_* : \mathbf{fHA} \to \mathbf{fES}$ and $\circ^* : \mathbf{fES} \to \mathbf{fHA}$ (as defined in Construction 2.7.1) so that they map to and from the desired categories.

Construction 5.1.1 : We extend the functors $\circ_* : \mathbf{fHA} \to \mathbf{fES}$ and $\circ^* : \mathbf{fES} \to \mathbf{fHA}$ to the categories **tHA** $\leftrightarrows$ **tES** as follows.

For **tHA** $\to$ **tES**, the object $\langle A, \wedge, \vee, \to, \blacklozenge, \Box, 0, 1\rangle$ maps to $\langle X, R^{\lhd}, R^{\rhd}, \Omega\rangle$ where

$$\langle X, R^{\rhd}, \Omega\rangle := \langle A, \wedge, \vee, \to, \Box, 0, 1\rangle_*$$

and for all $w, x \in X$,

$$x \, R^{\triangleleft} \, w \quad :\Longleftrightarrow \quad (\forall a \in A)(a \in w \Rightarrow \blacklozenge a \in x).$$

For **tES** $\to$ **tHA**, the object $\langle X, R^{\triangleleft}, R^{\triangleright}, \Omega \rangle$ maps to $\langle A, \wedge, \vee, \to, \blacklozenge, \Box, 0, 1 \rangle$ where

$$\langle A, \wedge, \vee, \to, \Box, 0, 1 \rangle := \langle X, R^{\triangleright}, \Omega \rangle^*$$

and for all $K \in A$,

$$\blacklozenge K := R^{\triangleright}[K] = \{x \in X \mid R^{\triangleleft}[x] \cap K \neq \varnothing\}.$$

The well-definedness of this extended functor will be shown in Lemmas 5.1.1 through 5.1.4, culminating in Corollary 5.1.1 which establishes a contravariant equivalence between the categories. Note that, when this well-definedness is established, it will be the case that $\circ_+$ now maps **tHA** $\to$ **tTran**.

The following lemma establishes the well-definedness of $\circ_* :$ **tHA** $\to$ **tES** on objects.

Lemma 5.1.1 : Given $\mathbb{A} \in$ **tHA**, we have $\mathbb{A}_* \in$ **tES**.

Proof : Let $\mathbb{X} := \langle X, R^{\triangleleft}, R^{\triangleright}, \Omega \rangle := \mathbb{A}_*$. Since it was established in [15] that $\langle X, R^{\triangleright}, \Omega \rangle \in$ **fES**, it remains only to show that $\mathbb{X}$ satisfies (tTran.2), (tES.o.3), and (tES.o.4).

(tTran.2) Assuming $x \, R^{\triangleright} \, y$, we show $y \, R^{\triangleleft} \, x$. Given $a \in x$, (tHA.o.4) implies that $\Box\blacklozenge a \in x$, implying that $\blacklozenge a \in y$. Now, assuming $y \, R^{\triangleleft} \, x$, we show $x \, R^{\triangleright} \, y$. Given $\Box a \in x$, we have $\blacklozenge\Box a \in y$, implying, by (tHA.o.5), that $a \in y$.

(tES.o.3) Given $K \in \mathsf{ClopUp}(\mathbb{X})$, we know, by Fact 2.7.1, that there is some $a \in \mathbb{A}$ such that $K = \pi a$. We claim that $R^{\triangleright}[K] = \pi\blacklozenge a$. Note that if this is the case, then $R^{\triangleright}[K] \in \mathsf{ClopUp}(\mathbb{X})$ as desired. ($\subseteq$) Given $x \in R^{\triangleright}[K]$, we have $w \in K$ and $w \, R^{\triangleright} \, x$, implying that $x \, R^{\triangleleft} \, w$. Since $w \in \pi a$, we have $a \in w$, implying $\blacklozenge a \in x$, which implies that $x \in \pi\blacklozenge a$. ($\supseteq$) Given $x \in \pi\blacklozenge a$, we find a prime filter w such that $w \in \pi a = K$ and $w \, R^{\triangleright} \, x$. Consider the filter $\uparrow a$ and the ideal $\blacklozenge^{-1}[-x]$. Assume, toward a contradiction, that these sets are *not* disjoint. Then there is some $b \in \uparrow a \cap \blacklozenge^{-1}[-x]$, implying that $a \leqslant b$ and $\blacklozenge b \in -x$. Since $a \leqslant b$, we have $\blacklozenge a \leqslant \blacklozenge b$, so, since $-x$ is a downset, we have $\blacklozenge a \in -x$, implying $\blacklozenge a \notin x$, contradicting our assumption that $x \in \pi\blacklozenge a$. So, since $\uparrow a \cap \blacklozenge^{-1}[-x] = \varnothing$, we know, by the PFT, that there exists some $w \in \mathsf{PrFilt}(\mathbb{A})$ such that $\uparrow a \subseteq w$ and $w \cap \blacklozenge^{-1}[-x] = \varnothing$. Since $\uparrow a \subseteq w$, we have $w \in \pi a = K$. To see that $x \, R^{\triangleleft} \, w$, observe that given $\blacklozenge b \notin x$, we have $b \in \blacklozenge^{-1}[-x]$, implying $b \notin w$. So $w \, R^{\triangleright} \, x$ and $w \in K$, implying $x \in R^{\triangleright}[K]$.

(tES.o.4) Given $x \in \mathbb{X}$, we claim that $R^{\triangleleft}[x] = -\bigcup \pi[\blacklozenge^{-1}[-x]]$. Note that if this is the case, then since $\blacklozenge^{-1}[-x] \in \mathsf{Ideal}(\mathbb{A})$ (Fact 2.5.11), Fact 2.7.3 will imply that $\bigcup \pi[\blacklozenge^{-1}[-x]] \in \mathsf{OpUp}(\mathbb{X})$, implying that $R^{\lhd}[x] = -\bigcup \pi[\blacklozenge^{-1}[-x]] \in \mathsf{ClDown}(\mathbb{X})$.

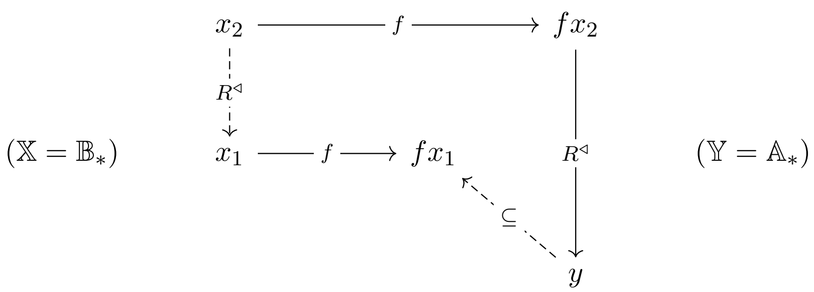
$$\begin{aligned} w \in R^{\lhd}[x] &\iff x\, R^{\lhd}\, w \\ &\iff (\forall a \in \mathbb{A})(a \in w \Rightarrow \blacklozenge a \in x) \\ &\iff (\neg\exists a \in \mathbb{A})(a \in w \text{ and } \blacklozenge a \in -x) \\ &\iff (\neg\exists a \in \mathbb{A})(a \in w \text{ and } a \in \blacklozenge^{-1}[-x]) \\ &\iff w \cap \blacklozenge^{-1}[-x] = \varnothing \\ &\iff w \notin \bigcup \pi[\blacklozenge^{-1}[-x]] \\ &\iff w \in -\bigcup \pi[\blacklozenge^{-1}[-x]] \end{aligned}$$

⊠

The following lemma establishes the well-definedness of $\circ_* : \mathbf{tHA} \to \mathbf{tES}$ on morphisms.

Lemma 5.1.2 : Given a **tHA**-morphism $h : \mathbb{A} \to \mathbb{B}$, the map $h_* : \mathbb{B}_* \to \mathbb{A}_*$ is a **tES**-morphism.

Proof : Let $\mathbb{X} := \mathbb{B}_*$ and $\mathbb{Y} := \mathbb{A}_*$ and $f := h_* : \mathbb{X} \to \mathbb{Y}$. Since (tES.m.1) was shown in [15], we again focus only on (tES.m.2). Letting $y \in \mathbb{Y}$ and $x_2 \in \mathbb{X}$ be such that $fx_2\, R^{\lhd}\, y$, we show that there exists an $x_1 \in \mathbb{X}$ such that $x_2\, R^{\lhd}\, x_1$ and $y \subseteq fx_1$.



Consider the filter $\uparrow h[y]$ and the ideal $\blacklozenge^{-1}[-x_2]$. Assume, toward a contradiction, that these sets are *not* disjoint. Then there is some $b \in \uparrow h[y] \cap \blacklozenge^{-1}[-x_2]$, implying that $a \in y$ and $ha \leqslant b$ and $\blacklozenge b \notin x_2$. Now since $ha \leqslant b$, we have $h\blacklozenge a = \blacklozenge ha \leqslant \blacklozenge b$ (using that fact that h is homomorphic over $\blacklozenge$). Also, since $a \in y$ and $fx_2\, R^{\lhd}\, y$, we have $\blacklozenge a \in fx_2 = h^{-1}[x_2]$, implying that $h\blacklozenge a \in x_2$. But this implies that $\blacklozenge b \in x_2$, contradicting our assumption that $\blacklozenge b \notin x_2$. So, since $\uparrow h[y] \cap \blacklozenge^{-1}[-x_2] = \varnothing$, we know, by the PFT, that there exists some $x_1 \in \mathbb{B}_*$ such that $\uparrow h[y] \subseteq x_1$ and $x_1 \cap \blacklozenge^{-1}[-x_2] = \varnothing$. Now we claim that $x_2\, R^{\lhd}\, x_1$ and $y \subseteq fx_1$. To see that $x_2\, R^{\lhd}\, x_1$, consider some $\blacklozenge b \notin x_2$ and observe that $b \in \blacklozenge^{-1}[-x_2]$, implying that $b \notin x_1$. To see the $y \subseteq fx_1$, consider some $a \in y$ and observe that $ha \in h[y] \subseteq \uparrow h[y] \subseteq x_1$, implying $a \in h^{-1}[x_1] = fx_1$. ⊠

The following lemma establishes the well-definedness of $\circ^* : \mathbf{tES} \to \mathbf{tHA}$ on objects.

Lemma 5.1.3 : Given $\mathbb{X} \in \mathbf{tES}$, we have $\mathbb{X}^* \in \mathbf{tHA}$.

Proof : Let $\mathbb{A} := \langle A, \cap, \cup, \to, \blacklozenge, \Box, \varnothing, X \rangle := \mathbb{X}^*$. Since it was established in [15] that $\langle A, \cap, \cup, \to, \Box, \varnothing, X \rangle \in \mathbf{fHA}$, it remains only to show that $\mathbb{A}$ satisfies (tHA.o.2) through (tHA.o.5).

To address well-definedness, note that $K \in \mathbb{X}^*$ implies $\blacklozenge K = R^{\triangleright}[K] \in \mathbb{X}^*$, implying that $\blacklozenge : \mathbb{X}^* \to \mathbb{X}^*$.

(tHA.o.2) $\blacklozenge\varnothing = \{x \in \mathsf{ClopUp}(\mathbb{X}) \mid R^{\triangleleft}[x] \cap \varnothing \neq \varnothing\} = \varnothing$.

(tHA.o.3) Given $K_1, K_2 \in \mathbb{A}$,

$$\begin{aligned}
\blacklozenge(K_1 \cup K_2) &= \{x \in \mathsf{ClopUp}(\mathbb{X}) \mid R^{\triangleleft}[x] \cap (K_1 \cup K_2) \neq \varnothing\} \\
&= \{x \in \mathsf{ClopUp}(\mathbb{X}) \mid (R^{\triangleleft}[x] \cap K_1) \cup (R^{\triangleleft}[x] \cap K_2) \neq \varnothing\} \\
&= \{x \in \mathsf{ClopUp}(\mathbb{X}) \mid R^{\triangleleft}[x] \cap K_1 \neq \varnothing \text{ or } R^{\triangleleft}[x] \cap K_2 \neq \varnothing\} \\
&= \{x \in \mathsf{ClopUp}(\mathbb{X}) \mid R^{\triangleleft}[x] \cap K_1 \neq \varnothing\} \\
&\quad \cup \{x \in \mathsf{ClopUp}(\mathbb{X}) \mid R^{\triangleleft}[x] \cap K_2 \neq \varnothing\} \\
&= \blacklozenge K_1 \cup \blacklozenge K_2.
\end{aligned}$$

(tHA.o.4) Given $K \in \mathbb{A}$, we show that $K \subseteq \Box\blacklozenge K$ via the contrapositive. Given $x \notin \Box\blacklozenge K$, we have $R^{\triangleright}[x] \not\subseteq \blacklozenge K$, implying $x\, R^{\triangleright}\, y$, (and, therefore $y\, R^{\triangleleft}\, x$,) as well as $y \notin \blacklozenge K$. Since $y \notin \blacklozenge K$, we have $R^{\triangleleft}[y] \cap K = \varnothing$, implying, since $x \in R^{\triangleleft}[y]$, that $x \notin K$.

(tHA.o.5) Given $K \in \mathbb{A}$, we show that $\blacklozenge\Box K \subseteq K$. Given $x \in \blacklozenge\Box K$, we have $R^{\triangleleft}[x] \cap \Box K \neq \varnothing$, implying $x\, R^{\triangleleft}\, w$, (and, therefore $w\, R^{\triangleright}\, x$,) as well as $w \in \Box K$. Since $w \in \Box K$, we have $R^{\triangleright}[w] \subseteq K$, implying, since $x \in R^{\triangleright}[w]$, that $x \in K$. ⊠

The following lemma establishes the well-definedness of $\circ^* : \mathbf{tES} \to \mathbf{tHA}$ on morphisms.

Lemma 5.1.4 : Given a $\mathbf{tES}$-morphism $f : \mathbb{X} \to \mathbb{Y}$, the map $f^* : \mathbb{Y}^* \to \mathbb{X}^*$ is a $\mathbf{tHA}$-morphism.

Proof : Let $\mathbb{A} := \mathbb{Y}^*$ and $\mathbb{B} := \mathbb{X}^*$ and $h := f^* : \mathbb{A} \to \mathbb{B}$. Since the fact that h is a $\mathbf{fHA}$-morphism was shown in [15], we focus only on showing h to be homomorphic over $\blacklozenge$. Letting $K \in \mathbb{A}$, we show that $h\blacklozenge K = \blacklozenge hK$.

($\subseteq$) Let $x_2 \in h\blacklozenge K = f^{-1}[\blacklozenge K]$, implying that $fx_2 \in \blacklozenge K$. This implies there is some $y \in \mathbb{Y}$ such that $y \in R^{\triangleleft}[fx_2] \cap K$, implying that $fx_2\, R^{\triangleleft}\, y$. Now since f is a

tES-morphism, there exists some $x_1 \in \mathbb{X}$ such that $x_2 \, R^{\lhd} \, x_1$ and $y \leqslant fx_1$. Also, since $y \leqslant fx_1$ and $y \in K$, we have $fx_1 \in K$, implying $x_1 \in f^{-1}[K] = hK$. So since $x_1 \in R^{\lhd}[x_2] \cap hK$, we have $x_2 \in \blacklozenge hK$.

($\supseteq$) Given $x_2 \in \blacklozenge hK$, we have $R^{\lhd}[x_2] \cap hK \neq \varnothing$, implying $x_2 \, R^{\lhd} \, x_1$ (equiv. $x_1 \, R^{\rhd} \, x_2$) and $x_1 \in hK = f^{-1}[K]$, implying $fx_1 \in K$. Since $x_1 \, R^{\rhd} \, x_2$, (fES.m.2) implies that $fx_1 \, R^{\rhd} \, fx_2$, implying $fx_1 \in R^{\lhd}[fx_2] \cap K$. This implies that $fx_2 \in \blacklozenge K$, finally implying that $x_2 \in f^{-1}[\blacklozenge K] = h\blacklozenge K$. ⊠

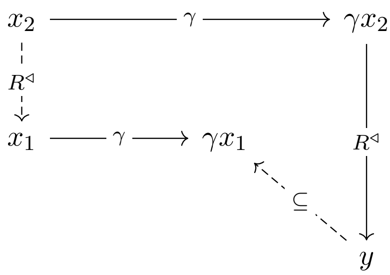
Lemmas 5.1.5 and 5.1.6 establish the fact that π and γ can still serve as our natural isomorphisms in the categories **tHA** and **tES** respectively.

Lemma 5.1.5 : Given $\mathbb{A} \in$ **tHA**, the map $\pi : \mathbb{A} \to \mathbb{A}_*^*$ is a **tHA**-morphism.

Proof : Since [15] establishes that π is a **fHA**-morphism, all the remains to show is that for all $a \in \mathbb{A}$, we have $\pi\blacklozenge a = \blacklozenge\pi a$. ($\supseteq$) Given $x \in \blacklozenge\pi a$, we have $R^{\lhd}[x] \cap \pi a \neq \varnothing$, so we let $x \, R^{\lhd} \, w$ and $w \in \pi a$. This implies that $a \in w$ implying that $\blacklozenge a \in x$, finally implying that $x \in \pi\blacklozenge a$. ($\subseteq$) Given $x \in \pi\blacklozenge a$, we have $\blacklozenge a \in x$. Now, by Fact 2.5.11, we have $\blacklozenge^{-1}[-x] \in \mathsf{Ideal}(\mathbb{A})$. Now we claim that $\uparrow a \cap \blacklozenge^{-1}[-x] = \varnothing$. If this were not the case, then we would have $a \leqslant b$ and $\blacklozenge b \notin x$. But by Fact 2.5.8, this would imply that $\blacklozenge a \leqslant \blacklozenge b$, implying that $\blacklozenge b \in x$, giving us a contradiction. Since $\uparrow a \cap \blacklozenge^{-1}[-x] = \varnothing$, the PFT implies that there exists some $w \in \mathbb{A}_*$ such that $\uparrow a \subseteq w$ and $w \cap \blacklozenge^{-1}[-x] = \varnothing$. Now we claim that $w \in R^{\lhd}[x] \cap \pi a$. Since $\uparrow a \subseteq w$, we have $a \in w$, implying $w \in \pi a$. And given $b \in w$, we have $b \notin \blacklozenge^{-1}[-x]$, implying $\blacklozenge b \notin -x$, implying $\blacklozenge b \in x$. This shows that $x \, R^{\lhd} \, w$. So since $R^{\lhd}[x] \cap \pi a \neq \varnothing$, we have $x \in \blacklozenge\pi a$ as desired. ⊠

Lemma 5.1.6 : Given $\mathbb{X} \in$ **tES**, the map $\gamma : \mathbb{X} \to \mathbb{X}_*^*$ is a **tES**-morphism.

Proof : Since [15] establishes that γ is a **fES**-morphism, all the remains to show is (tES.m.2), so we let $x_2 \in \mathbb{X}$ and $y \in \mathbb{X}_*^*$ and $\gamma x_2 \, R^{\lhd} \, y$ and find an $x_1 \in \mathbb{X}$ satisfying the following.



Assume, toward a contradiction, that no such x_1 exists, i.e.

$$(\neg\exists x_1 \in \mathbb{X}) \, (x_1 \in R^{\lhd}[x_2] \text{ and } y \subseteq \gamma x_1) \, .$$

It can be easily checked that $y \subseteq \gamma x_1 \Leftrightarrow x_1 \in \bigcap y$, so we have

$$(\neg\exists x_1 \in \mathbb{X})(x_1 \in R^{\triangleleft}[x_2] \text{ and } x_1 \in \bigcap y),$$

implying that $R^{\triangleleft}[x_2] \cap \bigcap y = \varnothing$. Since (tES.o.4) implies that $R^{\triangleleft}[x]$ is closed and y consists of clopen, and, therefore, *closed* subsets of $\mathbb{X}$, compactness implies that there is a finite subfamily $\{K_i\}_{i=1}^{n} \subseteq y$ such that

$$R^{\triangleleft}[x_2] \cap K_1 \cap \ldots \cap K_n = \varnothing.$$

Since $y \in \mathsf{PrFilt}(\mathbb{X}^*)$, we know that y is closed under finite meets, implying $K_\star \in y$ where $K_\star = K_1 \cap \ldots \cap K_n$. So we have $R^{\triangleleft}[x_2] \cap K_\star = \varnothing$, implying that $x_2 \notin \blacklozenge K_\star$, implying that $\blacklozenge K_\star \notin \gamma x_2$. But this gives us a contradiction, as it was assumed that $\gamma x_2 \, R^{\triangleleft} \, y$, but we have $K_\star \in y$ and $\blacklozenge K_\star \notin \gamma x_2$. ⊠

Combining Lemmas 5.1.1 through 5.1.6, we can state the following theorem.

Theorem 5.1.1 : The extended functors $◌_* : \mathbf{tHA} \to \mathbf{tES}$ and $◌^* : \mathbf{tES} \to \mathbf{tHA}$ are pseudo-inverse.

Proof : Having shown these functors to be well-defined (Lemmas 5.1.1 through 5.1.4), and that π is a $\mathbf{tHA}$-morphism and γ a $\mathbf{tES}$-morphism (Lemma 5.1.5 and Lemma 5.1.6 respectively), one need only reference the well-known fact that $\pi_{\mathbb{A}} : \mathbb{A} \to \mathbb{A}_*^*$ and $\gamma_{\mathbb{X}} : \mathbb{X} \to \mathbb{X}_*^*$ are bijections (and, therefore, in this context, isomorphisms) and that for all $\mathbb{A} \in \mathbf{tHA}$ and $\mathbb{X} \in \mathbf{tES}$, the following squares commute.

$$(\mathbf{tHA}) \quad \begin{array}{ccc} \mathbb{A} & \overset{\pi_{\mathbb{A}}}{\rightarrowtail\!\!\!\twoheadrightarrow} & \mathbb{A}_*^* \\ \downarrow h & & \downarrow h_*^* \\ \mathbb{B} & \overset{\pi_{\mathbb{B}}}{\rightarrowtail\!\!\!\twoheadrightarrow} & \mathbb{B}_*^* \end{array} \qquad \begin{array}{ccc} \mathbb{X} & \overset{\gamma_{\mathbb{X}}}{\rightarrowtail\!\!\!\twoheadrightarrow} & \mathbb{X}_*^* \\ \downarrow f & & \downarrow f_*^* \\ \mathbb{Y} & \overset{\gamma_{\mathbb{Y}}}{\rightarrowtail\!\!\!\twoheadrightarrow} & \mathbb{Y}_*^* \end{array} \quad (\mathbf{tES})$$

⊠

Finally, in light of Theorem 5.1.1, we can establish our duality.

Corollary 5.1.1 : The categories $\mathbf{tHA} \leftrightarrows \mathbf{tES}$ are contravariantly equivalent.

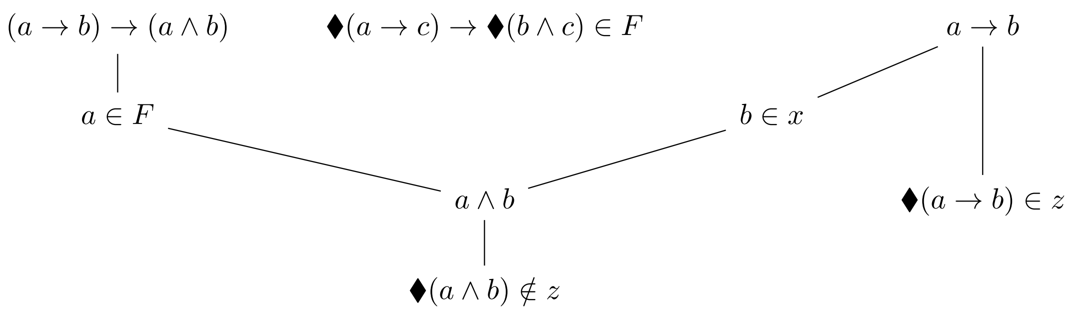
5.2 $\blacklozenge$-filters and closed archival upsets

This section extends the filter/closed-upset correspondence that is present in **HA** and its subvarieties (as discussed in §2.7 and formally established in Corollary 2.7.1) to the categories $\mathbf{tHA} \leftrightarrows \mathbf{tES}$. On these categories the correspondence will be shown to be between the $\blacklozenge$-filters of §3.2 and the closed archival upsets of §4.1.

We first establish the well-definedness of $\bigcap \pi[\circ]$ from the set of $\blacklozenge$-filters on a temporal Heyting algebra to the set of closed archival upsets on its dual space.

Lemma 5.2.1 : Given $\mathbb{A} \in \mathbf{tHA}$ and $F \in \blacklozenge\mathsf{Filt}(\mathbb{A})$, we have $\bigcap \pi[F] \in \mathsf{ClArcUp}(\mathbb{A}_*)$.

Proof : Let $\mathbb{X} := \mathbb{A}_*$. Since Corollary 2.7.1 implies that $\bigcap \pi[F] \in \mathsf{ClUp}(\mathbb{X})$, it remains only to show that $\bigcap \pi[F]$ is archival. Given $x, z \in \mathbb{X}$ such that $x \subseteq z$ and $x \notin \bigcap \pi[F] \ni z$, we show that there exists some $y \in \bigcap \pi[F]$ such that $x \subseteq y \, R^{\triangleright} \, z$. First note that since $x \notin \bigcap \pi[F] \ni z$, we have $x \neq z$, implying that $x \, R^{\triangleright} \, z$. Now consider the filter $[F \cup x)$ and the ideal $\blacklozenge^{-1}[-z]$. Assume, toward a contradiction, that these sets are *not* disjoint. This is the case if and only if there exists some $a \in F$ and some $b \in x$ such that $a \wedge b \in \blacklozenge^{-1}[-z]$, implying that $\blacklozenge(a \wedge b) \notin z$. Now since $b \in x$, we have $a \to b \in x$, implying, since $x \, R^{\triangleright} \, z$, that $\blacklozenge(a \to b) \in z$. Also, since $a \wedge (a \to b) = a \wedge b$, we have, by weakening, that $a \wedge (a \to b) \leqslant a \wedge b$, implying that $a \leqslant (a \to b) \to (a \wedge b)$, implying that $(a \to b) \to (a \wedge b) \in F$. Since $(a \to b) \to (a \wedge b) \in F \in \blacklozenge\mathsf{Filt}(\mathbb{A})$, we have $\blacklozenge(a \to b) \to \blacklozenge(a \wedge b) \in F \subseteq z$, implying, since $\blacklozenge(a \to b) \in z$, that $\blacklozenge(a \wedge b) \in z$, contradicting our assumption that $\blacklozenge(a \wedge b) \notin z$.



So the filter $[F \cup x)$ and the ideal $\blacklozenge^{-1}[-z]$ are indeed disjoint, implying, by the PFT, there exists some $y \in \mathbb{X}$ such that $[F \cup x) \subseteq y$ and $y \cap \blacklozenge^{-1}[-z] = \varnothing$. Now we claim that $y \in \bigcap \pi[F]$ and $x \subseteq y \, R^{\triangleright} \, z$. To see that $y \in \bigcap \pi[F]$ and $x \subseteq y$, simply observe that $F, x \subseteq [F \cup x) \subseteq y$. To see that the $y \, R^{\triangleright} \, z$, take some $\blacklozenge a \notin z$, implying that $a \in \blacklozenge^{-1}[-z]$. This implies that $z \notin y$, allowing us to conclude that $z \, R^{\triangleleft} \, y$, equiv. $y \, R^{\triangleright} \, z$. ⊠

Continuing, we establish the well-definedness of $\bigcap \gamma[\circ]$ from the set of closed archival upsets on a temporal Esakia space to the set of $\blacklozenge$-filters on its dual algebra.

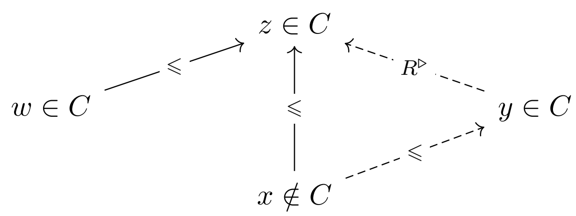
Lemma 5.2.2 : Given $\mathbb{X} \in \mathbf{tES}$ and $C \in \mathsf{ClArcUp}(\mathbb{X})$, we have $\bigcap \gamma[C] \in \blacklozenge\mathsf{Filt}(\mathbb{X}^*)$.

Proof : Let $\mathbb{A} := \mathbb{X}^*$. Since Corollary 2.7.1 implies that $\bigcap \gamma[C] \in \mathsf{Filt}(\mathbb{A})$, it remains only to show that $\bigcap \gamma[C]$ satisfies the additional condition. Assume, toward a contradiction, that $\bigcap \gamma[C]$ does *not* satisfy this condition. This implies there exists some $K_1, K_2 \in \mathbb{A}$ such that $K_1 \to K_2 \in \bigcap \gamma[C]$ but $\blacklozenge K_1 \to \blacklozenge K_2 \notin \bigcap \gamma[C]$, or, equivalently,

$C \subseteq K_1 \to K_2$ but $C \not\subseteq \blacklozenge K_1 \to \blacklozenge K_2$, implying there exists some $w \in \mathbb{X}$ such that

$$C \ni w \notin \blacklozenge K_1 \to \blacklozenge K_2 = -{\downarrow}(\blacklozenge K_1 \cap -\blacklozenge K_2).$$

Now since $w \in {\downarrow}(\blacklozenge K_1 \cap -\blacklozenge K_2)$, there exists some $z \in \mathbb{X}$ such that $w \leqslant z$ and $z \in \blacklozenge K_1$ and $z \notin \blacklozenge K_2$. And since $z \in \blacklozenge K_1$, there exists some $x \in \mathbb{X}$ such that $x \in K_1$ and $z\, R^{\triangleleft}\, x$ equiv. $x\, R^{\triangleright}\, z$. Also, since $z \notin \blacklozenge K_2$, we have $R^{\triangleleft}[z] \cap K_2 = \varnothing$, so $x \notin K_2$. Now since $x \leqslant x$ and $x \in K_1 \cap -K_2$, we have $x \in {\downarrow}(K_1 \cap -K_2)$, so $x \notin K_1 \to K_2$, implying $x \notin C$ (because $C \subseteq K_1 \to K_2$). Now since $x\, R^{\triangleright}\, z$, we have $x \leqslant z$, and since $w \leqslant z$ and $w \in C \in \mathsf{Up}(\mathbb{X})$, we have $z \in C$. So we have $x \leqslant z$ and $x \notin C \ni z$, implying, since C is archival, that there exists some $y \in C$ such that $x \leqslant y\, R^{\triangleright}\, z$.



However, since $y \geqslant x \in K_1 \in \mathsf{Up}(\mathbb{X})$, we have $y \in K_1$. And since $y \in R^{\triangleleft}[z]$ and $z \notin \blacklozenge K_2$, we have $y \notin K_2$. So $y \in K_1 \cap -K_2$, implying, since $y \leqslant y$, that $y \in {\downarrow}(K_1 \cap -K_2)$, further implying that $y \notin K_1 \to K_2$. But then $C \ni y \notin K_1 \to K_2$, implying that $C \not\subseteq K_1 \to K_2$, contradicting our assumption that $K_1 \to K_2 \in \bigcap \gamma[C]$. ⊠

We now state a result for the categories $\mathbf{tHA} \leftrightarrows \mathbf{tES}$ analogous to Corollary 2.7.1.

Theorem 5.2.1 : Given $\mathbb{A} \in \mathbf{tHA}$ and $\mathbb{X} \in \mathbf{tES}$,

$$\langle \blacklozenge\mathsf{Filt}(\mathbb{A}), \subseteq \rangle \cong^{\mathrm{POS}} \langle \mathsf{ClArcUp}(\mathbb{A}_*), \supseteq \rangle \qquad \langle \mathsf{ClArcUp}(\mathbb{X}), \subseteq \rangle \cong^{\mathrm{POS}} \langle \blacklozenge\mathsf{Filt}(\mathbb{X}^*), \supseteq \rangle.$$

Proof : This follows from Fact 2.7.2, Lemma 5.2.1, and Lemma 5.2.2. ⊠

We also state a result for the categories $\mathbf{tHA} \leftrightarrows \mathbf{tES}$ analogous to Corollary 2.7.2.

Theorem 5.2.2 : Given $\mathbb{A} \in \mathbf{tHA}$ and $\mathbb{X} \in \mathbf{tES}$,

$$\langle \mathsf{Cong}^{\mathrm{tHA}}(\mathbb{A}), \subseteq \rangle \cong^{\mathrm{POS}} \langle \mathsf{ClArcUp}(\mathbb{A}_*), \supseteq \rangle \qquad \langle \mathsf{ClArcUp}(\mathbb{X}), \subseteq \rangle \cong^{\mathrm{POS}} \langle \mathsf{Cong}^{\mathrm{tHA}}(\mathbb{X}^*), \supseteq \rangle.$$

Proof : This follows from Theorem 3.2.1 and Theorem 5.2.1. ⊠

Moving to the finite case, we state two propositions mirroring the two theorems above.

Proposition 5.2.1 : Given $\mathbb{A} \in \mathbf{tHA}_{\mathsf{fin}}$ and $\mathbb{X} \in \mathbf{tES}_{\mathsf{fin}}$,

$$\langle \blacklozenge \mathsf{Com}(\mathbb{A}), \leqslant \rangle \cong^{\mathrm{POS}} \langle \mathsf{ArcUp}(\mathbb{A}_*), \subseteq \rangle \qquad \langle \mathsf{ArcUp}(\mathbb{X}), \subseteq \rangle \cong^{\mathrm{POS}} \langle \blacklozenge \mathsf{Com}(\mathbb{X}^*), \subseteq \rangle.$$

Proof : This follows from Proposition 3.3.2, Theorem 5.2.1, and Fact 2.6.2. ☒

Proposition 5.2.2 : Given $\mathbb{A} \in \mathbf{tHA}_{\mathsf{fin}}$ and $\mathbb{X} \in \mathbf{tES}_{\mathsf{fin}}$,

$$\langle \mathsf{Cong}^{\mathrm{tHA}}(\mathbb{A}), \subseteq \rangle \cong^{\mathrm{POS}} \langle \mathsf{ArcUp}(\mathbb{A}_*), \supseteq \rangle \qquad \langle \mathsf{ArcUp}(\mathbb{X}), \subseteq \rangle \cong^{\mathrm{POS}} \langle \mathsf{Cong}^{\mathrm{tHA}}(\mathbb{X}^*), \supseteq \rangle.$$

Proof : This follows from Proposition 3.3.3 and Proposition 5.2.1. ☒

5.3 tHA-morphisms

To conclude the chapter, we characterise injective and surjective **tHA**-morphisms via their dual **tES**-morphisms. Since injective and surjective homomorphisms are well-understood and very prevalent in universal algebra, this provides us with a dual understanding of a great deal of the universal-algebraic theory of temporal Heyting algebras.

We begin with injective homomorphisms.

Theorem 5.3.1 : Given a **tHA**-morphism $h : \mathbb{A} \to \mathbb{B}$,

$$h \text{ injective} \iff h_* \text{ surjective}.$$

Proof : ($\Rightarrow$) To show that h_* is surjective, we take some $x \in \mathbb{A}_*$ and show that there exists some $y \in \mathbb{B}_*$ such that $h_* y = x$. Let $F := {\uparrow} h[x]$ and $I := {\downarrow} h[-x]$. We claim that $F \in \mathsf{Filt}(\mathbb{B})$ and $I \in \mathsf{Ideal}(\mathbb{B})$ and $F \cap I = \varnothing$. Clearly $F \in \mathsf{Up}(\mathbb{A})$ and $I \in \mathsf{Down}(\mathbb{A})$. Since $1 \in x$, we have $1 = h1 \in h[x] \subseteq {\uparrow} h[x] = F$, so $1 \in F$. Since $0 \in -x$, we have $0 = h0 \in h[-x] \subseteq {\downarrow} h[-x] = I$, so $0 \in I$. For $\wedge$-closedness, given $b_1, b_2 \in F = {\uparrow} h[x]$, we have $a_1, a_2 \in x$ such that $ha_1 \leqslant b_1$ and $ha_2 \leqslant b_2$, implying $h(a_1 \wedge a_2) = ha_1 \wedge ha_2 \leqslant b_1 \wedge b_2$. Since $a_1 \wedge a_2 \in x$, we have $h(a_1 \wedge a_2) \in h[x]$, so $b_1 \wedge b_2 \in {\uparrow} h[x] = F$. A symmetrical argument shows I is $\vee$-closed. Finally, if it were the case that $F \cap I \neq \varnothing$, we would have $a_1 \in x$ and $a_2 \in -x$ and $b \in \mathbb{B}$ such that $ha_1 \leqslant b \leqslant ha_2$. This would imply that $ha_1 \leqslant ha_2$, implying, by Fact 2.5.2, that $a_1 \leqslant a_2$, finally implying that $a_2 \in x$, contradicting our assumption that $a_2 \notin x$. Having shown $F \in \mathsf{Filt}(\mathbb{B})$ and $I \in \mathsf{Ideal}(\mathbb{B})$ and $F \cap I = \varnothing$, we can conclude, by the PFT, that there exists some $y \in \mathbb{B}_*$ such that $F \subseteq y$ and $y \cap I = \varnothing$. We now claim that $h_* y = x$. ($\subseteq$) Assume, toward a contradiction, that $h^{-1}[y] = h_* y \ni a \notin x$. This implies that $ha \in y \cap h[-x]$, contradicting the fact that $y \cap I = \varnothing$. ($\supseteq$) Given $a \in x$, we have $ha \in h[x] \subseteq {\uparrow} h[x] \subseteq y$, so $a \in h^{-1}[y] = h_* y$.

($\Leftarrow$) Let us assume that h_* is surjective and take $a_1, a_2 \in \mathbb{A}$ such that $a_1 \neq a_2$

and show that $ha_1 \neq ha_2$. We can assume, without loss of generality, that $a_1 \not\leq a_2$, implying that $\uparrow a_1 \cap \downarrow a_2 = \varnothing$, implying, by the PFT, that there exists some $x \in \mathbb{A}_*$ such that $\uparrow a_1 \subseteq x$ and $x \cap \downarrow a_2 = \varnothing$. In particular, this implies that $a_1 \in x \not\ni a_2$. Since h_* is surjective, we have some $y \in \mathbb{B}_*$ such that $x = h_* y = h^{-1}[y]$, so $a_1 \in h^{-1}[y] \not\ni a_2$. This implies that $ha_1 \in y \not\ni ha_2$, implying that $ha_1 \neq ha_2$. ⊠

We now characterise surjective homomorphisms.

Theorem 5.3.2 : Given a **tHA**-morphism $h : \mathbb{A} \to \mathbb{B}$,

$$h \text{ surjective} \iff h_* \text{ injective}.$$

Proof : ($\Rightarrow$) We assume h is surjective and take $y_1, y_2 \in \mathbb{B}_*$ such that $y_1 \neq y_2$ and show that $h_* y_1 \neq h_* y_2$. We can assume, without loss of generality, that $y_1 \not\leq y_2$, implying, by the PSA, that we have $K \in \mathsf{ClopUp}(\mathbb{B}_*)$ such that $y_1 \in K \not\ni y_2$. By Fact 2.7.1, this implies that we have $b \in \mathbb{B}$ such that $\pi b = K$, implying $y_1 \in \pi b \not\ni y_2$, implying $y_1 \ni b \notin y_2$. Since h is surjective, we have $a \in \mathbb{A}$ such that $ha = b$, implying that $y_1 \ni ha \notin y_2$. This implies that $h_* y_1 = h^{-1}[y_1] \ni a \notin h^{-1}[y_2] = h_* y_2$, implying $h_* y_1 \neq h_* y_2$.

($\Leftarrow$) We assume h_* is injective, take some $b \in \mathbb{B}$, and show that there is some $a \in \mathbb{A}$ such that $b = ha$. Now, recalling that the following square commutes, we have $h = \pi_{\mathbb{B}}^{-1} \circ h_*^* \circ \pi_{\mathbb{A}}$.

$$\begin{array}{ccc} \mathbb{A} & \overset{\pi_{\mathbb{A}}}{\rightarrowtail\!\!\twoheadrightarrow} & \mathbb{A}_*^* \\ \downarrow h & & \downarrow h_*^* \\ \mathbb{B} & \underset{\pi_{\mathbb{B}}}{\rightarrowtail\!\!\twoheadrightarrow} & \mathbb{B}_*^* \end{array}$$

So $b = ha$ if and only if $\pi b = h_*^*(\pi a)$. Now consider the sets $h_*[\pi b]$ and $h_*[-\pi b]$. By Fact 2.6.1, we know h_* is closed, so we know that both $h_*[\pi b]$ and $h_*[-\pi b]$ are closed in $\mathbb{A}_*$. Using (ES.m.3), it can be easily checked that $h_*[\pi b]$ is an upset and $h_*[-\pi b]$ is a downset. So we know that $h_*[\pi b] \in \mathsf{ClUp}(\mathbb{A}_*)$ and $h_*[-\pi b] \in \mathsf{ClDown}(\mathbb{A}_*)$, implying that $-h_*[-\pi b] \in \mathsf{OpUp}(\mathbb{A}_*)$. By Corollary 2.7.1 and Fact 2.7.3, that there exist $F \in \mathsf{Filt}(\mathbb{A})$ and $I \in \mathsf{Ideal}(\mathbb{A})$ such that

$$\bigcap \pi[F] = h_*[\pi b] \qquad \bigcup \pi[I] = -h_*[-\pi b].$$

Now I claim that $F \cap I \neq \varnothing$. For if we had $F \cap I \neq \varnothing$, then we would have some $x \in \mathbb{A}_*$ such that $F \subseteq x$ and $x \cap I = \varnothing$. This would mean that $x \in \bigcap \pi[F] = h_*[\pi b]$ and $x \notin \bigcup \pi[I]$, implying $x \in -\bigcup \pi[I] = h_*[-\pi b]$. By the injectivity of h_*, this would mean that $x = h_* y$ and $y \in \pi b \cap -\pi b$, which is a contradiction. So since $F \cap I \neq \varnothing$, we take $a \in F \cap I$ and claim that $\pi b = h_*^*(\pi a)$. ($\subseteq$) Given $y \in \pi b$, we have $h_* y \in h_*[\pi b] = \bigcap \pi[F]$. This implies that $F \subseteq h_* y$, implying that $a \in h_* y$, further implying that $h_* y \in \pi a$, finally implying that $y \in h_*^{-1}[\pi a] = h_*^*(\pi a)$ as desired. $(\supseteq)$ Given $y \in h_*^*(\pi a)$, we have $h_* y \in \pi a$, implying $a \in h_* y$, further implying that $h_* y \cap I \neq \varnothing$. This implies that $h_* y \in \bigcup \pi[I] = -h_*[-\pi b]$, implying that $h_* y \notin h_*[-\pi b]$. By the injectivity of h_*, this implies that $y \notin -\pi b$, finally implying that $y \in \pi b$. ☒

Chapter conclusion

Having established and studied a duality theory between the categories $\mathbf{tHA} \leftrightarrows \mathbf{tES}$, we leverage this duality to begin studying relational models of $\mathbf{tHC}$ in the following chapter.

Chapter 6

Relational models of tHC

In this chapter, we study the relational models of **tHC**. This method of studying **tHC** is more akin to the classic model-theoretic modal logic many readers will have come to know and love. We provide the following section-by-section outline.

- §6.1 We define a method for transforming algebraic models of **tHC** into relational models of **tHC**. We prove several results about the preservation and reflection of formula-satisfaction through this transformation.
- §6.2 We prove relational soundness and completeness for **tHC** and **tTran**.
- §6.3 We define the method of filtration on our models and use it to establish the FMP for relational models of **tHC** .

6.1 Starting from algebra

Readers may have noticed that in establishing Lemma 5.1.5, our proof had a very similar flavour to the relational semantics defined for $\blacklozenge\varphi$ in §2.3 if we take φ to be the clopen upset πa. This was far from a coincidence. In this section, we establish a means of doing relational modal logic on the dual space of a temporal Heyting algebra by formalising this intuition. This method, paired with our algebraic completeness result (Theorem 3.1.1), will be invaluable in establishing relational completeness in §6.2.

First, we establish that, given an algebraic valuation on some temporal Heyting algebra, we can always build a temporal intuitionistic Kripke model.

Notation 6.1.1 : Given $\mathbb{A} \in$ **tHA** and an algebraic valuation ν on $\mathbb{A}$,

$$\mathbb{M}^{\nu}_{\mathbb{A}} := \langle X, R^{\triangleleft}, R^{\triangleright}, \pi \circ \nu \rangle$$

where $\langle X, R^{\triangleleft}, R^{\triangleright} \rangle := \mathbb{A}_{+}$.

Lemma 6.1.1 : Given $\mathbb{A} \in$ **tHA** and an algebraic valuation ν on $\mathbb{A}$, we have $\mathbb{M}^{\nu}_{\mathbb{A}} \in$ **tIKM**.

Proof : Let $\langle X, R^{\triangleleft}, R^{\triangleright}\rangle := \mathbb{A}_+$. Since Lemma 5.1.1 establishes that $\mathbb{A}_+ \in$ **tTran** (tIKM.1), we need only show that $\langle X, \dot{R}^{\triangleright}, \pi \circ \nu\rangle \in$ **fIKM** (tIKM.2). Furthermore, since [15] establishes that $\langle X, R^{\triangleright}\rangle \in$ **Tran** (fIKM.1), all that remains to show is that $\langle X, \dot{R}^{\triangleright}, \pi \circ \nu\rangle \in$ **IKM** (fIKM.2). To see this fact, simply observe that, by Fact 2.2.2, $\langle X, \dot{R}^{\triangleright}\rangle \in$ **POS** and, by Fact 2.7.1, $\pi \circ \nu$ maps onto $\mathsf{ClopUp}(\mathbb{A}_*)$, implying that it maps into $\mathsf{Up}(\mathbb{A}_*)$. ☒

We now establish a result analogous to what is often referred to as the *Truth Lemma* [10, Lemma 4.21]. This establishes that a formula belongs to a prime filter if and only if the filter satisfies the formula with respect to the given relational semantics on the dual space. This proof typically requires a good deal of work, but it becomes completely painless with our having established Lemma 5.1.5.

Lemma 6.1.2 (Algebraic Truth Lemma) **:** Given $\mathbb{A} \in$ **tHA**, an algebraic valuation ν on $\mathbb{A}$, a formula $\varphi \in \mathcal{L}^{\mathsf{i}}_{\mathsf{t}}$, and $x \in \mathbb{M}^{\nu}_{\mathbb{A}}$,

$$\nu\varphi \in x \iff \langle \mathbb{M}^{\nu}_{\mathbb{A}}, x\rangle \models \varphi.$$

Proof : Arguing via induction on the complexity of φ, we skip the non-modal cases (as they are by definition or trivial to check) and focus on the cases where φ is of the form $\chi \to \psi$ or $\Box\chi$ or $\blacklozenge\chi$. Note that these proofs rely heavily on the fact that π and ν are both **tHA**-morphisms as established in Lemma 5.1.5.

$(\chi \to \psi)$

$$\begin{aligned}
\nu(\chi \to \psi) \in x &\iff x \in \pi\nu(\chi \to \psi)\\
&\iff x \in \pi\nu\chi \to \pi\nu\psi\\
&\iff x \notin {\downarrow}(\pi\nu\chi \cap -\pi\nu\psi)\\
&\iff (\neg\exists y)(x \subseteq y \text{ and } \ y \in \pi\nu\chi \text{ and } \ y \notin \pi\nu\psi)\\
&\iff (\forall y)(x \subseteq y \text{ and } \ y \in \pi\nu\chi \text{ implies } \ y \in \pi\nu\psi)\\
&\iff (\forall y)(x \subseteq y \text{ and } \ \nu\chi \in y \text{ implies } \ \nu\psi \in y)\\
&\iff (\forall y)(x \subseteq y \text{ and } \ \langle \mathbb{M}^{\nu}_{\mathbb{A}}, y\rangle \models \chi \text{ implies } \ \langle \mathbb{M}^{\nu}_{\mathbb{A}}, y\rangle \models \psi)\\
&\iff \langle \mathbb{M}^{\nu}_{\mathbb{A}}, x\rangle \models \chi \to \psi
\end{aligned}$$

($\Box\chi$)

$$\begin{aligned}
\nu\Box\chi \in x &\iff x \in \pi\nu\Box\chi \\
&\iff x \in \Box\pi\nu\chi \\
&\iff R^{\triangleright}[x] \subseteq \pi\nu\chi \\
&\iff (\forall y \in \mathbb{X})(x \, R^{\triangleright} \, y \text{ implies } y \in \pi\nu\chi) \\
&\iff (\forall y \in \mathbb{X})(x \, R^{\triangleright} \, y \text{ implies } \nu\chi \in y) \\
&\iff (\forall y \in \mathbb{X})(x \, R^{\triangleright} \, y \text{ implies } \langle \mathbb{M}^{\nu}_{\mathbb{A}}, y\rangle \models \chi) \\
&\iff \langle \mathbb{M}^{\nu}_{\mathbb{A}}, x\rangle \models \Box\chi
\end{aligned}$$

($\blacklozenge\chi$)

$$\begin{aligned}
\nu\blacklozenge\chi \in x &\iff x \in \pi\nu\blacklozenge\chi \\
&\iff x \in \blacklozenge\pi\nu\chi \\
&\iff R^{\triangleleft}[x] \cap \pi\nu\chi \neq \varnothing \\
&\iff (\exists w \in \mathbb{X})(x \, R^{\triangleleft} \, w \text{ and } w \in \pi\nu\chi) \\
&\iff (\exists w \in \mathbb{X})(x \, R^{\triangleleft} \, w \text{ and } \nu\chi \in w) \\
&\iff (\exists w \in \mathbb{X})(x \, R^{\triangleleft} \, w \text{ and } \langle \mathbb{M}^{\nu}_{\mathbb{A}}, w\rangle \models \chi) \\
&\iff \langle \mathbb{M}^{\nu}_{\mathbb{A}}, x\rangle \models \blacklozenge\chi
\end{aligned}$$

⊠

We now establish some results relating satisfaction on a temporal Heyting algebra and satisfaction on its dual frame. In particular, here we prove that every formula valid on the dual frame is also valid on the algebra, implying that our transformation is *truth-reflecting*.

Lemma 6.1.3 : Given $\mathbb{A} \in \mathbf{tHA}$ and $\varphi \in \mathcal{L}^{\mathsf{i}}_{\mathsf{t}}$,

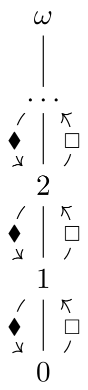
$$\mathbb{A}_+ \models \varphi \quad \Longrightarrow \quad \mathbb{A} \models \varphi.$$

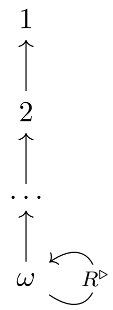
Proof : Let $\langle X, R^{\triangleleft}, R^{\triangleright}\rangle := \mathbb{A}_+$. Arguing via the contrapositive, we let $\mathbb{A} \not\models \varphi$, implying that we have an algebraic valuation on $\mathbb{A}$ such that $\nu\varphi \neq 1$. This implies, by Fact 2.7.1, that $\pi\nu\varphi \neq \pi 1 = X$, implying that there exists some $x \in X$ such that $x \notin \pi\nu\varphi$, further implying that $\nu\varphi \notin x$. By Lemma 6.1.2, this implies that $\langle \mathbb{M}^{\nu}_{\mathbb{A}}, x\rangle \not\models \varphi$. Since $\mathbb{M}^{\nu}_{\mathbb{A}} \in \mathbf{tIKM}$ (Lemma 6.1.1), we have $\mathbb{A}_+ \not\models \varphi$. ⊠

Given Lemma 6.1.3, we can modify our statement at the beginning of the current section to the following : « Given an algebraic valuation on some temporal Heyting algebra, we can always build a temporal intuitionistic Kripke model *that reflects the truths of* $\mathbb{A}$ ».

Unfortunately, the converse of Lemma 6.1.3 is not true in general as exemplified in the following example.

Example 6.1.1 : Consider, the following temporal Heyting algebra $\mathbb{A}$ (depicted on the left) and its dual temporal Esakia space (depicted on the right).





(As in Example 4.1.1, in the dual space, a is taken to mean $\uparrow a$ for all $a \in \mathbb{A}$.) Here we have $\Box\omega := \omega$ and $\blacklozenge\omega := \omega$ and $\mathsf{Refl}(\mathbb{A}_*) = \{\uparrow\omega\}$.

Consider, now, the so-called *Kuznetsov-Muravitsky formula* $\varphi := (\Box p \to p) \to p$ of [32, 37]. We claim that $\mathbb{A} \models \varphi$, but $\mathbb{A}_+ \not\models \varphi$.

To see that $\mathbb{A} \models \varphi$, let ν be an algebraic valuation on $\mathbb{A}$ and distinguish the cases where (1) $\nu p = \omega$ and (2) $\nu p \neq \omega$. In case (1), we have

$$\nu\varphi = (\Box\nu p \to \nu p) \to \nu p = (\Box\omega \to \omega) \to \omega = (\omega \to \omega) \to \omega = \omega \to \omega = \omega.$$

In case (2), we have

$$\nu\varphi = (\Box\nu p \to \nu p) \to \nu p = (\Box n \to n) \to n = ((n+1) \to n) \to n = n \to n = \omega.$$

So in either case we have $\nu\varphi = \omega$, implying that $\mathbb{A} \models \varphi$.

To see that $\mathbb{A}_+ \not\models \varphi$, consider some intuitionistic relational valuation ν such that $p \mapsto \{1, 2, \dots\}$. We claim that $\omega \not\models \varphi$ under this valuation. Since $\omega \not\models p$, it suffices to show that $\omega \models \Box p \to p$. We let $\omega \leqslant y$ and $y \models \Box p$ and show that $y \models p$. Since $\omega \not\models p$ and $\omega \in \mathsf{Refl}(\mathbb{A}_*)$, we have $\omega \not\models \Box p$, so it must be the case that $\omega \neq y$. But then $\omega < y$, implying that $y \in \{1, 2, \dots\}$, implying that $y \in \nu p$, implying that $y \models p$ as desired.

This example implies that our transformation is not *truth-preserving* in general. At the core of this Example 6.1.1 lies the fact that we have selected an intuitionistic relational valuation such that $\nu p \notin \mathsf{ClopUp}(\mathbb{A}_*)$. The key insight is that there are simply too many possible intuitionistic relational valuations on the dual frame for the converse of Lemma 6.1.3 to be true in general.

However, if $\mathbb{A}$ is finite, then both directions of Lemma 6.1.3 hold, as will be established in Lemma 6.1.5 with the help of Lemma 6.1.4.

Lemma 6.1.4 : Given $\langle X, R^{\triangleleft}, R^{\triangleright}, \nu\rangle \in \mathbf{tIKM}_{\mathsf{fin}}$, the map ν is an algebraic valuation on $\langle X, R^{\triangleleft}, R^{\triangleright}, \wp X\rangle^*$.

Proof : Let $\mathbb{A} := \langle X, R^{\triangleleft}, R^{\triangleright}, \wp X\rangle^*$. Beginning with well-definedness, we let $\varphi \in \mathsf{Term}(\mathbf{tHC})$ and show $\nu\varphi \in \mathsf{ClopUp}(\mathbb{X})$. Now since ν is an intuitionistic relational valuation, we have $\nu\varphi \in \mathsf{Up}(\mathbb{X})$. Since $\mathbb{X}$ is finite, Fact 2.6.2 implies that $\nu\varphi \in \mathsf{ClopUp}(\mathbb{X})$ as desired. It can be easily checked that ν is homomorphic over all operations. ⊠

Lemma 6.1.5 : Given $\mathbb{A} \in \mathbf{tHA}_{\mathsf{fin}}$ and $\varphi \in \mathcal{L}^{\mathsf{i}}_{\mathsf{t}}$,

$$\mathbb{A}_+ \models \varphi \quad\Longleftrightarrow\quad \mathbb{A} \models \varphi.$$

Proof : ($\Rightarrow$) This follows directly from Lemma 6.1.3. ($\Leftarrow$) Let $\langle X, R^{\triangleleft}, R^{\triangleright}\rangle := \mathbb{A}_+$ and observe that $\mathbb{A}^*_* = \langle X, R^{\triangleleft}, R^{\triangleright}, \wp X\rangle^*$. Arguing via the contrapositive, we let $\mathbb{A}_+ \not\models \varphi$, implying that we have $\nu : \mathsf{Prop} \to \mathsf{Up}(\mathbb{A}_+)$ such that $\langle X, R^{\triangleleft}, R^{\triangleright}, \nu\rangle \not\models \varphi$, implying that $\nu\varphi \neq X$. Since $\mathbb{A}$ is finite, we know, by Lemma 6.1.4, that $\nu : \mathsf{Term}(\mathbf{tHC}) \to \mathbb{A}^*_*$ is an algebraic valuation. Since X is the top element of $\mathbb{A}^*_*$, we have $\mathbb{A}^*_* \not\models \varphi$, implying $\mathbb{A} \not\models \varphi$. ⊠

So given a temporal Heyting algebra, we have, in general, a truth-reflecting means of transforming it into a temporal intuitionistic Kripke model. In the finite case, this transformation is truth-preserving as well.

6.2 Soundness and completeness

Now we establish soundness and completeness for relational models of **tHC**.

We first establish that all formulas in our logic **tHC** are valid on **tTran**.

Lemma 6.2.1 (Relational soundness) **:** Given $\varphi \in \mathcal{L}^{\mathsf{i}}_{\mathsf{t}}$,

$$\varphi \in \mathbf{tHC} \quad\Longrightarrow\quad \mathbf{tTran} \models \varphi.$$

Proof : Since we have $\mathbf{mHC} \dashv\vdash \mathbf{Tran}$ (by Fact 2.3.1), it suffices to show the validity of (tHC.1) through (tHC.4) on **tTran** and that (MP), (US), and (PD) preserve validity on **tTran**. Since (MP) and (US) are quite trivial to check, their proofs are omitted.

(tHC.1) Assume, toward a contradiction, that $\mathbf{tTran} \not\models \blacklozenge(p \vee q) \to (\blacklozenge p \vee \blacklozenge q)$, implying that we have some $x \in \mathbb{M} \in \mathbf{tIKM}$ such that $\langle \mathbb{M}, x\rangle \not\models \blacklozenge(p \vee q) \to (\blacklozenge p \vee \blacklozenge q)$. This implies there exists some $y \in \mathbb{M}$ such that $x \leqslant y$ and $y \models \blacklozenge(p \vee q)$ and $y \not\models \blacklozenge p \vee \blacklozenge q$. Since $y \models \blacklozenge(p \vee q)$, there exists some $w \in \mathbb{M}$ such that $y\ R^{\triangleleft}\ w$ and $w \models p \vee q$. We can assume, without loss of generality, that $w \models p$. But this implies that $y \models \blacklozenge p$, implying, by weakening, that $y \models \blacklozenge p \vee \blacklozenge q$, giving us a contradiction.

(tHC.2) Assume, toward a contradiction, that $\mathbf{tTran} \not\models \blacklozenge\bot \to \bot$, implying that we have some $x \in \mathbb{M} \in \mathbf{tIKM}$ such that $\langle \mathbb{M}, x\rangle \not\models \blacklozenge\bot \to \bot$. This implies there exists some $y \in \mathbb{M}$ such that $x \leqslant y$ and $y \models \blacklozenge\bot$ and $y \not\models \bot$. Since $y \models \blacklozenge\bot$, there exists some $w \in \mathbb{M}$ such that $y\ R^{\lhd}\ w$ and $w \models \bot$, giving us a contradiction.

(tHC.3) Assume, toward a contradiction, that $\mathbf{tTran} \not\models p \to \Box\blacklozenge p$, implying that we have some $x \in \mathbb{M} \in \mathbf{tIKM}$ such that $\langle \mathbb{M}, x\rangle \not\models p \to \Box\blacklozenge p$. This implies there exists some $y \in \mathbb{M}$ such that $x \leqslant y$ and $y \models p$ and $y \not\models \Box\blacklozenge p$. Since $y \not\models \Box\blacklozenge p$, we must have some $z \in \mathbb{M}$ such that $y\ R^{\rhd}\ z$ and $z \not\models \blacklozenge p$. But since $y\ R^{\rhd}\ z$, we have $z\ R^{\lhd}\ y$ implying, since $y \models p$, that $z \models \blacklozenge p$, giving us a contradiction.

(tHC.4) Assume, toward a contradiction, that $\mathbf{tTran} \not\models \blacklozenge\Box p \to p$, implying that we have some $x \in \mathbb{M} \in \mathbf{tIKM}$ such that $\langle \mathbb{M}, x\rangle \not\models \blacklozenge\Box p \to p$. This implies there exists some $y \in \mathbb{M}$ such that $x \leqslant y$ and $y \models \blacklozenge\Box p$ and $y \not\models p$. Since $y \models \blacklozenge\Box p$, there exists some $w \in \mathbb{M}$ such that $y\ R^{\lhd}\ w$ and $w \models \Box p$. But since $y\ R^{\lhd}\ w$, we have $w\ R^{\rhd}\ y$, implying, since $w \models \Box p$, that $y \models p$, giving us a contradiction.

(PD) Let us assume that $\mathbf{tTran} \models \varphi \to \chi$ and assume, toward a contradiction, that $\mathbf{tTran} \not\models \blacklozenge\varphi \to \blacklozenge\chi$, implying that we have some $x \in \mathbb{M} \in \mathbf{tIKM}$ such that $\langle \mathbb{M}, x\rangle \not\models \blacklozenge\varphi \to \blacklozenge\chi$. This implies there exists some $y \in \mathbb{M}$ such that $x \leqslant y$ and $y \models \blacklozenge\varphi$ and $y \not\models \blacklozenge\chi$. Since $y \models \blacklozenge\varphi$, there exists some $w \in \mathbb{M}$ such that $y\ R^{\lhd}\ w$ and $w \models \varphi$. But since $\mathbf{tIKM} \models \varphi \to \chi$, we have $w \models \varphi \to \chi$, implying that $w \models \chi$, finally implying that $y \models \blacklozenge\chi$, giving us a contradiction. ⊠

We now establish the completeness of **tHC** with respect to **tTran**. The reader should note that completeness can also be achieved —without ever having studied algebraic models of **tHC**— via the well-known method of *canonical models* [10, §4.2] as we have checked that the logic **tHC** is, indeed, *canonical*. However, given that we already have algebraic completeness and a truth-reflecting transformation from algebraic models to relational models, we have opted to pursue what could be called « completeness via representation ».

Lemma 6.2.2 (Relational completeness) **:** Given $\varphi \in \mathcal{L}^{\mathsf{i}}_{\mathsf{t}}$,

$$\mathbf{tTran} \models \varphi \implies \varphi \in \mathbf{tHC}.$$

Proof : Arguing via the contrapositive, let $\varphi \notin \mathbf{tHC}$. Given Theorem 3.1.1, this implies that there exists some $\mathbb{A} \in \mathbf{tHA}$ such that $\mathbb{A} \not\models \varphi$, implying, by Lemma 6.1.3, that $\mathbb{A}_+ \not\models \varphi$. Since $\mathbb{A}_+ \in \mathbf{tTran}$ (Lemma 5.1.1), we have $\mathbf{tTran} \not\models \varphi$. ⊠

We can now state our relational soundness and completeness result.

Corollary 6.2.1 : **tHC** $\dashv\vdash$ **tTran**.

Proof : This follows directly from Lemmas 6.2.1 and 6.2.2. ⊠

6.3 Finite model property for relational models

Here we establish the FMP for relational models of **tHC** via the well-known method of relational *filtration* [10, Definition 2.36]. The idea here is to define a class-operation on relational models that shrinks them to a finite cardinality while preserving and reflecting enough structure that a formula refuted on the original model is still refuted on the finite model.

It should be noted that an algebraic analogue of filtration exists and has been studied in the context of superintuitionistic logics [3], so it is entirely likely that the FMP for algebraic models of **tHC** could have been established without having established **tHC** $\dashv\vdash$ **tTran** in §6.2. However, we believe that future investigators of **tHC** are more likely to be familiar with and interested in relational filtration as opposed to algebraic. For this reason, the FMP is established relationally and the algebraic FMP follows as essentially a corollary in Theorem 7.1.1.

Definition 6.3.1 (Quotientable) **:** Given a set of formulas Σ, we call Σ *quotientable* if Σ is finite and closed under subformulas.

There are many ways of preforming filtration. A look at the definition of a filtration reveals that they need only meet a few basic requirements ; specifically, they must preserve existing structure while not adding so much structure that they contradict formula-satisfaction in the original model. With two relations in play —or *three* if you count the implicit $\leqslant$—, one could write a good deal solely on what sizes and combinations of relation-filtrations preserve the property of being a temporal intuitionistic Kripke model. Such an investigation falls outside the scope of the current text, so we have opted to define and work with only the *smallest transitive filtration* [2, p. 4], i.e. the transitive closure of the filtration that relates the fewest elements of the filtrated models, adding the least amount of structure while ensuring transitivity. Since our interest in filtration is purely extrinsic (motivated by a desire to establish the FMP), this will be more than sufficient for our purposes. An in-depth investigation into filtration methods on temporal intuitionistic Kripke models is included in Chapter 8 as potential future work.

Construction 6.3.1 (Smallest transitive filtration) **:** Given $\mathbb{M} := \langle X, R^{\triangleleft}, R^{\triangleright}, \nu \rangle \in$ **tIKM** and a quotientable set $\Sigma \subseteq \mathcal{L}_{\mathsf{t}}^{\mathsf{i}}$, we construct a model $\mathbb{M}_{\Sigma} \in$ **tIKM**$_{\mathsf{fin}}$.

We define the binary relation $\sim$ on $\mathbb{X}$ as follows.

$$x \sim y \quad :\Longleftrightarrow \quad (\forall \varphi \in \Sigma)(\langle \mathbb{M}, x \rangle \models \varphi \Leftrightarrow \langle \mathbb{M}, y \rangle \models \varphi)$$

It is clear that $\sim$ is an equivalence relation. We let $X_\Sigma := X/{\sim}$ and we use the shorthand $[x] := [x]_\sim = \{y \in X \mid x \sim y\}$.

We then define

$$\begin{aligned}[x]\, r^{\lhd}_{\Sigma}\, [w] \quad &:\Longleftrightarrow \quad (\exists x', w' \in X)(x \sim x'\, R^{\lhd}\, w' \sim w)\\ [x]\, r^{\rhd}_{\Sigma}\, [y] \quad &:\Longleftrightarrow \quad (\exists x', y' \in X)(x \sim x'\, R^{\rhd}\, y' \sim y)\end{aligned}$$

and define $R^{\lhd}_{\Sigma}$ and $R^{\rhd}_{\Sigma}$ to be the transitive closures of $r^{\lhd}_{\Sigma}$ and $r^{\rhd}_{\Sigma}$ respectively.

Next we define $\nu_\Sigma : \Sigma \to X_\Sigma$ by the rule

$$p \longmapsto \{[x] \in X_\Sigma \mid x \in \nu p\}.$$

Finally, we define $\mathbb{M}_\Sigma := \langle X_\Sigma, R^{\lhd}_{\Sigma}, R^{\rhd}_{\Sigma}, \nu_\Sigma\rangle$. It will be shown in Lemma 6.3.3 that $\mathbb{M}_\Sigma$ is indeed a temporal intuitionistic Kripke model.

Having defined the smallest transitive filtration, we prove two lemmas that will aid us in proving that the class **tIKM** is closed under the class-operation $(-)_\Sigma$.

Lemma 6.3.1 : Given $x, y \in \mathbb{M} \in$ **tIKM**, a quotientable set $\Sigma \subseteq \mathcal{L}^{\mathsf{i}}_{\mathsf{t}}$, and $\varphi \in \Sigma$,

$$\langle \mathbb{M}, x\rangle \models \varphi \text{ and } [x]\, R^{\rhd}_{\Sigma}\, [y] \implies \langle \mathbb{M}, y\rangle \models \varphi.$$

Proof : Given $[x]\, R^{\rhd}_{\Sigma}\, [y]$, we have a finite path $[x]\, r^{\rhd}_{\Sigma} \cdots r^{\rhd}_{\Sigma}\, [y]$. We proceed via induction on the length of the path. (1) Here we have $x \sim x'\, R^{\rhd}\, y' \sim y$, implying that $x' \leqslant y'$. Since $\varphi \in \Sigma$ and $x \sim x'$, we have $x' \models \varphi$. Since $x' \models \varphi$ and $x' \leqslant y'$, we have $y' \models \varphi$. And, finally, since $y' \models \varphi$ and $y' \sim y$, we have $y \models \varphi$. $(n+1)$ Here we have $[x]\, r^{\rhd}_{\Sigma} \cdots r^{\rhd}_{\Sigma}\, [z_n]\, r^{\rhd}_{\Sigma}\, [y]$, implying that $z_n \models \varphi$ and $z_n \sim z'_n\, R^{\rhd}\, y' \sim y$. By the same argument as the previous case, we have $y \models \varphi$. ⊠

Lemma 6.3.2 : Given $x, y \in \mathbb{M} \in$ **tIKM**, a quotientable set $\Sigma \subseteq \mathcal{L}^{\mathsf{i}}_{\mathsf{t}}$, and $\varphi \in \Sigma$,

$$[x]\, r^{\rhd}_{\Sigma}\, [y] \iff [y]\, r^{\lhd}_{\Sigma}\, [x].$$

Proof : We have $[x]\, r^{\rhd}_{\Sigma}\, [y]$ if and only if we have $x \sim x'\, R^{\rhd}\, y' \sim y$, which is the case if and only if $y \sim y'\, R^{\lhd}\, x' \sim x$, which is the case if and only iff $y\, r^{\lhd}_{\Sigma}\, x$. ⊠

We now establish that performing the smallest transitive filtration of a temporal intuitionistic Kripke model results in a finite temporal intuitionistic Kripke model.

Lemma 6.3.3 : Given $\mathbb{M} \in$ **tIKM** and a quotientable set $\Sigma \subseteq \mathcal{L}^{\mathsf{i}}_{\mathsf{t}}$, we have $\mathbb{M}_\Sigma \in$ **tIKM**$_{\mathsf{fin}}$.

Proof : Let $\langle X, R^{\triangleleft}, R^{\triangleright}, \nu \rangle := \mathbb{M}$. First, to see that the resulting structure is finite, consult the simple argument in [10, Proposition 2.38]. Now we move to the preservation of relational structure. To show that $\mathbb{M}_{\Sigma} \in$ **tIKM**, we must show (Tran.1), (Tran.2), (tTran.2), and (fIKM.2).

(Tran.1) Observe that $\langle X_{\Sigma}, R^{\triangleright}_{\Sigma} \rangle$ is transitive by construction as it is the transitive closure of $\langle X_{\Sigma}, r^{\triangleright}_{\Sigma} \rangle$.

(Tran.2) We let $[x]\ R^{\triangleright}_{\Sigma}\ [y]$ and $[y]\ R^{\triangleright}_{\Sigma}\ [x]$ and show that $[x] = [y]$, i.e. $x \sim y$, which is the case if and only if $(\forall \varphi \in \Sigma)(x \models \varphi \Leftrightarrow y \models \varphi)$. But if $x \models \varphi$, then Lemma 6.3.1 implies that $y \models \varphi$. Likewise, if $y \models \varphi$, Lemma 6.3.1 implies that $x \models \varphi$. So we have $x \sim y$, implying $[x] = [y]$.

(tTran.2) We have $[x]\ R^{\triangleright}_{\Sigma}\ [y]$ if and only if there is a finite path $[x]\ r^{\triangleright}_{\Sigma}\ [z_2]\ r^{\triangleright}_{\Sigma} \cdots r^{\triangleright}_{\Sigma}$ $[z_{n-1}]\, r^{\triangleright}_{\Sigma}\, [y]$. By Lemma 6.3.2, this is the case if and only if $[y]\, r^{\triangleleft}_{\Sigma}\, [z_{n-1}]\, r^{\triangleleft}_{\Sigma} \cdots r^{\triangleleft}_{\Sigma}\, [z_2]\, r^{\triangleleft}_{\Sigma}\, [x]$, which is the case if and only if $[y]\ R^{\triangleleft}_{\Sigma}\ [x]$.

(fIKM.2) Since $\langle X_{\Sigma}, R^{\triangleright}_{\Sigma} \rangle \in$ **Tran** implies $\langle X_{\Sigma}, \dot{R}^{\triangleright}_{\Sigma} \rangle \in$ **POS** (Fact 2.2.2), we need only show (IKM.2) : that ν_{Σ} is an intuitionistic relational valuation on $\langle X_{\Sigma}, \dot{R}^{\triangleright}_{\Sigma} \rangle$. Given $p \in \mathsf{Prop} \cap \Sigma$ and $[x] \in \nu_{\Sigma}(p)$ and $[x]\ \dot{R}^{\triangleright}_{\Sigma}\ [y]$, distinguish two cases : (1) $[x] = [y]$ and (2) $[x]\, R^{\triangleright}_{\Sigma}\, [y]$. Note that, in either case, $[x] \in \nu_{\Sigma}(p)$ implies $x \in \nu p$. (1) In this case, we have $x \sim y$, so $x \in \nu p$ implies $y \in \nu p$, implying $[y] \in \nu_{\Sigma}(p)$ as desired. (2) In this case, we have $x \models p$ and $[x]\ R^{\triangleright}_{\Sigma}\ [y]$, implying, by Lemma 6.3.1, that $y \models p$, implying $[y] \in \nu_{\Sigma}(p)$ as desired. ⊠

Here we establish some facts about the relational structure that is preserved and reflected by the smallest transitive filtration of a temporal intuitionistic Kripke model.

Lemma 6.3.4 : Given $\langle X, R^{\triangleleft}, R^{\triangleright}, \nu \rangle \in$ **tIKM** and a quotientable set $\Sigma \subseteq \mathcal{L}^{\mathsf{i}}_{\mathsf{t}}$, the following hold for all $w, x, y \in X$.

1. $x\ R^{\triangleright}\ y$ implies $[x]\ R^{\triangleright}_{\Sigma}\ [y]$
2. $[x]\ R^{\triangleright}_{\Sigma}\ [y]$ implies $(\forall \Box\varphi \in \Sigma)(x \models \Box\varphi \Rightarrow y \models \varphi)$
3. $x\ R^{\triangleleft}\ w$ implies $[x]\ R^{\triangleleft}_{\Sigma}\ [w]$
4. $[x]\ R^{\triangleleft}_{\Sigma}\ [w]$ implies $(\forall \blacklozenge\varphi \in \Sigma)(w \models \varphi \Rightarrow x \models \blacklozenge\varphi)$
5. $x \leqslant y$ implies $[x] \leqslant_{\Sigma} [y]$
6. $[x] \leqslant_{\Sigma} [y]$ implies $(\forall \varphi \in \Sigma)(x \models \varphi \Rightarrow y \models \varphi)$

Proof : (1) Given $xR^{\triangleright}y$, we have $x \sim xR^{\triangleright}y \sim y$, so we have $[x]r^{\triangleright}_{\Sigma}[y]$, implying $[x]R^{\triangleright}_{\Sigma}[y]$.

(2) We assume that $[x]\ R^{\triangleright}_{\Sigma}\ [y]$ and $x \models \Box\varphi$ and show that $y \models \varphi$. Given $[x]\ R^{\triangleright}_{\Sigma}\ [y]$, we have some finite path $[x]\ r^{\triangleright}_{\Sigma} \cdots r^{\triangleright}_{\Sigma}\ [y]$. We proceed via induction on the length of the path. (1) Here we have $x \sim x'\ R^{\triangleright}\ y' \sim y$. Since $x \models \Box\varphi$, we have $x' \models \Box\varphi$. Since $x'\ R^{\triangleright}\ y'$,we have $y' \models \varphi$. Finally, since $y' \sim y$, we have $y \models \varphi$. $(n+1)$ Here we have $[x]\ r^{\triangleright}_{\Sigma}\ [z_2]\ r^{\triangleright}_{\Sigma} \cdots r^{\triangleright}_{\Sigma}\ [z_n]\ r^{\triangleright}_{\Sigma}\ [y]$, implying that $z_n \models \varphi$ and $[z_n]\ R^{\triangleright}_{\Sigma}\ [y]$, implying, by Lemma 6.3.1, that $y \models \varphi$.

(3) Given $x\ R^{\triangleleft}\ w$, we have $x \sim x\ R^{\triangleleft}\ w \sim w$, so we have $[x]\ r^{\triangleleft}_{\Sigma}\ [w]$, implying $[x]\ R^{\triangleleft}_{\Sigma}\ [w]$.

(4) We assume that $[x]\ R^{\triangleleft}_{\Sigma}\ [w]$ and $w \models \varphi$ and show that $x \models \blacklozenge\varphi$. Given $[x]\ R^{\triangleleft}_{\Sigma}\ [w]$, we have some finite path $[x]\ r^{\triangleleft}_{\Sigma} \cdots r^{\triangleleft}_{\Sigma}\ [w]$. We proceed via induction on the length of the path. (1) Here we have $x \sim x'\ R^{\triangleleft}\ w' \sim w$. Since $w \models \varphi$, we have $w' \models \varphi$. Since $x'\ R^{\triangleleft}\ w'$, we have $x' \models \blacklozenge\varphi$. Finally, since $x' \sim x$, we have $x \models \blacklozenge\varphi$. $(n+1)$ Here we have $[x]\ r^{\triangleleft}_{\Sigma}\ [z_2]\ r^{\triangleleft}_{\Sigma} \cdots r^{\triangleleft}_{\Sigma}\ [z_n]\ r^{\triangleleft}_{\Sigma}\ [w]$, implying that $z_2 \models \blacklozenge\varphi$ and $[x]\ R^{\triangleleft}_{\Sigma}\ [z_2]$. This implies that $[z_2]\ R^{\triangleright}_{\Sigma}\ [x]$, implying, by Lemma 6.3.1, that $x \models \blacklozenge\varphi$.

(5) Given $x \leqslant y$, we distinguish the cases where (a) $x = y$ and (b) $x < y$. (a) If $x = y$, then $[x] = [y]$, implying $[x]\ \dot{R}^{\triangleright}_{\Sigma}\ [y]$, equiv. $[x] \leqslant_{\Sigma} [y]$. (b) If $x < y$, we have $x\ R^{\triangleright}\ y$, implying, by (1), that $[x]\ R^{\triangleright}_{\Sigma}\ [y]$, implying that $[x]\ \dot{R}^{\triangleright}_{\Sigma}\ [y]$, equiv. $[x] \leqslant_{\Sigma} [y]$.

(6) We assume that $[x] \leqslant_{\Sigma} [y]$ and $x \models \varphi$ and show that $y \models \varphi$. Given $[x] \leqslant_{\Sigma} [y]$, we distinguish the cases where (a) $[x] = [y]$ and (b) $[x] < [y]$. (a) If $[x] = [y]$, we have $x \sim y$, implying, since $x \models \varphi$, that $y \models \varphi$. (b) Here we have $[x]\ R^{\triangleright}_{\Sigma}\ [y]$ and $x \models \varphi$, implying, by Lemma 6.3.1, that $y \models \varphi$. ⊠

We now establish a correspondence between the formulas satisfied at a point and those satisfied by its corresponding point in the filtrated model.

Theorem 6.3.1 (Filtration Theorem) **:** Given $x \in \mathbb{M} \in$ **tIKM**, a quotientable set $\Sigma \subseteq \mathcal{L}^{\mathsf{i}}_{\mathsf{t}}$, and $\varphi \in \Sigma$,

$$\langle \mathbb{M}, x \rangle \models \varphi \quad \Longleftrightarrow \quad \langle \mathbb{M}_{\Sigma}, [x] \rangle \models \varphi.$$

Proof : We proceed via induction on φ. Now if φ has non-modal semantics, this follows either by definition or by a trivial argument, so we consider the cases $\rightarrow$, $\blacklozenge$, and $\Box$ $(\rightarrow)$ Having shown in Lemma 6.3.4 that $\leqslant_{\Sigma}$ is a valid *intuitionistic filtration* [3, Definition 2.1], this follows from the « Filtration Lemma » of [3, Lemma 2.3]. ($\blacklozenge$ and $\Box$) Having shown in Lemma 6.3.4 that $R^{\triangleleft}_{\Sigma}$ and $R^{\triangleright}_{\Sigma}$ are valid *modal filtrations* [10, Definition 2.36], this follows from the « Filtration Theorem » of [10, Theorem 2.39]. ⊠

Having established these preservation results, we can prove the FMP for relational models of **tHC**.

Theorem 6.3.2 (Finite model property : relational models) **:** $\mathbf{tHC} \dashv\vdash \mathbf{tTran}_{\mathsf{fin}}$.

Proof : ($\vdash$) Arguing via the contrapositive, we let $\varphi \notin \mathbf{tHC}$, implying, by Corollary 6.2.1, there exists some $x \in \mathbb{M} \in \mathbf{tIKM}$ such that $\langle \mathbb{M}, x \rangle \not\models \varphi$. If we let

$$\Sigma := \{\chi \in \mathcal{L}^{\mathsf{i}}_{\mathsf{t}} \mid \chi \text{ is a subformula of } \varphi\},$$

then Theorem 6.3.1 implies that $\langle \mathbb{M}_\Sigma, [x] \rangle \not\models \varphi$, implying $\mathbb{M}_\Sigma \not\models \varphi$, implying, by Lemma 6.3.3, that $\mathbf{tTran}_{\mathsf{fin}} \not\models \varphi$. ($\dashv$) Since $\mathbf{tTran}_{\mathsf{fin}} \subseteq \mathbf{tTran}$, Corollary 6.2.1 implies that $\mathbf{tTran}_{\mathsf{fin}} \models \mathbf{tHC}$. ⊠

Chapter conclusion

The relational theory developed in the current chapter will allow us, in the coming chapter, to establish the FMP for algebraic models of $\mathbf{tHC}$. In combination with a subsequent characterisation of the simple and subdirectly-irreducible elements of $\mathbf{tHA}$, this will enable us to prove a final relational completeness result for $\mathbf{tHC}$ in the final section of the text.

Chapter 7

Duality applications for tHA

This chapter applies results established in Chapter 3 through Chapter 6 to study properties of the variety **tHA**, and, thereby, the logic **tHC**. It contains the main results of the current text in §7.2 through §7.4. We provide the following section-by-section outline.

§7.1 We establish the FMP for algebraic models of **tHC** and use it to arrive at a stronger algebraic soundness and completeness result.

§7.2 We characterise simple temporal Heyting algebras both lattice-theoretically and order-topologically (in both the general and finite cases).

§7.3 We characterise subdirectly-irreducible temporal Heyting algebras both lattice-theoretically and order-topologically (in both the general and finite cases).

§7.4 We prove a final relational completeness result combining finiteness and Z-rootedness.

7.1 Finite model property for algebraic models

Here we leverage the FMP for relational models of **tHC** to establish the FMP for algebraic models of **tHC**. Readers curious as to why the FMP was not established purely algebraically should reference the comments at the beginning of §6.3

Theorem 7.1.1 (Finite model property : algebraic models) **:** $\mathbf{tHC} \dashv\vdash \mathbf{tHA}_{\mathsf{fin}}$.

Proof : ($\vdash$) Arguing via the contrapositive, we let $\varphi \notin \mathbf{tHC}$, implying, by Theorem 6.3.2, there exists some $\langle X, R^{\triangleleft}, R^{\triangleright}\rangle \in \mathbf{tTran}_{\mathsf{fin}}$ such that $\langle X, R^{\triangleleft}, R^{\triangleright}\rangle \not\models \varphi$. Fact 2.6.3 implies that $\mathbb{X} := \langle X, R^{\triangleleft}, R^{\triangleright}, \wp X\rangle \in \mathbf{tES}_{\mathsf{fin}}$, implying that $(\mathbb{X}^*)_+ \not\models \varphi$. This implies, by Lemma 6.1.5, that $\mathbb{X}^* \not\models \varphi$ and $\mathbb{X}^* \in \mathbf{tHA}_{\mathsf{fin}}$, implying that $\mathbf{tHA}_{\mathsf{fin}} \not\models \varphi$. ($\dashv$) Since $\mathbf{tHA}_{\mathsf{fin}} \subseteq \mathbf{tHA}$, Theorem 3.1.1 implies that $\mathbf{tHA}_{\mathsf{fin}} \models \mathbf{tHC}$. ⊠

We can apply Theorem 7.1.1 to arrive at an even stronger version of Theorem 3.1.2.

Theorem 7.1.2 : $\mathbf{tHC} \dashv\vdash \mathbf{tHA}_{\mathsf{fsi}}$.

Proof : ($\vdash$) Arguing via the contrapositive, we let $\varphi \notin \mathbf{tHC}$, implying, by Theorem 7.1.1, there exists some $\mathbb{A} \in \mathbf{tHA}_{\mathsf{fin}}$ such that $\mathbb{A} \not\models \varphi \approx \top$. By [13, Corollary 8.7], we know that $\mathbb{A}$ is isomorphic to the subdirect product of a finite set of finite subdirectly-irreducible algebras in $\mathbf{tHA}$. This implies that $\mathbb{A} \cong \mathbb{B}$ and $\mathbb{B} \in \mathsf{S}(\mathbb{C})$ and $\mathbb{C} \in \mathsf{P}(\{\mathbb{D}_i\}_{i=1}^{n})$ where $\mathbb{D}_i \in \mathbf{tHA}_{\mathsf{fsi}}$. Since $\mathbb{A} \not\models \varphi \approx \top$, we have $\mathbb{B} \not\models \varphi \approx \top$, implying, by Fact 2.4.3, that $\mathbb{C} \not\models \varphi \approx \top$, finally implying that there exists some $\mathbb{D}_k$ such that $\mathbb{D}_k \not\models \varphi \approx \top$. Since $\mathbb{D}_k \in \mathbf{tHA}_{\mathsf{fsi}}$, we have $\mathbf{tHA}_{\mathsf{fsi}} \not\models \varphi$. ($\dashv$) Since $\mathbf{tHA}_{\mathsf{fsi}} \subseteq \mathbf{tHA}$, Theorem 3.1.1 implies that $\mathbf{tHA}_{\mathsf{fsi}} \models \mathbf{tHC}$. ⊠

7.2 Simple algebras

We now apply the theory developed throughout this text to characterise *simple* temporal Heyting algebras both lattice-theoretically and order-topologically as in [46, Theorem 1] and [8, Theorem 2].

Theorem 7.2.1 : Given $\mathbb{A} \in \mathbf{tHA}$, the following are equivalent.

1. $\mathbb{A}$ is simple
2. $\blacklozenge\mathsf{Filt}(\mathbb{A}) = \{\{1\}, \mathbb{A}\}$
3. $\mathbb{A}_*$ is topo-connected

Proof : Let $\mathbb{X} := \mathbb{A}_*$.

($1 \Leftrightarrow 2$) This follows directly from Theorem 3.2.1.

($2 \Rightarrow 3$) Here we take some $x, y \in \mathbb{X}$ and show that y is topo-reachable from x. Since $\blacklozenge\mathsf{Filt}(\mathbb{A}) = \{\{1\}, \mathbb{A}\}$, Theorem 5.2.1 implies

$$\mathsf{ClArcUp}(\mathbb{A}) = \left\langle \bigcap \pi[\{1\}], \bigcap \pi[\mathbb{A}] \right\rangle = \langle \mathbb{X}, \varnothing \rangle.$$

This implies that $\widehat{x} = \mathbb{X}$, implying that $y \in \widehat{x}$, finally implying that y is topo-reachable from x.

($2 \Leftarrow 3$) Arguing via the contrapositive, suppose we have $\blacklozenge\mathsf{Filt}(\mathbb{A}) \neq \{\{1\}, \mathbb{A}\}$, implying that there is some $F \in \blacklozenge\mathsf{Filt}(\mathbb{A})$ such that $\{1\} \subsetneq F \subsetneq \mathbb{A}$. This implies, by

Theorem 5.2.1, that $\bigcap \pi[F] \in \mathsf{ClArcUp}(\mathbb{X})$ and

$$\mathbb{X} = \bigcap \pi[\{1\}] \supsetneq \bigcap \pi[F] \supsetneq \bigcap \pi[\mathbb{A}] = \varnothing.$$

Since $\mathbb{X} \supsetneq \bigcap \pi[F] \supsetneq \varnothing$, we have some $x, y \in \mathbb{X}$ such that $x \in \bigcap \pi[F] \not\ni y$. Since $\bigcap \pi[F] \in \mathsf{ClArcUp}(\mathbb{X})$, Lemma 4.2.1 implies that we have $\widehat{x} \subseteq \bigcap \pi[F] \not\ni y$, implying that $y \notin \widehat{x}$, further implying that y is *not* topo-reachable from x, finally implying that $\mathbb{X}$ is *not* topo-connected. ⊠

If we restrict our focus to the finite case, we can state an element-wise and frame-theoretic analogue to Theorem 7.2.1.

Theorem 7.2.2 : Given $\mathbb{A} \in \mathbf{tHA}_{\mathsf{fin}}$, the following are equivalent.

1. $\mathbb{A}$ is simple
2. $\blacklozenge\mathsf{Com}(\mathbb{A}) = \{1, 0\}$
3. $\mathbb{A}_*$ is Z-connected

Proof :

$$\begin{array}{ccc} & (1) & \\ \text{Thm. 7.2.1} \Updownarrow & & \Updownarrow \text{Thm. 7.2.1} \\ \blacklozenge\mathsf{Filt}(\mathbb{A}) = \{\{1\}, \mathbb{A}\} & \overset{\text{Thm. 7.2.1}}{\Longleftrightarrow} & \mathbb{A}_* \text{ topo-connected} \\ \text{Prp. 3.3.2} \Updownarrow & & \Updownarrow \text{Cor. 4.4.2} \\ (2) & & (3) \end{array}$$

⊠

7.3 Subdirectly-irreducible algebras

We now apply the theory developed throughout this text to characterise *subdirectly-irreducible* temporal Heyting algebras both lattice-theoretically and order-topologically as in [46, Corollary 3] and [8, Theorem 1].

Theorem 7.3.1 : Given $\mathbb{A} \in \mathbf{tHA}$, the following are equivalent.

1. $\mathbb{A}$ is subdirectly-irreducible
2. $\blacklozenge\mathsf{Filt}(\mathbb{A})$ has a second-least element

3. $\mathsf{ToRo}(\mathbb{A}_*)$ is non-empty and open

Proof : Let $\mathbb{X} := \mathbb{A}_*$.

$(1 \Leftrightarrow 2)$ This follows directly from Theorem 3.2.1.

$(2 \Rightarrow 3)$ Let F be the second-least $\blacklozenge$-filter. Note that this implies that $\{1\} \subsetneq F$ and that $\bigcap \pi[F]$ is the second-greatest closed archival upset (by Theorem 5.2.1). We claim that

$$\bigcap \pi[F] = -\mathsf{ToRo}(\mathbb{X}).$$

Note that if this is the case, then, by Theorem 5.2.1, we have $-\mathsf{ToRo}(\mathbb{X}) \in \mathsf{ClArcUp}(\mathbb{X})$, implying that $\mathsf{ToRo}(\mathbb{X})$ is open. Also, since $\{1\} \subsetneq F$, we'll have $\bigcap \pi[F] \subsetneq \bigcap \pi[\{1\}] = \mathbb{X}$, implying that $-\mathsf{ToRo}(\mathbb{X}) = \bigcap \pi[F] \subsetneq \mathbb{X}$ and, therefore, $\varnothing \subsetneq \mathsf{ToRo}(\mathbb{X})$, implying that $\mathsf{ToRo}(\mathbb{X})$ is non-empty as desired. So we show that $\bigcap \pi[F] = -\mathsf{ToRo}(\mathbb{X})$. $(\subseteq)$ Given $x \in \bigcap \pi[F]$, we have $\widehat{x} \subseteq \bigcap \pi[F] \subsetneq \mathbb{X}$, implying that $x \notin \mathsf{ToRo}(\mathbb{X})$. $(\supseteq)$ Given $x \in -\mathsf{ToRo}(\mathbb{X})$, we have $\widehat{x} \subsetneq \mathbb{X}$, implying, since $\bigcap \pi[F]$ is the second-greatest closed archival upset, that $\widehat{x} \subseteq \bigcap \pi[F]$, implying that $x \in \bigcap \pi[F]$.

$(2 \Leftarrow 3)$ Assume toward a contradiction, that $\mathsf{ToRo}(\mathbb{A})$ is non-empty and open, but $\blacklozenge\mathsf{Filt}(\mathbb{A})$ does *not* have a second-least element, implying that

$$(\forall F \neq \{1\})(\exists F' \neq \{1\})(F \not\subseteq F')$$

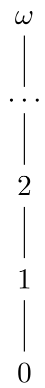
(where F, F' are taken to range over $\blacklozenge\mathsf{Filt}(\mathbb{A})$). By Theorem 5.2.1, this implies that

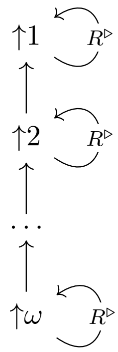
$$(\forall C \neq \mathbb{X})(\exists C' \neq \mathbb{X})(C' \not\subseteq C)$$

(where C, C' are taken to range over $\mathsf{ClArcUp}(\mathbb{X})$). But observe that if $\mathsf{ToRo}(\mathbb{X})$ is non-empty and open, then $-\mathsf{ToRo}(\mathbb{X})$ is non-total and closed, implying, by Lemma 4.2.2, that $-\mathsf{ToRo}(\mathbb{X}) \in \mathsf{ClArcUp}(\mathbb{X})$. So there exists some $C' \in \mathsf{ClArcUp}(\mathbb{X})$ such that $\mathbb{X} \neq C' \not\subseteq -\mathsf{ToRo}(\mathbb{X})$, implying that we have $C' \ni x \notin -\mathsf{ToRo}(\mathbb{X})$. But this implies that $x \in \mathsf{ToRo}(\mathbb{X})$, implying that $\widehat{x} = \mathbb{X}$, implying, since $x \in C'$, that $\mathbb{X} = \widehat{x} \subseteq C'$, contradicting the fact that $C' \neq \mathbb{X}$. ⊠

It had been posited in previous work [1] that given $\mathbb{A} \in$ **tHA**, we have $\mathbb{A}$ subdirectly-irreducible if and only if $\mathsf{ToRo}(\mathbb{A}_*)$ is non-empty. While Theorem 7.3.1 confirms the forward direction of this statement, Example 7.3.1 provides a counterexample to the backward direction, i.e., an $\mathbb{A} \in$ **tHA** such that $\mathsf{ToRo}(\mathbb{A}_*)$ non-empty, but $\mathbb{A}$ *not* subdirectly-irreducible.

Example 7.3.1 : Consider the algebra $\mathbb{A} \in$ **tHA** (depicted on the left) and its dual $\mathbb{A}_* \in$ **tES** (depicted on the right).





Here we have $\Box : a \mapsto a$ and $\blacklozenge : a \mapsto a$ and $\mathsf{Refl}(\mathbb{A}_*) = \mathbb{A}_*$.

It can be checked that $\mathsf{ToRo}(\mathbb{A}_*) = \{\uparrow\omega\}$ (as $\widehat{\uparrow n} = \uparrow n \subsetneq \mathbb{X}$), so $\mathsf{ToRo}(\mathbb{A}_*)$ is non-empty. Note, however, that $\mathsf{ToRo}(\mathbb{A}_*)$ is *not* open. For if $\mathsf{ToRo}(\mathbb{A}_*)$ *were* open, it would belong to $\mathsf{OpDown}(\mathbb{A}_*)$, which would mean $-\mathsf{ToRo}(\mathbb{A}_*)$ would belong to $\mathsf{ClUp}(\mathbb{A}_*)$, which would imply, by Fact 2.7.2, that there is some $F \in \mathsf{Filt}(\mathbb{A})$ such that $-\mathsf{ToRo}(\mathbb{A}_*) = \bigcap \pi[F] = \{x \in \mathbb{A}_* \mid F \subseteq x\}$. But one can see quite clearly that there is no such filter F which is a subset of all $\uparrow n$ but not a subset of $\uparrow\omega$.

It can also be checked that $\blacklozenge\mathsf{Com}(\mathbb{A}) = \mathbb{A}$ (as $a \wedge \blacklozenge b = a \wedge b = \blacklozenge(a \wedge b)$), implying, by Proposition 3.3.1, that $(\forall a \in \mathbb{A})(\uparrow a \in \blacklozenge\mathsf{Filt}(\mathbb{A}))$, implying that $\mathbb{A}$ *cannot* have a second-least $\blacklozenge$-filter, finally implying, by Theorem 3.2.1, that $\mathbb{A}$ is *not* subdirectly-irreducible.

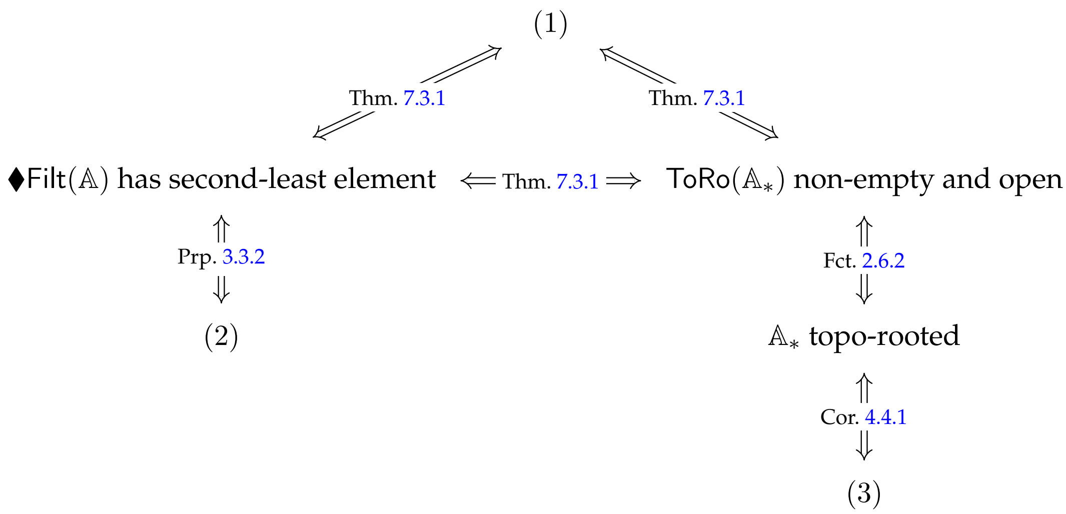
So here we have $\mathsf{ToRo}(\mathbb{A}_*)$ non-empty, but $\mathbb{A}$ *not* subdirectly-irreducible.

If we restrict our focus to the finite case, we can state an element-wise and frame-theoretic analogue to Theorem 7.3.1.

Theorem 7.3.2 : Given $\mathbb{A} \in \mathbf{tHA}_{\mathsf{fin}}$, the following are equivalent.

1. $\mathbb{A}$ is subdirectly-irreducible
2. $\mathbb{A}$ has a $\blacklozenge$-opremum
3. $\mathbb{A}_*$ is Z-rooted

Proof :



⊠

7.4 A final completeness result

When possible, it is desirable to combine finiteness (achieved by the FMP) and the frame-condition dual to subdirect-irreducibility to arrive at a simple class of frames for our logic. This allows us, when working with relational models of our logic, to not only restrict our consideration to finite frames, but finite frames of a certain shape. In the case of **HC**, we can do just this, proving that **HC** is sound and complete with respect to the class of finite, rooted posets [44, Theorem 6.12] (where rootedness corresponds to subdirect-irreducibility in the finite case [5, Theorem 2.3.16]). Here, we state an analogous result for our logic **tHC**.

Theorem 7.4.1 : tHC $\dashv\vdash$ The class of finite Z-rooted temporal transits.

Proof : Let the above-described class be denoted by $\mathbf{Z}$. ($\vdash$) Arguing via the contrapositive, we let $\varphi \notin \mathbf{tHC}$, implying, by Theorem 7.1.2, there exists some $\mathbb{A} \in \mathbf{tHA}_{\mathsf{fsi}}$ such that $\mathbb{A} \not\models \varphi$. By Theorem 7.3.2, this implies that $\mathbb{A}_*$ is Z-rooted. By Lemma 6.1.3, we have $\mathbb{A}_* \not\models \varphi$. So since $\mathbb{A}_*$ is a finite Z-rooted temporal transit, we can conclude that $\mathbf{Z} \not\models \varphi$. ($\dashv$) Since $\mathbf{Z} \subseteq \mathbf{tTran}$, Corollary 6.2.1 implies that $\mathbf{Z} \models \mathbf{tHC}$. ⊠

Chapter 8

Conclusion and future work

In this thesis, we studied the logic $\mathbf{tHC}$ by developing an Esakia duality between its corresponding category of algebras and the category of temporal Esakia spaces, allowing us to subsequently develop a theory of the relational models of the logic. We utilised the developed duality theory to dually characterise simple and subdirectly-irreducible temporal Heyting algebras. In combination with the established FMP, this allowed us to state a completeness theorem combining finiteness and the frame-condition corresponding to the subdirect-irreducibility of temporal Heyting algebras, giving us a simple class of frames, limited in both size and shape, for our logic $\mathbf{tHC}$.

To conclude this thesis, we state several ideas resulting from the theory developed in this thesis that could be investigated in future work on $\mathbf{tHC}$.

1. Remark 3.3.2 points out the analogy between open elements of closure algebras and ⧫-compatible elements of temporal Heyting algebras. It is well-known that open elements of a closure algebra $\mathbb{C} := \langle \mathbb{B}, \Diamond, \Box \rangle$ form a Heyting algebra that is *not*, in general, a sub-Heyting algebra of $\mathbb{B}$ (the Boolean algebra reduct of $\mathbb{C}$) [23, Proposition 2.2.4]. On this Heyting algebra, we have $a \to b := \Box(\neg a \vee b)$. We conjecture that there may be additional structure definable on the ⧫-compatible elements ; in particular, that a *co-Heyting algebra* may be definable. The topological representation of the co-implication would likely be $\uparrow(K_1 \cap -K_2)$ (where $K_1, K_2 \in \mathsf{ClopArcUp}(\mathbb{A}_*)$), but it is not clear how this would be represented algebraically. This problem reduces to defining a co-implication $a - b$ on $a, b \in \blacklozenge\mathsf{Com}(\mathbb{A})$ such that $\pi(a - b) = \uparrow(\pi a \cap -\pi a)$.

2. In Chapter 4, we define two notions of « reachability » on temporal Esakia spaces and show that they coincide in the finite case. « Topo-reachability » was given in terms of closed archival upsets and « Z-reachability » was given in terms of a zig-zagging relation, but only defined in the finite case. We believe that the relation Z could be extended to the infinite case, using a transfinite definition along the lines

of $Z_\lambda := \bigcup_{\alpha<\lambda} Z_\alpha$ for limit ordinals λ. Indeed, this was investigated and several of the lemmas in §4.3 can still be proven, but the main difficulty lies in making the topology behave well with this relation. If we are to maintain coincidence with « topo-reachability » in the infinite case, we want $Z[S]$ to be closed (to correspond to $\widehat{S}$), but Z is defined using unions, so proving closedness is very non-trivial. Defining Z in the infinite case and showing coincidence with $\trianglelefteq$ would likely require the development of a great deal of additional theory on temporal Esakia spaces.

3. In §6.3, we define the smallest transitive filtration on a temporal intuitionistic Kripke model and show that **tIKM** is closed under this class-operation $(-)_\Sigma$. As mentioned immediately before Construction 6.3.1, we opted to work with only the smallest transitive filtration because our goal was to establish the FMP for relational models. It is an open question as to what other methods of filtration can be defined without leaving the class **tIKM**. In particular, since we have two relations $R^\triangleleft$ and $R^\triangleright$, it is not clear which filtrations preserve (tTran.2). Also, it is not clear how large a filtration of $R^\triangleright$ can be such that $\dot{R}^\triangleright_\Sigma$ is still a valid intuitionistic filtration of $\leqslant := \dot{R}^\triangleright$.

4. In [24, Corollary 21], Esakia established a *modal companionship* between the intuitionistic modal logic **mHC** and the classical modal logic **K4.Grz**[1]. For reference, the connection between these logics was then studied in great depth in [38]. The modal companionship was accomplished by combining the Gödel translation [6, Definition 33] with the so-called « splitting map » [24, p. 357] studied by Boolos, Goldblatt, and Kuznetsov. The resulting map # commutes with the connectives $\{\top, \bot, \wedge, \vee, \Box\}$ and is otherwise defined as follows.

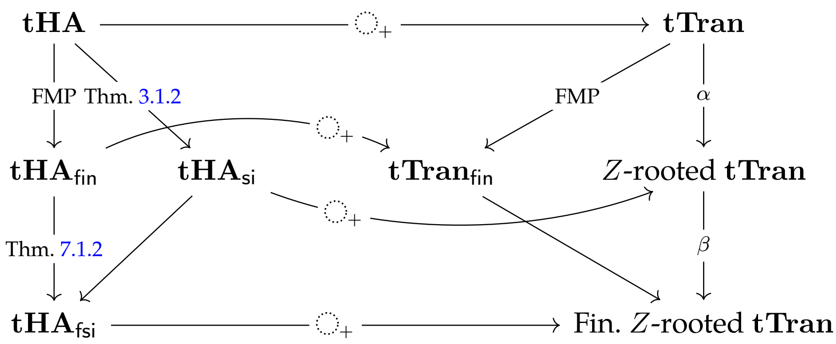
$$\#p := p \wedge \Box p \qquad \#(\varphi \to \chi) := (\#\varphi \to \#\chi) \wedge \Box(\#\varphi \to \#\chi)$$

Future work could extend this translation to the language $\mathcal{L}_t^i$ by defining $\#\blacklozenge\varphi := \blacklozenge\#\varphi$ and establishing a modal companionship with a temporal version of **K4.Grz**. A temporal version of this logic was briefly mentioned in [34, p. 203] and appears to be a good candidate for modal companionship with **tHC**.

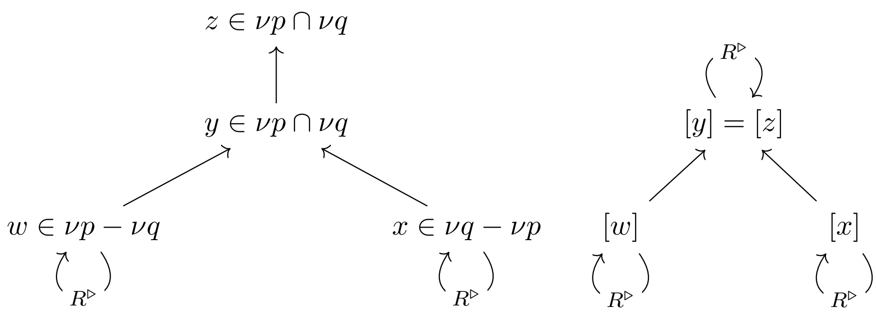
5. To arrive at Theorem 7.4.1, we essentially took the route of refuting φ on an algebra $\mathbb{A}$, then refuting it on a finite algebra $\mathbb{B}$, the refuting it on a subdirectly-irreducible $\mathbb{C}$ that is used to generate $\mathbb{B}$, then transferring this refutation to its dual frame $\mathbb{C}_+$, which we knew to be finite and

[1] In other literature, this logic is sometimes known as *weak Grzegorczyk logic* and denoted by **wGrz** [34].

Z-rooted. Essentially, we took the route « ◌$_+$ ∘ Thm. 7.1.2 ∘ FMP ».



It is not clear if there exist class-operations α and β such that the other paths become available. In particular, does there exist a truth-reflecting α that makes a temporal transit Z-rooted ? This could be seen as analogous to the well-known model-operation of taking « generated submodels » [10, Definition 2.5], which forces a model to become rooted. Also interesting : does there exist a truth-reflecting β that makes a Z-rooted temporal transit finite *without* breaking Z-rootedness ? This would likely be some kind of filtration designed to preserve the relation Z, but the smallest transitive filtration does *not* preserve Z or Z-rootedness in general.



In the model on the left, we have $y\ Z\ w$, and $y\ Z\ x$ but if we perform the smallest transitive filtration through $\Sigma := \{p, q\}$ (depicted on the right), then we do *not* have $[y]\ Z\ [w]$ or $[y]\ Z\ [x]$ because $[y] \in ([w], [y]] \cap \mathsf{Refl}(\mathbb{X}_\Sigma)$ and likewise for $[x]$. Indeed, the original model is Z-connected and the filtrated model is not even Z-rooted. So it is an open problem how to preserve Z in a filtration and, more generally, how to define α and β so that the paths « $b \circ \alpha \circ$ ◌$_+$ » and « $\beta \circ$ ◌$_+$ ∘ Thm. 3.1.2 » become available to us.

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Appendix A

`thacheck` : a symbolic model checker for tHA

We briefly present a symbolic model checker for temporal Heyting algebras called `thacheck`, authored in Python 3.12 and available for reference and use at the following url.

https://gitlab.com/dqalombardi/thacheck/

This checker proved to be very valuable for the preliminary work on this thesis when it was used to generate all temporal Heyting algebras of cardinalities less than or equal to 5. This provided a good deal of initial intuition as to what these algebras as well as their $\blacklozenge$-filters and $\blacklozenge$-compatible elements look like. It was also used to keep a database of temporal Heyting algebras and periodically test hypotheses against them. This saved a great deal of time that would have been spent trying to prove facts that were very quickly shown to have a counterexample among the algebras in this database.

This package is very extensible ; it can be used as a model checker for any class of algebras that the user might want to define. In particular, `BoundedLattice`, `HeytingAlgebra`, `FrontalHeytingAlgebra`, and `TemporalHeytingAlgebra` are all already included in the package. In addition, the functors $\circ_*$ and $\circ^*$ could be easily added to the package as the notion of a « prime filter » is already present as well as the logic necessary to define relations and generate topologies. This addition would allow us, in light of Lemma 6.1.5, to consider this a *relational* symbolic model checker as well.

We give a brief tour of the package and how it can be used.

The source directory is as follows.

```
src/thacheck
|-- exceptions.py
|-- language
|   |-- connectives.py
|   |-- equations
|   |   |-- equation.py
|   |   `-- equations.py
|   |-- formula.py
|   `-- variables
|       |-- free_variable_store.py
|       |-- variable_database.py
|       `-- variable.py
|-- logging.py
|-- models
|   |-- algebras
|   |   |-- algebra.py
|   |   |-- bounded_lattice.py
|   |   |-- frontal_heyting_algebra.py
|   |   |-- heyting_algebra.py
|   |   |-- lattice.py
|   |   `-- temporal_heyting_algebra.py
|   |-- element.py
|   `-- poset.py
|-- set_theory.py
`-- types
    |-- serialisable.py
    `-- types.py
```

Users can define equations using the `Equation` class. Below, we show how (fHA.o.3) can be defined equationally, i.e. $p \wedge \Box p = p$, and stored in the variable `F2`. Indeed, all of the axioms for **BDL**, **fHA**, and **tHA** have been included in the enum `Equations`.

```
_p1 = FreeVariableStore.get("p", 1)
_P1 = Formula(node=_p1, subformulas=None)
_GP1 = Formula(
    node=Connectives.G,
    subformulas=(_P1,),
)
F2 = Equation(
    lhs=Formula(
        node=Connectives.MEET,
        subformulas=(
            _P1,
            _GP1,
        ),
    ),
    rhs=_P1,
)
```

Users can then define algebras by defining their elements and operations.

```
class TemporalHeytingAlgebra(FrontalHeytingAlgebra):

    def __init__(
        self,
        elements: FrozenSet[Element],
        meet_map: BinaryOperation,
        join_map: BinaryOperation,
        imp_map: BinaryOperation,
        p_map: UnaryOperation,
        g_map: UnaryOperation,
        bot_map: NullaryOperation,
        top_map: NullaryOperation,
        check: bool = False,
        name: str = None,
    ) -> None:
```

Here, `p_map` refers to $\blacklozenge$ and `g_map` refers to $\Box$ (following the tradition of denoting a past diamond and a future box by P and G respectively). If the keyword argument `check=True` is passed, then the symbolic model checker will check that all axioms are satisfied on the passed structure, i.e. that it is indeed a temporal Heyting algebra.

At the core of how the checker works lies the `satisfies` method, which is inherited by any subclass of `Algebra`.

```
def satisfies(self, equation: Equation, assignment:
    AssignmentMap) -> bool:
    lhs_evaluation = self.evaluate(formula=equation.lhs,
        assignment=assignment)
    rhs_evaluation = self.evaluate(formula=equation.rhs,
        assignment=assignment)
    return lhs_evaluation == rhs_evaluation
```

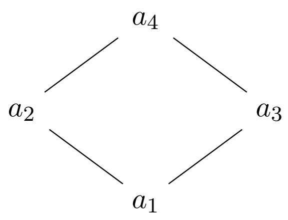
This will check if a given `assignment` (a map $\mathsf{Prop} \to \mathbb{A}$) makes the `lhs` and `rhs` of an `Equation` equal.

To see if an equation is valid on an algebra regardless of the valuation (as in Definition 2.4.5), the user can use the method `validates`, which generates all possible assignments on $\mathbb{A}$ and tests them using `satisfies`.

```
def validates(self, equation: Equation) -> bool:
    for assignment in self.get_assignments(equation.variables
        ):
        if not self.satisfies(equation, assignment):
            return False
    return True
```

Finally, we recognise that defining an algebra by defining all of its operations can be tedious, so we mention that a user can also provide a minimal ordering and use the method `Poset.from_view_map` to generate a poset. This will take the reflexive, transitive closure of whatever ordering is passed. If this poset is a bounded distributive lattice, then the user can then pass this poset to `HeytingAlgebra.from_poset`, which will return a `HeytingAlgebra` (since all finite bounded distributive lattice are Heyting algebras).

Thus, to generate the Heyting algebra



one need only define

$$a_1 < a_2 \qquad a_1 < a_3 \qquad a_2 < a_4 \qquad a_3 < a_4.$$

Once the user has this `HeytingAlgebra`, they can pass its operations (`meet_map`, `join_map`, etc.) along with their manually-defined `p_map` and `g_map`, to the constructor of the `TemporalHeytingAlgebra` and begin checking their equations.