A Constructive Small Object Argument

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

The small object argument is an important tool in homotopy theory and recently also in logic. Originally proved by Quillen in 1967, it has evolved significantly over time. In recent work Bourke and Garner proved the most general version of the small object argument thus far, using the notion of cofibrant generation by a double category. Moreover, they proved that this version is sufficient to generate all interesting algebraic weak factorisation systems. However, their proof is not constructively valid. In this thesis we prove a constructive version of Bourke and Garner's small object argument by restricting to the finitary case. We introduce the notion of a finitary algebraic weak factorisation system, and we identify conditions on a double category under which it generates a finitary AWFS. Two crucial steps in the proof are constructing the free algebra for a finitary pointed endofunctor, and constructing colimits in the category of finitary monads on a cocomplete category. We show that our result is an important step in building a constructive model of homotopy type theory based on effective Kan fibrations, a line of research recently initiated by van den Berg and Faber.

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1. Introduction

1.1. Small Object Argument

Weak Factorisation Systems. A *weak factorisation system* (WFS) on a category $\mathcal{C}$ consists of two classes of morphisms, a left class $\mathcal{L}$ and a right class $\mathcal{R}$, satisfying certain axioms, for example that each morphism in $\mathcal{C}$ factors as a left map followed by a right map. These systems arise naturally in many areas of mathematics and are particularly ubiquitous in homotopy theory. For instance every model category has two interacting weak factorisation systems [13, 20]. The *small object argument*, originally proved by Quillen in 1967 [19], is an important tool to generate such systems. Starting with a set $\mathcal{I}$ of morphisms in $\mathcal{C}$ subject to certain smallness conditions, it states that the weak factorisation system 'freely generated' by $\mathcal{I}$ exists; this is called the WFS *cofibrantly generated* by $\mathcal{I}$. The proof relies on a transfinite construction to factorise a morphism into a left and a right map.

Over the years the theory of weak factorisation systems has evolved significantly, which has resulted in the 'algebraization' of the concept of a weak factorisation system. That is, a more structured version of the concept where *properties* are turned into *algebraic structure*. This led to the concept of *natural weak factorisation systems*, introduced by Grandis and Tholen in 2006 [11]. Instead of a left class $\mathcal{L}$ and a right class $\mathcal{R}$, we now have a category of coalgebras L-**Coalg** and a category of algebras R-**Alg** for some comonad L and monad R. Subsequently the concept *algebraic weak factorisation systems* (AWFS) arose [21]. This is a natural weak factorisation system with an additional distributivity law of the comonad over the monad.

This algebraic version of a weak factorisation system has several advantages. Firstly it makes the theory more constructive, as it leads to explicit lifts, explicit factorisations, and explicit structure that shows that a morphism is a left or right map. Secondly it captures more examples than the non-algebraic variant, making it more general. For instance the split Grothendieck fibrations occur as the class of right maps of an AWFS, but not of a WFS. Other examples include classical homotopical structures [10, 21] and also several variants of fibrations emerging in type theory and homotopy type theory [2, 9]. Thirdly, it has better categorical properties. For example it comes with a category of left maps closed under colimits and a category of right maps closed under limits.

The Algebraic Small Object Argument. Along with this evolution of weak factorisation systems, the small object argument has evolved as well. Indeed, since this is the standard tool to generate weak factorisation systems, one would desire an algebraic variant of this as well. This adaptation was done by Garner in 2009 [10], when he proved the *algebraic small object argument*. This uses the notion of cofibrant generation by a small category instead of a set. He shows that under a suitable size condition,[1] every small category generates an accessible algebraic weak factorisation system, that is, an AWFS $(\mathcal{C}, \mathsf{L}, \mathsf{R})$ where the underlying monad R and comonad L preserve κ-filtered colimits for some regular cardinal κ. However, it was noted in a subsequent paper by Bourke and Garner [7] that the notion of cofibrant generation by a category is insufficient to capture all interesting AWFS. They introduce the notion of cofibrant generation by a *double category*. They then prove an even more general version of the small object argument, which states that under a suitable size condition any small double category generates an accessible AWFS. Moreover, they also prove a converse result: every accessible AWFS is generated by some small double category, thus proving that this version of the small object argument is sufficient to generate all interesting examples.

Constructivity. What is still missing in this picture is a *constructive* small object argument. The constructive issue here is that the theory of ordinals and cardinals is classical, and is not well-developed constructively. In their proof, Bourke and Garner rely on Kelly's monumental work [15] for the existence of several free structures using transfinite constructions that are not constructively valid (using for instance the fact that the ordinals are well-ordered).

The aim of this thesis is to prove a constructive version of Bourke and Garner's algebraic small object argument for double categories. We circumvent the constructive issues by introducing the notion of a *finitary AWFS*. This is an AWFS $(\mathcal{C}, \mathsf{L}, \mathsf{R})$ where the underlying monad R and comonad L are finitary, i.e. preserve ω-filtered colimits. We then proceed to prove a finitary version of the small object argument: we identify conditions under which a double category generates a finitary AWFS. More specifically, we prove the following theorem.

Theorem (finitary small object argument)**.** *Let $\mathcal{C}$ be a locally small, (countably) cocomplete category,*[2] *and let $U : \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ be a double functor subject to the following conditions*

1. *$\mathbb{J}$ is small,*
2. *the object Uj is ω-compact*[3] *for every object $j \in \mathcal{J}_0$.*

Then the AWFS cofibrantly generated by U exists and is finitary.

It appears as Theorem 6 in this thesis. Moreover, the proof we provide is constructive. In the rest of the introduction, we illustrate the usefulness of the above theorem by showing that it is an important step in the development of a constructive model of homotopy type theory based on effective Kan fibrations. Let us first give a brief introduction to constructive mathematics.

[1] The base category $\mathcal{C}$ is locally presentable.

[2] In this thesis, we show the theorem for cocomplete categories, but inspecting the proof shows that it can easily be adapted to the case of a countably cocomplete category.

[3] That is, the functor $\hom_{\mathcal{C}}(Uj, -) : \mathcal{C} \to \mathbf{Sets}$ is finitary.

1.2. Constructive Mathematics

Constructive mathematics is distinguished from classical mathematics by the logic that it is based on: intuitionistic logic instead of classical logic. Certain axioms that are true in classical logic, such as the law of the excluded middle, are not valid intuitionistically. The program of defining the foundations of mathematics on intuitionistic logic was started by L.E.J. Brouwer and further developed by his student Arend Heyting. The standard explanation of intuitionistic logic is the *BHK interpretation*, which emphasises the notion of a proof as an effective procedure. For example 'there exists' is interpreted as 'we can construct'. This point of view is strengthened by the *Curry–Howard isomorphism*, which establishes a one-to-one correspondence between proofs in intuitionistic propositional logic and terms in lambda calculus. This enables one to transfer between both worlds, which led to the interpretation of 'propositions as types' and 'proofs as programs', thus showing that intuitionistic logic and computation are fundamentally linked. Therefore, providing a constructive proof of a theorem has the following advantages over a classical proof:

- By the above correspondence, a constructive proof has a *computational content*, we could in principle carry out the proof algorithmically. This makes constructive proofs particularly well-suited for implementation in theorem provers such as Coq.
- Since the axiom system of intuitionistic logic is a subset of classical logic, a constructive proof is 'stronger' and thus more general. They can be internalised to arbitrary Grothendieck toposes (not just **Sets**).

In this thesis, we work in Aczel's constructive set theory **CZF** [18], which is a subsystem of classical **ZF**, Zermelo-Fraenkel set theory (without choice). This means that we cannot make use of certain classical principles that are common in mathematical arguments. One example is the *least number principle*

$$A \subseteq \mathbb{N}, \exists x \in \mathbb{N}(x \in A) \Rightarrow A \text{ has a smallest element.}$$

This is not constructively true, since there might not be an effective procedure to find such a smallest element. The theory of ordinals and cardinals is also not well-developed constructively. For instance, it is not true constructively that the ordinals are totally ordered, since it is not decidable whether $\alpha < \beta, \alpha = \beta$, or $\alpha > \beta$. This makes most transfinite arguments constructively invalid. However, induction on the natural numbers is constructively valid. In Chapter 4 we make use of this; we work with the ordinal $\omega + \omega$, for which the order is decidable.

1.3. A Constructive Model of Homotopy Type Theory

The classical Kan–Quillen model structure on the category of simplicial sets [19] is a model category which is seen as the cornerstone of modern simplicial homotopy theory, it provides a combinatorial model of homotopy theory. Kan fibrations play an important role in this: they are the fibrations in the Kan–Quillen model structure, they serve as combinatorial analogs of Serre fibrations of topological spaces.

One step in the construction of the Kan–Quillen model structure is to prove that Kan fibrations form the right class of a weak factorisation system. Specifically, one can show that

$$(\text{Anodyne morphisms}, \text{Kan fibrations})$$

is a weak factorisation system. The proof of this relies on the Quillen small object argument.

Though they originated in homotopy theory, weak factorisation systems have now become important in logic as well, in particular for models of (homotopy) type theory [1]. Voevodsky [14] constructed a model of homotopy type theory based on the Kan–Quillen model structure. However, his proof is not constructive. Indeed, it was shown by Bezem, Coquand and Parmann [4] that it is constructively unprovable that Kan fibrations are closed under pushforward along Kan fibrations. Since the type families are interpreted as Kan fibrations in Voevodsky's model, it is crucial that one can prove this property. In an attempt to circumvent this problem, roughly two alternative approaches arose. Firstly, to abandon simplicial sets and to turn to cubical sets. Researchers in this direction have been able to constructively build a model of homotopy type theory in cubical sets [8]. Secondly, to stick with simplicial sets, but to introduce an alternative notion of a Kan fibration, an 'effective Kan fibration' [2]. We zoom in a bit further on this approach in the next section.

1.4. Effective Kan Fibrations

In [2] van den Berg and Faber introduce the notion of an *effective Kan fibration* as a tool to constructively build a model of homotopy type theory.

In [3] van den Berg and Geerligs show that the effective Kan fibrations are the right class of an *algebraic weak factorisation system* (AWFS). However, this result relies on the small object argument for double categories as presented in [5] (equivalent to the one presented by Bourke and Garner in [7]), which is not constructively valid. The question of whether a constructive proof exists of their result is left as an open problem.

Their construction is along the following lines.

- A *decidable sieve* of Δ_n is a sieve $S \subseteq \Delta_n$ such that for any $0 \leq m \leq n$ and any $p : \Delta_m \to \Delta_n$ it is decidable whether or not p factors through S.
- They define a double category over $\mathbb{S}\mathbf{q}(\mathbf{sSet})$, $\mathbb{D}_\ell \to \mathbb{S}\mathbf{q}(\mathbf{sSet})$, where the objects of $\mathbb{D}_\ell$ are the decidable sieves. They show that $\mathbf{RLP}(\mathbb{D}_\ell)$ is exactly the category of effective Kan fibrations.
- Finally they apply [5, Proposition 18] to conclude that $(\mathbf{LLP}(\mathbf{RLP}(\mathbb{D}_\ell)), \mathbf{RLP}(\mathbb{D}_\ell))$ is an AWFS, which means that the effective Kan fibrations are the right class in an AWFS.

For the last step, note that the category $\mathbb{D}_\ell$ is small and the category **sSet** is locally presentable, so that the conditions of the small object argument for double categories are satisfied. In fact, even stronger conditions apply. We have that $\mathbb{D}_\ell$ is countable, and moreover that every decidable sieve is a finite colimit of representables.

Corollary 4 then implies that the conditions of Theorem 6 are satisfied, whence in order to provide a constructive proof of the fact that the effective Kan fibrations are the right class of an AWFS, it suffices to provide a constructive proof of Theorem 6.

1.5. Contribution and Structure of this Thesis

Goal. The aim of this thesis is to provide a constructive proof of a finitary version of the small object argument, as stated in Theorem 6, and thereby to provide a solution to the open problem left in [3]. Since this version is sufficient to show that the effective Kan fibrations are the right class of an AWFS, this is an important step in building a constructive model of homotopy type theory based on effective Kan fibrations.

For the outline of the proof of Theorem 6, we follow the paper of Bourke and Garner [7]. The goal is thus twofold. Firstly, to adapt the parts that are constructively valid in this proof to the finitary case and to present them in a more detailed way. Secondly, to identify the non-constructive parts in the proof, and to replace these with constructively valid proofs, adapted to the finitary version of the small object argument.

Bourke and Garner's proof proceeds roughly in three steps:

Firstly they prove a Beck theorem for AWFS, which characterises the image of the semantics functor $(-)\text{-}\mathbb{A}\mathbf{lg} : \mathbf{AWFS}_{\mathrm{lax}} \to \mathbf{DBL}^{\mathbf{2}}$. This is helpful in the proof of the small object argument since it allows one to prove that a double category is the right class of an AWFS by checking the conditions of the Beck theorem.

Secondly they prove a small object argument for categories, which is equivalent to the algebraic small object argument proved by Garner [10]. The crucial ingredient here is the existence of the free algebra for an accessible pointed endofunctor, for which they invoke Kelly [15, Theorem 22.3]. The proof uses a transfinite argument which is not constructively valid.

Thirdly they prove a small object argument for double categories. The crucial ingredient here is the existence of coequalisers of accessible monads. For this they use another result by Kelly [15, Theorem 27.1], which is also not constructively valid.

Contributions. The original contributions of this thesis can be summarised as follows:

- We introduce the notion of a finitary AWFS, and we formulate a finitary version of the small object argument for categories (Theorem 5) and for double categories (Theorem 6).

- We adapt the proof of Bourke and Garner's small object argument [7] to this finitary case.

- We identify the non-constructive parts of this proof, and replace these with constructive proofs. This can be summarised in two major steps:
  (i) Constructing the free algebra on a finitary pointed endofunctor (Theorem 2). Our proof of this is based on [6, Appendix A], and is adapted to the finitary case.
  (ii) Constructing colimits in the category of finitary monads on a cocomplete category (Theorem 4). Our proof strategy is based on [15], again adapted to the finitary case. The proof relies on the existence of the free algebra on a pointed endofunctor, for which we use (i). Throughout the proof we explicitly keep track of the construction and we introduce the notion of an *effective repletion* to avoid the use of the axiom of choice.

Structure. In Chapter 2 we recall the preliminaries about algebraic weak factorisation systems and double categories.

In Chapter 3 we recall the Beck theorem for AWFS proved by Bourke and Garner [7]. The proof of this is already constructively valid, we fill in the details left out in [7].

In Chapter 4 we provide a constructive proof that the free algebra on a finitary pointed endofunctor (P, ρ) exists.

In Chapter 5 we show that the construction in Chapter 4 also provides the free monad T on a pointed endofunctor, namely the monad induced by the free-forgetful adjunction of the free algebra on (P, ρ). This was proven by Kelly [15], and we check that the proof is constructively valid. Moreover, if P is finitary, then so is T. Subsequently we use our result from Chapter 4 to constructively prove that the category of finitary monads on a cocomplete category is cocomplete, which is a finitary version of [15, Theorem 27.1]. A corollary of our construction is that the algebra functor which assigns to a monad its category of algebras preserves limits, something that we need in the proof of Theorem 6 in Chapter 7.

In Chapter 6 we recall the notion of cofibrant generation by a category, and we give a constructive proof of the finitary small object argument for categories. We depend on Chapter 4 for a constructive proof of the existence of the free algebra for a finitary pointed endofunctor, and on Chapter 5 to show that the resulting monad is again finitary, and free on a pointed endofunctor.

In Chapter 7 we then give a proof of Theorem 6 which is constructively valid. The proof relies on the finitary small object argument for categories as proven in Chapter 6, and depends on Chapter 5 to obtain coequalisers in the category of finitary monads which are moreover preserved by the algebra functor.

Finally, in Chapter 8, we end with a summary of the thesis and some directions for future research.

2. Preliminaries

In this chapter we give a brief introduction to the theory of algebraic weak factorisation systems, a more detailed treatment can be found in [22].

2.1. Smallness in Categories

We first recall certain smallness conditions on a category. Let κ be a regular cardinal. A category $\mathcal{C}$ is κ-filtered if every diagram $\mathcal{I} \to \mathcal{C}$ with $|\mathcal{I}_1| < \kappa$ has a cocone.[1] A *κ-filtered colimit* is one over a κ-filtered category.

Definition 1. Let $\mathcal{C}$ be a category, X an object in $\mathcal{C}$ and κ a regular cardinal. We say that

(i) X is *κ-compact* if the corepresentable functor

$$\hom_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathbf{Sets}$$

preserves κ-filtered colimits.

(ii) X is *small* if it is κ-compact for some regular cardinal κ.

A subclass $S \hookrightarrow \mathrm{Ob}(\mathcal{C})$ *generates* $\mathcal{C}$ if every object in $\mathcal{C}$ is a colimit of objects in S, that is, the colimit over a small diagram

$$D \to \mathcal{C}^0 \hookrightarrow \mathcal{C},$$

where $\mathcal{C}^0$ is the full subcategory on S. A category $\mathcal{C}$ is *locally presentable* if it is locally small, has all small colimits, and if there exists a set $S \hookrightarrow \mathrm{Ob}(\mathcal{C})$ of small objects that generates $\mathcal{C}$.

We call a presheaf *finitely generated* if it can be written as a finite colimit of representables.

We call a functor that preserves all filtered colimits a *finitary functor*. We call a monad R finitary if its underlying endofunctor R is finitary.

2.2. Algebraic Weak Factorisation Systems

Algebraic weak factorisation systems were originally introduced by Grandis and Tholen [11] (under the name natural weak factorisation systems) as a more structured version

[1] We may just write 'filtered' for ω-filtered.

of weak factorisation systems. The first ingredient is a *functorial factorisation system* on a category $\mathcal{C}$. This is a functor $F : \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{3}}$ which is a section of the composition functor $\circ : \mathcal{C}^{\mathbf{3}} \to \mathcal{C}^{\mathbf{2}}$. Here $\mathcal{C}^{\mathbf{2}}$ is the arrow category of $\mathcal{C}$ and $\mathcal{C}^{\mathbf{3}}$ is the category of composable pairs of arrows in $\mathcal{C}$. Note that such a functor consists of three 'components' $F = (L, E, R)$, where an arrow $f : X \to Y$ is sent to the pair

$$X \xrightarrow{Lf} Ef \xrightarrow{Rf} Y.$$

Since F is a section of the composition functor, we have $Rf \circ Lf = f$, so that indeed we have a factorisation of f. A functorial factorisation comes equipped with two natural transformations $\eta : 1 \Rightarrow R$ and $\epsilon : L \Rightarrow 1$ whose components at f are given respectively by

$$\begin{tikzcd} X \arrow[r, "Lf"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\eta_f}{\Longrightarrow}"] & Ef \arrow[d, "Rf"] \\ Y \arrow[r, "1"'] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "1"] \arrow[d, "Lf"'] \arrow[dr, phantom, "\overset{\epsilon_f}{\Longrightarrow}"] & X \arrow[d, "f"] \\ Ef \arrow[r, "Rf"'] & Y \end{tikzcd}$$

Hence we have a pointed endofunctor (R, η) and a copointed endofunctor (L, ϵ). We now define an algebraic weak factorisation system as follows.

Definition 2. An algebraic weak factorisation system (AWFS) on $\mathcal{C}$ consists of

- A functorial factorisation (L, E, R) on $\mathcal{C}$,
- An extension of (L, ϵ) to a comonad $\mathsf{L} = (L, \epsilon, \delta)$,
- An extension of (R, η) to a monad $\mathsf{R} = (R, \eta, \mu)$,
- The following square

$$\begin{tikzcd} Ef \arrow[r, "\delta_f"] \arrow[d, "LRf"'] & ELf \arrow[d, "RLf"] \\ ERf \arrow[r, "\mu_f"'] & Ef \end{tikzcd}$$

  constitutes a distributive law $LR \Rightarrow RL$ of the comonad L over the monad R. This is equivalent to the much simpler *Garner equation* [2],

$$\delta_f \circ \mu_f = \mu_{LF} \circ E(\delta_f, \mu_f) \circ \delta_{Rf}.$$

We call an AWFS *accessible* if its comonad L and monad R preserve κ-filtered colimits for some regular cardinal κ. We call an AWFS *finitary* if its comonad and monad are finitary.

Every algebraic weak factorisation system has an underlying weak factorisation system. This can be recovered in two ways. Firstly we can take $(\mathcal{L}, \mathcal{R}) = (L\text{-}\mathbf{Map}, R\text{-}\mathbf{Map})$, where L-**Map** is the class of maps in $\mathcal{C}$ that admit a coalgebra structure for the copointed endofunctor (L, ϵ) and R-**Map** is the class of maps in $\mathcal{C}$ that admit an algebra structure for the pointed endofunctor (R, η). Secondly, we can take L-**Coalg** and R-**Alg**, the coalgebras for the comonad L and the algebras for the monad R and close these classes under retracts. The following proposition states that these two are equivalent.

Proposition 1. *Let $(\mathcal{C}, \mathsf{L}, \mathsf{R})$ be an AWFS, then*

(i) $(L\text{-}\mathbf{Map}, R\text{-}\mathbf{Map})$ *is a weak factorisation system,*

(ii) L-**Map** *is the retract closure of* L-**Coalg** *and* R-**Map** *is the retract closure of* R-**Alg**.

Proof. See for instance [21]. □

For an AWFS to be finitary, it is sufficient that either its monad or its comonad is finitary, as the next lemma states.

Lemma 1. *Let $(\mathcal{C}, \mathsf{L}, \mathsf{R})$ be an AWFS. If R is a finitary monad or L is a finitary comonad, then the AWFS is finitary.*

Proof. A map $f : X \to Y$ in $\mathcal{C}^{\mathbf{2}}$ factors as $X \overset{Lf}{\to} Ef \overset{Rf}{\to} Y$. Since $\mathcal{C}^{\mathbf{2}}$ is a presheaf category, colimits are calculated pointwise. Hence L is finitary if and only if $E : \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ is finitary, and similarly for R. □

Given the factorisation of a morphism f in $\mathcal{C}$ as $Rf \circ Lf$, note that Rf underlies the free R-algebra $\boldsymbol{Rf} = (Rf, \mu_f)$ and Lf underlies the cofree L-coalgebra $\boldsymbol{Lf} = (Lf, \delta_f)$. Since the free R-algebra functor $\mathcal{C}^{\mathbf{2}} \to \mathsf{R}\text{-}\mathbf{Alg} : f \mapsto (Rf, \mu_f)$ is left adjoint to the forgetful functor $\mathsf{R}\text{-}\mathbf{Alg} \to \mathcal{C}^{\mathbf{2}}$, we have that for any R-algebra $\boldsymbol{g} = (g, \alpha)$ and morphism $(h, k) : f \to g$ in $\mathcal{C}^{\mathbf{2}}$ there is a unique arrow ℓ making the left diagram commute

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "Lf"'] \arrow[dd, "f"'] & & C \arrow[dd, "g"] \\ & Ef \arrow[ur, dashed, "\exists!\ell"'] \arrow[d, "Rf"'] & \\ B \arrow[r, "1"'] & B \arrow[r, "k"'] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "1"] \arrow[d, "Lg"'] & A \arrow[d, "g"] \\ Eg \arrow[r, "Rg"'] \arrow[ur, dashed, "\alpha"] & B \end{tikzcd} \tag{2.1}$$

and such that (ℓ, k) is an algebra morphism $\boldsymbol{Rf} \to \boldsymbol{g}$. In particular, taking $f = g$ and (h, k) the identity, there is a unique filler ℓ for the diagram on the right such that $(\ell, 1)$ is an algebra morphism $\boldsymbol{Rg} \to \boldsymbol{g}$. But α itself has these properties. This shows that an R-algebra is uniquely determined by its liftings against L-coalgebras. The dual fact can also be stated for L-coalgebras.

Before we turn to morphisms of AWFS, we need the following lemma which states that there is a one-to-one correspondence between lax monad morphisms over a functor T and liftings of T to the category of algebras.

Lemma 2 [12, Lemma 1]**.** *Let* $\mathsf{H} = (H, \eta, \mu)$ *and* $\mathsf{K} = (K, \iota, \nu)$ *be two monads on categories* $\mathcal{C}$ *and* $\mathcal{D}$ *respectively. There is a natural one-to-one correspondence between liftings of* T *to the categories of algebras,*

$$\begin{tikzcd} \mathsf{H}\text{-}\mathbf{Alg} \arrow[r, "\overline{T}"] \arrow[d, "U^H"'] & \mathsf{K}\text{-}\mathbf{Alg} \arrow[d, "U^K"] \\ \mathcal{C} \arrow[r, "T"'] & \mathcal{D} \end{tikzcd}$$

and natural transformations $\lambda : KT \to TH$ *making* $(T, \lambda) : \mathsf{H} \to \mathsf{K}$ *into a lax monad morphism in the sense of [23], that is, making the following diagram commute*

$$\begin{tikzcd} T \arrow[r, "\iota_T"] \arrow[dr, "T\eta"'] & KT \arrow[d, "\lambda"] & KKT \arrow[l, "\nu_T"'] \arrow[d, "K\lambda"] \\ & TH & KTH \arrow[d, "\lambda_H"] \\ & & THH. \arrow[ul, "T\mu"] \end{tikzcd} \tag{2.2}$$

Proof. Assume we have a lax monad morphism $(T, \lambda) : \mathsf{H} \to \mathsf{K}$. For an H-algebra $\xi : HX \to X$ we define

$$\overline{T}(HX \xrightarrow{\xi} X) = KTX \xrightarrow{\lambda_X} THX \xrightarrow{T\xi} TX.$$

To see that this is indeed a K-algebra, we observe the following two diagrams

$$\begin{tikzcd} TX \arrow[r, "\iota_{TX}"] \arrow[dr, "T\eta_X"] \arrow[ddr, "1"'] & KTX \arrow[d, "\lambda_X"] \\ & THX \arrow[d, "T\xi"] \\ & TX. \end{tikzcd} \qquad \begin{tikzcd} KKTX \arrow[r, "K\lambda_X"] \arrow[dd, "\nu_{TX}"'] & KTHX \arrow[r, "KT\xi"] \arrow[d, "\lambda_{HX}"'] & KTX \arrow[d, "\lambda_X"] \\ & THHX \arrow[r, "TH\xi"] \arrow[d, "T\mu_X"'] & THX \arrow[d, "T\xi"] \\ KTX \arrow[r, "\lambda_X"'] & THX \arrow[r, "T\xi"'] & TX. \end{tikzcd}$$

In the left diagram, the upper triangle commutes since λ is a lax monad morphism, and the lower triangle commutes since ξ is an H-algebra. In the right diagram, the left square commutes since λ is a lax monad morphism, the upper right square commutes by naturality of λ, and the lower right square commutes since ξ is an H-algebra.

Conversely, suppose we have a lifting $\overline{T} : \mathsf{H}\text{-}\mathbf{Alg} \to \mathsf{K}\text{-}\mathbf{Alg}$ which commutes with the forgetful functors. Note that we have the free-forgetful adjunctions $F_K \dashv U_K$ and

$F_H \dashv U_H$. We firstly define $\overline{\lambda} : F_K T \to \overline{T} F_H$ as the transpose of the map $T \xrightarrow{T\eta} TH = TU_H F_H = U_K \overline{T} F_H$. Next we define λ as

$$KT = U_K F_K T \xrightarrow{U_K \overline{\lambda}} U_K \overline{T} F_H = TU_H F_H = TH.$$

We leave it to the reader to verify that this is indeed a lax monad morphism and that the two constructions are mutually inverse. $\square$

Given two AWFS (L, R) and $(\mathsf{L}', \mathsf{R}')$ on categories $\mathcal{C}$ and $\mathcal{D}$ respectively, a morphism $(F, \alpha) : (\mathcal{C}, \mathsf{L}, \mathsf{R}) \to (\mathcal{D}, \mathsf{L}', \mathsf{R}')$ consists of a functor $F : \mathcal{C} \to \mathcal{D}$ and a natural family of maps $\alpha_f : E'Ff \to FEf$ such that for every morphism $f : X \to Y$ the following diagram commutes

$$\begin{tikzcd}
 & FX \arrow[dl, "L'Ff"'] \arrow[dr, "FLf"] & \\
E'Ff \arrow[rr, "\alpha_f"] \arrow[dr, "R'Ff"'] & & FEf \arrow[dl, "FRf"] \\
 & FY &
\end{tikzcd}$$

Moreover we require the induced map $(\alpha, 1) : R'F^{\mathbf{2}} \Rightarrow F^{\mathbf{2}}R$ to be a lax monad morphism and similarly the induced map $(1, \alpha) : L'F^{\mathbf{2}} \Rightarrow F^{\mathbf{2}}L$ to be an oplax comonad morphism. This means that these natural transformations satisfy unit and associativity conditions. For the former, we have the following two commutative diagrams

$$\begin{tikzcd}
 & R'F^{\mathbf{2}} \arrow[dd, "{(\alpha,1)}"] \\
F^{\mathbf{2}} \arrow[ur, "\eta' F^{\mathbf{2}}"] \arrow[dr, "F^{\mathbf{2}}\eta"'] & \\
 & F^{\mathbf{2}}R
\end{tikzcd}
\qquad
\begin{tikzcd}
 & R'F^{\mathbf{2}}R \arrow[r, "{(\alpha,1)R}"] & F^{\mathbf{2}}RR \arrow[dd, "F^{\mathbf{2}}\mu"] \\
R'R'F^{\mathbf{2}} \arrow[ur, "{R'(\alpha,1)}"] \arrow[dr, "\mu' F^{\mathbf{2}}"'] & & \\
 & R'F^{\mathbf{2}} \arrow[r, "{(\alpha,1)}"'] & F^{\mathbf{2}}R.
\end{tikzcd} \tag{2.3}$$

For the oplax comonad morphism, we require the following two diagrams to commute

$$\begin{tikzcd}
 & L'F^{\mathbf{2}} \arrow[dl, "\epsilon' F^{\mathbf{2}}"'] \arrow[dd, "{(1,\alpha)}"] \\
F^{\mathbf{2}} & \\
 & F^{\mathbf{2}}L \arrow[ul, "F^{\mathbf{2}}\epsilon"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & L'F^{\mathbf{2}}L \arrow[r, "{(1,\alpha)L}"] & F^{\mathbf{2}}LL \\
L'L'F^{\mathbf{2}} \arrow[ur, "{L'(1,\alpha)}"] & & \\
 & L'F^{\mathbf{2}} \arrow[ul, "\delta' F^{\mathbf{2}}"] \arrow[r, "{(1,\alpha)}"'] & F^{\mathbf{2}}L \arrow[uu, "F^{\mathbf{2}}\delta"']
\end{tikzcd} \tag{2.4}$$

A transformation $(F,\alpha) \Rightarrow (G,\beta)$ between two morphisms of AWFS is a natural transformation $\mu : F \Rightarrow G$ such that for every $f : X \to Y$ the following square commutes

$$\begin{tikzcd} E'Ff \arrow[r, "\alpha_f"] \arrow[d, "E'(\mu_X,\mu_Y)"'] & FEf \arrow[d, "\mu_{Ef}"] \\ E'Gf \arrow[r, "\beta_f"'] & GEf. \end{tikzcd}$$

These components form a 2-category $\mathbf{AWFS}_{\mathrm{lax}}$. Note that we have a functor $\mathbf{AWFS}_{\mathrm{lax}} \to \mathbf{Cat}^{\mathbf{2}}$ which sends an AWFS $(\mathcal{C},\mathsf{R},\mathsf{L})$ to the forgetful functor $U^R : \mathsf{R}\text{-}\mathbf{Alg} \to \mathcal{C}^{\mathbf{2}}$. To see that this is indeed a functor, suppose we have a morphism $(F,\alpha) : (\mathcal{C},\mathsf{L},\mathsf{R}) \to (\mathcal{D},\mathsf{L}',\mathsf{R}')$ of AWFS. Then since F underlies a monad morphism, by Lemma 2 the functor $F^{\mathbf{2}} : \mathcal{C}^{\mathbf{2}} \to \mathcal{D}^{\mathbf{2}}$ lifts to a functor $\overline{F} : \mathsf{R}\text{-}\mathbf{Alg} \to \mathsf{R}'\text{-}\mathbf{Alg}$ as in the following diagram

$$\begin{tikzcd} \mathsf{R}\text{-}\mathbf{Alg} \arrow[r, dashed, "\overline{F}"] \arrow[d, "U^R"'] & \mathsf{R}'\text{-}\mathbf{Alg} \arrow[d, "U^{R'}"] \\ \mathcal{C}^{\mathbf{2}} \arrow[r, "F^{\mathbf{2}}"'] & \mathcal{D}^{\mathbf{2}}. \end{tikzcd}$$

$\overline{F}$ sends an R-algebra (f,s) to the R'-algebra (Ff,t) where $t = (t_0, \mathrm{id}_{FY})$ with t_0 defined as the composition

$$E'Ff \xrightarrow{\alpha_f} FEf \xrightarrow{Fs_0} FX.$$

In the next section we introduce lifting structures, it turns out that there is a close connection between lifting structures and R-algebras, L-coalgebras.

2.3. Lifting Structures

Given a commutative square $\sigma : g \to f$, we call σ a *lifting problem* for g and f, and a diagonal filler φ of σ a *solution* to the lifting problem σ:

$$\begin{tikzcd} \cdot \arrow[r, "\sigma_0"] \arrow[d, "g"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "\sigma_1"'] \arrow[ur, dashed, "\varphi"] & \cdot \end{tikzcd}$$

We say that f has the *right lifting property* (RLP) against g if for every lifting problem $\sigma : g \to f$ a solution exists. Equivalently, we say that g has the left lifting property against f (LLP). For a class of morphisms $\mathcal{I}$ in $\mathcal{C}$, we write ${}^{\pitchfork}\mathcal{I}$ for the class of maps in $\mathcal{C}$ having the LLP against all maps in $\mathcal{I}$ and $\mathcal{I}^{\pitchfork}$ for the class of all maps having the RLP against all maps in $\mathcal{I}$.

Let $\mathcal{C}$ be a category and $m : \mathcal{I} \to \mathcal{C}^{\mathbf{2}}$ be a functor. We define the category of right lifting structure $\mathcal{I}^{\pitchfork}$ as follows.

- Objects are pairs (f, φ), where $f : X \to Y$ is a morphism in $\mathcal{C}$ and φ is an operation that picks a lift $\varphi_i(\sigma)$ for every lifting problem $\sigma : m(i) \to f$,

$$\begin{tikzcd} \cdot \arrow[r, "\sigma_0"] \arrow[d, "m(i)"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "\sigma_1"'] \arrow[ur, dashed] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r, "m(\alpha)_0"] \arrow[d, "m(j)"'] & \cdot \arrow[r, "\sigma_0"] \arrow[d, "m(i)"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "m(\alpha)_1"'] \arrow[urr, dashed] & \cdot \arrow[r, "\sigma_1"'] \arrow[ur, dashed] & \cdot \end{tikzcd}$$

  subject to the condition that for any $\alpha : j \to i$ in $\mathcal{I}$, we have that $\varphi_i(\sigma) \circ m(\alpha)_1 = \varphi_j(\sigma \circ m(\alpha))$.

- A morphism $(f, \varphi) \to (g, \psi)$ is a map $\tau : f \to g$ such that for any lifting problem $\sigma : m(i) \to f$, we have $\psi_i(\tau \circ \sigma) = \tau_0 \circ \varphi_i(\sigma)$.

$$\begin{tikzcd} \cdot \arrow[r, "\sigma_0"] \arrow[d, "m(i)"'] & \cdot \arrow[r, "\tau_0"] \arrow[d, "f"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "\sigma_1"'] \arrow[ur, dashed] \arrow[urr, dashed] & \cdot \arrow[r, "\tau_1"'] & \cdot \end{tikzcd}$$

Let $(\mathcal{C}, \mathsf{L}, \mathsf{R})$ be an AWFS. Given an L-**Map** (f, λ) and an R-**Map** (g, ρ), we have a canonical lifting operation

$$\Phi_{\boldsymbol{f},\boldsymbol{g}}(\sigma) = \rho \circ E(\sigma) \circ \lambda, \tag{2.5}$$

as illustrated in the following diagram

$$\begin{tikzcd} A \arrow[rrr, "\sigma_0"] \arrow[dr, "Lf"] \arrow[ddd, "f"'] & & & C \arrow[ddd, "g"] \arrow[dl, "Lg"'] \\ & E(f) \arrow[r, "E(\sigma_0,\sigma_1)"] \arrow[ddl, "Rf"'] & E(g) \arrow[ur, dashed, "\rho"'] \arrow[ddr, "Rg"] & \\ & & & \\ B \arrow[rrr, "\sigma_1"'] \arrow[uur, dashed, "\lambda"'] & & & D. \end{tikzcd}$$

This gives us a map $\Phi : R\text{-}\mathbf{Map} \to \mathsf{L}\text{-}\mathbf{Coalg}^{\pitchfork}$. This map is an isomorphism with inverse given by $(g, \varphi) \mapsto (g, \varphi_{\boldsymbol{Lg},\boldsymbol{g}}(\epsilon_g))$, where $\boldsymbol{Lg} = (Lg, \delta_g)$ is the cofree L-coalgebra on g.

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "Lg"'] & X \arrow[d, "g"] \\ Ef \arrow[r, "Rg"'] \arrow[ur, dashed, "\rho"] & Y \end{tikzcd} \tag{2.6}$$

The next lemma characterises the image of the induced map $\mathsf{R}\text{-}\mathbf{Alg} \to \mathsf{L}\text{-}\mathbf{Coalg}^{\pitchfork}$.

Lemma 3 [7, Lemma 1]**.** *The functor $\Phi : \mathsf{R}\text{-}\mathbf{Alg} \to \mathsf{L}\text{-}\mathbf{Coalg}^{\pitchfork}$ is injective on objects, fully faithful, and its image is given by exactly those lifting structures (g, φ) for which we have*

$$\varphi_{Lf}(u, v) \circ \mu_f = \varphi_{LRf}(\varphi_{Lf}(u, v), v \circ \mu_f), \tag{2.7}$$

as depicted in the following two diagrams,

$$
\begin{tikzcd}
A \arrow[r, "1"] \arrow[d, "Lf"'] & A \arrow[r, "u"] \arrow[dd, "Lf"] & Y \arrow[dd, "g"] \\
Ef \arrow[d, "LRf"'] & & \\
ERf \arrow[r, "\mu_f"'] \arrow[uurr, dashed] & Ef \arrow[r, "v"'] \arrow[uur, dashed] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "Lf"'] & Y \arrow[dd, "g"] \\
Ef \arrow[d, "LRf"'] \arrow[ur, dashed] & \\
ERf \arrow[r, "v \circ \mu_f"'] \arrow[uur, dashed] & X
\end{tikzcd}
$$

Note that we have $\mu_f \circ LRf = 1_{Ef}$ by one of the unit laws of a monad.

Proof. See [7] or [22]. □

In the next section we introduce double categories and we show that the functor $\mathbf{AWFS}_{\mathrm{lax}} \to \mathbf{Cat^2}$ can in fact be seen as a functor $\mathbf{AWFS}_{\mathrm{lax}} \to \mathbf{DBL^2}$, that is, the forgetful functor upgrades to a double functor $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$.

2.4. Double Categories

A *double category* $\mathbb{A}$ is an internal category in **Cat**. This means that $\mathbb{A}$ consists of two categories $\mathcal{A}_0$ and $\mathcal{A}_1$. The former consists of *objects* and *horizontal arrows*, while the latter consists of *vertical arrows* and *squares*. Moreover, $\mathbb{A}$ comes equipped with a composition, domain, codomain, and identity arrow, as shown in the following diagram.

$$
\mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_1 \xrightarrow{\ \circ\ } \mathcal{A}_1 \overset{\xrightarrow{\mathrm{dom}}}{\underset{\xrightarrow[\mathrm{cod}]{}}{\xleftarrow{\mathrm{id}}}} \mathcal{A}_0 .
$$

A double functor $F : \mathbb{C} \to \mathbb{D}$ is a pair of functors $F_0 : \mathcal{C}_0 \to \mathcal{D}_0$, $F_1 : \mathcal{C}_1 \to \mathcal{D}_1$ which is compatible with squares, that is, it sends a square $\alpha : f \to g$ to

$$
\begin{tikzcd}
F_0(sf) \arrow[r, "F_0(s\alpha)"] \arrow[d, "F_1 f"'] \arrow[dr, phantom, "\overset{F_1\alpha}{\Longrightarrow}"] & F_0(sg) \arrow[d, "F_1 g"] \\
F_0(tf) \arrow[r, "F_0(t\alpha)"'] & F_0(tg)
\end{tikzcd} .
$$

A typical example of a double category is the *category of squares* on any category $\mathcal{C}$. The objects are objects of $\mathcal{C}$, the horizontal arrows as well as the vertical arrows are arrows in $\mathcal{C}$, and the squares are commutative squares in $\mathcal{C}$. It is denoted by $\mathbb{S}\mathbf{q}(\mathcal{C})$. We define a *concrete double category* over $\mathcal{C}$ as a functor $V : \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ such that V_0 is the identity and V_1 is faithful.

Given an AWFS (L, R) on a category $\mathcal{C}$ we may consider the category $\mathsf{R}\text{-}\mathbf{Alg}$ as a double category, denoted as $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg}$, as follows.

- Objects are object of $\mathcal{C}$,
- Horizontal morphisms are morphisms of $\mathcal{C}$,
- Vertical morphisms are R-algebras,
- Squares are maps of R-algebras.

Given two composable R-algebras $(g : X \to Y, \alpha : Rg \to g)$ and $(h : Y \to Z, \beta : Rh \to h)$, we need to define the vertical composition of these, i.e. we need to exhibit an R-algebra structure $\kappa : R(hg) \to hg$ on the composition $hg : X \to Z$,

$$\begin{tikzcd} E(hg) \arrow[r, "\kappa"] \arrow[d, "R(hg)"'] & X \arrow[d, "hg"] \\ Z \arrow[r, "1_Z"'] & Z \end{tikzcd}$$

We define κ as the following composition

$$E(hg) \xrightarrow{\delta_{hg}} EL(hg) \xrightarrow{E(1,\beta\circ E(g,1))} E(g) \xrightarrow{\alpha} X,$$

where $\delta : L \to L^2$ is the comonad multiplication. Denote this vertical composition by $\beta \bullet \alpha$. It can be verified that this vertical composition works well with composition of squares. That is, if we have two morphisms of R-algebras $(u, v) : (f, \alpha) \to (h, \gamma)$ and $(v, w) : (g, \beta) \to (k, \lambda)$, then (u, w) is a morphism $(gf, \beta \bullet \alpha) \to (kh, \lambda \bullet \gamma)$. Hence we indeed get a double category R-$\mathbb{A}\mathbf{lg}$. Dually we can also define the double category L-$\mathbb{C}\mathbf{oalg}$. Note that, due to Lemma 3 every R-algebra is uniquely associated with a lifting structure against L-coalgebras. One can verify that the canonical lifting structure associated with the vertical composition $\boldsymbol{h} \cdot \boldsymbol{g}$ is given by

$$\Phi_{\boldsymbol{f},\boldsymbol{h}\cdot\boldsymbol{g}}(u, v) = \Phi_{\boldsymbol{f},\boldsymbol{g}}(u, \Phi_{\boldsymbol{f},\boldsymbol{h}}(gu, v)), \tag{2.8}$$

the 'stepwise lift' of f against h and g,

$$\begin{tikzcd} {} \arrow[r, "u"] \arrow[dd, "f"'] & {} \arrow[d, "g"] \\ & {} \arrow[d, "h"] \\ {} \arrow[r, "v"'] \arrow[ur, dashed] \arrow[uur, dashed] & {} \end{tikzcd} .$$

The forgetful functor $U^R : \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ is a concrete double category over $\mathcal{C}$.

Lemma 4. *Let $(\mathcal{C}, \mathsf{L}, \mathsf{R})$ be an AWFS, and consider the double categories R-$\mathbb{A}\mathbf{lg}$ and L-$\mathbb{C}\mathbf{oalg}$. For every $f : X \to Y$ in $\mathcal{C}$ the following two diagrams are squares of algebras* *and coalgebras respectively*

$$
\begin{tikzcd}
Ef \arrow[r, "\delta_f"] \arrow[dd, "Rf"'] & ELf \arrow[d, "RLf"] \\
 & Ef \arrow[d, "Rf"] \\
Y \arrow[r, "1"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "1"] \arrow[d, "Lf"'] & X \arrow[dd, "Lf"] \\
Ef \arrow[d, "LRf"'] & \\
ERf \arrow[r, "\mu_f"'] & Ef
\end{tikzcd}
$$

Proof. We focus on the right diagram. By the dual of Lemma 2.7, the functor L-**Coalg** $\to$ ${}^{\pitchfork}\mathsf{R}$-**Alg** is fully faithful and injective on objects. Thus, it is sufficient to show that its image $(1, \mu_f)$ is a morphism in the category ${}^{\pitchfork}\mathsf{R}$-**Alg** between the lifting structures induced by $\boldsymbol{LRf} \cdot \boldsymbol{Lf}$ and $\boldsymbol{Lf}$. Note that by the dual of (2.8) the lifting operation associated to $\boldsymbol{LRf} \cdot \boldsymbol{Lf}$ is the stepwise lifting operation

$$\Phi_{\boldsymbol{LRf}\cdot\boldsymbol{Lf},\boldsymbol{g}}(\sigma_0, \sigma_1) = \Phi_{\boldsymbol{LRf},\boldsymbol{g}}(\Phi_{\boldsymbol{Lf},\boldsymbol{g}}(\sigma_0, \sigma_1 \circ LRf), \sigma_1). \tag{2.9}$$

Now let $(u, v) : \boldsymbol{Lf} \to \boldsymbol{g}$ be a lifting problem with $\boldsymbol{g}$ an R-algebra. To show that $(1, \mu_f)$ is a morphism of lifting structures, we need to show that

$$\Phi_{\boldsymbol{LRf}\cdot\boldsymbol{Lf},\boldsymbol{g}}(u, v \circ \mu_f) = \Phi_{\boldsymbol{Lf},\boldsymbol{g}}(u, v) \circ \mu_f.$$

But by (2.9) this is precisely (2.7) in Lemma 3. Thus, the statement that the right diagram is a square of algebras is a reformulation of (2.7). Dually we have that the square on the left is a square of coalgebras. $\square$

In the next proposition we show that the 'semantics functor' is in fact a functor to $\mathbf{DBL}^{\mathbf{2}}$, the arrow category of double categories. Firstly we need a lemma that shows how the canonical lifting operation interacts with morphisms of AWFS.

Lemma 5. *Let* $(F, \alpha) : (\mathcal{C}, \mathsf{L}, \mathsf{R}) \to (\mathcal{D}, \mathsf{L}', \mathsf{R}')$ *be a morphism of AWFS. For each morphism* f *in* $\mathcal{C}$*,* R*-algebra* $\boldsymbol{g} = (g, p)$*, and morphism* $(u, v) : Lf \to g$ *we have*

$$F\Phi_{\boldsymbol{Lf},\boldsymbol{g}}(u, v) \circ \alpha_f = \Phi_{\boldsymbol{L'Ff},\boldsymbol{Fg}}(Fu, Fv \circ \alpha_f),$$

where $\boldsymbol{Lf} = (Lf, \delta_f)$ *is the cofree* L*-coalgebra.*

Proof. We calculate:

$$
\begin{aligned}
F\Phi_{\boldsymbol{Lf},\boldsymbol{g}}(u, v) \circ \alpha_f &= F(p \circ E(u, v) \circ \delta_f) \circ \alpha_f && \text{(Def. } \Phi\text{)} \\
&= Fp \circ FE(u, v) \circ F\delta_f \circ \alpha_f && \text{(Functoriality } F\text{)} \\
&= Fp \circ FE(u, v) \circ \alpha_{Lf} \circ E'(1, \alpha_f) \circ \delta'_{Ff} && \text{(2.4)} \\
&= Fp \circ \alpha_g \circ E'(Fu, Fv) \circ E'(1, \alpha_f) \circ \delta'_{Ff} && \text{(Naturality } \alpha\text{)} \\
&= Fp \circ \alpha_g \circ E'(Fu, Fv \circ \alpha_f) \circ \delta'_{Ff} && \text{(Functoriality } E'\text{)} \\
&= \Phi_{\boldsymbol{L'Ff},\boldsymbol{Fg}}(Fu, Fv \circ \alpha_f) && \text{(Def. } \Phi\text{)}
\end{aligned}
$$

$\square$

Proposition 2. *We have a functor* $(-)\text{-}\mathbb{A}\mathbf{lg} : \mathbf{AWFS}_{\text{lax}} \to \mathbf{DBL^2}$ *which sends an AWFS* (L, R) *to the forgetful double functor* $U^R : \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$*. This is called the (right)* semantics functor*. Dually we have the (left) semantics functor*

$$(-)\text{-}\mathbb{C}\mathbf{oalg} : \mathbf{AWFS}_{\text{oplax}} \to \mathbf{DBL^2}\,.$$

Proof. In order to prove the proposition, we need to show that the lifted functor $\overline{F}$: $\mathsf{R}\text{-}\mathbf{Alg} \to \mathsf{R}'\text{-}\mathbf{Alg}$ is in fact a double functor

$$\begin{tikzcd}
\mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \arrow[r, dashed, "\overline{F}"] \arrow[d, "U^R"'] & \mathsf{R}'\text{-}\mathbb{A}\mathbf{lg} \arrow[d, "U^{R'}"] \\
\mathbb{S}\mathbf{q}(\mathcal{C}) \arrow[r, "F^{\mathbf{2}}"'] & \mathbb{S}\mathbf{q}(\mathcal{D}).
\end{tikzcd}$$

That is, we need to show that F preserves vertical composition. So let $\boldsymbol{g} = (g : X \to Y, p)$ and $\boldsymbol{h} = (h : Y \to Z, q)$ be two R-algebras, with vertical composition $\boldsymbol{h} \cdot \boldsymbol{g} = (hg, q \bullet p)$. We want to show that $\overline{F}(\boldsymbol{h} \cdot \boldsymbol{g}) = \overline{F}(\boldsymbol{h}) \cdot \overline{F}(\boldsymbol{g})$. by (2.6) it is sufficient to show that the corresponding lifting operations are equal for the following square

$$\begin{tikzcd}
FX \arrow[r, "1_{FX}"] \arrow[d, "L'F(hg)"'] & FX \arrow[d, "F(hg)"] \\
E'F(hg) \arrow[r, "R'F(hg)"'] \arrow[ur, dashed] & FZ
\end{tikzcd}$$

To this end, firstly by Lemma 2.4 we have that for each morphism f, R-algebra $\boldsymbol{g}$, and morphism $(u, v) : Lf \to g$,

$$F\Phi_{\boldsymbol{Lf},\boldsymbol{g}}(u, v) \circ \alpha_f = \Phi_{\boldsymbol{L'Ff},\boldsymbol{Fg}}(Fu, Fv \circ \alpha_f).$$

Now using this together with (2.8) we calculate:

$$\begin{aligned}
\Phi_{\boldsymbol{L'Ff},F(\boldsymbol{h}\cdot\boldsymbol{g})}(Fu, Fv \circ \alpha_{hg}) &= F\Phi_{\boldsymbol{Lf},\boldsymbol{h}\cdot\boldsymbol{g}}(u, v) \circ \alpha_{hg} \\
&= F(\Phi_{\boldsymbol{Lf},\boldsymbol{g}}(u, \Phi_{\boldsymbol{Lf},\boldsymbol{h}}(gu, v)) \circ \alpha_{hg} \\
&= \Phi_{\boldsymbol{L'Ff},\boldsymbol{Fg}}(Fu, F\Phi_{\boldsymbol{Lf},h}(gu, v) \circ \alpha_{hg}) \\
&= \Phi_{\boldsymbol{L'Ff},\boldsymbol{Fg}}(Fu, \Phi_{\boldsymbol{L'Ff},\boldsymbol{Fh}}(F(gu) \circ Fv \circ \alpha_{hg})) \\
&= \Phi_{\boldsymbol{L'Ff},\boldsymbol{Fh}\cdot\boldsymbol{Fg}}(Fu, Fv \circ \alpha_{hg}).
\end{aligned}$$

Finally, if we instantiate this with $f = hg$, $(u, v) = (1, Rhg)$, and use the fact that $FR(hg) \circ \alpha_{hg} = R'F(hg)$, we find that

$$\Phi_{\boldsymbol{L'F(hg)},F(\boldsymbol{h}\cdot\boldsymbol{g})}(1, R'F(hg)) = \Phi_{\boldsymbol{L'F(hg)},\boldsymbol{Fh}\cdot\boldsymbol{Fg}}(1, R'F(hg)),$$

as desired. □

As a result of Proposition 2 we have a left and a right semantics functor

$$(-)\text{-}\mathbb{C}\mathbf{oalg} : \mathbf{AWFS}_{\text{oplax}} \to \mathbf{DBL^2}, \qquad (-)\text{-}\mathbb{A}\mathbf{lg} : \mathbf{AWFS}_{\text{lax}} \to \mathbf{DBL^2}\,.$$

In the next chapter, we prove a Beck theorem for AWFS, which characterises the image of these functors.

3. A Beck Theorem for AWFS

In this chapter we recall a Beck theorem for AWFS (Theorem 1) which was proved by Bourke and Garner [7]. Whereas the original Beck monadicity theorem (Theorem 7, Appendix A.2) characterises the essential image of the algebra functor $(-)\text{-}\mathbf{Alg}$: $\mathbf{Mnd}(\mathcal{C})^{\mathrm{op}} \to \mathbf{Cat}\,/\mathcal{C}$, the Beck theorem for AWFS characterises the essential image of the semantics functor $(-)\text{-}\mathbb{A}\mathbf{lg} : \mathbf{AWFS}_{\mathrm{lax}} \to \mathbf{DBL^2}$, which sends an AWFS to its right class. Proving that a small double category $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ generates an AWFS then reduces to verifying that $\mathbb{J}^{\pitchfork} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ satisfies the two conditions of the Beck theorem. It is an important tool to streamline the proof of the small object argument.

This chapter is based on [7, Section 3]. We fill in the details omitted there.

3.1. Semantics Functors

Firstly, we show that the semantics functors are in fact fully faithful.

Proposition 3 [7, Proposition 2]**.** *The functors* $(-)\text{-}\mathbb{C}\mathbf{oalg} : \mathbf{AWFS}_{\mathrm{oplax}} \to \mathbf{DBL^2}$ *and* $(-)\text{-}\mathbb{A}\mathbf{lg} : \mathbf{AWFS}_{\mathrm{lax}} \to \mathbf{DBL^2}$ *are fully faithful.*

Proof. By duality, it is sufficient to consider the case $(-)\text{-}\mathbb{A}\mathbf{lg}$. We need to show that the functor induces a bijection

$$\hom_{\mathbf{AWFS}_{\mathrm{lax}}}((\mathcal{C}, \mathsf{L}, \mathsf{R}), (\mathcal{D}, \mathsf{L}', \mathsf{R}')) \cong \hom_{\mathbf{DBL^2}}(\mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \overset{U^R}{\to} \mathbb{S}\mathbf{q}(\mathcal{C}), \mathsf{R}'\text{-}\mathbb{A}\mathbf{lg} \overset{U^{R'}}{\to} \mathbb{S}\mathbf{q}(\mathcal{D})).$$

Assume we have a morphism as on the left,

$$\begin{tikzcd} \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \arrow[r, dashed, "\overline{F}"] \arrow[d, "U^R"'] & \mathsf{R}'\text{-}\mathbb{A}\mathbf{lg} \arrow[d, "U^{R'}"] \\ \mathbb{S}\mathbf{q}(\mathcal{C}) \arrow[r, "\mathbb{S}\mathbf{q}(F)"'] & \mathbb{S}\mathbf{q}(\mathcal{D}). \end{tikzcd} \qquad \begin{tikzcd} \mathsf{R}\text{-}\mathbf{Alg} \arrow[r, dashed, "\overline{F}"] \arrow[d, "U^R"'] & \mathsf{R}'\text{-}\mathbf{Alg} \arrow[d, "U^{R'}"] \\ \mathcal{C}^{\mathbf{2}} \arrow[r, "F^{\mathbf{2}}"'] & \mathcal{D}^{\mathbf{2}}. \end{tikzcd}$$

Note that every functor $\mathbb{S}\mathbf{q}(\mathcal{C}) \to \mathbb{S}\mathbf{q}(\mathcal{D})$ must be of the form $\mathbb{S}\mathbf{q}(F)$ for some functor $F : \mathcal{C} \to \mathcal{D}$. By looking at the action on vertical arrows, we get an induced morphism as on the right above. By Lemma 2 this is induced by a unique lax monad morphism $\lambda : R'F^{\mathbf{2}} \to F^{\mathbf{2}}R$. The unit condition implies that we have the following commutative diagram on the left

$$\begin{tikzcd}
FX \arrow[r, "L'Ff"] \arrow[d, "Ff"'] \arrow[rr, bend left, "FLf"] & EFf \arrow[r, "\lambda_0"] \arrow[d, "R'Ff"] & FEf \arrow[d, "FRf"] \\
FY \arrow[r, "1"'] \arrow[rr, bend right, "1"'] & FY \arrow[r, "\lambda_1"'] & FY
\end{tikzcd}
\qquad
\begin{tikzcd}
E'Ff \arrow[r, "\alpha_f"] \arrow[d, "E'F(u,v)"'] & FEf \arrow[d, "FE(u,v)"] \\
E'Fg \arrow[r, "\alpha_g"'] & FEg
\end{tikzcd}$$

It follows that λ is of the form $\lambda = (\alpha, 1)$ for natural maps α_f, with the property that $\alpha_f \circ L'Ff = FLf$ for all $f \in \mathcal{C}^{\mathbf{2}}$. Naturality of α means that for every morphism $(u, v) : f \to g$ we have a commutative square as on the right. In particular, if we take the morphism $(f, 1) : f \to 1$ and use that $Rf = E(f, 1)$, we obtain that $FRf \circ \alpha_f = R'Ff$. It remains to check that $(1, \alpha) : L'F^{\mathbf{2}} \to F^{\mathbf{2}}L$ is a lax comonad morphism $\mathsf{L} \to \mathsf{L}'$ over $F^{\mathbf{2}}$. Note that the counit condition is given exactly by $FRf \circ \alpha_f = R'Ff$. It remains to show the condition for the comultiplication. To this end, we observe the following two diagrams

$$\begin{tikzcd}
E'Ff \arrow[r, "\delta'_{Ff}"] \arrow[dd, "R'Ff"'] & E'L'Ff \arrow[r, "E'(1,\alpha_f)"] \arrow[d, "R'L'Ff"'] & E'FLf \arrow[r, "\alpha_{Lf}"] \arrow[d, "R'FLf"'] & FELf \arrow[d, "FRLf"] \\
 & E'Ff \arrow[r, "\alpha_f"'] \arrow[d, "R'Ff"'] & FEf \arrow[r, "1"'] \arrow[d, "FRf"'] & FEf \arrow[d, "FRf"] \\
FBFB \arrow[r, "1"'] & FB \arrow[r, "1"'] & FB \arrow[r, "1"'] & FB
\end{tikzcd}
\qquad
\begin{tikzcd}
E'Ff \arrow[r, "\alpha_f"] \arrow[dd, "R'Ff"'] & FEf \arrow[r, "F\delta_f"] \arrow[dd, "FRf"'] & FELf \arrow[d, "FRLf"] \\
 & & FEf \arrow[d, "FRf"] \\
FB \arrow[r, "1"'] & FB \arrow[r, "1"'] & FB
\end{tikzcd}$$

All squares in these diagrams are morphisms of R-algebras. Indeed, the leftmost square and the rightmost square are so by Lemma 4; the other squares are because $(\alpha, 1)$ is a lax monad morphism. Hence, the outer squares are also maps of R-algebras. Now we claim that if we compose both squares on the left with the unit morphism $(L'Ff, 1) : Ff \to R'Ff$, they have the same composite, namely $FLLf$. Note that the following diagram commutes

$$\begin{tikzcd}
FX \arrow[r, "1"] \arrow[d, "L'L'Ff"'] & FX \arrow[r, "1"] \arrow[d, "L'FLf"] & FX \arrow[d, "FLLf"] \\
E'L'Ff \arrow[r, "E'(1,\alpha_f)"'] & E'FLf \arrow[r, "\alpha_{Lf}"'] & FELf
\end{tikzcd}$$

The left square is the morphism $L'(1, \alpha_f)$, whereas the right square is the counit condition for Lf. Using this and the fact that $\delta_g \circ Lg = LLg$ for all g, we compute:

$$\begin{aligned}
\alpha_{Lf} \circ E'(1, \alpha_f) \circ \delta'_{Ff} \circ L'Ff &= \alpha_{Lf} \circ E'(1, \alpha_f) \circ L'L'Ff \\
&= \alpha_{Lf} \circ L'FLf \\
&= FLLf
\end{aligned}$$

$$
\begin{aligned}
&= F(\delta_f \circ Lf) \\
&= F\delta_f \circ FLf \\
&= F\delta_f \circ \alpha_f \circ L'Ff.
\end{aligned}
$$

But then it follows by freeness of $R'Ff$, as in (2.1), that we have

$$\alpha_{Lf} \circ E'(1, \alpha_f) \circ \delta'_{Ff} = F\delta_f \circ \alpha_f.$$

We conclude that $(1, \alpha)$ is an oplax comonad morphism, which completes the proof. $\square$

The following proposition shows that constructing an AWFS for a given monad R on $\mathcal{C}^{\mathbf{2}}$ is equivalent to equipping the category of R-algebras with a vertical composition.

Proposition 4 [7, Proposition 4]**.** *Let* R *be a monad on* $\mathcal{C}^{\mathbf{2}}$ *over the codomain functor. Then the right semantics functor* $(-)\text{-}\mathbb{A}\mathbf{lg} : \mathbf{AWFS}_{\mathrm{lax}} \to \mathbf{DBL}^{\mathbf{2}}$ *induces a bijection between AWFS* $(\mathcal{C}, \mathsf{L}, \mathsf{R})$ *and extensions of* $\mathsf{R}\text{-}\mathbf{Alg}$ *to a concrete double category of* $\mathcal{C}$*.*

Proof. Suppose we have a vertical composition law on R-algebras $\boldsymbol{h}, \boldsymbol{g} \mapsto \boldsymbol{h} * \boldsymbol{g}$ which is associative, unital, and compatible with R-algebra morphisms. We show that there is a canonical AWFS (L, R) associated with this. We define the comonad $\mathsf{L} = (L, \delta, \epsilon)$ as follows. $L : \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ is defined by sending a morphism f to the upper map of the unit of R, $\eta_f : f \to Rf$. We define the counit $\epsilon_f := (1, Rf) : Lf \to f$ in the usual way. This gives us a pointed endofunctor (L, ϵ); it remains to define the comultiplication $\delta : L \to L^2$. Using diagram (2.1), we let δ_f be the unique arrow making the following diagram commute

$$
\begin{tikzcd}
X \arrow[rr, "LLf"] \arrow[dr, "Lf"'] \arrow[ddd, "f"'] & & ELf \arrow[d, "RLf"] \\
& Ef \arrow[ur, dashed, "\delta_f"'] \arrow[dd, "Rf"'] & Ef \arrow[dd, "Rf"] \\
& & \\
Y \arrow[r, "1"'] & Y \arrow[r, "1"'] & Y
\end{tikzcd}
$$

such that $(\delta_f, 1) : Rf \to Rf * RLf$ is an algebra morphism. This implies that $\gamma \circ E(\delta_f, 1) = \delta_f \circ \mu_f$ where γ is the vertical composition of the algebras $\boldsymbol{RLf}$ and $\boldsymbol{Rf}$. Note that indeed, if $*$ would arise from an AWFS, then this square is exactly the left square in Lemma 4, and hence it would be the unique possible choice for δ_f. We now firstly show that the (lower components of) δ form a natural transformation $\delta : E \to EL$. To the end, let $f : X \to Y$ be a morphism in $\mathcal{C}$. We must show that for any morphism $(u, v) : f \to g$ the following diagram commutes

$$
\begin{tikzcd}
Ef \arrow[r, "E(u,v)"] \arrow[d, "\delta_f"'] & Eg \arrow[d, "\delta_g"] \\
ELf \arrow[r, "EL(u,v)"'] & ELg.
\end{tikzcd}
$$

To show this, we observe the following two diagrams

$$\begin{tikzcd}
X \arrow[r, "Lf"] \arrow[dd, "f"'] & Ef \arrow[r, "{E(u,v)}"] \arrow[dd, "Rf"'] & Eg \arrow[r, "\delta_g"] \arrow[dd, "Rg"] & ELg \arrow[d, "RLg"] \\
 & & & Eg \arrow[d, "Rg"] \\
Y \arrow[r, "1"'] & Y \arrow[r, "v"'] & B \arrow[r, "1"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "Lf"] \arrow[dd, "f"'] & Ef \arrow[r, "\delta_f"] \arrow[dd, "Rf"'] & ELf \arrow[r, "{EL(u,v)}"] \arrow[d, "RLf"'] & ELg \arrow[d, "RLg"] \\
 & & Ef \arrow[r, "{E(u,v)}"] \arrow[d, "Rf"'] & Eg \arrow[d, "Rg"] \\
Y \arrow[r, "1"'] & Y \arrow[r, "v"'] & B \arrow[r, "1"'] & B.
\end{tikzcd}$$

Note that every square is a morphism of R-algebras, and hence both outer squares are morphisms of R-algebras. We claim that the upper morphisms of both diagrams are equal. We compute:

$$\begin{aligned}
EL(u,v) \circ \delta_f \circ Lf &= EL(u,v) \circ LLf \\
&= LLg \circ u \\
&= \delta_g \circ Lg \circ u \\
&= \delta_g \circ E(u,v) \circ Lf.
\end{aligned}$$

For the first and the third equality we use the property of δ that comes with its definition. For the second and the last equality we use the fact that for every $(u,v) : \alpha \to \beta$, we have $E(u,v) \circ L\alpha = L\beta \circ u$. Now by freeness of $\boldsymbol{Rf}$ it follows that $\delta_g \circ E(u,v) = EL(u,v) \circ \delta_f$.

Next we will show the coassociativity axiom, so we need to show that $E(1,\delta_f) \circ \delta_f = \delta_{Lf} \circ \delta_f$. This time, we look at the following two diagrams

$$\begin{tikzcd}
Ef \arrow[r, "\delta_f"] \arrow[dd, "Rf"'] & ELf \arrow[r, "{E(1,\delta_f)}"] \arrow[d, "RLf"'] & ELLf \arrow[d, "RLLf"] \\
 & Ef \arrow[r, "\delta_f"] \arrow[d, "Rf"'] & ELf \arrow[d, "Rf*RLf"] \\
Y \arrow[r, "1"'] & Y \arrow[r, "1"'] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
Ef \arrow[r, "\delta_f"] \arrow[dd, "Rf"'] & ELf \arrow[r, "\delta_{Lf}"] \arrow[d, "RLf"'] & ELLf \arrow[d, "RLf*RLLf"] \\
 & Ef \arrow[r, "1"] \arrow[d, "Rf"'] & Ef \arrow[d, "Rf"] \\
Y \arrow[r, "1"'] & Y \arrow[r, "1"'] & Y.
\end{tikzcd}$$

Again, if we can show that their transposes along the free-forgetful adjunction are equal, we can conclude that they are equal. To this end, we calculate:

$$\begin{aligned}
E(1,\delta_f) \circ \delta_f \circ Lf &= E(1,\delta_f) \circ LLf \\
&= LLLf \\
&= \delta_{Lf} \circ LLf \\
&= \delta_{Lf} \circ \delta_f \circ Lf.
\end{aligned}$$

We leave the counit and distributivity axioms to the reader. It remains to show that the vertical composition induced by the AWFS[1] coincides with $*$. To the end, let h, g be two

[1] We denote this vertical composition with $\bullet$.

morphisms and denote $f = hg$. We look at the following two diagrams,

$$
\begin{tikzcd}
A \arrow[rr, "1"] \arrow[dr, "LLf"'] \arrow[dd, "Lf"'] & & A \arrow[d, "g"] \\
& ELf \arrow[ur, dashed, "\ell"'] \arrow[d, "RLf"'] & B \arrow[dd, "h"] \\
Ef \arrow[r, "1"'] \arrow[d, "Rf"'] & Ef \arrow[ur, dashed, "m"'] \arrow[d, "Rf"'] & \\
C \arrow[r, "1"'] & C \arrow[r, "1"'] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
Ef \arrow[r, "\delta_f"] \arrow[dd, "Rf"'] & ELf \arrow[r, "\ell"] \arrow[d, "RLf"'] & A \arrow[d, "g"] \\
& Ef \arrow[r, "m"] \arrow[d, "Rf"'] & B \arrow[d, "h"] \\
C \arrow[r, "1"'] & C \arrow[r, "1"'] & C.
\end{tikzcd}
$$

Using again diagram (2.1), the map m is the unique map induced by $(g, 1) : f \to h$, and the map ℓ is the unique map induced by $(1, m) : Lf \to g$. Hence, the two squares on the right in the right diagram are R-algebra morphisms. Moreover, the left square in the right diagram is an algebra morphism $Rf \to Rf * RLf$ by definition. It is also an algebra morphism $Rf \to Rf \bullet RLf$ by Lemma 4. But composing this with Lf gives the identity, $\ell \circ \delta_f \circ Lf = \ell \circ LLf = 1$. We conclude that $(\ell \circ \delta_f, 1) : Rf \to f$ is the algebra structure of both $h * g$ and $h \bullet g$, and thus they coincide. This concludes the proof. $\square$

3.2. Monads over the Codomain Functor

Before we can prove the Beck theorem, we still need a result which characterises when a monad is isomorphic to one over the codomain functor; this is the next proposition.

Proposition 5 [7, Proposition 5]**.** *A monad R on $\mathcal{C}^{\mathbf{2}}$ is isomorphic to a monad over the codomain functor if and only if:*

(a) Each identity map has an R-algebra structure, denoted by $\mathbf{1_X}$;

(b) For every $f : X \to Y$ there is an algebra map $(f, f) : \mathbf{1_X} \to \mathbf{1_Y}$;

(c) For every R-algebra $\boldsymbol{g} : A \to B$ there is an algebra map $(g, 1) : \boldsymbol{g} \to \mathbf{1_B}$.

Moreover, the algebra structure on identity arrows is unique.

Proof. Firstly, we recall that a monad R over the codomain means that we have a commutative diagram

$$
\begin{tikzcd}
\mathcal{C}^{\mathbf{2}} \arrow[rr, "R"] \arrow[dr, "\mathrm{cod}"'] & & \mathcal{C}^{\mathbf{2}} \arrow[dl, "\mathrm{cod}"] \\
& \mathcal{C} &
\end{tikzcd}
$$

and moreover that $\mathrm{cod}(\eta_f) = 1$ and $\mathrm{cod}(\mu_f) = 1$ for all $f \in \mathcal{C}^{\mathbf{2}}$.

Assume R is a monad on $\mathcal{C}^{\mathbf{2}}$ which is isomorphic to one over the codomain functor. Clearly the properties (a) - (c) are invariant under isomorphism, so we may assume that R itself is a monad over the codomain functor. From this, we see that there is a functor $E : \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$ such that R maps an object $f : X \to Y$ of $\mathcal{C}^{\mathbf{2}}$ to $Rf : Ef \to Y$, and a morphism $(u, v) : f \to g$ to $(E(u, v), v) : Rf \to Rg$,

$$\begin{tikzcd} Ef \arrow[r, "E(u,v)"] \arrow[d, "Rf"'] & Eg \arrow[d, "Rg"] \\ C \arrow[r, "v"'] & D \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_f"] \arrow[d, "f"'] & Ef \arrow[d, "Rf"] \\ Y \arrow[r, "1"'] & Y \end{tikzcd} \qquad \begin{tikzcd} ERf \arrow[r, "\mu_f"] \arrow[d, "RRf"'] & Ef \arrow[d, "Rf"] \\ Y \arrow[r, "1"'] & Y \end{tikzcd}$$

We claim that the unique algebra structure on 1_A is given by $(R(1_A), 1_A) : R(1_A) \to 1_A$. To prove this, firstly note that the unit for every f looks as the middle diagram above. In particular, we have that $R(1_A) \circ \eta_{1_A} = 1_A$. This shows that the unit condition is satisfied. For the multiplication, we have a diagram as on the right above. In particular, we have for $f = 1_A$ that $R(1_A) \circ \mu_{1_A} = RR(1_A)$. Combined with the fact that $R(R(1_A), 1_A) = (E(R(1_A), 1_A), 1_A) : RR(1_A) \to 1_A$, we calculate

$$R(1_A) \circ E(R(1_A), 1_A) = RR(1_A) = R(1_A) \circ \mu_{1_A}.$$

This shows the multiplication condition, and hence proves that 1_A has an algebra structure given by $(R(1_A), 1_A) : R(1_A) \to 1_A$. To see that this is unique, note that any R-algebra on 1_A has to satisfy the unit condition, as shown in the following diagram

$$\begin{tikzcd} A \arrow[r, "\eta_{1_A}"'] \arrow[d, "1_A"'] \arrow[rr, bend left, "1_A"] & E(1_A) \arrow[r, dashed] \arrow[d, "R(1_A)"] & A \arrow[d, "1_A"] \\ A \arrow[r, "1_A"] \arrow[rr, bend right, "1_A"'] & A \arrow[r, dashed] & A. \end{tikzcd}$$

Clearly this is only satisfied by $(R(1_A), 1_A)$ and is thus the unique R-algebra structure on 1_A. We show that with this algebra structure $\mathbf{1_A}$ conditions (b) and (c) are satisfied. Note that a morphism $(u, v) : f \to g$ in $\mathcal{C}^{\mathbf{2}}$ is an R-algebra morphism $(f, p) \to (g, k)$ if in addition we have $k \circ E(u, v) = u \circ p$. To see that $(f, f) : 1_A \to 1_B$ is an algebra morphism, Note that $R(f, f) : R(1_A) \to R(1_B)$ is given by

$$\begin{tikzcd} E(1_A) \arrow[r, "E(f,f)"] \arrow[d, "R(1_A)"'] & E(1_B) \arrow[d, "R(1_B)"] \\ A \arrow[r, "f"'] & B. \end{tikzcd}$$

Thus, we have $R(1_B) \circ E(f, f) = f \circ R(1_A)$, which shows that $(f, f) : \mathbf{1_A} \to \mathbf{1_B}$ is an algebra morphism. Lastly, let $\boldsymbol{g} = (g, p)$ be an R-algebra. We have to show that

$(g,1) : g \to 1_B$ is an algebra morphism. Applying R to the morphism $(g,1)$ gives a commutative diagram on the left below

$$\begin{tikzcd}
Eg \arrow[r, "E(g{,}1_B)"] \arrow[d, "Rg"'] & E(1_B) \arrow[d, "R(1_B)"] \\
B \arrow[r, "1_B"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
Eg \arrow[r, "p"] \arrow[d, "Rg"'] & A \arrow[d, "g"] \\
B \arrow[r, "1_B"'] & B.
\end{tikzcd}$$

Combining this with the algebra structure p as defined by the right diagram above, we have

$$R(1_B) \circ E(g, 1_B) = Rg = g \circ p,$$

which shows that $(g,1) : g \to 1_B$ is an algebra morphism.

Conversely, suppose R satisfies conditions (a) - (c). Let the unit map η_f be given by $(r,s) : f \to Rf$, as shown on the left below.

$$\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[d, "Rf"] \\
B \arrow[r, "s"'] & D
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "f"'] \arrow[rr, bend left, "f"] & C \arrow[r, "t"] \arrow[d, "Rf"] & B \arrow[d, "1_B"] \\
B \arrow[r, "s"'] \arrow[rr, bend right, "1_B"'] & D \arrow[r, "u"'] & B
\end{tikzcd}$$

We show that s is invertible. To this end, we note that since Rf underlies the free R-algebra on f, for the map $(f, 1_B) : f \to 1_B$ there is a unique R-algebra morphism $(t,u) : Rf \to 1_B$ making the diagram on the right above commute. It follows that $u \circ s = 1_B$, and hence it remains to show that $s \circ u = 1_D$. Using (b) we have an algebra map $Rf \xrightarrow{(t,u)} 1_B \xrightarrow{(s,s)} 1_D$, and using (c) we have an algebra map $(Rf, 1) : Rf \to 1_D$. Precomposing with the unit $(r,s) : f \to Rf$ we see that these two maps coincide, as shown in the following diagram

$$\begin{tikzcd}
A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[r, "t"] \arrow[d, "Rf"] \arrow[rr, bend left, "Rf"] & B \arrow[r, "s"] \arrow[d, "1_B"] & D \arrow[d, "1_D"] \\
B \arrow[r, "s"'] & D \arrow[r, "u"'] \arrow[rr, bend right, "1_D"'] & B \arrow[r, "s"'] & D
\end{tikzcd}$$

Indeed, we have $s \circ u \circ s = 1 \circ s = s$, and $s \circ t \circ r = s \circ f = Rf \circ r$. But this means, by the free-forgetful adjunction, that the original two maps coincide. In particular we have $s \circ u = 1_D$, and so indeed s is invertible. Hence we can define a functor $R' : \mathcal{C}^{\mathbf{2}} \to \mathcal{C}^{\mathbf{2}}$ as $R'f = u \circ Rf$. Then we have $(u,1) : Rf \overset{\cong}{\to} R'f$ for all $f \in \mathcal{C}^{\mathbf{2}}$. Now we can transport the monad structure of R to R' via this isomorphism. That is, we define the unit $\eta'_f : f \to R'f$ as $(r, 1_B)$, as shown on the left below.

$$
\begin{tikzcd}
A \arrow[r, "(\eta_f)_0"] \arrow[d, "f"'] & C \arrow[d, "Rf"] \\
B \arrow[r, "(\eta_f)_1"] \arrow[dr, "1_B"'] & D \arrow[d, "u"] \\
& B
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "(\mu_f)_0"] \arrow[d, "RRf"'] & C \arrow[d, "Rf"] \\
F \arrow[r, "(\mu_f)_1"] \arrow[d, "u' \circ Ru"'] & D \arrow[d, "u"] \\
B \arrow[r, "1_B"'] & B
\end{tikzcd}
$$

Note that the lower triangle commutes by definition of u. We define the multiplication $\mu'_f : R'R'f \to R'f$ as $\mu'_f = ((\mu_f)_0, 1_B)$, as shown on the right above. We use here that

$$R'R'f = u' \circ RR'f = u' \circ R(u \circ Rf) = u' \circ Ru \circ RRf.$$

To see that the lower square commutes in the right diagram, note first that by the monad axioms we have that $(\mu_f)_1 \circ (\eta_{Rf})_1 = 1_D$. Since u' is by definition the inverse of $(\eta_{Rf})_1$, it follows that $u' = (\mu_f)_1$. Thus we compute:

$$u' \circ Ru = (\mu_f)_1 \circ Ru = u \circ (\mu_f)_1.$$

Therefore we have that R' is a monad over the codomain functor, and thus conclude that R is isomorphic to a monad over the codomain functor. $\square$

3.3. Essential Image of the Semantics Functors

Now we are ready to state the main theorem of this chapter, the Beck theorem, which characterises the essential image of the semantics functor.

Theorem 1 [7, Theorem 6]**.** *The functor* $(-)\text{-}\mathbb{A}\mathbf{lg} : \mathbf{AWFS}_{\text{lax}} \to \mathbf{DBL^2}$ *has as its essential image exactly those concrete double categories* $V : \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ *such that:*

(i) The functor $V_1 : \mathcal{A}_1 \to \mathcal{C}^\mathbf{2}$ *on vertical arrows is strictly monadic;*

(ii) For each vertical arrow $\boldsymbol{f} : A \to B$ *in* $\mathbb{A}$*, the following is a square in* $\mathbb{A}$*,*

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\boldsymbol{f}"'] & B \arrow[d, "\mathbf{1}"] \\
B \arrow[r, "1"'] & B.
\end{tikzcd}
$$

Proof. Firstly we have to show that the forgetful functor $U^R : \mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ has properties (i) and (ii). Note that it is indeed a concrete double category. Clearly its action on vertical arrows, which is the forgetful functor $\mathsf{R}\text{-}\mathbf{Alg} \to \mathcal{C}^\mathbf{2}$, is strictly monadic.

Furthermore, property (ii) holds because of Proposition 5, since the monad R associated with an AWFS is indeed a monad over the codomain functor.

Conversely, suppose we have a concrete double category $V : \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ that satisfies both conditions of the theorem. Since V_1 is strictly monadic, it follows that $\mathcal{A}_1$ is isomorphic over $\mathcal{C}^{\mathbf{2}}$ to the category of algebras for some monad R, which is the monad that comes from the adjunction between V_1 and its left adjoint,

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[rr, "\cong"] \arrow[dr] & & \mathsf{R}\text{-}\mathbf{Alg} \arrow[dl] \\
& \mathcal{C}^2. &
\end{tikzcd}$$

Then it follows from assumption (ii) that this R satisfies part (c) of Proposition 5. Part (a) and (b) are satisfied because of the double categorical structure on $\mathbb{A}$. Indeed, for (a), let $X \in \mathcal{C}$. Then X is an object in $\mathcal{A}_0$ since V_0 is the identity. We have a functor $\mathrm{id} : \mathcal{A}_0 \to \mathcal{A}_1$, so 1_X is a vertical morphism, i.e. an object in $\mathcal{A}_1$. Since the above diagram commutes, we have that 1_X admits an R-algebra structure. Moreover, applying the functor $\mathrm{id} : \mathcal{A}_0 \to \mathcal{A}_1$ to the morphism $f : 1_X \to 1_Y$ in $\mathcal{C}^2$ (and thus in $\mathcal{A}_0$) and using that V_1 is faithful gives us a square $(f, f) : \mathbf{1_X} \to \mathbf{1_Y}$ in $\mathcal{A}_1$. Since the above diagram commutes, it follows that $(f, f) : 1_X \to 1_Y$ is an algebra map. Thus condition (b) is also satisfied.

Hence we have that R is isomorphic to a monad R' over the codomain functor. Now we can transport the double categorical structure along the isomorphisms $\mathcal{A}_1 \cong \mathsf{R}\text{-}\mathbf{Alg} \cong \mathsf{R}'\text{-}\mathbf{Alg}$. This yields a concrete double category $V' : \mathbb{A} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ over $\mathcal{C}$ which by Proposition 4 is in the image of the semantics functor. $\square$

A concrete double category is called *right-connected* if it has property (ii) above and *monadic right-connected* if it has both properties (i) and (ii). We can now combine the result of Theorem 1 and Proposition 3 to obtain the following result.

Corollary 1 [7, Corollary 7]**.** *The category* $\mathbf{AWFS}_{\mathrm{lax}}$ *is equivalent to the full subcategory of* $\mathbf{DBL}^{\mathbf{2}}$ *on the monadic right-connected concrete double categories.*

4. Free Algebras for Pointed Endofunctors

One crucial ingredient in the proof of the small object argument is the existence of the free algebra for an accessible pointed endofunctor. Bourke and Garner [7] rely on Kelly [15] for this step, using a transfinite argument that is not constructively valid. For our purposes, it suffices to show that the free algebra for a finitary pointed endofunctor exists. We give a constructive proof of this in the current chapter (Theorem 2). Our strategy is to use the notion of algebraic chains, originally introduced by Koubek and Reiterman [16]. This approach emphasises the explicit formulae involved by focusing not only on the free algebra but also on the free algebraic chain. We prove that for finitary functors, the induced algebraic chain stabilises at stage ω, leading to the free algebra $TX_\omega \to X_{\omega+1} \cong X_\omega$.

This chapter is an adaptation of [6, Appendix A] to the finitary case.

4.1. Algebraic Chains

An $\omega + \omega$-*chain* in $\mathcal{C}$ is a functor $X : \omega + \omega \to \mathcal{C}$ from the category $\omega + \omega$ to $\mathcal{C}$. A morphism of $\omega + \omega$-chains is a natural transformation between them.

Definition 3. Given a pointed endofunctor (T, η) on a category $\mathcal{C}$, an $\omega + \omega$-*algebraic chain* (X, x) is a $\omega + \omega$-chain X[1] equipped with maps $x_n : TX_n \to X_{n+1}$ for every $n \in \omega + \omega$ such that

(i) for all n,

$$\begin{tikzcd} X_n \arrow[r, "\eta X_n"] \arrow[dr, "j_n^{n+1}"'] & TX_n \arrow[d, "x_n"] \\ & X_{n+1} \end{tikzcd}$$

(ii) and for all $n < m$,

$$\begin{tikzcd} TX_n \arrow[r, "T(j_n^m)"] \arrow[d, "x_n"'] & TX_m \arrow[d, "x_m"] \\ X_{n+1} \arrow[r, "j_{n+1}^{m+1}"'] & X_{m+1} \end{tikzcd}$$

[1] from now one, we just write 'chain' and 'algebraic chain', leaving out the $\omega + \omega$.

commutes. Here we write j_n^{n+1} for $X(n \hookrightarrow n+1)$.

Thus the first few 'stages' of an algebraic chain can be represented by a (not necessarily commutative) diagram as shown below

$$\begin{tikzcd}
TX_0 \arrow[r, "T(j_0^1)"] & TX_1 \arrow[r, "T(j_1^2)"] & TX_2 \arrow[r, "T(j_2^3)"] & TX_3 \arrow[r] & \cdots \\
X_0 \arrow[u, "\eta_{X_0}"] \arrow[r, "j_0^1"'] & X_1 \arrow[u, "\eta_{X_1}"'] \arrow[r, "j_1^2"'] \arrow[from=ul, dashed, "x_0"] & X_2 \arrow[u, "\eta_{X_2}"'] \arrow[r, "j_2^3"'] \arrow[from=ul, dashed, "x_1"] & X_3 \arrow[u, "\eta_{X_3}"'] \arrow[r] \arrow[from=ul, dashed, "x_2"] & \cdots \arrow[from=ul, dashed, "x_3"]
\end{tikzcd}$$

A morphism $f : (X, x) \to (Y, y)$ of algebraic chains is a chain morphisms $f : X \to Y$ that commutes with the maps x_n and y_n for all n,

$$\begin{tikzcd}
TX_n \arrow[r, "T(f_n)"] \arrow[d, "x_n"'] & TY_n \arrow[d, "y_n"] \\
X_{n+1} \arrow[r, "f_{n+1}"'] & Y_{n+1}
\end{tikzcd}$$

These form a category which we denote as $T\text{-}\mathbf{Alg}_\omega$. It comes with a forgetful functor $V : T\text{-}\mathbf{Alg}_\omega \to \mathcal{C}$ which sends an algebraic chain (X, x) to X_0. Note that for condition (ii) of Definition 3 to hold for all $n, m \in \omega + \omega$ with $n < m$ it is sufficient that it holds for the two cases $m = n + 1$ and for the limit ordinal $m = \omega$. Now suppose we have a chain X with maps x_n satisfying condition (i). Then condition (ii) in the case $m = n+1$ becomes equivalent to the following diagram being a fork

$$\begin{tikzcd}
TX_n \arrow[r, shift left, "Tx_n \circ T\eta_{X_n}"] \arrow[r, shift right, "Tx_n \circ \eta_{TX_n}"'] & TX_{n+1} \arrow[r, "x_{n+1}"] & X_{n+2}
\end{tikzcd} \tag{4.1}$$

Indeed, using the fact that $x_n \circ \eta_{X_n} = j_n^{n+1}$ for all n, we have

$$\begin{aligned}
& x_{n+1} \circ T(j_n^{n+1}) = j_{n+1}^{n+2} \circ x_n \\
\iff\ & x_{n+1} \circ T(x_n \circ \eta_{X_n}) = x_{n+1} \circ \eta_{X_{n+1}} \circ x_n \\
\iff\ & x_{n+1} \circ Tx_n \circ T\eta_{X_n} = x_{n+1} \circ Tx_n \circ \eta_{TX_n},
\end{aligned}$$

where we use naturality of $\eta : 1 \to T$ in the last step. In the limit $m = \omega$ case the condition boils down to the following diagram being a fork

$$\begin{tikzcd}
TX_n \arrow[r, shift left, "Tj_n^m"] \arrow[r, shift right, "\eta_{X_m} \circ j_{n+1}^m \circ x_n"'] & TX_m \arrow[r, "x_m"] & X_{m+1}
\end{tikzcd} \tag{4.2}$$

for every $n < m$. Indeed, we have

$$x_m \circ Tj_n^m = x_m \circ \eta_{X_m} \circ j_{n+1}^m \circ x_n$$

$$\begin{aligned}
&\iff x_m \circ Tj_n^m = j_m^{m+1} \circ j_{n+1}^m \circ x_n \\
&\iff x_m \circ Tj_n^m = j_{n+1}^{m+1} \circ x_n.
\end{aligned}$$

If $\mathcal{C}$ has filtered colimits, then the latter condition is equivalent to the condition that the diagram

$$\begin{tikzcd}
\mathrm{colim}_{n<m} TX_n \arrow[r, shift left, "\langle Tj_n^m \rangle"] \arrow[r, shift right, "\langle \eta_{X_m} \circ j_{n+1}^m \circ x_n \rangle"'] & TX_m \arrow[r, "x_m"] & X_{m+1}
\end{tikzcd} \tag{4.3}$$

is a fork. The following propositions show that if $\mathcal{C}$ is cocomplete, then V has a left adjoint.

Proposition 6 [6, Proposition 21]**.** *If $\mathcal{C}$ is cocomplete, then the forgetful functor $V : T\text{-}\mathbf{Alg}_\omega \to \mathcal{C}$ has a left adjoint which sends an object $X \in \mathcal{C}$ to the algebraic chain $X_\bullet$ defined as follows:*

- *$X_0 = X, X_1 = TX, j_0^1 = \eta_X : X \to TX$ and $x_0 = 1 : TX \to TX$.*
- *At an ordinal of the form $n+2$, X_{n+2} is defined as the coequaliser*

$$\begin{tikzcd}
TX_n \arrow[r, shift left, "Tx_n \circ T\eta_{X_n}"] \arrow[r, shift right, "Tx_n \circ \eta_{TX_n}"'] & TX_{n+1} \arrow[r, "x_{n+1}"] & X_{n+2}
\end{tikzcd}$$

  and $j_{n+1}^{n+2} = x_{n+1} \circ \eta_{X_{n+1}}$.

- *At the limit ordinal $m = \omega$, we define $X_\omega = \mathrm{colim}_{n<\omega} X_n$ with the connecting maps j_n^m the colimit inclusions. We define $X_{\omega+1}$ as the coequaliser*

$$\begin{tikzcd}
\mathrm{colim}_{n<\omega} TX_n \arrow[r, shift left, "\langle Tj_n^\omega \rangle"] \arrow[r, shift right, "\langle \eta_{X_\omega} \circ j_{n+1}^\omega \circ x_n \rangle"'] & TX_\omega \arrow[r, "x_\omega"] & X_{\omega+1},
\end{tikzcd}$$

  with $j_m^{m+1} = x_m \circ \eta_{X_m}$.

Proof. We have to show that there is a natural bijection

$$\hom_{\mathsf{T}\text{-}\mathbf{Alg}_\omega}(X_\bullet, (Y, y)) \cong \hom_{\mathcal{C}}(X, Y_0).$$

Since the unit of the adjunction is the identity, it is sufficient to show that for every map $f : X \to Y_0 = V(Y, y)$ there is a unique map $\widetilde{f} : X_\bullet \to (Y, y)$ such that $\widetilde{f}_0 = f$. To define $\widetilde{f}_1$, we look at the square on the left below.

$$\begin{tikzcd}
TX \arrow[r, "Tf"] \arrow[d, "x_0 = 1"'] & TY_0 \arrow[d, "y_0"] \\
TX \arrow[r, "\widetilde{f}_1"'] & Y_1
\end{tikzcd}
\qquad
\begin{tikzcd}
TX_n \arrow[r, shift left, "Tx_n \circ T\eta_{X_n}"] \arrow[r, shift right, "Tx_n \circ \eta_{TX_n}"'] \arrow[d, "T\widetilde{f}_n"'] & TX_{n+1} \arrow[r, "x_{n+1}"] \arrow[d, "T\widetilde{f}_{n+1}"] & X_{n+2} \arrow[d, dashed, "\widetilde{f}_{n+2}"] \\
TY_n \arrow[r, shift left, "Ty_n \circ T\eta_{Y_n}"] \arrow[r, shift right, "Ty_n \circ \eta_{TY_n}"'] & TY_{n+1} \arrow[r, "y_{n+1}"] & Y_{n+2}
\end{tikzcd}$$

Since we want $\widetilde{f}$ to be a morphism of algebraic chains this has to commute, whence we are forced to put $\widetilde{f}_1 = y_0 \circ Tf$. In order to define $\widetilde{f}_{n+2}$, we look at the diagram on the right above. The upper row of the diagram is a coequaliser by definition, and the lower row is a fork by (4.1). Since both squares on the left commute, there exists a unique map $\widetilde{f}_{n+2} : X_{n+2} \to Y_{n+2}$ making the square on the right commute. This is exactly the condition for $\widetilde{f}$ being a morphism of algebraic chains, so we have a unique choice for $\widetilde{f}_{n+2}$. Now let m be the limit ordinal ω and suppose we have already defined the maps $\widetilde{f}_n : X_n \to Y_n$ for all $n < m$. Then we have a cocone with vertex Y_m, given by $(j_n^m \circ \widetilde{f}_n : X_n \to Y_m)_{n<m}$. We define $\widetilde{f}_m$ as the unique map $X_m = \operatorname{colim}_{n<m} X_n \to Y_m$ such that for every $n < m$ we have that $j_n^m \circ \widetilde{f}_n = \widetilde{f}_m \circ j_n^m$, which is exactly the condition we need to make $\widetilde{f}$ a chain morphism. Therefore it is our unique choice for $\widetilde{f}_m$. Lastly, to define $\widetilde{f}_{m+1} : X_{m+1} \to Y_{m+1}$ we use a similar argument as for the case $n+2$, namely by (4.3) there is a unique map $\widetilde{f}_{m+1}$ from the coequaliser X_{m+1} to Y_{m+1} satisfying $\widetilde{f}_{m+1} \circ x_m = y_m \circ T\widetilde{f}_m$. Hence, we get a morphism of algebraic chains $\widetilde{f} : X_\bullet \to Y$ with $\widetilde{f}_0 = f$, and since its definition was forced for every X_n, it follows that this is the unique morphism with this property, as desired. We leave it to the reader to check naturality. ☐

4.2. Existence of the Free Algebra

Note that the usual forgetful functor[2] $U : T\text{-}\mathbf{Alg} \to \mathcal{C}$ factors through the forgetful functor $V : T\text{-}\mathbf{Alg}_\omega \to \mathcal{C}$ by the functor $\Delta : T\text{-}\mathbf{Alg} \to T\text{-}\mathbf{Alg}_\omega$ which sends a T-algebra $(X, \gamma : TX \to X)$ to the constant algebraic chain with $X_n = X$ for every n, and $x_n = \gamma$ for every n. We say that an algebraic chain (X, x) *stabilises* at an ordinal n if for all $m > n$ the map $j_n^m : X_n \to X_m$ is invertible.

Lemma 6. *Let (X, x) be an algebraic chain which stabilises at n. Define*

$$\gamma_n := (j_n^{n+1})^{-1} \circ x_n : TX_n \to X_{n+1} \cong X_n.$$

Then (X_n, γ_n) is a T-algebra which is a reflection of (X, x) along Δ. Specifically, we have a bijection

$$\hom_{T\text{-}\mathbf{Alg}_\omega}((X, x), \Delta(A, a)) \cong \hom_{T\text{-}\mathbf{Alg}}((X_n, \gamma_n), (A, a)),$$

induced by sending a morphism $f : (X, x) \to \Delta(A, a)$ to $f_n : X_n \to A$.

Proof. Firstly note that (X_n, γ_n) is indeed a (T, η)-algebra on X_n since both triangles in the following diagram commute

$$\begin{tikzcd}
X_n \arrow[d, "\eta_{X_n}"'] \arrow[dr, "j_n^{n+1}"] \arrow[drr, "1"] & & \\
TX_n \arrow[r, "x_n"'] & X_{n+1} \arrow[r, "(j_n^{n+1})^{-1}"'] & X_n.
\end{tikzcd}$$

Now let $f : (X_n, \gamma_n) \to (A, a)$ be a T-algebra morphism. We show that this induces a unique morphism $(X, x) \to \Delta(A, a)$ of algebraic chains with $f_n = f$. Since we have

[2]Here we write T-**Alg** for the category of algebras for the pointed endofunctor (T, η).

$f_n := f : X_n \to A$, the rest of the morphism is uniquely determined. Indeed, by the fact that it has to be a chain morphism all the squares in the following diagram have to commute

$$\begin{tikzcd}
X_0 \arrow[r, "j_0^1"] \arrow[d, "f_0"] & X_1 \arrow[r, "j_1^2"] \arrow[d, "f_1"] & \dots \arrow[r] & X_n \arrow[r, "j_n^{n+1}"] \arrow[d, "f_n"] & X_{n+1} \arrow[r, "j_{n+1}^{n+2}"] \arrow[d, "f_{n+1}"] & \dots \\
A \arrow[r, "1"'] & A \arrow[r, "1"'] & \dots \arrow[r, "1"'] & A \arrow[r, "1"'] & A \arrow[r, "1"'] & \dots
\end{tikzcd}$$

This forces us to define:

$$f_k = \begin{cases} f_n \circ j_k^n, & k < n, \\ f_n \circ (j_n^k)^{-1}, & k > n. \end{cases}$$

It remains to show that the resulting map is a morphism of algebraic chains. We distinguish three cases. In the case that $k = n$, note that the following diagram commutes since f is a morphism of algebras,

$$\begin{tikzcd}
TX_n \arrow[rr, "T(f_n)"] \arrow[d, "x_n"'] \arrow[dr, "\gamma_n"] & & TA \arrow[d, "a"] \\
X_{n+1} \arrow[r, "(j_n^{n+1})^{-1}"'] & X_n \arrow[r, "f_n"'] & A.
\end{tikzcd}$$

so we have $f_{n+1} \circ x_n = a \circ T(f_n)$. In the case $k < n$, we look at the following diagram

$$\begin{tikzcd}
TX_k \arrow[r, "T(j_k^n)"] \arrow[d, "x_k"'] & TX_n \arrow[r, "T(f_n)"] \arrow[d, "x_n"] & TA \arrow[dd, "a"] \\
X_{k+1} \arrow[r, "j_{k+1}^{n+1}"'] \arrow[d, "1"'] & X_{n+1} \arrow[d, "(j_n^{n+1})^{-1}"] & \\
X_{k+1} \arrow[r, "j_{k+1}^n"'] & X_n \arrow[r, "f_n"'] & A
\end{tikzcd}$$

Since every inner square in the diagram commutes, the outer square also commutes, which shows that $f_{k+1} \circ x_k = a \circ T(f_k)$. The case $k > n$ is analogous.

Conversely, it is easy to see from the above considerations that if $f : (X, x) \to \Delta(A, a)$ is a morphism of algebraic chains, then $f_n : X_n \to A$ is an algebra morphism $(X_n, \gamma_n) \to (A, a)$. □

The following proposition shows that if the algebraic chain $X_\bullet$ stabilises, we can compute the free T-algebra on X.

Proposition 7 [6, Proposition 7]. *If $X_\bullet$ stabilises at n, then (X_n, γ_n) is the free T-algebra on X.*

Proof. To prove that (X_n, γ_n) is the free algebra on X, we need to show that for any T-algebra (A, a) there is a bijection

$$\hom_{T\text{-}\mathbf{Alg}}((X_n, \gamma_n), (A, a)) \cong \hom_{\mathcal{C}}(X, A).$$

To see this, we make use of the fact that U factors through V, and that we have already constructed a left adjoint of V in Proposition 6.

$$\begin{tikzcd}
T\text{-}\mathbf{Alg} \arrow[rr, "U"] \arrow[dr, "\Delta"'] & & \mathcal{C} \arrow[dl, shift right, "(-)_\bullet"'] \arrow[dl, phantom, "\perp"] \\
& T\text{-}\mathbf{Alg}_\omega \arrow[ur, shift right, "V"'] &
\end{tikzcd}$$

Using this fact, together with Lemma 6 for the algebraic chain $X_\bullet$, we have:

$$\begin{aligned}
\hom_{\mathcal{C}}(X, A) &\cong \hom_{\mathcal{C}}(X, V\Delta(A, a)) \\
&\cong \hom_{T\text{-}\mathbf{Alg}_\omega}(X_\bullet, \Delta(A, a)) \\
&\cong \hom_{T\text{-}\mathbf{Alg}}((X_n, \gamma), (A, a)).
\end{aligned}$$

□

The next proposition gives a sufficient criterion for the algebraic chain $X_\bullet$ to stabilise. Before we prove it, we prove a lemma that will be useful.

Lemma 7. *If $j_\omega^{\omega+1}$ is invertible in the algebraic chain $X_\bullet$, then j_ω^n is invertible for every $n \geq \omega$.*

Proof. Firstly note that

$$j_\omega^n = j_{n-1}^n \circ \ldots \circ j_{\omega+1}^{\omega+2} \circ j_\omega^{\omega+1}.$$

Hence, it is sufficient to show that j_n^{n+1} is invertible for every $n \geq \omega$. We do so by induction; the base case holds by assumption. Now assume j_n^{n+1} is invertible for some $n \geq \omega$. We show that then j_{n+1}^{n+2} is also invertible. To this end, we claim that the following diagram is a coequaliser diagram

$$\begin{tikzcd}
TX_n \arrow[r, shift left, "Tx_n \circ T\eta_{X_n}"] \arrow[r, shift right, "Tx_n \circ \eta_{TX_n}"'] & TX_{n+1} \arrow[r, "x_n \circ T(j_n^{n+1})^{-1}"] & X_{n+1}.
\end{tikzcd}$$

Firstly we show that it is a fork. We calculate, on the one hand

$$\begin{aligned}
x_n \circ T(j_n^{n+1})^{-1} \circ Tx_n \circ T\eta_{X_n} &= x_n \circ T((j_n^{n+1})^{-1} \circ x_n \circ \eta_{X_n}) \\
&= x_n \circ T((j_n^{n+1})^{-1} \circ j_n^{n+1}) \\
&= x_n \circ T(1_{X_n}) = x_n,
\end{aligned}$$

while on the other hand

$$\begin{aligned}
x_n \circ T(j_n^{n+1})^{-1} \circ Tx_n \circ \eta_{TX_n} &= x_n \circ T((j_n^{n+1})^{-1} \circ x_n) \circ \eta_{TX_n} \\
&= x_n \circ \eta_{X_n} \circ (j_n^{n+1})^{-1} \circ x_n
\end{aligned}$$

$$
\begin{aligned}
&= j_n^{n+1} \circ (j_n^{n+1})^{-1} \circ x_n \\
&= 1_{X_{n+1}} \circ x_n = x_n.
\end{aligned}
$$

For the second equality we have used the naturality diagram

$$
\begin{tikzcd}
TX_n \arrow[r, "\eta_{TX_n}"] \arrow[d, "(j_n^{n+1})^{-1} \circ x_n"'] & TTX_n \arrow[d, "T((j_n^{n+1})^{-1} \circ x_n)"] \\
X_n \arrow[r, "\eta_{X_n}"'] & TX_n.
\end{tikzcd}
$$

Now suppose we have a map $h : TX_{n+1} \to C$ which also coequalises the pair $Tx_n \circ T\eta_{X_n}$ and $Tx_n \circ \eta_{TX_n}$. We have to show that there is a unique map $\psi : X_{n+1} \to C$ such that the following diagram commutes

$$
\begin{tikzcd}
TX_n \arrow[r, shift left, "Tx_n \circ T\eta_{X_n}"] \arrow[r, shift right, "Tx_n \circ \eta_{TX_n}"'] & TX_{n+1} \arrow[r, "x_n \circ T(j_n^{n+1})^{-1}"] \arrow[dr, "h"'] & X_{n+1} \arrow[d, dashed, "\varphi"] \\
& & C
\end{tikzcd}
$$

We define $\varphi = h \circ \eta_{X_{n+1}}$. To show that this makes the triangle commute, we compute:

$$
\begin{aligned}
\varphi \circ x_n \circ T(j_n^{n+1})^{-1} &= h \circ \eta_{X_{n+1}} \circ x_n \circ T(j_n^{n+1})^{-1} \\
&= h \circ Tx_n \circ \eta_{TX_n} \circ T(j_n^{n+1})^{-1} \\
&= h \circ Tx_n \circ T\eta_{X_n} \circ T(j_n^{n+1})^{-1} \\
&= h \circ T(x_n \circ \eta_{X_n} \circ (j_n^{n+1})^{-1}) \\
&= h \circ T(j_n^{n+1} \circ (j_n^{n+1})^{-1}) \\
&= h \circ T(1_{X_{n+1}}) = h.
\end{aligned}
$$

For the second equality we use naturality of η, and for the third equality the coequaliser property of h. For uniqueness, suppose we have a morphism $\psi : X_{n+1} \to C$ for which the triangle commutes, i.e. we have $\psi \circ x_n \circ T(j_n^{n+1})^{-1} = h$. Then we have

$$
\begin{aligned}
\varphi &= h \circ \eta_{X_{n+1}} \\
&= \psi \circ x_n \circ T(j_n^{n+1})^{-1} \circ \eta_{X_{n+1}} \\
&= \psi \circ x_n \circ \eta_{X_n} \circ (j_n^{n+1})^{-1} \\
&= \psi \circ j_n^{n+1} \circ (j_n^{n+1})^{-1} \\
&= \psi \circ 1_{X_{n+1}} = \psi.
\end{aligned}
$$

Thus, we get an isomorphism $X_{n+1} \cong X_{n+2}$. Moreover, since the triangle on the left below commutes, this isomorphism is given by $j_{n+1}^{n+2} : X_{n+1} \to X_{n+2}$.

$$
\begin{tikzcd}
TX_{n+1} \arrow[r, "x_n \circ T(j_n^{n+1})^{-1}"] \arrow[dr, "x_{n+1}"'] & X_{n+1} \arrow[d, "j_{n+1}^{n+2}"] \\
& X_{n+2}
\end{tikzcd}
\qquad
\begin{tikzcd}
TX_{n+1} \arrow[r, "T(j_n^{n+1})^{-1}"] \arrow[d, "1"'] & TX_n \arrow[r, "x_n"] \arrow[d, "T(j_n^{n+1})"] & X_{n+1} \arrow[d, "j_{n+1}^{n+2}"] \\
TX_{n+1} \arrow[r, "1"'] & TX_{n+1} \arrow[r, "x_{n+1}"'] & X_{n+2}.
\end{tikzcd}
$$

Indeed, we observe that every square in the diagram on the right commutes, and thus the outer square commutes, which is the same as the triangle on the left. This shows that j^{n+2}_{n+1} is invertible, and therefore completes the proof. $\square$

Proposition 8 [6, Proposition 23]**.** *If T preserves the colimit $X_\omega = \mathrm{colim}_{n<\omega} X_n$, then $X_\bullet$ stabilises at ω.*

Proof. Firstly note that the maps $x_n : TX_n \to X_n$ for $n < \omega$ form a morphism between chains of length ω,

$$\begin{tikzcd}
TX_0 \arrow[r, "T(j_0^1)"] \arrow[d, "x_0"'] & TX_1 \arrow[r, "T(j_1^2)"] \arrow[d, "x_1"'] & TX_2 \arrow[r] \arrow[d, "x_2"'] & \dots \arrow[r] & \mathrm{colim}_{n<\omega} TX_n \arrow[d, dashed, "x'_\omega"] \\
X_1 \arrow[r, "j_1^2"'] & X_2 \arrow[r, "j_2^3"'] & X_3 \arrow[r] & \dots \arrow[r] & \mathrm{colim}_{n<\omega} X_n.
\end{tikzcd}$$

Thus we get a unique morphism between their colimits $\mathrm{colim}_{n<\omega} TX_n \to \mathrm{colim}_{n<\omega} X_n$ making the above diagram commute. Since T preserves the colimit X_ω, we have a canonical isomorphism $\mathrm{colim}_{n<\omega} TX_n \cong T(X_\omega)$ and thus we may consider the morphism x'_ω as a morphism $T(X_\omega) \to X_\omega$. We claim that the following diagram is a coequaliser

$$\begin{tikzcd}
\mathrm{colim}_{n<\omega} TX_n \arrow[r, shift left, "\langle Tj_n^\omega \rangle"] \arrow[r, shift right, "\langle \eta_{X_\omega} \circ j_{n+1}^\omega \circ x_n \rangle"'] & TX_\omega \arrow[r, "x'_\omega"] & X_\omega.
\end{tikzcd}$$

Firstly we show that it is a fork. We compute for $n < \omega$:

$$x'_\omega \circ Tj_n^\omega = j_{n+1}^\omega \circ x_n = x'_\omega \circ \eta_{X_\omega} \circ j_{n+1}^\omega \circ x_n.$$

Here we use that $x'_\omega \circ \eta_{X_\omega} = 1_{X_\omega}$. To prove this, note that both are maps out of a colimit, whence it suffices to show that they are equal upon precomposition with the colimit inclusion maps. We calculate

$$\begin{aligned}
x'_\omega \circ \eta_{X_\omega} \circ j_n^\omega &= x'_\omega \circ T(j_n^\omega) \circ \eta_{X_n} \\
&= j_{n+1}^\omega \circ x_n \circ \eta_{X_n} \\
&= j_{n+1}^\omega \circ j_n^{n+1} \\
&= j_n^\omega.
\end{aligned}$$

Here the first step uses naturality of η and the second step uses the definition of an algebraic chain. Next, we show that it also has the universal property of the coequaliser. Assume we have some morphism $h : TX_\omega \to C$ which coequalises the pair $\langle Tj_n^\omega \rangle$ and $\langle \eta_{X_\omega} \circ j_{n+1}^\omega \circ x_n \rangle$. We have to show that there exists a unique map $\varphi : X_\omega \to C$ such that $\varphi \circ x'_\omega = h$

$$\begin{tikzcd}
\mathrm{colim}_{n<\omega} TX_n \arrow[r, shift left, "\langle Tj_n^\omega \rangle"] \arrow[r, shift right, "\langle \eta_{X_\omega} \circ j_{n+1}^\omega \circ x_n \rangle"'] & TX_\omega \arrow[r, "x'_\omega"] \arrow[dr, "h"'] & X_\omega \arrow[d, dashed, "\varphi"] \\
& & C.
\end{tikzcd}$$

To define φ, note that we have a cocone $(h \circ Tj_n^\omega \circ \eta_{X_n})_{n\geq 1}$ on $(X_n)_{n<\omega}$ with vertex C. To see that this indeed forms a cocone, note that for any $k, n < \omega$ the following diagram commutes

$$\begin{tikzcd}
X_k \arrow[r, "\eta_{X_k}"] \arrow[d, "j_k^n"'] & TX_k \arrow[r, "Tj_k^\omega"] \arrow[d, "Tj_k^n"'] & TX_\omega \arrow[r, "h"] \arrow[d, "1"'] & C \arrow[d, "1"] \\
X_n \arrow[r, "\eta_{X_n}"'] & TX_n \arrow[r, "Tj_n^\omega"'] & TX_\omega \arrow[r, "h"'] & C.
\end{tikzcd}$$

Thus we get an induced map $\varphi : X_\omega \to C$ with the property that for all $n < \omega$,

$$\varphi \circ j_n^\omega = h \circ Tj_n^\omega \circ \eta_{X_n}.$$

We claim that $\varphi \circ x'_\omega = h$. In order to show this, firstly we show that $h \circ \eta_{X_\omega} = \varphi$. Since both are maps out of a colimit, it suffices to show that they coincide upon precomposition with the colimit inclusions. We have

$$h \circ \eta_{X_\omega} \circ j_n^\omega = h \circ Tj_n^\omega \circ \eta_{X_n} = \varphi \circ j_n^\omega.$$

So we have $h \circ \eta_{X_\omega} = \varphi$. Now to see that $\varphi \circ x'_\omega = h$ note again that both are maps out of a colimit, whence it suffices to show that they coincide upon precomposition with the colimit inclusions. We compute:

$$\begin{aligned}
h \circ Tj_n^\omega &= h \circ \eta_{X_\omega} \circ j_{n+1}^\omega \circ x_n \\
&= \varphi \circ j_{n+1}^\omega \circ x_n \\
&= \varphi \circ x'_\omega \circ Tj_n^\omega.
\end{aligned}$$

In the first equality we use the 'coequaliser property' of h. Lastly we need to show that φ is the unique map with this property. So suppose $\psi : X_\omega \to C$ also has the property that $\psi \circ x'_\omega = h$. To show that $\varphi = \psi$, it suffices to show that $\varphi \circ j_n^\omega = \psi \circ j_n^\omega$ for all $n < \omega$. We compute

$$\begin{aligned}
\varphi \circ j_n^\omega &= h \circ Tj_n^\omega \circ \eta_{X_n} \\
&= \psi \circ x'_\omega \circ Tj_n^\omega \circ \eta_{X_n} \\
&= \psi \circ x'_\omega \circ \eta_{X_\omega} \circ j_n^\omega \\
&= \psi \circ j_n^\omega.
\end{aligned}$$

Here we use that $x'_\omega \circ \eta_{X_\omega} = 1_{X_\omega}$, as shown before.

It follows that $X_\omega \cong X_{\omega+1}$, where we note that the isomorphism is given by $j_\omega^{\omega+1}$ since the following diagram commutes

$$\begin{tikzcd}
TX_\omega \arrow[r, "x'_\omega"] \arrow[dr, "x_\omega"'] & X_\omega \arrow[d, dashed, "j_\omega^{\omega+1}"] \\
& X_{\omega+1}
\end{tikzcd}$$

To see this, again one can calculate that both maps agree upon precomposition with the colimit inclusion. So we have that $j_\omega^{\omega+1}$ is invertible. An application of Lemma 7 now completes the proof. $\square$

We obtain the following theorem.

Theorem 2 [6, Theorem 24]**.** *Let* (T, η) *be a pointed endofunctor on a cocomplete category* $\mathcal{C}$*. If* T *preserves colimits of* ω*-chains (in particular if* T *is finitary), then the free* T*-algebra exists for every object* X *in* $\mathcal{C}$*. This extends to a functor* $F : \mathcal{C} \to T$-**Alg** *which is left adjoint to the forgetful functor* $U : T$-**Alg** $\to \mathcal{C}$*.*

Proof. Combining Proposition 7 and 8 shows that the free T-algebra exists for every object X in $\mathcal{C}$. This extends to a functor F, left adjoint to U, which is defined on morphisms in the usual way: we send $f : X \to Y$ to the induced map between their free T-algebras $\widetilde{f} : FX \to FY$. □

5. Free Monads and Finitary Monads

In this chapter we study the category of monads. In Chapter 4 we constructed the free algebra on a finitary pointed endofunctor (P, ρ), which induces a left adjoint to the forgetful functor (P, ρ)-$\mathbf{Alg} \to \mathcal{C}$. In this chapter we prove that the induced monad of this adjunction is in fact the free monad over the pointed endofunctor (P, ρ) and is finitary when P is finitary (Theorem 3). An important tool in the proof is the notion of an algebraically-free monad.

We will need these results in Chapter 6 to show that the generated AWFS is finitary and that its associated monad is free on a pointed endofunctor.

Lastly, in Section 4 of this chapter we give a constructive proof that the category of finitary monads on a cocomplete category is cocomplete (Theorem 4). We will need this in Chapter 6 to construct a coequaliser of finitary monads. Moreover, our construction implies that these colimits are 'algebraic', another fact that we need in Chapter 6.

Most of the results in this chapter occur in some form in Kelly [15], though the presentation here is quite different. We rephrase them and adapt certain results to the finitary case, we make sure that the results are constructively valid, and we fill in the details.

5.1. Category of Monads

Let us start by looking at the category of monads on a category $\mathcal{C}$. Recall that in Lemma 2 we defined a (lax) monad morphism $\mathsf{H} \to \mathsf{K}$ between two monads $\mathsf{H} = (H, \eta, \mu)$ and $\mathsf{K} = (K, \iota, \nu)$ on categories $\mathcal{C}$ and $\mathcal{D}$ respectively. This was defined as a tuple (T, λ) with a functor $T : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\lambda : KT \to TH$ which is compatible with the monad units and multiplications, i.e. making diagram (2.2) commute. However, when H and K are over the same category $\mathcal{C}$, then we take the oplax direction as standard (and T to be the identity). So concretely, a monad morphism $\mathsf{H} \to \mathsf{K}$ is a natural transformation $\lambda : H \to K$ which respects both the monad units and the monad multiplications, that is, the following two diagrams commute

$$\begin{tikzcd} 1 \arrow[d, "\eta"'] \arrow[dr, "\iota"] & \\ H \arrow[r, "\lambda"'] & K \end{tikzcd} \qquad \begin{tikzcd} HH \arrow[r, "H\lambda"] \arrow[d, "\mu"'] & HK \arrow[r, "\lambda_K"] & KK \arrow[d, "\iota"] \\ H \arrow[rr, "\lambda"'] & & K. \end{tikzcd}$$

We denote the category of monads over $\mathcal{C}$ by $\mathbf{Mnd}(\mathcal{C})$, and the full subcategory of finitary monads by $\mathbf{Mnd}_{\mathrm{f}}(\mathcal{C})$. Now by Lemma 2 we have that the assignment

$$(-)\text{-}\mathbf{Alg} : \mathbf{Mnd}(\mathcal{C})^{\mathrm{op}} \longrightarrow \mathbf{Cat}\,/\mathcal{C} : \mathsf{H} \longmapsto \mathsf{H}\text{-}\mathbf{Alg},$$

is part of a functor which is fully faithful.[1]

We also have the category $\mathbf{PtEnd}(\mathcal{C})$ of pointed endofunctors (P, ρ) over a category $\mathcal{C}$. A morphism between two pointed endofunctors is a natural transformation between them which respects the units, we call these *pointed transformations*. Clearly, we have a forgetful functor $\mathbf{Mnd}(\mathcal{C}) \to \mathbf{PtEnd}(\mathcal{C})$. We define what it means for a monad to be free over a pointed endofunctor, and algebraically-free over a pointed endofunctor.

Definition 4. Let $\mathcal{C}$ be a category. A monad $\mathsf{H} = (H, \eta, \mu)$ on $\mathcal{C}$ is called *free* over a pointed endofunctor (P, ρ) on $\mathcal{C}$ if it is the free object relative to the forgetful functor $\mathbf{Mnd}(\mathcal{C}) \to \mathbf{PtEnd}(\mathcal{C})$. A monad H is called *algebraically-free* over (P, ρ) if we have an isomorphism $\mathsf{H}\text{-}\mathbf{Alg} \cong (P, \rho)\text{-}\mathbf{Alg}$ over $\mathcal{C}$, where the former is the category of algebras for the monad H and the latter is the category of algebras for the pointed endofunctor (P, ρ).

We want to show that if a monad H is algebraically free over (P, ρ), then it is also free over (P, ρ). To this end, we need the following lemma.

Lemma 8. *Let (P, ρ) be a pointed endofunctor and $\mathsf{K} = (K, \iota, \nu)$ a monad on a category $\mathcal{C}$. Then there is a natural one-to-one correspondence between pointed transformations $(P, \rho) \to (K, \iota)$ and functors $\mathsf{K}\text{-}\mathbf{Alg} \to (P, \rho)\text{-}\mathbf{Alg}$ which commute with the forgetful functors. That is, we have a natural bijection*

$$\hom_{\mathbf{PtEnd}(\mathcal{C})}((P, \rho), (K, \iota)) \cong \hom_{\mathbf{Cat}\,/\mathcal{C}}(\mathsf{K}\text{-}\mathbf{Alg}, (P, \rho)\text{-}\mathbf{Alg}).$$

Proof. Firstly suppose we have a pointed transformation $\lambda : (P, \rho) \to (K, \iota)$. We define a functor $\mathsf{K}\text{-}\mathbf{Alg} \to (P, \rho)\text{-}\mathbf{Alg}$ as follows. On objects we map a K-algebra (X, α) to $(X, \alpha \circ \lambda_X)$. This is indeed a (P, ρ)-algebra since the diagram on the left commutes

$$\begin{tikzcd} X \arrow[r, "\rho_X"] \arrow[dr, "\iota_X"] \arrow[ddr, "1_X"'] & PX \arrow[d, "\lambda_X"] \\ & KX \arrow[d, "\alpha"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} PX \arrow[r, "Pf"] \arrow[d, "\lambda_X"'] & PY \arrow[d, "\lambda_Y"] \\ KX \arrow[r, "Kf"] \arrow[d, "\alpha"'] & KY \arrow[d, "\beta"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Indeed, the upper triangle commutes because λ is a pointed transformation, while the lower triangle commutes because α is a K-algebra. On morphisms, we map $f : (X, \alpha) \to (Y, \beta)$ to itself. This is a map of (P, ρ)-algebras because the diagram on the right commutes.

Conversely, suppose we have a functor $S : \mathsf{K}\text{-}\mathbf{Alg} \to (P, \rho)\text{-}\mathbf{Alg}$ which commutes with the forgetful functors. We define a natural transformation $\lambda : P \to K$ at $X \in \mathcal{C}$ as follows. Firstly note that we have the free K-algebra on X which is given by (KX, ν_X). Applying S and noting that S commutes with the forgetful functors, we get $S(KX, \nu_X) = (KX, \theta(X))$

[1] As mentioned previously, the 'op' comes from the fact that the direction of the monad changes from lax to oplax.

for some $\theta(X) : PKX \to KX$. Now we define $\lambda_X : PX \to KX$ as the following composition

$$PX \xrightarrow{P(\iota_X)} PKX \xrightarrow{\theta(X)} KX.$$

We need to show that this is indeed a pointed transformation between (P, ρ) and (K, ι). Note that since $(KX, \theta(X))$ is a (P, ρ)-algebra, we have that $\theta(X) \circ \rho_{KX} = 1_{KX}$. Now we have the following commutative diagram on the left below

$$\begin{tikzcd} X \arrow[r, "\rho_X"] \arrow[d, "\iota_X"'] & PX \arrow[d, "P(\iota_X)"] \\ KX \arrow[r, "\rho_{KX}"'] \arrow[dr, "1_{KX}"'] & PKX \arrow[d, "\theta(X)"] \\ & KX \end{tikzcd} \qquad \begin{tikzcd} PX \arrow[r, "P(\iota_X)"] \arrow[d, "Pf"'] & PKX \arrow[r, "\theta(X)"] \arrow[d, "PKf"] & KX \arrow[d, "Kf"] \\ PY \arrow[r, "P(\iota_Y)"'] & PKY \arrow[r, "\theta(Y)"'] & KY \end{tikzcd}$$

where the upper square commutes by naturality of $\rho : 1 \to P$. This shows that $\lambda_X \circ \rho_X = \iota_X$, and so λ is pointed. To show naturality, let $f : X \to Y$ be a morphism in $\mathcal{C}$. We look at the diagram on the right above. The left square commutes by naturality of ι. We claim that the right square also commutes. To see this, note that by naturality of ν we have a morphism $Kf : (KX, \nu_X) \to (KY, \nu_Y)$ in T-**Alg**. Applying S and using the fact that it commutes with the forgetful functors we have a morphism $Kf : (KX, \theta(X)) \to (KY, \theta(Y))$ in (P, ρ)-**Alg**. This shows that also the right square commutes and thus naturality holds.

To see that these operations are mutually inverse, firstly let $\lambda : (P, \rho) \to (K, \iota)$ be a pointed transformation. Then $S(\lambda)$ maps (KX, ν_X) to $(KX, \nu_X \circ \lambda_{KX})$, and thus the induced natural transformation $\lambda' : P \to K$ at X is given by

$$\lambda'_X = \nu_X \circ \lambda_{KX} \circ P(\iota_X) = \nu_X \circ K(\iota_X) \circ \lambda_X = \lambda_X,$$

where we use naturality of λ and the unit law of the monad K respectively. Conversely, let $S : \mathsf{K}$-**Alg** $\to (P, \rho)$-**Alg** be a functor over $\mathcal{C}$. Then the induced transformation $\lambda : P \to K$ is given at X by $\lambda_X = \theta(X) \circ P(\iota_X)$, with $\theta(X)$ defined by $S(KX, \nu_X) = (KX, \theta(X))$. Thus, the induced functor $S' : \mathsf{K}$-**Alg** $\to (P, \rho)$-**Alg** is given by $(X, \alpha) \mapsto (X, \alpha \circ \theta(X) \circ P(\iota_X))$. To see that this is equal to S, we note that $\alpha : KX \to X$ is a morphism in T-**Alg** between the T-algebras (KX, ν_X) and (X, α). Therefore its S image is a morphism in the category (P, ρ)-**Alg**, which means that the diagram below commutes

$$\begin{tikzcd} PKX \arrow[r, "P(\alpha)"] \arrow[d, "\theta(X)"'] & PX \arrow[d, "S(X,\alpha)"] \\ KX \arrow[r, "\alpha"'] & X. \end{tikzcd}$$

Using this we calculate:

$$S'(X, \alpha) = \alpha \circ \theta(X) \circ P(\iota_X) = S(X, \alpha) \circ P(\alpha) \circ P(\iota_X) = S(X, \alpha) \circ P(\alpha \circ \iota_X) = S(X, \alpha).$$

We leave it to the reader to verify that this bijection is natural. $\square$

Proposition 9 [15, Proposition 22.2]**.** *A monad* H *which is algebraically-free on a pointed endofunctor* (P, ρ) *is free on* (P, ρ).

Proof. Suppose H is algebraically-free on a pointed endofunctor (P, ρ). Thus by assumption we have an isomorphism $\mathsf{H}\text{-}\mathbf{Alg} \cong (P, \rho)\text{-}\mathbf{Alg}$ over $\mathcal{C}$. To show that it is free on (P, ρ), let $\mathsf{K} = (K, \iota, \nu)$ be an arbitrary monad. We have the following string of bijections

$$\begin{aligned} \hom_{\mathbf{Mnd}(\mathcal{C})}(\mathsf{H}, \mathsf{K}) &\cong \hom_{\mathbf{Cat}/\mathcal{C}}(\mathsf{K}\text{-}\mathbf{Alg}, \mathsf{H}\text{-}\mathbf{Alg}) && \text{(Lemma 2)} \\ &\cong \hom_{\mathbf{Cat}/\mathcal{C}}(\mathsf{K}\text{-}\mathbf{Alg}, (P, \rho)\text{-}\mathbf{Alg}) && \text{(by assumption)} \\ &\cong \hom_{\mathbf{PtEnd}(\mathcal{C})}((P, \rho), (K, \iota)). && \text{(Lemma 8)} \end{aligned}$$

□

5.2. Existence of the Free Monad

Proposition 20 states that for a monad T the forgetful functor $U^T : \mathsf{T}\text{-}\mathbf{Alg} \to \mathcal{C}$ creates all limits that exist in the codomain and creates colimits which are preserved by T. The situation for a pointed endofunctor is similar: the forgetful functor $U : (P, \rho)\text{-}\mathbf{Alg} \to \mathcal{C}$ creates all colimits that exist in its codomain and are preserved by the corresponding endofunctor P, as is proved in the next lemma.

Lemma 9. *Let* (P, ρ) *be a pointed endofunctor on some category* $\mathcal{C}$. *Then the forgetful functor* $U : (P, \rho)\text{-}\mathbf{Alg} \to \mathcal{C}$ *creates all colimits that exist in its codomain and are preserved by* P.

Proof. Let $X : \mathcal{I} \to (P, \rho)\text{-}\mathbf{Alg}$ be a functor and (C, σ) be a colimiting cocone for $U \circ X$. We denote the P-algebra $X(i)$ as (X_i, α_i). We need to construct a (P, ρ)-algebra structure on C, i.e. a map $\alpha : PC \to C$ such that $\alpha \circ \rho_C = 1_C$. By assumption P preserves the colimit of $U \circ X$, so the cocone $(PC, P(\sigma_i))$ is in fact colimiting. Therefore to give a morphism $PC \to C$ is equivalent to giving a compatible family of maps $\{PX_i \to C\}_{i \in \mathcal{I}}$. We take the family of maps $\{\sigma_i \circ \alpha_i\}_{i \in \mathcal{I}}$. Now for any morphism $f : i \to j$ in $\mathcal{I}$, since Xf is a map of algebras $(X_i, \alpha_i) \to (X_j, \alpha_j)$, we have that $Xf \circ \alpha_i = \alpha_j \circ P(Xf)$. Since (C, σ) is a cocone on $U \circ X$ we have $\sigma_j \circ Xf = \sigma_i$. Thus we compute:

$$\sigma_j \circ \alpha_j \circ P(Xf) = \sigma_j \circ Xf \circ \alpha_i = \sigma_i \circ \alpha_i.$$

It follows that $(C, \sigma_i \circ \alpha_i)_{i \in \mathcal{I}}$ a cocone on $P \circ U \circ X$. Therefore we have a unique map

$\alpha : PC \to C$ with the property that $\alpha \circ P(\sigma_i) = \sigma_i \circ \alpha_i$, as shown in the diagram below.

$$
\begin{tikzcd}
 & PC \arrow[d, dashed, "\alpha"] & \\
 & C & \\
X_i \arrow[ur, "\sigma_i"] \arrow[rr, "Xf"] & & X_j \arrow[ul, "\sigma_j"'] \\
PX_i \arrow[u, "\alpha_i"'] \arrow[rr, "P(Xf)"'] \arrow[uuur, bend left, "P(\sigma_i)"] & & PX_j \arrow[u, "\alpha_j"] \arrow[uuul, bend right, "P(\sigma_j)"']
\end{tikzcd}
$$

Thus, each map $\sigma_i : X_i \to C$ becomes a morphism of algebras $(X_i, \alpha_i) \to (C, \alpha)$ as soon as we can show that (C, α) is indeed an algebra for the pointed endofunctor (P, ρ), that is, $\alpha \circ \rho_C = 1_C$. Since both are maps out of a colimit, it suffices to show that they become equal after precomposing with the colimit inclusions. We calculate

$$\alpha \circ \rho_C \circ \sigma_i = \alpha \circ P(\sigma_i) \circ \rho_i = \sigma_i \circ \alpha_i \circ \rho_i = \sigma_i,$$

where the first equality is due to naturality of ρ, the second is by the property of α, and the third is because each (X_i, α_i) is an algebra for the pointed endofunctor (P, ρ).

It remains to verify that (C, α) together with the maps $\sigma_i : (X_i, \alpha_i) \to (C, \alpha)$ is the colimit of X in the category (P, ρ)-**Alg**; we leave this to the reader. □

We are now ready to prove the first main result of this chapter, namely the existence of the free monad over a pointed endofunctor under suitable conditions.

Theorem 3 [15, Theorem 22.3]**.** *Let (P, ρ) be a pointed endofunctor on a cocomplete category $\mathcal{C}$. If the forgetful functor $U : (P, \rho)$-**Alg** $\to \mathcal{C}$ has a left adjoint F, then the free monad $\mathsf{T} = (T, \eta, \mu)$ on (P, ρ) exists, and it is given by the monad induced by the adjunction $F \dashv U$. Moreover, if P is finitary, then so is the induced monad T.*

Proof. By Proposition 16 it follows that U creates coequalisers of U-split pairs. Since it also has a left adjoint F, it follows by Beck's monadicity theorem (Theorem 7 in Appendix A.2) that U is (strictly) monadic.[2] But then we have that the comparison functor

$$K : (P, \rho)\text{-}\mathbf{Alg} \to \mathsf{T}\text{-}\mathbf{Alg} : (X, \alpha) \mapsto (X, U(\epsilon_{(X,\alpha)}) : UFX \to X)$$

is an isomorphism. Thus, the monad T is algebraically-free over (P, ρ), and thus by Proposition 9 it is free on (P, ρ).

Now suppose that P is finitary. Then by Lemma 9 it follows that the forgetful functor U creates ω-filtered colimits since $\mathcal{C}$ has them, and hence U also preserves them. Since F is a left adjoint it preserves all colimits, and therefore $T = UF$ preserves ω-filtered colimits. □

[2]For a certain class of functors, the 'amnestic isofibrations', the notions of monadic and strictly monadic coincide; typical examples of such functors are forgetful functors.

Proposition 10. *The category* $\mathbf{PtEnd}(\mathcal{C})$ *of pointed endofunctors on a category* $\mathcal{C}$ *is cocomplete when* $\mathcal{C}$ *is cocomplete.*

Proof. Let $X : \mathcal{J} \to \mathbf{PtEnd}(\mathcal{C})$ be a diagram, and denote $X(j) = (T_j, \tau_j)$. We extend the diagram $\mathcal{J}$ to a new diagram $\mathcal{J}'$ by adding an initial object 0 to it. We define $X' : \mathcal{J}' \to \mathbf{End}(\mathcal{C})$ (where $\mathbf{End}(\mathcal{C})$ denotes the category of endofunctors on $\mathcal{C}$) as $X'(j) = T_j$, and $X'(0) = \mathrm{id}_{\mathcal{C}}$ the identity functor. We set $X'(! : 0 \to j) = \tau_j : 1 \to T_j$. Recall that in a functor category $[\mathcal{C}, \mathcal{D}]$, if $\mathcal{D}$ has (co)limits of shape $\mathcal{J}$, then so does $[\mathcal{C}, \mathcal{D}]$ for any category $\mathcal{C}$, and (co)limits are 'computed pointwise'. Thus we can take the colimit T of the diagram X' in $\mathbf{End}(\mathcal{C})$. Define $\tau : 1 \to T$ as $\tau = r_0$, the colimit inclusion of the identity functor $X'(0)$. We leave it to the reader to verify (T, τ) is the colimit of X in $\mathbf{PtEnd}(\mathcal{C})$. □

Corollary 2. *Let* $\mathcal{C}$ *be cocomplete, then the forgetful functor* $\mathbf{Mnd}_{\mathrm{f}}(\mathcal{C}) \to \mathbf{PtEnd}_{\mathrm{f}}(\mathcal{C})$ *mapping a finitary monad to its underlying pointed endofunctor has a left adjoint.*

Proof. Combine Theorems 2 and 3. □

5.3. Colimits of Finitary Monads

In this section, we construct colimits in the category of finitary monads, using the strategy of Kelly [15, Theorem 27.1]: taking the limit of the induced diagram of their algebra categories in $\mathbf{Cat}/\mathcal{C}$, and showing that the algebra functor $(-)\text{-}\mathbf{Alg} : \mathbf{Mnd}(\mathcal{C})^{\mathrm{op}} \to \mathbf{Cat}/\mathcal{C}$ creates limits. One step in the proof uses the existence of the free algebra for a finitary pointed endofunctor, for which we rely on Theorem 2 for a constructive proof. For the rest of the proof we explicitly keep track of the construction. Specifically, we introduce the notion of an *effective repletion* to avoid the use of the axiom of choice.

5.3.1. Well-pointed Endofunctors

Firstly, let us introduce well-pointed endofunctors. We call a pointed endofunctor (S, σ) *well-pointed* if $S\sigma = \sigma S : S \to S^2$. We need some technical lemmas about well-pointed endofunctors.

Lemma 10. *Suppose we have a pullback diagram of categories as on the left below*

$$
\begin{tikzcd}
\mathcal{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{F} \arrow[d] \\
\mathcal{C} \arrow[r, shift right, "V"'] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, shift right, "K"']
\end{tikzcd}
\qquad
\begin{tikzcd}
KV \arrow[r, "K\eta_V"] \arrow[d, "\epsilon"'] & KTV \arrow[d, "\mu"] \\
1 \arrow[r, "\rho"'] & P \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}
$$

with $K \dashv V$*, and with* $\mathcal{F} \cong (T, \eta)$*-Alg for some pointed endofunctor* (T, η) *on* $\mathcal{E}$*. Then* $\mathcal{D} \cong (P, \rho)$*-Alg, for some pointed endofunctor* (P, ρ) *on* $\mathcal{C}$*, where* P *and* ρ *are defined by the pushout on the right above. Moreover, if* (T, η) *is well-pointed, then so is* (P, ρ)*.*

Proof. An object of the pushout $\mathcal{D}$ is given by an object $X \in \mathcal{C}$ and T-algebra structure $\alpha : TVX \to VX$ subject to the unit condition as on the left below.

$$\begin{tikzcd} VX \arrow[r, "\eta_{VX}"] \arrow[dr, "1"'] & TVX \arrow[d, "\alpha"] \\ & VX \end{tikzcd} \qquad \begin{tikzcd} KVX \arrow[r, "K\eta_{VX}"] \arrow[dr, "\epsilon_X"'] & KTVX \arrow[d, "\overline{\alpha}"] \\ & X \end{tikzcd}$$

Transposing this diagram, we get the commutative diagram on the right for the map $\overline{\alpha} : KTVX \to X$. But by the universal property of a pushout this is the same as a map $PX \to X$ such that $\rho_X \circ \overline{\alpha} = 1$, i.e. an algebra structure for the pointed endofunctor (P, ρ).

Next, we show that (P, ρ) is well-pointed. That is, for every object A of $\mathcal{C}$ we need to show that $P\rho_A = \rho_{PA}$. Firstly, since (T, η) is well-pointed, we have that $T\eta_{VA} = \eta_{TVA}$ for all A in $\mathcal{C}$. Note that since $K \dashv V$, the map $\mu : KTV \to P$ induces a transpose map $\phi : TV \to VP$. We leave it to the reader to verify that this map satisfies the following commutative diagram

$$\begin{tikzcd} & V \arrow[dl, "\eta_V"'] \arrow[dr, "V\rho"] & \\ TV \arrow[rr, "\phi"'] & & VP. \end{tikzcd} \tag{5.1}$$

Now we calculate:

$$\begin{aligned} VP\rho_A \circ \phi_A &= \phi_{PA} \circ TV\rho_A && \text{(naturality of } \phi) \\ &= \phi_{PA} \circ T(\phi_A \circ \eta_{VA}) && (5.1) \\ &= \phi_{PA} \circ T\phi_A \circ T\eta_{VA} && \text{(functoriality of } T) \\ &= \phi_{PA} \circ T\phi_A \circ \eta_{TVA} && ((T, \eta) \text{ is well-pointed}) \\ &= \phi_{PA} \circ \eta_{VPA} \circ \phi_A && \text{(naturality of } \eta) \\ &= V\rho_{PA} \circ \phi_A. && (5.1) \end{aligned}$$

Transporting this equality along the adjunction we obtain $P\rho_A \circ \mu_A = \rho_{PA} \circ \mu_A$. By naturality of ρ we also have $P\rho_A \circ \rho_A = \rho_{PA} \circ \rho_A$. Hence, since $P\rho_A$ and ρ_{PA} become equal after precomposing with both pushout inclusion maps, it follows that they are equal, which finishes the proof. □

Lemma 11 [15, Proposition 5.2]**.** *Let (S, σ) be a well-pointed endofunctor on $\mathcal{C}$.*

(i) For any map $g : SB \to A$, let $f = g \circ \sigma_B : B \to A$. Then $Sf = \sigma_A \circ g : SB \to SA$.

(ii) For an object A in $\mathcal{C}$, A admits a (unique) (S, σ)-algebra structure if and only if σ_A is an isomorphism.

Proof. (i) We calculate:

$$Sf = Sg \circ S\sigma_B = Sg \circ \sigma_{SB} = \sigma_A \circ g.$$

In the second equality we use that S is well-pointed, while the third equality follows from naturality of σ.

(ii) Suppose A admits an (S, σ)-algebra structure $a : SA \to A$. Then $a \circ \sigma_A = 1_A$ by definition. But then also $\sigma_A \circ a = S(1_A) = 1_{SA}$ by part (i), so σ_A is an isomorphism. The converse is immediate. $\square$

Lemma 12 [15, Proposition 7.1]**.** *Let (S, σ) be a well-pointed endofunctor on $\mathcal{C}$ and $\phi : S \to T$ a natural transformation for which every ϕ_A is an epi. Then (T, τ) is a well-pointed endofunctor, with $\tau = \phi \circ \sigma$. Moreover, (T, τ)-**Alg** consists of those (S, σ)-algebras for which ϕ_A is an isomorphism.*

Proof. To show that (T, τ) is well-pointed, we need to show that $T\tau_A = \tau_{TA}$ for all objects A in $\mathcal{C}$. Note that by definition we have $\tau_A = \phi_A \circ \sigma_A$, and thus by Lemma 11 it follows that $S\tau_A = \sigma_{TA} \circ \phi_A$. We calculate:

$$\tau_{TA} \circ \phi_A = \phi_{TA} \circ \sigma_{TA} \circ \phi_A = \phi_{TA} \circ S\tau_A = T\tau_A \circ \phi_A.$$

here the last step follows from naturality of ϕ. We conclude that $T\tau_A = \tau_{TA}$ since ϕ_A is epi.

Now by Lemma 11 we have that A admits a (T, τ) algebra if and only if $\tau_A = \phi_A \circ \sigma_A$ is an isomorphism. We claim that this is the case if and only if A admits and (S, σ)-algebra structure and ϕ_A is an isomorphism. Suppose A admits an S-algebra structure and ϕ_A is an isomorphism. Then by Lemma 11 σ_A is an isomorphism, and hence so is τ_A. Conversely, suppose $\tau_A = \phi_A \circ \sigma_A$ is an isomorphism. Then σ_A has a left inverse $SA \to A$, which is thus a (S, σ)-algebra. Again by Lemma 11 it follows that σ_A is an isomorphism, and therefore ϕ_A is also an isomorphism. $\square$

5.3.2. The Category $T/\mathcal{C}$

Given an endofunctor $T : \mathcal{C} \to \mathcal{C}$, we have the comma category $T/\mathcal{C}$. Its objects are triples (A, a, B) with A, B objects in $\mathcal{C}$ and $a : TA \to B$ a morphism in $\mathcal{C}$. A morphism $(A, a, B) \to (A', a', B')$ consists of two maps $f : A \to A'$ and $g : B \to B'$ such that the following diagram commutes

$$\begin{tikzcd} TA \arrow[r, "Tf"] \arrow[d, "a"'] & TA' \arrow[d, "a'"] \\ B \arrow[r, "g"'] & B'. \end{tikzcd}$$

We have two forgetful functors U_1, U_2, mapping an object and a morphism to its first and second components respectively. We recall some facts about the category $T/\mathcal{C}$.

Lemma 13 [15, Section 14.1]**.** *Let $\mathcal{C}$ be cocomplete and $T : \mathcal{C} \to \mathcal{C}$ and endofunctor. Then $T/\mathcal{C}$ is cocomplete.*

Proof. A functor $\mathcal{J} \to T/\mathcal{C}$ is given by two functors $X, Y : \mathcal{J} \to \mathcal{C}$ together with a natural transformation $x : TX \to Y$. Its colimit is given by $(\operatorname{colim} X, a, B)$ where a, B are defined by the pushout

$$\begin{tikzcd} \operatorname{colim} TX \arrow[r, "\operatorname{colim} x"] \arrow[d, "\widetilde{T}"'] & \operatorname{colim} Y \arrow[d, "b"] \\ T \operatorname{colim} X \arrow[r, "a"'] & B \end{tikzcd} \tag{5.2}$$

where $\widetilde{T}$ is the canonical comparison map. We leave it to the reader to verify that this is indeed the colimit of the diagram $\mathcal{J} \to T/\mathcal{C}$. □

If we have a natural transformation $\alpha : T' \to T$, this induces a functor $\alpha^* : T/\mathcal{C} \to T'/\mathcal{C}$ which sends (A, a, B) to $(A, a \circ \alpha_A, B)$. It has a left adjoint $\alpha_* : T'/\mathcal{C} \to T/\mathcal{C}$ given by sending (C, c, D) to $(C, \bar{c}, \bar{D})$, defined by the following pushout

$$\begin{tikzcd} T'C \arrow[r, "c"] \arrow[d, "\alpha_C"'] & D \arrow[d, "\hat{c}"] \\ TC \arrow[r, "\bar{c}"'] & \bar{D}. \end{tikzcd}$$

Note that for the functor $T = 1$, we have that $1/\mathcal{C} \cong \mathcal{C}^{\mathbf{2}}$. Thus if we have a pointed endofunctor (T, τ), we have an adjunction

$$\begin{tikzcd} T/\mathcal{C} \arrow[r, bend right, "\tau^*"'] \arrow[r, phantom, "\perp"] & \mathcal{C}^{\mathbf{2}}. \arrow[l, bend right, "\tau_*"'] \end{tikzcd} \tag{5.3}$$

5.3.3. Effective Repletions

For a pointed endofunctor (T, τ), we have a full embedding (T, τ)-**Alg** $\hookrightarrow T/\mathcal{C}$ given by mapping an algebra (A, a) to (A, a, A) and a map of algebras f to the pair (f, f). So (T, τ)-**Alg** is a full subcategory of $T/\mathcal{C}$. Our next objective is to construct the repletion of (T, τ)-**Alg** in $T/\mathcal{C}$ as a category of algebras for a pointed endofunctor (S, σ) on $T/\mathcal{C}$. Let us first define what a repletion is.

Definition 5. A subcategory $\mathcal{D}$ of $\mathcal{C}$ is *replete* if for any object X in $\mathcal{D}$ and any isomorphism $f : X \cong Y$ in $\mathcal{C}$, both Y and f are also in $\mathcal{D}$. Any subcategory is contained in a smallest replete subcategory $\mathcal{D} \subseteq \operatorname{Repl}(\mathcal{D}) \subseteq \mathcal{C}$, called its *repletion*.

To make our proof constructive, we define an *effective repletion* of a subcategory $i : \mathcal{D} \hookrightarrow \mathcal{C}$ to be a repletion $\operatorname{Repl}(\mathcal{D})$ of $\mathcal{D}$ in $\mathcal{C}$, equipped with a map on objects $G : \operatorname{Repl}(\mathcal{D}) \to \mathcal{D}$ and for every $D \in \operatorname{Repl}(\mathcal{D})$ an isomorphism $\alpha_D : D \xrightarrow{\cong} iG(D)$.

Lemma 14. *Let* $(\operatorname{Repl}(\mathcal{D}), G, (\alpha_D)_{D \in \operatorname{Repl}(\mathcal{D})})$ *be an effective repletion of a full subcategory* $i : \mathcal{D} \hookrightarrow \mathcal{C}$*. Then* G *extends to a functor* $G : \operatorname{Repl}(\mathcal{D}) \to \mathcal{D}$ *which is left adjoint to the inclusion functor* i*.*

Proof. Since we are dealing with a full subcategory, the repletion $\mathrm{Repl}(\mathcal{D})$ is simply the full subcategory of $\mathcal{C}$ determined by those objects which are isomorphic to some object of $\mathcal{D}$. We make G functorial by sending a morphism $f : A \to B$ in $\mathrm{Repl}(\mathcal{D})$ to the unique map $Gf : GA \to GB$ making the following diagram commute

$$\begin{tikzcd} A \arrow[r, "\alpha_A", "\cong"'] \arrow[d, "f"'] & iGA \arrow[d, dashed, "iGf"] \\ B \arrow[r, "\cong", "\alpha_B"'] & iGB. \end{tikzcd}$$

Explicitly: $Gf = \alpha_B \circ f \circ \alpha_A^{-1}$. This is indeed a morphism in $\mathcal{D}$ since i is a full embedding. Note that the definition of G on morphisms ensures that $\alpha : 1 \xrightarrow{\cong} iG$ becomes a natural isomorphism. Therefore we get the following chain of natural bijections

$$\hom_{\mathcal{D}}(GC, D) \cong \hom_{\mathrm{Repl}(\mathcal{D})}(iGC, iD) \cong \hom_{\mathrm{Repl}(\mathcal{D})}(C, iD).$$

The first isomorphism follows from the fact that i is fully faithful, while the second isomorphism follows from the natural isomorphism $1 \cong iG$. We conclude that $G \dashv i$, as desired. $\square$

The full embedding (T,τ)-**Alg** $\hookrightarrow T/\mathcal{C}$ is not replete. Indeed, an object (A, a, B) is isomorphic to a (T,τ)-algebra if and only if the map $a \circ \tau_A$ is an isomorphism, as the next lemma shows.

Lemma 15. *Let (T,τ) be a pointed endofunctor on a category $\mathcal{C}$. An element (A, a, B) is isomorphic to a (T,τ) algebra if and only if $a \circ \tau_A$ is an isomorphism.*

Proof. Suppose first that (A, a, B) is isomorphic to a (T,τ)-algebra $c : TC \to C$, so that we have an isomorphism $(f, g) : (A, a, B) \to (C, c, C)$. Since we have two forgetful functors U_1, U_2 projecting a morphism on the first and second component, it follows that $f : A \to C$ and $g : B \to C$ are isomorphisms. We have the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "\tau_A"] \arrow[d, "\cong"', "f"] & TA \arrow[r, "a"] \arrow[d, "\cong"', "Tf"] & B \arrow[d, "\cong"', "g"] \\ C \arrow[r, "\tau_C"] \arrow[rr, bend right, "1_C"'] & TC \arrow[r, "c"] & C \end{tikzcd}$$

The left square commutes because of naturality of τ and the right square commutes since (f, g) is a morphism in $T/\mathcal{C}$. It follows that $a \circ \tau_A$ is an isomorphism with inverse $f^{-1} \circ g$.

Conversely, suppose (A, a, B) is an object in $T/\mathcal{C}$ such that $a \circ \tau_A$ is an isomorphism, with inverse $f : B \to A$. Then $f \circ a : TA \to A$ is a (T,τ)-algebra, and is isomorphic to (A, a, B) via the isomorphism $(1_A, f) : (A, a, B) \xrightarrow{\cong} (A, f \circ a, A)$. $\square$

In the next proposition we construct an endofunctor $S : T/\mathcal{C} \to T/\mathcal{C}$ together with a unit $\sigma : 1 \to S$ such that the category of algebras for the pointed endofunctor (S, σ) is the repletion of (T,τ)-**Alg**, which is in fact effective.

Proposition 11 [15, Theorem 14.4]**.** *Let $\mathcal{C}$ be a category and (T, τ) a pointed endofunctor on $\mathcal{C}$. There exists a well-pointed endofunctor (S, σ) on $T/\mathcal{C}$ such that (S, σ)-**Alg** is part of an effective repletion of (T, τ)-**Alg** in $T/\mathcal{C}$.*

Proof. We have an adjunction $\mathrm{cod} \dashv \mathrm{id} : \mathcal{C} \to \mathcal{C}^{\mathbf{2}}$. The induced monad of this adjunction R on $\mathcal{C}^{\mathbf{2}}$ has as its underlying endofunctor $R = \mathrm{id} \circ \mathrm{cod}$, which thus sends an object (A, a, B) to $(B, 1_B, B)$. Its unit $\rho : 1 \to R$ is given at (A, a, B) by $\rho_{(A,a,B)} = (a, 1)$. Now note that for an object (A, a, B) in $\mathcal{C}^{\mathbf{2}}$, the map $a : A \to B$ is an isomorphism if and only if (A, a, B) admits a (R, ρ)-algebra structure. By Lemma 15 we have that (A, a, B) in $T/\mathcal{C}$ is isomorphic to a (T, τ)-algebra if and only if $a \circ \tau_A$ is an isomorphism, and thus if and only if $\tau^*(A, a, B)$ is an (R, ρ)-algebra. In other words, we see that the repletion of (T, τ)-**Alg** in $T/\mathcal{C}$ is given by the pullback as on the left below.

$$
\begin{tikzcd}
\mathrm{Repl}((T,\tau)\text{-}\mathbf{Alg}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (R,\rho)\text{-}\mathbf{Alg} \arrow[d] \\
T/\mathcal{C} \arrow[r, shift right, "\tau^*"'] \arrow[r, phantom, "\perp"] & \mathcal{C}^{\mathbf{2}} \arrow[l, shift right, "\tau_*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\tau_*\tau^* \arrow[r, "\tau_*\rho\tau^*"] \arrow[d, "\epsilon"'] & \tau_* R \tau^* \arrow[d, "\psi"] \arrow[dl, phantom, "\ulcorner", very near end] \\
1 \arrow[r, "\sigma"'] & S
\end{tikzcd}
$$

It follows from Lemma 10 that $\mathrm{Repl}((T, \tau)\text{-}\mathbf{Alg}) \cong (S, \sigma)\text{-}\mathbf{Alg}$, where S and σ are constructed by the pushout as on the right above. It is not hard to check that (R, ρ) is well-pointed, and therefore (S, σ) is also well-pointed by Lemma 10.

Explicitly, the functor $S : T/\mathcal{C} \to T/\mathcal{C}$ maps an object (A, a, B) to (B, b, C), where $b : TB \to C$ is given by the coequaliser of the two parallel maps

$$
\begin{tikzcd}
TA \arrow[r, shift left, "\tau_{TA}"] \arrow[r, shift right, "T\tau_A"'] & T^2A \arrow[r, "Ta"] & TB.
\end{tikzcd}
\tag{5.4}
$$

The unit $\sigma : 1 \to S$ at (A, a, B) is given by $\sigma_{(A,a,B)} = (a \circ \tau_A, b \circ \tau_B)$.

To make the repletion effective, note that if (A, a, B) admits an (S, σ)-algebra $(\alpha_0, \alpha_1) : S(A, a, B) \to (A, a, B)$, it follow by Lemma 11 that $\sigma_{(A,a,B)}$ is an isomorphism with inverse (α_0, α_1). In particular $a \circ \tau_A$ is an isomorphism with inverse α_0. Thus (A, a, B) is isomorphic to the (T, τ)-algebra $\alpha_0 \circ a : TA \to A$, which gives us a map $G : (S, \sigma)\text{-}\mathbf{Alg} \to (T, \tau)\text{-}\mathbf{Alg} : (A, a, B) \mapsto (A, \alpha_0 \circ a, A)$. Moreover, we see from the proof of Lemma 11 that this isomorphism is witnessed by $\alpha_{(A,a,B)} = (1_A, \alpha_0)$. We conclude that we have an effective repletion. $\square$

Lemma 16. *For a (T, τ)-algebra $a : TA \to A$ (seen as the object (A, a, A) in $T/\mathcal{C}$), $a : TA \to A$ is a coequaliser of (5.4), whence $S(A, a, A) = (A, a, A)$.*

Proof. Since a is a (T, τ)-algebra we have $a \circ \tau_A = 1_A$. Thus the coequaliser of the two parallel maps in (5.4) boils down to the coequaliser of $Ta \circ \tau_{TA}$ and 1_{TA}, i.e. the coequaliser of $\tau_A \circ a$ and 1_{TA} by naturality. The map a coequalises these maps since $a \circ \tau_A = 1_A$. If $\gamma : TA \to X$ also coequalises these two maps, then we have a unique map $\phi : A \to X$ with $\phi \circ a = \gamma$, given by $\phi = \gamma \circ \tau_A$. $\square$

5.3.4. Constructing the Colimit

Now let $V : \mathcal{J} \to \mathbf{Mnd}_{\mathrm{f}}(\mathcal{C})$ be a diagram in the category of finitary monads (seen as a full subcategory of $\mathbf{Mnd}(\mathcal{C})$), we aim to construct its colimit. Denote the monad $V(j)$ as $\mathsf{T}_j = (T_j, \tau_j, \mu_j)$ for every $j \in \mathcal{J}$. Compose V^{op} with the algebra functor $(-)$-$\mathbf{Alg} : \mathbf{Mnd}(\mathcal{C})^{\mathrm{op}} \to \mathbf{Cat}\,/\mathcal{C}$, which gives us a diagram in $\mathbf{Cat}\,/\mathcal{C}$. Since $\mathbf{Cat}\,/\mathcal{C}$ is complete, we can take its limit, which we denote as V-$\mathbf{Alg}$. In fact we can give a concrete description of V-$\mathbf{Alg}$. An object is an object $X \in \mathcal{C}$, together with an algebra structure $a_j : T_j X \to X$ for the monad T_j for every $j \in \mathcal{J}$, such that the collection of this algebras is compatible with morphisms in $\mathcal{J}$. That is, for every $\alpha : i \to j$ in $\mathcal{J}$, the following diagram commutes

$$\begin{tikzcd} T_i X \arrow[rr, "V(\alpha)_X"] \arrow[dr, "a_i"'] & & T_j X \arrow[dl, "a_j"] \\ & X. & \end{tikzcd}$$

We call such an object a V-algebra. A morphism of V-algebras $f : X \to X'$ is a map of T_j-algebras for every $j \in \mathcal{J}$. We write $Q_j : V$-$\mathbf{Alg} \to \mathsf{T_j}$-$\mathbf{Alg}$ for the projection functors. Since $(-)$-$\mathbf{Alg} : \mathbf{Mnd}(\mathcal{C})^{\mathrm{op}} \to \mathbf{Cat}\,/\mathcal{C}$ is fully faithful, for any monad P on $\mathcal{C}$ we have

$$\hom_{\mathbf{Mnd}(\mathcal{C})}(\mathsf{T_j}, \mathsf{P}) \cong \hom_{\mathbf{Cat}\,/\mathcal{C}}(\mathsf{P}\text{-}\mathbf{Alg}, \mathsf{T_j}\text{-}\mathbf{Alg}).$$

Thus, a cocone $r = (r_j : \mathsf{T}_j \to \mathsf{P})_{j \in \mathcal{J}}$ induces a cone $(\overline{r_j} : \mathsf{P}\text{-}\mathbf{Alg} \to \mathsf{T_j}\text{-}\mathbf{Alg})_{j \in \mathcal{J}}$ in $\mathbf{Cat}\,/\mathcal{C}$ and thus a unique morphism $\overline{r} : \mathsf{P}$-$\mathbf{Alg} \to V$-$\mathbf{Alg}$ over $\mathcal{C}$ making the following diagram commute for any $\alpha : j \to i$ in $\mathcal{J}$.

$$\begin{tikzcd} & \mathsf{P}\text{-}\mathbf{Alg} \arrow[d, dashed, "\overline{r}"] \arrow[ddl, bend right, "\overline{r_i}"'] \arrow[ddr, bend left, "\overline{r_j}"] & \\ & V\text{-}\mathbf{Alg} \arrow[dl, "Q_i"'] \arrow[dr, "Q_j"] & \\ \mathsf{T_i}\text{-}\mathbf{Alg} \arrow[rr, "\overline{V(\alpha)}"'] & & \mathsf{T_j}\text{-}\mathbf{Alg} \end{tikzcd}$$

Moreover, by full- and faithfulness of the $(-)$-$\mathbf{Alg}$ functor it follows easily that any map P-$\mathbf{Alg} \to V$-$\mathbf{Alg}$ over $\mathcal{C}$ is of this form. We say that the monad P with the cocone r is the *algebraic colimit* of $V : \mathcal{J} \to \mathbf{Mnd}(\mathcal{C})$ if the induced map $\overline{r} : \mathsf{P}$-$\mathbf{Alg} \to V$-$\mathbf{Alg}$ is an isomorphism.

Proposition 12 [15, Proposition 26.2]**.** *When the algebraic colimit of $V : \mathcal{J} \to \mathbf{Mnd}(\mathcal{C})$ exists, it is also the colimit of V in the ordinary sense.*

Proof. By definition P-$\mathbf{Alg}$ is the limit of the diagram $(\mathsf{T_j}\text{-}\mathbf{Alg})_{j \in \mathcal{J}}$ in $\mathbf{Cat}\,/\mathcal{C}$. Since the functor $(-)$-$\mathbf{Alg} : \mathbf{Mnd}(\mathcal{C})^{\mathrm{op}} \to \mathbf{Cat}\,/\mathcal{C}$ is fully faithful (see Section 5.1) it reflects limits, whence we conclude that P is the colimit of the diagram $(\mathsf{T}_j)_{j \in \mathcal{J}}$ in $\mathbf{Mnd}(\mathcal{C})$. $\square$

Note that we have a forgetful functor $U : V\text{-}\mathbf{Alg} \to \mathcal{C}$. If we show that U is monadic, then we have $V\text{-}\mathbf{Alg} \cong \mathsf{P}\text{-}\mathbf{Alg}$ over $\mathcal{C}$ for some monad P, namely the monad induced by the adjunction. Then P the algebraic colimit of V, and hence by Proposition 12 the colimit of V. Thus we aim to show that U is monadic. To apply Beck's monadicity theorem (Theorem 7), we need to check that U creates coequalisers of U-split pairs and has a left adjoint. The former is not hard to check, and we deal with this in the next lemma.

Lemma 17. *The forgetful functor $U : V\text{-}\mathbf{Alg} \to \mathcal{C}$ creates coequalisers of U-split pairs.*

Proof. Suppose we have a U-split pair $f, g : (A, (a_j)) \to (B, (b_j))$ in $V\text{-}\mathbf{Alg}$. Then the bottom line of the diagram below is a split coequaliser in $\mathcal{C}$. Since split coequalisers are absolute, it is preserved by every T_j, and therefore the middle and the top row are also a split coequaliser. Hence, we get a unique arrow $c_j : T_jC \to C$ making the lower part of the diagram commute

$$\begin{tikzcd}
T_jT_jA \arrow[r, shift left, "T_j^2 f"] \arrow[r, shift right, "T_j^2 g"'] \arrow[d, "T_ja_j"'] & T_jT_jB \arrow[r, "T_j^2 e"] \arrow[d, "T_jb_j"] & T_jT_jC \arrow[d, dashed, "T_jc_j"'] \arrow[dd, dashed, bend left=60, "c_j\circ(\mu_j)_C"] \\
T_jA \arrow[r, shift left, "T_j f"] \arrow[r, shift right, "T_j g"'] \arrow[d, "a_j"'] & T_jB \arrow[r, "T_je"'] \arrow[d, "b_j"] & T_jC \arrow[d, dashed, "c_j"'] \\
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "e"'] & C
\end{tikzcd}$$

The fact that the lower right square commutes means that e is a T_j-algebra morphism $(B, b_j) \to (C, c_j)$. It remains to show that c_j satisfies the unit condition and multiplication condition. For the former, we have to verify that $c_j \circ (\tau_j)_C = 1_C$. To this end, it is sufficient to show that both maps become equal after precomposition with e (e being a coequaliser). We compute:

$$c_j \circ (\tau_j)_C \circ e = c_j \circ T_je \circ (\tau_j)_B = e \circ b_j \circ (\tau_j)_B = e,$$

where we use naturality of $\tau_j : 1 \to T_j$ in the first equality, commutativity of the bottom right square in the second equality, and the fact that b_j satisfies the unit condition in the third equality. For the multiplication condition we need to verify that $c_j \circ (\mu_j)_C = c_j \circ T_j(c_j)$. This follows from the fact that both maps satisfy the property of the unique map $T_jT_jC \to C$ which arises from the fact that $e \circ b_j \circ T_jb_j$ coequalises $T_j^2 f$ and $T_j^2 g$, as shown in the diagram above. Thus we get a T_j-algebra structure on C for every $j \in \mathcal{J}$. We leave it to the reader to verify that these algebra structures are compatible with morphisms in $\mathcal{J}$, and that the resulting object is indeed the coequaliser of f, g in $V\text{-}\mathbf{Alg}$. This completes the proof that U creates U-split coequalisers. $\square$

It remains to show that U has a left adjoint F, i.e. that the free V-algebra exists. The colimit of V is then given by the monad P arising from the adjunction $F \dashv U$. By Lemma 10 the category of pointed endofunctors has colimits. Let (T, τ) be the colimit of the diagram $(T_j, \tau_j)_{j \in \mathcal{J}}$ of pointed endofunctors, with colimit inclusions $r_j : T_j \to T$. Note that giving a (T, τ)-algebra $a : TA \to A$ is the same as giving component maps $a_j = a \circ (r_j)_A : T_jA \to A$ for each j, subject to the compatibility conditions $a_i = a_j \circ V(\alpha)_A$ for every map $\alpha : i \to j$ in $\mathcal{J}$, and the unit condition $a_j \circ (\tau_j)_A = 1_A$ for all $j \in \mathcal{J}$. Indeed, since the colimit T is calculated pointwise we have that TA is the colimit of $(T_jA)_{j \in \mathcal{J}}$ with colimit inclusions $(r_j)_A$, as shown in the following diagram

$$\begin{tikzcd}
& A & \\
& TA \arrow[u, dashed, "a"] & \\
T_iA \arrow[uur, bend left, "a_i"] \arrow[ur, "(r_i)_A"'] \arrow[rr, "V(\alpha)_A"'] & & T_jA. \arrow[uul, bend right, "a_j"'] \arrow[ul, "(r_j)_A"]
\end{tikzcd}$$

The unit condition $a \circ \tau_A = 1_A$ is equivalent to $a_j \circ (\tau_j)_A = 1_A$ for all $j \in \mathcal{J}$. It follows that V-**Alg** is a full subcategory of (T, τ)-**Alg**, consisting of exactly those T-algebras $a : TA \to A$ for which the components $a_j : T_jA \to A$ not only satisfy the unit condition, but also the multiplication condition for the monad T_j. We thus have the full inclusions

$$V\text{-}\mathbf{Alg} \subset (T, \tau)\text{-}\mathbf{Alg} \subset T/\mathcal{C}. \tag{5.5}$$

We shall now construct a pointed endofunctor $(L, \lambda) : T/\mathcal{C} \to T/\mathcal{C}$ such that (L, λ)-**Alg** is the repletion of V-**Alg** in $T/\mathcal{C}$. Explicitly, we define $L(A, a, B) = (B, c, D)$, where $c : TB \to D$ is defined as the joint coequaliser for all $j \in \mathcal{J}$ of the parallel pairs

$$\begin{tikzcd}
& T_jA \arrow[dr, "T_j(\tau_j)_A"] & & & \\
T_j^2A \arrow[ur, "(\mu_j)_A"] \arrow[rr, "1"'] & & T_j^2A \arrow[r, "T_ja_j"] & T_jB \arrow[r, "(r_j)_B"] & TB.
\end{tikzcd} \tag{5.6}$$

where $a_j = a \circ (r_j)_A$ as before. L acts on morphisms in the obvious way.

Lemma 18. *A (T, τ)-algebra $a : TA \to A$ is a V-algebra if and only if it coequalises (5.6) for all $j \in \mathcal{J}$.*

Proof. We have seen that a (T, τ) algebra (A, a) is the same as a collection of algebras $a_j : T_jA \to A$ for the pointed endofunctor (T_j, τ_j) which are compatible with the morphisms in $\mathcal{J}$. Thus, such an element is a V-algebra if and only if it also satisfies the multiplication condition $a_j \circ (\mu_j)_A = a_j \circ T_ja_j$ for every $j \in \mathcal{J}$. Since each a_j satisfies the unit condition we have $T_ja_j \circ T_j(\tau_j)_A = 1_{T_jA}$, and thus it satisfies the multiplication condition exactly when a coequalises the diagram (5.6) for every $j \in \mathcal{J}$. $\square$

Consider the following diagram

$$\begin{tikzcd}
T_jA \arrow[r, shift left, "T_j(\tau_j)_A"] \arrow[r, shift right, "(\tau_j)_{T_jA}"'] \arrow[d, "(r_j)_A"'] & T_j^2A \arrow[r, "T_ja_j"] \arrow[d, "(r_jr_j)_A"] & T_jB \arrow[d, "(r_j)_B"] \\
TA \arrow[r, shift left, "T\tau_A"] \arrow[r, shift right, "\tau_{TA}"'] & T^2A \arrow[r, "Ta"'] & TB.
\end{tikzcd}$$

Every one of the three squares commutes. Since $(\mu_j)_A \circ (\tau_j)_{T_jA} = 1_{T_jA}$ the upper row[3] is given by pre-composing the diagram (5.6) with $(\tau_j)_{T_jA}$. It therefore follows that $c : TB \to D$ coequalises the upper row of the diagram, and therefore also the lower row for every $j \in \mathcal{J}$. Moreover, since the $(r_j)_A$ are colimit inclusions for all $j \in \mathcal{J}$ they are jointly epimorphic. Therefore c coequalises $Ta \circ T\tau_A$ and $Ta \circ \tau_{TA}$, and thus it factors through the coequaliser $b : TB \to C$ as defined in the construction of the well-pointed endofunctor (S, σ) in (5.4). So we have $c = qb$ for a regular epimorphism $q : C \to D$. This induces an epimorphic natural transformation $\phi : S \to L$, with $\phi_{(A,a,B)} = (1, q)$. Now applying Lemma 12 gives us a well-pointed endofunctor (L, λ) with $\lambda = \phi \circ \sigma$. We claim that its category of algebras is the repletion of V-**Alg** in $T/\mathcal{C}$.

Proposition 13 [15, Section 27.1]**.** *(L, λ)-**Alg** is part of an effective repletion of V-**Alg** in $T/\mathcal{C}$.*

Proof. We have the following commutative diagram of embeddings

$$\begin{tikzcd}
 & (T,\tau)\text{-}\mathbf{Alg} \arrow[r] & (S,\sigma)\text{-}\mathbf{Alg} \arrow[dr] & \\
V\text{-}\mathbf{Alg} \arrow[ur] \arrow[dr] & & & T/\mathcal{C}. \\
 & (L,\lambda)\text{-}\mathbf{Alg} \arrow[urr] & &
\end{tikzcd} \tag{5.7}$$

The embedding V-**Alg** $\to (T, \tau)$-**Alg** is the one from (5.5), the embedding (T, τ)-**Alg** $\to$ (S, σ)-**Alg** follows from Lemma 11, the fact that (S, σ)-**Alg** is the repletion of (T, τ)-**Alg**. The embeddings (S, σ)-**Alg** $\to T/\mathcal{C}$ and (L, λ)-**Alg** $\to T/\mathcal{C}$ are just the forgetful functors. Lastly, the embedding V-**Alg** $\to (L, \lambda)$-**Alg** sends $(A, (a_j : T_jA \to A))$ to (A, a, A), where $a : TA \to A$ is the unique map induced by the $(a_j)_{j\in\mathcal{J}}$. We have to equip it with an action $S(A, a, A) = (A, c, D) \to (A, a, A)$. The first component is the identity map 1_A. The second component is a map from the joint coequaliser D of (5.6) to A. Such a map exists since the map $a : TA \to A$ also coequalises this diagram for every j by Lemma 18. Hence we get a unique map $g : D \to A$ such that $gc = a$, thus making $(1_A, g) : S(A, a, A) \to (A, a, A)$ into a morphism in $T/\mathcal{C}$.

[3] We call the parallel pair $(r_j)_B \circ T_ja_j \circ T_j(\tau_j)_A$, $(r_j)_B \circ T_ja_j \circ (\tau_j)_{T_jA}$ the "upper row", and the parallel pair $Ta \circ T\tau_A \circ (r_j)_A$, $Ta \circ \tau_{TA} \circ (r_j)_A$ the "lower row".

Now to show that (L, λ)-**Alg** is the repletion of V-**Alg** we want to show that an element (A, a, B) is isomorphic to a V-algebra if and only if it admits an (L, λ)-algebra structure. We have seen in diagram (5.7) that every V-algebra admits an (L, λ)-algebra structure; and thus if (A, a, B) is isomorphic to a V-algebra, so does (A, a, B).

For the converse, suppose (A, a, B) admits an (L, λ)-algebra structure $(\alpha_0, \alpha_1) : L(A, a, B) \to (A, a, B)$. By Lemma 12 we have that (A, a, B) admits an (S, σ)-algebra structure and that $\phi_{(A,a,B)}$ is an isomorphism. Explicitly this induced (S, σ)-action is given by $(\alpha_0, \alpha_1 q) : S(A, a, B) \to (A, a, B)$, with $S(A, a, B) = (B, b, C)$, as defined in (5.4). By Lemma 11 we have that (S, σ)-**Alg** is the repletion of (T, τ)-**Alg**, and that (A, a, B) is isomorphic to the (T, τ)-algebra $(A, \alpha_0 \circ a)$, witnessed by $(1, \alpha_0) : (A, a, B) \overset{\cong}{\to} (A, \alpha_0 \circ a, A)$. We also recall from Lemma 11 that $\sigma_{(A,a,B)} = (a \circ \tau_A, b \circ \tau_B)$ is an isomorphism with inverse $(\alpha_0, \alpha_1 q)$, so that $a \circ \tau_A$ is an isomorphism with inverse α_0 and $b \circ \tau_B$ is an isomorphism with inverse $\alpha_1 \circ q$. By Lemma 18 it follows that the (T, τ)-algebra $(A, \alpha_0 \circ a)$ is in V-**Alg** as soon as we can show that $\alpha_0 \circ a$ coequalises (5.6) for all $j \in \mathcal{J}$. To see this, recall that $\phi_{(A,a,B)} : S(A, a, B) \to T(A, a, B)$ is given by $(1, q) : (B, b, C) \to (B, c, D)$, where $q : C \to D$ is the unique map induced by the coequaliser $b : TB \to C$, since $c : TB \to D$ also coequalises the maps in (5.4). Since $\phi_{(A,a,B)}$ is an iso, q is also an iso. But then it follows that b also coequalises the pairs in (5.6) for all $j \in \mathcal{J}$. Now consider the following diagram

$$\begin{tikzcd}
 & T_jA \arrow[dr, "T_j(\tau_j)_A"] & & & & \\
T_j^2A \arrow[ur, "(\mu_j)_A"] \arrow[rr, "1"'] & & T_j^2A \arrow[r, "T_ja_j"] \arrow[dr, "T_j(\alpha_0 \circ a)_j"'] & T_jB \arrow[r, "(r_j)_B"] \arrow[d, "T_j(\alpha_0)", "\cong"'] & TB \arrow[r, "b"] \arrow[d, "T\alpha_0", "\cong"'] & C \arrow[d, "\alpha_0\alpha_1 q", "\cong"'] \\
 & & & T_jA \arrow[r, "(r_j)_A"'] & TA \arrow[r, "\alpha_0 \circ a"'] & A.
\end{tikzcd}$$

The rightmost square is the composition of the maps

$$S(A, a, B) \xrightarrow{(\alpha_0, \alpha_1 q)} (A, a, B) \xrightarrow[\cong]{(1, \alpha_0)} (A, \alpha_0 \circ a, A).$$

The square left to it commutes because of naturality of $r_j : T_j \to T$. The triangle to the left of that commutes because

$$(\alpha_0 \circ a)_j = \alpha_0 \circ a \circ (r_j)_A = \alpha_0 \circ a_j.$$

Thus the diagram commutes and the three vertical maps are isomorphisms because $\alpha_0, \alpha_1 \circ q$ are isomorphisms. Hence since b coequalises the upper row, $\alpha_0 \circ a$ coequalises the lower row, which is exactly (5.6) for the algebra $\alpha_0 \circ a$. An application of Lemma 18 then finishes the proof.

Tracing the construction explicitly, we see that the repletion is effective in a similar manner as in Proposition 11, namely $G : (L, \lambda)$-**Alg** $\to V$-**Alg** sends an (L, λ)-algebra $(\alpha_0, \alpha_1) : L(A, a, B) \to (A, a, B)$ to the V-algebra induced by the (T, τ)-algebra $\alpha_0 \circ a : TA \to A$. The isomorphism $\alpha_{(A,a,B)}$ is given by $(1, \alpha_0) : (A, a, B) \overset{\cong}{\to} (A, \alpha_0 \circ a, A)$. $\square$

The last ingredient we need in order to prove that the category of finitary monads has colimits is the fact that L is finitary, which we prove in Lemma 20. Before we prove this lemma, we need the following technical lemma which implies as a special case that a colimit of finitary functors on a cocomplete category is finitary.

Lemma 19. *Let $\boldsymbol{F} : \mathcal{I} \to [\mathcal{C}, \mathcal{D}]$ be a diagram of functors and $\boldsymbol{X} : \mathcal{J} \to \mathcal{C}$ a diagram for which the colimit exists in $\mathcal{C}$ and is preserved by all the functors F_i. Suppose that $\mathcal{D}$ has colimits of type $\mathcal{I}, \mathcal{J}$, and $\mathcal{I} \times \mathcal{J}$. Then the colimit $\operatorname{colim}_{i \in \mathcal{I}} F_i$ exists in $[\mathcal{C}, \mathcal{D}]$, and also preserves the colimit of $\boldsymbol{X}$.*

Proof. Firstly recall that if $\mathcal{D}$ has (co)limits of shape $\mathcal{I}$, then so does $[\mathcal{C}, \mathcal{D}]$ for any category $\mathcal{C}$, and (co)limits are 'computed pointwise'. Therefore the colimit of F of $\boldsymbol{F}$ exists and we have

$$F(C) = (\operatorname{colim}_{i \in \mathcal{I}} F_i)(C) \cong \operatorname{colim}_{i \in \mathcal{I}} F_i(C).$$

Let A be the following composition of functors

$$\mathcal{I} \times \mathcal{J} \xrightarrow{\boldsymbol{F} \times \boldsymbol{X}} [\mathcal{C}, \mathcal{D}] \times \mathcal{C} \xrightarrow{\text{ev}} \mathcal{D} : (i, j) \longmapsto F_i(X_j).$$

It is well-known that limits commute with limits and dually colimits commute with colimits; that is

$$\operatorname{colim}_{i \in \mathcal{I}} \operatorname{colim}_{j \in \mathcal{J}} A(i, j) \cong \operatorname{colim}_{(i,j) \in \mathcal{I} \times \mathcal{J}} A(i, j) \cong \operatorname{colim}_{j \in \mathcal{J}} \operatorname{colim}_{i \in \mathcal{I}} A(i, j)$$

if all the relevant colimits exist. Denote X for the colimit of $\boldsymbol{X}$. To show that F preserves X, we compute:

$$\begin{aligned}
F(\operatorname{colim}_{j \in \mathcal{J}} X_j) &= (\operatorname{colim}_{i \in \mathcal{I}} F_i)(X) \\
&\cong \operatorname{colim}_{i \in \mathcal{I}} F_i(X) \\
&= \operatorname{colim}_{i \in \mathcal{I}} F_i(\operatorname{colim}_{j \in \mathcal{J}} X_j) \\
&\cong \operatorname{colim}_{i \in \mathcal{I}} \operatorname{colim}_{j \in \mathcal{J}} F_i(X_j) \\
&\cong \operatorname{colim}_{j \in \mathcal{J}} \operatorname{colim}_{i \in \mathcal{I}} F_i(X_j) \\
&\cong \operatorname{colim}_{j \in \mathcal{J}} (\operatorname{colim}_{i \in \mathcal{I}} F_i)(X_j) \\
&= \operatorname{colim}_{j \in \mathcal{J}} F(X_j).
\end{aligned}$$

We leave it to the reader to verify that this isomorphism is natural and that F also preserves the colimit inclusions. □

Corollary 3. *Let $\mathcal{C}$ be a cocomplete category and $(T_i)_{i \in \mathcal{I}}$ a diagram of finitary functors. Then $\operatorname{colim}_{i \in \mathcal{I}} T_i$ exists and is also finitary.*

Lemma 20. *The functor $L : T/\mathcal{C} \to T/\mathcal{C}$ is finitary if $\mathcal{C}$ is cocomplete.*

Proof. Firstly, since every (T_j, τ_j) is finitary and $\mathcal{C}$ is cocomplete, it follows by Corollary 3 that (T, τ) is finitary (since T is just the colimit of underlying the endofunctors T_j, as can be seen in the proof of Lemma 10). Now suppose we have a diagram $F : \mathcal{I} \to T/\mathcal{C}$ with $\mathcal{I}$ filtered, and which has a colimit in $T/\mathcal{C}$. Then F consists of two functors $X, Y : \mathcal{I} \to \mathcal{C}$ together with a natural transformation $x : TX \to Y$. We denote $F(i) = (X_i, x_i, Y_i)$. By Lemma 13 we know its colimit is $(\operatorname{colim} X, a, B)$ where a, B are defined by the pushout in (5.2). But since T is finitary it preserves the colimit of X, and thus $\widetilde{T}$ is an isomorphism. But since (5.2) is a pushout diagram, it follows that b is also an isomorphism. Hence the colimit of F is given by the pointwise colimit $(A, a, B) = (\operatorname{colim} X, \operatorname{colim} x, \operatorname{colim} Y)$[4]. Now we see that $L(\operatorname{colim} X, \operatorname{colim} x, \operatorname{colim} Y) = (\operatorname{colim} Y, c, D)$, where $c : T(\operatorname{colim} Y) \to D$ is the joint coequaliser for all $j \in \mathcal{J}$ of the parallel pairs

$$\begin{tikzcd}
 & T_j(\operatorname{colim} X) \arrow[dr, "T_j(\tau_j)_A"] & & & \\
T_j^2(\operatorname{colim} X) \arrow[ur, "(\mu_j)_A"] \arrow[rr, "1"'] & & T_j^2(\operatorname{colim} X) \arrow[r, "T_j a_j"] & T_j(\operatorname{colim} Y) \arrow[r, "(r_j)_B"] & T(\operatorname{colim} Y),
\end{tikzcd}$$

But since T and every T_j preserves filtered colimits, it follows that $c : T(\operatorname{colim} Y) \to D$ is in fact, up to isomorphism, the joint coequaliser of the parallel pairs

$$\begin{tikzcd}
 & \operatorname{colim} T_j X \arrow[dr, "\operatorname{colim} T_j(\tau_j)_X"] & & & \\
\operatorname{colim} T_j^2 X \arrow[ur, "\operatorname{colim}(\mu_j)_X"] \arrow[rr, "1"'] & & \operatorname{colim} T_j^2 X \arrow[r] & \operatorname{colim} T_j Y \arrow[r] & \operatorname{colim} TY.
\end{tikzcd} \tag{5.8}$$

On the other hand, we have $LF(i) = (Y_i, c_i, D_i)$, where $c_i : T(Y_i) \to D_i$ is given by the joint coequaliser for all $j \in \mathcal{J}$ of the parallel pairs

$$\begin{tikzcd}
 & T_j(X_i) \arrow[dr, "T_j(\tau_j)_{X_i}"] & & & \\
T_j^2(X_i) \arrow[ur, "(\mu_j)_{X_i}"] \arrow[rr, "1"'] & & T_j^2(X_i) \arrow[r, "T_j(x_i)_j"] & T_j(Y_i) \arrow[r, "(r_j)_{Y_i}"] & T(Y_i),
\end{tikzcd}$$

Again using that the colimit is calculated pointwise, we have an isomorphism $\operatorname{colim} LF \cong (\operatorname{colim} Y, \operatorname{colim} c_i, \operatorname{colim} D_i)$. But since the colimit functor itself is a left adjoint (to the diagonal functor), it preserves colimits and in particular coequalisers. Thus, we obtain that the map $\operatorname{colim} c_i : \operatorname{colim} T(Y_i) \to \operatorname{colim} D_i$ is a joint coequaliser of the diagram (5.8). But as we have seen $c : T(\operatorname{colim} Y) \to D$ is up to isomorphism also a joint coequaliser of this diagram. Therefore we obtain an isomorphism $f : \operatorname{colim} D_i \to D$ making the

[4] Formally, by $(\operatorname{colim} X, \operatorname{colim} x \circ \widetilde{T}^{-1}, \operatorname{colim} Y)$, but for readability we will just use $\operatorname{colim} x$ since this is 'isomorphic' to $\operatorname{colim} x \circ \widetilde{T}^{-1}$

following diagram commute:

$$\begin{tikzcd}
T(\operatorname{colim} Y) \arrow[r, "\cong"] \arrow[d, "="'] \arrow[rr, bend right, "c"', start anchor=south east, end anchor=south west, shift right=6] & \operatorname{colim} TY \arrow[r, "\operatorname{colim} c_i"] \arrow[d, "="'] & \operatorname{colim} D_i \arrow[d, "\cong"', "f"] \\
T(\operatorname{colim} Y) \arrow[r, "\cong"] & \operatorname{colim} TY \arrow[r] & D.
\end{tikzcd}$$

This shows that $(1, f) : \operatorname{colim} LF \overset{\cong}{\to} L(\operatorname{colim} F)$ is an isomorphism, as desired. $\square$

Theorem 4. *Let $\mathcal{C}$ be cocomplete, then the category of finitary monads $\mathbf{Mnd}_{\mathrm{f}}(\mathcal{C})$ is also cocomplete. Moreover, colimits are algebraic. That is, if T is the colimit of a diagram of monads $(\mathsf{T}_j)_{j\in\mathcal{J}}$, then T-$\mathbf{Alg}$ is the limit of the diagram of categories $(\mathsf{T}_j\text{-}\mathbf{Alg})_{j\in\mathcal{J}}$ in $\mathbf{Cat}\,/\mathcal{C}$, and vice versa: the limit of the diagram $(\mathsf{T}_j\text{-}\mathbf{Alg})_{j\in\mathcal{J}}$ is of the form T-$\mathbf{Alg}$ for some monad T which is the colimit of $(\mathsf{T}_j)_{j\in\mathcal{J}}$. In other words, the functor $(-)$-$\mathbf{Alg}$: $\mathbf{Mnd}_{\mathrm{f}}(\mathcal{C})^{\mathrm{op}} \to \mathbf{Cat}\,/\mathcal{C}$ creates and preserves limits.*

Proof. Let $V : \mathcal{J} \to \mathbf{Mnd}_{\mathrm{f}}(\mathcal{C})$ be a diagram in the category of finitary monads; we aim to construct its colimit. Denote the monad $V(j)$ by $\mathsf{T}_j = (T_j, \tau_j, \mu_j)$ for every $j \in \mathcal{J}$. We have seen in this section that if the forgetful functor V-$\mathbf{Alg} \to \mathcal{C}$ has a left adjoint, then the algebraic colimit P of $\mathcal{J}$ exists, and is by Lemma 12 also the ordinary colimit of V in the category of monads. Thus the proof is complete if we can show that $U : V\text{-}\mathbf{Alg} \to \mathcal{C}$ has a left adjoint and that P is finitary. We factor the forgetful functor as follows

$$\begin{tikzcd}
V\text{-}\mathbf{Alg} \arrow[r, hook, shift right, "i"'] \arrow[r, phantom, "\perp"] & (L,\lambda)\text{-}\mathbf{Alg} \arrow[l, shift right, "G"'] \arrow[r, hook, shift right, "U"'] \arrow[r, phantom, "\perp"] & T/\mathcal{C} \arrow[l, shift right, "F"'] \arrow[r, shift right, "\tau^*"'] \arrow[r, phantom, "\perp"] & \mathcal{C}^{\mathbf{2}} \arrow[l, shift right, "\tau_*"'] \arrow[r, shift right, "\mathrm{dom}"'] \arrow[r, phantom, "\perp"] & \mathcal{C}. \arrow[l, shift right, "\mathrm{id}"']
\end{tikzcd} \tag{5.9}$$

Here (T, τ) is an endofunctor on $\mathcal{C}$ and (L, λ) is an endofunctor on $T/\mathcal{C}$, as constructed earlier in this section. The adjunction $G \dashv i$ follows from Proposition 13 in combination with Lemma 14. The adjunction $F \dashv U$ follows from Theorem 2. Indeed, by Lemma 13 the category $T/\mathcal{C}$ is cocomplete, and by Lemma 20 L is finitary, so the conditions of the theorem are satisfied and we have that the free (L, λ)-algebra exists. The adjunction $\tau_* \dashv \tau^*$ is the one from (5.3). Lastly the adjunction $\mathrm{id} \dashv \mathrm{dom}$ is the well-known identity domain adjunction. It is easily verified that the composition of these four maps is indeed the forgetful functor V-$\mathbf{Alg} \to \mathcal{C}$, and therefore we conclude that it has a left adjoint. It remains to show that the induced monad P from this adjunction is finitary. Let us rewrite diagram (5.9) as

$$\begin{tikzcd}
V\text{-}\mathbf{Alg} \arrow[r, hook, shift right, "i"'] \arrow[r, phantom, "\perp"] & (L,\lambda)\text{-}\mathbf{Alg} \arrow[l, shift right, "G"'] \arrow[r, hook, shift right, "U'"'] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift right, "F'"']
\end{tikzcd}$$

with U' and F' the composition of the relevant functors in (5.9). It follows that the underlying endofunctor of the monad P is given by $P = U'iGF'$. By Lemma 20 L is finitary and by Lemma 13 $T/\mathcal{C}$ is cocomplete. Thus it follows from Lemma 9 that U is finitary. The functor $\mathrm{dom} \circ \tau^*$ projects an element (A, a, B) onto its first coordinate A, so it follows by the explicit form of the colimit in $T/\mathcal{C}$ (Lemma 13) that this preserves all colimits. F' also preserves all colimits since it is a left adjoint. Therefore we have that $U'F'$ is finitary.

By Lemma 14 we have a natural isomorphism $iG \cong 1$, and therefore it follows that $P = U'iGF' \cong U'F'$ is finitary, which completes the proof. □

6. Cofibrant Generation by a Category

In this chapter we prove the finitary small object argument for categories (Theorem 5). The proof uses the Beck theorem of Chapter 3: we prove that $\mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ generates an AWFS by verifying the conditions of the Beck theorem for $\mathcal{J}^{\pitchfork} \to \mathbb{S}\mathbf{q}(\mathcal{C})$. We depend on Chapter 4 for a constructive proof of the existence of the free algebra for a finitary pointed endofunctor. We rely on Chapter 5 for a constructive proof that the induced monad R is finitary and is free on a pointed endofunctor.

This chapter is based on [7, Section 5], and is adapted to the finitary case.

6.1. Lifting Operations

Given two categories $\mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ and $\mathcal{K} \to \mathcal{C}^{\mathbf{2}}$, a $(\mathcal{J}, \mathcal{K})$-*lifting operation* is defined as a natural family of functions $\varphi_{j,k}$ which picks, for every $j \in \mathcal{J}$, $k \in \mathcal{K}$, and morphism $(u, v) : Uj \to Vk$ a lift, as shown in the diagram below

$$\begin{tikzcd}
\operatorname{dom} Uj \arrow[r, "u"] \arrow[d, "Uj"'] & \operatorname{dom} Vk \arrow[d, "Vk"] \\
\operatorname{cod} Uj \arrow[r, "v"'] \arrow[ur, dashed, "\varphi_{j,k}(u,v)"] & \operatorname{cod} Vk.
\end{tikzcd}$$

An example we have seen before is the canonical $(\mathsf{L}\text{-}\mathbf{Coalg}, \mathsf{R}\text{-}\mathbf{Alg})$-lifting operation (2.5) for an AWFS. The assignment of $(\mathcal{J}, \mathcal{K})$ to the collection of its $(\mathcal{J}, \mathcal{K})$-lifting operations is the object part of a functor

$$\mathbf{Lift} : (\mathbf{Cat} /\mathcal{C}^{\mathbf{2}})^{\mathrm{op}} \times (\mathbf{Cat} /\mathcal{C}^{\mathbf{2}})^{\mathrm{op}} \to \mathbf{Sets}\,.$$

This induces the following adjunction.

Proposition 14 [10, Proposition 3.8]**.** *Let $\mathcal{C}$ be a category. Each functor* $\mathbf{Lift}(\mathcal{J}, -)$ *and* $\mathbf{Lift}(-, \mathcal{K})$ *is representable, whence we have an adjunction* ${}^{\pitchfork}(-) \dashv (-)^{\pitchfork}$*.*

$$\begin{tikzcd}
(\mathbf{Cat} /\mathcal{C}^{\mathbf{2}})^{\mathrm{op}} \arrow[r, shift left, "{}^{\pitchfork}(-)"] \arrow[r, phantom, "\perp"] & \mathbf{Cat} /\mathcal{C}^{\mathbf{2}}. \arrow[l, shift left, "(-)^{\pitchfork}"]
\end{tikzcd} \tag{6.1}$$

Proof. The functor $\mathbf{Lift}(\mathcal{J}, -) : (\mathbf{Cat} /\mathcal{C}^{\mathbf{2}})^{\mathrm{op}} \to \mathbf{Sets}$ is represented by the category $\mathcal{J}^{\pitchfork} \to \mathcal{C}^{\mathbf{2}}$, as introduced in Section 2.3. For more details we refer to [10] or [22]. □

If we have a $(\mathcal{J}, g)$ and a $(\mathcal{J}, h)$ lifting operation for some $g : A \to B$ and $h : B \to C$ in $\mathcal{C}^{\mathbf{2}}$, then the composition hg also bears a canonical lifting operation which is defined by taking the 'stepwise lift', as in

$$\varphi_{j,hg}(u, v) = \varphi_{j,g}(u, \varphi_{j,h}(gu, v)).$$

This provides a vertical composition for the category $\mathcal{J}^{\pitchfork} \to \mathcal{C}^{\mathbf{2}}$ and turns it into a concrete double category $\mathcal{J}^{\pitchfork\!\!\pitchfork} \to \mathbb{S}\mathbf{q}(\mathcal{C})$, where we equip the identity map with the unique lifting structure. We now define what a cofibrantly generated AWFS is.

Definition 6 (Cofibrant generation)**.** An AWFS $(\mathsf{C}, \mathsf{L}, \mathsf{R})$ is *cofibrantly generated* by a small category $\mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ if $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \cong \mathcal{J}^{\pitchfork\!\!\pitchfork}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$. If $\mathcal{J}$ is large, we say that it is *class-cofibrantly generated* by $\mathcal{J} \to \mathcal{C}^{\mathbf{2}}$.

6.2. Split Epimorphisms

Let $\mathcal{C}$ be a cocomplete category. We can form the category $\mathbf{SplEpi}(\mathcal{C})$ of split epimorphisms in $\mathcal{C}$. An object in this category is a pair consisting of a morphism $f : A \to B$, together with a section $p : B \to A$. A morphism $(f, p) \to (g, q)$ is a morphism $f \to g$ which commutes with the sections, as shown in the diagram on the left below

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, shift right, "f"'] & C \arrow[d, shift right, "g"'] \\ B \arrow[r, "v"'] \arrow[u, shift right, "p"'] & D \arrow[u, shift right, "q"'] \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "m"] \arrow[dr, "1"'] & 0 \arrow[d, "e"] \arrow[dr, "me"] & \\ & 1 \arrow[r, "m"'] & 0. \end{tikzcd}$$

Since we can compose sections, we have a vertical composition on split epis, and hence get a concrete double category $\mathbb{S}\mathbf{plEpi}(\mathcal{C})$ over $\mathcal{C}$. Clearly this double category is right-connected. Hence, by Theorem 1 it follows that $\mathbb{S}\mathbf{plEpi}(\mathcal{C})$ is the right class in an AWFS if we can show that the forgetful functor $U : \mathbf{SplEpi}(\mathcal{C}) \to \mathcal{C}^{\mathbf{2}}$ is strictly monadic. Note that we can view $\mathbf{SplEpi}(\mathcal{C})$ as the presheaf category $\mathcal{C}^{\mathcal{S}}$, where $\mathcal{S}$ is the free split epimorphism and shown on the right above. Under this identification the functor U becomes $\mathcal{C}^{j} : \mathcal{C}^{\mathcal{S}} \to \mathcal{C}^{\mathbf{2}}$, where $j : \mathbf{2} \to \mathcal{S}$ is the inclusion which selects the arrow e.

Lemma 21. *The forgetful functor $U : \mathbf{SplEpi}(\mathcal{C}) \to \mathcal{C}^{\mathbf{2}}$ strictly creates colimits.*

Proof. Suppose we have a diagram $X : \mathcal{I} \to \mathbf{SplEpi}(\mathcal{C})$ with colimiting cocone (D, σ) for the diagram UX in $\mathcal{C}^{\mathbf{2}}$. For every $i \in \mathcal{I}$ we denote $X_i : A_i \to B_i$, with section $s_i : B_i \to A_i$, and we denote $\sigma_i = (u_i, v_i) : X_i \to D$. We need to show that the map $D : A \to B$ also has a section. To do so, note that since colimits are computed pointwise in presheaf categories, we have that $(B, v_i)_{i \in \mathcal{I}}$ is the colimit of the diagram $(B_i)_{i \in \mathcal{I}}$ in $\mathcal{C}$.

We claim that $(A, u_i \circ s_i)_{i\in\mathcal{I}}$ is also a cocone for this diagram.

$$\begin{tikzcd}
B_i \arrow[r, "v_i"] \arrow[d, "s_i"'] & B \arrow[d, dashed, "s"] & B_j \arrow[l, "v_j"'] \arrow[d, "s_j"] \\
A_i \arrow[r, "u_i"] \arrow[d, "X_i"'] & A \arrow[d, "D"] & A_j \arrow[l, "u_j"'] \arrow[d, "X_j"] \\
B_i \arrow[r, "v_i"'] & B & B_j \arrow[l, "v_j"]
\end{tikzcd}
\qquad
\begin{tikzcd}
& A & \\
A_i \arrow[ur, "u_i"] \arrow[rr, "X(\alpha)_0"] \arrow[d, shift right, "X_i"'] & & A_j \arrow[ul, "u_j"'] \arrow[d, shift right, "X_j"'] \\
B_i \arrow[rr, "X(\alpha)_1"'] \arrow[u, shift right, "s_i"'] & & B_j \arrow[u, shift right, "s_j"']
\end{tikzcd}$$

To see this, let $\alpha : i \to j$ be a morphism in $\mathcal{I}$. We need to show that $u_j \circ s_j \circ X(\alpha)_1 = u_i \circ s_i$. To see this, we look at the diagram on the right above. The square commutes because $X(\alpha)$ is a map of split epis. The triangle commutes because (D, σ) is a cocone on UX. Thus we have

$$u_j \circ s_j \circ X(\alpha)_1 = u_j \circ X(\alpha)_0 \circ s_i = u_i \circ s_i.$$

So we get an induced unique map $s : B \to A$ with the property that $s \circ v_i = u_i \circ s_i$ for all $i \in \mathcal{I}$, as shown on the right above. Moreover, we claim that s is indeed a section of D, i.e. that $D \circ s = 1_B$. To see this, note that both are maps out of a colimit, so it suffices to show that they agree after precomposition with the colimit inclusions. We compute:

$$D \circ s \circ v_i = D \circ u_i \circ s_i = v_i \circ X_i \circ s_i = v_i.$$

So we see that D is a split epi and moreover that $\sigma_i : X_i \to D$ commutes with the sections. It remains to show that (D, σ) is indeed the colimit of $X : \mathcal{I} \to \mathbf{SplEpi}(\mathcal{C})$; we leave this to the reader. □

Proposition 15. *Let $\mathcal{C}$ be a cocomplete category. Then $\mathbf{SplEpi}(\mathcal{C}) \cong (R, \eta)$-$\mathbf{Alg}$ for some monad $\mathsf{R} = (R, \eta, \mu)$ which is algebraically-free over its underlying pointed endofunctor. Moreover R is cocontinuous.*

Proof. By Lemma 21 U strictly creates colimits, so by the Beck monadicity theorem (Theorem 7) it is (strictly) monadic as soon as it has a left adjoint $\mathcal{C}^{\mathbf{2}} \to \mathbf{SplEpi}(\mathcal{C})$. That is, we have to show that the free split epimorphism on an arrow f exists. Since $\mathcal{C}$ is cocomplete, it has binary coproducts. The free split epi Rf on $f : A \to B$ is given by $\langle f, 1\rangle : A + B \to B$, with section the inclusion $\iota_B : B \to A + B$. Indeed, one can verify that for every morphism $(u, v) : f \to g$ where g is a split epi with section q, there is a unique map $(Rf, \iota_B) \to (g, q)$ given by the diagram on the left

$$\begin{tikzcd}
A + B \arrow[r, "\langle u,qv\rangle"] \arrow[d, shift right, "\langle f,1\rangle"'] & C \arrow[d, shift right, "g"'] \\
B \arrow[r, "v"'] \arrow[u, shift right, "\iota_B"'] & D \arrow[u, shift right, "q"']
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\iota_A"] \arrow[d, "f"'] & A + B \arrow[d, shift right, "\langle f,1\rangle"'] \\
B \arrow[r, "1"'] & B. \arrow[u, shift right, "\iota_B"']
\end{tikzcd}$$

Thus $U : \mathbf{SplEpi}(\mathcal{C}) \to \mathcal{C}^{\mathbf{2}}$ has a left adjoint F, with unit $f \to Rf$ given by the diagram on the right above. It follows that U is strictly monadic, and thus by Theorem 1 we have

$\mathbf{SplEpi}(\mathcal{C}) \cong \mathsf{R}\text{-}\mathbf{Alg}$ over $\mathcal{C}^{\mathbf{2}}$ for some monad $\mathsf{R} = (R, \eta, \mu)$ which is the right class of an AWFS. We recall from the proof of Theorem 1 that the monad R of the resulting AWFS is in fact the monad induced by the adjunction $F \dashv U$. So $R = UF$ and the unit η is given by the unit of the adjunction, that is, $\eta_f = (\iota_A, 1)$ as on the right above.

We now claim that in fact $(R, \eta)\text{-}\mathbf{Alg} \cong \mathbf{SplEpi}(\mathcal{C})$, that is, $\mathbf{SplEpi}(\mathcal{C})$ is not only isomorphic to the algebras for the monad R, but also to the algebras of the underlying pointed endofunctor (R, η). To see this, note that every split epimorphism has an algebra structure for the monad R, and thus certainly also an (R, η)-algebra structure. For the converse, let $f : A \to B$ be a morphism in $\mathcal{C}$ with an (R, η)-algebra structure $\alpha : Rf \to f$. Then $\alpha \circ \eta_f = 1$, so the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "\iota_A"] \arrow[d, "f"'] \arrow[rr, bend left, "1_A"] & A + B \arrow[r, "\alpha_0"] \arrow[d, shift right, "{\langle f,1 \rangle}"'] & A \arrow[d, "f"] \\
B \arrow[r, "1_B"'] \arrow[rr, bend right, "1_B"'] & B \arrow[r, "\alpha_1"'] \arrow[u, shift right, "\iota_B"'] & B
\end{tikzcd}$$

It follows that $\alpha_1 = 1_B$ and therefore $\alpha_0 \circ \iota_B$ is a section of f. Thus f is a split epi. We conclude that the monad R is algebraically-free (and thus free by Proposition 9) over its underlying pointed endofunctor (R, η).

Finally, we have that $R = UF$ preserves colimits. Indeed, F is a left adjoint, so it preserves colimits. Lemma 21 shows that U strictly creates colimits, and since $\mathcal{C}^{\mathbf{2}}$ is cocomplete (since $\mathcal{C}$ is cocomplete), it also preserves them. So we conclude that R is cocontinuous. $\square$

6.3. The Finitary Small Object Argument for Categories

In order to prove the small object argument for categories, we need a few lemmas that are needed in the main proof. Firstly we need a small lemma about natural transformations between adjoints.

Lemma 22. *If we have two adjunctions $F_1 \dashv G_1, F_2 \dashv G_2$ and a natural transformation $\mu : G_1 \to G_2$, then we get an induced natural transformation between the left adjoints $\overline{\mu} : F_2 \to F_1$, which we call its* mate.

$$\begin{tikzcd}
F_1 \arrow[r, phantom, "\dashv"] & G_1 \arrow[d, "\mu"] \\
F_2 \arrow[u, dashed, "\overline{\mu}"] \arrow[r, phantom, "\dashv"] & G_2
\end{tikzcd}$$

Proof. We define $\overline{\mu}$ as the composition

$$F_2 \xrightarrow{F_2\eta} F_2G_1F_1 \xrightarrow{F_2\mu_{F_1}} F_2G_2F_1 \xrightarrow{\epsilon_{F_1}} F_1$$

where $\eta : 1 \to G_1 F_1$ is the unit of the first adjunction and $\epsilon : F_2 G_2 \to 1$ is the counit of the second adjunction. $\square$

We need the following proposition to invoke Beck's monadicity theorem in the proof of the small object argument for categories.

Proposition 16. *Let (P, ρ) be a pointed endofunctor on a category $\mathcal{C}$. Then the forgetful functor $U : (P, \rho)$-Alg $\to \mathcal{C}$ creates U-split coequalisers.*

Proof. Similar to Lemma 17. $\square$

We need one more lemma about the preservation of colimits.

Lemma 23. *Let $F : \mathcal{C} \to \mathcal{D}^{\mathbf{2}}$ be a functor. If* $\mathrm{dom} \circ F$ *and* $\mathrm{cod} \circ F$ *preserve colimits of type $\mathcal{I}$, then so does F.*

Proof. Let $D : \mathcal{I} \to \mathcal{C}^{\mathbf{2}}$ be a diagram in $\mathcal{C}$ for which the colimit exists and is preserved by $\mathrm{dom} \circ F$ and $\mathrm{cod} \circ F$. Then we have canonical isomorphisms $\mathrm{colim}_{i \in \mathcal{I}} \mathrm{dom}\, F(D_i) \cong \mathrm{dom}\, F(\mathrm{colim}_{i \in \mathcal{I}} D_i)$ and $\mathrm{colim}_{i \in \mathcal{I}} \mathrm{cod}\, F(D_i) \cong \mathrm{cod}\, F(\mathrm{colim}_{i \in \mathcal{I}} D_i)$. Since $\mathcal{D}^{\mathbf{2}}$ is a presheaf category, we know that colimits are computed pointwise, which implies that (the vertex of) $\mathrm{colim}_{i \in \mathcal{I}} F(D_i)$ is given by the map

$$\mathrm{colim}_{i \in \mathcal{I}} \mathrm{dom}\, F(D_i) \longrightarrow \mathrm{colim}_{i \in \mathcal{I}} \mathrm{cod}\, F(D_i),$$

which is the unique map out of $\mathrm{colim}_{i \in \mathcal{I}} \mathrm{dom}\, F(D_i)$ induced by the maps

$$\mathrm{dom}\, F(D_i) \xrightarrow{F(D_i)} \mathrm{cod}\, F(D_i) \xrightarrow{\iota_i^{cod}} \mathrm{colim}_{i \in \mathcal{I}} \mathrm{cod}\, F(D_i).$$

Now consider the following diagram

$$\begin{tikzcd}
\mathrm{dom}\, F(D_i) \arrow[r, "\iota_i^{\mathrm{dom}}"'] \arrow[d, "F(D_i)"'] \arrow[rr, bend left, "\mathrm{dom}\, F(\iota_i)"] & \mathrm{colim}_{i \in \mathcal{I}} \mathrm{dom}\, F(D_i) \arrow[r, "\cong"] \arrow[d, dashed, "\mathrm{colim}_{i \in \mathcal{I}} F(D_i)"] & \mathrm{dom}\, F(\mathrm{colim}_{i \in \mathcal{I}} D_i) \arrow[d, "F(\mathrm{colim}_{i \in \mathcal{I}} D_i)"] \\
\mathrm{cod}\, F(D_i) \arrow[r, "\iota_i^{\mathrm{cod}}"] \arrow[rr, bend right, "\mathrm{cod}\, F(\iota_i)"'] & \mathrm{colim}_{i \in \mathcal{I}} \mathrm{cod}\, F(D_i) \arrow[r, "\cong"] & \mathrm{cod}\, F(\mathrm{colim}_{i \in \mathcal{I}} D_i)
\end{tikzcd}$$

Because the outer diagram, the left square, and the top and bottom triangles commute, the map $F(\mathrm{colim}_{i \in \mathcal{I}} D_i)$ composed with the two isomorphisms is also a map $\phi : \mathrm{colim}_{i \in \mathcal{I}} \mathrm{dom}\, F(D_i) \longrightarrow \mathrm{colim}_{i \in \mathcal{I}} \mathrm{cod}\, F(D_i)$ with the property that $\phi \circ \iota_i^{\mathrm{dom}} = \iota_i^{\mathrm{cod}} \circ F(D_i)$. Thus by unicity ϕ equals $\mathrm{colim}_{i \in \mathcal{I}} F(D_i)$, that is, the right triangle commutes as well. It follows that $F(\mathrm{colim}_{i \in \mathcal{I}} D_i)$ is colimiting, as desired. $\square$

We are now ready to prove the finitary small object argument for categories.

Theorem 5. *Let $\mathcal{C}$ be a locally small and cocomplete category and let $U : \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ be a small category over $\mathcal{C}^{\mathbf{2}}$ such that for every $j \in \mathcal{J}$ the domain and the codomain of Uj are ω-compact. Then the AWFS $(\mathcal{C}, \mathsf{L}, \mathsf{R})$ cofibrantly generated by U exists and is finitary. Moreover, the monad R is algebraically-free on a finitary pointed endofunctor.*

Proof. Firstly, we claim that $V : \mathcal{J}^{⫛} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ is right-connected. To see this, let $\varphi_{-,g}$ be a vertical map in $\mathcal{J}^{⫛}$, that is, a morphism $g : A \to B$ in $\mathcal{C}$ equipped with a lifting structure against $\mathcal{J}$. We need to show that $(g, 1) : \varphi_{-,g} \to \varphi_{-,1}$ is a square, i.e. a morphism of lifting structures. So suppose we have a morphism $(u, v) : m(j) \to g$. Then we have:

$$g \circ \varphi_{j,g}(u, v) = v = \varphi_{j,1}(gu, v).$$

Hence, by Theorem 1 it is sufficient to show that $V_1 : \mathcal{J}^{\pitchfork} \to \mathcal{C}^{\mathbf{2}}$ is strictly monadic. To show this, firstly we note that an object of $\mathcal{J}^{\pitchfork}$ is a map g together with a section φ_{-g} of the natural transformation

$$\psi_{-g} : \hom_{\mathcal{C}}(\operatorname{cod} U-, \operatorname{dom} g) \Rightarrow \hom_{\mathcal{C}^{\mathbf{2}}}(U-, g) : \mathcal{J}^{\mathrm{op}} \to \mathbf{Sets}\,.$$

which sends a map $\phi : \operatorname{cod} Uj \to \operatorname{dom} g$ to the morphism $(\phi \circ Uj, g \circ \phi) : Uj \to g$. A section therefore provides for every $j \in \mathcal{J}$ and morphism $\sigma : Uj \to g$ a lift of the diagram on the left below

$$\begin{tikzcd} \operatorname{dom} Uj \arrow[r, "\sigma_0"] \arrow[d, "Uj"'] & \operatorname{dom} g \arrow[d, "g"] \\ \operatorname{cod} Uj \arrow[r, "\sigma_1"'] \arrow[ur, dashed, "\varphi_{j,g}(\sigma)"] & \operatorname{cod} g \end{tikzcd} \qquad \begin{tikzcd} \hom_{\mathcal{C}^{\mathbf{2}}}(Ui, g) \arrow[r, "\varphi_{i,g}"] & \hom_{\mathcal{C}}(\operatorname{cod} Ui, \operatorname{dom} g) \\ \hom_{\mathcal{C}^{\mathbf{2}}}(Uj, g) \arrow[r, "\varphi_{j,g}"'] \arrow[u] & \hom_{\mathcal{C}}(\operatorname{cod} Ui, \operatorname{dom} g). \arrow[u] \end{tikzcd}$$

Moreover, naturality ensures that for every $\alpha : i \to j$ in $\mathcal{J}$ we have a commutative diagram as on the right above. So we have

$$\varphi_{i,g}(\sigma \circ U\alpha) = \varphi_{j,g}(\sigma) \circ U(\alpha)_1,$$

which is precisely the horizontal condition. A morphism $(g, \varphi_{-g}) \to (h, \varphi_{-h})$ in $\mathcal{J}^{\pitchfork}$ is then a map $(u, v) : g \to h$ in $\mathcal{C}^{\mathbf{2}}$ for which the induced map $\psi_{-g} \to \psi_{-h}$ in $[\mathcal{J}^{\mathrm{op}}, \mathbf{Sets}]^{\mathbf{2}}$ commutes with the sections, as shown in the following diagram

$$\begin{tikzcd} \hom_{\mathcal{C}}(\operatorname{cod} Uj, \operatorname{dom} g) \arrow[r] \arrow[d, shift right, "\psi_{j,g}"'] & \hom_{\mathcal{C}}(\operatorname{cod} Uj, \operatorname{dom} h) \arrow[d, shift left, "\psi_{j,h}"] \\ \hom_{\mathcal{C}^{\mathbf{2}}}(Uj, g) \arrow[r] \arrow[u, dashed, shift right, "\varphi_{j,g}"'] & \hom_{\mathcal{C}^{\mathbf{2}}}(Uj, h). \arrow[u, dashed, shift left, "\varphi_{j,h}"] \end{tikzcd}$$

This means that for every lifting problem $\sigma : Uj \to g$, we have

$$u \circ \varphi_{j,g}(\sigma) = \varphi_{j,h}(u\sigma_0, v\sigma_1),$$

which is indeed a map between lifting structures as we have seen before. Hence, we have the following pullback

$$\begin{tikzcd} \mathcal{J}^{\pitchfork} \arrow[r] \arrow[d, "V_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathbf{SplEpi}([\mathcal{J}^{\mathrm{op}}, \mathbf{Sets}]) \arrow[d] \\ \mathcal{C}^{\mathbf{2}} \arrow[r, "\psi"'] & [\mathcal{J}^{\mathrm{op}}, \mathbf{Sets}]^{\mathbf{2}} \end{tikzcd}$$

We show that ψ has a left adjoint. To do so, firstly we claim that the composition of ψ with the domain and codomain functor both have a left adjoint,

$$\mathcal{C}^{\mathbf{2}} \xrightarrow{\psi} [\mathcal{J}^{\mathrm{op}}, \mathbf{Sets}]^{\mathbf{2}} \underset{\mathrm{cod}}{\overset{\mathrm{dom}}{\rightrightarrows}} [\mathcal{J}^{\mathrm{op}}, \mathbf{Sets}].$$

Note that $\mathrm{cod} \circ \psi : \mathcal{C}^{\mathbf{2}} \to [\mathcal{J}^{\mathrm{op}}, \mathbf{Sets}]$ is given by

$$(\mathrm{cod} \circ \psi)(g)(j) = \hom_{\mathcal{C}^{\mathbf{2}}}(Uj, g).$$

We recognise this as the right adjoint of the Kan extension of $U : \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ along the Yoneda embedding (see Theorem 9, in Appendix A.3). Thus, $\mathrm{cod} \circ \psi$ has a left adjoint F_1, as on the left below.

$$\begin{tikzcd} \mathcal{J} \arrow[r, "y"] \arrow[dr, "U"'] & [\mathcal{J}^{\mathrm{op}}, \mathbf{Sets}] \arrow[d, shift right, "F_1"'] \arrow[d, phantom, "\dashv"] \\ & \mathcal{C}^{\mathbf{2}} \arrow[u, shift right, "\psi\cdot\mathrm{cod}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{J} \arrow[r, "y"] \arrow[dr, "\mathrm{id}\cdot\mathrm{cod}\cdot U"'] & [\mathcal{J}^{\mathrm{op}}, \mathbf{Sets}] \arrow[d, shift right, "F_2"'] \arrow[d, phantom, "\dashv"] \\ & \mathcal{C}^{\mathbf{2}} \arrow[u, shift right, "\psi\cdot\mathrm{dom}"'] \end{tikzcd}$$

For $\mathrm{dom} \circ \psi$, we have

$$(\mathrm{dom} \circ \psi)(g)(j) = \hom_{\mathcal{C}}(\mathrm{cod}\, Uj, \mathrm{dom}\, g) \cong \hom_{\mathcal{C}^{\mathbf{2}}}(\mathrm{id}_{\mathrm{cod}\, Uj}, g),$$

where the isomorphism follows from the adjunction $\mathrm{id} \dashv \mathrm{dom} : \mathcal{C}^{\mathbf{2}} \to \mathcal{C}$. Again we recognise this as the right adjoint of the Kan extension $\mathrm{id} \circ \mathrm{cod} \circ U : \mathcal{J} \to \mathcal{C}^{\mathbf{2}}$ and therefore $\mathrm{dom} \circ \psi$ has a left adjoint F_2. Now by Lemma 22 the natural transformation $\psi : \mathrm{dom} \circ \psi \to \mathrm{cod} \circ \psi$ induces a natural transformation between its left adjoints $\alpha : F_1 \to F_2$. Now we claim that the left adjoint K of ψ is given by sending $f : X \to Y$ to the pushout of $F_1 f : F_1 X \to F_1 Y$ along $\alpha_X : F_1 X \to F_2 X$,

$$\begin{tikzcd} F_1 X \arrow[r, "\alpha_X"] \arrow[d, "F_1 f"'] & F_2 X \arrow[d] \\ F_1 Y \arrow[r] \arrow[ur, phantom, very near end, "\ulcorner"] & Kf \end{tikzcd}$$

To show this, we have to prove that

$$\hom_{\mathcal{C}^{\mathbf{2}}}(Kf, g) \cong \hom_{[\mathcal{J}^{\mathrm{op}}, \mathbf{Sets}]^{\mathbf{2}}}(f, \psi_g).$$

Assume we have a morphism $\sigma : f \to \psi_g$ in $[\mathcal{J}^{\mathrm{op}}, \mathbf{Sets}]^{\mathbf{2}}$, as shown on the left below.

$$\begin{tikzcd} X \arrow[r, "\sigma_0"] \arrow[d, "f"'] & \operatorname{dom} \psi_g \arrow[d, "\psi_g"] \\ Y \arrow[r, "\sigma_1"'] & \operatorname{cod} \psi_g \end{tikzcd} \qquad \begin{tikzcd} F_1 X \arrow[r, "\alpha_X"] \arrow[d, "F_1 f"'] & F_2 X \arrow[d, "\overline{\sigma_0}"] \\ F_1 Y \arrow[r, "\overline{\sigma_1}"'] & g \end{tikzcd}$$

This diagram transposes under the adjunctions $F_1 \dashv \psi \circ \operatorname{cod}$ and $F_2 \dashv \psi \circ \operatorname{dom}$ to the diagram on the right above. But a commutative diagram of this form is the same as a map from the pullback Kf to g. So indeed we have a bijection $\hom_{\mathcal{C}^{\mathbf{2}}}(Kf, g) \cong \hom_{[\mathcal{J}^{\mathrm{op}}, \mathbf{Sets}]^{\mathbf{2}}}(f, \psi_g)$ which is natural.

Next we claim that ψ preserves ω-filtered colimits. To see this, fix $j \in \mathcal{J}$. Firstly we claim that the domain and codomain of

$$\psi_{j-} : \hom_{\mathcal{C}}(\operatorname{cod} Uj, \operatorname{dom} -) \Rightarrow \hom_{\mathcal{C}^{\mathbf{2}}}(Uj, -)$$

preserve ω-filtered colimits. By assumption, $\operatorname{cod} Uj$ is ω-compact so that $\hom_{\mathcal{C}}(\operatorname{cod} Uj, -)$ preserves filtered colimits. But the functor $\operatorname{dom} : \mathcal{D}^{\mathbf{2}} \to \mathcal{D}$ also preserves colimits (for any category $\mathcal{D}$), because $\mathcal{D}^{\mathbf{2}}$ is a presheaf category and thus colimits are computed pointwise. Thus we have that $\hom_{\mathcal{C}}(\operatorname{cod} Uj, \operatorname{dom} -)$ preserves filtered colimits. Similarly, since both the domain and codomain of Uj are ω-compact and $\mathcal{C}^{\mathbf{2}}$ is a presheaf category, we have that $\hom_{\mathcal{C}^{\mathbf{2}}}(Uj, -)$ preserves filtered colimits. Thus, by Lemma 23 it follows that every ψ_{j-} preserves filtered colimits. Lastly, again using that colimits are computed pointwise in functor categories, we conclude that ψ preserves filtered colimits.

By Proposition 15 the category $\mathbf{SplEpi}([\mathcal{J}^{\mathrm{op}}, \mathbf{Sets}])$[1] can be identified with the category (T, η)-$\mathbf{Alg}$ for some cocontinuous pointed endofunctor (T, η), which moreover extends to a monad $\mathsf{T} = (T, \eta, \mu)$ that is algebraically-free over its underlying pointed endofunctor. Hence it follows by Lemma 10 that $\mathcal{J}^{\pitchfork}$ is isomorphic over $\mathcal{C}^{\mathbf{2}}$ to the category of algebras for the pointed endofunctor (P, ρ) given by the pushout on the left below.

$$\begin{tikzcd} K\psi \arrow[r, "K\eta_\psi"] \arrow[d, "\epsilon"'] & KT\psi \arrow[d] \\ 1 \arrow[r, "\rho"'] \arrow[ur, phantom, "\ulcorner", near end] & P \end{tikzcd} \qquad \begin{tikzcd} \mathcal{J}^{\pitchfork} \arrow[rr, "\cong"] \arrow[dr, "V_1"'] & & (P, \rho)\text{-}\mathbf{Alg} \arrow[dl, "U"] \\ & \mathcal{C}^{\mathbf{2}} & \end{tikzcd}$$

Therefore the situation is as shown on the right above, and we can identify V_1 with the forgetful functor $U : (P, \rho)$-$\mathbf{Alg} \to \mathcal{C}^{\mathbf{2}}$. By Proposition 16 it follows that U creates coequalisers of U-split pairs. By Beck's monadicity theorem (Theorem 7 in Appendix A.2) it follows that U (and therefore V_1) is (strictly) monadic as soon as we show that it has a left adjoint, that is, we have to show that the free (P, ρ) algebra exists for an arbitrary $f \in \mathcal{C}^{\mathbf{2}}$. But this follows from Theorem 2 as soon as we can show that the

[1] Note that $[\mathcal{J}^{\mathrm{op}}, \mathbf{Sets}]$ is a presheaf category and thus cocomplete.

conditions of the theorem are satisfied. Firstly, since $\mathcal{C}$ is cocomplete, so is $\mathcal{C}^{\mathbf{2}}$. Secondly, We need to show that P is finitary. But note that ψ, K, and T are all finitary. ψ as we have shown above, K because it is a left adjoint, and T because it is cocontinuous by Proposition 15. Hence P is a pushout of finitary functors, whence by Corollary 3 itself finitary. Thus U has a left adjoint F and so V_1 has a left adjoint F_1, and it follows that V_1 is (strictly) monadic.

Lastly, in order to show that the resulting AWFS is finitary and that R is free on a pointed endofunctor, we recall from the proof of Theorem 1 that the monad R of the resulting AWFS is in fact the monad induced by the adjunction $F_1 \dashv V_1$. Thus it follows by Theorem 3 that R is finitary and is the algebraically-free monad on the finitary pointed endofunctor (P, ρ). An application of Lemma 1 then shows that L is also finitary, and thus we conclude that the induced AWFS is finitary.

□

7. Cofibrant Generation by a Double Category

It turns out that cofibrant generation by categories is not sufficient to generate all interesting AWFS. There are examples such as the lalis AWFS on **Cat** [7] which is not generated by a small category.

To capture all interesting AWFS, we need to turn to the concept of cofibrant generation by a double category. In this chapter, we constructively prove our main theorem: the finitary small object argument for double categories (Theorem 6). The proof relies on the Beck theorem (Theorem 1), the small object argument for categories (Theorem 5), and on Chapter 5 to obtain a coequaliser of monad maps, and the fact that this induces an equaliser diagram under the algebra functor.

7.1. Double Categorical Lifting Operations

Let $U : \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ and $V : \mathbb{K} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ be two double functors. We define a $(\mathbb{J}, \mathbb{K})$-*lifting operation* to be a $(\mathcal{J}_1, \mathcal{K}_1)$-lifting operation which is also compatible with vertical composition. That is,

$$\varphi_{j,\ell\cdot k}(u,v) = \varphi_{j,k}(u, \varphi_{j,\ell}(Vk \cdot u, v)), \quad \text{and} \quad \varphi_{j\cdot i,k}(u,v) = \varphi_{j,k}(\varphi_{i,k}(u, v \cdot Uj), v)$$

for all vertically composable arrows j, i in $\mathbb{J}$ and ℓ, k in $\mathbb{K}$. Diagrammatically, we have

$$\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[dd, "Uj"'] & \cdot \arrow[d, "Vk"] \\
& \cdot \arrow[d, "V\ell"] \\
\cdot \arrow[r, "v"'] \arrow[uur, dashed] \arrow[ur, dashed] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
\cdot \arrow[r, "u"] \arrow[d, "Ui"'] & \cdot \arrow[dd, "Vk"] \\
\cdot \arrow[d, "Uj"'] \arrow[ur, dashed] & \\
\cdot \arrow[r, "v"'] \arrow[uur, dashed] & \cdot
\end{tikzcd}$$

An example of this is the lifting operation associated with an AWFS, which is a $(\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg}, \mathsf{R}\text{-}\mathbb{A}\mathbf{lg})$-lifting operation. We have a functor

$$\mathsf{Lift} : (\mathbf{DBL} \,/\, \mathbb{S}\mathbf{q}(\mathcal{C}))^{\mathrm{op}} \times (\mathbf{DBL} \,/\, \mathbb{S}\mathbf{q}(\mathcal{C}))^{\mathrm{op}} \to \mathbf{Sets},$$

which sends $\mathbb{J}, \mathbb{K}$ to the collection of $(\mathbb{J}, \mathbb{K})$-lifting operations. This induces the following adjunction.

Proposition 17. *Let $\mathcal{C}$ be a category. Each functor* $\mathsf{Lift}(\mathbb{J}, -)$ *and* $\mathbf{Lift}(-, \mathbb{K})$ *is representable, whence we have an adjunction* ${}^{\pitchfork}(-) \dashv (-)^{\pitchfork}$

$$\begin{tikzcd}
(\mathbf{DBL} \,/\, \mathbb{S}\mathbf{q}(\mathcal{C}))^{\mathrm{op}} \arrow[r, shift left=1.5ex, "{}^{\pitchfork}(-)"] \arrow[r, phantom, "\perp"] & \mathbf{DBL} \,/\, \mathbb{S}\mathbf{q}(\mathcal{C}). \arrow[l, shift left=1.5ex, "(-)^{\pitchfork}"]
\end{tikzcd}$$

Proof. The functor $\mathsf{Lift}(\mathbb{J}, -) : (\mathbf{DBL} \,/ \mathcal{C}^{\mathbf{2}})^{\mathrm{op}} \to \mathbf{Sets}$ is represented by the double category $\mathbb{J}^{\pitchfork} \to \mathcal{C}^{\mathbf{2}}$, which is defined as follows.

- Objects are objects in $\mathcal{C}$;
- Horizontal arrows are morphisms in $\mathcal{C}$;
- Vertical arrows are pairs (g, φ_{-g}) where g is a morphism in $\mathcal{C}$ and φ a *right lifting structure* of g against $\mathbb{J}$, that is, an operation φ which chooses for every vertical morphism $j \in \mathbb{J}$ and square $\sigma : Uj \to g$ a solution $\varphi_{j,g}(\sigma)$

$$\begin{tikzcd}
\cdot \arrow[r, "\sigma_0"] \arrow[d, "Uj"'] & \cdot \arrow[d, "g"] \\
\cdot \arrow[r, "\sigma_1"'] \arrow[ur, dashed] & \cdot
\end{tikzcd}$$

  subject to the following conditions

  1. *Horizontal condition*: for any square $\alpha : j \to i$ in $\mathbb{J}$, we have that $\varphi_i(\sigma) \circ U(\alpha)_1 = \varphi_j(\sigma \circ U(\alpha))$.

$$\begin{tikzcd}
\cdot \arrow[r, "U(\alpha)_0"] \arrow[d, "Uj"'] & \cdot \arrow[r, "\sigma_0"] \arrow[d, "U(i)"] & \cdot \arrow[d, "f"] \\
\cdot \arrow[r, "U(\alpha)_1"'] \arrow[urr, dashed] & \cdot \arrow[r, "\sigma_1"'] \arrow[ur, dashed] & \cdot
\end{tikzcd}$$

  2. *Vertical condition:* for any vertical composition $j \cdot i$ in $\mathbb{J}$ and square $\sigma : U(j \cdot i) \to f$, we have $\varphi_{j \cdot i}(\sigma) = \varphi_j(\varphi_i(\sigma_0, \sigma_1 \circ U(j)), \sigma_1)$.

$$\begin{tikzcd}
\cdot \arrow[r, "\sigma_0"] \arrow[d, "Ui"'] & \cdot \arrow[dd, "g"] \\
\cdot \arrow[d, "Uj"'] \arrow[ur, dashed] & \\
\cdot \arrow[r, "\sigma_1"'] \arrow[uur, dashed] & \cdot
\end{tikzcd}$$

  vertical composition is given by the 'stepwise' composition of lifting structures as described in Section 6.1;

- Squares are morphisms of lifting structures as before. That is, a square $\tau : (f, \varphi) \to (g, \psi)$ is a commutative square as shown on the right of the following diagram

$$\begin{tikzcd} {} \arrow[r, "u"] \arrow[d, "Ui"'] & {} \arrow[r, "\tau_0"] \arrow[d, "f"] & {} \arrow[d, "g"] \\ {} \arrow[r, "v"'] \arrow[ur, dashed] \arrow[urr, dashed] & {} \arrow[r, "\tau_1"'] & {} \end{tikzcd}$$

such that for any $i \in \mathbb{J}$ and $(u, v) : Ui \to f$ we have that $\psi_i(\tau_0 \circ u, \tau_1 \circ v) = \tau_0 \circ \varphi_i(u, v)$. It follows easily from this definition that vertical composition of such squares is compatible with the composition operation on vertical morphisms.

For more details on the proof we refer to [7] or [22]. $\square$

In Definition 6 we defined cofibrant generation by a category. We can now extend this definition to double categories. We say that an AWFS (L, R) is *cofibrantly generated by a small double category* $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ if there is an isomorphism of double categories $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \cong \mathbb{J}^{\pitchfork}$ over $\mathbb{S}\mathbf{q}(\mathcal{C})$. If this holds for a large double category $\mathbb{J}$, then we say that (L, R) is *class-cofibrantly generated* by $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$.

Proposition 18. *Let* (L, R) *be an AWFS. Then we have an isomorphism of double categories* $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \cong (\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg})^{\pitchfork}$ *and dually* $\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg} \cong {}^{\pitchfork}(\mathsf{R}\text{-}\mathbb{A}\mathbf{lg})$. *In other words,* (L, R) *is class-cofibrantly generated by* $\mathsf{L}\text{-}\mathbb{C}\mathbf{oalg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ *and class-fibrantly generated by* $\mathsf{R}\text{-}\mathbb{A}\mathbf{lg} \to \mathbb{S}\mathbf{q}(\mathcal{C})$*.*

Proof. See [7] or [22]. $\square$

7.2. The Finitary Small Object Argument for Double Categories

We are now ready to prove the main result of this thesis, the finitary small object argument for double categories.

Theorem 6. *Let* $\mathcal{C}$ *be a locally small, cocomplete category, and let* $U : \mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ *be a double functor subject to the following conditions*

1. $\mathbb{J}$ *is small,*

2. *the object* Uj *is* ω*-compact for every object* $j \in \mathcal{J}_0$*.*

Then the AWFS cofibrantly generated by U *exists and is finitary.*

Proof. We write $\mathcal{J}_2 = \mathcal{J}_1 \times_{\mathcal{J}_0} \mathcal{J}_1$ for the category of vertically composable pairs in $\mathbb{J}$, and let $m : \mathcal{J}_2 \to \mathcal{J}_1$ be the composition. Then we have a commutative triangle as on the left

$$\begin{tikzcd} \mathcal{J}_2 \arrow[rr, "m"] \arrow[dr, "U_1 m"'] & & \mathcal{J}_1 \arrow[dl, "U_1"] \\ & \mathcal{C}^{\mathbf{2}} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{J}_1^{\pitchfork} \arrow[rr, "m^{\pitchfork}"] \arrow[dr] & & \mathcal{J}_2^{\pitchfork} \arrow[dl] \\ & \mathbb{S}\mathbf{q}(\mathcal{C}). & \end{tikzcd}$$

This induces a morphism $m^{\pitchfork} : \mathcal{J}_1^{\pitchfork} \to \mathcal{J}_2^{\pitchfork}$ of concrete double categories over $\mathbb{S}\mathbf{q}(\mathcal{C})$ as on the right above, which sends a $(\mathcal{J}_1, g)$-lifting operation φ_{-g} to the $(\mathcal{J}_2, g)$-lifting operation ψ_{-g}, defined as on the left below:

$$\psi_{(j,i),g}(u, v) = \varphi_{j\cdot i,g}(u, v) \qquad\qquad \theta_{(j,i),g}(u, v) = \varphi_{j,g}(\varphi_{i,g}(u, v \cdot Uj), v).$$

We can define an additional double functor of concrete double categories over $\mathcal{C}$, $\delta_{\mathbb{J}} : \mathcal{J}_1^{\pitchfork} \to \mathcal{J}_2^{\pitchfork}$ which sends a $(\mathcal{J}_1, g)$-lifting operation φ_{-g} to the $(\mathcal{J}_2, g)$-lifting operation θ_{-g} as defined on the right above. Now note that the inclusion $\mathbb{J}^{\pitchfork} \to \mathcal{J}_1^{\pitchfork}$ is the equaliser of this parallel pair of morphisms, since this is exactly the subcategory of $\mathcal{J}_1^{\pitchfork}$ of lifting structures also satisfying the vertical condition, i.e. $\psi_{(j,i),g}(u, v) = \theta_{(j,i),g}(u, v)$ for all j, i, g, u, v. Thus, we have an equaliser diagram of concrete double categories over $\mathcal{C}$ as shown below

$$\begin{tikzcd}
\mathbb{J}^{\pitchfork} \arrow[r] \arrow[dr] & \mathcal{J}_1^{\pitchfork} \arrow[r, shift left, "m^{\pitchfork}"] \arrow[r, shift right, "\delta_{\mathbb{J}}"'] \arrow[d] & \mathcal{J}_2^{\pitchfork} \arrow[dl] \\
& \mathbb{S}\mathbf{q}(\mathcal{C}). &
\end{tikzcd}$$

Since $\mathcal{J}_1^{\pitchfork}$ and $\mathcal{J}_2^{\pitchfork}$ are both right-connected, $\mathbb{J}^{\pitchfork}$ will also be right-connected. Thus by Theorem 1 the AWFS cofibrantly generated by $\mathbb{J}$ exists precisely when $(\mathbb{J}^{\pitchfork})_1 \to \mathcal{C}^{\mathbf{2}}$ is strictly monadic. Since $\mathcal{J}_1, \mathcal{J}_2$ are small and since Uj is ω-compact for every object $j \in \mathcal{J}_0$ (in particular the domain and codomain of the image of every arrow in $\mathcal{J}_1$ and $\mathcal{J}_2$ is ω-compact), by Theorem 5 there exist AWFS $(\mathsf{L}_1, \mathsf{R}_1)$ and $(\mathsf{L}_2, \mathsf{R}_2)$ such that $\mathcal{J}_1^{\pitchfork} \cong \mathsf{R}_1\text{-}\mathbf{Alg}$ and $\mathcal{J}_2^{\pitchfork} \cong \mathsf{R}_2\text{-}\mathbf{Alg}$, where R_1 and R_2 are finitary monads which are free on a pointed endofunctor. Thus the maps $m^{\pitchfork}$ and $(\delta_{\mathbb{J}})_1$ induce via the isomorphisms two maps $s^*, t^* : \mathsf{R}_1\text{-}\mathbf{Alg} \to \mathsf{R}_2\text{-}\mathbf{Alg}$. In Section 5.1 we noted that the assignment

$$\mathbf{Mnd}(\mathcal{C})^{\mathrm{op}} \longrightarrow \mathbf{Cat}/\mathcal{C} : \mathsf{H} \longmapsto \mathsf{H}\text{-}\mathbf{Alg}$$

is fully faithful. It follows that these maps are in fact induced by unique monad maps $s, t : \mathsf{R}_2 \to \mathsf{R}_1$. By Theorem 4 we can take the coequaliser (R, q) of s and t, and by Corollary 3 it follows that R is also finitary. It also follows from Theorem 4 that q^* is the equaliser of s^* and t^*, and hence we have a commutative diagram as below

$$\begin{tikzcd}
(\mathbb{J}^{\pitchfork})_1 \arrow[r] \arrow[d, "\cong"'] & \mathcal{J}_1^{\pitchfork} \arrow[r, shift left, "m^{\pitchfork}"] \arrow[r, shift right, "(\delta_{\mathbb{J}})_1"'] \arrow[d, "\cong"'] & \mathcal{J}_2^{\pitchfork} \arrow[d, "\cong"] \\
\mathsf{R}\text{-}\mathbf{Alg} \arrow[r, "q^*"'] & \mathsf{R}_1\text{-}\mathbf{Alg} \arrow[r, shift left, "s^*"] \arrow[r, shift right, "t^*"'] & \mathsf{R}_2\text{-}\mathbf{Alg}.
\end{tikzcd}$$

So we have $(\mathbb{J}^{\pitchfork})_1 \cong \mathsf{R}\text{-}\mathbf{Alg}$ over $\mathcal{C}^{\mathbf{2}}$, and we conclude that $(\mathbb{J}^{\pitchfork})_1 \to \mathcal{C}^{\mathbf{2}}$ is strictly monadic for a finitary monad R. Moreover, by Lemma 1 it follows that the comonad L of the generated AWFS is also finitary, which concludes the proof. $\square$

7.3. Presheaf Categories

To obtain a corollary about presheaf categories, we need the following lemma.

Lemma 24. *Let $\mathbb{C}$ be a small category and consider its category of presheaves $\widehat{\mathbb{C}}$. If X is a finitely generated presheaf, then X is ω-compact.*

Proof. We have to show that the functor $\hom_{\widehat{\mathbb{C}}}(X, -) : \widehat{\mathbb{C}} \to \mathbf{Sets}$ preserves ω-filtered colimits. Let $\mathcal{J}$ be a filtered category and $Y : \mathcal{J} \to \widehat{\mathbb{C}}$ a diagram of presheaves with colimit $\operatorname{colim}_{j\in\mathcal{J}} Y(j)$. Since X is finitely generated we can write $X \cong \operatorname{colim}_{i\in I_0} yC_i$ with I_0 finite.[1] We calculate:

$$\begin{aligned}
\hom_{\widehat{\mathbb{C}}}(X, \operatorname{colim}_{j\in\mathcal{J}} Y(j)) &\cong \hom_{\widehat{\mathbb{C}}}(\operatorname{colim}_{i\in I_0} yC_i, \operatorname{colim}_{j\in\mathcal{J}} Y(j)) \\
&\cong \lim_{i\in I_0} \hom_{\widehat{\mathbb{C}}}(yC_i, \operatorname{colim}_{j\in\mathcal{J}} Y(j)) \\
&\cong \lim_{i\in I_0} (\operatorname{colim}_{j\in\mathcal{J}} Y(j))(C_i) \\
&\cong \lim_{i\in I_0} \operatorname{colim}_{j\in\mathcal{J}} Y(j)(C_i) \\
&\cong \operatorname{colim}_{j\in\mathcal{J}} \lim_{i\in I_0} Y(j)(C_i) \\
&\cong \operatorname{colim}_{j\in\mathcal{J}} \lim_{i\in I_0} \hom_{\widehat{\mathbb{C}}}(yC_i, Y(j)) \\
&\cong \operatorname{colim}_{j\in\mathcal{J}} \hom_{\widehat{\mathbb{C}}}(\operatorname{colim}_{i\in I_0} yC_i, Y(j)) \\
&\cong \operatorname{colim}_{j\in\mathcal{J}} \hom_{\widehat{\mathbb{C}}}(X, Y(j)).
\end{aligned}$$

Here (2) and (7) follow from the universal property of the colimit, (3) and (6) follow from the Yoneda lemma, (4) follows from the fact that colimits of presheaves are computed pointwise, and (5) follows from the fact that in **Sets**, filtered colimits commute with finite limits (see e.g. [17, §IX.2 Thm. 1]). The composition gives the desired isomorphism

$$\hom_{\widehat{\mathbb{C}}}(X, \operatorname{colim}_{j\in\mathcal{J}} Y(j)) \cong \operatorname{colim}_{j\in\mathcal{J}} \hom_{\widehat{\mathbb{C}}}(X, Y(j)).$$

We leave it to the reader to verify that $\hom_{\widehat{\mathbb{C}}}(X, -) : \widehat{\mathbb{C}} \to \mathbf{Sets}$ also preserves the colimit inclusions of the colimit of Y. □

Corollary 4. *Let $\mathbb{C}$ be a small category and let $U : \mathbb{J} \to \mathbb{S}\mathbf{q}(\widehat{\mathbb{C}})$ be a double functor subject to the following conditions*

1. *$\mathbb{J}$ is small,*
2. *Uj is finitely generated for every object $j \in \mathcal{J}_0$.*

Then the AWFS cofibrantly generated by U exists and is finitary.

Proof. Note that $\widehat{\mathbb{C}}$ is locally small since $\mathbb{C}$ is small. Moreover, a presheaf category is always cocomplete. By Lemma 24 every Uj is ω-compact. Thus we can apply Theorem 6 to conclude that the AWFS cofibrantly generated by U exists and is finitary. □

[1] We denote $y : \mathbb{C} \hookrightarrow \widehat{\mathbb{C}}$ for the Yoneda embedding and yC for the representable functor $\hom_{\mathbb{C}}(-, C) : \mathbb{C}^{\mathrm{op}} \to \mathbf{Sets}$.

8. Conclusion

In [7] Bourke and Garner prove the most general version of the small object argument thus far, a small object argument for double categories. Moreover, they prove a converse result which implies that this version is sufficient to generate all interesting AWFS. However, their proof relies several times on the work of Kelly [15] for the existence of certain free structures, which are obtained using transfinite constructions that are not constructively valid. In this thesis we proved a constructive version of Bourke and Garner's small object argument. In order to do so, we restricted our attention to the finitary case. We introduced the notion of a finitary AWFS, and we identified conditions under which a small double category generates a finitary AWFS. This resulted in a finitary version of the small object argument, as stated in Theorem 6, for which we gave a constructive proof in this thesis. This theorem is sufficient to solve an open problem left in [3]: a constructive proof of the fact that the effective Kan fibrations are the right class of an AWFS. The proof of Theorem 6 has proceeded in several steps, below we give a brief overview of the steps we took in the proof.

Step 1. Proving the *Beck theorem* for AWFS (Theorem 1, Chapter 3). This theorem characterises the essential image of the semantics functor $(-)\text{-}\mathbb{A}\mathbf{lg} : \mathbf{AWFS}_{\text{lax}} \to \mathbf{DBL}^{\mathbf{2}}$, which sends an AWFS to its right class. Thus proving that a small double category $\mathbb{J} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ generates an AWFS reduces to verifying that $\mathbb{J}^{\pitchfork} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ satisfies the two conditions of the Beck theorem.

Step 2. Constructing the free algebra for a finitary pointed endofunctor (Theorem 2, Chapter 4). To this end, we used the notion of *algebraic chains*, originally introduced by Koubek and Reiterman [16]. This approach emphasises the explicit formulae involved by focusing not only on the free algebra but also on the free algebraic chain. We proved that for finitary functors, the induced algebraic chain stabilises at stage ω, leading to the free algebra $TX_\omega \to X_{\omega+1} \cong X_\omega$.

Step 3. Constructing colimits in the category of finitary monads on a cocomplete category (Theorem 4, Chapter 5). In Step 2 we constructed the free algebra on a finitary pointed endofunctor, which induces a left adjoint to the forgetful functor $(P, \rho)\text{-}\mathbf{Alg} \to \mathcal{C}$. We proved that the induced monad of this adjunction is in fact the free monad over the pointed endofunctor (P, ρ) and is finitary if P is finitary (Theorem 3). Subsequently we constructed colimits in the category of finitary monads on a cocomplete category, which we need in Step 5. From our construction it follows that the algebra functor (assigning to a monad its category of algebras) preserves limits, another fact that we need in Step 5.

Step 4. Proving the finitary small object argument for categories (Theorem 5, Chapter 6). In the proof we used the Beck theorem of Chapter 3. The major constructive issue in the original proof by Bourke and Garner is the existence of the free algebra on a pointed endofunctor. We relied on Chapter 4 for a constructive proof of its existence for a finitary pointed endofunctor. We used a result of Chapter 5 for a constructive proof that the induced AWFS is finitary, and that its monad is free on a pointed endofunctor.

Step 5. Finally, we constructively proved Theorem 6, the finitary small object argument for double categories (Chapter 7). To show that $\mathbb{J}^{\pitchfork} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ generates an AWFS we verified the conditions of the Beck Theorem. To this end we exhibited $\mathbb{J}^{\pitchfork} \to \mathbb{S}\mathbf{q}(\mathcal{C})$ as an equaliser of $\mathcal{J}_1^{\pitchfork}$ and $\mathcal{J}_2^{\pitchfork}$ for two suitable categories $\mathcal{J}_1$ and $\mathcal{J}_2$ and maps between them. Subsequently we used Step 4 to show that these two categories generate a finitary AWFS. We then used Step 3 to obtain the coequaliser of the induced monad maps, and the fact that this induces an equaliser diagram under the algebra functor. This then shows that $(\mathbb{J}^{\pitchfork})_1 \cong \mathsf{R}\text{-}\mathbf{Alg}$ over $\mathcal{C}^{\mathbf{2}}$, which is sufficient to verify the conditions of the Beck theorem and thus completes the proof.

Based on the work in this thesis we have identified some directions for future research, which are listed below.

- We have given a constructive proof of the small object argument for double categories. A next step could be to formalise the proof in a theorem prover such as Coq.
- In our main theorem we imposed certain restrictions on the double category to make sure that we stayed in 'finitary territory'. An interesting question for future research is whether these conditions can be relaxed, while still maintaining a constructive small object argument. One could also investigate if a similar result holds if the base category is locally finitely presentable.
- In [7], Bourke and Garner prove a converse result to the small object argument: every accessible AWFS is generated by a small double category (under a suitable size condition). We could ask ourselves the same question for finitary AWFS: can we find a converse result? Is every finitary AWFS generated by a double category as in Theorem 6, or are these conditions too strict?
- The crucial step in the proof of Theorem 6 was to show that the functor $(\mathbb{J}^{\pitchfork})_1 \to \mathcal{C}^{\mathbf{2}}$ is strictly monadic, in particular that it has a left adjoint. We did so using a rather indirect argument, by using the small object argument for categories. An interesting next step is to give an explicit description of this adjoint.

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A. Category Theory Background

A.1. Monads and Comonads

Definition 7. A monad on a category $\mathcal{C}$ is a triple $\mathsf{T} = (T, \mu, \eta)$ with $T : \mathcal{C} \to \mathcal{C}$ an endofunctor, multiplication $\mu : T^2 \to T$, and unit $\eta : 1_{\mathcal{C}} \to T$, such that the following two diagrams commute

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu_T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta_T"] \arrow[dr, "1_T"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "1_T"] \\ & T & \end{tikzcd}$$

Dually, we have the concept of a comonad

Definition 8. A comonad on a category $\mathcal{C}$ is a triple $\mathsf{L} = (L, \delta, \epsilon)$ with $L : \mathcal{C} \to \mathcal{C}$ an endofunctor, comultiplication $\delta : L \to L^2$, and counit $\epsilon : L \to 1_{\mathcal{C}}$, such that the following two diagrams commute

$$\begin{tikzcd} L \arrow[r, "\delta"] \arrow[d, "\delta"'] & L^2 \arrow[d, "L\delta"] \\ L^2 \arrow[r, "\delta_L"'] & L^3 \end{tikzcd} \qquad \begin{tikzcd} & L \arrow[dl, "1_L"'] \arrow[d, "\delta"] \arrow[dr, "1_L"] & \\ L & L^2 \arrow[l, "\epsilon_L"] \arrow[r, "L\epsilon"'] & L \end{tikzcd}$$

We want to investigate in which ways a monad arises from an adjunction.

Definition 9. Given a monad $\mathsf{T} = (T, \mu, \eta)$, we define the category T-**Alg** of Eilenberg-Moore algebras for the monad T or T-algebras, as follows.

- *Objects* are pairs (X, h) with $X \in \mathcal{C}$ and $h : TX \to X$, satisfying

$$\begin{tikzcd} T^2X \arrow[r, "Th"] \arrow[d, "\mu_X"'] & TX \arrow[d, "h"] \\ TX \arrow[r, "h"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "1_X"'] & TX \arrow[d, "h"] \\ & X \end{tikzcd}$$

- *Morphisms* $(X, h) \to (Y, k)$ are morphisms $f : X \to Y$ in $\mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd} TX \arrow[r, "Tf"] \arrow[d, "h"'] & TY \arrow[d, "k"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

We have a functor $F^T : \mathcal{C} \to \mathsf{T}\text{-}\mathbf{Alg}$ which sends X to the 'free' algebra $(TX, \mu_X : T^2X \to TX)$. Dually, we have the concept of a coalgebra for a comonad.

Definition 10. Given a comonad (L, δ, ϵ), we define the category $\mathsf{L}\text{-}\mathbf{Coalg}$ of coalgebras for the comonad L or L-coalgebras, as follows.

- *Objects* are pairs (X, h) with $X \in \mathcal{C}$ and $h : X \to LX$, satisfying

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "h"'] & LX \arrow[d, "\delta_X"] \\ LX \arrow[r, "Lh"'] & L^2X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "1_X"'] & LX \arrow[d, "\epsilon_X"] \\ & X \end{tikzcd}$$

- *Morphisms* $(X, h) \to (Y, k)$ are morphisms $f : X \to Y$ in $\mathcal{C}$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "h"'] & Y \arrow[d, "k"] \\ LX \arrow[r, "Lf"'] & LY \end{tikzcd}$$

Dually, we have a free functor $F^L : \mathcal{C} \to \mathsf{L}\text{-}\mathbf{Coalg}$ which sends X to the 'free' coalgebra $(LX, \delta_X : LX \to L^2X)$.

Every adjunction induces a monad, as stated in the following proposition.

Proposition 19. *Suppose we have an adjunction*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

Then we have an induced monad $\mathsf{T} = (T, \eta, \mu)$ *on* $\mathcal{C}$*, where* $T = GF$*,* η *is the unit of the adjunction, and* $\mu : T^2 \Rightarrow T$ *is defined by* $\mu_C = G(\epsilon_{FC})$ *for every* C *in* $\mathcal{C}$ *(with* ϵ *the counit of the adjunction).*

A.2. Beck's Monadicity Theorem

In the previous section we have seen the concept of monads and monad algebras. The assigment that sends a monad T to its category of algebras $\mathsf{T}\text{-}\mathbf{Alg}$ over $\mathcal{C}$ is in fact part of a functor $\mathbf{Mnd}(\mathcal{C})^{\mathrm{op}} \to \mathbf{Cat}/\mathcal{C}$. Beck's monadicity theorem characterises the essential image of this functor. In this section we collect some definitions and results that are related to Beck's monadicity theorem.

Definition 11. A functor $F : \mathcal{C} \to \mathcal{D}$ *creates limits of type* $\mathcal{I}$ if for every functor $M : \mathcal{I} \to \mathcal{C}$ and limiting cone (D, σ) for FM in $\mathcal{D}$ there is a unique cone (C, τ) for M in $\mathcal{C}$ which is taken by F to (D, σ) and which is limiting for M in $\mathcal{D}$. We say that F *creates limits* if F creates limits of every small type $\mathcal{I}$.

Proposition 20. *Let* T *be a monad. The forgetful functor* $U^T : \mathsf{T}\text{-}\mathbf{Alg} \to \mathcal{C}$ *creates limits of every type. Moreover it creates every colimit which is preserved by* T.

Definition 12. Suppose we have an adjunction

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

Let T be the induced monad on $\mathcal{C}$, where $T = GF$. The comparison functor $K : \mathcal{D} \to \mathsf{T}\text{-}\mathbf{Alg}$ is defined by sending an object $D \in \mathcal{D}$ to the T-algebra $GFG(D) \xrightarrow{G(\epsilon_D)} G(D)$, and an arrow $f : D \to D'$ to Gf. The functor G is called *(strictly) monadic* if K is an equivalence (isomorphism).

Proposition 21. *Let* $G : \mathcal{D} \to \mathcal{C}$ *be monadic, then* G *creates limits.*

Definition 13. Let $\mathcal{C}$ be a category

1. A *fork* is a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "e"] & C. \end{tikzcd}$$

2. A *split coequaliser* is a fork together with morphisms $s : C \to B$ and $t : B \to A$

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[l, bend right, "t"'] \arrow[r, "e"] & C. \arrow[l, bend right, "s"'] \end{tikzcd}$$

with s and t section of e and f respectively, and with $gt = se$. This is equivalent to the statement that $(f, e) : g \to e$ has a section in $\mathcal{C}^{\mathbf{2}}$.

3. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. A parallel pair $f, g : A \to B$ in $\mathcal{C}$ is called G*-split* if the pair Gf, Gg has a split coequaliser in $\mathcal{C}$.

4. The functor $G : \mathcal{D} \to \mathcal{C}$ is said to *create coequalisers of* G*-split pairs* if for any such G-split pair, there exists a coequaliser e of f and g in $\mathcal{D}$ which is preserved by G, and moreover any fork in $\mathcal{D}$ whose image in $\mathcal{C}$ is a split coequaliser must itself be a coequaliser.

Theorem 7 (Beck's monadicity theorem, [17, Section VI.7])**.** *A functor* $G : \mathcal{D} \to \mathcal{C}$ *is monadic if and only if*

(i) G *has a left adjoint, and*

(ii) G *creates coequalisers of* G*-split pairs.*

A.3. Kan Extensions

The Kan extension is a universal construction to extend one functor along another functor. It was famously referred to by MacLane as "the notion that subsumes all the other fundamental concepts of category theory" [17]. Indeed, concepts such as limits, adjoints, and the Yoneda lemma can be shown to be a specific instance of a Kan extension.

Definition 14. Let $F : \mathcal{C} \to \mathcal{E}$ and $K : \mathcal{C} \to \mathcal{D}$ be functors. A *left Kan extension* of F along K is a functor $\mathrm{Lan}_K F : \mathcal{D} \to \mathcal{E}$ equipped with a natural transformation $\eta : F \Rightarrow \mathrm{Lan}_K F \circ K$ such that any other pair $(G : \mathcal{D} \to \mathcal{E}, \gamma : F \Rightarrow GK)$, γ factors uniquely through η.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{E} \\
\mathcal{D} \arrow[ur, dashed, "\mathrm{Lan}_K F"'] &
\end{tikzcd}$$

That is, there exists a unique natural transformation $\alpha : \mathrm{Lan}_K F \Rightarrow G$ such that

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{E} \arrow[d, "1"] \arrow[dl, phantom, "\Swarrow\eta"] \\
\mathcal{D} \arrow[r, "\mathrm{Lan}_K F"] \arrow[d, "1"'] & \mathcal{E} \arrow[d, "1"] \arrow[dl, phantom, "\Swarrow\exists!\alpha"] \\
\mathcal{D} \arrow[r, "G"'] & \mathcal{E}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "K"'] & \mathcal{E} \arrow[d, "1"] \arrow[dl, phantom, "\Swarrow\gamma"] \\
\mathcal{D} \arrow[r, "G"'] & \mathcal{E}
\end{tikzcd}$$

Dually, a *right Kan extension* of F along K is a functor $\mathrm{Ran}_K : \mathcal{D} \to \mathcal{E}$ together with a natural transformation $\epsilon : \mathrm{Ran}_K F \circ K \Rightarrow F$ such that for any such pair $(H : \mathcal{D} \to \mathcal{E}, \delta : HK \Rightarrow F)$ there exists a unique factorisation $\beta : H \Rightarrow \mathrm{Ran}_K F$ of δ through ϵ.

The left and right Kan extension can be characterised as the left and right adjoint of the composition functor, as the following theorem states.

Theorem 8. *Let $K : \mathcal{C} \to \mathcal{D}$ be a functor which induces the composition functor $- \circ K : [\mathcal{D}, \mathcal{E}] \to [\mathcal{C}, \mathcal{E}]$. If for every $F : \mathcal{C} \to \mathcal{E}$ the Kan extension exists, then* $\mathrm{Lan}_K -$ *and* $\mathrm{Ran}_K -$ *are respectively the left and the right adjoint of $- \circ K$.*

$$\begin{tikzcd}
{[\mathcal{D}, \mathcal{E}]} \arrow[r, "- \circ K" description] & {[\mathcal{C}, \mathcal{E}]} \arrow[l, bend right=40, "\mathrm{Lan}_K -"'] \arrow[l, bend left=40, "\mathrm{Ran}_K -"] \arrow[l, phantom, shift right=2.5ex, "\perp"] \arrow[l, phantom, shift left=2.5ex, "\perp"]
\end{tikzcd}$$

The following proposition gives a sufficient condition for the existence of Kan extensions.

Proposition 22 [17, Section X.3, Corollary 2]**.** *If $\mathcal{C}$ is small and $\mathcal{D}$ is complete, then any functor $F : \mathcal{C} \to \mathcal{D}$ has a right Kan extension along any functor $K : \mathcal{C} \to \mathcal{E}$. Dually, if $\mathcal{C}$ is small and $\mathcal{D}$ is cocomplete, then any functor $F : \mathcal{C} \to \mathcal{D}$ has a left Kan extension along any functor $K : \mathcal{C} \to \mathcal{E}$.*

One instance of a Kan extension that occurs often in practise is the Kan extension of a colimit preserving functor along the Yoneda embedding $y : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$. Specifically, we have the following theorem.

Theorem 9. *Let $\mathcal{C}$ be a small category and $y : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ the Yoneda embedding. Given a category $\mathcal{E}$ which is locally small and cocomplete and a functor $F : \mathcal{C} \to \mathcal{E}$, there is a colimit preserving functor $F_! : \widehat{\mathcal{C}} \to \mathcal{E}$ such that $F_! \circ y \cong F$:*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "y"'] & \mathcal{D} \\
\widehat{\mathcal{C}}. \arrow[ur, dashed, "F_!"'] &
\end{tikzcd}$$

Moreover $F_!$ is, up to natural isomorphism, the unique colimit preserving functor with this property. In addition, $F_!$ has a right adjoint $F^* : \widehat{\mathcal{C}} \to \mathcal{D}$ given by $F^*(D)(C) = \hom_{\mathcal{D}}(FC, D)$.