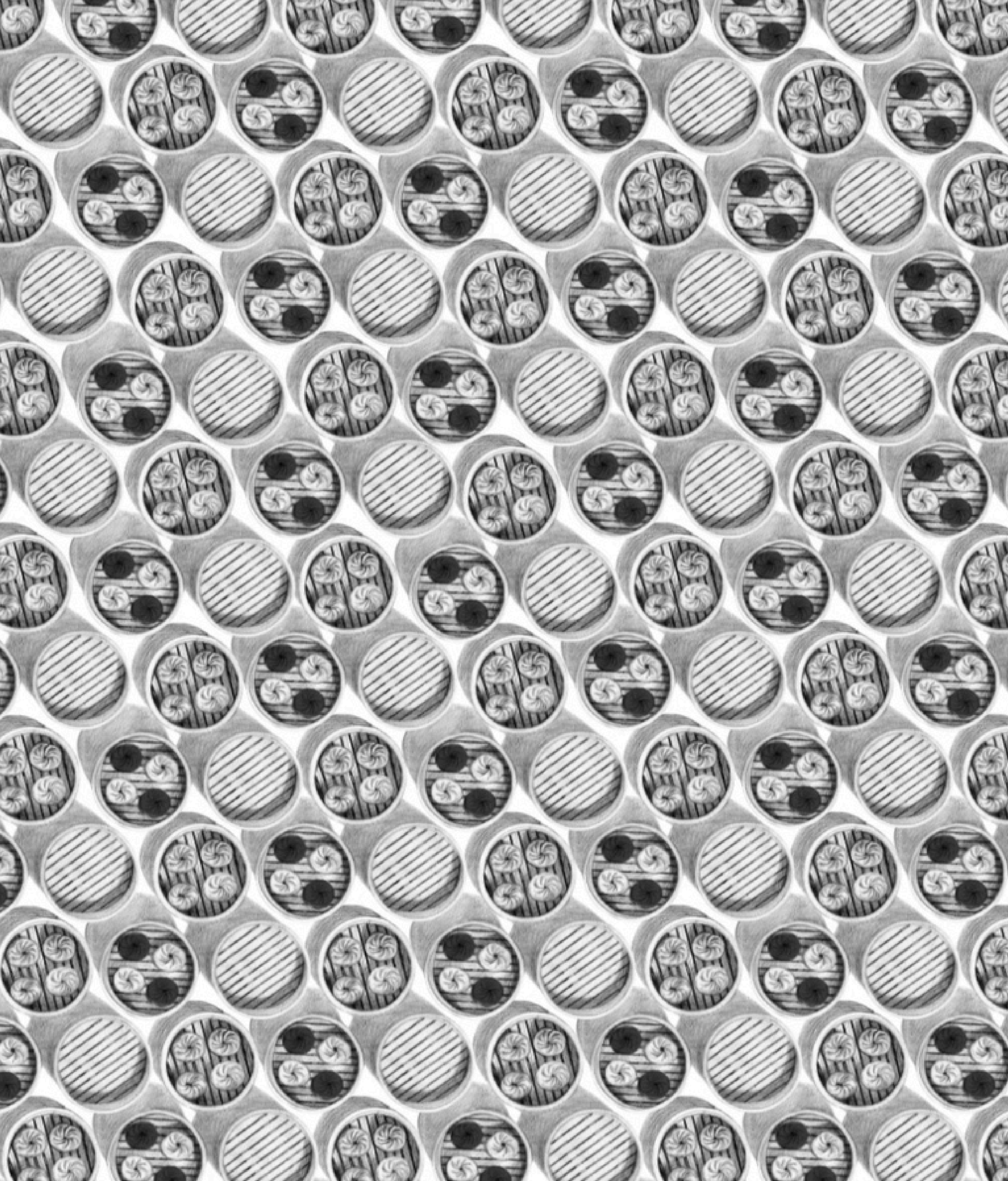

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

Aleksi Anttila

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Not Nothing
Nonemptiness in Team Semantics

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Lumille ja Naolle

Contents

Acknowledgments

First, a few words about how this thesis came into being: although the thesis was finished at the ILLC, I began work on it at the Department of Mathematics and Statistics, University of Helsinki, under the supervision of Fan Yang. Following Fan's departure to Utrecht University, I first followed her on an extended research visit to Utrecht and eventually transferred to the ILLC at the end of 2023, with Robert van Rooij, Maria Aloni, and Fan serving as (co)promotors. For a period of a few months between my Utrecht research visit and my transfer to Amsterdam, I was supervised by Jouko Väänänen and funded by his ERC grant.

I am grateful, most of all, to the *de facto* supervisors of this thesis, Fan Yang and Maria Aloni, for all their support, guidance, and advice. The cover photo and illustration are a tribute to Fan and Maria—they depict verifiers, falsifiers, and zero-models (as in Maria's [6]), with bamboo steamers playing the part of teams, as per Fan's Chinese translation of 'team' as '屉' (one possible translation of 'steamer' being '蒸屉').

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Aleksi Anttila
Amsterdam, February 2025

Chapter Sources and Author Contributions

This thesis is based on the following papers/manuscripts. In all cases, the authors are listed alphabetically and authorship is shared equally. All authors for a given project contributed equally to all aspects of the project unless specified otherwise. Chapters 2 and 5, which are based on published/accepted papers, reproduce those papers with some very minor additions and modifications. The other chapters, which are based on preprints/unpublished manuscripts, may differ substantially from any possible future publications based on their content.

Chapter 2 is based on:

Maria Aloni, Aleksi Anttila, and Fan Yang. "State-Based Modal Logics for Free Choice". In: *Notre Dame J. Formal Logic* (2024), pp. 1–47. URL: `https://doi.org/10.1215/00294527-2024-0027`

This paper, in turn, is partly based on Anttila's M.Sc. thesis [12], jointly supervised by Aloni and Yang. (It should be emphasized here that the paper contains a significant number of results which are not in the M.Sc. thesis, including the specific axiomatization which was the main goal of this project, and the material was completely rewritten.) Most of the writing is due to Anttila. This work has been presented (by Anttila) at Logic Colloquium, 2021; Workshop on Logics of Dependence and Independence, ESSLLI, 2021; and Scandinavian Logic Symposium, 2022.

Chapter 3 is based on:

Aleksi Anttila and Søren Brinck Knudstorp. *Convex Propositional and Modal Team Logics*. Manuscript. 2024

Knudstorp first showed that the logics $\mathrm{PL}_{\vee}(\mathrm{NE})$ and BSML are convex and came up with the expressive completeness proofs for these union-closed convex logics included in Section 3.3; Anttila then developed the convex logics in Section 3.2 and showed their expressive completeness by building on the ideas in Knudstorp's proof. The authors jointly developed and polished this material, and added the material on uniform definability and uniform extensions in Section 3.4.

Chapter 4 is based on:

Aleksi Anttila. *Further remarks on the dual negation in team logics*. 2024. arXiv: `2410.07067 [math.LO]`. URL: `https://arxiv.org/abs/2410.07067`

This work has been presented at Dutch Logic PhD Day, 2023; and the 4th Tsinghua Interdisciplinary Workshop on Logic, Language, and Meaning, 2024

Chapter 5 is based on:

Aleksi Anttila, Matilda Häggblom, and Fan Yang. "Axiomatizing modal inclusion logic and its variants". In: *Arch. Math. Logic* (2024). Forthcoming. arXiv: `2312.02285 [math.LO]`. URL: `https://arxiv.org/abs/2312.02285`

This paper, in turn, is partly based on Häggblom's M.Sc. thesis [63], jointly supervised by Yang and Anttila. (It should be emphasized here that although the paper is based on Häggblom's M.Sc. thesis, the authors contributed equally to this project. As with Chapter 2, the paper contains a significant number of results which are not in the M.Sc. thesis, and the material was completely rewritten.) This work has been presented (by Häggblom) at Logic Colloquium, 2023.

Chapter 6 is based on:

Aleksi Anttila, Rosalie Iemhoff, and Fan Yang. *Deep-inference Sequent Calculi for Propositional Team Logics*. Manuscript. 2024

The main idea for the system and most of the writing is due to Anttila. This work has been presented (by Anttila) at PhDs in Logic, 2023.

The samenvatting was translated from the abstract using ChatGPT, with corrections by Robin Martinot and Robert van Rooij.

Chapter 1

Overview

This dissertation collects together five papers which focus primarily on the expressive power, axiomatizations, and proof theory of propositional and modal logics with *team semantics*.

In team semantics, formulas are interpreted with respect to sets of evaluation points called *teams* rather than single evaluation points as in standard Tarskian semantics. The basic idea can be adapted to multiple settings: teams can be for instance, sets of first-order assignments, sets of propositional valuations, or sets of possible worlds. Team semantics was introduced by Hodges [78, 79] to provide a compositional semantics for Hintikka and Sandu's *independence-friendly logic* (IF) [77, 75]. It was later further developed by Väänänen in his work on *dependence logic* (D) [119]; this gave rise to a research programme focused in particular on so-called *logics of dependency*—logics with team semantics such as dependence logic, *inclusion logic* [51, 54], and *independence logic* [59] furnished with *atoms of dependency* expressing that certain relationships hold between the values of the elements in a team. Independently, Ciardelli, Groenendijk, and Roelofsen (building in particular on ideas in dynamic semantics [68, 61, 117, 123]) developed *inquisitive logic* [60, 41, 38, 35] which also essentially employs team semantics (see [133, 34]).

Teams can have different interpretations—they can represent, for instance, databases, data sets, and information states; accordingly, team semantics and team-based logics have found application in many different fields. The atoms of dependency mentioned above, for instance, correspond to dependency notions studied in relational database theory, and axiomatizations and implication problems involving these notions can be studied in the setting of team logics [65, 66]. Other areas these logics have been applied to include formal semantics and philosophical logic [38, 33, 67, 101, 7]), quantum information theory [83, 4, 1], and social choice theory [102].

One of the primary motivations for the development and study of logics with team semantics has been the capacity of teams to represent, in a perspicuous and simple manner, phenomena that would be difficult or impossible to model (at least in an equally satisfactory way) in a setting based on standard singleton evaluation points. By way of example, let us consider *propositional dependence logic* [139]. This logic extends classical propositional logic with *dependence atoms* $=(p,q)$ expressing that the truth value of q is *functionally dependent* on that of p. Consider the following table, which represents a propositional team $t = \{v_1, v_2, v_3\}$—a set of valuations v_1, v_2, v_3 over a finite set $\{p,q,r\}$ of propositional variables:

	p	q	r
v_1	1	0	1
v_2	1	0	0
v_3	0	1	0

The atom $=(p,q)$ is true in this team because the value of q is functionally dependent on that of p in that there is a function f from the set $t[p] = \{v(p) \mid v \in t\}$ of truth values of p in the team to the set of truth values of q such that $f(v(p)) = v(q)$ for all $v \in t$: for any valuation v with $v(p) = 1$, we have $v(q) = 0$, and for any valuation v with $v(p) = 0$, we have $v(q) = 1$. On the other hand, the atom $=(p,r)$ is not true, because if $v(p) = 1$, $v(r)$ might be 1 or 0.

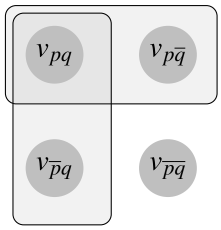
A dependence atom, then, expresses directly that such a functional dependence fact holds at a (team-semantic) evaluation point, and this is only meaningful due to the fact that the evaluations points are sets rather than singletons: a single valuation assigns a single value to each propositional variable, so functional dependence can never fail at a single valuation.

In other words, the semantics of the dependence atom leverages the nature of the evaluation points as sets to directly express a phenomenon—functional dependence—that can be naturally represented using sets. Many other team-semantic atoms and connectives function in a similar way: their truth conditions at an evaluation point (a team) make use not only of the values assigned to variables at that point, but also the second dimension provided by the nature of the evaluation point as a set—a multitude—and the structure of the set itself (its cardinality, etc.), as well as the interactions between these dimensions (as we saw in the dependence atom example: $=(p,q)$ is true in a team just in case any pair of elements in the team which agree on the truth value of p also agree on the truth value of q).

Let us consider another example. *Propositional inquisitive logic* [41, 38, 35] employs the *global* or *inquisitive disjunction* $\mathbin{\rotatebox[origin=c]{90}{$\geqslant$}}$ (see also [2]) to formalize questions: $p \mathbin{\rotatebox[origin=c]{90}{$\geqslant$}} \neg p$, for instance, represents the question as to whether or not p is the case. This disjunction has the standard disjunction semantics, defined with respect to teams:

$$t \vDash \varphi \mathbin{\rotatebox[origin=c]{90}{$\geqslant$}} \psi \iff t \vDash \varphi \text{ or } t \vDash \psi.$$

Valuations represent possible states of affairs or possible worlds in the familiar way, whereas teams—sets of valuations—represent *information states*: if, for instance, it is raining according to all worlds in a team/information state, then according to the information embodied in this state, it is raining. A propositional variable p is satisfied by a team just in case it is satisfied by all valuations in the team; its satisfaction by a team represents the fact that the sentence corresponding to p is established in the corresponding information state as described above. (A negated variable $\neg p$, similarly, is satisfied by a team just in case it is satisfied by all valuations in the team in the usual way.) On the other hand, if a question such as $p \mathbin{\vee\!\!\!\vee} \neg p$ is satisfied by a team, this represents the fact that the question can be truthfully resolved given the information in the state—the information in the state implies that one of the possible answers to the question is true. The formula $p \mathbin{\vee\!\!\!\vee} \neg p$ is satisfied by a team just in case p is satisfied by the team, or $\neg p$ is—the question as to whether p can be truthfully resolved in an information state just in case either the information state establishes that p, or it establishes that not p. Consider the following figure, where the nodes are valuations (labelled according to what they make true/false: $v_{p\bar{q}}(p) = 1$ and $v_{p\bar{q}}(q) = 0$, etc.), and the encircled areas are teams:



Here the team $\{v_{pq}, v_{p\bar{q}}\}$ satisfies p because all valuations in the team evaluate p as true. The corresponding information state therefore establishes that p, and hence the question $p \mathbin{\vee\!\!\!\vee} \neg p$ can be resolved in the affirmative in this state. The team $\{v_{pq}, v_{\bar{p}q}\}$ does not satisfy p (because it contains the valuation $v_{\bar{p}q}$, with $v_{\bar{p}q}(p) = 0$), and it does not satisfy $\neg p$ (because it contains the valuation v_{pq}, with $v_{pq}(\neg p) = 0$), and hence there is not enough information in the corresponding state to resolve this question.

The information state-interpretation leverages the nature of the evaluation points as sets in that sets can incorporate multiple possibilities, thus representing uncertainty or lack of information. A single possible world assigns a single definite truth value to each propositional variable, which represents every fact being settled in such a world: for each propositional variable, either $v(p) = 1$, or $v(p) = 0$ (whence $v(\neg p) = 1$). As we saw above, a team need not satisfy one of p and $\neg p$ because it can contain possibilities instantiating both options—this represents uncertainty concerning p.

The interpretation of $\mathbin{\vee\!\!\!\vee}$ builds on the information-state interpretation and exploits the sethood of teams in a further way: notice that given the standard Tarskian semantics, there is only one way for a formula α to be satisfied in that there is a single maximal information state $\{v \mid v \vDash \alpha\}$ that establishes α. All further refinements of this state—all subsets of the set—also establish α, and hence whatever information they add is superfluous to the truth value of α. For a question such as $p \mathbin{\vee\!\!\!\vee} \neg p$, on the other hand, there is no such single maximal information state—a team satisfies $p \mathbin{\vee\!\!\!\vee} \neg p$ just in case it is either a subset of $\{v \mid v \vDash p\}$, or a subset of $\{v \mid v \vDash \neg p\}$. The two ways in which the formula can be satisfied represent the two distinct ways the question can be resolved—the two possible answers to the question.

Now, one basic structural feature of sets is emptiness/nonemptiness: a team, unlike a valuation, can be empty. Four of the five papers in this dissertation focus mainly on logics leveraging this particular feature of teams in some way, so it functions as a loose guiding theme for this dissertation and overview. The two most prominent types of team-based logic introduced above which inspired much of the subsequent work in team semantics—dependence logic and inquisitive logic—do not exploit this feature in any significant way (in a sense we explain directly below). There are, however, a large number of logics that do make significant use of emptiness/nonemptiness, and they come with their own conceptual and technical motivations for embracing the use of this feature.

Before discussing logics which do make use of emptiness/nonemptiness, let us explain the sense in which dependence logic and inquisitive logic do not make use of this particular structural feature of sets. This is reflected in the *team-semantic closure properties* satisfied by the formulas of these logics (or, simply, by these logics). The closure properties describe how formulas of team-based logics interact with the structural properties of teams, and they play an important conceptual and technical role in the study of these logics—as we discuss presently, many of the results in this dissertation concern expressive completeness results with respect to classes of team properties satisfying certain closure properties. The closure properties of dependence logic and inquisitive logic relevant to our present purposes are the *empty team property*—each formula of either of these logics is true in the empty team—and the *downward closure property*: if a formula of one of these logics is true in a team t, it is also true in all subteams of t.

The empty team property directly reflects the fact that the emptiness/nonemptiness of the evaluation team is not exploited in the semantics of these logics in a way that would allow formulas to express that the evaluation team is nonempty. More simply, nonemptiness cannot be expressed in the languages of these logics because all their formulas are true in the empty team. Due to this property, the empty team, from the point of view of these logics, is simply a state of absurdity in which even contradictions are true.[1] One of the reasons these logics abide by the empty team property has to do with the conceptual difficulties associated with apprehending the empty team as anything but an absurdity state that makes everything true. Consider, for instance, the following passage from Rönnholm [109, p. 192]:

> By interpreting teams as epistemic sets of uncertainty, the empty team does not make much sense. This is because (at least) the “real” state of affairs should always be included in the set of all possibilities. Moreover,

[1] It is interesting to note that $\bot$, the syntactic representation of contradiction, while true in no evaluation point whatsoever in classical logics with standard Tarskian semantics, has the following truth conditions in these logics, and indeed in most team-based logics: $\bot$ is true in t just in case $t = \varnothing$.

by interpreting teams as databases, the empty team would mean *not having a database at all...*

While the empty team property can be seen as reflecting the fact that the emptiness/nonemptiness of the evaluation team itself is not exploited in the semantics of these logics, downward closure, on the other hand, reflects the fact that the truth conditions of these logics do not include conditions to the effect that in order for a formula to be true in the evaluation team, the team must contain elements (or non-empty subsets) satisfying certain properties. This type of condition can also be seen as a condition involving emptiness/nonemptiness, albeit in a less direct way.

Let us consider an example. Many of the papers in this dissertation are concerned with operators which can be used to represent *epistemic modals* such as the 'might' in 'It might be raining'. One such operator is the operator $\triangledown$ (introduced in [73], and studied in Chapter 5), which has the following truth conditions:

$$t \vDash \triangledown\varphi \iff [t = \varnothing \text{ or } \exists s \subseteq t : s \neq \varnothing \text{ and } s \vDash \varphi].$$

That is, $\triangledown\varphi$ is true in a team just in case the team is empty, or it contains a nonempty subteam that satisfies φ. How does this represent an epistemic modal? Recalling the information state-interpretation of teams, the idea is that if, for instance, there are some worlds in an information state according to which it is raining—or, equivalently, there is a nonempty substate of the state consisting of such worlds—then, for all that one knows given the information embodied in the state, it might be raining.[2] It is easy to see that while $\triangledown\varphi$ always has the empty team property, we have, for instance, that $\triangledown p$ is not downward closed (where, as above, p is true in a team t just in case it is true in all $v \in t$). Therefore, operators such as $\triangledown$ are not expressible in dependence logic or in inquisitive logic, which are downward closed.

In many frameworks (see, e.g., [35, 123, 25, 67]), downward closure is called *persistency*. This has to do with the fact that, on the information state-interpretation, downward closure corresponds to the following property of the information or content communicated by formulas: if one is in an information state that establishes some specific piece of information, then the process of becoming more informed by ruling out more possibilities (i.e., moving to a substate of the initial information state) does not cause one to lose the piece of information that had already been established. The content communicated by formulas which are not downward closed is not persistent in this way—we discuss this briefly in Chapter 4.

We now move on to logics which do make use (in the sense characterized above) of the emptiness/nonemptiness feature of teams. As expected, these logics break the empty team property or downward closure, or both. We mention one important family of logics from the literature before discussing the logics we focus on in this dissertation.

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[2]It seems that the first account of epistemic modals along these lines was in Veltman's dynamic framework in [123]. See also [117, 129, 25, 73, 107, 140, 67]. We consider multiple different 'might'-operators in Chapters 3, 4, and 5.

Consider the *Boolean* or *contradictory negation* $\sim$, with the semantics

$$t \vDash \sim \varphi \iff t \nvDash \varphi.$$

That is, the Boolean negation of φ is true in a team just in case φ is not true—this negation is the classical negation with respect to teams. Clearly, for instance, $\sim p$ neither has the empty team property nor is downward closed. The Boolean negation was essentially considered already by Hintikka, who developed an extension of independence-friendly logic (IF) with this negation in order to compensate for some of the apparent shortcomings of the *dual negation* of IF [75] (discussed in Chapter 4). Väänänen, similarly, introduced *team logic*[3], an extension of first-order dependence logic (D) with $\sim$. This connective is much more powerful than the others we consider here (clearly it can capture much more than only emptiness/nonemptiness phenomena); for instance, while D is expressively equivalent to existential second-order order logic over sentences [48, 124, 119], team logic is equivalent to full second-order logic [91]. For more on logics featuring this negation, see [96, 97].

Two of the logics of dependency mentioned above, inclusion logic and independence logic, have the empty team property but are not downward closed. Their semantics include conditions of the kind described above that mandate the existence of elements satisfying certain properties in teams. For instance, in Chapter 5 we study *extended modal inclusion logic* (introduced in [73]). This logic extends classical modal logic with *inclusion atoms* $\alpha \subseteq \beta$ (where α and β are classical formulas, interpretable on single worlds), where

$$t \vDash \alpha \subseteq \beta \iff [\forall w \in t : \exists v \in t : w \vDash \alpha \iff v \vDash \beta].$$

That is, $\alpha \subseteq \beta$ is true in a team t just in case for all worlds w in t, there exists a world v in t that gives the same truth value to β as w does to α. These atoms import the notion of *inclusion dependencies* from database theory into team semantics (just as dependence atoms may be seen as importing the notion of functional dependencies). While the contribution of nonemptiness in the semantics of these atoms is less direct, we will see in Chapter 5 that this logic is equivalent in expressive power to an extension of classical modal logic with the operator $\triangledown$ discussed above (indeed, observe that for classical α, $\triangledown\alpha$ is definable using these atoms: $\triangledown\alpha \equiv \top \subseteq \alpha$.)

Finally, and most crucially for our purposes, we have the *nonemptiness atom* NE (introduced in [121, 140])—which this dissertation is named after—with the semantics:

$$t \vDash \mathrm{NE} \iff t \neq \varnothing.$$

That is, NE is true in a team just in case the team is nonempty. This atom, then, isolates the emptiness/nonemptiness feature of teams, and expresses nothing but nonemptiness

[3]The term 'team logic' is ambiguous. In this dissertation, we generally use it to refer to any logic primarily intended to be interpreted using team semantics.

facts. Given that, as discussed above, it is not immediately clear from the interpretations of team semantics what it would mean for the empty team to fail to make something true, not much use has been made in the literature of this atom.

Recently, however, Aloni [6] introduced *bilateral state-based modal logic*, an extension of classical modal logic (interpreted on teams) with NE. Aloni employs BSML to account for *free choice* inferences (see, e.g., [128, 87]) and related linguistic phenomena. In these inferences, conjunctive meanings are unexpectedly derived from disjunctive sentences; for instance:

You may go to the beach or to the cinema.
$\rightsquigarrow$ You may go to the beach and you may go to the cinema.

That is, upon hearing the first of these sentences, people are inclined to infer the second. On Aloni's account, speakers, when interpreting language, create mental structures that represent reality; and inferences such as the above are due to a tendency in human cognition to disregard empty structures in this process of interpretation (the *neglect-zero tendency*). This tendency is modelled using a *pragmatic enrichment function* which, given a formula α of classical modal logic, recursively appends '$\wedge$NE' to each subformula of α. We discuss this account in more detail in Chapter 2.

The bulk of this dissertation is dedicated to the study of the logic and expressive import of nonemptiness, carried out via an investigation of the mathematical properties of BSML and related logics. Most of the results are technical[4]; they concern, as mentioned above, the expressive power, axiomatizations, and proof theory of these logics.

An important role is played by *expressive completeness theorems* with respect to the team-semantic closure properties (such as downward closure). To give an example of such a theorem, propositional dependence logic is expressively complete with respect to the class of all downward-closed *team properties* with the empty team property [139] (where a team property is a class of teams). This means that the class of properties definable by formulas of propositional dependence logic (where φ defines the property consisting of all teams in which φ is true) is precisely the class of all downward-closed properties with the empty team property (the definitions of the closure properties can be extended to team properties in a natural and obvious way). These theorems allow for concise and tractable characterizations and classification of these logics, and provide an effective tool, as we will see, for proving definability results, completeness of axiomatizations (see, e.g., [139, 140]), and other properties such as uniform interpolation [43].

We conclude by giving a brief summary of the contents of each paper.

In the first paper, *State-based Modal Logics for Free Choice* [7] (Chapter 2), we focus on Aloni's BSML. We introduce two extensions of BSML: $\text{BSML}^{\oslash}$, or BSML with

[4] Chapter 4 also includes some conceptual discussion; note in particular that we touch there on the failure of persistency (Sections 4.4.2 and 4.4.3), and, somewhat indirectly, on the interpretation of NE and hence on what it might mean to simply express that a team is nonempty (Section 4.4.3).

the *emptiness operator* $\oslash$, an operator which can be used to cancel the effects of NE, and BSML$^{\mathbb{V}}$, or BSML with the global disjunction $\mathbb{V}$. We prove expressive completeness theorems for these extensions: we show that BSML$^{\mathbb{V}}$ is expressively complete for the class of all modal team properties invariant under bounded team bisimulation. We also show that BSML$^{\oslash}$ is expressively complete for the class of all *union-closed* modal properties invariant under bounded team bisimulation, where a formula is union closed if its truth in a nonempty collection of teams implies its truth in the union of the collection (and similarly for team properties).

Building on the propositional axiomatizations in [140], and following a strategy commonly used for axiomatizing propositional and modal team-based logics (see, e.g., [139, 140]), we make use of normal forms obtained from the expressive completeness theorems to provide natural deduction axiomatizations for the extensions as well as for BSML.

While we obtain expressive completeness theorems for extensions of BSML in the first paper, the expressive power of BSML itself is left as an open problem in that paper. This is solved in the second paper, *Convex Propositional and Modal Team Logics* [16] (Chapter 3). In this paper, we focus more generally on expressive completeness results involving the closure property of *convexity*—a formula is convex just in case if it is true in two teams, then it is also true in all the teams between these two teams with respect to set inclusion. While convexity has been extensively studied in formal semantics (and other fields) [19, 23, 55], it has thus far not received much attention in the team semantics literature (with the exception of [72, 36]). Convexity is a natural extension of downward closure to a setting in which the empty team property fails in that a formula is downward closed if and only if it is convex and additionally has the property that if it is true in any team whatsoever, it is also true in the empty team, and hence has the empty team property.

We show that BSML is expressively complete for the class of all convex and union-closed modal properties (invariant under bounded bisimulation), and we show an analogous result for the propositional fragment of BSML as well as for another modal extension of this fragment.

We also introduce propositional and modal logics which we prove to be expressively complete with respect to the class of *all* convex propositional/modal properties. The connective involved in expressing emptiness/nonemptiness facts in these convex logics is a variant $\blacklozenge$ of the might-operator $\triangledown$ discussed above.

There is an interesting sense, related to the failure of *closure under uniform substitution* in team-based logics, in which one of our novel convex propositional logics extends propositional dependence logic, and in which another of them extends propositional inquisitive logic; we generalize the notion of *uniform definability* [32, 52, 133, 135, 37, 72] from the team semantics literature to make this notion of extension precise.

In the third paper, *Further Remarks on the Dual Negation in Team Logics* [13] (Chapter 4), we take a closer look at the *bilateral negation* of BSML. This is essentially the same notion as the *dual* or *game-theoretical negation* of independence-friendly logic and early formulations of dependence logic. In IF and D, the dual negation exhibits an extreme degree of semantic indeterminacy in that for any pair of sentences φ and ψ of IF/D, if φ and ψ are *incompatible* in the sense that they share no models, there is a sentence θ of IF/D such that $\varphi \equiv \theta$ and $\psi \equiv \neg\theta$ (as shown originally by Burgess [29] in the context of the prenex fragment of *Henkin quantifier logic* [74], which is equivalent in expressive power to IF/D [48, 124, 75, 119]; and later generalized to arbitrary formulas by Kontinen and Väänänen [94]).

The most natural analogue of Burgess' notion of incompatibility in the setting of modal team semantics of BSML is what we call $\bot$*-incompatibility*: two formulas φ and ψ are $\bot$-incompatible just in case if they are both jointly true in a team, the team must be empty, i.e., $\varphi, \psi \vDash \bot$. While the negation $\neg$ of BSML does conform to this notion of incompatibility in the sense that φ and $\neg\varphi$ are always $\bot$-incompatible, it can be shown that a BSML-formula and its negation also conform to another, stronger notion of incompatibility: φ and ψ are *ground-incompatible* if the truth of φ in t and of ψ in s implies that the intersection of t and s is empty. Ground-incompatibility is a notion that is sensitive to the emptiness/nonemptiness information in formulas which are not downward closed: we show that while ground-incompatibility is equivalent to $\bot$-incompatibility in a downward-closed setting, it is strictly stronger when downward closure fails (that is, a ground-incompatible pair is always $\bot$-incompatible, but the converse implication fails in a non-downward-closed setting). We proceed to show that an analogue of Burgess' theorem holds for BSML if we employ ground-incompatibility as our notion of incompatibility, while no such analogue holds if $\bot$-incompatibility is used.

We then go on to prove analogues of Burgess' theorem for multiple different propositional and modal team-based logics, defining new notions of incompatibility as required. These notions are, more properly, *pair properties*: classes of ordered pairs of team properties. Together with its converse, a Burgess-theorem analogue can be seen as an expressive completeness theorem with respect to the relevant pair property; we make this precise by defining the notion of *bicompleteness*.

We also consider some intuitive interpretations of the incompatibility notions we define, connecting some of them with *epistemic contradictions* such as 'It is raining but it might not be raining', and some with what we call *pragmatic contradictions*—contradictions obtained using Aloni's pragmatic enrichment function.

In the fourth article, *Axiomatizing Modal Inclusion Logic and its Variants* [14] (Chapter 5), we consider extended modal inclusion logic, already introduced above—the extension of classical modal logic with inclusion atoms $\alpha \subseteq \beta$ (a logic which has the empty team property, but breaks downward closure). We recall the expressive completeness theorem for this logic (it is complete for the class of all union-closed modal properties with the empty team property which are invariant under bounded bisimulation) given in [73], and refine this proof slightly using formulas defined in [89]. We then leverage expressive completeness to provide a natural deduction axiomatization for this logic, making use of essentially the same strategy as that employed in Chapter 2. We similarly recall (likewise from [73]) the expressive completeness theorem for the extension of classical modal logic with the ∇-operator, which is expressively equivalent to extended modal inclusion logic, and axiomatize this extension as well. Finally, we introduce yet another variant $\dot{\nabla}$ of ∇, and show the expressive completeness of and axiomatize the extension of classical modal logic with $\dot{\nabla}$ (this extension is also expressively equivalent to modal inclusion logic).

In the fifth article, *A Deep-Inference Sequent Calculus for a Propositional Team Logic* [15] (Chapter 6), we provide a sequent calculus for PL($\vvee$) (see, e.g., [112, 139, 33, 138]), the team-based propositional logic with both the *split disjunction* $\vee$ and the global disjunction $\vvee$—a logic similar to both propositional dependence logic and propositional inquisitive logic. This logic has the empty team property and is downward closed.

While there are many natural deduction- and Hilbert-style axiomatizations of propositional team logics in the literature (see, e.g., [41, 112, 107, 139, 33, 140, 96, 39, 137]), the development of sequent calculus systems and of proof theory in general for these logics has been slower. One of the main reasons for this is that typically these logics are not closed under uniform substitution. Many proof-theoretic techniques depend on the universal applicability of rules, but given the failure of closure under uniform substitution in team logics, the axiomatizations of these logics typically feature rules with syntactic restrictions which are not sound for all formulas.

Our solution to this issue is to employ a standard classical G3-style sequent calculus for the fragment of PL($\vvee$) corresponding to classical propositional logic (the $\vvee$-free fragment), with syntactic restrictions limiting some of these rules to classical formulas. This classical subsystem is supplemented with *deep-inference* (see, e.g., [62, 27, 28]) rules for the global disjunction $\vvee$—that is, rules which allow one to introduce the global disjunction (almost) anywhere within a formula, rather than only as its main connective. The deep-inference rules allow us to work around the difficulties engendered by the syntactically restricted rules. The resulting system is a simple minimal extension of the classical G3-style subsystem, and we show that it satisfies various desirable properties: it admits height-preserving weakening, contraction and inversion; it supports a procedure for constructing cutfree proofs and countermodels similar to that for G3cp; and cut elimination holds as a corollary of cut elimination for the G3-style subsystem together with a normal form theorem for cutfree derivations in the system.

Chapter 2

State-based Modal Logics for Free Choice

This chapter is based on:

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Abstract We study the mathematical properties of bilateral state-based modal logic (BSML), a modal logic employing state-based semantics (also known as team semantics), which has been used to account for free choice inferences and related linguistic phenomena. This logic extends classical modal logic with a nonemptiness atom which is true in a state if and only if the state is nonempty. We introduce two extensions of BSML and show that the extensions are expressively complete, and develop natural deduction axiomatizations for the three logics.

2.1 Introduction

In this article, we study *bilateral state-based modal logic* (BSML), a modal logic employing *team semantics*. BSML was introduced by Aloni in [6] to account for *free choice* (FC) inferences. In FC-inferences, conjunctive meanings are unexpectedly derived from disjunctive sentences:

(1) *Free choice* (see Kamp [87], von Wright [128])

(a) You may go to the beach or to the cinema.
$\rightsquigarrow$ You may go to the beach and you may go to the cinema.

(b) $\Diamond(b \vee c) \rightsquigarrow \Diamond b \wedge \Diamond c$

The novel hypothesis at the core of the account in [6] is that FC and other related inferences are a consequence of a tendency in human cognition to disregard structures

[■,■,■]	[■,□,■]	[ ]; [△,△,△]; [◇,▲,♠]
(a) Verifier	(b) Falsifier	(c) Zero-models

Figure 2.1: Models for the sentence *Every square is black.*

that verify sentences by virtue of some empty configuration (*neglect-zero tendency*); see Bott, Schlotterbeck, and Klein [26]. Models that verify sentences by virtue of an empty witness-set are called *zero-models* (see Figure 2.1 for an illustration).

BSML is introduced as a framework allowing for the formalization of the neglect-zero tendency and the rigorous study of its impact on linguistic interpretation. It extends classical modal logic (ML) with a special *nonemptiness atom* (NE) requiring the adoption of *team semantics*. In team semantics (introduced originally by Hodges [78, 79] and developed further in the literature on *dependence logic* by Väänänen [119, 120]), formulas are interpreted with respect to sets of evaluation points called *teams*, rather than single points. In the type of modal team semantics employed by BSML, teams are *sets of possible worlds*. In [6], teams are also called *states* since they represent speakers' information states; we will use these terms interchangeably. The atom NE (introduced by Yang and Väänänen in [121] and [140]) is true (or *supported*) in a state if and only if the state is not empty: $s \vDash \text{NE}$ iff $s \neq \varnothing$. This atom is used, in [6], to define a *pragmatic enrichment function* $[\ ]^+$ whose core effect is to disallow zero-models; it is then shown that pragmatic enrichment yields nontrivial effects including the prediction (given certain preconditions) of both narrow-scope ($\Diamond(b \vee c) \rightsquigarrow \Diamond b \wedge \Diamond c$) and wide-scope ($\Diamond b \vee \Diamond c \rightsquigarrow \Diamond b \wedge \Diamond c$) FC-inferences, as well as their cancellation under negation (see (2) below). See Figure 2.2 for illustrations of zero-models and the narrow-scope FC-prediction in BSML.[1] See [6] for more discussion and for comparisons with other accounts of FC.

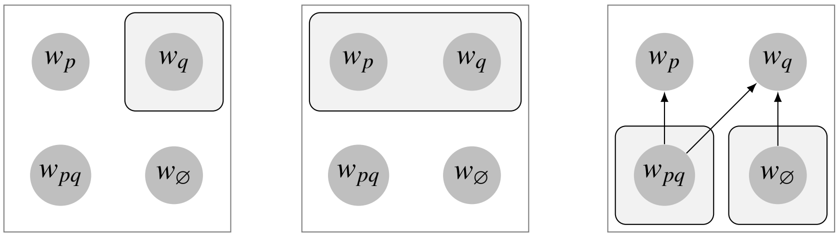
(2) *Cancellation of* FC *under negation* (see Alonso-Ovalle [10])

(a) You are not allowed to eat the cake or the ice cream.
$\rightsquigarrow$ You are not allowed to eat either one.

(b) $\neg\Diamond(\alpha \vee \beta) \rightsquigarrow \neg\Diamond\alpha \wedge \neg\Diamond\beta$

BSML is closely related to modal logics in the lineage of dependence logic (see [119]), such as modal dependence (see Hella et al. [71], Väänänen [120], Yang [134]),

[1]The state $\{w_q\}$ in Figure 2.2(a) supports $p \vee q$. This is because we can find substates of $\{w_q\}$ supporting each disjunct: $\{w_q\}$ itself supports q, and the empty state vacuously supports p. But $\{w_q\}$ is a zero-model for $p \vee q$ because no *nonempty* substate supports the first disjunct. Pragmatic enrichment has the effect of ruling out this zero-model—$\{w_q\}$ does not support $[p \vee q]^+$. The state $\{w_p, w_q\}$ in Figure 2.2(b) is a nonzero verifier for $p \vee q$, so it supports $[p \vee q]^+$. The state $\{w_\varnothing\}$ in Figure 2.2(c) is a zero-model for $\Diamond(p \vee q)$, and it does not support $\Diamond p \wedge \Diamond q$. The state $\{w_{pq}\}$, on the other hand, is a nonzero verifier for $\Diamond(p \vee q)$. We will see that such a verifier must also verify $\Diamond p \wedge \Diamond q$—in other words, narrow-scope FC is predicted by the fact that $[\Diamond(p \vee q)]^+ \vDash \Diamond p \wedge \Diamond q$. (See Section 2.2 for the definition of the pragmatic enrichment function and more details.)



(a) $\{w_q\} \not\models [p \vee q]^+$; zero-model for $p \vee q$

(b) $\{w_p, w_q\} \models [p \vee q]^+$; non-zero verifier for $p \vee q$

(c) $\{w_\varnothing\} \not\models [\Diamond(p \vee q)]^+$; $\{w_{pq}\} \models [\Diamond(p \vee q)]^+$

Figure 2.2: Zero-models and FC in BSML. Worlds are labelled according to what is true in them: w_p stands for a world where only p is true, w_{pq} for one in which only p and q are, etc. The accessibility relation is depicted using arrows.

inclusion (see Anttila, Häggblom, and Yang [14], Hella and Stumpf [73], Kontinen et al. [89]), independence (see Kontinen et al. [90]), and team logic (see [89], Lück [97], Müller [99]). It is also similar to modal inquisitive logic (see Ciardelli [34, 35]). However, the specific combination of connectives (NE, $\vee, \Diamond, \neg$) on which the account of FC in [6] depends is unique to BSML. Let us briefly discuss each of these in turn.

The most crucial component in the account of FC is the nonemptiness atom NE. This atom also sets BSML apart from dependence and inquisitive logic in that formulas containing NE might not be *downward closed* and might not have the *empty state* (or empty team) property. Logics which violate these properties are commonly known as *team logics* (see, e.g., [119, 140, 97]). Most team logics incorporate the *Boolean negation* ~ ($s \models{\sim} \varphi$ iff $s \not\models \varphi$). This negation is very strong in terms of expressive power: the extension of classical modal logic ML with ~ (known as *modal team logic* (MTL)) can express all modally definable state properties (see [89]). BSML, being essentially ML together with NE, is a more modest extension. The propositional fragment of BSML (i.e., classical propositional logic extended with NE) was studied already in [140].

The disjunction $\vee$ in BSML is the standard disjunction in the dependence logic lineage, commonly referred to as the *tensor disjunction* or *local disjunction*. Inquisitive logic makes use of a different disjunction, the *global disjunction* $\mathbin{\vee\!\!\!\vee}$ (also known as the *inquisitive disjunction*):

$$s \models \varphi \vee \psi \quad \text{iff} \quad \text{there are } t, u \subseteq s \text{ s.t. } s = t \cup u \text{ and } t \models \varphi \text{ and } u \models \psi;$$
$$s \models \varphi \mathbin{\vee\!\!\!\vee} \psi \quad \text{iff} \quad s \models \varphi \text{ or } s \models \psi.$$

The predictions in [6] rely on the use of $\vee$, but it is interesting to note that alternative accounts of FC have been given in inquisitive semantics, making use of $\mathbin{\vee\!\!\!\vee}$ instead of $\vee$ (see, e.g., Aloni [5], Aloni and Ciardelli [8], Nygren [101]; see also [6] for a comparison).

The modalities $\Diamond$ and $\Box$ of BSML (essential for the FC-predictions) are equivalent to ones considered in an early version of modal inquisitive logic (see [34]), and are closely related to the modalities in possibility semantics (see Humberstone [81]). They are distinct from the standard modalities of modal team semantics—we refer the reader to Anttila [12] for a comparison.

For its predictions involving negation (such as (2)), BSML relies on a *bilateral* semantics: in addition to support/assertion conditions, each formula is also provided with *anti-support*/rejection conditions. The semantics of the *bilateral negation* $\neg$ are then defined using the anti-support conditions.[2]

In this article, we study the logical properties of BSML, as well as two new extensions of BSML: BSML with the global disjunction $\mathbin{\mathbb{V}}$ (BSML$^{\mathbb{V}}$), and BSML with the novel *emptiness operator* $\oslash$ (BSML$^{\oslash}$). The emptiness operator is a natural counterpart to NE which can be used to cancel out its effects: a state s supports $\oslash\varphi$ if and only if s supports φ or s is empty ($s \vDash \oslash\varphi$ iff $s \vDash \varphi$ or $s = \varnothing$), meaning that, for instance, $\oslash$NE is always supported. While these extensions may prove to have interesting applications of their own (we return to this in the conclusion), our introduction of them is motivated primarily by technical considerations: as we show in the first part of the article, each extension is expressively complete for a natural class of state properties. Moreover, in the second part of the article we axiomatize each of the three logics, and the expressive completeness of the extensions plays a crucial role in our proof of the completeness of these axiomatizations.

In the first part, then, we focus on expressive power. We characterize the expressiveness of our logics in terms of the notion of *state (or team) bisimulation* as introduced in the literature on modal team semantics (see [71, 89]; cf. *inquisitive bisimulation* in Ciardelli and Otto [40]). We show that BSML$^{\mathbb{V}}$ is expressively complete for the class of all state properties invariant under bounded bisimulation (meaning that BSML$^{\mathbb{V}}$ and MTL are equivalent in expressive power; see [89]), and that BSML$^{\oslash}$ is expressively complete for the class of all *union-closed* state properties invariant under bounded bisimulation. These results build on similar results in, for instance, [71], [73], [89], and Sevenster [114]. BSML, as we will demonstrate, is union closed but cannot express all union-closed properties.

In the second part, we develop natural deduction axiomatizations for each of the three logics. These systems build on systems presented in [140] for logics which are essentially the propositional fragments of BSML and BSML$^{\mathbb{V}}$. For other similar natural deduction axiomatizations for modal state-based logics, see, for instance, [14] and [134].

[2]Bilateralism is typically associated with inferentialism in the proof-theoretic tradition (see, e.g., Price [104], Smiley [115], Rumfitt [110], Restall [108], Wansing [125], Incurvati and Schlöder [86], and Wansing and Ayhan [126]; see also Schroeder-Heister [113] for an overview). Other systems employing bilateral semantics include first-degree entailment logic (FDE) (see Anderson and Belnap [11], Dunn [46], and Belnap [21, 22]) and truthmaker semantics (see van Fraassen [122], and Fine [49]). Recently, Aher [3] and Willer [127] presented bilateral accounts of FC in an inquisitive and a dynamic setting, respectively. The bilateral negation in BSML is essentially the same notion as the *dual* or *game-theoretic negation* of *independence-friendly logic* (see Hintikka and Sandu [77], and Hintikka [75]) and some formulations of dependence logic (see [119, 120]).

The article is structured as follows. In Section 2.2, we define the syntax and semantics of BSML and the extensions, and discuss some basic properties of these logics. In Section 2.3, we show the expressive power results described above. We also derive some simple consequences of these results such as the finite model property (for each of the logics) and uniform interpolation (for $\mathrm{BSML}^{\mathbb{V}}$ and $\mathrm{BSML}^{\oslash}$). In Section 2.4, we present our natural deduction axiomatizations, and in Section 2.5 we conclude by noting some open problems. Our work is partly based on Anttila's M.Sc. thesis [12] (supervised by Aloni and Yang), which contains preliminary versions of some of the results.

2.2 Bilateral State-Based Modal Logic

In this section, we define the syntax and semantics of BSML and its extensions, discuss the basic properties of these logics, and recall standard notions and results from team semantics.

2.2.1. DEFINITION (Syntax). Fix a (countably infinite) set Prop of propositional variables. The set of formulas of *bilateral state-based modal logic* BSML is generated by

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Diamond\varphi \mid \mathrm{NE},$$

where $p \in \mathsf{Prop}$. *Classical modal logic* ML is the NE-free fragment of BSML.

$\mathrm{BSML}^{\mathbb{V}}$ is BSML extended with the *global disjunction* $\mathbb{V}$: the set of formulas of $\mathrm{BSML}^{\mathbb{V}}$ is generated by the definition for BSML augmented with the case $\varphi \mathbb{V} \varphi$.

$\mathrm{BSML}^{\oslash}$ is BSML extended with the *emptiness operator* $\oslash$: the set of formulas of $\mathrm{BSML}^{\oslash}$ is generated by the definition for BSML augmented with the case $\oslash\varphi$.

Throughout the article, we use the first Greek letters $\alpha, \beta, \gamma, \ldots$ to stand for formulas of ML (also called *classical formulas*). We write $\mathsf{P}(\varphi)$ for the set of propositional variables in φ and $\varphi(p_1, \ldots, p_n)$ if $\mathsf{P}(\varphi) \subseteq \{p_1, \ldots, p_n\}$. We write $\varphi(\psi/p)$ for the result of replacing all occurrences of p in φ by ψ.

2.2.2. DEFINITION. A *(Kripke) model* (over $\mathsf{X} \subseteq \mathsf{Prop}$) is a triple $M = (W, R, V)$, where

- W is a nonempty set, whose elements are called *(possible) worlds*;
- $R \subseteq W \times W$ is a binary relation, called the *accessibility relation*;
- $V : \mathsf{X} \to \wp(W)$ is a function, called the *valuation*.

We call a subset $s \subseteq W$ of W a *state* on M.

For any world w in M, define, as usual, $R[w] := \{v \in W \mid wRv\}$. Similarly, for any state s on M, define $R[s] := \bigcup_{w \in s} R[w]$.

In the standard world-based Kripke semantics for modal logic, formulas are evaluated with respect to worlds: one writes $M, w \vDash \varphi$ if φ is true in the world w in the model M. In our state-based semantics, formulas are instead evaluated with respect to states. We will also make use of two fundamental semantic notions—*support* and *anti-support*—rather than one. As noted in Section 2.1, states in BSML represent speakers' information states. Support of a formula by (or in) a state represents that what the formula expresses is assertible given the information in the state; anti-support, similarly, represents rejectability in a state.

2.2.3. DEFINITION (Semantics). For a model $M = (W, R, V)$ over X, a state s on M, and formula φ with $\mathsf{P}(\varphi) \subseteq \mathsf{X}$, the notions of φ being *supported/anti-supported* by s in M, written $M, s \vDash \varphi / M, s \dashv \varphi$ (or simply $s \vDash \varphi / s \dashv \varphi$), are defined recursively as follows:

$$
\begin{array}{lll}
M,s \vDash p & :\Longleftrightarrow & \text{for all } w \in s : w \in V(p); \\
M,s \dashv p & :\Longleftrightarrow & \text{for all } w \in s : w \notin V(p); \\
\\
M,s \vDash \bot & :\Longleftrightarrow & s = \varnothing; \\
M,s \dashv \bot & & \text{always the case;} \\
\\
M,s \vDash \mathrm{NE} & :\Longleftrightarrow & s \neq \varnothing; \\
M,s \dashv \mathrm{NE} & :\Longleftrightarrow & s = \varnothing; \\
\\
M,s \vDash \neg\varphi & :\Longleftrightarrow & M,s \dashv \varphi; \\
M,s \dashv \neg\varphi & :\Longleftrightarrow & M,s \vDash \varphi; \\
\\
M,s \vDash \varphi \wedge \psi & :\Longleftrightarrow & M,s \vDash \varphi \text{ and } M,s \vDash \psi; \\
M,s \dashv \varphi \wedge \psi & :\Longleftrightarrow & \text{there exist } t,u \text{ s.t. } s = t \cup u \text{ and } M,t \dashv \varphi \text{ and } M,u \dashv \psi; \\
\\
M,s \vDash \varphi \vee \psi & :\Longleftrightarrow & \text{there exist } t,u \text{ s.t. } s = t \cup u \text{ and } M,t \vDash \varphi \text{ and } M,u \vDash \psi; \\
M,s \dashv \varphi \vee \psi & :\Longleftrightarrow & M,s \dashv \varphi \text{ and } M,s \dashv \psi; \\
\\
M,s \vDash \varphi \mathbin{\mathbb{V}} \psi & :\Longleftrightarrow & M,s \vDash \varphi \text{ or } M,s \vDash \psi; \\
M,s \dashv \varphi \mathbin{\mathbb{V}} \psi & :\Longleftrightarrow & M,s \dashv \varphi \text{ and } M,s \dashv \psi; \\
\\
M,s \vDash \Diamond\varphi & :\Longleftrightarrow & \text{for all } w \in s \text{ there exists } t \subseteq R[w] \text{ s.t. } t \neq \varnothing \text{ and } M,t \vDash \varphi; \\
M,s \dashv \Diamond\varphi & :\Longleftrightarrow & \text{for all } w \in s : M,R[w] \dashv \varphi; \\
\\
M,s \vDash \oslash\varphi & :\Longleftrightarrow & M,s \vDash \varphi \text{ or } s = \varnothing; \\
M,s \dashv \oslash\varphi & :\Longleftrightarrow & M,s \dashv \varphi.
\end{array}
$$

For a set Φ of formulas, we write $M,s \vDash \Phi$ if $M,s \vDash \varphi$ for all $\varphi \in \Phi$. We say that Φ *entails* ψ, written $\Phi \vDash \psi$, if for all models M and states s on M: $M,s \vDash \Phi$ implies $M,s \vDash \psi$. We also write simply $\varphi_1,\dots,\varphi_n \vDash \psi$ for $\{\varphi_1,\dots,\varphi_n\} \vDash \psi$, and $\Phi,\varphi \vDash \psi$ for $\Phi \cup \{\varphi\} \vDash \psi$. We write $\vDash \varphi$ for $\varnothing \vDash \varphi$ and in this case say that φ is *valid*. If both $\varphi \vDash \psi$ and $\psi \vDash \varphi$, then we write $\varphi \equiv \psi$ and say that φ and ψ are *equivalent*. If both $\varphi \equiv \psi$ and $\neg\varphi \equiv \neg\psi$, then φ and ψ are said to be *strongly equivalent*, written $\varphi \boxminus \psi$.

The box modality $\Box$ is defined as the dual of the diamond: $\Box\varphi := \neg\Diamond\neg\varphi$; the resulting support/anti-support clauses are:

$$\begin{aligned}
M,s \vDash \Box\varphi \quad &\iff \quad \text{for all } w \in s : M,R[w] \vDash \varphi;\\
M,s \dashv \Box\varphi \quad &\iff \quad \text{for all } w \in s \text{ there exists } t \subseteq R[w] \text{ s.t. } t \neq \varnothing \text{ and } M,t \dashv \varphi.
\end{aligned}$$

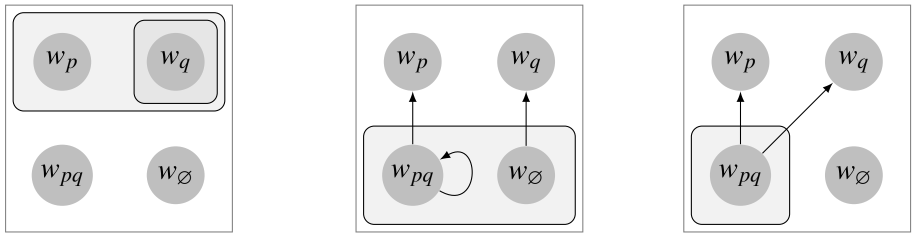
We refer to the atom $\bot$ as the *weak contradiction*. The original syntax for BSML in [6] does not include $\bot$, but rather defines it as $\bot := p \wedge \neg p$ for some fixed p. Including $\bot$ in our syntax allows us to simplify parts of our exposition. We also define the *strong contradiction* $⫫ := \bot \wedge \mathrm{NE}$, and the *strong tautology* $⫪ := \neg\bot$. The weak contradiction is supported only in the empty state, the strong contradiction in no state, and the strong tautology in all states. The atom NE (supported in all nonempty states) can also be viewed as the *weak tautology* $\top$; accordingly, we let $\top := \mathrm{NE}$. Note that we have the following equivalences:

$$\neg\bot \equiv ⫪, \qquad \neg ⫪ \equiv \bot, \qquad \neg ⫫ \equiv ⫪, \qquad \neg\top \equiv \bot.$$

We use these contradictions and tautologies to interpret the empty disjunctions and conjunction:

$$\bigvee\varnothing := \bot, \qquad \bigvee\!\!\!\bigvee\varnothing := ⫫, \qquad \bigwedge\varnothing := ⫪.$$

We now give some examples to illustrate the semantics—consider Figure 2.3. A (local) disjunction $\varphi \vee \psi$ is supported in s if s can be split into two states t and u such that $t \vDash \varphi$ and $u \vDash \psi$. Note that one or both of these substates might be empty; for instance, in Figure 2.3(a) we have $\{w_q\} \vDash p \vee q$. One can force these substates to be nonempty using NE: we have $\{w_q\} \nvDash (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$, whereas $\{w_p, w_q\} \vDash (p \wedge \mathrm{NE}) \vee (q \wedge \mathrm{NE})$. A global disjunction, on the other hand, is supported if either disjunct is supported: $\{w_q\} \vDash (p \wedge \mathrm{NE}) \mathbin{\vee\!\!\!\vee} (q \wedge \mathrm{NE})$, whereas $\{w_p, w_q\} \nvDash (p \wedge \mathrm{NE}) \mathbin{\vee\!\!\!\vee} (q \wedge \mathrm{NE})$. The emptiness operator $\oslash$ is essentially a restricted variant of $\mathbin{\vee\!\!\!\vee}$ in that $\oslash\varphi \equiv \varphi \mathbin{\vee\!\!\!\vee} \bot$: $\{w_q\} \vDash \oslash(p \wedge \mathrm{NE}) \vee \oslash(q \wedge \mathrm{NE})$ and $\{w_p, w_q\} \vDash \oslash(p \wedge \mathrm{NE}) \vee \oslash(q \wedge \mathrm{NE})$. A diamond formula $\Diamond\varphi$ is supported in s if for each world w in s there is a nonempty substate t of the state of all worlds $R[w]$ accessible from w that supports φ; for example, in Figure 2.3(b) we have $s_b \vDash \Diamond q$ and $s_b \nvDash \Diamond p$. A box formula $\Box\varphi$ is supported in s if for each $w \in s$, $R[w]$ as a whole supports φ: $s_b \nvDash \Box q$ and $s_b \vDash \Box p \vee \Box q$. It is easy to verify that the conjunction and (local) disjunction distribute over the global disjunction: $\varphi \wedge (\psi \mathbin{\vee\!\!\!\vee} \chi) \equiv (\varphi \wedge \psi) \mathbin{\vee\!\!\!\vee} (\varphi \wedge \chi)$ and $\varphi \vee (\psi \mathbin{\vee\!\!\!\vee} \chi) \equiv (\varphi \vee \psi) \mathbin{\vee\!\!\!\vee} (\varphi \vee \chi)$; whereas the modalities do not distribute over $\mathbin{\vee\!\!\!\vee}$, as, for example, $s_b \vDash \Box(p \mathbin{\vee\!\!\!\vee} q)$ but $s_b \nvDash \Box p \mathbin{\vee\!\!\!\vee} \Box q$. We have instead $\Diamond(\varphi \mathbin{\vee\!\!\!\vee} \psi) \equiv \Diamond\varphi \vee \Diamond\psi$ and $\Box(\varphi \mathbin{\vee\!\!\!\vee} \psi) \equiv \Box\varphi \vee \Box\psi$.



(a) $\{w_q\} \vDash p \vee q$; $\{w_p, w_q\} \vDash (p \wedge \text{NE}) \vee (q \wedge \text{NE})$ (b) $s_b \vDash \Diamond q$ (c) $s_c \vDash [\Diamond(p \vee q)]^+$

Figure 2.3: Examples of the semantics

Let us also revisit the pragmatic enrichment function $[\ ]^+ : \text{ML} \to \text{BSML}$ described in Section 2.1, which can now be formally and recursively defined as:

$$
\begin{array}{llll}
[p]^+ & := & p \wedge \text{NE}; & \\
[\circ\alpha]^+ & := & \circ([\alpha]^+) \wedge \text{NE} & \text{for } \circ \in \{\neg, \Diamond, \Box\}; \\
[\alpha \triangle \beta]^+ & := & ([\alpha]^+ \triangle [\beta]^+) \wedge \text{NE} & \text{for } \triangle \in \{\wedge, \vee\}.
\end{array}
$$

It is easy to see that $[\ ]^+$ disallows zero-models as described in Section 2.1. Consider, for instance, $[p \vee q]^+ = ((p \wedge \text{NE}) \vee (q \wedge \text{NE})) \wedge \text{NE} \equiv (p \wedge \text{NE}) \vee (q \wedge \text{NE})$. The state $\{w_q\}$ is a zero-model for $p \vee q$ since $\{w_q\} \vDash p \vee q$ but $\{w_q\} \nvDash (p \wedge \text{NE}) \vee (q \wedge \text{NE})$, whereas $\{w_p, w_q\}$ is a nonzero verifier since $\{w_p, w_q\} \vDash (p \wedge \text{NE}) \vee (q \wedge \text{NE})$. For a formula $\Diamond(\alpha \vee \beta)$ as in the antecedent of a narrow-scope FC-inference, we have $[\Diamond(\alpha \vee \beta)]^+ \equiv \Diamond((\alpha \wedge \text{NE}) \vee (\beta \wedge \text{NE})) \wedge \text{NE}$. It is then also easy to see that $[\Diamond(\alpha \vee \beta)]^+ \vDash \Diamond\alpha \wedge \Diamond\beta$—a nonzero verifier for $\Diamond(\alpha \vee \beta)$ must verify $\Diamond\alpha \wedge \Diamond\beta$. In Figure 2.3, s_b is a zero-model for $\Diamond(p \vee q)$ and s_c is a nonzero verifier: we have $s_b \vDash p \vee q$, $s_b \nvDash \Diamond((p \wedge \text{NE}) \vee (q \wedge \text{NE}))$, and $s_b \nvDash \Diamond p \wedge \Diamond q$; whereas $s_c \vDash \Diamond((p \wedge \text{NE}) \vee (q \wedge \text{NE}))$ and $s_c \vDash \Diamond p \wedge \Diamond q$.

Similarly, to many other logics employing team semantics (see, e.g., Ciardelli and Roelofsen [41], Yang and Väänänen [140]), BSML and its extensions do not admit uniform substitution: $\varphi(p) \vDash \psi(p)$ need not imply $\varphi(\chi/p) \vDash \psi(\chi/p)$. For instance, we have $\vDash p \vee \neg p$ and $p \vee p \vDash p$, whereas $\nvDash \text{NE} \vee \neg \text{NE}$ and $(p \mathbin{\veebar} \neg p) \vee (p \mathbin{\veebar} \neg p) \nvDash p \mathbin{\veebar} \neg p$.

Closely related is the failure of preservation of equivalence under replacement: $\varphi \equiv \psi$ need not imply $\theta(\varphi/p) \equiv \theta(\psi/p)$. For instance, we have $\neg(\varphi \vee \psi) \equiv \neg(\varphi \mathbin{\veebar} \psi)$ but $\neg\neg(\varphi \vee \psi) \equiv \varphi \vee \psi \not\equiv \varphi \mathbin{\veebar} \psi \equiv \neg\neg(\varphi \mathbin{\veebar} \psi)$, and $\bot \equiv \neg \text{NE}$ but $\neg\bot \equiv \top \not\equiv \text{NE} \equiv \neg\neg \text{NE}$. This failure is occasioned by the bilateral negation: it is easy to show that if p is not within the scope of $\neg$, then it does follow that $\theta(\varphi/p) \equiv \theta(\psi/p)$. Strong equivalence, on the other hand, *is* preserved under replacement, as we prove in the following proposition (cf. a similar replacement theorem in [119, Lemma 3.25]). We write $\varphi[\chi]$ (and sometimes simply $[\chi]$ or χ) to refer to a specific occurrence of the subformula χ in φ, and $\varphi[\psi/\chi]$ for the result of replacing this occurrence of χ in φ with ψ. This notation is also extended to sequences of occurrences $\varphi[\chi_1, \ldots, \chi_n]$ and sequences or replacements $\varphi[\psi_1/\chi_1, \ldots, \psi_n/\chi_n]$ in the obvious way.

2.2.4. PROPOSITION (Replacement with respect to strong equivalence). *For any formulas θ, φ, and ψ, if $\varphi \boxminus \psi$, then $\theta[\varphi/p] \boxminus \theta[\psi/p]$.*

Proof:
This is by induction on θ. We only give a detailed proof for a few interesting cases.

Case 1: $\theta = \neg\eta$. By the induction hypothesis, $\eta[\varphi/p] \boxminus \eta[\psi/p]$, so $\neg\eta[\varphi/p] \equiv \neg\eta[\psi/p]$ and $\neg\neg\eta[\varphi/p] \equiv \eta[\varphi/p] \equiv \eta[\psi/p] \equiv \neg\neg\eta[\psi/p]$. Therefore, $\neg\eta[\varphi/p] \boxminus \neg\eta[\psi/p]$.

Case 2: $\theta = \Diamond\eta$. By the induction hypothesis, we have $\eta[\varphi/p] \boxminus \eta[\psi/p]$. To show $\Diamond\eta[\varphi/p] \vDash \Diamond\eta[\psi/p]$, assume $s \vDash \Diamond\eta[\varphi/p]$. Then for all $w \in s$ there is a nonempty $t \subseteq R[w]$ such that $t \vDash \eta[\varphi/p]$. But then also for each such t we have $t \vDash \eta[\psi/p]$, so $s \vDash \Diamond\eta[\psi/p]$. The proof of $\Diamond\eta[\psi/p] \vDash \Diamond\eta[\varphi/p]$ is analogous. To show $\neg\Diamond\eta[\varphi/p] \vDash \neg\Diamond\eta[\psi/p]$, assume $s \vDash \neg\Diamond\eta[\varphi/p]$. Then $R[w] \vDash \neg\eta[\varphi/p]$ for all $w \in s$. But then also $R[w] \vDash \neg\eta[\psi/p]$ for all $w \in s$, so $s \vDash \neg\Diamond\eta[\psi/p]$. The proof of $\neg\Diamond\eta[\psi/p] \vDash \neg\Diamond\eta[\varphi/p]$ is analogous.

Case 3: $\theta = \oslash\eta$. By the induction hypothesis, we have $\eta[\varphi/p] \boxminus \eta[\psi/p]$. We then have $\oslash\eta[\varphi/p] \equiv \eta[\varphi/p] \vee\!\!\!\vee \bot \equiv \eta[\psi/p] \vee\!\!\!\vee \bot \equiv \oslash\eta[\psi/p]$ and $\neg\oslash\eta[\varphi/p] \equiv \neg\eta[\varphi/p] \equiv \neg\eta[\psi/p] \equiv \neg\oslash\eta[\psi/p]$. □

An easy induction shows that a formula φ and its negation $\neg\varphi$ are *incompatible* in the sense that if $M, s \vDash \varphi$ and $M, t \vDash \neg\varphi$, then $s \cap t = \varnothing$ (for a proof, see [12, Proposition 3.3.9]). It also appears (see Anttila [13], or Theorem 4.2.19 in Chapter 4) that one can prove the converse of (a reformulation of) this fact: if $M, s \vDash \varphi$ and $M, t \vDash \psi$ implies $s \cap t = \varnothing$, then there is a formula θ such that $\varphi \equiv \theta$ and $\psi \equiv \neg\theta$ (cf. the similar results for Henkin sentences and their contraries in Burgess [29] and for dependence logic with the dual negation in Kontinen and Väänänen [94]).

As usual, each formula can be transformed into an equivalent formula in *negation normal form*; that is, a formula in each occurrence of negation directly precedes a propositional variable p or the atom NE, or is part of the defined connective □.

2.2.5. FACT (Negation normal form). For any formula φ, there exists a formula φ' in negation normal form such that $\varphi \equiv \varphi'$. Moreover, if $\varphi \in$ BSML, then $\varphi \boxminus \varphi'$.

Proof:
Follows easily from the following (strong) equivalences:

$$\neg\neg\varphi \boxminus \varphi; \qquad \neg(\varphi \wedge \psi) \boxminus \neg\varphi \vee \neg\psi; \qquad \neg(\varphi \vee\!\!\!\vee \psi) \equiv \neg\varphi \wedge \neg\psi;$$
$$\neg\Diamond\varphi \boxminus \Box\neg\varphi; \qquad \neg(\varphi \vee \psi); \boxminus \neg\varphi \wedge \neg\psi; \qquad \neg\oslash\varphi \equiv \neg\varphi.$$
□

Throughout the article, we make extensive use of the following properties.

2.2.6. DEFINITION. We say that a formula φ

- is *downward closed*, provided that $[M,s \models \varphi$ and $t \subseteq s] \Longrightarrow M,t \models \varphi$;
- is *union closed*, provided that $[M,s \models \varphi$ for all $s \in S \neq \varnothing] \Longrightarrow M,\bigcup S \models \varphi$;
- has the *empty state property*, provided that $M,\varnothing \models \varphi$ for all M;
- is *flat*, provided that $M,s \models \varphi \iff M,\{w\} \models \varphi$ for all $w \in s$.

It is easy to check that a formula is flat if and only if it is downward closed, union closed, and has the empty state property. A simple induction shows the following.

2.2.7. FACT. If a formula φ does not contain NE, then it is downward closed and has the empty state property. If φ does not contain $\vee\!\!\!\vee$, then it is union closed.

In particular, formulas of BSML and BSML$^{\oslash}$ are union closed; formulas of ML are flat (as they are downward and union closed, and have the empty state property).

It is also easy to verify that $\Diamond\varphi$ and $\Box\varphi$ are always flat. It follows from the flatness of ML-formulas that the state-based semantics for ML correspond with the standard (single-world) Kripke semantics in the sense of the following fact.

2.2.8. FACT. For any formula $\alpha \in$ ML,

$$M,s \models \alpha \quad\iff\quad M,\{w\} \models \alpha \text{ for all } w \in s \quad\iff\quad M,w \models \alpha \text{ for all } w \in s.$$

Proof:
The first equivalence follows from flatness, and the second from $\{w\} \models \alpha$ iff $w \models \alpha$ (proved by an easy induction). □

As an immediate consequence of Fact 2.2.8, we have that for any set $\Delta \cup \{\alpha\}$ of ML-formulas, $\Delta \models \alpha$ in the state-semantics sense if and only if $\Delta \models \alpha$ in the usual single-world-semantics sense. For this reason, when discussing ML-formulas, we simply use the notation $\models$ to refer to both state-based and world-based entailment (similarly for $\equiv$).

2.3 Expressive Power and Normal Forms

In this section, we study the expressive power of BSML and its extensions. We measure the expressive power of the logics in terms of the *properties*—sets of pointed models—they can express. It is well known that the expressive power of classical modal logic ML is characterized by *bisimulation-invariance*: the properties expressible in ML are invariant (closed) under bisimulation. This also holds in the state-based modal setting: our logics are invariant under *state bisimulation*. In Section 2.3.1, we introduce state bisimulation and prove bisimulation-invariance; in Section 2.3.2, we apply this notion to show expressive completeness results for our logics.

2.3.1 Bisimulation

We briefly recall some standard results concerning bisimulation for classical modal logic and standard Kripke semantics. (For more comprehensive discussion, we refer the reader to, for instance, Blackburn, de Rijke, and Venema [24], Goranko and Otto [56]).

Throughout this section, we fix a finite set $\mathsf{X} \subseteq \mathsf{Prop}$ of propositional variables. We omit mention of X whenever doing so does not result in confusion in order to keep our notation light. A *pointed model* (over X) is a pair (M,w), where M is a model over X and $w \in W$.

2.3.1. DEFINITION (Bisimilarity). For any $k \in \mathbb{N}$ and any (M,w) (over $\mathsf{Y} \supseteq \mathsf{X}$) and (M',w') (over $\mathsf{Y}' \supseteq \mathsf{X}$), we write $M,w \rightleftharpoons_k^{\mathsf{X}} M',w'$ (or simply $w \rightleftharpoons_k w'$) and say that (M,w) and (M',w') are X,*k-bisimilar* if the following recursively defined relation holds:

- $M,w \rightleftharpoons_0^{\mathsf{X}} M,w :\Longleftrightarrow$ for all $p \in \mathsf{X}$ we have $M,w \vDash p$ iff $M',w' \vDash p$.
- $M,w \rightleftharpoons_{k+1}^{\mathsf{X}} M',w' :\Longleftrightarrow M,w \rightleftharpoons_0 M',w'$ and
  (a) [forth] for all $v \in R[w]$ there is a $v' \in R'[w']$ such that $M,v \rightleftharpoons_k M',v'$;
  (b) [back] for all $v' \in R'[w']$ there is a $v \in R[w]$ such that $M,v \rightleftharpoons_k M',v'$.

The *modal depth* $md(\varphi)$ of a formula φ is defined recursively as:

- $md(p) = md(\textsc{ne}) = 0$;
- $md(\neg\varphi) = md(\oslash\varphi) = md(\varphi)$;
- $md(\varphi \wedge \psi) = md(\varphi \vee \psi) = md(\varphi \mathbin{\veebar} \psi) = \max\{md(\varphi), md(\psi)\}$;
- $md(\Diamond\varphi) = md(\varphi) + 1$.

We say that (M,w) and (M,w') are X,*k-equivalent* ($k \in \mathbb{N}$), written $M,w \equiv_k^{\mathsf{X}} M',w'$ (or simply $w \equiv_k w'$), if for all $\alpha(\mathsf{X}) \in \mathrm{ML}$ with $md(\alpha) \le k$: $M,w \vDash \alpha \iff M',w' \vDash \alpha$.

2.3.2. DEFINITION (Hintikka formulas). Let $k \in \mathbb{N}$, and let (M,w) be a pointed model over $\mathsf{Y} \supseteq \mathsf{X}$. We define the *kth Hintikka formula* $\chi_{M,w}^{\mathsf{X},k}$ (or simply χ_w^k) of (M,w) recursively as:

$$\chi_{M,w}^{\mathsf{X},0} := \bigwedge\{p \mid p \in \mathsf{X}, w \in V(p)\} \wedge \bigwedge\{\neg p \mid p \in \mathsf{X}, w \notin V(p)\},$$
$$\chi_{M,w}^{\mathsf{X},k+1} := \chi_w^k \wedge \bigwedge_{v \in R[w]} \Diamond \chi_v^k \wedge \Box \bigvee_{v \in R[w]} \chi_v^k.$$

It is easy to see that there are only finitely many nonequivalent kth Hintikka formulas for a given finite X—this is why we may assume that the conjunction and the disjunction in $\chi_{M,w}^{\mathsf{X},k+1}$ are finite and hence that $\chi_{M,w}^{\mathsf{X},k+1}$ is well defined.

2.3.3. THEOREM (see [24, 56]). *We have*

$$w \equiv_k w' \iff w \rightleftharpoons_k w' \iff w' \vDash \chi^k_w \iff \chi^k_w \equiv \chi^k_{w'}.$$

In the context of standard Kripke semantics, we measure the expressive power of a logic in terms of the classes of pointed models expressible in the logic.

2.3.4. DEFINITION. A *property* (over X) is a class of pointed models over X. Each formula α *expresses* a property (over $\mathsf{X} \supseteq \mathsf{P}(\varphi)$)

$$[\![\alpha]\!]_{\mathsf{X}} := \{(M,w) \text{ over } \mathsf{X} \mid M,w \vDash \alpha\}.$$

We say that a logic L is *expressively complete* for a class of properties $\mathbb{P}$, written $[\![L]\!] = \mathbb{P}$, if for each finite X, the class $\mathbb{P}_{\mathsf{X}}$ of properties over X in $\mathbb{P}$ is precisely the class of properties over X expressible by formulas of L with, that is, if

$$\mathbb{P}_{\mathsf{X}} = [\![L]\!]_{\mathsf{X}} := \{[\![\alpha]\!]_{\mathsf{X}} \mid \alpha \in L\}.$$

A property $\mathcal{P}$ is *invariant under* X,k-bisimulation if $(M,w) \in \mathcal{P}$ and $M,w \rightleftharpoons^{\mathsf{X}}_k M',w'$ imply $(M',w') \in \mathcal{P}$. A property $\mathcal{P}$ over X is *invariant under bounded bisimulation* if $\mathcal{P}$ is invariant under X,k-bisimulation for some $k \in \mathbb{N}$. Theorem 2.3.3 allows us to prove a world-based expressive completeness theorem for ML: ML can express all world-based properties that are invariant under bounded bisimulation.

2.3.5. THEOREM (World-based expressive completeness of ML).

$$[\![\mathrm{ML}]\!] = \{\mathcal{P} \mid \mathcal{P} \textit{ is invariant under bounded bisimulation}\}.$$

Proof:
For the inclusion $\subseteq$, for any $\alpha \in \mathrm{ML}$, we have that $[\![\alpha]\!]$ is invariant under k-bisimulation for $k = md(\alpha)$ by Theorem 2.3.3. For the converse inclusion $\supseteq$, for any $\mathcal{P}$ invariant under k-bisimulation, it follows easily from Theorem 2.3.3 that

$$M',w' \vDash \bigvee_{(M,w)\in\mathcal{P}} \chi^k_w \iff (M',w') \in \mathcal{P} \quad \text{(i.e., } \mathcal{P} = [\![\bigvee_{(M,w)\in\mathcal{P}} \chi^k_w]\!] \in [\![\mathrm{ML}]\!]).$$

Note that since there are only finitely many nonequivalent Hintikka formulas χ^k_w for a given finite X, we may assume that the disjunction in the above formula $\bigvee_{(M,w)\in\mathcal{P}} \chi^k_w$ is finite and hence that the formula is well defined. □

The formula $\bigvee_{(M,w)\in\mathcal{P}} \chi^k_w$ in the above proof can be viewed as a characteristic formula for $\mathcal{P}$. The proof also yields a disjunctive normal form for formulas of ML: given that $[\![\alpha]\!]$ is invariant under k-bisimulation for $k = md(\alpha)$, $[\![\alpha]\!] = [\![\bigvee_{(M,w)\in[\![\alpha]\!]} \chi^{md(\alpha)}_w]\!]$, that is, $\alpha \equiv \bigvee_{(M,w)\in[\![\alpha]\!]} \chi^{md(\alpha)}_w$.

We now introduce state-based analogues of the preceding notions and results. A *pointed (state) model* (over X) is a pair (M,s), where M is a model over X and s is a state on M. For a given $k \in \mathbb{N}$, (M,s) and (M,s') are X,k-*equivalent* (in a logic L), written $M,s \equiv_k^{\mathsf{X}} M',s'$, if for all $\varphi(\mathsf{X}) \in L$ with $md(\varphi) \leq k : M,s \models \varphi \iff M',s' \models \varphi$.

State-based bisimulation (first introduced in [71, 89]) is a natural generalization of the world-based notion: two pointed state models are bisimilar if for each world in one there is a bisimilar world in the other, and vice versa.

2.3.6. DEFINITION (State bisimilarity). For any $k \in \mathbb{N}$, (M,s) (over $\mathsf{Y} \supseteq \mathsf{X}$) and (M',s') (over $\mathsf{Y}' \supseteq \mathsf{X}$) are X,$k$-*bisimilar*, written $M,s \rightleftharpoons_k^{\mathsf{X}} M',s'$ (or simply $s \rightleftharpoons_k s'$), if

- [forth] for each $w \in s$ there is some $w' \in s'$ such that $M,w \rightleftharpoons_k M',w'$;
- [back] for each $w' \in s'$ there is some $w \in s$ such that $M,w \rightleftharpoons_k M',w'$.

State bisimulation is clearly a conservative extension of world-based bisimulation in that $\{w\} \rightleftharpoons_k \{w'\}$ iff $w \rightleftharpoons_k w'$. We thus use the same symbol $\rightleftharpoons_k$ for both world-based and state based bisimulation, and we sometimes write simply $w \rightleftharpoons_k s'$ for $\{w\} \rightleftharpoons_k s'$. It is easy to see that for any $k \in \mathbb{N}$, if $s \rightleftharpoons_k s'$, then $s \rightleftharpoons_n s'$ for all $n < k$ (for $w \rightleftharpoons_k w'$ similarly implies $w \rightleftharpoons_n w'$ for all $n < k$).

We list below some useful properties of state bisimulation (see [71] for a proof).

2.3.7. LEMMA. *If $s \rightleftharpoons_{k+1} s'$, then:*

(i) $R[s] \rightleftharpoons_k R'[s']$;

(ii) for all $t,u \subseteq s$ such that $s = t \cup u$ there are $t',u' \subseteq s'$ such that $s' = t' \cup u'$, $t \rightleftharpoons_{k+1} t'$, and $u \rightleftharpoons_{k+1} u'$.

As expected, state k-bisimilarity implies k-equivalence in each of the three logics.

2.3.8. THEOREM (Bisimulation invariance). *If $s \rightleftharpoons_k s'$, then $s \equiv_k s'$.*

Proof:
For any φ, we show by induction on φ that if $s \rightleftharpoons_k s'$ for $k = md(\varphi)$, then $s \models \varphi$ iff $s' \models \varphi$. By Fact 2.2.5, we may assume that φ is in negation normal form. Most cases can be found in [71]; note that the $\vee$-case follows by Lemma 2.3.7(ii). The case $\varphi = \oslash \psi$ follows by $\oslash \psi \equiv \psi \vee\!\!\!\vee \neg \mathrm{NE}$ and the other cases. We now show the cases for the modalities. We only give the detailed proof for one direction of each implication.

Case 1: $\varphi = \Diamond \psi$. Suppose that $s \rightleftharpoons_k s'$ for $k = md(\Diamond \psi) = md(\psi) + 1$, and $s \models \Diamond \psi$. Let $w' \in s'$. By $s \rightleftharpoons_k s'$, there is some $w \in s$ such that $w \rightleftharpoons_k w'$. Since $s \models \Diamond \psi$, there is a nonempty $t \subseteq R[w]$ such that $t \models \psi$. By $w \rightleftharpoons_k w'$, we obtain $R[w] \rightleftharpoons_{k-1} R'[w']$ by Lemma 2.3.7(i). Then by Lemma 2.3.7(ii), there is some $t' \subseteq R'[w']$ such that $t \rightleftharpoons_{k-1} t'$. By the induction hypothesis, $t' \models \psi$. Since $t \neq \varnothing$ and $t \rightleftharpoons_{k-1} t'$, we also have $t' \neq \varnothing$. Hence, $s' \models \Diamond \psi$.

Case 2: $\varphi = \Box\psi$. Suppose that $s \rightleftharpoons_k s'$ for $k = md(\Box\psi) = md(\psi) + 1$, and $s \vDash \Box\psi$. Let $w' \in s'$. By $s \rightleftharpoons_k s'$, there is a $w \in s$ such that $w \rightleftharpoons_k w'$. Since $s \vDash \Box\psi$, we have $R[w] \vDash \psi$. By $w \rightleftharpoons_k w'$, we obtain $R[w] \rightleftharpoons_{k-1} R'[w']$ by Lemma 2.3.7(i). Then by the induction hypothesis, $R'[w'] \vDash \psi$, so $s' \vDash \Box\psi$. □

2.3.2 Expressive Completeness

We move on to the main results concerning the expressive power of our logics. In the classical world-based setting we measured the expressive power of a logic in terms of the classes of pointed models expressible in the logic; we now use classes of pointed state models.

2.3.9. DEFINITION. A *state property* (over X) is a class of pointed state models over X. Each formula φ *expresses* a property (over $\mathsf{X} \supseteq \mathsf{P}(\varphi)$)

$$\|\varphi\|_{\mathsf{X}} := \{(M,s) \text{ over } \mathsf{X} \mid M,s \vDash \varphi\}.$$

We say that a logic L is *expressively complete* for a class of properties $\mathbb{P}$, written $\|L\| = \mathbb{P}$, if for each finite X, the class $\mathbb{P}_{\mathsf{X}}$ of properties over X in $\mathbb{P}$ is precisely the class of properties over X expressible by formulas of L, that is, if

$$\mathbb{P}_{\mathsf{X}} = \|L\|_{\mathsf{X}} := \{\|\varphi\|_{\mathsf{X}} \mid \varphi \in L\}.$$

We say that a state property $\mathcal{P}$ is *invariant under* X, k*-bisimulation (*$k \in \mathbb{N}$*)* if

$$[M,s \rightleftharpoons_k^{\mathsf{X}} M',s'] \implies (M',s') \in \mathcal{P},$$

and that $\mathcal{P}$ over X is *invariant under bounded bisimulation* if $\mathcal{P}$ is invariant under X, k-bisimulation for some $k \in \mathbb{N}$. Similarly, $\mathcal{P}$ is *union closed* if

$$[(M,s) \in \mathcal{P} \text{ for all } s \in S \neq \varnothing] \implies (M, \bigcup S) \in \mathcal{P},$$

and $\mathcal{P}$ is *flat* if

$$(M,s) \in \mathcal{P} \iff (M,\{w\}) \in \mathcal{P} \quad \text{for all } w \in s.$$

Clearly, $\mathcal{P}$ is flat if and only if $\mathcal{P}$ is union closed and *downward closed* (i.e., $(M,s) \in \mathcal{P}$ and $t \subseteq s$ imply $(M,t) \in \mathcal{P}$), and has the *empty state property* (i.e., $(M,\varnothing) \in \mathcal{P}$ for all M).

We aim to show three results. First, $\mathrm{BSML}^{\mathbb{V}}$ is expressively complete for

$$\mathbb{B} := \{\mathcal{P} \mid \mathcal{P} \text{ is invariant under bounded bisimulation}\}.$$

Second, $\mathrm{BSML}^{\oslash}$ is complete for

$$\mathbb{U} := \{\mathcal{P} \mid \mathcal{P} \text{ is union closed and invariant under bounded bisimulation}\}.$$

Third, although the properties expressed by formulas of BSML are union closed and invariant under bounded bisimulation, BSML is not complete for $\mathbb{U}$. Using the notions we have introduced, it is also easy to show that ML is complete for

$$\mathbb{F} := \{\mathcal{P} \mid \mathcal{P} \text{ is flat and invariant under bounded bisimulation}\}.$$

We include below a brief proof of this folklore result as seeing it alongside the other expressive completeness theorems is instructive. Writing $\|L_1\| \subset \|L_2\|$ if $\|L_1\|_{\mathsf{X}} \subseteq \|L_2\|_{\mathsf{X}}$ for all finite X and $\|L_2\|_{\mathsf{Y}} \nsubseteq \|L_1\|_{\mathsf{Y}}$ for some finite Y, we therefore have

$$\|\text{ML}\| \subset \|\text{BSML}\| \subset \|\text{BSML}^{\oslash}\| \subset \|\text{BSML}^{\mathbb{V}}\|.$$

We begin by using Hintikka formulas $\chi_w^k \in \text{ML}$ for pointed models to construct Hintikka formulas $\chi_s^k \in \text{ML}$ and $\theta_s^k \in \text{BSML}$ for pointed state models.

2.3.10. DEFINITION (Hintikka formulas for states). For any pointed state model (M, s) over $\mathsf{Y} \supseteq \mathsf{X}$ and any $k \in \mathbb{N}$, the *kth Hintikka formula* $\chi_{M,s}^{\mathsf{X},k}$ (or simply χ_s^k) and the *kth strong Hintikka formula* $\theta_{M,s}^{\mathsf{X},k}$ (or simply θ_s^k) of (M, s) are defined as

$$\chi_{M,s}^{\mathsf{X},k} := \bigvee_{w \in s} \chi_w^k,$$
$$\theta_{M,s}^{\mathsf{X},k} := \bigvee_{w \in s} (\chi_w^k \wedge \text{NE}).$$

Recall that we stipulate that $\bigvee \varnothing = \bot$. Since there are only finitely many nonequivalent χ_w^k for a given finite X, we may assume that the disjunctions in θ_s^k and χ_w^k are finite, and hence that the formulas are well defined. These formulas function as characteristic formulas for states in the sense of the following proposition.

2.3.11. PROPOSITION.

(i) $M', s' \vDash \chi_s^k \iff$ *there is a state* $t \subseteq s$ *such that* $M, t \rightleftharpoons_k M', s'$.

(ii) $M', s' \vDash \theta_s^k \iff M, s \rightleftharpoons_k M', s'$.

Proof:
We prove the two items simultaneously. If $s = \varnothing$, then $\theta_\varnothing^k = \chi_\varnothing^k = \bot$, and $s' \vDash \bot \iff s' = \varnothing \iff s \rightleftharpoons_k s' \iff \exists t \subseteq s : t \rightleftharpoons_k s'$. Now assume that $s \neq \varnothing$.

$\Longleftarrow$: We first prove (i). Suppose that $t \rightleftharpoons_k s'$ for some $t \subseteq s$. Then for any $w' \in s'$, there exists $w \in t \subseteq s$ such that $w \rightleftharpoons_k w'$, which, by Theorem 2.3.3, implies that $w' \vDash \chi_w^k$ and so $w' \vDash \chi_s^k$. Since the classical formula χ_s^k is flat (see Fact 2.2.7), we conclude that $s' \vDash \chi_s^k$.

For (ii), suppose $s \rightleftharpoons_k s'$. First observe that the back condition for $s \rightleftharpoons_k s'$ implies that there is some $t \subseteq s$ such that $t \rightleftharpoons_k s'$. Thus, by (i), we have $s' \vDash \chi_s^k$, which means that for every $w \in s$, there exists $s'_w \subseteq s'$ such that $s' = \bigcup_{w \in s} s'_w$ and $s'_w \vDash \chi_w^k$. Meanwhile,

Hintikka formula	Logic	Characterization effect	Defined in
χ^k_w	ML	$w' \vDash \chi^k_w \iff w \rightleftharpoons_k w'$	Definition 2.3.2
$\chi^k_s = \bigvee_{w\in s}\chi^k_w$	ML	$s' \vDash \chi^k_s \iff t \rightleftharpoons_k s'$ for some $t \subseteq s$	Definition 2.3.10
$\theta^k_s = \bigvee_{w\in s}(\chi^k_w \wedge \text{NE})$	BSML	$s' \vDash \theta^k_s \iff s \rightleftharpoons_k s'$	Definition 2.3.10

Normal form	Logic	Characterization effect	Defined in
$\nu^k_{\mathcal{P}} = \bigvee_{(M,s)\in\mathcal{P}}\chi^k_s$	ML	$M,s \vDash \nu^k_{\mathcal{P}} \iff (M,s)\in\mathcal{P}$ $(\mathcal{P}\in\mathbb{F})$	Definition 2.3.12
$\xi^k_{\mathcal{P}} = \bigvee\!\!\!\bigvee_{(M,s)\in\mathcal{P}}\theta^k_s$	$\text{BSML}^{\mathbb{V}}$	$M,s \vDash \xi^k_{\mathcal{P}} \iff (M,s)\in\mathcal{P}$ $(\mathcal{P}\in\mathbb{B})$	Definition 2.3.14
$\zeta^k_{\mathcal{P}} = \bigvee_{(M,s)\in\mathcal{P}}\oslash\theta^k_s$	$\text{BSML}^{\oslash}$	$M,s \vDash \zeta^k_{\mathcal{P}} \iff (M,s)\in\mathcal{P}$ $(\mathcal{P}\in\mathbb{U}^{\varnothing}\subseteq\mathbb{U})$	Definition 2.3.16
$\text{NE}\wedge\zeta^k_{\mathcal{P}}$	$\text{BSML}^{\oslash}$	$M,s \vDash \text{NE}\wedge\zeta^k_{\mathcal{P}} \iff (M,s)\in\mathcal{P}$ $(\mathcal{P}\in\mathbb{U}^{\text{NE}}\subseteq\mathbb{U})$	Definition 2.3.16

Table 2.2: Characteristic formulas

by the forth condition for $s \rightleftharpoons_k s'$, we know that for every $w \in s$, there exists $w' \in s'$ such that $w \rightleftharpoons_k w'$, which implies that $w' \vDash \chi^k_w$ by Theorem 2.3.3. Thus, we may assume without loss of generality that each $s'_w \neq \varnothing$ (for we can simply include the world w' in s'_w), giving that $s'_w \vDash \chi^k_w \wedge \text{NE}$. Hence, $s' \vDash \theta^k_s$.

$\Longrightarrow$: For (i), suppose $s' \vDash \chi^k_s$. Then, for each $w \in s$, there exists a subset $s'_w \subseteq s'$ such that $s' = \bigcup_{w\in s} s'_w$ and $s'_w \vDash \chi^k_w$. By the empty state property of classical formulas and Theorem 2.3.3, we have that either $s'_w = \varnothing$ or $w \rightleftharpoons_k s'_w$. Let $t = \{w \in s \mid s'_w \neq \varnothing\} \subseteq s$. Clearly, the forth condition for $t \rightleftharpoons_k s'$ is satisfied. To verify the back condition, observe that for any $v' \in s' = \bigcup_{w\in s} s'_w$, there exists $w \in s$ such that $v' \in s'_w \neq \varnothing$. Thus, $w \in t$ and $w \rightleftharpoons_k s'_w$, whereby $w \rightleftharpoons_k v'$.

For (ii), suppose $s' \vDash \theta^k_s$. Then, for every $w \in s$, there exists $s'_w \subseteq s'$ such that $s' = \bigcup_{w\in s} s'_w$ and $s'_w \vDash \chi^k_w \wedge \text{NE}$. We then have $s'_w \neq \varnothing$ and $w \rightleftharpoons_k s'$ (by Theorem 2.3.3 again). Analogous to the above proof for (i), we define $t = \{w \in s \mid s'_w \neq \varnothing\} \subseteq s$ and obtain $t \rightleftharpoons_k s'$ by the same argument. But since now each $s'_w \neq \varnothing$, we have actually $t = s$ and thus $s \rightleftharpoons_k s'$. □

The formulas χ^k_s are discussed also in [71], and the formulas χ^k_s and θ^k_s are modal versions of the propositional characteristic formulas defined by the third author and Väänänen in [139] and [140]. Table 2.2 presents a summary of the different characteristic formulas we consider.

Table 2.2 also illustrates the main results of the remaining part of this section: we will use Hintikka formulas χ^k_s and θ^k_s for states to construct characteristic formulas for different types of state properties, and the formula for a given type of property will, in turn, be used to show that the logic it belongs to is expressively complete with respect to that type of property. Each of the characteristic formulas for a type of property also exemplifies the disjunctive normal form for the logic in question. Similar disjunctive normal forms are studied in the modal context in, for example, [71], [73], and [89]; and in the propositional context in, for example, [41], [139], and [140].

We now give the proofs of our expressive completeness results in order. We start with the folklore result that ML is complete for $\mathbb{F}$. Hereafter, we often abbreviate a pointed state model (M,s) in a property $\mathcal{P}$ simply as s.

2.3.12. DEFINITION (Characteristic formulas for properties in $\mathbb{F}$). For any $\mathcal{P} \in \mathbb{F}_{\mathsf{X}}$ and $k \in \mathbb{N}$, the *kth characteristic formula* $\nu_{\mathcal{P}}^{\mathsf{X},k} \in \mathrm{ML}$ of $\mathcal{P}$ is defined as

$$\nu_{\mathcal{P}}^{\mathsf{X},k} := \bigvee_{s \in \mathcal{P}} \chi_s^k.$$

As before, the formula is clearly well defined given a specific finite X.

2.3.13. PROPOSITION (ML is expressively complete for $\mathbb{F}$).

$$\|\mathrm{ML}\| = \mathbb{F} := \{\mathcal{P} \mid \mathcal{P} \textit{ is flat and invariant under bounded bisimulation}\}.$$

Proof:
The inclusion $\subseteq$ follows from Theorem 2.3.8 and Fact 2.2.7. For the converse inclusion $\supseteq$, we show that for any $\mathcal{P} \in \mathbb{F}$ invariant under k-bisimulation,

$$M', s' \vDash \bigvee_{(M,s)\in\mathcal{P}} \chi_s^k \iff (M',s') \in \mathcal{P} \quad \text{(i.e., } \mathcal{P} = \|\nu_{\mathcal{P}}^k\| \in \|\mathrm{ML}\|\text{)}.$$

$\Longleftarrow$: Suppose $s' \in \mathcal{P}$. Since $s' \rightleftharpoons_k s'$, we have $s' \vDash \chi_{s'}^k$ by Proposition 2.3.11(i), which implies that $s' \vDash \bigvee_{s\in\mathcal{P}} \chi_s^k$ by the empty state property of χ_s^k.

$\Longrightarrow$: Suppose $s' \vDash \bigvee_{s\in\mathcal{P}} \chi_s^k$. Then for each $s \in \mathcal{P}$, there exists $t_s' \subseteq s'$ such that $s' = \bigcup_{s\in\mathcal{P}} t_s'$ and $t_s' \vDash \chi_s^k$. The latter means, by Proposition 2.3.11(i), that there exists $t \subseteq s$ such that $t_s' \rightleftharpoons_k t$. Since $\mathbb{F}$ is downward closed and invariant under k-bisimulation, we have $t_s' \in \mathcal{P}$ for each $s \in \mathcal{P}$. Finally, since $\mathcal{P}$ is union closed, and also using the fact that since $\mathcal{P}$ has the empty state property, it is not the empty property, we conclude that $s' = \bigcup_{s\in\mathcal{P}} t_s' \in \mathcal{P}$. □

The theorem also yields an alternative normal form for formulas in ML: for an arbitrary ML-formula α, since $\|\alpha\|$ is invariant under k-bisimulation for $k = md(\alpha)$, we have $\|\alpha\| = \|\nu_{\|\alpha\|}^{md(\alpha)}\|$, that is, $\alpha \equiv \bigvee_{s\in\|\alpha\|} \chi_s^{md(\alpha)}$.

Next, we show the first of our main results: $\mathrm{BSML}^{\mathbb{V}}$ is complete for $\mathbb{B}$, and is thus the expressively strongest logic. The proof is similar to the one given for ML and $\mathbb{F}$, but the addition of NE and $\mathbb{V}$ allows us to fully characterize states and to capture all properties invariant under bounded bisimulation.

2.3.14. DEFINITION (Characteristic formulas for properties in $\mathbb{B}$). For any $\mathcal{P} \in \mathbb{B}_{\mathsf{X}}$ and $k \in \mathbb{N}$, the *kth characteristic formula* $\xi_{\mathcal{P}}^{\mathsf{X},k} \in \mathrm{BSML}^{\mathbb{V}}$ of $\mathcal{P}$ is defined as

$$\xi_{\mathcal{P}}^{\mathcal{X},k} := \bigvee\!\!\!\!\bigvee_{s\in\mathcal{P}} \theta_s^k.$$

Recall that we stipulate that $\bigvee\!\!\!\!\bigvee \varnothing = \bot\!\!\!\bot$.

2.3.15. THEOREM ($\mathrm{BSML}^{\mathbb{V}}$ is expressively complete for $\mathbb{B}$).

$$\|\mathrm{BSML}^{\mathbb{V}}\| = \mathbb{B} = \{\mathcal{P} \mid \mathcal{P} \textit{ is invariant under bounded bisimulation}\}.$$

Proof:
The inclusion $\subseteq$ follows from Theorem 2.3.8. For the nontrivial inclusion $\supseteq$, we show that for any $\mathcal{P} \in \mathbb{B}$ invariant under k-bisimulation,

$$M', s' \vDash \bigvee\!\!\!\!\bigvee_{(M,s)\in\mathcal{P}} \theta_s^k \iff (M', s') \in \mathcal{P} \quad (\text{i.e., } \mathcal{P} = \|\xi_{\mathcal{P}}^k\| \in \|\mathrm{BSML}^{\mathbb{V}}\|).$$

If $\mathcal{P} = \varnothing$, then $\xi_{\mathcal{P}}^k = \bot\!\!\!\bot$, and thus $\mathcal{P} = \|\bot\!\!\!\bot\| = \|\xi_{\mathcal{P}}^k\|$. Now, suppose $\mathcal{P} \neq \varnothing$. If $s' \vDash \bigvee\!\!\!\!\bigvee_{s\in\mathcal{P}} \theta_s^k$, then $s' \vDash \theta_s^k$ for some $s \in \mathcal{P}$. By Proposition 2.3.11(ii), we have $s \rightleftharpoons_k s'$, which implies that $s' \in \mathcal{P}$ by invariance under k-bisimulation. Conversely, if $s' \in \mathcal{P}$, since $s' \rightleftharpoons_k s'$, we have $s' \vDash \theta_{s'}^k$ by Proposition 2.3.11(ii) and so $s' \vDash \bigvee\!\!\!\!\bigvee_{s\in\mathcal{P}} \theta_s^k$. □

As above, this also provides us with a disjunctive normal form: for each $\mathrm{BSML}^{\mathbb{V}}$-formula φ: $\|\varphi\| = \|\xi_{\|\varphi\|}^{md(\varphi)}\|$, that is, $\varphi \equiv \bigvee\!\!\!\!\bigvee_{s\in\|\varphi\|} \theta_s^{md(\varphi)}$.

The expressive completeness of $\mathrm{BSML}^{\oslash}$ for $\mathbb{U}$ is proved in a similar way, but there is a small twist in this case. First define

$$\mathbb{U}^{\varnothing} := \{\mathcal{P} \in \mathbb{U} \mid \mathcal{P} \text{ has the empty state property}\},$$
$$\mathbb{U}^{\mathrm{NE}} := \{\mathcal{P} \in \mathbb{U} \mid \mathcal{P} \text{ does not have the empty state property}\}$$

so that $\mathbb{U} = \mathbb{U}^{\varnothing} \cup \mathbb{U}^{\mathrm{NE}}$. Note that if $\mathcal{P} \in \mathbb{U}^{\mathrm{NE}}$ so that for some M we have $(M, \varnothing) \notin \mathcal{P}$, then in fact by invariance under k-bisimulation $(M', \varnothing) \notin \mathcal{P}$ for all M'.

2.3.16. DEFINITION (Characteristic formulas for properties in $\mathbb{U}$). For any $\mathcal{P} \in \mathbb{U}_{\mathsf{X}}$ and $k \in \mathbb{N}$, let

$$\zeta_{\mathcal{P}}^{\mathsf{X},k} := \bigvee_{s\in\mathcal{P}} \oslash \theta_s^k.$$

If $\mathcal{P} \in \mathbb{U}^{\varnothing}$, then the *characteristic formula* of $\mathcal{P}$ is $\zeta_{\mathcal{P}}^k$. If $\mathcal{P} \in \mathbb{U}^{\mathrm{NE}}$, then the *characteristic formula* of $\mathcal{P}$ is $\mathrm{NE} \wedge \zeta_{\mathcal{P}}^k$.

2.3.17. THEOREM ($\mathrm{BSML}^{\oslash}$ is expressively complete for $\mathbb{U}$).

$$\begin{aligned}\|\mathrm{BSML}^{\oslash}\| &= \mathbb{U}\\ &= \{\mathcal{P} \mid \mathcal{P} \textit{ is union closed and invariant under bounded bisimulation}\}.\end{aligned}$$

Proof:
The inclusion ⊆ follows from Fact 2.2.7 and Theorem 2.3.8. For the nontrivial inclusion ⊇, we first show that for any $\mathcal{P} \in \mathbb{U}^{\varnothing}$ invariant under k-bisimulation,

$$M', s' \vDash \bigvee_{(M,s)\in\mathcal{P}} \oslash\theta_s^k \iff (M', s') \in \mathcal{P} \quad \text{(i.e., } \mathcal{P} = \|\zeta_{\mathcal{P}}^k\| \in \|\mathrm{BSML}^{\oslash}\|\text{).}$$

$\Longleftarrow$: Suppose that $s' \in \mathcal{P} \in \mathbb{U}^{\varnothing}$. Since $s' \rightleftharpoons_k s'$, by Proposition 2.3.11(ii), we have $s' \vDash \theta_{s'}^k$, whereby $s' \vDash \oslash\theta_{s'}^k$. Hence, $s' \vDash \bigvee_{s\in\mathcal{P}} \oslash\theta_s^k$ by the empty state property of $\oslash\theta_s^k$.

$\Longrightarrow$: Suppose that $s' \vDash \bigvee_{s\in\mathcal{P}} \oslash\theta_s^k$. Then for each $s \in \mathcal{P}$, there exists $t_s' \subseteq s'$ such that $s' = \bigcup_{s\in\mathcal{P}} t_s'$ and $t_s' \vDash \oslash\theta_s^k$. The latter implies, by Proposition 2.3.11(ii), that $t_s' = \varnothing$ or $t_s' \rightleftharpoons_k s$. Since $\mathcal{P} \in \mathbb{U}^{\varnothing}$ has the empty state property, and since it is invariant under k-bisimulation, we have $t_s' \in \mathcal{P}$ for each $s \in \mathcal{P}$. Finally, since $\mathcal{P}$ is union closed, and also using the fact that since $\mathcal{P}$ has the empty state property, it is not the empty property, we conclude that $s' = \bigcup_{s\in\mathcal{P}} t_s' \in \mathcal{P}$.

Next, we show that for any $\mathcal{P} \in \mathbb{U}^{\mathrm{NE}}$ invariant under k-bisimulation,

$$M', s' \vDash \mathrm{NE} \wedge \bigvee_{(M,s)\in\mathcal{P}} \oslash\theta_s^k \iff (M', s') \in \mathcal{P}$$
$$\text{(i.e., } \mathcal{P} = \|\mathrm{NE} \wedge \zeta_{\mathcal{P}}^k\| \in \|\mathrm{BSML}^{\oslash}\|\text{).}$$

If $\mathcal{P} = \varnothing$, then $\zeta_{\mathcal{P}}^k = \bot$, and thus $\mathcal{P} = \|⫫\| = \|\mathrm{NE} \wedge \bot\| = \|\mathrm{NE} \wedge \zeta_{\mathcal{P}}^k\|$. Now, suppose $\mathcal{P} \neq \varnothing$.

$\Longrightarrow$: Suppose $s' \in \mathcal{P} \in \mathbb{U}^{\mathrm{NE}}$. By the same argument as in the $\mathbb{U}^{\varnothing}$-case, we obtain $s' \vDash \bigvee_{s\in\mathcal{P}} \oslash\theta_s^k$. Moreover, since $\mathcal{P} \in \mathbb{U}^{\mathrm{NE}}$, we have $s' \neq \varnothing$ so $s' \vDash \mathrm{NE}$.

$\Longleftarrow$: Suppose $s' \vDash \mathrm{NE} \wedge \bigvee_{s\in P} \oslash\theta_s^k$. As in the $\mathbb{U}^{\varnothing}$-case, for each $s \in \mathcal{P}$, there exists $t_s' \subseteq s'$ such that $s' = \bigcup_{s\in\mathcal{P}} t_s'$ and $t_s' = \varnothing$ or $t_s' \rightleftharpoons_k s$. Let $\mathcal{Q} := \{s \in \mathcal{P} \mid t_s' \rightleftharpoons_k s\}$ so that $s' = \bigcup_{s\in\mathcal{Q}} t_s'$. Since $\mathcal{P}$ is invariant under k-bisimulation, we have $t_s' \in \mathcal{P}$ for each $s \in \mathcal{Q}$, and since $s' \vDash \mathrm{NE}$, we have $s' \neq \varnothing$ so that $\mathcal{Q} \neq \varnothing$. Finally, since $\mathcal{P}$ is union closed and $\mathcal{Q} \neq \varnothing$, we conclude $s' = \bigcup_{s\in\mathcal{Q}} t_s' \in \mathcal{P}$. □

Again the proof yields a disjunctive normal form: for any BSML$^{\oslash}$-formula φ, either $\varphi \equiv \zeta_{\|\varphi\|}^{md(\varphi)}$ or $\varphi \equiv \mathrm{NE} \wedge \zeta_{\|\varphi\|}^{md(\varphi)}$.

Finally, we remark that while BSML is union closed and invariant under bisimulation, it is not complete for $\mathbb{U}$. To prove this, we show first that for BSML-formulas, the empty state property implies the downward closure property. Note that while converse might also appear to hold, it does not—for instance, the formula ⫫ is downward closed but does not have the empty state property.

2.3.18. Lemma. *For any $\varphi \in$ BSML, if φ has the empty state property, then φ is downward closed.*

Proof:
This is by induction on φ (assumed to be in negation normal form). We only show the nontrivial cases.

Let $\varphi = \psi \wedge \chi$. If $M, \varnothing \vDash \psi \wedge \chi$, then $M, \varnothing \vDash \psi$ and $M, \varnothing \vDash \chi$. Therefore, by the induction hypothesis, ψ and χ are downward closed. So if $s \vDash \psi \wedge \chi$ and $t \subseteq s$, then $t \vDash \psi$ and $t \vDash \chi$, and therefore $t \vDash \psi \wedge \chi$.

Let $\varphi = \psi \vee \chi$. If $M, \varnothing \vDash \psi \vee \chi$, then clearly $M, \varnothing \vDash \psi$ and $M, \varnothing \vDash \chi$. So by the induction hypothesis, ψ and χ are downward closed. If $s \vDash \psi \vee \chi$, and $t \subseteq s$, then there are s_1, s_2 such that $s = s_1 \cup s_2$, $s_1 \vDash \psi$ and $s_2 \vDash \chi$. By downward closure, we have $t \cap s_1 \vDash \psi$ and $s \cap t_2 \vDash \chi$ and therefore $t = (t \cap s_1) \cup (t \cap s_2) \vDash \psi \vee \chi$. □

2.3.19. FACT (BSML is not expressively complete for $\mathbb{U}$).

$$\begin{aligned} \|\text{BSML}\| &\subset \mathbb{U} \\ &= \{\mathcal{P} \mid \mathcal{P} \text{ is invariant under bounded bisimulation and union closed}\}. \end{aligned}$$

Proof:
The inclusion $\subseteq$ follows from Fact 2.2.7 and Theorem 2.3.8. For inequality, let

$$\mathcal{P} := \|(p \wedge \text{NE}) \vee (\neg p \wedge \text{NE})\| \cup \|\bot\|.$$

Clearly, $\mathcal{P} \in \mathbb{U}$. Assume for contradiction that $\|\varphi\| = \mathcal{P}$ for some φ in BSML. Then φ has the empty state property so it is downward closed by Lemma 2.3.18. Let $(M, s) \in \|(p \wedge \text{NE}) \vee (\neg p \wedge \text{NE})\| \subseteq \mathcal{P}$. Then there are t, u such that $s = t \cup u$, $t \vDash p \wedge \text{NE}$, and $u \vDash \neg p \wedge \text{NE}$. By $\|\varphi\| = \mathcal{P}$, we have $s \vDash \varphi$. By downward closure, $t \vDash \varphi$, so $(M, t) \in \mathcal{P}$. But $t \nvDash (p \wedge \text{NE}) \vee (\neg p \wedge \text{NE})$ and $t \nvDash \bot$; a contradiction. □

The property $\mathcal{P} \in \mathbb{U}$ in the above proof is clearly expressible in the expressively complete logic $\text{BSML}^{\oslash}$ as $\mathcal{P} = \|\oslash((p \wedge \text{NE}) \vee (\neg p \wedge \text{NE}))\|$.

We conclude this section by noting some consequences of the expressive power results. First, it is shown by D'Agostino in [43] that any state-based modal logic with the locality property and which is also *forgetting* enjoys *uniform interpolation*. Each of our logics has the locality property, and it follows from our expressive completeness results that $\text{BSML}^{\mathbb{V}}$ and $\text{BSML}^{\oslash}$ are forgetting. Therefore $\text{BSML}^{\mathbb{V}}$ and $\text{BSML}^{\oslash}$ enjoy uniform interpolation. (See [43] for details; for the locality requirement, see Yang [137].)

Second, the finite model property for our logics follows from that for ML, by a simple argument that makes use of disjoint unions of models. The *disjoint union* $\biguplus_{i \in I} M_i$ of the models $\{M_i \mid i \in I \neq \varnothing\}$ is defined as usual. Recall in particular that the domain of the disjoint union is defined as $\bigcup_{i \in I}(W_i \times \{i\})$. We also extend the notion to state properties in a natural way: the *disjoint union* of a nonempty property $\mathcal{P} = \{(M_i, s_i) \mid i \in I\}$ is $\biguplus \mathcal{P} := (\biguplus_{i \in I} M_i, \biguplus_{i \in I} s_i)$, where $\biguplus_{i \in I} s_i := \bigcup_{i \in I}(s_i \times \{i\})$. To simplify notation, we define the disjoint union $\biguplus \varnothing$ of the empty property to be $(M, \varnothing)$ for some fixed M. We refer to an element (w, i) in the disjoint union as simply w, and similarly for states. The following is standard (see, e.g., [24, 56]).

2.3.20. PROPOSITION. *For all $i \in I, w \in W_i, k \in \mathbb{N}, \mathsf{X} \subseteq \mathsf{Prop} : M_i, w \rightleftharpoons_k^{\mathsf{X}} \biguplus_{i\in I} M_i, w$. Consequently, for all $s \subseteq W_i : M_i, s \rightleftharpoons_k \biguplus_{i\in I} M_i, s$.*

2.3.21. PROPOSITION (Finite model property). *If $\nvDash \varphi$, then there is some finite model M and state s such that $M, s \nvDash \varphi$.*

Proof:
Let (M,s) be such that $M, s \nvDash \varphi$, and let $\mathsf{X} = \mathsf{P}(\varphi)$ and $k = md(\varphi)$. If $s = \varnothing$, then we clearly have $M', s \nvDash \varphi$ for all finite M'. Now assume that $s \neq \varnothing$. Since there are only finitely many X, k-bisimilarity types (or Hintikka formulas), we can pick a finite $t \subseteq s$ such that $t \rightleftharpoons_k^{\mathsf{X}} s$. Clearly for any $w \in t$ we have $\chi_w^k \not\equiv \bot$; therefore, by the finite model property for ML, there is some finite (M_w, v_w) such that $M_w, v_w \vDash \chi_w^k$. Then by Theorem 2.3.3, $M_w, v_w \rightleftharpoons_k M, w$. By Proposition 2.3.20, $\biguplus_{w\in t} M_w, v_w \rightleftharpoons_k M_w, v_w$ for all $w \in t$; then also $\biguplus_{w\in t} M_w, v_w \rightleftharpoons_k M, w$ for all $w \in t$, so $\biguplus_{w\in t} M_w, \{v_w \mid w \in t\} \rightleftharpoons_k M, t$, whence $\biguplus_{w\in t} M_w, \{v_w \mid w \in t\} \nvDash \varphi$ by Theorem 2.3.8, where $\biguplus_{w\in t} M_w$ is clearly finite. □

Together with the completeness results to be proved in Section 2.4, this implies that our logics BSML, BSML$^{\oslash}$, and BSML$^{\mathbb{V}}$ are decidable.

2.4 Axiomatizations

In this section, we introduce sound and complete natural deduction systems for BSML and its extensions. The strategy we employ in proving completeness makes essential use of the expressive power results in Section 2.3: for the expressively complete logics BSML$^{\mathbb{V}}$ and BSML$^{\oslash}$, we show that each formula is provably equivalent to some formula in disjunctive normal form—completeness then reduces to showing it for formulas in normal form. As for BSML, for which no similar normal form is available, our strategy involves simulating the BSML$^{\mathbb{V}}$-normal form by using sets of BSML-formulas. For this reason we first axiomatize the extensions.

2.4.1 BSML$^{\mathbb{V}}$

We first focus on the strongest logic BSML$^{\mathbb{V}}$. The propositional fragment of BSML$^{\mathbb{V}}$ corresponds to the logic PT$^+$ studied in [140] (with the caveat that the negation in PT$^+$ is not the bilateral negation). Our system for BSML$^{\mathbb{V}}$ is essentially an extension (and simplification) of the propositional system for PT$^+$ presented in [140], with additional rules for the bilateral negation and the modalities.

Before we present the system, let us issue a word of caution. Recall from Section 2.2 that BSML and its extensions are not closed under uniform substitution. Due to this, our systems will *not* admit the usual *uniform substitution* rule. Note in particular that in the presentation of our rules, the metavariables α and β range, as before, exclusively over formulas of ML.

Most rules in the system for $\mathrm{BSML}^{\mathbb{V}}$ are also rules in the systems for BSML and $\mathrm{BSML}^{\oslash}$. For ease of reference, we first present these shared rules; all rules involving $\mathbb{V}$ are grouped together at the end.

2.4.1. DEFINITION (Natural deduction system for $\mathrm{BSML}^{\mathbb{V}}$). The following rules comprise a natural deduction system for $\mathrm{BSML}^{\mathbb{V}}$.

(a) Rules for $\wedge$:

$$\frac{\begin{array}{cc} D_1 & D_2 \\ \varphi & \psi \end{array}}{\varphi \wedge \psi}\ \wedge\mathsf{I} \qquad \frac{\begin{array}{c} D \\ \varphi \wedge \psi \end{array}}{\varphi}\ \wedge\mathsf{E} \qquad \frac{\begin{array}{c} D \\ \varphi \wedge \psi \end{array}}{\psi}\ \wedge\mathsf{E}$$

(b) Rules for $\neg$:

$$\frac{\begin{array}{c} [\alpha] \\ D \\ \bot \end{array}}{\neg\alpha}\ \neg\mathsf{I}(*) \qquad \frac{\begin{array}{cc} D_1 & D_2 \\ \alpha & \neg\alpha \end{array}}{\beta}\ \neg\mathsf{E} \qquad \begin{array}{c} D \\ \neg\neg\varphi \\ \hline\hline \varphi \end{array}\ \neg\neg\mathsf{E}$$

$$\begin{array}{c} D \\ \neg(\varphi \wedge \psi) \\ \hline\hline \neg\varphi \vee \neg\psi \end{array}\ \mathsf{DM}_{\wedge} \qquad \begin{array}{c} D \\ \neg(\varphi \vee \psi) \\ \hline\hline \neg\varphi \wedge \neg\psi \end{array}\ \mathsf{DM}_{\vee} \qquad \begin{array}{c} D \\ \neg\mathrm{NE} \\ \hline\hline \bot \end{array}\ \neg\mathrm{NE}\mathsf{E}$$

$(*)$ The undischarged assumptions in D do not contain NE.

(c) Rules for $\vee$:

$$\frac{\begin{array}{c} D \\ \varphi \end{array}}{\varphi \vee \psi}\ \vee\mathsf{I}(*) \qquad \frac{\begin{array}{c} D \\ \varphi \end{array}}{\varphi \vee \varphi}\ \vee\mathsf{W} \qquad \frac{\begin{array}{c} D \\ \varphi \vee \psi \end{array}}{\psi \vee \varphi}\ \mathsf{Com}\vee \qquad \frac{\begin{array}{c} D \\ \varphi \vee (\psi \vee \chi) \end{array}}{(\varphi \vee \psi) \vee \chi}\ \mathsf{Ass}\vee$$

$$\frac{\begin{array}{ccc} & [\varphi] & [\psi] \\ D & D_1 & D_2 \\ \varphi \vee \psi & \chi & \chi \end{array}}{\chi}\ \vee\mathsf{E}(\dagger,\ddagger) \qquad \frac{\begin{array}{cc} & [\psi] \\ D & D_1 \\ \varphi \vee \psi & \chi \end{array}}{\varphi \vee \chi}\ \vee\mathsf{Mon}(\dagger)$$

$(*)$ ψ does not contain NE.
$(\dagger)$ The undischarged assumptions in D_1, D_2 do not contain NE.
$(\ddagger)$ χ does not contain $\mathbb{V}$.

(d) Rules for $\bot$ and NE:

$$\frac{\begin{array}{c} D \\ \bot \vee \varphi \end{array}}{\varphi}\ \bot\mathsf{E} \qquad \frac{\begin{array}{c} D \\ \perp\!\!\!\perp \vee \varphi \end{array}}{\psi}\ \perp\!\!\!\perp\mathsf{Ctr}$$

(e) Basic modal rules:

$$\dfrac{\begin{array}{c}[\varphi]\\ D\\ \psi\end{array}\qquad\begin{array}{c}D_1\\ \Diamond\varphi\end{array}}{\Diamond\psi}\ \Diamond\mathsf{Mon}(*)\qquad\qquad \dfrac{\begin{array}{c}[\varphi_1]\cdots[\varphi_n]\\ D\\ \psi\end{array}\qquad\begin{array}{c}D_1\\ \Box\varphi_1\end{array}\quad\cdots\quad\begin{array}{c}D_n\\ \Box\varphi_n\end{array}}{\Box\psi}\ \Box\mathsf{Mon}(*)$$

$(*)$ D does not contain undischarged assumptions.

$$\begin{array}{c}D\\ \neg\Diamond\varphi\\ \hline\hline \Box\neg\varphi\end{array}\ \mathsf{Inter}\Diamond\Box$$

(f) Rules governing the interaction of the modalities and connectives:

$$\dfrac{\begin{array}{c}D\\ \Diamond(\varphi\vee(\psi\wedge\mathrm{NE}))\end{array}}{\Diamond\psi}\ \Diamond\mathsf{Sep}\qquad\qquad \dfrac{\begin{array}{c}D_1\\ \Diamond\varphi\end{array}\qquad\begin{array}{c}D_2\\ \Diamond\psi\end{array}}{\Diamond(\varphi\vee\psi)}\ \Diamond\mathsf{Join}$$

$$\dfrac{\begin{array}{c}D\\ \Box(\varphi\wedge\mathrm{NE})\end{array}}{\Diamond\varphi}\ \Box\mathsf{Inst}\qquad\qquad \dfrac{\begin{array}{c}D_1\\ \Box\varphi\end{array}\qquad\begin{array}{c}D_2\\ \Diamond\psi\end{array}}{\Box(\varphi\vee\psi)}\ \Box\Diamond\mathsf{Join}$$

(g) Propositional rules involving $\vee\!\!\!\vee$ ($\mathrm{BSML}^{\vee\!\!\!\vee}$-specific):

$$\dfrac{\begin{array}{c}D\\ \varphi\end{array}}{\varphi\vee\!\!\!\vee\psi}\ \vee\!\!\!\vee\mathsf{I}\qquad \dfrac{\begin{array}{c}D\\ \psi\end{array}}{\varphi\vee\!\!\!\vee\psi}\ \vee\!\!\!\vee\mathsf{I}\qquad \dfrac{\begin{array}{c}D\\ \varphi\vee\!\!\!\vee\psi\end{array}\qquad\begin{array}{c}[\varphi]\\ D_1\\ \chi\end{array}\qquad\begin{array}{c}[\psi]\\ D_2\\ \chi\end{array}}{\chi}\ \vee\!\!\!\vee\mathsf{E}$$

$$\dfrac{\begin{array}{c}D\\ \varphi\vee(\psi\vee\!\!\!\vee\chi)\end{array}}{(\varphi\vee\psi)\vee\!\!\!\vee(\varphi\vee\chi)}\ \mathsf{Distr}\vee\vee\!\!\!\vee\qquad \begin{array}{c}D\\ \neg(\varphi\vee\!\!\!\vee\psi)\\ \hline\hline \neg\varphi\wedge\neg\psi\end{array}\ \mathsf{DM}_{\vee\!\!\!\vee}\qquad \dfrac{}{\bot\vee\!\!\!\vee\mathrm{NE}}\ \mathrm{NE}\mathsf{I}$$

(h) Modal rules for $\vee\!\!\!\vee$ ($\mathrm{BSML}^{\vee\!\!\!\vee}$-specific):

$$\begin{array}{c}D\\ \Diamond(\varphi\vee\!\!\!\vee\psi)\\ \hline\hline \Diamond\varphi\vee\Diamond\psi\end{array}\ \mathsf{Conv}\Diamond\vee\!\!\!\vee\vee\qquad\qquad \begin{array}{c}D\\ \Box(\varphi\vee\!\!\!\vee\psi)\\ \hline\hline \Box\varphi\vee\Box\psi\end{array}\ \mathsf{Conv}\Box\vee\!\!\!\vee\vee$$

We write $\Phi\vdash_{\mathrm{BSML}^{\vee\!\!\!\vee}}\psi$ (or simply $\Phi\vdash\psi$) if ψ is derivable from formulas in Φ in the system. We also write $\Phi,\varphi_1,\dots,\varphi_n\vdash\psi$ for $\Phi\cup\{\varphi_1,\dots,\varphi_n\}\vdash\psi$. We say that φ and ψ are *provably equivalent*, written $\varphi\dashv\vdash\psi$, if $\varphi\vdash\psi$ and $\psi\vdash\varphi$.

The conjunction $\wedge$ and global disjunction $\vee\!\!\!\vee$ have the standard introduction and elimination rules. The rules involving $\vee$ must be constrained due to the failure of downward/union closure occasioned by the presence of NE and $\vee\!\!\!\vee$. For instance, $p\not\models p\vee\mathrm{NE}$, so unconstrained $\vee$-introduction is not sound. To ensure soundness, the introduced disjunct must have the empty state property, and so in $\vee$I we require that the introduced disjunct does not contain NE. Alternatively, the introduced disjunct may simply be the premise again, as in $\vee$-weakening $\vee$W. $\vee$-elimination is similarly constrained to ensure that the subderivations D_1 and D_2 do not depend on formulas which are not downward closed, and to ensure that the consequent χ of the rule is union closed. The commutativity and associativity of $\vee$, and the distributivity of $\vee$ over $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ need to be included as rules; for the derivable algebraic properties of the connectives, see Proposition 2.4.6 below. The rules for $\neg$ include both constrained standard rules as well as rules corresponding to the equivalences noted in Section 2.2. The propositional rules involving $\bot$ and NE—$\bot$E, $⫫$Ctr ($⫫$-contraction), NEI and $\neg$NEE—are self-explanatory. Note that with $\bot$ we have *ex falso* with respect to classical formulas, and with $⫫$, we have it with respect to all formulas.

2.4.2. LEMMA. $⫫ \vdash \varphi$.

Proof:
$⫫ \vdash ⫫ \vee \bot$ by $\vee$I, and then $⫫ \vee \bot \vdash \varphi$ by $⫫$Ctr. □

The basic modal rules (e) are standard. While the modalities do not distribute over $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, a weaker distributivity in which $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ is switched with $\vee$—the conversion rules Conv$\Diamond\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}\vee$ and Conv$\Box\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}\vee$—does hold. The $\Diamond$-separation rule $\Diamond$Sep corresponds to FC-entailment for pragmatically enriched formulas as described in Sections 2.1 and 2.2. The $\Box$-instantiation rule $\Box$Inst characterizes the fact that $\Box\varphi$ implies $\Diamond\varphi$ in case accessible worlds exist. The two join rules $\Diamond$Join and $\Box\Diamond$Join allow one to graft together witnessing accessible states.

2.4.3. THEOREM (Soundness of $\mathrm{BSML}^{\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}}$). *If* $\Phi \vdash \psi$, *then* $\Phi \vDash \psi$.

Proof:
This is by induction on the length k of possible derivations of $\Phi \vdash \psi$. We only include the cases for the novel rules involving modalities, as well as for $\vee$E; most of the other cases can be found in [140]. The base case ($k = 1$) is simple (see [140]). For the inductive case, assume the result holds for all derivations of length at most k. We consider the different possibilities for what the final rule in the derivation $\Phi \vdash \psi$ can be.

$\vee$E: Assume that we have derivations of length at most k witnessing $\Phi \vdash \varphi \vee \psi$; $\Phi_1, \varphi \vdash \chi$; and $\Phi_2, \psi \vdash \chi$; that for all $\eta \in \Phi_1 \cup \Phi_2$, η does not contain NE, and that χ does not contain $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$. By the induction hypothesis, $\Phi \vDash \varphi \vee \psi$; $\Phi_1, \varphi \vDash \chi$; and $\Phi_2, \psi \vDash \chi$. We show that $\Phi, \Phi_1, \Phi_2 \vDash \chi$. Assume that $s \vDash \Phi \cup \Phi_1 \cup \Phi_2$. Then $s \vDash \varphi \vee \psi$, so there are some t_1, t_2 such that $s = t_1 \cup t_2$; $t_1 \vDash \varphi$; and $t_2 \vDash \psi$. Since no $\eta \in \Phi_1 \cup \Phi_2$ contains NE, each such η is downward closed. So $t_1 \vDash \eta_1$ for all $\eta_1 \in \Phi_1$, and $t_2 \vDash \eta_2$ for all $\eta_2 \in \Phi_2$. Thus $t_1 \vDash \chi$ and $t_2 \vDash \chi$. Since χ does not contain $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, it is union closed; therefore, $s \vDash \chi$.

Conv$\Diamond\mathbin{\vee\!\!\!\vee}\vee$: It suffices to show that $\Diamond(\varphi \mathbin{\vee\!\!\!\vee} \psi) \equiv \Diamond\varphi \vee \Diamond\psi$; $\dashv$ is easy. For $\vDash$, let $s \vDash \Diamond(\varphi \mathbin{\vee\!\!\!\vee} \psi)$. If $s = \varnothing$, then clearly $s \vDash \Diamond\varphi \vee \Diamond\psi$. Otherwise if $s \neq \varnothing$, let $w \in s$. Then there is a nonempty $t \subseteq R[w]$ such that $t \vDash \varphi \mathbin{\vee\!\!\!\vee} \psi$, so that $t \vDash \varphi$ or $t \vDash \psi$. Letting

$$s_1 := \{w \in s \mid \exists t \subseteq R[w] : t \neq \varnothing \text{ and } t \vDash \varphi\} \qquad \text{and}$$
$$s_2 := \{w \in s \mid \exists t \subseteq R[w] : t \neq \varnothing \text{ and } t \vDash \psi\},$$

we have $s = s_1 \cup s_2$. Clearly $s_1 \vDash \Diamond\varphi$ and $s_2 \vDash \Diamond\psi$, so $s \vDash \Diamond\varphi \vee \Diamond\psi$.

Conv$\Box\mathbin{\vee\!\!\!\vee}\vee$: This is analogous to Conv$\Diamond\mathbin{\vee\!\!\!\vee}\vee$.

$\Diamond$Sep: It suffices to show that $\Diamond(\varphi \vee (\psi \wedge \text{NE})) \vDash \Diamond\psi$; let $s \vDash \Diamond(\varphi \vee (\psi \wedge \text{NE}))$. If $s = \varnothing$, then clearly $s \vDash \Diamond\psi$. If $s \neq \varnothing$, let $w \in s$. Then there is a nonempty $t \subseteq R[w]$ such that $t \vDash \varphi \vee (\psi \wedge \text{NE})$. Therefore, there are t_1, t_2 such that $t = t_1 \cup t_2$; $t_1 \vDash \varphi$; and $t_2 \vDash \psi \wedge \text{NE}$. Then $t_2 \neq \varnothing$ and $t_2 \vDash \psi$, and clearly $t_2 \subseteq R[w]$. Thus $s \vDash \Diamond\psi$.

$\Diamond$Join: It suffices to show that $\Diamond\varphi, \Diamond\psi \vDash \Diamond(\varphi \vee \psi)$; let $s \vDash \Diamond\varphi$ and $s \vDash \Diamond\psi$. If $s = \varnothing$, then clearly $s \vDash \Diamond(\varphi \vee \psi)$. Otherwise if $s \neq \varnothing$, let $w \in s$. Since $s \vDash \Diamond\varphi$ and $s \vDash \Diamond\psi$, there are nonempty $t_1, t_2 \subseteq R[w]$ such that $t_1 \vDash \varphi$ and $t_2 \vDash \psi$. Therefore, $t_1 \cup t_2 \vDash \varphi \vee \psi$. Clearly $t_1 \cup t_2$ is nonempty and $t_1 \cup t_2 \subseteq R[w]$, so $s \vDash \Diamond(\varphi \vee \psi)$.

$\Box$Inst: It suffices to show that $\Box(\varphi \wedge \text{NE}) \vDash \Diamond\varphi$; let $s \vDash \Box(\varphi \wedge \text{NE})$. If $s = \varnothing$, clearly $s \vDash \Diamond\varphi$. If $s \neq \varnothing$, let $w \in s$. Then $R[w] \vDash \varphi \wedge \text{NE}$, so $R[w] \neq \varnothing$, and therefore $s \vDash \Diamond\varphi$.

$\Box\Diamond$Join: It suffices to show $\Box\varphi, \Diamond\psi \vDash \Box(\varphi \vee \psi)$; let $s \vDash \Box\varphi$ and $s \vDash \Diamond\psi$. If $s = \varnothing$, then clearly $s \vDash \Box(\varphi \vee \psi)$. If $s \neq \varnothing$, let $w \in s$. Since $s \vDash \Box\varphi$, we have $R[w] \vDash \varphi$, and since $s \vDash \Diamond\psi$, there is some nonempty $t \subseteq R[w]$ such that $t \vDash \psi$. Since $R[w] = R[w] \cup t$, we have $R[w] \vDash \varphi \vee \psi$, and so $s \vDash \Box(\varphi \vee \psi)$. □

As expected, a limited replacement lemma holds for our system (cf. Proposition 2.2.4, the semantic replacement lemma with respect to strong equivalence).

2.4.4. LEMMA (Replacement). *Suppose that θ contains a specific occurrence $\theta[p]$ of p which is not in the scope of $\neg$ (unless the $\neg$ forms part of $\Box = \neg\Diamond\neg$). Then $\varphi \vdash \psi$ implies $\theta[\varphi/p] \vdash \theta[\psi/p]$. In particular, if $\varphi \dashv\vdash \psi$, then $\theta[\varphi/p] \dashv\vdash \theta[\psi/p]$.*

Proof:
This is by a routine induction on θ, where the modality cases are proved by applying $\Diamond$Mon and $\Box$Mon. □

We showed in Fact 2.2.5 that every formula is equivalent semantically to one in negation normal form. This fact also holds in the proof system.

2.4.5. PROPOSITION (Negation normal form). *Every formula φ is provably equivalent to a formula in negation normal form.*

Proof:
The result follows by repeated applications of $\neg\neg\varphi \dashv\vdash \varphi$ (by $\neg\neg$E); $\neg(\varphi \wedge \psi) \dashv\vdash \neg\varphi \vee \neg\psi$, $\neg(\varphi \vee \psi) \dashv\vdash \neg\varphi \wedge \neg\psi$ and $\neg(\varphi \mathbin{\vee\!\!\!\vee} \psi) \dashv\vdash \neg\varphi \wedge \neg\psi$ (by the DM-rules); and

$\neg\Diamond\varphi \dashv\vdash \Box\neg\varphi$ (by Inter$\Diamond\Box$). □

Standard commutativity, associativity and distributivity laws for $\wedge$ and $\vee\!\!\!\vee$ can be derived in the usual way in our system. Laws involving $\vee$ hold with some restrictions.

2.4.6. PROPOSITION.

(i) $\varphi\wedge(\psi\vee\chi)\vdash(\varphi\wedge\psi)\vee(\varphi\wedge\chi)$ *if* φ *does not contain* NE, *and* $(\varphi\wedge\psi)\vee(\varphi\wedge\chi)\vdash\varphi\wedge(\psi\vee\chi)$ *if* φ *does not contain* $\vee\!\!\!\vee$;

(ii) $\varphi\vee(\psi\wedge\chi)\vdash(\varphi\vee\psi)\wedge(\varphi\vee\chi)$;

(iii) $\varphi\vee\!\!\!\vee(\psi\vee\chi)\vdash(\varphi\vee\!\!\!\vee\psi)\vee(\varphi\vee\!\!\!\vee\chi)$;

(iv) $\varphi\vee(\psi\vee\!\!\!\vee\chi)\dashv\vdash(\varphi\vee\psi)\vee\!\!\!\vee(\varphi\vee\chi)$.

Proof:
This is easy. □

We often apply the above three basic lemmas (Lemmas 2.4.4–2.4.6) without explicit reference to them. Note that all results in this section which do not involve $\vee\!\!\!\vee$ (such as (i) and (ii) of Proposition 2.4.6) also hold for $\mathrm{BSML}^{\oslash}$ and BSML. Some such results are given $\mathrm{BSML}^{\vee\!\!\!\vee}$-specific derivations in this section. The $\mathrm{BSML}^{\oslash}$- and BSML-derivations can be found in the sequel.

We now move on to prove the completeness theorem for the system. Our strategy consists in showing that each formula is provably equivalent to one in disjunctive normal form—as stated in the following lemma—and then making use of the semantic and proof-theoretic properties of formulas in this form.

2.4.7. LEMMA. *For each* $\varphi\in\mathrm{BSML}^{\vee\!\!\!\vee}$, *each* $k\geq md(\varphi)$, *and each finite* $\mathsf{X}\supseteq\mathsf{P}(\varphi)$, *there is some property* $\mathcal{P}$ *over* X *such that*

$$\varphi\dashv\vdash\mathop{\vee\!\!\!\vee}_{s\in\mathcal{P}}\theta_s^{\mathsf{X},k}.$$

The proof of the above lemma is involved. We withhold this proof for now, and first present the main completeness argument.

As a first step, let us note that our system is a conservative extension of the smallest normal logic **K**—it is easy to see by inspecting our rules, that, for instance, the axioms and rules of the Hilbert-style system for **K** are derivable (see also [134] for a detailed proof of the analogous fact for a similar system). Therefore, since **K** is complete with respect to the class of all Kripke models, the same holds for the classical fragment of our system.

2.4.8. PROPOSITION (Classical completeness). *For any* $\Delta\cup\{\alpha\}\subseteq\mathrm{ML}:\Delta\vDash\alpha$ *if and only if* $\Delta\vdash\alpha$.

As an easy corollary, the kth Hintikka formulas of k-bisimilar pointed models are provably equivalent.

2.4.9. LEMMA.

(i) If $w \rightleftharpoons_k w'$, *then* $\chi^k_w \dashv\vdash \chi^k_{w'}$.

(ii) If $s \rightleftharpoons_k s'$, *then* $\chi^k_s \dashv\vdash \chi^k_{s'}$.

Proof:
For (i), we have $\chi^k_w \equiv \chi^k_{w'}$ by Theorem 2.3.3. Thus $\chi^k_w \dashv\vdash \chi^k_{w'}$ follows from Proposition 2.4.8. Item (ii) follows from (i) and Proposition 2.4.8. □

Note that, strictly speaking, Hintikka formulas are only guaranteed to be defined for finite pointed models; for an infinite pointed model, we in effect choose some finite k-bisimilar pointed model and treat the kth Hintikka formula of this finite model as that of the infinite model. The lemma above ensures that our choice of finite representative does not matter proof-theoretically and hence that our use of these representatives in this section is admissible. It follows from results we show that similar provable equivalence results hold for all the characteristic formulas we make use of, and hence that in all cases the use of these representatives is admissible. We now show that the strong Hintikka formulas of two bisimilar pointed models are likewise provably equivalent.

2.4.10. LEMMA. *If* $s \rightleftharpoons_k s'$, *then* $\theta^k_s \dashv\vdash \theta^k_{s'}$.

Proof:
Suppose that $s \rightleftharpoons_k s'$. The two directions are symmetric; we only give the detailed proof for $\theta^k_s \vdash \theta^k_{s'}$, that is, $\bigvee_{w\in s}(\chi^k_w \wedge \mathrm{NE}) \vdash \bigvee_{w'\in s'}(\chi^k_{w'} \wedge \mathrm{NE})$. If $s = \varnothing$, then clearly $s' = \varnothing$, so that $\theta^k_s = \theta^k_{s'} = \bot$ and $\theta^k_s \vdash \theta^k_{s'}$. Otherwise if $s \neq \varnothing$, then for any $w \in s$, there exists by assumption a (nonempty) substate $s'_w \subseteq s'$ such that $w \rightleftharpoons_k w'$ for all $w' \in s'_w$. We may assume that s'_w is the maximal such substate. By Lemma 2.4.9, we have that for any $w' \in s'_w$, $\chi^k_w \vdash \chi^k_{w'}$ implying $\chi^k_w \wedge \mathrm{NE} \vdash \chi^k_{w'} \wedge \mathrm{NE}$. Now, by repeatedly applying the rule $\vee$W, we obtain $\chi^k_{w'} \wedge \mathrm{NE} \vdash \bigvee_{u\in s'_w}(\chi^k_{w'} \wedge \mathrm{NE})$. Since w' is also k-bisimilar to every element $u \in s'_w$, we have by Lemma 2.4.9 again that $\chi^k_{w'} \wedge \mathrm{NE} \vdash \chi^k_u \wedge \mathrm{NE}$. Thus, by $\vee$Mon, we obtain $\bigvee_{u\in s'_w}(\chi^k_{w'} \wedge \mathrm{NE}) \vdash \bigvee_{u\in s'_w}(\chi^k_u \wedge \mathrm{NE})$. Putting all these together, we derive $\chi^k_w \wedge \mathrm{NE} \vdash \bigvee_{u\in s'_w}(\chi^k_u \wedge \mathrm{NE})$; that is, $\chi^k_w \wedge \mathrm{NE} \vdash \theta_{s'_w}$.

Hence, by repeatedly applying $\vee$Mon and $\vee$E, we obtain $\bigvee_{w\in s}(\chi^k_w \wedge \mathrm{NE}) \vdash \bigvee_{w\in s} \theta_{s'_w} \vdash \theta_{\bigcup_{w\in s} s'_w}$. Observe that since $s \rightleftharpoons_k s'$, we have $s' = \bigcup_{w\in s} s'_w$, whereby $\theta_{\bigcup_{w\in s} s'_w} = \theta_{s'}$. Finally, we conclude that $\theta_s \vdash \theta_{s'}$. □

On the other hand, if two states are not k-bisimilar, their strong Hintikka formulas are contradictory.

2.4.11. LEMMA. *If* $s \not\rightleftharpoons_k s'$, *then* $\theta^k_s, \theta^k_{s'} \vdash \bot\!\!\!\bot$.

To prove this fact, we need two lemmas. The first is an analogous result for Hintikka formulas for worlds.

2.4.12. LEMMA. *If $w \not\rightleftharpoons_k w'$ for all $w' \in s'$, then $\bigvee_{w' \in s'} \chi^k_{w'} \vdash \neg \chi^k_w$.*

Proof:
For any $w' \in s'$, we have $w' \nvDash \chi^k_w$ by Theorem 2.3.3 and therefore $w' \vdash \neg \chi^k_w$ by Proposition 2.4.8. So $\bigvee_{w' \in s'} \chi^k_{w'} \vdash \neg \chi^k_w$ by $\vee$E. □

The second lemma states a basic fact concerning the strong contradiction.

2.4.13. LEMMA. $(\alpha \wedge \text{NE}) \vee \varphi, \neg \alpha \vdash \perp\!\!\!\perp$.

Proof:
We have that

$$
\begin{aligned}
((\alpha \wedge \text{NE}) \vee \varphi) \wedge \neg \alpha \quad &\vdash \quad ((\alpha \wedge \text{NE}) \wedge \neg \alpha) \vee (\varphi \wedge \neg \alpha) && \text{(Prop. 2.4.6)} \\
&\vdash \quad (\bot \wedge \text{NE}) \vee (\varphi \wedge \neg \alpha) && (\neg \text{E}) \\
&\vdash \quad \perp\!\!\!\perp && (\perp\!\!\!\perp \text{Ctr})
\end{aligned}
$$

□

Proof of Lemma 2.4.11:
If $s \not\rightleftharpoons_k s'$, then either there is a $w \in s$ such that $w \not\rightleftharpoons_k w'$ for all $w' \in s'$, or there is a $w' \in s'$ such that $w \not\rightleftharpoons_k w'$ for all $w \in s$. We may without loss of generality assume the former. Then

$$
\begin{aligned}
\theta^k_s \wedge \theta^k_{s'} \quad &= \quad \theta^k_s \wedge \bigvee_{w' \in s'} (\chi^k_{w'} \wedge \text{NE}) \\
&\vdash \quad \theta^k_s \wedge \bigvee_{w' \in s'} \chi^k_{w'} \\
&\vdash \quad ((\chi^k_w \wedge \text{NE}) \vee \theta^k_{s \setminus \{w\}}) \wedge \neg \chi^k_w && \text{(Lemma 2.4.12)} \\
&\vdash \quad \perp\!\!\!\perp && \text{(Lemma 2.4.13)}
\end{aligned}
$$

□

We show also that strong Hintikka formulas $\theta^{\mathsf{X},k}_s$ are monotone with respect to the parameters k and X in the sense of the following lemma.

2.4.14. LEMMA.

(i) If $n \leq k$, then $\theta^k_s \vdash \theta^n_s$.

(ii) If $\mathsf{Y} \subseteq \mathsf{X}$, then $\theta^{\mathsf{X},k}_s \vdash \theta^{\mathsf{Y},k}_s$.

Proof:
It follows from Theorem 2.3.3 that $\chi_w^k \vDash \chi_w^n$ and $\chi_w^{\mathsf{X},k} \vDash \chi_w^{\mathsf{Y},k}$ for any $w \in s$. Therefore, $\chi_w^k \vdash \chi_w^n$ and $\chi_w^{\mathsf{X},k} \vdash \chi_w^{\mathsf{Y},k}$ by Proposition 2.4.8. Then also $\chi_w^k \wedge \text{NE} \vdash \chi_w^n \wedge \text{NE}$ and $\chi_w^{\mathsf{X},k} \wedge \text{NE} \vdash \chi_w^{\mathsf{Y},k} \wedge \text{NE}$, and the results therefore follow by repeated applications of $\vee$Mon. □

Our final lemma leading to the completeness proof concerns some standard properties of the disjunction elimination rule.

2.4.15. LEMMA.

(i) *If* $\Phi, \mathbb{V}_{j \in J} \varphi_j \nvdash \psi$*, then* $\Phi, \varphi_j \nvdash \psi$ *for some* $j \in J$.

(ii) *If* $\{\mathbb{V}_{j \in J_i} \varphi_j \mid i \in I\} \nvdash \psi$*, then* $\{\varphi_i \mid i \in I\} \nvdash \psi$ *for some collection* $\{\varphi_i \mid i \in I\}$ *such that for each* $i \in I$*,* $\varphi_i \in \{\varphi_j \mid j \in J_i\}$.

Proof:
See, for example, Ciardelli, Iemhoff, and Yang [39]. Item (i) is proved easily by applying $\mathbb{V}$E. For (ii), choose an enumeration of I; the result then follows by an inductive proof making use of (i). □

We are now ready to prove completeness. Our proof is similar to the one given in [140] for the completeness of the system PT$^+$, though note that the proof in [140] is for weak completeness; here we directly prove strong completeness.

2.4.16. THEOREM (Completeness of BSML$^{\mathbb{V}}$). *If* $\Phi \vDash \psi$*, then* $\Phi \vdash \psi$.

Proof:
Assume that $\Phi \nvdash \psi$. We show that $\Phi \nvDash \psi$. Put $\Phi = \{\varphi_i \mid i \in I\}$. For each $i \in I$, let $k_i := \max\{md(\varphi_i), md(\psi)\}$, and $\mathsf{X}_i := \mathsf{P}(\varphi_i) \cup \mathsf{P}(\psi)$. By Lemma 2.4.7,

$$\varphi_i \dashv\vdash \mathbb{V}_{s \in \mathcal{P}_i} \theta_s^{\mathsf{X}_i, k_i} \tag{2.1}$$

for some properties $\mathcal{P}_i$ over X_i. Then also $\{\mathbb{V}_{s \in \mathcal{P}_i} \theta_s^{\mathsf{X}_i, k_i} \mid i \in I\} \nvdash \psi$. By Lemma 2.4.15, there is a collection of formulas $\Phi' = \{\theta_{s_i}^{\mathsf{X}_i, k_i} \mid s_i \in \mathcal{P}_i \text{ and } i \in I\}$ such that $\Phi' \nvdash \psi$.

Observe that $\Phi' \nvDash \bot\!\!\!\bot$, since otherwise clearly for some $\theta_{s_i}^{\mathsf{X}_i, k_i}, \theta_{s_j}^{\mathsf{X}_j, k_j} \in \Phi'$ we have $s_i \not\rightleftarrows_m^{\mathsf{M}} s_j$ for $m = \min\{k_i, k_j\}$ and $\mathsf{M} = \mathsf{X}_i \cap \mathsf{X}_j$, so that by Lemma 2.4.11, $\theta_{s_i}^{\mathsf{M},m}, \theta_{s_j}^{\mathsf{M},m} \vdash \bot\!\!\!\bot$. Then by Lemma 2.4.14, $\theta_{s_i}^{\mathsf{X}_i, k_i}, \theta_{s_j}^{\mathsf{X}_j, k_j} \vdash \bot\!\!\!\bot$, so that $\Phi' \vdash \bot\!\!\!\bot$, whence $\Phi' \vdash \psi$ by Lemma 2.4.2; a contradiction.

Thus, we can let t be such that $t \vDash \Phi'$. By (2.1) and soundness, we have $\Phi' \vDash \varphi_i$ for each $i \in I$. Therefore, $t \vDash \Phi$. To show that $\Phi \nvDash \psi$, it suffices to show that $t \nvDash \psi$. Assume otherwise. Take an $i \in I$. By Lemma 2.4.7, we have

$$\psi \dashv\vdash \mathbb{V}_{r \in \mathcal{Q}} \theta_r^{\mathsf{X}_i, k_i} \tag{2.2}$$

for some property $\mathcal{Q}$ over X_i. Since $t \vDash \psi$, by (2.2) and soundness, we have $t \vDash \theta_r^{\mathsf{X}_i,k_i}$ for some $r \in \mathcal{Q}$. Meanwhile, $t \vDash \Phi'$ implies $t \vDash \theta_{s_i}^{\mathsf{X}_i,k_i}$. Thus, by Proposition 2.3.11(ii), we have $r \rightleftharpoons_{k_i}^{\mathsf{X}_i} t \rightleftharpoons_{k_i}^{\mathsf{X}_i} s_i$, whence $\theta_{s_i}^{\mathsf{X}_i,k_i} \vdash \theta_r^{\mathsf{X}_i,k_i}$ by Lemma 2.4.10. Then by $\mathbb{V}$ I and (2.2), $\theta_{s_i}^{\mathcal{X}_i,k_i} \vdash \mathbb{V}_{r\in\mathcal{Q}} \theta_r^{\mathsf{X}_i,k_i} \vdash \psi$. But then $\Phi' \vdash \psi$, which is a contradiction. □

We derive the compactness of BSML$^{\mathbb{V}}$ as well as that of the weaker logics BSML BSML$^{\oslash}$ as a corollary of strong completeness. The remainder of this section is devoted to the proof of Lemma 2.4.7 (provable equivalence of the normal form). We start with two technical lemmas concerning the behavior of NE in disjunctions.

2.4.17. LEMMA. $\varphi \vee (\psi \wedge \text{NE}) \dashv\vdash (\varphi \vee (\psi \wedge \text{NE})) \wedge \text{NE}$.

Proof:
$\dashv$ follows by $\wedge$E. We give a BSML$^{\mathbb{V}}$-specific derivation for $\vdash$:

	$\varphi \vee (\psi \wedge \text{NE})$	
$\vdash$	$(\varphi \vee (\psi \wedge \text{NE})) \wedge (\bot \mathbin{\mathbb{V}} \text{NE})$	(NEI)
$\vdash$	$((\varphi \vee (\psi \wedge \text{NE})) \wedge \bot) \mathbin{\mathbb{V}} ((\varphi \vee (\psi \wedge \text{NE})) \wedge \text{NE})$	
$\vdash$	$((\varphi \wedge \bot) \vee ((\psi \wedge \text{NE}) \wedge \bot)) \mathbin{\mathbb{V}} ((\varphi \vee (\psi \wedge \text{NE})) \wedge \text{NE})$	(Prop. 2.4.6)
$\vdash$	$((\varphi \wedge \bot) \vee (\psi \wedge \bot\!\!\!\bot)) \mathbin{\mathbb{V}} ((\varphi \vee (\psi \wedge \text{NE})) \wedge \text{NE})$	
$\vdash$	$((\varphi \vee (\psi \wedge \text{NE})) \wedge \text{NE}) \mathbin{\mathbb{V}} ((\varphi \vee (\psi \wedge \text{NE})) \wedge \text{NE})$	($\bot\!\!\!\bot$Ctr)
$\vdash$	$(\varphi \vee (\psi \wedge \text{NE})) \wedge \text{NE}$	

□

2.4.18. LEMMA. $\varphi \vee \psi \vdash \varphi \mathbin{\mathbb{V}} \psi \mathbin{\mathbb{V}} ((\varphi \wedge \text{NE}) \vee ((\psi \wedge \text{NE}))$.

Proof:
We have that

$\varphi \vee \psi$	$\vdash$	$(\varphi \wedge (\bot \mathbin{\mathbb{V}} \text{NE})) \vee \psi$	(NEI, $\vee$Mon)
	$\vdash$	$((\varphi \wedge \bot) \mathbin{\mathbb{V}} (\varphi \wedge \text{NE})) \vee \psi$	
	$\vdash$	$((\varphi \wedge \bot) \vee \psi) \mathbin{\mathbb{V}} ((\varphi \wedge \text{NE}) \vee \psi)$	(Distr $\vee\mathbb{V}$)
	$\vdash$	$\psi \mathbin{\mathbb{V}} ((\varphi \wedge \text{NE}) \vee \psi)$	($\bot$E)
	$\vdash$	$\psi \mathbin{\mathbb{V}} ((\varphi \wedge \text{NE}) \vee (\psi \wedge (\bot \mathbin{\mathbb{V}} \text{NE})))$	(NEI, $\vee$Mon)
	$\vdash$	$\psi \mathbin{\mathbb{V}} ((\varphi \wedge \text{NE}) \vee ((\psi \wedge \bot) \mathbin{\mathbb{V}} (\psi \wedge \text{NE})))$	
	$\vdash$	$\psi \mathbin{\mathbb{V}} ((\varphi \wedge \text{NE}) \vee (\psi \wedge \bot)) \mathbin{\mathbb{V}} ((\varphi \wedge \text{NE}) \vee (\psi \wedge \text{NE}))$	(Distr $\vee\mathbb{V}$)
	$\vdash$	$\psi \mathbin{\mathbb{V}} \varphi \mathbin{\mathbb{V}} ((\varphi \wedge \text{NE}) \vee (\psi \wedge \text{NE}))$	($\bot$E)

□

The following lemma further characterizes the interactions between NE and the disjunctions. From item (iv) below, it follows that formulas which are in ML-normal form $\bigvee_{(M,w)\in[\![\alpha]\!]}\chi^k_w$ can be converted into BSML$^{\mathbb{V}}$-normal form $\mathbb{V}_{\mathcal{P}\subseteq[\![\alpha]\!]}\bigvee_{w\in\mathcal{P}}(\chi^k_w\wedge \text{NE})$.

2.4.19. LEMMA.

(i) $\text{NE},\bigvee_{i\in I}\varphi_i\vdash \mathbb{V}_{\varnothing\neq J\subseteq I}\bigvee_{j\in J}(\varphi_j\wedge\text{NE})$.

(ii) $\text{NE}\wedge\bigvee_{i\in I}\alpha_i\dashv\vdash \mathbb{V}_{\varnothing\neq J\subseteq I}\bigvee_{j\in J}(\alpha_j\wedge\text{NE})$.

(iii) $\bigvee_{i\in I}\varphi_i\vdash \mathbb{V}_{J\subseteq I}\bigvee_{j\in J}(\varphi_j\wedge\text{NE})$.

(iv) $\bigvee_{i\in I}\alpha_i\dashv\vdash \mathbb{V}_{J\subseteq I}\bigvee_{j\in J}(\alpha_j\wedge\text{NE})$.

Proof:
(i) This is by induction on $k=|I|$. If $k=0$, then $\bigvee\varnothing=\bot$, and by Lemma 2.4.2 we have $\text{NE}\wedge\bot\vdash\psi$ for any ψ.

If $k=1$, then $\bigvee\{\varphi\}=\varphi$, and $\varphi,\text{NE}\vdash\varphi\wedge\text{NE}$ by $\wedge$I; and $\mathbb{V}\{\bigvee\{\varphi\wedge\text{NE}\}\}=\varphi\wedge\text{NE}$.

If $k=2$, then by Lemma 2.4.18 we have $\varphi\vee\psi\vdash\varphi\mathbin{\mathbb{V}}\psi\mathbin{\mathbb{V}}((\varphi\wedge\text{NE})\vee(\psi\wedge\text{NE}))$, so that clearly $\text{NE},\varphi\vee\psi\vdash(\varphi\wedge\text{NE})\mathbin{\mathbb{V}}(\psi\wedge\text{NE})\mathbin{\mathbb{V}}((\varphi\wedge\text{NE})\vee(\psi\wedge\text{NE}))$.

For the case $|I|=k+1$, by the induction hypothesis,

$$\text{NE},\bigvee_{i\in(I\setminus\{x\})}\varphi_i\vdash\mathbb{V}_{\varnothing\neq J\subseteq(I\setminus\{x\})}\bigvee_{j\in J}(\varphi_j\wedge\text{NE}),\tag{2.3}$$

where $x\in I$. Then we have that

$$\begin{aligned}
&\text{NE},\bigvee_{i\in I}\varphi_i\\
\vdash\ &\text{NE}\wedge(\varphi_x\vee\bigvee_{i\in(I\setminus\{x\})}\varphi_i)\\
\vdash\ &(\varphi_x\wedge\text{NE})\mathbin{\mathbb{V}}(\text{NE}\wedge\bigvee_{i\in(I\setminus\{x\})}\varphi_i)\mathbin{\mathbb{V}}((\varphi_x\wedge\text{NE})\vee(\text{NE}\wedge\bigvee_{i\in(I\setminus\{x\})}\varphi_i)) && \text{(Case } k=2)\\
\vdash\ &(\varphi_x\wedge\text{NE})\mathbin{\mathbb{V}}(\mathbb{V}_{\varnothing\neq J\subseteq(I\setminus\{x\})}\bigvee_{j\in J}(\varphi_j\wedge\text{NE}))\mathbin{\mathbb{V}}((\varphi_x\wedge\text{NE})\vee(\mathbb{V}_{\varnothing\neq J\subseteq(I\setminus\{x\})}\bigvee_{j\in J}(\varphi_j\wedge\text{NE})) && (2.3)\\
\vdash\ &(\varphi_x\wedge\text{NE})\mathbin{\mathbb{V}}(\mathbb{V}_{\varnothing\neq J\subseteq(I\setminus\{x\})}\bigvee_{j\in J}(\varphi_j\wedge\text{NE}))\mathbin{\mathbb{V}}\mathbb{V}_{\varnothing\neq J\subseteq(I\setminus\{x\})}((\varphi_x\wedge\text{NE})\vee\bigvee_{j\in J}(\varphi_j\wedge\text{NE})) && (\mathsf{Distr}\vee\mathbb{V})\\
\vdash\ &\mathbb{V}_{\varnothing\neq J\subseteq I}\bigvee_{j\in J}(\varphi_j\wedge\text{NE})
\end{aligned}$$

(ii) $\vdash$ by (i). $\dashv$: For any nonempty $J \subseteq I$, we have that $\bigvee_{j\in J}(\alpha_j \wedge \text{NE}) \vdash \text{NE}$ by Lemma 2.4.17. We also have that for any $j \in J$, $\alpha_j \wedge \text{NE} \vdash \bigvee_{i\in I}\alpha_i$ by $\wedge\mathsf{E}$ and $\vee\mathsf{I}$. Therefore, $\bigvee_{j\in J}(\alpha_j \wedge \text{NE}) \vdash \bigvee_{i\in I}\alpha_i$ by $\vee\mathsf{E}$. And so $\bigvee\!\!\!\bigvee_{\varnothing\neq J\subseteq I}\bigvee_{j\in J}(\alpha_j \wedge \text{NE}) \vdash \text{NE} \wedge \bigvee_{i\in I}\alpha_i$ by $\mathbin{\vee\!\!\!\vee}\mathsf{E}$.

(iii) We have that

$$\begin{aligned}
\bigvee_{i\in I}\varphi_i &\vdash (\bot \mathbin{\vee\!\!\!\vee} \text{NE}) \wedge \bigvee_{i\in I}\varphi_i && (\text{NE}\mathsf{I})\\
&\vdash (\bot \wedge \bigvee_{i\in I}\varphi_i) \mathbin{\vee\!\!\!\vee} (\text{NE} \wedge \bigvee_{i\in I}\varphi_i) &&\\
&\vdash \bot \mathbin{\vee\!\!\!\vee} \bigvee\!\!\!\bigvee_{\varnothing\neq J\subseteq I}\bigvee_{j\in J}(\varphi_j \wedge \text{NE}) && ((\text{i}))\\
&\vdash \bigvee\!\!\!\bigvee_{J\subseteq I}\bigvee_{j\in J}(\varphi_j \wedge \text{NE}) && (\bigvee\varnothing = \bot)
\end{aligned}$$

(iv) $\vdash$ by (iii). $\dashv$: We have that

$$\begin{aligned}
\bigvee\!\!\!\bigvee_{J\subseteq I}\bigvee_{j\in J}(\alpha_j \wedge \text{NE}) &\vdash \bot \mathbin{\vee\!\!\!\vee} \bigvee\!\!\!\bigvee_{\varnothing\neq J\subseteq I}\bigvee_{j\in J}(\alpha_j \wedge \text{NE}) && (\bigvee\varnothing = \bot)\\
&\vdash \bot \mathbin{\vee\!\!\!\vee} (\text{NE} \wedge \bigvee_{i\in I}\alpha_i) && ((\text{ii}))\\
&\vdash \bigvee_{i\in I}\alpha_i && (\mathbin{\vee\!\!\!\vee}\mathsf{E}, \text{Prop. } 2.4.8)
\end{aligned}$$

□

It now follows that each classical formula is equivalent to one in $\text{BSML}^{\mathbin{\vee\!\!\!\vee}}$-normal form, which, in our inductive proof for Lemma 2.4.10, takes care of the case in which the formula is classical.

2.4.20. LEMMA. *For each $\alpha \in \text{ML}$, each $k \geq md(\alpha)$, and each finite $\mathsf{X} \supseteq \mathsf{P}(\alpha)$, there is some property $\mathcal{P}$ over X such that*

$$\alpha \dashv\vdash \bigvee\!\!\!\bigvee_{s\in\mathcal{P}} \theta_s^{\mathsf{X},\mathsf{k}}.$$

Proof:
$[\![\alpha]\!]$ (over X) is invariant under k-bisimulation so by the proof of Theorem 2.3.5, we have $\alpha \equiv \bigvee_{w\in[\![\alpha]\!]}\chi_w^k$. Then by Proposition 2.4.8, $\alpha \dashv\vdash \bigvee_{w\in[\![\alpha]\!]}\chi_w^k$, and therefore by Lemma 2.4.19(iv):

$$\alpha \dashv\vdash \bigvee_{w\in[\![\alpha]\!]}\chi_w^k \dashv\vdash \bigvee\!\!\!\bigvee_{\mathcal{P}\subseteq[\![\alpha]\!]}\bigvee_{w\in\mathcal{P}}(\chi_w^k \wedge \text{NE}) = \bigvee\!\!\!\bigvee_{s\in\wp([\![\alpha]\!])}\bigvee_{w\in s}\theta_s^k,$$

where $\wp([\![\alpha]\!])$ is the state property $\{\biguplus\{(M,\{w\}) \mid (M,w) \in \mathcal{P}\} \mid \mathcal{P} \subseteq [\![\alpha]\!]\}$. □

Now, for the modality cases in our inductive proof for Lemma 2.4.10, we need to show normal form provable equivalence for formulas of the forms $\Diamond\xi_{\mathcal{P}}^k$ and $\Box\xi_{\mathcal{P}}^k$.

These formulas are flat, so by Proposition 2.3.13 we have $\Diamond\xi^k_{\mathcal{P}} \equiv \alpha$ and $\Box\xi^k_{\mathcal{P}} \equiv \beta$ for some $\alpha, \beta \in \mathrm{ML}$. Given Lemma 2.4.20, it is thus sufficient to show that $\Diamond\xi^k_{\mathcal{P}} \dashv\vdash \alpha$ and $\Box\xi^k_{\mathcal{P}} \dashv\vdash \beta$; that is, the modality cases can be reduced to the case for classical formulas. As will become clear later, this further reduces to showing that formulas of the form $\Diamond\theta^k_s$ or $\Box\theta^k_s$ are provably equivalent to classical formulas. We now work towards proving this result. First, two technical lemmas are in order.

2.4.21. LEMMA. $\Diamond\varphi \vdash \Diamond(\varphi \wedge \mathrm{NE})$.

Proof:
We give a $\mathrm{BSML}^{\mathbb{V}}$-specific derivation:

$$\begin{array}{rcll}
\Diamond\varphi & \vdash & \Diamond(\varphi \wedge (\bot \mathbb{V} \mathrm{NE})) & (\mathrm{NEI}) \\
& \vdash & \Diamond((\varphi \wedge \bot) \mathbb{V} (\varphi \wedge \mathrm{NE})) & \\
& \vdash & \Diamond(\bot \mathbb{V} (\varphi \wedge \mathrm{NE})) & \\
& \vdash & \Diamond\bot \vee \Diamond(\varphi \wedge \mathrm{NE}) & (\Diamond\mathbb{V}\vee\mathsf{Conv}) \\
& \vdash & \bot \vee \Diamond(\varphi \wedge \mathrm{NE}) & (\text{Prop. 2.4.8}, \vee\mathsf{Mon}) \\
& \vdash & \Diamond(\varphi \wedge \mathrm{NE}) & (\bot\mathsf{E})
\end{array}$$

□

2.4.22. LEMMA. *For all φ which do not contain $\mathbb{V}$ and for all ψ:*

(i) $\Diamond((\varphi \wedge \mathrm{NE}) \vee \psi) \dashv\vdash \Diamond(\varphi \vee \psi) \wedge \Diamond\varphi$;

(ii) $\Box((\varphi \wedge \mathrm{NE}) \vee \psi) \dashv\vdash \Box(\varphi \vee \psi) \wedge \Diamond\varphi$.

Proof:
(i) $\vdash$: By $\Diamond$Sep. $\dashv$: We have that

$$\begin{array}{rcll}
\Diamond(\varphi \vee \psi) \wedge \Diamond\varphi & \vdash & \Diamond(\varphi \vee \psi) \wedge \Diamond(\varphi \wedge \mathrm{NE}) & (\text{Lemma 2.4.21}) \\
& \vdash & \Diamond(\varphi \vee (\varphi \wedge \mathrm{NE}) \vee \psi) & (\Diamond\mathsf{Join}) \\
& \vdash & \Diamond(((\varphi \vee (\varphi \wedge \mathrm{NE})) \wedge \mathrm{NE}) \vee \psi) & (\text{Lemma 2.4.17}, \vee\mathsf{Mon}) \\
& \vdash & \Diamond((\varphi \wedge \mathrm{NE}) \vee \psi) & (\vee\mathsf{E})
\end{array}$$

(ii) $\vdash$: We have that

$$\begin{array}{rcll}
\Box((\varphi \wedge \mathrm{NE}) \vee \psi) & \vdash & \Box(\varphi \vee \psi) \wedge \Box((\varphi \wedge \mathrm{NE}) \vee \psi) & \\
& \vdash & \Box(\varphi \vee \psi) \wedge \Box(((\varphi \wedge \mathrm{NE}) \vee \psi) \wedge \mathrm{NE}) & (\text{Lemma 2.4.17}) \\
& \vdash & \Box(\varphi \vee \psi) \wedge \Diamond((\varphi \wedge \mathrm{NE}) \vee \psi) & (\Box\mathsf{Inst}) \\
& \vdash & \Box(\varphi \vee \psi) \wedge \Diamond\varphi & (\Diamond\mathsf{Sep})
\end{array}$$

$\dashv$: Similar to $\dashv$ of (i), using $\Box\Diamond$Join instead of $\Diamond$Join. □

We now show that formulas of the form $\Diamond\theta^k_s$ or $\Box\theta^k_s$ are provably equivalent to classical formulas.

2.4.23. LEMMA.

(i) $\Diamond\theta^k_s \dashv\vdash \Diamond\chi^k_s \wedge \bigwedge_{w\in s}\Diamond\chi^k_w$.

(ii) $\Box\theta^k_s \dashv\vdash \Box\chi^k_s \wedge \bigwedge_{w\in s}\Diamond\chi^k_w$.

Proof:
(i) If $s=\varnothing$, then $\Diamond\theta^k_s = \Diamond\bot = \Diamond\chi^k_s \dashv\vdash \Diamond\chi^k_s \wedge \top\!\!\top = \Diamond\chi^k_s \wedge \bigwedge_{w\in s}\Diamond\chi^k_w$. Otherwise if $s\neq\varnothing$, let $\theta^k_s = (\chi^k_{w_1}\wedge\mathrm{NE})\vee\cdots\vee(\chi^k_{w_n}\wedge\mathrm{NE})$. Then by Lemma 2.4.22(i),

$$\begin{aligned}
\Diamond\theta^k_s &= \Diamond\bigvee_{w\in s}(\chi^k_w\wedge\mathrm{NE})\\
&\dashv\vdash \Diamond(\bigvee_{i=1}^{n-1}(\chi^k_{w_i}\wedge\mathrm{NE})\vee\chi^k_{w_n})\wedge\Diamond\chi^k_{w_n}\\
&\dashv\vdash \Diamond(\bigvee_{i=1}^{n-2}(\chi^k_{w_i}\wedge\mathrm{NE})\vee\chi^k_{w_{n-1}}\vee\chi^k_{w_n})\wedge\Diamond\chi^k_{w_{n-1}}\wedge\Diamond\chi^k_{w_n}\\
&\ \ \vdots\\
&\dashv\vdash \Diamond\chi^k_s\wedge\bigwedge_{w\in s}\Diamond\chi^k_w
\end{aligned}$$

(ii) This is similar to (i), using Lemma 2.4.22(ii) instead of Lemma 2.4.22(i). □

We are finally ready to prove Lemma 2.4.7, the normal form lemma.

Proof of Lemma 2.4.7:
This is by induction on the complexity of φ. By Proposition 2.4.5, we may assume that φ is in negation normal form. Let $k\geq md(\varphi)$.

If $\varphi = p$ or $\varphi=\neg p$, we apply Lemma 2.4.20. If $\varphi=\neg\mathrm{NE}$, we have $\neg\mathrm{NE}\dashv\vdash\bot$ by NEE, and the result then follows by Lemma 2.4.20. If $\varphi=\mathrm{NE}$, we have that

$$\begin{aligned}
\mathrm{NE} &\dashv\vdash \mathrm{NE}\wedge(p\vee\neg p) && \text{(Prop. 2.4.8)}\\
&\dashv\vdash \mathrm{NE}\wedge\mathbin{\backslash\!\!\!\bigvee}_{s\in\mathcal{P}}\theta^k_s && \text{(Lemma 2.4.20)}\\
&\dashv\vdash \mathrm{NE}\wedge(\bot\mathbin{\backslash\!\!\!\vee}\mathbin{\backslash\!\!\!\bigvee}_{s\neq\varnothing\in\mathcal{P}}\theta^k_s) &&\\
&\dashv\vdash (\mathrm{NE}\wedge\bot)\mathbin{\backslash\!\!\!\vee}\mathbin{\backslash\!\!\!\bigvee}_{s\neq\varnothing\in\mathcal{P}}\theta^k_s && \text{(Prop. 2.4.6, Lemma 2.4.17)}\\
&\dashv\vdash \mathbin{\backslash\!\!\!\bigvee}_{s\in\mathcal{P}}\theta^k_s && \text{(Lemma 2.4.2)}
\end{aligned}$$

If $\varphi=\psi\wedge\chi$, then $k\geq md(\varphi)=\max\{md(\psi),md(\chi)\}$, so by the induction hypothesis there are $\mathcal{P}$ and $\mathcal{Q}$ such that $\psi\dashv\vdash\mathbin{\backslash\!\!\!\bigvee}_{s\in\mathcal{P}}\theta^k_s$ and $\chi\dashv\vdash\mathbin{\backslash\!\!\!\bigvee}_{t\in\mathcal{Q}}\theta^k_t$. Let $\mathcal{R}:=\{r\mid\exists s\in\mathcal{P},\exists t\in\mathcal{Q}: s\rightleftharpoons_k r\rightleftharpoons_k t\}$. We have that

$$\psi\wedge\chi \dashv\vdash \mathbin{\backslash\!\!\!\bigvee}_{s\in\mathcal{P}}\theta^k_s\wedge\mathbin{\backslash\!\!\!\bigvee}_{t\in\mathcal{Q}}\theta^k_t$$

$$
\begin{aligned}
&\dashv\vdash \quad \bigvee\!\!\!\bigvee_{s\in\mathcal{P}} \bigvee\!\!\!\bigvee_{t\in\mathcal{Q}} (\theta_s^k \wedge \theta_t^k)\\
&\dashv\vdash \quad \bigvee\!\!\!\bigvee_{s\in\mathcal{P}\cap\mathcal{R}} \bigvee\!\!\!\bigvee_{t\in\mathcal{Q}\cap[s]_k} (\theta_s^k \wedge \theta_t^k) \mathbin{\vee\!\!\!\vee} \bigvee\!\!\!\bigvee_{s\in\mathcal{P}\cap\mathcal{R}} \bigvee\!\!\!\bigvee_{t\in\mathcal{Q}\setminus[s]_k} (\theta_s^k \wedge \theta_t^k)\\
&\qquad \mathbin{\vee\!\!\!\vee} \bigvee\!\!\!\bigvee_{s\in\mathcal{P}\setminus\mathcal{R}} \bigvee\!\!\!\bigvee_{t\in\mathcal{Q}} (\theta_s^k \wedge \theta_t^k)\\
&\qquad\qquad \text{(where } [s]_k \text{ denotes the } \rightleftharpoons_k\text{-equivalence class of } s)\\
&\dashv\vdash \quad \bigvee\!\!\!\bigvee_{s\in\mathcal{P}\cap\mathcal{R}} \bigvee\!\!\!\bigvee_{t\in\mathcal{Q}\cap[s]_k} (\theta_s^k \wedge \theta_t^k) \mathbin{\vee\!\!\!\vee} (\bigvee\!\!\!\bigvee_{s\in\mathcal{P}\cap\mathcal{R}} \bigvee\!\!\!\bigvee_{t\in\mathcal{Q}\setminus[s]_k} \bot\!\!\!\bot) \mathbin{\vee\!\!\!\vee} \bigvee\!\!\!\bigvee_{s\in\mathcal{P}\setminus\mathcal{R}} \bigvee\!\!\!\bigvee_{t\in\mathcal{Q}} \bot\!\!\!\bot\\
&\qquad\qquad \text{(Lemmas 2.4.11 and 2.4.2)}\\
&\dashv\vdash \quad \bigvee\!\!\!\bigvee_{s\in\mathcal{P}\cap\mathcal{R}} \bigvee\!\!\!\bigvee_{t\in\mathcal{Q}\cap[s]_k} (\theta_s^k \wedge \theta_t^k) \qquad \text{(Lemma 2.4.2)}\\
&\dashv\vdash \quad \bigvee\!\!\!\bigvee_{s\in\mathcal{R}} \theta_s^k \qquad \text{(Lemma 2.4.10)}
\end{aligned}
$$

If $\varphi = \psi \vee \chi$, then $k \geq md(\varphi) = \max\{md(\psi), md(\chi)\}$, and by the induction hypothesis we have properties $\mathcal{P}, \mathcal{Q}$ as in the conjunction case. Then

$$
\begin{aligned}
\psi \vee \chi \quad &\dashv\vdash \quad \bigvee\!\!\!\bigvee_{s\in\mathcal{P}} \theta_s^k \vee \bigvee\!\!\!\bigvee_{t\in\mathcal{Q}} \theta_t^k\\
&\dashv\vdash \quad \bigvee\!\!\!\bigvee_{s\in\mathcal{P}} \bigvee\!\!\!\bigvee_{t\in\mathcal{Q}} (\theta_s^k \vee \theta_t^k) \qquad \text{(Prop. 2.4.6)}\\
&\dashv\vdash \quad \bigvee\!\!\!\bigvee_{s\in\mathcal{P}} \bigvee\!\!\!\bigvee_{t\in\mathcal{Q}} \theta_{s\uplus t}^k
\end{aligned}
$$

If $\varphi = \psi \mathbin{\vee\!\!\!\vee} \chi$, then the result follows immediately by the induction hypothesis.

If $\varphi = \Diamond \psi$, then $k-1 \geq md(\varphi) - 1 = md(\psi)$, so by the induction hypothesis there is a property $\mathcal{P}$ such that $\psi \dashv\vdash \bigvee\!\!\!\bigvee_{s\in\mathcal{P}} \theta_s^{k-1}$. Thus,

$$
\begin{aligned}
\Diamond \psi \quad &\dashv\vdash \quad \Diamond \bigvee\!\!\!\bigvee_{s\in\mathcal{P}} \theta_s^{k-1}\\
&\dashv\vdash \quad \bigvee_{s\in\mathcal{P}} \Diamond \theta_s^{k-1} \qquad (\mathsf{Conv}\Diamond \mathbin{\vee\!\!\!\vee} \vee)\\
&\dashv\vdash \quad \bigvee_{s\in\mathcal{P}} (\Diamond \chi_s^{k-1} \wedge \bigwedge_{w\in s} \Diamond \chi_w^{k-1}) \qquad \text{(Lemma 2.4.23(i))}
\end{aligned}
$$

This formula is classical and of modal depth at most k, so we are done by Lemma 2.4.20.

The case $\varphi = \Box \psi$ is similar to the case for $\Diamond \psi$, except we use $\mathsf{Conv}\Box \mathbin{\vee\!\!\!\vee} \vee$ instead of $\mathsf{Conv}\Diamond \mathbin{\vee\!\!\!\vee} \vee$ and Lemma 2.4.23(ii) instead of Lemma 2.4.23(i). □

2.4.2 BSML$^{\oslash}$

For the BSML$^{\oslash}$-system, we remove the rules concerning $\mathbin{\vee\!\!\!\vee}$ from the BSML$^{\mathbin{\vee\!\!\!\vee}}$-system and add rules for the emptiness operator $\oslash$. Recall that $\oslash$ corresponds essentially to one specific type of global disjunction, as $\oslash\psi \equiv \psi \mathbin{\vee\!\!\!\vee} \bot$. The introduction rules for $\oslash$ capture (the right-to-left entailment in) this equivalence by imitating instances of $\mathbin{\vee\!\!\!\vee}$-introduction which yield $\psi \vdash \psi \mathbin{\vee\!\!\!\vee} \bot$ and $\bot \vdash \psi \mathbin{\vee\!\!\!\vee} \bot$. The elimination rules for $\oslash$ capture something stronger than (the left-to-right entailment in) $\oslash\psi \equiv \psi \mathbin{\vee\!\!\!\vee} \bot$: we also encode the fact that $\vee$, $\wedge$, and $\oslash$ distribute over $\mathbin{\vee\!\!\!\vee}$—and that $\neg$ and $\Diamond$ do not—directly into these rules. To that end, we call a formula occurrence $[\psi]$ $\mathbin{\vee\!\!\!\vee}$-*distributive* in χ if $[\psi]$ is not within the scope of any $\neg$ or $\Diamond$ in χ (where recall that $\Box$ is an abbreviation of $\neg\Diamond\neg$). For example, $[p]$ is not $\mathbin{\vee\!\!\!\vee}$-distributive in $\Box(p \vee q) = \neg\Diamond\neg(p \vee q)$ but it *is* $\mathbin{\vee\!\!\!\vee}$-distributive in the subformula $p \vee q$. Given the pertinent equivalence and distributivity facts, we have that if $[\oslash\psi]$ is $\mathbin{\vee\!\!\!\vee}$-distributive in φ, then $\varphi \equiv \varphi[\psi \mathbin{\vee\!\!\!\vee} \bot/\oslash\psi] \equiv \varphi[\psi/\oslash\psi] \mathbin{\vee\!\!\!\vee} \varphi[\bot/\oslash\psi]$ and $\Diamond\varphi \equiv \Diamond\varphi[\psi \mathbin{\vee\!\!\!\vee} \bot/\oslash\psi] \equiv \Diamond(\varphi[\psi/\oslash\psi] \mathbin{\vee\!\!\!\vee} \varphi[\bot/\oslash\psi]) \equiv \Diamond\varphi[\psi/\oslash\psi] \vee \Diamond\varphi[\bot/\oslash\psi]$ (and similarly for $\Box$). The elimination rules for $\oslash$ capture (the left-to-right entailments in) these equivalences by imitating the $\mathbin{\vee\!\!\!\vee}$-rules as applied to formulas of the form $\varphi[\psi/\oslash\psi] \mathbin{\vee\!\!\!\vee} \varphi[\bot/\oslash\psi]$ and $\Diamond/\Box(\varphi[\psi/\oslash\psi] \mathbin{\vee\!\!\!\vee} \varphi[\bot/\oslash\psi])$.

2.4.24. DEFINITION (Natural deduction system for BSML$^{\oslash}$). The natural deduction system for BSML$^{\oslash}$ includes all rules not involving $\mathbin{\vee\!\!\!\vee}$ from the system for BSML$^{\mathbin{\vee\!\!\!\vee}}$ (boxes (a)–(f)) and the following rules for $\oslash$:

$$\frac{}{\oslash \mathrm{NE}}\ \oslash\mathrm{NEI} \qquad \frac{\begin{matrix} D \\ \bot \end{matrix}}{\oslash\varphi}\ \oslash\mathsf{I} \qquad \frac{\begin{matrix} D \\ \varphi \end{matrix}}{\oslash\varphi}\ \oslash\mathsf{I}$$

$$\frac{\begin{matrix} D \\ \varphi \end{matrix} \qquad \begin{matrix} [\varphi[\psi/\oslash\psi]] \\ D_1 \\ \chi \end{matrix} \qquad \begin{matrix} [\varphi[\bot/\oslash\psi]] \\ D_2 \\ \chi \end{matrix}}{\chi}\ \oslash\mathsf{E}(*) \qquad \begin{matrix} D \\ \neg\oslash\varphi \\ \hline\hline \neg\varphi \end{matrix}\ \neg\oslash\mathsf{E}$$

$$\frac{\begin{matrix} D \\ \Diamond\varphi \end{matrix}}{\Diamond\varphi[\psi/\oslash\psi] \vee \Diamond\varphi[\bot/\oslash\psi]}\ \Diamond\oslash\mathsf{E}(*) \qquad \frac{\begin{matrix} D \\ \Box\varphi \end{matrix}}{\Box\varphi[\psi/\oslash\psi] \vee \Box\varphi[\bot/\oslash\psi]}\ \Box\oslash\mathsf{E}(*)$$

$(*)$ $[\oslash\psi]$ is $\mathbin{\vee\!\!\!\vee}$-distributive in φ.

The introduction rules $\oslash$NEI and $\oslash$I imitate the BSML$^{\mathbin{\vee\!\!\!\vee}}$-rules NEI and $\mathbin{\vee\!\!\!\vee}$I, respectively. The elimination rule $\oslash$E imitates the rule $\mathbin{\vee\!\!\!\vee}$E; and $\Diamond\oslash$E and $\Box\oslash$E imitate $\Diamond\mathbin{\vee\!\!\!\vee}\vee$Conv and $\Box\mathbin{\vee\!\!\!\vee}\vee$Conv, respectively. The anti-support clause for $\oslash\varphi$ is characterized by the elimination rule $\neg\oslash$E. The soundness of these new rules follows from the equivalences noted above.

2.4.25. THEOREM (Soundness of BSML$^{\oslash}$). *If* $\Phi \vdash \psi$, *then* $\Phi \vDash \psi$.

To provide a simple demonstration of the new rules, we derive the useful fact that $\oslash$ cancels out the effects of appending $\wedge$NE to classical formulas.

2.4.26. LEMMA. $\alpha \dashv\vdash \oslash(\alpha \wedge \mathrm{NE})$.

Proof:
$\dashv$: $\bot \vdash \alpha$ by $\neg$I and $\neg$E, and $\alpha \wedge \mathrm{NE} \vdash \alpha$ so $\oslash(\alpha \wedge \mathrm{NE}) \vdash \alpha$ by $\oslash$E. $\vdash$: $\alpha \vdash \alpha \wedge \oslash\mathrm{NE}$ by $\oslash$NEI. We have $\alpha \wedge \oslash\mathrm{NE}[\mathrm{NE}/\oslash\mathrm{NE}] = \alpha \wedge \mathrm{NE}$, and $\alpha \wedge \mathrm{NE} \vdash \oslash(\alpha \wedge \mathrm{NE})$ by $\oslash$I. On the other hand, $\alpha \wedge \oslash\mathrm{NE}[\bot/\oslash\mathrm{NE}] = \alpha \wedge \bot$, and $\alpha \wedge \bot \vdash \bot \vdash \oslash(\alpha \wedge \mathrm{NE})$ by $\oslash$I. Therefore, $\oslash\mathrm{NE} \wedge \alpha \vdash \oslash(\alpha \wedge \mathrm{NE})$ by $\oslash$E. □

Our strategy for proving completeness is similar to that used in Section 2.4.1. We again require a crucial normal form provable equivalence result. This is stated in the next lemma; the proof is once more withheld until the end of the section.

2.4.27. LEMMA. *For each $\varphi \in \mathrm{BSML}^{\oslash}$ and each $k \geq md(\varphi)$, there is some property $\mathcal{P}$ such that*

$$\varphi \dashv\vdash \bigvee_{s\in\mathcal{P}} \oslash\theta_s^k \qquad or \qquad \varphi \dashv\vdash \mathrm{NE} \wedge \bigvee_{s\in\mathcal{P}} \oslash\theta_s^k.$$

A number of lemmas which were proved specifically for $\mathrm{BSML}^{\mathbb{V}}$ in Section 2.4.1 also hold for $\mathrm{BSML}^{\oslash}$, and their $\mathrm{BSML}^{\oslash}$-proofs are similar. In particular, Lemma 2.4.4 (replacement) follows by a similar inductive argument, where the new case $\theta = \oslash\eta$ is proved by applying $\oslash$E and $\oslash$I. Proposition 2.4.5 (negation normal form) is proved by a similar argument, using the additional equivalence $\neg\oslash\varphi \dashv\vdash \neg\varphi$ (by $\neg\oslash$E).

Now, observe that a formula in $\mathrm{BSML}^{\oslash}$-normal form can be converted into an equivalent formula in the normal form of the expressively stronger logic $\mathrm{BSML}^{\mathbb{V}}$, as stated in the following lemma (we omit the easy proof).

2.4.28. LEMMA. $\bigvee_{s\in\mathcal{P}} \oslash\theta_s^k \equiv \bigvee\!\!\!\bigvee_{\mathcal{Q}\subseteq\mathcal{P}} \theta^k_{\uplus\mathcal{Q}}$.

While the global disjunction $\mathbb{V}$ is not in the language of $\mathrm{BSML}^{\oslash}$, we show in the next lemma that the formulas $\bigvee_{s\in\mathcal{P}} \oslash\theta_s^k$ and $\bigvee\!\!\!\bigvee_{\mathcal{Q}\subseteq\mathcal{P}} \theta^k_{\uplus\mathcal{Q}}$ are also proof-theoretically interchangeable in the sense that if rules for both $\oslash$ and $\mathbb{V}$ were available, then the two formulas would derive the same consequences and be derivable from the same premises. Item (i) below corresponds to the entailment $\bigvee\!\!\!\bigvee_{\mathcal{Q}\subseteq\mathcal{P}} \theta^k_{\uplus\mathcal{Q}} \vDash \bigvee_{s\in\mathcal{P}} \oslash\theta_s^k$, and item (ii) simulates the other direction $\bigvee_{s\in\mathcal{P}} \oslash\theta_s^k \vDash \bigvee\!\!\!\bigvee_{\mathcal{Q}\subseteq\mathcal{P}} \theta^k_{\uplus\mathcal{Q}}$.

2.4.29. LEMMA.

(i) For any $\mathcal{Q} \subseteq \mathcal{P}$, we have $\theta^k_{\uplus\mathcal{Q}} \vdash \bigvee_{s\in\mathcal{P}} \oslash\theta_s^k$.

(ii) Suppose that $\bigvee_{s\in\mathcal{P}} \oslash\theta_s^k$ is $\mathbb{V}$-distributive in φ. If $\Phi, \varphi[\theta^k_{\uplus\mathcal{Q}}/\bigvee_{s\in\mathcal{P}} \oslash\theta_s^k] \vdash \psi$ for all $\mathcal{Q} \subseteq \mathcal{P}$, then $\Phi, \varphi \vdash \psi$.

Proof:
(i) Clearly, $\theta^k_{\uplus\mathcal{Q}} \dashv\vdash \bigvee_{s\in\mathcal{Q}}\theta^k_s$. Next, we derive $\bigvee_{s\in\mathcal{Q}}\theta^k_s \vdash \bigvee_{s\in\mathcal{Q}}\oslash\theta^k_s$ by $\oslash$I and $\vee$Mon. Since $\bigvee_{s\in\mathcal{Q}}\oslash\theta^k_s \vdash \bigvee_{s\in\mathcal{Q}}\oslash\theta^k_s \vee \bigvee_{t\in\mathcal{P}\setminus\mathcal{Q}}\bot$ (by $\vee$I), we obtain $\bigvee_{s\in\mathcal{Q}}\oslash\theta^k_s \vdash \bigvee_{s\in\mathcal{Q}}\oslash\theta^k_s \vee \bigvee_{t\in\mathcal{P}\setminus\mathcal{Q}}\oslash\theta^k_t$ by $\oslash$I—that is, $\bigvee_{s\in\mathcal{Q}}\oslash\theta^k_s \vdash \bigvee_{s\in\mathcal{P}}\oslash\theta^k_s$. Putting all these together, we conclude that $\theta^k_{\uplus\mathcal{Q}} \vdash \bigvee_{s\in\mathcal{P}}\oslash\theta^k_s$.

(ii) Let $\mathcal{P}=\{s_1,\dots,s_n\}$, and thus $\zeta^k_{\mathcal{P}} = \bigvee_{s\in\mathcal{P}}\oslash\theta^k_s = \oslash\theta^k_{s_1}\vee\cdots\vee\oslash\theta^k_{s_n}$. Observe that for every $\mathcal{Q}\subseteq\mathcal{P}$, we have $\theta^k_{\uplus\mathcal{Q}} \dashv\vdash \bigvee_{s_i\in\mathcal{Q}}\theta^k_{s_i} \dashv\vdash \bigvee_{i=1}^n\tau_i$, where $\tau_i=\theta^k_{s_i}$ if $s_i\in\mathcal{Q}$, and $\tau_i=\bot$ if $s_i\notin\mathcal{Q}$, by $\vee$I and $\bot$E. Now, by assumption, we have $\Phi,\varphi[\theta^k_{\uplus\varnothing}/\zeta^k_{\mathcal{P}}]\vdash\psi$ and $\Phi,\varphi[\theta^k_{\uplus\{s_n\}}/\zeta^k_{\mathcal{P}}]\vdash\psi$. Since $\theta^k_{\uplus\varnothing}\dashv\vdash\bigvee_{i=1}^n\bot$ and $\theta^k_{\uplus\{s_n\}}\dashv\vdash(\bigvee_{i=1}^{n-1}\bot)\vee\theta^k_{s_n}$, we obtain

$$\Phi,\varphi[(\bigvee_{i=1}^{n-2}\bot)\vee\bot\vee\bot/\zeta^k_{\mathcal{P}}]\vdash\psi \qquad \text{and} \qquad \Phi,\varphi[(\bigvee_{i=1}^{n-2}\bot)\vee\bot\vee\theta^k_{s_n}/\zeta^k_{\mathcal{P}}]\vdash\psi.$$

Then by $\oslash$E, we obtain

$$\Phi,\varphi[(\bigvee_{i=1}^{n-2}\bot)\vee\bot\vee\oslash\theta^k_{s_n}/\zeta^k_{\mathcal{P}}]\vdash\psi. \tag{2.4}$$

Next, by assumption again, $\Phi,\varphi[\theta^k_{\uplus\{s_{n-1}\}}/\zeta^k_{\mathcal{P}}]\vdash\psi$ and $\Phi,\varphi[\theta^k_{\uplus\{s_{n-1},s_n\}}/\zeta^k_{\mathcal{P}}]\vdash\psi$, which imply

$$\Phi,\varphi[(\bigvee_{i=1}^{n-2}\bot)\vee\theta^k_{s_{n-1}}\vee\bot/\zeta^k_{\mathcal{P}}]\vdash\psi \qquad \text{and} \qquad \Phi,\varphi[(\bigvee_{i=1}^{n-2}\bot)\vee\theta^k_{s_{n-1}}\vee\theta^k_{s_n}/\zeta^k_{\mathcal{P}}]\vdash\psi.$$

Again, by $\oslash$E, we obtain

$$\Phi,\varphi[(\bigvee_{i=1}^{n-2}\bot)\vee\theta^k_{s_{n-1}}\vee\oslash\theta^k_{s_n}/\zeta^k_{\mathcal{P}}]\vdash\psi. \tag{2.5}$$

Applying $\oslash$E to (2.4) and (2.5), we obtain $\Phi,\varphi[(\bigvee_{i=1}^{n-2}\bot)\vee\oslash\theta^k_{s_{n-1}}\vee\oslash\theta^k_{s_n}/\zeta^k_{\mathcal{P}}]\vdash\psi$. Repeating this argument, we eventually get $\Phi,\varphi\vdash\psi$, as required. □

Given this proof-theoretic correspondence between the normal forms, we can show completeness for BSML$^{\oslash}$ using an argument that is similar to that used for BSML$^{\mathbb{V}}$. This time we first establish weak completeness, and then derive the strong completeness theorem as a corollary of weak completeness and compactness. (The strategy we employed for the direct proof of the strong completeness of BSML$^{\mathbb{V}}$ does not work for BSML$^{\oslash}$ due to the structure of the BSML$^{\oslash}$-normal form.)

2.4.30. THEOREM (Weak completeness of BSML$^{\oslash}$). *If $\varphi\vDash\psi$, then $\varphi\vdash\psi$.*

Proof:
Let $k=\max\{md(\varphi),md(\psi)\}$. By Lemma 2.4.27,

$$\varphi\dashv\vdash\eta_\varphi\wedge\bigvee_{s\in\mathcal{P}}\oslash\theta^k_s \qquad \text{and} \qquad \psi\dashv\vdash\eta_\psi\wedge\bigvee_{t\in\mathcal{Q}}\oslash\theta^k_t, \tag{2.6}$$

where each of η_φ and η_ψ is either NE or (for notational convenience) $\top\!\!\!\top$. Since $\varphi \vDash \psi$, $\eta_\varphi = \top\!\!\!\top$ implies $\eta_\psi = \top\!\!\!\top$. Each of the remaining possibilities—that is, (a) $\eta_\varphi = \eta_\psi = \top\!\!\!\top$; (b) $\eta_\varphi = \eta_\psi = \text{NE}$; (c) $\eta_\varphi = \text{NE}$ and $\eta_\psi = \top\!\!\!\top$—implies by (2.6) and soundness,

$$\bigvee_{s\in\mathcal{P}} \oslash\theta^k_s \vDash \bigvee_{t\in\mathcal{Q}} \oslash\theta^k_t,$$

so that by Lemma 2.4.28,

$$\bigvee\!\!\!\!\bigvee_{\mathcal{P}'\subseteq\mathcal{P}} \theta^k_{\biguplus\mathcal{P}'} \vDash \bigvee\!\!\!\!\bigvee_{\mathcal{Q}'\subseteq\mathcal{Q}} \theta^k_{\biguplus\mathcal{Q}'}. \tag{2.7}$$

Let $\mathcal{P}' \subseteq \mathcal{P}$, and $u \vDash \theta^k_{\biguplus\mathcal{P}'}$. By (2.7), we have $u \vDash \bigvee\!\!\!\bigvee_{\mathcal{Q}'\subseteq\mathcal{Q}} \theta^k_{\biguplus\mathcal{Q}'}$. Thus, $u \vDash \theta^k_{\biguplus\mathcal{Q}'}$ for some $\mathcal{Q}' \subseteq \mathcal{Q}$. By Proposition 2.3.11(ii), $\biguplus\mathcal{P}' \rightleftharpoons_k u \rightleftharpoons_k \biguplus\mathcal{Q}'$. Then by Lemma 2.4.10, $\theta^k_{\biguplus\mathcal{P}'} \vdash \theta^k_{\biguplus\mathcal{Q}'}$. By Lemma 2.4.29(i), $\theta^k_{\biguplus\mathcal{Q}'} \vdash \bigvee_{t\in\mathcal{Q}} \oslash\theta^k_t$, so also $\theta^k_{\biguplus\mathcal{P}'} \vdash \bigvee_{t\in\mathcal{Q}} \oslash\theta^k_t$. Hence, by Lemma 2.4.29(ii), $\bigvee_{s\in\mathcal{P}} \oslash\theta^k_s \vdash \bigvee_{t\in\mathcal{Q}} \oslash\theta^k_t$, and therefore $\varphi \vdash \psi$. □

2.4.31. COROLLARY (Strong completeness of BSML$^\oslash$). *If* $\Phi \vDash \psi$*, then* $\Phi \vdash \psi$*.*

Proof:
This is by Theorem 2.4.30 and compactness. □

In the remainder of this section we give the inductive proof of Lemma 2.4.27 (normal form provable equivalence). As in Section 2.4.1, we first show normal form provable equivalence for classical formulas.

2.4.32. LEMMA. *For each* $\alpha \in \text{ML}$ *and each* $k \geq md(\alpha)$*, there is some property* $\mathcal{P}$ *such that*

$$\alpha \dashv\vdash \bigvee_{s\in\mathcal{P}} \oslash\theta^k_s.$$

Proof:
Since $[\![\alpha]\!]$ is invariant under k-bisimulation, by the proof of Theorem 2.3.5, we have $\alpha \equiv \bigvee_{w\in[\![\alpha]\!]} \chi^k_w$. Then by Proposition 2.4.8, $\alpha \dashv\vdash \bigvee_{w\in[\![\alpha]\!]} \chi^k_w$, and by Lemma 2.4.26 and $\vee$Mon,

$$\bigvee_{w\in[\![\alpha]\!]} \chi^k_w \dashv\vdash \bigvee_{w\in[\![\alpha]\!]} \oslash(\chi^k_w \wedge \text{NE}) = \bigvee_{\{w\}\in\wp([\![\alpha]\!])} \oslash\theta^k_{\{w\}}.$$ □

Again following the argument in Section 2.4.1, recall that in our inductive proof of Lemma 2.4.27, another important step consists in showing that formulas of the form $\Diamond\theta^k_s$ or $\Box\theta^k_s$ are provably equivalent to classical formulas. In Section 2.4.1, this was proved in Lemma 2.4.23, which made use of Lemma 2.4.22. The two lemmas leading to Lemma 2.4.22 (namely, Lemmas 2.4.17 and 2.4.21) were given, in Section 2.4.1, only $\mathrm{BSML}^{\mathbb{V}}$-specific derivations. We now show that these two lemmas can also be derived in the system for $\mathrm{BSML}^{\oslash}$; we may then also make use of Lemmas 2.4.22 and 2.4.23 in $\mathrm{BSML}^{\oslash}$.

Proof of Lemma 2.4.17 (in $\mathrm{BSML}^{\oslash}$**):**

We show the nontrivial direction $\varphi \vee (\psi \wedge \mathrm{NE}) \vdash (\varphi \vee (\psi \wedge \mathrm{NE})) \wedge \mathrm{NE}$. First, $\varphi \vee (\psi \wedge \mathrm{NE}) \vdash (\varphi \vee (\psi \wedge \mathrm{NE})) \wedge \oslash \mathrm{NE}$ by $\oslash \mathrm{NE}$I. Then, $(\varphi \vee (\psi \wedge \mathrm{NE})) \wedge \bot \vdash (\varphi \wedge \bot) \vee (\psi \wedge \mathrm{NE} \wedge \bot)$ by Proposition 2.4.6 and $(\varphi \wedge \bot) \vee (\psi \wedge \mathrm{NE} \wedge \bot) \vdash (\varphi \wedge \bot) \vee \mathbb{\bot} \vdash (\varphi \vee (\psi \wedge \mathrm{NE})) \wedge \mathrm{NE}$ by $\mathbb{\bot}$Ctr. Obviously, $(\varphi \vee (\psi \wedge \mathrm{NE})) \wedge \mathrm{NE} \vdash (\varphi \vee (\psi \wedge \mathrm{NE})) \wedge \mathrm{NE}$; thus we have $\varphi \vee (\psi \wedge \mathrm{NE}) \vdash (\varphi \vee (\psi \wedge \mathrm{NE})) \wedge \mathrm{NE}$ by $\oslash$E. □

Proof of Lemma 2.4.21 (in $\mathrm{BSML}^{\oslash}$**):**

We derive $\Diamond \varphi \vdash \Diamond(\varphi \wedge \mathrm{NE})$ as follows:

$$
\begin{array}{lcll}
\Diamond \varphi & \vdash & \Diamond(\varphi \wedge \oslash \mathrm{NE}) & (\oslash \mathrm{NE}\mathsf{I}) \\
& \vdash & \Diamond(\varphi \wedge \mathrm{NE}) \vee \Diamond(\varphi \wedge \bot) & (\Diamond \oslash \mathsf{E}) \\
& \vdash & \Diamond(\varphi \wedge \mathrm{NE}) \vee \bot & (\text{Prop. } 2.4.8) \\
& \vdash & \Diamond(\varphi \wedge \mathrm{NE}) & (\bot \mathsf{E})
\end{array}
$$

□

With all the pieces at hand, we are now ready to give the full proof of Lemma 2.4.27.

Proof of Lemma 2.4.27:

This is by induction on the complexity of φ (assumed without loss of generality to be in negation normal form). Let $k \geq md(\varphi)$.

If $\varphi = p$ or $\varphi = \neg p$ or $\varphi = \neg \mathrm{NE}$, then the result follows by Lemma 2.4.32 and $\neg \mathrm{NE}$E. If $\varphi = \mathrm{NE}$, then we have $\mathrm{NE} \dashv\vdash \mathrm{NE} \wedge (p \vee \neg p)$ by Proposition 2.4.8, and $\mathrm{NE} \wedge (p \vee \neg p) \dashv\vdash \mathrm{NE} \wedge \bigvee_{s \in \mathcal{P}} \oslash \theta_s^k$ by Lemma 2.4.32.

If $\varphi = \psi \wedge \chi$, then $k \geq md(\varphi) = \max\{md(\psi), md(\chi)\}$, so by the induction hypothesis there are properties $\mathcal{P}, \mathcal{Q}$ such that

$$
\psi \dashv\vdash \eta_\psi \wedge \bigvee_{s \in \mathcal{P}} \oslash \theta_s^k \qquad \text{and} \qquad \chi \dashv\vdash \eta_\chi \wedge \bigvee_{t \in \mathcal{Q}} \oslash \theta_t^k, \tag{2.8}
$$

where each of η_ψ and η_χ is either $\top\!\!\top$ or NE. Let

$$
\mathcal{R} := \{ r \mid \biguplus \mathcal{P}' \rightleftharpoons_k r \rightleftharpoons_k \biguplus \mathcal{Q}' \text{ for some } \mathcal{P}' \subseteq \mathcal{P} \text{ and some } \mathcal{Q}' \subseteq \mathcal{Q} \}.
$$

We will show that

$$
\bigvee_{s \in \mathcal{P}} \oslash \theta_s^k \wedge \bigvee_{t \in \mathcal{Q}} \oslash \theta_t^k \dashv\vdash \bigvee_{r \in \mathcal{R}} \oslash \theta_r^k, \tag{2.9}
$$

from which the result will follow, since if $\eta_\psi = \eta_\chi = \top\!\!\top$, then $\varphi \wedge \psi \dashv\vdash \bigvee_{r \in \mathcal{R}} \oslash \theta_r^k$; and if $\eta_\psi = \mathrm{NE}$ or $\eta_\chi = \mathrm{NE}$, then $\varphi \wedge \psi \dashv\vdash \mathrm{NE} \wedge \bigvee_{s \in \mathcal{P}} \oslash \theta_s^k \wedge \bigvee_{t \in \mathcal{Q}} \oslash \theta_t^k \dashv\vdash \mathrm{NE} \wedge \bigvee_{r \in \mathcal{R}} \oslash \theta_r^k$.

Now, for the direction $\vdash$ of (2.9), by Lemma 2.4.29(ii), it suffices to show that $\theta^k_{\biguplus \mathcal{P}'}, \theta^k_{\biguplus \mathcal{Q}'} \vdash \bigvee_{s \in \mathcal{R}} \oslash \theta_s^k$ for all $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{Q}' \subseteq \mathcal{Q}$. If $\biguplus \mathcal{P}' \not\rightleftharpoons_k \biguplus \mathcal{Q}'$, then $\theta^k_{\biguplus \mathcal{P}'}, \theta^k_{\biguplus \mathcal{Q}'} \vdash$

$\perp\!\!\!\perp \vdash \bigvee_{r\in\mathcal{R}} \oslash \theta_r^k$ by Lemmas 2.4.11 and 2.4.2. If $\biguplus\mathcal{P}' \rightleftharpoons_k \biguplus\mathcal{Q}' \rightleftharpoons_k u \in \mathcal{R}$, then $\theta^k_{\biguplus\mathcal{P}'} \dashv\vdash \theta^k_u$ by Lemma 2.4.10. Next, we derive by applying $\oslash$I and $\vee$I that $\theta^k_u \vdash \oslash\theta^k_u \vee \bot \vdash \oslash\theta^k_u \vee \bigvee_{u\neq r\in\mathcal{R}} \oslash\theta^k_r \vdash \bigvee_{r\in\mathcal{R}} \oslash\theta^k_r$.

For the converse direction $\dashv$ of (2.9), we only give the detailed proof for $\bigvee_{r\in\mathcal{R}} \oslash\theta^k_r \vdash \bigvee_{s\in\mathcal{P}} \oslash\theta^k_s$. Given Lemma 2.4.29(ii), it suffices to show that $\theta^k_{\biguplus\mathcal{R}'} \vdash \bigvee_{s\in\mathcal{P}} \oslash\theta^k_s$ for all $\mathcal{R}' \subseteq \mathcal{R}$. Observe that $\mathcal{R}' = \{r \mid r \rightleftharpoons_k \biguplus\mathcal{P}' \text{ and } \mathcal{P}' \in \mathcal{X}\}$ for some $\mathcal{X} \subseteq \wp(\mathcal{P})$ and $\biguplus\mathcal{R}' \rightleftharpoons_k \biguplus\bigcup\mathcal{X}$. Thus, $\theta^k_{\biguplus\mathcal{R}'} \dashv\vdash \theta^k_{\biguplus\bigcup\mathcal{X}}$ by Lemma 2.4.10. Since $\bigcup\mathcal{X} \subseteq \mathcal{P}$, we have by Lemma 2.4.29(i) that $\theta^k_{\biguplus\bigcup\mathcal{X}} \vdash \bigvee_{s\in\mathcal{P}} \oslash\theta^k_s$.

If $\varphi = \psi \vee \chi$, then $k \geq md(\varphi) = \max\{md(\psi), md(\chi)\}$, so by the induction hypothesis we have (2.8). If $\eta_\psi = \mathrm{NE}$ and $\eta_\chi = \top\!\!\!\top$, let

$$\mathcal{R} := \{\biguplus\mathcal{P}' \uplus \biguplus\mathcal{Q}' \mid \mathcal{P}' \subseteq \mathcal{P} \text{ s.t. } \biguplus\mathcal{P}' \not\rightleftharpoons_k (M,\varnothing), \text{ and } \mathcal{Q}' \subseteq \mathcal{Q}\},$$

where M is some arbitrary model. We show that $\psi \vee \chi \dashv\vdash \mathrm{NE} \wedge \bigvee_{r\in\mathcal{R}} \oslash\theta^k_r$.

For the direction $\vdash$, given Lemma 2.4.29(ii), it suffices to show that for all $\mathcal{P}' \subseteq \mathcal{P}$ and $\mathcal{Q}' \subseteq \mathcal{Q}$, $(\mathrm{NE} \wedge \theta^k_{\biguplus\mathcal{P}'}) \vee \theta^k_{\biguplus\mathcal{Q}'} \vdash \mathrm{NE} \wedge \bigvee_{r\in\mathcal{R}} \oslash\theta^k_r$. If $\biguplus\mathcal{P}' \rightleftharpoons_k (M,\varnothing)$, then $\theta^k_{\biguplus\mathcal{P}'} \dashv\vdash \bot$ and we have $(\mathrm{NE} \wedge \theta^k_{\biguplus\mathcal{P}'}) \vee \theta^k_{\biguplus\mathcal{Q}'} \vdash \perp\!\!\!\perp \vee \theta^k_{\biguplus\mathcal{Q}'} \vdash \mathrm{NE} \wedge \bigvee_{r\in\mathcal{R}} \oslash\theta^k_r$ by $\perp\!\!\!\perp$Ctr. Otherwise $\biguplus\mathcal{P}' \not\rightleftharpoons_k \varnothing$, and thus $\biguplus\mathcal{P}' \uplus \biguplus\mathcal{Q}' \in \mathcal{R}$. We have

$$(\mathrm{NE} \wedge \theta^k_{\biguplus\mathcal{P}'}) \vee \theta^k_{\biguplus\mathcal{Q}'} \vdash \theta^k_{\biguplus\mathcal{P}'} \vee \theta^k_{\biguplus\mathcal{Q}'} \dashv\vdash \theta^k_{\biguplus\mathcal{P}' \uplus \biguplus\mathcal{Q}'} \vdash \bigvee_{r\in\mathcal{R}} \oslash\theta^k_r,$$

by $\oslash$I and $\vee$I, and $(\mathrm{NE} \wedge \theta^k_{\biguplus\mathcal{P}'}) \vee \theta^k_{\biguplus\mathcal{Q}'} \vdash \mathrm{NE}$ by Lemma 2.4.17.

For the direction $\dashv$, given Lemma 2.4.29(ii), it suffices to show that for all $\mathcal{R}' \subseteq \mathcal{R}$, $\mathrm{NE} \wedge \theta^k_{\biguplus\mathcal{R}'} \vdash (\mathrm{NE} \wedge \bigvee_{s\in\mathcal{P}} \oslash\theta^k_s) \vee \bigvee_{t\in\mathcal{Q}} \oslash\theta^k_t$. If $\mathcal{R}' = \varnothing$, then $\mathrm{NE} \wedge \theta^k_{\biguplus\mathcal{R}'} = \mathrm{NE} \wedge \bot = \perp\!\!\!\perp$, and the implication follows by Lemma 2.4.2. Otherwise

$$\mathcal{R}' = \{\biguplus\mathcal{P}' \uplus \biguplus\mathcal{Q}' \mid \mathcal{P}' \in \mathcal{X} \text{ and } \mathcal{Q}' \in \mathcal{Y}\}$$

for some nonempty $\mathcal{X} \subseteq \wp(\mathcal{P})$ and nonempty $\mathcal{Y} \subseteq \wp(\mathcal{Q})$ with $\biguplus\mathcal{P}' \not\rightleftharpoons_k (M,\varnothing)$ for all $\mathcal{P}' \in \mathcal{X}$. Since $\biguplus\mathcal{R}' \rightleftharpoons_k \biguplus\bigcup\mathcal{X} \uplus \biguplus\bigcup\mathcal{Y}$, by Lemma 2.4.10 we have

$$\theta^k_{\biguplus\mathcal{R}'} \dashv\vdash \theta^k_{\biguplus\bigcup\mathcal{X} \uplus \biguplus\bigcup\mathcal{Y}} \dashv\vdash \theta^k_{\biguplus\bigcup\mathcal{X}} \vee \theta^k_{\biguplus\bigcup\mathcal{Y}}.$$

Observe that $\biguplus\bigcup\mathcal{X} \not\rightleftharpoons_k (M,\varnothing)$, which implies $\theta^k_{\biguplus\bigcup\mathcal{X}} \neq \bot$, so that $\theta^k_{\biguplus\bigcup\mathcal{X}} \vdash \mathrm{NE} \wedge \theta^k_{\biguplus\bigcup\mathcal{X}}$ by Lemma 2.4.17. Thus, $\theta^k_{\biguplus\bigcup\mathcal{X}} \vee \theta^k_{\biguplus\bigcup\mathcal{Y}} \vdash (\mathrm{NE} \wedge \theta^k_{\biguplus\bigcup\mathcal{X}}) \vee \theta^k_{\biguplus\bigcup\mathcal{Y}}$. Finally, we derive by Lemma 2.4.29(i) that $(\mathrm{NE} \wedge \theta^k_{\biguplus\bigcup\mathcal{X}}) \vee \theta^k_{\biguplus\bigcup\mathcal{Y}} \vdash (\mathrm{NE} \wedge \bigvee_{s\in\mathcal{P}} \oslash\theta^k_s) \vee \bigvee_{t\in\mathcal{Q}} \oslash\theta^k_t$.

The other cases are similar; for example, if $\eta_\psi = \eta_\chi = \mathrm{NE}$, one uses

$$\mathcal{R} := \{\biguplus\mathcal{P}' \uplus \biguplus\mathcal{Q}' \mid \mathcal{P}' \subseteq \mathcal{P} \text{ and } \mathcal{Q}' \subseteq \mathcal{Q} \text{ s.t. } \biguplus\mathcal{P}' \not\rightleftharpoons_k \varnothing \text{ and } \biguplus\mathcal{Q}' \not\rightleftharpoons_k \varnothing\},$$

and if $\eta_\psi = \eta_\chi = \top\!\!\!\top$, we clearly have $\varphi \vee \psi \dashv\vdash \bigvee_{u\in\mathcal{P}\cup\mathcal{Q}} \oslash\theta^k_u$.

If $\varphi = \oslash\psi$, then $k \geq md(\varphi) = md(\psi)$, so by the induction hypothesis we have the equivalence for ψ as in (2.8). We show that $\oslash(\eta_\psi \wedge \bigvee_{s\in\mathcal{P}} \oslash\theta^k_s) \dashv\vdash \bigvee_{s\in\mathcal{P}} \oslash\theta^k_s$. For the direction $\vdash$, we have $\eta_\psi \wedge \bigvee_{s\in\mathcal{P}} \oslash\theta^k_s \vdash \bigvee_{s\in\mathcal{P}} \oslash\theta^k_s$ and $\bot \vdash \bigvee_{s\in\mathcal{P}} \bot \vdash \bigvee_{s\in\mathcal{P}} \oslash\theta^k_s$ by $\oslash$I. Thus we conclude that $\oslash(\eta_\psi \wedge \bigvee_{s\in\mathcal{P}} \oslash\theta^k_s) \vdash \bigvee_{s\in\mathcal{P}} \oslash\theta^k_s$ by $\oslash$E. For the converse direction $\dashv$, if $\eta_\psi = \mathbb{T}$, we have $\bigvee_{s\in\mathcal{P}} \oslash\theta^k_s \dashv\vdash \mathbb{T} \wedge \bigvee_{s\in\mathcal{P}} \oslash\theta^k_s \vdash \oslash(\mathbb{T} \wedge \bigvee_{s\in\mathcal{P}} \oslash\theta^k_s)$ by $\oslash$I. If $\eta_\psi = \mathrm{NE}$, we have $\bigvee_{s\in\mathcal{P}} \oslash\theta^k_s \vdash \bigvee_{s\in\mathcal{P}} \oslash\theta^k_s \wedge \oslash\mathrm{NE}$ by $\oslash$NEI. Then since $\bigvee_{s\in\mathcal{P}} \oslash\theta^k_s \wedge \mathrm{NE} \vdash \oslash(\bigvee_{s\in\mathcal{P}} \oslash\theta^k_s \wedge \mathrm{NE})$ by $\oslash$I and $\bigvee_{s\in\mathcal{P}} \oslash\theta^k_s \wedge \bot \vdash \bot \vdash \oslash(\bigvee_{s\in\mathcal{P}} \oslash\theta^k_s \wedge \mathrm{NE})$ by $\oslash$I, we have $\bigvee_{s\in\mathcal{P}} \oslash\theta^k_s \wedge \oslash\mathrm{NE} \vdash \oslash(\bigvee_{s\in\mathcal{P}} \oslash\theta^k_s \wedge \mathrm{NE})$ by $\oslash$E.

If $\varphi = \Diamond\psi$, then $k-1 \geq md(\varphi)-1 = md(\psi)$, so by the induction hypothesis there is a property $\mathcal{P}$ such that $\psi \dashv\vdash \eta_\psi \wedge \bigvee_{s\in\mathcal{P}} \oslash\theta^{k-1}_s$, where $\eta_\psi = \mathbb{T}$ or $\eta_\psi = \mathrm{NE}$. Let $n = k-1$. By Lemma 2.4.32, it suffices to show that ψ is equivalent to some classical formula of modal depth at most $k = n+1$. Now, if $\eta_\psi = \mathrm{NE}$, we have $\Diamond\psi \dashv\vdash \Diamond(\eta_\psi \wedge \bigvee_{s\in\mathcal{P}} \oslash\theta^n_s) \dashv\vdash \Diamond\bigvee_{s\in\mathcal{P}} \oslash\theta^n_s$ by Lemma 2.4.21; the same equivalence holds also when $\eta_\psi = \mathbb{T}$. It is then sufficient to show that

$$\Diamond \bigvee_{s\in\mathcal{P}} \oslash\theta^n_s \dashv\vdash \bigvee_{\mathcal{P}'\subseteq\mathcal{P}} \Diamond\theta^n_{\uplus\mathcal{P}'}, \tag{2.10}$$

for then we have $\bigvee_{\mathcal{P}'\subseteq\mathcal{P}} \Diamond\theta^n_{\uplus\mathcal{P}'} \dashv\vdash \bigvee_{\mathcal{P}'\subseteq\mathcal{P}} (\Diamond\chi^n_{\uplus\mathcal{P}'} \wedge \bigwedge_{w\in\uplus\mathcal{P}'} \Diamond\chi^n_w)$ by Lemma 2.4.23(i).

To prove the direction $\dashv$ of (2.10), for any $\mathcal{P}' \subseteq \mathcal{P}$ we have $\theta^n_{\uplus\mathcal{P}'} \vdash \bigvee_{s\in\mathcal{P}} \oslash\theta^n_s$ by Lemma 2.4.29(i), so $\Diamond\theta^n_{\uplus\mathcal{P}'} \vdash \Diamond\bigvee_{s\in\mathcal{P}} \oslash\theta^n_s$ by $\Diamond$Mon. The result then follows by $\vee$E. For the converse direction $\vdash$, let $\mathcal{P} = \{s_1, \dots, s_m\}$. Repeatedly applying $\Diamond\oslash$E gives

$$\begin{aligned}
\Diamond \bigvee_{i=1}^{m} \oslash\theta^n_{s_i} &\vdash \Diamond(\theta^n_{s_1} \vee \oslash\theta^n_{s_2} \vee \cdots \vee \oslash\theta^n_{s_m}) \vee \Diamond(\bot \vee \oslash\theta^n_{s_2} \vee \cdots \vee \oslash\theta^n_{s_m}) \\
&\vdash \Diamond(\theta^n_{s_1} \vee \theta^n_{s_2} \vee \oslash\theta^n_{s_3} \vee \cdots \vee \oslash\theta^n_{s_m}) \vee \Diamond(\theta^n_{s_1} \vee \bot \vee \oslash\theta^n_{s_3} \vee \cdots \vee \oslash\theta^n_{s_m}) \\
&\quad \vee \Diamond(\bot \vee \theta^n_{s_2} \vee \oslash\theta^n_{s_3} \vee \cdots \vee \oslash\theta^n_{s_m}) \vee \Diamond(\bot \vee \bot \vee \oslash\theta^n_{s_3} \vee \cdots \vee \oslash\theta^n_{s_m}) \\
&\ \vdots \\
&\vdash \bigvee_{\tau_1\in\{s_1,\varnothing\}} \cdots \bigvee_{\tau_m\in\{s_m,\varnothing\}} \Diamond(\theta^n_{\tau_1} \vee \cdots \vee \theta^n_{\tau_m}). \qquad (\text{recall } \theta^n_\varnothing = \bot)
\end{aligned}$$

For each sequence $\tau_1, \dots, \tau_m$ as above, putting $\mathcal{P}' = \{\tau_i \mid \tau_i = s_i,\ 1 \leq i \leq m\}$, it is easy to see that $\Diamond(\theta^n_{\tau_1} \vee \cdots \vee \theta^n_{\tau_m}) \dashv\vdash \Diamond\bigvee_{\tau_i\in\mathcal{P}'} \theta^n_{\tau_i} \dashv\vdash \Diamond\theta^n_{\uplus\mathcal{P}'}$. Thus,

$$\bigvee_{\tau_1\in\{s_1,\varnothing\}} \cdots \bigvee_{\tau_m\in\{s_m,\varnothing\}} \Diamond(\theta^n_{\tau_1} \vee \cdots \vee \theta^n_{\tau_m}) \vdash \bigvee_{\mathcal{P}'\subseteq\mathcal{P}} \Diamond\theta^n_{\uplus\mathcal{P}'}.$$

If $\varphi = \Box\psi$, then $n = k-1 \geq md(\varphi)-1 = md(\psi)$, so by the induction hypothesis, we have the same normal form equivalence for ψ as in the diamond case. Again, by Lemma 2.4.32 it suffices to show that $\Box\psi$ is provably equivalent to a classical formula of modal depth at most k. For the case $\eta_\psi = \mathbb{T}$, we have $\Box\psi \dashv\vdash \Box\bigvee_{s\in\mathcal{P}} \oslash\theta^n_s$. One can show that $\Box\bigvee_{s\in\mathcal{P}} \oslash\theta^n_s$ is provably equivalent to a classical formula of modal depth at most k with a proof analogous to that given for $\Diamond\bigvee_{s\in\mathcal{P}} \oslash\theta^n_s$ in the case for $\varphi = \Diamond\psi$ (using $\Box$Mon, $\Box\oslash$E, and Lemma 2.4.23(ii) instead of $\Diamond$Mon, $\Diamond\oslash$E, and Lemma 2.4.23(i), respectively). For the case $\eta_\psi = \mathrm{NE}$, we have $\Box\psi \dashv\vdash \Box(\mathrm{NE} \wedge \bigvee_{s\in\mathcal{P}} \oslash\theta^n_s) \dashv\vdash \Box\bigvee_{s\in\mathcal{P}} \oslash\theta^n_s \wedge \Diamond\bigvee_{s\in\mathcal{P}} \oslash\theta^n_s$ by Lemma 2.4.22(ii). We have already shown that each conjunct in this formula is provably equivalent to a classical formula. □

2.4.3 BSML

In this final section, we axiomatize our base logic BSML. As with $BSML^{\mathbb{V}}$ and PT^+, the propositional fragment of BSML corresponds to the logic CPL^+ studied and axiomatized in [140] (with the caveat that CPL^+ does not feature a bilateral negation). To construct the system for BSML, we remove all rules involving $\mathbb{V}$ from the system for $BSML^{\mathbb{V}}$, and add rules which simulate the removed rules.

2.4.33. DEFINITION (Natural deduction system for BSML). The natural deduction system for BSML includes all rules not involving $\mathbb{V}$ from the system for $BSML^{\mathbb{V}}$ (boxes (a)–(f)) and the following rules simulating the behavior of $\mathbb{V}$:

$$\dfrac{\begin{matrix} D \\ \varphi \end{matrix} \quad \begin{matrix} [\varphi[\psi \wedge \mathrm{NE}/\psi]] \\ D_1 \\ \chi \end{matrix} \quad \begin{matrix} [\varphi[\psi \wedge \bot/\psi]] \\ D_2 \\ \chi \end{matrix}}{\chi}\ \bot\mathrm{NE}\mathsf{Trs}(*)$$

$$\dfrac{\begin{matrix} D \\ \Diamond\varphi \end{matrix}}{\Diamond\varphi[\psi \wedge \mathrm{NE}/\psi] \vee \Diamond\varphi[\psi \wedge \bot/\psi]}\ \Diamond\bot\mathrm{NE}\mathsf{Trs}(*)$$

$$\dfrac{\begin{matrix} D \\ \Box\varphi \end{matrix}}{\Box\varphi[\psi \wedge \mathrm{NE}/\psi] \vee \Box\varphi[\psi \wedge \bot/\psi]}\ \Box\bot\mathrm{NE}\mathsf{Trs}(*)$$

$(*)$ $[\psi]$ is $\mathbb{V}$-distributive in φ.

Recall that $[\psi]$ is $\mathbb{V}$-distributive in χ if $[\psi]$ is not within the scope of any $\neg$ or $\Diamond$ in χ. So given $\psi \equiv \psi \wedge (\mathrm{NE} \mathbb{V} \bot)$, and given that $\wedge$ and $\vee$ distribute over $\mathbb{V}$, if $[\psi]$ is $\mathbb{V}$-distributive in φ, then $\varphi \equiv \varphi[\psi \wedge (\mathrm{NE} \mathbb{V} \bot)/\psi] \equiv \varphi[\psi \wedge \mathrm{NE}/\psi] \mathbb{V} \varphi[\psi \wedge \bot/\psi]$. The $\bot\mathrm{NE}$-translation rule $\bot\mathrm{NE}\mathsf{Trs}$ captures this equivalence by simulating $\mathbb{V}$-elimination as applied to $\varphi[\psi \wedge \mathrm{NE}/\psi] \mathbb{V} \varphi[\psi \wedge \bot/\psi]$. The rule $\Diamond\bot\mathrm{NE}\mathsf{Trs}$ captures the equivalence $\Diamond\varphi \equiv \Diamond(\varphi[\psi \wedge \mathrm{NE}/\psi] \mathbb{V} \varphi[\psi \wedge \bot/\psi]) \equiv \Diamond\varphi[\psi \wedge \mathrm{NE}/\psi] \vee \Diamond\varphi[\psi \wedge \bot/\psi]$ by simulating $\Diamond\mathbb{V}\vee\mathsf{Conv}$; similarly for $\Box\bot\mathrm{NE}\mathsf{Trs}$ and $\Box\mathbb{V}\vee\mathsf{Conv}$. The soundness of the new rules follows from these equivalences, and so we have the following result.

2.4.34. THEOREM (Soundness of BSML). *If* $\Phi \vdash \psi$, *then* $\Phi \vDash \psi$.

Observe that the proofs given in Section 2.4.1 for Lemma 2.4.4 (replacement) and Proposition 2.4.5 (negation normal form) also suffice to establish these results for the BSML-system. In what follows we can thus always assume that formulas are in negation normal form.

We now turn to the proof of the completeness theorem for the system. From our expressivity analysis in Section 2.3, we know that every formula φ in BSML (being also a formula in the stronger logic $\mathrm{BSML}^{\mathbb{V}}$) is equivalent to a formula in $\mathrm{BSML}^{\mathbb{V}}$-normal form: $\varphi \equiv \mathbb{V}_{s\in\|\varphi\|}\theta_s^k$, where $k = md(\varphi)$. While the global disjunction in the normal form is not in the language of BSML, we will still be able to make use of this disjunctive normal form in our completeness proof, as we were also able to do in our proof for $\mathrm{BSML}^{\oslash}$. Adapting a similar strategy employed in [140] for CPL^+, we associate with each formula φ a set of BSML-formulas φ^f of certain syntactic form, called *realizations* of φ. The realizations of φ correspond to the disjuncts θ_s^k in the normal form for φ—we will show that $\varphi \equiv \mathbb{V}_{s\in\|\varphi\|}\theta_s^k \equiv \mathbb{V}\{\varphi^f \mid \varphi^f$ is a realization for $\varphi\}$ (cf. Lemma 2.4.28)—and φ can be simulated in the completeness proof using its realizations: we also show that φ and $\mathbb{V}\{\varphi^f \mid \varphi^f$ is a realization for $\varphi\}$ are proof-theoretically interchangeable (cf. Lemma 2.4.29).

Realizations for φ are defined by replacing $\mathbb{V}$-distributive occurrences $[\eta]$ in φ by disjuncts θ_s^k in their respective normal forms $\mathbb{V}_{\|\eta\|}\theta_s^k$ (where $k \geq md(\varphi)$). The equivalence between φ and $\mathbb{V}\{\varphi^f \mid \varphi^f$ is a realization for $\varphi\}$ then follows by the $\mathbb{V}$-distributivity of these occurrences. In order to secure the proof-theoretic interchangeability result, we require each η to be replaced to be either classical or NE—we first essentially show that the required results hold for formulas that can be constructed using only such occurrences (that is, they hold for formulas of the form $\eta_1 \circ_1 \eta_2 \circ_2 \cdots \circ_{n-1} \eta_n$ where each $\eta_i \in \mathrm{ML}$ or $\eta_i = \mathrm{NE}$, and each $\circ_i$ is either $\wedge$ or $\vee$), and later show that every formula is provably equivalent to a formula of this form.

Formally, we define realizations relative to $\mathbb{V}$-distributive sequences, and we show that our results hold with respect to such sequences. A *$\mathbb{V}$-distributive sequence* of φ is a sequence $\mathsf{a} = \langle \eta_1, \ldots, \eta_n \rangle$ of distinct $\mathbb{V}$-distributive occurrences of formulas η_i in φ such that each $\eta_i \in \mathrm{ML}$ or $\eta_i = \mathrm{NE}$ for all $1 \leq i \leq n$.

Let $\mathsf{a} = \langle \eta_1, \ldots, \eta_n \rangle$ be a $\mathbb{V}$–distributive sequence of φ, and let $\mathsf{X} \supseteq \mathsf{P}(\varphi)$ be some finite set of propositional variables. A *realizing function over* a *and* X is a function $f : \{1, \ldots, n\} \to \|\top\!\!\top\|_{\mathsf{X}}$ such that $f_i \vDash \eta_i$ for all $1 \leq i \leq n$, where $\|\top\!\!\top\|_{\mathsf{X}} = \{(M,s) \mid M,s \vDash \top\!\!\top\}$ is the class of all pointed state models over X. For any $k \geq md(\varphi)$, the *(k,f)-realization of* φ, written $\varphi^{k,f}$ (or simply φ^f), is the BSML-formula

$$\varphi[\theta_{f_1}^k/\eta_1, \ldots, \theta_{f_n}^k/\eta_n],$$

where recall that each $\theta_{f_i}^k$ is the strong Hintikka formula for the state f_i. We write $\mathcal{F}^{\mathsf{X},\mathsf{a}}$ (or simply $\mathcal{F}^{\mathsf{a}}$) for the set of all realizing functions over a and X.

Most results in this section hold for all finite $\mathsf{X} \supseteq \mathsf{P}(\varphi)$ and all $k \geq md(\varphi)$ where φ is clear from the context; we usually omit mention of these conditions. As promised, each formula is equivalent to the global disjunction of all its realizations.

2.4.35. LEMMA. *Let* a *be a* $\mathbb{V}$*-distributive sequence of* φ*. Then* $\varphi \equiv \mathbb{V}_{f\in\mathcal{F}^{\mathsf{a}}}\varphi^f$.

Proof:
By the proof of Theorem 2.3.15, $\eta_i \equiv \mathbb{V}_{s\in\|\eta_i\|}\theta_s^k$ for each i. The result then follows since a is $\mathbb{V}$-distributive, and $\wedge$ and $\vee$ distribute over $\mathbb{V}$. □

As explained above, we will also show that in the system for BSML, every formula is proof-theoretically interchangeable with the global disjunction of its realizations. This is expressed formally in the following lemma.

2.4.36. LEMMA. *Let* a *be a* $\mathbb{V}$*-distributive sequence of* φ*. Then*

(i) $\varphi^f \vdash \varphi$ *for all* $f \in \mathcal{F}^{\mathsf{a}}$*, and*

(ii) if $\Phi, \varphi^f \vdash \psi$ *for all* $f \in \mathcal{F}^{\mathsf{a}}$*, then* $\Phi, \varphi \vdash \psi$*.*

The proof of the above requires several technical lemmas. For item (i), we need to show that Lemma 2.4.17 is also derivable in the system for BSML.

Proof of Lemma 2.4.17 (in BSML**):**
We prove $\varphi \vee (\psi \wedge \mathrm{NE}) \dashv\vdash (\varphi \vee (\psi \wedge \mathrm{NE})) \wedge \mathrm{NE}$. Let $\chi := \varphi \vee (\psi \wedge \mathrm{NE})$. Then $\chi[\chi \wedge \bot/\chi] = (\varphi \vee (\psi \wedge \mathrm{NE})) \wedge \bot \vdash (\varphi \wedge \bot) \vee (\psi \wedge \mathrm{NE} \wedge \bot)$ by Proposition 2.4.6, and $(\varphi \wedge \bot) \vee (\psi \wedge \mathrm{NE} \wedge \bot) \vdash (\varphi \wedge \bot) \vee \bot\!\!\!\bot \vdash (\varphi \vee (\psi \wedge \mathrm{NE})) \wedge \mathrm{NE}$ by $\bot\!\!\!\bot$Ctr. And $\chi[\chi \wedge \mathrm{NE}/\chi] = (\varphi \vee (\psi \wedge \mathrm{NE})) \wedge \mathrm{NE} \vdash (\varphi \vee (\psi \wedge \mathrm{NE})) \wedge \mathrm{NE}$, and so we have $\varphi \vee (\psi \wedge \mathrm{NE}) \vdash (\varphi \vee (\psi \wedge \mathrm{NE})) \wedge \mathrm{NE}$ by $\bot\mathrm{NE}$Trs. □

Proof of Lemma 2.4.36(i):
By Lemma 2.4.4, it suffices to show that $\theta^k_{f_i} \vdash \eta_i$ for all $1 \leq i \leq n$. If $\eta_i = \mathrm{NE}$, then $f_i \neq \varnothing$ so $\theta^k_{f_i}$ is not $\bot$. We have $\theta^k_{f_i} = \bigvee_{w \in f_i}(\chi^k_w \wedge \mathrm{NE}) \vdash \mathrm{NE}$ by Lemma 2.4.17. If $\eta_i = \alpha \in \mathrm{ML}$, we clearly have that $\theta^k_{f_i} \vdash \chi^k_{f_i}$. Since $f_i \vDash \alpha$, we have $f_i \subseteq [\![\alpha]\!]$ by flatness, which implies $\chi^k_{f_i} \vDash \chi^k_{[\![\alpha]\!]} \equiv \alpha$. Therefore, $\chi^k_{f_i} \vdash \alpha$ by Proposition 2.4.8. □

To prove Lemma 2.4.36(ii), we first show that for a $\mathbb{V}$-distributive sequence $\mathsf{a} = \langle \eta \rangle$ of length 1, the entailment $\varphi[\eta] \vDash \varphi[\top\!\!\!\top \wedge \eta/\eta] \vDash \varphi[\mathbb{V}_{s \in \|\top\!\!\!\top\|_{\mathsf{X}}} \theta^{\mathsf{X},k}_s \wedge \eta/\eta]$ can be simulated in our system in the sense of the following lemma.

2.4.37. LEMMA. *Let* $[\eta]$ *be* $\mathbb{V}$*-distributive in* φ*. Let* $k \in \mathbb{N}$*, and let* $\mathsf{X} \subseteq \mathsf{Prop}$ *be finite. If* $\Phi, \varphi[\theta^{\mathsf{X},k}_s \wedge \eta/\eta] \vdash \psi$ *for all* $s \in \|\top\!\!\!\top\|_{\mathsf{X}}$*, then* $\Phi, \varphi \vdash \psi$*.*

Proof:
Consider $\chi^k_{[\![\top\!\!\!\top]\!]} := \bigvee_{w \in [\![\top\!\!\!\top]\!]_{\mathsf{X}}} \chi^k_w$, where $[\![\top\!\!\!\top]\!]_{\mathsf{X}} = \{w \mid w \vDash \top\!\!\!\top\}$. Since there are only finitely many nonequivalent kth Hintikka formulas χ^k_w over X, we may assume without loss of generality that $\chi^k_{[\![\top\!\!\!\top]\!]} = \chi^k_{w_1} \vee \cdots \vee \chi^k_{w_n}$, for some worlds $w_1, \ldots, w_n$ from some models $M_1, \ldots, M_n$. Clearly, $\chi^k_{[\![\top\!\!\!\top]\!]} \equiv \top\!\!\!\top$, which implies $\vdash \chi^k_{[\![\top\!\!\!\top]\!]}$ by Proposition 2.4.8. Thus, we have $\varphi \vdash \varphi[\chi^k_{[\![\top\!\!\!\top]\!]} \wedge \eta/\eta]$ by Lemma 2.4.4.

Now, for any $\tau_1, \ldots, \tau_n \in \{\mathrm{NE}, \bot\}$, consider the state $s = \{w_i \mid \tau_i = \mathrm{NE}\}$. By applying $\bot$E and $\vee$I, we derive $\bigvee^n_{i=1}(\chi^k_{w_i} \wedge \tau_i) \dashv\vdash \bigvee_{w_i \in s}(\chi^k_{w_i} \wedge \mathrm{NE}) = \theta^k_s$. Thus, we have $\varphi[\bigvee^n_{i=1}(\chi^k_{w_i} \wedge \tau_i) \wedge \eta/\eta] \dashv\vdash \varphi[\theta^k_s \wedge \eta/\eta]$. Therefore, by assumption we have

$$\Phi, \varphi[\big((\chi^k_{w_1} \wedge \bot) \vee \cdots \vee (\chi^k_{w_n} \wedge \bot)\big) \wedge \eta/\eta] \vdash \psi \qquad \text{and}$$

$$\Phi, \varphi[((\chi^k_{w_1} \wedge \bot) \vee \cdots \vee (\chi^k_{w_n} \wedge \mathrm{NE})) \wedge \eta/\eta] \vdash \psi,$$

from which $\Phi, \varphi[((\chi^k_{w_1} \wedge \bot) \vee \cdots \vee (\chi^k_{w_{n-1}} \wedge \bot) \vee \chi^k_{w_n}) \wedge \eta/\eta] \vdash \psi$ follows by $\bot$NETrs. Repeating this argument, we finally get $\Phi, \varphi[\chi^k_{[\![\top]\!]} \wedge \eta/\eta] \vdash \psi$, whence $\Phi, \varphi \vdash \psi$. □

Next, in (i) of the following lemma ((ii) is used later) we show that for a $\mathbb{V}$-distributive sequence $\mathsf{a} = \langle \eta_1, \dots, \eta_n \rangle$ of arbitrary length, we can simulate the entailment

$$\varphi[\mathbb{V}_{s \in \|\top\|_{\mathcal{X}}} \theta^{\mathcal{X},k}_s \wedge \eta_1/\eta_1, \dots, \mathbb{V}_{s \in \|\top\|_{\mathcal{X}}} \theta^{\mathcal{X},k}_s \wedge \eta_n/\eta_n] \vDash \mathbb{V}_{f \in \mathcal{F}^{\mathsf{X},\mathsf{a}}} \varphi^{k,f}.$$

2.4.38. LEMMA. *Let* a *be a* $\mathbb{V}$*-distributive sequence of* φ. *Then:*

(i) for any $s_1, \dots, s_n \in \|\top\|_{\mathsf{X}}$, *there is some* $f \in \mathcal{F}^{\mathsf{a}}$ *such that*

$$\varphi[\theta^{\mathsf{X},k}_{s_1} \wedge \eta_1/\eta_1, \dots, \theta^{\mathsf{X},k}_{s_n} \wedge \eta_n/\eta_n] \vdash \varphi^{k,f};$$

(ii) for any $f \in \mathcal{F}^{\mathsf{a}}$, *there are some* $s_1, \dots, s_n \in \|\top\|_{\mathsf{X}}$ *such that*

$$\varphi[\theta^{\mathsf{X},k}_{s_1} \wedge \eta_1/\eta_1, \dots, \theta^{\mathsf{X},k}_{s_n} \wedge \eta_n/\eta_n] \vdash \varphi^{k,f}.$$

Proof:
(i) We first give the proof for the case when $\mathsf{a} = \langle \eta_1 \rangle$ is a sequence of length 1. Let $(M_1, s_1) \in \|\top\|_{\mathsf{X}}$. Case 1: $\eta_1 = \mathrm{NE}$. If $s_1 \neq \varnothing$, let $f_1 := (M_1, s_1)$. Since $s_1 \vDash \mathrm{NE}$, we have $f \in \mathcal{F}^{\mathsf{a}}$. We derive $\varphi[\theta^k_{s_1} \wedge \mathrm{NE}/\mathrm{NE}] \vdash \varphi[\theta^k_{s_1}/\mathrm{NE}] = \varphi^f$ by Lemma 2.4.4. If $s_1 = \varnothing$, pick any (N, t) such that $t \neq \varnothing$ and let $f_1 := (N, t)$. Since $t \vDash \mathrm{NE}$, we have $f \in \mathcal{F}^{\mathsf{a}}$. Now, we derive $\theta^k_{s_1} \wedge \mathrm{NE} = \bot \wedge \mathrm{NE} = \bot\!\!\!\bot \vdash \theta^k_t$ by Lemma 2.4.2. Thus, $\varphi[\theta^k_{s_1} \wedge \mathrm{NE}/\mathrm{NE}] \vdash \varphi[\theta^k_t/\mathrm{NE}] = \varphi^f$ follows from Lemma 2.4.4.

Case 2: $\eta_1 = \alpha \in \mathrm{ML}$. Let $r = \{w \in s_1 \mid w \vDash \alpha\}$. Define $f_1 := (M_1, r)$, which yields a realizing function over $\langle \eta_1 \rangle$ (as $r \vDash \alpha$). We show that $\varphi[\theta^k_{s_1} \wedge \alpha/\alpha] \vdash \varphi^f$, that is, $\varphi[\theta^k_{s_1} \wedge \alpha/\alpha] \vdash \varphi[\theta^k_r/\alpha]$. By Lemma 2.4.4, it suffices to show that $\theta^k_{s_1} \wedge \alpha \vdash \theta^k_r$. Now, by Proposition 2.4.6, we derive

$$\theta^k_{s_1} \wedge \alpha \vdash (\theta^k_r \vee \theta^k_{s_1 \setminus r}) \wedge \alpha \vdash \theta^k_r \vee (\theta^k_{s_1 \setminus r} \wedge \alpha) \vdash \theta^k_r \vee (\chi^k_{s_1 \setminus r} \wedge \alpha).$$

Clearly, $\chi^k_{s_1 \setminus r} \vDash \neg\alpha$ and thus $\chi^k_{s_1 \setminus r} \vdash \neg\alpha$ by Proposition 2.4.8. Thus, we further derive $\theta^k_r \vee (\chi^k_{s_1 \setminus r} \wedge \alpha) \vdash \theta^k_r \vee (\neg\alpha \wedge \alpha) \vdash \theta^k_r \vee \bot \vdash \theta^k_r$ by $\bot$E. We thus conclude that $\theta^k_{s_1} \wedge \alpha \vdash \theta^k_r$.

Now let $\mathsf{a} = \langle \eta_1, \dots, \eta_n \rangle$ with $n > 1$. Let $\varphi' := \varphi[\theta^k_{s_2} \wedge \eta_2/\eta_2, \dots, \theta^k_{s_n} \wedge \eta_n/\eta_n]$ and $\mathsf{a}' := \langle \eta_1 \rangle$. Applying what we just proved for φ' and $\mathbf{a}'$ we have $\varphi'[\theta^k_{s_1} \wedge \eta_1/\eta_1] \vdash \varphi'^g$ for some $g \in \mathcal{F}^{\mathsf{a}'}$. Repeating this argument n times, we can find some $f \in \mathcal{F}^{\mathsf{a}}$ such that $\varphi[\theta^k_{s_1} \wedge \eta_1/\eta_1, \dots, \theta^k_{s_n} \wedge \eta_n/\eta_n] \vdash \varphi^f$.

(ii) Define $(M_i, s_i) := f_i$. Since $\theta^k_{s_i} \wedge \eta_i \vdash \theta^k_{s_i}$ for each i, we derive by Lemma 2.4.4 that $\varphi[\theta^k_{s_1} \wedge \eta_1/\eta_1, \dots, \theta^k_{s_n} \wedge \eta_n/\eta_n] \vdash \varphi[\theta^k_{s_1}/\eta_1, \dots, \theta^k_{s_n}/\eta_n] = \varphi^f$. □

Putting together Lemmas 2.4.37 and 2.4.38(i), we can simulate $\varphi \vDash \bigvee\!\!\!\!\bigvee_{f \in \mathcal{F}^{\mathsf{a}}} \varphi^f$; that is, we can prove Lemma 2.4.36(ii).

Proof of Lemma 2.4.36(ii):

Assume that $\Phi, \varphi^f \vdash \psi$ for all $f \in \mathcal{F}^{\mathsf{a}}$. By multiple applications of Lemma 2.4.37, it suffices to show that $\Phi, \varphi[\theta^k_{s_1} \wedge \eta_1/\eta_1, \dots, \theta^k_{s_n} \wedge \eta_n/\eta_n] \vdash \psi$ for all $s_1, \dots, s_n \in \|\top\!\!\top\|_{\mathsf{X}}$. This follows by Lemma 2.4.38(i) and our assumption. □

We have now settled that relative to $\bigvee\!\!\!\!\bigvee$-distributive sequences, each formula is proof-theoretically interchangeable with the global disjunction of all its realizations. For a formula φ that can be constructed solely using the occurrences in such a sequence a, the realizations over a correspond with (and are, in fact, provably equivalent to) the disjuncts θ^k_s in the normal form $\bigvee\!\!\!\!\bigvee_{s \in \|\varphi\|} \theta^k_s$ for φ. We must now show that every formula is provably equivalent to some formula of this form. Formally, we say that a $\bigvee\!\!\!\!\bigvee$-distributive sequence $\mathsf{a} = \langle \eta_1, \dots, \eta_n \rangle$ of a formula is in addition a *$\bigvee\!\!\!\!\bigvee$-distributive partition* of φ if φ is of the form $[\eta_1] \circ_1 [\eta_2] \circ_2 \cdots \circ_{n-1} [\eta_n]$, where each $\circ_i$ is either $\wedge$ or $\vee$. We show the following.

2.4.39. LEMMA. *For all φ there is a φ' such that φ' has a partition and $\varphi \dashv\vdash \varphi'$.*

Before we prove this lemma, let us confirm that realizations over $\bigvee\!\!\!\!\bigvee$-distributive partitions are indeed provably equivalent to strong Hintikka formulas θ^k_s. Hereafter, to simplify notation, we write $\theta^k_{\mathbf{0}} := \perp\!\!\!\perp$ and view $\mathbf{0}$ as a null state such that $\mathbf{0} \vDash \perp\!\!\!\perp$.

2.4.40. LEMMA. *Let a be a $\bigvee\!\!\!\!\bigvee$-distributive partition of φ. Then for all $f \in \mathcal{F}^{\mathsf{X},\mathsf{a}}$, we have that $\varphi^{k,f} \dashv\vdash \theta^{\mathsf{X},k}_s$ for some $s \in \|\top\!\!\top\|_{\mathsf{X}}$ or $s = \mathbf{0}$.*

Proof:

It suffices to show that for all states s and t (including the null state), $\theta^k_s \wedge \theta^k_t \dashv\vdash \theta^k_u$ and $\theta^k_s \vee \theta^k_t \dashv\vdash \theta^k_v$ for some (null or nonnull) states u and v. If one of s and t is $\mathbf{0}$, then $\theta^k_s \wedge \theta^k_t \dashv\vdash \theta^k_{\mathbf{0}}$ and $\theta^k_s \vee \theta^k_t \dashv\vdash \theta^k_{\mathbf{0}}$ by Lemma 2.4.2 and $\perp\!\!\!\perp$Ctr. If neither state is $\mathbf{0}$, then $\theta^k_s \vee \theta^k_t \dashv\vdash \theta^k_{s \uplus t}$. For $\theta^k_s \wedge \theta^k_t$, if $s \not\rightleftharpoons_k t$, then $\theta^k_s \wedge \theta^k_t \dashv\vdash \theta^k_{\mathbf{0}}$ by Lemmas 2.4.11 and 2.4.2; if $s \rightleftharpoons_k t$, then $\theta^k_s \dashv\vdash \theta^k_t$ by Lemma 2.4.10, so clearly $\theta^k_s \wedge \theta^k_t \dashv\vdash \theta^k_s$. □

We now turn to Lemma 2.4.39. This result is, as it turns out, an easy corollary of the fact that all formulas of the form $\Diamond\varphi$ or $\Box\varphi$ are provably equivalent to classical formulas. To establish this fact, we first recall that in Lemma 2.4.23 in Section 2.4.1, we proved that formulas of the form $\Diamond\theta^k_s$ or $\Box\theta^k_s$ are provably equivalent to classical formulas. Lemma 2.4.23 depends on Lemma 2.4.22, which in turn depends on Lemmas 2.4.17 and 2.4.21, which were given BSML$^{\bigvee\!\!\!\!\bigvee}$-specific derivations in Section 2.4.1. We have already shown Lemma 2.4.17 for BSML, and directly below we show Lemma 2.4.21; we may then also make use of Lemmas 2.4.22 and 2.4.23 in BSML.

Proof of Lemma 2.4.21 (in BSML):

By $\Diamond\bot$NETrs, $\Diamond\varphi \vdash \Diamond(\varphi \wedge \mathrm{NE}) \vee \Diamond(\varphi \wedge \bot)$. We have $\Diamond(\varphi \wedge \mathrm{NE}) \vee \Diamond(\varphi \wedge \bot) \vdash \Diamond(\varphi \wedge$

$\mathrm{NE}) \vee \Diamond\bot \vdash \Diamond(\varphi \wedge \mathrm{NE}) \vee \bot$ by Proposition 2.4.8, and finally $\Diamond(\varphi \wedge \mathrm{NE}) \vee \bot \vdash \Diamond(\varphi \wedge \mathrm{NE})$ by $\bot$E. □

We also need the following modal analogue of Lemma 2.4.37.

2.4.41. LEMMA. *Let $[\eta]$ be $\mathbb{v}$-distributive in φ. For any $k \in \mathbb{N}$ and any finite $\mathsf{X} \subseteq \mathsf{Prop}$:*

$$\Diamond\varphi \vdash \bigvee_{s \in \|\mathbb{T}\|_{\mathsf{X}}} \Diamond\varphi[\theta_s^{\mathsf{X},k} \wedge \eta/\eta] \qquad \textit{and} \qquad \Box\varphi \vdash \bigvee_{s \in \|\mathbb{T}\|_{\mathsf{X}}} \Box\varphi[\theta_s^{\mathsf{X},k} \wedge \eta/\eta].$$

Proof:
We prove the result for $\Diamond\varphi$. Consider $\chi^k_{[\![\mathbb{T}]\!]} = \chi^k_{w_1} \vee \cdots \vee \chi^k_{w_n}$, where $w_1, \ldots, w_n$ are some worlds from some models $M_1, \ldots, M_n$. Since $\vdash \chi^k_{[\![\mathbb{T}]\!]}$, we derive $\Diamond\varphi \vdash \Diamond\varphi[\chi^k_{[\![\mathbb{T}]\!]} \wedge \eta/\eta]$ by Lemma 2.4.4. Put $\varphi' := \varphi[\chi^k_{[\![\mathbb{T}]\!]} \wedge \eta/\eta]$. By applying $\Diamond\bot\mathrm{NE}$Trs repeatedly, we derive

$$\begin{aligned}
\Diamond\varphi' \quad &\vdash \quad \Diamond\varphi'[\chi^k_{w_1} \wedge \mathrm{NE}/\chi^k_{w_1}] \vee \Diamond\varphi'[\chi^k_{w_1} \wedge \bot/\chi^k_{w_1}] \\
&\vdash \quad \Diamond\varphi'[\chi^k_{w_1} \wedge \mathrm{NE}/\chi^k_{w_1}, \chi^k_{w_2} \wedge \mathrm{NE}/\chi^k_{w_2}] \vee \Diamond\varphi'[\chi^k_{w_1} \wedge \mathrm{NE}/\chi^k_{w_1}, \chi^k_{w_2} \wedge \bot/\chi^k_{w_2}] \vee \\
&\qquad \Diamond\varphi'[\chi^k_{w_1} \wedge \bot/\chi^k_{w_1}, \chi^k_{w_2} \wedge \mathrm{NE}/\chi^k_{w_2}] \vee \Diamond\varphi'[\chi^k_{w_1} \wedge \bot/\chi^k_{w_1}, \chi^k_{w_2} \wedge \bot/\chi^k_{w_2}] \\
&\ \ldots \\
&\vdash \quad \bigvee_{\tau_1,\ldots,\tau_n \in \{\mathrm{NE},\bot\}} \Diamond\varphi'[\chi^k_{w_1} \wedge \tau_1/\chi^k_{w_1}, \ldots, \chi^k_{w_n} \wedge \tau_n/\chi^k_{w_n}] \\
&\vdash \quad \bigvee_{\tau_1,\ldots,\tau_n \in \{\mathrm{NE},\bot\}} \Diamond\varphi'[(\chi^k_{w_1} \wedge \tau_1) \vee \cdots \vee (\chi^k_{w_n} \wedge \tau_n)/\chi^k_{[\![\mathbb{T}]\!]}] \\
&\vdash \quad \bigvee_{\tau_1,\ldots,\tau_n \in \{\mathrm{NE},\bot\}} \Diamond\varphi[\bigvee_{i=1}^{n}(\chi^k_{w_i} \wedge \tau_i) \wedge \eta/\eta]
\end{aligned}$$

Now, consider each disjunct of the above formula, with some arbitrary fixed $\tau_1, \ldots, \tau_n \in \{\mathrm{NE}, \bot\}$. Let $s = \{w_i \mid \tau_i = \mathrm{NE}\} \subseteq \{w_1, \ldots, w_n\}$. By applying $\bot$E and $\vee$I, we derive $\bigvee_{i=1}^n(\chi^k_{w_i} \wedge \tau_i) \dashv\vdash \bigvee_{w_i \in s}(\chi^k_{w_i} \wedge \mathrm{NE}) = \theta^k_s$, whence by Lemma 2.4.4, $\Diamond\varphi[\bigvee_{i=1}^n(\chi^k_{w_i} \wedge \tau_i) \wedge \eta/\eta] \vdash \Diamond\varphi[\theta^k_s \wedge \eta/\eta]$. We then clearly also have $\bigvee_{\tau_1,\ldots,\tau_n \in \{\mathrm{NE},\bot\}} \Diamond\varphi[\bigvee_{i=1}^n(\chi^k_{w_i} \wedge \tau_i) \wedge \eta/\eta] \vdash \bigvee_{s \subseteq \{w_1,\ldots,w_n\}} \Diamond\varphi[\theta^k_s \wedge \eta/\eta]$. Observe that $\|\mathbb{T}\|_{\mathsf{X}}$ is (modulo k-bisimulation) $\{(M_1 \uplus \cdots \uplus M_n, s) \mid s \subseteq \{w_1, \ldots, w_n\}\}$. Thus, we have $\bigvee_{s \subseteq \{w_1,\ldots,w_n\}} \Diamond\varphi[\theta^k_s \wedge \eta/\eta] \vdash \bigvee_{s \in \|\mathbb{T}\|_{\mathsf{X}}} \Diamond\varphi[\theta^k_s \wedge \eta/\eta]$. Putting all these together, we obtain $\Diamond\varphi \vdash \bigvee_{s \in \|\mathbb{T}\|_{\mathsf{X}}} \Diamond\varphi[\theta^k_s \wedge \eta/\eta]$.

The $\Box\varphi$-result is proved analogously, using $\Box\bot\mathrm{NE}$Trs in place of $\Diamond\bot\mathrm{NE}$Trs. □

We can now show that formulas of the form $\Diamond\varphi$ or $\Box\varphi$ are provably equivalent to classical formulas, and then derive the partition lemma as a corollary.

2.4.42. LEMMA. *$\Diamond\varphi$ and $\Box\varphi$ are provably equivalent to classical formulas.*

Proof:
We prove the two results simultaneously but only give the details for $\Diamond\varphi$; the details for $\Box\varphi$ are similar. We first prove that the result holds in case φ has a partition a. Let $k \geq md(\varphi)$, and let $\mathsf{X} \supseteq \mathsf{P}(\varphi)$ be finite. We first show that

$$\Diamond\varphi \dashv\vdash \bigvee_{f\in\mathcal{F}^{\mathsf{X},\mathsf{a}}} \Diamond\varphi^{k,f}.$$

For the direction $\dashv$, for a given $f \in \mathcal{F}^{\mathsf{a}}$, we have $\varphi^f \vdash \varphi$ by Lemma 2.4.36(i), which implies $\Diamond\varphi^f \vdash \Diamond\varphi$; therefore, $\bigvee_{f\in\mathcal{F}^{\mathsf{a}}} \Diamond\varphi^f \vdash \Diamond\varphi$ follows by $\vee$E. For the converse direction $\vdash$, by repeated applications of Lemma 2.4.41, we have

$$\Diamond\varphi \vdash \bigvee_{s_1\in\|\pi\|_{\mathsf{X}}} \cdots \bigvee_{s_n\in\|\pi\|_{\mathsf{X}}} \Diamond\varphi[\theta^k_{s_1}\wedge\eta_1/\eta_1,\ldots,\theta^k_{s_n}\wedge\eta_n/\eta_n].$$

By Lemma 2.4.38(i) and (ii) together with $\vee$Mon and $\Diamond$Mon, we derive

$$\bigvee_{s_1\in\|\pi\|_{\mathsf{X}}} \cdots \bigvee_{s_n\in\|\pi\|_{\mathsf{X}}} \Diamond\varphi[\theta^k_{s_1}\wedge\eta_1/\eta_1,\ldots,\theta^k_{s_n}\wedge\eta_n/\eta_n] \vdash \bigvee_{f\in\mathcal{F}^{\mathsf{a}}} \Diamond\varphi^f.$$

It now suffices to show that $\bigvee_{f\in\mathcal{F}^{\mathsf{a}}} \Diamond\varphi^f$ is provably equivalent to a classical formula. We show this by showing that each disjunct $\Diamond\varphi^f$ is so. By Lemma 2.4.40, we have $\varphi^f \dashv\vdash \theta^k_s$ for some state s. Thus, also $\Diamond\varphi^f \dashv\vdash \Diamond\theta^k_s$. If $s = \mathbf{0}$, then $\Diamond\theta^k_s = \Diamond\perp\!\!\!\perp \dashv\vdash \Diamond\bot \dashv\vdash \bot$ by Proposition 2.4.8 and Lemma 2.4.21. If $s \neq \mathbf{0}$, then by Lemma 2.4.23(i), we know that $\Diamond\theta^k_s$ is provably equivalent to a classical formula.

We now show the general case by induction on the modal depth of φ. Note that by $\neg$NEE, we may assume without loss of generality that $\varphi = \varphi(\bot/\neg\mathrm{NE})$. If $md(\varphi) = 0$, then there is clearly a partition of φ, so the result follows from what we just proved. If $md(\varphi) = n+1$, then for all subformulas of φ of form $\Diamond\psi$ or $\Box\psi$, we have $md(\psi) \leq n$, and thus by the induction hypothesis $\Diamond\psi \dashv\vdash \alpha_{\Diamond\psi}$ and $\Box\psi \dashv\vdash \alpha_{\Box\psi}$ for some $\alpha_{\Diamond\psi}, \alpha_{\Box\psi} \in$ ML. By Lemma 2.4.4, $\varphi \dashv\vdash \varphi' := \varphi[\alpha_{\bigcirc_1\psi_1}/\bigcirc_1\psi_1,\ldots,\alpha_{\bigcirc_n\psi_n}/\bigcirc_n\psi_n]$, where $[\bigcirc_1\psi_1],\ldots,[\bigcirc_n\psi_n]$ are all the subformula occurrences of the form $\Diamond\psi$ or $\Box\psi$ in φ such that no $[\bigcirc_i\psi_i]$ is a suboccurrence of any other occurrence of the form $\Diamond\psi$ or $\Box\psi$ in φ. The formula φ' clearly has a partition, so $\Diamond\varphi' \dashv\vdash \alpha \in$ ML; then also $\Diamond\varphi \dashv\vdash \alpha$. □

Proof of Lemma 2.4.39:
Apply Lemma 2.4.42, $\neg$NEE, and Lemma 2.4.4 to replace all subformula occurrences of the form $\Diamond\psi$, $\Box\psi$, or $\neg$NE in φ with classical formulas. □

We are now ready to prove the completeness theorem for the BSML-system.

2.4.43. THEOREM (Completeness of BSML). *If* $\Phi \vDash \psi$*, then* $\Phi \vdash \psi$.

Proof:
Assume that $\Phi \nvdash \psi$. We show that $\Phi \nvDash \psi$. Let $\Phi = \{\varphi_i \mid i \in I\}$. By Lemma 2.4.39,

we may assume that each φ_i has a partition a_i and ψ has a partition a. For each i, let $k_i := \max\{md(\varphi_i), md(\psi)\}$ and $\mathsf{X}_i := \mathsf{P}(\varphi_i) \cup \mathsf{P}(\psi)$. Now, for an arbitrary $i \in I$, since $\Phi \setminus \{\varphi_i\}, \varphi_i \nvdash \psi$, by Lemma 2.4.36(ii), we know there is a realizing function $f_i \in \mathcal{F}^{\mathsf{X}_i,\mathsf{a}_i}$ such that $\Phi \setminus \{\varphi_i\}, \varphi_i^f \nvdash \psi$. Continuing to argue in a similar way, one can find for each $i \in I$ a realizing function $f_i \in \mathcal{F}^{\mathsf{X}_i,\mathsf{a}_i}$ such that $\{\varphi_i^{k_i,f_i} \mid f_i \in \mathcal{F}^{\mathsf{X}_i,\mathsf{a}_i}, i \in I\} \nvdash \psi$. By Lemma 2.4.40, each $\varphi_i^{k_i,f_i}$ is provably equivalent to $\theta_{s_i}^{\mathsf{X}_i,k_i}$ for some s_i. Thus, $\Phi' = \{\theta_{s_i}^{\mathsf{X}_i,k_i} \mid i \in I\} \nvdash \psi$ as well.

Observe that $\Phi' \nvDash \perp\!\!\!\perp$, since otherwise either $s_i = \mathbf{0}$ for some $i \in I$, or $s_i \not\rightleftharpoons_m^{\mathsf{M}} s_j$ for some $i, j \in I$ and for $m = \min\{k_i, k_j\}$ and $\mathsf{M} = \mathsf{X}_i \cap \mathsf{X}_j$. In the former case, $\theta_{s_i}^{\mathsf{X}_i,k_i} = \perp\!\!\!\perp$ and thus $\Phi' \vdash \psi$; a contradiction. In the latter case, $\theta_{s_i}^{\mathsf{M},m}, \theta_{s_j}^{\mathsf{M},m} \vdash \perp\!\!\!\perp$ by Lemma 2.4.11, so that by Lemma 2.4.14, $\theta_{s_i}^{\mathsf{X}_i,k_i}, \theta_{s_j}^{\mathsf{X}_j,k_j} \vdash \perp\!\!\!\perp$. Thus, $\Phi' \vdash \psi$; a contradiction again.

Now, let t be such that $t \vDash \Phi'$. For each $i \in I$, we have $t \vDash \theta_{s_i}^{k_i} \dashv\vdash \varphi_i^{f_i}$, and thus $t \vDash \varphi_i^{f_i}$ by soundness. By Lemma 2.4.35, we further conclude that $t \vDash \varphi_i$, whereby $t \vDash \Phi$. To show that $\Phi \nvDash \psi$, it then suffices to show that $t \nvDash \psi$. Assume otherwise. Take $i \in I$. By Lemma 2.4.35, $t \vDash \psi^{k_i,g}$ for some $g \in \mathcal{F}^{\mathsf{X}_i,\mathsf{a}}$. In view of Lemma 2.4.40, we have $t \vDash \theta_r^{\mathsf{X}_i,k_i}$ for some (nonnull) state r with $\psi^{k_i,g} \dashv\vdash \theta_r^{\mathsf{X}_i,k_i}$. On the other hand, since $t \vDash \Phi'$, we have $t \vDash \theta_{s_i}^{\mathsf{X}_i,k_i}$. Then by Proposition 2.3.11(ii), we have $r \rightleftharpoons_{k_i}^{\mathsf{X}_i} t \rightleftharpoons_{k_i}^{\mathsf{X}_i} s_i$. Therefore, by Lemma 2.4.10, $\theta_{s_i}^{\mathsf{X}_i,k_i} \vdash \theta_r^{\mathsf{X}_i,k_i} \vdash \psi^{k_i,g}$. As $\psi^{k_i,g} \vdash \psi$ (by Lemma 2.4.36(i)), we are forced to conclude that $\Phi' \vdash \psi$, which is a contradiction. □

2.5 Conclusion

In this article, we presented natural deduction axiomatizations for BSML and the two extensions we introduced, $\mathrm{BSML}^{\mathbb{V}}$ (BSML with the global/inquisitive disjunction $\mathbb{V}$) and $\mathrm{BSML}^{\oslash}$ (BSML with the novel emptiness operator $\oslash$). We also proved the expressive completeness of the two extensions: $\mathrm{BSML}^{\mathbb{V}}$ is expressively complete for the class of all state properties invariant under bounded bisimulation; $\mathrm{BSML}^{\oslash}$ for the class of all union-closed state properties invariant under bounded bisimulation. We conclude by noting an additional preliminary result known to us, and listing possible directions for further investigation.

We saw that BSML is union closed but not expressively complete for union-closed properties. It appears, however, that one can find a different natural class of state properties for which BSML *is* complete: according to a very recent unpublished result (see Knudstorp [88], and Anttila and Knudstorp [16]; or Section 3.3.2 in Chapter 3), BSML is complete for the class of all properties $\mathcal{P}$ such that $\mathcal{P}$ is invariant under bounded bisimulation, union-closed, and also *convex*, where $\mathcal{P}$ is convex if $[(M,s) \in \mathcal{P}$ and $(M,t) \in \mathcal{P}$ and $t \subseteq u \subseteq s]$ implies $(M,u) \in \mathcal{P}$.

The semantics of the quantifiers in first-order dependence logic are analogous to the semantics of the modalities used in modal dependence logic, which are distinct from the BSML-modalities $\Diamond$ and $\Box$. Defining first-order quantifiers whose semantics are analogous to those of $\Diamond$ and $\Box$ may yield first-order state-based logics with interesting properties.

In inquisitive semantics, the global disjunction $\mathbb{V}$ is used to model the meanings of questions. Similarly, in the context of the extension $\text{BSML}^{\mathbb{V}}$, this disjunction might allow one to model questions and their interaction with free choice phenomena (3), and to arrive at a full account of examples of overt free choice cancellation (4), which were left open in [6].

3. Can I have coffee or tea? $\Diamond(c \vee t) \mathbb{V} \neg\Diamond(c \vee t)$
   (a) Yes $\rightsquigarrow$ you can have coffee and you can have tea.
   (b) No $\rightsquigarrow$ you cannot have either.
4. You may have coffee or tea, I don't know which.

Finally, given that the emptiness operator $\oslash$ is a natural counterpart to NE and that it can be used to cancel out the effects of pragmatic enrichment in the sense that $\oslash(\alpha \wedge \text{NE}) \equiv \alpha$, investigating the applications of this operator in formal semantics may yield some interesting results.

Chapter 3
Convex Propositional and Modal Team Logics

This chapter is based on:

Aleksi Anttila and Søren Brinck Knudstorp. *Convex Propositional and Modal Team Logics*. Manuscript. 2024

Abstract We prove expressive completeness results for convex propositional and modal team logics, where a logic is convex if, for each of its formulas, if the formula is true in two teams, then it is also true in all the teams between these two teams with respect to set inclusion. We introduce multiple propositional/modal logics which are expressively complete for the class of all convex propositional/modal team properties. We also answer an open question concerning the expressive power of classical propositional logic with team semantics extended with the nonemptiness atom NE—we show that this logic expressively complete for the class all of convex and union-closed propositional team properties. A modal analogue of this result additionally yields an expressive completeness theorem for Aloni's Bilateral State-based Modal Logic. There is a specific sense in which one of the novel propositional convex logics extends propositional dependence logic, and in which another of them extends propositional inquisitive logic. This sense is related to the notion of uniform definability studied in the team semantics literature. We introduce a generalization of uniform definability and define distinct notions of extension making use of this generalization in order to clarify and make precise the notion of extension pertaining to the convex logics, as well as to succintly express a number of facts we discuss relating to definability of connectives.

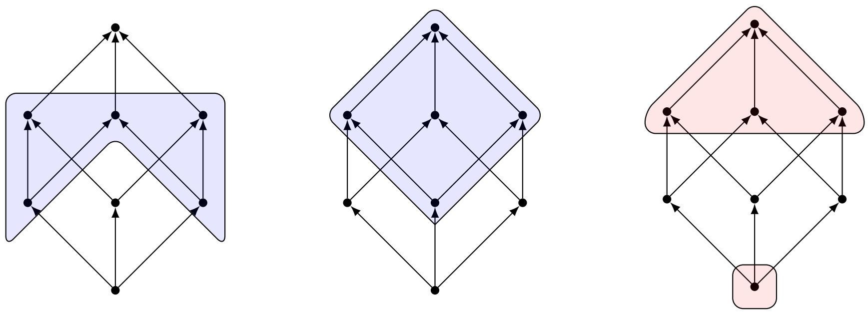
3.1 Introduction

In *team semantics*—originally introduced by Hodges [78, 79] to provide a compositional semantics for Hintikka and Sandu's *independence-friendly logic* [77, 75] and later refined by Väänänen in his work on *dependence logic* [119]; also independently developed as a semantics for *inquisitive logic* chiefly by Ciardelli, Groenendijk, and Roelofsen [41, 38, 35]—formulas are interpreted with respect to sets of evaluation points called *teams*, as opposed to single evaluation points as in standard Tarskian semantics. In propositional team semantics, teams are sets of propositional valuations; in modal team semantics [120], teams are sets of possible worlds; etc. We refer to logics which are primarily intended to be interpreted using team semantics as *team logics*.

Team-semantic *closure properties* such as *downward closure* (a formula φ is downward closed just in case its truth in a team implies truth in all subteams—$[t \vDash \varphi$ and $s \subseteq t] \implies s \vDash \varphi)$ and *union closure* (φ is union closed iff given a nonempty collections of teams T, if $t \vDash \varphi$ for all $t \in T$, then $\bigcup T \vDash \varphi$) play an important role in the study of team logics. They allow for concise and tractable characterizations and classification of these logics, and provide an effective tool for proving definability results, completeness of axiomatizations (see, e.g., [139, 140]), and other properties such as uniform interpolation [43]. In addition to their useful formal properties, the closure properties are also conceptually suggestive—for instance, on the common interpretation of team logics in which teams represent *information states*, a formula φ being downward closed can be thought of as representing the fact that the kind of information or content expressed by φ is *persistent* in the sense that if φ is established in an information state t ($t \vDash \varphi$), then moving from t to a more informed state s by ruling out some possibilities ($s \subseteq t$) does not invalidate the information that φ ($s \vDash \varphi$) (see, e.g., [123, 38, 129, 25, 6]). (Compare the sentence 'It is raining' and the sentence 'It might be raining' with the *epistemic modality* 'might'. The former is persistent: if I know that it is raining—if my information state truthfully establishes that it is raining—then there is no further information that could invalidate 'It is raining'. 'It might be raining', on the other hand, is not persistent: if, for all that I know, it might be raining, and I learn that it is, in fact, not raining, the information or content expressed by 'It might be raining' is invalidated by the further information.)

In this paper, we focus on the *convexity* closure property: a formula φ is convex if its truth in teams s and t implies its truth in all teams u with $s \subseteq u \subseteq t$ (see Figure 3.1). Intuitively, φ is convex if there are no "gaps" in the (team-based) property $\|\varphi\| = \{t \mid t \vDash \varphi\}$ expressed by φ—the meaning of φ is continuous (convexity is sometimes referred to as *continuity*, as in [23]). Convexity can also be seen as a natural minimal generalization of downward closure: clearly, if a formula is downward closed, it is also convex. Shifting from a downward-closed setting to a convex one can potentially allow for the expression of many additional interesting and natural properties; for instance, the epistemic modality-example from above, while not expressible by any downward-closed formula, can, as we will see, be formalized using a simple convex formula.

The notion of convexity defined above is a particularization to the setting of propositional team semantics of a more general notion of gaplessness in meanings. There have been many proposals to the effect that such an absence of gaps is a common or even essential feature of the meanings of simple lexicalized expressions—that convexity constitutes a linguistic or cognitive universal of some kind. Some prominent early



(a) Convex (b) Convex (and upward-directed) (c) Not convex (but upward-directed)

Figure 3.1: Examples of convex/non-convex subsets of a lattice. Note that in a powerset lattice, upward-directedness corresponds to union closure.

examples of such claims include Barwise and Cooper's monotonicity constraint for the meanings of simple noun phrases [19]; van Benthem's expectation that "reasonable" quantifiers exhibit convexity [23]; and Gärdenfors' proposal that concepts conform to convexity [55].

In spite of its naturalness and prominence in the literature, convexity has not received much attention in the study of team logics (with the exception of the recent [72, 36]). In this paper, we aim to further the understanding of convexity in team semantics by proving *expressive completeness theorems* for propositional and modal team logics with respect to convex classes of properties.

In the first part of the paper, we introduce propositional and modal logics which we prove to be expressively complete with respect to the class of all convex propositional/modal properties—that is, we show that all formulas of these logics are convex, and that all convex properties can be expressed by formulas of these logics.

Most team logics are conservative extensions of well-known logics with standard single-evaluation-point Tarskian semantics, in the sense that these logics include a fragment such that for each formula α of the fragment,

$$t \vDash \alpha \iff v \vDash \alpha \text{ for all } v \in t;$$

that is, α is true (according to the team semantics of the logic in question) in a team t just in case it is true in all elements of the team (according to the standard semantics of the fragment); and the set of connectives of the fragment is *functionally complete* for the logic with the standard semantics. We are concerned in particular with propositional logics which extend classical propositional logic and modal logics which extend the smallest normal modal logic **K**—we call each of these fragments the *classical basis* of any team logic that extends it.

The standard classical basis in the team logic literature in the lineage of Väänänen's dependence logic includes the *split disjunction* $\vee$ (also known as the *tensor disjunction* and the *local disjunction*). We show that in a setting in which all convex properties are definable, $\vee$ does not preserve convexity—that is, one can find convex formulas φ and ψ such that $\varphi \vee \psi$ is not convex. To obtain logics which are complete for all convex properties, we must therefore break with this lineage and either modify the split disjunction or opt for a different classical basis. The first of our propositional convex logics, *convex propositional dependence logic* $\mathrm{PL}_{\dot\vee}(=\!(\cdot), \blacklozenge)$, extends one of the most prominent propositional team logics, *propositional dependence logic* $\mathrm{PL}_{\vee}(=\!(\cdot))$ (expressively complete for all downward-closed properties with the *empty team property*) [139] with an operator $\blacklozenge$ corresponding to the epistemic 'might' discussed above, and replaces the $\vee$ of $\mathrm{PL}_{\vee}(=\!(\cdot))$ with a variant $\dot\vee$ (also employed in Hodges' original formulation of team semantics [78]) designed to force downward closure and hence convexity.

Propositional inquisitive logic $\mathrm{PL}_{\to}(\vee\!\!\!\vee)$ is another of the most of prominent propositional team logics. Like propositional dependence logic, $\mathrm{PL}_{\to}(\vee\!\!\!\vee)$ is also expressively complete for the class of all downward-closed propositional properties with the empty team property, making it another candidate for a team logic with a natural convex extension. The classical basis of $\mathrm{PL}_{\to}(\vee\!\!\!\vee)$ (unlike that of $\mathrm{PL}_{\vee}(=\!(\cdot))$) does preserve convexity, and we show that $\mathrm{PL}_{\to}(\blacklozenge)$, the extension of this classical basis with the $\blacklozenge$-operator, is expressively complete for all convex propositional properties. $\mathrm{PL}_{\to}(\vee\!\!\!\vee)$ extends its classical basis with the *inquisitive disjunction* $\vee\!\!\!\vee$ (also known as the *global disjunction*) which, like the split disjunction, does not preserve convexity in a convex setting; we also introduce a convex variant of $\mathrm{PL}_{\to}(\vee\!\!\!\vee)$, *convex inquisitive logic* $\mathrm{PL}_{\to}(\dot{\vee\!\!\!\vee}, \blacklozenge)$, which incorporates a variant $\dot{\vee\!\!\!\vee}$ of $\vee\!\!\!\vee$, which, similarly to $\dot\vee$, forces convexity by forcing downward closure; and show that this variant is, like $\mathrm{PL}_{\dot\vee}(=\!(\cdot), \blacklozenge)$ and $\mathrm{PL}_{\to}(\blacklozenge)$, complete for the class of all convex properties.

We then move to the setting of modal team semantics; as with $\vee$ and $\vee\!\!\!\vee$, we show that the standard diamond modality $\Diamond\!\!\!\!\Diamond$ (the *global diamond*) employed in the modal team logics literature in the dependence logic lineage [120, 73, 90] fails to preserve convexity in a convex setting. To obtain modal team logics complete for the class of all (bisimulation-invariant) convex modal properties, we instead extend our convex propositional logics with the *flat modalities* $\Diamond$ and $\Box$ employed in some formulations of modal inquisitive logic [34] and, more recently, in Aloni's Bilateral State-based Modal Logic (BSML) [6].

In the second part of the paper, we focus on a subclass of convex properties: convex union-closed properties. Union-closed logics such as *inclusion logic* [51, 137] are another prominent family of team logics. We show that in the more restricted union-closed setting, the split disjunction $\vee$ *does* preserve convexity, so we may make use of the standard dependence logic classical basis in formulating a logic expressively complete for this class of properties. Indeed, we answer a problem that was left open in [140] by showing that $\mathrm{PL}_{\vee}(\mathrm{NE})$, the extension of this basis with the *nonemptiness atom* NE—true in a team just in case the team is nonempty—is complete for all convex and union-closed propositional properties. A modal analogue of this yields an expressive completeness result for BSML; this answers another open problem [7]. As with $\vee$, the global diamond $\Diamond\!\!\!\!\diamond$ preserves convexity in a union-closed setting, and we show that the extension of $\mathrm{PL}_{\vee}(\mathrm{NE})$ with the $\Diamond\!\!\!\!\diamond$ and global box $\boxdot$ is also complete for all convex and union-closed modal properties.

The facts we show concerning $\vee$, $\setminus\!\!\!\!/$, and $\Diamond\!\!\!\!\diamond$ are related to the *failure of uniform substitution* exhibited by many team logics. The notion of *uniform definability* [32, 52, 133, 135, 37, 72] arises from this failure, and our facts concerning $\vee$, $\setminus\!\!\!\!/$, and $\Diamond\!\!\!\!\diamond$ also imply facts about the uniform definability of these connectives in the logics we consider. In the final part of this paper, we define a generalization of uniform definability and use this generalization to formulate multiple senses in which one team logic may be said to extend another team logic. These notions of extension, together with the facts concerning $\vee$ and $\setminus\!\!\!\!/$, then allow us to articulate more precisely the sense in which our convex logics extend the downward-closed logics $\mathrm{PL}_{\vee}(=(\cdot))$ and $\mathrm{PL}_{\rightarrow}(\setminus\!\!\!\!/)$.

The paper is structured as follows. In Section 3.2, we work in the convex setting. We introduce the propositional logics $\mathrm{PL}_{\vee}(=(\cdot),\blacklozenge)$, $\mathrm{PL}_{\rightarrow}(\setminus\!\!\!\!/,\blacklozenge)$, and $\mathrm{PL}_{\rightarrow}(\blacklozenge)$, and show that each of them is expressively complete with respect to the class of all convex propositional properties. We then show that modal extensions of these logics are expressively complete with respect to the class of all convex modal properties invariant under bounded bisimulation. We further show that the disjunctions $\vee$ and $\setminus\!\!\!\!/$ and the modality $\Diamond\!\!\!\!\diamond$ can break convexity in a convex setting. In Section 3.3, we move on to the convex and union-closed setting. We show that the logic $\mathrm{PL}_{\vee}(\mathrm{NE})$ is expressively complete with respect to the class of all convex and union-closed propositional properties, and that two distinct modal extensions of $\mathrm{PL}_{\vee}(\mathrm{NE})$ are expressively complete with respect to the class of all convex and union-closed modal properties invariant under bounded bisimulation. In Section 3.4, we define a generalization of uniform definability and use this notion to distinguish multiple senses in which one team logic can extend another. In Section 3.5, we conclude by listing some open problems.

3.2 Convex Properties

In Section 3.2.1, we introduce three logics which we show to be expressively complete with respect to the class of all convex properties. In Section 3.2.2, we introduce modal extensions of these logics and show modal analogues of the expressive completeness result.

3.2.1 Propositional Properties

We define the syntax and semantics of the different classical bases we consider, of our three convex logics, and of propositional dependence logic and propositional inquisitive logic; recall basic facts about propositional team semantics and team-semantic closure properties; show that the tensor disjunction $\vee$ and the global disjunction $\setminus\!\!\!\!/$ fail to preserve convexity in a convex setting; and show our expressive completeness result for $\mathrm{PL}_{\vee}(\mathrm{NE})$.

Downward-closed logic	Classical basis	Convex variant/logic	Classical basis
$\mathrm{PL}_{\vee}(=(\cdot))$	$\mathrm{PL}_{\vee}$	$\mathrm{PL}_{\dot\vee}(=(\cdot),\blacklozenge)$	$\mathrm{PL}_{\dot\vee}$
$\mathrm{PL}_{\to}(\vee\!\!\!\vee)$	$\mathrm{PL}_{\to}$	$\mathrm{PL}_{\to}(\dot{\vee\!\!\!\vee},\blacklozenge)$	$\mathrm{PL}_{\to}$
		$\mathrm{PL}_{\to}(\blacklozenge)$	$\mathrm{PL}_{\to}$

Table 3.1: The logics and their classical bases

Preliminaries

Propositional dependence logic $\mathrm{PL}_{\vee}(=(\cdot))$ is an extension of classical propositional logic with *dependence atoms* $=(p_1,\dots,p_n,p)$; our convex variant $\mathrm{PL}_{\dot\vee}(=(\cdot),\blacklozenge)$ extends classical propositional logic with both dependence atoms and the *epistemic might*-operator $\blacklozenge$. The fragment $\mathrm{PL}_{\vee}$ of $\mathrm{PL}_{\vee}(=(\cdot))$ corresponding to classical propositional logic (the *classical basis* of $\mathrm{PL}_{\vee}(=(\cdot))$) is different from that of $\mathrm{PL}_{\dot\vee}(=(\cdot),\blacklozenge)$ ($\mathrm{PL}_{\dot\vee}$), with the former featuring the *split disjunction* $\vee$, and the latter a variant $\dot\vee$ of $\vee$. *Propositional inquisitive logic* $\mathrm{PL}_{\to}(\vee\!\!\!\vee)$ extends its classical basis $\mathrm{PL}_{\to}$ with the *global* or *inquisitive disjunction* $\vee\!\!\!\vee$. Our convex variant $\mathrm{PL}_{\to}(\dot{\vee\!\!\!\vee},\blacklozenge)$ extends this basis with the operator $\blacklozenge$ and a variant $\dot{\vee\!\!\!\vee}$ of $\vee\!\!\!\vee$; and our third convex logic $\mathrm{PL}_{\to}(\blacklozenge)$, with only the operator $\blacklozenge$. See Table 3.1.

Fix a countably infinite set of proposition letters P.

3.2.1. DEFINITION (Syntax). The formulas of *classical propositional logic (with* $\vee$*/with* $\dot\vee$*/with* $\to$*)* $\mathrm{PL}_{\vee}/\mathrm{PL}_{\dot\vee}/\mathrm{PL}_{\to}$ are given by the BNF-grammars

$$\alpha ::= p \mid \bot \mid \neg\alpha \mid \alpha\wedge\alpha \mid \alpha\vee\alpha; \qquad (\mathrm{PL}_{\vee})$$

$$\alpha ::= p \mid \bot \mid \neg\alpha \mid \alpha\wedge\alpha \mid \alpha\dot\vee\alpha; \qquad (\mathrm{PL}_{\dot\vee})$$

$$\alpha ::= p \mid \bot \mid \alpha\wedge\alpha \mid \alpha\to\alpha; \qquad (\mathrm{PL}_{\to})$$

where $p\in\mathsf{P}$. The formulas of *propositional dependence logic* $\mathrm{PL}_{\vee}(=(\cdot))$/*convex propositional dependence logic* $\mathrm{PL}_{\dot\vee}(=(\cdot),\blacklozenge)$/*propositional inquisitive logic* $\mathrm{PL}_{\to}(\vee\!\!\!\vee)$/*convex propositional inquisitive logic* $\mathrm{PL}_{\to}(\dot{\vee\!\!\!\vee},\blacklozenge)$/$\mathrm{PL}_{\to}(\blacklozenge)$ are given by the BNF-grammars

$$\varphi ::= p \mid \bot \mid \neg\alpha \mid \varphi\wedge\varphi \mid \varphi\vee\varphi \mid =(p_1,\dots,p_n,p); \qquad (\mathrm{PL}_{\vee}(=(\cdot)))$$

$$\varphi ::= p \mid \bot \mid \neg\beta \mid \varphi\wedge\varphi \mid \varphi\dot\vee\varphi \mid =(p_1,\dots,p_n,p) \mid \blacklozenge\varphi; \qquad (\mathrm{PL}_{\dot\vee}(=(\cdot),\blacklozenge))$$

$$\varphi ::= p \mid \bot \mid \varphi\wedge\varphi \mid \varphi\to\varphi \mid \varphi\vee\!\!\!\vee\varphi; \qquad (\mathrm{PL}_{\to}(\vee\!\!\!\vee))$$

$$\varphi ::= p \mid \bot \mid \varphi\wedge\varphi \mid \varphi\to\varphi \mid \varphi\dot{\vee\!\!\!\vee}\varphi \mid \blacklozenge\varphi; \qquad (\mathrm{PL}_{\to}(\dot{\vee\!\!\!\vee},\blacklozenge))$$

$$\varphi ::= p \mid \bot \mid \varphi\wedge\varphi \mid \varphi\to\varphi \mid \blacklozenge\varphi; \qquad (\mathrm{PL}_{\to}(\blacklozenge))$$

where $p,p_1,\dots,p_n\in\mathsf{P}$, $\alpha\in\mathrm{PL}_{\vee}$, and $\beta\in\mathrm{PL}_{\dot\vee}$.

We use the first Greek letters α and β to range exclusively over classical formulas (formulas of $\mathrm{PL}_{\vee}/\mathrm{PL}_{\dot\vee}/\mathrm{PL}_{\dot\vee}$). We write $\mathsf{P}(\varphi)$ for the set of proposition letters appearing in φ, and $\varphi(\mathsf{X})$ if $\mathsf{P}(\varphi)\subseteq\mathsf{X}\subseteq\mathsf{P}$. We write $\varphi(\psi_1/p_1,\dots,\psi_n/p_n)$ for the result of replacing all occurrences of p_i in φ by ψ_i, for $1\le i\le n$.

A (propositional) *team* t *with domain* $\mathsf{X} \subseteq \mathsf{P}$ (or a *team over* X) is a set of valuations with domain X: $t \subseteq 2^{\mathsf{X}}$.

3.2.2. DEFINITION (Semantics). Given a team t over X, the *truth* of a formula $\varphi(\mathsf{X})$ in t (written $t \vDash \varphi$) is defined by the following recursive clauses:

$$
\begin{array}{lll}
t \vDash p & :\Longleftrightarrow & v(p) = 1 \textit{ for all } v \in t. \\
t \vDash \bot & :\Longleftrightarrow & t = \varnothing. \\
t \vDash \neg\alpha & :\Longleftrightarrow & \{v\} \nvDash \alpha \textit{ for all } v \in t. \\
t \vDash \varphi \wedge \psi & :\Longleftrightarrow & t \vDash \varphi \textit{ and } t \vDash \psi. \\
t \vDash \varphi \vee \psi & :\Longleftrightarrow & \textit{there exist } s,u \textit{ such that } s \vDash \varphi,\ u \vDash \psi, \textit{ and } t = s \cup u. \\
t \vDash \varphi \mathbin{\dot\vee} \psi & :\Longleftrightarrow & \textit{there exist } s,u \textit{ such that } s \vDash \varphi,\ u \vDash \psi, \textit{ and } t \subseteq s \cup u. \\
t \vDash \varphi \to \psi & :\Longleftrightarrow & \text{for all } s \subseteq t:\ \text{if } s \vDash \varphi \text{ then } s \vDash \psi. \\
t \vDash =\!(p_1, \dots, p_n, p) & :\Longleftrightarrow & \forall v, w \in t : [\forall 1 \leq i \leq n : v(p_i) = w(p_i)] \implies v(p) = w(p). \\
t \vDash \blacklozenge\varphi & :\Longleftrightarrow & \textit{there exists } s \subseteq t \textit{ such that } s \neq \varnothing \textit{ and } s \vDash \varphi. \\
t \vDash \varphi \mathbin{\vee\!\!\!\vee} \psi & :\Longleftrightarrow & t \vDash \varphi \text{ or } t \vDash \psi. \\
t \vDash \varphi \mathbin{\dot\vee\!\!\!\vee} \psi & :\Longleftrightarrow & \text{there exists } s \supseteq t \text{ such that } s \vDash \varphi \text{ or } s \vDash \psi.
\end{array}
$$

We say that a set of formulas Γ *entails* φ, written $\Gamma \vDash \varphi$, if for all teams t, if $t \vDash \gamma$ for all $\gamma \in \Gamma$, then $t \vDash \varphi$. We write simply $\varphi_1, \dots, \varphi_n \vDash \varphi$ for $\{\varphi_1, \dots, \varphi_n\} \vDash \varphi$ and $\vDash \varphi$ for $\varnothing \vDash \varphi$, where $\varnothing$ is the empty set of formulas. If both $\varphi \vDash \psi$ and $\psi \vDash \varphi$, we say that φ and ψ are *equivalent*, and write $\varphi \equiv \psi$.

We also define (in $\mathrm{PL}_{\to}$ and its extensions) $\neg\varphi := \varphi \to \bot$; (in all logics) $\top := \neg\bot$; (in logics with $\blacklozenge$) $\perp\!\!\!\perp := \blacklozenge\bot$; (in the classical bases) $\alpha \vee \beta := \neg(\neg\alpha \wedge \neg\beta)$. It can be verified that the truth conditions of the same symbol are always the same, regardless of the logic. Note that in $\mathrm{PL}_{\to}$ and its extensions, $t \vDash \neg\varphi \iff \forall s \subseteq t : [s \vDash \varphi \text{ implies } s = \varnothing]$. Note also that $t \vDash \top$ is always the case and $t \vDash \perp\!\!\!\perp$ is never the case. We stipulate that $\bigvee\varnothing := \bot$, $\dot{\bigvee}\varnothing := \bot$, $\bigwedge\varnothing := \top$, $\bigvee\!\!\!\bigvee\varnothing := \bot$, $\dot{\bigvee}\!\!\!\bigvee\varnothing := \bot$.

There are many sets of connectives which are *functionally complete* for classical propositional logic. One can define team-based versions of these sets of connectives to obtain different versions of team-based classical propositional logic such as the classical bases above. Whereas, due to the functional completeness of the different alternatives, the choice of connectives often makes no substantial difference when one is studying the properties of classical propositional logic on its own, this choice may become more significant when the logic is extended with nonclassical connectives. This is the case in our setting: the choice of classical basis of a team logic alters its expressive power, as we see below.

Propositional dependence logic $\mathrm{PL}_{\vee}(=\!(\cdot))$ uses the classical basis $\mathrm{PL}_{\vee}$ with the split disjunction $\vee$, where $\varphi \vee \psi$ is true in a team t just in case the team can be split into subteams s and u such that φ is true in s and ψ is true in u. $\mathrm{PL}_{\vee}$ is the standard classical propositional basis in the dependence logic literature (being used, for instance, in propositional dependence logic [139] and propositional inclusion logic [137]), and $\vee$ is the canonical 'classical' disjunction in this literature (in the sense of Fact 3.2.5, below). Note that one of the two subteams can be empty; for instance, we have that $\{v_{p\overline{q}}\} \vDash p \vee \neg p$ (where $v_{p\overline{q}}(p) = 1$ and $v_{p\overline{q}}(q) = 0$) because $\{v_{p\overline{q}}\} = \{v_{p\overline{q}}\} \cup \varnothing$ where $\{v_{p\overline{q}}\} \vDash p$ and $\varnothing \vDash \neg p$. $\mathrm{PL}_{\vee}(=(\cdot))$ extends $\mathrm{PL}_{\vee}$ with dependence atoms $=(p_1, \ldots, p_n, p)$. The intuitive meaning of $=(p_1, \ldots, p_n, p)$ is that the truth values of $p_1, \ldots, p_n$ jointly determine the truth value of p. A unary dependence atom $=(p)$ is called a *constancy atom*—$=(p)$ is true in a team just in case the truth value of p is constant across the team ($t \vDash =(p)$ iff $[\forall v \in t : v(p) = 1 \text{ or } \forall v \in t : v(p) = 0]$).

The classical basis $\mathrm{PL}_{\dot{\vee}}$ replaces $\vee$ in $\mathrm{PL}_{\vee}$ with the variant $\dot{\vee}$ (used in Hodges' original formulation of team semantics [78]), where $\varphi \dot{\vee} \psi$ is true in a team t just in case t is a subteam of the union of some s which makes φ and some u which makes ψ true. This variant always preserves convexity, whereas $\vee$ does not, as we see below. Our convex variant of dependence logic $\mathrm{PL}_{\dot{\vee}}(=(\cdot), \blacklozenge)$ extends $\mathrm{PL}_{\dot{\vee}}$ with the epistemic might-operator $\blacklozenge$, where $\blacklozenge\varphi$ is true in t just in case t contains a nonempty subteam in which φ is true. Its name is due to the fact that similar operators have been used to model the meanings of epistemic modalities such as the 'might' in 'it might be raining' (see, e.g., [123, 129, 25]). The idea is that a team represents an information state, and if an information state contains some possible states of affairs in which it is raining ($t \vDash \blacklozenge r$), then the information state does not rule out the proposition that it is raining, and hence, for all that one knows given the information in the state, it might be raining.

$\mathrm{PL}_{\rightarrow}$, featuring the *intuitionistic implication*—where $t \vDash \varphi \rightarrow \psi$ just in case whenever φ is true in a subteam of t, so is ψ— is the classical basis for propositional inquisitive logic $\mathrm{PL}_{\rightarrow}(\mathbb{V})$. $\mathrm{PL}_{\rightarrow}(\mathbb{V})$ extends $\mathrm{PL}_{\rightarrow}$ with the *inquisitive* or *global* disjunction $\mathbb{V}$ which has the classical disjunction satisfaction clause (with respect to teams), but which, as we will see, behaves nonclassically. The inquisitive disjunction is used to model the meanings of question in inquisitive logic/semantics; for instance, $p \mathbb{V} q$ represents the question 'p or q?'—the question is true (or *supported*) in a team just in case one of its answers is true. $\mathrm{PL}_{\rightarrow}(\dot{\mathbb{V}}, \blacklozenge)$ replaces $\mathbb{V}$ in $\mathrm{PL}_{\rightarrow}(\mathbb{V})$ with $\dot{\mathbb{V}}$, which, similarly to $\dot{\vee}$, guarantees the preservation of convexity, whereas $\mathbb{V}$ does not. $\mathrm{PL}_{\rightarrow}(\blacklozenge)$, the $\dot{\mathbb{V}}$-free fragment of $\mathrm{PL}_{\rightarrow}(\dot{\mathbb{V}}, \blacklozenge)$, is also sufficiently strong to capture all convex properties (we introduce the stronger logic $\mathrm{PL}_{\rightarrow}(\dot{\mathbb{V}}, \blacklozenge)$ because there is an interesting sense in which $\mathrm{PL}_{\rightarrow}(\dot{\mathbb{V}}, \blacklozenge)$ extends $\mathrm{PL}_{\rightarrow}(\mathbb{V})$, whereas it is an open question whether $\mathrm{PL}_{\rightarrow}(\blacklozenge)$ also extends $\mathrm{PL}_{\rightarrow}(\mathbb{V})$ in this way—see Section 3.4).

3.2.3. DEFINITION (Closure properties). We say that

φ is *downward closed*	iff	$[t \vDash \varphi$ and $s \subseteq t] \Rightarrow s \vDash \varphi$;
φ is *upward closed*	iff	$[t \vDash \varphi$ and $s \supseteq t] \Rightarrow s \vDash \varphi$;
φ is *convex*	iff	$[t \vDash \varphi$, $s \vDash \varphi$, and $s \subseteq u \subseteq t] \Rightarrow u \vDash \varphi$;
φ is *union closed*	iff	$[t \vDash \varphi$ for all $t \in T \neq \varnothing] \Rightarrow \bigcup T \vDash \varphi$;
φ has the *empty team property*	iff	$\varnothing \vDash \varphi$;
φ is *flat*	iff	$t \vDash \varphi \Leftrightarrow [\{v\} \vDash \varphi$ for all $v \in t]$.

It is straightforward to verify that:

3.2.4. FACT.

(i) φ is flat iff φ is downward and union closed, and φ has the empty team property.

(ii) φ is downward closed iff φ is convex and [if $\varphi \not\equiv \perp\!\!\!\perp$, then $\varnothing \vDash \varphi$].

(iii) $\varphi(\mathsf{X})$ is upward closed iff φ is convex and [if $\varphi \not\equiv \perp\!\!\!\perp$, then $2^{\mathsf{X}} \vDash \varphi$].

Classical formulas (formulas of $\mathrm{PL}_{\vee}$/$\mathrm{PL}_{\dot\vee}$/$\mathrm{PL}_{\rightarrow}$) are flat, and we also have:

3.2.5. FACT. For classical α:

$$t \vDash \alpha \iff \{v\} \vDash \alpha \text{ for all } v \in t \iff v \vDash \alpha \text{ for all } v \in t.$$

Here $\vDash$ on the right is the usual single-valuation truth relation for classical propositional logic. Therefore, extensions of our classical bases are also conservative extensions of classical logic—we have that for any set $\Gamma \cup \{\alpha\}$ of classical formulas,

$$\Gamma \vDash \alpha \iff \Gamma \vDash_c \alpha,$$

where $\vDash_c$ is the entailment relation for single-valuation semantics. Given these facts, we use the notations $\{v\} \vDash \alpha$ and $v \vDash \alpha$ interchangeably whenever α is classical, and similarly for $\Gamma \vDash \alpha$ and $\Gamma \vDash_c \alpha$.

Formulas with $\blacklozenge$ may clearly violate downward closure and the empty team property. Formulas of each of our nonclassical logics need not be union closed. Consider, for instance, $\blacklozenge p \rightarrow q$. We have that $\{v_{pq}\} \vDash \blacklozenge p \rightarrow q$ and $\{v_{\overline{pq}}\} \vDash \blacklozenge p \rightarrow q$, but $\{v_{pq}, v_{\overline{pq}}\} \nvDash \blacklozenge p \rightarrow q$. Similarly, for $\varphi :== (q, p,)/\varphi := p \mathbin{\vee\!\!\!\vee} \neg p/\varphi := p \mathbin{\dot\vee\!\!\!\vee} \neg p$ we have $\{v_{pq}\} \vDash \varphi$ and $\{v_{\overline{p}q}\} \vDash \varphi$, but $\{v_{pq}, v_{\overline{p}q}\} \nvDash \varphi$. However, we do have:

3.2.6. PROPOSITION. *Formulas of* $\mathrm{PL}_{\vee}(=(\cdot))$ *and* $\mathrm{PL}_{\rightarrow}(\mathbin{\vee\!\!\!\vee})$ *are downward closed (and hence also convex). Formulas of* $\mathrm{PL}_{\dot\vee}(=(\cdot), \blacklozenge)$, $\mathrm{PL}_{\rightarrow}(\mathbin{\dot\vee\!\!\!\vee}, \blacklozenge)$, *and* $\mathrm{PL}_{\rightarrow}(\blacklozenge)$ *are convex.*

Proof:
By induction on the structure of formulas φ. Most cases are straightforward—note in particular that $=(p_1, \ldots, p_n, p)$, $\varphi \rightarrow \psi$, $\varphi \vee \psi$, and $\varphi \mathbin{\vee\!\!\!\vee} \psi$ are always downward closed, and hence convex, and that $\blacklozenge\varphi$ is always upward closed, and hence convex. □

As mentioned above, whereas the variant $\dot\vee$ always preserves convexity, the split disjunction $\vee$ need not do so, and in fact, as implied by the fact below, no logic expressively complete for the class of all convex properties can incorporate $\vee$—this is why we swap $\vee$ for $\dot\vee$ in our convex variant of dependence logic $\mathrm{PL}_{\dot\vee}(=(\cdot), \blacklozenge)$. The situation with $\mathbin{\vee\!\!\!\vee}$ and $\mathbin{\dot\vee\!\!\!\vee}$ is analogous.

3.2.7. FACT. There are convex φ, ψ such that $\varphi \vee \psi$ is not convex. Similarly, there are convex φ, ψ such that $\varphi \mathbin{\vee\!\!\!\vee} \psi$ is not convex.

Proof:
For the first part, let $\varphi := (((p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE})) \to \bot) \wedge \blacklozenge r$. Clearly the first conjunct is downward closed (and hence convex) and the second is upward closed (and hence convex); therefore, since conjunction preserves convexity, φ is convex. Now observe that $\{v_{\bar{p}r}\} \vDash \varphi \vee \varphi$; $\{v_{\bar{p}r}, v_{p\bar{r}}\} \nvDash \varphi \vee \varphi$; and $\{v_{\bar{p}r}, v_{p\bar{r}}, v_{pr}\} \vDash \varphi \vee \varphi$, where $\{v_{\bar{p}r}\} \subseteq \{v_{\bar{p}r}, v_{p\bar{r}}\} \subseteq \{v_{\bar{p}r}, v_{p\bar{r}}, v_{pr}\}$.

For the second part, note that p and $\blacklozenge q$ are convex. We have $\{v_{p\bar{q}}\} \vDash p \mathbin{\vee\!\!\vee} \blacklozenge q$; $\{v_{p\bar{q}}, v_{\overline{pq}}\} \nvDash p \mathbin{\vee\!\!\vee} \blacklozenge q$; and $\{v_{p\bar{q}}, v_{\overline{pq}}, v_{pq}\} \vDash p \mathbin{\vee\!\!\vee} \blacklozenge q$, and also that $\{v_{p\bar{q}}\} \subseteq \{v_{p\bar{q}}, v_{\overline{pq}}\} \subseteq \{v_{p\bar{q}}, v_{\overline{pq}}, v_{pq}\}$. □

The above is related to the fact that, as with many team-based logics, the nonclassical logics we consider are not *closed under uniform substitution*: $\varphi \vDash \psi$ need not imply $\varphi(\chi/p) \vDash \psi(\chi/p)$. For instance, we have $p \vee p \vDash p$, but $(p \mathbin{\vee\!\!\vee} \neg p) \vee (p \mathbin{\vee\!\!\vee} \neg p) \nvDash (p \mathbin{\vee\!\!\vee} \neg p)$ (consider the team $\{v_p, v_{\bar{p}}\}$), and $p \wedge (q \vee r) \vDash (p \wedge q) \vee (p \wedge r)$, but $\blacklozenge\top \wedge (q \vee r) \nvDash (\blacklozenge\top \wedge q) \vee (\blacklozenge\top \wedge r)$ (consider the team $\{v_{q\bar{r}}\}$). We return to Fact 3.2.7 and its connection with closure under uniform substitution in Section 3.4.

Expressive Completeness

We measure the expressive power of the logics in terms of the properties—classes of teams—expressible in them.

3.2.8. DEFINITION (Properties and Expressive Completeness). A *(propositional team) property* over X is a class of (propositional) teams over X. For each formula $\varphi(\mathsf{X})$, we denote by $\|\varphi\|_{\mathsf{X}}$ (or simply $\|\varphi\|$) the property over X *expressed* by φ:

$$\|\varphi\|_{\mathsf{X}} := \{t \in 2^{\mathsf{X}} \mid t \vDash \varphi\}.$$

Given a class of properties $\mathbb{P}$ and $\mathsf{X} \subseteq \mathsf{P}$, we let

$$\mathbb{P}_{\mathsf{X}} := \{\mathcal{P} \text{ is a property over } \mathsf{X} \mid \mathcal{P} \in \mathbb{P}\}.$$

We say that a logic L is *expressively complete* for a class of properties $\mathbb{P}$, written $\|\mathrm{L}\| = \mathbb{P}$, if for each finite $\mathsf{X} \subseteq \mathsf{P}$,

$$\|\mathrm{L}\|_{\mathsf{X}} := \{\|\varphi\|_{\mathsf{X}} \mid \varphi \text{ is a formula of L}\} = \mathbb{P}_{\mathsf{X}}.$$

That is, L is expressively complete for $\mathbb{P}$ if ($\subseteq$) each property $\|\varphi\|$ definable by a formula φ of L is in $\mathbb{P}$, and ($\supseteq$) each property in $\mathbb{P}$ over a finite X is definable by a formula of L. We also write $\|\mathrm{L}\| \subseteq \mathbb{P}$ to mean that for each finite $\mathsf{X} \subseteq \mathsf{P}$, $\|\mathrm{L}\|_{\mathsf{X}} \subseteq \mathbb{P}_{\mathsf{X}}$, etc.

The definition of closure properties is extended to team properties in the obvious way. For instance, a property $\mathcal{P}$ is downward closed if $[t \in \mathcal{P}$ and $s \subseteq t]$ implies $s \in \mathcal{P}$. Let $\mathbb{C}/\mathbb{CU}/\mathbb{DE}/\mathbb{U}/\mathbb{F}$ be the class of all convex/convex and union-closed/downward-closed and empty-team-property/upward-closed/flat properties, respectively.

We show that each of $\mathrm{PL}_{\vee}(=(\cdot),\blacklozenge)$, $\mathrm{PL}_{\to}(\mathbb{V},\blacklozenge)$, and $\mathrm{PL}_{\to}(\blacklozenge)$ is complete for $\mathbb{C}$, i.e., $\|\mathrm{PL}_{\vee}(=(\cdot),\blacklozenge)\| = \|\mathrm{PL}_{\to}(\mathbb{V},\blacklozenge)\| = \|\mathrm{PL}_{\to}(\blacklozenge)\| = \mathbb{C}$. Note that have already shown, in Proposition 3.2.6, that $\|\mathrm{PL}_{\vee}(=(\cdot),\blacklozenge)\| \subseteq \mathbb{C}$, $\|\mathrm{PL}_{\to}(\mathbb{V},\blacklozenge)\| \subseteq \mathbb{C}$, and $\|\mathrm{PL}_{\to}(\blacklozenge)\| \subseteq \mathbb{C}$.

To show the other direction, we construct, for each property $\mathcal{P}$ in $\mathbb{C}_{\mathsf{X}}$, a formula in $\|\mathrm{PL}_{\vee}(=(\cdot),\blacklozenge)\|_{\mathsf{X}}$ that expresses that $\mathcal{P}$: a characteristic formula for $\mathcal{P}$—and similarly for the other logics. We begin by recalling characteristic formulas for valuations and teams $\chi_v^{\mathsf{X}}, \chi_t^{\mathsf{X}} \in \mathrm{PL}_{\vee}/\mathrm{PL}_{\vee}/\mathrm{PL}_{\to}$ from the literature (see, e.g., [41, 71, 139, 140]). Fix a finite $\mathsf{X} \subseteq \mathsf{P}$. For a valuation v, let

$$\chi_v^{\mathsf{X}} := \bigwedge\{p \mid p \in \mathsf{X}, v(p) = 1\} \wedge \bigwedge\{\neg p \mid p \in \mathsf{X}, v(p) = 0\}.$$

It is then easy to see that:

$$w \vDash \chi_v^{\mathsf{X}} \iff w \upharpoonright \mathsf{X} = v \upharpoonright \mathsf{X},$$

and if $w, v \in 2^{\mathsf{X}}$, then $w \vDash \chi_v^{\mathsf{X}} \iff w = v$. We usually write simply χ_v. For a team t, we let:

$$\chi_t^{\mathsf{X}} := \bigvee_{v \in t} \chi_v^{\mathsf{X}}.$$

Then:

$$s \vDash \chi_t^{\mathsf{X}} \iff s \upharpoonright \mathsf{X} \subseteq t \upharpoonright \mathsf{X}, \qquad \text{(where for a team } u,\ u \upharpoonright \mathsf{X} := \{v \upharpoonright \mathsf{X} \mid v \in u\})$$

and if $t, s \subseteq 2^{\mathsf{X}}$, then $s \vDash \chi_t^{\mathsf{X}} \iff s \subseteq t$. Again, we usually write simply χ_t. Note that since for a given finite X, there are only finitely many χ_v^{X}, we may assume the disjunction in χ_t^{X} to be finite and therefore for the formula to be well-defined.[1] Observe also that we have used the defined disjunction $\vee = \neg\wedge\neg$ (available in each of our logics) in the definition of χ_t; it is, however, easy to check that $\bigvee_{v\in t} \chi_v \equiv \bigvee_{v \in t} \chi_v \equiv \mathbb{V}_{v\in t} \chi_v$.

It is instructive to present our construction of the characteristic formulas for properties in a schematic manner. Note first that the empty property $\mathcal{P} = \varnothing$ is convex, and that it is expressible in each of logics using the formula(s) $\perp\!\!\!\perp$. As for nonempty properties, we construct, in each of our logics, for such property $\mathcal{P}$, a formula $\chi_{\mathcal{P}}^{\mathbb{D}}$ such that

$$t \vDash \chi_{\mathcal{P}}^{\mathbb{D}} \iff \exists s \in \mathcal{P} : t \subseteq s, \tag{$*$}$$

and a formula $\chi_{\mathcal{P}}^{\mathbb{U}}$ such that

$$t \vDash \chi_{\mathcal{P}}^{\mathbb{U}} \iff \exists s \in \mathcal{P} : t \supseteq s.$$

The formulas $\chi_{\mathcal{P}}^{\mathbb{D}}$ are characteristic formulas for nonempty downward-closed properties: observe that for nonempty downward-closed $\mathcal{P}$, $\left\|\chi_{\mathcal{P}}^{\mathbb{D}}\right\| = \mathcal{P}$; similarly the formulas $\chi_{\mathcal{P}}^{\mathbb{U}}$ are characteristic formulas for nonempty upward-closed properties. Using these formulas, we construct characteristic formulas for nonempty convex properties as follows:

[1] More precisely, what we do is choose, for each infinite t, some finite s such that $\{\chi_v^{\mathsf{X}} \mid v \in t\} = \{\chi_v^{\mathsf{X}} \mid v \in s\}$ to act as the representative of t, and define $\chi_t^{\mathsf{X}} := \chi_s^{\mathsf{X}}$. Similar remarks apply to the characteristic formulas for properties defined below—observe that this is what allows us to treat all properties over a finite X as if they were finite.

3.2.9. LEMMA. *Let* $\mathsf{X} \subseteq \mathsf{P}$ *be finite, and for each* $\mathcal{P} \neq \varnothing$ *over* X*, let* $\chi_{\mathcal{P}}^{\mathsf{X},\mathbb{D}}$ *and* $\chi_{\mathcal{P}}^{\mathsf{X},\mathbb{U}}$ *be such that for any* $t \subseteq 2^{\mathsf{X}}$*,* $t \vDash \chi_{\mathcal{P}}^{\mathsf{X},\mathbb{D}} \iff \exists s \in \mathcal{P} : t \subseteq s$ *and* $t \vDash \chi_{\mathcal{P}}^{\mathsf{X},\mathbb{U}} \iff \exists s \in \mathcal{P} : t \supseteq s$*. Then for any convex* $\mathcal{P} \neq \varnothing$ *over* X*,* $\left\| \chi_{\mathcal{P}}^{\mathsf{X},\mathbb{D}} \wedge \chi_{\mathcal{P}}^{\mathsf{X},\mathbb{U}} \right\|_{\mathsf{X}} = \mathcal{P}$*.*

Proof:

$\supseteq$: For any $t \in \mathcal{P}$, $t \subseteq t \subseteq t$, whence $t \vDash \chi_{\mathcal{P}}^{\mathbb{D}} \wedge \chi_{\mathcal{P}}^{\mathbb{U}}$.

$\subseteq$: If $t \vDash \chi_{\mathcal{P}}^{\mathbb{D}} \wedge \chi_{\mathcal{P}}^{\mathbb{U}}$, then for some $u, s \in \mathcal{P}$, $s \subseteq t \subseteq u$, whence $t \in \mathcal{P}$ by convexity. □

We first construct the formulas $\chi_{\mathcal{P}}^{\mathbb{U}}$. These can be constructed in the same manner in each of our logics:

3.2.10. LEMMA. *Let* $\mathsf{X} \subseteq \mathsf{P}$ *be finite. For* $\mathcal{P} = \{t_1, \ldots, t_n\} \neq \varnothing$ *over* X*, let*

$$\chi_{\mathcal{P}}^{\mathsf{X},\mathbb{U}} := \bigwedge_{v_1 \in t_1, \ldots, v_n \in t_n} \blacklozenge(\chi_{v_1}^{\mathsf{X}} \vee \ldots \vee \chi_{v_n}^{\mathsf{X}}).$$

Then for any $t \subseteq 2^{\mathsf{X}}$*,* $t \vDash \chi_{\mathcal{P}}^{\mathsf{X},\mathbb{U}} \iff \exists t_i \in \mathcal{P} : t \supseteq t_i$*.*

Proof:

$\Longleftarrow$: Let $t_i \in \mathcal{P}$ be such that $t \supseteq t_i$. For each $v_i \in t_i$, we have that $v_i \vDash \chi_{v_i}$, so that also $v_i \vDash \chi_{v_1} \vee \ldots \vee \chi_{v_n}$ for any $v_1 \in t_1 \ldots v_{i-1} \in t_{i-1}, v_{i+1} \in t_{i+1}, \ldots v_n \in t_n$. Therefore, since $v_i \in t_i \subseteq t$, $t \vDash \blacklozenge(\chi_{v_1} \vee \ldots \vee \chi_{v_n})$. Repeating, this argument, we have $t \vDash \bigwedge_{v_1 \in t_1, \ldots, v_n \in t_n} \blacklozenge(\chi_{v_1} \vee \ldots \vee \chi_{v_n})$.

$\Longrightarrow$: Let $t \vDash \bigwedge_{v_1 \in t_1, \ldots, v_n \in t_n} \blacklozenge(\chi_{v_1} \vee \ldots \vee \chi_{v_n})$. If $\mathcal{P}$ has the empty team property, we have $\varnothing \subseteq t$ where $\varnothing \in \mathcal{P}$; we may therefore assume $\mathcal{P}$ does not have the empty team property, whence $t_i \neq \varnothing$ for all $1 \leq i \leq n$, whence $\bigwedge_{v_1 \in t_1, \ldots, v_n \in t_n} \blacklozenge(\chi_{v_1} \vee \ldots \vee \chi_{v_n})$ is not simply $\top$ (recall that $\bigwedge \varnothing = \top$). Assume for contradiction that $t_i \nsubseteq t$ for all $t_i \in \mathcal{P}$. Then for each $t_i \in \mathcal{P}$ there is some $w_i \in t_i$ such that $w_i \notin t$. By $t \vDash \bigwedge_{v_1 \in t_1, \ldots, v_n \in t_n} \blacklozenge(\chi_{v_1} \vee \ldots \vee \chi_{v_n})$, there is, for each $v_1 \in t_1, \ldots, v_n \in t_n$, a nonempty $t_{v_1, \ldots, v_n} \subseteq t$ such that $t_{v_1, \ldots, v_n} \vDash \chi_{v_1} \vee \ldots \vee \chi_{v_n}$. Therefore, in particular, there is a nonempty $t_{w_1, \ldots, w_n} \subseteq t$ such that $t_{w_1, \ldots, w_n} \vDash \chi_{w_1} \vee \ldots \vee \chi_{w_n}$. Then $t_{w_1, \ldots, w_n} \subseteq \bigcup_{1 \leq i \leq n} \{w_i\}$ and $t_{w_1, \ldots, w_n} \neq \varnothing$. Given $t_{w_1, \ldots, w_n} \subseteq t$ we have $t \cap \bigcup_{1 \leq i \leq n} \{w_i\} \neq \varnothing$, contradicting the fact that $w_i \notin t$ for all $t_i \in \mathcal{P}$. □

We now turn to the formulas $\chi_{\mathcal{P}}^{\mathbb{D}}$. We will construct these in a distinct manner in each of the three logics $\mathrm{PL}_{\vee}(=(\cdot), \blacklozenge)$, $\mathrm{PL}_{\rightarrow}(\mathbin{\rotatebox[origin=c]{180}{$\wedge$}}\!\!\!\!\vee, \blacklozenge)$ and $\mathrm{PL}_{\rightarrow}(\blacklozenge)$. We begin constructing these formulas by recalling the following expressive completeness results for propositional dependence logic and propositional inquisitive logic:

3.2.11. THEOREM ([41, 139]). *Each of* $\mathrm{PL}_{\vee}(=(\cdot))$ *and* $\mathrm{PL}_{\rightarrow}(\mathbb{V})$ *is expressively complete for the class of all downward-closed properties with the empty team property:*

$$\|\mathrm{PL}_{\vee}(=(\cdot))\| = \|\mathrm{PL}_{\rightarrow}(\mathbb{V})\| = \mathbb{DE}.$$

We make use of the characteristic formulas used to prove the above and the similarity between these logics and our convex logics to construct the formulas $\chi^{\mathbb{D}}_{\mathcal{P}}$. Note that although the formulas $\chi^{\mathbb{D}}_{\mathcal{P}}$ are characteristic formulas for nonempty downward-closed properties (equivalently, for properties in $\mathbb{DE}$), it need not be the case that any given characteristic formula $\chi_{\mathcal{P}}$ for a property in $\mathbb{DE}$ satisfies property $(*)$. However, it turns out that by manipulating the formulas of $\mathrm{PL}_{\vee}(=(\cdot))$/$\mathrm{PL}_{\rightarrow}(\mathbin{\vee\!\!\vee})$ used to prove Theorem 3.2.11, we are in each case able find similar formulas $\chi^{\mathbb{D}}_{\mathcal{P}}$ with the desired property $(*)$ in our convex logics.

Let us first consider the characteristic formulas in $\mathrm{PL}_{\rightarrow}(\mathbin{\vee\!\!\vee})$, which we will use to construct $\chi^{\mathbb{D}}_{\mathcal{P}}$ in $\mathrm{PL}_{\rightarrow}(\mathbin{\vee\!\!\vee}, \blacklozenge)$ and $\mathrm{PL}_{\rightarrow}(\blacklozenge)$. These formulas are of the form $\bigvee\!\!\!\bigvee_{s\in\mathcal{P}}\chi^{\mathsf{X}}_{s}$. It is easy to see that these formulas already satisfy property $(*)$; it therefore suffices to show that we can find equivalent formulas in $\mathrm{PL}_{\rightarrow}(\mathbin{\vee\!\!\vee}, \blacklozenge)$ and $\mathrm{PL}_{\rightarrow}(\blacklozenge)$. In the former case this is trivial given that for any downward-closed (and hence for any classical) φ and ψ, we have $\varphi \mathbin{\vee\!\!\vee} \psi \equiv \varphi \mathbin{\dot{\vee}\!\!\vee} \psi$.

3.2.12. LEMMA. *Let* $\mathsf{X} \subseteq \mathsf{P}$ *be finite. For* $\mathcal{P} \neq \varnothing$ *over* X*, let*

$$\chi^{\mathcal{X},\mathbb{D}}_{\mathsf{P}} := \dot{\bigvee}\!\!\!\bigvee_{s\in\mathcal{P}}\chi^{\mathsf{X}}_{s}.$$

Then for any $t \subseteq 2^{\mathsf{X}}$, $t \models \chi^{\mathsf{X},\mathbb{D}}_{\mathcal{P}} \iff \exists s \in \mathcal{P} : t \subseteq s$.

Proof:
Immediate from the definitions and the fact that $s \models \chi_s$. □

As for $\mathrm{PL}_{\rightarrow}(\blacklozenge)$, we show below that there is a formula equivalent to $\bigvee\!\!\!\bigvee_{s\in\mathcal{P}}\chi_s$ in $\mathrm{PL}_{\rightarrow}(\blacklozenge)$ by showing that the global disjunction of any collection of flat formulas is definable in $\mathrm{PL}_{\rightarrow}(\blacklozenge)$. Note that since $\mathrm{PL}_{\rightarrow}(\mathbin{\vee\!\!\vee}, \blacklozenge)$ is a syntactic extension of $\mathrm{PL}_{\rightarrow}(\blacklozenge)$, the below also yields a formula in $\mathrm{PL}_{\rightarrow}(\mathbin{\vee\!\!\vee}, \blacklozenge)$ which is equivalent to $\bigvee\!\!\!\bigvee_{s\in\mathcal{P}}\chi_s$, whence the disjunction $\mathbin{\vee\!\!\vee}$ and the formula $\dot{\bigvee}\!\!\!\bigvee_{s\in\mathcal{P}}\chi_s$ defined above are not required to prove the expressive completeness of $\mathrm{PL}_{\rightarrow}(\mathbin{\vee\!\!\vee}, \blacklozenge)$. We discuss our reasons for defining this extension with a disjunction which does not in this case yield an increase in expressive power in Section 3.4.

3.2.13. PROPOSITION. *For any nonempty collection* $\{\varphi_i\}_{i\in I}$ *of flat formulas,*

$$\bigwedge_{i\in I}((\bigwedge_{j\in I\setminus\{i\}} \blacklozenge\neg\varphi_j) \rightarrow \varphi_i) \equiv \bigvee\!\!\!\bigvee_{i\in I}\varphi_i \equiv \dot{\bigvee}\!\!\!\bigvee_{i\in I}\varphi_i.$$

Proof:
We show the first equivalence.

$\models$: Let $t \models \bigwedge_{i\in I}((\bigwedge_{j\in I\setminus\{i\}} \blacklozenge\neg\varphi_j) \rightarrow \varphi_i)$ and assume for contradiction that $t \not\models \varphi_i$ for each $i \in I$. By flatness, for each $i \in I$ there is some $v_i \in t$ such that $\{v_i\} \not\models \varphi_i$ whence also $\{v_i\} \models \neg\varphi_i$. Then for each $i \in I$, $t \models \blacklozenge\neg\varphi_i$. By $t \models (\bigwedge_{j\in I\setminus\{i\}} \blacklozenge\neg\varphi_i) \rightarrow \varphi_i$, we have $t \models \varphi_i$ for all $i \in I$, a contradiction. So for some $i \in I$, $t \models \varphi_i$, whence $t \models \bigvee\!\!\!\bigvee_{i\in I}\varphi_i$.

$\dashv$: Let $t \vDash \bigvee\!\!\!\!\bigvee_{i\in I}\varphi_i$. We can then fix some $i \in I$ such that $t \vDash \varphi_i$. Let $s \subseteq t$ be such that $s \vDash \bigwedge_{j\in I\setminus\{i\}} \blacklozenge\neg\varphi_j$. By downward closure, also $s \vDash \varphi_i$, so $t \vDash (\bigwedge_{j\in I\setminus\{i\}} \blacklozenge\neg\varphi_j) \to \varphi_i$. Now let $k \in I$ be such that $k \neq i$. Let $s \subseteq t$, and assume for contradiction that $s \vDash \bigwedge_{j\in I\setminus\{k\}} \blacklozenge\neg\varphi_j$. Then $s \vDash \blacklozenge\neg\varphi_i$, so there is some $u \subseteq s$ such that $u \neq \varnothing$ and $u \vDash \neg\varphi_i$. But we have $u \subseteq s \subseteq t$, whence by downward closure, $u \vDash \varphi_i$. Then by $u \vDash \neg\varphi_i$ we have $u = \varnothing$, a contradiction. Therefore, vacuously $t \vDash (\bigwedge_{j\in I\setminus\{k\}} \blacklozenge\neg\varphi_j) \to \varphi_k$. □

It remains only to show that we can define formulas $\chi^{\mathbb{D}}_{\mathcal{P}}$ with property $(*)$ in $\mathrm{PL}_{\veebar}(=(\cdot),\blacklozenge)$. We make use of the characteristic formulas of $\mathrm{PL}_{\veebar}(=(\cdot))$ used to prove Theorem 3.2.11. These are defined by first letting, for each finite $\mathsf{X} \subseteq \mathsf{P}$, $\gamma^{\mathsf{X}}_0 := \bot$, $\gamma^{\mathsf{X}}_1 := \bigwedge_{p\in\mathsf{X}} {=}(p)$, and for $n \geq 2$, $\gamma^{\mathsf{X}}_n := \bigvee_n \gamma_1$. Then it is easy to see that for $t \subseteq 2^{\mathsf{X}}$, we have $t \vDash \gamma^{\mathsf{X}}_n \iff |t| \leq n$, where $|t|$ is the size of t. One then defines, for each nonempty $t \subseteq 2^{\mathsf{X}}$, $\xi^{\mathsf{X}}_t := \gamma^{\mathsf{X}}_{|t|-1} \vee \chi^{\mathsf{X}}_{\|\top\|_{\mathsf{X}}\setminus t}$. It can then be shown that for $t \subseteq 2^{\mathsf{X}}$, $t \vDash \xi^{\mathsf{X}}_s \iff s \nsubseteq t$. Finally, the characteristic formula for a $\mathcal{P} \in \mathbb{DE}$ over X is given by $\bigwedge_{s\in\|\top\|_{\mathsf{X}}\setminus\mathcal{P}} \xi^{\mathsf{X}}_s$. Now, it can be verified that if $\mathcal{P}$ is not downward closed, its characteristic formula as defined here need not have property $(*)$. Given the properties of the formulas ξ_s, we do, however, have:

3.2.14. LEMMA ([139]). *Let $\mathsf{X} \subseteq \mathsf{P}$ be finite. For $\mathcal{P}$ over X such that $\varnothing \notin \mathcal{P}$, and $t \subseteq 2^{\mathsf{X}}$:*

$$t \vDash \bigwedge_{s\in\mathcal{P}} \xi^{\mathsf{X}}_s \iff s \nsubseteq t \textit{ for all } s \in \mathcal{P}.$$

We use this to construct our formulas $\chi^{\mathbb{D}}_{\mathcal{P}}$ with property $(*)$:

3.2.15. LEMMA. *Let $\mathsf{X} \subseteq \mathsf{P}$ be finite. For $\mathcal{P} \neq \varnothing$ over X, let*

$$\chi^{\mathsf{X},\mathbb{D}}_{\mathcal{P}} := \bigwedge_{u\in\mathcal{Q}} \xi'^{\mathsf{X}}_u,$$

where $\xi'^{\mathsf{X}}_u \in \mathrm{PL}_{\veebar}(=(\cdot),\blacklozenge)$ is defined by replacing each $\vee$ in ξ^{X}_u with $\veebar$, and $\mathcal{Q} := \{u \subseteq 2^{\mathsf{X}} \mid u \nsubseteq s \text{ for all } s \in \mathcal{P}\}$. Then for any $t \subseteq 2^{\mathsf{X}}$, $t \vDash \chi^{\mathsf{X},\mathbb{D}}_{\mathcal{P}} \iff \exists s \in \mathcal{P} : t \subseteq s$.

Proof:
Note that for all downward-closed formulas φ and ψ, $\varphi \vee \psi \equiv \varphi \veebar \psi$, and that, since $\mathcal{P} \neq \varnothing$, we have $\varnothing \notin \mathcal{Q}$. Therefore, by Lemma 3.2.14, we have that $t \vDash \chi^{\mathbb{D}}_{\mathcal{P}} \iff u \nsubseteq t$ for all $u \in \mathcal{Q}$. We show that $u \nsubseteq t$ for all $u \in \mathcal{Q} \iff \exists s \in \mathcal{P} : t \subseteq s$.

$\Longrightarrow$: Assume for contradiction that there is no $s \in \mathcal{P}$ such that $t \subseteq s$. Then $t \in \mathcal{Q}$, contradicting $u \nsubseteq t$ for all $u \in \mathcal{Q}$.

$\Longleftarrow$: Let $s \in \mathcal{P}$ be such that $t \subseteq s$. Assume for contradiction that there is some $u \in \mathcal{Q}$ such that $u \subseteq t$. Then $u \subseteq t \subseteq s$, contradicting the definition of $\mathcal{Q}$. □

And we are done:

3.2.16. THEOREM. *Each of* $\mathrm{PL}_{\veebar}(=(\cdot),\blacklozenge)$, $\mathrm{PL}_{\to}(\bigvee\!\!\!\!\bigvee,\blacklozenge)$, *and* $\mathrm{PL}_{\to}(\blacklozenge)$ *is expressively complete for the class of all convex properties:*

$$\|\mathrm{PL}_{\veebar}(=(\cdot),\blacklozenge)\| = \|\mathrm{PL}_{\to}(\bigvee\!\!\!\!\bigvee,\blacklozenge)\| = \|\mathrm{PL}_{\to}(\blacklozenge)\| = \mathbb{C}.$$

Proof:
The direction $\|\mathrm{L}\| \subseteq \mathbb{C}$ is by Proposition 3.2.6 for each relevant L. For the direction $\mathbb{C} \subseteq \|\mathrm{L}\|$, let $\mathcal{P} \in \mathbb{C}_{\mathsf{X}}$. If $\mathcal{P} = \varnothing$, then $\mathcal{P} = \|\perp\!\!\!\perp\|_{\mathsf{X}} \in \|\mathrm{L}\|_{\mathsf{X}}$. If $\mathcal{P} \neq \varnothing$, the result follows by Lemmas/Propositions 3.2.9, 3.2.10, 3.2.12, 3.2.13, and 3.2.15. □

3.2.2 Modal Properties

We define two distinct modal extensions of each of our classical bases. In each case, one extension uses what we will call the *flat modalities* $\Diamond$ and $\Box$, also used in early versions of modal inquisitive logic [34] and in Aloni's Bilateral State-based Modal Logic [6]; the other the *global modalities* ◈ and ⧈, used in modal dependence logic and other modal logics of dependence [120, 73, 90]. We call each of these new classical bases $\mathrm{ML}_{i,j}$, where $i \in \{\vee, \dot{\vee}, \to\}$ and $j \in \{\Diamond, \text{◈}\}$, *classical modal logic*; they are defined in the obvious way. The modal extensions $\mathrm{ML}_{\dot{\vee},j}(=\!(\cdot), \blacklozenge)$, $\mathrm{ML}_{\to,j}(\text{⩖}, \blacklozenge)$ and $\mathrm{ML}_{\to,j}(\blacklozenge)$, where $j \in \{\Diamond, \text{◈}\}$ are likewise defined in the obvious way, except we define dependence atoms in $\mathrm{ML}_{\dot{\vee},\Diamond}(=\!(\cdot), \blacklozenge)$ as follows:

$$=\!(\alpha_1, \dots, \alpha_n, \alpha),$$

where $\alpha, \alpha_1, \dots, \alpha_n, \alpha \in \mathrm{ML}_{\dot{\vee},\Diamond}$ (and similarly for $\mathrm{ML}_{\dot{\vee},\text{◈}}(=\!(\cdot), \blacklozenge)$). That is, we now allow all classical formulas to appear in dependence atoms. (This leads to an increase in expressive power; for instance, it can be shown that the analogue of Theorem 3.2.11 holds for $\mathrm{ML}_{\vee,\text{◈}}(=\!(\cdot))$ with these extended dependence atoms, but not for the variant which only allows propositional variables in dependence atoms [47, 71].)

In this section, we show modal analogues of our expressive completeness theorems for the extensions of our convex logics with the flat modalities, and we show that no such analogues can be obtained for the extensions with the global modalities.

Modal team logics are interpreted on teams in standard Kripke models.

3.2.17. DEFINITION. A *(Kripke) model* (over $\mathsf{X} \subseteq \mathsf{P}$) is a triple $M = (W, R, V)$, where

- W is a nonempty set, whose elements are called *(possible) worlds*;
- $R \subseteq W \times W$ is a binary relation, called the *accessibility relation*;
- $V : \mathsf{X} \to \wp(W)$ is a function, called the *valuation*.

We call a subset $t \subseteq W$ of W a *team* on M.

For any world w in M, define, as usual, $R[w] := \{v \in W \mid wRv\}$. Similarly, for any team t on M, define $R[t] := \bigcup_{w \in t} R[w]$ and $R^{-1}[t] := \{v \in W \mid \exists w \in t : vRw\}$. We write tRs and say that s is a *successor team* of t if $s \subseteq R[t]$ and $t \subseteq R^{-1}[s]$.

The modal semantics for most connectives are the obvious analogues of their propositional semantics (or instance, a team t on M makes p true—written $M, t \vDash p$—just in case $t \subseteq V(p)$). We only explicitly give the semantics for the dependence atoms and the modalities.

$$
\begin{array}{lll}
M,t \vDash =\!(\alpha_1,\dots,\alpha_n,\alpha) & :\Longleftrightarrow & \forall v,w \in t : [\forall 1 \le i \le n : \{v\} \vDash \alpha_i \iff \{w\} \vDash \alpha_i] \Longrightarrow \\
 & & [\{v\} \vDash \alpha \iff \{w\} \vDash \alpha]. \\
M,t \vDash \Diamond\varphi & :\Longleftrightarrow & \forall w \in t : \exists s \subseteq R[w] : s \neq \varnothing \text{ and } M,s \vDash \varphi. \\
M,t \vDash \Box\varphi & :\Longleftrightarrow & \forall w \in t : M,R[s] \vDash \varphi. \\
M,t \vDash \diamondcdot\varphi & :\Longleftrightarrow & \exists s \subseteq W : tRs \text{ and } M,s \vDash \varphi. \\
M,t \vDash \boxdot\varphi & :\Longleftrightarrow & M,R[t] \vDash \varphi.
\end{array}
$$

The semantics of the local modalities $\Diamond$ and $\Box$ are defined by stating that a condition that applies to worlds must hold for all worlds in a team; this clearly makes all formulas $\Diamond\varphi$ and $\Box\varphi$ flat. The global modalities $\diamondcdot$ and $\boxdot$, on the other hand, make use of conditions which apply globally to teams.

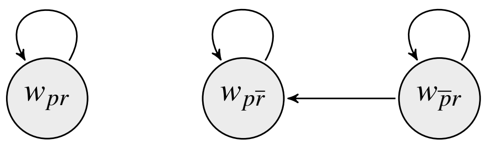
We define the modal analogues of the closure properties (Definition 3.2.3) in the obvious way. It is then easy to see that the modal analogues of Facts 3.2.4 and Fact 3.2.5 (for each of our new classical bases) hold. Given the flatness of $\Diamond\varphi$ and $\Box\varphi$, an easy extension of Proposition 3.2.6 yields:

3.2.18. PROPOSITION. *Formulas of* $\mathrm{ML}_{\vee,\Diamond}(=\!(\cdot),\blacklozenge)$*,* $\mathrm{ML}_{\to,\Diamond}(\mathbin{\vee\!\!\!\vee},\blacklozenge)$ *and* $\mathrm{ML}_{\to,\Diamond}(\blacklozenge)$ *are convex.*

The extensions with the global modalities are, however, not convex, as the fact below shows. The fact further shows that, as with $\vee$ and $\mathbin{\vee\!\!\!\vee}$, the global diamond $\diamondcdot$ does not preserve convexity in a convex setting and so no logic with $\diamondcdot$ can be complete for the class of all convex modal properties.

3.2.19. FACT. There are (i) $\psi \in \mathrm{ML}_{\to,\diamondcdot}(\blacklozenge)$ and (ii) $\chi \in \mathrm{ML}_{\vee,\diamondcdot}(=\!(\cdot),\blacklozenge)$ that are not convex. There is (iii) a convex φ such that $\diamondcdot\varphi$ is not convex.

Proof:
Consider the formula $\varphi := ((\blacklozenge p \wedge \blacklozenge\neg p) \to \bot) \wedge \blacklozenge r$ and observe that $\varphi \equiv =\!(p) \wedge \blacklozenge r$. It is easy to see that φ is convex; to show (i–iii) it therefore suffices to show that $\diamondcdot\varphi$ is not convex. Consider the following model $M = (W,R,V)$ (with R represented using arrows):



We have $M,\{w_{\overline{p}r}\} \vDash \varphi$ and $\{w_{\overline{p}r}\}R\{w_{\overline{p}r}\}$, whence $M,\{w_{\overline{p}r}\} \vDash \diamondcdot\varphi$. We also have $M,\{w_{pr},w_{p\overline{r}}\} \vDash \varphi$ and $\{w_{pr},w_{p\overline{r}},w_{\overline{p}r}\}R\{w_{pr},w_{p\overline{r}}\}$, whence $M,\{w_{pr},w_{p\overline{r}},w_{\overline{p}r}\} \vDash \diamondcdot\varphi$. But we have $M,\{w_{p\overline{r}},w_{\overline{p}r}\} \nvDash \diamondcdot\varphi$. For the only $t \subseteq W$ such that $\{w_{p\overline{r}},w_{\overline{p}r}\}Rt$ are

$\{w_{p\bar{r}}, w_{\bar{p}r}\}$ and $\{w_{p\bar{r}}\}$, and for neither of these do we have $t \vDash \varphi$. □

We move on to the modal analogues of Theorem 3.2.16 for the logics with the flat modalities $\mathrm{ML}_{\vee,\Diamond}(=(\cdot), \blacklozenge)$, $\mathrm{ML}_{\to,\Diamond}(\mathbb{V}, \blacklozenge)$ and $\mathrm{ML}_{\to,\Diamond}(\blacklozenge)$. We omit the details: given Proposition 3.2.18, the proofs of these results are almost completely analogous to that of Theorem 3.2.16 (with one departure, which we comment on below). One can define natural analogues of team properties, expressive completeness, and the formulas χ_v^{X} and χ_t^{X} in the modal setting (see [71, 89] for details). One can also define a notion of bisimulation appropriate for modal team semantics (see [71, 89]), and it is easy to show that each of our logics is invariant under this notion of bisimulation (cf. the bisimulation invariance results for similar logics in [73, 7, 14]). The one departure which must be made from the strategy followed in Section 3.2.1 is in defining the modal analogues of the formulas γ_n^{X} used in the proof of Lemma 3.2.15. These require a more complicated definition; see [71] for details. The modal analogue of proof of Theorem 3.2.16 yields:

3.2.20. THEOREM. *Each of* $\mathrm{ML}_{\vee,\Diamond}(=(\cdot), \blacklozenge)$, $\mathrm{ML}_{\to,\Diamond}(\mathbb{V}, \blacklozenge)$, *and* $\mathrm{ML}_{\to,\Diamond}(\blacklozenge)$ *is expressively complete for the class of all convex modal properties invariant under bounded bisimulation.*

3.3 Convex and Union-closed Properties

In Section 3.3.1, we show that the logic $\mathrm{PL}_{\vee}(\mathrm{NE})$ is expressively complete with respect to the class of all convex and union-closed propositional properties. In Section 3.3.2, we show modal analogues of this result for two distinct modal extensions of $\mathrm{PL}_{\vee}(\mathrm{NE})$.

3.3.1 Propositional Properties

$\mathrm{PL}_{\vee}(\mathrm{NE})$ is an extension of the classical basis $\mathrm{PL}_{\vee}$ with the *nonemptiness atom* NE, true in a team just in case the team is nonempty:

$$t \vDash \mathrm{NE} \iff t \neq \varnothing.$$

We define $\top$, $\alpha \vee \beta$, $\bigvee\varnothing$ and $\bigwedge\varnothing$ as before, and now define $\bot\!\!\!\bot := \bot \wedge \mathrm{NE}$ and $\blacklozenge\varphi := (\varphi \wedge \mathrm{NE}) \vee \top$. It is easy to see that the truth conditions of the same symbol are still always the same, regardless of the logic.

$\mathrm{PL}_{\vee}(\mathrm{NE})$ clearly violates downward closure and the empty team property, but we do have convexity and union closure—interestingly, in contrast with Fact 3.2.7, the split disjunction $\vee$ does preserve convexity in a convex and union-closed setting.

3.3.1. PROPOSITION. *Formulas of* $\mathrm{PL}_{\vee}(\mathrm{NE})$ *are convex and union closed.*

Proof:
By induction on the structure of formulas φ. Most cases are straightforward—note in particular that NE is upward closed and therefore also convex.

We only explicitly show that $\varphi \vee \psi$ is convex whenever φ and ψ are convex and union closed. Let $s \vDash \varphi \vee \psi$, $t \vDash \varphi \vee \psi$, and $s \subseteq u \subseteq t$. Then $s = s_\varphi \cup s_\psi$ and $t = t_\varphi \cup t_\psi$ where $t_\varphi \vDash \varphi$, etc. Let $u_\varphi := (s_\varphi \cup t_\varphi) \cap u$ and $u_\psi := (s_\psi \cup t_\psi) \cap u$. We have $s_\varphi \cup t_\varphi \vDash \varphi$ and $s_\psi \cup t_\psi \vDash \psi$ by union closure, whence $u_\varphi \vDash \varphi$ by convexity since $s_\varphi \subseteq u_\varphi \subseteq s_\varphi \cup t_\varphi$. Similarly, $u_\psi \vDash \psi$. Clearly $u = u_\varphi \cup u_\psi$, whence $u \vDash \varphi \vee \psi$. □

The above shows $\|\mathrm{PL}_\vee(\mathrm{NE})\| \subseteq \mathbb{CU}$. We now further show—solving a problem that was left open in [140]—that $\|\mathrm{PL}_\vee(\mathrm{NE})\| = \mathbb{CU}$—that is, that $\mathrm{PL}_\vee(\mathrm{NE})$ is expressively complete for $\mathbb{CU}$.

We prove this in two distinct ways—it is instructive to see both of these proofs as they break down the characteristics of union-closed convex properties in different ways and feature distinct (if similar) characteristic formulas. For the first proof we use the formula $\perp\!\!\!\perp = \bot \wedge \mathrm{NE}$ for the empty property, and construct, for each nonempty property $\mathcal{P}$, a formula $\chi^{\mathbb{U}}_{\mathcal{P}}$ such that

$$t \vDash \chi^{\mathbb{U}}_{\mathcal{P}} \iff \exists s \in \mathcal{P} : t \supseteq s,$$

and a formula $\chi^{\mathbb{F}}_{\mathcal{P}}$ such that

$$t \vDash \chi^{\mathbb{F}}_{\mathcal{P}} \iff \exists s \in \mathcal{P} : t \subseteq \bigcup \mathcal{P}.$$

As before, the formulas $\chi^{\mathbb{D}}_{\mathcal{P}}$ are characteristic formulas for nonempty downward-closed properties; the formulas $\chi^{\mathbb{F}}_{\mathcal{P}}$ are characteristic formulas for flat properties: for flat $\mathcal{P}$, $\|\chi^{\mathbb{F}}_{\mathcal{P}}\| = \mathcal{P}$. Using these formulas, we construct characteristic formulas for nonempty union-closed convex properties as follows:

3.3.2. LEMMA. *Let* $\mathsf{X} \subseteq \mathsf{P}$ *be finite, and for each* $\mathcal{P} \neq \varnothing$ *over* X, *let* $\chi^{\mathsf{X},\mathbb{F}}_{\mathcal{P}}$ *and* $\chi^{\mathsf{X},\mathbb{U}}_{\mathcal{P}}$ *be such that for any* $t \subseteq 2^{\mathsf{X}}$, $t \vDash \chi^{\mathsf{X},\mathbb{F}}_{\mathcal{P}} \iff t \subseteq \bigcup \mathcal{P}$ *and* $t \vDash \chi^{\mathsf{X},\mathbb{U}}_{\mathcal{P}} \iff \exists s \in \mathcal{P} : t \supseteq s$. *Then for any union-closed convex* $\mathcal{P} \neq \varnothing$ *over* X, $\left\|\chi^{\mathsf{X},\mathbb{D}}_{\mathcal{P}} \wedge \chi^{\mathsf{X},\mathbb{U}}_{\mathcal{P}}\right\|_{\mathsf{X}} = \mathcal{P}$.

Proof:
$\supseteq$: For any $t \in \mathcal{P}$, $t \subseteq t \subseteq \bigcup \mathcal{P}$, whence $t \vDash \chi^{\mathbb{F}}_{\mathcal{P}} \wedge \chi^{\mathbb{U}}_{\mathcal{P}}$.

$\subseteq$: If $t \vDash \chi^{\mathbb{F}}_{\mathcal{P}} \wedge \chi^{\mathbb{U}}_{\mathcal{P}}$, then for some $s \in \mathcal{P}$, $s \subseteq t \subseteq \bigcup \mathcal{P}$. We have $\bigcup \mathcal{P} \in \mathcal{P}$ by union closure and the fact that $\mathcal{P} \neq \varnothing$, whence $t \in \mathcal{P}$ by convexity. □

Clearly given $\blacklozenge\varphi \equiv (\varphi \wedge \mathrm{NE}) \vee \top$, we can construct the formulas $\chi^{\mathbb{U}}_{\mathcal{P}}$ analogously to how we did in Lemma 3.2.10. As for the formulas $\chi^{\mathbb{F}}_{\mathcal{P}}$, we use the following (which can used to prove that $\mathrm{PL}_\vee$ is expressively complete for the class of all flat properties [140]):

3.3.3. LEMMA. *Let* $\mathsf{X} \subseteq \mathsf{P}$ *be finite. For* $\mathcal{P}$ *over* X*, let*

$$\chi_{\mathcal{P}}^{\mathsf{X},\mathbb{F}} := \bigvee_{s \in \mathcal{P}} \chi_s^{\mathsf{X}}.$$

Then for any $t \subseteq 2^{\mathsf{X}}$, $t \vDash \chi_{\mathcal{P}}^{\mathsf{X},\mathbb{F}} \iff t \subseteq \bigcup \mathcal{P}$.

Proof:
Letting $\mathcal{P} = \{s_1, \dots, s_n\}$, we have that

$$t \vDash \bigvee_{s \in \mathcal{P}} \chi_s \iff [\exists t_1, \dots, t_n : t = \bigcup_{i=1}^{n} t_i \text{ and } t_i \vDash \chi_{s_i} \text{ for each } 1 \leq i \leq n]$$
$$\iff [\exists t_1, \dots, t_n : t = \bigcup_{i=1}^{n} t_i \text{ and } t_i \subseteq s_i \text{ for each } 1 \leq i \leq n] \iff t \subseteq \bigcup \mathcal{P}. \qquad \square$$

We have finished proving the required lemmas for the first proof. In the second proof, we make a case distinction as to whether or not $\mathcal{P}$ has the empty team property.

3.3.4. LEMMA. *For each finite* X *and each* $\mathcal{P} = \{t_1, \dots, t_n\} \in \mathbb{CU}_{\mathsf{X}}$*:*

(a) If $\mathcal{P}$ *is the empty property,* $\mathcal{P} = \|\perp\!\!\!\perp\|_{\mathsf{X}}$.

(b) If $\mathcal{P}$ *has the empty team property,* $\mathcal{P} = \|\bigvee_{s \in \mathcal{P}} \chi_s\|_{\mathsf{X}}$.

(c) If $\mathcal{P} \neq \varnothing$ *and* $\mathcal{P}$ *does not have the empty team property,*

$$\mathcal{P} = \left\| \bigvee_{v_1 \in t_1, \dots, v_n \in t_n} ((\chi_{v_1} \vee \dots \vee \chi_{v_n}) \wedge \mathrm{NE}) \right\|_{\mathsf{X}}.$$

Proof:
Item (a) is obvious. For item (b), note that by Fact 3.2.4, $\mathcal{P}$ is downward closed and hence also flat. By Lemma 3.3.3, $t \vDash \bigvee_{s \in \mathcal{P}} \chi$ iff $t \subseteq \bigcup \mathcal{P}$, and, if $\mathcal{P}$ is flat, clearly $t \subseteq \bigcup \mathcal{P}$ iff $t \in \mathcal{P}$.

We now show item (c). For the direction $\subseteq$, let $t_i \in \mathcal{P}$. For each $v_i \in t_i$, we have that $\{v_i\} \vDash \chi_{v_i} \wedge \mathrm{NE}$, so that also (using the empty team property of classical formulas), $\{v_i\} \vDash (\chi_{v_1} \vee \dots \vee \chi_{v_n}) \wedge \mathrm{NE}$ for any $v_1 \in t_1 \dots v_{i-1} \in t_{i-1}, v_{i+1} \in t_{i+1}, \dots v_n \in t_n$. Therefore,

$$\{v_i\} \vDash \bigvee_{v_1 \in t_1} \dots \bigvee_{v_{i-1} \in t_{i-1}} \bigvee_{v_{i+1} \in t_{i+1}} \dots \bigvee_{v_n \in t_n} ((\chi_{v_1} \vee \dots \vee \chi_{v_n}) \wedge \mathrm{NE}),$$

whence $t_i = \bigcup_{v_i \in t_i} \{v_i\} \vDash \bigvee_{v_1 \in t_1, \dots, v_n \in t_n} ((\chi_{v_1} \vee \dots \vee \chi_{v_n}) \wedge \mathrm{NE})$.

For the direction $\supseteq$, let $s \vDash \bigvee_{v_1 \in t_1, \dots, v_n \in t_n} ((\chi_{v_1} \vee \dots \vee \chi_{v_n}) \wedge \mathrm{NE})$. By the fact that $\mathcal{P}$ does not have the empty team property, no $t_i \in \mathcal{P}$ is the empty team. We then have that $s = \bigcup_{v_1 \in t_1, \dots, v_n \in t_n} t_{v_1, \dots, v_n}$ where $t_{v_1, \dots, v_n} \vDash (\chi_{v_1} \vee \dots \vee \chi_{v_n}) \wedge \mathrm{NE}$. (Note that if $\mathcal{P}$ did have the empty team property, we would have $\bigvee_{v_1 \in t_1, \dots, v_n \in t_n} ((\chi_{v_1} \vee \dots \vee \chi_{v_n}) \wedge \mathrm{NE}) \equiv \bigvee \varnothing \equiv \bot$.)

We show that $s \subseteq \bigcup \mathcal{P}$ and that $t_i \subseteq s$ for some $t_i \in \mathcal{P}$. Since $\bigcup \mathcal{P} \in \mathcal{P}$ by the union closure of $\mathcal{P}$ (as well as the fact that $\mathcal{P} \neq \varnothing$), we will then have $t_i \in \mathcal{P}$ by the convexity of $\mathcal{P}$.

$s \subseteq \bigcup \mathcal{P}$: We have that $s = \bigcup_{v_1 \in t_1, \ldots, v_n \in t_n} t_{v_1,\ldots,v_n}$, and for each $t_{v_1,\ldots,v_n}$, by $t_{v_1,\ldots,v_n} \vDash (\chi_{v_1} \vee \ldots \vee \chi_{v_n}) \wedge \mathrm{NE}$ we have that $t_{v_1,\ldots,v_n} \subseteq \bigcup_{1 \leq i \leq n} \{v_i\} \subseteq \bigcup \mathcal{P}$.

$t_i \subseteq s$ for some $t_i \in \mathcal{P}$: Assume for contradiction that $t_i \nsubseteq s$ for all $t_i \in \mathcal{P}$. Then for each $t_i \in \mathcal{P}$ there is some $w_i \in t_i$ such that $w_i \notin s$. We have that $t_{w_1,\ldots,w_n} \vDash (\chi_{w_1} \vee \ldots \vee \chi_{w_n}) \wedge \mathrm{NE}$ so $t_{w_1,\ldots,w_n} \subseteq \bigcup_{1 \leq i \leq n} \{w_i\}$ and $t_{w_1,\ldots,w_n} \neq \varnothing$. We also have $t_{w_1,\ldots,w_n} \subseteq s$, so $s \cap \bigcup_{1 \leq i \leq n} \{w_i\} \neq \varnothing$, contradicting the fact that $w_i \notin s$ for all $t_i \in \mathcal{P}$. □

Putting together Proposition 3.3.1, the Lemmas 3.3.2, 3.2.10, and 3.3.3 (for the first proof), and Lemma 3.3.4 (for the second proof), we have shown:

3.3.5. THEOREM. $\mathrm{PL}_{\vee}(\mathrm{NE})$ *is expressively complete for convex union-closed properties:*

$$\|\mathrm{PL}_{\vee}(\mathrm{NE})\| = \mathbb{CU}.$$

To conclude this section, let us comment on the relationship between the connectives we have studied which break downward closure—NE and $\blacklozenge$. As we have observed, in a setting with NE and $\vee$ (and $\top$ and $\wedge$), $\blacklozenge\varphi$ is definable as $(\varphi \wedge \mathrm{NE}) \vee \top$. On the other hand, NE is definable using $\blacklozenge$ (and $\top$) as $\blacklozenge\top$. It clearly follows that the extension of $\mathrm{PL}_{\vee}$ with $\blacklozenge$ is also expressively complete for $\mathbb{CU}$. However, $\blacklozenge$ is stronger than NE in that swapping out $\blacklozenge$ with NE in any of our convex logics yields a logic that, while convex, is no longer complete for $\mathbb{C}$—an easy induction shows that these logics are *downward closed modulo the empty team*: for each formula φ of one of the logics, if $t \vDash \varphi$, $s \subseteq t$, and $s \neq \varnothing$, then $s \vDash \varphi$. Clearly there are convex properties which are not downward closed modulo the empty team (e.g., $\|\blacklozenge p \wedge \blacklozenge \neg p\| \in \mathbb{C}$).

3.3.2 Modal Properties

We define the syntax and the semantics of the modal extensions $\mathrm{ML}_{\vee,\Diamond}(\mathrm{NE})$ and $\mathrm{ML}_{\vee,\circledDiamond}(\mathrm{NE})$ in the obvious way. In contrast with the extensions featuring the global modalities defined in Section 3.2.2, the extension $\mathrm{ML}_{\vee,\circledDiamond}(\mathrm{NE})$ is convex— the situation with $\circledDiamond$ is analogous to that with $\vee$ in that whereas $\circledDiamond$ does not preserve convexity in a convex setting, it does preserve convexity in a convex and union-closed setting:

3.3.6. PROPOSITION. *Formulas of* $\mathrm{ML}_{\vee,\Diamond}(\mathrm{NE})$ *and* $\mathrm{ML}_{\vee,\circledDiamond}(\mathrm{NE})$ *are convex and union closed.*

Proof:
By induction on the structure of formulas φ. Most cases are straightforward—note in particular that $\Diamond\varphi$ and $\Box\varphi$ are flat and therefore convex and union closed. We only show explicitly that (i) $\circledDiamond\varphi$ is union closed provided that φ is union closed; and (ii) $\circledDiamond\varphi$ is convex provided that φ is convex and union closed.

For (i), let $T \neq \varnothing$ be such that for all $t \in T$, $M, t \vDash \diamondsuit\varphi$. Then for each $t \in T$, there is an $s_t \subseteq W$ such that tRs_t and $M, s_t \vDash \varphi$. By union closure, $M, \bigcup_{t\in T} s_t \vDash \varphi$. By $s_t \subseteq R[t]$ for all $t \in T$, it follows that $\bigcup_{t\in T} s_t \subseteq \bigcup_{t\in T} R[t] = R[\bigcup T]$, and by $t \subseteq R^{-1}[s_t]$ for all $t \in T$, it follows that $\bigcup T = \bigcup_{t\in T} t \subseteq \bigcup_{t\in T} R^{-1}[s_t] = R^{-1}[\bigcup_{t\in T} s_t]$; therefore $\bigcup T R \bigcup_{t\in T} s_t$ whence $M, \bigcup T \vDash \diamondsuit\varphi$.

For (ii), let $M, t \vDash \varphi$ and $M, s \vDash \varphi$ and $s \subseteq u \subseteq t$. Then there are $t', s' \subseteq W$ such that tRt', $sR's$, $M, t' \vDash \varphi$, and $M, s' \vDash \varphi$. By union closure, $M, t' \cup s' \vDash \varphi$. Define $u' := \{w \in t' \cup s' \mid \exists v \in u : vRw\}$. We will show (a) uRu' and (b) $s' \subseteq u'$. Then by (b) we will have $s' \subseteq u' \subseteq t' \cup s'$ whence by convexity, $M, u' \vDash \varphi$; so that by (a) we will have $M, u \vDash \diamondsuit\varphi$.

For (a), clearly $u' \subseteq R[u]$. To show $u \subseteq R^{-1}[u']$, let $v \in u$. Then $v \in t \subseteq R^{-1}[t']$, so there is a $w \in t'$ such that vRw. But then $w \in u'$, whence $v \in R^{-1}[u']$. Therefore $u \subseteq R^{-1}[u']$, and so uRu'.

For (b), let $w \in s'$. Since $s' \subseteq R[s]$, there is a $v \in s$ such that vRw. Then $v \in u$, so $w \in u'$. □

We can further show the modal analogue of Theorem 3.3.5 for each of these extensions. As in Section 3.2.2, we omit the details; the proofs are completely analogous to the propositional proof.

3.3.7. THEOREM. *Each of* $\mathrm{ML}_{\vee,\diamond}(\mathrm{NE})$ *and* $\mathrm{ML}_{\vee,\diamondsuit}(\mathrm{NE})$ *is expressively complete for the class of all convex union-closed modal properties invariant under bounded bisimulation.*

Aloni's Bilateral State-based Modal (BSML) is essentially $\mathrm{ML}_{\vee,\diamond}(\mathrm{NE})$ extended with a *bilateral negation* which does not affect the expressive power of the logic (see [7, 13]). Therefore, the above also establishes that BSML is expressively complete for the class of all convex union-closed modal properties invariant under bounded bisimulation; this solves a problem that was left open in [7].

3.4 Uniform Definability and Uniform Extensions

In this section, we generalize the notion of uniform definability [32, 52, 133, 135, 37, 72] from the team semantics literature, use this generalization to articulate multiple senses of what it means for one team logic to extend another, and apply these definitions to clarify the relationships between the logics we have studied as well as the downward-closed logics $\mathrm{PL}_{\vee}(=(\cdot))$ and $\mathrm{PL}_{\rightarrow}(\mathbin{\vee\!\!\!\vee})$.

Recall that each of $\mathrm{PL}_{\vee}(=(\cdot))$ and $\mathrm{PL}_{\rightarrow}(\mathbin{\vee\!\!\!\vee})$ is expressively complete for the class of all downward-closed properties with the empty team property, whence they are expressively equivalent in the sense that whatever property is expressible in one is also expressible in the other: $\|\mathrm{PL}_{\vee}(=(\cdot))\| = \|\mathrm{PL}_{\rightarrow}(\mathbin{\vee\!\!\!\vee})\| = \mathbb{DE}$. Given that the split disjunction $\vee$ preserves downward closure and the empty team property (in that for any φ and ψ with these properties, $\varphi \vee \psi$ also has these properties), this also has the consequence not only that for any $\varphi, \psi \in \mathrm{PL}_{\vee}(=(\cdot))$ we have $\|\varphi \vee \psi\| \in \|\mathrm{PL}_{\rightarrow}(\mathbin{\vee\!\!\!\vee})\|$, but also that for any $\varphi, \psi \in \mathrm{PL}_{\rightarrow}(\mathbb{V})$ we have $\|\varphi \vee \psi\| \in \|\mathrm{PL}_{\rightarrow}(\mathbb{V})\|$—and more generally, each property expressed by a split disjunction of downward-closed formulas is expressible in $\mathrm{PL}_{\rightarrow}(\mathbb{V})$. However, even though, for each particular split disjunction $\varphi \vee \psi$ of $\mathrm{PL}_{\rightarrow}(\mathbb{V})$-formulas, there is some $\mathrm{PL}_{\rightarrow}(\mathbb{V})$-formula $\theta_{\varphi \vee \psi}$ that is equivalent to $\varphi \vee \psi$, it can be shown [37] that $\vee$ is not *uniformly definable* in $\mathrm{PL}_{\rightarrow}(\mathbb{V})$ in that there is no *context* $\theta_{\vee}[\cdot_1, \cdot_2]$ of $\mathrm{PL}_{\rightarrow}(\mathbb{V})$ (where a context is a formula $\theta_{\vee}(\cdot_1, \cdot_2, p_1, \ldots, p_n)$ with designated atoms $\cdot_i$) such that for any $\varphi, \psi \in \mathrm{PL}_{\rightarrow}(\mathbb{V})$, $\varphi \vee \psi \equiv \theta_{\vee}[\varphi, \psi]$ (where $\theta_{\vee}[\varphi, \psi] := \theta_{\vee}(\varphi/\cdot_1, \psi/\cdot_2)$). Similarly, it has been shown [135] that neither $\mathbb{V}$ nor $\rightarrow$ is uniformly definable in $\mathrm{PL}_{\vee}(=(\cdot))$ (while, again, each $\|\varphi \mathbb{V} \psi\|$ and each $\|\varphi \rightarrow \psi\|$ is expressible in $\mathrm{PL}_{\vee}(=(\cdot))$).

This disconnect between expressibility and uniform definability is only possible due to the failure of closure of uniform substitution in these logics. Clearly, if each connective of one propositional team logic L_1 is uniformly definable in the other L_2 and vice versa, each property expressible in one is also expressible in the other: $\|\mathrm{L}_1\| = \|\mathrm{L}_2\|$. And assuming closure under uniform substitution, $\|\mathrm{L}_1\| = \|\mathrm{L}_2\|$ implies the uniform definability of each connective of one logic in the other. For if L_1 can express, say, $\|p \circ q\|$ where $\circ$ is a binary connective of L_2, then there is[2] a context $\theta_{\circ}[\cdot_1, \cdot_2]$ of L_1 such that $\theta_{\circ}[p, q] \equiv p \circ q$. Then for any formulas φ, ψ of L_1, by closure under uniform substitution, also $\theta_{\circ}[\varphi, \psi] \equiv \varphi \circ \psi$.

Now, to connect this discussion with our concerns, let us first recall Fact 3.2.7, and restate and reprove this fact in a more abstract manner. Given properties $\mathcal{P}$ and $\mathcal{Q}$, we write $\mathcal{P} \vee \mathcal{Q} := \{t \cup s \mid t \in \mathcal{P} \text{ and } s \in \mathcal{Q}\}$ and $\mathcal{P} \mathbb{V} \mathcal{Q} := \mathcal{P} \cup \mathcal{Q}$; and given classes of properties $\mathbb{P}$ and $\mathbb{Q}$, we write $\mathbb{P} \vee \mathbb{P} \subseteq \mathbb{Q}$ if $\mathcal{P}, \mathcal{Q} \in \mathbb{P}_{\mathsf{X}}$ implies $\mathcal{P} \vee \mathcal{Q} \in \mathbb{Q}_{\mathsf{X}}$, and similarly for $\mathbb{P} \mathbb{V} \mathbb{P} \subseteq \mathbb{Q}$.

3.4.1. FACT. For any logic L, (i) if $\mathbb{C} \vee \mathbb{C} \subseteq \|\mathrm{L}\|$, then $\|\mathrm{L}\| \nsubseteq \mathbb{C}$, and (ii) if $\mathbb{C} \mathbb{V} \mathbb{C} \subseteq \|\mathrm{L}\|$, then $\|\mathrm{L}\| \nsubseteq \mathbb{C}$.

Proof:
(i) Let $\mathcal{P} := \{\{v_1\}, \{v_2, v_3\}\}$ and $\mathcal{Q} := \{\{v_1\}\}$ (where the v_i are valuations). Then $\mathcal{P}, \mathcal{Q} \in \mathbb{C}$, but $\mathcal{P} \vee \mathcal{Q} = \{\{v_1\}, \{v_1, v_2, v_3\}\} \notin \mathbb{C}$.

(ii) Let $\mathcal{P} := \{\{v_1, v_2, v_3\}\}$ and $\mathcal{Q} := \{\{v_1\}\}$. Then $\mathcal{P}, \mathcal{Q} \in \mathbb{C}$, but we have $\mathcal{P} \mathbb{V} \mathcal{Q} = \{\{v_1\}, \{v_1, v_2, v_3\}\} \notin \mathbb{C}$. □

That is, if each split disjunction of convex formulas/properties is expressible in L, then L is not convex (and similarly for $\mathbb{V}$). Given this fact and Theorem 3.2.16, there are, for instance, formulas φ, ψ of $\mathrm{PL}_{\veebar}(=(\cdot), \blacklozenge)$ such that $\|\varphi \vee \psi\| \notin \|\mathrm{PL}_{\veebar}(=(\cdot), \blacklozenge)\|$—it is not the case that each split disjunction of $\mathrm{PL}_{\veebar}(=(\cdot), \blacklozenge)$-formulas is expressible in $\mathrm{PL}_{\veebar}(=(\cdot), \blacklozenge)$; *a fortiori*, the split disjunction is not uniformly definable in $\mathrm{PL}_{\veebar}(=(\cdot), \blacklozenge)$.

However, it is not difficult to see, given our results, that $\mathbb{D} \vee \mathbb{D} \subseteq \|\mathrm{PL}_{\veebar}(=(\cdot), \blacklozenge)\|$. Furthermore, given that for downward-closed φ and ψ, $\varphi \vee \psi \equiv \varphi \veebar \psi$, the split disjunction

[2] We are assuming here that L_1 and L_2 have the *locality property*: for any formula φ of one of these logics, if $t \upharpoonright \mathsf{P}(\varphi) = s \upharpoonright \mathsf{P}(\varphi)$, then $t \vDash \varphi \iff s \vDash \varphi$. Some team logics, interestingly, lack this property—see, e.g., [51].

of any two downward-closed formulas *is* definable by the context $\theta_{\vee}[\cdot_1, \cdot_2] := \cdot_1 \veebar \cdot_2$ in $\mathrm{PL}_{\veebar}(=(\cdot), \blacklozenge)$. Therefore, in particular, the split disjunction restricted to $\|\mathrm{PL}_{\vee}(=(\cdot))\| = \mathbb{DE}$ is definable in this sense in $\mathrm{PL}_{\veebar}(=(\cdot), \blacklozenge)$, and in fact each connective of $\mathrm{PL}_{\vee}(=(\cdot))$ is definable in $\mathrm{PL}_{\veebar}(=(\cdot), \blacklozenge)$ when restricted to $\|\mathrm{PL}_{\vee}(=(\cdot))\|$. And so, while $\mathrm{PL}_{\veebar}(=(\cdot), \blacklozenge)$ is not a syntactic extension of $\mathrm{PL}_{\vee}(=(\cdot))$, and while it is not an extension in the familiar sense in which each connective of $\mathrm{PL}_{\vee}(=(\cdot))$ is definable in $\mathrm{PL}_{\veebar}(=(\cdot), \blacklozenge)$[3] (since $\vee$ is not uniformly definable in $\mathrm{PL}_{\veebar}(=(\cdot), \blacklozenge)$), it is not only an expressive extension of $\mathrm{PL}_{\vee}(=(\cdot))$ (in that $\|\mathrm{PL}_{\vee}(=(\cdot))\| \subseteq \|\mathrm{PL}_{\veebar}(=(\cdot), \blacklozenge)\|$), but also an extension in the deeper sense that every function/context definable in $\mathrm{PL}_{\vee}(=(\cdot))$ (whose domain is a subset of some Cartesian power of $\|\mathrm{PL}_{\vee}(=(\cdot))\|$) is definable in $\mathrm{PL}_{\veebar}(=(\cdot), \blacklozenge)$. As defined in the literature, the concept of uniform definability in a logic L applies only to definability as restricted to $\|\mathrm{L}\|$. In order to make our observations precise, we must therefore generalize this notion; this is the aim of the rest of this section.

An n-ary *context* $\theta[\cdot_1, \dots, \cdot_n]$ for a logic L is a formula θ of L with distinguished atoms $\cdot_1, \dots, \cdot_n$ (it may also contain other atoms such as propositional variables). Given formulas $\varphi_1, \dots, \varphi_n$, we write $\theta[\varphi_1, \dots, \varphi_n]$ for the formula $\theta(\varphi_1/\cdot_1, \dots, \varphi_n/\cdot_n)$. Each n-ary logical connective $\circ$ clearly defines a context $\circ[\cdot_1, \dots, \cdot_n] := \circ(\cdot_1, \dots, \cdot_n)$. We use the symbol for a logical connective $\circ$ to refer interchangeably both to the connective and to the context it defines.

We restrict our attention to logics whose contexts (and connectives) are *compositional* in that given classes of propositional properties $\mathbb{P}_1, \dots, \mathbb{P}_n, \mathbb{P}_{n+1}$ and a finite $\mathsf{X} \subseteq \mathsf{P}$, each such n-ary context θ defines a function $\theta_{\mathsf{X}}^{\mathbb{P}_1, \dots, \mathbb{P}_n, \mathbb{P}_{n+1}} : \mathbb{P}_{1\mathsf{X}} \times \dots \times \mathbb{P}_{n\mathsf{X}} \to \mathbb{P}_{n+1\mathsf{X}}$ (or simply θ) given by

$$\theta_{\mathsf{X}}^{\mathbb{P}_1, \dots, \mathbb{P}_n, \mathbb{P}_{n+1}}(\|\varphi_1\|_{\mathsf{X}}, \dots, \|\varphi_n\|_{\mathsf{X}}) = \|\varphi_{n+1}\|_{\mathsf{X}}$$

for any $\varphi_1, \dots, \varphi_n, \varphi_{n+1}$ with $\mathsf{P}(\varphi_i) \subseteq \mathsf{X}$ and $\|\varphi_i\| \subseteq \mathbb{P}_i$ for $1 \leq i \leq n+1$. [4] We also use a logical connective $\circ$ to refer to each such function it defines. We abbreviate $\theta_{\mathsf{X}}^{\mathbb{P}, \dots, \mathbb{P}, \mathbb{P}}$ as $\theta_{\mathsf{X}}^{\mathbb{P}}$ (or simply $\theta^{\mathbb{P}}$).

We now recall again the notion of uniform definability from the literature.

3.4.2. DEFINITION (Uniform definability). An n-ary logical connective $\circ$ is *uniformly definable* in a logic L if there exists an n-ary context $\theta_\circ$ of L such that for all $\varphi_1, \dots \varphi_n \in$ L, $\circ(\varphi_1, \dots, \varphi_n) \equiv \theta_\circ[\varphi_1, \dots, \varphi_n]$.

Equivalently, using the functions above, $\circ$ is uniformly definable in L if there is a context $\theta_\circ$ in L such that for all finite $\mathsf{X} \subseteq \mathsf{P}$, $\circ_{\mathsf{X}}^{\|\mathrm{L}\|} = \theta_{\circ\mathsf{X}}^{\|\mathrm{L}\|}$ (which we may write simply as $\circ^{\|\mathrm{L}\|} = \theta_\circ^{\|\mathrm{L}\|}$).

Generalizing this notion as described above, we have:

[3] It is in this sense that, for instance, we would say that the variant of classical propositional logic featuring the connectives $\neg$ and $\wedge$ extends the fragment of classical logic featuring only the connective $\vee$.

[4] We assume here that if φ is a formula of two distinct logics, then the property $\|\varphi\|_{\mathsf{X}}$ defined by φ according to the semantics of one of these logics is identical to the property as defined according to the semantics of the other logic.

3.4.3. DEFINITION (Generalized uniform definability). An n-ary logical connective (or context) $\circ$ is *uniformly definable with respect to* $\mathbb{P}_1, \dots, \mathbb{P}_n, \mathbb{P}_{n+1}$ in a logic L if there exists an n-ary context $\theta_\circ$ for L such that for all finite $\mathsf{X} \subseteq \mathsf{P}$, $\circ_{\mathsf{X}}^{\mathbb{P}_1,\dots,\mathbb{P}_n,\mathbb{P}_{n+1}} = \theta_{\circ\mathsf{X}}^{\mathbb{P}_1,\dots,\mathbb{P}_n,\mathbb{P}_{n+1}}$. We say that $\circ$ is *uniformly definable with respect to* $\mathbb{P}$ in L if it is uniformly definable with respect to $\mathbb{P}, \dots, \mathbb{P}, \mathbb{P}$ in L.

We say that $\circ$ is *locally uniformly definable* in L if it is uniformly definable in L with respect to $\|\mathrm{L}\|$, and that it is *globally uniformly definable* in L if it is uniformly definable in L with respect to $\mathbb{A}$, where $\mathbb{A}$ is the class of all properties.

Note that uniform definability as in Definition 3.4.2 is the same as local uniform definability.

We are now ready to define the notion of extension described above—*inner local uniform extension*. We will also define/recall other interesting notions of extension to contrast with this notion. The notions are listed in order of strength—we have:

syntactic extension $\implies$ global uniform extension $\implies$ outer local uniform extension $\implies$ inner local uniform extension $\implies$ expressive extension

Let L_1 and L_2 be team logics.

Syntactic extension: We say that L_1 is a *syntactic extension* of L_2 if the set of logical connectives of L_2 is a subset of that of L_1.

Example: $\mathrm{PL}_{\dot\vee}(=(\cdot), \blacklozenge)$ is a syntactic extension of $\mathrm{PL}_{\dot\vee}$, but not of $\mathrm{PL}_{\vee}(=(\cdot))$.

Global uniform extension: L_1 is a *global uniform extension* of L_2 if each logical connective of L_2 is globally uniformly definable in L_1.

Example: Since $\mathrm{NE} \equiv \blacklozenge\top$, $\mathrm{PL}_{\rightarrow}(\blacklozenge)$ is clearly a global uniform extension (but not a syntactic extension) of the extension $\mathrm{PL}_{\rightarrow}(\mathrm{NE})$ of $\mathrm{PL}_{\rightarrow}$ with NE.

Outer local uniform extension: L_1 is an *outer local uniform extension* of L_2 if each logical connective of L_2 is locally uniformly definable in L_1.

Example: Consider the extension $\mathrm{PL}_{\dot\vee}(=(\cdot), \perp\!\!\!\perp)$ of $\mathrm{PL}_{\dot\vee}$ with dependence atoms and $\perp\!\!\!\perp$, where $\perp\!\!\!\perp$ is now a primitive with its usual semantics. It is not difficult to check (given Theorem 3.2.11) that $\|\mathrm{PL}_{\dot\vee}(=(\cdot), \perp\!\!\!\perp)\| = \mathbb{D}$, whence $\mathbb{D} = \|\mathrm{PL}_{\dot\vee}(=(\cdot), \perp\!\!\!\perp)\| \supset \|\mathrm{PL}_{\vee}(=(\cdot))\| = \mathbb{DE}$. As we have observed before, $\vee^{\mathbb{D}} = \dot\vee^{\mathbb{D}}$, whence $\vee$ is locally uniformly definable in $\mathrm{PL}_{\dot\vee}(=(\cdot), \perp\!\!\!\perp)$. Clearly, therefore, $\mathrm{PL}_{\dot\vee}(=(\cdot), \perp\!\!\!\perp)$ is an outer local uniform extension of $\mathrm{PL}_{\vee}(=(\cdot))$. Note, however, that given Fact 3.4.1, there can be no context $\theta_\vee$ of $\mathrm{PL}_{\dot\vee}(=(\cdot), \perp\!\!\!\perp)$ such that $\theta_\vee^{\mathbb{C}} = \vee^{\mathbb{C}}$; therefore there can also be no such context $\theta_\vee$ such that $\theta_\vee^{\mathbb{A}} = \vee^{\mathbb{A}}$. And so $\mathrm{PL}_{\dot\vee}(=(\cdot), \perp\!\!\!\perp)$ is not a global uniform extension of $\mathrm{PL}_{\vee}(=(\cdot))$.

Inner local uniform extension: L_1 is an *inner local uniform extension* of L_2 if each logical connective $\circ$ of L_2 is uniformly definable in L_1 with respect to $\|\mathrm{L}_2\|$.

Examples: Given that $\vee^{\mathbb{DE}} = \dot{\vee}^{\mathbb{DE}}$ and $\vee\!\!\!\vee^{\mathbb{DE}} = \dot{\vee\!\!\!\vee}^{\mathbb{DE}}$, we have that $\vee$ is uniformly definable in $PL_{\dot{\vee}}(=(\cdot), \blacklozenge)$ with respect to $\mathbb{DE} = \|PL_{\vee}(=(\cdot))\|$, and that $\vee\!\!\!\vee$ is uniformly definable in $PL_{\rightarrow}(\vee\!\!\!\vee, \blacklozenge)$ with respect to $\mathbb{DE} = \|PL_{\rightarrow}(\vee\!\!\!\vee)\|$. It is then easy to see that $PL_{\dot{\vee}}(=(\cdot), \blacklozenge)$ is an inner local uniform extension of $PL_{\vee}(=(\cdot))$, and similarly with $PL_{\rightarrow}(\vee\!\!\!\vee, \blacklozenge)$ and $PL_{\rightarrow}(\vee\!\!\!\vee)$. Given Fact 3.4.1, however, $PL_{\dot{\vee}}(=(\cdot), \blacklozenge)$ is not an outer local uniform extension of $PL_{\vee}(=(\cdot))$, and $PL_{\rightarrow}(\vee\!\!\!\vee, \blacklozenge)$ is not an outer local uniform extension of $PL_{\rightarrow}(\vee\!\!\!\vee)$.

Expressive extension: L_1 is an *expressive extension* of L_2 if $\|L_2\| \subseteq \|L_1\|$.

Examples: $PL_{\vee}(=(\cdot))$ is an expressive extension of $PL_{\rightarrow}(\vee\!\!\!\vee)$ (and vice versa), but not of $PL_{\dot{\vee}}(=(\cdot), \blacklozenge)$. We mentioned above that $\vee\!\!\!\vee$ and $\rightarrow$ are not (locally) uniformly definable in $PL_{\vee}(=(\cdot))$. They are therefore not uniformly definable in $PL_{\vee}(=(\cdot))$ with respect to $\|PL_{\rightarrow}(\vee\!\!\!\vee)\| = \|PL_{\vee}(=(\cdot))\|$; therefore, $PL_{\vee}(=(\cdot))$ is not an inner local uniform extension of $PL_{\rightarrow}(\vee\!\!\!\vee)$.

We noted above that in a setting in which closure under uniform substitution holds, expressive equivalence and uniform definability of connectives collapse into one another. Now observe further that, essentially by the same argument, the notions of expressive extension, inner/outer local uniform extension and global uniform extension are all equivalent in such a setting.

To repeat what was already mentioned above, if L_1 is an inner local uniform extension of L_2, then each function definable only by the means available in L_2 (in that such a function is definable by a context of L_2 and its domain is a subset of some Cartesian power of $\|L_2\|$) is also definable by a context of L_1. It is for this reason that we call $PL_{\dot{\vee}}(=(\cdot), \blacklozenge)$ a convex variant of $PL_{\vee}(=(\cdot))$, and $PL_{\rightarrow}(\vee\!\!\!\vee, \blacklozenge)$ a convex variant of $PL_{\rightarrow}(\vee\!\!\!\vee)$; it is not clear to us whether $PL_{\rightarrow}(\blacklozenge)$ is also an inner local uniform extension of $PL_{\rightarrow}(\vee\!\!\!\vee)$ (see Open problem 3 below), and we therefore refrain from calling it a convex variant of $PL_{\rightarrow}(\vee\!\!\!\vee)$.

Let us conclude by noting one result and three open problems concerning uniform definability and inner local uniform extensionhood in our logics.

As mentioned above, it is shown in [135] that $\rightarrow$ is not uniformly definable in $PL_{\vee}(=(\cdot))$. Essentially the same proof shows that $\rightarrow$ is also not uniformly definable in $PL_{\dot{\vee}}(=(\cdot), \blacklozenge)$, whence $PL_{\dot{\vee}}(=(\cdot), \blacklozenge)$ is not an inner local uniform extension of $PL_{\rightarrow}(\vee\!\!\!\vee, \blacklozenge)$ or of $PL_{\rightarrow}(\blacklozenge)$.

Open problem(s) 1: Is $\dot{\vee}$ uniformly definable in $PL_{\rightarrow}(\blacklozenge)$ or in $PL_{\rightarrow}(\vee\!\!\!\vee, \blacklozenge)$? (Is $PL_{\dot{\vee}}(=(\cdot), \blacklozenge)$ an inner local uniform extension of $PL_{\rightarrow}(\blacklozenge)$/$PL_{\rightarrow}(\vee\!\!\!\vee, \blacklozenge)$?) It might be possible to adapt the proof from [37] showing that $\vee$ is not uniformly definable in $PL_{\rightarrow}(\vee\!\!\!\vee)$ to show that this is not the case.

(Note that dependence atoms are uniformly definable in both $PL_{\rightarrow}(\vee\!\!\!\vee, \blacklozenge)$ and in $PL_{\rightarrow}(\blacklozenge)$. For $PL_{\rightarrow}(\vee\!\!\!\vee, \blacklozenge)$, we have (following the definition of dependence atoms in $PL_{\rightarrow}(\vee\!\!\!\vee)$, which follows essentially from [2]):

$$=(p_1, \ldots, p_n, p) \equiv ((p_1 \vee\!\!\!\vee \neg p_1) \wedge \ldots (p_n \vee\!\!\!\vee \neg p_n)) \rightarrow (p \vee\!\!\!\vee \neg p).$$

For $\mathrm{PL}_{\to}(\blacklozenge)$, we use the above in conjunction with Proposition 3.2.13.)

Open problem(s) 2: Is $\dot{\mathbin{\vee\!\!\!\vee}}$ uniformly definable in $\mathrm{PL}_{\dot\vee}(=(\cdot),\blacklozenge)$? It might be possible to adapt the proof from [135] showing that $\mathbin{\vee\!\!\!\vee}$ is not uniformly definable in $\mathrm{PL}_{\vee}(=(\cdot))$ to show that this is not the case.

Open problem(s) 3: Is $\mathbin{\vee\!\!\!\vee}$ uniformly definable with respect to $\mathbb{DE}$ in $\mathrm{PL}_{\to}(\blacklozenge)$? (Is $\mathrm{PL}_{\to}(\blacklozenge)$ an inner local uniform extension of $\mathrm{PL}_{\to}(\mathbin{\vee\!\!\!\vee})$?) It seems likely that if one solves this, one also solves whether $\dot{\mathbin{\vee\!\!\!\vee}}$ is uniformly definable in $\mathrm{PL}_{\to}(\blacklozenge)$. (Is $\mathrm{PL}_{\to}(\blacklozenge)$ an inner local uniform extension of $\mathrm{PL}_{\to}(\dot{\mathbin{\vee\!\!\!\vee}},\blacklozenge)$?) Observe that in Proposition 3.2.13, we only showed that $\mathbin{\vee\!\!\!\vee}$ and $\dot{\mathbin{\vee\!\!\!\vee}}$ are uniformly definable in $\mathrm{PL}_{\to}(\blacklozenge)$ with respect to $\mathbb{F}$.

3.5 Conclusion

In this paper, we introduced the logics $\mathrm{PL}_{\dot\vee}(=(\cdot),\blacklozenge)$, $\mathrm{PL}_{\to}(\dot{\mathbin{\vee\!\!\!\vee}},\blacklozenge)$ and $\mathrm{PL}_{\to}(\blacklozenge)$, which we showed to be expressively complete for the class of all convex propositional properties. We also introduced modal extensions of these logics and showed modal analogues of the expressive completeness theorems for the extensions of the convex logics with the flat modalities $\Diamond$ and $\Box$. We then examined some union-closed convex logics from the literature: we showed that the propositional logic $\mathrm{PL}_{\vee}(\mathrm{NE})$ is expressively complete for the class of all union-closed convex propositional properties, and we showed modal analogues of this result for the modal extensions $\mathrm{ML}_{\vee,\Diamond}(\mathrm{NE})$, $\mathrm{ML}_{\vee,◈}(\mathrm{NE})$ and BSML of $\mathrm{PL}_{\vee}(\mathrm{NE})$. We also showed that while the split disjunction $\vee$ and the global diamond ◈ preserve convexity in a convex and union-closed setting, they do not preserve convexity in general. Similarly, while the global disjunction $\mathbin{\vee\!\!\!\vee}$ preserves downward closure (and hence convexity) in a downward-closed setting, it does not preserve convexity in general. Finally, we generalized the notion of uniform definability from the literature in order to articulate the sense in which our convex variant $\mathrm{PL}_{\dot\vee}(=(\cdot),\blacklozenge)$ of dependence logic $\mathrm{PL}_{\vee}(=(\cdot))$ our convex variant $\mathrm{PL}_{\to}(\dot{\mathbin{\vee\!\!\!\vee}},\blacklozenge)$ of inquisitive logic extend their downward-closed counterparts. The convex variants are inner local uniform extensions of the downward-closed counterparts: each connective of the counterpart is uniformly definable in the convex variant with respect to class of properties expressible in the counterpart.

We note two more open problems to add to the list in Section 3.4:

Open problem(s) 4: Axiomatizations of the logics studied. There is a commonly-used strategy for finding axiomatizations of propositional and modal team logics (see, for instance, [41, 139, 140, 134, 137]) that makes heavy use of the characteristic formulas for properties provided by expressive completeness results such as those established in this paper. Given our expressive completeness results, we expect it to be possible to use this strategy to construct axiomatizations of the logics we have studied. It would be particularly interesting to see axiomatizations of the convex propositional logics, to see how these axiomatizations differ from those of the downward-closed logics $\mathrm{PL}_{\to}(\mathbin{\vee\!\!\!\vee})$ and $\mathrm{PL}_{\vee}(=(\cdot))$. (Note that $\mathrm{PL}_{\vee}(\mathrm{NE})$ has already been axiomatized in [140], and $\mathrm{ML}_{\Diamond,\vee}(\mathrm{NE})$ and BSML in [7]. An extension of $\mathrm{ML}_{◈,\vee}(\mathrm{NE})$ has been axiomatized in [12].)

Open problem(s) 5: First-order dependence logic (D) coincides in expressive power with the downward-monotone (or downward-closed) fragment of existential second-order logic (ESO) in the sense that D and this fragment define the same class of team properties [93]. Similarly, we have, for instance, that first-order *independence logic* coincides with full ESO [51]; D with the *Boolean negation* coincides with full second-order logic (SO) [91], as does first-order independence logic with $\rightarrow$ [132, 133]; D with $\rightarrow$ coincides with the downward-closed fragment of SO [132, 133]; and there are other team logics coinciding in this way with the union-closed fragment of ESO [80], as well as with both full first-order logic and its downward-closed fragment [95]. Is it possible to construct a convex variant of first-order dependence logic (or indeed any natural team logic, perhaps dissimilar to dependence logic) which coincides in this sense with the convex fragment of ESO? Similarly, can we find a natural team logic equivalent to the convex union-closed fragment of ESO? How about the convex (and convex and union-closed) fragments of first-order and second-order logic?

Chapter 4
Further Remarks on the Dual Negation in Team Logics

This chapter is based on:

Aleksi Anttila. *Further remarks on the dual negation in team logics*. 2024. arXiv: `2410.07067 [math.LO]`. URL: `https://arxiv.org/abs/2410.07067`

Abstract The dual or game-theoretical negation ¬ of independence-friendly logic (IF) and dependence logic (D) exhibits an extreme degree of semantic indeterminacy in that for any pair of sentences φ and ψ of IF/D, if φ and ψ are incompatible in the sense that they share no models, there is a sentence θ of IF/D such that $\varphi \equiv \theta$ and $\psi \equiv \neg\theta$ (as shown originally by Burgess in the equivalent context of the prenex fragment of Henkin quantifier logic). We show that by adjusting the notion of incompatibility employed, analogues of this result can be established for a number of modal and propositional team logics, including Aloni's bilateral state-based modal logic, Hawke and Steinert-Threlkeld's semantic expressivist logic for epistemic modals, as well as propositional dependence logic with the dual negation. Together with its converse, a result of this type can be seen as an expressive completeness theorem with respect to the relevant incompatibility notion; we formulate a notion of expressive completeness for pairs of properties to make this precise.

4.1 Introduction

4.1.1 The Dual Negation and Burgess' Theorem

Henkin quantifier logic (H) extends classical first-order logic (FO) with *Henkin quantifiers* [74] (also known as *branching* or *partially ordered quantifiers*) such as the following:

$$\begin{pmatrix} \forall x & \exists y \\ \forall u & \exists v \end{pmatrix} \varphi(x,y,u,v). \tag{4.1}$$

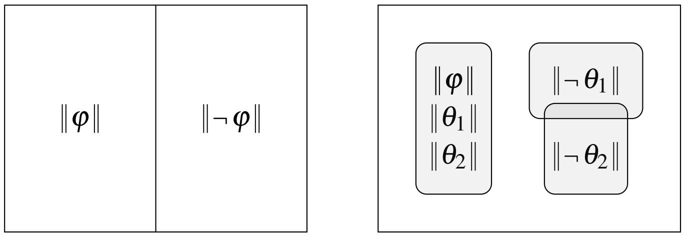
The intuitive meaning of (4.1) is that the value of y depends only on that of x, and the value of v only on that of u—(4.1) is equivalent to the existential second-order sentence $\exists f \exists g \forall x \forall u \varphi(x, f(x), u, g(u))$ (cf. the FO-sentence $\forall x \exists y \forall u \exists v \varphi(x,y,u,v)$). Hintikka and Sandu's *independence-friendly logic* (IF) [77, 75] similarly extends FO with *slashed quantifiers* $(\exists y/\forall x)$, with the intended meaning of $(\exists y/\forall x)$ being that the value of y must be chosen independently of the value of x (the IF version of (4.1) is $\forall x \exists y \forall u (\exists v/\forall x) \varphi(x,y,u,v)$). Finally, *dependence logic* (D) [119], Väänänen's refinement of IF, replaces the slashed quantifiers with *dependence atoms*: the atom $=(x_1,\ldots,x_n,y)$ asserts that the value of y is functionally determined by those of $x_1,\ldots,x_n$ (the dependence logic version of (4.1) is $\forall x \exists y \forall u \exists v (\varphi(x,y,u,v) \wedge =(u,v))$).

The prenex fragment of H (denoted H_p; the formulas of H_p are of the prenex form $Q\varphi$, where the prefix Q is a set of quantifiers with a partial order and the matrix φ is a quantifier-free FO-formula) is expressively equivalent to existential second-order logic Σ^1_1 [124, 48], and the same holds for the full logics IF [75] and D [119]. These logics are therefore not closed under classical negation $\sim$ (where $\sim \varphi$ is true just in case φ is false).[1] The original semantics for IF were game-theoretical, with the meaning of $(\exists y/\forall x)$ captured with a game rule to the effect that the player choosing the value of y must do so without knowing the value of x—the game is one of *imperfect information* (whereas in the game for FO the players have perfect information). This naturally led to the adoption (in both IF and D) of what is known as the *dual* or *game-theoretical* negation $\neg$. The rule associated with this negation in the semantic game is simply the standard negation rule (also used in the game-theoretical semantics for FO) whereby the players switch their verifier/falsifier roles[2], and indeed, IF and D are conservative extensions of FO, with the dual negation extending the classical negation of FO.

In the context of the full logics, however, the dual negation exhibits non-classical behavior. Most germane to our purposes is the fact that *preservation of equivalence under replacement* fails in negated contexts—for instance, $\varphi \equiv \psi$ need not imply $\neg\varphi \equiv \neg\psi$. (A concrete example: in dependence logic we have $\neg =(x,y) \equiv \bot$, but $\neg\neg =(x,y) \equiv =(x,y) \not\equiv \bot \equiv \neg\neg\bot$.) Put another way, each sentence φ defines a class of models $\|\varphi\| = \{\mathcal{M} \mid \mathcal{M} \vDash \varphi\}$ which we may think of as the meaning of φ, and whereas negating a sentence in FO corresponds to the semantic operation of complementation on these classes of models, the dual negation fails to correspond to any operation on such

[1] The fact that Σ^1_1 is not closed under $\sim$ follows, for instance, from the compactness of Σ^1_1 together with the fact that one can define a Σ^1_1-sentence which is true in a model just in case the model is infinite.

[2] The truth conditions of the dual negation can equivalently be obtained by first defining the conditions for negated classical atoms by translating the standard Tarskian truth conditions for such negated atoms to the semantic framework one is working with (e.g. game-theoretic-semantics or team semantics), and then defining those for complex formulas by stipulating that the following dual (or almost dual) equivalences are to hold: $\varphi \equiv \neg\neg\varphi$; $\neg(\varphi \wedge \psi) \equiv \neg\varphi \vee \neg\psi$; $\neg(\varphi \vee \psi) \equiv \neg\varphi \wedge \neg\psi$; $\neg\forall x\varphi \equiv \exists x \neg\varphi$; $\neg\exists x\varphi \equiv \forall x \neg\varphi$; $\neg(\exists y/\forall x)\varphi \equiv \forall x \neg\varphi$ (in IF); and $\neg =(x_1,\ldots,x_n,y) \equiv \bot$ (in D).



(a) FO (b) H_p

Figure 4.1: Failure of determination: in FO, $\|\neg\varphi\|$ is the complement of $\|\varphi\|$. In H_p, given only $\|\varphi\|$, $\|\neg\varphi\|$ can be any class disjoint with $\|\varphi\|$ that is expressible in H_p.

classes: the class of models $\|\varphi\|$ of φ does not determine the class of models $\|\neg\varphi\|$ of its dual negation. This does not mean that $\|\neg\varphi\|$ is fully unconstrained: one can show that φ and $\neg\varphi$ must be incompatible in that they share no models ($\|\varphi\| \cap \|\neg\varphi\| = \varnothing$), with the effect that despite some standard inferential principles involving negation (such as negation introduction $\Gamma, \varphi \vDash \bot \implies \Gamma \vDash \neg\varphi$) being invalidated due to the lack of determination, others (such as *ex falso* $\varphi, \neg\varphi \vDash \psi$) remain valid.

Burgess [29], however, showed that this failure of determination is extreme in the sense that incompatibility is the *only* constraint that $\|\varphi\|$ places on $\|\neg\varphi\|$. That is, he showed (in the context of H_p[3]) that for any sentences φ, ψ, if φ and ψ share no models ($\|\varphi\| \cap \|\psi\| = \varnothing$), then there is a sentence θ such that $\varphi \equiv \theta$ and $\psi \equiv \neg\theta$ ($\|\varphi\| = \|\theta\|$ and $\|\psi\| = \|\neg\theta\|$). So knowing only $\|\varphi\|$ (without knowing the sentence φ) tells us nothing whatsoever about $\|\neg\varphi\|$, save for the fact that the two classes are disjoint and the fact that $\|\neg\varphi\|$ is expressible in H_p: for a given $\|\varphi\|$, take any class $\mathcal{X}$ expressible in H_p and disjoint with $\|\varphi\|$. By Burgess' theorem, there is then a θ with $\|\varphi\| = \|\theta\|$ and $\mathcal{X} = \|\neg\theta\|$, so for all we know, $\|\neg\varphi\|$ might be $\mathcal{X}$. See Figure 4.1. Dechesne [44] later reformulated the result for IF. Kontinen and Väänänen [94] (working in D and IF) generalized the theorem to arbitrary formulas (that may contain free variables). Mann (in [98]) proved an analogue of the result for sentences for the *perfect-recall* fragment of IF.[4]

4.1.2 Burgess Theorems for Modal and Propositional Team Logics

In this paper, we prove analogues of Burgess' theorem for a number of modal and propositional *team logics*.

Team logics are logics such as dependence logic which are primarily intended to

[3]Burgess proved his result for sentence of H_p and their *contraries*, where the contrary of $Q\varphi$ is obtained by swapping all quantifiers in Q with their duals ($\forall$ with $\exists$ and vice versa), and negating the FO matrix φ. This is equivalent to the dual negation in that given a sentence φ of H_p and an equivalent sentence ψ of D or IF, the contrary of φ and the dual negation of ψ are also equivalent.

[4]See also [20], which corrects for the failure of determination in that it introduces an extension of IF featuring a negation with a natural game-theoretic interpretation which does not exhibit this failure.

be interpreted using *team semantics*. In team semantics (introduced by Hodges [78, 79] to provide a compositional semantics for IF, and independently developed as a semantics for *inquisitive logic* chiefly by Ciardelli, Groenendijk, and Roelofsen [41, 38, 35]), formulas are evaluated with respect to *sets* of evaluation points called *teams* rather than single points, as in standard Tarskian semantics (so, for instance, in first-order team semantics, teams are sets of variable assignments; in modal team semantics [120], teams are sets of possible worlds, etc.).[5]

Contemporary team logics in the lineage of dependence logic typically do not incorporate the dual negation. However, interest in this negation has recently been reinvigorated due to team logics in the philosophical logic and formal semantics literature which employ *bilateral* notions of negation similar to the dual negation. We are concerned with two such logics in particular: Aloni's *bilateral state-based modal logic* (BSML) [6], and Hawke and Steinert-Threlkeld's semantic expressivist logic for epistemic modals [67]. These are both propositional modal team logics developed to account for linguistic phenomena such as *free choice inferences* and *epistemic contradictions*. The semantics of each of these logics can be formulated bilaterally—that is, using both a positive primitive semantic relation $\vDash$ (interpreted as assertibility), as well as a negative primitive relation $=\!\!|$ (interpreted as rejectability). The bilateral semantics is used, in each of these logics, to define a *bilateral negation* $\neg$: for a team s, $s \vDash \neg\varphi$ iff $s =\!\!| \varphi$, and $s =\!\!| \neg\varphi$ iff $s \vDash \varphi$. The resulting bilateral negations are very similar to the dual negation in that they exhibit failure of determination, and, as we will show, an analogue of Burgess' theorem holds for each of them.

In the first part of this paper, we show the analogue for BSML (we additionally show one for $\mathrm{BSML}^{\mathbb{V}}$, the extension of BSML with the *global* or *inquisitive disjunction* $\mathbb{V}$). This demonstrates, as noted above, that the negation in BSML can be seen as a type of dual negation, and establishes that what we will call Burgess theorems can also be obtained in the propositional modal setting. BSML and $\mathrm{BSML}^{\mathbb{V}}$ differ substantially from D and IF not only in being propositional and modal rather than first order, but also in that formulas of D and IF are *downward closed*—a formula is downward closed if its truth in a team t implies truth in all subteams $s \subseteq t$ of t—whereas those of BSML and $\mathrm{BSML}^{\mathbb{V}}$ need not be. This gives rise to an interesting technical detail: we identify

[5]The term 'team logic' is from Väänänen [119], who originally used it to refer to the extension D with the *Boolean* or *contradictory negation* $\sim$ (where $\sim\varphi$ is true in a team iff φ is false), and it is now frequently used to refer to any logic employing team semantics that incorporates $\sim$. The term is used both in this sense and in the more general sense (whereby 'team logics' are logics intended to be interpreted using team semantics) in the literature.

We noted earlier that D and IF are not closed under $\sim$, and referred to $\sim$ as the 'classical negation'. This name is appropriate when discussing the relationship between D, IF, and Σ_1^1 because on the level on which D and IF correspond to Σ_1^1 (essentially the level of teams), $\sim$ corresponds to the standard, classical negation of second-order logic. In the context of team semantics, the name 'classical negation' would potentially be confusing because there is another level (essentially the level of assignments or worlds—the elements of teams) on which $\sim$ behaves nonclassically (on this level it is the dual negation rather than $\sim$ which corresponds, in the FO-fragments of D and IF, to the classical negation of FO). See Fact 4.2.4 and Section 4.3.1.

two distinct natural incompatibility notions which are equivalent to one another (and to the notion employed by Burgess) in a downward-closed setting, but which come apart when downward closure fails (cf. [97, Section 6.4]). We show that the weaker of these notions ($\bot$-*incompatibility*) is not sufficiently strong to yield a Burgess theorem for BSML or $\mathrm{BSML}^{\mathbb{V}}$, but that the stronger notion (*ground-incompatibility*) does suffice.

Burgess observed that his theorem can be seen as a result on the expressive power of H_p:

> The Enderton-Walkoe theorem says that for any PC [pseudo-elementary (i.e., Σ^1_1-definable) class of models], call it K, there is a Henkin sentence θ such that $K = \|\theta\|$. The corollary just proved allows this theorem to be strengthened to say that for any two disjoint PCs, call them K_0 and K_1, there is a Henkin sentence θ such that $K_0 = \|\theta\|$ and $K_1 = \|\neg\theta\|$. [29, Notation amended.]

Dechesne [44] similarly formulates the theorem as an expressivity result for pairs. We develop this idea further by combining it with our distinct notions of incompatibility, which are essentially properties of pairs (of classes of pointed models, in the modal setting): a Burgess theorem, together with its converse, can be seen as establishing that the logic in question is expressively complete for the class of pairs satisfying the relevant pair property/incompatibility notion.[6] For instance, the results described above imply that whereas D is expressively complete both for relevant pairs that are $\bot$-incompatible as well as those that ground-incompatible, BSML is only complete for ground-incompatible relevant pairs. We formulate a notion of expressive completeness for pairs (*bicompleteness*) to succinctly describe these results.

Whereas the logics in the first part of the paper employ modal team semantics—team semantics on Kripke models—in the second part of the paper, we consider logics with dual-like negations interpreted using propositional team semantics. The simpler propositional setting allows us to readily prove multiple Burgess/bicompleteness theorems for multiple notions of incompatibility[7], giving us a better handle on the notion of bicompleteness as well as on the diverse behavior this type of negation can be induced to exhibit. We show Burgess theorems for Hawke and Steinert-Threlkeld's logic HS [67] (which we treat here as a propositional logic despite its featuring a modality since it is interpreted on propositional teams); for the propositional fragments of BSML, and $\mathrm{BSML}^{\mathbb{V}}$ (both introduced in [140]); and for *propositional dependence logic* [139] with the dual negation, the propositional version of D. These theorems require the introduction of yet more pair properties/incompatibility notions. While our proofs for the modal logics (as well as that for the propositional fragment of $\mathrm{BSML}^{\mathbb{V}}$) are analogous to Burgess'[8] (these results are essentially corollaries of interpolation for

[6] I am grateful to Søren Brinck Knudstorp for suggesting this idea to me.

[7] We will speak interchangeably of incompatibility notions and pair properties, but it should be noted that many of the pair properties we identify do not correspond to any concept of 'incompatibility' as ordinarily conceived.

[8] This is why we choose to first work in the more complicated modal setting.

classical modal/propositional logic; Burgess', similarly, is a corollary of interpolation for FO), we employ different methods to prove the other propositional theorems: the HS-result follows easily from the basic properties of the negation in HS, while the results for the propositional fragment of BSML and propositional dependence logic are shown to follow from the expressive completeness theorems for these logics. We also demonstrate, in the second part, what happens when the notion of bicompleteness is applied to propositional logics which do not exhibit failure of determination (for instance, classical propositional logic interpreted on teams), and provide an example of maximal failure of determination by introducing a logic which is bicomplete for all propositional pair properties.

While we distinguish the different pair properties/incompatibility notions for technical reasons, many of the notions we consider are also conceptually suggestive. In the third part of the paper, we sketch some possible intuitive interpretations of some of the notions.

We prove the BSML- and $\mathrm{BSML}^{\mathbb{V}}$-theorems in Section 4.2 and the propositional theorems in Section 4.3. We consider possible interpretations of the incompatibility notions in Section 4.4. We conclude, in Section 4.5, with some discussion of the results. In the remainder of this introduction, I comment briefly on how Burgess' theorem has been interpreted.

4.1.3 Remarks on Burgess' Theorem

Burgess himself intended his theorem to serve, in part, as a point against IF and Hintikka's philosophical ambitions. He writes:

> In recent years Hintikka and co-workers have revived a variant version of the logic of Henkin sentences under the label "independence-friendly" logic, have restated many theorems about existential second-order sentences for this "new" logic, and have made very large claims about the philosophical importance of the theorems thus restated. In discussion, pro and con, of such philosophical claims it has not been sufficiently emphasized that contrariety [dual negation], the only kind of "negation" available, fails to correspond to any operation on classes of models. For this reason it seemed worthwhile to set down, in the form of the corollary above, a clear statement of just how total the failure is. [29]

Accordingly, the result is occasionally cited as attesting that the behaviour of the negation is anomalous or problematic (as in [82]). A common gloss has it that Burgess establishes or shows that the dual negation is "not a semantic operation".

I do not intend to assess the merits of Burgess' argument, or to argue for or against the dual negation here. What I do hope to do is to point out why the common characterization of the remark is potentially misleading in multiple ways.[9]

[9] It is also worth pointing out, in connection with the passage from Burgess concerning Hintikka

My first qualm with the gloss is that it is not Burgess' theorem that establishes the "non-semantic" nature of the dual negation. The failure of the negation to correspond to any operation on classes of models is an easily observable, simple fact (recall our dependence logic example: $\neg =\!(x,y) \equiv \bot$, but $\neg\neg =\!(x,y) \not\equiv \neg\neg\bot$). The theorem is a further fact, characterized by Burgess as concerning the degree of this failure. Indeed, observe that in the passage above, the theorem functions as a kind of *reductio* of the failure itself: the (implied) problem with the negation is its failure to correspond to any operation on classes of models. The theorem merely serves to demonstrate the extreme level of this failure, and this extremeness is, in turn, implied to constitute part of what Burgess appears to view as the absurd consequences of adopting an operator that fails to so correspond. The failure itself is independent of the theorem. The results in this paper help to reinforce this point: we prove variants of Burgess' theorem with respect to multiple distinct pair properties/notions of incompatibility, which we may think of as constituting a way of measuring the degree of failure of correspondence in that a Burgess theorem with respect to a weaker notion of incompatibility corresponds to a higher degree of failure (see Section 4.5)—we have, then, failure of determination in multiple different logics, regardless of whether Burgess' original theorem holds for that logic (and also regardless of whether some variant of the theorem can be established), with the Burgess theorems providing insight into the degree and nature of the failure in each case.

Second, as essentially already pointed out by Hodges [78], and later by many others [119, 44, 94, 58] (and as is already implicit in our discussion of bicompleteness), there is a sense in which the dual negation is a semantic operation, which, while perhaps trivial, should not be ignored. Indeed, Hodges' [78] goal in developing team semantics was to demonstrate that logics of imperfect information such as IF could be given a semantics in which all connectives are semantic operations in the sense that replacing a subformula occurrence with one with the same meaning does not change the meaning of a formula. Instead of taking the meaning of φ to be $\|\varphi\|$, let it be represented by the pair $(\|\varphi\|, \|\neg\varphi\|)$. Then the meaning of $\neg\varphi$ is determined by that of φ since to obtain the former from the latter, one need only flip the elements of the pair: $(\|\neg\varphi\|, \|\varphi\|)$.

One could of course perform a similar trick with any other operator or connective no matter its semantics, but there are potentially good reasons for insisting that, applied to the negation, this move is not entirely *ad hoc*. For instance, on the view known both

and IF, that Hintikka did also consider an extension of IF with the Boolean/contradictory negation ~ (*extended independence-friendly logic*), and that he ultimately viewed each negation as indispensable [75, 76]. Consider, for instance, the following passage [75, p. 154]:

> ...*in any sufficiently rich language, there will be two different notions of negation present.* Or if you prefer a different formulation, our ordinary concept of negation is intrinsically ambiguous. The reason is that one of the central things we certainly want to express in our language is the contradictory negation. But ... a contradictory negation is not self-sufficient. In order to have actual rules for dealing with negation, one must also have the dual negation present, however implicitly.

as *rejectivism* and as *bilateralism* (see, for instance, [104, 105, 115, 110]), the speech act of rejection is not reducible to that of assertion—these notions should be treated as being on par, with neither being conceptually reducible to the other. If one endorses this view, it is natural to conceive of the pair $(\|\varphi\|^+, \|\varphi\|^-)$ as the full meaning of φ, where $\|\varphi\|^+$ denotes the models in which in φ is assertible, and $\|\varphi\|^-$ those in which it is rejectable. And if, as in BSML, one additionally associates rejectability with a negation operator (cf. varieties of rejectivism with a specialized rejection operator such as [115, 110]), one may equate $(\|\varphi\|^+, \|\varphi\|^-)$ with $(\|\varphi\|, \|\neg\varphi\|)$.[10] Whether it is possible to formulate a tenable form of rejectivism featuring the dual negation in this way is beyond the scope of this paper; at any rate, given the clear similarity between Hodges' notion of meaning and that of the rejectivists, it seemed worthwhile to make this connection explicit (rejectivism has not thus far, to my knowledge, been discussed in the literature on the dual negation). It should also be noted that, as alluded to in endnote 8, bilateral notions of meaning of this kind have been proposed for reasons other than rejectivism.

4.2 Burgess Theorems for BSML and $\mathrm{BSML}^{\mathbb{V}}$

In this section, we list the required preliminaries concerning the logics BSML and $\mathrm{BSML}^{\mathbb{V}}$ and team semantics (Section 4.2.1); reformulate Burgess' theorem as an expressive completeness theorem for pairs satisfying a specific notion of incompatibility (Section 4.2.2); distinguish between multiple notions of incompatibility and examine how they are related (Section 4.2.2); and show Burgess theorems for BSML and $\mathrm{BSML}^{\mathbb{V}}$ using some of these incompatibility notions (Section 4.2.3).

4.2.1 Preliminaries

We first define the syntax and semantics of BSML and $\mathrm{BSML}^{\mathbb{V}}$ (from [6, 7]); then recall the definitions of standard team-semantic closure properties and list some results relating these properties to our logics (from [7, 16]); and, finally, recall basic notions and results from modal logic (see, e.g., [24, 56]), together with team-based analogues (from [71, 7]).

[10]The bilateral semantics of BSML and Hawke and Steinert-Threlkeld's logic are not motivated by rejectivism *per se*; the primary intended function of bilateralism in these logics is, rather, to allow these logics to correctly capture and predict linguistic data. Aloni [6] makes a point in defence of failure of replacement by drawing on these empirical considerations:

> As we will see, however, this non-classical behavior is precisely what we need to explain the effects of pragmatic enrichment in negative contexts. [...] when replacement under $\neg$ holds, the rejection conditions are derivable from the support conditions, and so without failure of replacement bilateralism would not give different predictions from unilateral systems and so it would be empirically unjustified.

BSML is an extension of classical modal logic with the *nonemptiness atom* NE (introduced in [121, 140]), which is true in a team just in case the team is nonempty. BSML$^{\mathbb{V}}$ is BSML extended with the *global* or *inquisitive disjunction* $\mathbb{V}$, used in inquisitive logic [41, 38, 35] (see Sections 4.3.1 and 4.4.4) to model the meanings of questions.

4.2.1. DEFINITION (Syntax of ML, BSML, and BSML$^{\mathbb{V}}$). Fix a (countably infinite) set Prop of propositional variables. The set of formulas of *bilateral state-based modal logic* BSML is generated by:

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \Diamond\varphi \mid \text{NE}$$

where $p \in \mathsf{Prop}$. *Classical modal logic* ML is the NE-free fragment of BSML. BSML$^{\mathbb{V}}$ is the extension of BSML with the binary connective $\mathbb{V}$.

We use the first Greek letters α, β to refer exclusively to formulas of ML (also called *classical formulas*). We write $\mathsf{P}(\varphi)$ for the set of propositional variables in φ.

A Kripke model $M = (W, R, V)$ over $\mathsf{X} \subseteq \mathsf{Prop}$ is defined as usual, where in particular, $V : \mathsf{X} \to \wp(W)$. We call a subset $s \subseteq W$ of W a (modal) *team* on M. For any world w in M, define, as usual, $R[w] := \{v \in W \mid wRv\}$. Similarly, for any team s on M, define $R[s] := \bigcup_{w \in s} R[w]$.

4.2.2. DEFINITION (Semantics of ML, BSML and BSML$^{\mathbb{V}}$). For a model $M = (W, R, V)$ over X, a team s on M, and formula φ with $\mathsf{P}(\varphi) \subseteq \mathsf{X}$, the notions of φ being *supported by/anti-supported by* s in M, written $M, s \vDash \varphi / M, s \dashv \varphi$ (or simply $s \vDash \varphi / s \dashv \varphi$), are defined recursively as follows:

$M, s \vDash p$	$:\Longleftrightarrow$	for all $w \in s : w \in V(p)$;
$M, s \dashv p$	$:\Longleftrightarrow$	for all $w \in s : w \notin V(p)$;
$M, s \vDash \bot$	$:\Longleftrightarrow$	$s = \varnothing$;
$M, s \dashv \bot$		always the case;
$M, s \vDash \text{NE}$	$:\Longleftrightarrow$	$s \neq \varnothing$;
$M, s \dashv \text{NE}$	$:\Longleftrightarrow$	$s = \varnothing$;
$M, s \vDash \neg\varphi$	$:\Longleftrightarrow$	$M, s \dashv \varphi$;
$M, s \dashv \neg\varphi$	$:\Longleftrightarrow$	$M, s \vDash \varphi$;
$M, s \vDash \varphi \wedge \psi$	$:\Longleftrightarrow$	$M, s \vDash \varphi$ and $M, s \vDash \psi$;
$M, s \dashv \varphi \wedge \psi$	$:\Longleftrightarrow$	there exist t, u s.t. $s = t \cup u$ and $M, t \dashv \varphi$ and $M, u \dashv \psi$;
$M, s \vDash \varphi \vee \psi$	$:\Longleftrightarrow$	there exist t, u s.t. $s = t \cup u$ and $M, t \vDash \varphi$ and $M, u \vDash \psi$;

$$M,s \dashv \varphi \vee \psi \quad :\Longleftrightarrow \quad M,s \dashv \varphi \text{ and } M,s \dashv \psi;$$

$$M,s \vDash \varphi \mathbin{\mathbb{V}} \psi \quad :\Longleftrightarrow \quad M,s \vDash \varphi \text{ or } M,s \vDash \psi;$$
$$M,s \dashv \varphi \mathbin{\mathbb{V}} \psi \quad :\Longleftrightarrow \quad M,s \dashv \varphi \text{ and } M,s \dashv \psi;$$

$$M,s \vDash \Diamond \varphi \quad :\Longleftrightarrow \quad \text{for all } w \in s \text{ there exists } t \subseteq R[w] \text{ s.t. } t \neq \varnothing \text{ and } M,t \vDash \varphi;$$
$$M,s \dashv \Diamond \varphi \quad :\Longleftrightarrow \quad \text{for all } w \in s : M,R[w] \dashv \varphi.$$

We also refer to support by s as *truth in s* for convenience, although we caution that most interpretations of team logics rely on distinguishing the way in which a team satisfies a formula (often called 'support') from the way in which an element of a team satisfies one (often called 'truth'; see Fact 4.2.4). We write $\varphi \vDash \psi$ and say φ *entails* ψ if for all M and all s on M, $M,s \vDash \varphi$ implies $M,s \vDash \psi$. If both $\varphi \vDash \psi$ and $\psi \vDash \varphi$, then we write $\varphi \equiv \psi$ and say that φ and ψ are *equivalent*. If both $\varphi \equiv \psi$ and $\neg\varphi \equiv \neg\psi$, then φ and ψ are said to be *bi-equivalent* or *strongly equivalent*, written $\varphi \equiv^{\pm} \psi$.

We refer to the atom $\bot$ as the *weak contradiction*. We also define the *strong contradiction* $\bot\!\!\!\bot := \bot \wedge \text{NE}$, and $\top := \neg\bot$. The weak contradiction is true only in the empty team; the strong contradiction in no team; $\top$ in all teams. Note that *ex falso* with respect to $\bot\!\!\!\bot$ holds for all formulas φ: $\bot\!\!\!\bot \vDash \varphi$; with respect to $\bot$ it holds only for formulas ψ with the empty team property (see below): $\bot \vDash \psi$ if $M,\varnothing \vDash \psi$ for all M. We let $\bigvee\varnothing := \bot$; $\mathbb{V}\varnothing := \bot\!\!\!\bot$; and $\Box\varphi := \neg\Diamond\neg\varphi$. Observe that $\neg(\varphi \vee \psi) \equiv \neg\varphi \wedge \neg\psi \equiv \neg(\varphi \mathbin{\mathbb{V}} \psi)$; $\neg\neg\varphi \equiv \varphi$; and $\neg\Diamond\varphi \equiv \Box\neg\varphi$. It follows from these equivalences that each formula is equivalent to one in negation normal form.

4.2.3. DEFINITION (Closure properties). We say that a formula φ

- is *downward closed*, provided $[M,s \vDash \varphi \text{ and } t \subseteq s] \Longrightarrow M,t \vDash \varphi$;
- is *convex*, provided $[M,s \vDash \varphi;\ M,t \vDash \varphi; \text{ and } t \subseteq u \subseteq s] \Longrightarrow M,u \vDash \varphi$;
- is *union closed*, provided $[M,s \vDash \varphi \text{ for all } s \in S \neq \varnothing] \Longrightarrow M,\bigcup S \vDash \varphi$;
- has the *empty team property*, provided $M,\varnothing \vDash \varphi$ for all M;
- is *flat*, provided $M,s \vDash \varphi \iff M,\{w\} \vDash \varphi$ for all $w \in s$.

It is easy to check that a formula is flat if and only if it is downwards closed and union closed, and has the empty team property. One can show by induction that, in the context of the connectives we have introduced, all $\mathbb{V}$-free formulas (in particular, all formulas of BSML) are union closed and convex, and that all NE-free formulas are downward closed and have the empty team property (so that all formulas of ML are flat). Formulas of BSML or $\text{BSML}^{\mathbb{V}}$ clearly need not be downward closed or have the empty team property (consider $p \wedge \text{NE}$); formulas of $\text{BSML}^{\mathbb{V}}$ need not be union closed or convex (consider $(a \wedge ((p \wedge \text{NE}) \vee \top)) \mathbin{\mathbb{V}} (b \wedge ((q \wedge \text{NE}) \vee \top))$. Another easy induction shows:

4.2.4. FACT (Conservativity of ML over standard semantics). For any $\alpha \in$ ML:

$$M, s \vDash \alpha \iff M, \{w\} \vDash \alpha \text{ for all } w \in s \iff M, w \vDash \alpha \text{ for all } w \in s,$$

where $\vDash$ on the right is the standard single-world truth relation for ML. This implies that for $\alpha, \beta \in$ ML: $\alpha \vDash \beta$ in the team-semantics sense iff $\alpha \vDash \beta$ in the usual single-world-semantics sense; we may therefore use the symbol '$\vDash$' to refer interchangeably to either type of entailment when discussing ML-formulas.

We write $\chi[\xi]$ to refer to a specific occurrence of the subformula ξ in χ, and $\chi[\psi/\xi]$ for the result of replacing this occurrence of ξ in χ with φ. Our Burgess theorems will build on the fact that preservation of equivalence under replacement fails in negative contexts: if $\chi[p]$ is within the scope of an odd number of $\neg$ in χ, $\varphi \equiv \psi$ need not imply $\chi[\varphi/p] \equiv \chi[\psi/p]$. For instance, $\bot \equiv \neg\text{NE}$ but $\neg\bot \equiv \top \not\equiv \text{NE} \equiv \neg\neg\text{NE}$, and $\neg(\varphi \vee \psi) \equiv \neg(\varphi \mathbin{\mathbb{V}} \psi)$ but $\neg\neg(\varphi \vee \psi) \equiv \varphi \vee \psi \not\equiv \varphi \mathbin{\mathbb{V}} \psi \equiv \neg\neg(\varphi \mathbin{\mathbb{V}} \psi)$. Bi-equivalence, on the other hand, is preserved under replacement: if $\varphi \equiv^{\pm} \psi$, then $\chi[\varphi/p] \equiv^{\pm} \chi[\psi/p]$. (This implies that if we take meanings to be represented by pairs $\|\varphi\|^{\pm} = (\|\varphi\|, \|\neg\varphi\|)$—see below for the definition of this notation—the semantics is compositional in Hodges' sense. See Section 4.1.3.) And, crucially, replacement does hold in non-negated and other positive contexts:

4.2.5. FACT (Replacement in positive contexts). If $\chi[p]$ is in the scope of an even number of $\neg$ in χ, then $\varphi \equiv \psi$ implies $\chi[\varphi/p] \equiv \chi[\psi/p]$.

We officially reserve the term *dual negation* to refer to any negation operator that engenders failure of preservation of equivalence under replacement in negated contexts (as noted above, in BSML and BSML$^{\mathbb{V}}$ this failure occurs only in *negative* negated contexts—contexts in the scope of odd number of negations). However, for simplicity, we also occasionally use this term to refer to any negation operator with bilateral semantics (such as the negation of ML as defined above).

A *pointed model* over X is a pair (M, w) where M is a model over X and $w \in W$. We define *k-bisimilarity* (for $k \in \mathbb{N}$) between pointed models with respect to a set of propositional variables X in the usual way:

- $M, w \rightleftharpoons_0^{\mathsf{X}} M, w :\iff$ for all $p \in \mathsf{X}$ we have $M, w \vDash p$ iff $M', w' \vDash p$.
- $M, w \rightleftharpoons_{k+1}^{\mathsf{X}} M', w' :\iff M, w \rightleftharpoons_0 M', w'$ and
  - [forth] for all $v \in R[w]$ there is a $v' \in R'[w']$ such that $M, v \rightleftharpoons_k M', v'$;
  - [back] for all $v' \in R'[w']$ there is a $v \in R[w]$ such that $M, v \rightleftharpoons_k M', v'$,

and we write $M, w \rightleftharpoons_k^{\mathsf{X}} M', w'$ (or simply $w \rightleftharpoons^k w'$) if (M, w) and (M', w') are k-bisimilar with respect to X. The *modal depth* $md(\varphi)$ of a formula φ is defined as usual; note that we put $md(\text{NE}) := 0$ and $md(\varphi \mathbin{\mathbb{V}} \psi) := max\{md(\varphi), md(\psi)\}$. We say that (M, w) and (M, w') are X, k*-equivalent*, written $M, w \equiv_k^{\mathsf{X}} M', w'$ (or simply $w \equiv_k w'$), if for all

$\alpha(\mathsf{X}) \in \mathrm{ML}$ with $md(\alpha) \leq k$: $M, w \vDash \alpha \iff M', w' \vDash \alpha$. We define the *k-th Hintikka formula* or *characteristic formula* $\chi^{\mathsf{X},k}_{M,w} \in \mathrm{ML}$ (or simply χ^k_w) of (M,w) as follows:

$$\chi^{\mathsf{X},0}_{M,w} := \bigwedge\{p \mid p \in \mathsf{X}, w \in V(p)\} \wedge \bigwedge\{\neg p \mid p \in \mathsf{X}, w \notin V(p)\};$$
$$\chi^{\mathsf{X},k+1}_{M,w} := \chi^k_w \wedge \bigwedge_{v \in R[w]} \Diamond \chi^k_v \wedge \Box \bigvee_{v \in R[w]} \chi^k_v.$$

We then have that:

4.2.6. THEOREM. *(See, for instance, [24, 56].)*

$$w \equiv_k w' \iff w \rightleftharpoons_k w' \iff w' \vDash \chi^k_w \iff \chi^k_w \equiv \chi^k_{w'}.$$

Similarly, a *pointed (team) model* over X is a pair (M,s) where M is a model over X and s is a team on M. For a given $k \in \mathbb{N}$, (M,s) and (M,s') are X, *k-equivalent* in a logic L, written $M, s \equiv^{\mathsf{X}}_k M', s'$, if for all $\varphi(\mathsf{X}) \in L$ with $md(\varphi) \leq k : M, s \vDash \varphi \iff M', s' \vDash \varphi$. It can be shown that $s \equiv^{\mathsf{X}}_k s'$ in BSML iff $s \equiv^{\mathsf{X}}_k s'$ in $\mathrm{BSML}^{\mathbb{V}}$, so we use $s \equiv^{\mathsf{X}}_k s'$ to refer to equivalence in either of these two logics. We say that (M,s) and (M',s') are *(team) k-bisimilar* with respect to X, and write $M, s \rightleftharpoons^{\mathsf{X}}_k M', s'$ (or simply $s \rightleftharpoons_k s'$) if for each $w \in s$ there is some $w' \in s'$ such that $w \rightleftharpoons_k w'$; and for each $w' \in s'$ there is some $w \in s$ such that $w \rightleftharpoons_k w'$. We define the *(weak) k-th Hintikka formula* $\chi^{\mathsf{X},s}_{M,s}$ (or simply χ^k_s) of (M,s) by $\chi^{\mathsf{X},k}_{M,s} := \bigvee_{w \in s} \chi^k_w$. We have:

4.2.7. THEOREM. *[71, 7]*

$$[s \equiv_k s' \iff s \rightleftharpoons_k s'] \quad \textit{and} \quad [s' \vDash \chi^k_s \iff s' \rightleftharpoons_k t \textit{ for some } t \subseteq s].$$

A *(world) property* (over X) is a class of pointed models over X. Each formula $\alpha \in \mathrm{ML}$ *expresses* a property $[\![\alpha]\!]_{\mathsf{X}}$ (or simply $[\![\alpha]\!]$) over any $\mathsf{X} \supseteq \mathsf{P}(\alpha)$, where

$$[\![\alpha]\!]_{\mathsf{X}} := \{(M,w) \text{ over } \mathsf{X} \mid M, w \vDash \alpha\}.$$

A *(team) property* (over X) is a class of pointed team models over X. Each formula φ *expresses* a property $\|\varphi\|_{\mathsf{X}}$ (or simply $\|\varphi\|$) over any $\mathsf{X} \supseteq \mathsf{P}(\varphi)$, where

$$\|\varphi\|_{\mathsf{X}} := \{(M,s) \text{ over } \mathsf{X} \mid M, s \vDash \varphi\}.$$

The *ground team* of a team property $\mathcal{P}$, denoted $\dot{\bigcup}\mathcal{P}$, is the world property

$$\dot{\bigcup}\mathcal{P} := \{(M,w) \mid \exists (M,s) \in \mathcal{P} : w \in s\},$$

and the *ground team* of a formula φ (over X), denoted $|\varphi|_{\mathsf{X}}$, is defined by $|\varphi|_{\mathsf{X}} := \dot{\bigcup}\|\varphi\|_{\mathsf{X}}$, so that

$$|\varphi|_{\mathsf{X}} = \{(M,w) \mid \exists (M,s) \in \|\varphi\|_{\mathsf{X}} : w \in s\}.$$

(Note that $\biguplus\mathcal{P}$ is essentially a team modulo bisimulation[11], whence the name.)[12] Observe that for classical α, $[\![\alpha]\!] = |\alpha|$.

We say that a logic L is *expressively complete* for a class of (team) properties $\mathbb{P}$, written $\|L\| = \mathbb{P}$, if for each finite X, the class $\mathbb{P}_{\mathsf{X}}$ of properties over X in $\mathbb{P}$ is precisely the class of properties over X expressible by formulas of L, that is, if

$$\|L\|_{\mathsf{X}} = \mathbb{P}_{\mathsf{X}}, \text{ where } \|L\|_{\mathsf{X}} := \{\|\varphi\|_{\mathsf{X}} \mid \varphi \in L\}.$$

The definition of closure properties (4.2.3) is extended to team properties in the obvious way; for instance, a property $\mathcal{P}$ is *union closed* just in case $[(M,s) \in \mathcal{P}$ for all $s \in S \neq \varnothing]$ implies $(M, \bigcup S) \in \mathcal{P}$. We additionally say that $\mathcal{P}$ is *invariant under* X, k*-bisimulation* $(k \in \mathbb{N})$ if $[(M,s) \in \mathcal{P}$ and $M,s \rightleftharpoons_k^{\mathsf{X}} M',s']$ implies $(M',s') \in \mathcal{P}$, and that $\mathcal{P}$ over X is *invariant under bounded bisimulation* if $\mathcal{P}$ is invariant under X, k-bisimulation for some $k \in \mathbb{N}$.

4.2.8. THEOREM (Expressive completeness of BSML and $\mathrm{BSML}^{\mathbb{V}}$). *[16, 7]*

[11] $\biguplus\mathcal{P}$ is bisimilar to the pointed model $(\uplus\{M' \mid (M',w') \in \biguplus\mathcal{P}\}, \{w' \mid (M',w') \in \biguplus\mathcal{P}\})$, where $\uplus$ denotes disjoint union, in the sense that for each $(M,w) \in \biguplus\mathcal{P}$ there is a $w' \in \{w' \mid (M',w') \in \biguplus\mathcal{P}\}$ such that (M,w) is bisimilar to $(\uplus\{M' \mid (M',w') \in \biguplus\mathcal{P}\}, w')$, and vice versa.

[12] While the other notions in Section 4.2.1 are from the literature, the term 'ground team' is introduced in this paper. The notion itself is, however, essentially identical or very similar to a number of other notions in the literature. In this endnote we list some of these notions to stave off confusion, to help the reader make connections, and also to highlight the fact that ground teams (and hence potentially also the incompatibility notions making use of ground teams—see Sections 4.2.2 and 4.3) can (at least in some settings) be given meaningful, interesting interpretations. (We will discuss some possible interpretations of ground teams and the incompatibility notions in more detail in Section 4.4.)

Hodges [78] defines (a first-order analogue) of the following notion: the *flattening operator* $\downarrow$ is such that $\|\downarrow\varphi\| = \{(M,t) \mid \forall w \in t : \exists (M,s) \in \|\varphi\| : w \in s\}$. In other words, the flattening $\downarrow\varphi$ of φ expresses the property consisting of teams consisting of elements of $|\varphi|$, the ground team of φ. This is essentially the power set of the ground team—it is the flat team property corresponding to the world property $|\varphi|$. Hodges' flattening is similar to but distinct from Väänänen's [119] *flattening* φ^f of φ (see Sections 4.3.3 and 4.3.4)—φ^f is, roughly, φ with each nonclassical subformula ψ replaced with a classical subformula α such that $\psi \vDash \alpha$ and $|\psi| = |\alpha|$.

In inquisitive semantics/logic (see [41, 42, 38, 35], as well as Sections 4.3.1 and 4.4.4; we consider here only the propositional setting for simplicity), the union $\bigcup\mathcal{P}$ of a property $\mathcal{P}$ is referred to and represents the *informative content* of the (inquisitive) proposition represented by $\mathcal{P}$ (in the propositional setting—see Section 4.3—we define $\biguplus\mathcal{P} := \bigcup\mathcal{P}$, so $\bigcup\mathcal{P}$ is the same notion as the ground team of $\mathcal{P}$). Similarly, the informative content of φ is $\bigcup\|\varphi\|$. There is an operator on propositions ! such that $!\mathcal{P} = \wp(\bigcup\mathcal{P})$ and a corresponding connective ! such that $\|!\varphi\| = \wp(\bigcup\|\varphi\|)$, and hence $!\varphi$ is equivalent to the Hodges' flattening $\downarrow\varphi$ of φ. Applied to $\mathcal{P}/\varphi$, the operator/connective yields a proposition/formula that has the same informative content as $\mathcal{P}/\varphi$, but with no *inquisitive content*. The set of worlds w such that $\{w\} \vDash \varphi$ is referred to as the *truth-set* of φ (as in BSML, the fundamental semantic notion, defined with respect to teams, is *support*; truth is defined as support with respect to singleton teams). Due to the downward closure of inquisitive logic, the truth-set of φ is equal to its ground team/informative content. The truth-set/informative content of φ can also represent the information that φ presupposes, if φ expresses a question. Inquisitive logic also features a notion analogous to Väänänen's flattening of φ; this is called the *classical variant* of φ.

(i) BSML *is expressively complete for the class* $\{\mathcal{P} \mid \mathcal{P}$ *is convex, union closed, and invariant under bounded bisimulation*$\}$.

(ii) $\text{BSML}^{\mathbb{V}}$ *is expressively complete for the class* $\{\mathcal{P} \mid \mathcal{P}$ *is invariant under bounded bisimulation*$\}$.

4.2.2 Bicompleteness and Incompatibility

In this section, we reformulate Burgess' theorem as an expressive completeness result for pairs satisfying a specific notion of incompatibility, or pair property. We then distinguish between multiple different notions of incompatibility, examine how they are related, and show that some of them are not sufficiently strong to yield Burgess theorems for BSML and $\text{BSML}^{\mathbb{V}}$ due to the failure of downward closure and the empty team property in these logics.

Burgess [29] formulates his result as follows: if two sentences φ and ψ are incompatible in that they share no models, then there is a sentence θ such that $\varphi \equiv \theta$ and $\psi \equiv \neg\theta$. Kontinen and Väänänen [94] (working in D and IF, whose formulas are downward closed and have the empty team property), make use of an equivalent notion of incompatibility, generalized to make it applicable to arbitrary formulas (that may contain free variables): the formulas φ and ψ are incompatible if, for any given model and (first-order) team on that model, if φ and ψ are both true in the team, the team must be empty.[13] Let us say that two formulas which are incompatible in this way are $\bot$*-incompatible*. In our modal setting:

4.2.9. DEFINITION ($\bot$-incompatibility). φ_0 and φ_1 are $\bot$*-incompatible* ($\bot$*-I*) if $[s \vDash \varphi_0$ and $s \vDash \varphi_1]$ implies $s = \varnothing$ (equivalently, if $\varphi_0, \varphi_1 \vDash \bot$).

Table 4.1 lists all the pair properties/incompatibility notions we consider in this paper, formulated for the simpler propositional setting (see Section 4.3). Figure 4.2 displays some of the implications between the incompatibility notions.

We noted in Section 4.1 that a sentence (of D or IF) and its dual negation share no models. The analogous result also holds for arbitrary formulas: any formula φ and its dual negation $\neg\varphi$ are $\bot$-incompatible: $\varphi, \neg\varphi \vDash \bot$. Observe that this fact immediately yields the converse of Kontinen and Väänänen's Burgess theorem: for any formulas φ and ψ, if there is a formula θ such that $\varphi \equiv \theta$ and $\psi \equiv \neg\theta$, then since θ and $\neg\theta$ are $\bot$-incompatible, φ and ψ must also be $\bot$-incompatible.

We may view a Burgess theorem, together with its converse, as a type of expressive completeness result. In our modal setting, we let $\|\varphi\|_{\mathsf{X}}^{\pm,\neg}$ (or simply $\|\varphi\|_{\mathsf{X}}^{\pm}$) denote the pair $(\|\varphi\|_{\mathsf{X}}, \|\neg\varphi\|_{\mathsf{X}})$. A *pair property* $\mathscr{P}$ over X is a class of pairs of team properties $(\mathcal{P}, \mathcal{Q})$, where $\mathcal{P}$ and $\mathcal{Q}$ are over X. We say that a logic L is *bicomplete* for a pair

[13] For sentences, Kontinen and Väänänen's notion is equivalent to Burgess'. This follows from the fact that a sentence, in first-order team semantics, is defined to be true in a model just in case it is true in all first-order teams on the model, and this is equivalent to it being true in some nonempty team on the model.

Pair property	Definition(s)	`Bicomplete` logics
$\bot$-incompatible ($\bot$-I)	• $[s \vDash \varphi_0$ and $s \vDash \varphi_1] \implies s = \varnothing$ • $\varphi_0, \varphi_1 \vDash \bot$ • NE-incompatible or $\perp\!\!\!\perp$-incompatible	D, IF
Ground-incompatible (G-I)	• $[s \vDash \varphi_0$ and $t \vDash \varphi_1] \implies s \cap t = \varnothing$ • $\lvert\varphi_0\rvert \cap \lvert\varphi_1\rvert = \varnothing$	D, IF, BSML, BSML$^{\mathbb{V}}$, PL(NE, $\mathbb{V}$)
$\perp\!\!\!\perp$-incompatible ($\perp\!\!\!\perp$-I)	• φ_0 and φ_1 never jointly true • $\varphi_0, \varphi_1 \vDash \perp\!\!\!\perp$ • $\lVert\varphi_0\rVert \cap \lVert\varphi_1\rVert = \varnothing = \lVert\perp\!\!\!\perp\rVert$	
NE-incompatible (NE-I)	• $[s \vDash \varphi_0$ and $s \vDash \varphi_1] \iff s = \varnothing$ • $\varphi_0, \varphi_1 \vDash \bot$ and $\varphi_0, \varphi_1 \nvDash \perp\!\!\!\perp$ • $\lVert\varphi_0\rVert \cap \lVert\varphi_1\rVert = \{\varnothing\} = \lVert\bot\rVert$	D, IF
World-incompatible (W-I)	• $\{w\} \vDash \varphi_0 \iff \{w\} \nvDash \varphi_1$	PL(=(·)), PL
Team-incompatible (T-I)	• $s \vDash \varphi_i \iff s \nvDash \varphi_{1-i}$ • $\lVert\varphi_i\rVert = \lVert\top\rVert \setminus \lVert\varphi_{1-i}\rVert$	PL(∼) (w.r.t. ∼)
Flat-incompatible (F-I)	• world-incompatible and φ_0, φ_1 flat • $\lVert\varphi_i\rVert = \wp(\lvert\top\rvert \setminus \lvert\varphi_{1-i}\rvert)$ • $\lVert\varphi_i\rVert = \{s \mid t \vDash \varphi_{1-i} \implies s \cap t = \varnothing\}$ • $\lVert\varphi_i\rVert = \bigcup\{\mathcal{P} \subseteq \lVert\top\rVert \mid \mathcal{P}, \lVert\varphi_{1-i}\rVert$ G-I$\}$	PL
φ_1 down-set incompatibility (D-I) of φ_0	• $s \vDash \varphi_1 \iff$ $[[t \vDash \varphi_0$ and $t \subseteq s] \implies t = \varnothing]$	InqB (w.r.t. $\neg_i$), PL
Down-set incompatibilities (on either side) (E-D-I)	• φ_1 D-I of φ_0 or φ_0 D-I of φ_1	HS, PL
Ground-complementary (G-C)	• $\lvert\varphi_i\rvert = \lvert\top\rvert \setminus \lvert\varphi_{1-i}\rvert$	PL(=(·)), PL
Ground-complementary modulo $\perp\!\!\!\perp$ (G-C mod $\perp\!\!\!\perp$)	• $\lvert\varphi_i\rvert = \lvert\top\rvert \setminus \lvert\varphi_{1-i}\rvert$ or $\varphi_0 \equiv \perp\!\!\!\perp$ or $\varphi_1 \equiv \perp\!\!\!\perp$	PL(NE), PL(=(·)), PL
All pairs		PL(NE*, $\mathbb{V}$)

Table 4.1: Pair properties/incompatibility notions, formulated for the propositional setting. Each bulleted (•) item is a definition for the relevant notion; all definitions for a given notion are equivalent.

property $\mathscr{P}$ (with respect to $\neg$), written $\lVert L\rVert^{\pm,\neg} = \mathscr{P}$ (or simply $\lVert L\rVert^{\pm} = \mathscr{P}$), if for all finite $\mathsf{X} \subseteq \mathsf{Prop}$,

$$\lVert \mathrm{L}\rVert^{\pm,\neg}_{\mathsf{X}} = \mathscr{P}_{\mathsf{X}}, \text{ where}$$
$$\lVert \mathrm{L}\rVert^{\pm,\neg}_{\mathsf{X}} := \{(\lVert\varphi\rVert^{\pm,\neg}_{\mathsf{X}} \mid \varphi \in \mathrm{L}\} \text{ and } \mathscr{P}_{\mathsf{X}} := \{(\mathcal{P},\mathcal{Q}) \in \mathscr{P} \mid \mathcal{P},\mathcal{Q} \text{ over } \mathsf{X}\},$$

and we say it is *bicomplete modulo expressive power* (or simply `bicomplete`, written in monospace font) for $\mathscr{P}$ (with respect to $\neg$) if it is bicomplete for $\mathscr{P}_{\mathrm{L}} := \mathscr{P} \cap \{(\lVert\varphi\rVert, \lVert\psi\rVert) \mid \varphi, \psi \in L\}$ (with respect to $\neg$). A Burgess theorem for an incompatibil-

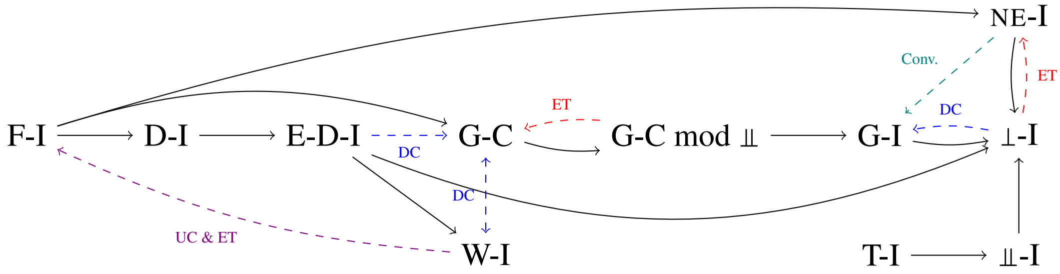


Figure 4.2: Some implications among the incompatibility notions. Black arrows indicate implications which always hold. Dashed arrows in other colors indicate implications conditional on closure properties (e.g., DC = downward closure, ET = empty team property, etc.). Implications which trivialize the notions have been omitted; for instance, in a setting with the empty team property, no pair can be team-incompatible, so team-incompatibility with the empty team property implies all other notions. Observe also that if a pair φ_0, φ_1 is NE-incompatible, then both formulas have the empty team property. Similarly flat-incompatibility forces the flatness of both φ_0 and φ_1, while D-I forces the downward closure and the empty team property of φ_1. Each of team-incompatibility and $\perp\!\!\!\perp$-incompatibility is inconsistent with both formulas having the empty team property.

ity notion (pair property) $\mathscr{P}$ yields an inclusion (abusing our notation) $\mathscr{P}_L \subseteq \|L\|^{\pm}$, and its converse yields the converse inclusion $\|L\|^{\pm} \subseteq \mathscr{P}_L$; together, then, they constitute a `bicompleteness` theorem. Say that L_1 and L_2 are *(expressively) bi-equivalent* or *strongly equivalent* (with respect to $\neg_1$ and $\neg_2$) if $\|L_1\|_{\mathsf{X}}^{\pm,\neg_1} = \|L_2\|_{\mathsf{X}}^{\pm,\neg_2}$ for all finite $\mathsf{X} \subseteq \mathsf{Prop}$.

We extend the definition of $\perp$-incompatibility to pairs of properties in the obvious way: $\mathcal{P}$ and $\mathcal{Q}$ are $\perp$*-incompatible* if $(M,s) \in \mathcal{P} \cap \mathcal{Q}$ implies $s = \varnothing$. The first-order analogues of these notions (formulated for all formulas rather than only sentences) give us the following reformulation of Kontinen and Väänänen's result:

4.2.10. THEOREM (Bicompleteness of D and IF). *[94] Each of the logics D and IF is `bicomplete` for $\perp$-incompatible pairs.*

Let us now consider the logics BSML and $\text{BSML}^{\mathbb{V}}$. While it does hold, in these logics, that a formula and its dual negation are $\perp$-incompatible, $\perp$-incompatibility does not yield a Burgess theorem or bicompleteness theorem for either of these logics. To show why, we make use of the following proposition:

4.2.11. PROPOSITION. *If $s \vDash \varphi$ and $t \vDash \neg\varphi$, then $s \cap t = \varnothing$.*

Proof:
By induction on the complexity of φ, which we may assume to be in negation normal form. We only show some cases; for the rest, see [12, Proposition 3.3.9].

If $\varphi = \psi \vee \chi$, we have $s = s_1 \cup s_2$ where $s_1 \vDash \psi$ and $s_2 \vDash \chi$, and $t \dashv \psi \vee \chi$, whence $t \dashv \psi$ and $t \dashv \chi$, and therefore $t \vDash \neg \psi$ and $t \vDash \neg \chi$. By the induction hypothesis, $s_1 \cap t = s_1 \cap t = \varnothing$, so that also $s \cap t = (s_1 \cup s_2) \cap t = (s_1 \cap t) \cup (s_2 \cap t) = \varnothing$.

If $\varphi = \Diamond \psi$, assume for contradiction that $s \cap t \neq \varnothing$. Let $w \in s \cap t$. By $s \vDash \Diamond \varphi$, there is a nonempty $t \subseteq R[w]$ such that $t \vDash \psi$. By $t \vDash \neg \Diamond \psi$, we have $t \dashv \Diamond \psi$ whence $R[w] \dashv \psi$ so that also $R[w] \vDash \neg \psi$. But then by the induction hypothesis, $t = t \cap R[w] = \varnothing$, contradicting the fact that t is nonempty. □

Now let $\varphi := p$ and $\psi := ((p \wedge \text{NE}) \vee (\neg p \wedge \text{NE}))$. Then $\varphi, \psi \vDash \bot\!\!\!\bot \vDash \bot$, so φ and ψ are $\bot$-incompatible. A Burgess theorem employing $\bot$-incompatibility would then produce a θ such that $\varphi \equiv \theta$ and $\psi \equiv \neg \theta$. But consider the teams $\{w_p\}$ and $\{w_p, w_{\neg p}\}$ (where $w_p \vDash p$, etc.). We have $\{w_p\} \vDash \varphi$, so $\{w_p\} \vDash \theta$; and $\{w_p, w_{\neg p}\} \vDash \psi$, so $\{w_p, w_{\neg p}\} \vDash \neg \theta$. Therefore, by Proposition 4.2.11, $\{w_p\} \cap \{w_p, w_{\neg p}\} = \{w_p\} = \varnothing$, a contradiction.

Drawing on Proposition 4.2.11, we can define an incompatibility notion more appropriate for BSML and $\text{BSML}^{\mathbb{V}}$:

4.2.12. DEFINITION (Ground-incompatibility). φ_0 and φ_1 are *ground-incompatible (G-I)* if $[s \vDash \varphi_0$ and $t \vDash \varphi_1]$ implies $s \cap t = \varnothing$ (equivalently, if $|\varphi_0|_{\mathsf{X}} \cap |\varphi_1|_{\mathsf{X}} = \varnothing = |\bot|_{\mathsf{X}}$ for any $\mathsf{X} \supseteq \mathsf{P}(\varphi_0) \cup \mathsf{P}(\varphi_1)$).

We will show in Section 4.2.3 that ground-incompatibility does yield Burgess theorems for BSML and $\text{BSML}^{\mathbb{V}}$. It is easy to see that ground-incompatibility is strictly stronger than $\bot$-incompatibility:

4.2.13. FACT. If φ and ψ are ground-incompatible, then they are $\bot$-incompatible, and the converse implication does not hold in general.

Proof:
Given ground-incompatible φ and ψ, if there is some s such that $s \vDash \varphi$ and $s \vDash \psi$, this implies $s = \varnothing$ whence $\varphi, \psi \vDash \bot$. Otherwise, if there is no such s, then $\varphi, \psi \vDash \bot\!\!\!\bot \vDash \bot$. Either way, $\varphi, \psi \vDash \bot$.

We have already seen that the converse does not hold (in the example directly below Proposition 4.2.11). □

The analogous fact clearly also holds in the first-order setting. This means that the Burgess theorems for D and IF also hold with respect to ground-incompatibility. In general, if a Burgess theorem holds with respect to a given incompatibility notion—a given pair property—it also holds with respect to all stronger notions: if $\mathscr{P}_1 \subseteq \mathscr{P}_2$, then $\mathscr{P}_{2\mathrm{L}} \subseteq \|\mathrm{L}\|^{\pm}$ implies $\mathscr{P}_{1\mathrm{L}} \subseteq \|\mathrm{L}\|^{\pm}$. In a downward-closed setting (such as that of D and IF), ground-incompatibility is in fact equivalent to $\bot$-incompatibility:

4.2.14. FACT. For downward-closed φ and ψ: φ and ψ are ground-incompatible iff they are $\bot$-incompatible.

Proof:
Given downward-closed and $\bot$-incompatible φ and ψ, if $s \vDash \varphi$ and $t \vDash \psi$, then by downward closure, $s \cap t \vDash \varphi \wedge \psi$, whence $s \cap t = \varnothing$. □

This, in turn, means that the converses of the Burgess theorems for D and IF also hold with respect to ground-incompatibility, and hence that D and IF are also `bicomplete` for ground-incompatible pairs. In general we clearly have that if the converse of a Burgess theorem holds with respect to a given notion, it also holds with respect to all weaker notions: if $\mathscr{P}_1 \subseteq \mathscr{P}_2$, then $\|\mathrm{L}\|^{\pm} \subseteq \mathscr{P}_{1\mathrm{L}}$ implies $\|\mathrm{L}\|^{\pm} \subseteq \mathscr{P}_{2\mathrm{L}}$.

So $\bot$-incompatibility and ground-incompatibility are equivalent in a downward-closed setting, but come apart when downward closure fails. Are there any natural interpretations of $\bot$-/ground-incompatibility that would also yield some intuitive understanding of this fact? In Section 4.4, we provide rough sketches of some possible sets of interpretations. It will be helpful, for the discussion in Section 4.4, to further dissect $\bot$-incompatibility into two more natural notions of incompatibility; this we do now.

4.2.15. DEFINITION ($\bot\!\!\!\bot$-incompatibility and NE-incompatibility).

(i) φ_0 and φ_1 are $\bot\!\!\!\bot$*-incompatible* ($\bot\!\!\!\bot$*-I*) if there is no team (including the empty team) that makes both φ_0 and φ_1 true (equivalently, if $\varphi_0, \varphi_1 \vDash \bot\!\!\!\bot$; equivalently, if $\|\varphi_0\|_{\mathsf{X}} \cap \|\varphi_1\|_{\mathsf{X}} = \varnothing = \|\bot\!\!\!\bot\|_{\mathsf{X}}$ for any $\mathsf{X} \supseteq \mathsf{P}(\varphi_0) \cup \mathsf{P}(\varphi_1)$).

(ii) *Incompatible in non-empty teams* or NE*-incompatible* (NE*-I*) if $[[s \vDash \varphi_0$ and $s \vDash \varphi_1]$ if and only if $s = \varnothing]$ (equivalently, if $\varphi_0, \varphi_1 \vDash \bot$ and $\varphi_0, \varphi_1 \nvDash \bot\!\!\!\bot$; equivalently, if $\|\varphi_0\|_{\mathsf{X}} \cap \|\varphi_1\|_{\mathsf{X}} = \|\bot\|_{\mathsf{X}}$ for any $\mathsf{X} \supseteq \mathsf{P}(\varphi_0) \cup \mathsf{P}(\varphi_1)$).

Clearly φ and ψ are $\bot$-incompatible if and only if they are $\bot\!\!\!\bot$-incompatible or they are NE-incompatible, and they are NE-incompatible if and only if they are $\bot$-incompatible and each of them has the empty team property. Therefore, in a setting with both downward closure and the empty team property such as that of D/IF, ground-incompatibility, $\bot$-incompatibility and NE-incompatibility are all equivalent (and so D and IF are also `bicomplete` for NE-incompatible pairs). We further have:

4.2.16. FACT. For convex φ and ψ, if φ and ψ are NE-incompatible, then they are ground-incompatible. NE-incompatibility does not imply ground-incompatibility in general.

Proof:
If φ and ψ are NE-incompatible, then each has the empty team property. The empty team property together with convexity implies downward closure, so the first claim now follows by Fact 4.2.14. For the second claim, consider the formulas $\neg p \vee\!\!\!\vee ((p \wedge \mathrm{NE}) \vee (\neg p \wedge \mathrm{NE}))$ and p. □

In particular, NE-incompatibility implies ground-incompatibility in BSML. The Burgess theorem with respect to ground-incompatibility in Section 4.2.3 will, therefore, also imply a Burgess theorem with respect to NE-incompatibility for BSML. The converse of this theorem clearly does not hold, since, for instance, for $\varphi := \mathrm{NE}$, φ and $\neg\varphi$ are not NE-incompatible.

4.2.3 The Theorems

In this section, we prove the Burgess- and bicompleteness theorems for BSML and BSML$^{\mathbb{V}}$.

Burgess' original result is essentially a corollary of Craig's interpolation for classical first-order logic. Similarly, our results follow from interpolation for classical modal logic ML (that is, interpolation for the smallest normal modal logic K—as with the other logics in this paper, we use 'ML' to refer both to the language as well as the logic consisting of all ML-formulas which are valid in all Kripke frames). This allows us to prove the following variant of interpolation for ground-incompatible formulas (c.f. the similar results for IF [75, pp. 59–60] and D [119, Theorem 6.7]; and note that restricted to ML-formulas, the following is equivalent to interpolation for ML/K).

4.2.17. THEOREM (Separation theorem). *If φ and ψ are ground-incompatible, then there is a classical formula $\gamma \in \mathrm{ML}$ such that $\mathsf{P}(\gamma) \subseteq \mathsf{P}(\varphi) \cap \mathsf{P}(\psi)$, $\varphi \vDash \gamma$, and $\psi \vDash \neg\gamma$.*

Proof:
Take $\alpha := \bigvee_{u \in \|\varphi\|_{\mathsf{P}(\varphi)}} \chi_u^{\mathsf{P}(\varphi),md(\varphi)}$. We show that $\varphi \vDash \alpha$. If there are no (M,t) such that $M,t \vDash \varphi$, we have $\alpha = \bot$ and $\varphi \equiv \perp\!\!\!\perp \vDash \bot$. Otherwise let $M,t \vDash \varphi$, where (M,t) is over $\mathsf{X} \supseteq \mathsf{P}(\varphi)$. This implies $M',t \vDash \varphi$, where M' is the restriction of M to $\mathsf{P}(\varphi)$, i.e., $(M',t) \in \|\varphi\|_{\mathsf{P}(\varphi)}$. Clearly $M,t \rightleftharpoons_{\mathsf{P}(\varphi)}^{md(\varphi)} M',t$ so $M,t \vDash \chi_{M',t}^{\mathsf{P}(\varphi),md(\varphi)}$ by Theorem 4.2.7. Then by the empty team property of ML, $M,t \vDash \alpha$. Similarly, for $\beta := \bigvee_{u \in \|\psi\|} \chi_u^{\mathsf{P}(\psi),md(\psi)}$, we have $\psi \vDash \beta$.

We now show that α and β are ground-incompatible, so let $M,s \vDash \alpha$ and $M,t \vDash \beta$, where M is over $\mathsf{X} \supseteq \mathsf{P}(\varphi) \cup \mathsf{P}(\psi)$, and assume for contradiction that $w \in s \cap t$. This implies that there are $(M_\varphi, u_\varphi) \in \|\varphi\|_{\mathsf{P}(\varphi)}$ and $w_\varphi \in u_\varphi$ such that $w \vDash \chi_{w_\varphi}^{\mathsf{P}(\varphi),md(\varphi)}$; and $(M_\psi, u_\psi) \in \|\psi\|_{\mathsf{P}(\psi)}$ and $w_\psi \in u_\psi$ such that $w \vDash \chi_{w_\psi}^{\mathsf{P}(\psi),md(\psi)}$. By Theorem 4.2.6, $w \rightleftharpoons_{md(\varphi)}^{\mathsf{P}(\varphi)} w_\varphi$ and $w \rightleftharpoons_{md(\psi)}^{\mathsf{P}(\psi)} w_\psi$. Let $(M_\varphi^*, u_\varphi^*)$ be a pointed model over X defined by taking (M_φ, u_φ) and assigning arbitrary valuations for all $p \in \mathsf{X} \setminus \mathsf{P}(\varphi)$, and similarly for (M_ψ^*, u_ψ^*). Let M' be the disjoint union $M \uplus M_\varphi^* \uplus M_\psi^*$, and let $u'_\varphi := (u_\varphi^* \setminus \{w_\varphi^*\}) \cup \{w\}$ and $u'_\psi := (u_\psi^* \setminus \{w_\psi^*\}) \cup \{w\}$. Clearly $M', u'_\varphi \rightleftharpoons_{md(\varphi)}^{\mathsf{P}(\varphi)} M_\varphi, u_\varphi$ and $M', u'_\psi \rightleftharpoons_{md(\psi)}^{\mathsf{P}(\psi)} M_\psi, u_\psi$, whence $u'_\varphi \vDash \varphi$ and $u'_\psi \vDash \psi$ by Theorem 4.2.7. But we have $w \in u'_\varphi \cap u'_\psi$, contradicting the assumption that φ and ψ are ground-incompatible.

We have shown $|\alpha|_{\mathsf{X}} \cap |\beta|_{\mathsf{X}} = \varnothing$; therefore also $[\![\alpha]\!]_{\mathsf{X}} \cap [\![\beta]\!]_{\mathsf{X}} = \varnothing$, and so we have $\alpha \vDash \neg\beta$. By interpolation for ML, there is an ML-formula γ such that $\alpha \vDash \gamma \vDash \neg\beta$ and

$\mathsf{P}(\gamma) = \mathsf{P}(\alpha) \cap \mathsf{P}(\beta) = \mathsf{P}(\varphi) \cap \mathsf{P}(\psi)$. Then also $\varphi \vDash \alpha \vDash \gamma$ and $\psi \vDash \beta \vDash \neg\gamma$. □

The modal complications in the proof above might distract from the structure of the argument. In the analogous proof for the propositional setting (see Section 4.3 for details on this setting), to show $|\alpha|_{\mathsf{X}} \cap |\beta|_{\mathsf{X}} = \varnothing$, one simply notes that $|\varphi|_{\mathsf{P}(\varphi)} = |\alpha|_{\mathsf{P}(\varphi)}$ and $|\psi|_{\mathsf{P}(\psi)} = |\beta|_{\mathsf{P}(\psi)}$, which implies $|\varphi|_{\mathsf{X}} = |\alpha|_{\mathsf{X}}$ and $|\psi|_{\mathsf{X}} = |\beta|_{\mathsf{X}}$, whence $|\alpha|_{\mathsf{X}} \cap |\beta|_{\mathsf{X}} = \varnothing$. (In the modal case, we have $|\varphi|_{\mathsf{P}(\varphi)} \subseteq |\alpha|_{\mathsf{P}(\varphi)}$, but $|\varphi|_{\mathsf{P}(\varphi)} \supseteq |\alpha|_{\mathsf{P}(\varphi)}$ might only hold modulo bisimulation.)

We require one more simple lemma.

4.2.18. LEMMA. *For any formula φ, there is a formula φ' such that $\varphi' \equiv \varphi$, and $\neg\varphi'$ has the empty team property.*

Proof:
Define φ' by putting φ in negation normal form and then replacing each occurrence of $\neg$NE with $\bot$. An easy induction shows that φ' is as required. □

We can now prove the Burgess theorems for BSML and $\text{BSML}^{\mathbb{V}}$. We give one version of the proof which is completely analogous to Burgess' and which works for both logics, and then provide an alternative proof for $\text{BSML}^{\mathbb{V}}$. The alternative proof does not make essential use of modal operators, and it is easy to see that its propositional analogue establishes the analogous result for the propositional fragment of $\text{BSML}^{\mathbb{V}}$. The propositional analogue of this result does not hold for the propositional fragment of BSML; see Section 4.3.3. We show both the Burgess theorems as well as their converses.

4.2.19. THEOREM (Burgess theorems for BSML and $\text{BSML}^{\mathbb{V}}$). *In both of the logics* BSML *and* $\text{BSML}^{\mathbb{V}}$*, the following are equivalent:*

(i) φ and ψ are ground-incompatible.

(ii) There is a formula θ such that $\varphi \equiv \theta$ and $\psi \equiv \neg\theta$ (and $\mathsf{P}(\theta) = \mathsf{P}(\varphi) \cup \mathsf{P}(\psi)$).

Proof:
(ii) $\Longrightarrow$ (i) by Proposition 4.2.11. (i) $\Longrightarrow$ (ii): Let $\theta_0 := \Diamond(\bot\!\!\!\bot \vee \neg\bot\!\!\!\bot)$ so that $\theta_0 \equiv \Diamond\bot\!\!\!\bot \equiv \bot$ and $\neg\theta_0 \equiv \Box\neg(\bot\!\!\!\bot \vee \neg\bot\!\!\!\bot) \equiv \Box(\neg\bot\!\!\!\bot \wedge \bot\!\!\!\bot) \equiv \Box\bot\!\!\!\bot \equiv \bot$. By Lemma 4.2.18, let φ' be such that $\varphi \equiv \varphi'$ and $\neg\varphi'$ has the empty team property, and similarly for ψ'. Let $\varphi_0 := \varphi' \vee \theta_0$ and $\psi_0 := \psi' \vee \theta_0$ so that $\varphi_0 \equiv \varphi$ and $\neg\varphi_0 \equiv \bot$ (clearly $\neg\varphi_0 \equiv \neg\varphi' \wedge \neg\theta_0 \vDash \bot$, and by the empty team property of $\neg\varphi'$, we also have $\bot \vDash \neg\varphi' \wedge \neg\theta_0$), and similarly for ψ_0. By Theorem 4.2.17, let $\gamma \in \text{ML}$ be such that $\varphi_0 \vDash \gamma$ and $\psi_0 \vDash \neg\gamma$. Finally, let $\theta := (\varphi_0 \wedge \gamma) \vee \neg\psi_0$. Then:

$$\begin{array}{lllllll}\theta & = & (\varphi_0 \wedge \gamma) \vee \neg\psi_0 & \equiv & \varphi_0 \vee \bot & \equiv & \varphi; \\ \neg\theta & = & \neg((\varphi_0 \wedge \gamma) \vee \neg\psi_0) & \equiv & (\neg\varphi_0 \vee \neg\gamma) \wedge \psi_0 & \equiv & (\bot \vee \neg\gamma) \wedge \psi_0 \quad \equiv \quad \psi.\end{array}$$

Alternatively, for $\varphi, \psi \in \mathrm{BSML}^{\mathbb{V}}$, by Theorem 4.2.17 we can let γ be such that $\varphi \vDash \gamma$ and $\psi \vDash \neg\gamma$. Let $\varphi_{⫫} := \neg\varphi \vee ⫫$ and $\psi_{⫫} := \neg\psi \vee ⫫$ so that $\varphi_{⫫} \equiv ⫫$ and $\neg\varphi_{⫫} \equiv \varphi$, and similarly for $\psi_{⫫}$. Finally, let $\theta := \neg(\varphi_{⫫} \mathbin{\mathbb{V}} \neg(\psi_{⫫} \mathbin{\mathbb{V}} \gamma))$. Then:

$$\begin{array}{lllllllll} \theta & = & \neg(\varphi_{⫫} \mathbin{\mathbb{V}} \neg(\psi_{⫫} \mathbin{\mathbb{V}} \gamma)) & \equiv & \neg\varphi_{⫫} \wedge (\psi_{⫫} \mathbin{\mathbb{V}} \gamma) & \equiv & \varphi \wedge \gamma & \equiv & \varphi; \\ \neg\theta & = & \neg\neg(\varphi_{⫫} \mathbin{\mathbb{V}} \neg(\psi_{⫫} \mathbin{\mathbb{V}} \gamma)) & \equiv & \varphi_{⫫} \mathbin{\mathbb{V}} (\neg\psi_{⫫} \wedge \neg\gamma) & \equiv & \psi \wedge \neg\gamma & \equiv & \psi. \end{array} \qquad \square$$

Burgess' proof (as well as those of Dechesne, Kontinen and Väänänen, and Mann) likewise makes use of a formula θ_0 such that $\theta_0 \equiv \bot \equiv \neg\theta_0$; let us briefly discuss the sentence used by Burgess (originally due to Väänänen) to allow for easy comparison with our θ_0. Formulated in IF, this sentence is $\theta_0 := \forall x(\exists y/\forall x)(x = y)$. This is equivalent to the Σ^1_1-sentence $\exists f \forall x(x = f(x))$, and its dual negation is equivalent to the FO-sentence $\exists x \forall y(x \neq y)$. Clearly, then, θ_0 is true in a model just in case the domain of the model has less than two elements; and $\neg\theta_0$ is never true. Therefore, ignoring empty models and models of size one, we have $\theta_0 \equiv \bot \equiv \neg\theta_0$.

We extend the definition of ground-incompatibility to pairs of properties in the obvious way: $\mathcal{P}$ and $\mathcal{Q}$ are *ground-incompatible* if $\dot{\bigcup} P \cap \dot{\bigcup} \mathcal{Q} = \varnothing$. By Theorems 4.2.8 and 4.2.19,

4.2.20. COROLLARY (Bicompleteness of BSML and $\mathrm{BSML}^{\mathbb{V}}$).

(i) BSML *is bicomplete for* $\{(\mathcal{P}, \mathcal{Q}) \mid \mathcal{P}, \mathcal{Q}$ *are convex, union closed, and invariant under bounded bisimulation;* $\mathcal{P}, \mathcal{Q}$ *are ground-incompatible*$\}$.

(ii) $\mathrm{BSML}^{\mathbb{V}}$ *is bicomplete for* $\{(\mathcal{P}, \mathcal{Q}) \mid \mathcal{P}, \mathcal{Q}$ *are invariant under bounded bisimulation;* $\mathcal{P}, \mathcal{Q}$ *are ground-incompatible*$\}$.

(iii) Each of BSML *and* $\mathrm{BSML}^{\mathbb{V}}$ *is* `bicomplete` *for ground-incompatible pairs.*

Note that a Burgess theorem can be viewed as a kind of strengthening of the relevant Separation theorem (4.2.17): given ground-incompatible φ and ψ, there is a θ such that not only $\varphi \vDash \theta$ and $\psi \vDash \neg\theta$, but also $\theta \vDash \varphi$ and $\neg\theta \vDash \psi$.[14] However, to get also these converse entailments, one must go from $\mathsf{P}(\theta) \subseteq \mathsf{P}(\varphi) \cap \mathsf{P}(\psi)$ to $\mathsf{P}(\theta) = \mathsf{P}(\varphi) \cup \mathsf{P}(\psi)$.

4.3 Burgess Theorems for Propositional Team Logics

In this section, we prove Burgess/bicompleteness theorems for Hawke and Steinert-Threlkeld's semantic expressivist logic for epistemic modals HS (Section 4.3.2); for PL(NE) and PL(NE, $\mathbb{V}$), the propositional fragments of BSML and $\mathrm{BSML}^{\mathbb{V}}$, respectively (Section 4.3.3); as well as for propositional dependence logic $\mathrm{PL}(=\!(\cdot))$ (Section 4.3.4). We also comment (Section 4.3.1) on the bicompleteness of team logics with

[14] I am grateful to Maria Aloni for pointing this out to me.

negations that do not exhibit failure of replacement: team-based classical propositional logic PL, propositional inquisitive logic InqB, and the extension PL($\sim$) of PL with the Boolean negation $\sim$ (also known as *propositional team logic*—see endnote 5), and give an example (in Section 4.3.3) of a logic bicomplete for all pairs.

4.3.1 Bicompleteness Without Failure of Replacement

The notion of bicompleteness can also be applied to logics in which replacement of equivalents under negation does not fail. We briefly consider three such logics by way of illustration.

4.3.1. DEFINITION (Syntax of PL, PL($\sim$), and InqB). The set of formulas of *classical propositional logic* PL is generated by:

$$\alpha ::= p \mid \bot \mid \neg\alpha \mid (\alpha \wedge \alpha) \mid (\alpha \vee \alpha)$$

where $p \in \mathsf{Prop}$, and Prop is a countable set of propositional variables, as before.

The set of formulas of *propositional logic with the Boolean negation* PL($\sim$) is generated by:

$$\varphi ::= p \mid \bot \mid \neg\alpha \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid \sim\varphi$$

where $p \in \mathsf{Prop}$ and $\alpha \in$ PL.

The set of formulas of *propositional inquisitive logic* InqB is generated by:

$$\varphi ::= p \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \to \varphi) \mid (\varphi \vee\!\!\!\vee \varphi)$$

where $p \in \mathsf{Prop}$.

We use α, β to refer exclusively to formulas of PL as well as the $\vee\!\!\!\vee$-free fragment of InqB (*classical formulas*). In InqB, let $\neg_i \varphi := \varphi \to \bot$ and $\bigvee\!\!\!\!\bigvee \varnothing := \bot$. As before, let $\top := \neg\bot$ or $\top := \neg_i \top$, and $\bigvee \varnothing := \bot$.

A (propositional) *team* is a set of valuations $s \subseteq 2^{\mathsf{X}}$, where $\mathsf{X} \subseteq \mathsf{Prop}$. We formulate the semantics of PL and PL($\sim$) in terms of support/anti-support conditions as in BSML. The conditions for the atoms and connectives of PL are the obvious propositional analogues of their modal support/anti-support conditions. The support condition for the Boolean negation is given by

$$s \vDash \sim\varphi :\iff s \nvDash \varphi.$$

The Boolean negation is not permitted to occur in the scope of the dual negation, so it does not have anti-support conditions.[15]. We only formulate support conditions for InqB; the support conditions for $\to$, the *intuitionistic implication*, are as follows:

$$s \vDash \varphi \to \psi :\iff \forall t \subseteq s : [t \vDash \varphi \implies t \vDash \psi].$$

[15]One could of course formulate anti-support conditions for $\sim$ (i.e., truth/support conditions for $\neg\sim\varphi$), but it is not clear what they should be. The anti-support conditions for the connectives in BSML are

The resulting support conditions for $\neg_i$, the *intuitionistic negation*, are:

$$s \vDash \neg_i \varphi \iff \forall t \subseteq s : [t \vDash \varphi \implies t = \varnothing].$$

We define the propositional analogues of the team semantic closure properties, team properties, expressive completeness, and the notations $[\![\alpha]\!]_X$, $\|\varphi\|_X$, $|\varphi|_X$, etc. in the obvious way. Note that now $\biguplus\mathcal{P} = \bigcup\mathcal{P}$. We let $\|\varphi\|_X^{\pm,\neg} := (\|\varphi\|_X, \|\neg\varphi\|_X)$, $\|\varphi\|_X^{\pm,\sim} := (\|\varphi\|_X, \|\sim\varphi\|_X)$, and $\|\varphi\|_X^{\pm,\neg_i} := (\|\varphi\|_X, \|\neg_i\varphi\|_X)$.

As with ML, PL-formulas are flat, and the analogue of Fact 4.2.4 holds for these formulas. PL is indeed, therefore, simply classical propositional logic with team semantics, and the negation $\neg$ is the classical negation with respect to worlds. PL$(\sim)$ additionally incorporates the Boolean negation, which is essentially the classical negation with respect to teams. InqB features a negation $\neg_i$ defined in terms of the intuitionistic implication $\rightarrow$. Note that, unlike with all other negations considered in this paper, double negation elimination is not valid for $\neg_i$ (as, for instance, $\neg_i\neg_i(p \mathbin{\vee\!\!\!\vee} q) \equiv p \vee q \nvDash p \mathbin{\vee\!\!\!\vee} q$). The $\mathbin{\vee\!\!\!\vee}$-free fragment of InqB is an alternative way of formulating team-based classical propositional logic; one can show that this fragment is flat and that an analogue of 4.2.4 holds. Formulas InqB are downward closed and have the empty team property.

The usual formulations of PL and PL$(\sim)$ (as well as those of the logics in Sections 4.3.3 and 4.3.4) in the literature do not include the dual negation; instead, these formulations feature what we can call the *restricted classical negation* $\neg_c$. This negation may only precede classical formulas, and its semantics are given by:

$$s \vDash \neg_c \alpha :\iff \forall w \in s : \{w\} \nvDash \alpha.$$

One can show that for all classical α (where α may also include $\neg, \neg_i, \neg_c$): $\neg\,\alpha \equiv \neg_i\,\alpha \equiv \neg_c\,\alpha$. These formulations are therefore expressively equivalent in the usual sense (of being able to express the same properties) to the present formulations; let us express this by writing, for instance, $\|\mathrm{PL}_\neg\| = \|\mathrm{PL}_{\neg_c}\|$. Since, in PL and PL$(\sim)$, the dual negation only appears in front of classical formulas, the $\neg_c$-formulations of these logics are also bi-equivalent to the present ones (with respect to both $\neg, \neg_c$ as well as $\sim, \sim$): $\|\mathrm{PL}_\neg\|^{\pm,\neg} = \|\mathrm{PL}_{\neg_c}\|^{\pm,\neg_c}$; $\|\mathrm{PL}(\sim)_\neg\|^{\pm,\neg} = \|\mathrm{PL}(\sim)_{\neg_c}\|^{\pm,\neg_c}$; and $\|\mathrm{PL}(\sim)_\neg\|^{\pm,\sim} = \|\mathrm{PL}(\sim)_{\neg_c}\|^{\pm,\sim}$. This second fact does not, however, hold for the logics we introduce in Sections 4.3.3 and 4.3.4—in these logics, the dual negation may occur in front of non-classical formulas, so the dual-negation formulations allow one to capture more pairs than the $\neg_c$-formulations. So for instance, for the propositional fragment of BSML, which we

motivated by empirical considerations, whereas those in all other logics we consider ultimately derive from the game-theoretic semantics for IF and D. In these semantics, the dual negation corresponds to an in-game move in which the players switch their verifier/falsifier roles. On the other hand, in the extensions of IF and D with $\sim$, $\sim\varphi$ is true just in case the verifier does not have a winning strategy for φ. The Boolean negation is evaluated from outside the game, so there is no way to interpret $\neg\sim\varphi$ in these semantics. For more discussion, see [75, 119]. See also [20], where the extension of D with $\sim$ is provided with game-theoretic semantics in an indirect way, and Section 4.3.2, featuring the logic HS, which has a bilateral negation and in which $\sim\varphi$ is uniformly definable (using the HS-definition of $\sim\varphi$ from Section 4.3.2, we get $s \dashv\sim\varphi \iff \forall t \subseteq s : t \vDash \varphi$)

denote by PL(NE), while we do have $\|\mathrm{PL}(\mathrm{NE})_{\neg}\| = \|\mathrm{PL}(\mathrm{NE})_{\neg_c}\|$, it is not the case that $\|\mathrm{PL}(\mathrm{NE})_{\neg}\|^{\pm,\neg} = \|\mathrm{PL}(\mathrm{NE})_{\neg_c}\|^{\pm,\neg_c}$. We prove Burgess theorems for the dual-negation formulations; no such theorems hold for the $\neg_c$-formulations (we will clarify below what we mean by 'Burgess theorems' in this context).

Returning to the properties of the logics PL, PL(~) and InqB, it can be shown:

4.3.2. THEOREM (Expressive completeness of PL, InqB, and PL(~)). *[41, 140]*

(i) PL *is expressively complete for* $\{\mathcal{P} \mid \mathcal{P} \text{ is flat}\}$.

(ii) InqB *is expressively complete for* $\{\mathcal{P} \mid \mathcal{P} \text{ is downward closed and has the empty team property}\}$.

(iii) PL(~) *is expressively complete for the class of all team properties.*

We introduce the following natural incompatibility notions corresponding to the two types of classical negation ¬ and ~:

4.3.3. DEFINITION (World-incompatibility and Team-incompatibility).

(i) φ_0 and φ_1 are *world-incompatible (W-I)* if $\{w\} \vDash \varphi_0 \iff \{w\} \nvDash \varphi_1$.

(ii) φ_0 and φ_1 are *team-incompatible (T-I)* if $s \vDash \varphi_0 \iff s \nvDash \varphi_1$ (equivalently, if $\|\varphi_0\|_{\mathsf{X}} = \|\top\|_{\mathsf{X}} \setminus \|\varphi_1\|_{\mathsf{X}}$ for all $\mathsf{X} \supseteq \mathsf{P}(\varphi_0) \cup \mathsf{P}(\varphi_1)$).

Observe that, e.g., p and $\neg p$ are world-incompatible but not team-incompatible (since $\varnothing \vDash p \wedge \neg p$), whereas p and $\sim p$ are both team-incompatible as well as world-incompatible. These notions yield bicompleteness theorems for PL and PL(~), respectively, but we also introduce a stronger notion for PL; this will allow us to differentiate between the pairs expressible (modulo expressive power) in PL from those expressible in propositional dependence logic (Section 4.3.4).

4.3.4. DEFINITION (Flat-incompatibility). φ_0 and φ_1 are *flat-incompatible (F-I)* if they are world-incompatible and flat; or, equivalently, if any of the following holds for all $\mathsf{X} \supseteq \mathsf{P}(\varphi) \cup \mathsf{P}(\psi)$ and $i \in \{0,1\}$:

(i) $\|\varphi_i\|_{\mathsf{X}} = \wp(|\top|_{\mathsf{X}} \setminus |\varphi_{1-i}|_{\mathsf{X}})$.

(ii) $\|\varphi_i\|_{\mathsf{X}} = \{s \subseteq 2^{\mathsf{X}} \mid t \vDash \varphi_{1-i} \implies s \cap t = \varnothing\}$.

(iii) $\|\varphi_i\|_{\mathsf{X}} = \bigcup\{\mathcal{P} \subseteq 2^{2^{\mathsf{X}}} \mid \mathcal{P} \text{ and } \|\varphi_{1-i}\|_{\mathsf{X}} \text{ are ground-incompatible}\}$.

Observe that p and $\neg p$ are flat-incompatible, whereas p and $\sim p$ are not.

As for the intuitionistic negation $\neg_i$, the truth conditions yield the following incompatibility notion:

4.3.5. DEFINITION ((One-sided) Down-set Incompatibility). φ_1 is a *down-set incompatibility (D-I)* of φ_0 if [$s \vDash \varphi_1$ iff [for all $t \subseteq s$, $t \vDash \varphi_0$ implies $t = \varnothing$]].

Unlike the other notions we consider, this is not symmetric—note, for instance, that $\neg p \wedge \neg q$ (which is equivalent to $\neg_i(p \vee\!\!\!\vee q)$) is a down-set incompatibility of $p \vee\!\!\!\vee q$, but not vice versa: we have that $[t \subseteq \{w_{p,\neg q}, w_{\neg p,q}\}$ and $t \vDash \neg p \wedge \neg q]$ implies $t = \varnothing$, but $\{w_{p,\neg q}, w_{\neg p,q}\} \nvDash p \vee\!\!\!\vee q$. This reflects the fact that double negation elimination is not sound for $\neg_i$. It is easy to check that D-I implies world-incompatibility, and that in a flat setting (such as that of PL), flat-incompatibility, D-I, and world-incompatibility are equivalent; we will therefore have that PL is `bicomplete` for each of the corresponding pair properties. That D-I implies W-I tracks the fact that in InqB the intuitionistic negation $\neg_i$ behaves classically with respect to worlds/singletons; for instance, double negation elimination holds with respect to singletons: $\{w\} \vDash \neg_i \neg_i \varphi \iff \{w\} \vDash \varphi$.

Extending these incompatibility notions to pairs of properties in the obvious way, it is easy to see that the following bicompleteness results hold:

4.3.6. COROLLARY (Bicompleteness of PL, InqB, and PL($\sim$)).

(i) PL *is bicomplete for F-I/W-I/D-I pairs of flat properties, and it is therefore also* `bicomplete` *for F-I/W-I/D-I pairs (with respect to* $\neg$*).*

(ii) InqB *is bicomplete for D-I pairs of downward closed properties with the empty team property, and it is therefore also* `bicomplete` *for D-I pairs (with respect to* $\neg_i$*).*

(iii) PL($\sim$) *is both bicomplete and* `bicomplete` *for T-I pairs (with respect to* $\sim$*).*

Inclusions such as $\{(\mathcal{P},\mathcal{Q}) \mid (\mathcal{P},\mathcal{Q})$ world-incompatible$\}_{\text{PL}} \subseteq \|\text{PL}\|^{\pm,\neg}$, which form part of the `bicompleteness` facts above, are of the same form as Burgess theorems. However, world-incompatibility relativized to pairs of flat properties is a pair property such that the first element of the pairs $(\mathcal{P},\mathcal{Q})$ in the property determines the second element: $\mathcal{Q} = \wp(\|\top\| \setminus \bigcup\mathcal{P})$. Given the converse inclusion, then, this inclusion is trivial: if $(\mathcal{P},\mathcal{Q}) \in \{(\mathcal{P},\mathcal{Q}) \mid (\mathcal{P},\mathcal{Q})$ world-incompatible$\}_{\text{PL}}$, then $\mathcal{Q} = \wp(\|\top\| \setminus \bigcup\mathcal{P})$ and $\mathcal{P},\mathcal{Q} \in \|\text{PL}\|$, whence $\mathcal{P} = \|\alpha\|$ and $\mathcal{Q} = \|\beta\|$ where $\alpha,\beta \in \text{PL}$. By the converse inclusion, $\|\alpha\|^{\pm} \in \{(\mathcal{P},\mathcal{Q}) \mid (\mathcal{P},\mathcal{Q})$ world-incompatible$\}_{\text{PL}}$, whence $\|\neg\alpha\| = \wp(\|\top\| \setminus \bigcup\|\alpha\|) = \wp(\|\top\| \setminus \bigcup\mathcal{P}) = \mathcal{Q} = \|\beta\|$, so $(\mathcal{P},\mathcal{Q}) = (\|\alpha\|,\|\beta\|) = (\|\alpha\|,\|\neg\alpha\|) \in \|\text{PL}\|^{\pm,\neg}$. The same holds for all incompatibility notions considered in this section. Symmetric reasoning shows that the Burgess inclusion is similarly trivial if the second element always determines the first. In order to keep the term ‘Burgess theorem’ meaningful, let us reserve it for a theorem which shows an inclusion of this sort for a property of pairs containing at least one pair whose first element does not determine the second, and at least one pair whose second element does not determine the first. Note that this allows for a Burgess theorem with respect to a property of pairs such that in each pair, either the first element determines the second or the second the first (as long as the first does not always determine the second, etc.). We will see such a theorem in Section 4.3.2.

4.3.2 Hawke and Steinert-Threlkeld's Logic

Hawke and Steinert-Threlkeld's Logic HS [67] is essentially PL together with a operator $\diamondslash$ intended to represent epistemic modals such as the 'might' in the sentence 'It might be raining' (cf. the operator $\blacklozenge$ in Section 4.4.2). The anti-support clauses for the conjunction and disjunction are different from those defined in Section 4.3.1; we will therefore use distinct symbols for the conjunction and disjunction of HS to avoid confusion.

4.3.7. DEFINITION (Syntax of HS). The set of formulas of HS is generated by:

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid (\varphi \mathbin{\dot\wedge} \varphi) \mid (\varphi \mathbin{\dot\vee} \varphi) \mid \diamondslash\varphi$$

where $p \in \mathsf{Prop}$.

As before, let $\top := \neg\bot$. The support clauses for p, $\bot$, and $\neg\varphi$ are as before; the support clauses for $\dot\wedge$ and $\dot\vee$ are the same as those for $\wedge$ and $\vee$, respectively. For $\diamondslash$, we have:

$$s \vDash \diamondslash\varphi :\iff s \not\dashv \varphi.$$

The anti-support clause for $\neg\varphi$ is as before. For all formulas φ whose main connective is not $\neg$:

$$s \dashv \varphi :\iff \forall t \subseteq s : [t \vDash \varphi \implies t = \varnothing].$$

Observe that this anti-support clause is equivalent to our usual one if φ is p or $\bot$.

We have made two alterations to the system as presented in [67]: first, we have formulated the semantics in terms of support and anti-support conditions (Hawke and Steinert-Therlkeld's semantics are essentially bilateral, but not formulated in terms of support/anti-support conditions). Second, we have added the constant $\bot$. This addition does not substantially change the logic—$\bot$ is strongly uniformly definable in the $\bot$-free fragment using any given $p \in \mathsf{Prop}$ in that $\bot \equiv^{\pm} p \mathbin{\dot\wedge} \neg p$—but it allows us to secure a simple expressive completeness theorem for the logic (by giving us the means to express all team properties over the empty set of propositional variables).

Many of the propositional analogues of the results in Section 4.2.1 hold for this logic; we note in particular that the natural analogue of Fact 4.2.4 holds for the $\diamondslash$-free fragment, meaning that this fragment is expressively equivalent (in the usual positive sense) to PL. The expressive completeness theorem mentioned above follows from this together with the fact that we can uniformly define $\sim$ in HS:

4.3.8. THEOREM (Expressive completeness of HS). HS *is expressively complete for the class of all team properties.*

Proof:
Observe that $s \vDash \diamondslash\neg\varphi \iff s \not\dashv \neg\varphi \iff s \nvDash \varphi \iff s \vDash \sim\varphi$. The result now follows from Theorem 4.3.2 (iii) together with the fact that the $\diamondslash$-free fragment of HS is expressively equivalent to PL. □

Replacement of equivalents fails in negated contexts; note, for instance, that $\neg p \equiv \neg \diamondsuit p$, but $\neg\neg p \equiv p \not\equiv \diamondsuit p \equiv \neg\neg\diamondsuit p$. The incompatibility notion appropriate for HS is a kind of weaker two-sided version of down-set incompatibility:

4.3.9. DEFINITION ((Either-sided) Down-set Incompatibility). φ_0 and φ_1 are *down-set incompatible (on either side) (E-D-I)* if φ_0 is a down-set incompatibility of φ_1 or φ_1 is a down-set incompatibility of φ_0.

4.3.10. PROPOSITION. *For any $\varphi \in \mathrm{HS}$, φ and $\neg\varphi$ are down-set incompatible (on either side).*

Proof:
Case 1: φ is of the form $\neg^n \psi$, where $\neg^n$ denotes a string of symbols $\neg$ of length n, and n is even (possibly 0); and the main connective of ψ is not $\neg$. Then $\varphi \equiv \psi$ and $\neg\varphi \equiv \neg\psi$. We have $s \vDash \neg\psi$ iff $\forall t \subseteq s : [t \vDash \psi \implies t = \varnothing]$, and so $\neg\psi$ is a down-set incompatibility of ψ; similarly $\neg\varphi$ is a down-set incompatibility of φ.

Case 2: φ is of the form $\neg^n \psi$, where $\neg^n$ and ψ are as above, but n is odd. Then $\varphi \equiv \neg\psi$ and $\neg\varphi \equiv \psi$. Similarly to the above, we now have that φ is a down-set incompatibility of $\neg\varphi$. □

Note that while the first element of a D-I pair determines the second, it is not the case that the first element of an E-D-I pair always determines the second (recall our example from above: $\neg p \equiv \neg\diamondsuit p$, but $\neg\neg p \equiv p \not\equiv \diamondsuit p \equiv \neg\neg\diamondsuit p$, so an E-D-I pair with first element $\|\neg p\|$ does not determine the second element) or that the second element always determines the first. It is, however, the case that for any given E-D-I pair, either the first element determines the second, or vice versa.

Clearly D-I implies E-D-I, and it is also easy to see that, like D-I, E-D-I implies W-I. For other implications, see Figure 4.2. Note that in a flat context, E-D-I is equivalent to F-I as well as D-I, so that PL is also `bicomplete` for E-D-I pairs.

We now show Burgess and bicompleteness theorems for HS using E-D-I. This time our proof will not be analogous to Burgess'; rather, the result follows easily from the basic properties of the negation.

4.3.11. THEOREM (Burgess theorem for HS). *In* HS*, the following are equivalent:*

(i) φ and ψ are down-set incompatible (on either side).

(ii) There is a formula θ such that $\varphi \equiv \theta$ and $\psi \equiv \neg\theta$ (and $\mathsf{P}(\theta) \subseteq \mathsf{P}(\varphi) \cup \mathsf{P}(\psi)$).

Proof:
(ii) $\implies$ (i) by Proposition 4.3.10. (i) $\implies$ (ii): If ψ is D-I of φ, let $\theta := \varphi \wedge \top$. Then $\theta \equiv \varphi$ and

$$s \vDash \neg\theta \iff \forall t \subseteq s : [t \vDash \varphi \wedge \top \implies t = \varnothing] \iff \forall t \subseteq s : [t \vDash \varphi \implies t = \varnothing],$$

so that $\neg\theta \equiv \psi$. Else if φ is D-I of ψ, let $\theta := \neg(\psi \wedge \top)$. Similarly to the previous case, $\theta \equiv \varphi$ and $\neg\theta \equiv \psi$. □

4.3.12. Corollary (Bicompleteness of HS). HS *is both bicomplete and* `bicomplete` *for E-D-I pairs.*

Note that by Corollaries 4.3.6 (iii) and 4.3.12, we have $\|\mathrm{PL}(\sim)\| = \|\mathrm{HS}\|$, whereas $\|\mathrm{PL}(\sim)\|^{\pm,\sim} \neq \|\mathrm{HS}\|^{\pm,\neg}$.

Interestingly, Hawke and Steinert-Threlkeld also consider a variant of HS which uses $\wedge$ and $\vee$ instead of $\mathbin{\bar\wedge}$ and $\mathbin{\bar\vee}$, but they dismiss it since the pairs it produces are not $\bot$-incompatible (among other reasons): "the altered system egregiously allows for the assertibility of contradictions" [67]. (HS allows for a formula and its negation to be simultaneously true in the empty team, but in the variant this also holds for nonempty teams. An example from [67]: $\neg(\Diamond p \wedge \Diamond \neg p) \wedge (\Diamond p \wedge \Diamond \neg p)$, which is true in any team s such that $s \not\models p$ and $s \not\models \neg p$.) The bicompleteness of this variant is an open problem.

4.3.3 The Propositional Fragments of BSML and $\mathrm{BSML}^{\mathbb{V}}$

Let $\mathrm{PL}(\mathrm{NE})$ (the propositional fragment of BSML) and $\mathrm{PL}(\mathrm{NE}, \mathbb{V})$ (the propositional fragment of $\mathrm{BSML}^{\mathbb{V}}$) be the extension of PL with NE, and with NE and $\mathbb{V}$, respectively. In this section, we show that while $\mathrm{PL}(\mathrm{NE}, \mathbb{V})$ is, like BSML and $\mathrm{BSML}^{\mathbb{V}}$, `bicomplete` for ground-incompatible pairs, $\mathrm{PL}(\mathrm{NE})$ is `bicomplete` for pairs conforming to a stronger notion of incompatibility. We also note that a variant of the logic $\mathrm{PL}(\mathrm{NE}, \mathbb{V})$ is both bicomplete and `bicomplete` for all pairs.

Define, as before, $\bot\!\!\!\bot := \bot \wedge \mathrm{NE}$ and $\bigvee\varnothing := \bot\!\!\!\bot$. The support and anti-support conditions are again the obvious propositional analogues of the modal ones. The propositional analogues of all results in Section 4.2.1 hold; we note in particular that each formula is equivalent to one in negation normal form, that replacement holds in positive contexts, and that we have the following propositional analogue of Theorem 4.2.8:

4.3.13. Theorem (Expressive completeness of $\mathrm{PL}(\mathrm{NE})$ and $\mathrm{PL}(\mathrm{NE}, \mathbb{V})$). *[16, 140]*

(i) $\mathrm{PL}(\mathrm{NE})$ *is expressively complete for* $\{\mathcal{P} \mid \mathcal{P}$ *is convex and union closed*$\}$.

(ii) $\mathrm{PL}(\mathrm{NE}, \mathbb{V})$ *is expressively complete for the class of all team properties.*

As already noted in Section 4.2.3, the $\mathrm{BSML}^{\mathbb{V}}$ Burgess theorem makes no essential use of the modal operators, and so the propositional analogues of Theorem 4.2.17 and Theorem 4.2.19 (using the second proof given for this theorem) give us:

4.3.14. Theorem (Burgess theorem for $\mathrm{PL}(\mathrm{NE}, \mathbb{V})$). *In* $\mathrm{PL}(\mathrm{NE}, \mathbb{V})$*, the following are equivalent:*

(i) φ *and* ψ *are ground-incompatible.*

(ii) There is a formula θ *such that* $\varphi \equiv \theta$ *and* $\psi \equiv \neg\theta$ *(and* $\mathsf{P}(\theta) = \mathsf{P}(\varphi) \cup \mathsf{P}(\psi)$*).*

4.3.15. Corollary (Bicompleteness of $\mathrm{PL}(\mathrm{NE}, \mathbb{V})$). $\mathrm{PL}(\mathrm{NE}, \mathbb{V})$ *is both bicomplete and* `bicomplete` *for ground-incompatible pairs.*

However, the propositional analogue of Theorem 4.2.19 does not hold for PL(NE). To show why, we recursively define the *flattening*[16] $\varphi^f \in$ PL of $\varphi \in$ PL(NE) as follows: $p^f := p$; $\bot^f := \bot$; $\text{NE}^f := \top$; $(\neg\varphi)^f = \neg\varphi^f$; $(\varphi\wedge\psi)^f := \varphi^f\wedge\psi^f$; and $(\varphi\vee\psi)^f := \varphi^f\vee\psi^f$. Note that since φ^f is classical, we have $|\varphi^f| = [\![\varphi^f]\!]$. We require the following lemma for the sequel:

4.3.16. LEMMA. *Let φ' be the negation normal form of $\varphi \in$ PL(NE). Then $\varphi^f \equiv \varphi'^f$.*

Proof:
By induction on the complexity of φ. The cases for the atoms, $\wedge$, and $\vee$ are all immediate. The cases for negated atoms are likewise immediate.

If $\varphi = \neg\neg\psi$, then $\varphi' = \psi'$. By the induction hypothesis, $\psi^f \equiv \psi'^f$. We then have $\varphi^f = \neg\neg\psi^f \equiv \psi^f \equiv \psi'^f = \varphi'^f$.

If $\varphi = \neg(\psi\wedge\chi)$, then $\varphi' = (\neg\psi)'\vee(\neg\chi)'$. By the induction hypothesis, $(\neg\psi)^f \equiv (\neg\psi)'^f$ and $(\neg\chi)^f \equiv (\neg\chi)'^f$. We then have

$$\varphi^f = \neg(\psi^f\wedge\chi^f) \equiv \neg\psi^f\vee\neg\chi^f = (\neg\psi)^f\vee(\neg\chi)^f \equiv (\neg\psi)'^f\vee(\neg\chi)'^f = \varphi'^f.$$

The case for $\varphi = \neg(\psi\vee\chi)$ is similar to that for $\varphi = \neg(\psi\wedge\chi)$. □

Using which we can show:

4.3.17. LEMMA. *For all $\varphi \in$ PL(NE) and all $\mathsf{X} \supseteq \mathsf{P}(\varphi)$, either $|\varphi|_{\mathsf{X}} = |\varphi^f|_{\mathsf{X}}$, or $\varphi \equiv \perp\!\!\!\perp$.*

Proof:
By induction on the complexity of φ, which we may assume to be in negation normal form by Lemma 4.3.16. The base cases—those for atoms and negated atoms—are obvious; note in particular that $|\text{NE}| = |\top| = |\text{NE}^f|$ since for any w, $w \in \{w\} \vDash \text{NE}$, and $|\neg\text{NE}| = |\bot| = |\neg\top| = |(\neg\text{NE})^f|$.

Consider $\varphi = \psi\wedge\chi$. If either $\psi \equiv \perp\!\!\!\perp$ or $\chi \equiv \perp\!\!\!\perp$, then $\psi\wedge\chi \equiv \perp\!\!\!\perp$. Otherwise we have both $|\psi| = |\psi^f|$ and $|\chi| = |\chi^f|$. If there is no s such that $s \vDash \psi\wedge\chi$, then $\psi\wedge\chi \equiv \perp\!\!\!\perp$, and we are done; we may therefore assume there is s with $s \vDash \psi\wedge\chi$. We now show $|\psi\wedge\chi| = |\psi|\cap|\chi|$. $|\psi\wedge\chi| \subseteq |\psi|\cap|\chi|$ is immediate; for the converse inclusion, let $w \in |\psi|\cap|\chi|$. Then $w \in t \vDash \psi$ and $w \in u \vDash \chi$. Let s be such that $s \vDash \psi\wedge\chi$. By union closure, $s\cup t \vDash \psi$ and $s\cup u \vDash \chi$. Since $s \subseteq s\cup\{w\} \subseteq s\cup t$ and $s \subseteq s\cup\{w\} \subseteq s\cup u$, by convexity we have $s\cup\{w\} \vDash \psi$ and $s\cup\{w\} \vDash \chi$, so $w \in |\psi\wedge\chi|$. We then have

$$|\psi\wedge\chi| = |\psi|\cap|\chi| = |\psi^f|\cap|\chi^f| = [\![\psi^f]\!]\cap[\![\chi^f]\!] = [\![\psi^f\wedge\chi^f]\!] = [\![(\psi\wedge\chi)^f]\!] = |(\psi\wedge\chi)^f|.$$

Now consider $\varphi = \psi\vee\chi$. If either $\psi \equiv \perp\!\!\!\perp$ or $\chi \equiv \perp\!\!\!\perp$, then $\psi\vee\chi \equiv \perp\!\!\!\perp$. Otherwise we have both $|\psi| = |\psi^f|$ and $|\chi| = |\chi^f|$. We show $|\psi\vee\chi| = |\psi|\cup|\chi|$. For $|\psi\vee\chi| \subseteq |\psi|\cup|\chi|$, if $w \in |\psi\vee\chi|$, then $w \in s \vDash \psi\vee\chi$. We have that $s = s_1\cup s_2$ where $s_1 \vDash \psi$ and $s_2 \vDash \chi$, so $w \in s_1$

[16] The name 'flattening' is taken from Väänänen, who defines an analogous notion for D in [119, p. 42]. See also endnote 10.

or $w \in s_2$. Either way, $w \in |\psi| \cup |\chi|$. To show the converse inclusion, let $w \in |\psi| \cup |\chi|$. Then $w \in s$ where $s \vDash \psi$ or $s \vDash \chi$; assume without loss of generality that $s \vDash \psi$. Since $\chi \not\equiv \bot\!\!\!\bot$, there is t such that $t \vDash \chi$. Then $s \cup t \vDash \psi \vee \chi$, so $s \in |\psi \vee \chi|$. We then have:

$$|\psi \vee \chi| = |\psi| \cup |\chi| = |\psi^f| \cup |\chi^f| = [\![\psi^f]\!] \cup [\![\chi^f]\!] = [\![\psi^f \vee \chi^f]\!] = [\![(\psi \vee \chi)^f]\!] = |(\psi \vee \chi)^f|.$$

□

Note that the BSML-analogue of Lemma 4.3.17 fails, since, for instance, extending the definition of flattening in the obvious way, we would have $|\Box \bot\!\!\!\bot| = |\bot|$ and $\Box \bot\!\!\!\bot \not\equiv \bot\!\!\!\bot$, but $|(\Box \bot\!\!\!\bot)^f| = |\Box(\bot \wedge \top)| = |\Box \bot| \neq |\bot|$.

Now assume for contradiction that the propositional analogue of Theorem 4.2.19 holds for PL(NE), i.e., that for any ground-incompatible φ and ψ, there is a θ such that $\varphi \equiv \theta$ and $\psi \equiv \neg \theta$. Let $\varphi := p$ and $\psi := \neg p \wedge q$. Clearly these are ground-incompatible, so there is a θ such that $\varphi \equiv \theta$ and $\psi \equiv \neg \theta$. By Lemma 4.3.17,

$$|\varphi| = |\theta| = |\theta^f| = [\![\theta^f]\!] = [\![\top]\!] \setminus [\![\neg \theta^f]\!] = |\top| \setminus |\neg \theta^f| = |\top| \setminus |(\neg \theta)^f| = |\top| \setminus |\neg \theta| = |\top| \setminus |\psi|.$$

But this leads to contradiction, since, for instance, $w_{\neg p, \neg q} \notin |\varphi|$ and $w_{\neg p, \neg q} \notin |\psi|$.

PL(NE)-formulas and their negations conform, instead, to the following incompatibility notion.

4.3.18. DEFINITION (Ground-complementariness (modulo $\bot\!\!\!\bot$)).

(i) φ_0 and φ_1 are *ground-complementary* if $|\varphi_i|_{\mathsf{X}} = |\top|_{\mathsf{X}} \setminus |\varphi_{1-i}|_{\mathsf{X}}$ for all $i \in \{0,1\}$ and $\mathsf{X} \supseteq \mathsf{P}(\varphi) \cup \mathsf{P}(\psi)$.

(ii) φ_0 and φ_1 are *ground-complementary modulo* $\bot\!\!\!\bot$ if

(a) they are ground-complementary, (b) $\varphi_0 \equiv \bot\!\!\!\bot$, or (c) $\varphi_1 \equiv \bot\!\!\!\bot$.

Similarly, let $\mathcal{P}$ and $\mathcal{Q}$ be *ground-complementary (modulo* $\bot\!\!\!\bot$*)* if $\bigcup \mathcal{P} = |\top| \setminus \bigcup \mathcal{Q}$ (or $\mathcal{P} = \varnothing$ or $\mathcal{Q} = \varnothing$). We have:

4.3.19. PROPOSITION. *For any $\varphi \in$ PL(NE), φ and $\neg \varphi$ are ground-complementary modulo* $\bot\!\!\!\bot$.

Proof:
If $\varphi \equiv \bot\!\!\!\bot$ or $\neg \varphi \equiv \bot\!\!\!\bot$, we are done. Otherwise by Lemma 4.3.17,

$$|\varphi| = |\varphi^f| = [\![\varphi^f]\!] = [\![\top]\!] \setminus [\![\neg \varphi^f]\!] = |\top| \setminus |\neg \varphi^f| = |\top| \setminus |(\neg \varphi)^f| = |\top| \setminus |\neg \varphi|.$$ □

Note that we have the following connection between ground-complementariness and world-incompatibility:

4.3.20. FACT. For downward-closed φ, ψ: φ and ψ are world-incompatible if and only if they are ground-complementary. Neither implication holds for all formulas.

Proof:
Let φ and ψ be world-incompatible, and let $w \in |\varphi|$. Assume for contradiction that $w \in |\psi|$. We then have that $w \in s \vDash \varphi$ and $w \in t \vDash \psi$, whence by downward closure, $\{w\} \vDash \varphi$ and $\{w\} \vDash \psi$, contradicting world-incompatibility. So $|\varphi| \subseteq |\top| \setminus |\psi|$. Now let $w \in |\top| \setminus |\psi|$. By world-incompatibility, either $\{w\} \vDash \varphi$ or $\{w\} \vDash \psi$. The latter clearly implies $w \in |\psi|$, so we must have $\{w\} \vDash \varphi$. Therefore, $|\top| \setminus |\psi| \subseteq |\varphi|$ and so $|\varphi| = |\top| \setminus |\psi|$.

Conversely, let φ and ψ be ground-complementary. Let $\{w\} \vDash \varphi$ and assume for contradiction that $\{w\} \vDash \psi$. Then $w \in |\varphi| \cap |\psi|$, contradicting ground-complementariness. So $\{w\} \vDash \varphi \implies \{w\} \nvDash \psi$. Now let $\{w\} \nvDash \psi$. Assume for contradiction that $w \in |\psi|$. Then $w \in t \vDash \psi$, so by downward closure, $\{w\} \vDash \psi$, contradicting our assumption. So $w \notin |\psi|$, whence $w \in |\varphi|$ by ground-complementariness.

To see why world-incompatibility does not imply ground-complementariness in general, consider, for instance, the pair $\top$ and $(p \wedge \text{NE}) \vee (\neg p \wedge \text{NE})$, or the pair q and $\neg q \mathbin{\veebar} ((p \wedge \text{NE}) \vee (\neg p \wedge \text{NE}))$.

To see why ground-complementariness does not imply world-incompatibility in general, consider, for instance, the pair $\bot$ and $(p \wedge \text{NE}) \vee (\neg p \wedge \text{NE})$, or the pair $q \wedge ((p \wedge \text{NE}) \vee (\neg p \wedge \text{NE}))$ and $\neg q \wedge ((p \wedge \text{NE}) \vee (\neg p \wedge \text{NE}))$. □

Furthermore, for formulas with the empty team property, G-C modulo $\perp\!\!\!\perp$ and G-C are clearly equivalent. Therefore, given the fact above, G-C modulo $\perp\!\!\!\perp$, G-C, and W-I are all equivalent for downward-closed properties with the empty team property. In a flat context, each of these is also equivalent to each of F-I, D-I, and E-D-I, so that PL is also `bicomplete` for G-C/G-C mod $\perp\!\!\!\perp$ pairs.

We now show a Burgess theorem for PL(NE) using ground-complementariness modulo $\perp\!\!\!\perp$. As in the preceding section, our proof will not be analogous to Burgess'; this time we derive our result by making a small modification to the proof of Theorem 4.3.13 (i). This proof makes use of the propositional analogues $\chi_w^{\mathsf{X}} := \bigwedge\{p \mid p \in \mathsf{X}, w \vDash p\} \wedge \bigwedge\{\neg p \mid p \in \mathsf{X}, w \nvDash p\}$ and $\chi_s^{\mathsf{X}} := \bigvee_{w \in s} \chi_w^{\mathsf{X}}$ of Hintikka formulas. As with Hintikka formulas, we have $w' \vDash \chi_w^{\mathsf{X}} \iff w' \upharpoonright \mathsf{X} = w \upharpoonright X$ (so that $w' \vDash \chi_w^{\mathsf{X}} \iff w' = w$ if $w, w' \in 2^{\mathsf{X}}$) and $s' \vDash \chi_s^{\mathsf{X}} \iff s' \upharpoonright \mathsf{X} = t$ for some $t \subseteq s \upharpoonright X$, where $s \upharpoonright \mathsf{X} := \{v \upharpoonright \mathsf{X} \mid v \in s\}$, etc. (so that $s' \vDash \chi_s^{\mathsf{X}} \iff s' \subseteq s$ if $s, s' \subseteq 2^{\mathsf{X}}$). Given a property $\mathcal{P} = \{t_1, \ldots, t_n\}$, define

$$\delta_{\mathcal{P}}^{\mathsf{X}} := \bigwedge_{w_1 \in t_1, \ldots, w_n \in t_n} (((\chi_{w_1}^{\mathsf{X}} \vee \ldots \vee \chi_{w_n}^{\mathsf{X}}) \wedge \text{NE}) \vee \top).$$

Theorem 4.3.13 (i) essentially follows from:

4.3.21. PROPOSITION. *[16] For any nonempty, convex, and union-closed property* $\mathcal{P} = \{t_1, \ldots, t_n\}$ *over* $\mathsf{X} \subseteq \mathsf{Prop}$ *(where* X *is finite),* $\mathcal{P} = \|\bigvee_{t \in \mathcal{P}} \chi_t^{\mathsf{X}} \wedge \delta_{\mathcal{P}}^{\mathsf{X}}\|_{\mathsf{X}}$.

Using which we get:

4.3.22. PROPOSITION. *For any nonempty, convex, and union-closed properties* $\mathcal{P} = \{t_1, \ldots, t_n\}$, $\mathcal{Q} = \{s_1, \ldots, s_m\}$ *over* $\mathsf{X} \subseteq \mathsf{Prop}$ *(where* X *is finite) such that* $\mathcal{P}, \mathcal{Q}$ *are ground-complementary modulo* $\perp\!\!\!\perp$, $(\mathcal{P}, \mathcal{Q}) = \|(\bigvee_{t \in \mathcal{P}} \chi_t^{\mathsf{X}} \wedge \delta_{\mathcal{P}}^{\mathsf{X}}) \vee \neg \delta_{\mathcal{Q}}^{\mathsf{X}}\|_{\mathsf{X}}^{\pm}$.

Proof:
It is easy to check that $\neg\,\delta_{\mathcal{Q}} \equiv \bot$, whence

$$(\bigvee_{t\in\mathcal{P}} \chi_t \wedge \delta_{\mathcal{P}}) \vee \neg\,\delta_{\mathcal{Q}} \equiv (\bigvee_{t\in\mathcal{P}} \chi_t \wedge \delta_{\mathcal{P}}) \vee \bot \equiv \bigvee_{t\in\mathcal{P}} \chi_t \wedge \delta_{\mathcal{P}},$$

and therefore, by Proposition 4.3.21, $\|\bigvee_{t\in\mathcal{P}} \chi_t \wedge \delta_{\mathcal{P}} \vee \neg\,\delta_{\mathcal{Q}}\| = \mathcal{P}$. It remains to show that $\|\neg((\bigvee_{t\in\mathcal{P}} \chi_t \wedge \delta_{\mathcal{P}}) \vee \neg\,\delta_{\mathcal{Q}})\| = \mathcal{Q}$. We have:

$$\neg((\bigvee_{t\in\mathcal{P}} \chi_t \wedge \delta_{\mathcal{P}}) \vee \neg\,\delta_{\mathcal{Q}}) \equiv (\neg\bigvee_{t\in\mathcal{P}} \chi_t \vee \neg\,\delta_{\mathcal{P}}) \wedge \delta_{\mathcal{Q}} \equiv (\neg\bigvee_{t\in\mathcal{P}} \chi_t \vee \bot) \wedge \delta_{\mathcal{Q}} \equiv \neg\bigvee_{t\in\mathcal{P}} \chi_t \wedge \delta_{\mathcal{Q}}.$$

Observe that $\bigcup\mathcal{P} = |\bigvee_{t\in\mathcal{P}} \chi_t| = [\![\bigvee_{t\in\mathcal{P}} \chi_t]\!]$, and similarly for $\mathcal{Q}$. Since $\mathcal{P}$ and $\mathcal{Q}$ are ground-complementary modulo $\perp\!\!\!\perp$, and since they are nonempty, $\bigcup\mathcal{Q} = |\top| \setminus \bigcup\mathcal{P}$. We therefore have:

$$[\![\neg\bigvee_{t\in\mathcal{P}} \chi_t]\!] = [\![\top]\!] \setminus [\![\bigvee_{t\in\mathcal{P}} \chi_t]\!] = [\![\top]\!] \setminus \bigcup\mathcal{P} = \bigcup\mathcal{Q} = [\![\bigvee_{s\in\mathcal{Q}} \chi_s]\!],$$

from which it follows by flatness that $\|\neg\bigvee_{t\in\mathcal{P}} \chi_t\| = \|\bigvee_{s\in\mathcal{Q}} \chi_s\|$. We therefore have

$$\neg((\bigvee_{t\in\mathcal{P}} \chi_t \wedge \delta_{\mathcal{P}}) \vee \neg\,\delta_{\mathcal{Q}}) \equiv \bigvee_{s\in\mathcal{Q}} \chi_s \wedge \delta_{\mathcal{Q}},$$

so that by Proposition 4.3.21, $\|\neg((\bigvee_{t\in\mathcal{P}} \chi_t \wedge \delta_{\mathcal{P}}) \vee \neg\,\delta_{\mathcal{Q}})\| = \mathcal{Q}$. □

The Burgess and bicompleteness theorems now follow easily:

4.3.23. THEOREM (Burgess theorem for PL(NE)). *In* PL(NE), *the following are equivalent:*

(i) φ *and* ψ *are ground-complementary modulo* $\perp\!\!\!\perp$.

(ii) There is a formula θ *such that* $\varphi \equiv \theta$ *and* $\psi \equiv \neg\,\theta$ *(and* $\mathsf{P}(\theta) = \mathsf{P}(\varphi) \cup \mathsf{P}(\psi)$*).*

Proof:
(ii) $\Longrightarrow$ (i) by Proposition 4.3.19. For (i) $\Longrightarrow$ (ii), if $\psi \equiv \perp\!\!\!\perp$, let $\theta := \varphi \wedge \neg\perp\!\!\!\perp$. Then $\theta \equiv \varphi \wedge (\neg\bot \vee \neg\mathrm{NE}) \equiv \varphi \wedge (\top \vee \bot) \equiv \varphi$ and $\neg\,\theta \equiv \neg\,\varphi \vee \perp\!\!\!\perp \equiv \perp\!\!\!\perp \equiv \psi$. The case for $\varphi \equiv \perp\!\!\!\perp$ is similar. If $\varphi \not\equiv \perp\!\!\!\perp$ and $\psi \not\equiv \perp\!\!\!\perp$, let $\mathsf{X} := \mathsf{P}(\varphi) \cup \mathsf{P}(\psi)$, and let $\theta := (\bigvee_{t\in\mathcal{P}} \chi_t^{\mathsf{X}} \wedge \delta_{\mathcal{P}}^{\mathsf{X}}) \vee \neg\,\delta_{\mathcal{Q}}^{\mathsf{X}}$. By Proposition 4.3.22, θ is as desired. □

4.3.24. COROLLARY (Bicompleteness of PL(NE)). PL(NE) *is bicomplete for*

$$\{(\mathcal{P},\mathcal{Q}) \mid \mathcal{P},\mathcal{Q} \text{ are convex and union closed; } \mathcal{P},\mathcal{Q} \text{ are ground-complementary modulo } \perp\!\!\!\perp\},$$

and hence `bicomplete` *for pairs which are ground-complementary modulo* $\perp\!\!\!\perp$.

We conclude this section by showing that a variant of $\mathrm{PL}(\mathrm{NE}, \mathbin{\vee\!\!\!\vee})$ is both bicomplete and `bicomplete` for all pairs. Let $\mathrm{PL}(\mathrm{NE}^*, \mathbin{\vee\!\!\!\vee})$ be $\mathrm{PL}(\mathrm{NE}, \mathbin{\vee\!\!\!\vee})$ with NE swapped out for an atom NE^* with the following support/anti-support clauses: $s \vDash \mathrm{NE}^* :\Longleftrightarrow s \neq \varnothing$ and $s \dashv \mathrm{NE}^* :\Longleftrightarrow s \neq \varnothing$.[17] Clearly many of the properties of $\mathrm{PL}(\mathrm{NE}, \mathbin{\vee\!\!\!\vee})$ are preserved in this variant: we still have a negation normal form, replacement in non-negated contexts, and expressive completeness for all properties.

We define $\top := \neg\top$, $\perp\!\!\!\perp^* := \neg((\mathrm{NE}^* \mathbin{\vee\!\!\!\vee} \bot) \vee \top)$, and $\mathrm{NE}^{*-} := \neg((\mathrm{NE}^* \wedge \perp\!\!\!\perp^*) \mathbin{\vee\!\!\!\vee} \bot)$. One can then check that $\perp\!\!\!\perp^* \equiv^{\pm} \perp\!\!\!\perp$ and $\mathrm{NE}^{*-} \equiv^{\pm} \mathrm{NE}$.

4.3.25. THEOREM (Burgess theorem for $\mathrm{PL}(\mathrm{NE}^*, \mathbin{\vee\!\!\!\vee})$). *For any $\varphi, \psi \in \mathrm{PL}(\mathrm{NE}^*, \mathbin{\vee\!\!\!\vee})$, there is a $\theta \in \mathrm{PL}(\mathrm{NE}^*, \mathbin{\vee\!\!\!\vee})$ such that $\varphi \equiv \theta$ and $\psi \equiv \neg\theta$.*

Proof:
Let φ', ψ' be the negation normal forms of φ, ψ respectively. Let φ^* be the result of replacing each occurrence of NE^* in φ' with NE^{*-}, and each occurrence of $\neg\mathrm{NE}^*$ with $\bot$, and similarly for ψ^*. We then have, as in Lemma 4.2.18, that $\varphi^* \equiv \varphi$, and that $\neg\varphi^*$ has the empty team property, and similarly for ψ^*. Now let $\varphi_\top := \neg(\neg\varphi^* \mathbin{\vee\!\!\!\vee} \neg(\mathrm{NE}^* \mathbin{\vee\!\!\!\vee} \bot))$, and similarly for $\psi_\top$. Then:

$$\begin{array}{lllllllll} \varphi_\top & = & \neg(\neg\varphi^* \mathbin{\vee\!\!\!\vee} \neg(\mathrm{NE}^* \mathbin{\vee\!\!\!\vee} \bot)) & \equiv & \varphi^* \wedge (\mathrm{NE}^* \mathbin{\vee\!\!\!\vee} \bot) & \equiv & \varphi^* & \equiv & \varphi; \\ \neg\varphi_\top & \equiv & \neg\varphi^* \mathbin{\vee\!\!\!\vee} \neg(\mathrm{NE}^* \mathbin{\vee\!\!\!\vee} \bot) & \equiv & \neg\varphi^* \mathbin{\vee\!\!\!\vee} (\mathrm{NE}^* \wedge \top) & \equiv & \top; & & \end{array}$$

where the final equivalence holds because $\neg\varphi^*$ has the empty team property. Similarly, $\psi_\top \equiv \psi$ and $\neg\psi_\top \equiv \top$. Finally, let $\theta := \varphi_\top \mathbin{\vee\!\!\!\vee} (\perp\!\!\!\perp^* \vee \neg\psi_\top)$. Then:

$$\begin{array}{lllllll} \theta & = & \varphi_\top \mathbin{\vee\!\!\!\vee} (\perp\!\!\!\perp^* \vee \neg\psi_\top) & \equiv & \varphi_\top & \equiv & \varphi; \\ \neg\theta & = & \neg(\varphi_\top \mathbin{\vee\!\!\!\vee} (\perp\!\!\!\perp^* \vee \neg\psi_\top)) & \equiv & \neg\varphi_\top \wedge \neg(\perp\!\!\!\perp^* \vee \neg\psi_\top) & \equiv & \top \wedge (\neg\perp\!\!\!\perp^* \wedge \psi_\top) \\ & \equiv & \top \wedge (\top \wedge \psi_\top) & \equiv & \psi_\top & \equiv & \psi. \end{array} \qquad \square$$

In particular, for any φ there is a θ such that $\theta \equiv \varphi \equiv \neg\theta$.

4.3.26. COROLLARY (Bicompleteness of $\mathrm{PL}(\mathrm{NE}^*, \mathbin{\vee\!\!\!\vee})$). $\mathrm{PL}(\mathrm{NE}^*, \mathbin{\vee\!\!\!\vee})$ *is both bicomplete and* `bicomplete` *for all pairs.*

4.3.4 Propositional Dependence Logic

Propositional dependence logic $\mathrm{PL}(=(\cdot))$ is PL extended with the $n+1$-ary ($n \geq -1$) connectives $=(p_1 \ldots, p_n, q)$, where $p_1, \ldots, p_n, q \in \mathsf{Prop}$. The formulas $=(p_1 \ldots, p_n, q)$ are called *dependence atoms*. In this section, we show that $\mathrm{PL}(=(\cdot))$ is `bicomplete` for ground-complementary pairs (recall that, in contrast, first-order dependence logic D is `bicomplete` for $\bot$-incompatible pairs).

[17] NE^* was introduced by Tomasz Klochowicz in unpublished work.

The support/anti-support conditions for dependence atoms are as follows:

$$s \vDash =(p_1 \ldots, p_n, q) \quad :\Longleftrightarrow \quad \forall v, w \in s : [v \vDash p_i \iff w \vDash p_i \text{ for all } \forall 1 \le i \le n] \implies [v \vDash q \iff w \vDash q];$$
$$s \dashv =(p_1 \ldots, p_n, q) \quad :\Longleftrightarrow \quad s = \varnothing.$$

In other words, a dependence atom $=(p_1 \ldots, p_n, q)$ is true/supported in a team s if, whenever two valuations v and w in s agree on the truth values of all the p_i, they also agree on the truth value of q—in other words, the values of $p_i, \ldots, p_n$ jointly determine the value of q in any valuation in the team. $=(p_1 \ldots, p_n, q)$ is anti-supported in a team just in case the team is empty.[18] We call a unary dependence atom $=(p)$ a *constancy atom*. $=(p)$ expresses that the value of p is constant in all valuations in a team.

The propositional analogues of many results in Section 4.2.1 also hold in the context of $\mathrm{PL}(=(\cdot))$; we note in particular that each formula is equivalent to one in negation normal form, and that replacement holds in positive contexts. As with first-order dependence logic D, formulas of $\mathrm{PL}(=(\cdot))$ are downward closed and have the empty team property, which, by Fact 4.2.14 and the definition of NE-incompatibility, means that ground-incompatibility, $\bot$-incompatibility, and $\varnothing$-incompatibility are equivalent for these formulas. Formulas of $\mathrm{PL}(=(\cdot))$ need not be union closed: consider $=(p)$. We have the following expressive completeness theorem:

4.3.27. THEOREM (Expressive completeness of $\mathrm{PL}(=(\cdot))$). *[139]* $\mathrm{PL}(=(\cdot))$ *is expressively complete for* $\{\mathcal{P} \mid \mathcal{P}$ *is downward closed and has the empty team property*$\}$.

In contrast with first-order dependence logic D, a Burgess theorem that employs $\bot$-incompatibility does not hold for $\mathrm{PL}(=(\cdot))$. We can show this, as we did the analogous fact for PL(NE), using flattenings of formulas. Define the flattening $\varphi^f \in \mathrm{PL}$ of $\varphi \in \mathrm{PL}(=(\cdot))$ as in Section 4.3.3, together with $=(p_1 \ldots, p_n, q)^f := \top$. As in Section 4.3.3, it can be shown:

[18]The anti-support clause for $=(p_1 \ldots, p_n, q)$ might seem puzzling at first glance. What is the reason, one might ask, for adopting the clause we use rather than, say, $s \dashv =(p_1 \ldots, p_n, q)$ iff $s \nvDash =(p_1 \ldots, p_n, q)$? (Pietro Galliani, in recent unpublished work [53], considers a variant of D with a first-order version of this anti-support clause.) The clause we use has been adapted from the analogous first-order clause for D in [119]. This first-order clause was, in turn, chosen (it seems to me) for the following the reasons: it (unlike the variant clause above) preserves the equivalence between IF and D, as well as the empty team property and downward closure. Väänänen [119, p. 24] also offers the following explanation (notation amended; adapted for the propositional setting):

> Why not allow $s \dashv =(p_1 \ldots, p_n, q)$ for non-empty s? The reason is that if we negate "for all $w, w' \in s$ such that $w \vDash p_1 \iff w' \vDash p_1, \ldots, w \vDash p_n \iff w' \vDash p_n$, we have $w \vDash q \iff w' \vDash q$", maintaining the analogy with [$s \dashv p$ iff for all $w \in s : w \nvDash p$], we get "for all $w, w' \in s$ we have $w \vDash p_1 \iff w' \vDash p_1, \ldots, w \vDash p_n \iff w' \vDash p_n$ and $w \vDash q \not\Longleftrightarrow w' \vDash q$", which is only possible if $s = \varnothing$.

Note also that Väänänen's clause preserves the ground-incompatibility and NE-incompatibility of φ and $\neg\varphi$, whereas the variant above does not.

4.3.28. LEMMA. *Let φ' be the negation normal form of $\varphi \in \mathrm{PL}(=(\cdot))$. Then $\varphi^f \equiv \varphi'^f$.*

And we have:

4.3.29. LEMMA. *For all $\varphi \in \mathrm{PL}(=(\cdot))$ and all $\mathsf{X} \supseteq \mathsf{P}(\varphi)$, $|\varphi|_{\mathsf{X}} = |\varphi^f|_{\mathsf{X}}$.*

Proof:
By induction on the complexity of φ, which we may assume to be in negation normal form by Lemma 4.3.28. The base cases—those for atoms and negated atoms—are obvious; note that $|=(p_1 \dots, p_n, q)| = |\top| = |=(p_1 \dots, p_n, q)^f|$ since $w \in \{w\} \vDash =(p_1 \dots, p_n, q)$ for any w, and $|\neg =(p_1 \dots, p_n, q)| = |\bot| = |(\neg =(p_1 \dots, p_n, q))^f|$.

Consider $\varphi = \psi \wedge \chi$. We show $|\psi \wedge \chi| = |\psi| \cap |\chi|$; the result will then follow as in the proof of Lemma 4.3.17. $|\psi \wedge \chi| \subseteq |\psi| \cap |\chi|$ is immediate; for the converse inclusion, let $w \in |\psi| \cap |\chi|$. Then $w \in t \vDash \psi$ and $w \in u \vDash \chi$. By downward closure, $\{w\} \vDash \psi$ and $\{w\} \vDash \chi$, so $w \in |\psi \wedge \chi|$.

Now consider $\varphi = \psi \vee \chi$. We show $|\psi \vee \chi| = |\psi| \cup |\chi|$; the result will then follow as in the proof of Lemma 4.3.17. $|\psi \vee \chi| \subseteq |\psi| \cup |\chi|$ follows as in the proof of Lemma 4.3.17. For the converse inclusion, let $w \in |\psi| \cup |\chi|$. Then $w \in s$ where $s \vDash \psi$ or $s \vDash \chi$; assume without loss of generality that $s \vDash \psi$. By the empty team property, $s = s \cup \varnothing \vDash \psi \vee \chi$, so $w \in |\psi \vee \chi|$. □

One can then show that a Burgess theorem employing $\bot$-incompatibility would lead to contradiction using essentially the same argument as that used for the analogous fact in Section 4.3.3. We could instead show a Burgess theorem for $\mathrm{PL}(=(\cdot))$ using ground-complementariness modulo $\perp\!\!\!\perp$, as we did for PL(NE), but, as can be gleaned from the lemma above, or, alternatively, from the fact that formulas of $\mathrm{PL}(=(\cdot))$ have the empty team property, 'modulo $\perp\!\!\!\perp$' is redundant here—we may simply use ground-complementariness. Note that since $\mathrm{PL}(=(\cdot))$ is downward closed, by Fact 4.3.20 we could also equivalently use world-incompatibility. We have:

4.3.30. PROPOSITION. *For any $\varphi \in \mathrm{PL}(=(\cdot))$, φ and $\neg\varphi$ are ground-complementary.*

Proof:
By Lemma 4.3.29; almost the same as the proof of Lemma 4.3.19. □

We now show a Burgess theorem for $\mathrm{PL}(=(\cdot))$ using ground-complementariness. As with PL(NE), we prove this by modifying the proof of the relevant expressive completeness theorem, Theorem 4.3.27. For further details on what follows, see [139].

For a finite $\mathsf{X} \subseteq \mathsf{Prop}$, define $\gamma_0^{\mathsf{X}} := \bot$, $\gamma_1^{\mathsf{X}} := \bigwedge_{p \in \mathsf{X}} =(p)$, and for $n \geq 2$, $\gamma_n^{\mathsf{X}} := \bigvee_n \gamma_1$. Then for $s \subseteq 2^{\mathsf{X}}$, we have $s \vDash \gamma_n^{\mathsf{X}} \iff |s| \leq n$, where $|s|$ is the size of s. Note that $\neg\gamma_0^{\mathsf{X}} \equiv \top$ and $\neg\gamma_n^{\mathsf{X}} \equiv \bot$ for $n \geq 1$.

For a nonempty $s \subseteq 2^{\mathsf{X}}$, define $\xi_s^{\mathsf{X}} := \gamma_{|s|-1}^{\mathsf{X}} \vee \chi_{|\top|_{\mathsf{X}} \setminus s}^{\mathsf{X}}$. It can be shown that for $t \subseteq 2^{\mathsf{X}}$, $t \vDash \xi_s^{\mathsf{X}} \iff s \nsubseteq t$. Note that if $|s| = 1$, then $\xi_s^{\mathsf{X}} \equiv \chi_{|\top|_{\mathsf{X}} \setminus s}^{\mathsf{X}}$ so $|\xi_s^{\mathsf{X}}|_{\mathsf{X}} = |\chi_{|\top|_{\mathsf{X}} \setminus s}^{\mathsf{X}}|_{\mathsf{X}}$, and if $|s| > 2$, then $|\xi^{\mathsf{X}}_s|_{\mathsf{X}} = |\top|_{\mathsf{X}}$. Note also that if $|s| = 1$ (say $s = \{w\}$), then $\neg\xi^{\mathsf{X}}_s \equiv \top \wedge \neg\chi^{\mathsf{X}}_{|\top|_{\mathsf{X}}\setminus s} \equiv \chi^{\mathsf{X}}_s = \chi^{\mathsf{X}}_w$ and if $|s| > 1$, then $\neg\xi^{\mathsf{X}}_s \equiv \bot$.

Theorem 4.3.27 essentially follows from:

4.3.31. PROPOSITION. *[139] For any downward-closed property $\mathcal{P} = \{t_1, \dots, t_n\}$ with the empty team property over $\mathsf{X} \subseteq \mathsf{Prop}$ (where X is finite), $\mathcal{P} = \|\bigwedge_{s\in\|\top\|_{\mathsf{X}}\setminus\mathcal{P}} \xi^{\mathsf{X}}_s\|_{\mathsf{X}}$.*

For a property $\mathcal{P} = \{t_1, \dots, t_n\}$ over a finite X, define $\mathcal{P}^1 := \{s \in \mathcal{P} \mid |s| = 1\}$ and $\mathcal{P}^{>1} := \{s \in \mathcal{P} \mid |s| > 1\}$. We have the following corollary of Proposition 4.3.31:

4.3.32. PROPOSITION. *For any downward-closed properties $\mathcal{P} = \{t_1, \dots, t_n\}$ and $\mathcal{Q} = \{s_1, \dots, s_m\}$ with the empty team property over $\mathsf{X} \subseteq \mathsf{Prop}$ (where X is finite) such that $\mathcal{P}, \mathcal{Q}$ are ground-complementary,*

$$(\mathcal{P}, \mathcal{Q}) = \left\| \bigwedge_{s\in\|\top\|_{\mathsf{X}}\setminus\mathcal{P}} \xi^{\mathsf{X}}_s \vee \neg \bigwedge_{s\in(\|\top\|_{\mathsf{X}}\setminus\mathcal{Q})^{>1}} \xi^{\mathsf{X}}_s \right\|^{\pm}_{\mathsf{X}}.$$

Proof:

It is easy to check that $\neg\bigwedge_{s\in(\|\top\|\setminus\mathcal{Q})^{>1}} \xi_s \equiv \bot$, whence

$$\bigwedge_{s\in\|\top\|\setminus\mathcal{P}} \xi_s \vee \neg \bigwedge_{s\in(\|\top\|\setminus\mathcal{Q})^{>1}} \xi_s \equiv \bigwedge_{s\in\|\top\|\setminus\mathcal{P}} \xi_s \vee \bot \equiv \bigwedge_{s\in\|\top\|\setminus\mathcal{P}} \xi_s,$$

and therefore by Proposition 4.3.31, $\|\bigwedge_{s\in\|\top\|\setminus\mathcal{P}} \xi_s \vee \neg\bigwedge_{s\in(\|\top\|\setminus\mathcal{Q})^{>1}} \xi_s\| = \mathcal{P}$. It remains to show $\|\neg(\bigwedge_{s\in\|\top\|\setminus\mathcal{P}} \xi_s \vee \neg\bigwedge_{s\in(\|\top\|\setminus\mathcal{Q})^{>1}} \xi_s)\| = \mathcal{Q}$. We first prove:

$$\text{For any team } s,\ \chi^{\mathsf{X}}_s \equiv \bigwedge_{w\in|\top|_{\mathsf{X}}\setminus s} \chi^{\mathsf{X}}_{|\top|_{\mathsf{X}}\setminus\{w\}}. \tag{4.2}$$

Proof of (4.2):

$$\chi_s = \bigvee_{v\in s} \chi_v \equiv \bigwedge_{w\in|\top|\setminus s} \neg\chi_w \equiv \bigwedge_{w\in|\top|\setminus s} \bigvee_{w'\in|\top|\setminus\{w\}} \chi_{w'} \equiv \bigwedge_{w\in|\top|\setminus s} \chi_{|\top|\setminus\{w\}}. \qquad \dashv$$

We can now prove $\|\neg(\bigwedge_{s\in\|\top\|\setminus\mathcal{P}} \xi_s \vee \neg\bigwedge_{s\in(\|\top\|\setminus\mathcal{Q})^{>1}} \xi_s)\| = \mathcal{Q}$. We have:

$$\begin{aligned}
& \neg(\bigwedge_{s\in\|\top\|\setminus\mathcal{P}} \xi_s \vee \neg \bigwedge_{s\in(\|\top\|\setminus\mathcal{Q})^{>1}} \xi_s) \\
\equiv\ & \neg \bigwedge_{s\in\|\top\|\setminus\mathcal{P}} \xi_s \wedge \bigwedge_{s\in(\|\top\|\setminus\mathcal{Q})^{>1}} \xi_s \\
\equiv\ & \neg(\bigwedge_{s\in(\|\top\|\setminus\mathcal{P})^1} \xi_s \wedge \bigwedge_{s\in(\|\top\|\setminus\mathcal{P})^{>1}} \xi_s) \wedge \bigwedge_{s\in(\|\top\|\setminus\mathcal{Q})^{>1}} \xi_s \\
\equiv\ & (\neg \bigwedge_{s\in(\|\top\|\setminus\mathcal{P})^1} \xi_s \vee \neg \bigwedge_{s\in(\|\top\|\setminus\mathcal{P})^{>1}} \xi_s) \wedge \bigwedge_{s\in(\|\top\|\setminus\mathcal{Q})^{>1}} \xi_s
\end{aligned}$$

$$\begin{aligned}
&\equiv (\neg \bigwedge_{s\in(\|\top\|\setminus\mathcal{P})^1} \xi_s \vee \bot) \wedge \bigwedge_{s\in(\|\top\|\setminus\mathcal{Q})^{>1}} \xi_s \\
&\equiv \neg \bigwedge_{s\in(\|\top\|\setminus\mathcal{P})^1} \xi_s \wedge \bigwedge_{s\in(\|\top\|\setminus\mathcal{Q})^{>1}} \xi_s \\
&\equiv \bigvee_{s\in(\|\top\|\setminus\mathcal{P})^1} \neg\xi_s \wedge \bigwedge_{s\in(\|\top\|\setminus\mathcal{Q})^{>1}} \xi_s \\
&\equiv \bigvee_{s\in(\|\top\|\setminus\mathcal{P})^1} \chi_s \wedge \bigwedge_{s\in(\|\top\|\setminus\mathcal{Q})^{>1}} \xi_s \\
&\equiv \bigvee_{\{w\}\in(\|\top\|\setminus\mathcal{P})^1} \chi_w \wedge \bigwedge_{s\in(\|\top\|\setminus\mathcal{Q})^{>1}} \xi_s \\
&\equiv \chi_{|\top|\setminus\bigcup\mathcal{P}} \wedge \bigwedge_{s\in(\|\top\|\setminus\mathcal{Q})^{>1}} \xi_s && \text{(left-to-right by downward closure of } \mathcal{P}) \\
&\equiv \bigwedge_{w\in\bigcup\mathcal{P}} \chi_{|\top|\setminus\{w\}} \wedge \bigwedge_{s\in(\|\top\|\setminus\mathcal{Q})^{>1}} \xi_s && (4.2) \\
&\equiv \bigwedge_{w\in\bigcup\mathcal{P}} \xi_{\{w\}} \wedge \bigwedge_{s\in(\|\top\|\setminus\mathcal{Q})^{>1}} \xi_s \\
&\equiv \bigwedge_{w\in|\top|\setminus\bigcup\mathcal{Q}} \xi_{\{w\}} \wedge \bigwedge_{s\in(\|\top\|\setminus\mathcal{Q})^{>1}} \xi_s && (\mathcal{P}, \mathcal{Q} \text{ ground-complementary}) \\
&\equiv \bigwedge_{\{w\}\in(\|\top\|\setminus\mathcal{Q})^1} \xi_{\{w\}} \wedge \bigwedge_{s\in(\|\top\|\setminus\mathcal{Q})^{>1}} \xi_s && \text{(right-to-left by downward closure of } \mathcal{Q}) \\
&\equiv \bigwedge_{s\in\|\top\|\setminus\mathcal{Q}} \xi_s
\end{aligned}$$

The result now follows by Proposition 4.3.31. □

And so we have our Burgess and bicompleteness theorems:

4.3.33. THEOREM (Burgess theorem for PL(=(·))). *In the logic* PL(=(·))*, the following are equivalent:*

(i) φ *and* ψ *are ground-complementary.*

(ii) There is a formula θ *such that* $\varphi \equiv \theta$ *and* $\psi \equiv \neg\theta$ *(and* $\mathsf{P}(\theta) = \mathsf{P}(\varphi) \cup \mathsf{P}(\psi)$*).*

Proof:
(ii) $\Longrightarrow$ (i) by Proposition 4.3.30. For (i) $\Longrightarrow$ (ii), let $\mathsf{X} := \mathsf{P}(\varphi) \cup \mathsf{P}(\psi)$, and let $\theta := \bigwedge_{s\in\|\top\|_{\mathsf{X}}\setminus\mathcal{P}} \xi_s^{\mathsf{X}} \vee \neg \bigwedge_{s\in(\|\top\|_{\mathsf{X}}\setminus\mathcal{Q})^{>1}} \xi_s^{\mathsf{X}}$. By Proposition 4.3.32, θ is as desired. □

4.3.34. COROLLARY (Bicompleteness of PL(=(·))). PL(=(·)) *is bicomplete for*

$$\{(\mathcal{P},\mathcal{Q}) \mid \mathcal{P},\mathcal{Q} \textit{ are downward closed and have the empty team property; } \mathcal{P} \textit{ and } \mathcal{Q} \textit{ are G-C/G-C mod } \perp\!\!\!\perp\textit{/W-I}\},$$

and hence `bicomplete` *for pairs which are G-C/G-C mod ⫫/W-I.*

Note that by Corollaries 4.3.6 (ii) and 4.3.34, we have $\|\text{InqB}\| = \|\text{PL}(=(\cdot))\|$, but $\|\text{InqB}\|^{\pm,\neg_i} \neq \|\text{PL}(=(\cdot))\|^{\pm,\neg}$.

To conclude, let us note that it is trivial to find a propositional logic which, like D in the first-order setting, is `bicomplete` for $\perp$-incompatible pairs, $\varnothing$-incompatible pairs, and ground-incompatible pairs: for instance, one could extend $\text{PL}(=(\cdot))$ with a constant θ_0 such that $s \vDash \theta_0 \iff s \dashv \theta_0 \iff s = \varnothing$. It is not clear whether there is some interesting and non-*ad hoc* propositional logic `bicomplete` for these pairs.

4.4 Interpretations of the Incompatibility Notions

In this section, we provide sketches of some possible intuitive interpretations of the incompatibility notions, drawing mainly on [116, 123, 129, 130, 42, 25, 107, 38, 35, 67, 6, 45]. (I am particularly indebted to Yalcin [129, 130]; I borrow both from his analysis as well as his perspicuous explanations.) Our main focus will be on the relationship between $\perp$-incompatibility and ground-incompatibility (Facts 4.2.13 and 4.2.14), and the related notions of $\perp\!\!\!\perp$-incompatibility and NE-incompatibility (and Fact 4.2.16), although we will also comment briefly on some of the other notions. Our discussion must be brief so we ignore many potential complications[19]; the aim is provide some basis for further interpretations, and to give the reader some intuitive grasp on our technical notions.

4.4.1 Information States; Factual Information; Persistency

A common interpretation of team logics is one in which possible worlds/propositional valuations represent possible states of affairs, and teams represent information states. A valuation determines a collection of formulas which are true according to the valuation, and this collection stands in for the state of affairs in which precisely whatever is represented by the formulas in the collection obtains. A set of these valuations, or a propositional team (see Section 4.3), then corresponds to an information state in that if one knows that the valuation representing the actual world is one of the worlds in a team s, one knows that whatever holds in all worlds in s must be the case. For instance, if it is raining in all worlds in s, the information embodied in s includes the information that it is raining. A modal team also incorporates information about what is necessary/possible (or obligatory/permissible, etc.) given that the actual world is in the state. The empty information state is a state of absurdity—a state which establishes inconsistent information.

[19]Note in particular that while much of our presentation follows that of Yalcin [129, 130], we make no principled distinctions between the *compositional semantic value* of a linguistic expression and its *informational content* or its *object of assertion*; these distinctions (see, for instance, [129, 130, 131, 67]) are key for Yalcin. We use the terms 'information' and 'content' to refer, essentially, to (possibly pragmatically enriched) compositional semantic values.

Each formula determines a constraint on information states which we may think of as the information or content expressed by the formula: a state satisfies the constraint corresponding to a given formula just in case the team which represents the state supports the formula. A classical formula determines a constraint on possible worlds in a similar fashion; the constraint it determines on states is parasitic on this constraint on worlds: each world in the state must satisfy the constraint on worlds (as per Fact 4.2.4). The type of information expressed by classical propositional formulas—call this *factual information*—is, then, information concerning which facts (as captured by valuations) hold in a given world or according to a given information state. Similarly, classical modal formulas express information concerning which facts hold at accessible worlds (*modal factual information*). As we have seen, classical formulas are downward closed. Downward closure may be thought of as corresponding to the fact that as one becomes more informed (by ruling out certain possibilities), then, at least as far this type of information is concerned, one does not lose the information one has already acquired. (Downward closure and analogous notions are, accordingly, often called *persistency* [35, 123, 25, 67].) If s supports the assertion that it is raining, moving to a subteam t of s by ruling out some possible worlds does not erase this information: t also supports the assertion that it is raining.

This is the basic picture. It already allows us to indicate in broad strokes what some of our notions are expressing. ψ is a down-set incompatibility of φ just in case one whenever an information state establishes that ψ ($s \vDash \psi$), φ is ruled out in that there is no further information or content that could consistently establish φ (there is no $t \subseteq s$ such that $t \neq \varnothing$ and $t \vDash \varphi$). Trivially, φ and ψ are team-incompatible if one is established in a state iff the other is not—if the constraint on states determined by ψ is (expressible as) the classical negation (on states) of that determined by φ, and vice versa. Analogously for world-incompatibility and constraints on worlds. φ and ψ are flat-incompatible (φ and ψ are W-I, and both formulas are flat) just in case the world-constraint determined by one is the classical negation (on worlds) of the world-constraint determined by the other (W-I); and they both express essentially factual information: the state-constraints they determine are parasitic on their world-constraints (flat).

However, for most of the other notions we consider, it is helpful, before providing interpretations of these notions, to first formulate what we take properly state-based constraints to represent—what is the nature of the *non-factual information/content* which is not expressible by classical formulas? In Section 4.2.2, we observed that $\bot$-incompatibility and ground-incompatibility are equivalent for downward-closed formulas, but when downward closure fails, there are some pairs which are $\bot$-incompatible but which fail to be ground-incompatible (Facts 4.2.13 and 4.2.14). We additionally noted that each $\bot$-incompatible pair is either $\perp\!\!\!\perp$-incompatible or NE-incompatible, and that NE-incompatibility entails ground-incompatibility in a convex setting (Fact 4.2.16). There are, accordingly, two types of pairs which are $\bot$-incompatible but not ground-incompatible: $\perp\!\!\!\perp$-I but not G-I, and NE-I but not G-I, with the latter type only possible when convexity is violated. We will set as our goal, in the remainder of this section, to augment the basic picture with interpretations of non-classical formulas in a way that allows us to make intuitive sense of these facts. In Section 4.4.2, we focus on $\perp\!\!\!\perp$-I but not G-I pairs. We assume a convex setting, and examine an expressivist epistemic interpretation for formulas which fail downward closure (due essentially to Yalcin [129, 130] as well as Hawke and Steinert-Threlkeld [67]); on this interpretation, $\perp\!\!\!\perp$-I but not G-I pairs represent *epistemic contradictions*. We also comment briefly on what this interpretation tells us about G-C (mod $\perp\!\!\!\perp$) pairs. In Section 4.4.3, we again assume convexity, and consider a pragmatic interpretation (essentially due to Aloni [6]) which leads to $\perp\!\!\!\perp$-I but not G-I pairs being *pragmatic contradictions*. In Section 4.4.4, we build on the interpretation of Section 4.4.2 to also provide an account of NE-I but not G-I pairs. We adapt the interpretation of the inquisitive disjunction $\vee\!\!\!\vee$ and the notion of *inquisitive content* from inquisitive semantics [38, 35], and conclude that, assuming the context of the rest of our interpretations, NE-I but not G-I pairs are examples of a specific type of contradiction with inquisitive content.

4.4.2 Expressivist Epistemic Information; Factual Contradictions vs. Epistemic Contradictions

(We work in the convex setting of BSML in this section.) Not all types of information need be persistent. Define an *epistemic might* operator $\blacklozenge$ as follows: $\blacklozenge\varphi := (\varphi \wedge \text{NE}) \vee \top$. Then is $\blacklozenge\varphi$ supported by a team just in case the team contains a nonempty subteam supporting φ. This operator can be used to represent the 'might' in sentences such as 'It might not be raining': if there is a nonempty subteam t of s consisting of worlds in which it is not raining ($s \vDash \blacklozenge\neg r$), then, for all that one knows given the information embodied in s, it might not be raining. The information that something might be the case *can* come to be invalidated upon receipt of further information: if one moves from s to a subteam u consisting only of worlds in which it is raining (that is, say, if one's information state supports the assertion that it might not be raining, and one then learns that it is raining), $\blacklozenge\neg r$ is no longer supported. (For similar accounts in the literature, see, e.g., [129, 130, 25, 107, 67]; it seems the first account along these lines was Veltman's [123].)

Whereas the type of constraint on information states corresponding to factual information is local in the sense that it is parasitic on a constraint on worlds, the type of constraint on information states determined by $\blacklozenge\varphi$ is global: the state as a whole must have a non-empty φ-substate. The constraint also pertains only to information states, rather than to both information states and worlds: it does not seem to make sense, for instance, to speak of a (fully determined) state of affairs in which might be raining. In this sense, the type of information expressed by $\blacklozenge\varphi$ is non-factual: it does not serve to demarcate the collection of possible states of affairs in which some specific facts hold, and an assertion of $\blacklozenge\varphi$ does not purport to (directly) help one's interlocutors better situate the actual world in the space of possibilities; rather, it only indicates that whatever state of information conforms to it is such that φ cannot be ruled out (which state this is taken to apply to—often the speaker's information state—is determined pragmatically via the context). Following Yalcin's [129, 130] and Hawke and Steinert-Threlkeld's [67] expressivism about epistemic modality, let us call the type of information expressed by $\blacklozenge\varphi$ *expressivist epistemic information*. Compare 'It might be raining' $\blacklozenge r$ with 'For all I know, it is raining', which we might formalize, using an epistemic Kripke modality $\Diamond_a$ tracking what is compatible with what the utterer agent a knows in the usual way, as $\Diamond_a r$. The latter (expressing what we can call *factual epistemic information*) does directly express a constraint on worlds: it excludes all worlds in which a's epistemic possibilities do not include r-worlds.[20] Indeed, the information that *for all agent a knows, it is raining* is persistent—it is for instance not invalidated by the information that it is not raining. One might pragmatically infer, from agent a's utterance of (something formalized as) $\blacklozenge r$, that $\Diamond_a r$, but the formula only directly serves to describe a constraint on information states. See [129, 130] for more on this distinction.

In this interpretation we have, then, factual information expressible using classical (downward closed) formulas and expressivist epistemic information expressed using $\blacklozenge$ (which breaks downward closure). In this context, we may think of the ground team $|\varphi|$ of φ as representing the factual information expressed by φ: the ground-team of φ is a classical proposition which is true precisely in all the possible worlds which compose the information states in which φ in assertible, and which does not directly communicate any expressivist epistemic information.[21] 'It might be raining' expresses no factual information since $|\blacklozenge r| = |\top|$—knowing that it might be raining gives one no information as to whether it actually is raining. Similarly, the factual information expressed by 'it is not hailing but it might be raining' $\neg h \land \blacklozenge r$ is simply that it is not hailing: $|\neg h \land \blacklozenge r| = |\neg h|$.

We may, accordingly, think of ground-incompatibility as contradiction in factual

[20]Note that epistemic puzzles such as the muddy children puzzle [17] turn on the factuality of the type of information expressed by formulas of the form $\neg\Box_a\varphi$.

[21]The ground team can be thought of as indirectly communicating expressivist epistemic information via the modal operators. For instance, consider $\Diamond(\blacklozenge r \land \blacklozenge h)$. The ground team $|\Diamond(\blacklozenge r \land \blacklozenge h)|$ excludes all worlds which do not see an information state that supports $\blacklozenge r \land \blacklozenge h$—all worlds in which it is not possible (in whatever way $\Diamond$ represents) that it simultaneously might (expressivist-epistemically) be the case that r and might be the case that h. In the modal setting, the ground team therefore communicates not only factual information concerning what is the case in the actual world and factual modal information concerning what is factually the case in accessible worlds, but also factual (in the sense of defining a constraint on possible worlds) modal information concerning what might (expressivist-epistemically) be the case in modally accessible states. This is essentially because the semantics of the modalities we employ make use of state-based support and anti-support conditions, and so they are sensitive to constraints on states (such as those generated by $\blacklozenge$), but the constraints the modalities themselves define are constraints on worlds. To make some intuitive sense of the formula above, assume we are defining rules for a system which deals in weather probabilities using a modality $\Diamond$. We can then take, say, $\neg\Diamond(r \land \neg c)$ to describe a rule to the effect that no information state of the system in which it is raining and there are no clouds is permissible/possible, whereas $\Diamond(\blacklozenge r \land \blacklozenge h)$ states that being in a state according to which it might rain and it might hail is permissible (and grants the system factual information concerning which kinds of information states are accessible from the actual world).

information: if φ and ψ are ground-incompatible ($|\varphi| \cap |\psi| = \varnothing$), the factual information expressed by φ rules out that expressed by ψ, and vice versa. On the other hand, it seems that a $\bot$-incompatibility that is not also a ground-incompatibility (which, assuming we are in a convex setting, must be a $⫫$-incompatibility by Fact 4.2.16) such as the pair r, $\blacklozenge \neg r$ (we have $r, \blacklozenge \neg r \vDash ⫫ \vDash \bot$ and $|r| \cap |\blacklozenge \neg r| = |r| \neq \varnothing$) would then have to be a contradiction involving expressivist epistemic information, or an *epistemic contradiction* in the sense of [129, 130]. Indeed, $r \wedge \blacklozenge \neg r$ corresponds to the classic epistemic contradiction 'it is raining and it might not be raining', and other typical examples similarly clearly constitute pairs which are $⫫$-I but not G-I.[22] We then have the following intuitive explanation for Facts 4.2.13 and 4.2.14: G-I—factual contradictoriness—and $\bot$-I—contradictoriness in some more general sense—come apart when expressive epistemic information is admitted since in addition to conflicts in purely fact-based constraints we now have also conflicts also involving expressive epistemic constraints, which include the typical examples of epistemic contradictions. In a setting with only factual, downward-closed information the latter type of conflict does not arise, so G-I and $\bot$-I coincide.

Given the association between the failure of downward closure and expressive epistemic information, one might be tempted to further stipulate that two formulas are in epistemic contradiction precisely when they are $⫫$-incompatible (setting aside formulas which are themselves equivalent to $⫫$, the only way two formulas can be $⫫$-I is if one of them is not downward closed). Note, however, that $r \wedge \blacklozenge q, \neg r \vDash ⫫$, yet clearly with this pair the incompatibility arises due to r and $\neg r$ being factually contradictory. Perhaps the most we can say is that if a pair is $⫫$-I, the pair is contradictory and at least one of the formulas involved expresses expressivist epistemic information.

To conclude this section, let us also comment briefly on what it means for two formulas φ and ψ to be G-C ($|\varphi| = |\top| \setminus |\psi|$ and $|\psi| = |\top| \setminus |\varphi|$) (or G-C mod $⫫$) in the context of this interpretation. We can say that either one of the formulas is (if it is equivalent to $⫫$), by itself, a contradiction that incorporates expressive epistemic information in some way, or that the factual information expressed by the two formulas is not only contradictory, but also classically complementary: there is a classical formula α such that α expresses the same factual information as φ, and $\neg \alpha$ expresses the same factual information as ψ.

We can also say, given Lemmas 4.3.17 and 4.3.29, that, in the logics PL(NE) and PL(=(·)) (as long as φ is not equivalent to $⫫$) the flattening φ^f of φ can properly be regarded as the classical variant of φ (as per the name given to the analogue of flatten-

[22]Compare epistemic contradictions with Moorean sentences such as 'It is raining and I do not know that it is raining', which we might formalize as $r \wedge \neg \Box_a r$. This sentence entails a ground-incompatibility on the pragmatic assumption that if agent a asserts r, they are also thereby asserting $\Box_a r$. It is therefore, by our lights, not a contradiction involving expressive epistemic information, but rather a kind of pragmatic factual contradiction, dovetailing Yalcin's [129, 130] distinction between epistemic contradictions and Moorean sentences. One might also arguably pragmatically infer $r \wedge \neg \Box_a r$ from a's utterance of $r \wedge \blacklozenge \neg r$, so one might argue that $r \wedge \blacklozenge \neg r$ involves the same kind of pragmatic contradiction as $r \wedge \neg \Box_a r$ in addition to being an epistemic contradiction.

ing in inquisitive logic; see endnote 10) in that φ and φ^f express the same information regarding valuations/worlds. Propositions 4.3.19 and 4.3.30, then, say that in PL(NE) and PL$(=(\cdot))$ the dual negation behaves classically at least to the extent that (as long as $\varphi \neq \perp\!\!\perp$ and $\psi \neq \perp\!\!\perp$) the classical variant of $\neg\varphi$ is simply the classical negation of the classical variant of φ, and on the level of valuations/worlds, there is no failure of determination.

4.4.3 Pragmatic Information via the Neglect-zero Tendency; Factual Contradictions vs. Pragmatic Contradictions

(We again work in BSML.) Let us consider another possible type of non-downward-closed information. Aloni and her collaborators [6, 45] use BSML to account for *modal disjunction inferences* such as the following:

(1) a. Sue went to the beach or to the cinema.
$\rightsquigarrow$ *The speaker deems it possible that Sue went to the beach and the speaker deems it possible that Sue went to the cinema.*
b. $b \vee c \rightsquigarrow \Diamond_s b \wedge \Diamond_s c$

Aloni's proposal is that speakers, when interpreting language, create mental structures that represent reality; and that inferences like (1) are due to a tendency in human cognition to disregard empty structures in this process of interpretation (the *neglect-zero tendency*). This tendency is modelled using a *pragmatic enrichment function* $[\ ]^+ : \mathrm{ML} \to \mathrm{BSML}$ which, given a classical formula α, recursively appends $\wedge\mathrm{NE}$ to each subformula of α (for instance, $[b \vee c]^+ = ((b \wedge \mathrm{NE}) \vee (c \wedge \mathrm{NE})) \wedge \mathrm{NE}$). One can then check that, for instance, the team $\{w_b\}$ is such that $\{w_b\} \vDash b \vee c$ but $\{w_b\} \nvDash [b \vee c]^+$—the team only supports $b \vee c$ by virtue of the fact that the empty team $\varnothing$ supports c, whereas it does not support $[b \vee c]^+$ since pragmatic enrichment rules out empty verifying structures (*zero-models*).

The modal disjunction inference above can then be accounted for as follows. We model the speaker's information state using a pointed model (M, t) with an accessibility relation R_s corresponding to a modality $\Diamond_s$ tracking what is epistemically possible given the speaker's knowledge. R_s must be such that for all $w \in t$, $R[w] = t$ (in Aloni's terminology, R_s is *state-based* in t)—this tracks the facts that R_s is intended to be an epistemic accessibility relation, and that t is the speaker's information state (note that if R_s is state-based in (M, t), then $M, t \vDash \Diamond_s \varphi$ iff $M, t \vDash \blacklozenge\varphi$). In such a pointed model, $[b \vee c]^+$ is assertible just in case $\Diamond_s b \wedge \Diamond_s c$ is as well—restricting entailment to pointed models with state-based accessibility relations R_s, we have $[b \vee c]^+ \vDash \Diamond_s b \wedge \Diamond_s c$.

Formulas which are not pragmatically enriched (these are all classical formulas—downward-closed formulas with the empty team property), then, express information which does not incorporate any pragmatic component of the kind encoded by the pragmatic enrichment function, whereas pragmatically enriched formulas (these are non-downward-closed formulas without the empty team property) express information which does have this kind of component. This *pragmatic information* is global and non-factual in a similar way as the expressive epistemic information of the previous section.

A sentence such as

(2) a. Sue went to the beach or to the cinema, but she did not go to the cinema.
b. $(b \vee c) \wedge \neg c$

could, then, be classified as a *pragmatic contradiction*, or a contradiction involving pragmatic content, in that there is no contradiction on the non-pragmatic level—$(b \vee c) \wedge \neg c \not\models \bot$—but there is one when pragmatic content is taken into account: $[(b \vee c) \wedge \neg c]^+ \models \perp\!\!\!\perp \models \bot$. Many pragmatic contradictions, on this definition, (including the present example) involve pairs that are $\bot$-I but not G-I (and hence $\perp\!\!\!\perp$-I, assuming convexity), and, assuming a setting with only ML-formulas and pragmatically enriched formulas, all such pairs must clearly be pragmatic contradictions. When no pragmatic enrichment is present (when all formulas are downward closed and have the empty team property), there are no pragmatic contradictions, so ground-incompatibility and $\bot$-incompatibility coincide (Fact 4.2.14). We can think of ground-incompatibility as corresponding to contradiction in factual information as before, and of $\bot$-incompatibility without ground-incompatibility as contradiction involving pragmatic information.

Note, however, that on this definition of pragmatic contradiction, there are some pragmatic contradictions in BSML involving ground-incompatible and hence factually contradictory pairs. This type of contradiction can be generated, for instance, by the type of sentence involved in *free choice inferences* (see, e.g., [128, 87]) such as the following:

(3) a. You may go to the beach or to the cinema.
$\rightsquigarrow$ You may go to the beach and you may go to the cinema.
b. $\Diamond(b \vee c) \rightsquigarrow \Diamond b \wedge \Diamond c$

As with modal disjunction inferences, Aloni [6] accounts for inferences such as (3) using pragmatic enrichment—the following entailment holds: $[\Diamond(b \vee c)]^+ \models \Diamond b \wedge \Diamond c$ (where $\Diamond$ is deontic).

Now consider:

(4) a. You may go to the beach or to the cinema, but you may not go to the cinema.
b. $\Diamond(b \vee c) \wedge \neg\Diamond c$
c. $[\Diamond(b \vee c) \wedge \neg\Diamond c]^+$

By our definition, this is a pragmatic contradiction: $\Diamond(b \vee c) \wedge \neg\Diamond c \not\models \bot$ and $[\Diamond(b \vee c) \wedge \neg\Diamond c]^+ \models \perp\!\!\!\perp \models \bot$. However, the relevant pair in this case is a ground-incompatibility ($|[\Diamond(b \vee c)]^+| \cap |[\neg\Diamond c]^+| = \varnothing$; this is because if $w \in s \models [\Diamond(b \vee c)]^+$, then there must be a $t \subseteq R[w]$ such that $t \models c \wedge \mathrm{NE}$, but if also $w \in u \models [\neg\Diamond c]^+$, we must have $R[w] \models \neg c$, conflicting with the existence of a t as above), indicating, by the lights of our interpretation, that it involves a contradiction in factual information.

To clarify, our interpretation gives rise to the following characterization of this situation: the pair in question is a contradiction in factual information (in the sense we have been considering: arising from a conflict in constraints on possible worlds), with the factual information in question engendered by pragmatic enrichment (and so it is in this sense a pragmatic contradiction). The modalities in BSML are sensitive to restrictions on states such as those generated by pragmatic enrichment, but they define constraints on worlds (see endnote 19). (4) c defines a constraint on worlds that involves restrictions (which incorporate some pragmatic content) on the kinds of states that can be deontically accessible from a world if it is to meet the constraint. The restrictions on accessible states are mutually exclusive due to the pragmatic content (we must have both $\exists t \subseteq R[w] : t \vDash c \wedge \text{NE}$; and $R[w] \vDash \neg c$; it seems to the interpreter that the speaker is simultaneously claiming that there are non-empty accessible cinema-states, and that there no non-empty accessible cinema-states), and so the factual constraint is impossible to meet; therefore (4) c is a factual contradiction.

In this example, we also have NEs outside the scope of the modalities, and hence (4) c also places a non-factual pragmatic constraint on information states. This results in (4) c being not only a factual contradiction (a ground-incompatibility), but also a contradiction involving pragmatic information in some way (a $⫫$-incompatibility; cf. the pair $r \wedge \blacklozenge q, \neg r$ in Section 4.4.2). In what way? One way to conceive of NE as it figures in pragmatically enriched formulas is as an explicit marker of the pragmatic understanding that speakers usually represent themselves as not uttering contradictions (in Aloni's [6] succinct formulation, 'avoid $\bot$'). The interpreter applies this understanding to all structures involved in interpreting (4), and, as we have just seen, applying it in the scope of the modals leads to the conclusion that the speaker has expressed something contradictory: $\Diamond((b \wedge \text{NE}) \vee (c \wedge \text{NE})) \wedge \neg \Diamond(c \wedge \text{NE}) \vDash \bot$. Applying 'avoid $\bot$' outside the scope of the modals then engenders a further pragmatic violation: the speaker is supposedly presenting themselves as uttering no contradictions ($\wedge$NE), but they just expressed a contradiction ($\bot$). We may think of this further pragmatic violation as being represented by $⫫$.

4.4.4 Contradictions with Inquisitive Content

How about NE-I but not G-I pairs such as $\neg p \mathbin{\vee\!\!\!\vee} ((p \wedge \text{NE}) \vee (\neg p \wedge \text{NE}))$ and p? By Fact 4.2.16, such pairs require violation of convexity—in order to consider these pairs, we therefore now move to the non-convex setting of $\text{PL}(\text{NE}, \mathbin{\vee\!\!\!\vee})$ (we omit the modalities to simplify our discussion; we also assume that all formulas are in negation normal form, again for the sake of simplicity). We extend the picture of Section 4.4.2 by adapting the interpretation of $\mathbin{\vee\!\!\!\vee}$ from inquisitive logic/semantics [38, 35] to provide an interpretation of NE-I but not G-I pairs.

Inquisitive semantics provides an account of the meanings of questions using teams (information states). On this account, while to understand the meaning of a declarative sentence (or *statement*) is to understand when the sentence is true—to know its truth conditions—to understand the meaning of a question is to understand what information it would take to resolve the question—to know its *resolution conditions*. Truth conditions and resolution conditions may both be represented using support conditions: a statement is supported by s if it is true in s, while a question is supported by s if it is resolved in s (s supporting φ may be thought of as s *settling* or *establishing* φ in a general sense applicable to both statements and questions).

Each resolution to a question is taken to be expressed by a statement, so the resolution conditions of a question are of the form: one of the statements expressing a resolution to this question is true. This is modelled using the global/inquisitive disjunction: $p \mathbin{\vee\!\!\!\vee} \neg p$ corresponds to the question whether p or $\neg p$ is the case; it is resolved (supported) just in case either p is true (supported) or $\neg p$ is true. Intuitively, the characteristic which distinguishes questions from statements on this account is that whereas a statement can only be settled in one way in an information state (a statement is settled in s just in case it is true according to the information in s), a question can be settled in multiple ways, with each distinct resolution to a question providing a distinct way of settling the question (if an expression does not raise the question as to which of its resolutions is true—if it has only one resolution—it is not a question).

Formally, we recursively define the set of *resolutions* $\mathcal{R}(\varphi)$ of $\varphi \in \mathrm{PL}(\mathrm{NE}, \mathbin{\vee\!\!\!\vee})$ in negation normal form as follows: $\mathcal{R}(\psi) := \psi$ if ψ is an atom or a negated atom; $\mathcal{R}(\psi \wedge / \vee \chi) := \{\psi' \wedge / \vee \chi' \mid \psi' \in \mathcal{R}(\psi), \chi' \in \mathcal{R}(\chi)\}$; $\mathcal{R}(\psi \mathbin{\vee\!\!\!\vee} \chi) := \mathcal{R}(\psi) \cup \mathcal{R}(\chi)$. Note that clearly each resolution is $\mathbin{\vee\!\!\!\vee}$-free (and hence union closed and convex); and that we allow for equivalent but distinct resolutions. One can show that:

4.4.1. PROPOSITION. *For each $\varphi \in \mathrm{PL}(\mathrm{NE}, \mathbin{\vee\!\!\!\vee})$ (in negation normal form), $\varphi \equiv \mathop{\vee\!\!\!\vee} \mathcal{R}(\varphi)$.*

Proof:
Follows from the fact that $\wedge$ and $\vee$ distribute over $\mathbin{\vee\!\!\!\vee}$—see [7, 35]. □

We will say that φ is a *statement* if $|\mathcal{R}(\varphi)| = 1$, and that it is a *question* if $|\mathcal{R}(\varphi)| > 1$. We now have, in addition to informative content (comprising both factual information and expressive epistemic information) also *inquisitive content*: φ has inquisitive content just in case it is a question. Say that φ has *trivial inquisitive content* if it is a union-closed and convex question, and that it has *non-trivial inquisitive content* if it is a question that is either not union closed or not convex (examples of questions with trivial inquisitive content: $p \mathbin{\vee\!\!\!\vee} (p \wedge (q \vee \neg q))$; $(p \wedge q) \mathbin{\vee\!\!\!\vee} (p \wedge \mathrm{NE})$). (These definitions are based on the fact that union-closed and convex formulas are equivalent to ones which do not contain the question-forming connective $\mathbin{\vee\!\!\!\vee}$ (Theorems 4.2.8 (i) and 4.3.13 (i)); we therefore take it that they do not express genuine inquisitive content. On the other hand, if we assume that $\mathbin{\vee\!\!\!\vee}$ only contributes inquisitive content, any formula not expressible in the fragment of the language not containing $\mathbin{\vee\!\!\!\vee}$ must express non-trivial inquisitive content.)[23]

[23]The notions introduced in Section 4.4.4 are adapted from notions used in inquisitive logic, but many of our definitions (including those of statement and question) depart from those most commonly

We will take it that the ground team $|\varphi|$ of φ still expresses the factual information conveyed by φ—in effect, moving from $\|\varphi\|$ to $|\varphi|$ now cancels both inquisitive content and expressive epistemic information. The factual information expressed by a question φ amounts to the constraint that in each world forming part of an information state in which φ is resolved, (the factual component of) at least one of the resolutions of φ is true: by Proposition 4.4.1, for each $\varphi \in \mathrm{PL}(\mathrm{NE}, \mathbin{\setminus\!\!\!/})$ (in negation normal form), $|\varphi| = |\mathbin{\setminus\!\!\!/}\mathcal{R}(\varphi)| = \bigcup_{\varphi' \in \mathcal{R}(\varphi)} |\varphi'|$, so $w \in |\varphi|$ iff $w \in |\varphi'|$ for some $\varphi' \in \mathcal{R}(\varphi)$. This can be thought of as representing the fact that the question as to which element of $\mathcal{R}(\varphi)$ is true would seem to presuppose that at least one of the elements of $\mathcal{R}(\varphi)$ is indeed true—the factual information expressed by a question is the factual information that the question presupposes.

Now note that:

4.4.2. FACT. If $\varphi, \psi \in \mathrm{PL}(\mathrm{NE}, \mathbin{\setminus\!\!\!/})$ are NE-I and not G-I, then

(i) $\varphi \wedge \psi$ has trivial inquisitive content;

(ii) for each $\varphi' \wedge \psi' \in \mathcal{R}(\varphi \wedge \psi)$, the pair φ', ψ' is either G-I or $\perp\!\!\!\perp$-I; and

(iii) there is at least one $\varphi' \wedge \psi' \in \mathcal{R}(\varphi \wedge \psi)$ such that φ', ψ' are G-I and not $\perp\!\!\!\perp$-I, and at least one $\varphi' \wedge \psi' \in \mathcal{R}(\varphi \wedge \psi)$ such that φ', ψ' are $\perp\!\!\!\perp$-I and not G-I.

Proof:
(i) First assume for contradiction that $\varphi \wedge \psi$ is not a question. Then $|\mathcal{R}(\varphi \wedge \psi)| = 1$, whence also $|\mathcal{R}(\varphi)| = |\mathcal{R}(\psi)| = 1$. By Proposition 4.4.1, $\varphi \equiv \mathbin{\setminus\!\!\!/}\mathcal{R}(\varphi)$. Since $|\mathcal{R}(\varphi)| = 1$, φ is simply equivalent to the only element of $\mathcal{R}(\varphi)$, and is therefore convex. Similarly for ψ. Then by Fact 4.2.16, φ and ψ cannot NE-I without being G-I, contradicting our assumptions. Therefore, $\varphi \wedge \psi$ must be a question. Since φ, ψ are NE-I we have $\varphi \wedge \psi \equiv \bot$, so $\varphi \wedge \psi$ is union-closed and convex, and hence has trivial inquisitive content.

(ii) First assume for contradiction that for some $\varphi' \wedge \psi' \in \mathcal{R}(\varphi \wedge \psi)$, φ', ψ' are not $\bot$-I. Then there is a non-empty team s such that $s \vDash \varphi' \wedge \psi'$. But then by Proposition 4.4.1, also $s \vDash \varphi \wedge \psi$, contradicting the fact that φ and ψ are NE-I. So we must have that φ', ψ' are $\bot$-I. Then φ', ψ' are either NE-I or $\perp\!\!\!\perp$-I. Since $\varphi' \in \mathcal{R}(\varphi)$ and $\psi' \in \mathcal{R}(\psi)$, these formulas are convex; therefore, if they are NE-I, they must also be G-I by Fact 4.2.16.

(iii) Since φ, ψ are not $\perp\!\!\!\perp$-I, there is some s such that $s \vDash \varphi \wedge \psi$. By Proposition 4.4.1, also $s \vDash \mathbin{\setminus\!\!\!/}\mathcal{R}(\varphi \wedge \psi)$, so $s \vDash \varphi' \wedge \psi'$ for some $\varphi' \wedge \psi' \in \mathcal{R}(\varphi \wedge \psi)$. But then φ', ψ' are not $\perp\!\!\!\perp$-I, and by (ii) they are G-I.

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used in inquisitive logic. One reason for this is that the definitions used in inquisitive logic typically assume a downward-closed setting; another is that providing a syntactic rather than semantic criterion for questionhood and considering semantic questionhood separately via the distinction between trivial and nontrivial inquisitive content makes it easier to formulate our characterization of NE-I but not G-I pairs. It may also be surprising that we have not introduced a notion of *alternatives*. The standard treatment of alternatives assumes downward closure; it should be possible to formulate an analogue of this notion which works in our non-downward setting, but this is orthogonal to our aims here.

On the other hand, since φ, ψ are not G-I, there is some $w \in |\varphi| \cap |\psi|$, so that $w \in t \vDash \varphi$ and $w \in u \vDash \psi$ for some t and u. By Proposition 4.4.1, $t \vDash \bigvee\!\!\!\bigvee \mathcal{R}(\varphi)$ and $u \vDash \bigvee\!\!\!\bigvee \mathcal{R}(\psi)$, so for some $\varphi'' \in \mathcal{R}(\varphi)$ and some $\psi'' \in \mathcal{R}(\psi)$ we have $t \vDash \varphi''$ and $u \vDash \psi''$. But then we have $\varphi'' \wedge \psi'' \in \mathcal{R}(\varphi \wedge \psi)$, and $w \in |\varphi''| \cap |\psi''|$ so that φ'', ψ'' are not G-I. By (ii), φ'', ψ'' must be $\perp\!\!\!\perp$-I. □

We therefore have the following characterization of NE-I but not G-I pairs such as $p \vee\!\!\!\vee \blacklozenge q, \neg p \wedge \neg q$. At least one member of the pair must be a question (otherwise both are convex, contradicting Fact 4.2.16). Settling both elements of the pair simultaneously (that is, resolving the member that is a question, and either resolving or establishing the truth of the other member) in a consistent manner—in a non-empty information state—is not possible because the pair is $\bot$-I. This is because each possible way of resolving both members simultaneously (each $\varphi' \wedge \psi' \in \mathcal{R}(\varphi \wedge \psi)$) is either a factual contradiction (G-I and not $\perp\!\!\!\perp$-I), or an epistemic contradiction ($\perp\!\!\!\perp$-I and not G-I). But it is not the case that the factual information expressed by either of the members rules out the other (the pair is not G-I), or that the conflict between the pair is solely expressive-epistemic in nature (there is some $\varphi' \wedge \psi' \in \mathcal{R}(\varphi \wedge \psi)$ such that φ', ψ' are G-I and not $\perp\!\!\!\perp$-I). Rather, simultaneous settling of the pair involves resolving inquisitive content ($\varphi \wedge \psi$ is a question), with some possible resolutions being factual contradictions and some possible resolutions being epistemic contradictions. The pair expresses inquisitive content in that we can formulate ways of simultaneously settling both members (the resolutions of $\varphi \wedge \psi$) and there are more than one of these ways (again, $\varphi \wedge \psi$ is a question by our syntactic definition), but the inquisitive content is ultimately trivial because the pair is contradictory: simultaneous settling in a consistent manner is not possible, and there is (semantically) only one way of resolving $\varphi \wedge \psi$—by being in the inconsistent state $\varnothing$.

We also present a toy natural language example for concreteness, although we concede that our formalization of this example is potentially controversial. Consider the following question-answer pair: ‘Is Mary already home, or is it possible that she is still at the airport?’ ‘She’s on the train.’ We formalize these as $h \vee\!\!\!\vee \blacklozenge a$ and $\neg h \wedge \neg a$, respectively. This pair of formulas is an NE-incompatibility, and the resolutions of $(h \vee\!\!\!\vee \blacklozenge a) \wedge (\neg h \wedge \neg a)$ are $h \wedge (\neg h \wedge \neg a)$ and $\blacklozenge a \wedge (\neg h \wedge \neg a)$. The first resolution $h \wedge (\neg h \wedge \neg a)$ involves a G-I (and not $\perp\!\!\!\perp$-I) pair, and hence a factual contradiction: it is not possible to simultaneously resolve the question with the answer that Mary is at home, and to accommodate the information that Mary is on the train (and hence not at home or at the airport) because this would involve a contradiction in factual information. The second resolution $\blacklozenge a \wedge (\neg h \wedge \neg a)$ involves a $\perp\!\!\!\perp$-I and not G-I pair and hence an epistemic contradiction: it cannot be the case, according to any information state, both that Mary might be at the airport and that she is on the train. The pair is therefore contradictory ($\bot$-I)—one cannot consistently simultaneously resolve the question as presented and accept the statement.

4.5 Concluding Remarks

In this paper, we proved analogues of Burgess' theorem for BSML and $\mathrm{BSML}^{\mathbb{V}}$, for the propositional fragments of these logics, for Hawke and Steinert-Threlkeld's semantic expressivist logic for epistemic modals, as well as for propositional dependence logic with the dual negation. We saw that the notion of incompatibility employed to secure a Burgess theorem had to be adjusted according to the logic in question, formulated notions suitable for each of the logics we considered, and examined the relationships between these notions. We defined the notion of bicompleteness to succinctly describe our results: a logic is `bicomplete` with respect to a class of pairs of the relevant sort and conforming to a particular incompatibility notion just in case both a Burgess theorem employing that incompatibility notion as well as its converse hold for the logic. We also applied the notion of bicompleteness to logics which do not exhibit failure of determination of negated meanings by positive meanings, and gave an example of a logic which is `bicomplete` for all pairs.

The incompatibility notions and bicompleteness are interesting in their own right, and might warrant further study. We conclude with some final remarks on these notions, and on the interpretation and potential uses of our results.

We noted in Section 4.1 that Burgess characterized his theorem as concerning the degree of failure of contrariness/the dual negation (in H_p, and equivalently in IF and in D) to correspond to any operation on classes of models. One can think of our `bicompleteness` results and incompatibility notions as giving us a way of measuring this degree in different logics, with `bicompleteness` with respect to a weaker notion of incompatibility corresponding to a stronger degree of failure. On one end of the scale we have the classical PL, which exhibits no such failure—the class of teams $\|\varphi\|$ on which φ is true determines that $\|\neg\varphi\|$ on which $\neg\varphi$ is true since, as per flat-incompatibility, $\|\neg\varphi\| = \wp(|\top| \setminus |\varphi|)$—moreover, $\|\neg\varphi\|$ also determines $\|\varphi\|$: $\|\varphi\| = \wp(|\top| \setminus |\neg\varphi|)$. To get `bicompleteness` for InqB, we must go from the stronger flat-incompatibility to the weaker D-I $\|\neg\varphi\| = \{s \mid [t \subseteq s \text{ and } t \vDash \varphi] \implies t = \varnothing\}$—$\|\varphi\|$ still determines $\|\neg\varphi\|$, but the converse is no longer true. Further along the scale, we have mild failure of determination: for HS and E-D-I, it is not the case that $\|\varphi\|$ always determines $\|\neg\varphi\|$ or vice versa, but it is always the case that one of the two determines the other. Propositional dependence logic violates determination in a different way with ground-complements $|\neg\varphi| = |\top| \setminus |\varphi|$. Neither $\|\varphi\|$ nor $\|\neg\varphi\|$ determines the other, but $\|\varphi\|$ does determine the ground team $|\neg\varphi|$ of the negation, and similarly for $\|\neg\varphi\|$ and $|\varphi|$. Even further along we have first-order dependence logic D, and we must now weaken ground-complements to $\bot$-incompatibility $\varphi, \neg\varphi \vDash \bot$. Properties and ground teams are no longer determined, but we do at least know that $\|\varphi\|$ and $\|\neg\varphi\|$ are disjoint modulo the empty team. At the very far end of the scale we have $\mathrm{PL}(\mathrm{NE}^*, \mathbb{V})$, `bicomplete` for all pairs: $\|\varphi\|$ does not constrain $\|\neg\varphi\|$ in any way.

We saw that dual negations in seemingly similar logics can conform to very different incompatibility notions. For instance, whereas BSML is `bicomplete` for ground-incompatible pairs—pairs $(\mathcal{P}, \mathcal{Q})$ in which $\mathcal{P}$ places only a very weak constraint on $\mathcal{Q}$

($\bigcup \mathcal{P} \cap \bigcup \mathcal{Q} = \varnothing$)—simply removing the modalities gives us a logic (PL(NE)) which is `bicomplete` for pairs which are ground-complements modulo $\perp\!\!\!\perp$—pairs $(\mathcal{P}, \mathcal{Q})$ such that, as long as $\mathcal{P} \neq \varnothing \neq \mathcal{Q}$, all the valuation-level information concerning $\mathcal{P}$ can be recovered from $\mathcal{Q}$ ($\bigcup \mathcal{P} = |\top| \setminus \bigcup \mathcal{Q}$). Similarly, whereas first-order dependence logic D is `bicomplete` for $\perp$-incompatible pairs, the natural propositional analogue PL$(=(\cdot))$ is `bicomplete` for ground-complementary pairs.

On the one hand, these types of results are to be expected whenever the semantics of the negation $\neg\varphi$ of φ are defined in the bilateral way we have been observing, with the support conditions of $\neg\varphi$ not depending in a uniform manner on the positive semantic value $\|\varphi\|$ of the argument φ, but depending, rather, on the anti-support conditions specific to the main connective or atom of φ. The pair properties that a given logic conforms to are a function of each of the anti-support clauses; if new atoms or connectives are added, their anti-support clauses must be configured to guarantee or bring about whatever properties are desired (for instance, switching out NE, in PL(NE, $\vee\!\!\!\vee$) with the atom NE*, whose semantics clearly do not conform to any of our incompatibility notions, allowed us to violate all of these notions).[24] On the other hand, however, the results mentioned above demonstrate the surprising ways in which the anti-support conditions of different connectives can interact to violate aspects of determination which the conditions of any given connective do not violate on their own. If we add the BSML-modalities to PL, we have ML, which, it is easy to see, is `bicomplete` for flat-incompatible pairs. If, instead of the modalities, we add NE, we will thereby violate determination of $\|\neg\varphi\|$, but will still have determination of $|\neg\varphi|$. But if we add both the modalities and NE, we have BSML, which violates determination in a more radical way.

The standard expressive completeness theorems with respect to team-semantic closure properties (such as 4.2.8, 4.3.2, 4.3.13, and 4.3.27) play an important role in the study of team logics, allowing for concise and tractable characterizations of these logics, and providing useful resources for axiomatization completeness proofs (see, e.g., [139, 140]) and the proofs of other properties such as uniform interpolation [43]. Are there any similar technical applications for bicompleteness theorems? It seems more

[24] The dependence of the semantics of the dual negation on the main logical symbol of the argument is obvious in the bilateral support/anti-support semantics we have been considering. It is also obvious in the team semantics for IF [78] and D [119], and Väänänen's team-based game semantics for D [119, Section 5.2]—each of these systems features two clauses/rules for each connective or atom, one applied when the symbol in question appears in a positive context, the other when it appears in a negative context. However, it should be noted that this dependence is far less clear in the original assignment-based game semantics for IF [75] and D [119, Section 5.3]. This type of semantics has a single game rule for each connective or atom, and the behavior of the negation is brought about by the interaction of these rules. The team semantics for these logics (whether bilateral or game-theoretical), accordingly, also make it easier to see why Burgess theorems hold, and enable one to easily adjust the properties of the dual negation as described in the main text. It seems that making similar adjustments would be more challenging in an assignment-, world-, or valuation-based game semantics, and it is not clear to me whether one can formulate (reasonable) world- or valuation-based game semantics for the logics we consider in this paper.

likely to me that theorems of this type would find conceptual application—we have observed for instance, that we may think of them as providing a measure of the degree of failure of determination of negative meanings by positive ones, and that which incompatibility notions a given logic conforms to can provide insight into which types of information are clashing in a contradiction $\varphi \wedge \neg \varphi$ of the logic. To that end, it may prove interesting and fruitful to establish `bicompleteness` results for more logics in the philosophical logic and formal semantics literature featuring bilateral negations, including logics which do not employ team semantics.

Chapter 5

Axiomatizing Modal Inclusion Logic and its Variants

This chapter is based on:

Aleksi Anttila, Matilda Häggblom, and Fan Yang. "Axiomatizing modal inclusion logic and its variants". In: *Arch. Math. Logic* (2024). Forthcoming. arXiv: `2312.02285 [math.LO]`. URL: `https://arxiv.org/abs/2312.02285`

Abstract We provide a complete axiomatization of modal inclusion logic—team-based modal logic extended with inclusion atoms. We review and refine an expressive completeness and normal form theorem for the logic, define a natural deduction proof system, and use the normal form to prove completeness of the axiomatization. Complete axiomatizations are also provided for two other extensions of modal logic with the same expressive power as modal inclusion logic: one augmented with a might-operator and the other with a single-world variant of the might-operator.

5.1 Introduction

In this article, we axiomatize modal inclusion logic and two other expressively equivalent logics. Modal inclusion logic extends the usual modal logic with *inclusion atoms*—non-classical atoms the interpretation of which requires the use of *team semantics*. In team semantics—introduced by Hodges [78, 79] to provide a compositional semantics for Hintikka and Sandu's *independence-friendly logic* [77, 75], and developed further by Väänänen in his work on *dependence logic* [119]—formulas are interpreted with respect to sets of evaluation points called *teams*, as opposed to single evaluation points. In Hodges' and Väänänen's first-order setting, teams are sets of variable assignments; we will mainly work in *modal team semantics* (first considered in [120]), in which teams are sets of possible worlds in a Kripke model. The shift to teams enables one to express that certain relationships hold between the truth/assignment values obtained by formulas/variables in the worlds/assignments in a team, rendering team-based logics generally more expressive than their singularly evaluated counterparts. The articulation of these relationships is typically accomplished by furnishing team-based logics with various *atoms of dependency* expressing different kinds of relationships. The first such atoms to be considered were Väänänen's *dependence atoms* [119, 120]. Another example of note are Grädel and Väänänen's *independence atoms* [59].

Inclusion atoms were introduced by Galliani in [51] to import the notion of *inclusion dependencies* from database theory (see, e.g., [30]) into team semantics. In the modal setting, an inclusion atom is a formula of the form $\mathsf{a} \subseteq \mathsf{b}$, where a and b are finite sequences of formulas of modal logic of the same length. The atom $\mathsf{a} \subseteq \mathsf{b}$ is satisfied in a team (i.e., a set of possible worlds) if any sequence of truth values assigned to the formulas in a by some world in the team is also assigned to the sequence b by some world in the team. Consider a database in which various facts about the stores in some town have been collected. Each row (world) of the database contains data about a single store, and each column (propositional symbol) corresponds to a category of collected data such as whether a store sells flowers: if a store sells flowers, then the row for the store has a 1 in the `Sell_flowers` column. If the team consisting of all rows corresponding to stores in some neighbourhood satisfies the atom `Sell_flowers` $\subseteq$ `Sell_mulch`, we know that if there is a store in the neighbourhood that sells flowers, there is one that sells mulch (and that if there is one that does not sell flowers, there is one that does not sell mulch). The atom $\top\bot \subseteq$ `Sell_flowers Sell_food`, on the other hand, would express that there is a store that sells flowers but not food.

Our main focus in this article is on modal logic extended with inclusion atoms, or *modal inclusion logic*, which we will denote by ML($\subseteq$). The expressive power of ML($\subseteq$) has been studied in [73, 89] and its complexity in [69, 70], but the logic has so far not been axiomatized. We fill this gap in the literature by providing a sound and complete natural deduction system for the logic. Our system is an extension of the system for propositional inclusion logic introduced in [137].

In addition to ML($\subseteq$), we axiomatize two other logics with the same expressive power: ML($\triangledown$) and ML($\dot\triangledown$), or modal logic extended with a *might-operator* $\triangledown$ (first considered in the team semantics literature in [73]) and with a *singular might-operator* $\dot\triangledown$ that we introduce in this article, respectively. The names reflect the fact that similar operators have been used to model the meanings of epistemic possibility modalities such as the "might" in "It might be raining"—see, for instance, [129, 123].

Each of these three logics—ML($\subseteq$), ML($\triangledown$), and ML($\dot\triangledown$)—is *closed under unions*: if a formula is true in all teams in a nonempty collection of teams, it is true in the team formed by union of the collection. The most well-known team-based logics such as dependence logic are not union closed, but union-closed logics have recently been receiving more attention in the literature (see, e.g., [6, 7, 54, 57, 69, 70, 73, 80, 136, 137]). Hella and Stumpf proved in [73] that each of ML($\subseteq$) and ML($\triangledown$) is expressively complete for team properties that are closed under unions and bounded bisimulations, and which contain the empty team. We revisit this proof and also show that $\mathrm{ML}(\dot{\triangledown})$ is complete for this class of properties. These proofs of expressive completeness also yield normal forms for each of the three logics. The normal forms play a crucial role in the completeness proofs for our axiomatizations.

The structure of the article is as follows: in Section 5.2, we define the syntax and semantics of modal inclusion logic $\mathrm{ML}(\subseteq)$ and the might-operator logics $\mathrm{ML}(\triangledown)$ and $\mathrm{ML}(\dot{\triangledown})$, and discuss some basic properties of these logics. In Section 5.3 we prove the expressive completeness results discussed above. In Section 5.4, we introduce natural deduction systems for each of the logics and show that they are complete. We conclude the article and discuss directions for possible further research in Section 5.5. In an appendix (Section 5.6) we provide a translation of $\mathrm{ML}(\subseteq)$ into first-order inclusion logic.

This article is partly based on Häggblom's master's thesis [63], supervised by Yang and Anttila, which contains preliminary versions of some of the results.

5.2 Preliminaries

In this section, we define the syntax and semantics of modal inclusion logic $\mathrm{ML}(\subseteq)$ and the two might-operator logics $\mathrm{ML}(\triangledown)$ and $\mathrm{ML}(\dot{\triangledown})$. We also discuss some basic properties of these logics.

Fix a (countably infinite) set Prop of propositional symbols.

5.2.1. DEFINITION. The syntax for the usual modal logic (ML) is given by the grammar:

$$\alpha ::= p \mid \bot \mid \neg\alpha \mid (\alpha \vee \alpha) \mid (\alpha \wedge \alpha) \mid \Diamond\alpha \mid \Box\alpha,$$

where $p \in \mathsf{Prop}$. Define $\top := \neg\bot$. We also call formulas of ML *classical formulas*. Throughout the article, we reserve the first Greek letters α and β for classical formulas.

The syntax for modal inclusion logic ($\mathrm{ML}(\subseteq)$) is given by:

$$\varphi ::= p \mid \bot \mid (\alpha_1 \dots \alpha_n \subseteq \beta_1 \dots \beta_n) \mid \neg\alpha \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \Diamond\varphi \mid \Box\varphi,$$

where $p \in \mathsf{Prop}$, and $\alpha, \alpha_i, \beta_i$ (for all $1 \le i \le n$) range over classical formulas. We write $\mathsf{Prop}(\varphi)$ for the set of propositional symbols appearing in φ, and $\varphi(\mathsf{X})$ if $\mathsf{Prop}(\varphi) \subseteq \mathsf{X} \subseteq \mathsf{Prop}$.

The above syntax of $\mathrm{ML}(\subseteq)$ deserves some comment. First, we only allow negation to occur in front of classical formulas. Next, our inclusion atoms $\alpha_1 \dots \alpha_n \subseteq \beta_1 \dots \beta_n$ with α_i and β_j being (possibly complex) classical formulas are known in the literature also as *extended inclusion atoms*, and the logic $\mathrm{ML}(\subseteq)$ defined above is sometimes also referred to as *extended modal inclusion logic*—in contexts which use this terminology, a distinction is drawn between extended inclusion atoms and inclusion atoms *simpliciter*, the latter referring to atoms which may only contain propositional symbols as subformulas (modal inclusion logic *simpliciter*, in these contexts, is then the variant which only allows these simpler atoms). In this article we only study the extended variant. We do not allow nested inclusion atoms; for example, $p \subseteq (p \subseteq q)$ is not a formula of $\mathrm{ML}(\subseteq)$. We usually omit the parentheses around inclusion atoms, and stipulate higher precedence for the inclusion symbol $\subseteq$ than the other connectives; for instance, $p \wedge q \subseteq r$ has the subformula $q \subseteq r$.

Modal inclusion logic is interpreted on standard Kripke models, but we use teams (sets of worlds) rather than single worlds as points of evaluation. A *(Kripke) model* $M = (W, R, V)$ (over $\mathsf{X} \subseteq \mathsf{Prop}$) consists of a set W of *possible worlds*, a binary *accessibility relation* $R \subseteq W \times W$ and a *valuation function* $V : \mathsf{X} \to \mathcal{P}(W)$. A *team* T of M is a subset $T \subseteq W$ of the set of worlds in M. The *image* of T (denoted $R[T]$) and the *preimage* of T (denoted $R^{-1}[T]$) are defined as

$$R[T] = \{v \in W \mid \exists w \in T : wRv\}, \text{ and}$$

$$R^{-1}[T] = \{w \in W \mid \exists v \in T : wRv\}.$$

We say that a team S of M is a *successor team* of T, written TRS, if

$$S \subseteq R[T] \text{ and } T \subseteq R^{-1}[S];$$

that is, every world in S is accessible from a world in T, and every world in T has an accessible world in S.

5.2.2. DEFINITION. For any Kripke model M over X and team T of M, the satisfaction relation $M, T \vDash \varphi$ (or simply $T \vDash \varphi$) for an $\mathrm{ML}(\subseteq)$-formula $\varphi(\mathsf{X})$ is given inductively by the following clauses:

$$\begin{aligned}
M, T \vDash p &\iff T \subseteq V(p).\\
M, T \vDash \bot &\iff T = \varnothing.\\
M, T \vDash \alpha_1 \dots \alpha_n \subseteq \beta_1 \dots \beta_n &\iff \text{for all } w \in T, \text{ there exists } v \in T \text{ such that for all}\\
&\qquad 1 \leq i \leq n,\ M, \{w\} \vDash \alpha_i \text{ iff } M, \{v\} \vDash \beta_i.\\
M, T \vDash \neg \alpha &\iff M, \{w\} \nvDash \alpha \text{ for all } w \in T.\\
M, T \vDash \varphi \vee \psi &\iff M, T_1 \vDash \varphi \text{ and } M, T_2 \vDash \psi \text{ for some } T_1, T_2 \subseteq T\\
&\qquad \text{such that } T_1 \cup T_2 = T.\\
M, T \vDash \varphi \wedge \psi &\iff M, T \vDash \varphi \text{ and } M, T \vDash \psi.\\
M, T \vDash \Diamond \varphi &\iff M, S \vDash \varphi \text{ for some } S \text{ such that } TRS.\\
M, T \vDash \Box \varphi &\iff M, R[T] \vDash \varphi.
\end{aligned}$$

We say that a set of formulas Γ *entails* φ, written $\Gamma \vDash \varphi$, if for all models M and teams T of M, if $M, T \vDash \gamma$ for all $\gamma \in \Gamma$, then $M, T \vDash \varphi$. We write simply $\varphi_1, \dots, \varphi_n \vDash \varphi$ for $\{\varphi_1, \dots, \varphi_n\} \vDash \varphi$ and $\vDash \varphi$ for $\varnothing \vDash \varphi$, where $\varnothing$ is the empty set of formulas. If both $\varphi \vDash \psi$ and $\psi \vDash \varphi$, we say that φ and ψ are *equivalent*, and write $\varphi \equiv \psi$.

We often write $T \models \varphi$ instead of $M,T \models \varphi$.

It is easy to verify that formulas of ML($\subseteq$) have the following properties:

Union closure: If $M,T_i \models \varphi$ for all $i \in I \neq \emptyset$, then $M, \bigcup_{i\in I} T_i \models \varphi$.

Empty Team Property: $M, \emptyset \models \varphi$ for all models M.

Classical formulas α (i.e., formulas of ML) are, in addition, *downward closed*, meaning that $M,T \models \alpha$ implies $M,S \models \alpha$ for all $S \subseteq T$. It is easy to show that a formula has the downward closure property, union closure property and the empty team property if and only if it has the *flatness property*:

Flatness: $M,T \models \varphi$ iff $M,\{w\} \models \varphi$ for every $w \in T$.

Classical formulas are thus flat. It is also straightforward to verify that for any classical formula α,

$$M,\{w\} \models \alpha \iff M,w \models \alpha,$$

where $\models$ on the right is the usual single-world-based satisfaction relation for ML. It follows from this that ML (with team semantics) coincides with the usual single-world based modal logic, and hence that ML($\subseteq$) is *conservative* over the usual modal logic.

5.2.3. PROPOSITION. *For any set* $\Gamma \cup \{\alpha\}$ *of* ML*-formulas,*

$$\Gamma \models \alpha \iff \Gamma \models^c \alpha,$$

where $\models^c$ *is the usual entailment relation for* ML *(over the single-world semantics).*

Given these facts, we use the notations $M,\{w\} \models \alpha$ and $M,w \models \alpha$ interchangeably whenever α is classical, and similarly for $\Gamma \models \alpha$ and $\Gamma \models^c \alpha$.

Let us briefly comment on the truth conditions for the modalities. Recall our store database example from Section 5.1. If the database is also equipped with an accessibility relation detailing the (epistemically) possible future inventories of each store, we could use a modal statement such as $\diamondsuit(\top\bot \subseteq \texttt{Sell\_flowers}\,\texttt{Sell\_food})$ to express that there might in the future be a store that sells flowers and does not sell food. Note that if S is a successor team of T, there may, for any given world w in T, be multiple worlds accessible to w in S. Interpretations of the modalities must take this into account—for instance, in our database example, a successor team may contain multiple inconsistent records for each store.

One may also consider alternative team-semantic truth conditions for the diamond, such as the following (called the *strict semantics* for the diamond, whereas the semantics above are the *lax semantics*):

$$T \models \diamondsuit_s \varphi \iff \exists f : T \to R[T] \text{ s.t. } \forall w \in T : wRf(w) \text{ and } f[T] \models \varphi,$$

where $f[T] = \{f(w) \mid w \in T\}$. With the strict semantics, ML($\subseteq$) is no longer union closed, and it also fails to be bisimulation-invariant for the notion of team bisimulation established in the literature.[1] In this article we only consider the lax semantics. For more on strict and lax semantics, see [51, 137, 69, 70].[2] For other alternative sets of team-semantic truth conditions for the modalities, see [42, 34, 6, 7].

We next discuss *primitive inclusion atoms*—inclusion atoms of a specific restricted form which play a crucial role in our expressive completeness results. We often abbreviate a sequence $\langle\alpha_1,\dots,\alpha_n\rangle$ of classical formulas as a; similarly, b is short for $\langle\beta_1,\dots,\beta_n\rangle$, etc. We also often write x for a sequence $\langle x_1,\dots,x_n\rangle$ in which each x_i is one of the constants $\bot$ or $\top$. The *arity* of an inclusion atom $\mathsf{a}\subseteq\mathsf{b}$ is defined as the length $|\mathsf{a}|$ of the sequence a. *Primitive inclusion atoms* are inclusion atoms of the form $\mathsf{x}\subseteq\mathsf{a}$—that is, they are atoms with only the constants $\bot$ and $\top$ on the left-hand side. We additionally call primitive inclusion atoms $\top\dots\top\subseteq\alpha_1\dots\alpha_n$ with only the constant $\top$ on the left-hand side *top inclusion atoms*. Primitive inclusion atoms $\mathsf{x}\subseteq\mathsf{a}$ are clearly *upward closed* (modulo the empty team property), i.e., for any nonempty teams T and S, if $T\vDash\mathsf{x}\subseteq\mathsf{a}$ and $S\supseteq T$, then $S\vDash\mathsf{x}\subseteq\mathsf{a}$.

Primitive inclusion atoms are relatively tractable, as the next proposition shows. Hereafter, for any $x\in\{\top,\bot\}$, we write $\alpha^\top$ for α, and $\alpha^\bot$ for $\neg\alpha$. For $x_1,\dots,x_n\in\{\top,\bot\}$, we abbreviate $\alpha_1^{x_1}\wedge\dots\wedge\alpha_n^{x_n}$ as a^{x}.

5.2.4. PROPOSITION. *For any nonempty team T,*

$$T\vDash\mathsf{x}\subseteq\mathsf{a} \textit{ iff there exists } v\in T \textit{ such that } v\vDash\mathsf{a}^{\mathsf{x}}.$$

Proof:
Suppose $T\neq\varnothing$ and $T\vDash\mathsf{x}\subseteq\mathsf{a}$. Let $w\in T$. Then there is a $v\in T$ such that $w\vDash x_i$ iff $v\vDash\alpha_i$ for all $i=1,\dots,n$. Let $i\in\{1,\dots,n\}$. If $x_i=\top$, then $w\vDash x_i$, so $v\vDash\alpha_i$. Hence $v\vDash\alpha_i^\top$. If $x_i=\bot$, then $w\nvDash x_i$, so $v\nvDash\alpha_i$. Hence $v\vDash\alpha_i^\bot$. So for all $i=1,\dots,n$, $v\vDash\alpha_i^{x_i}$. We conclude $v\vDash\mathsf{a}^{\mathsf{x}}$. The other direction is similar. □

5.2.5. COROLLARY. $\bot\subseteq\alpha\equiv\top\subseteq\neg\alpha$.

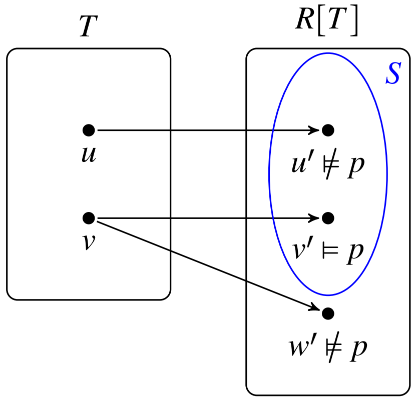
The following example illustrates an interesting consequence of the upward closure (modulo the empty team property) of primitive inclusion atoms in our modal setting.

5.2.6. EXAMPLE. Consider the model $M=(W,R,V)$ as illustrated in the figure below, where the relation R is represented by the arrows, and $V(p)=\{v'\}$. Consider the teams $T=\{u,v\}$ and $S=\{u',v'\}$. Since $v'\in S$ and $v'\vDash p$, by Proposition 5.2.4, $S\vDash\top\subseteq p$. It is also easy to verify that TRS, and hence $T\vDash\Diamond(\top\subseteq p)$. Similarly $R[T]\vDash\top\subseteq p$, whereby

[1]For the failure of union closure, consider the formula $\varphi:=\Diamond_s(pq\subseteq rs)$ and a model with $W=\{w_1,w_2,w_3,w_4\}$, $R=\{(w_1,w_1),(w_2,w_2),(w_2,w_3),(w_4,w_4)\}$, and $V(p)=\{w_1,w_2\}$, $V(q)=\{w_3,w_4\}$, $V(r)=\{w_2\}$, $V(s)=\{w_3\}$. We have $\{w_1,w_2\}\vDash\varphi$ and $\{w_2,w_4\}\vDash\varphi$, but $\{w_1,w_2,w_4\}\nvDash\varphi$. For the failure of bisimulation invariance, see Section 5.3.

[2]In the literature, the term "strict semantics" typically refers to the adoption of different truth conditions not only for the diamond but also for the disjunction.

$T \vDash \boxdot(\top \subseteq p)$. In fact, we have in general $\diamondsuit(\mathsf{x} \subseteq \mathsf{a}) \vDash \boxdot(\mathsf{x} \subseteq \mathsf{a})$ for an arbitrary primitive inclusion atom $\mathsf{x} \subseteq \mathsf{a}$. Moreover, observe that the subteam $\{u\}$ of T does not satisfy either $\diamondsuit(\top \subseteq p)$ or $\boxdot(\top \subseteq p)$, illustrating the failure of downward closure in ML($\subseteq$).



It was proved in [137] in the propositional context that arbitrary inclusion atoms can be defined in terms of primitive inclusion atoms, and further in terms of unary primitive inclusion atoms (inclusion atoms of the form $\top \subseteq \alpha$ or $\bot \subseteq \alpha$). This result can be extended to our current modal context by the same argument.

5.2.7. Lemma ([137]). *Let* a, b *be sequences of* ML*-formulas, and let* x, y *be sequences each of whose elements is either* $\top$ *or* $\bot$*. Then*

(i) $\bigwedge_{\mathsf{x} \in \{\top,\bot\}^{|\mathsf{a}|}} (\neg \mathsf{a}^{\mathsf{x}} \vee \mathsf{x} \subseteq \mathsf{b}) \equiv \mathsf{a} \subseteq \mathsf{b}$.

(ii) $\mathsf{xy} \subseteq \mathsf{ab} \equiv \mathsf{x} \subseteq \mathsf{a} \wedge ((\mathsf{y} \subseteq \mathsf{b} \wedge \mathsf{a}^{\mathsf{x}}) \vee \neg \mathsf{a}^{\mathsf{x}})$.

Given Corollary 5.2.5, we can further conclude that an arbitrary inclusion atom can be defined in terms of unary top inclusion atoms $\top \subseteq \alpha$. Indeed, we will see in Section 5.3 that every formula of ML($\subseteq$) is equivalent to one in a normal form which contains no inclusion atoms save for unary top inclusion atoms.

In addition to modal inclusion logic, we also consider two other extensions of the usual modal logic: ML($\triangledown$), or modal logic with the *might-operator* $\triangledown$, and ML($\dot\triangledown$), or modal logic with the *singular might-operator* $\dot\triangledown$. We define the team semantics of these operators as follows:

$$M, T \vDash \triangledown\varphi \iff T = \varnothing \text{ or there exists a nonempty } S \subseteq T \text{ such that } M, S \vDash \varphi.$$
$$M, T \vDash \dot\triangledown\varphi \iff T = \varnothing \text{ or there exists } w \in T \text{ such that } M, \{w\} \vDash \varphi.$$

The operator $\triangledown$ was first considered in the team semantics literature in [73] and the other operator $\dot\triangledown$ is introduced in the present article, but very similar operators have been employed in philosophical logic and formal semantics to model the meanings of epistemic possibility modalities such as the "might" in "It might be raining" (see, for instance [129, 123]). One can also make sense of this interpretation in the team semantics setting. A team T can be conceived of as corresponding to an *information state*: to know that the actual world v is one of the worlds in T is to know that whatever holds in all worlds in T must be the case. If it is raining in all worlds in T, this information state supports the assertion that it is raining. If, on the other hand, there is some nonempty subteam S of T consisting of worlds in which it is not raining, then for all that one knows given the information embodied in T, it might not be raining. Both operators may be thought of as expressing this kind of epistemic modality, with some subtle differences in meaning between the two (note, for instance, that if $\dot\nabla\varphi$ appears within the scope of another $\dot\nabla$ in a propositional (non-modal) formula, it can be substituted *salva veritate* with φ, whereas the analogous fact does not hold for ∇).[3]

The singular might $\dot\nabla\varphi$ clearly entails the (non-singular) might $\nabla\varphi$; the converse direction $\nabla\varphi \vDash \dot\nabla\varphi$ holds whenever φ is downward closed, and in particular whenever φ is classical. For classical formulas α, both $\nabla\alpha$ and $\dot\nabla\alpha$ coincide with the unary top inclusion atom $\top \subseteq \alpha$.

5.2.8. FACT. For any ML-formula α, $\top \subseteq \alpha \equiv \dot\nabla\alpha \equiv \nabla\alpha$.

In general, the two might-operators behave differently; in particular, there are subtle differences when the operators are iterated: $\nabla(\nabla\varphi \wedge \nabla\psi) \equiv \nabla\varphi \wedge \nabla\psi$, whereas $\dot\nabla\varphi \wedge \dot\nabla\psi \nvDash \dot\nabla(\dot\nabla\varphi \wedge \dot\nabla\psi)$; $\dot\nabla(\varphi \wedge \dot\nabla\psi) \equiv \dot\nabla(\varphi \wedge \psi)$, whereas $\nabla(\varphi \wedge \nabla\psi) \nvDash \nabla(\varphi \wedge \psi)$.

As is the case with ML($\subseteq$), the logics ML(∇) and ML($\dot\nabla$) are union closed and have the empty team property; we further show in Section 5.3 that the three logics are in fact expressively equivalent.

As with many other logics with team semantics, the three logics we consider in this article are not closed under uniform substitution. Recall that a *substitution* σ for a logic $\mathcal{L}$ is a mapping from the set of formulas $\mathcal{L}$ to itself that commutes with the connectives (and connective-like atoms such inclusion atoms) of $\mathcal{L}$. We say that $\mathcal{L}$ is *closed under* substitution σ if for any set $\Gamma \cup \{\varphi\}$ of formulas of $\mathcal{L}$,

$$\Gamma \vDash \varphi \quad \text{implies} \quad \{\sigma(\gamma) \mid \gamma \in \Gamma\} \vDash \sigma(\varphi).$$

A logic $\mathcal{L}$ is *closed under uniform substitution* if it is closed under all substitutions. Note that due to our syntactic restrictions, any map mapping a ML($\subseteq$)-formula $p \subseteq q$ to a non-formula $(r \subseteq s) \subseteq q$ is not considered a valid substitution for ML($\subseteq$). This does not mean that substitution into inclusion atoms is disallowed in general: a valid substitution might map $p \subseteq q$ to $(r \wedge s) \subseteq q$.

To see why ML($\subseteq$) is not closed under uniform substitution, note that clearly $(p \vee \neg p) \wedge q \vDash (p \wedge q) \vee (\neg p \wedge q)$ holds. But when we substitute $\top \subseteq p$ for q on both sides,

[3]The requirement in the truth conditions that $\nabla\varphi$ and $\dot\nabla\varphi$ also be true in the empty team is added to preserve the empty team property. One may also consider variants of these operators without this requirement. The nonempty variant of $\dot\nabla$ is sometimes called the *exists operator* E and is discussed in, for instance, [89, 97]. Logics with both the nonempty variant ∇_{NE} of ∇ and the disjunction $\vee$ may be thought of as variants of logics which include $\vee$ and the *nonemptiness atom* NE (where $T \vDash \mathrm{NE}$ iff $T \neq \varnothing$) due to the equivalences $\nabla_{\mathrm{NE}}\varphi \equiv (\varphi \wedge \mathrm{NE}) \vee \top$ and $\mathrm{NE} \equiv \nabla_{\mathrm{NE}}\top$; for more on these logics, see [140, 7].

the entailment no longer holds: $(p \vee \neg p) \wedge \top \subseteq p \nvDash (p \wedge \top \subseteq p) \vee (\neg p \wedge \top \subseteq p)$. For a counterexample to the entailment, consider the team $R[T] = \{u', v', w'\}$ from Example 5.2.6. Clearly $R[T] \vDash (p \vee \neg p) \wedge \top \subseteq p$, but there are no subteams $T_1, T_2 \subseteq R[T]$ such that $T_1 \cup T_2 = R[T]$ with $T_1 \vDash p \wedge \top \subseteq p$ and $T_2 \vDash \neg p \wedge \top \subseteq p$. Similar counterexamples can also be found for ML($\triangledown$) and ML($\overline{\triangledown}$): in the above example, instead of using the top inclusion atom $\top \subseteq p$, we can equivalently (by Fact 5.2.8) use the formula $\triangledown p$ or the formula $\overline{\triangledown} p$.

Nevertheless, the three logics we consider in the article are closed under *classical substitutions*, namely, substitutions σ such that $\sigma(p)$ is a classical formula for all $p \in$ Prop. We now prove this by using the same strategy as in [140], where it was proved that a number of propositional team logics (including propositional inclusion logic) are closed under classical substitutions.

5.2.9. LEMMA. *Let φ be a formula in* ML($\subseteq$)*,* ML($\triangledown$)*, or* ML($\overline{\triangledown}$)*, and σ a classical substitution for the relevant logic. For any model $M = (W, R, V)$ over $\mathsf{X} \supseteq \mathsf{Prop}(\sigma(\varphi))$ and team T of M,*

$$M, T \vDash \sigma(\varphi) \iff M_\sigma, T \vDash \varphi,$$

where $M_\sigma = (W, R, V_\sigma)$ is any model over $\mathsf{Y} \supseteq \mathsf{Prop}(\varphi)$ satisfying $M_\sigma, w \vDash p$ iff $M, w \vDash \sigma(p)$, for all $p \in \mathsf{Y}$ and $w \in W$.

Proof:
We prove the lemma by induction on φ. The case in which $\varphi = p$ follows by flatness of $\sigma(p)$. The other cases follow by the induction hypothesis. In particular, for $\varphi = \diamondsuit\psi$, we have $M, T \vDash \diamondsuit\sigma(\psi)$ if and only if there is a team S such that TRS and $M, S \vDash \sigma(\psi)$. By the induction hypothesis, this is the case if and only if there is a team S such that TRS and $M_\sigma, S \vDash \psi$, which holds if and only if $M_\sigma, T \vDash \diamondsuit\psi$. □

5.2.10. THEOREM. *Let $\Gamma \cup \{\varphi\}$ be a set of formulas in* ML($\subseteq$)*,* ML($\triangledown$)*, or* ML($\overline{\triangledown}$)*. For any classical substitution σ for the relevant logic, if $\Gamma \vDash \varphi$, then $\{\sigma(\gamma) | \gamma \in \Gamma\} \vDash \sigma(\varphi)$.*

Proof:
If $\Gamma \vDash \varphi$, then for any model M and team T of M,

$$\begin{aligned} M, T \vDash \sigma(\gamma) \text{ for all } \gamma \in \Gamma &\implies M_\sigma, T \vDash \gamma \text{ for all } \gamma \in \Gamma && \text{(Lemma 5.2.9)} \\ &\implies M_\sigma, T \vDash \varphi && \text{(By assumption)} \\ &\implies M, T \vDash \sigma(\varphi) && \text{(Lemma 5.2.9)} \end{aligned}$$

Hence $\{\sigma(\gamma) | \gamma \in \Gamma\} \vDash \sigma(\varphi)$. □

As with standard modal logic, one can provide a first-order translation of ML($\subseteq$) and its variants; see [89] for a translation (via normal form) of many modal team-based logics into classical first-order logic, and see Section 5.6 for a translation of ML($\subseteq$) into first-order inclusion logic.

5.3 Expressive Completeness and Normal Forms

It was proved in [73] that each of $\mathrm{ML}(\subseteq)$ and $\mathrm{ML}(\triangledown)$ is expressively complete with respect to the class $\mathbb{U}$ of union-closed modally definable team properties with the empty team property. In this section, we review this result, and show that our new variant $\mathrm{ML}(\triangledown)$ has the same expressive power. The proofs of these results also yield a normal form for each logic. These normal forms are crucial for proving the completeness of our axiomatizations for the logics—see Section 5.4.

As in the single-world setting, in the team-based setting modal definability can be characterized by bisimulation invariance: the modally definable properties are precisely those invariant (i.e., closed) under bisimulation. We begin by recalling the standard notion of bisimulation as well as that of Hintikka formulas—characteristic formulas which serve to capture bisimilarity in the syntax of ML—and then proceed to construct analogous notions for teams.

Throughout this section, we make use of a fixed finite set $\mathsf{X} \subseteq \mathsf{Prop}$ of propositional symbols; we often omit mention of X in definitions and results to keep notation light. If M is a Kripke model (over X) and $w \in W$, we call (M,w) a *pointed model* (over X).

5.3.1. DEFINITION. For any (M,w), (M',w'), and $k \in \mathbb{N}$, (M,w) and (M',w') are X,k-bisimilar, written $M,w \rightleftharpoons_k^{\mathsf{X}} M',w'$ (or simply $w \rightleftharpoons_k w'$), if the following recursively defined relation holds:

(i) $M,w \rightleftharpoons_0^{\mathsf{X}} M',w'$ iff $M,w \vDash p \iff M',w' \vDash p$ for all $p \in \mathsf{X}$.

(ii) $M,w \rightleftharpoons_{k+1}^{\mathsf{X}} M',w'$ if $M,w \rightleftharpoons_0^{\mathsf{X}} M',w'$ and:

(Forth condition) For every world v of M with wRv there is a world v' of M' with $w'Rv'$ such that $M,v \rightleftharpoons_k^{\mathsf{X}} M',v'$.

(Back condition) For every world v' of M' with $w'Rv'$ there is a world v of M with wRv such that $M,v \rightleftharpoons_k^{\mathsf{X}} M',v'$.

The *modal depth* $md(\varphi)$ of a formula φ is defined by the following clauses:

$$\begin{aligned}
md(p) &= md(\bot) = 0,\\
md(\neg\alpha) &= md(\alpha),\\
md(\psi_1 \vee \psi_2) &= md(\psi_1 \wedge \psi_2) = \max\{md(\psi_1), md(\psi_2)\},\\
md(\Diamond\psi) &= md(\Box\psi) = md(\psi)+1, \text{ and}\\
md(\alpha_1 \dots \alpha_n \subseteq \beta_1 \dots \beta_n) &= \max\{md(\alpha_1), \dots md(\alpha_n), md(\beta_1) \dots md(\beta_n)\}.
\end{aligned}$$

We say that two pointed models (M,w) and (M',w') are X,k*-equivalent*, written $M,w \equiv_k^{\mathsf{X}} M',w'$ (or simply $w \equiv_k w'$), if they satisfy the same ML-formulas with propositional symbols among those in X up to modal depth k, i.e., if $M,w \vDash \alpha$ iff $M',w' \vDash \alpha$ for every $\alpha(\mathsf{X}) \in \mathrm{ML}$ with $md(\alpha) \leq k$.

5.3.2. DEFINITION. Let $k \in \mathbb{N}$ and let (M,w) be a pointed model over $\mathsf{Y} \supseteq \mathsf{X}$. The kth *Hintikka formula* $\chi^{\mathsf{X},k}_{M,w}$ (or simply χ^k_w) of (M,w) is defined recursively by:

$$\chi^{\mathsf{X},0}_{M,w} := \bigwedge\{p \mid p \in \mathsf{X} \text{ and } w \in V(p)\} \wedge \bigwedge\{\neg p \mid p \in \mathsf{X} \text{ and } w \notin V(p)\};$$

$$\chi^{\mathsf{X},k+1}_{M,w} := \chi^k_{M,w} \wedge \bigwedge_{v \in R[w]} \Diamond \chi^k_{M,v} \wedge \Box \bigvee_{v \in R[w]} \chi^k_{M,v}.$$

It is not hard to see that there are only finitely many non-equivalent kth Hintikka formulas for a given finite X. This is why we may assume that the conjunction and the disjunction in $\chi^{\mathsf{X},k+1}_{M,w}$ are finite and hence that $\chi^{\mathsf{X},k+1}_{M,w}$ is well-defined[4].

5.3.3. THEOREM (See, e.g., [56]). *For any $k \in \mathbb{N}$ and any pointed models (M,w) and (M',w'):*

$$w \equiv_k w' \iff w \rightleftharpoons_k w' \iff w' \models \chi^k_w.$$

We now define team-based analogues to the preceding world-based notions. A *model with a team* (over X) is a pair (M,T), where M is a model (over X) and T is a team of M. Team bisimulation (introduced in [71, 89]) is a straightforward generalization of world bisimulation:

5.3.4. DEFINITION. For any (M,T) and (M',T'), and any $k \in \mathbb{N}$, (M,T) and (M',T') are (team) X,k-bisimilar, written $M,T \rightleftharpoons^{\mathsf{X}}_k M',T'$ (or simply $T \rightleftharpoons_k T'$), if:

(Forth condition) For every $w \in T$ there exists a $w' \in T'$ such that $w \rightleftharpoons^{\mathsf{X}}_k w'$.

(Back condition) For every $w' \in T'$ there exists a $w \in T$ such that $w \rightleftharpoons^{\mathsf{X}}_k w'$.

Clearly, both world and team k-bisimulation are equivalence relations. Moreover, if $w \rightleftharpoons_k w'$, then $w \rightleftharpoons_n w'$ for all $n \leq k$, and similarly if $T \rightleftharpoons_k T'$, then $T \rightleftharpoons_n T'$ for all $n \leq k$. The following lemma lists some further simple facts about team bisimulation.

5.3.5. LEMMA ([71]). *If $T \rightleftharpoons_{k+1} T'$, then*

(i) For every S such that TRS, there is a S' such that $T'R'S'$ and $S \rightleftharpoons_k S'$.

(ii) $R[T] \rightleftharpoons_k R'[T']$.

(iii) For all $T_1, T_2 \subseteq T$ such that $T_1 \cup T_2 = T$ there are $T'_1, T'_2 \subseteq T'$ such that $T'_1 \cup T'_2 = T'$ and $T_i \rightleftharpoons_{k+1} T'_i$ for $i \in \{1,2\}$.

[4] To be more precise, this assumption amounts to the following: if $R[w]$ is infinite, we choose a finite set $T \subseteq R[w]$ such that for all $v \in R[w]$, there is a $v' \in T$ with $\chi^{\mathsf{X},k}_{M,v} \equiv \chi^{\mathsf{X},k}_{M,v'}$, and define $\chi^{\mathsf{X},k+1}_{M,w}$ using this finite representative set T in place of $R[w]$. Clearly, the specific choice of this finite representative set T makes no difference for the Hintikka formula. Similar remarks apply to other finiteness assumptions made in the sequel.

We say that the models with teams (M,T) and (M',T') are X,k-*equivalent* (in a logic $\mathcal{L}$), written $M,T \equiv_k^{\mathsf{X}} M',T'$ (or simply $T \equiv_k T'$), if for every formula $\varphi(\mathsf{X})$ in $\mathcal{L}$ with $md(\varphi) \leq k$, $M,T \vDash \varphi$ iff $M',T' \vDash \varphi$. It will follow from results we show that $T \equiv_k T'$ in any of the logics we consider iff $T \equiv_k T'$ in either of the other logics; we therefore simply write $T \equiv_k T$ without specifying the logic. It is easy to show that k-bisimilarity implies k-equivalence for all three logics:

5.3.6. THEOREM (Bisimulation invariance). *If $T \rightleftharpoons_k T'$, then $T \equiv_k T'$.*

Proof:
We show by induction on the complexity of formulas φ in ML($\subseteq$), ML($\triangledown$), or ML($\triangledown$) that if $T \rightleftharpoons_k T'$, then $T \equiv_k T'$ for $k = md(\varphi)$. See [73] for details. In particular, items (i), (ii) and (iii) in Lemma 5.3.5 are used in the diamond, box and disjunction cases, respectively. For $\varphi = \triangledown\psi$, we have $T \vDash \triangledown\psi$ iff $T = \varnothing$ or $\{w\} \vDash \psi$ for some $w \in T$. If $T = \varnothing$, then by $T \rightleftharpoons_k T'$ also $T' = \varnothing$ so that $T' \vDash \triangledown\psi$. Otherwise by $T \rightleftharpoons_k T'$ we have $\{w\} \rightleftharpoons_k \{w'\}$ for some $w' \in T'$, and then $\{w'\} \vDash \psi$ by the induction hypothesis, whence $T' \vDash \triangledown\psi$. The other direction is similar. □

Let us also demonstrate why ML($\subseteq$), ML($\triangledown$), and ML($\triangledown$) with the strict semantics for the diamond (see Section 5.2) are not bisimulation-invariant for the notion of team bisimulation we have adopted. Consider a model $M = (W,R,V)$ over $\{p\}$ where $W = \{w_1,w_2,w_3\}$; $R = \{(w_1,w_1),(w_1,w_3),(w_2,w_2),(w_2,w_3)\}$; and $V(p) = \{w_3\}$. Clearly $w_1 \rightleftharpoons_k w_2$ for all $k \in \mathbb{N}$, so that also $\{w_1\} \rightleftharpoons_k \{w_1,w_2\}$ for all $k \in \mathbb{N}$. Defining $f(w_1) := w_3$ and $f(w_2) := w_2$, we have $f[\{w_1,w_2\}] \vDash \top \subseteq p \wedge \bot \subseteq p$ so that $\{w_1,w_2\} \vDash \Diamond_s(\top \subseteq p \wedge \bot \subseteq p)$. Clearly $\{w_1\} \nvDash \Diamond_s(\top \subseteq p \wedge \bot \subseteq p)$, so $\{w_1\} \,{}_s\!\not\equiv_1 \{w_1,w_2\}$ where ${}_s\!\equiv_1$ is 1-equivalence defined with respect to the strict semantics. An analogous argument can be conducted in ML($\triangledown$) as well as in ML($\triangledown$).

With a suitable notion of bisimilarity at hand, we now proceed to show our expressive completeness results. We measure the expressive power of the logics in terms of the *properties* they can express. A *(team) property* (over X) is a class of models with teams (over X). For each formula $\varphi(\mathsf{X})$ we denote by $\|\varphi\|_{\mathsf{X}}$ (or simply $\|\varphi\|$) the property (over X) expressed or defined by φ, i.e.,

$$\|\varphi\|_{\mathsf{X}} := \{(M,T) \text{ over } \mathsf{X} \mid M,T \vDash \varphi\}.$$

A property $\mathcal{P}$ is *invariant under* X,k-*bisimulation* if $(M,T) \in \mathcal{P}$ and $M,T \rightleftharpoons_k^{\mathsf{X}} M',T'$ imply that $(M',T') \in \mathcal{P}$; is *closed under unions* if $(M,T_i) \in \mathcal{P}$ for all $i \in I \neq \varnothing$ implies that $(M,\bigcup_{i\in I} T_i) \in \mathcal{P}$; and has the *empty team property* if $(M,\varnothing) \in \mathcal{P}$ for all M.

We say that a logic $\mathcal{L}$ is *expressively complete* for a class of properties $\mathbb{P}$, written $\mathbb{P} = \|\mathcal{L}\|$, if for each finite $\mathsf{X} \subseteq \mathsf{Prop}$, the class $\mathbb{P}_{\mathsf{X}}$ of properties over X in $\mathbb{P}$ is precisely the class of properties over X definable by formulas in $\mathcal{L}$, i.e., if

$$\mathbb{P}_{\mathsf{X}} = \{\|\varphi\|_{\mathsf{X}} \mid \varphi \in \mathcal{L} \text{ and } \mathsf{Prop}(\varphi) = \mathsf{X}\}.$$

Our goal is to show that each of the three logics is expressively complete for the class $\mathbb{U}$ of properties $\mathcal{P}$ such that $\mathcal{P}$ is union closed; $\mathcal{P}$ has the empty team property; and $\mathcal{P}$ is invariant under bounded bisimulation, meaning that if $\mathcal{P}$ is a property over X, then $\mathcal{P}$ is invariant under X,k-bisimulation for some $k \in \mathbb{N}$. That is, we show:

$$\mathbb{U} = \|\mathrm{ML}(\subseteq)\| = \|\mathrm{ML}(\triangledown)\| = \|\mathrm{ML}(\triangledown)\|.$$

Clearly, any property $\|\varphi\|_{\mathsf{X}}$ expressible in any of the three logics is union closed and has the empty team property, and by Theorem 5.3.6 it is also invariant under X,k-bisimulation for some $k \in \mathbb{N}$ (take any $k \geq md(\varphi)$). Therefore, we have $\{\|\varphi\|_{\mathsf{X}} \mid \varphi \in \mathcal{L}$ where $\mathsf{Prop}(\varphi) = \mathsf{X}\} \subseteq \mathbb{U}_{\mathsf{X}}$ for $\mathcal{L}$ being any of the three logics. For the converse inclusion, we must show that any property $\mathcal{P} \in \mathbb{U}_{\mathsf{X}}$ is definable by some formula in each of the logics. Let us focus on $\mathrm{ML}(\subseteq)$ first.

As a first step, we will construct characteristic formulas in $\mathrm{ML}(\subseteq)$ for teams analogous to the Hintikka formulas (characteristic formulas for worlds) defined above. These formulas and the accompanying lemma are essentially as in [73]; we have simplified them somewhat using an idea presented in [89].

5.3.7. LEMMA. *For any $k \in \mathbb{N}$ and any (M,T) over X, there are formulas $\eta^{\mathsf{X},k}_{M,T} \in \mathrm{ML}$ and $\zeta^{\mathsf{X},k}_{M,T}, \theta^{\mathsf{X},k}_{M,T} \in \mathrm{ML}(\subseteq)$ such that:*

(i) $M',T' \models \zeta^{\mathsf{X},k}_{M,T}$ iff $T' = \varnothing$ or for all $w \in T$ there is a $w' \in T'$ such that $M,w \rightleftharpoons^{\mathsf{X}}_{k} M',w'$.

(ii) $M',T' \models \eta^{\mathsf{X},k}_{M,T}$ iff for all $w' \in T'$ there is a $w \in T$ such that $M,w \rightleftharpoons^{\mathsf{X}}_{k} M',w'$.

(iii) $M',T' \models \theta^{\mathsf{X},k}_{M,T}$ iff $M,T \rightleftharpoons^{\mathsf{X}}_{k} M',T'$ or $T' = \varnothing$.

Proof:

(i) Let

$$\zeta^{\mathsf{X},k}_{M,T} := \bigwedge_{w \in T} (\top \subseteq \chi^{\mathsf{X},k}_{M,w}),$$

where we stipulate $\bigwedge \varnothing = \top$. Since there are only a finite number of non-equivalent kth Hintikka formulas for a given finite X, we can assume that the conjunction $\bigwedge_{w \in T}(\top \subseteq \chi^{\mathsf{X},k}_{M,w})$ is finite and that $\zeta^{k}_{M,T}$ is well-defined. If $T' \neq \varnothing$, then:

$$\begin{aligned}
T' \models \bigwedge_{w \in T} (\top \subseteq \chi^k_w) &\iff \forall w \in T : T' \models \top \subseteq \chi^k_w \\
&\iff \forall w \in T\ \forall v' \in T'\ \exists w' \in T' : v' \models \top \iff w' \models \chi^k_w \\
&\iff \forall w \in T\ \exists w' \in T' : w' \models \chi^k_w && \text{(Since } T' \neq \varnothing) \\
&\iff \forall w \in T\ \exists w' \in T' : w \rightleftharpoons_k w'. && \text{(Theorem 5.3.3)}
\end{aligned}$$

(ii) Let

$$\eta^{\mathsf{X},k}_{M,T} := \bigvee_{w\in T} \chi^{\mathsf{X},k}_{M,w},$$

where we stipulate $\bigvee \varnothing = \bot$.

$$\begin{aligned} T' \vDash \bigvee_{w\in T} \chi^k_w &\iff T' = \bigcup_{w\in T} T'_w \text{ where } T'_w \vDash \chi^k_w \\ &\iff \forall w' \in T' \; \exists w \in T : w' \vDash \chi^k_w && \text{(By flatness)} \\ &\iff \forall w' \in T' \; \exists w \in T : w \rightleftharpoons_k w'. && \text{(Theorem 5.3.3)} \end{aligned}$$

(iii) Let

$$\theta^{\mathsf{X},k}_{M,T} := \eta^{\mathsf{X},k}_{M,T} \wedge \zeta^{\mathsf{X},k}_{M,T}.$$

If $T' = \varnothing$, then $T' \vDash \theta^k_T$ by the empty team property. Else if $T' \neq \varnothing$, then $T' \vDash \eta^k_T \wedge \zeta^k_T$ if and only if the back and forth conditions in Definition 5.3.4 hold (by items (i) and (ii)), which is the case if and only if $T \rightleftharpoons_k T'$. □

We call the characteristic formulas θ^k_T for teams obtained in Lemma 5.3.7 (iii) *Hintikka formulas for teams*. As with standard Hintikka formulas, it is easy to see that for a given finite X, there are only a finite number of non-equivalent kth Hintikka formulas for teams.

We are now ready to show also the inclusion $\mathbb{U}_{\mathsf{X}} \subseteq \{\|\varphi\|_{\mathsf{X}} \mid \varphi \in \mathrm{ML}(\subseteq) \text{ and } \mathsf{Prop}(\varphi) = \mathsf{X}\}$.

5.3.8. THEOREM ([73]). $\mathbb{U} = \|\mathrm{ML}(\subseteq)\|$. *That is, for each finite* $\mathsf{X} \subseteq \mathsf{Prop}$, $\mathbb{U}_{\mathsf{X}} = \{\mathcal{C}$ *over* $\mathsf{X} \mid \mathcal{C}$ *is union closed, has the empty team property, and is invariant under* $\rightleftharpoons^{\mathsf{X}}_k$ *for some* $k \in \mathbb{N}\} = \{\|\varphi\|_{\mathsf{X}} \mid \varphi \in \mathrm{ML}(\subseteq)$ *and* $\mathsf{Prop}(\varphi) = \mathsf{X}\}$.

Proof:
The direction $\supseteq$ follows by Theorem 5.3.6 and the fact that formulas in $\mathrm{ML}(\subseteq)$ are union closed and have the empty team property.

For the direction $\subseteq$, let $\mathcal{C} \in \mathbb{U}_{\mathsf{X}}$, and let $k \in \mathbb{N}$ be such that $\mathcal{C}$ is invariant under $\rightleftharpoons^{\mathsf{X}}_k$. Let

$$\varphi' := \bigvee_{(M,T)\in\mathcal{C}} \theta^{\mathsf{X},k}_{M,T},$$

where $\theta^{\mathsf{X},k}_{M,T}$ is defined as in the proof of Lemma 5.3.7 (iii). Since there are only finitely many non-equivalent kth Hintikka formulas for teams for a given finite X, we may assume the disjunction in φ' to be finite and φ' to be well-defined. We show $\mathcal{C} = \|\varphi'\|_{\mathsf{X}}$. Note that since both $\mathcal{C}$ and $\|\varphi'\|_{\mathsf{X}}$ have the empty team property, neither is empty.

Let $(M,T) \in \mathcal{C}$. Clearly $T \rightleftharpoons_k T$, so by Lemma 5.3.7 (iii), $T \vDash \theta^k_T$, whence $T \vDash \varphi'$.

Now let $M', T' \vDash \varphi'$. Then there are subteams $T'_T \subseteq T'$ such that $T' = \bigcup_{(M,T)\in\mathcal{C}} T'_T$ and $M', T'_T \vDash \theta^k_T$. By Lemma 5.3.7 (iii) it follows that for a given $(M,T) \in \mathcal{C}$, either $M,T \rightleftharpoons_k M', T'_T$ or $T'_T = \varnothing$. If $T'_T = \varnothing$, then by the empty team property $(M', T'_T) \in \mathcal{C}$. If

$M, T \rightleftharpoons_k M', T'_T$, then since $\mathcal{C}$ is invariant under k-bisimulation, $(M', T'_T) \in \mathcal{C}$. So for all $(M,T) \in \mathcal{C}$ we have $(M', T'_T) \in \mathcal{C}$; then since $\mathcal{C}$ is closed under unions, we conclude that $(M', T') \in \mathcal{C}$. □

It follows from the proof of Theorem 5.3.8 that each ML($\subseteq$)-formula is equivalent to a formula in the normal form

$$\bigvee_{(M,T)\in\mathcal{C}} \theta^k_T = \bigvee_{(M,T)\in\mathcal{C}} \Big(\bigvee_{w\in T} \chi^k_w \wedge \bigwedge_{w\in T} (\top \subseteq \chi^k_w)\Big). \tag{NF}$$

Note that in this normal form, the extended inclusion atoms $\top \subseteq \chi^k_w$ play a substantial role that cannot be replicated using inclusion atoms *simpliciter*, i.e., inclusion atoms $p_1 \dots p_n \subseteq q_1 \dots q_n$ with propositional symbols only. It was observed in [137] that the variant of propositional inclusion logic with only these simpler inclusion atoms is strictly less expressive than extended propositional inclusion logic, as, e.g., in one propositional symbol p, the former contains only one (trivial) inclusion atom $p \subseteq p$ and is thus flat, while the latter can express non-flat properties using extended inclusion atoms such as $\top \subseteq p$ and $\bot \subseteq p$. The analogous fact clearly also holds in the modal case.

Observe also that only unary top inclusion atoms (atoms of the form $\top \subseteq \alpha$) are required in the above normal form to express any given property in $\mathbb{U}$. Given that $\top \subseteq \alpha \equiv \triangledown\alpha \equiv \mathbin{\triangledown\mkern-14mu\vee}\alpha$ (Fact 5.2.8), we can derive the expressive completeness of ML($\triangledown$) as well as that of ML($\mathbin{\triangledown\mkern-14mu\vee}$) as a corollary to Theorem 5.3.8.

5.3.9. THEOREM. $\mathbb{U} = \|\text{ML}(\triangledown)\| = \|\text{ML}(\mathbin{\triangledown\mkern-14mu\vee})\|$.

And we clearly have the following normal forms for these logics:

$$\bigvee_{(M,T)\in\mathcal{C}} \Big(\bigvee_{w\in T} \chi^k_w \wedge \bigwedge_{w\in T} \triangledown\chi^k_w\Big). \tag{$\triangledown$NF}$$

$$\bigvee_{(M,T)\in\mathcal{C}} \Big(\bigvee_{w\in T} \chi^k_w \wedge \bigwedge_{w\in T} \mathbin{\triangledown\mkern-14mu\vee}\chi^k_w\Big). \tag{$\mathbin{\triangledown\mkern-14mu\vee}$NF}$$

The expressive completeness of ML($\triangledown$) was also proved in [73]; we have added the result for ML($\mathbin{\triangledown\mkern-14mu\vee}$). Let us briefly comment on how the structure of the normal form motivates the introduction of the operator $\mathbin{\triangledown\mkern-14mu\vee}$. Given a team T, the formula $\zeta^k_T = \bigwedge_{w\in T}(\top \subseteq \chi^k_w)$ is intended to express the forth-condition of $\rightleftharpoons_k$ from T to a team T' in the sense that $T' \vDash \zeta^k_T$ iff for all $w \in T$ there is a $w' \in T$ such that $w \rightleftharpoons_k w'$ (see the proof of Lemma 5.3.7 (i)). The formula $\bigwedge_{w\in T} \mathbin{\triangledown\mkern-14mu\vee}\chi^k_w$ articulates precisely what T' needs to satisfy in order to fulfill this condition. Inclusion atoms are clearly stronger than is strictly necessary if we only wish to express this condition. The operator $\triangledown$ is also *prima facie* stronger than $\mathbin{\triangledown\mkern-14mu\vee}$ in the sense that for any $\varphi \in \text{ML}(\mathbin{\triangledown\mkern-14mu\vee})$, there is an $\alpha \in \text{ML}$ such that $\mathbin{\triangledown\mkern-14mu\vee}\varphi \equiv \mathbin{\triangledown\mkern-14mu\vee}\alpha$,[5] whereas, for instance, there is no $\alpha \in \text{ML}$ such that $\triangledown(\triangledown p \wedge \triangledown q) \equiv \triangledown\alpha$—

[5]This follows from the fact that ML is expressively complete for the class of flat properties invariant under k-bisimulation for some $k \in \mathbb{N}$ (see, e.g., [134]). A team property $\mathcal{P}$ is *flat* if $(M,T) \in \mathcal{P} \iff (M,\{w\}) \in \mathcal{P}$ for all $w \in T$. Given any $\varphi \in \text{ML}(\mathbin{\triangledown\mkern-14mu\vee})$, $\{(M,T) \mid M,\{w\} \vDash \varphi$ for all $w \in T\}$ is clearly flat and invariant under k-bisimulation for $k = md(\varphi)$, so by the expressive completeness fact we have $\{(M,T) \mid M,\{w\} \vDash \varphi$ for all $w \in T\} = \|\alpha\|$ for some $\alpha \in \text{ML}$, and then $\mathbin{\triangledown\mkern-14mu\vee}\varphi \equiv \mathbin{\triangledown\mkern-14mu\vee}\alpha$.

the singular might, in a sense, cancels out non-classical content within its scope. It is interesting, therefore, that ML($\triangledown$) has the same expressive power as the other two logics.

One important corollary to the results in this section is that the logics we consider are compact. This follows from the fact that any team property invariant under bounded bisimulation can be expressed in classical first-order logic—see [89]. (It should be noted that for many team-based logics, compactness formulated in terms of satisfiability need not coincide with compactness formulated in terms of entailment, but the logics we consider are compact in both senses; see Section 5.6 for further discussion.) It also follows readily from the expressive completeness results that each of the three logics admits uniform interpolation. It was proved in [43] that any team-based modal logic which is *local* and *forgetting* admits uniform interpolation. The logics we consider are all local, and it follows from the expressive completeness theorems that they are also forgetting; they therefore admit uniform interpolation. We refer the reader to [43] for the detailed proof and more discussion about the notion of uniform interpolation; see [137] for clarification on the role of locality.

5.4 Axiomatizations

In this section, we introduce sound and complete natural deduction systems for the logics ML($\subseteq$), ML($\triangledown$) and $\mathcal{ML}(\triangledown)$. We prove the completeness of the axiomatizations by means of a strategy commonly used for propositional and modal team-based logics which involves showing that each formula is provably equivalent to its normal form. This strategy is used in [137] for the propositional inclusion logic system; we also adapt many of the lemmas and other details of the proof presented in [137].

5.4.1 ML($\subseteq$)

We start by axiomatizing our core logic, modal inclusion logic ML($\subseteq$). Our natural deduction system for ML($\subseteq$) comprises standard rules for connectives and modalities for the smallest normal modal logic $\mathcal{K}$, modified to account for special features of our setting such as the failure of downward closure, as well as rules governing the behaviour of inclusion atoms and their interaction with the other connectives. Note that since ML($\subseteq$) is not closed under uniform substitution (see Section 5.2), the system does not admit the usual uniform substitution rule.

5.4.1. Definition. The natural deduction system for ML($\subseteq$) consists of all axioms and rules presented in Tables 5.1-5.3. We write $\Gamma \vdash_{\mathrm{ML}(\subseteq)} \varphi$ (or simply $\Gamma \vdash \varphi$) if φ is derivable from formulas in Γ using this system. We write $\varphi \vdash \psi$ for $\{\varphi\} \vdash \psi$ and $\vdash \varphi$ for $\varnothing \vdash \varphi$, and say that φ and ψ are *provably equivalent*, denoted by $\varphi \dashv\vdash \psi$, if $\varphi \vdash \psi$ and $\psi \vdash \varphi$.

$$
\frac{\begin{array}{c}[\alpha]\\ D_0\\ \bot\end{array}}{\neg\alpha}\,\neg\mathrm{I}_{(1)}
\qquad
\frac{\begin{array}{c}[\neg\alpha]\\ D_0\\ \bot\end{array}}{\alpha}\,\mathrm{RAA}_{(1)}
\qquad
\frac{\begin{array}{cc}D_0 & D_1\\ \alpha & \neg\alpha\end{array}}{\varphi}\,\neg\mathrm{E}
$$

$$
\frac{\begin{array}{c}D\\ \varphi\end{array}}{\varphi\vee\psi}\,\vee\mathrm{I}
\qquad
\frac{\begin{array}{c}D\\ \psi\end{array}}{\varphi\vee\psi}\,\vee\mathrm{I}
\qquad
\frac{\begin{array}{ccc} & [\varphi] & [\psi]\\ D & D_0 & D_1\\ \varphi\vee\psi & \chi & \chi\end{array}}{\chi}\,\vee\mathrm{E}_{(1)}
$$

$$
\frac{\begin{array}{cc}D_0 & D_1\\ \varphi & \psi\end{array}}{\varphi\wedge\psi}\,\wedge\mathrm{I}
\qquad
\frac{\begin{array}{c}D\\ \varphi\wedge\psi\end{array}}{\varphi}\,\wedge\mathrm{E}
\qquad
\frac{\begin{array}{c}D\\ \varphi\wedge\psi\end{array}}{\psi}\,\wedge\mathrm{E}
$$

$$
\frac{\begin{array}{c}D\\ \neg\boxdot\alpha\end{array}}{\Diamond\neg\alpha}\,\Diamond\boxdot\mathrm{Inter}
\qquad
\frac{\begin{array}{c}D\\ \Diamond\neg\alpha\end{array}}{\neg\boxdot\alpha}\,\boxdot\Diamond\mathrm{Inter}
\qquad
\frac{\begin{array}{c}D\\ \Diamond(\varphi\vee\psi)\end{array}}{\Diamond\varphi\vee\Diamond\psi}\,\Diamond\vee\mathrm{Distr}
$$

$$
\frac{\begin{array}{cccc}[\varphi_1]\ \dots\ [\varphi_n] & & & \\ D_0 & & & \\ \ddots\vdots\,\cdot^{\cdot^{\cdot}} & D_1 & \cdots & D_n\\ \psi & \boxdot\varphi_1 & \cdots & \boxdot\varphi_n\end{array}}{\boxdot\psi}\,\boxdot\mathrm{Mon}_{(2)}
\qquad
\frac{\begin{array}{cc}[\varphi] & \\ D_0 & D_1\\ \psi & \Diamond\varphi\end{array}}{\Diamond\psi}\,\Diamond\mathrm{Mon}_{(2)}
$$

(1) The undischarged assumptions in D_0 and D_1 are ML-formulas.
(2) D_0 has no undischarged assumptions.

Table 5.1: Rules for classical modal logic.

Table 5.1 lists the rules which operate only on the connectives and operators of ML. The soundness of the disjunction elimination rule as well as that of the negation rules involving assumptions requires that the undischarged assumptions in the subderivations are downward closed, hence the restriction to ML-formulas.

Table 5.2 lists the propositional rules for inclusion atoms. All rules in this table are adapted from rules or results in the propositional inclusion logic system in [137], but we have considerably simplified the propositional system (see Proposition 5.4.4, in which we show that the full set of rules from [137] is derivable in our system). The rule $\subseteq_\neg\mathrm{E}$ captures the fact that $\mathsf{x}\subseteq\mathsf{a}$ and $\neg\mathsf{a}^{\mathsf{x}}$ lead to a contradiction, since (for a nonempty team) $\mathsf{x}\subseteq\mathsf{a}$ implies that a^{x} is true somewhere in the team (see Proposition 5.2.4), while $\neg\mathsf{a}^{\mathsf{x}}$ implies that a^{x} is not true anywhere in the team. The rules $\subseteq$Ext and $\subseteq$Rdt allow us to reduce arbitrary inclusion atoms to equivalent formulas in which

$$\dfrac{}{\mathsf{a}\subseteq\mathsf{a}}\subseteq\mathrm{Id}\qquad\qquad \dfrac{\begin{matrix}D_0 & D_1\\ \alpha^{x} & \mathsf{b}\subseteq\mathsf{c}\end{matrix}}{x\mathsf{b}\subseteq\alpha\mathsf{c}}\subseteq\mathrm{Exp}$$

$$\dfrac{\begin{matrix}D_0 & D_1\\ \neg\mathsf{a}^{\mathsf{x}} & \mathsf{x}\subseteq\mathsf{a}\end{matrix}}{\varphi}\subseteq_{\neg}\mathrm{E}\qquad\qquad \dfrac{\begin{matrix} & [\mathsf{x}\subseteq\mathsf{a}] & [\psi]\\ D & D_0 & D_1\\ \mathsf{x}\subseteq\mathsf{a}\vee\psi & \chi & \chi\end{matrix}}{\chi}\vee_{\subseteq}\mathrm{E}$$

$$\dfrac{\begin{matrix}D\\ \bigwedge_{\mathsf{x}\in\{\top,\bot\}^{|\mathsf{a}|}}(\neg\mathsf{a}^{\mathsf{x}}\vee\mathsf{x}\subseteq\mathsf{b})\end{matrix}}{\mathsf{a}\subseteq\mathsf{b}}\subseteq\mathrm{Ext}\qquad\qquad \dfrac{\begin{matrix}D\\ \mathsf{a}\subseteq\mathsf{b}\end{matrix}}{\neg\mathsf{a}^{\mathsf{x}}\vee\mathsf{x}\subseteq\mathsf{b}}\subseteq\mathrm{Rdt}$$

$$\dfrac{\begin{matrix}D_0 & D_1 & \cdots & D_n\\ \varphi\vee\psi & \mathsf{x}_1\subseteq\mathsf{a}_1 & \dots & \mathsf{x}_n\subseteq\mathsf{a}_n\end{matrix}}{((\varphi\vee\mathsf{a}_1^{\mathsf{x}_1}\vee\cdots\vee\mathsf{a}_n^{\mathsf{x}_n})\wedge\mathsf{x}_1\subseteq\mathsf{a}_1\wedge\cdots\wedge\mathsf{x}_n\subseteq\mathsf{a}_n)\vee\psi}\subseteq\mathrm{Distr}$$

Table 5.2: Rules for inclusion.

$$\dfrac{\begin{matrix}D\\ \Diamond(\mathsf{x}\subseteq\mathsf{a})\end{matrix}}{\top\subseteq\Diamond\mathsf{a}^{\mathsf{x}}}\Diamond_{\subseteq}\mathrm{Distr}\qquad\qquad \dfrac{\begin{matrix}D\\ \top\subseteq\Diamond\mathsf{a}^{\mathsf{x}}\end{matrix}}{\Box(\mathsf{x}\subseteq\mathsf{a})}\Diamond\Box_{\subseteq}\mathrm{Exc}$$

$$\dfrac{\begin{matrix}D_0 & D_1\\ \top\subseteq\Diamond\beta & \Box(\mathsf{x}\subseteq\mathsf{a})\end{matrix}}{\top\subseteq\Diamond\mathsf{a}^{\mathsf{x}}}\Box\Diamond_{\subseteq}\mathrm{Exc}\qquad\qquad \dfrac{\begin{matrix} & [\top\subseteq\Diamond\mathsf{a}^{\mathsf{x}}] & [\Box\psi]\\ D & D_0 & D_1\\ \Box(\mathsf{x}\subseteq\mathsf{a}\vee\psi) & \chi & \chi\end{matrix}}{\chi}\Box\vee_{\subseteq}\mathrm{E}$$

$$\dfrac{\begin{matrix}D_0 & D_1 & \cdots & D_n\\ \Diamond\varphi & x_1\subseteq\Diamond\alpha_1 & \dots & x_n\subseteq\Diamond\alpha_n\end{matrix}}{\Diamond((\varphi\vee\alpha_1^{x_1}\vee\ldots\vee\alpha_n^{x_n})\wedge x_1\subseteq\alpha_1\wedge\ldots x_n\subseteq\alpha_n)}\subseteq_{\Diamond}\mathrm{Distr}$$

Table 5.3: Rules for modal operators and inclusion.

all non-classical subformulas are primitive inclusion atoms (see Lemma 5.2.7) and to utilize the properties of classical formulas to derive properties of inclusion atoms (see Proposition 5.4.4). The rule $\vee_{\subseteq}$E models the upward closure (modulo the empty team property) of primitive inclusion atoms: if $T\vDash\mathsf{x}\subseteq\mathsf{a}\vee\psi$ and $T\nvDash\psi$, then $T=S\cup U$ where $S\vDash\mathsf{x}\subseteq\mathsf{a}$, $U\vDash\psi$ and $S\neq\varnothing$, whence also $T\vDash\mathsf{x}\subseteq\mathsf{a}$. Primitive inclusion atoms are not downward closed so we do not in general have the converse direction: $(\varphi\vee\psi)\wedge\mathsf{x}\subseteq\mathsf{a}\nvDash$

$(\varphi \wedge \mathsf{x} \subseteq \mathsf{a}) \vee \psi$. However, $(\varphi \vee \psi) \wedge \mathsf{x} \subseteq \mathsf{a} \vDash ((\varphi \vee \mathsf{a}^{\mathsf{x}}) \wedge \mathsf{x} \subseteq \mathsf{a}) \vee \psi$ does hold, as reflected in the rule $\subseteq$Distr.

The rules in Table 5.3 concern modal operators and inclusion atoms. The rule $\Diamond_{\subseteq}$Distr allows us to distribute the diamond over the inclusion atom. The converse direction of this rule is not sound, as, e.g., $\top \subseteq \Diamond p \nvDash \Diamond(\top \subseteq p)$ (consider a team with a world $w \vDash \Diamond p$ but without a successor team). If we ensure that there is a successor team by requiring $\Diamond\varphi$ to hold for some φ, the converse is sound. The rule $\subseteq_{\Diamond}$Distr generalizes this fact, and indeed we have:

$$\Diamond\varphi, \top \subseteq \Diamond \mathsf{a}^{\mathsf{x}} \vdash \Diamond((\varphi \vee (\mathsf{a}^{\mathsf{x}})^{\top}) \wedge \top \subseteq \mathsf{a}^{\mathsf{x}}) \qquad (\subseteq_{\Diamond}\text{Distr})$$

$$\vdash \Diamond(\top \subseteq \mathsf{a}^{\mathsf{x}}) \qquad (\Diamond\text{Mon})$$

The rule $\Diamond\Box_{\subseteq}$Exc allows us to derive a box formula from a top inclusion formula with a diamond formula on the right. The converse is not sound—consider a nonempty team T with $R[T] = \varnothing$. By the empty team property, $T \vDash \Box(\top \subseteq \alpha)$, whereas $T \nvDash \top \subseteq \Diamond\alpha$, since the worlds in T have no accessible worlds. We can ensure that $R[T] \neq \varnothing$ in case $T \neq \varnothing$ by requiring $\top \subseteq \Diamond\beta$ to hold for some β—this yields the rule $\Box\Diamond_{\subseteq}$Exc. The rule $\Box\vee_{\subseteq}$E is similar to $\vee_{\subseteq}$E in Table 5.2, but it applies to box formulas.

5.4.2. THEOREM (Soundness). *If* $\Gamma \vdash \varphi$, *then* $\Gamma \vDash \varphi$.

Proof:
Soundness proofs for most rules in Tables 5.1 and 5.2 can be found in [137]. The soundness of $\Diamond_{\subseteq}$Distr, $\Box\Diamond_{\subseteq}$Exc and $\Diamond\Box_{\subseteq}$Exc (Table 5.3) is easy to show using Proposition 5.2.4. We only give detailed proofs for $\subseteq_{\Diamond}$Distr and $\Box\vee_{\subseteq}$E (Table 5.3).

($\subseteq_{\Diamond}$Distr) It suffices to show $\Diamond\varphi \wedge \bigwedge_{1\le i\le n}(x_i \subseteq \Diamond\alpha_i) \vDash \Diamond((\varphi \vee \bigvee_{1\le i\le n}\alpha_i^{x_i}) \wedge \bigwedge_{1\le i\le n}(x_i \subseteq \alpha_i))$. Suppose that $T \vDash \Diamond\varphi \wedge \bigwedge_{i\in I}(x_i \subseteq \Diamond\alpha_i)$. If $T = \varnothing$, it satisfies the conclusion by the empty team property so we may suppose $T \neq \varnothing$. Then there is some S such that TRS and $S \vDash \varphi$, and for each $1 \le i \le n$, $T \vDash x_i \subseteq \Diamond\alpha_i$. If $x_i = \top$, then there is some $v_i \in R[T]$ such that $v_i \vDash \alpha_i$, i.e., $v_i \vDash \alpha_i^{x_i}$. If, on the other hand, $x_i = \bot$, then by Proposition 5.2.4 there is some $w_i \in T$ such that $w_i \vDash \neg\Diamond\alpha_i$, i.e., $w_i \vDash \Box\neg\alpha_i$. By TRS there is a world $v_i \in R[T]$ such that $w_i R v_i$, and since $w_i \vDash \Box\neg\alpha_i$, we have that $v_i \vDash \neg\alpha_i$, i.e., $v_i \vDash \alpha_i^{x_i}$. Let $S' := S \cup \{v_i\}_{1\le i\le n}$. Clearly $S' \vDash (\varphi \vee \bigvee_{1\le i\le n}\alpha_i^{x_i}) \wedge \bigwedge_{1\le i\le n}(x_i \subseteq \alpha_i)$. Since $S \subseteq S' \subseteq R[T]$ and TRS, we have that TRS', so $T \vDash \Diamond((\varphi \vee \bigvee_{1\le i\le n}\alpha_i^{x_i}) \wedge \bigwedge_{1\le i\le n}(x_i \subseteq \alpha_i))$.

($\Box\vee_{\subseteq}$E) Suppose that $\Gamma, \top \subseteq \Diamond\mathsf{a}^{\mathsf{x}} \vDash \chi$, and $\Gamma, \Box\psi \vDash \chi$. Let $T \vDash \Box(\mathsf{x} \subseteq \mathsf{a} \vee \psi)$ and $T \vDash \gamma$ for all $\gamma \in \Gamma$. We show that $T \vDash \chi$.

We have that $R[T] \vDash \mathsf{x} \subseteq \mathsf{a} \vee \psi$, so $R[T] = T_1 \cup T_2$ where $T_1 \vDash \mathsf{x} \subseteq \mathsf{a}$ and $T_2 \vDash \psi$. If $T_1 = \varnothing$, then $T_2 = R[T]$, so $R[T] \vDash \psi$, whence $T \vDash \Box\psi$; therefore also $T \vDash \chi$. Else if $T_1 \neq \varnothing$, then by $T_1 \vDash \mathsf{x} \subseteq \mathsf{a}$ and Proposition 5.2.4 it follows that there is some $v \in T_1$ such that $v \vDash \mathsf{a}^{\mathsf{x}}$. By $v \in R[T]$, there is a $w \in T$ with wRv, whence $w \vDash \Diamond\mathsf{a}^{\mathsf{x}}$. Therefore $T \vDash \top \subseteq \Diamond\mathsf{a}^{\mathsf{x}}$, whence also $T \vDash \chi$. □

It is easy to verify that the rules in Table 5.1 constitute a system that is equivalent to other axiomatizations of $\mathcal{K}$,[6] and so, given soundness, Proposition 5.2.3, and the fact that $\mathcal{K}$ is complete for the class of all Kripke models, we have:

5.4.3. PROPOSITION (Classical completeness). *Let $\Gamma \cup \{\alpha\}$ be a set of* ML*-formulas. Then*

$$\Gamma \vDash \alpha \iff \Gamma \vdash_{\mathrm{ML}(\subseteq)} \alpha.$$

In the following proposition we derive some useful properties of inclusion atoms. These properties allow us to manipulate the atoms with ease, and we occasionally make use of them without explicit reference to this proposition. All items in the proposition were included as rules in the propositional system in [137]; the proposition shows that the simpler propositional system yielded by the propositional fragment of the rules in Tables 5.1 and 5.2 suffices.[7]

5.4.4. PROPOSITION.

(i) $\mathsf{a} \subseteq \mathsf{b}, \mathsf{b} \subseteq \mathsf{c} \vdash \mathsf{a} \subseteq \mathsf{c}$.

(ii) $\mathsf{a}_0\mathsf{a}_1\mathsf{a}_2 \subseteq \mathsf{b}_0\mathsf{b}_1\mathsf{b}_2 \vdash \mathsf{a}_1\mathsf{a}_0\mathsf{a}_2 \subseteq \mathsf{b}_1\mathsf{b}_0\mathsf{b}_2$.

(iii) $\mathsf{a}_0\mathsf{a}_1 \subseteq \mathsf{b}_0\mathsf{b}_1 \vdash \mathsf{a}_0\mathsf{a}_0\mathsf{a}_1 \subseteq \mathsf{b}_0\mathsf{b}_0\mathsf{b}_1$.

(iv) $\mathsf{ab} \subseteq \mathsf{cd} \vdash \mathsf{a} \subseteq \mathsf{c}$.

(v) $\mathsf{a} \subseteq \mathsf{b}, \alpha(\mathsf{b}) \vdash \alpha(\mathsf{a})$*, where α, or $\alpha(\mathsf{c})$, is classical and propositional, the propositional symbols and constants occurring in $\alpha(\mathsf{c})$ are among those in* c*, and $\alpha(\mathsf{a})$ and $\alpha(\mathsf{b})$ denote the result of replacing each element of* c *in α with the corresponding element in* a *and* b*, respectively.*

Proof:

(i) By $\subseteq$Ext, it suffices to derive $\neg \mathsf{a}^{\mathsf{x}} \vee \mathsf{x} \subseteq \mathsf{c}$ for all $\mathsf{x} \in \{\top, \bot\}^{|\mathsf{a}|}$. For a given x, we have $\mathsf{a} \subseteq \mathsf{b} \vdash \neg \mathsf{a}^{\mathsf{x}} \vee \mathsf{x} \subseteq \mathsf{b}$ and $\mathsf{b} \subseteq \mathsf{c} \vdash \neg \mathsf{b}^{\mathsf{x}} \vee \mathsf{x} \subseteq \mathsf{c}$ by $\subseteq$Rdt. By $\vee_{\subseteq}$E it now suffices to show:

(a) $\neg \mathsf{a}^{\mathsf{x}} \vdash \neg \mathsf{a}^{\mathsf{x}} \vee \mathsf{x} \subseteq \mathsf{c}$.

[6] For a proof that a similar system is equivalent to a Hilbert-style system for $\mathcal{K}$, see [134]. Note that some of the rules in Table 5.1 such as $\circledcirc\vee$Distr are derivable for the classical fragment of our axiomatization and are therefore not necessary for this equivalence.

[7] The system in [137] did not feature our $\subseteq_{\neg}$E as a rule; it was instead shown to be derivable in that system using a rule corresponding to our Proposition 5.4.4 (v). An alternative simplified propositional system would then replace $\subseteq_{\neg}$E with Proposition 5.4.4 (v). The rule corresponding to Proposition 5.4.4 (v) was originally introduced in the first-order system in [64].

(b) $x \subseteq b, \neg b^x \vdash \neg a^x \vee x \subseteq c$.

(c) $x \subseteq c \vdash \neg a^x \vee x \subseteq c$.

We have that (a) and (c) follow by $\vee$I, and (b) by $\subseteq_\neg$E.

(ii) By $\subseteq$Ext, it suffices to derive $\neg a_1 a_0 a_2{}^{x_1 x_0 x_2} \vee x_1 x_0 x_2 \subseteq b_1 b_0 b_2$ for all $x_1 x_0 x_2 \in \{\top, \bot\}^{|a_1 a_0 a_2|}$. For a given $x_1 x_0 x_2$, we have $a_0 a_1 a_2 \subseteq b_0 b_1 b_2 \vdash \neg a_0 a_1 a_2{}^{x_0 x_1 x_2} \vee x_0 x_1 x_2 \subseteq b_0 b_1 b_2$ by $\subseteq$Rdt. By Proposition 5.4.3 we have that $\neg a_0 a_1 a_2{}^{x_0 x_1 x_2} \dashv\vdash \neg a_1 a_0 a_2{}^{x_1 x_0 x_2}$, and therefore also

$$\neg a_0 a_1 a_2{}^{x_0 x_1 x_2} \vee x_0 x_1 x_2 \subseteq b_0 b_1 b_2 \vdash \neg a_1 a_0 a_2{}^{x_1 x_0 x_2} \vee x_0 x_1 x_2 \subseteq b_0 b_1 b_2. \quad (*)$$

We also have $b_1 b_0 b_2 \subseteq b_1 b_0 b_2$ by $\subseteq$Id, and then $\neg b_1 b_0 b_2{}^{x_1 x_0 x_2} \vee x_1 x_0 x_2 \subseteq b_1 b_0 b_2$ by $\subseteq$Rdt. By Proposition 5.4.3, we have $\neg b_0 b_1 b_2{}^{x_0 x_1 x_2} \dashv\vdash \neg b_1 b_0 b_2{}^{x_1 x_0 x_2}$ so that

$$\neg b_1 b_0 b_2{}^{x_1 x_0 x_2} \vee x_1 x_0 x_2 \subseteq b_1 b_0 b_2 \vdash \neg b_0 b_1 b_2{}^{x_0 x_1 x_2} \vee x_1 x_0 x_2 \subseteq b_1 b_0 b_2. \quad (**)$$

By $\vee_\subseteq$E applied to $(*)$ and $(**)$ it now suffices to show:

(a) $\neg a_1 a_0 a_2{}^{x_1 x_0 x_2} \vdash \neg a_1 a_0 a_2{}^{x_1 x_0 x_2} \vee x_1 x_0 x_2 \subseteq b_1 b_0 b_2$.

(b) $x_0 x_1 x_2 \subseteq b_0 b_1 b_2, \neg b_0 b_1 b_2{}^{x_0 x_1 x_2} \vdash \neg a_1 a_0 a_2{}^{x_1 x_0 x_2} \vee x_1 x_0 x_2 \subseteq b_1 b_0 b_2$.

(c) $x_1 x_0 x_2 \subseteq b_1 b_0 b_2 \vdash \neg a_1 a_0 a_2{}^{x_1 x_0 x_2} \vee x_1 x_0 x_2 \subseteq b_1 b_0 b_2$.

We have that (a) and (c) follow by $\vee$I, and (b) by $\subseteq_\neg$E.

(iii) Similar to (ii).

(iv) By $\subseteq$Ext, it suffices to derive $\neg a^x \vee x \subseteq c$ for all $x \in \{\top, \bot\}^{|a|}$. For a given x, we have $\bigwedge_{y \in \{\top,\bot\}^{|b|}} (\neg ab^{xy} \vee xy \subseteq cd)$ by $\subseteq$Rdt. By repeatedly applying $\vee_\subseteq$E, it suffices to show:

(a) $xy \subseteq cd \vdash \neg a^x \vee x \subseteq c$ for any given $y \in \{\top, \bot\}^{|b|}$.

(b) $\bigwedge_{y \in \{\top,\bot\}^{|b|}} \neg ab^{xy} \vdash \neg a^x \vee x \subseteq c$.

For (b), by Proposition 5.4.3 we have $\vdash \bigvee_{y \in \{\top,\bot\}^{|b|}} b^y$, and by Proposition 5.4.3 and $\vee$I, $\bigwedge_{y \in \{\top,\bot\}^{|b|}} \neg ab^{xy}, \bigvee_{y \in \{\top,\bot\}^{|b|}} b^y \vdash \neg a^x \vee x \subseteq c$ (for a given y, $\neg ab^{xy} = \neg(a^x \wedge b^y) \dashv\vdash \neg a^x \vee \neg b^y$, and $\neg a^x \vee \neg b^y, b^y \vdash \neg a^x$).

For (a), we have $c \subseteq c$ by $\subseteq$Id, and then $\neg c^x \vee x \subseteq c$ by $\subseteq$Rdt. By Proposition 5.4.3, we have $\neg c^x \vee x \subseteq c \vdash \bigwedge_{z \in \{\top,\bot\}^{|b|}} \neg cd^{xz} \vee x \subseteq c$ (for a given z, we have $\neg c^x \vdash \neg c^x \vee \neg d^z \dashv\vdash \neg(c^x \wedge d^z) = \neg cd^{xz}$). By $\vee_\subseteq$E it therefore suffices to show:

(c) $x \subseteq c \vdash \neg a^x \vee x \subseteq c$.

(d) $xy \subseteq cd, \bigwedge_{z \in \{\top,\bot\}^{|b|}} \neg cd^{xz} \vdash \neg a^x \vee x \subseteq c$ for any given $y \in \{\top, \bot\}^{|b|}$.

We have that (c) follows by $\vee$I, and (d) by $\subseteq_\neg$E.

(v) Let $n := |\mathsf{a}| = |\mathsf{b}| = |\mathsf{c}|$. Define a function $f : \{\bot, \top\}^n \to \{\bot, \top\}$ by $f(\mathsf{y}) = \top :\iff w \vDash \alpha(\mathsf{y})$ for some w (note that since the propositional symbols and constants appearing in $\alpha(\mathsf{c})$ are among those in c, the definition of f is independent of the choice of w). From $\mathsf{a} \subseteq \mathsf{b}$ we have $\bigwedge_{\{\mathsf{x} \in \{\top,\bot\}^n | f(\mathsf{x}) = \bot\}}(\neg \mathsf{a}^{\mathsf{x}} \vee \mathsf{x} \subseteq \mathsf{b})$ by $\subseteq$Rdt. By repeatedly applying $\vee_{\subseteq}$E, it suffices to show:

(a) $\bigwedge_{\{\mathsf{x} \in \{\top,\bot\}^n | f(\mathsf{x}) = \bot\}} \neg \mathsf{a}^{\mathsf{x}} \vdash \alpha(\mathsf{a})$.

(b) $\alpha(\mathsf{b}), \mathsf{x} \subseteq \mathsf{b} \vdash \alpha(\mathsf{a})$ for any given $\mathsf{x} \in \{\top, \bot\}^n$ with $f(\mathsf{x}) = \bot$.

To show (a) and (b), we show (c):

(c) $\bigwedge_{\{\mathsf{x} \in \{\top,\bot\}^n | f(\mathsf{x}) = \bot\}} \neg \mathsf{c}^{\mathsf{x}} \equiv \alpha(\mathsf{c})$.

For the left-to-right direction, let $w \vDash \bigwedge_{\{\mathsf{x} \in \{\top,\bot\}^n | f(\mathsf{x}) = \bot\}} \neg \mathsf{c}^{\mathsf{x}}$ and assume for contradiction that $w \nvDash \alpha(\mathsf{c})$. Then for $\mathsf{y} \in \{\bot, \top\}^n$ given by $y_i = \top \iff w \vDash c_i$ we have $w \nvDash \alpha(\mathsf{y})$ so $f(\mathsf{y}) = \bot$, whence $w \vDash \neg \mathsf{c}^{\mathsf{y}}$. But clearly by the definition of y we have $w \vDash \mathsf{c}^{\mathsf{y}}$, a contradiction.

For the right-to-left direction, let $w \vDash \alpha(\mathsf{c})$ and let $\mathsf{x} \in \{\top, \bot\}^n$ be such that $f(\mathsf{x}) = \bot$. Then for some x_i we must have $w \vDash c_i \iff x_i = \bot$ (lest we have $w \vDash \alpha(\mathsf{x})$ whence $f(\mathsf{x}) = \top$, contradicting $f(\mathsf{x}) = \bot$), and therefore $w \vDash \neg \mathsf{c}^{\mathsf{x}}$.

Given (c), (a) follows immediately by Proposition 5.4.3.

For (b), by (c) and Proposition 5.4.3, $\alpha(\mathsf{b})$ gives us $\neg \mathsf{b}^{\mathsf{x}}$. By $\subseteq_{\neg}$E applied to $\neg \mathsf{b}^{\mathsf{x}}$ and $\mathsf{x} \subseteq \mathsf{b}$, we have $\alpha(\mathsf{a})$. □

We move on to the completeness proof. Our strategy involves showing that each ML($\subseteq$)-formula is provably equivalent to a formula in the normal form presented in Section 5.3. Once all formulas are in normal form, completeness follows from the semantic and proof-theoretic properties of formulas in this form. That is, we show:

5.4.5. LEMMA (Provable equivalence of the normal form). *For any formula $\varphi(\mathsf{X})$ in* ML($\subseteq$) *and $k \geq md(\varphi)$, there is a (finite, nonempty) property $\mathcal{C}$ (over X) such that*

$$\varphi \dashv\vdash \bigvee_{(M,T) \in \mathcal{C}} \theta^{\mathsf{X},k}_{M,T}, \qquad i.e., \qquad \varphi \dashv\vdash \bigvee_{(M,T) \in \mathcal{C}} \Big(\bigvee_{w \in T} \chi^{\mathsf{X},k}_{M,w} \wedge \bigwedge_{w \in T} (\top \subseteq \chi^{\mathsf{X},k}_{M,w})\Big).$$

The proof of the above lemma is more involved. We postpone it for now, and first focus on how it allows us to prove completeness. To that end, note the following immediate consequence of classical completeness: the kth Hintikka formulas of two k-bisimilar pointed models are provably equivalent.

5.4.6. LEMMA. *If $M, w \rightleftharpoons_k M', u$, then $\chi^k_{M,w} \dashv\vdash \chi^k_{M',u}$.*

Proof:
By Theorem 5.3.3, $w \rightleftharpoons_k u$ implies $w \equiv_k u$ and then $\chi^k_w \dashv\vdash \chi^k_u$ by Proposition 5.4.3. □

Recall from footnote 4 in Section 5.3 that in defining the $(k+1)$th Hintikka formula $\chi^{\mathsf{X},k+1}_{M,w}$, we choose an arbitrary finite set T of representatives from the possibly infinite set $R[w]$. The above lemma shows that two k-bisimilar worlds give rise to provably equivalent Hintikka formulas, so the specific choice of the representatives also makes no difference in our proof system. Therefore, we may continue our practice of using arbitrary finite representatives in the definition of $\chi^{\mathsf{X},k+1}_{M,w}$ in the context of the proof system. Similar results hold for other formulas employing finite representatives, though we henceforth omit explicit discussion of these representatives. Our next aim is to prove the analogous result for Hintikka formulas for teams—that is:

5.4.7. LEMMA. *If $M,T \rightleftharpoons_k M',S$, then $\theta^k_{M,T} \dashv\vdash \theta^k_{M',S}$.*

In order to establish this, we first prove some results concerning primitive inclusion atoms. We also show another unrelated result concerning primitive inclusion atoms ((iii) in the lemma below) which is required in the sequel.

5.4.8. LEMMA.

(i) $\alpha_1,\dots,\alpha_n \vdash \top^n \subseteq \alpha_1\dots\alpha_n$, *where* $\top^n$ *denotes the sequence* $\top,\dots,\top$ *of length n.*

(ii) *If* $\alpha_i \vdash \beta_i$, *then* $\top^n \subseteq \alpha_1\dots\alpha_i\dots\alpha_n \vdash \top^n \subseteq \alpha_1\dots\beta_i\dots\alpha_n$.

(iii) $\top^n \subseteq \alpha_1^{x_1}\dots\alpha_n^{x_n} \vdash x_1\dots x_n \subseteq \alpha_1\dots\alpha_n$.

Proof:

(i) $\top \subseteq \top$ by $\subseteq$Id; using α_n and $\top \subseteq \top$ we get $\top\top \subseteq \alpha_n\top$ by $\subseteq$Exp; and then $\top\top \subseteq \alpha_n\top \vdash \top \subseteq \alpha_n$ by Proposition 5.4.4 (iv). We repeat the $\subseteq$Exp-step with $\alpha_{n-1},\dots,\alpha_1$.

(ii) By Proposition 5.4.3, $\vdash \neg(\alpha_1\wedge\dots\wedge\alpha_n)\vee(\alpha_1\wedge\dots\wedge\alpha_n)$. From the assumption $\alpha_i \vdash \beta_i$ together with item (i) we have that $\alpha_1\wedge\dots\wedge\alpha_n \vdash \alpha_1\wedge\dots\wedge\beta_i\wedge\dots\wedge\alpha_n \vdash \top^n \subseteq \alpha_1\dots\beta_i\dots\alpha_n$. Thus $\neg(\alpha_1\wedge\dots\wedge\alpha_n)\vee(\alpha_1\wedge\dots\wedge\alpha_n) \vdash \neg(\alpha_1\wedge\dots\wedge\alpha_n)\vee(\top^n \subseteq \alpha_1\dots\beta_i\dots\alpha_n)$. By $\vee_{\subseteq}$E, it suffices to show:

(a) $\top^n \subseteq \alpha_1\dots\alpha_n, \neg(\alpha_1\wedge\dots\wedge\alpha_n) \vdash \top^n \subseteq \alpha_1\dots\beta_i\dots\alpha_n$

(b) $\top^n \subseteq \alpha_1\dots\alpha_n, \top^n \subseteq \alpha_1\dots\beta_i\dots\alpha_n \vdash \top^n \subseteq \alpha_1\dots\beta_i\dots\alpha_n$

We have that (a) follows by $\subseteq_\neg$E, and (b) is immediate.

(iii) By $\subseteq$Id, we have $\alpha_1\dots\alpha_n \subseteq \alpha_1\dots\alpha_n$ and then by $\subseteq$Rdt, $\neg(\alpha_1\dots\alpha_n^{x_1\dots x_n})\vee x_1\dots x_n \subseteq \alpha_1\dots\alpha_n$. By $\vee_{\subseteq}$E it suffices to show $\neg(\alpha_1\dots\alpha_n^{x_1\dots x_n}), \top^n \subseteq \alpha_1^{x_1}\dots\alpha_n^{x_n} \vdash x_1\dots x_n \subseteq \alpha_1\dots\alpha_n$ and $x_1\dots x_n \subseteq \alpha_1\dots\alpha_n \vdash x_1\dots x_n \subseteq \alpha_1\dots\alpha_n$. The latter is immediate; the former follows by $\subseteq_\neg$E since $\alpha_1\dots\alpha_n^{x_1\dots x_n} = (\alpha_1^{x_1}\dots\alpha_n^{x_n})^{\top^n}$. □

Proof of Lemma 5.4.7:
Suppose that $T \rightleftharpoons_k S$. If $T = S = \varnothing$, then $\theta^k_T = \theta^k_S$. Otherwise, both T and S are nonempty. By symmetry it suffices to show $\theta^k_T \vdash \theta^k_S$. We have that $\theta^k_T \vdash \bigvee_{w\in T}\chi^k_w$ by $\wedge$E. Let $w \in T$. By $T \rightleftharpoons_k S$, there is some $u \in S$ such that $w \rightleftharpoons_k u$, so that by Lemma 5.4.6, $\chi^k_w \dashv\vdash \chi^k_u$. Therefore, by $\vee$I and $\vee$E, $\bigvee_{w\in T}\chi^k_w \vdash \bigvee_{u\in S}\chi^k_u$. Now let $u \in S$. By $T \rightleftharpoons_k S$, there is some $w_u \in T$ such that $w_u \rightleftharpoons_k u$, so that by Lemma 5.4.6, $\chi^k_{w_u} \dashv\vdash \chi^k_u$. By $\wedge$E and Lemma 5.4.8 (ii), $\theta^k_T \vdash \top \subseteq \chi^k_{w_u} \vdash \top \subseteq \chi^k_u$. Therefore by $\wedge$I, $\theta^k_T \vdash \bigwedge_{u\in S}(\top \subseteq \chi^k_u)$, so that $\theta^k_T \vdash \bigvee_{u\in S}\chi^k_u \wedge \bigwedge_{u\in S}(\top \subseteq \chi_u) = \theta^k_S$. □

Next we prove some important semantic facts concerning the relationship between formulas in normal form and disjoint unions. The *disjoint union* $\biguplus_{i\in I} M_i$ of the models M_i ($i \in I$) is defined as usual, where in particular we recall that the domain of the disjoint union is defined to be $\biguplus_{i\in I} W_i = \bigcup_{i\in I}(W_i \times \{i\})$. We also extend the notion to properties: the *disjoint union* of a property $\mathcal{C} = \{(M_i, T_i) \mid i \in I\}$ is $\biguplus\mathcal{C} = (\biguplus_{i\in I} M_i, \biguplus_{i\in I} T_i)$, where $\biguplus_{i\in I} T_i = \bigcup_{i\in I}(T_i \times \{i\})$. To simplify notation, we define the disjoint union $\biguplus\varnothing$ of the empty property to be $(M^*, \varnothing)$ for some fixed model M^*. We refer to a world (w,i) in a disjoint union as simply w, and to a team $T \times \{i\}$ as simply T. The following result is standard (see, for instance, [24]):

5.4.9. PROPOSITION. *For any $i \in I$, any $w \in W_i$, and any $k \in \mathbb{N}$: $M_i, w \rightleftharpoons_k \biguplus_{i\in I} M_i, w$. Therefore, for any $T \subseteq W_i$: $M_i, T \rightleftharpoons_k \biguplus_{i\in I} M_i, T$.*

We have:

5.4.10. LEMMA. *$M, S \vDash \bigvee_{(M',T)\in\mathcal{C}} \theta^k_{M',T}$ iff $M, S \rightleftharpoons_k \biguplus\mathcal{C}'$ for some $\mathcal{C}' \subseteq \mathcal{C}$.*

Proof:
$\Longrightarrow$: If $M, S \vDash \bigvee_{(M',T)\in\mathcal{C}} \theta^k_{M',T}$, then $mS = \bigcup_{(M',T)\in\mathcal{C}} S_{M',T}$ where $S_{M',T} \vDash \theta^k_{M',T}$. By Lemma 5.3.7 (iii), either $S_{M',T} \rightleftharpoons_k T$ or $S_{M',T} = \varnothing$. Let $\mathcal{C}' := \{(M',T) \mid S_{M',T} \rightleftharpoons_k T\}$. Then $S = \bigcup_{(M',T)\in\mathcal{C}} S_{M',T} = \bigcup_{(M',T)\in\mathcal{C}'} S_{M',T}$. By Proposition 5.4.9, we have:

$$M, S = M, \bigcup_{(M',T)\in\mathcal{C}'} S_{M',T} \rightleftharpoons_k \biguplus\{(M',T) \mid T \in \mathcal{C}'\} = \biguplus\mathcal{C}'.$$

Note that if $S_{M',T} = \varnothing$ for all $(M',T) \in \mathcal{C}$, then $S = \varnothing$ and $\biguplus\mathcal{C}' = \biguplus\varnothing = (M^*, \varnothing)$ by our convention.

$\Longleftarrow$: Let $\mathcal{C}' = \{(M_i, T_i) \mid i \in I\}$. If $M, S \rightleftharpoons_k \biguplus\mathcal{C}'$, then by Lemma 5.3.5 (iii), there are subteams $S_i \subseteq S$ such that $\bigcup_{i\in I} S_i = S$ and $M, S_i \rightleftharpoons_k \biguplus_{j\in I} M_j, T_i$. By Proposition 5.4.9, $\biguplus_{j\in I} M_j, T_i \rightleftharpoons_k M_i, T_i$ so that also $M, S_i \rightleftharpoons_k M_i, T_i$. By Lemma 5.3.7 (iii), $M, S_i \vDash \theta^k_{M_i,T_i}$ so that $M, S \vDash \bigvee_{(M_i,T_i)\in\mathcal{C}'} \theta^k_{M_i,T_i}$, and by the empty team property, $M, S \vDash \bigvee_{(M',T)\in\mathcal{C}} \theta^k_{M',T}$. □

5.4.11. COROLLARY. *$\bigvee_{(M,T)\in\mathcal{C}} \theta^k_{M,T} \vDash \bigvee_{(M',S)\in\mathcal{D}} \theta^k_{M',S}$ iff for all $(M'',U) \in \mathcal{C}$, $M'', U \rightleftharpoons_k \biguplus\mathcal{D}_{(M'',U)}$ for some $\mathcal{D}_{(M'',U)} \subseteq \mathcal{D}$.*

Proof:
$\Longrightarrow$: Let $(M'',U) \in \mathcal{C}$. By Lemma 5.3.7 (iii), $U \vDash \theta^k_U$. By the empty team property, $U \vDash \bigvee_{(M,T)\in\mathcal{C}} \theta^k_T$, and then by assumption, $U \vDash \bigvee_{(M',S)\in\mathcal{D}} \theta^k_S$. The result now follows by Lemma 5.4.10.

$\Longleftarrow$: Let $M''',Y \vDash \bigvee_{(M,T)\in\mathcal{C}} \theta^k_T$. By Lemma 5.4.10, $M''',Y \rightleftharpoons_k \biguplus\mathcal{C}'$ for some $\mathcal{C}' \subseteq \mathcal{C}$. By assumption, for each $(M'',U) \in \mathcal{C}'$ there is some $\mathcal{D}_{(M'',U)} \subseteq \mathcal{D}$ such that $M'',U \rightleftharpoons_k \biguplus\mathcal{D}_{(M'',U)}$. Then $M''',Y \rightleftharpoons_k \biguplus\{\biguplus\mathcal{D}_{(M'',U)} \mid (M'',U) \in \mathcal{C}'\} \rightleftharpoons_k \biguplus\bigcup_{(M'',U)\in\mathcal{C}'} \mathcal{D}_{(M'',U)}$, so that by Lemma 5.4.10, $Y \vDash \bigvee_{(M',S)\in\mathcal{D}} \theta^k_S$. □

The final lemma required for the completeness proof is a proof-theoretic analogue of (one direction of) Lemma 5.4.10.

5.4.12. LEMMA. $\theta^k_{\biguplus\mathcal{D}} \vdash \bigvee_{(M,S)\in\mathcal{D}} \theta^k_{M,S}$.

Proof:
If $\mathcal{D}$ is empty, then $\theta^k_{\biguplus\mathcal{D}} \dashv\vdash \bot \vdash \bigvee_{(M,S)\in\mathcal{D}} \theta^k_S$. Otherwise let $\mathcal{D} = \{S_1,\dots,S_n\}$. We have:

$$\theta^k_{\biguplus\mathcal{D}} = \bigvee_{w\in\biguplus\mathcal{D}} \chi^k_w \wedge \bigwedge_{w\in\biguplus\mathcal{D}} (\top \subseteq \chi^k_w) = \bigvee_{(M,S)\in\mathcal{D}} \bigvee_{w\in S} \chi^k_w \wedge \bigwedge_{(M,S)\in\mathcal{D}} \bigwedge_{w\in S} (\top \subseteq \chi^k_w),$$

from which we derive:

$$\begin{aligned}
&\vdash \Big(\bigvee_{w\in S_1} \chi^k_w \vee \bigvee_{2\le i\le n} \bigvee_{w\in S_i} \chi^k_w\Big) \wedge \bigwedge_{w\in S_1} (\top \subseteq \chi^k_w) \wedge \bigwedge_{2\le i\le n} \bigwedge_{w\in S_i} (\top \subseteq \chi^k_w) \\
&\vdash \Big(\Big(\Big(\bigvee_{w\in S_1} \chi^k_w \vee \bigvee_{w\in S_1} \chi^k_w\Big) \wedge \bigwedge_{w\in S_1} (\top \subseteq \chi^k_w)\Big) \vee \bigvee_{2\le i\le n} \bigvee_{w\in S_i} \chi^k_w\Big) \wedge \bigwedge_{2\le i\le n} \bigwedge_{w\in S_i} (\top \subseteq \chi^k_w) \quad (\subseteq\text{Distr}) \\
&\vdash \Big(\theta^k_{S_1} \vee \bigvee_{2\le i\le n} \bigvee_{w\in S_i} \chi^k_w\Big) \wedge \bigwedge_{2\le i\le n} \bigwedge_{w\in S_i} (\top \subseteq \chi^k_w) \\
&\ \vdots \\
&\vdash \theta^k_{S_1} \vee \dots \vee \theta^k_{S_n}.
\end{aligned}$$

□

5.4.13. THEOREM (Completeness). *If* $\Gamma \vDash \psi$*, then* $\Gamma \vdash \psi$.

Proof:
Suppose that $\Gamma \vDash \psi$. Since ML($\subseteq$) is compact, there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vDash \psi$. It suffices to show that $\varphi \vdash \psi$ where $\varphi = \bigwedge_{\gamma\in\Gamma_0} \gamma$.

Let $k \geq \max\{md(\varphi), md(\psi)\}$. By Lemma 5.4.5,

$$\varphi \dashv\vdash \bigvee_{(M,T)\in\mathcal{C}} \theta^k_{M,T} \text{ and } \psi \dashv\vdash \bigvee_{(M',S)\in\mathcal{D}} \theta^k_{M',S}.$$

for some finite nonempty properties $\mathcal{C}$ and $\mathcal{D}$. By soundness and $\varphi \vDash \psi$,

$$\bigvee_{(M,T)\in\mathcal{C}} \theta^k_{M,T} \vDash \bigvee_{(M',S)\in\mathcal{D}} \theta^k_{M',S}.$$

Let $(M,T)\in\mathcal{C}$. By Corollary 5.4.11, $M,T \rightleftharpoons_k \biguplus\mathcal{D}_T$ for some $\mathcal{D}_T \subseteq \mathcal{D}$. By Lemma 5.4.7, Lemma 5.4.12, and $\vee$I,

$$\theta^k_{M,T} \vdash \theta^k_{\biguplus\mathcal{D}_T} \vdash \bigvee_{(M',S)\in\mathcal{D}_T} \theta^k_{M',S} \vdash \bigvee_{(M',S)\in\mathcal{D}} \theta^k_{M',S}.$$

By $\vee$E we get that $\bigvee_{(M,T)\in\mathcal{C}} \theta^k_{M,T} \vdash \bigvee_{(M',S)\in\mathcal{D}} \theta^k_{M',S}$, and conclude that $\varphi \vdash \psi$. □

We dedicate the rest of this subsection to the proof of Lemma 5.4.5: provable equivalence of the normal form. We first prove some technical lemmas. The rules $\vee_\subseteq$E and $\boxdot\vee_\subseteq$E can be generalized as follows:

5.4.14. LEMMA. *(i) If*

(a) $\Gamma, \mathsf{x}_1 \subseteq \mathsf{a}_1, \ldots, \mathsf{x}_k \subseteq \mathsf{a}_k \vdash \chi$ *and*

(b) $\Gamma, \psi \vdash \chi$,

then $\Gamma, (\mathsf{x}_1 \subseteq \mathsf{a}_1 \wedge \cdots \wedge \mathsf{x}_k \subseteq \mathsf{a}_k) \vee \psi \vdash \chi$.

(ii) Let I be a finite index set and for each $i \in I$, let ι_i be a conjunction of finitely many primitive inclusion atoms. If for every nonempty $J \subseteq I$,

$$\Gamma, \bigvee_{j\in J} \varphi_j, \bigwedge_{j\in J} \iota_j \vdash \chi,$$

then $\Gamma, \bigvee_{i\in I}(\varphi_i \wedge \iota_i) \vdash \chi$.

(iii) If

(a) $\Gamma, \top \subseteq \diamondsuit\mathsf{a}_1, \ldots, \top \subseteq \diamondsuit\mathsf{a}_k \vdash \chi$ *and*

(b) $\Gamma, \boxdot\psi \vdash \chi$,

then $\Gamma, \boxdot((\mathsf{x}_1 \subseteq \mathsf{a}_1 \wedge \cdots \wedge \mathsf{x}_k \subseteq \mathsf{a}_k) \vee \psi) \vdash \chi$.

(iv) Let I be a finite index set and for each $i \in I$, let ι_i be a conjunction of finitely many primitive inclusion atoms. For $\iota_i = \bigwedge_{k\in K_i}(\mathsf{x}_k \subseteq \mathsf{a}_k)$, define $\iota_{\diamondsuit i} = \bigwedge_{k\in K_i}(\top \subseteq \diamondsuit\mathsf{a}_k^{\mathsf{x}_k})$. If for every nonempty $J \subseteq I$

$$\Gamma, \boxdot\bigvee_{j\in J} \varphi_j, \bigwedge_{j\in J} \iota_{\diamondsuit j} \vdash \chi,$$

then $\Gamma, \boxdot\bigvee_{i\in I}(\varphi_i \wedge \iota_i) \vdash \chi$.

Proof:
We prove (i) and (ii); the proofs of (iii) and (iv) are similar.

(i) We have that $(\mathsf{x}_1 \subseteq \mathsf{a}_1 \wedge \cdots \wedge \mathsf{x}_k \subseteq \mathsf{a}_k) \vee \psi \vdash ((\mathsf{x}_1 \subseteq \mathsf{a}_1 \wedge \cdots \wedge \mathsf{x}_{k-1} \subseteq \mathsf{a}_{k-1}) \vee \psi) \wedge (\mathsf{x}_k \subseteq \mathsf{a}_k \vee \psi)$ by $\vee$E, $\vee$I, $\wedge$E, and $\wedge$I. To show $\Gamma, (\mathsf{x}_1 \subseteq \mathsf{a}_1 \wedge \cdots \wedge \mathsf{x}_{k-1} \subseteq \mathsf{a}_{k-1}) \vee \psi, \mathsf{x}_k \subseteq \mathsf{a}_k \vee \psi \vdash \chi$, by $\vee_{\subseteq}$E it suffices to show:

(a) $\Gamma, (\mathsf{x}_1 \subseteq \mathsf{a}_1 \wedge \cdots \wedge \mathsf{x}_{k-1} \subseteq \mathsf{a}_{k-1}) \vee \psi, \psi \vdash \chi$ and

(b) $\Gamma, (\mathsf{x}_1 \subseteq \mathsf{a}_1 \wedge \cdots \wedge \mathsf{x}_{k-1} \subseteq \mathsf{a}_{k-1}) \vee \psi, \mathsf{x}_k \subseteq \mathsf{a}_k \vdash \chi$.

Showing $\Gamma, \psi \vdash \chi$ suffices for (a). As for (b), similarly to the above, we have $((\mathsf{x}_1 \subseteq \mathsf{a}_1 \wedge \cdots \wedge \mathsf{x}_{k-1} \subseteq \mathsf{a}_{k-1}) \vee \psi) \wedge \mathsf{x}_k \subseteq \mathsf{a}_k \vdash ((\mathsf{x}_1 \subseteq \mathsf{a}_1 \wedge \cdots \wedge \mathsf{x}_{k-2} \subseteq \mathsf{a}_{k-2}) \vee \psi) \wedge \mathsf{x}_k \subseteq \mathsf{a}_k \wedge (\mathsf{x}_{k-1} \subseteq \mathsf{a}_{k-1} \vee \psi)$. To show (b), it therefore suffices, by $\vee_{\subseteq}$E, to show:

(c) $\Gamma, (\mathsf{x}_1 \subseteq \mathsf{a}_1 \wedge \cdots \wedge \mathsf{x}_{k-2} \subseteq \mathsf{a}_{k-2}) \vee \psi, \mathsf{x}_k \subseteq \mathsf{a}_k, \psi \vdash \chi$ and

(d) $\Gamma, (\mathsf{x}_1 \subseteq \mathsf{a}_1 \wedge \cdots \wedge \mathsf{x}_{k-2} \subseteq \mathsf{a}_{k-2}) \vee \psi, \mathsf{x}_k \subseteq \mathsf{a}_k, \mathsf{x}_{k-1} \subseteq \mathsf{a}_{k-1} \vdash \chi$.

Showing $\Gamma, \psi \vdash \chi$ would again suffice for (c). Continuing in the same manner, one eventually finds that it suffices to show $\Gamma, \psi \vdash \chi$ and $\Gamma, \mathsf{x}_1 \subseteq \mathsf{a}_1, \ldots, \mathsf{x}_k \subseteq \mathsf{a}_k \vdash \chi$.

(ii) Let $I = \{1, \ldots, n\}$. We have that $\bigvee_{1 \le i \le n-1}(\varphi_i \wedge \iota_i) \vdash (\bigvee_{1 \le i \le n-1}(\varphi_i \wedge \iota_i) \vee \varphi_n) \wedge (\bigvee_{1 \le i \le n-1}(\varphi_i \wedge \iota_i) \vee \iota_n)$ by $\vee$E, $\vee$I, $\wedge$E, and $\wedge$I. To show $\Gamma, \bigvee_{1 \le i \le n-1}(\varphi_i \wedge \iota_i) \vee \varphi_n, \bigvee_{1 \le i \le n-1}(\varphi_i \wedge \iota_i) \vee \iota_n \vdash \chi$, by (i) it suffices to show:

(a) $\Gamma, \bigvee_{1 \le i \le n-1}(\varphi_i \wedge \iota_i) \vee \varphi_n, \bigvee_{1 \le i \le n-1}(\varphi_i \wedge \iota_i) \vdash \chi$ and

(b) $\Gamma, \bigvee_{1 \le i \le n-1}(\varphi_i \wedge \iota_i) \vee \varphi_n, \iota_n \vdash \chi$.

By $\vee$I, showing $\Gamma, \bigvee_{1 \le i \le n-1}(\varphi_i \wedge \iota_i) \vdash \chi$ suffices for (a). Furthermore, by a similar manipulation as used above, (c) suffices for (a), and (d) suffices for (b).

(c) $\Gamma, \bigvee_{1 \le i \le n-2}(\varphi_i \wedge \iota_i) \vee \varphi_{n-1}, \bigvee_{1 \le i \le n-2}(\varphi_i \wedge \iota_i) \vee \iota_{n-1} \vdash \chi$.

(d) $\Gamma, \bigvee_{1 \le i \le n-2}(\varphi_i \wedge \iota_i) \vee \varphi_{n-1} \vee \varphi_n, \bigvee_{1 \le i \le n-2}(\varphi_i \wedge \iota_i) \vee \varphi_n \vee \iota_{n-1}, \iota_n \vdash \chi$.

By (i), to show (c) it suffices to show (e) and (f), and to show (d) it suffices to show (g) and (h):

(e) $\Gamma, \bigvee_{1 \le i \le n-2}(\varphi_i \wedge \iota_i) \vee \varphi_{n-1}, \bigvee_{1 \le i \le n-2}(\varphi_i \wedge \iota_i) \vdash \chi$.

(f) $\Gamma, \bigvee_{1 \le i \le n-2}(\varphi_i \wedge \iota_i) \vee \varphi_{n-1}, \iota_{n-1} \vdash \chi$.

(g) $\Gamma, \bigvee_{1 \le i \le n-2}(\varphi_i \wedge \iota_i) \vee \varphi_{n-1} \vee \varphi_n, \bigvee_{1 \le i \le n-2}(\varphi_i \wedge \iota_i) \vee \varphi_n, \iota_n \vdash \chi$.

(h) $\Gamma, \bigvee_{1 \le i \le n-2}(\varphi_i \wedge \iota_i) \vee \varphi_{n-1} \vee \varphi_n, \iota_{n-1}, \iota_n \vdash \chi$.

Showing that $\Gamma, \bigvee_{1 \le i \le n-2}(\varphi_i \wedge \iota_i) \vdash \chi$ would suffice to show (e), and showing that $\Gamma, \bigvee_{1 \le i \le n-2}(\varphi_i \wedge \iota_i) \vee \varphi_n, \iota_n \vdash \chi$ would suffice to show (g). Continuing in the same manner, one eventually finds that it suffices to show $\Gamma, \bigvee_{j \in J} \varphi_j, \bigwedge_{j \in J} \iota_j \vdash \chi$ for every nonempty $J \subseteq I$. □

Next, we show that the Hintikka formulas of non-bisimilar pointed models/models with teams are contradictory in our proof system.

5.4.15. LEMMA.

(i) If $M,w \not\rightleftharpoons_k M',u$, then $\chi^k_{M,w}, \chi^k_{M',u} \vdash \bot$.

(ii) If $M,T \not\rightleftharpoons_k M',S$, then $\theta^k_{M,T}, \theta^k_{M',S} \vdash \bot$.

Proof:

(i) Assume for contradiction that $\chi^k_w, \chi^k_u \nvdash \bot$. By Proposition 5.4.3, there is some model M'' and a nonempty team T of M'' such that $M'',T \vDash \chi^k_w$ and $M'',T \vDash \chi^k_u$. By flatness, $M'',v \vDash \chi^k_w$ and $M'',v \vDash \chi^k_u$ for all $v \in T$, from which it follows by Theorem 5.3.3 that $M,w \rightleftharpoons_k M'',v \rightleftharpoons_k M',u$, a contradiction.

(ii) W.l.o.g., we assume that there is some $w \in T$ such that $M,w \not\rightleftharpoons_k M',u$ for all $u \in S$. By item (i), $\chi^k_w, \chi^k_u \vdash \bot$ for all $u \in S$, whence by $\vee$E, $\bigvee_{u\in S} \chi^k_u, \chi^k_w \vdash \bot$. By $\neg$I, we derive $\bigvee_{u\in S} \chi^k_u \vdash \neg \chi^k_w$. Then by $\wedge$E, we have $\theta^k_S \vdash \bigvee_{u\in S} \chi^k_u \vdash \neg \chi^k_w$. We also derive $\theta^k_T \vdash \top \subseteq \chi^k_w$ by $\wedge$E. Finally, we use $\subseteq_\neg$E to derive $\top \subseteq \chi^k_w, \neg \chi^k_w \vdash \bot$. □

Finally, we note the following simple consequence of classical completeness:

5.4.16. LEMMA. *If $M,w \vDash \alpha$, then $\chi^k_{M,w} \vdash \alpha$, where $k \geq md(\alpha)$.*

Proof:
By Proposition 5.4.3, it suffices to show $\chi^k_w \vDash \alpha$. Let $T \vDash \chi^k_w$ so that $u \vDash \chi^k_w$ for all $u \in T$ by flatness. By Theorem 5.3.3, it follows that $w \equiv_k u$ whence also $u \vDash \alpha$ for all $u \in T$. Using flatness again, we conclude $T \vDash \alpha$. □

We are now ready to prove the main lemma.

Proof of Lemma 5.4.5:
By induction on φ.

- If $\varphi = \alpha \in$ ML and $k \geq md(\alpha)$, letting $\mathcal{D} := \{(M,w) \mid w \vDash \alpha\}$, by a standard normal form result for ML (see, e.g., [56]), we have that $\alpha \equiv \bigvee_{(M,w)\in\mathcal{D}} \chi^k_w$. Then by Proposition 5.4.3, $\alpha \dashv\vdash \bigvee_{(M,w)\in\mathcal{D}} \chi^k_w$. If $\mathcal{D} = \varnothing$, we have $\alpha \dashv\vdash \bigvee_{(M,w)\in\mathcal{D}} \chi^k_w = \bot = \theta^k_{(M^*,\varnothing)}$. Otherwise, we show that $\bigvee_{(M,w)\in\mathcal{D}} \chi^k_w \dashv\vdash \bigvee_{(M,w)\in\mathcal{D}} \theta^k_{\{w\}} = \bigvee_{(M,w)\in\mathcal{D}} (\chi^k_w \wedge \top \subseteq \chi^k_w)$. The direction $\dashv$ is easy. For the other direction $\vdash$, we derive $\chi^k_w \vdash \chi^k_w \wedge \top \subseteq \chi^k_w$ by Lemma 5.4.8 (i); the result then follows by $\vee$E and $\vee$I.
- Let $\varphi = \psi_1 \vee \psi_2$ and let $k \geq md(\varphi)$. Then $k \geq md(\psi_1), md(\psi_2)$, so by the induction hypothesis there are nonempty $\mathcal{C}, \mathcal{D}$ such that $\psi_1 \dashv\vdash \bigvee_{(M,T)\in\mathcal{C}} \theta^k_T$ and $\psi_2 \dashv\vdash \bigvee_{(M',S)\in\mathcal{D}} \theta^k_S$. Clearly $\psi_1 \vee \psi_2 \dashv\vdash \bigvee_{(M'',U)\in\mathcal{C}\cup\mathcal{D}} \theta^k_U$.

- Let $\varphi = \psi_1 \wedge \psi_2$ and let $k \geq md(\varphi)$. Then $k \geq md(\psi_1), md(\psi_2)$, so by the induction hypothesis there are nonempty $\mathcal{C}, \mathcal{D}$ such that $\psi_1 \dashv\vdash \bigvee_{(M,T)\in\mathcal{C}} \theta^k_T$ and $\psi_2 \dashv\vdash \bigvee_{(M',S)\in\mathcal{D}} \theta^k_S$. Let

$$\mathcal{Y} = \{\biguplus\mathcal{C}' \mid \mathcal{C}' \subseteq \mathcal{C} \text{ and } \biguplus\mathcal{C}' \rightleftharpoons_k \biguplus\mathcal{D}', \text{ for some } \mathcal{D}' \subseteq \mathcal{D}\},$$

that is, $\mathcal{Y}$ contains each disjoint union of (models with) teams in $\mathcal{C}$ which is k-bisimilar to some disjoint union of teams in $\mathcal{D}$. By union closure and bisimulation invariance, both ψ_1 and ψ_2 will hold in each team in $\mathcal{Y}$, and it is also easy to see that each team in which both ψ_1 and ψ_2 hold must be k-bisimilar to some team in $\mathcal{Y}$. We show $\psi_1 \wedge \psi_2 \dashv\vdash \bigvee_{(M'',X)\in\mathcal{Y}} \theta^k_X$.

($\vdash$) By Lemma 5.4.14 (ii) it suffices to show

$$\bigvee_{(M,T)\in\mathcal{C}'} \bigvee_{w\in T} \chi^k_w, \bigwedge_{(M,T)\in\mathcal{C}'} \bigwedge_{w\in T} (\top \subseteq \chi^k_w), \bigvee_{(M',S)\in\mathcal{D}'} \bigvee_{w\in S} \chi^k_w, \bigwedge_{(M',S)\in\mathcal{D}'} \bigwedge_{w\in S} (\top \subseteq \chi^k_w)$$
$$\vdash \bigvee_{(M'',X)\in\mathcal{Y}} \theta^k_X,$$

for all nonempty $\mathcal{C}' \subseteq \mathcal{C}$ and all nonempty $\mathcal{D}' \subseteq \mathcal{D}$. This reduces to showing $\theta_{\biguplus\mathcal{C}'}, \theta_{\biguplus\mathcal{D}'} \vdash \bigvee_{(M'',X)\in\mathcal{Y}} \theta^k_X$ for all nonempty $\mathcal{C}' \subseteq \mathcal{C}$ and all nonempty $\mathcal{D}' \subseteq \mathcal{D}$. For a given $\mathcal{C}'$ and $\mathcal{D}'$, if $\biguplus\mathcal{C}' \not\rightleftharpoons_k \biguplus\mathcal{D}'$, then $\theta_{\biguplus\mathcal{C}'}, \theta_{\biguplus\mathcal{D}'} \vdash \bot \vdash \bigvee_{(M'',X)\in\mathcal{Y}} \theta^k_X$ by Lemma 5.4.15 (ii). If $\biguplus\mathcal{C}' \rightleftharpoons_k \biguplus\mathcal{D}'$, then $\biguplus\mathcal{C}' \in \mathcal{Y}$, whence $\theta_{\biguplus\mathcal{C}'} \vdash \bigvee_{(M'',X)\in\mathcal{Y}} \theta^k_X$ by $\vee$I.

($\dashv$) Let $\biguplus\mathcal{C}' \in \mathcal{Y}$. By Lemma 5.4.12, we have $\theta^k_{\biguplus\mathcal{C}'} \vdash \bigvee_{(M,T)\in\mathcal{C}'} \theta^k_T$, and by $\vee$I, $\bigvee_{(M,T)\in\mathcal{C}'} \theta^k_T \vdash \bigvee_{(M,T)\in\mathcal{C}} \theta^k_T \vdash \psi_1$. Similarly $\theta^k_{\biguplus\mathcal{C}'} \vdash \psi_2$, so by $\vee$E, $\bigvee_{\biguplus\mathcal{C}'\in\mathcal{Y}} \theta^k_{\biguplus\mathcal{C}'} \vdash \psi_1 \wedge \psi_2$.

- Let $\varphi = \mathsf{a} \subseteq \mathsf{b}$, where $\mathsf{a} = \alpha_1 \ldots \alpha_n$ and $\mathsf{b} = \beta_1 \ldots \beta_n$, and let $k \geq md(\varphi)$. By $\subseteq$Rdt and $\subseteq$Ext,

$$\mathsf{a} \subseteq \mathsf{b} \dashv\vdash \bigwedge_{\mathsf{x}\in\{\top,\bot\}^{|\mathsf{a}|}} (\neg\, \mathsf{a}^{\mathsf{x}} \vee \mathsf{x} \subseteq \mathsf{b}).$$

Given the induction cases for ML-formulas, conjunction and disjunction, it therefore suffices to show that each primitive inclusion atom is provably equivalent to a formula in the normal form. We show that $\mathsf{x} \subseteq \mathsf{b} \dashv\vdash \bigvee_{(M,T)\in\mathcal{Y}} \theta^k_T$, where

$$\mathsf{Y} = \{(M,T) \mid \exists w \in T \text{ such that } w \vDash \mathsf{b}^{\mathsf{x}}\}.$$

Note that the provable equivalence we wish to show corresponds to the semantic fact in Proposition 5.2.4.

($\vdash$) Let $\mathcal{M} := \{(M',w) \mid w \vDash \top\}$. Then $\vDash \bigvee_{(M',w)\in\mathcal{M}} \chi^k_w$ so that $\vdash \bigvee_{(M',w)\in\mathcal{M}} \chi^k_w$ by Proposition 5.4.3. We have $\chi^k_w \vdash \bigvee_{(M',w)\in\mathcal{M}} (\chi^k_w \wedge \top \subseteq \chi^k_w)$ by Lemma 5.4.8 (i) and

$\vee$I, so $\vdash \bigvee_{(M',w)\in\mathcal{M}}(\chi^k_w \wedge \top \subseteq \chi^k_w)$ by $\vee$E. To show $\bigvee_{(M',w)\in\mathcal{M}}(\chi^k_w \wedge \top \subseteq \chi^k_w), \mathsf{x} \subseteq \mathsf{b} \vdash \bigvee_{(M,T)\in\mathcal{Y}} \theta^k_T$, by Lemma 5.4.14 (ii) it suffices to show that for all nonempty teams S,

$$\bigvee_{w\in S}\chi^k_w, \bigwedge_{w\in S}(\top \subseteq \chi^k_w), \mathsf{x}\subseteq \mathsf{b} \vdash \bigvee_{(M,T)\in\mathcal{Y}} \theta^k_T.$$

If $S \in \mathcal{Y}$, the result follows by $\vee$I. If $S \notin \mathcal{Y}$, then for any $w \in S$, $w \vDash \neg\,\mathsf{b}^{\mathsf{x}}$. We have that $md(\neg\,\mathsf{b}^{\mathsf{x}}) = md(\mathsf{x}\subseteq\mathsf{b}) \leq md(\mathsf{a}\subseteq\mathsf{b}) = md(\varphi) \leq k$, whence $\chi^k_w \vdash \neg\,\mathsf{b}^{\mathsf{x}}$ by Lemma 5.4.16. Therefore $\bigvee_{w\in S}\chi^k_w \vdash \neg\,\mathsf{b}^{\mathsf{x}}$ by $\vee$E. By $\subseteq_\neg$E, we have $\neg\,\mathsf{b}^{\mathsf{x}}, \mathsf{x}\subseteq\mathsf{b} \vdash \bigvee_{(M,T)\in\mathcal{Y}} \theta^k_T$.

($\dashv$) Let $(M,T) \in \mathcal{Y}$. Then there is some $w \in T$ such that $w \vDash \mathsf{b}^{\mathsf{x}}$ so that since $md(\mathsf{b}^{\mathsf{x}}) = md(\mathsf{x}\subseteq\mathsf{b}) \leq md(\mathsf{a}\subseteq\mathsf{b}) = md(\varphi) \leq k$, we have $\chi^k_w \vdash \mathsf{b}^{\mathsf{x}}$ by Lemma 5.4.16. We have $\theta^k_T \vdash \top \subseteq \chi^k_w$ by $\wedge$E, and $\top \subseteq \chi^k_w \vdash \top \subseteq \mathsf{b}^{\mathsf{x}}$ by $\chi^k_w \vdash \mathsf{b}^{\mathsf{x}}$ and Lemma 5.4.8 (ii). Then:

$$\begin{aligned} \top \subseteq \mathsf{b}^{\mathsf{x}} &\vdash \top^{|\mathsf{b}|} \subseteq \mathsf{b}^{\mathsf{x}} \dots \mathsf{b}^{\mathsf{x}} && \text{(Prop. 5.4.4 (iii))} \\ &\vdash \top^{|\mathsf{b}|} \subseteq \beta_1^{x_1} \dots \beta_n^{x_n} && \text{(Lemma 5.4.8 (ii))} \\ &\vdash x_1 \dots x_n \subseteq \beta_1 \dots \beta_n. && \text{(Lemma 5.4.8 (iii))} \end{aligned}$$

where $\mathsf{x}\subseteq\mathsf{b} = x_1\dots x_n \subseteq \beta_1\dots\beta_n$. Therefore $\bigvee_{(M,T)\in\mathcal{Y}} \theta^k_T \vdash \mathsf{x}\subseteq\mathsf{b}$ by $\vee$E.

· Let $\varphi = \Diamond\psi$ and let $k \geq md(\varphi)$. Then $n := k-1 \geq md(\psi)$ and by the induction hypothesis there is a nonempty $\mathcal{D}$ such that $\psi \dashv\vdash \bigvee_{(M,T)\in\mathcal{D}} \theta^n_T$. By $\Diamond$Mon, $\Diamond\psi \dashv\vdash \Diamond\bigvee_{(M,T)\in\mathcal{D}} \theta^n_T$. We show that

$$\Diamond \bigvee_{(M,T)\in\mathcal{D}} \theta^n_T \dashv\vdash \bigvee_{(M,T)\in\mathcal{D}} \Diamond\theta^n_T \dashv\vdash \bigvee_{(M,T)\in\mathcal{D}} (\Diamond\bigvee_{w\in T}\chi^n_w \wedge \bigwedge_{w\in T}(\top \subseteq \Diamond\chi^n_w)). \tag{5.1}$$

The modal depth of the formula on the very right is $\leq n+1 = k$, so the result then follows by the induction cases for ML-formulas, inclusion atoms, conjunction, and disjunction.

The first equivalence in (5.1) follows from the more general equivalence $\Diamond\varphi \vee \Diamond\psi \dashv\vdash \Diamond(\varphi\vee\psi)$, whose direction $\dashv$ can be derived by $\Diamond\vee$Distr, and the converse direction $\vdash$ by applying $\vee$E to $\Diamond\varphi \vdash \Diamond(\varphi\vee\psi)$ and $\Diamond\psi \vdash \Diamond(\varphi\vee\psi)$ (which are given by $\Diamond$Mon).

For the second equivalence in (5.1), by $\vee$I and $\vee$E, it suffices to derive $\Diamond\theta^n_T \dashv\vdash \Diamond\bigvee_{w\in T}\chi^n_w \wedge \bigwedge_{w\in T}(\top\subseteq\Diamond\chi^n_w)$ for each $T \in \mathcal{D}$, i.e.,

$$\Diamond(\bigvee_{w\in T}\chi^n_w \wedge \bigwedge_{w\in T}(\top\subseteq\chi^n_w)) \dashv\vdash \Diamond\bigvee_{w\in T}\chi^n_w \wedge \bigwedge_{w\in T}(\top\subseteq\Diamond\chi^n_w). \tag{5.2}$$

Intuitively, if a team S satisfies the formula on the right in (5.2), this means that S has a successor team that is a subset (modulo bisimulation) of T (captured by the left conjunct), and that all elements in T can be seen from S (captured by the right conjunct). Combining these facts, one gets that S has the team T (modulo bisimulation) as a successor team—which is what the formula $\diamondsuit\theta^n_T$ on the left of the equivalence expresses.

Now, the direction $\vdash$ of (5.2) can be derived by:

$$\diamondsuit\Big(\bigvee_{w\in T}\chi^n_w \wedge \bigwedge_{w\in T}(\top\subseteq\chi^n_w)\Big) \vdash \diamondsuit\bigvee_{w\in T}\chi^n_w \wedge \bigwedge_{w\in T}\diamondsuit(\top\subseteq\chi^n_w) \qquad (\diamondsuit\text{Mon})$$
$$\vdash \diamondsuit\bigvee_{w\in T}\chi^n_w \wedge \bigwedge_{w\in T}(\top\subseteq\diamondsuit\chi^n_w), \qquad (\diamondsuit_{\subseteq}\text{Distr})$$

while the other direction $\dashv$ is derived by:

$$\diamondsuit\bigvee_{w\in T}\chi^n_w \wedge \bigwedge_{w\in T}(\top\subseteq\diamondsuit\chi^n_w) \vdash \diamondsuit\Big(\Big(\bigvee_{w\in T}\chi^n_w \vee \bigvee_{w\in T}\chi^n_w\Big) \wedge \bigwedge_{w\in T}(\top\subseteq\chi^n_w)\Big) \qquad (\subseteq_{\diamondsuit}\text{Distr})$$
$$\vdash \diamondsuit\Big(\bigvee_{w\in T}\chi^n_w \wedge \bigwedge_{w\in T}(\top\subseteq\chi^n_w)\Big). \qquad (\diamondsuit\text{Mon})$$

· Let $\varphi = \boxdot\psi$ and let $k \geq md(\varphi)$. Then $n = k-1 \geq md(\psi)$, so by the induction hypothesis there is a nonempty $\mathcal{D}$ such that $\psi \dashv\vdash \bigvee_{(M,T)\in\mathcal{D}}\theta^n_T$. By $\boxdot$Mon, we have that $\boxdot\psi \dashv\vdash \boxdot\bigvee_{(M,T)\in\mathcal{D}}\theta^n_T$. We show that

$$\boxdot\bigvee_{(M,T)\in\mathcal{D}}\theta^n_T \dashv\vdash \bigvee_{\mathcal{C}\subseteq\mathcal{D}}\Big(\boxdot\bigvee_{w\in\biguplus\mathcal{C}}\chi^n_w \wedge \bigwedge_{w\in\biguplus\mathcal{C}}(\top\subseteq\diamondsuit\chi^n_w)\Big).$$

The modal depth of the formula on the right is $\leq n+1 = k$, so the result then follows by the induction cases for ML-formulas, inclusion atoms, conjunction, and disjunction. Intuitively, if a team S satisfies the formula on the left of the above equivalence, this means that $R[S]$ is bisimilar to the (disjoint) union of some teams in $\mathcal{D}$, i.e., $R[S] \rightleftharpoons_n \biguplus\mathcal{C}$ for some $\mathcal{C}\subseteq\mathcal{D}$ (see Lemma 5.4.10). In other words, by the forth condition, $R[S]$ is bisimilar to a subteam of $\biguplus\mathcal{C}$ (which is captured by the left conjunct of the disjunct corresponding to $\mathcal{C}$ of the formula on the right of the equivalence), and, by the back condition, each world in $\biguplus\mathcal{C}$ can be seen from S (which is captured by the right conjunct of the relevant disjunct).

($\vdash$) By Lemma 5.4.14 (iv) it suffices to show

$$\boxdot\bigvee_{(M,T)\in\mathcal{E}}\bigvee_{w\in T}\chi^n_w, \bigwedge_{(M,T)\in\mathcal{E}}\bigwedge_{w\in T}(\top\subseteq\diamondsuit\chi^n_w) \vdash \bigvee_{\mathcal{C}\subseteq\mathcal{D}}\Big(\boxdot\bigvee_{w\in\biguplus\mathcal{C}}\chi^n_w \wedge \bigwedge_{w\in\biguplus\mathcal{C}}(\top\subseteq\diamondsuit\chi^n_w)\Big),$$

for all nonempty $\mathcal{E}\subseteq\mathcal{D}$. This reduces to showing

$$\boxdot\bigvee_{w\in\biguplus\mathcal{E}}\chi^n_w, \bigwedge_{w\in\biguplus\mathcal{E}}(\top\subseteq\diamondsuit\chi^n_w) \vdash \bigvee_{\mathcal{C}\subseteq\mathcal{D}}\Big(\boxdot\bigvee_{w\in\biguplus\mathcal{C}}\chi^n_w \wedge \bigwedge_{w\in\biguplus\mathcal{C}}(\top\subseteq\diamondsuit\chi^n_w)\Big),$$

for all nonempty $\mathcal{E}\subseteq\mathcal{D}$, which is given by $\vee$I.

($\dashv$) Let $\mathcal{C}\subseteq\mathcal{D}$. We have:

$$\boxdot\bigvee_{w\in\biguplus\mathcal{C}}\chi^n_w \wedge \bigwedge_{w\in\biguplus\mathcal{C}}(\top\subseteq\diamondsuit\chi^n_w)$$

$$
\begin{aligned}
&\vdash && \boxdot \bigvee_{w \in \biguplus \mathcal{C}} \chi_w^n \wedge \bigwedge_{w \in \biguplus \mathcal{C}} \boxdot(\top \subseteq \chi_w^n) && (\diamondsuit\boxdot_{\subseteq}\text{Exc})\\
&\vdash && \boxdot\Big(\bigvee_{w \in \biguplus \mathcal{C}} \chi_w^n \wedge \bigwedge_{w \in \biguplus \mathcal{C}} (\top \subseteq \chi_w^n)\Big) && (\boxdot\text{Mon})\\
&= && \boxdot\, \theta^n_{\biguplus \mathcal{C}} && \\
&\vdash && \boxdot \bigvee_{(M,T)\in \mathcal{C}} \theta_T^n && (\text{Lemma 5.4.12}, \boxdot\text{Mon})\\
&\vdash && \boxdot \bigvee_{(M,T)\in \mathcal{D}} \theta_T^n && (\vee\text{I}, \boxdot\text{Mon})
\end{aligned}
$$

The result then follows by $\vee$E. □

5.4.2 ML($\triangledown$) and ML($\overline{\triangledown}$)

By adapting relevant rules from the system for modal inclusion logic ML($\subseteq$), we obtain sound and complete systems for the two expressively equivalent might-operator logics ML($\triangledown$) and ML($\overline{\triangledown}$).

5.4.17. DEFINITION. The natural deduction system for ML($\triangledown$)/ML($\overline{\triangledown}$) consists of the classical rules in Table 5.1 and the $\triangledown/\overline{\triangledown}$-rules in Table 5.4.

Most rules in Table 5.4 correspond to rules or derivable results for the ML($\subseteq$)-system applied to inclusion atoms of the form $\top \subseteq \alpha$ (recall the equivalence $\top \subseteq \alpha \equiv \bullet\alpha$, where $\bullet \in \{\triangledown, \overline{\triangledown}\}$). Given that there is no syntactic restriction on what may appear in the scope of $\bullet$ (unlike with inclusion atoms of the corresponding form), these rules may now be generalized to also apply to formulas $\bullet\varphi$ where φ is non-classical; this has been done whenever the generalization in question is sound. The rules $\bullet_{\neg}$E, $\vee_{\bullet}$E, $\bullet$Distr, $\boxdot\diamondsuit_{\bullet}$Exc, $\diamondsuit\boxdot_{\bullet}$Exc, $\boxdot\vee_{\bullet}$E, $\diamondsuit_{\bullet}$Distr, and $\bullet_{\diamondsuit}$Distr correspond to the rules $\subseteq_{\neg}$E, $\vee_{\subseteq}$E, $\subseteq$Distr, $\boxdot\diamondsuit_{\subseteq}$E, $\diamondsuit\boxdot_{\subseteq}$E, $\boxdot\vee_{\subseteq}$E, $\diamondsuit_{\subseteq}$Distr, and $\subseteq_{\diamondsuit}$Distr, respectively. The rule $\bullet$I corresponds to Lemma 5.4.8 (i), and $\bullet$Mon (restricted to classical formulas) corresponds to Lemma 5.4.8 (ii).

For both systems, we add a rule $\bullet\vee$Distr asserting the distributivity of $\bullet$ over $\vee$. The rules $\triangledown$Join and $\overline{\triangledown}\wedge$Simpl reflect the entailments pointed out in Section 5.2 and serve to differentiate the systems. The $\overline{\triangledown}$-version of the rule $\triangledown$E is also sound, and it is derivable using $\overline{\triangledown}\wedge$Simpl and $\overline{\triangledown}$Mon.

5.4.18. THEOREM (Soundness). *If* $\Gamma \vdash \varphi$, *then* $\Gamma \vDash \varphi$.

Proof:
Most cases are analogous to those for ML($\subseteq$). We only prove some of the more interesting cases. By the empty team property, it suffices to check soundness for an arbitrary nonempty team T.

$$\frac{\begin{matrix} D_0 & & D_1 \\ \nabla\varphi & & \nabla\psi \end{matrix}}{\nabla((\varphi\vee\psi)\wedge\nabla\varphi\wedge\nabla\psi)}\,\nabla\text{Join} \qquad \frac{\begin{matrix} D \\ \nabla\nabla\varphi \end{matrix}}{\nabla\varphi}\,\nabla\text{E} \qquad \frac{\begin{matrix} D \\ \dot{\nabla}(\nabla\varphi\wedge\psi) \end{matrix}}{\dot{\nabla}(\varphi\wedge\psi)}\,\dot{\nabla}\wedge\text{Simpl}$$

$$\frac{\begin{matrix} & & [\varphi] \\ D & & D_0 \\ \bullet\varphi & & \psi \end{matrix}}{\bullet\psi}\,\bullet\text{Mon}^{(1)} \qquad \frac{\begin{matrix} D \\ \bullet(\varphi\vee\psi) \end{matrix}}{\bullet\varphi\vee\bullet\psi}\,\bullet\vee\text{Distr} \qquad \frac{\begin{matrix} D \\ \alpha \end{matrix}}{\bullet\alpha}\,\bullet\text{I} \qquad \frac{\begin{matrix} D_0 & & D_1 \\ \neg\alpha & & \bullet\alpha \end{matrix}}{\varphi}\,\bullet_\neg\text{E}$$

$$\frac{\begin{matrix} & & [\bullet\varphi] & & [\psi] \\ D & & D_0 & & D_1 \\ \bullet\varphi\vee\psi & & \chi & & \chi \end{matrix}}{\chi}\,\vee_\bullet\text{E} \qquad \frac{\begin{matrix} D_0 & & D_1 & & \cdots & & D_n \\ \varphi\vee\psi & & \bullet\chi_1 & & \ldots & & \bullet\chi_n \end{matrix}}{((\varphi\vee\chi_1\vee\cdots\vee\chi_n)\wedge\bullet\chi_1\wedge\cdots\wedge\bullet\chi_n)\vee\psi}\,\bullet\text{Distr}$$

(1) The undischarged assumptions in D_0 are ML-formulas.

$$\frac{\begin{matrix} D_0 & & D_1 \\ \bullet\Diamonddot\psi & & \boxdot\bullet\varphi \end{matrix}}{\bullet\Diamonddot\varphi}\,\boxdot\Diamonddot_\bullet\text{Exc} \qquad \frac{\begin{matrix} D \\ \bullet\Diamonddot\varphi \end{matrix}}{\boxdot\bullet\varphi}\,\Diamonddot\boxdot_\bullet\text{Exc}$$

$$\frac{\begin{matrix} D \\ \Diamonddot\bullet\varphi \end{matrix}}{\bullet\Diamonddot\varphi}\,\Diamonddot_\bullet\text{Distr} \qquad \frac{\begin{matrix} & & [\bullet\Diamonddot\varphi] & & [\boxdot\psi] \\ D & & D_0 & & D_1 \\ \boxdot(\bullet\varphi\vee\psi) & & \chi & & \chi \end{matrix}}{\chi}\,\boxdot\vee_\bullet\text{E}$$

$$\frac{\begin{matrix} D_0 & & D_1 & & \cdots & & D_n \\ \Diamonddot\varphi & & \bullet\Diamonddot\psi_1 & & \ldots & & \bullet\Diamonddot\psi_n \end{matrix}}{\Diamonddot((\varphi\vee\psi_1\vee\cdots\vee\psi_n)\wedge\bullet\psi_1\wedge\cdots\wedge\bullet\psi_n)}\,\bullet_{\Diamonddot}\text{Distr}$$

Table 5.4: Rules for ML($\bullet$) with $\bullet\in\{\nabla,\dot{\nabla}\}$.

($\bullet$Mon) Let $T\vDash\bullet\varphi$ and $T\vDash\gamma$ for all $\gamma\in\Gamma$, and assume that $\Gamma,\varphi\vDash\psi$, where Γ consists of ML-formulas. Then there is a nonempty (singleton) subteam $T'\subseteq T$ such that $T'\vDash\varphi$. By downward closure of the formulas in Γ, we have $T'\vDash\gamma$ for all $\gamma\in\Gamma$. It follows that $T'\vDash\psi$, and hence $T\vDash\bullet\psi$.

($\bullet\vee$Distr) Let $T\vDash\bullet(\varphi\vee\psi)$. Then there is a nonempty (singleton) subteam $T'\subseteq T$ such that $T'\vDash\varphi\vee\psi$. Then there are $T_1,T_2\subseteq T'$ such that $T_1\cup T_2=T'$, $T_1\vDash\varphi$ and $T_2\vDash\psi$. W.l.o.g. suppose that T_1 is nonempty. Then $T\vDash\bullet\varphi$ so that $T\vDash\bullet\varphi\vee\bullet\psi$.

($\triangledown$E) Let $T \models \triangledown\triangledown\varphi$. Then there are nonempty subteams $T_1, T_2 \subseteq T$ such that $T_1 \subseteq T_2 \subseteq T$ and $T_1 \models \varphi$. Thus $T \models \triangledown\varphi$.

($\triangledown$Join) Let $T \models \triangledown\varphi \wedge \triangledown\psi$. Then there are nonempty subteams $T_1, T_2 \subseteq T$ such that $T_1 \models \varphi$ and $T_2 \models \psi$. Then $T' = T_1 \cup T_2$ is a nonempty subteam of T such that $T' \models \varphi \vee \psi$, $T' \models \triangledown\varphi$ and $T' \models \triangledown\psi$. Thus $T \models \triangledown((\varphi \vee \psi) \wedge \triangledown\varphi \wedge \triangledown\psi)$.

($\triangledown\wedge$Simpl) Let $T \models \triangledown(\triangledown\varphi \wedge \psi)$. Then there is a $w \in T$ such that $\{w\} \models \psi$ and $\{w\} \models \triangledown\varphi$, from which it follows that $\{w\} \models \varphi$. Hence $T \models \triangledown(\varphi \wedge \psi)$. □

5.4.19. PROPOSITION.

(i) $\bullet\varphi \vee \bullet\psi \dashv\vdash \bullet(\varphi \vee \psi)$.

(ii) $\bullet(\varphi \wedge \psi) \vdash \bullet\varphi \wedge \bullet\psi$.

Proof:
The direction $\vdash$ in item (i) follows by $\vee$E, $\bullet$Mon and $\vee$I, the other direction is by $\bullet\vee$Distr. Item (ii) follows by $\bullet$Mon, $\wedge$E and $\wedge$I. □

We show completeness using the same strategy as used for ML($\subseteq$). Many of the details are analogous to parts of the ML($\subseteq$)-proof; we omit most of these details and focus on the steps that specifically concern the might-operators.

5.4.20. LEMMA (Provable equivalence of the normal form). *For any φ in* ML($\bullet$) *and $k \geq md(\varphi)$, there is a (finite, nonempty) property $\mathcal{C}$ such that*

$$\varphi \dashv\vdash \bigvee_{(M,T)\in\mathcal{C}} (\bigvee_{w\in T} \chi_w^k \wedge \bigwedge_{w\in T} \bullet\chi_w^k).$$

Proof:
By induction on φ. All inductive cases except the one for $\bullet$ are similar to the corresponding proofs for ML($\subseteq$) (see Lemma 5.4.5). Inspecting the proof of Lemma 5.4.5, one sees that with the exception of the case for inclusion atoms, the lemma is derivable using rules which have analogues in the ML($\bullet$)-systems, together with the proof-theoretic results in Proposition 5.4.3 and Lemmas 5.4.6, 5.4.7, 5.4.8 (i), 5.4.8 (ii), 5.4.12, 5.4.14, 5.4.15, and 5.4.16. These results, in turn, are also (with the exception of Lemmas 5.4.8 (i), 5.4.8 (ii)) derivable using rules which have analogues in the ML($\bullet$)-system. Any steps making use of Lemma 5.4.8 (i) can be replaced by applications of $\bullet$I; similarly with Lemma 5.4.8 (ii) and $\bullet$Mon. We add the cases for $\bullet$.

Let $\varphi = \bullet\psi$. We derive

$$\bullet\psi \dashv\vdash \bullet \bigvee_{(M,T)\in\mathcal{C}} (\bigvee_{w\in T} \chi_w^k \wedge \bigwedge_{w\in T} \bullet\chi_w^k) \quad \text{(Induction hypothesis, } \bullet \text{ Mon)}$$

$$\dashv\vdash \bigvee_{(M,T)\in\mathcal{C}} \bullet(\bigvee_{w\in T} \chi_w^k \wedge \bigwedge_{w\in T} \bullet\chi_w^k) \quad \text{(Prop. 5.4.19 (i))}$$

We will now show:

(a) $\bigvee_{(M,T)\in\mathcal{C}} \triangledown(\bigvee_{w\in T}\chi_w^k \wedge \bigwedge_{w\in T}\triangledown\chi_w^k) \dashv\vdash \bigvee_{(M,T)\in\mathcal{C}}\bigwedge_{w\in T}\triangledown\chi_w^k$ and

(b) $\bigvee_{(M,T)\in\mathcal{C}} \dot{\triangledown}(\bigvee_{w\in T}\chi_w^k \wedge \bigwedge_{w\in T}\dot{\triangledown}\chi_w^k) \dashv\vdash \bigvee_{(M,T)\in\mathcal{C}}\dot{\triangledown}\bigwedge_{w\in T}\chi_w^k$ and

(c) every formula in the form $\bullet\alpha$ is provably equivalent to a formula in normal form.

The result will then follow by the induction cases for conjunction and disjunction.

We first show (a). By $\vee$E and $\vee$I, it suffices to show that $\triangledown(\bigvee_{w\in T}\chi_w^k \wedge \bigwedge_{w\in T}\triangledown\chi_w^k) \dashv\vdash \bigwedge_{w\in T}\triangledown\chi_w^k$ for an arbitrary $(M,T)\in\mathcal{C}$. For the direction $\vdash$, we derive $\triangledown(\bigvee_{w\in T}\chi_w^k \wedge \bigwedge_{w\in T}\triangledown\chi_w^k) \vdash \triangledown\bigwedge_{w\in T}\triangledown\chi_w^k \vdash \bigwedge_{w\in T}\triangledown\triangledown\chi_w^k \vdash \bigwedge_{w\in T}\triangledown\chi_w^k$ by $\triangledown$Mon, $\wedge$E, Proposition 5.4.19 (ii), and $\triangledown$E. The direction $\dashv$ follows by $\triangledown$Join.

We now show (b). As above, it suffices to show that $\dot{\triangledown}(\bigvee_{w\in T}\chi_w^k \wedge \bigwedge_{w\in T}\dot{\triangledown}\chi_w^k) \dashv\vdash \dot{\triangledown}\bigwedge_{w\in T}\chi_w^k$ for an arbitrary $(M,T)\in\mathcal{C}$. For the direction $\vdash$, we derive $\dot{\triangledown}(\bigvee_{w\in T}\chi_w^k \wedge \bigwedge_{w\in T}\dot{\triangledown}\chi_w^k) \vdash \dot{\triangledown}\bigwedge_{w\in T}\dot{\triangledown}\chi_w^k \vdash \dot{\triangledown}\bigwedge_{w\in T}\chi_w^k$ by $\dot{\triangledown}$Mon, $\wedge$E and $\dot{\triangledown}\wedge$Simpl. For the direction $\dashv$, we note that $\bigwedge_{w\in T}\chi_w^k \vdash \bigvee_{w\in T}\chi_w^k \wedge \bigwedge_{w\in T}\dot{\triangledown}\chi_w^k$ by $\wedge$E, $\vee$I, and $\dot{\triangledown}$I. Therefore $\dot{\triangledown}\bigwedge_{w\in T}\chi_w^k \vdash \dot{\triangledown}(\bigvee_{w\in T}\chi_w^k \wedge \bigwedge_{w\in T}\dot{\triangledown}\chi_w^k)$ by $\dot{\triangledown}$Mon.

The proof of (c) is analogous to the proof of the primitive inclusion atom case in Lemma 5.4.5 (with an application of $\bullet$Mon replacing the appeal to Lemma 5.4.8 (ii)). □

5.4.21. THEOREM (Completeness). *If* $\Gamma \vDash \psi$, *then* $\Gamma \vdash \psi$.

Proof:
Similar to the proof of Theorem 5.4.13. Inspecting this proof, one sees that it can be conducted using only rules which have analogues in the ML($\bullet$)-systems, together with the proof-theoretic results in Proposition 5.4.3 and Lemmas 5.4.6, 5.4.7, 5.4.8 (i), 5.4.8 (ii), 5.4.12 and 5.4.5. We have proved an analogue to Lemma 5.4.5 in Lemma 5.4.20, and as noted above, analogues to the other required results can be obtained in ML($\bullet$). □

5.5 Concluding Remarks and Directions for Further Research

In this article, we have addressed a recognized gap in the literature on team-based modal logics by presenting an axiomatization for modal inclusion logic ML($\subseteq$). This logic, together with modal dependence logic and modal independence logic, are commonly considered to be the core team-based modal logics. While modal dependence logic has already been axiomatized in previous work [134], modal independence logic (see [90]) as well as propositional independence logic are still missing an axiomatization.

We also studied two other union-closed extensions of modal logic—the two might-operator logics ML($\triangledown$) and ML($\dot{\triangledown}$). The logics ML($\subseteq$) and ML($\triangledown$) were shown to be expressively complete for the same class of properties and hence expressively equivalent in [73]; we reviewed and refined this result and showed that the new variant $\mathrm{ML}(\dot{\triangledown})$ with the singular might-operator $\dot{\triangledown}$ is likewise expressively complete for this class of properties. We also provided axiomatizations for $\mathrm{ML}(\triangledown)$ and $\mathrm{ML}(\dot{\triangledown})$. Note that one can obtain expressive completeness results, axiomatizations, and completeness proofs for the propositional variants of the might-operator logics via a straightforward adaptation of the results in this article.

All our axiomatizations are presented in natural deduction style—as opposed to the Hilbert style commonly used for modal logics—mainly because we do not have an *implication* connective in the languages of the logics we consider. To turn our natural deduction rules into Hilbert-style axioms, one could consider extending the languages with an implication. The implication would have to preserve union closure and the empty team property, and presumably also satisfy other desiderata for an implication such as the deduction theorem. Whether such a team-based implication exists is currently unknown. To better study the proof-theoretic properties of these logics, it would also be desirable to introduce sequent calculi for them. To this end, finding more elegant versions of the proof systems presented in this article might be useful. A point of difficulty could be that the logics do not admit uniform substitution, although sequent calculi have been developed for other team-based logics—see, e.g., [50, 31, 100].

We observed in Section 5.3 that with the strict semantics for the diamond, the three logics are not invariant under the notion of team bisimulation established in the literature. This raises the question whether one can formulate a relation between models with teams—a *strict* team bisimulation—that would be strong enough to ensure invariance with respect to strict semantics but that would still respect the local character of modal logic and not imply first-order invariance.

As we now have a better understanding of the logical properties of these three union closed team-based modal logics, it is natural to ask about their possible applications in other fields. In connection with this, we note that certain closely related logics have recently found application in formal semantics. The two-sorted first-order team semantics framework in [9] employs first-order inclusion atoms together with existential quantification to represent epistemic modalities in a manner similar to how the might-operators function. In [6], the usual modal logic is extended with a *nonemptiness atom* NE satisfied by all but the empty team; this logic (which is union closed but does not have the empty team property, and is in some ways very similar to the might-operator logics—see footnote 3) is then used to account for *free choice inferences* and related natural-language phenomena.

5.6 Appendix: Translation of ML(⊆) into First-order Inclusion Logic.

In this appendix, we provide a translation of ML(⊆) into first-order inclusion logic ($FO(\subseteq)$). Before we do so, however, let us issue a note of caution. One way to utilize the standard translation of ML into classical first-order logic is to derive the compactness of ML from that of first-order logic. The article [134], similarly, provides a translation from modal dependence logic into first-order dependence logic, and uses this translation and the fact that first-order dependence logic is compact [106] to conclude that modal dependence logic is likewise compact. The presentation in [134] is erroneous because it conflates compactness formulated in terms of satisfiability (if each finite subset of Γ is satisfiable, then Γ is satisfiable) with compactness formulated in terms of entailment (if $\Gamma \vDash \varphi$, then there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vDash \varphi$). These notions need not coincide for a logic that is not closed under classical negation, and, indeed, while first-order dependence logic is satisfiability-compact, it is not entailment-compact. The same applies to $FO(\subseteq)$, which is shown to be satisfiability-compact in [106], but which is not entailment-compact. [8] One ought not, therefore, use the translation in this section to argue that ML(⊆) is entailment-compact. As noted in Section 5.3, however, we can derive both the satisfiability- and the entailment-compactness of ML(⊆) (as well as those of its variants, and those of modal dependence logic) from the fact, proved in [89], that team properties invariant under bounded bisimulation can be expressed in classical first-order logic.

We briefly recall the syntax and semantics of $FO(\subseteq)$; for detailed discussion of the logic, see., e.g., [51]. Fix an infinite set Var of first-order variables, and a first-order vocabulary τ. The set of first-order τ-terms is defined as usual. The syntax for $FO(\subseteq)$ over τ is given by:

$$\varphi ::= t_1 = t_2 \mid R(t_1,\dots,t_n) \mid \vec{x} \subseteq \vec{y} \mid \neg t_1 = t_2 \mid \neg R(t_1,\dots,t_n) \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \exists x\varphi \mid \forall x\varphi,$$

where each t_i is a τ-term, R is an n-ary relation symbol in τ, and $\vec{x}$ and $\vec{y}$ are two finite sequences of variables of the same length. Define, as usual, $\bot := \forall x(\neg x = x)$ and $\top := \forall x(x = x)$.

Let $\mathcal{M}$ be a τ-model with domain W. An assignment over W with domain $\mathsf{V} \subseteq \mathsf{Var}$ is a function $s : \mathsf{V} \to W$. We abbreviate $s(\vec{x}) := \langle s(x_1),\dots,s(x_n)\rangle$, where $\vec{x} = \langle x_1,\dots,x_n\rangle$. Given $a \in W$, the modified assignment $s(a/x)$ is defined as $s(a/x)(y) := a$ for $y = x$, and $s(a/x)(y) := s(y)$ otherwise. Given $\mathsf{V} \subseteq Var$, a (first-order) team X of $\mathcal{M}$ with domain V is a set of assignments $s : \mathsf{V} \to W$. Given $A \subseteq W$ and a team X of $\mathcal{M}$, we let $X(A/x) := \{s(a/x) \mid s \in X, a \in A\}$.

[8] To see why, consider the $FO(\subseteq)$-sentence $\varphi_{\mathsf{nwf}} := \exists x \exists y(x \subseteq y \wedge x < y)$ where $<$ is a binary relation symbol. Now φ_{nwf} expresses that there is an infinite descending $<$-chain, i.e., $<$ is not well-founded (see, e.g., [136] for details). Let φ_{lo} be a first-order sentence expressing that $<$ is a strict linear order with a (unique) greatest element. For each $n \in \mathbb{N}$, define the first-order sentence $\varphi_n := \exists! z \forall x((x = z \vee x < z) \wedge \exists y_1 \dots \exists y_n(y_n < \dots < y_1 < z))$, expressing that there is a $<$-chain of length n below the greatest element. Thus $\{\varphi_{\mathsf{lo}}, \varphi_n \mid n \in \mathbb{N}\} \vDash \varphi_{\mathsf{nwf}}$, but the entailment does not hold for any finite subset of $\{\varphi_{\mathsf{lo}}, \varphi_n \mid n \in \mathbb{N}\}$.

The team semantics of $FO(\subseteq)$ is given by (here we follow the convention for first-order team-based logics and write the team on the right-hand side of the satisfaction symbol):

$$\begin{aligned}
\mathcal{M} \vDash_X \alpha &\iff \text{for all } s \in X, \mathcal{M} \vDash_s \alpha, \text{ if } \alpha \text{ is a first-order literal;}\\
\mathcal{M} \vDash_X \vec{x} \subseteq \vec{y} &\iff \text{for all } s \in X \text{ there exists } s' \in X \text{ such that } s(\vec{x}) = s'(\vec{y});\\
\mathcal{M} \vDash_X \varphi \vee \psi &\iff \text{there exist } Y, Z \subseteq X \text{ such that } X = Y \cup Z, \mathcal{M} \vDash_Y \varphi \text{ and } \mathcal{M} \vDash_Z \psi;\\
\mathcal{M} \vDash_X \varphi \wedge \psi &\iff \mathcal{M} \vDash_X \varphi \text{ and } \mathcal{M} \vDash_X \psi;\\
\mathcal{M} \vDash_X \exists x \varphi &\iff \text{there exists a function } F : X \to \mathcal{P}(W) \setminus \{\varnothing\} \text{ such that}\\
&\qquad \mathcal{M} \vDash_{X(F/x)} \varphi, \text{ where } X(F/x) := \{s(a/x) \mid s \in X, a \in F(s)\};\\
\mathcal{M} \vDash_X \forall x \varphi &\iff \mathcal{M} \vDash_{X(W/x)} \varphi.
\end{aligned}$$

It is easy to check that all formulas of $FO(\subseteq)$ are union closed and have the empty team property, and that formulas without inclusion atoms (formulas of FO) are additionally downward closed and flat (the first-order versions of these notions are defined analogously to the modal versions).

5.6.1. DEFINITION. For any formula φ in $\mathrm{ML}(\subseteq)$, its standard translation $ST_x(\varphi)$ into first-order inclusion logic $FO(\subseteq)$ (with respect to the first-order variable x) is defined inductively as follows:

$$\begin{aligned}
ST_x(p) &:= Px;\\
ST_x(\bot) &:= \bot;\\
ST_x(\neg\alpha) &:= \neg ST_x(\alpha);\\
ST_x(\varphi \wedge \psi) &:= ST_x(\varphi) \wedge ST_x(\psi);\\
ST_x(\varphi \vee \psi) &:= ST_x(\varphi) \vee ST_x(\psi);\\
ST_x(\Diamond\varphi) &:= \exists y(xRy \wedge ST_y(\varphi));\\
ST_x(\Box\varphi) &:= \forall y(\neg xRy \vee (xRy \wedge ST_y(\varphi)));\\
ST_x(\mathsf{a} \subseteq \mathsf{b}) &:= \bigwedge_{\mathsf{z} \in \{\top,\bot\}^{|\mathsf{a}|}} (ST_x(\neg \mathsf{a}^{\mathsf{z}}) \vee \exists y(y \subseteq x \wedge ST_y(\mathsf{b}^{\mathsf{z}}))).
\end{aligned}$$

For $\mathsf{X} \subseteq \mathsf{Prop}$, let σ_{X} be the first-order signature containing a binary relation symbol R_0 and a unary relation symbol P for each $p \in \mathsf{X}$. There is a one-to-one correspondence between Kripke models over X and first-order σ_{X}-structures: $M = (W, R, V)$ over X corresponds to the σ_{X}-structure $\mathcal{M} = (W, R_0^{\mathcal{M}}, \{P^{\mathcal{M}}\}_{p \in \mathsf{X}})$, where $P^{\mathcal{M}} = V(p)$ is the interpretation of each unary P, and where $R_0^{\mathcal{M}} = R$.

We conclude with a lemma establishing that the translation functions as desired.

5.6.2. LEMMA. *Let φ be a formula in modal inclusion logic* $\mathrm{ML}(\subseteq)$. *For any Kripke model M and team T of M, and any variable x,*

$$M, T \vDash \varphi \iff \mathcal{M} \vDash_{T_x} ST_x(\varphi),$$

where $T_x = \{\{(x, w)\} | w \in T\}$ is a first-order team with domain $\{x\}$.

Proof:
The proof is by induction on φ; we refer to [97] and add the case for $\mathsf{a} \subseteq \mathsf{b}$.

We want to show that $M, T \vDash \mathsf{a} \subseteq \mathsf{b} \iff \mathcal{M}, T_x \vDash \bigwedge_{\mathsf{z} \in \{\top,\bot\}^{|\mathsf{a}|}} (ST_x(\neg \mathsf{a}^{\mathsf{z}}) \vee \exists y(y \subseteq x \wedge ST_y(\mathsf{b}^{\mathsf{z}}))$. By Lemma 5.2.7 (i), we have $\mathsf{a} \subseteq \mathsf{b} \equiv \bigwedge_{\mathsf{z} \in \{\top,\bot\}^{|\mathsf{a}|}} (\neg \mathsf{a}^{\mathsf{z}} \vee \mathsf{z} \subseteq \mathsf{b})$. It therefore suffices to show that for any $\mathsf{z} \in \{\top,\bot\}^{|\mathsf{a}|}$, $M, T \vDash \mathsf{z} \subseteq \mathsf{b} \iff \mathcal{M} \vDash_{T_x} \exists y(y \subseteq x \wedge ST_y(\mathsf{b}^{\mathsf{z}}))$, for then the result follows by the other induction cases. Let $\mathsf{z} \in \{\top,\bot\}^{|\mathsf{a}|}$.

The case in which $T = \varnothing$ is trivial. Suppose that $T \neq \varnothing$ whence also $T_x \neq \varnothing$. For the left-to-right direction, by Proposition 5.2.4 there is a $w \in T$ such that $\{w\} \vDash \mathsf{b}^{\mathsf{z}}$. By the induction hypothesis, $\mathcal{M} \vDash_{\{w\}_x} ST_x(\mathsf{b}^{\mathsf{z}})$. It is easy to verify that this is equivalent to $\mathcal{M} \vDash_{T_x(\{w\}/y)} ST_y(\mathsf{b}^{\mathsf{z}})$. Define a function $F : T_x \to \mathcal{P}(W) \setminus \{\varnothing\}$ by $F(s) = \{w\}$ for all $s \in T_x$. Clearly $\mathcal{M} \vDash_{T_x(F/y)} ST_y(\mathsf{b}^{\mathsf{z}})$. Since $w \in T$, $\mathcal{M} \vDash_{T_x(F/y)} y \subseteq x$, whence $\mathcal{M} \vDash_{T_x} \exists y(y \subseteq x \wedge ST_y(\mathsf{b}^{\mathsf{z}}))$.

For the other direction, we have that there is a function $F : T_x \to \mathcal{P}(W) \setminus \{\varnothing\}$, such that $\mathcal{M} \vDash_{T_x(F/y)} (y \subseteq x \wedge ST_y(\mathsf{b}^{\mathsf{z}}))$. We have that $T_x \neq \varnothing$ whence also $T_x(F/y) \neq \varnothing$ so let $s \in T_x(F/y)$. By $\mathcal{M} \vDash_{T_x(F/y)} y \subseteq x$, there is a $s' \in T_x(F/y)$ with $s(y) = s'(x)$. We must have $s'(x) = w \in T$; then $s = s''(w/y)$ for some $s'' \in T_x$. By downward closure, $\mathcal{M} \vDash_{\{s''(w/y)\}} ST_y(\mathsf{b}^{\mathsf{z}})$. It follows that $\mathcal{M} \vDash_{\{w\}_y} ST_y(\mathsf{b}^{\mathsf{z}})$. By the induction hypothesis, $M, \{w\} \vDash \mathsf{b}^{\mathsf{z}}$, so that by Proposition 5.2.4, $M, T \vDash \mathsf{z} \subseteq \mathsf{b}$. □

Chapter 6

A Deep-Inference Sequent Calculus for a Propositional Team Logic

This chapter is based on:

Aleksi Anttila, Rosalie Iemhoff, and Fan Yang. *Deep-inference Sequent Calculi for Propositional Team Logics*. Manuscript. 2024

Abstract We introduce a sequent calculus for the propositional team logic with both the split disjunction and the inquisitive disjunction consisting of a G3-style system for classical propositional logic together with deep-inference rules for the inquisitive disjunction. We show that the system satisfies various desirable properties: it admits height-preserving weakening, contraction and inversion; it supports a procedure for constructing cutfree proofs and countermodels similar to that for G3cp; and cut elimination holds as a corollary of cut elimination for the G3-style subsystem together with a normal form theorem for cutfree derivations in the system.

6.1 Introduction

Logics such as *dependence logic* [119] and *inquisitive logic* [41] are typically interpreted using *team semantics* [78, 79]: formulas are interpreted with respect to sets of evaluation points (valuations/assignments/possible worlds) called *teams*, rather than single evaluation points as in the usual Tarskian semantics. Team semantics was originally introduced by Hodges [78, 79] to provide a compositional semantics for Hintikka and Sandu's *independence-friendly logic* [77, 75]; independently, Ciardelli, Groenendijk, and Roelofsen developed *inquisitive logic* [60, 41, 38, 35] which also essentially employs team semantics (see [133, 34]). The use of teams allows for simple and natural ways of formalizing notions such as question meaning (the question as to whether p is the case can be formalized in propositional inquisitive logic as $p \mathbin{\mathbb{V}} \neg p$, where $\mathbin{\mathbb{V}}$ is the *inquisitive* or *global disjunction*) and dependence ("the value of q functionally depends on the value of p" can be formalized in propositional dependence logic [139] using a *dependence atom* as $=(p,q)$). We will refer to logics primarily intended to be interpreted using team semantics as *team logics*.

In this paper, we focus on propositional team logics. While there are a great number of natural deduction- and Hilbert-style axiomatizations of propositional team logics in the literature [41, 112, 107, 139, 33, 140, 96, 39, 137], the development of sequent calculus systems and of proof theory in general for these logics has been slower. The sequent calculi that have been constructed have all been for variants of propositional inquisitive logic; these include multiple labelled systems [111, 31, 100, 18] as well as the multi-type display calculus in [50]. There is additionally one natural deduction system with normalization [100].[1]

As observed in [50], one of the main difficulties in providing standard sequent calculi for team logics is that typically these logics are not *closed uniform substitution*. Due to this failure, axiomatizations for team logics typically feature rules that may only be applied to some subclass of formulas, and these axiomatizations do not admit the usual uniform substitution rule. Many proof-theoretic techniques depend on the universal applicability of the rules, so it is not immediately obvious how to apply these techniques to most team logics (we discuss the difficulties with cut elimination in a setting with restricted rules in some more detail in Section 6.7)—often some specialized machinery has to be introduced to handle this issue. For instance, the construction of the multi-type display calculus in [50] involves the introduction of a new language featuring two types of formulas for the team logic axiomatized, with closure under substitution holding within each of these types. It is also not even clear how a sequent should be interpreted in the setting of team semantics—there are, for instance, multiple disjunctions available to interpret the commas in the succedent Δ of a sequent $\Gamma \Rightarrow \Delta$.

In this paper, we introduce a sequent calculus with cut elimination (i.e., cut is admissible in the cutfree fragment) for the propositional team logic $\mathrm{PL}(\mathbin{\mathbb{V}})$ (studied in, for instance, [139, 112, 33, 138]). This logic features both the *split disjunction* $\vee$ (also known as the *tensor* or *local disjunction*)—this is the canonical disjunction employed in dependence logic and other logics in the lineage of dependence logic—and the *inquisitive disjunction* $\mathbin{\mathbb{V}}$ already mentioned above, which is used in inquisitive logic to model the meanings of questions. The semantics for these connectives are as follows (here t is a propositional team—a set of valuations):

$t \vDash \varphi \vee \psi$	iff	there are s, u such that $t = s \cup u$, $s \vDash \varphi$, and $u \vDash \psi$
$t \vDash \varphi \mathbin{\mathbb{V}} \psi$	iff	$t \vDash \varphi$ or $t \vDash \psi$

That is, a split disjunction $\varphi \vee \psi$ is true in a team just in case the team can be split into two subteams, with each disjunct being true in at least one of the subteams; and an

[1] These systems are all for *propositional inquisitive logic* InqB [41]. There is also an earlier labelled system [111] for an early version propositional inquisitive logic [60] interpreted on ordered pairs (essentially teams of size 2) rather than on teams in general.

inquisitive disjunction is true in a team just in case one of the disjuncts is. The logic PL($\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$) is a conservative extension of classical propositional logic PL, with the split disjunction $\vee$ extending the classical disjunction.

The calculus we construct features a standard Gentzen-style system for PL with some syntactic restrictions to the effect that certain active formulas and context sets must be classical ($\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$-free). Adapting an idea from the natural deduction systems in [139, 140], this Gentzen-style system is supplemented with *deep-inference* (see, e.g., [62, 27, 28]) rules for the inquisitive disjunction $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$—that is, rules which allow one to introduce the inquisitive disjunction (almost) anywhere within a formula, rather than only as its main connective. The deep-inference rules allow for cutfree completeness of the system and for many standard proof-theoretic techniques to be applied despite the limited applicability of the restricted rules: essentially, cutfree proofs can be constructed by first constructing cutfree classical proofs, and then inserting inquisitive disjunctions as required; and procedures involving the commuting of sequents which depend on the universal applicability of the rules (such as cut elimination) can be conducted in such a way that they only involve commuting sequents in the classical part of the calculus, in which the rules *are* universally applicable. The cutfree fragment of the calculus has a weak subformula property; we define the relevant notion of weak subformula by generalizing the notion of *resolutions* from inquisitive logic [35].

Our aim was to develop a simple system with cut elimination that departs as little as possible from a Gentzen-style calculus. Our deep-inference approach accomplishes this in the following ways.

First, we avoid both importing the semantics into the system in the form of labels (like the labelled systems in [31, 100, 18]) and extending the syntax of the logic (like the multi-type display calculus in [50]).

Second, our system consists of only a single pair of rules for each connective: an introduction (right) rule and an elimination (left) rule (this is in contrast with the frequently very complicated natural deduction systems for team logics, including the natural deduction system for PL($\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$) [139]).

Third, the system simply extends a well-known Gentzen-style system for PL (a variant of G3cp without the implication rules—see, e.g., [118]) with rules for the inquisitive disjunction $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$—this means that the fact, mentioned above, that PL($\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$) is an extension of PL with $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ is directly reflected in a straightforward way in the calculus, and the calculus allows us to see immediately and transparently exactly what is required to be added to an axiomatization of PL to axiomatize PL($\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$). It should be pointed out that this structure is possible due to a key design decision: instead of interpreting a sequent $\Gamma \Rightarrow \Delta$ as $\bigwedge\Gamma \vDash \bigvee\!\!\!\!\bigvee\Delta$ (i.e., taking the sequent $\Gamma \Rightarrow \Delta$ to be valid just in case whenever each formula in Γ is true in a team t, at least one formula of Δ is true in t), we interpret $\Gamma \Rightarrow \Delta$ as $\bigwedge\Gamma \vDash \bigvee\Delta$ (i.e., $\Gamma \Rightarrow \Delta$ is valid just in case whenever each formula in Γ is true in a team t, there is, for each $\varphi \in \Delta$, a t_φ such that $t_\varphi \vDash \varphi$; and $t = \bigcup_{\varphi\in\Delta} t_\varphi$). That is, we interpret the comma in the succedent of a sequent not as the inquisitive disjunction $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$, but, rather, as the split disjunction $\vee$. To reiterate, while the inquisitive disjunction $\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}$ has the standard disjunction semantics (with respect to teams), it is the split disjunction $\vee$ that extends the classical disjunction of PL; therefore, in order to employ, as we have done, a calculus for PL in the team setting without making extensive changes, the succedent comma must be interpreted as $\vee$. The labelled systems in [31, 100, 18], in contrast, interpret the succedent comma as $\mathbin{\rotatebox[origin=c]{90}{$\geqslant$}}$; the multi-type display calculus in [50] essentially interprets the succedent comma as $\vee$ for one type of the system, and as $\mathbin{\rotatebox[origin=c]{90}{$\geqslant$}}$ for the other.

Fourth and finally, due to the third point above, we can show many proof-theoretic results for our systems as easy extensions or corollaries of the analogous results for the classical Gentzen-style system. The following results in this paper are examples of this: the admissibility (with some restrictions) of height-preserving weakening, contraction and inversion (Section 6.5), the G3cp-style proof of cutfree completeness and countermodel construction in Section 6.6; and our cut elimination procedure (Section 6.7).

The system also has other interesting features. For instance, due to the decision to interpret the succedent comma as the split disjunction, there is a correspondence between certain structural rules of the calculus and certain *team-semantic closure properties*—semantic properties of formulas which play an important role in the study of team logics. The closure properties which have structural correspondents in our system are the *empty team property* and *union closure*. The logic PL($\mathbin{\rotatebox[origin=c]{90}{$\geqslant$}}$) has the empty team property (meaning that the empty team satisfies all PL($\mathbin{\rotatebox[origin=c]{90}{$\geqslant$}}$)-formulas), and its classical fragment is union closed (meaning that the truth of a classical formula in a collection of teams implies its truth in the union of the collection); PL($\mathbin{\rotatebox[origin=c]{90}{$\geqslant$}}$) as a whole is not union closed. The empty team property corresponds to the soundness of weakening on the right, and union closure corresponds to the soundness of contraction on the right. Therefore, weakening on the right is sound for all PL($\mathbin{\rotatebox[origin=c]{90}{$\geqslant$}}$)-formulas (and indeed admissible in the cutfree fragment of our system for all PL($\mathbin{\rotatebox[origin=c]{90}{$\geqslant$}}$)-formulas), whereas contraction on the right is only guaranteed to be sound (and admissible in the cutfree fragment) for classical formulas.

The paper is structured as follows. In Section 6.2.1, we define the syntax and semantics of the logic PL($\mathbin{\rotatebox[origin=c]{90}{$\geqslant$}}$) and recall some basic facts about team semantics and this logic. In Section 6.2.2, we adapt the notion of *resolutions* from inquisitive logic to our setting—resolutions turn out to be a convenient tool for describing how our system functions. In Section 6.3, we introduce our deep-inference sequent calculus GT for PL($\mathbin{\rotatebox[origin=c]{90}{$\geqslant$}}$). We also provide additional motivation for the deep-inference approach by discussing why a straightforward sequent calculus-translation of the natural deduction system for PL($\mathbin{\rotatebox[origin=c]{90}{$\geqslant$}}$) would not be cutfree complete, and how the deep-inference rules provide for a natural cutfree extension of this translation. In Section 6.4, we generalize the notion of resolutions to what we call *partial resolutions*, and use partial resolutions to define a weak subformula property for the cutfree fragment of our system as well as to prove that the system is cutfree complete (that is, we provide in this section a semantic proof of cut elimination). In Section 6.5 we prove some basic properties of the system: the admissibility (with some restrictions) of height-preserving weakening, contraction, and inversion in the cutfree fragment. In Section 6.6, we make use of the (semantic) invertibility of the rules in our system to provide a procedure for constructing cutfree proofs and countermodels that is similar to the analogous procedure for G3cp (this yields a second semantic proof of cut elimination). In Section 6.7, we give a syntactic proof of cut elimination—a cut elimination procedure. This makes use of a *normal form theorem* for derivations (or what is known in the deep inference-literature as a *decomposition theorem*): each derivation in the cutfree fragment of GT can be transformed into a derivation in which one first applies only rules within the classical subsystem of GT, then applies only the right deep-inference rule for $\vee\!\!\vee$, and finally applies only the left deep-inference rule for $\vee\!\!\vee$. Using this theorem, each cut in GT can be transformed into cuts within the classical subsystem of GT, which can then be eliminated using the cut elimination procedure for this subsystem. In Section 6.8, we define a variant of GT featuring independent-context rules instead of shared-context rules with syntactic restrictions. Section 6.9 concludes with some discussion concerning the applicability of the deep-inference approach to other team logics.

6.2 Preliminaries

6.2.1 Syntax, Semantics, and Closure Properties

We fix a (countably infinite) set Prop of propositional variables.

6.2.1. DEFINITION (Syntax). The set of formulas α of *classical propositional logic* PL is generated by:

$$\alpha ::= p \mid \bot \mid \neg\alpha \mid \alpha\wedge\alpha \mid \alpha\vee\alpha$$

where $p \in \mathsf{Prop}$.

The set of formulas φ of *propositional logic with the inquisitive disjunction* PL($\vee\!\!\vee$) is generated by:

$$\varphi ::= \alpha \mid \varphi\wedge\varphi \mid \varphi\vee\varphi \mid \varphi\vee\!\!\vee\varphi$$

where $\alpha \in$ PL.

We let $\top := \neg\bot$, $\bigvee\varnothing := \bot$, and $\bigwedge\varnothing := \top$. $\vee$ is the *split disjunction* (also known as the *tensor disjunction* and the *local disjunction*) and $\vee\!\!\vee$ is the *inquisitive disjunction* (also known as the *global disjunction*). We write $\mathsf{P}(\varphi)$ for the set of propositional variables appearing in φ, and $\varphi(p_1,\dots,p_n)$ if $\{p_1,\dots,p_n\}\subseteq\mathsf{P}(\varphi)$. For a set/multiset of formulas Γ, we let $\mathsf{P}(\Gamma) := \bigcup_{\varphi\in\Gamma}\mathsf{P}(\Gamma)$. We also call formulas of PL *classical formulas*. We reserve the first lowercase Greek letters α,β for classical formulas, and the uppercase Greek letters $\Xi,\Lambda,\Theta,\Omega$ for multisets of classical formulas. The set of *subformulas* of a formula is defined in the standard way.

Let $\mathsf{N}\subseteq\mathsf{Prop}$. An *team with domain* N is a set $t\subseteq 2^{\mathsf{N}}$ of valuations over N.

6.2.2. DEFINITION (Semantics). For any formula φ and any team t with domain $\supseteq$ $\mathsf{P}(\varphi)$, the satisfaction relation $t\vDash\varphi$ is defined inductively by:

- $t \vDash p$ iff for all $v \in t$, $v(p) = 1$;
- $t \vDash \bot$ iff $t = \varnothing$;
- $t \vDash \neg\alpha$ iff for all $v \in t$, $\{v\} \nvDash \alpha$;
- $t \vDash \varphi \wedge \psi$ iff $t \vDash \varphi$ and $t \vDash \psi$;
- $t \vDash \varphi \vee \psi$ iff there exist $r, s \subseteq t$ such that $t = r \cup s$, $r \vDash \varphi$ and $s \vDash \psi$;
- $t \vDash \varphi \mathbin{\vee\!\!\!\vee} \psi$ iff $t \vDash \varphi$ or $t \vDash \psi$.

For any multiset $\Gamma \cup \varphi$ of formulas, we write $t \vDash \Gamma$ if $t \vDash \psi$ for all $\psi \in \Gamma$, and we write $\Gamma \vDash \varphi$ if $t \vDash \Gamma$ implies $t \vDash \varphi$. We write $\varphi \vDash \psi$ for $\{\varphi\} \vDash \psi$. We write $\vDash \varphi$ if $t \vDash \varphi$ for all teams t with domain $\supseteq \mathsf{P}(\varphi)$.

Let φ be a formula. We define the following standard *closure properties*:

Empty team property $\varnothing \vDash \varphi$.

Downward closure If $t \vDash \varphi$ and $s \subseteq t$, then $s \vDash \varphi$.

Union closure If $t \vDash \varphi$ and $s \vDash \varphi$, then $t \cup s \vDash \varphi$.

Flatness $t \vDash \varphi \iff$ for all $v \in t$, $\{v\} \vDash \varphi$.

It is easy to see that φ is flat iff it satisfies the empty team, downwards-closure, and union-closure properties.

6.2.3. PROPOSITION. *All formulas of* PL($\mathbin{\vee\!\!\!\vee}$) *satisfy the empty team and downward-closure properties. All formulas of* PL *additionally satisfy the union-closure property and hence also the flatness property.*

We also have that the team semantics of formulas of PL on singletons coincide with their standard single-valuation semantics: $\{v\} \vDash \alpha \iff v \vDash \alpha$. We therefore have that for any $\alpha \in$ PL:

$$t \vDash \alpha \iff \{v\} \vDash \alpha \text{ for all } v \in t \iff v \vDash \alpha \text{ for all } v \in t.$$

6.2.4. COROLLARY. *For any set/multiset* $\Lambda \cup \{\alpha\}$ *of classical formula,* $\Lambda \vDash^c \alpha \iff \Lambda \vDash \alpha$*, where* $\vDash^c$ *stands for the usual (single-valuation) entailment relation.*

To see why formulas including $\mathbin{\vee\!\!\!\vee}$ might fail to be union closed, consider $p \mathbin{\vee\!\!\!\vee} \neg p$. If $v_p \vDash p$ and $v_{\overline{p}} \vDash \neg p$, then $\{v_p\} \vDash p \mathbin{\vee\!\!\!\vee} \neg p$ and $\{v_{\overline{p}}\} \vDash p \mathbin{\vee\!\!\!\vee} \neg p$, but $\{v_p, v_{\overline{p}}\} \vDash p \mathbin{\vee\!\!\!\vee} \neg p$.

PL($\mathbin{\vee\!\!\!\vee}$) does not admit uniform substitution: writing $\varphi(\chi/p)$ for the result of replacing all occurrences of p in $\varphi(p)$ with χ, it is not the case that $\varphi(p) \vDash \psi(p)$ implies $\varphi(\chi/p) \vDash \psi(\chi/p)$. For instance, we have $p \vee p \vDash p$, but $(p \mathbin{\vee\!\!\!\vee} \neg p) \vee (p \mathbin{\vee\!\!\!\vee} \neg p) \nvDash p \mathbin{\vee\!\!\!\vee} \neg p$.

6.2.2 Resolutions and Normal Form

In this section, we adapt the notion of *resolutions* from inquisitive logic [35], and list pertinent results. The proofs of these results are analogous to their proofs in propositional inquisitive logic; see [35] for details.

As discussed in Section 6.1, the inquisitive disjunction $\mathbin{\mathbb{V}}$ is used in inquisitive logic as a question-forming connective—for instance, the question as to whether or not p is the case is represented by $p \mathbin{\mathbb{V}} \neg p$. Resolutions serve mainly a technical purpose, but it is helpful to think of the resolutions of a formula φ as representing the answers to φ (the different possible ways of truthfully resolving φ) whenever φ represents a question. If φ does not contain instances of $\mathbin{\mathbb{V}}$, it does not represent a question, and its only resolution is φ itself. In Section 6.4, we generalize the notion of resolutions to *partial resolutions*, and use partial resolutions to prove a weak subformula property for and the cutfree completeness of our sequent calculus.

6.2.5. DEFINITION (Resolutions). The set $\mathcal{R}(\varphi)$ of *resolutions* of a formula φ is defined recursively as follows:

- $\mathcal{R}(p) \coloneqq \{p\}$;
- $\mathcal{R}(\bot) \coloneqq \{\bot\}$;
- $\mathcal{R}(\neg\alpha) \coloneqq \{\neg\beta \mid \beta \in \mathcal{R}(\alpha)\}$;
- $\mathcal{R}(\varphi \wedge \psi) \coloneqq \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\varphi) \text{ and } \beta \in \mathcal{R}(\psi)\}$;
- $\mathcal{R}(\varphi \vee \psi) \coloneqq \{\alpha \vee \beta \mid \alpha \in \mathcal{R}(\varphi) \text{ and } \beta \in \mathcal{R}(\psi)\}$;
- $\mathcal{R}(\varphi \mathbin{\mathbb{V}} \psi) \coloneqq \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$.

Clearly for $\varphi \in \mathrm{PL}(\mathbin{\mathbb{V}})$, $\mathcal{R}(\varphi) \subseteq \mathrm{PL}$; and for $\alpha \in \mathrm{PL}$, $\mathcal{R}(\alpha) = \{\alpha\}$.

6.2.6. DEFINITION (Resolutions for multisets). A *resolution function for a multiset of formulas* $\Gamma \subseteq \mathrm{PL}(\mathbin{\mathbb{V}})$ is a map $f : \Gamma \to \mathrm{PL}$ such that for each $\varphi \in \Gamma$, $f(\varphi) \in \mathcal{R}(\varphi)$. The set (of multisets) $\mathcal{R}(\Gamma)$ of *resolutions* of Γ is:

$$\mathcal{R}(\Gamma) \coloneqq \{f[\Gamma] \mid f \text{ if a resolution function for } \Gamma\}.$$

As above, for $\Gamma \subseteq \mathrm{PL}(\mathbin{\mathbb{V}})$, $\mathcal{R}(\Gamma) \subseteq \mathrm{PL}$; and for $\Lambda \subseteq \mathrm{PL}$, $\mathcal{R}(\Lambda) = \{\Lambda\}$.

Each formula is equivalent to the inquisitive disjunction of its resolutions (and hence to an inquisitive disjunction of classical formulas), and similarly for multisets of formulas.

6.2.7. PROPOSITION (Normal form). $[\varphi \equiv \mathbin{\mathbb{V}}\mathcal{R}(\varphi)]$ *and* $[t \vDash \Gamma \iff t \vDash \mathbin{\mathbb{V}}\mathcal{R}(\Gamma)]$.

Proof:
Follows from the fact that $\wedge$ and $\vee$ distribute over $\mathbin{\vee\!\!\!\vee}$. □

The *Split property* is a generalization of the disjunction property (with respect to $\mathbin{\vee\!\!\!\vee}$): a classical multiset of formulas entails a $\mathbin{\vee\!\!\!\vee}$-disjunction just in case it entails one of the disjuncts.

6.2.8. PROPOSITION (Split property). *For* $\Lambda \subseteq \mathrm{PL}$ *and* $\varphi, \psi \in \mathrm{PL}(\mathbin{\vee\!\!\!\vee})$*:*

$$\Lambda \vDash \varphi \mathbin{\vee\!\!\!\vee} \psi \iff [\Lambda \vDash \varphi \text{ or } \Lambda \vDash \psi].$$

Proof:
By the union closure of PL and the downward closure of $\mathrm{PL}(\mathbin{\vee\!\!\!\vee})$. □

Finally, a multiset of formulas Γ entails ψ just in case each resolution of Γ entails some resolution of ψ:

6.2.9. THEOREM (Resolution theorem).

$$\Gamma \vDash \psi \iff \textit{for each } \Lambda \in \mathcal{R}(\Gamma) \textit{ there is some } \alpha \in \mathcal{R}(\psi) \textit{ such that } \Lambda \vDash \alpha.$$

Proof:
By Propositions 6.2.7 and 6.2.8. □

6.3 The System GT

We introduce the sequent calculus GT for PL ('G' stands for Gentzen, 'T' for team). This is a G3-style system (structural rules are incorporated into the logical rules; see Section 6.8 for an alternative system featuring some explicit structural rules) with deep-inference style rules for the inquisitive disjunction $\mathbin{\vee\!\!\!\vee}$.

Given formulas $\varphi_1, \ldots, \varphi_n$ and multisets of formulas Γ and Δ, we use the notation $\varphi_1, \ldots, \varphi_n, \Gamma$ to denote the multiset $\{\varphi_1, \ldots, \varphi_n\} \cup \Gamma$, and the notation Γ, Δ to denote the multiset $\Gamma \cup \Delta$ (in both cases, $\cup$ denotes the multiset union operation). A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite multisets of formulas. In $\Gamma \Rightarrow \Delta$, Γ is the *antecedent* and Δ is the *succedent*. The intended interpretation of $\Gamma \Rightarrow \Delta$ is $\bigwedge \Gamma \vDash \bigvee \Delta$; we say that $\Gamma \Rightarrow \Delta$ is *valid* if $\bigwedge \Gamma \vDash \bigvee \Delta$. Let us emphasize that the commas in the succedent of a sequent $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \varphi_m$ are interpreted as the split disjunction $\vee$ (while the commas in the antecedent are interpreted as conjunction $\wedge$); the sequent $\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \varphi_m$ is valid just in case for any team t, if $t \vDash \varphi_1 \wedge \ldots \wedge \varphi_n$, then $t = t_1 \cup \ldots \cup t_m$, where $t_1 \vDash \psi_1, \ldots, t_m \vDash \psi_m$.

Our system makes use of a notion of formula contexts to implement deep-inference rules. A *(formula) context* $\varphi\{\cdot_1\} \ldots \{\cdot_n\}$ is a formula φ with designated atoms $\cdot_1 \ldots \cdot_n$ (each of which occurs only once in φ). We do not allow contexts in which any $\cdot_i$ is in the scope of a negation $\neg$. We write $\varphi\{\psi_1\}\ldots\{\psi_n\}$ for the result of replacing $\cdot_1,\ldots,\cdot_n$ in $\varphi\{\cdot_1\}\ldots\{\cdot_n\}$ with $\psi_1,\ldots,\psi_n$, respectively. For instance, if $\psi\{\cdot\} = p\wedge(\cdot \mathbin{\vee\!\!\!\vee} r)$, then $\psi\{q\vee p\} = p\wedge((q\vee p)\mathbin{\vee\!\!\!\vee} r)$.

6.3.1. DEFINITION (The sequent calculus GT).

Axioms

$$\Gamma, p \Rightarrow p, \Delta \ \ \mathsf{At} \qquad\qquad \Gamma, \bot \Rightarrow \Delta \ \ \mathsf{L}\bot$$

Logical rules

$$\frac{\Gamma \Rightarrow \alpha, \Delta}{\Gamma, \neg\alpha \Rightarrow \Delta}\,\mathsf{L}\neg \qquad\qquad \frac{\Gamma, \alpha \Rightarrow \Delta}{\Gamma \Rightarrow \neg\alpha, \Delta}\,\mathsf{R}\neg$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi\wedge\psi \Rightarrow \Delta}\,\mathsf{L}\wedge \qquad\qquad \frac{\Gamma \Rightarrow \varphi, \Lambda \qquad \Gamma \Rightarrow \psi, \Lambda}{\Gamma \Rightarrow \varphi\wedge\psi, \Lambda, \Delta}\,\mathsf{R}\wedge$$

$$\frac{\Gamma, \varphi \Rightarrow \Lambda \qquad \Gamma, \psi \Rightarrow \Lambda}{\Gamma, \varphi\vee\psi \Rightarrow \Lambda, \Delta}\,\mathsf{L}\vee \qquad\qquad \frac{\Gamma \Rightarrow \varphi, \psi, \Delta}{\Gamma \Rightarrow \varphi\vee\psi, \Delta}\,\mathsf{R}\vee$$

$$\frac{\Gamma, \chi\{\varphi_L\} \Rightarrow \Delta \qquad \Gamma, \chi\{\varphi_R\} \Rightarrow \Delta}{\Gamma, \chi\{\varphi_L \mathbin{\vee\!\!\!\vee} \varphi_R\} \Rightarrow \Delta}\,\mathsf{L}\mathbin{\vee\!\!\!\vee} \qquad\qquad \frac{\Gamma \Rightarrow \chi\{\varphi_i\}, \Delta}{\Gamma \Rightarrow \chi\{\varphi_L \mathbin{\vee\!\!\!\vee} \varphi_R\}, \Delta}\,\mathsf{R}\mathbin{\vee\!\!\!\vee}$$

Cut

$$\frac{\Gamma \Rightarrow \varphi, \Delta \qquad \Pi, \varphi \Rightarrow \Sigma}{\Pi, \Gamma \Rightarrow \Delta, \Sigma}\,\mathsf{Cut}$$

The sequent(s) above the line in each rule (application) are called the *premise(s)* of that rule (application); the sequent below the line is the *conclusion* of the rule. The formulas $\varphi, \psi, \alpha, \chi\{\varphi_i\}$ in the premise(s) are the *active formulas* of the rule; the newly introduced formula in the conclusion is the *principal formula* of the rule. Γ, Δ, Λ are *side formulas* or the *left/right context ((multi)set)* (left: Γ; right: Δ, Λ). In $\mathsf{L}\mathbin{\vee\!\!\!\vee}$ and $\mathsf{R}\mathbin{\vee\!\!\!\vee}$, we call the subformula(s) φ_i in the premise(s) the *active subformulas* of the rule and the subformula $\varphi_L \mathbin{\vee\!\!\!\vee} \varphi_R$ in the conclusion the *principal subformula* of the rule; for all other rules, their active subformula(s) are the same as their active formula(s), and similarly for their principal subformula(s). The formula φ in Cut is the *cutformula*, and we say Cut is a *cut on* φ.[2]

The rules $\mathsf{L}\neg$, $\mathsf{R}\neg$, $\mathsf{R}\wedge$, and $\mathsf{L}\vee$ feature syntactic restrictions. The restrictions on the negation rules are due to the corresponding restriction in the syntax of PL($\mathbin{\vee\!\!\!\vee}$). The

[2]We will often refer to formula occurrences simply as formulas for simplicity, and disambiguate when it aids clarity.

restrictions that the premise right context in $\mathsf{R}\wedge$ and $\mathsf{L}\vee$ be classical are required for soundness—note, for instance, that $\mathsf{L}\vee$ without the restriction would not be sound, essentially due to the failure of union closure of non-classical formulas: $p \vDash p \varovee \neg p$ and $\neg p \vDash p \varovee \neg p$, but $p \vee \neg p \nvDash p \varovee \neg p$. The restriction to a classical (and hence union-closed) premise right context ensures soundness. These two rules also feature implicit weakening (Δ is added to the right context) to make up for the loss in strength incurred due to the syntactic restrictions—they allow us to prove that right weakening is admissible in the system (see Lemma 6.5.1).

Relatedly, observe that with the comma on the right of a sequent corresponding to the split disjunction, we have the following correspondences between structural rules on the right and team-semantic closure properties. Right weakening $[\vDash \Gamma \Rightarrow \Delta, \xi$ implies $\vDash \Gamma \Rightarrow \Delta, \xi, \xi]$ is sound for ξ if ξ has the empty team property; and right contraction $[\vDash \Gamma \Rightarrow \Delta, \xi, \xi$ implies $\vDash \Gamma \Rightarrow \Delta, \xi]$ is sound for ξ if ξ is union closed. In our setting, all formulas have the empty team property and classical formulas are union closed; accordingly, right weakening is sound for all formulas, and right contraction is sound for classical formulas. To see why right contraction is not sound in general with respect to formulas which are not union closed, note that $(p \varovee \neg p) \vee (p \varovee \neg p) \Rightarrow (p \varovee \neg p), (p \varovee \neg p)$ is valid whereas $(p \varovee \neg p) \vee (p \varovee \neg p) \Rightarrow (p \varovee \neg p)$ is not.

$\mathsf{L}\varovee$ and $\mathsf{R}\varovee$ are our deep-inference rules. They are standard classical disjunction rules, strengthened to allow the active/principal subformula(s) to appear within any position (not in the scope of a negation) in the active/principle formula(s). The soundness of these rules follows from the fact that $\wedge$ and $\vee$ distribute over $\varovee$. Example application:

$$\frac{\Gamma, p \wedge (q \vee r) \Rightarrow \Delta \qquad \Gamma, p \wedge (s \varovee (q \wedge \neg p)) \Rightarrow \Delta}{\Gamma, p \wedge ((q \vee r) \varovee (s \varovee (q \wedge \neg p)) \Rightarrow \Delta}\ \mathsf{L}\varovee$$

Given any set of sequent calculus rules C, we let C also denote the system consisting of the rules in C. Let $\mathsf{G3cp}$ (for PL) be the system $\mathsf{GT} \setminus \{\mathsf{L}\varovee, \mathsf{R}\varovee\}$. This is a G3-style cutfree complete system for PL—the soundness (given the standard semantics for classical propositional logic) and cutfree completeness of this system follow easily from that for the usual $\mathsf{G3cp}$ (see, e.g., [118]). We write C^- for the Cut-free fragment of C, i.e., $\mathsf{C}^- := \mathsf{C} \setminus \mathsf{Cut}$. We write $\vdash_{\mathsf{C}} \Gamma \Rightarrow \Delta$ (or simply $\vdash \Gamma \Rightarrow \Delta$ when the system is clear from the context) if there is a derivation of $\Gamma \Rightarrow \Delta$ in calculus C. If $\mathcal{D}$ is a derivation of $\Gamma \Rightarrow \Delta$ in calculus C, we say that $\mathcal{D}$ *witnesses* $\vdash_{\mathsf{C}} \Gamma \Rightarrow \Delta$. We define the *height* of a derivation $\mathcal{D}$ inductively: if $\mathcal{D}$ consists of a single axiom, the height of $\mathcal{D}$ is one; otherwise the height of $\mathcal{D}$ is 1 plus the maximum of the heights of the subderivations ending in the premises of the final rule application in $\mathcal{D}$.

We now verify that the system is sound given the team semantics for $\mathrm{PL}(\varovee)$.

6.3.2. THEOREM (Soundness of GT). $\vdash \Gamma \Rightarrow \Delta$ *implies* $\bigwedge \Gamma \vDash \bigvee \Delta$.

Proof:
By induction on the height of derivations; we separate into different cases based on the final rule applied in the derivation.

- At: $p \vDash p \vee \bigvee\Delta$ by $p \vDash p$ and the empty team property.
- L⊥: $\bot \vDash \bigvee\Delta$ by the empty team property.
- L¬: Assume $\bigwedge\Gamma \vDash \alpha \vee \bigvee\Delta$. We want to show $\neg\alpha \wedge \bigwedge\Gamma \vDash \bigvee\Delta$, so let $t \vDash \neg\alpha \wedge \bigwedge\Gamma$. By $\bigwedge\Gamma \vDash \alpha \vee \bigvee\Delta$, we have $t \vDash \alpha \vee \bigvee\Delta$, so $t = t_\alpha \cup t_\Delta$ where $t_\alpha \vDash \alpha$ and $t_\Delta \vDash \bigvee\Delta$. By $t_\alpha \vDash \alpha$, $t \vDash \neg\alpha$, and the flatness of α and $\neg\alpha$, we have that for each $v \in t_\alpha$, $v \vDash \alpha \wedge \neg\alpha$; therefore $t_\alpha = \varnothing$, whence $t = t_\Delta \vDash \bigvee\Delta$.
- R¬: Assume $\bigwedge\Gamma \wedge \alpha \vDash \bigvee\Delta$. We want to show $\bigwedge\Gamma \vDash \neg\alpha \vee \bigvee\Delta$, so let $t \vDash \bigwedge\Gamma$. Let $t_\alpha := \{v \in t \mid v \vDash \alpha\}$ and $t_{\neg\alpha} := t \setminus t_\alpha$. By the flatness of α we have $t_\alpha \vDash \alpha$, and by downward closure we have $t_\alpha \vDash \bigwedge\Gamma$; therefore by $\bigwedge\Gamma \wedge \alpha \vDash \bigvee\Delta$, we have $t_\alpha \vDash \bigvee\Delta$. By the flatness of $\neg\alpha$, we have $t_{\neg\alpha} \vDash \neg\alpha$, so that by $t = t_{\neg\alpha} \cup t_\alpha$, we have $t \vDash \neg\alpha \vee \bigvee\Delta$.
- L∧: Follows immediately from the induction hypothesis.
- R∧: Assume $\bigwedge\Gamma \vDash \varphi \vee \bigvee\Lambda$ and $\bigwedge\Gamma \vDash \psi \vee \bigvee\Lambda$. We want to show $\bigwedge\Gamma \vDash (\varphi \wedge \psi) \vee \bigvee\Lambda$, so let $t \vDash \bigwedge\Gamma$. Then $t \vDash \varphi \vee \bigvee\Lambda$ and $t \vDash \psi \vee \bigvee\Lambda$, so $t = t_\varphi \cup t_{\Lambda 1}$ where $t_\varphi \vDash \varphi$ and $t_{\Lambda 1} \vDash \bigvee\Lambda$, and $t = t_\psi \cup t_{\Lambda 2}$ where $t_\psi \vDash \psi$ and $t_{\Lambda 2} \vDash \bigvee\Lambda$. Then also $t = (t_\varphi \cap t_\psi) \cup t_{\Lambda 1} \cup t_{\Lambda 2}$. By downward closure, $t_\varphi \cap t_\psi \vDash \varphi \wedge \psi$, and by the union closure of $\bigvee\Lambda$, we have $t_{\Lambda 1} \cup t_{\Lambda 2} \vDash \bigvee\Lambda$. Therefore, $t \vDash (\varphi \wedge \psi) \vee \bigvee\Lambda$.
- L∨: Assume $\bigwedge\Gamma \wedge \varphi \vDash \bigvee\Lambda$ and $\bigwedge\Gamma \wedge \psi \vDash \bigvee\Lambda$. We want to show $\bigwedge\Gamma \wedge (\varphi \vee \psi) \vDash \bigvee\Lambda$, so let $t \vDash \bigwedge\Gamma \wedge (\varphi \vee \psi)$. Then $t = t_\varphi \cup t_\psi$ where $t_\varphi \vDash \varphi$ and $t_\psi \vDash \psi$, and $t \vDash \bigwedge\Gamma$. By downward closure, $t_\varphi \vDash \bigwedge\Gamma$ so that $t_\varphi \vDash \bigvee\Lambda$; similarly $t_\psi \vDash \bigvee\Lambda$. By the union closure of $\bigvee\Lambda$, we have $t \vDash \bigvee\Lambda$.
- R∨: Follows immediately from the induction hypothesis.
- R⩔: Follows immediately from the induction hypothesis and the fact that $\wedge$ and $\vee$ distribute over $\mathbin{\vee\!\!\!\vee}$.
- L⩔: Assume $\bigwedge\Gamma \wedge \chi\{\varphi_L\} \vDash \bigvee\Delta$ and $\bigwedge\Gamma \wedge \chi\{\varphi_R\} \vDash \bigvee\Delta$. We want to show $\bigwedge\Gamma \wedge \chi\{\varphi_L \mathbin{\vee\!\!\!\vee} \varphi_R\} \vDash \bigvee\Delta$, so let $t \vDash \bigwedge\Gamma \wedge \chi\{\varphi_L \mathbin{\vee\!\!\!\vee} \varphi_R\}$. Since $\wedge$ and $\vee$ distribute over $\mathbin{\vee\!\!\!\vee}$, $\chi\{\varphi_L \mathbin{\vee\!\!\!\vee} \varphi_R\} \equiv \chi\{\varphi_L\} \mathbin{\vee\!\!\!\vee} \chi\{\varphi_R\}$, so $t \vDash \chi\{\varphi_L\}$ or $\chi\{\varphi_R\}$. In either case, by our assumptions $t \vDash \bigvee\Delta$.
- Cut: Assume $\bigwedge\Gamma \vDash \varphi \vee \bigvee\Delta$ and $\bigwedge\Pi \wedge \varphi \vDash \bigvee\Sigma$. We want to show $\bigwedge\Gamma \wedge \bigwedge\Pi \vDash \bigvee\Delta \vee \bigvee\Sigma$, so let $t \vDash \bigwedge\Gamma \wedge \bigwedge\Pi$. By $\bigwedge\Gamma \vDash \varphi \vee \bigvee\Delta$, $t \vDash \varphi \vee \bigvee\Delta$, whence $t = t_\varphi \cup t_\Delta$, where $t_\varphi \vDash \varphi$ and $t_\Delta \vDash \bigvee\Delta$. By downward closure, $t_\varphi \vDash \bigvee\Pi$, whence by $\bigwedge\Pi \wedge \varphi \vDash \bigvee\Sigma$, $t_\varphi \vDash \bigvee\Sigma$. Therefore, $t \vDash \bigvee\Delta \vee \bigvee\Sigma$. □

6.3.3. THEOREM (Cut elimination for G3cp). *[118] If $\mathcal{D}$ witnesses $\vdash_{\mathsf{G3cp}} \Xi \Rightarrow \Lambda$, there is an effective procedure for transforming $\mathcal{D}$ into a derivation $\mathcal{D}'$ witnessing* $\vdash_{\mathsf{G3cp}^-} \Xi \Rightarrow \Lambda$.

6.3.4. COROLLARY (Cutfree classical completeness). $\bigwedge\Xi \vDash \bigvee\Lambda$ *implies* $\vdash_{\mathsf{G3cp}^-} \Xi \Rightarrow \Lambda$ *(whence also* $\vdash_{\mathsf{GT}^-} \Xi \Rightarrow \Lambda$ *and* $\vdash_{\mathsf{GT}} \Xi \Rightarrow \Lambda$*).*

Why Deep Inference?

One possible natural translation of the natural deduction rules for PL($\mathbin{\vee\!\!\!\vee}$) [139] into sequent calculus rules would give us the system $\mathsf{G3cp} \cup \{\mathsf{L}\mathbin{\vee\!\!\!\vee}^-, \mathsf{R}\mathbin{\vee\!\!\!\vee}^-, \mathsf{LDstr}, \mathsf{RDstr}\}$, where:

$$\frac{\Gamma, \varphi_L \Rightarrow \Delta \qquad \Gamma, \varphi_R \Rightarrow \Delta}{\Gamma, \varphi_L \mathbin{\vee\!\!\!\vee} \varphi_R \Rightarrow \Delta}\ \mathsf{L}\mathbin{\vee\!\!\!\vee}^- \qquad\qquad \frac{\Gamma \Rightarrow \varphi_i, \Delta}{\Gamma \Rightarrow \varphi_L \mathbin{\vee\!\!\!\vee} \varphi_R, \Delta}\ \mathsf{R}\mathbin{\vee\!\!\!\vee}^-$$

$$\frac{\Gamma, \varphi \vee \psi_L \Rightarrow \Delta \qquad \Gamma, \varphi \vee \psi_R \Rightarrow \Delta}{\Gamma, \varphi \vee (\psi_L \mathbin{\vee\!\!\!\vee} \psi_R) \Rightarrow \Delta}\ \mathsf{LDstr} \qquad \frac{\Gamma \Rightarrow \varphi \vee (\psi_L \mathbin{\vee\!\!\!\vee} \psi_R), \Delta}{\Gamma \Rightarrow (\varphi \vee \psi_L) \mathbin{\vee\!\!\!\vee} (\varphi \vee \psi_R), \Delta}\ \mathsf{RDstr}$$

This system is not cutfree complete. For instance, the following valid sequent is not derivable:

$$(r \wedge x) \vee (((p \wedge x) \mathbin{\vee\!\!\!\vee} (q \wedge x)) \vee (y \wedge x)) \Rightarrow (x \wedge (r \vee (p \vee y))) \mathbin{\vee\!\!\!\vee} (x \wedge (r \vee (q \vee y))).$$

There is no rule of this system that any putative derivation of this sequent could have concluded with. Any putative derivation could not have concluded with either of the distributivity rules because the putative principal formula of such a rule application would not be of the right form in either case. It could not have concluded with $\mathsf{R}\mathbin{\vee\!\!\!\vee}^-$ or $\mathsf{L}\vee$ because none of the possible premises of such a rule application are valid. Finally, it could not have concluded with any other rule because the main connective of the putative principal formula of such a rule application would not be the connective of that rule.

The problem is that the distributivity rules do not reach deep enough into the active/principal formula (we conjecture that one can formulate a notion of context depth and show that a system with the distributivity rules is cutfree complete for formulas in which $\mathbin{\vee\!\!\!\vee}$ only appears up to certain depth). The deep-inference rules $\mathsf{L}\mathbin{\vee\!\!\!\vee}$ and $\mathsf{R}\mathbin{\vee\!\!\!\vee}$ fix this issue and have the added benefit of allowing for a system consisting only of a single introduction (right) and single elimination (left) rule for each connective.

6.4 Partial Resolutions and Cutfree Completeness

GT^- has a weak subformula property. We can formulate this property in terms of what we will call *partial resolutions*—a generalization of resolutions (Section 6.2.2). Partial resolutions also yield an easy way of showing the cutfree completeness of GT.

In order to define partial resolutions, it is helpful to first define some more notation for the type of substitutions carried out via the deep inference rules. A $\mathbin{\vee\!\!\!\vee}$*-labelled formula (occurrence)* is a formula occurrence φ together with a function f mapping each occurrence of $\mathbin{\vee\!\!\!\vee}$ in φ to $\mathbb{N}$. Examples: $p \mathbin{\vee\!\!\!\vee}_0 (q \mathbin{\vee\!\!\!\vee}_1 r)$; $(p \mathbin{\vee\!\!\!\vee}_2 x) \mathbin{\vee\!\!\!\vee}_2 (q \mathbin{\vee\!\!\!\vee}_1 r)$. We will frequently refer to $\mathbb{V}$-labelled occurrences as simply formula occurrences or formulas, and switch between labelled and unlabelled occurrences when it is convenient to do so. The $\mathbb{V}$*-labelling* $\varphi_{\mathbb{V}}$ of a formula occurrence φ is the $\mathbb{V}$-labelled formula occurrence consisting of the formula φ and the function mapping each occurrence of $\mathbb{V}$ in φ according to its position in the sequence of such occurrences; e.g., if $\varphi = p \mathbb{V} (q \mathbb{V} r)$, then $\varphi_{\mathbb{V}} = p \mathbb{V}_0 (q \mathbb{V}_1 r)$. Below, we will use $\mathbb{V}$-labellings $\varphi_{\mathbb{V}}$ to define other $\mathbb{V}$-labelled formulas. We use the notation $\psi[\chi_1 \mathbb{V}_i \chi_2]^{\varphi}$ (or simply $\psi[\chi_1 \mathbb{V}_i \chi_2]$ or $\psi[i]^{\varphi}$ or $\psi[i]$) to refer to a subformula occurrence $\chi_L \mathbb{V}_i \chi_R$ in ψ with $\varphi_{\mathbb{V}}$-label i. We let $\psi[i/j]^{\varphi}$ (or simply $\psi[i/j]$), where $i \in \{L, R\}$, denote the result of replacing each $\psi[\chi_L \mathbb{V}_j \chi_R]^{\varphi}$ in ψ (if any such exist) with χ_i. For instance, for φ as above, let $\psi := \varphi[R/1]^{\varphi} = p \mathbb{V}_0 r$. Then $\psi[0]^{\varphi} = \psi$, $\psi[L/0] = p$, and $\psi[L/1] = \psi$. We let $|\varphi|_{\mathbb{V}}$ denote the number of occurrences of $\mathbb{V}$ in φ.

6.4.1. DEFINITION (Partial resolutions). The set $\mathcal{PR}_n(\varphi)$ of *partial resolutions of* φ *of degree* $n \in \{0, \dots, |\varphi|_{\mathbb{V}}\}$ is defined by

$$\begin{aligned}\mathcal{PR}_n(\varphi) := \{\varphi[i_1/j_1]^{\varphi} \dots [i_n/j_n]^{\varphi} \mid & \{i_x\}_{1 \le x \le n} \subseteq \{L, R\}, \\ & \{j_x\}_{1 \le x \le n} \subseteq \{0, \dots, |\varphi|_{\mathbb{V}} - 1\}, \\ & [x \neq y \text{ implies } j_x \neq j_y]\}.\end{aligned}$$

The set $\mathcal{PR}(\varphi)$ of *partial resolutions of* φ is defined by

$$\mathcal{PR}(\varphi) := \bigcup_{0 \le n \le |\varphi|_{\mathbb{V}}} \mathcal{PR}_n(\varphi).$$

Example: for $\varphi_{\mathbb{V}} = p \mathbb{V}_0 (q \mathbb{V}_1 r)$,

$$\begin{aligned}\mathcal{PR}_0(\varphi) =& \{\varphi\} = \{p \mathbb{V}_0 (q \mathbb{V}_1 r)\}; \\ \mathcal{PR}_1(\varphi) =& \{\varphi[L/0], \varphi[R/0], \varphi[L/1], \varphi[R/1]\} \\ =& \{p, q \mathbb{V}_1 r, p \mathbb{V}_0 q, p \mathbb{V}_0 r\}; \\ \mathcal{PR}_2(\varphi) =& \{\varphi[L/0][L/1], \varphi[L/0][R/1], \varphi[R/0][L/1], \varphi[R/0][R/1], \\ & \varphi[L/1][L/0], \varphi[L/1][R/0], \varphi[R/1][L/0], \varphi[R/1][R/0]\} \\ =& \{p, q, r\}.\end{aligned}$$

It is easy to see that $\mathcal{PR}_{|\varphi|_{\mathbb{V}}}(\varphi) = \mathcal{R}(\varphi)$; we generalize this in Proposition 6.4.5 below. The guiding intuition behind partial resolutions is that if $\mathbb{V}$ represents a question-forming operator as in inquisitive logic, each formula in $\mathcal{PR}_n(\varphi)$ represents an attempt to resolve n "questions" in φ.[3] The formulas in $\mathcal{R}(\varphi) = \mathcal{PR}_{|\varphi|_{\mathbb{V}}}(\varphi)$, then, resolve all the questions in φ.

[3] The notion of "question" here is such that each occurrence $\mathbb{V}$ corresponds to one question. The formula $a \mathbb{V}_0 (b \mathbb{V}_1 c)$ (which could be thought to correspond roughly to "Is Mary in Amsterdam or is she in Beijing or in Copenhagen?") has two questions. Each of the partial resolutions a ("Mary is in Amsterdam") and $b \mathbb{V}_1 c$ ("Is Mary in Beijing or in Copenhagen?") resolves question 0. The first partial resolution a, since it itself contains no questions, also constitutes a (full) resolution to φ.

We may now formulate the weak subformula property for GT^-. Observe that using labelled formulas we may reformulate the deep-inference rules $\mathsf{L}\mathbin{\mathbb{V}}$ and $\mathsf{R}\mathbin{\mathbb{V}}$ as follows:

$$\frac{\Gamma,\chi[L/j]^{\varphi}\Rightarrow\Delta \qquad \Gamma,\chi[R/j]^{\varphi}\Rightarrow\Delta}{\Gamma,\chi\Rightarrow\Delta}\,\mathsf{L}\mathbin{\mathbb{V}} \qquad\qquad \frac{\Gamma\Rightarrow\chi[i/j]^{\varphi},\Delta}{\Gamma\Rightarrow\chi,\Delta}\,\mathsf{R}\mathbin{\mathbb{V}}$$

The following are therefore immediate from the shape of the rules:

6.4.2. LEMMA.

(i) If $\Gamma\Rightarrow\chi[i/j]^{\varphi},\Delta$*, then* $\Gamma\Rightarrow\chi,\Delta$.

(ii) If $\Gamma,\chi[L/j]^{\varphi}\Rightarrow\Delta$ *and* $\Gamma,\chi[R/j]^{\varphi}\Rightarrow\Delta$*, then* $\Gamma,\chi\Rightarrow\Delta$.

6.4.3. PROPOSITION (Partial resolution subformula property). *For any derivation* $\mathcal{D}$ *witnessing* $\Gamma\vdash_{\mathsf{GT}^-}\Delta$*, each formula occurring in* $\mathcal{D}$ *is a subformula of a partial resolution of some formula occurring in* Γ,Δ.

It is also easy to see from the reformulation above—together with the fact that $\mathcal{PR}_{|\varphi|_{\mathbb{V}}}(\varphi)=\mathcal{R}(\varphi)$—that the cutfree completeness of GT follows essentially immediately from the resolution theorem (Theorem 6.2.9) in conjunction with cutfree classical completeness (Corollary 6.3.4). To make this more precise, we now also define partial resolutions for multisets. Given a multiset $\Gamma=\{\varphi_0,\ldots,\varphi_n\}$, we assume Γ comes with some canonical ordering of its elements and write $\vec{\Gamma}=\langle\varphi_0,\ldots,\varphi_n\rangle$ for the sequence corresponding to this canonical ordering. We write $\Gamma[i/j]^{\varphi}_k$ (or simply $\Gamma[i/j]_k$, or $\Gamma[i/j]_{\varphi_k}$) for the multiset consisting of the elements of the sequence $\langle\varphi_0,\ldots,\varphi_k[i/j]^{\varphi},\ldots,\varphi_n\rangle$, where $\vec{\Gamma}=\langle\varphi_0,\ldots,\varphi_n\rangle$. E.g., for $\varphi=p\mathbin{\mathbb{V}}_0(q\mathbin{\mathbb{V}}_1 r)$ and $\psi=p\mathbin{\mathbb{V}}_0 r=\psi[0]^{\varphi}$ and $\vec{\Gamma}=\langle\varphi,\varphi,\psi\rangle$, we have $\Gamma[L/0]^{\varphi}_2=\{\varphi,\varphi,p\}$; $\Gamma[L/1]_1=\{\varphi,p\mathbin{\mathbb{V}}_0 q,\psi\}$; and $\Gamma[L/1]_3=\Gamma$. We let $|\Gamma|_{\mathbb{V}}:=\sum_{\varphi\in\Gamma}|\varphi|_{\mathbb{V}}$.

6.4.4. DEFINITION (Partial resolutions for multisets). The set (of multisets) $\mathcal{PR}_n(\Gamma)$ of *partial resolutions of* $\Gamma=\{\varphi_0,\ldots,\varphi_q\}$ *of degree* $n\in\{0,\ldots,|\Gamma|_{\mathbb{V}}\}$ is defined by

$$\begin{aligned}\mathcal{PR}_n(\Gamma):=\{\Gamma[i_1/j_1]^{\varphi^1}_{k_1}\ldots[i_n/j_n]^{\varphi^n}_{k_n}\ |\ &\text{for each } x\in\{1,\ldots,n\}:\varphi^x\in\Gamma,\\ &\{i_x\}_{1\le x\le n}\subseteq\{L,R\},\\ &\{k_x\}_{1\le x\le n}\subseteq\{0,\ldots,q\},\\ &\text{for each } w\in\{0,\ldots,q\}:\\ &\{j_x\mid\varphi^x=\varphi_w\}_{1\le x\le n}\subseteq\{0,\ldots,|\varphi_w|_{\mathbb{V}}-1\}\text{ and}\\ &[[x\neq y\text{ and }\varphi^x=\varphi^y=\varphi_w]\text{ implies } j_x\neq j_y]\}.\end{aligned}$$

The set (of multisets) $\mathcal{PR}(\varphi)$ of *partial resolutions of* φ is defined by

$$\mathcal{PR}(\varphi):=\bigcup_{0\le n\le|\Gamma|_{\mathbb{V}}}\mathcal{PR}_n(\Gamma).$$

Example: for $\Gamma = \{\varphi_0, \varphi_1\} = \{p \mathbin{\mathbb{V}}_0 (q \mathbin{\mathbb{V}}_1 r), s \mathbin{\mathbb{V}}_0 r\}$, where we have $\vec{\Gamma} = \langle \varphi_0, \varphi_1 \rangle = \langle p \mathbin{\mathbb{V}}_0 (q \mathbin{\mathbb{V}}_1 r), s \mathbin{\mathbb{V}}_0 r \rangle$,

$$\begin{aligned}
\mathcal{PR}_0(\Gamma) =& \{\Gamma\} = \{\{p \mathbin{\mathbb{V}}_0 (q \mathbin{\mathbb{V}}_1 r), s \mathbin{\mathbb{V}}_0 r\}\};\\
\mathcal{PR}_1(\Gamma) =& \{\Gamma[L/0]_0, \Gamma[R/0]_0, \Gamma[L/1]_0, \Gamma[R/1]_0, \Gamma[L/0]_1, \Gamma[R/0]_1\}\\
=& \{\{p, s \mathbin{\mathbb{V}}_0 r\}, \{q \mathbin{\mathbb{V}}_1 r, s \mathbin{\mathbb{V}}_0 r\}, \{p \mathbin{\mathbb{V}}_0 q, s \mathbin{\mathbb{V}}_0 r\}, \{p \mathbin{\mathbb{V}}_0 r, s \mathbin{\mathbb{V}}_0 r\},\\
& \{p \mathbin{\mathbb{V}}_0 (q \mathbin{\mathbb{V}}_1 r), s\}, \{p \mathbin{\mathbb{V}}_0 (q \mathbin{\mathbb{V}}_1 r), r\}\};\\
\mathcal{PR}_2(\Gamma) =& \{\{p, s \mathbin{\mathbb{V}}_0 r\}, \{p, s\}, \{p, r\}, \{q, s \mathbin{\mathbb{V}}_0 r\}, \{r, s \mathbin{\mathbb{V}}_0 r\}, \{q \mathbin{\mathbb{V}}_1 r, s\},\\
& \{q \mathbin{\mathbb{V}}_1 r, r\}, \{p \mathbin{\mathbb{V}}_0 q, s\}, \{p \mathbin{\mathbb{V}}_0 q, r\}, \{p \mathbin{\mathbb{V}}_0 r, s\}, \{p \mathbin{\mathbb{V}}_0 r, r\}\};\\
\mathcal{PR}_3(\Gamma) =& \{\{p, s\}, \{p, r\}, \{q, s\}, \{q, r\}, \{r, s\}, \{r, r\}\} = \mathcal{R}(\Gamma).
\end{aligned}$$

One can check that $\mathcal{PR}_n(\{\varphi\}) = \{\{\psi\} \mid \psi \in \mathcal{PR}_n(\varphi)\}$, so this notion generalizes that for formulas. It is easy to see that we have the following generalization of the fact we observed above:

6.4.5. PROPOSITION. $\mathcal{PR}_{|\Gamma|_{\mathbb{V}}}(\Gamma) = \mathcal{R}(\Gamma)$.

And we have:

6.4.6. LEMMA.

(i) *For any* $n \in \{0, \ldots, |\Delta_{\mathbb{V}}| - 1\}$*, if there is a* $\Delta_{n+1} \in \mathcal{PR}_{n+1}(\Delta)$ *such that* $\vdash_{\mathsf{GT}^-} \Gamma \Rightarrow \Delta_{n+1}$*, then there is a* $\Delta_n \in \mathcal{PR}_n(\Delta)$ *such that* $\vdash_{\mathsf{GT}^-} \Gamma \Rightarrow \Delta_n$*.*

(ii) *For any* $n \in \{0, \ldots, |\Delta_{\mathbb{V}}| - 1\}$*, if for all* $\Gamma_{n+1} \in \mathcal{PR}_{n+1}(\Delta)$ *we have* $\vdash_{\mathsf{GT}^-} \Gamma_{n+1} \Rightarrow \Delta$*, then for all* $\Gamma_n \in \mathcal{PR}_n(\Gamma)$ *we have* $\vdash_{\mathsf{GT}^-} \Gamma_n \Rightarrow \Delta$*.*

Proof:
Follows by the definition of partial resolutions together with Lemma 6.4.2. □

6.4.7. THEOREM (Cutfree completeness). $\bigwedge \Gamma \vDash \bigvee \Delta$ *implies* $\vdash_{\mathsf{GT}^-} \Gamma \Rightarrow \Delta$.

Proof:
By the resolution theorem (Theorem 6.2.9), for each $\Xi \in \mathcal{R}(\Gamma)$, there is some $\alpha \in \mathcal{R}(\bigvee \Delta)$ such that $\Xi \vDash \alpha$, i.e., there is some $\Lambda \in \mathcal{R}(\Delta)$ such that $\bigwedge \Xi \vDash \bigvee \Lambda$. By cutfree classical completeness (Corollary 6.3.4), also $\vdash_{\mathsf{GT}^-} \Xi \Rightarrow \Lambda$. So by Proposition 6.4.5, for each $\Xi \in \mathcal{PR}_{|\Gamma|_{\mathbb{V}}}(\Gamma)$, there is some $\Lambda \in \mathcal{PR}_{|\Delta|_{\mathbb{V}}}(\Delta)$ such that $\vdash_{\mathsf{GT}^-} \Xi \Rightarrow \Lambda$. By repeated applications of Lemma 6.4.6 (i), for each $\Xi \in \mathcal{PR}_{|\Gamma|_{\mathbb{V}}}(\Gamma)$, $\vdash_{\mathsf{GT}^-} \Xi \Rightarrow \Delta$. By repeated applications of Lemma 6.4.6 (ii), $\vdash_{\mathsf{GT}^-} \Gamma \Rightarrow \Delta$. □

6.5 Weakening, Inversion, Contraction

We mentioned in Section 6.3 that in our setting, right weakening is sound for all formulas, whereas right contraction is only (guaranteed to be) sound for union-closed (and hence classical) formulas. In this section, we further show that (left and right) weakening is height-preserving admissible in GT^-; that inversion for all rules except $\mathsf{R}{\vee\!\!\!\vee}$ is height-preserving admissible; and that left contraction is height-preserving admissible with respect to all formulas, whereas right contraction is only (guaranteed to be) admissible with respect to classical formulas.

Let $\vdash^n_{\mathsf{GT}^-} \Gamma \Rightarrow \Delta$ (or simply $\vdash^n \Gamma \Rightarrow \Delta$) denote the fact that there is a GT^--derivation of $\Gamma \Rightarrow \Delta$ of height at most n.

6.5.1. LEMMA (Height-preserving weakening). *Both left and right weakening are height-preserving admissible:*

- *Left weakening:* $\vdash^n \Gamma \Rightarrow \Delta$ *implies* $\vdash^n \Gamma, \xi \Rightarrow \Delta$.
- *Right weakening:* $\vdash^n \Gamma \Rightarrow \Delta$ *implies* $\vdash^n \Gamma \Rightarrow \xi, \Delta$.

Proof:
By induction on n. For both items, if $n = 1$, the derivation $\mathcal{D}$ witnessing the antecedent of the item consists of a single axiom, and then the consequent of the item follows by the same axiom. For the inductive step, we assume both items hold for n and prove that they hold for $n+1$. Letting R denote the final rule applied in $\vdash^{n+1} \Gamma \Rightarrow \Delta$, if R is not $\mathsf{R}\wedge$ or $\mathsf{L}\vee$, both items follow by applying the induction hypothesis to the (subderivation(s) ending in the) premise(s) of R, and then applying R. If R is $\mathsf{R}\wedge$ or $\mathsf{L}\vee$, left weakening follows by the induction hypothesis and R, as above, and right weakening follows immediately by the implicit weakening in R. For instance, if R is $\mathsf{R}\wedge$, it is of the form:

$$\frac{\Gamma \Rightarrow \varphi, \Lambda \qquad \Gamma \Rightarrow \psi, \Lambda}{\Gamma \Rightarrow \varphi \wedge \psi, \Lambda, \Delta'}\ \mathsf{R}\wedge$$

Left weakening: By the induction hypothesis, $\vdash^n \Gamma, \xi \Rightarrow \varphi, \Lambda$ and $\vdash^n \Gamma, \xi \Rightarrow \psi, \Lambda$, whence by $\mathsf{R}\wedge$, we have $\vdash^{n+1} \Gamma, \xi \Rightarrow \varphi \wedge \psi, \Lambda, \Delta'$.

Right weakening: By $\mathsf{R}\wedge$ applied to the premises, $\vdash^{n+1} \Gamma \Rightarrow \xi, \varphi \wedge \psi, \Lambda, \Delta'$. □

Height-preserving inversion is admissible for all rules except for $\mathsf{R}{\vee\!\!\!\vee}$; $\mathsf{R}{\vee\!\!\!\vee}$ instead has an inversion-like property which corresponds to the Split property (Proposition 6.2.8). The failure of invertibility of $\mathsf{R}{\vee\!\!\!\vee}$ is as desired: the inverted rule is not sound as, for instance, $p \vee\!\!\!\vee \neg p \Rightarrow p \vee\!\!\!\vee \neg p$ is valid whereas $p \vee\!\!\!\vee \neg p \Rightarrow p$ is not. Observe that for $\mathsf{L}\vee$ and $\mathsf{R}\vee$, the inverted rules, unlike the original rules, have no syntactic restrictions and feature no implicit weakening (whereas the inversion-like property for $\mathsf{R}{\vee\!\!\!\vee}$ has a syntactic restriction on the left context).

Note that in the following proof we use the fact that if a formula is of the form $\varphi\{\psi_1\}\{\psi_2\}$, this indicates that neither the subformula occurrence ψ_1 replacing $\cdot_1$ in

$\varphi\{\cdot_1\}\{\cdot_2\}$ nor the occurrence ψ_2 replacing $\cdot_2$ is a suboccurrence of the other in the formula $\varphi\{\psi_1\}\{\psi_2\}$. For instance, for $\varphi_1\{\cdot_1\}\{\cdot_2\} = p \wedge \cdot_1 \wedge \cdot_2$ we have $\varphi_1\{q\}\{r\} = p \wedge q \wedge r$, in which neither q nor r is a suboccurrence of the other. We also use the following convention: if we know that ψ_2 is a suboccurrence of ψ_1 in $\varphi\{\psi_1\}$, we write $\varphi\{\psi_1'\{\psi_2\}\}$, where $\psi_1'\{\cdot\}$ is constructed in the obvious way. For instance, for $\varphi\{\cdot\} = p \wedge \cdot$ and $\psi = q \wedge r$, we have that r is a suboccurrence of $\varphi\{\psi\} = p \wedge q \wedge r$, whence we write $\varphi\{\psi_1'\{r\}\}$, where $\psi_1' = q \wedge \cdot$.

6.5.2. LEMMA (Height-preserving inversion).

(L¬) $\vdash^n \Gamma, \neg\alpha \Rightarrow \Delta$ *implies* $\vdash^n \Gamma \Rightarrow \alpha, \Delta$;

(R¬) $\vdash^n \Gamma \Rightarrow \neg\alpha, \Delta$ *implies* $\vdash^n \Gamma, \alpha \Rightarrow \Delta$;

(L∧) $\vdash^n \Gamma, \varphi \wedge \psi \Rightarrow \Delta$ *implies* $\vdash^n \Gamma, \varphi, \psi \Rightarrow \Delta$;

(R∧) $\vdash^n \Gamma \Rightarrow \varphi \wedge \psi, \Delta$ *implies* $\vdash^n \Gamma \Rightarrow \varphi, \Delta$ *and* $\vdash^n \Gamma \Rightarrow \psi, \Delta$;

(L∨) $\vdash^n \Gamma, \varphi \vee \psi \Rightarrow \Delta$ *implies* $\vdash^n \Gamma, \varphi \Rightarrow \Delta$ *and* $\vdash^n \Gamma, \psi \Rightarrow \Delta$;

(R∨) $\vdash^n \Gamma \Rightarrow \varphi \vee \psi, \Delta$ *implies* $\vdash^n \Gamma \Rightarrow \varphi, \psi, \Delta$;

(L⩔) $\vdash^n \Gamma, \chi\{\varphi_L \mathbin{⩔} \varphi_R\} \Rightarrow \Delta$ *implies* $\vdash^n \Gamma, \chi\{\varphi_L\} \Rightarrow \Delta$ *and* $\vdash^n \Gamma, \chi\{\varphi_R\} \Rightarrow \Delta$;

(R⩔) $\vdash^n \Xi \Rightarrow \chi\{\varphi_L \mathbin{⩔} \varphi_R\}, \Delta$ *implies* $\vdash^n \Xi \Rightarrow \chi\{\varphi_L\}, \Delta$ *or* $\vdash^n \Xi \Rightarrow \chi\{\varphi_R\}, \Delta$.

Proof:
Each item is proved by induction on n. For each item, if $n = 1$, the derivation $\mathcal{D}$ witnessing the antecedent of the item consists of a single axiom. In all cases, the consequent then follows by the same axiom. We now assume that each item holds for n and prove that it holds for $n+1$. We separate subcases for each item as follows (note that not all subcases exist for each item):

Case 1: The active formula of the inversion (e.g., the formula $\neg\alpha$ in the item L¬) is not principal in the final rule R applied in $\mathcal{D}$.

Case 1.1: The active formula of the inversion is in the left/right context of each premise of R. In this case, the result follows by the induction hypothesis applied to the (subderivation(s) ending in the) premise(s) of R followed by application(s) of R. For instance, for the item R∧, if R is also R∧, it is, in this case, of the form:

$$\frac{\Gamma \Rightarrow \eta, \varphi \wedge \psi, \Lambda' \qquad \Gamma \Rightarrow \xi, \varphi \wedge \psi, \Lambda'}{\Gamma \Rightarrow \eta \wedge \xi, \varphi \wedge \psi, \Lambda', \Delta'}\ \mathsf{R}\wedge,$$

where $\varphi \wedge \psi, \Lambda' = \Lambda$ is the right context of each premise of R. Applying the induction hypothesis to the (subderivations ending in the) premises yields $\vdash^n \Gamma \Rightarrow \eta, \varphi, \Lambda'$; $\vdash^n \Gamma \Rightarrow \eta, \psi, \Lambda'$; $\vdash^n \Gamma \Rightarrow \xi, \varphi, \Lambda'$; and $\vdash^n \Gamma \Rightarrow \xi, \psi, \Lambda'$. Then two applications of R∧ yield $\vdash^{n+1} \Gamma \Rightarrow \eta \wedge \xi, \varphi, \Lambda', \Delta'$ and $\vdash^{n+1} \Gamma \Rightarrow \eta \wedge \xi, \psi, \Lambda', \Delta'$.

Case 1.2: The active formula of the inversion is introduced by the implicit weakening in R. In this case, the right context of the conclusion of R is of the form $\Lambda, \varphi_a, \Delta'$, where φ_a is the active formula of the inversion, and the right context of each premise is Λ. The result follows by an application of R to the premises with a conclusion right context of the form $\Lambda, \varphi_p, \Delta'$, where φ_p is a principal formula of the inversion. For instance, for the item $\mathsf{R}\wedge$, if R is also $\mathsf{R}\wedge$, it is, in this case, of the form:

$$\dfrac{\Gamma \Rightarrow \eta, \Lambda \qquad \Gamma \Rightarrow \xi, \Lambda}{\Gamma \Rightarrow \eta \wedge \xi, \Lambda, \varphi \wedge \psi, \Delta'}\ \mathsf{R}\wedge$$

By applying $\mathsf{R}\wedge$ to the premises, we get $\vdash^{n+1} \Gamma \Rightarrow \eta \wedge \xi, \Lambda, \varphi, \Delta'$ and $\vdash^{n+1} \Gamma \Rightarrow \eta \wedge \xi, \Lambda, \psi, \Delta'$.

Case 2: The active formula of the inversion is principal in R.

Case 2.1: The active subformula of the inversion (e.g., the formula $\neg\alpha$ in the item $\mathsf{L}\neg$, and the formula $\varphi_L \mathbin{\mathbb{V}} \varphi_R$ in $\mathsf{L}\mathbb{V}$) is the active subformula of R. In this case, the (subderivation(s) ending in the) premise(s) of R already yield the desired result (if R is $\mathsf{R}\wedge$ or $\mathsf{L}\vee$, right weakening must also be applied to the premises). For instance, for the item $\mathsf{R}\wedge$, R must in this case also be $\mathsf{R}\wedge$, and it is of the form:

$$\dfrac{\Gamma \Rightarrow \varphi, \Lambda \qquad \Gamma \Rightarrow \psi, \Lambda}{\Gamma \Rightarrow \varphi \wedge \psi, \Lambda, \Delta'}\ \mathsf{R}\wedge$$

We get $\vdash^{n} \Gamma \Rightarrow \varphi, \Lambda, \Delta'$ and $\vdash^{n} \Gamma \Rightarrow \psi, \Lambda, \Delta'$ (whence also $\vdash^{n+1} \Gamma \Rightarrow \varphi, \Lambda, \Delta'$ and $\vdash^{n+1} \Gamma \Rightarrow \psi, \Lambda, \Delta'$) by applying height-preserving right weakening (Lemma 6.5.1) to the premises.

Case 2.2: The active subformula of the inversion is not the active subformula of R. In this case, the result follows by the induction hypothesis applied to the premises of R followed by an application (or multiple applications) of R. For instance, for the item $\mathsf{L}\vee$, R must in this case be $\mathsf{L}\mathbb{V}$, and we may assume without loss of generality that it is of the form:

$$\dfrac{\Gamma, \varphi\{\varphi_L\} \vee \psi \Rightarrow \Delta \qquad \Gamma, \varphi\{\varphi_R\} \vee \psi \Rightarrow \Delta}{\Gamma, \varphi\{\varphi_L \mathbin{\mathbb{V}} \varphi_R\} \vee \psi \Rightarrow \Delta}\ \mathsf{L}\mathbb{V}$$

Applying the induction hypothesis to the premises yields $\vdash^{n} \Gamma, \varphi\{\varphi_L\} \Rightarrow \Delta$; $\vdash^{n} \Gamma, \psi \Rightarrow \Delta$; and $\vdash^{n} \Gamma, \varphi\{\varphi_R\} \Rightarrow \Delta$. Then we already have $\vdash^{n+1} \Gamma, \psi \Rightarrow \Delta$, and an application of $\mathsf{L}\mathbb{V}$ yields $\vdash^{n+1} \Gamma, \varphi\{\varphi_L \mathbin{\mathbb{V}} \varphi_R\} \Rightarrow \Delta$.

For the items $\mathsf{L}\mathbb{V}$ and $\mathsf{R}\mathbb{V}$, the subcases of Case 2.2 in which R is one of the deep-inference rules follow essentially as above, but this is less easy to see due to the complicated structure of the formulas involved, and there are some minor additional complications. We now provide some further details on these subcases. In each of these subcases, the principal formula of both the inversion and R is $\chi\{\varphi_L \mathbin{\mathbb{V}} \varphi_R\}$, and the principal subformula $\varphi_L \mathbin{\mathbb{V}} \varphi_R$ of the inversion is not the principal subformula $\xi_L \mathbin{\mathbb{V}} \xi_R$ of R. The formula $\chi\{\varphi_L \mathbin{\mathbb{V}} \varphi_R\}$ is then of one of the following forms: (i) $\chi\{\varphi_L \mathbin{\mathbb{V}} \varphi_R\}\{\xi_L \mathbin{\mathbb{V}} \xi_R\}$; (ii) $\chi\{(\varphi_L \mathbin{\mathbb{V}} \varphi_R)'\{\xi_L \mathbin{\mathbb{V}} \xi_R\}\}$; or (iii) $\eta\{(\xi_L \mathbin{\mathbb{V}} \xi_R)'\{\varphi_L \mathbin{\mathbb{V}} \varphi_R\}\}$.

Case 2.2 of item $\mathsf{L}\mathbb{V}$; R is $\mathsf{L}\mathbb{V}$. In all of the subcases (i), (ii), (iii), the result follows essentially, as above, by applying the induction hypothesis to the premises and then applying *R*. We provide the details for (iii). We may assume without loss of generality that $\eta\{(\xi_L \vee\!\!\vee \xi_R)'\{\varphi_L \vee\!\!\vee \varphi_R\}\} = \eta\{\xi_L'\{\varphi_L \vee\!\!\vee \varphi_R\} \vee\!\!\vee \xi_R\}$. Then *R* is of the form:

$$\frac{\Gamma, \eta\{\xi_L'\{\varphi_L \vee\!\!\vee \varphi_R\}\} \Rightarrow \Delta \qquad \Gamma, \eta\{\xi_R\} \Rightarrow \Delta}{\Gamma, \eta\{\xi_L'\{\varphi_L \vee\!\!\vee \varphi_R\} \vee\!\!\vee \xi_R\} \Rightarrow \Delta}\ \mathsf{L}\vee\!\!\vee$$

By the induction hypothesis applied to the left premise, $\vdash^n \Gamma, \eta\{\xi_L'\{\varphi_L\}\} \Rightarrow \Delta$ and $\vdash^n \Gamma, \eta\{\xi_L'\{\varphi_R\}\} \Rightarrow \Delta$, and from the right premise we have $\vdash^n \Gamma, \eta\{\xi_R\} \Rightarrow \Delta$. Then by $\mathsf{L}\vee\!\!\vee$, we have $\vdash^{n+1} \Gamma, \eta\{\xi_L'\{\varphi_L\} \vee\!\!\vee \xi_R\} \Rightarrow \Delta$ (i.e., $\vdash^{n+1} \Gamma, \chi\{\varphi_L\} \Rightarrow \Delta$) and $\vdash^{n+1} \Gamma, \eta\{\xi_L'\{\varphi_R\} \vee\!\!\vee \xi_R\} \Rightarrow \Delta$ (i.e., $\vdash^{n+1} \Gamma, \chi\{\varphi_R\} \Rightarrow \Delta$).

Case 2.2 of item $\mathsf{R}\vee\!\!\vee$; *R* is $\mathsf{R}\vee\!\!\vee$. In all of the subcases (i), (ii), (iii), the result follows essentially, as above, by applying the induction hypothesis to the premises and then applying *R*. We provide the details for (iii). We may assume without loss of generality that $\eta\{(\xi_L \vee\!\!\vee \xi_R)'\{\varphi_L \vee\!\!\vee \varphi_R\}\} = \eta\{\xi_L'\{\varphi_L \vee\!\!\vee \varphi_R\} \vee\!\!\vee \xi_R\}$. There are two subcases. In the first, *R* is of the form:

$$\frac{\Xi \Rightarrow \eta\{\xi_L'\{\varphi_L \vee\!\!\vee \varphi_R\}\}, \Delta}{\Xi \Rightarrow \eta\{\xi_L'\{\varphi_L \vee\!\!\vee \varphi_R\} \vee\!\!\vee \xi_R\}, \Delta}\ \mathsf{R}\vee\!\!\vee$$

By the induction hypothesis applied to the premise, we have $\vdash^n \Xi \Rightarrow \eta\{\xi_L'\{\varphi_L\}\}, \Delta$ or $\vdash^n \Xi \Rightarrow \eta\{\xi_L'\{\varphi_R\}\}, \Delta$. If the former, then $\vdash^{n+1} \Xi \Rightarrow \eta\{\xi_L'\{\varphi_L\} \vee\!\!\vee \xi_R\}, \Delta$ (i.e., $\vdash^{n+1} \Xi \Rightarrow \chi\{\varphi_L\}, \Delta$) by $\mathsf{R}\vee\!\!\vee$; if the latter, then $\vdash^{n+1} \Xi \Rightarrow \eta\{\xi_L'\{\varphi_R\} \vee\!\!\vee \xi_R\}, \Delta$ (i.e., $\vdash^{n+1} \Xi \Rightarrow \chi\{\varphi_R\}, \Delta$) by $\mathsf{R}\vee\!\!\vee$.

In the second subcase, *R* is of the form

$$\frac{\Xi \Rightarrow \eta\{\xi_R\}, \Delta}{\Xi \Rightarrow \eta\{\xi_L'\{\varphi_L \vee\!\!\vee \varphi_R\} \vee\!\!\vee \xi_R\}, \Delta}\ \mathsf{R}\vee\!\!\vee$$

Then by $\mathsf{R}\vee\!\!\vee$ applied to the premise, $\vdash^{n+1} \Xi \Rightarrow \eta\{\xi_L'\{\varphi_L\} \vee\!\!\vee \xi_R\}, \Delta$ (i.e., $\vdash^{n+1} \Xi \Rightarrow \chi\{\varphi_L\}, \Delta$). □

We have concluded the proof, but it is worth making more explicit why the induction goes through for item $\mathsf{R}\vee\!\!\vee$ in the cases in which *R* has two premises. Note first that *R* cannot be $\mathsf{L}\vee\!\!\vee$ since the left context of the conclusion of *R* must be classical (in order to serve as the antecedent for item $\mathsf{R}\vee\!\!\vee$). So *R* must be $\mathsf{R}\wedge$ or $\mathsf{L}\vee$. If the principal formula $\chi\{\varphi_L \vee\!\!\vee \varphi_R\}$ of the inversion is a side formula of *R*, it must then have been introduced via the implicit weakening in *R*, and we proceed as in Case 1.2 (instead of introducing $\chi\{\varphi_L \vee\!\!\vee \varphi_R\}$, we introduce $\chi\{\varphi_L\}$ or $\chi\{\varphi_R\}$). If, on the other hand, $\chi\{\varphi_L \vee\!\!\vee \varphi_R\}$ is an active formula of *R*, *R* must be $\mathsf{R}\wedge$, and we may assume without loss of generality that $\chi\{\varphi_L \vee\!\!\vee \varphi_R\} = \chi_1\{\varphi_L \vee\!\!\vee \varphi_R\} \wedge \chi_2$ and that *R* is of the form:

$$\frac{\Xi \Rightarrow \chi_1\{\varphi_L \vee\!\!\vee \varphi_R\}, \Lambda \qquad \Xi \Rightarrow \chi_2, \Lambda}{\Xi \Rightarrow \chi_1\{\varphi_L \vee\!\!\vee \varphi_R\} \wedge \chi_2, \Lambda, \Delta'}\ \mathsf{R}\wedge,$$

We proceed essentially as in Case 2.2, by first applying the induction hypothesis to the left premise and then applying *R*.

6.5.3. LEMMA (Height-preserving contraction). *Left contraction is height-preserving admissible, and right contraction is height-preserving admissible with respect to classical formulas:*

- *Left contraction:* $\vdash^n \Gamma, \xi, \xi \Rightarrow \Delta$ *implies* $\vdash^n \Gamma, \xi \Rightarrow \Delta$.
- *Right contraction:* $\vdash^n \Gamma \Rightarrow \alpha, \alpha, \Delta$ *implies* $\vdash^n \Gamma \Rightarrow \alpha, \Delta$.

Proof:
By simultaneous induction on n. For both items, if $n = 1$, the derivation $\mathcal{D}$ witnessing the antecedent of the item consists of a single axiom, and then the consequent of the item follows by the same axiom. We now assume that both items hold for n and prove that they hold for $n+1$. Let R denote the final rule applied in $\mathcal{D}$. We separate subcases for each item as follows (note that not all subcases exist for each item):

Case 1: The formula (occurrence) to be contracted is not the principal formula of R. In this case, the result follows by the induction hypothesis applied to the (subderivation(s) ending in the) premise(s) of R followed by an application of R. For instance, for right contraction, if R is $\mathsf{L}\neg$, it is of the form:

$$\dfrac{\Gamma' \Rightarrow \beta, \alpha, \alpha, \Delta}{\Gamma', \neg\beta \Rightarrow \alpha, \alpha, \Delta}\ \mathsf{L}\neg$$

By the induction hypothesis applied to the premise followed by an application of $\mathsf{L}\neg$, we have $\vdash^{n+1} \Gamma', \neg\beta \Rightarrow \alpha, \Delta$.

For right contraction, when R is $\mathsf{L}\vee$ or $\mathsf{R}\wedge$, Case 1 has two subcases.

Case 1.1: Both of the formula occurrences α to be contracted are in the right context of each premise of R. The result follows as in Case 1. For instance, if R is $\mathsf{R}\wedge$, it is of the form:

$$\dfrac{\Gamma \Rightarrow \alpha, \alpha, \varphi, \Lambda \qquad \Gamma \Rightarrow \alpha, \alpha, \psi, \Lambda}{\Gamma \Rightarrow \alpha, \alpha, \varphi \wedge \psi, \Lambda, \Delta'}\ \mathsf{R}\wedge$$

By the induction hypothesis applied to the premises followed by an application of $\mathsf{R}\wedge$, we have $\vdash^{n+1} \Gamma \Rightarrow \alpha, \varphi \wedge \psi, \Lambda, \Delta'$.

Case 1.2: At least one of the formula occurrences α to be contracted is introduced by the implicit weakening in R. In this case, we replace the application of R by another application in which one fewer occurrence is introduced. For instance, if R is $\mathsf{R}\wedge$, it is of one of the following forms:

$$\dfrac{\Gamma \Rightarrow \varphi, \Lambda \qquad \Gamma \Rightarrow \psi, \Lambda}{\Gamma \Rightarrow \alpha, \alpha, \varphi \wedge \psi, \Lambda, \Delta'}\ \mathsf{R}\wedge \qquad \dfrac{\Gamma \Rightarrow \alpha, \varphi, \Lambda \qquad \Gamma \Rightarrow \alpha, \psi, \Lambda}{\Gamma \Rightarrow \alpha, \alpha, \varphi \wedge \psi, \Lambda, \Delta'}\ \mathsf{R}\wedge$$

In either case, by $\mathsf{R}\wedge$ applied to the premises, $\vdash^{n+1} \Gamma \Rightarrow \alpha, \varphi \wedge \psi, \Lambda, \Delta'$.

Case 2: The formula occurrence to be contracted is the principal formula of R. In this case, the result follows by application(s) of the inversion lemma to the (subderivations ending in the) premise(s), followed by application(s) of the induction hypothesis (for one of the two items), followed by an application of R. We give three examples.

For left contraction, if R is $\mathsf{L}\neg$, we have that $\xi = \neg\beta$ and R is of the form:

$$\frac{\Gamma, \neg\beta \Rightarrow \beta, \Delta}{\Gamma, \neg\beta, \neg\beta \Rightarrow \Delta}\ \mathsf{L}\neg$$

By the $\mathsf{L}\neg$-case of Lemma 6.5.2 applied to the premise, $\vdash^n \Gamma \Rightarrow \beta, \beta, \Delta$. Then by the induction hypothesis (for right contraction), we have $\vdash^n \Gamma \Rightarrow \beta, \Delta$, whence by $\mathsf{L}\neg$, we have $\vdash^{n+1} \Gamma, \neg\beta \Rightarrow \Delta$.

For right contraction, if R is $\mathsf{R}\vee$, we have that $\alpha = \beta_1 \vee \beta_2$ and R is of the form:

$$\frac{\Gamma \Rightarrow \beta_1, \beta_2, \beta_1 \vee \beta_2, \Delta}{\Gamma \Rightarrow \beta_1 \vee \beta_2, \beta_1 \vee \beta_2, \Delta}\ \mathsf{R}\vee$$

By the $\mathsf{R}\vee$-case of Lemma 6.5.2 applied to the premise, $\vdash^n \Gamma \Rightarrow \beta_1, \beta_2, \beta_1, \beta_2, \Delta$. Then by the induction hypothesis, we obtain first $\vdash^n \Gamma \Rightarrow \beta_1, \beta_1, \beta_2, \Delta$ and then $\vdash^n \Gamma \Rightarrow \beta_1, \beta_2, \Delta$, whence by $\mathsf{R}\vee$, we have $\vdash^{n+1} \Gamma \Rightarrow \beta_1 \vee \beta_2, \Delta$.

For left contraction, if R is $\mathsf{L}\mathbb{V}$, we have that $\xi = \chi\{\varphi_L \mathbb{V} \varphi_R\}$ and R is of the form:

$$\frac{\Gamma, \chi\{\varphi_L \mathbb{V} \varphi_R\}, \chi\{\varphi_L\} \Rightarrow \Delta \qquad \Gamma, \chi\{\varphi_L \mathbb{V} \varphi_R\}, \chi\{\varphi_R\} \Rightarrow \Delta}{\Gamma, \chi\{\varphi_L \mathbb{V} \varphi_R\}, \chi\{\varphi_L \mathbb{V} \varphi_R\} \Rightarrow \Delta}\ \mathsf{L}\mathbb{V}$$

By the $\mathsf{L}\mathbb{V}$-case of Lemma 6.5.2 applied to the premises, $\vdash^n \Gamma, \chi\{\varphi_L\}, \chi\{\varphi_L\} \Rightarrow \Delta$ and $\vdash^n \Gamma \Rightarrow \chi\{\varphi_R\}, \chi\{\varphi_R\} \Rightarrow \Delta$. Then by the induction hypothesis, $\vdash^n \Gamma, \chi\{\varphi_L\} \Rightarrow \Delta$ and $\vdash^n \Gamma, \chi\{\varphi_R\} \Rightarrow \Delta$, whence by $\mathsf{L}\mathbb{V}$, we have $\vdash^{n+1} \Gamma, \chi\{\varphi_L \mathbb{V} \varphi_R\} \Rightarrow \Delta$.

Note that for right contraction, if R is $\mathsf{R}\mathbb{V}$, there is no Case 2 since the principal formula of R is not classical and hence cannot be the target formula occurrence of the contraction. □

6.6 Countermodel Construction

Due to the (semantic) invertibility of the rules, together with the Split property (Proposition 6.2.8), there is a simple way of proving the cutfree completeness of GT that is similar to one for G3cp [118, pp. 104–105]: roughly, given a sequent S, one applies inverted versions of the rules until one arrives at *atomic sequents*—sequents consisting solely of propositional variables and $\bot$. Given semantic invertibility and Split, if S is valid, so are these atomic sequents, whence they are axioms; one will then have constructed a derivation of S consisting of these axioms and the non-inverted versions of the inverted rules applied in the process. Conversely, if S is invalid, some of these atomic sequents must also be invalid, and these invalid atomic sequents then enable one to construct a countermodel for S.

6.6.1. LEMMA.

- *There is an effective procedure for constructing a countermodel to the conclusion of any rule apart from* $\mathsf{R}\wedge$, $\mathsf{L}\vee$ *and* $\mathsf{R}\mathbb{V}$ *from a countermodel to any premise of that rule.*

- R$\wedge$*: There is an effective procedure for constructing a countermodel to* $\Gamma \Rightarrow \varphi \wedge \psi, \Lambda$ *from a countermodel to either* $\Gamma \Rightarrow \varphi, \Lambda$ *or* $\Gamma \Rightarrow \psi, \Lambda$.
- L$\vee$*: There is an effective procedure for constructing a countermodel to* $\Gamma, \varphi \vee \psi \Rightarrow \Lambda$ *from a countermodel to either* $\Gamma, \varphi \Rightarrow \Lambda$ *or* $\Gamma, \psi \Rightarrow \Lambda$.
- R$\mathbin{\vee\!\!\!\vee}$*: There is an effective procedure for constructing a countermodel to* $\Xi \Rightarrow \chi\{\psi_L \mathbin{\vee\!\!\!\vee} \psi_R\}$ *given countermodels to both* $\Xi \Rightarrow \chi\{\psi_L\}, \Delta$ *and* $\Xi \Rightarrow \chi\{\psi_R\}, \Delta$.

Proof:

- L$\neg$: Let $t \vDash \bigwedge\Gamma$ and $t \nvDash \alpha \vee \bigvee\Delta$. Let $t_{\neg\alpha} := \{v \in t \mid v \vDash \neg\alpha\}$; this is our countermodel. By downward closure, $t_{\neg\alpha} \vDash \bigwedge\Gamma \wedge \neg\alpha$. And if $t_{\neg\alpha} \vDash \bigvee\Delta$, then $t \vDash \alpha \vee \bigvee\Delta$, a contradiction; therefore $t_{\neg\alpha} \nvDash \bigvee\Delta$.
- R$\neg$: Let $t \vDash \bigwedge\Gamma \wedge \alpha$ and $t \nvDash \bigvee\Delta$. t is our countermodel. Clearly $t \vDash \bigwedge\Gamma$. Assume for contradiction that $t \vDash \neg\alpha \vee \bigvee\Delta$. Then $t = t_{\neg\alpha} \cup t_\Delta$ where $t_{\neg\alpha} \vDash \neg\alpha$ and $t_\Delta \vDash \bigvee\Delta$. But then by the flatness of α and $\neg\alpha$, we have that for each $v \in t_{\neg\alpha}$, $v \vDash \alpha \wedge \neg\alpha$; therefore $t_{\neg\alpha} = \varnothing$, whence $t = t_\Delta \vDash \bigvee\Delta$, a contradiction. Therefore, $t \nvDash \neg\alpha \vee \bigvee\Delta$.
- L$\wedge$: Immediate from the countermodel to the premise.
- R$\wedge$: Assume $t \vDash \bigwedge\Gamma$ and $t \nvDash \varphi \vee \bigvee\Lambda$. t is our countermodel: clearly $t \nvDash (\varphi \wedge \psi) \vee \bigvee\Lambda$. Similarly for the other premise.
- L$\vee$: Assume $t \vDash \bigwedge\Gamma \wedge \varphi$ and $t \nvDash \bigvee\Lambda$. t is our countermodel: by the empty team property, $t \vDash \bigwedge\Gamma \wedge (\varphi \vee \psi)$. Similarly for the other premise.
- R$\vee$: Immediate from the countermodel to the premise.
- L$\mathbin{\vee\!\!\!\vee}$: Assume $t \vDash \bigwedge\Gamma \wedge \chi\{\varphi_L\}$ and $t \nvDash \bigvee\Delta$. t is our countermodel: clearly $t \vDash \bigwedge\Gamma \wedge \chi\{\varphi_L \mathbin{\vee\!\!\!\vee} \varphi_R\}$. Similarly for the other premise.
- R$\mathbin{\vee\!\!\!\vee}$: Assume $t_1 \vDash \bigwedge\Xi$ and $t_1 \nvDash \chi\{\varphi_L\}, \Delta$ and $t_2 \vDash \bigwedge\Xi$ and $t_2 \nvDash \chi\{\varphi_R\}, \Delta$. $t_1 \cup t_2$ is our countermodel. By the union closure of $\bigwedge\Xi$, $t_1 \cup t_2 \vDash \bigwedge\Xi$. By downward closure, $t_1 \cup t_2 \nvDash \chi\{\varphi_L\}, \Delta$ and $t_1 \cup t_2 \nvDash \chi\{\varphi_R\}, \Delta$ so $t_1 \cup t_2 \nvDash \chi\{\varphi_L \mathbin{\vee\!\!\!\vee} \varphi_R\}, \Delta$. $\square$

We introduce some metalanguage notation for convenience. We use $\mathbin{\wedge\!\!\!\wedge}$ as a metalanguage 'and', and $\mathbin{\vee\!\!\!\vee}$ as a metalanguage 'or'. We call a collection of sequents joined together by the metalanguage connectives a *Boolean combinations of sequents*. For instance, $\Gamma \Rightarrow \varphi, \Lambda \mathbin{\wedge\!\!\!\wedge} \Gamma \Rightarrow \psi, \Lambda$ is a Boolean combination of sequents. We will use these combinations to express derivability and semantic invertibility facts in the following manner:

R$\wedge$ $(\Gamma \Rightarrow \varphi, \Lambda \mathbin{\wedge\!\!\!\wedge} \Gamma \Rightarrow \psi, \Lambda) \dashv\vdash_{\mathsf{R}\wedge} \Gamma \Rightarrow \varphi \wedge \psi, \Lambda$;

R$\Vee$ $(\Xi \Rightarrow \chi\{\varphi_L\},\Delta \mathrel{\vee\!\!\vee} \Xi \Rightarrow \chi\{\varphi_R\},\Delta) \dashv\vdash_{\mathsf{R}\Vee} \Xi \Rightarrow \chi\{\varphi_L \Vee \varphi_R\},\Delta$.

The left-to-right direction of the first line above reformulates the rule R$\wedge$ (without implicit weakening): given both of the sequents $\Gamma \Rightarrow \varphi,\Lambda$ and $\Gamma \Rightarrow \psi,\Lambda$, one can derive the sequent $\Gamma \Rightarrow \varphi \wedge \psi,\Lambda$ using only R$\wedge$. The right-to-left direction expresses the semantic invertibility of R$\wedge$ (restricted to classical right contexts): if $\Gamma \Rightarrow \varphi \wedge \psi,\Lambda$ is valid, then both $\Gamma \Rightarrow \varphi,\Lambda$ and $\Gamma \Rightarrow \psi,\Lambda$ are valid. Similarly, the second line expresses that one can derive $\Xi \Rightarrow \chi\{\varphi_L \Vee \varphi_R\},\Delta$ via R$\Vee$ using either $\Xi \Rightarrow \chi\{\varphi_L\},\Delta$ or $\Xi \Rightarrow \chi\{\varphi_R\},\Delta$, and that if $\Xi \Rightarrow \chi\{\varphi_L \Vee \varphi_R\},\Delta$ is valid, so is either $\Xi \Rightarrow \chi\{\varphi_L\},\Delta$ or $\Xi \Rightarrow \chi\{\varphi_R\},\Delta$.

We say that a Boolean combination $\mathcal{S}$ of sequents is *valid* if, replacing each valid sequent in $\mathcal{S}$ with 1 and each invalid sequent with 0, the Boolean function determined by the metalanguage connectives in $\mathcal{S}$ outputs 1. We write $\mathcal{S}_1 \vDash \mathcal{S}_2$ to denote the fact that the validity of $\mathcal{S}_1$ implies the validity of $\mathcal{S}_2$. We say that a subcombination of $\mathcal{S}$ is a *witnessing combination* for $\mathcal{S}$ if the validity of the subcombination entails the validity of $\mathcal{S}$ (e.g., $\mathcal{S}_1$ is a witnessing combination for $\mathcal{S}_1 \mathrel{\vee\!\!\vee} \mathcal{S}_2$). We write $\mathcal{S}_1 \vdash_{\mathsf{C}} \mathcal{S}_2$ to denote the fact that there is some witnessing combination for $\mathcal{S}_1$ such that given derivation(s) in GT^- for the sequents in $\mathcal{S}_1$, one can extend these derivations using rules in C to construct derivation(s) of a witnessing combination for $\mathcal{S}_2$. We write $\mathsf{S}_1 \,{}_{\mathsf{C}_2}\!\dashv\vdash_{\mathsf{C}_1} \mathcal{S}_2$ if $\mathcal{S}_1 \vdash_{\mathsf{C}_1} \mathcal{S}_2$ and $\mathcal{S}_2 \vdash_{\mathsf{C}_2} \mathcal{S}_1$, and $\mathcal{S}_1 \dashv\vdash_{\mathsf{C}} \mathcal{S}_2$ if $\mathcal{S}_1 \vdash_{\mathsf{C}_1} \mathcal{S}_2$ and $\mathcal{S}_2 \vDash \mathcal{S}_1$. We write simply $\vdash$ for $\vdash_{\mathsf{GT}^-}$. We write $\vdash \mathcal{S}$ if there are derivations for the sequents in a witnessing combination for $\mathcal{S}$ (from axioms) in GT^-. We say that a derivation $\mathcal{D}$ *witnesses* $\vdash_{\mathsf{C}_1} \mathcal{S}_1 \vdash_{\mathsf{C}_2} \mathcal{S}_2 \vdash_{\mathsf{C}_3} \ldots \vdash_{\mathsf{C}_n} \mathcal{S}_n$ if $\mathcal{D}$ consists of derivations of the sequents in a witnessing combination for $\mathcal{S}_1$ using rules in C_1, extensions of these derivations using rules in C_2 such that the combined derivations constitute derivations of the sequents in a witnessing combination for $\mathcal{S}_2$, and so on ($\mathcal{D}$ *witnessing* $\mathcal{S}_1 \vdash_{\mathsf{C}_2} \mathcal{S}_2 \vdash_{\mathsf{C}_3} \ldots \vdash_{\mathsf{C}_n} \mathcal{S}_n$ is defined in a similar way).

The following list of results (given, for instance, by the rules, by Lemma 6.5.2, and by soundness) summarizes the facts we make use of in our proof of the cutfree completeness/countermodel construction theorem (Theorem 6.6.2):

L$\neg$ $\Gamma \Rightarrow \alpha,\Delta \dashv\vdash_{\mathsf{L}\neg} \Gamma, \neg\alpha \Rightarrow \Delta$;

R$\neg$ $\Gamma,\alpha \Rightarrow \Delta \dashv\vdash_{\mathsf{R}\neg} \Gamma \Rightarrow \neg\alpha,\Delta$;

L$\wedge$ $\Gamma,\varphi,\psi \Rightarrow \Delta \dashv\vdash_{\mathsf{L}\wedge} \Gamma,\varphi \wedge \psi \Rightarrow \Delta$;

R$\wedge$ $(\Gamma \Rightarrow \varphi,\Lambda \mathrel{\wedge\!\!\wedge} \Gamma \Rightarrow \psi,\Lambda) \dashv\vdash_{\mathsf{R}\wedge} \Gamma \Rightarrow \varphi \wedge \psi,\Lambda$;

L$\vee$ $(\Gamma,\varphi \Rightarrow \Lambda \mathrel{\wedge\!\!\wedge} \Gamma,\psi \Rightarrow \Lambda) \dashv\vdash_{\mathsf{L}\vee} \Gamma,\varphi \vee \psi \Rightarrow \Lambda$;

R$\vee$ $\Gamma \Rightarrow \varphi,\psi,\Delta \dashv\vdash_{\mathsf{R}\vee} \Gamma \Rightarrow \varphi \vee \psi,\Delta$;

L$\Vee$ $(\Gamma,\chi\{\varphi_L\} \Rightarrow \Delta \mathrel{\wedge\!\!\wedge} \Gamma,\chi\{\varphi_L\} \Rightarrow \Delta) \dashv\vdash_{\mathsf{L}\Vee} \Gamma,\chi\{\varphi_L \Vee \varphi_R\} \Rightarrow \Delta$;

R$\Vee$ $(\Xi \Rightarrow \chi\{\varphi_L\},\Delta \mathrel{\vee\!\!\vee} \Xi \Rightarrow \chi\{\varphi_R\},\Delta) \dashv\vdash_{\mathsf{R}\Vee} \Xi \Rightarrow \chi\{\varphi_L \Vee \varphi_R\},\Delta$.

6.6.2. THEOREM (Cutfree completeness and countermodel construction). *There is an effective procedure that, given a sequent $\Gamma \Rightarrow \Delta$, yields a cutfree derivation of $\Gamma \Rightarrow \Delta$ if $\Gamma \Rightarrow \Delta$ is valid, and yields a countermodel to $\Gamma \Rightarrow \Delta$ if $\Gamma \Rightarrow \Delta$ is not valid.*

Proof:
By applying $\mathsf{L}\varovee$ in reverse, and by the semantic invertibility of $\mathsf{L}\varovee$, we can find $\Xi_1; \ldots; \Xi_n$ such that

$$\bigwedge_{1\leq i\leq n} (\Xi_i \Rightarrow \Delta) \dashv\vdash_{\mathsf{L}\varovee} \Gamma \Rightarrow \Delta.$$

By applying $\mathsf{R}\varovee$ in reverse, and by the Split property (Proposition 6.2.8), we can find $\Lambda_{11}; \ldots; \Lambda_{nm_n}$ such that

$$\bigwedge_{1\leq i\leq n} \bigvee_{1\leq j\leq m_i} (\Xi_i \Rightarrow \Lambda_{ij}) \dashv\vdash_{\mathsf{R}\varovee} \bigwedge_{1\leq i\leq n} (\Xi_i \Rightarrow \Delta).$$

Finally, by applying rules in $\mathsf{G3cp}^-$ in reverse, and by the semantic invertibility of $\mathsf{G3cp}^-$, we can find atomic sequents $\Xi_{111} \Rightarrow \Lambda_{111}; \ldots; \Xi_{nm_nq_{nm_n}} \Rightarrow \Lambda_{nm_nq_{nm_n}}$ such that

$$\bigwedge_{1\leq i\leq n} \bigvee_{1\leq j\leq m_i} \bigwedge_{1\leq k\leq q_{ij}} (\Xi_{ijk} \Rightarrow \Lambda_{ijk}) \dashv\vdash_{\mathsf{G3cp}^-} \bigwedge_{1\leq i\leq n} \bigvee_{1\leq j\leq m_i} (\Xi_i \Rightarrow \Lambda_{ij}).$$

Therefore,

$$\bigwedge_{1\leq i\leq n} \bigvee_{1\leq j\leq m_i} \bigwedge_{1\leq k\leq q_{ij}} (\Xi_{ijk} \Rightarrow \Lambda_{ijk}) \dashv\vdash \Gamma \Rightarrow \Delta.$$

Then if $\Gamma \Rightarrow \Delta$ is valid, so is

$$\bigwedge_{1\leq i\leq n} \bigvee_{1\leq j\leq m_i} \bigwedge_{1\leq k\leq q_{ij}} (\Xi_{ijk} \Rightarrow \Lambda_{ijk}),$$

whence we must have

$$\vdash \bigwedge_{1\leq i\leq n} \bigvee_{1\leq j\leq m_i} \bigwedge_{1\leq k\leq q_{ij}} (\Xi_{ijk} \Rightarrow \Lambda_{ijk})$$

by the axioms, whence $\vdash \Gamma \Rightarrow \Delta$. Conversely, if $\Gamma \Rightarrow \Delta$ is invalid, then by the soundness of GT^-, we also have that

$$\bigwedge_{1\leq i\leq n} \bigvee_{1\leq j\leq m_i} \bigwedge_{1\leq k\leq q_{ij}} (\Xi_{ijk} \Rightarrow \Lambda_{ijk})$$

is invalid. Since the sequents in this combination are atomic, it is clear how to find countermodels for the invalid sequents in this combination; Lemma 6.6.1 then yields a procedure for constructing a countermodel to $\Gamma \Rightarrow \Delta$ from these countermodels. □

Let us consider an example. We write

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \qquad \Gamma_2 \Rightarrow \Delta_2}{\Gamma_3 \Rightarrow \Delta_3}\,\mathrm{R}$$

to denote $(\Gamma_1 \Rightarrow \Delta_1 \Wedge \Gamma_2 \Rightarrow \Delta_2) \dashv\vdash_R \Gamma_3 \Rightarrow \Delta_3$ for all rules R other than $\mathsf{R}{\vee\!\!\vee}$ (and similarly for single-premise rules), and we write

$$\dfrac{\Xi \Rightarrow \chi\{\varphi_L\},\Delta \qquad \Xi \Rightarrow \chi\{\varphi_R\},\Delta}{\Xi \Rightarrow \chi\{\varphi_L \vee\!\!\vee \varphi_R\},\Delta}\ \mathsf{R}{\vee\!\!\vee}$$

to denote $(\Xi \Rightarrow \chi\{\varphi_L\},\Delta \Vee \Xi \Rightarrow \chi\{\varphi_R\},\Delta) \dashv\vdash_{\mathsf{L}\vee\!\!\vee} \Xi \Rightarrow \chi\{\varphi_L \vee\!\!\vee \varphi_R\},\Delta$. Valid sequents are blue; invalid sequents are red; countermodels are written above the sequent arrows. The following demonstrates how the procedure yields a countermodel $\{v_p, v_{\overline{p}}\}$ to the invalid sequent $p \vee\!\!\vee (p \vee \neg p) \Rightarrow p \vee\!\!\vee \neg p$.

$$\dfrac{\dfrac{p \Rightarrow p \qquad \dfrac{p,p \overset{\{v_p\}}{\Rightarrow}}{p \overset{\{v_p\}}{\Rightarrow} \neg p}\,\mathsf{R}\neg}{p \Rightarrow p \vee\!\!\vee \neg p}\,\mathsf{R}{\vee\!\!\vee} \qquad \dfrac{\dfrac{p \Rightarrow p \qquad \dfrac{\overset{\{v_{\overline{p}}\}}{\Rightarrow} p,p}{\neg p \overset{\{v_{\overline{p}}\}}{\Rightarrow} p}\,\mathsf{L}\neg}{p \vee \neg p \overset{\{v_{\overline{p}}\}}{\Rightarrow} p}\,\mathsf{L}\vee \qquad \dfrac{\dfrac{p,p \overset{\{v_p\}}{\Rightarrow}}{p \overset{\{v_p\}}{\Rightarrow} \neg p}\,\mathsf{R}\neg \qquad \dfrac{\dfrac{p \Rightarrow p}{\neg p, p \Rightarrow}\,\mathsf{L}\neg}{\neg p \Rightarrow \neg p}\,\mathsf{R}\neg}{p \vee \neg p \overset{\{v_p\}}{\Rightarrow} \neg p}\,\mathsf{L}\vee}{p \vee \neg p \overset{\{v_p, v_{\overline{p}}\}}{\Rightarrow} p \vee\!\!\vee \neg p}\,\mathsf{R}{\vee\!\!\vee}}{p \vee\!\!\vee (p \vee \neg p) \overset{\{v_p, v_{\overline{p}}\}}{\Rightarrow} p \vee\!\!\vee \neg p}\,\mathsf{L}{\vee\!\!\vee}$$

Or, in our inline notation,

$$((p \Rightarrow p \Vee p,p \Rightarrow) \Wedge ((p \Rightarrow p \Wedge \Rightarrow p,p) \Vee (p,p \Rightarrow \Wedge p \Rightarrow p))$$
$$\dashv\vdash_{\mathsf{G3cp}^-}$$
$$((p \Rightarrow p \Vee p \Rightarrow \neg p) \Wedge (p \vee \neg p \Rightarrow p \Vee p \vee \neg p \Rightarrow \neg p))$$
$$\dashv\vdash_{\mathsf{R}\vee\!\!\vee}$$
$$(p \Rightarrow p \vee\!\!\vee \neg p \Wedge p \vee \neg p \Rightarrow p \vee\!\!\vee \neg p)$$
$$\dashv\vdash_{\mathsf{L}\vee\!\!\vee}$$
$$p \vee\!\!\vee (p \vee \neg p) \Rightarrow p \vee\!\!\vee \neg p$$

6.7 Cut Elimination and Derivation Normal Form

In this section, we prove a normal form result for cutfree derivations and use this normal form result to describe how the cut elimination procedure for G3cp yields one for GT.

Let us introduce some standard terminology. The *level* of a cut (i.e., an application of Cut) is the sum of the heights of the deductions of its premises. The *rank* of a cut on φ is $|\varphi|$, the number of symbols in φ. The *cutrank* $cr(\mathcal{D})$ of a deduction $\mathcal{D}$ is the maximum of the ranks of the cutformulas occurring in $\mathcal{D}$.

The standard approach to cut elimination fails due to the syntactic restrictions on the rules. Consider, for instance, the following case in a prospective standard cut elimination proof: the rightmost cut of maximal rank and maximal level among cuts of the same rank is such that neither premise is an axiom; the cutformula is not principal on the left; the final rule applied to get the left premise is $\mathsf{L}\vee$. In the setting of G3cp, this cut would look as follows:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Xi,\beta_1 \Rightarrow \alpha,\Lambda} \qquad \overset{\mathcal{D}_2}{\Xi,\beta_2 \Rightarrow \alpha,\Lambda}}{\Xi,\beta_1 \vee \beta_2 \Rightarrow \alpha,\Lambda}\,\mathsf{L}\vee \qquad \overset{\mathcal{D}_3}{\Theta,\alpha \Rightarrow \Omega}}{\Xi,\Theta,\beta_1 \vee \beta_2 \Rightarrow \Lambda,\Omega}\,\mathsf{Cut}$$

In the cut elimination procedure for G3cp, we would commute the cut upwards to obtain a derivation with the same cutrank as before ($|\alpha|$), but with either a lower maximal level among cuts of rank $|\alpha|$, or one fewer cut of maximal level of rank $|\alpha|$:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Xi,\beta_1 \Rightarrow \alpha,\Lambda} \qquad \overset{\mathcal{D}_3}{\Theta,\alpha \Rightarrow \Omega}}{\Xi,\Theta,\beta_1 \Rightarrow \Lambda,\Omega}\,\mathsf{Cut} \qquad \dfrac{\overset{\mathcal{D}_2}{\Xi,\beta_2 \Rightarrow \alpha,\Lambda} \qquad \overset{\mathcal{D}_3}{\Theta,\alpha \Rightarrow \Omega}}{\Xi,\Theta,\beta_2 \Rightarrow \Lambda,\Omega}\,\mathsf{Cut}}{\Xi,\Theta,\beta_1 \vee \beta_2 \Rightarrow \Lambda,\Omega}\,\mathsf{L}\vee$$

In the general setting of GT, the cut would look as follows (in the case in which the cutformula α is not part of the right context set Δ generated by the implicit weakening):

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Gamma,\psi_1 \Rightarrow \alpha,\Lambda} \qquad \overset{\mathcal{D}_2}{\Gamma,\psi_2 \Rightarrow \alpha,\Lambda}}{\Gamma,\psi_1 \vee \psi_2 \Rightarrow \alpha,\Lambda,\Delta}\,\mathsf{L}\vee \qquad \overset{\mathcal{D}_3}{\Pi,\alpha \Rightarrow \Sigma}}{\Gamma,\Pi,\psi_1 \vee \psi_2 \Rightarrow \Lambda,\Delta,\Sigma}\,\mathsf{Cut}$$

We cannot freely commute the cut upwards because Σ might not be classical. If we tried to do so, we would end up with:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Gamma,\psi_1 \Rightarrow \alpha,\Lambda} \qquad \overset{\mathcal{D}_3}{\Pi,\alpha \Rightarrow \Sigma}}{\Gamma,\Pi,\psi_1 \Rightarrow \Lambda,\Sigma}\,\mathsf{Cut} \qquad \dfrac{\overset{\mathcal{D}_2}{\Gamma,\psi_2 \Rightarrow \alpha,\Lambda} \qquad \overset{\mathcal{D}_3}{\Pi,\alpha \Rightarrow \Sigma}}{\Gamma,\Pi,\psi_2 \Rightarrow \Lambda,\Sigma}\,\mathsf{Cut}}{\Gamma,\Pi,\psi_1 \vee \psi_2 \Rightarrow \Lambda,\Delta,\Sigma}\,\#\mathsf{L}\vee$$

Here the application of L$\vee$ is not legitimate because the right context set Λ,Σ of the premises might not be classical.

We show instead that each cut can be transformed into a cut on classical sequents. This follows from a derivation normal form theorem for the cutfree system: any cutfree derivation of $\Gamma \Rightarrow \Delta$ can be transformed into a cutfree derivation in which one first derives classical sequents whose antecedents are the resolutions of Γ and whose succedents are resolutions of Δ, and then applies the deep-inference rules to derive $\Gamma \Rightarrow \Delta$. (This is type of theorem is also known as a *decomposition theorem* in the deep-inference literature [27].) Before stating the theorem, we give an example. The theorem transforms the following derivation

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{\mathcal{D}_1}{x,\neg x \vee (\neg q \vee p), q \Rightarrow p}}{x,\neg x \vee (\neg q \vee p), q \Rightarrow p \mathbin{\vee\!\!\vee} r}\,\mathsf{R}\mathbin{\vee\!\!\vee} \qquad \dfrac{\overset{\mathcal{D}_2}{x,\neg x \vee (\neg q \vee r), q \Rightarrow r}}{x,\neg x \vee (\neg q \vee r), q \Rightarrow p \mathbin{\vee\!\!\vee} r}\,\mathsf{R}\mathbin{\vee\!\!\vee}}{x,\neg x \vee (\neg q \vee (p \mathbin{\vee\!\!\vee} r)), q \Rightarrow p \mathbin{\vee\!\!\vee} r}\,\mathsf{L}\mathbin{\vee\!\!\vee}}{x,\neg x \vee (\neg q \vee (p \mathbin{\vee\!\!\vee} r)) \Rightarrow p \mathbin{\vee\!\!\vee} r, \neg q}\,\mathsf{R}\neg}{x,\neg x \vee (\neg q \vee (p \mathbin{\vee\!\!\vee} r)) \Rightarrow (p \mathbin{\vee\!\!\vee} r) \vee \neg q}\,\mathsf{R}\vee}{x \wedge (\neg x \vee (\neg q \vee (p \mathbin{\vee\!\!\vee} r))) \Rightarrow (p \mathbin{\vee\!\!\vee} r) \vee \neg q}\,\mathsf{L}\wedge$$

into the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{x,\neg x\vee(\neg q\vee p),q\Rightarrow p}}{x,\neg x\vee(\neg q\vee p)\Rightarrow p,\neg q}\,\mathsf{R}\neg}{x,\neg x\vee(\neg q\vee p)\Rightarrow p\vee\neg q}\,\mathsf{R}\vee}{x\wedge(\neg x\vee(\neg q\vee p))\Rightarrow p\vee\neg q}\,\mathsf{L}\wedge}{x\wedge(\neg x\vee(\neg q\vee p))\Rightarrow(p\mathbin{\vee\!\!\!\vee} r)\vee\neg q}\,\mathsf{R}\mathbin{\vee\!\!\!\vee} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_2}{x,\neg x\vee(\neg q\vee r),q\Rightarrow r}}{x,\neg x\vee(\neg q\vee r)\Rightarrow r,\neg q}\,\mathsf{R}\neg}{x,\neg x\vee(\neg q\vee r)\Rightarrow r\vee\neg q}\,\mathsf{R}\vee}{x\wedge(\neg x\vee(\neg q\vee r))\Rightarrow r\vee\neg q}\,\mathsf{L}\wedge}{x\wedge(\neg x\vee(\neg q\vee r))\Rightarrow(p\mathbin{\vee\!\!\!\vee} r)\vee\neg q}\,\mathsf{R}\mathbin{\vee\!\!\!\vee}$$

$$\overline{\quad x\wedge(\neg x\vee(\neg q\vee(p\mathbin{\vee\!\!\!\vee} r)))\Rightarrow(p\mathbin{\vee\!\!\!\vee} r)\vee\neg q\quad}\,\mathsf{L}\mathbin{\vee\!\!\!\vee}$$

where notice that $\mathcal{R}(x\wedge(\neg x\vee(\neg q\vee(p\mathbin{\vee\!\!\!\vee} r)))) = \{x\wedge(\neg x\vee(\neg q\vee p)), x\wedge(\neg x\vee(\neg q\vee r))\}$ and that $\mathcal{R}((p\mathbin{\vee\!\!\!\vee} r)\vee\neg q) = \{p\vee\neg q, r\vee\neg q\}$.

6.7.1. THEOREM (Derivation normal form). *There is an effective procedure (that is height-preserving) transforming any derivation witnessing* $\vdash_{\mathsf{GT}^-}\Gamma\Rightarrow\Delta$ *into a derivation witnessing*

$$\vdash_{\mathsf{G3cp}^-}\ \bigwedge\!\!\!\!\bigwedge_{\Xi\in\mathcal{R}(\Gamma)}(\Xi\Rightarrow f[\Xi])$$

$$\vdash_{\mathcal{R}\mathbin{\vee\!\!\!\vee}}\ \bigwedge\!\!\!\!\bigwedge_{\Xi\in\mathcal{R}(\Gamma)}(\Xi\Rightarrow\Delta)$$

$$\vdash_{\mathsf{L}\mathbin{\vee\!\!\!\vee}}\ \Gamma\Rightarrow\Delta,$$

where $f:\mathcal{R}(\Gamma)\to\mathcal{R}(\Delta)$. *We say that a derivation of* $\Gamma\Rightarrow\Delta$ *of the form above is in* normal form.

Proof:
It suffices to show that applications of $\mathsf{R}\mathbin{\vee\!\!\!\vee}$ can be commuted below applications of rules in G3cp, and that applications $\mathsf{L}\mathbin{\vee\!\!\!\vee}$ can be commuted below applications of all other rules (in a way that preserves height); it is then easy to see that the rest follows by the definition of resolutions. We show this by induction on the height n of derivations. If $n = 1$, there is nothing to show. We now assume the result holds for n and prove that it holds for $n+1$ (note that most cases do not need to make use of the induction hypothesis). We only explicitly show that applications of the $\mathbin{\vee\!\!\!\vee}$-rules can be commuted below applications of $\mathsf{R}\wedge$; that applications of $\mathsf{L}\mathbin{\vee\!\!\!\vee}$ can be commuted below applications of $\mathsf{R}\mathbin{\vee\!\!\!\vee}$; and that applications of $\mathsf{L}\mathbin{\vee\!\!\!\vee}$ can be commuted below one another. The rest of the cases are similar. In most cases below we write only the relevant rule applications, and omit the rest of the derivation.

We show that the $\mathbin{\vee\!\!\!\vee}$-rules can be commuted below $\mathsf{R}\wedge$.

$\mathsf{R}\mathbin{\vee\!\!\!\vee}$: The principal formula of $\mathsf{R}\mathbin{\vee\!\!\!\vee}$ cannot be a side formula of $\mathsf{R}\wedge$ since the former must be nonclassical and the latter classical. It is therefore an active formula of $\mathsf{R}\wedge$, and we may assume that the relevant part of $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\Gamma\Rightarrow\chi\{\varphi_i\},\Lambda}{\Gamma\Rightarrow\chi\{\varphi_L\mathbin{\vee\!\!\!\vee}\varphi_R\},\Lambda}\,\mathsf{R}\mathbin{\vee\!\!\!\vee}\quad \Gamma\Rightarrow\psi,\Lambda}{\Gamma\Rightarrow\chi\{\varphi_L\mathbin{\vee\!\!\!\vee}\varphi_R\}\wedge\psi,\Lambda,\Delta}\,\mathsf{R}\wedge$$

This is transformed into:

$$\dfrac{\dfrac{\Gamma \Rightarrow \chi\{\varphi_i\}, \Lambda \qquad \Gamma \Rightarrow \psi, \Lambda}{\Gamma \Rightarrow \chi\{\varphi_i\} \wedge \psi, \Lambda, \Delta}\,\mathsf{R}\wedge}{\Gamma \Rightarrow \chi\{\varphi_L \Vvee \varphi_R\} \wedge \psi, \Lambda, \Delta}\,\mathsf{R}\Vvee$$

$\mathsf{L}\Vvee$: The relevant part of $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Gamma, \chi\{\varphi_L\} \Rightarrow \varphi, \Lambda} \qquad \overset{\mathcal{D}_2}{\Gamma, \chi\{\varphi_R\} \Rightarrow \varphi, \Lambda}}{\Gamma, \chi\{\varphi_L \Vvee \varphi_R\} \Rightarrow \varphi, \Lambda}\,\mathsf{L}\Vvee \qquad \overset{\mathcal{D}_3}{\Gamma, \chi\{\varphi_L \Vvee \varphi_R\} \Rightarrow \psi, \Lambda}}{\Gamma, \chi\{\varphi_L \Vvee \varphi_R\} \Rightarrow \varphi \wedge \psi, \Lambda, \Delta}\,\mathsf{R}\wedge$$

By the induction hypothesis, we may assume that either $\mathcal{D}_3$ consists of a single axiom, or the final rule applied in $\mathcal{D}_3$ is $\mathsf{L}\Vvee$, the principal formula of the final rule application is $\chi\{\varphi_L \Vvee \varphi_R\}$, and the principal subformula is $\varphi_L \Vvee \varphi_R$. In the former case, $\Gamma, \chi\{\varphi_L\} \Rightarrow \psi, \Lambda$ and $\Gamma, \chi\{\varphi_R\} \Rightarrow \psi, \Lambda$ are also axioms, and the above is transformed into:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Gamma, \chi\{\varphi_L\} \Rightarrow \varphi, \Lambda} \qquad \Gamma, \chi\{\varphi_L\} \Rightarrow \psi, \Lambda}{\Gamma, \chi\{\varphi_L\} \Rightarrow \varphi \wedge \psi, \Lambda, \Delta}\,\mathsf{R}\wedge \qquad \dfrac{\overset{\mathcal{D}_2}{\Gamma, \chi\{\varphi_R\} \Rightarrow \varphi, \Lambda} \qquad \Gamma, \chi\{\varphi_R\} \Rightarrow \psi, \Lambda}{\Gamma, \chi\{\varphi_R\} \Rightarrow \varphi \wedge \psi, \Lambda, \Delta}\,\mathsf{R}\wedge}{\Gamma, \chi\{\varphi_L \Vvee \varphi_R\} \Rightarrow \varphi \wedge \psi, \Lambda, \Delta}\,\mathsf{L}\Vvee$$

In the latter case, we have:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Gamma, \chi\{\varphi_L\} \Rightarrow \varphi, \Lambda} \qquad \overset{\mathcal{D}_2}{\Gamma, \chi\{\varphi_R\} \Rightarrow \varphi, \Lambda}}{\Gamma, \chi\{\varphi_L \Vvee \varphi_R\} \Rightarrow \varphi, \Lambda}\,\mathsf{L}\Vvee \qquad \dfrac{\overset{\mathcal{D}_4}{\Gamma, \chi\{\varphi_L\} \Rightarrow \psi, \Lambda} \qquad \overset{\mathcal{D}_5}{\Gamma, \chi\{\varphi_R\} \Rightarrow \psi, \Lambda}}{\Gamma, \chi\{\varphi_L \Vvee \varphi_R\} \Rightarrow \psi, \Lambda}\,\mathsf{L}\Vvee}{\Gamma, \chi\{\varphi_L \Vvee \varphi_R\} \Rightarrow \varphi \wedge \psi, \Lambda, \Delta}\,\mathsf{R}\wedge$$

This is transformed into:

$$\dfrac{\dfrac{\overset{\mathcal{D}_1}{\Gamma, \chi\{\varphi_L\} \Rightarrow \varphi, \Lambda} \qquad \overset{\mathcal{D}_4}{\Gamma, \chi\{\varphi_L\} \Rightarrow \psi, \Lambda}}{\Gamma, \chi\{\varphi_L\} \Rightarrow \varphi \wedge \psi, \Lambda, \Delta}\,\mathsf{R}\wedge \qquad \dfrac{\overset{\mathcal{D}_2}{\Gamma, \chi\{\varphi_R\} \Rightarrow \varphi, \Lambda} \qquad \overset{\mathcal{D}_5}{\Gamma, \chi\{\varphi_R\} \Rightarrow \psi, \Lambda}}{\Gamma, \chi\{\varphi_R\} \Rightarrow \varphi \wedge \psi, \Lambda, \Delta}\,\mathsf{R}\wedge}{\Gamma, \chi\{\varphi_L \Vvee \varphi_R\} \Rightarrow \varphi \wedge \psi, \Lambda, \Delta}\,\mathsf{L}\Vvee$$

We show that $\mathsf{L}\Vvee$ can be commuted below $\mathsf{R}\Vvee$. The relevant part of $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\Gamma, \chi\{\varphi_L\} \Rightarrow \eta\{\psi_i\}, \Delta \qquad \Gamma, \chi\{\varphi_R\} \Rightarrow \eta\{\psi_i\}, \Delta}{\Gamma, \chi\{\varphi_L \Vvee \varphi_R\} \Rightarrow \eta\{\psi_i\}, \Delta}\,\mathsf{L}\Vvee}{\Gamma, \chi\{\varphi_L \Vvee \varphi_R\} \Rightarrow \eta\{\psi_L \Vvee \psi_R\}, \Delta}\,\mathsf{R}\Vvee$$

This is transformed into:

$$\dfrac{\dfrac{\Gamma, \chi\{\varphi_L\} \Rightarrow \eta\{\psi_i\}, \Delta}{\Gamma, \chi\{\varphi_L\} \Rightarrow \eta\{\psi_L \Vvee \psi_R\}, \Delta}\,\mathsf{R}\Vvee \qquad \dfrac{\Gamma, \chi\{\varphi_R\} \Rightarrow \eta\{\psi_i\}, \Delta}{\Gamma, \chi\{\varphi_R\} \Rightarrow \eta\{\psi_L \Vvee \psi_R\}, \Delta}\,\mathsf{R}\Vvee}{\Gamma, \chi\{\varphi_L \Vvee \varphi_R\} \Rightarrow \eta\{\psi_L \Vvee \psi_R\}, \Delta}\,\mathsf{L}\Vvee$$

We show that applications of $\mathsf{L}\Vvee$ can be commuted around one another.

Case 1: The principal formula of the first application is a side formula of the second application. We may assume that the relevant part of $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\Gamma,\chi\{\varphi_L\},\eta\{\psi_L\}\Rightarrow\Delta \quad \Gamma,\chi\{\varphi_R\},\eta\{\psi_L\}\Rightarrow\Delta}{\Gamma,\chi\{\varphi_L\vee\varphi_R\},\eta\{\psi_L\}\Rightarrow\Delta}\,\mathsf{L}\vee \quad \Gamma,\chi\{\varphi_L\vee\varphi_R\},\eta\{\psi_R\}\Rightarrow\Delta}{\Gamma,\chi\{\varphi_L\vee\varphi_R\},\eta\{\psi_L\vee\psi_R\}\Rightarrow\Delta}\,\mathsf{L}\vee$$

This case is clearly analogous to the $\mathsf{L}\vee$-subcase of $\mathsf{R}\wedge$.

Case 2: The principal formula $\chi\{\varphi_L\vee\varphi_R\}$ of the first application is an active formula $\eta\{\psi_i\}$ of the second application (we assume $i = L$ without loss of generality). There are three subcases.

Case 2.1: $\chi\{\varphi_L\vee\varphi_R\} = \chi\{\varphi_L\vee\varphi_R\}\{\psi_L\}$. The relevant part of $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\Gamma,\chi\{\varphi_L\}\{\psi_L\}\Rightarrow\Delta \quad \Gamma,\chi\{\varphi_R\}\{\psi_L\}\Rightarrow\Delta}{\Gamma,\chi\{\varphi_L\vee\varphi_R\}\{\psi_L\}\Rightarrow\Delta}\,\mathsf{L}\vee \quad \Gamma,\chi\{\varphi_L\vee\varphi_R\}\{\psi_R\}\Rightarrow\Delta}{\Gamma,\chi\{\varphi_L\vee\varphi_R\}\{\psi_L\vee\psi_R\}\Rightarrow\Delta}\,\mathsf{L}\vee$$

This case is clearly analogous to the $\mathsf{L}\vee$-subcase of $\mathsf{R}\wedge$.

Case 2.2: $\chi\{\varphi_L\vee\varphi_R\} = \chi\{(\varphi_L\vee\varphi_R)'\{\psi_L\}\}$ (we assume without loss of generality that $\chi\{\varphi_L\vee\varphi_R\} = \chi\{\varphi_L'\{\psi_L\}\vee\varphi_R\}$). The relevant part of $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\Gamma,\chi\{\varphi_L'\{\psi_L\}\}\Rightarrow\Delta \quad \Gamma,\chi\{\varphi_R\}\Rightarrow\Delta}{\Gamma,\chi\{\varphi_L'\{\psi_L\}\vee\varphi_R\}\Rightarrow\Delta}\,\mathsf{L}\vee \quad \Gamma,\chi\{\varphi_L'\{\psi_R\}\vee\varphi_R\}\Rightarrow\Delta}{\Gamma,\chi\{\varphi_L'\{\psi_L\vee\psi_R\}\vee\varphi_R\}\Rightarrow\Delta}\,\mathsf{L}\vee$$

This case is similar to the $\mathsf{L}\vee$-subcase of $\mathsf{R}\wedge$.

Case 2.3: $\chi\{\varphi_L\vee\varphi_R\} = \eta\{\psi_L\} = \eta\{\psi_L'\{\varphi_L\vee\varphi_R\}\}$. The relevant part of $\mathcal{D}$ is of the form:

$$\dfrac{\dfrac{\Gamma,\eta\{\psi_L'\{\varphi_L\}\}\Rightarrow\Delta \quad \Gamma,\eta\{\psi_L'\{\varphi_R\}\}\Rightarrow\Delta}{\Gamma,\eta\{\psi_L'\{\varphi_L\vee\varphi_R\}\}\Rightarrow\Delta}\,\mathsf{L}\vee \quad \Gamma,\eta\{\psi_R\}\Rightarrow\Delta}{\Gamma,\eta\{\psi_L'\{\varphi_L\vee\varphi_R\}\vee\psi_R\}\Rightarrow\Delta}\,\mathsf{L}\vee$$

This is transformed into:

$$\dfrac{\dfrac{\Gamma,\eta\{\psi_L'\{\varphi_L\}\}\Rightarrow\Delta \quad \Gamma,\eta\{\psi_R\}\Rightarrow\Delta}{\Gamma,\eta\{\psi_L'\{\varphi_L\}\vee\psi_R\}\Rightarrow\Delta}\,\mathsf{L}\vee \quad \dfrac{\Gamma,\eta\{\psi_L'\{\varphi_R\}\}\Rightarrow\Delta \quad \Gamma,\eta\{\psi_R\}\Rightarrow\Delta}{\Gamma,\eta\{\psi_L'\{\varphi_R\}\vee\psi_R\}\Rightarrow\Delta}\,\mathsf{L}\vee}{\Gamma,\eta\{\psi_L'\{\varphi_L\vee\varphi_R\}\vee\psi_R\}\Rightarrow\Delta}\,\mathsf{L}\vee$$

□

Observe that derivations in normal form have a stronger subformula property than that of cutfree derivations in general (Proposition 6.4.3):

6.7.2. PROPOSITION (Normal form derivation partial resolution subformula property). *For any $\mathcal{D}$ in normal form witnessing $\Gamma\vdash_{\mathsf{GT}^-}\Delta$, each formula occurring in $\mathcal{D}$ is a partial resolution or a subformula of a resolution of some formula occurring in Γ,Δ.*

We are now ready to prove our cut elimination theorem.

6.7.3. THEOREM (Cut elimination). *If $\mathcal{D}$ witnesses $\vdash_{\mathsf{GT}} \Gamma \Rightarrow \Delta$, there is an effective procedure for transforming $\mathcal{D}$ into a derivation $\mathcal{D}'$ witnessing $\vdash_{\mathsf{GT}^-} \Gamma \Rightarrow \Delta$.*

Proof:
By induction on the cutrank, with a subinduction on the level of cuts. It suffices to show that we can transform a derivation $\mathcal{D}$ ending in a cut

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \Rightarrow \varphi, \Delta\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Pi, \varphi \Rightarrow \Sigma\end{array}}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \mathsf{Cut}$$

where $\mathcal{D}_1$ and $\mathcal{D}_2$ are cutfree into a cutfree derivation. By Theorem 6.7.1, we can replace $\mathcal{D}_1$, $\mathcal{D}_2$ with cutfree $\mathcal{D}_1'$, $\mathcal{D}_2'$ in normal form; i.e., $\mathcal{D}_1'$ witnesses

$$\begin{aligned}&\vdash_{\mathsf{G3cp}^-} \bigwedge\!\!\!\bigwedge_{\Xi \in \mathcal{R}(\Gamma)} (\Xi \Rightarrow f(\Xi))\\ &\vdash_{\mathcal{R}\mathbb{V}} \bigwedge\!\!\!\bigwedge_{\Xi \in \mathcal{R}(\Gamma)} (\Xi \Rightarrow \varphi, \Delta)\\ &\vdash_{\mathsf{L}\mathbb{V}} \Gamma \Rightarrow \varphi, \Delta,\end{aligned}$$

where $f : \mathcal{R}(\Gamma) \to \mathcal{R}(\varphi, \Delta)$, and $\mathcal{D}_2'$ witnesses

$$\begin{aligned}&\vdash_{\mathsf{G3cp}^-} \bigwedge\!\!\!\bigwedge_{(\Theta,\alpha) \in \mathcal{R}(\Pi,\varphi)} (\Theta, \alpha \Rightarrow g(\Theta, \alpha))\\ &\vdash_{\mathcal{R}\mathbb{V}} \bigwedge\!\!\!\bigwedge_{(\Theta,\alpha) \in \mathcal{R}(\Pi,\varphi)} (\Theta, \alpha \Rightarrow \Sigma)\\ &\vdash_{\mathsf{L}\mathbb{V}} \Pi, \varphi \Rightarrow \Sigma,\end{aligned}$$

where $g : \mathcal{R}(\Pi, \varphi) \to \mathcal{R}(\Sigma)$, each $\Theta \in \mathcal{R}(\Pi)$, and each $\alpha \in \mathcal{R}(\varphi)$.

Write $f(\Xi) = \alpha^\Xi, \Lambda^\Xi \in \mathcal{R}(\varphi, \Delta)$. For each $(\Xi, \Theta) \in \mathcal{R}(\Gamma, \Pi)$, we can find a subderivation $\mathcal{D}^\Xi$ of $\mathcal{D}_1'$ witnessing $\vdash_{\mathsf{G3cp}^-} \Xi \Rightarrow \Lambda^\Xi, \alpha^\Xi$ and a subderivation $\mathcal{D}^{\Xi,\Theta}$ of $\mathcal{D}_2'$ witnessing $\vdash_{\mathsf{G3cp}^-} \Theta, \alpha^\Xi \Rightarrow g(\Theta, \alpha^\Xi)$. We can then construct the following $\mathsf{G3cp}$-derivation:

$$\dfrac{\begin{array}{c}\mathcal{D}^\Xi\\ \Xi \Rightarrow \Lambda^\Xi, \alpha^\Xi\end{array} \qquad \begin{array}{c}\mathcal{D}^{\Xi,\Theta}\\ \Theta, \alpha^\Xi \Rightarrow g(\Theta, \alpha^\Xi)\end{array}}{\Xi, \Theta, \Rightarrow \Lambda^\Xi, g(\Theta, \alpha^\Xi)}\ \mathsf{Cut}$$

By classical cut elimination (Theorem 6.3.3), there is an effective procedure transforming this derivation into a derivation $\mathcal{D}_{\Xi,\Theta}$ witnessing $\vdash_{\mathsf{G3cp}^-} \Xi, \Theta, \Rightarrow \Lambda^\Xi, g(\Theta, \alpha^\Xi)$.

Combining the derivations $\mathcal{D}_{\Xi,\Theta}$ with subderivations derived from the subderivations of $\mathcal{D}_1'$ and $\mathcal{D}_2'$ which apply $\mathsf{R}\mathbb{V}$ to partial resolutions of Δ, Σ (changing the contexts as appropriate), we can construct derivations witnessing

$$\begin{aligned}&\vdash_{\mathsf{G3cp}^-} \bigwedge\!\!\!\bigwedge_{(\Xi,\Theta) \in \mathcal{R}(\Gamma,\Pi)} (\Xi, \Theta, \Rightarrow \Lambda^\Xi, g(\Theta, \alpha^\Xi))\\ &\vdash_{\mathcal{R}\mathbb{V}} \bigwedge\!\!\!\bigwedge_{(\Xi,\Theta) \in \mathcal{R}(\Gamma,\Pi)} (\Xi, \Theta, \Rightarrow \Delta, \Sigma)\end{aligned}$$

Finally, combining these derivations with subderivations derived from the subderivations of $\mathcal{D}_1'$ and $\mathcal{D}_2'$ which apply $\mathsf{L}\mathbin{\vee\!\!\!\vee}$ to partial resolutions of Γ, Π (changing the contexts as appropriate), we can construct a derivation witnessing

$$\vdash_{\mathsf{G3cp}^-} \bigwedge\!\!\!\bigwedge_{(\Xi,\Theta)\in\mathcal{R}(\Gamma,\Pi)} (\Xi, \Theta, \Rightarrow \Lambda^{\Xi}, g(\Theta, \alpha^{\Xi}))$$
$$\vdash_{\mathcal{R}\mathbin{\vee\!\!\!\vee}} \bigwedge\!\!\!\bigwedge_{(\Xi,\Theta)\in\mathcal{R}(\Gamma,\Pi)} (\Xi, \Theta, \Rightarrow \Delta, \Sigma)$$
$$\vdash_{\mathsf{L}\mathbin{\vee\!\!\!\vee}} \Gamma, \Pi \Rightarrow \Delta, \Sigma$$

□

Let us consider a simple example. We wish to eliminate the following cut from a derivation $\mathcal{D}$, where $\mathcal{D}_1$ and $\mathcal{D}_2$ are cutfree:

$$\dfrac{\begin{array}{c}\mathcal{D}_1\\ a\vee(p\mathbin{\vee\!\!\!\vee}q)\Rightarrow p\mathbin{\vee\!\!\!\vee}q, a\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ b, p\mathbin{\vee\!\!\!\vee}q\Rightarrow (b\wedge p)\mathbin{\vee\!\!\!\vee}(b\wedge q)\end{array}}{a\vee(p\mathbin{\vee\!\!\!\vee}q), b\Rightarrow a, (b\wedge p)\mathbin{\vee\!\!\!\vee}(b\wedge q)}\ \mathsf{Cut}$$

The normal form theorem transforms the derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ into derivations in normal form $\mathcal{D}_1'$ and $\mathcal{D}_2'$:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}^{a\vee p}\\ a\vee p\Rightarrow p, a\end{array}}{a\vee p\Rightarrow p\mathbin{\vee\!\!\!\vee}q, a}\ \mathsf{R}\mathbin{\vee\!\!\!\vee} \qquad \dfrac{\begin{array}{c}\mathcal{D}^{a\vee q}\\ a\vee q\Rightarrow q, a\end{array}}{a\vee q\Rightarrow p\mathbin{\vee\!\!\!\vee}q, a}\ \mathsf{R}\mathbin{\vee\!\!\!\vee}}{a\vee(p\mathbin{\vee\!\!\!\vee}q)\Rightarrow p\mathbin{\vee\!\!\!\vee}q, a}\ \mathsf{L}\mathbin{\vee\!\!\!\vee}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}^{a\vee p,b}\\ b, p\Rightarrow b\wedge p\end{array}}{b, p\Rightarrow (b\wedge p)\mathbin{\vee\!\!\!\vee}(b\wedge q)}\ \mathsf{R}\mathbin{\vee\!\!\!\vee} \qquad \dfrac{\begin{array}{c}\mathcal{D}^{a\vee q,b}\\ b, q\Rightarrow b\wedge q\end{array}}{b, q\Rightarrow (b\wedge p)\mathbin{\vee\!\!\!\vee}(b\wedge q)}\ \mathsf{R}\mathbin{\vee\!\!\!\vee}}{b, (p\mathbin{\vee\!\!\!\vee}q)\Rightarrow (b\wedge p)\mathbin{\vee\!\!\!\vee}(b\wedge q)}\ \mathsf{L}\mathbin{\vee\!\!\!\vee}$$

Combining the classical subderivations of $\mathcal{D}_1'$ and $\mathcal{D}_2'$, we have the following G3cp-derivations:

$$\dfrac{\begin{array}{c}\mathcal{D}^{a\vee p}\\ a\vee p\Rightarrow p, a\end{array} \qquad \begin{array}{c}\mathcal{D}^{a\vee p,b}\\ b, p\Rightarrow b\wedge p\end{array}}{a\vee p, b\Rightarrow a, b\wedge p}\ \mathsf{Cut} \qquad \dfrac{\begin{array}{c}\mathcal{D}^{a\vee q}\\ a\vee q\Rightarrow q, a\end{array} \qquad \begin{array}{c}\mathcal{D}^{a\vee q,b}\\ b, q\Rightarrow b\wedge q\end{array}}{a\vee q, b\Rightarrow a, b\wedge q}\ \mathsf{Cut}$$

By classical cut elimination, we may transform the first of these into a classical cutfree derivation $\mathcal{D}_{a\vee p,b}$ of $a\vee p, b\Rightarrow a, b\wedge p$, and similarly for the second. Combining these derivations with subderivations derived as appropriate from $\mathcal{D}_1'$ and $\mathcal{D}_2'$, we have the following cutfree derivation of $a\vee(p\mathbin{\vee\!\!\!\vee}q), b\Rightarrow (b\wedge p)\mathbin{\vee\!\!\!\vee}(b\wedge q)$:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_{a\vee p,b}\\ a\vee p, b\Rightarrow a, b\wedge p\end{array}}{a\vee p, b\Rightarrow a, (b\wedge p)\mathbin{\vee\!\!\!\vee}(b\wedge q)}\ \mathsf{R}\mathbin{\vee\!\!\!\vee} \qquad \dfrac{\begin{array}{c}\mathcal{D}_{a\vee q,b}\\ a\vee q, b\Rightarrow a, b\wedge q\end{array}}{a\vee q, b\Rightarrow a, (b\wedge p)\mathbin{\vee\!\!\!\vee}(b\wedge q)}\ \mathsf{R}\mathbin{\vee\!\!\!\vee}}{a\vee(p\mathbin{\vee\!\!\!\vee}q), b\Rightarrow a, (b\wedge p)\mathbin{\vee\!\!\!\vee}(b\wedge q)}\ \mathsf{L}\mathbin{\vee\!\!\!\vee}$$

Note that we have now the following counterpart to the Resolution theorem (Theorem 6.2.9) in our system:

6.7.4. COROLLARY (Derivability resolution theorem). *There is an effective procedure transforming any derivation $\mathcal{D}$ witnessing $\vdash_{\mathsf{GT}} \Gamma \Rightarrow \Delta$ into derivations $\mathcal{D}_\Xi$ ($\Xi \in \mathcal{R}(\Gamma)$) witnessing $\vdash_{\mathsf{GT}} \bigwedge\!\!\!\!\bigwedge_{\Xi \in \mathcal{R}(\Gamma)} (\Xi \Rightarrow f[\Xi])$ where $f : \mathcal{R}(\Gamma) \to \mathcal{R}(\Delta)$, and vice versa.*

Proof:
For the first direction, Theorem 6.7.3 yields a derivation witnessing $\vdash_{\mathsf{GT}^-} \Gamma \Rightarrow \Delta$, and the result then follows by Theorem 6.7.1. For the second direction, we can clearly construct a derivation witnessing $\vdash_{\mathsf{GT}} \Gamma \Rightarrow \Delta$ using the derivations $\mathcal{D}_\Xi$ and the rules $\mathsf{R}\mathbin{\rotatebox[origin=c]{180}{$\bigwedge\!\!\!\!\bigwedge$}}$ and $\mathsf{L}\mathbin{\rotatebox[origin=c]{180}{$\bigwedge\!\!\!\!\bigwedge$}}$. □

6.8 Variant System with Independent Contexts

Our system GT features shared contexts for the two-premise rules, and, as in $\mathsf{G3cp}$, the structural rules of contraction are implicit, having been 'absorbed' into these shared-context rules. In this section, we briefly present a variant of GT with independent contexts for $\mathsf{R}\wedge$ and $\mathsf{L}\vee$, and with explicit contraction rules. This variant features no syntactic restrictions on the rules $\mathsf{R}\wedge$ and $\mathsf{L}\vee$; the rules of the variant therefore make it explicit that the structural effect of the syntactic restrictions and the implicit weakening in $\mathsf{R}\wedge$ and $\mathsf{L}\vee$ is to allow for right weakening for all formulas while allowing for right contraction only for classical formulas.

6.8.1. DEFINITION (The sequent calculus GT'). The rules for GT' are as for GT except we remove the rules $\mathsf{R}\wedge$ and $\mathsf{L}\vee$, and add:

Logical rules

$$\frac{\Gamma_1, \varphi \Rightarrow \Delta_1 \qquad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, \varphi \vee \psi \Rightarrow \Delta_1, \Delta_2}\ \mathsf{L}\vee' \qquad\qquad \frac{\Gamma_1 \Rightarrow \varphi, \Delta_1 \qquad \Gamma_2 \Rightarrow \psi, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \varphi \wedge \psi, \Delta_1, \Delta_2}\ \mathsf{R}\wedge'$$

Structural rules

$$\frac{\Gamma, \varphi, \varphi \Rightarrow \Delta}{\Gamma, \varphi \Rightarrow \Delta}\ \mathsf{LC} \qquad\qquad \frac{\Gamma \Rightarrow \alpha, \alpha, \Delta}{\Gamma \Rightarrow \alpha, \Delta}\ \mathsf{RC}$$

It is interesting to note that the $\vee$- and $\wedge$-rules in this variant coincide with those of the *multiplicative disjunction (par)* and *conjunction (tensor)* in linear logic, respectively. Cf. [2], in which a connection is drawn between the semantics of the split disjunction $\vee$ and the multiplicative conjunction (recall also that $\vee$ is often called the *tensor disjunction*). The global disjunction, on the other hand, has the rules for the *additive disjunction (plus)*, modulo deep inference.

6.8.2. THEOREM (Soundness of GT'). $\vdash \Gamma \Rightarrow \Delta$ *implies* $\bigwedge \Gamma \vDash \bigvee \Delta$.

Proof:
By induction on the depth of derivations; we look at the final rule applied in the derivation and only consider the cases $\mathsf{L}\vee'$ and $\mathsf{R}\vee'$

- $\mathsf{L}\vee'$: Assume $\bigwedge \Gamma_1 \wedge \varphi \vDash \bigvee \Delta_1$ and $\bigwedge \Gamma_2 \wedge \psi \vDash \bigvee \Delta_2$. We want to show $\bigwedge \Gamma_1 \wedge \bigwedge \Gamma_2 \wedge (\varphi \vee \psi) \vDash \bigvee \Delta_1 \vee \bigvee \Delta_2$, so let $t \vDash \bigwedge \Gamma_1 \wedge \bigwedge \Gamma_2 \wedge (\varphi \vee \psi)$. Then $t = t_\varphi \cup t_\psi$ where $t_\varphi \vDash \varphi$ and $t_\psi \vDash \psi$, and $t \vDash \bigwedge \Gamma_1 \wedge \bigwedge \Gamma_2$. By downward closure, $t_\varphi \vDash \bigwedge \Gamma_1$ so that $t_\varphi \vDash \bigvee \Delta_1$; similarly $t_\psi \vDash \bigvee \Delta_2$, whence $t \vDash \bigvee \Delta_1 \vee \bigvee \Delta_2$.

- $\mathsf{R}\wedge$: Assume $\bigwedge \Gamma_1 \vDash \varphi \vee \bigvee \Delta_1$ and $\bigwedge \Gamma_2 \vDash \psi \vee \bigvee \Delta_2$. We want to show $\bigwedge \Gamma_1 \wedge \bigwedge \Gamma_2 \vDash (\varphi \wedge \psi) \vee \bigvee \Delta_1 \vee \bigvee \Delta_2$, so let $t \vDash \bigwedge \Gamma_1 \wedge \bigwedge \Gamma_2$. Then $t \vDash \varphi \vee \bigvee \Delta_1$ and $t \vDash \psi \vee \bigvee \Delta_2$, so $t = t_\varphi \cup t_{\Delta 1}$ where $t_\varphi \vDash \varphi$ and $t_{\Delta 1} \vDash \bigvee \Delta_1$, and $t = t_\psi \cup t_{\Delta 2}$ where $t_\psi \vDash \psi$ and $t_{\Delta 2} \vDash \bigvee \Delta_2$. Then also $t = (t_\varphi \cap t_\psi) \cup t_{\Delta 1} \cup t_{\Delta 2}$. By downward closure, $t_\varphi \cap t_\psi \vDash \varphi \wedge \psi$, whence, $t \vDash (\varphi \wedge \psi) \vee \bigvee \Delta_1 \vee \bigvee \Delta_2$. □

It is easy to see that the systems GT and GT' are equivalent:

6.8.3. PROPOSITION. $\vdash_{\mathsf{GT}^-} \Gamma \Rightarrow \Delta \iff \vdash_{\mathsf{GT}'^-} \Gamma \Rightarrow \Delta$.

Proof:
The direction $\Longleftarrow$ follows by soundness and the cutfree completeness of GT (Theorem 6.4.7). The direction $\Longrightarrow$ follows by the admissibility of $\mathsf{R}\vee$ and $\mathsf{L}\vee$ in GT'^-. For instance, we have:

$$\frac{\Gamma, \varphi \Rightarrow \Lambda \qquad \Gamma, \psi \Rightarrow \Lambda}{\Gamma, \Gamma, \varphi \vee \psi \Rightarrow \Lambda, \Lambda}\ \mathsf{L}\vee'$$

Clearly $\Gamma, \Gamma, \varphi \vee \psi \Rightarrow \Lambda, \Lambda \vdash_{\mathsf{LC}} \Gamma, \varphi \vee \psi \Rightarrow \Lambda, \Lambda \vdash_{\mathsf{RC}} \Gamma, \varphi \vee \psi \Rightarrow \Lambda$. As with GT (Lemma 6.5.1), it is easy to show that weakening is admissible in GT'^-, whence $\Gamma, \varphi \vee \psi \Rightarrow \Lambda \vdash_{\mathsf{GT}'^-} \Gamma, \varphi \vee \psi \Rightarrow \Lambda, \Delta$. □

6.9 Concluding Remarks

In this paper, we presented a sequent calculus GT for the team logic $\mathsf{PL}(\mathbin{\vee\!\!\!\vee})$ consisting of a subsystem $\mathsf{G3cp}$ for classical propositional logic together with deep-inference rules for the inquisitive disjunction $\mathbin{\vee\!\!\!\vee}$. We generalized the notion of resolutions from inquisitive logic to define the notion of partial resolutions, and used partial resolutions to define a weak subformula property for GT^- as well as to prove the cutfree completeness of GT. We showed that GT^- admits height-preserving weakening, contraction, and inversion; with the caveats that right contraction is only admissible for classical formulas (which is as expected, given the connection we observed between right contraction and union closure), and that $\mathsf{R}\mathbin{\vee\!\!\!\vee}$ is only 'invertible' in the sense of the Split property (Proposition 6.2.8). We provided a procedure for constructing cutfree derivations and countermodels in GT that is similar to the analogous procedure for G3cp. We proved a normal form theorem for cutfree derivations, using which we provided a cut elimination procedure for GT. Finally, we defined a variant of GT with independent-context rules instead of shared-context rules with restricted contexts for the connectives $\vee$ and $\wedge$, discussed the similarities between these variant and multiplicative linear logic, and showed that this variant is equivalent to GT.

Let us conclude by discussing some open problems, as well as the prospects of applying our deep-inference approach to provide sequent calculi for other team logics.

With regard to the first of these concerns, are there any further interesting consequences we can draw from the cutfree completeness of GT? To consider one example, can we leverage the cutfree completeness of GT to provide a constructive proof of Craig's/Lyndon's/uniform interpolation for PL($\mathbb{V}$), perhaps along the lines of [118, pp. 116–118], or [103, 85, 84]? (There is a semantic proof of uniform interpolation for PL($\mathbb{V}$) that follows essentially from results in [43].)

We can provide one limited negative result here: the procedure for proving Craig's and Lyndon's interpolation via sequent interpolants, as detailed in, for instance [118, pp. 116–118], fails for GT because the form of interpolation involved does not hold for PL($\mathbb{V}$). This form of interpolation states, roughly, that for each sequent $\Gamma \Rightarrow \Delta$, and each pair of partitions $\Gamma_1;\Gamma_2$ of Γ and $\Delta_1;\Delta_2$ of Δ, there is an interpolant φ in the common language of Γ_1,Δ_1 and Γ_2,Δ_2 such that $\Gamma_1 \Rightarrow \Delta_1,\varphi$ and $\Gamma_2,\varphi \Rightarrow \Delta_2$. Consider, however, the following valid sequent, where $?p := p \mathbb{V} \neg p$, etc.

$$?p, ?q \Rightarrow ?p \wedge ?q \wedge r, ?p \wedge ?q \wedge \neg r$$

If it were the case that this form of interpolation held for PL($\mathbb{V}$), there would be a formula φ such that the following sequents are valid: $?p \Rightarrow ?p \wedge ?q \wedge r, \varphi$ (whence $?p \vDash (?p \wedge ?q \wedge r) \vee \varphi$) and $?q, \varphi \Rightarrow ?p \wedge ?q \wedge \neg r$ (whence $?q \wedge \varphi \vDash ?p \wedge ?q \wedge \neg r$). Now consider the team $t := \{v_{p,q,r}, v_{p,\overline{q},r}, v_{p,q,\overline{r}}, v_{p,\overline{q},\overline{r}}\}$. Since $t \vDash ?p$, we must have $t \vDash (?p \wedge ?q \wedge r) \vee \varphi$, whence $t = t_1 \cup t_2$, where $t_1 \vDash ?p \wedge ?q \wedge r$ and $t_2 \vDash \varphi$. Clearly either $t_1 \subseteq \{v_{p,q,r}\}$ or $t_1 \subseteq \{v_{p,\overline{q},r}\}$, whence either $v_{p,\overline{q},r} \in t_2$ or $v_{p,q,r} \in t_2$. In the former case (the latter case is similar), by downward closure, $\{v_{p,\overline{q},r}\} \vDash \varphi$. Clearly also $\{v_{p,\overline{q},r}\} \vDash ?q$, whence $\{v_{p,\overline{q},r}\} \vDash ?p \wedge ?q \wedge \neg r$, but this clearly cannot be the case.

So the type of interpolation involved in the procedure in [118, pp. 116–118] fails; there might, however, be some other way of leveraging the system GT to prove interpolation results for PL($\mathbb{V}$).

We chose to employ only one set of deep-inference rules in order to keep our system as close to a Gentzen-style system as possible. However, it might be interesting to develop a system for PL($\mathbb{V}$) making more extensive use of deep-inference rules and techniques in order to make use of the powerful general results in the deep-inference literature.

With regard to extensions of our approach to other team logics, let us first note that given that resolutions are adapted from inquisitive logic, and that the analogues of the results in Section 6.2.2 hold for propositional inquisitive logic, we expect that our approach would also work for propositional inquisitive logic InqB (essentially PL($\mathbin{\vee\mkern-7mu\vee}$) without the split disjunction $\vee$, but with the so-called *intuitionistic implication* $\to$). However, given the fact that the inquisitive disjunction $\mathbin{\vee\mkern-7mu\vee}$ does not simply distribute over all other connectives in InqB (as it does in PL($\mathbin{\vee\mkern-7mu\vee}$)), which results in the resolutions in InqB having a more complicated structure than in PL($\mathbin{\vee\mkern-7mu\vee}$), it is not clear whether a deep-inference calculus along these lines for InqB would be as simple and appealing as the system in this paper. As for other team logics, we suspect, given our strong focus on the properties of a single team-semantic connective—the inquisitive disjunction $\mathbin{\vee\mkern-7mu\vee}$—as well as the apparent reliance of the cutfree completeness of our system on the Resolution theorem (Theorem 6.2.9), which depends, in turn, on the downward closure of PL($\mathbin{\vee\mkern-7mu\vee}$) as well as the union closure of the $\mathbin{\vee\mkern-7mu\vee}$-free fragment, we expect any application of this approach to logics which do not involve $\mathbin{\vee\mkern-7mu\vee}$ and which do not also share these additional features to involve some further innovations and/or stronger proof-theoretic machinery. There are, however, some other interesting team logics which do incorporate $\mathbin{\vee\mkern-7mu\vee}$ and do share these features (for instance, the extension of *Team Linear Temporal Logic* with $\mathbin{\vee\mkern-7mu\vee}$ in [92]).

To summarize, our approach was tailored for PL($\mathbin{\vee\mkern-7mu\vee}$), and the resulting system, with its simplicity and clarity, appears to be a good fit for this logic. Whether this approach can be generalised to other team logics remains to be explored. The intriguing links in our setting between structural properties and team-semantic closure properties do suggest that further investigation of team logics which differ in their closure properties from PL($\mathbin{\vee\mkern-7mu\vee}$) may prove fruitful; it might also be interesting to change the interpretation of the comma in the antecedent of a sequent to produce correspondences between team-semantic closure properties and structural properties on the left.

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Samenvatting

Dit proefschrift bundelt vijf artikelen die zich voornamelijk richten op de expressieve kracht en axiomatisaties van propositionele en modale logica's met teamsemantiek. De meeste van deze artikelen richten zich op omgevingen waarin de lege-team-eigenschap niet geldt—er zijn formules die niet waar zijn in het lege team—en waarin we feiten kunnen uitdrukken en kunnen redeneren over de leegheid en niet-leegheid van teams in de objecttaal.

Het eerste artikel bestudeert de wiskundige eigenschappen van de bilaterale state-gebaseerde modale logica (BSML), een modale teamlogica die is gebruikt om vrije keuzewijzen en gerelateerde linguïstische fenomenen te verklaren. Deze logica breidt de klassieke modale logica uit met een niet-leegheid-atoom NE dat waar is in een team dan en slechts dan als het team niet leeg is. In dit artikel introduceren we twee uitbreidingen van BSML, tonen we aan dat de uitbreidingen expressief compleet zijn, en ontwikkelen we axiomatisaties in natuurlijke-deductiesystemen voor de drie logica's.

In het tweede artikel bewijzen we expressieve volledigheidsresultaten voor convexe propositionele en modale teamlogica's, waarbij een logica convex is als voor elke formule geldt: als de formule waar is in twee teams, dan is de formule ook waar in alle teams tussen deze twee teams met betrekking tot de deelverzamelingrelatie. Convexiteit is een natuurlijke uitbreiding van de neerwaartse afsluiting naar een omgeving waarin de lege team-eigenschap niet geldt. We introduceren meerdere propositionele/modale logica's die expressief compleet zijn voor de klasse van alle convexe propositionele/modale teameigenschappen. We lossen ook een probleem op dat in het eerste artikel open was gebleven met betrekking tot de expressieve kracht van BSML en zijn propositionele fragment: we tonen aan dat dit propositionele fragment expressief compleet is voor de klasse van alle convexe en propositionele teameigenschappen die gesloten zijn onder vereniging, waarbij een modale analogie van dit resultaat een expressieve volledigheidsstelling voor BSML oplevert. We introduceren ook een generalisatie van uniforme definieerbaarheid en definiëren verschillende begrippen van uitbreiding door gebruik te maken van deze generalisatie om het gevoel te verduidelijken waarin onze nieuwe propositionele convexe logica's de propositionele afhankelijkheidslogica en propositionele *inquisitive logic* (propositionele logica die zich o.a. bezighoudt met de betekenis van vragen) uitbreiden.

In het derde artikel bestuderen we de eigenschappen van de negatie die in BSML wordt gebruikt, de bilaterale negatie $\neg$. Dit is in wezen dezelfde notie als de duale of speltheoretische negatie van onafhankelijkheidsvriendelijke logica (IF) en afhankelijkheidslogica (D). In IF en D vertoont de duale negatie een extreem hoge graad van semantische onbepaaldheid, aangezien voor elk paar zinnen φ en ψ van IF/D geldt dat: als φ en ψ onverenigbaar zijn in de zin dat ze geen modellen gemeen hebben, er een zin θ van IF/D bestaat zodat $\varphi \equiv \theta$ en $\psi \equiv \neg\theta$ (zoals oorspronkelijk aangetoond door Burgess in de equivalente context van het prenex-fragment van Henkin-kwantorlogica). We tonen aan dat door het begrip van onverenigbaarheid aan te passen (waardoor in sommige gevallen begrippen van onverenigbaarheid worden gegenereerd die beter geschikt zijn voor omgevingen waarin de lege team-eigenschap en neerwaartse afsluiting niet gelden dan de begrippen die uit neerwaarts gesloten omgevingen worden geïmporteerd), analogieën van dit resultaat voor een aantal modale en propositionele teamlogica's kunnen worden vastgesteld, waaronder BSML, de semantisch expressivistische logica van Hawke en Steinert-Threlkeld voor epistemische modaliteiten, evenals de propositionele afhankelijkheidslogica met de duale negatie. Samen met het omgekeerde van dit resultaat kan dit type resultaat worden gezien als een expressieve volledigheidsstelling met betrekking tot het relevante begrip van onverenigbaarheid; we formuleren een begrip van expressieve volledigheid voor paren van eigenschappen om dit precies te maken.

In het vierde artikel schakelen we over naar een omgeving die wel de lege team-eigenschap heeft, maar waarin neerwaartse afsluiting niet geldt. We bieden een volledige axiomatisatie van de modale inclusielogica – een teamgebaseerde modale logica uitgebreid met inclusie-atomen. We herzien en verfijnen een expressieve volledigheids- en normaalvormstelling voor de logica, definiëren een natuurlijke-deductie-bewijssysteem en gebruiken de normaalvorm om de volledigheid van de axiomatisatie te bewijzen. Volledige axiomatisaties worden ook gegeven voor twee andere uitbreidingen van de modale logica met dezelfde expressieve kracht als modale inclusielogica: de een verrijkt met een 'might'-operator en de andere met een enkele-wereldvariant van de 'might'-operator.

In het vijfde artikel introduceren we een sequentencalculus voor de propositionele teamlogica met zowel de gesplitste disjunctie als de *inquisitive* disjunctie, bestaande uit een G3-systeem voor de klassieke propositionele logica samen met *deep-inference* regels voor de *inquisitive* disjunctie. We tonen aan dat het systeem verschillende wenselijke eigenschappen heeft: het staat hoogte-behoudende verzwakking, contractie en inversie toe; het ondersteunt een procedure voor het construeren van snedevrije bewijzen en tegenmodellen, vergelijkbaar met die voor G3cp; en snede-eliminatie geldt als een corollarium van snede-eliminatie voor het G3-subsysteem samen met een normaalvormstelling voor snedevrije afleidingen in het systeem.

Abstract

This dissertation collects together five papers which focus primarily on the expressive power, axiomatizations, and proof theory of propositional and modal logics with team semantics. Most of these papers focus on settings in which the empty team property fails—that is, there are formulas which are not true in the empty team—and hence in which we are able to express facts and reason about the emptiness and nonemptiness of teams in the object language.

The first paper studies the mathematical properties of bilateral state-based modal logic (BSML), a modal team logic which has been used to account for free choice inferences and related linguistic phenomena. This logic extends classical modal logic with a nonemptiness atom NE which is true in a team if and only if the team is nonempty. In this paper, we introduce two extensions of BSML, show that the extensions are expressively complete, and develop natural deduction axiomatizations for the three logics.

In the second paper, we prove expressive completeness results for convex propositional and modal team logics, where a logic is convex if, for each of its formulas, if the formula is true in two teams, then it is also true in all the teams between these two teams with respect to set inclusion. Convexity is a natural generalization of the well-known property of downward closure to a setting in which the empty team property fails. We introduce multiple propositional/modal logics which are expressively complete for the class of all convex propositional/modal team properties. We also solve a problem that was left open in the first paper concerning the expressive power of BSML as well as its propositional fragment: we show that BSML is expressively complete for the class of all convex and union-closed modal team properties invariant under bounded bisimulation. We also introduce a generalization of uniform definability and define distinct notions of extension making use of this generalization in order to clarify the sense in which our novel propositional convex logics extend propositional dependence logic and propositional inquisitive logic.

In the third paper, we study the properties of the negation employed in BSML, the bilateral negation $\neg$. This is essentially the same notion as the dual or game-theoretical negation of independence-friendly logic (IF) and dependence logic (D). In IF and D,

the dual negation exhibits an extreme degree of semantic indeterminacy in that for any pair of sentences φ and ψ of IF/D, if φ and ψ are incompatible in the sense that they share no models, there is a sentence θ of IF/D such that $\varphi \equiv \theta$ and $\psi \equiv \neg\theta$ (as shown originally by Burgess in the equivalent context of the prenex fragment of Henkin quantifier logic). We show that by adjusting the notion of incompatibility employed (thus generating, in some cases, notions of incompatibility more suitable for settings in which the empty team property and downward closure fail than the notions imported from downward-closed settings), analogues of this result can be established for a number of modal and propositional team logics, including BSML and propositional dependence logic with the dual negation. Together with its converse, a result of this type can be seen as an expressive completeness theorem with respect to the relevant incompatibility notion; we formulate a notion of expressive completeness for pairs of properties to make this precise.

In the fourth paper, we move to a setting which does have the empty team property, but in which downward closure fails. We provide a complete axiomatization of modal inclusion logic—team-based modal logic extended with inclusion atoms. We review and refine an expressive completeness and normal form theorem for the logic, define a natural deduction proof system, and use the normal form to prove completeness of the axiomatization. Complete axiomatizations are also provided for two other extensions of modal logic with the same expressive power as modal inclusion logic: one augmented with a might-operator and the other with a single-world variant of the might-operator.

In the fifth paper, we introduce a sequent calculus for the propositional team logic with both the split disjunction and the inquisitive disjunction. The sequent calculus consists of a G3-style system for classical propositional logic together with deep-inference rules for the inquisitive disjunction. We show that the system satisfies various desirable properties: it admits height-preserving weakening, contraction and inversion; it supports a procedure for constructing cutfree proofs and countermodels similar to that for G3cp; and cut elimination holds as a corollary of cut elimination for the G3-style subsystem together with a normal form theorem for cutfree derivations in the system.

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