The Intuitionistic Conception of Logic*

Arend Heyting

1 The Logic of Being

Logic is often studied as a purely formal science, where the concern is not with the meaning of logical notions, but only with their formal properties. Therefore, one does not ask what it means for a proposition to be true or false, but only deals with the formal conditions under which one proposition can be deduced from other propositions. However, as soon as one wants to apply logic, one must address the question of the meaning of the word "true" and other logical terms, and is then led to statements such as: "A proposition is true if the state of affairs it expresses exists in the real world." The definition varies according to the philosopher's viewpoint, but it always presupposes a conception of reality; this amounts to saying that logic needs an ontology for its interpretation.

The difficulties encountered in interpreting implication are well known. In fact, in propositional logic, there is no proper room for implication, because each proposition is either true or false, and one cannot conceive how its truth could depend on that of other propositions. Thus, one is led to the definition of implication through truth and falsity: the proposition $p \to q$ is false if and only if p is true and q is false. On the one hand, this definition does not fit well with the intuitive idea of implication; on the other hand, its application depends on the notion of truth.

Therefore, propositional logic, in its traditional form, stands on its own only when treated as a purely formal calculus. As soon as one tries to interpret it, one must resort to metaphysics.

2 Modal Logic

Several logicians, dissatisfied with the interpretation of the implication I just discussed, have tried to provide one that better fits with intuition. Notably, C.I. Lewis has proposed to consider, besides material implication, the strict implication, whose definition is based on the notion of necessity. The proposition p strictly implies the proposition q if it is impossible that p is true and q is false. For Lewis, possibility is a primitive notion, and necessity can be definable using negation. However, to apply logic, we need to know what the words "necessary" and "possible" mean. One can say that a proposition is possible if it is not contradictory, but for this definition to have a clear meaning, the word "contradictory" must be taken in its logical sense, thus reducing the logic of strict implication to classical logic. This is not the goal of Lewis. As far as one can conclude from his examples, he takes the words "non-contradictory" in a rather vague sense explained by their everyday language usage. Analyzing this notion a bit, one sees that it is very complicated and gives rise to difficult problems.

Here is another definition of strict implication, suggested by Lewis's considerations: p strictly implies q if q expresses an effect of the cause expressed by p. For example: If it is dark in the room, I cannot read there. This definition adheres very closely to intuition, but it has the disadvantage of presupposing a theory of causality, and this theory, even more than that of existence, is a true battlefield in philosophy. It thus seems that the theory of strict implication, instead of clarifying the meaning of propositional logic, makes it depend on notions that are much more complicated and obscure.

---

*This is the English translation of the original publication: Arend Heyting, « La conception intuitionniste de la logique », *Les Études philosophiques*, Nouvelle Série, 11e Année, No. 2 (Avril/Juin 1956), pp. 226-233 © Presses Universitaires de France/Humensis, 1956. The translation was drafted by Claude-3.5-Sonnet-200k and carefully further edited by Philippe Balbiani, Hans van Ditmarsch, Dick de Jongh and Yanjing Wang. The editors thank the journal and the publisher for their approval and for their permission to put this version online.

3 Symbolic Logic

Are we thus reduced, in order to treat logic in an exact manner, to considering it as a formal calculus and renouncing any interpretation of this calculus? Symbolic logic (synonyms: mathematical logic, formal logic) is the subject of extensive studies today. It has established itself as a mathematical science, and its methods are modeled after mathematical methods. However, these methods have evolved considerably since the mid-19th century. Mathematics has gradually emancipated itself from its interpretation of reality; today, most mathematicians consider their science as purely formal. In France, the group of mathematicians who present themselves to the public under the collective name Bourbaki is publishing a treatise where mathematics is developed in this manner. From their point of view, mathematics consists of signs written on paper; everything else, all the ideas one connects to these signs, do not concern the mathematician as such, but only the physicist or technician who applies mathematics.

Logic has closely followed the evolution that led to this conception of mathematics. It has also established itself as a purely formal science, where one can operate with signs without concerning oneself with their meaning. Like mathematics, logic has gained in clarity and precision through the separation between the formal system and its interpretation. One could even say that in logic this separation has succeeded better than in mathematics, due to logic's simpler structure. Nevertheless, the question of application is more pressing for logic. One can easily conceive how mathematics applies to natural sciences, where experiments lead to numerical results; I have already emphasized that to explain the application of logic, one must first analyze the notions of true and false. Formal logic has contributed much to systematizing the laws of logic, but it is unable to clarify their application. Moreover, formal two-valued logic diverges greatly from the intuition of logical notions, such as, for example, that of implication.

4 The Logic of Knowing

To obtain a logic that is both better adapted to our intuitions and easier to apply, I choose as a starting point the fact that to apply a logical rule, we must know that the premises are true; then the rule teaches us that the conclusion is equally true. One almost always neglects the fact that in applications of logic, it is always about what we know and the conclusions we can draw from what we know.

One will undoubtedly object that by adopting this viewpoint, I base logic on the theory of knowledge, in which there is hardly more agreement between opinions than in metaphysics. I respond that one must choose some starting point; what matters is that the basic notions be as immediate as possible. However, at least for humans, knowing is more immediate than being, which manifests itself to them only through an analysis of knowing.

Another objection would be that nothing is gained by putting logic in relation with knowing instead of basing it on the notion of truth, because a proposition has a well-defined meaning only when one can know if it is true. In this way, by a detour, the notion of proposition is again based on that of truth. Nevertheless, in agreement with the fact that knowing is a more immediate notion than being, it is easier to specify the conditions under which one knows that a given proposition is true than to say exactly under what conditions it is true. More explicitly, to define the truth of a proposition, one has in many cases no other way than to enumerate the conditions under which one knows it to be true. Let us consider some examples. Everyone knows what information to gather to know if the proposition "The oldest resident of Paris is 101 years old" is true; it is precisely through this package of information that the meaning of the proposition is defined. The question is a bit more difficult for a statement like "All men are mortal." One can maintain that this proposition is analytic, the property of being mortal being part of the definition of man; one can also consider it as a statement verified by experience. In both cases, one indicates the conditions under which one knows that the proposition is true. For "All French are human," it is the first case that presents itself: without doubt it is part of the definition of being French that one is human. Let us now consider the syllogism: "All men are mortal. All French are human. Therefore: All French are mortal." It is clear that if I know that both premises are true, I know that the conclusion is true, and it is in this sense that the premises imply the conclusion. An analogous reasoning applies to *modus ponens*, for example: "If I have a headache, I cannot work. I have a headache. Therefore: I cannot work." I know from experience that the first premise is true; I directly experience the truth of the second. I therefore know that the conclusion is true.

In these examples, the difference between the logic of being and that of knowing is still not very important; the two logics are, so to speak, parallel. But from the moment that infinity plays a role, the difference becomes clearer. If a finite sequence of integers is given to me, I can know by simple inspection whether the number 5 is there or not; here again, knowing is right next to being. But if the given sequence is infinite, everything is different, because I no longer have the means to traverse the entire sequence. We are going to study more closely the difficulties caused by this circumstance.

5 Mathematical infinite

In mathematics, infinity first presents itself in the form of the sequence of natural numbers: 1, 2, 3, … I will simply say "number" instead of "natural number." What interests us is the significance of a statement about the existence of a number. This notion is difficult since it does not concern material existence. It is impossible to address here all the discussions about this concept; I only want to emphasize that the difficulties with the notion of mathematical existence suffice to explain why the logic of knowing was born as the logic of knowing of the infinite in mathematics. I just named another reason, namely that it is generally impossible to verify this existence by simple inspection.

To be more concrete, let us take an example. As we know, a number that has no divisors other than 1 and itself is called a prime number. (For example, 7 is prime because it has only 1 and 7 as divisors; 6 is not prime because, besides 1 and 6, it has the divisors 2 and 3.) Consider the following statement:

(A) Every number greater than 1 is either prime, or the sum of two prime numbers, or the sum of three prime numbers.

We do not know if (A) is true. For all numbers that have been tested, (A) has been found to be satisfied. (For example, $27 = 3 + 11 + 13$; $28 = 11 + 17$; 29 is prime; other decompositions are possible, such as $28 = 5 + 23$, $29 = 5 + 11 + 13$). Let us call "exceptional number" a number that does not satisfy (A), thus a number that is neither 1, nor prime, nor the sum of two or three prime numbers. As I just said, no exceptional number has ever been found, which does not exclude that one might exist among numbers not yet tested. Obviously (A) is equivalent to (A'):

(A') There exists no exceptional number.

Let us also study statements (B) and (B'):

(B) There exists an exceptional number.

(B') It is impossible that there exists no exceptional number.

In the logic of being, (B) and (B') are equivalent, because (B) is either true or false, thus if (B) cannot be false, (B) is true.

In the logic of knowing, we must first ask ourselves how we can know that an exceptional number exists. The simplest way is to effectively determine such a number. Consider therefore statements (C) and (D):

(C) I have effectively determined an exceptional number.

(D) I have reduced to a contradiction the supposition that no exceptional number exists.

In the logic of knowing (C) and (D) are not at all equivalent. In the case of (C) we know much more than in that of (D); in particular, in the case of (C) we know the exact value of an exceptional number, which in the case of (D) can remain completely unknown. We must therefore distinguish between the two statements; we are no longer permitted to express both of them by the same phrase: "There exists an exceptional number." The question of which of the two will be expressed by this phrase is a matter of terminology that does not touch the heart of the matter. However, there are reasons to consider "There exists an exceptional number" as synonymous with (C). It would be bizarre to define by a negative statement such as (D), existence, which, from the intuitive point of view, is the most positive notion of all. Moreover, although in mathematics one can reason in an abstract way about a number whose non-existence has been proved impossible, one can only use in calculations numbers whose value is known. We therefore adopt as a definition that (B) will mean the same as (C). More generally, if $P(x)$ is a predicate that is defined for natural numbers, "There exists a number x such that $P(x)$" will mean "We know how to compute a number x such that $P(x)$." One can consider this definition as a particular case of the "principle of positivity" which states as follows: Each mathematical or logical statement expresses the result of a construction.

6 Negation

A difficulty presents itself concerning the interpretation of negation. If (B) means the same thing as (C), one might be tempted to interpret "There exists no exceptional number" as meaning "I cannot compute an exceptional number." But this statement violates the principle of positivity. Moreover, it does not express a definitive result, because the fact that I don't know (at this moment) how to compute an exceptional number doesn't exclude that I might find one next week. To give the negative statement a meaning in the logic of knowing, we must indicate the construction by which we conclude the non-existence of an exceptional number. This construction can only be that of a contradiction. We are thus led to the following interpretation. "There exists no exceptional number" means: "One has derived a contradiction from the supposition that an exceptional number exists." In general, the negation of a proposition p will be interpreted as "One has derived a contradiction from the supposition that p." Deriving a contradiction is a construction; this definition of negation thus satisfies the principle of positivity.

Let's now compare statements (B) and (E):

(B) There exists an exceptional number.

(E) There exists no exceptional number.

According to our definitions, they have respectively the same meaning as (F) and (G):

(F) We know how to compute an exceptional number.

(G) We know how to derive a contradiction from the supposition that we have found an exceptional number.

There is no reason to assert that either (F) or (G) must be true. In fact, in the current state of science, neither (F) nor (G) is realized. This means that neither (B) nor (E) can be assumed to be true. In other words, the principle of the excluded middle is not valid. This result is less surprising than it appears at first sight, because it relies on the interpretations (F) and (G) that we have given to statements (B) and (E), which differ essentially from the usual interpretations in the logic of being.

It will be useful to guard against some misunderstandings. It would be incorrect to say that the principle of the excluded middle is false because that would mean it implies a contradiction. However, it is not contradictory that either (B) or (E) is true; we have only observed that in the current state of science, there is no reason to assert either one. This observation does not constitute a theorem of logic, just as the observation that a certain mathematical problem is not solved does not constitute a mathematical theorem. It would be equally incorrect to believe that the logic of knowing is a many-valued logic, where alongside true and false propositions one considers propositions that are neither true nor false, having some third logical value. There are only propositions, like that of the excluded middle, of which we do not know whether they are true or false and about whose truth we can consequently assert nothing. What gives rise to this misunderstanding is that one confuses considerations about logic with theorems of logic. That the principle of the excluded middle does not apply is an observation about logic, not a theorem of logic.

7 Implication

We will see that implication finds a very intuitive and natural interpretation in the logic of knowing.

Take as an example statement (H):

(H) If at least one exceptional number exists, then the smallest exceptional number is even.

Here is a proof of (H). Let a be an odd exceptional number. Then $a-3$ is neither 1, nor prime, nor the sum of two prime numbers. If $a-3$ were the sum of three prime numbers, one of them would be 2, so we would have $a-3=2+p+q$, with p and q prime. We would therefore have $a=5+p+q$, which would contradict the hypothesis that a is exceptional. I have proven that $a-3$ is an even exceptional number.

Since to every odd exceptional number a there corresponds an even exceptional number $a-3$ which is smaller, the smallest exceptional number must be even.

In analyzing this proof, we see that it consists of a construction which, starting from the hypothesis that we know an exceptional number a (or, what amounts to the same thing, that we have proven that an exceptional number exists), leads to a proof of the proposition "The smallest exceptional number is even." In general, one will prove the proposition "A implies B" by a construction that proves B under the hypothesis that a proof of A is given. The interpretation of "A implies B" in the logic of knowing is precisely this, that such a construction is known. The difficulties encountered in defining implication in the logic of being find their immediate solution here. While it is difficult to understand how the truth of one proposition can depend on the truth of another proposition, it is quite natural that the proof of one proposition depends on the proof of another proposition.

It was L.E.J. Brouwer who first understood that for mathematical infinity, the logic of knowing is most adequate, and it is he who introduced the expression "intuitionistic mathematics" to designate mathematics based on this logic.

As we have seen, the principle of the excluded middle is not valid in intuitionistic logic. Consequently, several other logical rules do not apply either. We can cite as an example that the double negation of a proposition p is not equivalent to p. Moreover, it is clear that (B'), which is the double negation of (B), is not equivalent to (B), because to be able to affirm (B) one must know an exceptional number, which is not necessary to be able to affirm (B'). The fact that several rules of traditional logic fail in intuitionistic logic is not an impoverishment, because this logic must be completed by rules, among others on the use of double negation, which in two-valued logic reduces to identities if one removes the double negations. It is not the aim of this article to enter into the details of the technique of intuitionistic logic. One will find some information on this together with a complete bibliography in my book *Les fondements des mathématiques* (Paris, Gauthier-Villars, 1955).