AKE-principles for deeply ramified fields

MSc Thesis *(Afstudeerscriptie)*

written by

Jonas van der Schaaf
(born January 12, 2002 in Zaanstad, Nederland)

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Date of the public defense:	**Members of the Thesis Committee:**
March 27, 2025	Dr. Benno van den Berg
	Prof. Dr. Franziska Jahnke
	Prof. Dr. Yde Venema
	Dr. Konstantinos Kartas

INSTITUTE FOR LOGIC, LANGUAGE AND COMPUTATION

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Chapter 1

Introduction

In the 1950s, Abraham Robinson proved a remarkable theorem: model completeness of the class of algebraically closed valued fields [Rob56]. This means that any embedding of algebraically closed valued fields is in fact an elementary embedding. This very strong result is analogous to the more well-known result that the theory of algebraically closed fields is the model completion of the theory of fields which one can deduce from quantifier elimination originally proved by Tarski [TZ12]. With this result he started research into the model theory of valued fields, a combination of number theory and logic that is still heavily studied to this day.

Another major milestone in the model theory of valued fields is a theorem proven by Ax and Kochen [AK65], and independently Ershov [Ers65]. They demonstrated that for a non-principal ultrafilter $\mathcal{U}$ on the set of primes we obtain an elementary equivalence

$$\prod_{\mathcal{U}} \mathbb{Q}_p \equiv \prod_{\mathcal{U}} \mathbb{F}_p\left(\left(t\right)\right).$$

This has many applications in number theory, the most notable one being an asymptotic proof of Artin's conjecture: for an integer $d \in \mathbb{N}$ and sufficiently large prime p and homogeneous polynomial over $\mathbb{Q}_p$ of degree d in more than d^2 variables has a root in $\mathbb{Q}_p$. The non-asymptotic result is called Artin's conjecture. Surprisingly, even though the asymptotic result is true, the general result for all p and d is false: in 1966, Terjanian gave examples for every prime of a polynomial p of sufficiently small degree and enough variables with no root in $\mathbb{Q}_p$ [Ter66].

The above theorem is actually a special case of a much stronger theorem, which is the first occurrence of what is now called an Ax-Kochen/Ershov (AKE) principle. It is a theorem about henselian valued fields whose residue field has characteristic 0. Let (K, v) and (L, w) be two such valued fields; if we write vK and wL for their value groups and Kv and Lw for their residue fields, then there is an elementary equivalence $(K, v) \equiv (L, w)$ if and only if $vK \equiv wL$ and $Kv \equiv Lw$.

We can easily see that this applies to the situation above: for any given prime p both $\mathbb{Q}_p$ and $\mathbb{F}_p\left(\left(t\right)\right)$ are henselian fields with residue field $\mathbb{F}_p$ and value group $\mathbb{Z}$. The ultraproduct of $\mathbb{F}_p\left(\left(t\right)\right)$ must be of characteristic 0, to this we can apply the AKE principle to get an elementary equivalence.

The classical “shape” of an AKE principle is the following: Given some elementary class $\mathcal{C}$ of valued fields containing $(K, v), (L, w)$ there is an equivalence

$$(K, v) \equiv (L, w) \iff vK \equiv wL \wedge Kv \equiv Lw,$$

where the left elementary equivalence is in the language of valued fields, and the right ones in the language of ordered abelian groups and fields respectively. We then say that $\mathcal{C}$ satisfies the AKE$_\equiv$ principle. We could replace $\equiv$ with other relations between models such as elementary substructure $\preceq$ or existential substructure $\preceq_\exists$ and obtain an AKE$_\preceq$ or AKE$_{\preceq_\exists}$ principle.

Ever since this result, many more AKE principles have been discovered for a variety of classes of valued fields. Even in recent years, a lot of new results have been proven in this area. Some of the most prominent results in this area are due to Franz-Viktor Kuhlmann, who has pioneered the model theory of (separably) tame valued fields. We will use these extensively throughout this thesis. A tame valued field is a valued field (K, v) which has the following properties:

1. the characteristic of the residue field Kv is some prime $p > 0$,
2. the residue field Kv is perfect,
3. it is algebraically maximal,
4. the value group vK is p-divisible.

This class of valued fields satisfy the AKE$_\preceq$ and AKE$_{\preceq_\exists}$ principles. If K in addition has characteristic p as well, it also satisfies the AKE$_\equiv$ principle [Kuh16]. For separably tame valued fields, similar results hold with some more caveats due to Kuhlmann and Pal [KP16]. We will elaborate on these results for separably tame valued fields further in this thesis.

Another interesting recent result is that of an AKE principle for henselian finitely ramified fields in [ADJ24]. Their result state that by adding certain predicates to the language of the residue field we can obtain an AKE principle for the class of valued fields of residue characteristic p such that the interval $(0, vp]$ is finite of size e: for henselian valued fields $(K, v), (L, w)$ of characteristic $(0, p)$ with the same finite ramification then the AKE$_\equiv$ principle holds except the elementary equivalence on residue fields is in this previously mentioned expanded structure.

The particular result we will expand upon in this thesis is the result of Jahnke and Kartas in [JK25]. They obtain AKE principles for perfectoid fields and their tilts. Their methods all take place in a setting with only perfect fields (in particular perfectoid fields). We extend some of their results to the deeply ramified (non-perfect) setting.

In chapter 2, we will give an introduction to (valued) field theory, as well as introduce some more advanced theorems which we will use in later parts of the thesis. Next, in chapter 3, we exhibit a proof of the AKE principle for separably tame valued fields for fields of possibly infinite p-degree.

Then in chapter 4, we will introduce our own work: we construct an elementary class which will serve as the class for which we exhibit our AKE principle. In particular, we prove the following proposition:

Theorem 3.1.2. *Let (K, v, t) be a pointed valued field of equal characteristic $p > 0$ which is henselian and deeply ramified. If $t \in \mathfrak{m}_v \setminus \{0\}$ then the property of $\mathcal{O}_v\left[\frac{1}{t}\right]$ being separably algebraically maximal is elementary.*

The class of pointed valued fields with this property is the class we are interested in. We will justify this in the second half of chapter 4, and give several ways of recognizing valued fields in the class.

Finally, in chapter 5 we will prove our various AKE principles. Most theorems we prove will be variants of the following theorem:

Theorem 4.1.1. *Let $(K, v) \subseteq (L_0, w_0), (L_1, w_1)$ be extensions of henselian deeply ramified valued fields such that*

(i) L_0/K and L_1/K are separable extensions,

(ii) L_0 and L_1 have the same (possibly infinite) p-degree,

(iii) there is some $t \in K^\times$ such that $vt > 0$ and the valuation rings $\mathcal{O}_v\left[\frac{1}{t}\right], \mathcal{O}_{w_0}\left[\frac{1}{t}\right], \mathcal{O}_{w_1}\left[\frac{1}{t}\right]$ are all algebraically maximal.

Then the following are equivalent:

(i) $(L_0, w_0) \equiv_{(K,v)} (L_1, w_1)$ in the language of valued fields,

(ii) $w_0 L_0 \equiv_{vK} w_1 L_1$ and $\mathcal{O}_{w_0}/t \equiv_{\mathcal{O}_v/t} \mathcal{O}_{w_1}/t$ in the language of ordered abelian groups and rings respectively.

In particular, we will show this result for elementary equivalence, elementary embedding, a modified version without base field, and finally the same result for existential embeddings.

Chapter 2

Preliminaries

In these preliminaries, we will discuss both some basics of field theory and a few more specific topics and theorems in valuation theory which we will use later. We will omit most proofs and refer either to [All09] or [EP05] for basic algebra and valued field theory.

2.1 Basic field theory

Definition 2.1.1 (Algebraic and separable extensions)**.** Let L/K be a field extension i.e. K is a subfield of L. It is called algebraic if for all $\alpha \in L$ there is a non-zero polynomial $p \in K[X]$ such that $p(\alpha) = 0$. The monic polynomial of smallest degree with this property is called the minimal polynomial, and we will write it as f_K^α.

An algebraic extension L/K is separable if for each $\alpha \in L$ with minimal polynomial f_K^α the root α is a simple root: if we compute $(f_K^\alpha)'$ as a formal derivative then $(f_K^\alpha)'(\alpha) \neq 0$.

Definition 2.1.2 (Perfect fields)**.** The following are equivalent properties for a field K:

1. Every finite field extension L/K is separable.
2. If K has characteristic $p > 0$ then the Frobenius morphism $x \mapsto x^p$ is surjective.

A field which satisfies this property is called perfect.

In general, for a ring R of characteristic p, we call R semi-perfect if the map $x \mapsto x^p$ is surjective.

Proof. We refer to [All09, Proposition 4.20]. □

Notice that in particular a field of characteristic 0 is always perfect. For a field of characteristic $p > 0$, we can measure how far away K is from being perfect using the p-degree or Ershov invariant:

Definition 2.1.3 (p-degree)**.** Let K be a field of characteristic $p > 0$. The p-degree e of K is then given by $\log_p([K : K^p])$ of the field extension K/K^p where K^p is the image of the Frobenius map

$x \mapsto x^p$. This is either a natural number $e \geq 1$ or infinity, where we don't distinguish between different cardinalities.

A subset $X \subseteq K$ is called p-independent if for all proper $X' \subsetneq X$ there is a proper inclusion $K^p(X') \subsetneq K^p(X)$. The size of the largest p-independent set (called a p-basis) is the same as the p-degree [Mac39].

Remark 2.1.4. A field K of characteristic $p > 0$ is perfect if and only if it has p-degree 1: it is perfect exactly if $K^p = K$.

There is a more general concept of separable extension for non-algebraic extensions which we will need in chapter 2 to generalize the AKE principle from [KP16] to infinite p-degree.

Definition 2.1.5 (Linearly disjoint and separable extensions)**.** Let $K \subseteq L, L' \subseteq \Omega$ be field extensions. Write $L'.L$ for the smallest subfield of Ω containing both. We say L' and L are linearly independent over K if the map from the tensor product $L` \otimes_K' L' \to \Omega$ is injective.

A general field extension L/K is called separable if the extensions L and K are linearly disjoint over K^p.

Theorem 2.1.6. *The following are equivalent for a field extension L/K:*

(i) L/K is separable,

(ii) Some p-basis of K is p-independent in L,

(iii) All p-bases of K are p-independent in L.

Proof. See [Mac39, Theorem 7 & 10]. □

Using these p-bases, we can define more notions related to separability which will be very useful later on.

Definition 2.1.7. A field extension L/K is separated if L/K is separable and both have the same p-degree. Equivalently, any p-basis of K is a p-basis of L.

We call an intermediate field $L/F/K$ coseparable if L/F is.

A field embedding $\varphi : L \hookrightarrow K$ is separable if $K/\varphi(L)$ is a separable field extension.

We apply this to get the following criterion for separable embeddings.

Corollary 2.1.8. Let L/K and K'/K be two separable field extensions with L/K separated. Then any embedding $L \hookrightarrow K'$ over K is separable.

Proof. Because L/K is separated, a p-basis for K is a p-basis of L. Embedding into K' means that this p-basis of L is still independent and hence K'/L is separable. □

We will use the following theorem to generate and verify some examples of fields that we are interested in later:

Theorem 2.1.9 (Fundamental theorem of Galois theory). *Let L/K be a finite (i.e. the dimension of L as K-vector space is finite), separable and normal (i.e. finite Galois) field extension. Then there is a bijective Galois correspondence between the lattice of intermediate fields $K \subseteq F \subseteq L$ and the lattice of subgroups of $\mathrm{Aut}(L/K)$, the group of automorphisms of L fixing K.*

In particular normal subgroups of $\mathrm{Aut}(L/K)$ correspond to normal intermediate extensions.

Proof. We refer to [All09, Section 6.2] and [All09, Proposition 6.15] for the statement on normal subgroups. □

A particular type of field extension we will use often is the class of Artin-Schreier extensions.

Definition 2.1.10 (Artin-Schreier extension). Let K be a field of characteristic $p > 0$ and $a \in K$ an element such that $f = x^p - x - a \in K[X]$ is irreducible. We will write ρ_a for the generator x and call the extension $K(\rho_a)/K$ an Artin-Schreier extension.

We will write $\wp : K \to K$ for the additive map $x \mapsto x^p - x$.

Artin-Schreier extensions have a lot of nice algebraic properties. We will discuss some of them below:

Lemma 2.1.11. *Let $a \in K$ be an element not in the image of $\wp$. Then $f = x^p - x - a$ is an irreducible polynomial over K and $K(\rho_a)$ is a Galois extension of degree p with Galois group $\mathbb{Z}/p\mathbb{Z}$. The roots of f in $K(\rho_a)$ are given by $\rho_a + i$ for $i \in \mathbb{F}_p$.*

Proof. First we show that $\rho_a + i$ are all roots of f:

$$\begin{aligned}(\rho_a + i)^p &= \rho_a^p + i^p \\ &= \rho_a + a + i \\ &= (\rho_a + i) + a.\end{aligned}$$

This gives p different roots of a polynomial of degree p which means that this must be all of them. Therefore, one root of f generates the entire splitting field, so f must be irreducible.

This extension is then separable because it is generated by ρ_a which is separable because $X^p - X - a$ has distinct roots. □

2.2 Valued Fields

Definition 2.2.1 (Valuation). Let K be a field and Γ an ordered abelian group. A valuation is a surjective group morphism $v : K^\times = K \setminus \{0\} \twoheadrightarrow \Gamma$ which has the following additional property for all $a, b \in K^\times$:

$$v(a + b) \geq \min\{va, vb\}.$$

We will additionally write $v0 = \infty$. Usually, we will write vK for the group Γ, also called the value group.

A valued field then consists of a tuple (K, v) consisting of a field and a valuation.

Example 2.2.2 (Gaussian valuation)**.** Let (K, v) be a valued field. We define the Gaussian valuation on the function field $K(X)$ to be the one induced by

$$v\left(\sum_{i=0}^{n} a_i X^i\right) = \min\{va_i : i \leq n\}$$

on $K[x]$.

Definition 2.2.3 (Valuation ring)**.** The valuation ring associated to a valued field (K, v) is the set

$$\mathcal{O}_v = \{x \in K : vx \geq 0\}.$$

It is closed under multiplication and contains 0 and 1, so it is a subring of K. The units of $\mathcal{O}_v$ are exactly those elements which have value 0. This means that the ideal

$$\mathfrak{m}_v = \{x \in \mathcal{O}_v : vx > 0\}$$

is the unique maximal ideal of $\mathcal{O}_v$, which must then be a local ring. We will write quotient $\mathcal{O}_v/\mathfrak{m}_v$ as Kv. This field will be called the residue field.

Theorem 2.2.4. *A valued field (K, v) has a natural topology which turns it into a topological ring.*

Proof. For $x \in K$ and $\varepsilon \in vK$, the open balls

$$B_\varepsilon(x) = \{y \in K : v(y - x) \geq \varepsilon\}$$

form the basis of a topology. To show that K is a topological ring with this topology note that multiplication and addition map open balls to open balls. □

In chapter 2, we will need that open balls in a valued field generate the field. We prove it here:

Lemma 2.2.5. *Let B be an open ball in a valued field (K, v). Then B generates K as a field.*

Proof. First, we may assume that $0 \in B$ by translation. We can rescale and assume that we look at $B_{\geq 0} = \mathcal{O}_v$ which clearly generates K as a field. □

2.3 Coarsening

One construction that we will use extensively throughout this thesis is called coarsening. It is a method to "split" a valuation into parts that might be easier to understand separately. A more thorough discussion and introduction can be found in [EP05].

Proposition 2.3.1 (Coarsening)**.** *Let (K, v) be a valued field with $\Delta \subseteq vK$ a convex subgroup of vK. Then there exist a valuation $w : K^\times \to vK/\Delta$ such that the residue field Kw is a valued field with valuation $\overline{v} : Kw^\times \to \Delta$ such that for $\overline{x} \in Kw$ we have $\overline{v}(\overline{x}) = vx$. This is the coarsening with respect to Δ.*

Proof. Because Δ is a convex subgroup the quotient vK/Δ is still ordered: we say $0 < a$ if and only if $\Delta < a$. In particular this has the property that $\overline{a} \leq \overline{b}$ if $a \leq b$:

$$\begin{aligned} a \leq b &\Rightarrow 0 \leq b - a \\ &\Rightarrow \overline{0} \leq \overline{b - a} \\ &\Rightarrow \overline{a} \leq \overline{b}. \end{aligned}$$

Write π for the quotient map $vK \twoheadrightarrow vK/\Delta$. Define $w = \pi \circ v$, we claim this is a valuation on K. Clearly the map $K^\times \twoheadrightarrow vK \twoheadrightarrow vK/\Delta$ is still a surjective morphism, therefore we only need to show that $w(x + y) \geq \min\{wx, wy\}$. This is pretty straightforward:

$$\begin{aligned} w(x+y) &= \overline{v(x+y)} \\ &\geq \overline{\min\{vx\}, \{vy\}} \\ &= \min\{\overline{vx}, \overline{vy}\} \\ &= \min\{wx, wy\}. \end{aligned}$$

Note that (K, w) is a valuation ring with $\mathfrak{m}_w = \{x \in K : vx > \Delta\}$.

We now show $(Kw, \overline{v})$ is a valued field as well.

We first show that the expression for $\overline{v}$ in the theorem statement is well-defined. Suppose we have $x, x' \in \mathcal{O}_w$ such that $\overline{x} = \overline{x'}$ in Kw. We claim that $vx = vx'$. Because $\overline{x - x'} = 0$ we obtain $x - x' > \Delta$, therefore $vx = v(x' + (x - x')) = \min\{vx', v(x - x')\} = vx'$ showing that this map is well-defined. It is surjective because for any $a \in \Delta$ and $x \in K$ with $vx = a$ we have that $x \in \mathcal{O}_w$ so $\overline{vx} = vx = a$.

The valuation is a group morphism because v is. For $\overline{x}, \overline{x'} \in Kw$ we have that

$$\begin{aligned} \overline{v}(\overline{x} + \overline{x'}) &= v(x + x') \\ &\geq \min\{vx, vx'\} \\ &= \min\{\overline{vx}, \overline{v}\overline{x'}\}. \end{aligned}$$

Therefore, $\overline{v}$ is in fact a valuation. □

Definition 2.3.2. We can show coarsenings in a diagram in the following manner: let (K, v) be a valued field with convex subgroup $\Delta \subseteq vK$. Then the coarsening is diagrammatically shown as a diagram of places:

$$K \xrightarrow{w} Kw \xrightarrow{\overline{v}} Kv$$

It turns out that most coarsenings do not change the induced topology on a valued field.

Theorem 2.3.3 (Induced topologies)**.** *Let (K, v) be valued field with a non-trivial coarsening w. Then w and v induce the same topology.*

Proof. We refer to [EP05, Theorem 2.3.4]. □

It is possible to repeatedly coarsen a valuation to get more complex diagrams of places.

Definition 2.3.4 (Standard decomposition)**.** Let (K, v) be a valued field with $t \in \mathfrak{m}_v \setminus \{0\}$. The standard decomposition with respect to t is a repeated coarsening.

We take the Δ^+ to be the smallest convex subgroup such that $vt \in \Delta^+$ and Δ^- the largest convex subgroup such that $vt \notin \Gamma$. We then first coarsen with respect to Δ^- to get the following coarsening:

$$K \xrightarrow{v^-} Kv^- \xrightarrow{\overline{v}} Kv$$

and then coarsen v^- with respect to Δ^+ to get nested coarsenings

$$K \xrightarrow{v^+} Kv^+ \xrightarrow{\overline{v^-}} Kv^- \xrightarrow{\overline{v}} Kv\ .$$

This is called the standard decomposition. The valuation v^+ has value group vK/Δ^+, $\overline{v^-}$ has Δ^+/Δ^- and $\overline{v^-}$ has Δ^+.

Remark 2.3.5. The valuation ring of v^+ is isomorphic to the localization $\mathcal{O}_v\left[\frac{1}{t}\right] \subseteq K$. It is the smallest valuation ring containing $\mathcal{O}_v$ such that t is invertible.

Similarly, the residue field of w is isomorphic to the fraction field of $\mathcal{O}_v/\sqrt{(t)}$. This ring is a domain because the prime ideals of $\mathcal{O}_v$ are linearly ordered.

2.4 Valued field extensions

Definition 2.4.1. Let (K, v) and (L, w) be valued fields such that $K \subseteq L$. We say that this is an inclusion or extension of valued fields if $\mathcal{O}_v \subseteq \mathcal{O}_w$ and $\mathfrak{m}_v = \mathfrak{m}_w \cap \mathcal{O}_v$ i.e. the inclusion $\mathcal{O}_v \to \mathcal{O}_w$ is a local morphism of rings.

An extension of valued fields $(K, v) \subseteq (L, w)$ induces an extension of residue field and value group: we get natural inclusion morphisms $vK \hookrightarrow vL$ and $Kv \hookrightarrow Lw$. Associated to such an inclusion we have two (possibly infinite) quantities: the ramification index and inertia degree.

Definition 2.4.2. Let $(K, v) \subseteq (L, w)$ be an inclusion of valued fields. The ramification index $e(w/v)$ of L/K is the cardinality of the group

$$e(w/v) = |wL/vK|$$

and the inertia degree $f(w/v)$ is the degree of the field extension

$$f(w/v) = [Lw : Kv].$$

If the valuation is clear from context, we will simply write e and f.

We can bound the inertia degree and ramification index using a theorem called the fundamental inequality.

Theorem 2.4.3 (Fundamental inequality)**.** *Let (K, v) be a valued field and L/K a finite field extension and $\{w_i\}_i$ the collection of extensions of v to L. Then*

$$\sum_i e(w_i/v) f(w_i/v) \leq [L : K].$$

Proof. We refer to [EP05, Theorem 3.3.4]. □

For some special valued fields the fundamental inequality is often an equality. These fields are often studied because they have nice model-theoretic properties.

Definition 2.4.4. A valued field (K, v) is called (separably) algebraically defectless if for all (separable) algebraic extensions the fundamental inequality is an equality.

Similarly, it is called (separably) algebraically maximal if all (separable) algebraic extensions L/K with $e = f = 1$ are trivial: they have $L = K$.

Later we will need some statements about extensions of valued fields such that the original field is dense as a topological space in the extension.

Lemma 2.4.5. *Let $(K, v) \subseteq (L, w)$ be a valued field extension such that K is dense in L. Let $t \in \mathfrak{m}_v \setminus \{0\}$ be any element. Then $\mathcal{O}_v/t \cong \mathcal{O}_w/t$.*

Proof. Clearly the map $\mathcal{O}_v/t \to \mathcal{O}_w/t$ is injective. We show it is also surjective. Take any $a \in \mathcal{O}_w$.

The field K is dense in L exactly if every $a \in L$ is a limit point of K. Then the open ball

$$B = \{b \in K : v(a - b) \geq vt\} \subseteq \mathcal{O}_w$$

must be non-empty. Take any element $a' \in B \cap \mathcal{O}_w$. Then $v(a - a') \geq t$ so $a - a' \in (t)$ and therefore $\overline{a} = \overline{a'} \in \mathcal{O}_w/t$. □

Corollary 2.4.6. Let (K, v) be a valued field with Artin-Schreier closure $(K^{\mathrm{AS}}, v^{\mathrm{AS}})$. Then $\mathcal{O}_{v^{\mathrm{AS}}}/t$ is semi-perfect.

Proof. By [Kuh10, Theorem 1.11] the field K^{AS} is dense in its perfect hull $((K^{\mathrm{AS}})^{\frac{1}{p^\infty}}, v')$ because the former is Artin-Schreier closed. From Lemma 2.4.5 we know $\mathcal{O}_{v^{\mathrm{AS}}}/t \cong \mathcal{O}_{v'}/t$. The latter is semi-perfect because it is a quotient of the semi-perfect ring $\mathcal{O}_{v'}$. □

This density is key in the following definition. In this thesis, we are interested in valued fields which are almost perfect, where almost means that we can arbitrarily approximate pth roots of any element.

Definition 2.4.7. A valued field (K, v) of characteristic p is called deeply ramified if K is dense in its perfect hull. Equivalently, it can be defined by the property that the completion $(\widehat{K}, v)$ is perfect.

2.5 Henselian valued fields

Definition 2.5.1. Let (K, v) be a valued field. It is called henselian if for all finite algebraic extensions L/K there is a unique extension of v to L.

The following is a basic result in valuation theory.

Proposition 2.5.2. *Let (K, v) be a valued field. Then the following are equivalent:*

1. *(K, v) is henselian.*
2. *For all $f \in \mathcal{O}_v$ and $a \in \mathcal{O}_v$ such that $\overline{f}(\overline{a}) = 0$ and $\overline{f}'(\overline{a}) \neq 0$ in Kv there is an element $b \in \mathcal{O}_v$ such that $\overline{b} = \overline{a}$ and $f(b) = 0$.*
3. *For any irreducible $f \in \mathcal{O}_v[X]$ which is non-constant on the residue field Kv, there is an irreducible $g \in \mathcal{O}_v[X]$ with $\overline{g} \in Kv[X]$ irreducible such that $\overline{f} = \overline{g}^s$.*

Proof. See [EP05, Theorem 4.1.3]. □

If (K, v) is a henselian valued field with a finite extension L/K we will often also write v for the (unique) valuation of v to L.

Since henselian valuations extend uniquely to any algebraic field extension, we get for any algebraic extension L/K the inequality

$$[L : K] \geq [Lv : Kv](vL : vK).$$

It turns out that we can add a correction term to this equation to make it an equality.

Theorem 2.5.3. *Let (K, v) be a henselian valued field and L/K a normal finite algebraic extension. Then there is some $n \in \mathbb{N}$ such that*

$$[L : K] = p^n[Lv : Kv](vL : vK).$$

Proof. For a proof see [Bou65, Page 190, Exercise 9]. □

The power p^n is the defect of the extension.

We will also use the following property of henselian valued fields, and a corollary of it:

Lemma 2.5.4 (Krasner's lemma)**.** *Let (K, v) be a henselian valued field with an algebraic separable algebraic extension L/K. Take $a \in L$ and let $f_K^a = (x - a_1) \cdots (x - a_n)$ be the minimal polynomial of a over K with $a = a_1$. If some $b \in L$ has the property that*

$$v(b - a_i) > \max\{v(a_j - a) : j \neq 1\}$$

then $K(a, b) = K(b)$. In particular $a \in K(b)$.

Proof. For a proof we refer to [EP05, Theorem 4.1.7]. □

Definition 2.5.5. For a separable polynomial f we write

$$\text{kras}(f) = \max\{v(\alpha - \alpha') : \alpha, \alpha' \text{ are distinct roots of } f\}.$$

It turns out that the roots of a polynomial are continuous in the coefficients of that polynomial: if two polynomials are similar, then their roots are also close together. This similarity of polynomials is expressed using the Gaussian valuation on $K[X]$.

Proposition 2.5.6. *Let (K, v) be a valued field $f \in K[X]$ be a separable monic polynomial. Then for every $\varepsilon \in vK$ there is some $\delta \in vK$ such that the following holds:*

If g is monic with $v(f - g) > \delta$ then $\deg f = \deg g$, and for each root α of f there is a root β of g such that $v(\alpha - \beta) \geq \varepsilon$. Moreover, if $\varepsilon > \mathrm{kras}(f)$ then the choice of β is unique and g is separable.

Proof. See [ĆKS23, Theorem 14]. □

Now combining Lemma 2.5.4 and Proposition 2.5.6 gives the following corollary:

Corollary 2.5.7. Let $f \in K[X]$ be monic, separable and irreducible. Then there is some $\delta \in vK$ such that for all monic polynomials g with $v(f - g) > \delta$ the polynomial g is irreducible and each root α of f there is a unique root β of g such that $K(\beta) = K(\alpha)$.

Proof. Let $\varepsilon > \mathrm{kras}(f)$, and pick δ as given by Proposition 2.5.6. Let α be a root of f and β the root of g such that $v(\alpha - \beta) \geq \varepsilon > \mathrm{kras}(f)$. Then by Krasner's lemma $K(\alpha) = K(\beta)$.

Because

$$\deg g = \deg f \leq [K(\alpha) : K] = [K(\beta) : K] \leq \deg g$$

the polynomial g must be irreducible. □

One concept we will often run into is that of an unramified extension. We will use a non-standard definition.

Definition 2.5.8. Let $(L, w)/(K, v)$ be an extension of henselian valued fields. We will call it unramified if $f(w/v) = [L : K]$ and Lw/Kv is a separable field extension. In this case, the fundamental inequality of Theorem 2.4.3 is in fact an equality.

2.6 Interpreting valued field extensions

The following section is an elaboration on an argument of Will Johnson in his PhD thesis "Fun with Fields" [Joh16]. We explain why certain properties of extensions of valued fields are definable.

Theorem 2.6.1. *Let (K, v) be a henselian valued field and L/K a finite extension. Then the following property is elementary: for all finite extensions L/K the unique extension (L, w) which have $e(w/v) = 1$, there is an $a \in L$ such that $L = K(a)$ and $0 \leq v(a) \leq v(t)$.*

In the following part we will need some notation to write polynomials over certain subsets of fields. For a subset $\mathfrak{a} \subseteq K$ we write $\mathfrak{a}[X]$ for the set of polynomials with coefficients in $\mathfrak{a}$. We will use the notation $\mathfrak{a}[X]_{\leq d}$ for the polynomials of degree at most d.

Lemma 2.6.2. *Let K be a field. Then field extensions L/K of fixed degree are elementarily interpretable. In particular, we can "quantify" over all (simple) field extensions of fixed degree.*

Proof. We first reduce to the case of simple field extensions. These are extensions generated by one element. For the reduction, note that a non-separable finite extension is still generated as a field by finitely many generators, so one can repeat this process to interpret finite non-separable field extensions instead.

Now let $K(\alpha)/K$ be a simple extension of degree d, which is also the degree of the minimal polynomial $f_K^\alpha \in K[X]$ of α. Then $K(\alpha) \cong K[X]/f_K^\alpha$.

We then know that all elements of $K(\alpha)$ are uniquely of the form $p(\alpha)$ for some $p \in K[X]_{\leq d}$. Therefore, the domain of our interpretations is the d-fold Cartesian product K^d. Addition of polynomials is just addition of the polynomials. For multiplication, we need to be a bit smarter. Multiplying two polynomials, while perfectly interpretable, adds their degrees, and we can only consider polynomials of degree smaller than d. For this we have one more trick up our sleeve: Euclidean division.

If p is a polynomial of degree $n > d$ then there are unique polynomials q, r with q of degree $n - d$ and r of degree $< d$ such that $p = f_K^\alpha q + r$. This is expressible using an elementary formula, so we can interpret multiplication in $K(\alpha)$.

In this interpretation, we can recover K as the constant polynomials.

To express all simple field extensions of degree d, one can quantify over the irreducible polynomials of degree d. Note that irreducibility of a polynomial is an elementary property, because we can express that there is a non-constant polynomial of smaller degree which divides it. $\square$

Lemma 2.6.3. *If (K, v) is a henselian valued field and L/K an algebraic extension, the following statements are equivalent for any $\alpha \in L$:*

(i) $\alpha \notin \mathcal{O}_L$,

(ii) there is a polynomial $p \in \mathfrak{m} \cap K[X]$ of degree at most L/K such that $p(\alpha) = 1$.

Proof. The implication (ii) $\Rightarrow$ (i) is clear. We assume $\alpha \in \mathcal{O}_v$ and show there is no such p. If there is such a $p = \sum_i a_i x^i \in \mathfrak{m} \cap K$ then by the strong triangle inequality $v(1) = v(p(\alpha)) \geq \min\{v(a_i) + i \cdot v(\alpha)\}i$. If $\alpha \in \mathcal{O}_v$ and $a_i \in \mathfrak{m}$ this is clearly impossible because $v(a_i) + i \cdot v(\alpha) > 0$ for all i.

For the converse we may assume that L/K is Galois. Consider all conjugates $\alpha = \alpha_1, \ldots, \alpha_n$. Then by henselianity $v(\alpha_i) < 0$ for all i. The minimal polynomial $f_K^\alpha = \sum_i a_i X^i$ factors as $\prod_i (X - \alpha_i)$. Then $a_0 = \pm \prod_i \alpha_i$ must have minimal valuation because $\sum_i v(\alpha_i)$ is smaller than summing over the valuations of some of the α_i.

Taking $p = -\frac{1}{a_0} f_K^\alpha + 1$, a simple calculation shows that $p \in \mathfrak{m}_v[X]$ and $p(\alpha) = 1$. $\square$

Using this, we can now prove our theorem.

Proof of Theorem 2.6.1. We define a set of formulas which will axiomatize this property.

In particular, for $n \in \mathbb{N}$ we define φ_n to be a formula quantifying over degree n extensions. Note that we can quantify over polynomials of a specific degree by quantifying over the coefficients. Multiplication and evaluation of polynomials can be expressed in formulas only using addition and multiplication. A polynomial $\sum_i a_i X^i$ is separable if and only if $a_i \neq 0$ for some i which is not a pth power.

Given some fixed extension L/K of degree n, we will write $\mathcal{O}_L(a)$ for the formula $\forall g \in \mathfrak{m}_v[X]_{\leq n} : g(a) \neq 1$.

$$\begin{aligned}\varphi_n = \forall f \in K[X]_n : & f \text{ is separable} \wedge f \text{ is irreducible} \\ & \to \exists a \in K[X]/f : \forall g \in K[X]_{<n} : g \neq 0 \to g(a) \neq 0 \\ & \quad \wedge \mathcal{O}_L(\delta a) \wedge \mathcal{O}_L(t/\delta a).\end{aligned}$$

Intuitively, this formula states the following: for all finite separable field extensions L/K, there is an $a \in L$ which is not the root of a polynomial of degree smaller than n (i.e. it generates L) of which the different δa has non-negative value and $t/\delta a$ does as well: $0 \leq v(\delta a) \leq v(t)$. □

Chapter 3

AKE principles for separably tame valued fields

In this chapter, we will discuss the model theory of separably tame valued fields as originally covered in [KP16]. We exhibit a yet unpublished proof by Anscombe to extend these results to infinite p-degree, which allows us to extend our results to infinite p-degree as well.

3.1 Relative Embedding Principles

Key to the AKE principles for separably tame valued fields are relative embedding principles. They allow for back-and-forth arguments to construct isomorphisms of valued fields.

Theorem 3.1.1 (Separable Relative Embedding Property)**.** *Let $(L, v), (K^*, v^*)$ be separably tame valued fields with a common separably tame subfield (K, v). If*

- *L/K is separable,*
- *(L, v) is $\aleph_0$-saturated,*
- *(K^*, v^*) is $|L|^+$-saturated,*
- *vL/vK is torsion-free and Lv/Kv is separable,*

then any embeddings of value groups $\sigma : vL \hookrightarrow v^*K^*$ and residue field $\rho : Lv \hookrightarrow K^*v^*$ over vK and Kv extend to a valued field morphism $(L, v) \to (K^*, v^*)$.

Remark 3.1.2. Any elementary extension of fields $K \preceq L$ is separable.

Now there is a nice way to remove the demand that L is $\aleph_0$-saturated.

Lemma 3.1.3. *Let N, M^* be structures of a language $\mathscr{L}$ and M^* is $\max(\mathscr{L}, |N|)^+$-saturated. If N embeds into an $\mathscr{L}$ structureM elementarily equivalent tot M then N embeds into M^*.*

Proof. The diagram of N is finitely satisfiable in M^* by assumption, the satisfiability of any finite collection of formulas $(\varphi_i(\overline{x}))_{i\leq n}$ is elementary because $\exists\overline{x}, \bigwedge_i \varphi(\overline{x})$ is a sentence satisfied in an elementarily equivalent model. So it is satisfiable in M^* because it is saturated of large enough size. $\square$

Corollary 3.1.4. Theorem 3.1.1 holds if (L, v) is not $\aleph_0$-saturated.

Proof. Let $(L, w), (K^*, v^*)$ be as in the theorem without (L, w) being $\aleph_0$-saturated. By taking an appropriate ultrapower we may assume that (K^*, v^*) is in fact $2^{|L|+}$-saturated. Then the same ultrapower of (L, w) is $\aleph_0$-saturated and the maps ρ, σ lift to embeddings of the value groups and residue fields of these ultrapowers as well.

Both are elementary equivalent to their smaller counterparts, so all conditions of Theorem 3.1.1 are now fulfilled. Therefore, we get an embedding

$$\prod_{\mathcal{U}}(L, w) \to \prod_{\mathcal{U}'}(K^*, v^*)$$

respecting the lift of ρ, σ, which restricts to an embedding

$$(L, w) \to \prod_{\mathcal{U}'}(K^*, v^*)$$

respecting ρ, σ. By the lemma above, we know that this embedding can in fact be chosen such that its image lies in (K^*, v^*). In fact, we can even choose it such that it respects ρ, σ: if we add a formula to the diagram that says $a = \sigma(a)$ for all $a \in vL$ and $x = \rho(x)$ for all $x \in Lv$ then this is still satisfied and hence can also be satisfied in (K^*, v^*). $\square$

3.2 Separably tame AKE for infinite degree of imperfection

We will now show a proof for the AKE-principle for separably tame valued fields which have infinite p-degree. It is a modification of the proof of Franz-Viktor Kuhlmann in [KP16]. The ideas here are due to Anscombe. We will prove the following strengthening of the separable relative embedding property:

Theorem 3.2.1 (Co-separable separable relative embedding property)**.** *Let $(L, w), (K^*, v^*)$ be equal-characteristic p separably tame valued field with a common separably tame subfield (K, v). If*

1. *L/K is separable,*
2. $\operatorname{impdeg} L/K \leq \operatorname{impdeg} K^*/K$,
3. *(K^*, v^*) is $|L|^+$-saturated,*
4. *wL/vK is torsion free and Lw/Kv is separable.*

Then any embeddings $\rho : Lw \to K^*v^*$ and $\sigma : wL \to v^*K^*$ over Kv and vK extend to an embedding $\imath : (L, w) \to (K^*, v^*)$ over (K, v) such that $K^*/\imath[L]$ is a separable extension.

First we state (quite) a few theorems, lemmas, and definitions which are necessary to prove a complex theorem such as the one above.

Theorem 3.2.2 (Separable going down). *Let $(L, w)/(K, v)$ be an extension of valued fields such that (L, w) is separably tame, K is relatively algebraically closed in L and Lw/Kv is algebraic. Then (K, v) is separably tame.*

Proof. For a proof see [Kuh16, Lemma 3.15]. □

Theorem 3.2.3 (Henselian rationality). *Let $(L, w)/(K, v)$ be an immediate function field of dimension 1 of (K, v) which is separably tame. Then there is some $b \in L$ such that $L \subseteq K(b)^h$. In particular if L is henselian then $L = K(b)^h$.*

Proof. We refer to [Kuh19, Proposition 5.7]. □

Theorem 3.2.4 (Strong inertial generation). *Let $(L, w)/(K, v)$ be a function field without transcendence defect with (K, v) defectless. If Lw/Kv is separable and wL/vK is torsion free then the extension of valued fields is strongly inertially generated.*

Proof. See [Kuh16, Theorem 1.5]. □

We will prove the theorem in three steps:

1. First we construct an intermediate field L_0/K such that L/L_0 has no transcendence defect and L_0 is relatively algebraically closed in L. We embed this into K^* by a separable embedding.
2. Secondly, we find L_1/L_0 such that L/L_1 is separated and embed it into K^* such that K^*/L_1 is separable.
3. Finally, we extend the embedding to the whole of L preserving the separability of the extension.

Proof. We may assume that all involved fields are saturated in a manner similar to Corollary 3.1.4. This gives us a cross-section of the residue map and a section of the valuation for K, L, K^* [vdDri+14, Lemma 7.9] [ADF23, Proposition 4.5]. In fact, we can pick those on L and K^* to be compatible with those on K. We call the cross-section $\zeta : Kv \to K$ and the section $\chi : vK \to K$, regardless of whether it is actually a cross-section of L or K.

Step 1: First we define L_0 to be the relative algebraic closure of the field $K(\chi[wL], \xi[Lw])$ in L. By construction, it does not have transcendence defect and so every finitely generated subextension is strongly inertially generated by Theorem 3.2.4. We now embed using compactness: by saturation L_0 embeds into K^* if and only if there is an extension of the structure of K^* satisfying the diagram of L_0. By compactness, we then only need to satisfy every finite subset of the diagram i.e. embed every finitely generated subextension. By [KP16, Lemma 3.5] we do in fact get these embeddings, so the full embedding also exists.

We also claim that L_0/K is a separated extension. First: the extension $K(\chi[wL], \xi[Lw])$ is separated because χ adds transcendental elements which have pth roots by p-divisibility. Similarly, ξ only adds transcendental with pth roots and separable-algebraic elements because Lw is perfect and L/K is separable respectively. This means that this extension is in fact separated. Now the relative algebraic closure L_0 is also separated because it is a purely algebraic separable extension. We conclude the embedding $L_0 \to K^*$ is necessarily separable by Corollary 2.1.8.

Now we note that L_0 is separably tame by Theorem 3.2.2 and L/L_0 is immediate by construction.

Step 2: Next, we find a (possibly transfinite) p-basis of L over L_0, call it $(b_\mu)_{\mu<\nu}$. We will embed each $L_{0,\mu} = L_0(b_\mu)^{\mathrm{rac}}_{\mu<\nu}$ into K^* using transfinite recursion such that the embedding is separable. This will be the intermediate field L_1.

Then afterwards by Corollary 2.1.8 any embedding $L \to K^*$ extending this map will be separable.

The base case $L_{0,0}$ is just L_0 and the limit case is also straightforward. Therefore, we only need to focus on the induction step.

Suppose we have a separable embedding $L_{0,\mu} \to K^*$. Take $c \in L_{0,\mu+1}$. By henselian rationality, there must be a $d \in L_{0,\mu+1}$ such that $L_{0,\mu}(d)^h = L_{0,\mu}(b_\mu, c)^h$. Our elements d and b_μ are p-interdependent, i.e. one is p-independent if the other is, so because b_μ is by assumption p-independent the element d is p-independent.

Choose a pseudo-Cauchy sequence $(d_\delta)_\delta \in L_{0,\mu}$ which has pseudo-limit d and no pseudo-limit in $KL_{0,\mu}$. By the proof of [KP16, Proposition 3.10] this sequence is of transcendental type

We then embed $L_{0,\mu}(d)^h$ into K^* over $L_{0,\mu}$ using this sequence. The quantifier free type of d is determined uniquely by the formulas $v(x - d_\delta) \geq \gamma_\delta$ for a sequence $(\gamma_\delta)_\delta$ [Kap, Theorem 2]. We can transfer these equations to K^* using our found embeddings. By saturation, we can satisfy this type in K^*, and thus we can embed $L_{0,\mu}(d)^h$ into K^* using the universal property of henselization.

We claim that we can do this such that the embedding is still separable as well. To do this, we show that this type can be satisfied by an element which is separable over $L_{0\mu}$. Take any finite set of formulas $v(x - d_\delta) \geq \gamma_\delta$ from earlier. Then the intersection of these balls is just the smallest of these balls. Because $(K^*)^p \cdot K_{0,\mu}$ cannot be the whole field K^* (for example because $\operatorname{impdeg} L_{0,\mu}/K < \operatorname{impdeg} L/K \leq \operatorname{impdeg} K^*/K$), there must be an element disjoint from this subfield contained in this ball, which is then p-independent of $K_{0,\mu}$ in K^*. We conclude that this extension of the embedding can therefore be chosen to be separable.

Step 3: Now finally, we can extend this embedding to the entirety of L in the same manner we have done before using transfinite pseudo-Cauchy sequences except we don't need to worry about separability any more: it is automatically separable because $L/L_{0,\mu}$ is a separated extension and $K^*/L_{0,\mu}$ is separable. □

Given this version of the embedding property, we can prove the $\mathrm{AKE}_\equiv$ property for the class of separably tame valued fields of the same p-degree.

Corollary 3.2.5. The class of separably tame valued field with the same (possibly infinite) p-degree satisfies the $\mathrm{AKE}_\equiv$ principle.

Proof. First, we may assume that the valued fields are non-trivially valued $vK = vL = 0$ because then the statement is immediate from the fact that $Kv = K \equiv L = Lw$.

Let $(K, v), (L, w)$ be two separably tame valued fields of characteristic p and p-degree e with elementarily equivalent value group and residue field. By absoluteness of the statement we may assume the generalized continuum hypothesis [HK23]. Therefore, we may assume that both are saturated of the same size. Then we in fact obtain isomorphisms $\rho : vK \xrightarrow{\sim} wL$ and $\sigma : Kv \xrightarrow{\sim} Lw$.

In the case of infinite p-degree, the saturation gives us a p-basis of cardinality $|K| = |L|$ of K and L.

Both contain the trivial valued field $\mathbb{F}_p$. We perform a back-and-forth argument over this using Theorem 3.2.1 to get an isomorphism of (K, v) and (L, w).

Let $(a_i)_i$ and $(b_j)_j$ be enumerations of K and L and construct the isomorphism compatible with ρ and σ by transfinite induction. Our assumptions in the induction step are that we work over a common valued subfield $(K', v') \subseteq (K, v), (L, w)$ such that both extensions are separable, the quotients of value groups are torsion free, the residue field extensions are separable and (K', v') is separably defectless. This base field will contain increasingly more a_i and b_j.

The base case of the induction is $\mathbb{F}_p \subseteq (K, v), (L, w)$ which clearly satisfies all of these criteria.

Now for the induction step. We may assume we want to embed an a_i, else we just swap K and L and proceed. Let $(K^*, v^*) \preceq (K, v)$ be a strict elementary substructure of smaller cardinality containing K' and a_i. Using Theorem 3.2.1, ρ and σ we embed it into L. We now take K^* as the common substructure for the next step.

The valued field (K^*, v^*) satisfies the induction hypothesis: it is an elementary substructure of K. Therefore, it is separably defectless K/K^* is separable the vK/v^*K^* is torsion free and Kv/K^*v^*. For L/K^* the same conclusions hold by Theorem 3.2.1 and the assumption that our embedding restricts to ρ and σ.

In the limit case, we just take the total embedding $K(a_i)_{i<\mu} \to L$ as induced by the union.

This process yields an isomorphism $(K, v) \cong (L, w)$ and so the two are elementarily equivalent. □

Remark 3.2.6 (Separable relative subcompleteness)**.** If we start with a different subfield than $\mathbb{F}_p$ which satisfies the same assumptions of the induction hypothesis, then we obtain an isomorphism over that subfield. This gives elementary equivalence over this subfield as well. This is called separable relative subcompleteness.

In the case of saturated models, elementary equivalence and isomorphisms are equivalent. We would like for the isomorphism of saturated models obtained from the AKE principle to be compatible with the isomorphisms of the residue fields. One can trace through the proof to see that the isomorphism constructed in Corollary 3.2.5 is compatible with the isomorphisms of residue field and value group.

Corollary 3.2.7. Let $(K, v) \subseteq (K^*, v^*)$ be separably tame valued fields and (K^*, v^*) saturated of size $|K|^+$ such that:

- The extension K^*/K is separable,
- The extension K^*v^*/Kv is separable and v^*K^*/vK is torsion free.

Then any pair of automorphisms

$$\begin{aligned} \rho :& K^*v^* \to K^*v^*, \\ \sigma :& v^*K^* \to v^*K^* \end{aligned}$$

fixing Kv and vK extend to an automorphism of (K^*, v^*) over (K, v).

Chapter 4

Elementary class

We now introduce the main object of study, a certain elementary class of valued fields. In this chapter, we will first introduce the class and show it is elementary, then we will justify why this class is the one we are interested in and give a way to "recognize" its members.

4.1 Axiomatizing separable-algebraic maximality

Lemma 4.1.1. *Let (K, v, t) be a deeply ramified henselian valued field of equicharacteristic $p > 0$ such that $\{vt^n = n \cdot vt : n \in \mathbb{N}\}$ is cofinal in vK. Then for any finite separable field extension L/K there is an $\alpha \in L$ such that $L = K(\alpha)$ and $0 \leq v\delta(a) \leq vt$.*

Proof. Let L/K be such an extension and take any generator $\alpha \in O_v$ which exists by separability. This will have $0 \leq v(\delta\alpha)$.

We get a diagram of valued fields

$$\begin{tikzcd} L \arrow[r] & L.K^{\frac{1}{p^\infty}} \\ K \arrow[u] \arrow[r] & K^{\frac{1}{p^\infty}} \arrow[u] \end{tikzcd}$$

The extension $K^{\frac{1}{p^\infty}}/K$ is immediate and purely inseparable, therefore it is linearly disjoint from L over K. Therefore, we get that $[L.K^{\frac{1}{p^\infty}} : K^{\frac{1}{p^\infty}}] = [L : K]$. In particular f_K^α is still irreducible over $K^{\frac{1}{p^\infty}}$ and $L.K^{\frac{1}{p^\infty}} = K^{\frac{1}{p^\infty}}(\alpha)$. By separability $\delta\alpha = (f_K^\alpha)'(\alpha) \neq 0$ so $v(\delta\alpha) < t^n$ for some $n \in \mathbb{N}$.

We claim that there is an $\alpha' \in K^{\frac{1}{p^\infty}}$ such that $L.K^{\frac{1}{p^\infty}} = K^{\frac{1}{p^\infty}}(\alpha')$ and $\delta\alpha' = \frac{1}{p^N}\delta\alpha \leq vt$. The p^Nth root $\sqrt[p^N]{\alpha}$ does the trick.

Writing $f_K^\alpha = \sum_{i=0}^k a_i x^i$ and naming its roots $\alpha = \alpha_1, \ldots, \alpha_k$, we see that the minimal polynomial of the generator $\sqrt[p^N]{\alpha}$ has roots $\sqrt[p^N]{\alpha} = \sqrt[p^N]{\alpha_1}, \ldots, \sqrt[p^N]{\alpha_n}$ and is given by $\sum_{i=0}^k \sqrt[p^N]{a_i} x^i$ with $a_k = 1$.

Therefore
$$\delta \sqrt[p^N]{\alpha} = \left(f_K^{\sqrt[p^N]{\alpha}}\right)' \left(\sqrt[p^N]{\alpha}\right) = \sqrt[p^N]{(f_K^{\alpha})'(\alpha)} = \sqrt[p^N]{\delta\alpha},$$
and also $K^{\frac{1}{p^\infty}}(\sqrt[p^N]{\alpha}) = K^{\frac{1}{p^\infty}}(\alpha)$ by a degree argument.

Now we approximate this root in K with sufficient precision and demonstrate that it yields the desired element. Let $\varepsilon > \frac{1}{p^N}\operatorname{kras}\left(f_K^{\sqrt[p^N]{\alpha}}\right)$ and δ be as given by Proposition 2.5.6. Let $g \in K[X]$ be a monic polynomial with $v\left(f_K^{\sqrt[p^N]{\alpha}} - g\right) > \delta$. Then for each $\sqrt[p^N]{a_i}$ there is a unique root β_i of g such that $v(\sqrt[p^N]{\alpha_i} - \beta_i) \geq \varepsilon$. We claim that $K(\beta) = L$ and $v\delta\beta = v\delta\sqrt[p^N]{\alpha} \leq vt$.

We know that $K(\beta) = L$ because
$$\begin{aligned} v(\beta^{p^N} - \alpha) &= p^N v(\beta - \sqrt[p^N]{\alpha}) \\ &> \frac{p^N}{p^N}\operatorname{kras}(f) \\ &= \operatorname{kras}(f), \end{aligned}$$
so we conclude by Lemma 2.5.4.

To show that $\sqrt[p^N]{\alpha}$ and β have the same different we use the ultrametric triangle inequality. First note that for $i \neq j$
$$v\left(\sqrt[p^N]{\alpha_i} - \beta_j\right) < \varepsilon \leq v\left(\sqrt[p^N]{\alpha_i} - \beta_i\right)$$
Hence, we claim that $v\left(\sqrt[p^N]{\alpha_i} - \sqrt[p^N]{\alpha_j}\right) = v(\beta_i - \beta_j)$. Take $i \neq j$, then
$$\begin{aligned} v(\beta_i - \beta_j) &= v\left(\beta_i - \sqrt[p^N]{\alpha_i} + \sqrt[p^N]{\alpha_i} - \beta_j\right) \\ &= \min\left\{v\left(\beta_i - \sqrt[p^N]{\alpha_i}\right),\ v\left(\sqrt[p^N]{\alpha_i} - \beta_j\right)\right\} \\ &= \min\left\{v\left(\beta_i - \sqrt[p^N]{\alpha_i}\right),\ v\left(\sqrt[p^N]{\alpha_i} - \sqrt[p^N]{\alpha_j} + \sqrt[p^N]{\alpha_j} - \beta_j\right)\right\} \\ &= \min\left\{v\left(\beta_i - \sqrt[p^N]{\alpha_i}\right),\ v\left(\sqrt[p^N]{\alpha_i} - \sqrt[p^N]{\alpha_j}\right),\ v\left(\sqrt[p^N]{\alpha_i} - \beta_j\right)\right\} \\ &= v\left(\sqrt[p^N]{\alpha_i} - \sqrt[p^N]{\alpha_j}\right). \end{aligned}$$

This means that we have
$$\begin{aligned} v\left(\delta\sqrt[p^N]{\alpha}\right) &= v\left(\prod_{j\neq i}\left(\sqrt[p^N]{\alpha_i} - \sqrt[p^N]{\alpha_j}\right)\right) \\ &= \sum_{j\neq i} v\left(\sqrt[p^N]{\alpha_i} - \sqrt[p^N]{\alpha_j}\right) \\ &= \sum_{j\neq i} v\left(\beta_i - \beta_j\right) \\ &= v(\delta\beta). \end{aligned}$$

□

Proposition 4.1.2. *Let (K, v) be a deeply ramified henselian valued field of equicharacteristic $p > 0$ with $t \in \mathfrak{m}_v \setminus \{0\}$. Then the following are equivalent*

(i) The valued field $\mathcal{O}_v\left[\frac{1}{t}\right]$ is separably defectless.

(ii) For every finite separable valued field extension $K' : K$ with $e(v'/v) = 1$ there is an $a \in O_{v'}$ such that $K' = K(a)$ and $0 \le v'(\delta a) \le vt$.

(iii) The valued field $\mathcal{O}_v\left[\frac{1}{t}\right]$ is separable-algebraically maximal.

Proof. First we prove that if $\mathcal{O}_v\left[\frac{1}{t}\right] = K$ all three statements are trivially true. This is immediately obvious for (i) and (iii). For (ii) this is true by Lemma 4.1.1.

Now we prove the equivalence. We will refer to the standard decomposition of K with respect to t a lot, so will give it for reference:

$$K \xrightarrow{v^+} Kv^+ \xrightarrow{\overline{v^-}} Kv^- \xrightarrow{\overline{v}} Kv$$

In particular if $\mathcal{O}_{v^+} \neq k$, then the residue field Kv^+ is perfect because (K, v) is deeply ramified.

$(i) \Rightarrow (ii)$ Let L/K be a separable valued field extension with $e = 1$. Then because v^+ is separably defectless we get that $[Lv^+ : Kv^+] = [L : K]$. This extension is separable because Kv^+ is perfect. Therefore, the extension is unramified. By [JK25, Lemma 4.1.2] we know that this means there is such a desired α.

$(ii) \Rightarrow (iii)$ Let L/K be an immediate extension with respect to v^+. By Theorem 2.5.3 we know that $[L : K] = p^n(v^+L : v^+K)[v^+L : v^+K]$ where p^n is the defect of the extension with respect to v^+. By assumption the extension was immediate so $[L : K] = p^n$. Then because p-divisible linearly ordered groups have no extensions of degree p^n we get that the extension L/K of the *entire* valuation v is also unramified.

This means that there is an a with $0 \le v(\delta a) \le vt$ and therefore the extension L/K is unramified with respect to v^+. This means the extension is trivial.

$(iii) \Rightarrow (i)$ Because Kv^+ is perfect, the valued field (K, v^+) is separable-algebraically maximal, and vK is p-divisible the valued field is separably tame [KP16, Lemma 2.14]. □

Corollary 4.1.3. The property of being contained in $\mathcal{C}$ is elementary.

Being deeply ramified [KR23, Theorem 1.2] and (ii) in equicharacteristic $p > 0$ are elementary. Then by Theorem 2.6.1 conclude that the class of pointed valued fields of equicharacteristic $p > 0$ (K, v, t) which are henselian, deeply ramified and with $\mathcal{O}_v\left[\frac{1}{t}\right]$ separably defectless is elementary.

We will write $\mathcal{C}_p$ for the class of all structures in this class. In the future, we will want to specialize to the elementary class of valued fields in $\mathcal{C}$ which have a fixed p-degree e. We will write $\mathcal{C}_{p,e}$ for this subclass.

We now give an example of a valued field which is not separably tame, but is contained in $\mathcal{C}$.

Example 4.1.4. Let p be an odd prime. The pointed valued field $(\mathbb{F}_p((t))^{\mathrm{AS}}, v, t)$ is in $\mathcal{C}$. It is deeply ramified and rank 1, so the coarsening $\mathcal{O}_v\left[\frac{1}{t}\right]$ is the trivial valuation ring which is defectless. The field itself however is not separably tame.

We claim that separable extension $F_p((t))(\sqrt{t})\left(\rho_{\frac{1}{\sqrt{t}}}\right)$ is linearly disjoint of $\mathbb{F}((t))^{\mathrm{AS}}$. This is sufficient to show that the composite extension $F_p((t))(\sqrt{t})\left(\rho_{\frac{1}{\sqrt{t}}}\right).\mathbb{F}_p((t))^{\mathrm{AS}}/\mathbb{F}_p((t))^{\mathrm{AS}}$ must have defect: both the value group and residue field of $\mathbb{F}_p((t))^{\mathrm{AS}}$ have no degree p extension. The value is $\frac{1}{p^\infty}\mathbb{Z}$ which is p-divisible, and the residue field ${\mathbb{F}_p}^{\mathrm{AS}}$ has no degree p-extensions either.

Now to show linear independence we use some basic Galois theory. In this case, it is sufficient to show that $F_p((t))(\sqrt{t})\left(\rho_{\frac{1}{\sqrt{t}}}\right) \cap \mathbb{F}_p((t))^{\mathrm{AS}}$ is $\mathbb{F}_p((t))$. Therefore, we assume that the intersection is a non-trivial intermediate extension as in the diagram below.

$$\begin{tikzcd}
 & \mathbb{F}_p((t))(\sqrt{t})\left(\rho_{\frac{1}{\sqrt{t}}}\right).\mathbb{F}_p((t))^{\mathrm{AS}} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{F}_p((t))(\sqrt{t})\left(\rho_{\frac{1}{\sqrt{t}}}\right) \arrow[dr, dash] & & \mathbb{F}_p((t))^{\mathrm{AS}} \arrow[dl, dash] \\
 & L \arrow[d, dash] & \\
 & \mathbb{F}_p((t)) &
\end{tikzcd}$$

The field extension $L/\mathbb{F}_p((t))$ must have degree p: $\mathbb{F}_p((t))^{\mathrm{AS}}/\mathbb{F}_p((t))$ is a tower of degree p extensions and $F_p((t))(\sqrt{t})/\mathbb{F}_p((t))$ has degree $2p$. This means that $L/\mathbb{F}_p((t))$ has degree p. It is also a Galois extension, because it is the intersection of two Galois extensions.

The Galois group $\mathrm{Gal}\left(\mathbb{F}_p((t))(\sqrt{t})/\mathbb{F}_p((t))\right)$ must be the dihedral group D_p of order $2p$, which is the only non-commutative group of order $2p$. This group has no normal subgroup of order 2, so by the fundamental theorem of Galois theory the non-trivial intermediate field L cannot exist.

This demonstrates that $\mathbb{F}_p((t))^{\mathrm{AS}}$ with the t-adic valuation is not separably defectless.

Not every deeply ramified valued field has a parameter t such that (K, v, t) is contained in $\mathcal{C}$. We can find such a field using [JK25, Observation 6.1.5].

Example 4.1.5. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and consider the field $K = \mathbb{F}_p((t))^{\mathcal{U}}$ once again for an odd prime p. We claim its perfect hull $K^{\frac{1}{p^\infty}}$ has no element x with the desired property. It is sufficient to only consider $x \in K$. This is because any element of $K^{\frac{1}{p^\infty}}$ is in the same Archimedean class as an element of K.

Let $(x_i)_{i\in\mathbb{N}} \in K$. Suppose that the coarsening $\mathcal{O}_v\left[\frac{1}{x}\right]$ of $K^{\frac{1}{p^\infty}}$ is algebraically maximal. We show it does not satisfy Item (ii)[1].

[1]Note that technically Item (ii) talks about deeply ramified valued fields. In this case the field is perfect, so separably algebraically maximality and algebraically maximality coincide and reduce to the original criterion in [JK25]

Choose a sequence $y = (y_i)_{i\in\mathbb{N}}$ such that $vy_i \geq n \cdot vx_i$, $vy_i < vy_j$ for $i < j$, and $p \nmid vy_i$ for all i. We show that $L = K^{\frac{1}{p^\infty}}(\rho_{\frac{1}{y}})$ gives a field extension with the desired property.

The value group of $\mathcal{O}_v\left[\frac{1}{x}\right]$ is p-divisible, so the extension $L/K^{\frac{1}{p^\infty}}$ is unramified Galois of degree p. Therefore, there must be an $a \in K$ with $va \geq -vt$ and $L = K(\rho_a)$.

Then we have that the following things are equivalent:

$$\begin{aligned}\frac{1}{y} - a \in \wp\left(K^{\frac{1}{p^\infty}}\right) &\Leftrightarrow y^{p^n} - a^{p^n} \in \wp(K) \text{ for some } n \in \mathbb{N}\\ &\Leftrightarrow y - a^{p^n} \in \wp(K) \text{ for some } n \in \mathbb{N}.\end{aligned}$$

The last equivalence holds because Artin-Schreier roots are additive and $a - a^p$ has a as Artin-Schreier root.

Then by Łos' theorem, the set

$$\left\{i : \frac{1}{y_i} - a^{p^n} \in \wp(K)\right\}$$

is an element of the ultrafilter $\mathcal{U}$. For sufficiently large i, $v\left(\frac{1}{y_i} - a^{p^n}\right) = v\left(\frac{1}{y_i}\right)$ which is not p-divisible. Therefore, $\frac{1}{y_i} - a^{p^n}$ cannot be in the image of $\wp$ because it is not p-divisible and no such a can exist. This gives a desired contradiction.

It turns out that this field does not admit any elementary extension which could be interpreted to be in $\mathcal{C}$ either. We show that no ultrapower of this field can have this property. By the Keisler-Shelah theorem [She71] this is sufficient to show that there is no elementary extension at all.

Let $\mathcal{U}$ be an ultrafilter on some set I, $(K', v') = (K, v)^{\mathcal{U}}$ and $x = (x_i)_{i\in I} \in \mathfrak{m}_{v'} \setminus \{0\}$. By Łos' theorem [TZ12], the set $X = \{i \in I : x_i \in \mathfrak{m}_v \setminus \{0\}\}$ is an element of $\mathcal{U}$. Then again by Łos' theorem and the above remark we can find a field extension L/K' of degree p which has defect for all $i \in X$ with respect to the coarsening[2] v^+ and hence L/K' has defect with respect to $\mathcal{O}_{v'}\left[\frac{1}{x}\right]$.

Example 4.1.6. We can also combine the proof strategy above with the example of the separable defect of $K = \mathbb{F}_p((t))^{\mathcal{U}\,\mathrm{AS}}$ for a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ to get a non-perfect field with no elementary extensions which can be interpreted to be in $\mathcal{C}$.

By the same reasoning as in Example 4.1.5, it suffices to find an upper bound on the smallest degree of separable defect extensions of $\mathbb{F}_p((t))^{\mathrm{AS}}$. Let $(x_i)_{i\in\mathbb{N}} \in \prod_{\mathcal{U}} \mathbb{F}_p((t))$ be an element of $\mathfrak{m} \setminus \{0\}$. Choose a sequence $y = (y_i)_{i\in\mathbb{N}}$ such that $vy_i > i \cdot vx_i$ and $2 \nmid vy_i$. The extension $K(\sqrt{y})(\rho_{\frac{1}{\sqrt{y}}})$ of degree $2p$ has separable defect with respect to $\mathcal{O}_v\left[\frac{1}{x}\right]$ by the same argument as in Example 4.1.4: $K(\sqrt{y})(\rho_{\frac{1}{\sqrt{y}}})/K(\sqrt{y})$ is an unramified Galois extension and hence descends to a Galois extension of degree p on the residue field. Because $K(\sqrt{y})/K$ is purely ramified, the extension of residue fields $K(\sqrt{t})(\rho_{\frac{1}{\sqrt{t}}})v^+/Kv^+$ is Galois of degree p as well: i.e. an Artin-Schreier extension. However, this residue field is Artin-Schreier closed, so there is no such extension. This gives us that this separable extension of degree $2p$ has defect.

[2]For each x_i individually.

4.2 More concrete descriptions

In [JK25], instead of considering deeply ramified valued fields, Jahnke and Kartas consider valued fields (K, v) of residue characteristic p such that $\mathcal{O}_v/p$ is semi-perfect. If K has characteristic 0, these notions are equivalent because in this case $\widehat{\mathcal{O}_v}/p \cong \mathcal{O}_v/p$. In equicharacteristic p, the notion of being deeply ramified is slightly weaker: we only ask that the completion of K is perfect, instead of K itself. We will give a justification why this is the correct generalization to the separably tame case. In doing so we will also find some ways to recognize valued fields in $\mathcal{C}$.

First we introduce a more concrete algebraic description of deeply ramified valued fields of equicharacteristic p.

Lemma 4.2.1. *Let (K, v) be a valued field of characteristic $p > 0$. Then K is deeply ramified if and only if for all $x \in \mathfrak{m}_v \setminus \{0\}$ the ring $\mathcal{O}_v/x$ is semi-perfect.*

Proof. The right to left implication is simple: $\mathcal{O}_v/x \cong \widehat{\mathcal{O}_v}/x$ and $\widehat{\mathcal{O}_v}$ is semi-perfect so $\mathcal{O}_v/x$ is as well.

For the other implication we show that for all $a \in \mathcal{O}_v$ and open balls around $\sqrt[p]{a}$ there is an element of $\mathcal{O}_v$ in it. Let $B_{>\varepsilon}(\sqrt[p]{a})$ be an open ball for some $\varepsilon \in vK$ and take x such that $p \cdot vx > \varepsilon$. Then there is a b such that $b^p - a \in (x^p)$. Then $v(b - \sqrt[p]{a}) \geq vx > \varepsilon$. ☐

One can weaken the criterium above to a slightly weaker statement.

Remark 4.2.2. It is in fact sufficient to have that $\mathcal{O}_v/x_i$ is semi-perfect for some sequence of x_i such that vx_i is cofinal in vK. This is because if $\mathcal{O}_v/x$ is semi-perfect and $vy \geq vx$ then there is a surjection $\mathcal{O}_v/x \twoheadrightarrow \mathcal{O}_v/y$, so this ring is semi-perfect as well.

Now using this equivalent algebraic property, we can relate deeply ramified coarsenings to the original valued field.

Lemma 4.2.3. *Let (K, v) be a valued field with $t \in \mathfrak{m}_v \setminus \{0\}$ such that $\mathcal{O}_v\left[\frac{1}{t}\right] \neq K$. If $\mathcal{O}\left[\frac{1}{t}\right]$ is deeply ramified, so is $\mathcal{O}_v$.*

Proof. By Remark 4.2.2 it is sufficient to show that for all $x \in \bigcap_{i \in \mathbb{N}}(t^i) \setminus \{0\}$ the ring $\mathcal{O}_v/x$ is semi-perfect.

For any such x the ring $\mathcal{O}_v\left[\frac{1}{t}\right]/(x^2)$ is semi-perfect by assumption. In particular for any $a \in \mathcal{O}_v$ there are $b, c \in \mathcal{O}_v$ such that

$$\left(\frac{b}{t^n}\right)^p = a + \frac{c}{t^m}x^2.$$

We will show that this element is contained in $\mathcal{O}_v$ and a pth root mod x. Notice that

$$\begin{aligned} v\left(\frac{c}{t^m}x^2\right) &= vc - m \cdot vt + 2 \cdot vx \\ &\geq vc + vx \\ &\geq vx \\ &\geq 0. \end{aligned}$$

We therefore see that

$$\begin{aligned} v\left(\frac{b}{t^n}\right) &= v\left(a + \frac{c}{t^n}x^2\right) \\ &\geq \min\left\{v(a), v\left(\frac{c}{t^n}x^2\right)\right\} \\ &\geq 0 \end{aligned}$$

and hence $\frac{b}{t^n} \in \mathcal{O}_v$. Now finally we have

$$\begin{aligned} v\left(\left(\frac{b}{t^n}\right)^p - a\right) &= v\left(\frac{c}{t^m}x^2\right) \\ &\geq x. \end{aligned}$$

This shows that

$$\left(\frac{b}{t^n}\right)^p - a \in (x)$$

showing that all elements of $\mathcal{O}_v/x$ have a pth root, so it is semi-perfect. □

Remark 4.2.4. There is also a topological argument for the above theorem: if $\mathcal{O}_v\left[\frac{1}{t}\right]$ is a non-trivial valuation ring then it induces the same topology as $\mathcal{O}_v$ by **??**. Therefore, K is dense in $K^{\frac{1}{p^\infty}}$ with respect to one valuation if and only if it is with respect to the other.

Now we are ready to reap the rewards of this algebraic manipulation. It turns out that for rank 1 fields, it is relatively easy to be contained in the class $\mathcal{C}$:

Lemma 4.2.5. *Let (K, v) be a valued field such that vK has no minimal positive element. If $x, y \in \mathfrak{m}_v$ are in the same Archimedean class then $\mathcal{O}_v/x$ is semi-perfect if and only if $\mathcal{O}_v/y$ is.*

Proof. We know that $\mathcal{O}_v/x$ is semi-perfect if and only if for all elementary extensions $(K', v') \succeq (K, v)$ the quotient $\mathcal{O}_v/\sqrt{(x)}$ is [JK25, Lemma 7.2.19]. Therefore, it suffices to show that $\sqrt{(x)} = \sqrt{(y)}$ in all elementary extensions. This is because x, y are in the same Archimedean class if and only if there are $n, m \in \mathbb{N}$ such that $n \cdot vx \geq vy$ and $m \cdot vy \geq vx$. This is also exactly the condition for $\sqrt{(x)} = \sqrt{(y)}$. These inequalities clearly lift to elementary extensions, so $\sqrt{(x)} = \sqrt{(y)}$ in all elementary extensions of (K, v). □

Corollary 4.2.6. If (K, v) has rank 1 and $\mathcal{O}_v/t$ is semi-perfect for some $t \in \mathfrak{m}_v \setminus \{0\}$ which is not of minimal positive value then $\mathcal{O}_v$ is deeply ramified.

Proof. If $\mathcal{O}_v/t$ is semi-perfect and $v(t)$ is not minimal positive, then vK must be p-divisible by the strong triangle inequality. By Lemma 4.2.5, the field is then deeply ramified. □

Proposition 4.2.7. *If (K, v) is a rank 1 valued field with some $t \in \mathfrak{m}_v \setminus \{0\}$ such that $\mathcal{O}_v/t$ is semi-perfect and $v(t)$ is not minimal positive. Then (K, v, t) is contained in the class $\mathcal{C}$.*

Proof. By the above corollary (K, v) is deeply ramified. The trivially valued field $\mathcal{O}_v\left[\frac{1}{t}\right] = K$ is separable-algebraically maximal. Therefore, (K, v, t) is contained in $\mathcal{C}$. □

For arbitrary valued fields, we obviously cannot use this rank 1 trick. We do however have some way to recognize the kind of fields we are interested in.

Proposition 4.2.8. *Let (K, v, t) be a pointed henselian valued field of equicharacteristic p such that vK is p-divisible and $t \in \mathfrak{m}_v \setminus \{0\}$. If there is some elementary extension $(K^*, v^*) \succeq (K, v)$ such that $\mathcal{O}_{v^*}\left[\frac{1}{t}\right]$ is non-trivial and separably tame, then (K, v) is deeply ramified and contained in $\mathcal{C}$.*

Proof. The valued ring $\mathcal{O}_{v^*}\left[\frac{1}{t}\right]$ is deeply ramified because it is separably tame and non-trivial [KR23, Theorem 1.2]. Then $\mathcal{O}_{v^*}$ is deeply ramified by Lemma 4.2.3. Being deeply ramified is an elementary property so $\mathcal{O}_v$ is deeply ramified as well.

By Proposition 4.1.2 the property that $\mathcal{O}_v\left[\frac{1}{t}\right]$ is separable-algebraically maximal is elementary for deeply ramified fields, so because the elementary extension (K^*, v^*) has this property, so does (K, v). □

It turns out that using the criteria above, it is much easier to tell whether $\aleph_1$-saturated fields are contained in $\mathcal{C}$. This is because it turns out we get just slightly more coarsenings with perfect residue.

Proposition 4.2.9. *If (K, v, t) is a pointed, $\aleph_1$-saturated valued field and $t \in \mathfrak{m} \setminus \{0\}$ such that $\mathcal{O}_v/t$ is semi-perfect and vK is p-divisible. Then it is contained in $\mathcal{C}$ if and only if $\mathcal{O}_v\left[\frac{1}{t}\right]$ is separable-algebraically maximal or, equivalently, separably defectless.*

Proof. The key insight here is that the residue field Kv^+ is also perfect, not just Kv^-. We prove this first. Consider the standard decomposition

$$K \xrightarrow{v^+} Kv^+ \xrightarrow{\overline{v^-}} Kv^- \xrightarrow{\overline{v}} Kv \ .$$

By [JK25, Lemma 2.3.7] the core valued field $(Kv^+, \overline{v^-})$ is spherically complete and therefore defectless. Let $x \in Kv^+$ be an element without pth root and consider the field extension $Kv^+\left(\sqrt[p]{x}\right)/Kv^+$. We claim that this extension is unramified: the value group of $\overline{v^-}$ is a quotient of a p-divisible group and hence p-divisible. The residue field already admits all pth roots by assumption, so this extension is immediate. By defectlessness we then get that $\sqrt[p]{x} \in Kv^+$. This is a contradiction, so the field is perfect.

This means that $\mathcal{O}_v\left[\frac{1}{t}\right]$ has perfect residue, is separable-algebraically maximal/separably defectless and p-divisible value group. Therefore, the valued field is separably tame.

By saturatedness, $\mathcal{O}_v\left[\frac{1}{t}\right]$ cannot be trivially valued. Thus, by Proposition 4.2.8 the pointed valued field (K, v, t) is contained in $\mathcal{C}$. □

Remark 4.2.10. If the pointed valued field $(K, v, t) \in \mathcal{C}$ is $\aleph_1$-saturated, then (K, v^+) cannot be trivially valued and therefore has perfect residue and is separably tame.

In the proof of the AKE principle next chapter, what we continuously use is the fact that the valued field $\mathcal{O}\left[\frac{1}{t}\right]$ is separably tame in saturated extensions. The corollary above gives us that any valued field with this property is contained in $\mathcal{C}$. This means that it is a pretty good guess for the largest class where the proof of the next section will work.

Chapter 5

Ax-Kochen/Ershov principle

We will now prove various version of the AKE principle for two fields in $\mathcal{C}$ if they have the same p-degree i.e. two fields in $\mathcal{C}_{p,e}$.

5.1 Elementary equivalence and substructure

Theorem 5.1.1. *Let $(K, v, t) \subseteq (L, w, t), (K', v', t)$ be pointed valued fields in the class $\mathcal{C}_p$ for a prime $p > 0$ such that $(L, w, t), (K', v', t) \in \mathcal{C}_{p,e}$ for a fixed p-degree e. Assuming*

(i) K'/K and L/K are separable,

(ii) $vK \preceq_\exists v'K'$,

the following are equivalent:

(i) $(L, w) \equiv_{(K,v)} (K', v')$,

(ii) $wL \equiv_{vK} v'K'$ and $\mathcal{O}_w/t \equiv_{\mathcal{O}_v/t} \mathcal{O}_{v'}/t$.

Proof. The claim $(i) \Rightarrow (ii)$ is trivially true.

By absoluteness of the statement we may assume the generalized continuum hypothesis [HK23]. Therefore, by Corollary 4.1.3 we may assume that (K', v') and (L, w) are saturated of size $|K|^+$ and (K, v) is $\aleph_1$-saturated.

The structures $\mathcal{O}_{v'}/t$ and $\mathcal{O}_w/t$ are elementarily equivalent over $\mathcal{O}_v/t$, and so by uniqueness of saturation we obtain an isomorphism $\mathcal{O}_{v'}/t \cong_{\mathcal{O}_v/t} \mathcal{O}_w/t$. The same holds for $v'K'$ and wL.

The isomorphism of rings above reduces to an isomorphism of valuation rings (i.e. an isomorphism of reduced structures)

$$\mathcal{O}_{\overline{v'}} = \mathcal{O}_{v'}/\sqrt{(t)} \cong_{\mathcal{O}_v/\sqrt{(t)}} \mathcal{O}_{w'}/\sqrt{(t)} = \mathcal{O}_{\overline{w}}$$

Therefore

$$\varphi : \left(K'(v')^-, \overline{v'}\right) \cong_{(Kv^-, \overline{v})} \left(Lw^-, \overline{w}\right)$$

as valued fields. We now prove that this extends to an isomorphism of our original valued fields over (K, v).

By $\aleph_1$-saturation of (K, v), the valued field $(Kv^+, \overline{v^-})$ is defectless [JK25, Lemma 2.3.7], with p-divisible value group and with perfect residue field hence it is tame. The composition of valued fields (K, v^-) is therefore also separably tame. Note that the same is true for the appropriate coarsenings of (L, w) and (K', v').

In order to do this, we add a predicate for and $\mathcal{O}_{v^-}$ and once again saturate our models in this new structure. The above remark about tameness is true in these saturated models as well: using the predicate we added this statement is elementary. The usefulness of these predicates is the fact that these coarsenings will now also be saturated. It is however not the case any more that these predicates correspond to the actual coarsenings $v^-, w^-, (v')^-$.

Now we can use Corollary 3.2.7 to lift isomorphisms to these finer coarsenings. By the AKE-principle for separably tame fields (Corollary 3.2.5) of the same p-degree, we see that

$$(K', v^-) \equiv (L, w^-)$$

and by saturation we get an isomorphism of these valued fields. Using Corollary 3.2.7 we can then actually take this isomorphism to be compatible with (K, v^-) and φ. We check all the requirements of the lemma: the (K, v^-) is saturated by a previous remark, K'/K is separable by assumption, the extension $(K'v')^-/Kv^-$ is separable because Kv^- is perfect, the quotient $(v')^-K'/v^-K$ is torsion free by [JK25, Lemma 2.2.2].

This proves the theorem. □

Corollary 5.1.2. If $(K, v, t) \subseteq (L, w, t)$ are both valued fields in $\mathcal{C}_{p,e}$ such that L/K is separable. Then the following are equivalent:

1. $(K, v) \preceq (L, w)$,
2. $vK \preceq vL$ and $\mathcal{O}_v/t \preceq \mathcal{O}_w/t$.

Proof. Take $(K', v') = (K, v)$. Then the statement is exactly Theorem 5.1.1. □

Theorem 5.1.3. *Let $(K, v, t), (L, w, s)$ be pointed valued fields in $\mathcal{C}_{p,e}$ of the same characteristic $p > 0$. If $(vK, vt) \equiv (wL, ws)$ and $\mathcal{O}_v/t \equiv \mathcal{O}_w/t$ then $(K, v) \equiv (L, w)$.*

Proof. The proof is analogous to that of Theorem 5.1.1 where we take the common substructure to be $(\mathbb{F}_p, v_{\text{triv}})$ with the trivial valuation. Each step is similar to the corresponding one in the proof of Theorem 5.1.1. At the end, we get isomorphisms of valued fields over $\mathbb{F}_p$.

The original intermediate isomorphism $\mathcal{O}_{v'}/\sqrt{(t)} \cong_{\mathcal{O}_v/\sqrt{(t)}} \mathcal{O}_w/\sqrt{(t)}$ now becomes an isomorphism over $\mathbb{F}_p$ instead of $\mathcal{O}_v/\sqrt{(t)}$. We can just as well apply Corollary 3.2.7 to this as well. Note that this works for $\mathbb{F}_p$ in particular because it is perfect and trivially valued (hence tame), and therefore any extension is separable and $vK/(v_{\text{triv}}\mathbb{F}_p) = vK$ is torsion free. □

Corollary 5.1.4. Let $K/\mathbb{F}_p(t)^h$ be an algebraic extension of valued fields with ramification such that the valuation ring $\mathcal{O}_v$ of K is semi-perfect mod t. Then we get an elementary embedding of valued fields

$$K \preceq K.\mathbb{F}_p((t)).$$

Proof. We claim that the inclusion of valued fields induces an isomorphism of value group and valuation rings mod t, and is separated. Then because both fields are deeply ramified and $\mathcal{O}_v\left[\frac{1}{t}\right]$ is trivially valued for both fields, we conclude that the embedding is elementary by Corollary 5.1.2.

Now we show all these claims.

We know that $\mathbb{F}_p(t)^h \subseteq \mathbb{F}_p((t))$ is an immediate extension. Therefore, the extension of value groups of $K.\mathbb{F}_p((t))/K$ is trivial as well.

We demonstrate that $\mathcal{O}_K/t = \mathcal{O}_{K.\mathbb{F}_p}((t))/t$ first for simple extensions $K/\mathbb{F}_p(t)^h$ and show we can repeat this process for arbitrary algebraic extension.

Let $\mathbb{F}_p(t)^h(a)$ be a simple algebraic extension. We claim that for all $f \in \mathbb{F}_p((t))(a)$, there is an $\widetilde{f} \in \mathbb{F}_p(t)^h(a)$ such that $v(f - f') \geq t$. The element f is a finite linear combination of powers of a and elements of $\mathbb{F}_p((t))$: $f = \sum_{i=0}^n x_i f^i$ for $x_i \in \mathbb{F}_p((t))$. Polynomials are continuous, and therefore there is some δ such that if $v(\widetilde{x}_i - x_i) \geq \delta$ and $v(\widetilde{f} - f) \geq \delta$ then

$$v\left(\sum_i x_i f^i - \sum_i \widetilde{x}_i f^i\right) \geq v(t).$$

Because $\mathbb{F}_p(t)^h$ is dense in $\mathbb{F}_p((t))$, we may take all $\widetilde{x}_i$ in $\mathbb{F}_p(t)^h$. This gives that $\mathcal{O}_K/t \to \mathcal{O}_{K.\mathbb{F}_p((t))}/t$ is a surjective map. It is injective by construction, so it is an isomorphism as well.

This works in general as well: if we want to show that some residue class of $\mathcal{O}_{K.\mathbb{F}_p((t))}/t$ is in the image of the inclusion, it suffices to consider a finite simple extension generated by an element of this residue class.

By assumption $v(t)$ is not minimal positive in vK. Then by Proposition 4.2.7 our field is deeply ramified.

Now all we need to show is that the extension is separated. To do this, there are two cases: $K/\mathbb{F}_p(t)^h$ is separable, or it is not.

If $K/\mathbb{F}_p(t)^h$ is separable-algebraic, then t is a p-basis of both K and $K.\mathbb{F}_p((t))$.

If K is perfect, then the extension is separable as well by definition of perfect fields. The only remaining case is if $K/\mathbb{F}_p(t)^h$ is non-perfect but non-separable. Then $\sqrt[p^n]{t}$ is a p-basis for some $n > 1$ and also in $K.\mathbb{F}_p((t))$.

This means the extension $K.\mathbb{F}_p((t))/K$ is always separable and so is an elementary embedding of valued fields. □

Corollary 5.1.5. The natural embedding $(\mathbb{F}_p(t)^h)^{\mathrm{AS}} \subseteq (\mathbb{F}_p(((t))))^{\mathrm{AS}}$ is elementary.

Proof. We show that $(\mathbb{F}_p(t)^h)^{\mathrm{AS}}.\mathbb{F}_p((t)) = \mathbb{F}_p((t))^{\mathrm{AS}}$ and use Corollary 5.1.4. To show the equality, we use Krasner's lemma (Lemma 2.5.4).

Clearly the former is contained in the latter. We show the first has all roots of Artin-Schreier roots. By the same argument as in Corollary 5.1.4, the inclusion $(\mathbb{F}_p(t)^h)^{\mathrm{AS}} \subseteq (\mathbb{F}_p(t)^h)^{\mathrm{AS}}.\mathbb{F}_p(t)^h$ has dense image. Now take any $f \in (\mathbb{F}_p(t)^h)^{\mathrm{AS}}.\mathbb{F}_p(t)^h$. By Lemma 2.5.4 it is sufficient to show that $X^p - X - \tilde{f}$ has a root for $\tilde{f}$ sufficiently close to f. We can get such an $\tilde{f}$ from $(\mathbb{F}_p(t)^h)^{\mathrm{AS}}$. □

Remark 5.1.6. An analogous statement to the one above for the inclusion $\mathbb{F}_p(t)^h \subseteq \mathbb{F}_p((t))$ is still one of the major open problems in the model theory of valued fields.

5.2 Existential substructures

We can also prove a similar AKE principle for existential embeddings. First we need an equivalent characterization for existential substructures.

Lemma 5.2.1. *Let $\mathscr{L}$ be some first order language with structures A, B. Then $A \preceq_\exists B$ if and only if there exists some elementary superstructure $C \succeq A$ such that B is a substructure of C over A.*

Proof. A more general statement can be found as an exercise in section 6.5 of [Hod93]. □

Using this, we are able to prove the desired theorem about existential substructures:

Theorem 5.2.2. *Let $(K, v) \subseteq (L, w)$ be a separable inclusion of deeply ramified valued fields in a fixed class $\mathcal{C}_{p,e}$. Then the following statements are equivalent:*

1. $(K, v) \preceq_\exists (L, w)$,
2. $vK \preceq_\exists wL$ *and* $\mathcal{O}_v/t \preceq_\exists \mathcal{O}_w/t$.

Proof. This proof is similar to that of Theorem 5.1.1. We may assume that L is saturated of size $|K|^+$.

Then can find a sufficiently saturated ultrapower (K^*, v^*) of (K, v) such that we have an embedding $\mathcal{O}_w/t \to \mathcal{O}_{v^*}/t$ and $wL \to v^*K^*$ over (K, v) using Lemma 5.2.1. Then the reduced ring $\mathcal{O}_w/\sqrt{(t)}$ embeds into $\mathcal{O}_{v^*}/\sqrt{(t)}$ over $\mathcal{O}_v/\sqrt{(t)}$ as well. This is exactly an embedding of valued fields $(Lw^-, \overline{w}) \to (K^*(v^*)^-, \overline{v^*})$ over (K, v).

$$\begin{tikzcd}
(Kv^-, \overline{v}) \arrow[r] \arrow[dr] & (Lw^-, \overline{w}) \arrow[d] \\
& (K^*(v^*)^-, \overline{v^*})
\end{tikzcd}$$

As before, the valued field $(Lv^+, \overline{v^-})$ is tame by saturation so the composition of valuations (L, v^-) is separably tame.

Now, we can use the separable relative embedding property to embed (L, w^-) into $(K^*, (v^*)^-)$ over (K, v^-). To see this note that by assumption L/K is separable and the residues of all non-trivial coarsenings are perfect because the valued fields are deeply ramified. The value group extension is torsion free because we assumed that $vK \preceq_\exists vL$.

Now, we obtain a commutative diagram using the separable relative embedding property

$$\begin{tikzcd}
(K, v^-) \arrow[r] \arrow[dr] & (L, w^-) \arrow[d] \\
& (K^*, (v^-)^*)
\end{tikzcd}$$

Because it is compatible with the embedding $O_w/t \to \mathcal{O}_{v^*}/t$ we actually get that this is an embedding with respect to the total valuations v, w, v^*. Now because $(K, v) \to (K^*, v^*)$ is elementary by construction, by Lemma 5.2.1 we must have that $(K, v) \preceq_\exists (L, w)$. $\square$

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