Chaos and Derivative Logic in Topological Dynamics

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

This thesis explores the connection between dynamical systems and logic. The relationship between the two subjects was first established by Artemov et al., who developed Tarski's idea of linking topology and logic in order to reason about dynamic topological systems. Their framework was later extended by Kremer, Mints and Fernández-Duque.

Expanding their framework, we introduce a non-deterministic generalisation of dynamical systems and a characterisation of chaos in such systems. Moreover, we provide axiomatisation for a natural sub-class of all topologically transitive non-deterministic dynamical systems.

Topological semantics with the Cantor derivative operator gives rise to derivative logics, also referred to as d-logics. These logics have not previously been studied in the framework of dynamical systems. We show that the logics **wK4C** and **GLC** both have the finite model property and are sound and complete with respect to the d-semantics in the deterministic setting. In particular, we prove that **wK4C** is the d-logic of all dynamic topological systems and **GLC** is the d-logic of all dynamic topological systems based on a scattered space.

The main contribution of this work is the foundation of a general proof method for finite model property and completeness of dynamic topological d-logics. Furthermore, such a result for **GLC** may constitute the first step towards a proof of completeness for the trimodal topo-temporal language with respect to a finite axiomatisation - something known to be impossible over the class of all spaces.

Contents

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Introduction

"I say unto you: one must still have chaos within oneself, to be able to give birth to a dancing star."

Friedrich Nietzsche,
Thus Spoke Zarathustra.

Topological dynamics is the study of dynamical systems from the viewpoint of general topology. In this sense, a dynamical system is a metric space with a continuous function that expresses the temporal relations between points in space. As temporal relations are functional, the system is in principle deterministic, meaning that each point leads to exactly one possible future. The founder of dynamical systems theory is considered to be the French mathematician Poincaré, who published his research in two monographs between the years 1892 and 1910 [34, 33]. In one subfield of dynamical systems theory, called chaos theory, dynamical systems are studied as objects in which irregularities and unpredictable behaviours are dictated by deterministic laws that are sensitive to initial conditions. In chaotic dynamical systems small changes in the initial conditions of the system can greatly affect the outcome, which diverges greatly. This is the case even though dynamical systems are deterministic and thus, in principal, one should be able to predict them entirely. Chaotic behaviours can be found in a variety of natural systems, while their effects and applications are well studied in physics [29], biology [13], economy [23], cryptography [3] and many other fields. The study of mathematical chaos theory is abundant and varied, and although Poincaré was the earliest proponent of the theory, the most prominent and universally accepted approach to chaos in topological dynamics was suggested by Devaney [14]. According to Devaney, a chaotic system could be described by having specific topological properties that are preserved under a certain type of morphisms called quasiconjugacies.

In academic fields like philosophy, computer science and in particular areas of the natural sciences, such as quantum mechanics, non-determinism retains important value. For some, it is a matter of convenience, while others rely exclusively on its existence. Although the topological setting of dynamical systems can certainly deal with non-deterministic constructions, to the best of the author's knowledge this line of research has not been studied in the context of topological dynamics. This rises some questions regarding dynamical systems and chaos. For instance, what new form will they take when some large number of alternative timelines become accesible? The extension of deterministic dynamical systems to non-deterministic ones will necessitate the use of relations rather than functions. One tool that adequately deals with relational topological structures such as these is modal logic. The connection between modal logic and dynamical systems has long been established. The majority of the mathematically valuable results are based on the topological completeness of the logic **S4** with respect to the real line $\mathbb{R}$. This result was first proved by McKinsey and Tarski [31]. The link between dynamical systems and logic was made by Artemov et al. [1] who proved that **S4C** is sound and complete with respect to the class of all dynamic topological systems. A logic for chaotic dynamical systems would be interesting for many reasons, both practical and purely theoretic in nature. Unfortunately, the properties that constitute chaos are too strong to be expressed by a limited language with only the functional and topological operators. In particular, it will require some stronger properties, such as the 'henceforth' operator from linear temporal logic [35] that was first considered in relation to dynamic topological systems by Kremer and Mints [25]. Fernández-Duque showed that the trimodal language that includes the 'henceforth' operator is not finitely axiomatisable for the class of all dynamic topological spaces [19]. A different approach will therefore have to be considered. One natural option would be to examine topological spaces with a relation instead of a function. This will form a non-deterministic system, for which one has to define the appropriate analogue of Devaney's chaos. Another possibility is to consider fragments of the trimodal topo-temporal language $\mathcal{L}_{\Box}^{\bigcirc[*]}$ alongside appropriate revisions that guarantee sufficient expressive power. For example, we may consider a limited topo-temporal language with a revision to the semantics that make it a bit more expressive. Finally, we may consider specific spaces to which finite axiomatisation could be found.

Dynamical systems are normally considered to not have isolated points. This is in order to avoid trivialities, as the world is in principle non-discrete and dynamical systems are often used to describe physical systems. Topological d-semantics uses the Cantor derivative operator instead of the closure operator. This semantics is more expressive and allows us to express properties such as a space being dense-in-itself – a property that is expressible by the formula $\Diamond\top$. We call the logics that use this interpretation d-logics. Their existence is well-known and they were first introduced by Esakia who showed that the d-logic **wK4** is sound and complete with respect to the class of all topological spaces [16]. These logics appear to be a natural choice for the axiomatisation of dynamical systems. However, there are no established results of completeness for such logics in the setting of dynamical systems, i.e. when a topological space is enriched with a continuous function. The most fundamental d-logic is **wK4C**, which consists of **wK4** and the temporal axioms for the continuous function f. A less basic example of a d-logic is **GLC**, which is the logic of all scattered spaces with a continuous function. It is a logic above **wK4C** and unlike any generic logic of the trimodal language $\mathcal{L}_{\Box}^{\bigcirc[*]}$, a complete finite axiomatisation for **GLC**, extended with axioms for the 'henceforth' operator, will not require changes to the trimodal language. This logic is of special interest to us as it may allow for the first finite axiomatisation and completeness results for a logic based on the trimodal topo-temporal language.

This thesis is structured as follows: in Chapter 1 we give the required definitions and notations for the comprehension of the thesis. We also provide some background of prior work on the topic of dynamic topological logics.

In Chapter 2 we introduce Devaney's chaos with its refinement and an alternative proof to Bank's canonical redundancy result, which makes an important implicit assumption explicit. We then introduce the structure preserving maps between dynamical systems called quasiconjugacies and their aid in proving chaos in dynamical systems. Next we provide a non-deterministic interpretation to chaos and show a similar redundancy theorem to the original result for deterministic chaos.

In Chapter 3 we prove that some important properties of dynamical systems are definable in our language while conjecturing that others cannot be expressed in it. We then show that quasiconjugacies are merely a weaker notion from p-morphisms. Finally, we prove completeness for a logic that expresses some of the properties of non-deterministic chaos.

In Chapter 4 we prove the finite model property, soundness and completeness for the d-logic **wK4C** with respect to the class of dynamic topological systems and for the d-logic **GLC** with respect to the class of dynamic topological systems based on a scattered space. We develop a proof technique that, given the right modifications, would work for any d-logic above **wK4C**. Moreover, we motivate the choice of proving completeness for **GLC** rather than any other logic over **wK4C**.

Finally, in Chapter 5 we discuss the results proven, address some of the weaknesses of the approaches in the thesis and provide the next directions for future work.

Chapter 1

Preliminaries

In section 1.1 we present basic concepts in topology that will be used in this thesis. Next we give some essential background on dynamical systems in section 1.2. In section 1.3 we present the trimodal topo-temporal language $\mathcal{L}_{\bigcirc}^{\Box[*]}$, which we will use with the topological semantics. We finish with an overview of prior work on dynamic topological logics.

1.1 Topology

Definition 1.1.1. (*topological space*) A topological space is a pair $\mathcal{X} = \langle X, \tau \rangle$, where X is a set of points, and τ is a subset of $\wp(X)$ that satisfies the following conditions:

- $X, \varnothing \in \tau$;
- if $U, V \in \tau$, then $U \cap V \in \tau$;
- if $\mathcal{U} \subseteq \tau$, then $\bigcup \mathcal{U} \in \tau$.

The elements of τ are called open sets, and we say that τ forms a topology on X. We will often denote such topology by τ_X. A complement of an open set is called a *closed* set.

Given any set $S \subseteq U$, its *interior* is defined as

$$Int(S) = \bigcup \{U \in \tau : U \subseteq S\}.$$

We define *closure* as its dual by

$$Cl(S) = \bigcap \{A \subseteq X : X - A \in \tau \textit{ and } S \subseteq A\}.$$

A subset $A \subseteq X$ is called *dense* if $Cl(A) = X$.

A subset $\mathcal{B} \subseteq \tau$ is called a basis of τ if every open $U \in \tau$ is a union of elements of $\mathcal{B}$. A subset $\mathcal{S} \subseteq \tau$ is called a subbasis of τ if every open $U \in \tau$ is a union of finite intersections of elements of $\mathcal{B}$.

Definition 1.1.2. (*openness and continuity*) A map $f : X \to Y$ between two topological spaces is called *open* if whenever $U \in \tau_X$, then $f(U) \in \tau_Y$. It is called *continuous* if whenever $U \in \tau_Y$, then $f^{-1}(U) \in \tau_X$. An open-continuous bijection is called a *homeomorphism*.

Besides topological spaces, we also have a special interest in metric spaces.

Definition 1.1.3. (*metric space*) Let X be a non-empty set and $d : X \times X \to \mathbb{R}$ a function such that:

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ iff $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

The third condition is also known as the *triangle inequality*. The map d is said to be a metric on X, $d(x, y)$ denotes the distance between x and y and the pair (X, d) is called a *metric space*.

Definition 1.1.4. (*open ball*) Let (X, d) be a metric space and r any positive number in $\mathbb{R}$. The set

$$B_r(x) = \{y : y \in X \textit{ and } d(x, y) < r\}$$

Is called the open ball of radius r around x.

Proposition 1.1.5. Let (X, d) be a metric space. Then the collection of open balls in (X, d) is a basis for a topology τ on X.

We say that the topology τ is *induced* by the metric d. If there exists a metric d inducing a topology τ on a set X, we say that (X, τ) is *metrizable*.

We introduce some important topological spaces that will be useful later.

Example 1.1.1. (*Euclidean space*) Let $\mathbb{R}$ denote the set of real numbers. We define the natural topology τ on $\mathbb{R}$ as the topology generated by the basis

$$\mathcal{B} = \{(a, b) : a < b \;\; \textit{and} \;\; a, b \in \mathbb{R}\},$$

which is the same as the basis of the open ball, where $a = c - r$, $b = c + r$ and c is the centre of the ball. This definition can easily be generalised to any euclidean space $\mathbb{R}^n$, where a set is open if and only if it contains an open ball around each of its points.

Similarly, the set of complex numbers $\mathbb{C}$ and $\mathbb{C}^n$ have a standard topology in which the basic open sets are open balls.

Example 1.1.2. (*Cantor space*) Recall that for an interval $[a, b]$ and a non-negative number c we have

$$[a, b] + c = [a + c, b + c] \quad \textit{and} \quad [a, b] \cdot c = [a \cdot c, b \cdot c]$$

We define the following collections of sets recursively:

- $S_0 = [0,1]$;
- $S_i = \frac{1}{3}S_{i-1} \cup \left(\frac{2}{3} + \frac{1}{3}S_{i-1}\right)$, for any $i \geq 1$.

The Cantor set is then defined as

$$\mathcal{C} = \bigcap_{i=1}^{\infty} S_i$$

A topological space is a *Cantor space* if it is homeomorphic to the Cantor set.

Definition 1.1.6. (*Cantor derivative*) Let $\mathcal{X} = \langle X, \tau \rangle$. Given $S \subseteq X$, the *Cantor derivative* of S is the set $d(S)$ of all limit points of S, i.e.

$$x \in d(S) \iff \forall U \in \tau \ s.t. \ x \in U : U \cap S \backslash \{x\} \neq \varnothing$$

Given subsets $A, B \subseteq X$ and a point $a \in X$, the Cantor derivative satisfies the following properties:

- $d(\varnothing) = \varnothing$
- $d(A \cup B) = d(A) \cup d(B)$
- $A \subseteq B$ implies $d(A) \subseteq d(B)$
- $Cl(A) = A \cup d(A)$
- $a \in d(A)$ implies $a \in d(A \backslash \{a\})$

In order to avoid ambiguity, we will often denote the Cantor derivative of a subset S of the topological space X by $d_X(S)$.

1.2 Dynamical Systems

We begin with a few definitions relevant to dynamical systems and give a few examples for well known dynamical systems.

Definition 1.2.1. (*dynamical system*) A dynamical system $\mathcal{X}_f$ is a triple $\langle X, d, f \rangle$ consisting of a metric space $\langle X, d \rangle$ and a continuous endomorphism $f : X \to X$.

We will often refer to dynamical systems as *dynamic metric systems* to make a clear distinction between them and *dynamic topological systems* of the form $\mathcal{X}_f = \langle X, \tau, f \rangle$, where $\langle X, \tau \rangle$ is a topological space and $f : X \to X$ is a continuous endomorphism.

For the following definitions, let $\mathcal{X}_f = \langle X, d, f \rangle$ be a dynamical system.

Definition 1.2.2. (*orbit*) For each $x \in X$ let the set

$$Orb_f(x) = \{x, f(x), f^2(x), \dots\}$$

be the *orbit* of x under f.

Definition 1.2.3. (*periodic point*) A point $x \in X$ is called periodic if there exists $n \geq 1$ such that $f^n(x) = x$.

The following are two examples of dynamical systems on the Euclidean plane.

Example 1.2.1. (*logistic map*) Let $\mu \in \mathbb{R}$. We define the logistic map $L_\mu : \mathbb{R} \to \mathbb{R}$ for each $\mu \in \mathbb{R}$ as $L_\mu(x) = \mu x(1-x)$. The map $L_{3.741}$ is shown in Figure 1.1.

Example 1.2.2. (*tent map*) Let $\mu \in \mathbb{R}$. We define the tent map $T_\mu : [0,1] \to [0,1]$ for each $\mu \in \mathbb{R}$ as

$$T(x) = \begin{cases} \mu x, & if \;\; x < \frac{1}{2} \\ \mu(1-x), & if \;\; x \geq \frac{1}{2} \end{cases} \quad .$$

The map $T_{1.9}$ is shown in Figure 1.1.

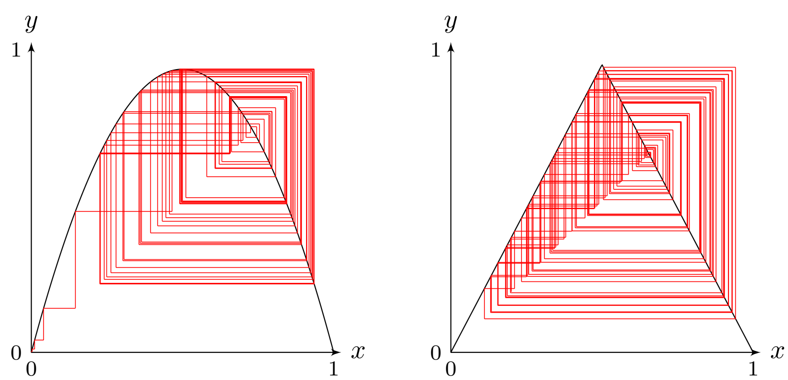

Figure 1.1: The first hundred iterations of the logistic map for $x = 0.00079$ and $\mu = 3.741$ (left); the first hundred iterations of the tent map for $x = 0.2$ and $\mu = 1.9$ (right).

The logistic map and the tent map are canonical examples for simple functions that given the right parameters range in their behaviour from predictable to chaotic.

1.3 Dynamic Topological Logics

In this section we introduce the trimodal topo-temporal language of dynamic topological logics along with its topological closure semantics. Since there is no known complete finite axiomatisation for that language, the discussion regarding the axiomatic systems of dynamic topological logics will be addressed in chapter 3.

Given a non-empty set PV of propositional variables, the language $\mathcal{L}_{\Box}^{\bigcirc[*]}$ is defined recursively as follows:

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Box\varphi \mid \bigcirc\varphi \mid [*]\varphi$$
$$p \in \mathsf{PV}.$$

It consists of the boolean connectives $\wedge$ and $\neg$, the temporal modalities $\bigcirc$ (next) and $[*]$ (henceforth) and the topological modality $\Box$ (interior) with its dual $\Diamond$ (closure). We denote by $\mathcal{L}_{\Box}^{\bigcirc}$ the language that includes only the modalities $\bigcirc$ and $\Box$, and by $\mathcal{L}_{\Box}^{[*]}$ the language that includes only the modalities $\Box$ and $[*]$.

A *dynamic topological system* (DTS) is a a triple $\mathcal{X}_f = \langle X, \tau, f\rangle$, where X is a set, τ is a topology on X and $f : X \to X$ is a continuous function on X. A *dynamic topological model* (DTM) is a quadruple $\mathcal{M}_f = \langle X, \tau, f, \nu\rangle$ where $\langle X, \tau, f\rangle$ is a DTS and $\nu : \mathsf{PV} \to \wp(X)$ is a valuation function assigning a subset of X to each propositional letter in PV.

Definition 1.3.1. (*c-semantics*) Given a DTM $\mathcal{M}_f = \langle X, \tau, f, \nu\rangle$ and a point $x \in X$, the satisfaction relation $\models$ is defined inductively as follows:

1. $\mathcal{M}_f, x \models p \iff x \in \nu(p)$;
2. $\mathcal{M}_f, x \models \neg\varphi \iff \mathcal{M}_f, x \not\models \varphi$;
3. $\mathcal{M}_f, x \models \varphi \wedge \psi \iff \mathcal{M}_f, x \models \varphi$ and $\mathcal{M}_f, x \models \psi$;
4. $\mathcal{M}_f, x \models \Box\varphi \iff \exists U \in \tau$ s.t. $x \in U$ and $\forall y \in U(\mathcal{M}_f, y \models \varphi)$,
   and therefore dually:
   $\mathcal{M}_f, x \models \Diamond\varphi \iff \forall U \in \tau$, if $x \in U$ then $\exists y \in U(\mathcal{M}_f, y \models \varphi)$;
5. $\mathcal{M}_f, x \models \bigcirc\varphi \iff \mathcal{M}_f, f(x) \models \varphi$;
6. $\mathcal{M}_f, x \models [*]\varphi \iff \mathcal{M}_f, f^n(x) \models \varphi$, for all $n \geq 0$.

We write $\mathcal{X}_f \models \varphi$ if φ is valid on $\mathcal{X}_f$, i.e. for any $x \in X$ and any valuation $\nu : \mathsf{PV} \to \wp(X)$, we have $\langle \mathcal{X}_f, \nu\rangle, x \models \varphi$. We will often abbreviate and write $x \models \varphi$ instead of $\mathcal{X}_f, x \models \varphi$ or $\mathcal{M}_f, x \models \varphi$, if no confusion may occur regarding which DTS or DTM is discussed. We may also abbreviate and write $\mathcal{X}_f, S \models \varphi$ instead of $\mathcal{X}_f, x \models \varphi$ for all $x \in S$. This is especially useful when talking about topological spaces.

We will often talk about the *relational semantics* of structures of the form
$$\mathfrak{M}_f = \langle W, R, f, V\rangle.$$

These are Kripke models enriched with a function f. We call such structures *dynamic Kripke models* (DKM). A *dynamic Kripke frame* (DKF) is a structure of the form $\mathfrak{F}_f = \langle W, R, f\rangle$. We denote the satisfaction relation of the relational semantics by $\models_r$. However, when no confusion may occur and when it is clear that we are talking about relational structures instead of topological spaces, we will simply use $\models$ instead of $\models_r$. The definitions of the relational and closure semantics are almost identical and differ only with respect to the topological operators $\Diamond$ and $\Box$. Given a dynamic Kripke model $\mathfrak{M}_f = \langle W, R, f, V\rangle$ and a point $w \in W$, the truth conditions of the topological operations are defined as

4'. $\mathfrak{M}_f, w \models_r \Box\varphi \iff \forall v \in W$, if wRv then $\mathfrak{M}_f, v \models \varphi$,

and therefore dually:

$\mathfrak{M}_f, w \models_r \Diamond\varphi \iff \exists v \in W$ s.t. wRv and $\mathfrak{M}_f, v \models \varphi$.

Let $\mathsf{DTL}_{\mathcal{T},\mathcal{F}}$ denote the set of valid formulas in all dynamic topological systems with a topological space from the class $\mathcal{T}$ and a continuous morphism from the class $\mathcal{F}$. Accordingly, $\mathsf{DTL} = \{\varphi : \models \varphi\}$ denotes the set of all validities in dynamic topological systems.

Given a relation R and a finite Kripke frame $\mathfrak{F}$ we define the R-depth $dpt_R(w)$ as the length of the longest strong R-path emanating at w_0, meaning $w_0 R w_1 R w_2 \ldots$, where $w_i R w_{i+1}$ but not $w_{i+1} R w_i$. We sometimes write $dpt_{[R]}(w)$ instead, where $[R]$ is the modal operator of the relation R.

1.3.1 Prior Work

The relation between modal logic and topology dates back to McKinsey and Tarski in the 1940s [31]. In fact, topological semantics generalises the now-popular Kripke semantics for **S4**, due to the following observation:

Definition 1.3.2. (*induced Alexandroff topology*) Let $\langle W, R\rangle$ be a Kripke frame and $S \subseteq W$ a set of points. Then S is open iff it is an R closed set, i.e. for every $x, y \in W$, if $x \in S$ and xRy then $y \in S$. We denote the R closed set of x by $\uparrow x$ and call it the upset of x. The family of open sets S induces a topology τ_R. For each Kripke frame we can construct such topology on the set W where the interior of each $S \subseteq W$ is defined as

$$Int(S) := \{w \in W : \forall v \in W, wRv \text{ implies } v \in S\}.$$

Under this definition, the intersection of arbitrary open sets is open and thus all such spaces are *Alexandroff spaces* [1]. Each Kripke frame $\langle W, R\rangle$ can therefore be associated with its corresponding Alexandroff space $\langle X_W, \tau_R\rangle$.

For the case were R is reflexive and transitive, we have a one-to-one correspondence between the Kripke frame and its corresponding Alexandroff space. This follows from a series of famous results by McKinsey and Tarski [31]. For any dense-in-itself metric space $\mathcal{X} = \langle X, d\rangle$ and a formula φ that does not contain the operators $\bigcirc$ or $[*]$, the following statements are equivalent:

1. $\varphi \in$ **S4**, where **S4** is the logic of all reflexive-transitive Kripke frames.
2. $\models \varphi$.
3. $\mathcal{X} \models \varphi$.
4. $\mathbb{R} \models \varphi$.
5. $\mathcal{Y} \models \varphi$, for any finite topological space $\mathcal{Y}$.

And the following equivalent that was proven by Kripke [26]:

6. $\mathcal{Y} \models \varphi$, for any Alexandroff space $\mathcal{Y}$.

In particular, we have the following important result:

Proposition 1.3.3. Every Kripke-complete logic above **S4** is also complete with respect to the topological c-semantics (topologically c-complete) [4].

Once we add the temporal operators $\bigcirc$ and $[*]$, matters become much more complex. For instance, the logic of all dynamic topological systems with homeomorphisms $\mathsf{DTL}_{\mathcal{H}}$ is known to be nonaxiomatisable [24], and the logic of all dynamic topological systems DTL is known to not have finite axiomatisation [19]. In fact, there are no known results for complete finite axiomatisation of the trimodal language $\mathcal{L}_{\Box}^{\bigcirc[*]}$. The closest achievement was made by Fernández-Duque [18] where infinite axiomatisation for DTL was introduced under a generalised interpretation of the closure operator $\Diamond$. This will be discussed in detail in Chapter 4. While in the past it was not known whether a complete finite axiomatisation for the trimodal language exists, advances were main on one particular fragment throughout the years.

Let **C** denote the set of axioms

1) $\bigcirc\Diamond p \to \Diamond\bigcirc p$ (continuity)

2) $\bigcirc(p \vee q) \equiv \bigcirc p \vee \bigcirc q$ ($\vee$-distributivity)

3) $\bigcirc\neg p \equiv \neg\bigcirc p$ (functionality)

and the necessitation rule for $\bigcirc$. In regards to the $\mathcal{L}_{\Box}^{\bigcirc}$ fragment of the trimodal language $\mathcal{L}_{\Box}^{\bigcirc[*]}$, we have a few important completeness results. For instance, Mints and Zhang proved completeness of **S4C** for continuous functions on Cantor space [32], and Fernández-Duque proved completeness of **S4C** for continuous functions on the real plane $\mathbb{R}^2$ [17].

Chapter 2

Chaos and Non-Deterministic Dynamical Systems

In this chapter we present Devaney's definition of chaos. We show that while this definition is redundant, it is not immediately obvious and one implicit assumption needs to be taken into account. We present a few more results about chaos and dynamical systems with relation to the shift space and chaos, and we introduce a non-deterministic interpretation to the originally deterministic dynamical systems and chaos.

2.1 Mathematical Chaos Theory

We will start by introducing a few important concepts that are required in order to define a chaotic dynamical system according to Devaney. The original definitions can be found in [14].

Let $\mathcal{X}_f = \langle X, d, f \rangle$ be a dynamic metric system.

Definition 2.1.1. (*Topological transitivity*) The map $f : X \to X$ is topologically transitive if for any nonempty $U, V \in \tau$ there exists $n \geq 0$ such that

$$f^n(U) \cap V \neq \varnothing.$$

Definition 2.1.2. (*Sensitive dependence on initial conditions*) The map $f : X \to X$ has sensitive dependence on initial conditions if there exists $\delta > 0$ such that for any $x \in X$ and $\varepsilon > 0$, there exists $y \in X$ with $d(x, y) < \varepsilon$ and

$$d(f^n(x), f^n(y)) > \delta, \text{ for some } n \geq 0.$$

We will often write that a dynamic metric space $\mathcal{X}_f$ is topologically transitive, has a dense set of periodic points, etc., in order to avoid ambiguity and to make sure that it is clear to which space we refer. With that in mind, such properties are properties of an endomorphism with respect to its domain.

We now have all the definitions needed in order to define a chaotic dynamic metric system according to Devaney.

Definition 2.1.3. (*Devaney's chaos - preliminary version*) A dynamic metric system $\mathcal{X} = \langle X, d, f\rangle$ is chaotic if the following conditions are satisfied:

i) $\mathcal{X}_f$ has a dense set of periodic points;

ii) $\mathcal{X}_f$ is topologically transitive;

iii) $\mathcal{X}_f$ has a sensitive dependence on initial conditions.

2.1.1 Refinement of Devaney's Chaos

In a classic paper by Banks et al. [2], they argue that (iii) follows from (i) and (ii), thus losing the very soul of Devaney's definition that, according to many, made it so intuitive and convincing. It is usually referred to as *the butterfly effect* [30]. Banks' result is widely used in the literature and so by *Devaney's chaos* people usually refer to the conditions (i) and (ii) alone. There is a small gap in Banks' theorem and it does not follow without an extra implicit assumption. Namely, X must be dense-in-itself, or at the very least there should be no point $x \in X$ such that $Orb_f(x) = X$. We first give an example for a dynamic metric system where (i) and (ii) hold, and (iii) fails to hold.

Proposition 2.1.4. There exists a dynamic metric system $\mathcal{X} = \langle X, d, f\rangle$ that has a dense set of periodic points, is topologically transitive, but does not have sensitive dependence on initial conditions.

Proof. Let $X := \{x_0, \dots, x_n\}$. Consider the discrete metric on X defined as $d(x, x) = 0$ and $d(x, y) = 1$ for each $x, y \in X$. Then the open ball $B_r(x_0) = \{x \in X : d(x_0, x) < r\}$ with radius $r > 0$ and a centre x_0 is

$$B_r(x_0) = \begin{cases} \{x_0\} & if \ \ 0 < r \leq 1 \\ X & if \ \ r > 1 \end{cases} ,$$

which induces the discrete topology. We define f to be the successor function, i.e. $f(x_i) = x_i + 1$ for each $i < n$ and $f(x_i) = x_0$ for $i = n$. It is clear that this dynamic metric system has a dense set of periodic points and is topologically transitive, however as there is no $y \neq x$ such that $d(x, y) < \varepsilon$ for $\varepsilon < 1$, then $\mathcal{X}_f$ does not have sensitive dependence on initial conditions. □

Although we showed that Devaney's definition of chaos is not redundant without a further assumption, it would be preferable to suppose that our metric space is dense-in-itself for several reasons. First, we want to be able to talk about general topological spaces and not only about metric spaces and this will only be possible if we drop the condition of sensitive dependence on initial conditions. This will become especially relevant to us since we want to build a formal system around these ideas and topological spaces are easier to work with and generalise these concepts on. Second, non-dense-in-itself spaces are not interesting for the project of dynamical systems. They are rather trivial and the assumption that a space is dense-in-itself, and often also compact (e.g. [21]), is commonly taken for granted in discussions about dynamical systems. As entities that reflect the universe in its physical continuity, this should be of no surprise.

We now prove that under the assumption that $\mathcal{X}_f$ is a dense-in-itself dynamic metric system, the third condition of chaos is indeed redundant.

Theorem 2.1.5. *Let $\mathcal{X}_f = \langle X, d, f \rangle$ be a dynamic metric system where $\mathcal{X}$ is dense-in-itself. Suppose $f : X \to X$ is topologically transitive and its set of periodic points is dense. Then f has sensitive dependence on initial conditions.*

Proof. Let d be a metric inducing a topology on X. We begin by proving that there exists $\gamma > 0$ such that for any $x \in X$ there exists a periodic point p for which

$$d(x, f^n(p)) \geq \gamma \text{ for all } n \geq 0 \ .$$

Since $\mathcal{X}$ is dense-in-itself, it follows that X is infinite. Take an arbitrary periodic point p_1, then it has a finite orbit. Since X is an infinite metric space, then the orbit of p_1 is not dense. Since the periodic points are dense in X, then there must be a periodic point $p_2 \neq p_1$. Since all orbits of periodic points are either the same or disjoint, then p_1 and p_2 are disjoint. Therefore, there must be two periodic points p_1, p_2 such that $Orb_f(p_1) \cap Orb_f(p_2) = \varnothing$. We set

$$\gamma := \min_{n,m \geq 0} \frac{d(f^n(p_1), f^m(p_2))}{2} > 0.$$

By the triangle inequality, it follows that for every $x \in X$ and for all $n \geq 0$ we have either $d(x, f^n(p_1)) \geq \gamma$ or $d(x, f^n(p_2)) \geq \gamma$.

We show that f has sensitive dependence on initial conditions for $\delta := \gamma/4$. Let $x \in X$ be an arbitrary point. From the result above, it follows that there exists a periodic point p such that

$$d(x, f^n p) \geq 4\delta \text{ for all } n \geq 0. \tag{2.1}$$

Since the periodic points of f are dense in X, then for any x we have a periodic point q with period m such that

$$d(x, q) < \varepsilon, \tag{2.2}$$

for any $\varepsilon > 0$ and in particular when $\varepsilon = \delta$. Clearly, there exists a neighbourhood N of p such that for any $y \in N$ we have $d(p, y) < \eta$ for some constant η. From the continuity of f, it follows that

$$d(f^n(p), f^n(y)) < \delta \text{ for all } n \leq m. \tag{2.3}$$

Now, by topological transitivity of f we are guaranteed to have a point z such that $d(x, z) < \varepsilon$ and $f^k(z) \in N$ for some $k \geq 0$. Let $j \geq 0$ fulfil $k \leq jm < k+m$, where m is the period of q. Then clearly, $d(f^{jm}(q), f^{jm}(z)) = d(q, f^{jm-k}f^k(z))$ since jm is a period of q and $k \leq jm$. Moreover, by the triangle inequality we have

$$d(x,q) + d(q, f^{jm-k}f^k(z)) + d(f^{jm-k}f^k(z), f^{jm-k}(p)) \geq d(x, f^{jm-k}(p)).$$

Finally, from (2.1)-(2.3) we derive that

$$d(x, f^{jm-k}(p)) - d(x,q) - d(f^{jm-k}f^k(z), f^{jm-k}(p)) > 4\delta - \delta - \delta = 2\delta,$$

and hence $d(f^{jm}(q), f^{jm}(z)) > 2\delta$. It follows that either $d(f^{jm}(x), f^{jm}(q)) > \delta$ or $d(f^{jm}(x), f^{jm}(z)) > \delta$, and since both $d(x,q) < \varepsilon$ and $d(x,z) < \varepsilon$, we conclude that f has sensitive dependence on initial conditions. □

We can therefore drop the third condition of chaos. Since that was the only condition that was specific to metric spaces, we drop the requirement of metric spaces and let chaos apply to any dynamic topological systems. We paraphrase the definition of chaos with that in mind.

Definition 2.1.6. (*Devaney's chaos*) A dense-in-itself dynamic topological system $\mathcal{X} = \langle X, \tau, f \rangle$ is chaotic if the following conditions are satisfied:

i) $\mathcal{X}_f$ has dense set of periodic points;

ii) $\mathcal{X}_f$ is topologically transitive.

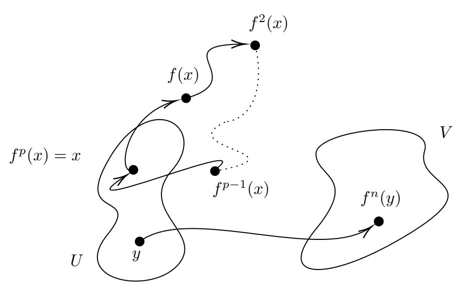

Figure 2.1: Devaney's Chaos.

From this point onward we drop the requirement of a metric space and consider general dynamic topological systems. One connection that is worth mentioning between chaos and the properties of dynamic topological systems is demonstrated by the following lemma:

Lemma 2.1.7. A dense-in-itself dynamic topological system $\mathcal{X}_f = \langle \mathcal{X}, \tau, f \rangle$ is chaotic if and only if for any non-empty $U, V \in \tau$ there exists a periodic point y and some $n \geq 0$ such that (i) $y \in U$ and (ii) $f^n(y) \in V$.

Proof. ($\Leftarrow$) Suppose that for any non-empty $U, V \in \tau$ there exists a periodic point y, such that $y \in U$ and $f^n(y) \in V$. Then all such periodic points are a subset of the set P_f of periodic points of the function f. From the fact that $y \in U$ for any y, it follows that P_f is dense. By $f^n(y) \in V$ we immediately get that $f^n(y) \in f^n(U) \cap V \neq \varnothing$ and therefore f is topologically transitive. It follows that $\mathcal{X}_f$ is chaotic.

($\Rightarrow$) Suppose $\mathcal{X}_f$ is chaotic. By topological transitivity, for any non-empty $U, V \in \tau$, $\exists n \geq 0$ and $\exists x \in U$ such that $f^n(x) \in V$. We set $O := f^{-n}(V) \cap U$. Note that O is open since U, V are open and from the continuity of f we get that $f^{-n}(V)$ is also open. Moreover, O is non-empty since x is an element of both U and $f^{-n}(V)$. Since the periodic points of f are dense in X, then there must exist a periodic point y such that $y \in O$. Since clearly, both $O \subseteq U$ and $f^n(O) \subseteq V$, then y is a periodic point such that (i) $y \in O \subseteq U$ and (ii) $f^n(y) \in f^n(O) \subseteq V$. Since $U, V \in \tau$ are arbitrary, this is true for any two open neighbourhoods as required. $\square$

To conclude this section, we provide an example of one particulary interesting chaotic map. Let $\langle x_n \rangle$ denote the sequence of points $x_0, \ldots, x_n$, and let

$$\Sigma_2 = \big\{ \langle x_n \rangle : \ x_n \in \{0, 1\} \big\},$$

be the set of all infinite binary sequences. A topology that is often associated with that space is the topology induced by the following metric:

$$d(x, y) = \sum_{n=0}^{\infty} \frac{|x_n - y_n|}{2^n}.$$

We will adopt a different approach that simplifies things and does not require the usage of metrics.

Let Σ_2^{fin} be the set of all finite binary sequences. Consider the topology τ_{Σ_2} that consists of all sets of the form:

$$C(x) = \{xy : x \in \Sigma_2^{fin} \textit{ and } y \in \Sigma_2\}.$$

The set $C(x)$ is called the *cylinder set* of x.

Proposition 2.1.8. $\langle \Sigma_2, \tau_{\Sigma_2} \rangle$ is homeomorphic to the Cantor set and is therefore a Cantor space.

Proof. This is a well-known result. It follows from a theorem by Brouwer [12] according to which any two non-empty compact Hausdorff spaces without isolated points and with countable bases consisting of clopen sets are homeomorphic to each other. $\square$

Finally, we define a function $\sigma : \Sigma_2 \to \Sigma_2$ as

$$\sigma(x_0, x_1, \ldots) = (x_1, x_2, \ldots).$$

We call this function the *shift map.*

Lemma 2.1.9. σ is continuous on $\langle \Sigma_2, \tau_{\Sigma_2} \rangle$.

Proof. Suppose $C(x) \in \tau_{\Sigma_2}$ is some open set. Then by definition

$$f^{-1}(C(x)) = \{\sigma^{-1}(x)y : y \in \Sigma_2\} = \bigcup_{x_i \in \sigma^{-1}(x)} C(x_i),$$

which is a union of cylinder sets and thus open. $\square$

Corollary 2.1.10. *The shift system $\mathcal{E}_2 = \langle \Sigma_2, \tau, \sigma \rangle$ is a dynamic topological system.*

The following proposition will be used to prove that $\mathcal{E}_2$ is chaotic:

Proposition 2.1.11. The orbit of $x \in \Sigma_2$ is dense iff every finite sequence appears in x.

Proof. Let $y \in \Sigma_2^{fin}$ be some finite binary sequence of length n. Suppose $Orb(x)$ is dense. Then there exists $k \geq 0$ such that $\sigma^k(x) \in C(y)$. Thus $\sigma^k(x)$ and y share the first n bits and so $y = (x_k, \ldots x_{k+n})$. It follows that x contains every binary sequence. The second direction is shown similarly. $\square$

Theorem 2.1.12. *The dynamic topological system $\mathcal{E}_2$ is chaotic.*

Proof. Let $C(x)$ and $C(y)$ be the open sets generated by the finite sequences x and y of lengths n and m respectively. We construct an infinite binary sequence z that contains every finite binary sequence ordered by length, i.e.

$$z = (0, 1 \textbf{,} 00, 01, 10, 11 \textbf{,} \ldots).$$

This is possible since the set of all finite binary sequences is countable. By Proposition 2.1.11, it follows that $Orb(z)$ is dense. Since z contains all strings of length n then there exists $k_1 \geq 0$ such that $\sigma^{k_1}(z)$ and x share the same first n bits and thus $\sigma^{k_1}(z) \in C(x)$. As before, since z contain any finite sequence, it also contains a sequence that is longer than n and shares the same first m bits with y. So there exists $k_2 \geq k_1$ such that $\sigma^{k_2}(z) \in C(y)$. Since $C(x), C(y)$ are arbitrary, then $\mathcal{E}_2$ is topologically transitive.

Next, let $C(x) \in \Sigma_2$ be a cylinder set where x is a finite binary sequence of length $m + 1$. Let

$$y := (x_0, \ldots x_m, x_0, \ldots, x_m, \ldots),$$

be a period of the $m + 1$ bits of x. So y is such that $x_i = y_i$ for $i = \{0, \ldots, m\}$. It follows that $y \in C(x)$. Since y is periodic and $C(x)$ is some arbitrary open set in τ_{Σ_2}, then $\mathcal{E}_2$ has a dense set of periodic points.

By Lemma 2.1.6 it follows that $\mathcal{E}_2$ is chaotic. $\square$

2.1.2 Quasiconjugacy

The structure preserving morphisms between dynamic topological systems are called *quasiconjugacies*. In this section we will examine their properties and demonstrate how they can be used to prove that dynamic topological systems are chaotic.

Definition 2.1.13. (*quasiconjugacy*) Suppose $\mathcal{X}_f = \langle X, \tau, f \rangle$ and $\mathcal{Y}_g = \langle Y, \upsilon, g \rangle$ are dynamic topological systems. The map g is *quasiconjugate* to f if there exists an image-dense continuous map $Q : X \to Y$ such that $g \circ Q = Q \circ f$, i.e. the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle Q} \\ Y & \xrightarrow{g} & Y \end{array}$$

If Q is a homeomorphism, then g and f are said to be *conjugate*.

Definition 2.1.14. A property S is preserved under quasiconjugacy if for every dynamic topological system $\mathcal{X}_f$ the property S holds for every dynamic topological system $\mathcal{Y}_g$ such that g is quasiconjugate to f.

We now show that the two conditions for a dynamic topological system to be chaotic are preserved under quasiconjugacy.

Proposition 2.1.15. Topological transitivity is preserved under quasiconjugacy.

Proof. Let g be quasiconjugate to f under $Q : X \to Y$. Let $U, V \in \upsilon$ be non-empty. Since Q is an image-dense continuous map, then $Q^{-1}(U)$ and $Q^{-1}(V)$ are non empty open sets in τ. Since $\mathcal{X}_f$ is topologically transitive, then $\exists x \in Q^{-1}(U)$ and $\exists n \geq 0$ such that $f^n(x) \in Q^{-1}(V)$. Accordingly, $Q(x) \in U$ and since g is quasiconjugate to f, then $g^n(Q(x)) = Q(f^n(x)) \in V$. $\square$

Proposition 2.1.16. Density of the set of periodic points is preserved under quasiconjugacy.

Proof. Let g be quasiconjugate to f under $Q : X \to Y$. Let $U \in \upsilon$ be non-empty. Since Q is image-dense continuous map then $Q^{-1}(U)$ is a non-empty open set in τ. Moreover, there exists a point $x \in U$ such that $f^n(x) = x$ for some $n > 0$. Accordingly, $Q(x) \in U$ and since g is quasiconjugate to f, then $g^n(Q(x)) = Q(f^n(x)) = Q(x)$. $\square$

Corollary 2.1.17. *Chaoticity of the system is preserved under quasiconjugacy.*

We will now show how we can use the notions of quasiconjugacy and the shift map to prove chaotic behaviour of other systems. We first introduce an important map called the *dyadic transformation*. It is simply defined by the function

$$f(x) = 2x \bmod 1 = \begin{cases} 2x, & if \;\; 0 \leq x < \frac{1}{2} \\ 2x - 1, & if \;\; \frac{1}{2} \leq x \leq 1 \end{cases} .$$

An illustration of the dyadic transformation for multiple iterated points is shown in Figure 2.2. We will demonstrate that the dyadic transformation on the unit circle is chaotic.

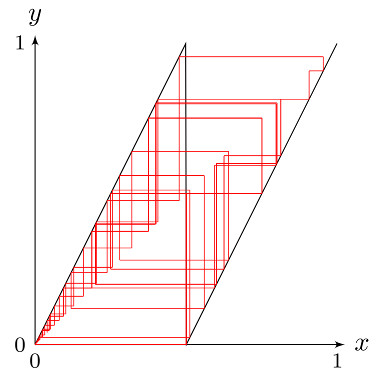

Figure 2.2: The first hundred iterations of the dyadic transformation for $x_n = 4 \cdot 0.1^n$, where $n \in \{1, \ldots, 5\}$.

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ denote the circle group. Given $0 \leq \alpha < 1$, we define the rotation map $R_\alpha : \mathbb{T} \to \mathbb{T}$ as

$$z \mapsto exp(2\pi i \alpha) \cdot z.$$

The map R_α rotates on the circle through an angle of $2\pi\alpha$. Recall that the quotient group $\mathbb{R}/\mathbb{Z}$ is an uncountable set such that $\mathbb{R}/\mathbb{Z} = \{\mathbb{Z} + r : r \in [0, 1)\}$ and from which all the resulting cosets are distinct. Note that the dyadic transformation is well defined for $\mathbb{R}/\mathbb{Z}$ since 1 and 0 are fixed-points of the function f. Moreover, since $\mathbb{R}/\mathbb{Z}$ is isomorphic to the circle group $\mathbb{T}$ we can view f multiplicatively as a map $f : \mathbb{T} \to \mathbb{T}$ such that $f(z) = z^2$.

We use the rotation map to create a dyadic transformation D by setting $z := exp(2\pi i \alpha)$ and

$$D : exp(2\pi i \alpha) \mapsto exp^2(2\pi i \alpha).$$

Recall that it is possible to represent α in its binary form as

$$\alpha = \frac{x_0}{2} + \cdots + \frac{x_n}{2^{n+1}} + \cdots = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}},$$

where $x_j \in \{0, 1\}$. We can therefore construct an infinite binary sequence $(x_0, x_1, x_2 \ldots)$ of numerators. We will use this fact in order to prove the following lemma:

Lemma 2.1.18. The dyadic transformation is chaotic.

Proof. We define a map $Q : \Sigma_2 \to \mathbb{T}$ as

$$Q(x_0, x_1, \dots) = exp\left(2\pi i \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}}\right).$$

Then we get

$$\begin{aligned} Q(\sigma(x_0, x_1, x_2, \dots)) &= Q(x_1, x_2, x_3, \dots) \\ &= exp\left(2\pi i \sum_{k=1}^{\infty} \frac{x_k}{2^k}\right). \end{aligned}$$

Moreover by the definition of the dyadic transformation

$$\begin{aligned} D(Q(x_0, x_1, x_2, \dots)) =& D\left(exp\left(2\pi i \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}}\right)\right) \\ =& exp\left(4\pi i \frac{x_0}{2} + 4\pi i \sum_{k=1}^{\infty} \frac{x_k}{2^{k+1}}\right) \\ =& exp\left(2\pi i \sum_{k=1}^{\infty} \frac{x_k}{2^k}\right). \end{aligned}$$

since $exp(x + y) = exp(x) \cdot exp(y)$ and $x_0 \in \{0, 1\}$.

It follows that $D \circ Q = Q \circ \sigma$. Since Q is surjective, its image is clearly dense. Hence, by Corollary 2.1.17 the dyadic transformation D is chaotic. $\square$

In this example the dyadic transformation plays a role of a shift on the binary representation. It is a common phenomenon that often occurs in dynamical systems and makes the notion of quasiconjugacy so essential to our investigation. There is an interesting relation between the dyadic transformation, the logistic map (Example 1.2.1) and the tent map (Example 1.2.2). The dyadic map is quasiconjugate to the logistic map with $\mu = 4$ and quasiconjugate to the unit-height tent map. Moreover, the logistic map with $\mu = 4$ and the tent map with $\mu = 2$ are conjugate and hence show similar behaviour under iteration. These are well known results that could be found for instance in [28].

Given the definition of a quasiconjugacy, we may view the collection of dynamic topological systems as a category in the natural way.

Definition 2.1.19. (*categorical interpretation*) The category **DTop** of dynamic topological systems consists of:

1. A collection $ob(\mathbf{DTop})$ of pairs $\langle \mathcal{X}, f \rangle$ consisting of a topological space $\mathcal{X} = \langle X, \tau \rangle$ and a continuous endomorphism f. We write $\mathcal{X}_f$ in short instead of $\langle \mathcal{X}, f \rangle$.

2. For each $\mathcal{X}_f, \mathcal{Y}_g \in ob(\mathbf{DTop})$ a collection $\mathbf{DTop}(\mathcal{X}_f, \mathcal{Y}_g)$ of morphisms $\delta : \mathcal{X}_f \to \mathcal{Y}_g$ that commute with the endomorphisms, i.e. $f \circ \delta = \delta \circ g$.

3. For each $\mathcal{X}_f, \mathcal{Y}_g, \mathcal{Z}_h \in ob(\mathbf{DTop})$ with $\delta : \mathcal{X}_f \to \mathcal{Y}_g$ and $\zeta : \mathcal{Y}_g \to \mathcal{Z}_h$ a composition $\zeta \circ \delta \in \mathbf{DTop}(\mathcal{X}_f, \mathcal{Z}_h)$.

In the next section we offer a non-deterministic interpretation to the concepts of dynamical systems and chaos.

2.2 Non-Deterministic Dynamical Systems

The concept of dynamical systems is exclusively discussed as a deterministic phenomenon in the literature of dynamical systems theory. In principle, deterministic systems do not entertain the idea of parallel timelines, but only one unique timeline in which the world can proceed. This is the standard approach in dynamical systems theory which offers a deterministic interpretation to the physical world. However, in other fields such as philosophy, non-determinism has long been considered a plausible ontology of the universe. For instance, this is the case in the study of metaphysics and free-will. Non-determinism was also considered in physics, as suggested by Popper [36], and in computer science as a theoretical tool in theories of computation such as automata, algorithms and complexity. It is therefore justified to try and establish a non-deterministic interpretation to dynamical systems theory and to chaos in particular. The first step will be to define the new framework with which we will be working, since a dynamic metric system consists of a function and functionality we no longer have. Instead, we will introduce a new structure called a non-deterministic dynamic metric system.

We first define the notion of a continuous relation.

Definition 2.2.1. (*continuous relation*) Let $\langle X, \tau \rangle$ be a topological space. A relation R is said to be continuous on X if whenever $U \in \tau$ then $R^{-1}(U) \in \tau$, where

$$R^{-1}(U) = \{v : vRu \text{ for some } u \in U\}.$$

There are several other possible definitions for continuous relations that can be found in the literature. One well-known example is hemicontinuity of multivalued functions [22]. We use the definition above as it most adequately serves our purpose and is compatible with the subsequent definitions.

Definition 2.2.2. (*non-deterministic dynamic metric system*) A non-deterministic dynamic metric system (NDMS) is a triple $\mathcal{X}_R = \langle X, d, R \rangle$ where X is a set of points, d is a metric on X and R is a continuous relation, where $R(x)$ is finite for each $x \in X$.

We can define the topological space induced by d in the usual way. This will result in the structure $\mathcal{X}_R = \langle X, \tau, R \rangle$ which is called *non-deterministic dynamic topological system* (NDTS).

The next step would be to provide a non-deterministic alternative to the properties that constitute chaos, i.e. dense set of periodic points, topological transitivity and sensitive dependence on initial conditions.

Definition 2.2.3. (*dense set of periodic points*) A point $x \in X$ is a periodic point if xR^nx for some $n \geq 1$. Let NP_R denote the set of all periodic points in $\mathcal{X}_R$. We say that $\mathcal{X}_R$ has a dense set of periodic points if

$$Cl(NP_R) = X$$

Definition 2.2.4. (*topological transitivity*) We say that $\mathcal{X}_R$ is topologically transitive if for all $U, V \in \tau$ there exist $u \in U$ and $v \in V$ such that uR^nv for some $n \geq 0$.

We will define the analogous non-deterministic version of sensitive dependence on initial conditions. We then prove that it is implied by the properties of a dense set of periodic points and topological transitivity given one additional constraint.

Definition 2.2.5. (*sensitive dependence on initial conditions*) Let $A, B \subseteq X$. Given a metric d, we define:

$$e_d(A, B) > \delta \iff \exists a \subseteq A, \exists b \subseteq B : d(a, b) > \delta.$$

We say that $\mathcal{X}_R$ has sensitive dependence on initial conditions if there exists $\delta > 0$ such that for any $x \in X$ and $\varepsilon > 0$ there exists $y \in X$ with $d(x, y) < \varepsilon$ and

$$e_d(R^n(x), R^n(y)) > \delta, \text{ for some } n \geq 0.$$

Definition 2.2.6. (*non-deterministic chaos*) Let $\mathcal{X}_R = \langle X, d, R \rangle$ be a dense-in-itself non-deterministic dynamic metric system. Then $\mathcal{X}_R$ is *chaotic* if the following conditions are satisfied.

i) $\mathcal{X}_R$ has a dense set of periodic points.

ii) $\mathcal{X}_R$ is topologically transitive.

iii) $\mathcal{X}_R$ has sensitive dependence on initial conditions.

We prove that given one additional constraint, the third condition turns redundant.

Theorem 2.2.7. *Let $\mathcal{X}_R = \langle X, d, R \rangle$ be a non-deterministic dynamic metric system where $\langle X, d \rangle$ is a dense-in-itself metric space. Suppose that the following conditions are fulfilled:*

1. *$\mathcal{X}_R$ has a dense set of periodic points;*
2. *$\mathcal{X}_R$ is topologically transitive;*
3. *There exist two periodic points $p_1, p_2 \in X$ such that*

$$R^n(p_1) \cap R^m(p_2) = \varnothing,$$

for all $n, m \geq 0$.

Then $\mathcal{X}_R$ has sensitive dependence on initial conditions.

Proof. Let d be a metric inducing a topology on X. We begin by proving that there exists $\gamma > 0$ such that for any $x \in X$ there exists a periodic point p for which

$$d(x, R^n(p)) \geq \gamma \text{ for all } n \geq 0.$$

Since $\langle X, d \rangle$ is a dense-in-itself metric space, it follows that X is infinite. By assumption, there are two periodic points p_1, p_2 such that $R^n(p_1) \cap R^m(p_2) = \varnothing$ for all $n, m \geq 0$. Since $R(x)$ is finite for any $x \in X$, we can set

$$\gamma := \min_{n,m \geq 0} \frac{d(R^n(p_1), R^m(p_2))}{2} > 0.$$

By the triangle inequality, it follows that for every $x \in X$ and for all $n \geq 0$ we have either $d(x, R^n(p_1)) \geq \gamma$ or $d(x, R^n(p_2)) \geq \gamma$.

We show that R has sensitive dependence on initial conditions for $\delta := \gamma/4$. Let $x \in X$ be an arbitrary point. From the result above, it follows that there exists a periodic point p such that

$$d(x, R^n(p)) \geq 4\delta \text{ for all } n \geq 0. \tag{2.4}$$

Since the periodic points are dense in R, then for any x we have a periodic point q such that $q \in R^m(q)$ for some $m > 0$ and

$$d(x, q) < \varepsilon, \tag{2.5}$$

for any $\varepsilon > 0$ and in particular when $\varepsilon = \delta$. Clearly, there exists a neighbourhood N of p such that for any $y \in N$ we have $d(p, y) < \eta$ for some constant η. From the continuity of R, it follows that

$$d(R^n(p), R^n(y)) < \delta \text{ for all } n \leq m. \tag{2.6}$$

Now, by topological transitivity of R we are guaranteed to have a point z such that $d(x, z) < \varepsilon$ and $R^k(z) \cap N \neq \varnothing$ for some $k \geq 0$. Let $z' \in R^k(z) \cap N$ and fix $j \geq 0$ that satisfies the equation $k \leq jm < k + m$. Then, by the triangle inequality we have

$$\begin{aligned} d(x, q) + d(q, R^{jm-k}(z')) + d(R^{jm-k}(z'), R^{jm-k}(p)) \\ \geq d(x, R^{jm-k}(p)). \end{aligned}$$

Finally, from (2.4)-(2.6) we derive that

$$\begin{aligned} d(x, R^{jm-k}(p)) - d(x, q) - d(R^{jm-k}(z'), R^{jm-k}(p)) \\ > 4\delta - \delta - \delta = 2\delta. \end{aligned}$$

Therefore, $d(q, R^{jm-k}(z')) > 2\delta$ and hence

$$e_d(R^{jm}(q), R^{jm}(z)) > 2\delta.$$

It follows that

$$e_d(R^{jm}(x), R^{jm}(q)) > \delta \ \text{ or } \ e_d(R^{jm}(x), R^{jm}(z)) > \delta.$$

Since both $d(x, q) < \varepsilon$ and $d(x, z) < \varepsilon$, we conclude that R has sensitive dependence on initial conditions. $\square$

This implies that from this point onward we can talk about chaos with respect to any non-deterministic dynamic topological system. This will be especially helpful since we would like to construct a formal system that describes such properties of dynamical systems. In principle, the standard language of dynamic topological systems cannot express metric properties such as sensitive dependence on initial conditions. In the next chapter we will attempt to construct such a formal system.

Chapter 3

Logic and Dynamical Systems

In this chapter we will attempt to construct a logical system that is able to capture the idea of chaos. In order to do that, we extend our language with the universal operator $\forall$, which results in the language $\mathcal{L}_{\Box\forall}^{\bigcirc[*]}$. Unfortunately, the language $\mathcal{L}_{\Box\forall}^{\bigcirc[*]}$ may very well be insufficient to express the property of having a dense set of periodic points. On the other hand, we will show that topological transitivity is expressible in that language. We start by defining the language.

Given a non-empty set PV of propositional variables the language $\mathcal{L}_{\Box\forall}^{\bigcirc[*]}$ is defined recursively as follows:

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Box\varphi \mid \bigcirc\varphi \mid [*]\varphi \mid \forall\varphi$$

$$p \in \mathsf{PV}.$$

Formulas of the form $\bigcirc\varphi$ have the reading 'In the next point in time φ holds'. Formulas of the form $[*]\varphi$ have the reading 'Henceforth φ holds'. The dual operator $\langle * \rangle := \neg[*]\neg$ has the reading 'Eventually φ holds'. Formulas of the form $\forall\varphi$ have the usual reading 'In all points φ holds' with its existential dual $\exists := \neg\forall\neg$. As in Definition 2.3.1, $\Box$ is the interior operator and its dual is the closure operator $\Diamond := \neg\Box\neg$.

We use the topological semantics presented in Definition 1.3.1 of the preliminaries with the following addition for the universal operator:

7. $\mathcal{M}_f, x \models \forall\varphi \iff \mathcal{M}_f, y \models \varphi$ for all $y \in X$.

Let $\mathcal{X}_f = \langle X, \tau, f \rangle$. We prove that the property of topological transitivity is expressible in our language.

Lemma 3.0.1. A dense-in-itself dynamic topological system $\mathcal{X}_f = \langle X, \tau, f \rangle$ is topologically transitive iff $\mathcal{X}_f \models \exists\Box p \to \Diamond\langle * \rangle p$.

Proof. (⇒) Suppose that $\mathcal{X}_f$ is topologically transitive and let $\nu : \mathsf{PV} \to \wp(X)$ be some valuation function. Suppose $\langle \mathcal{X}_f, \nu \rangle, x \models \exists \Box p$, for some $x \in X$. Then, there exists some $v \in X$ and $V \in \tau$ such that $v \in V$ and $V \subseteq \nu(p)$. Since $\mathcal{X}_f$ is topologically transitive, then for each $U \in \tau$ such that $x \in U$ there exists some $u \in U$ and $n \geq 0$ such that $f^n(u) \in V$. It follows that $\langle \mathcal{X}_f, \nu \rangle, f^n(u) \models p$ and since x and ν were arbitrary, then $\mathcal{X}_f \models \exists \Box p \to \Diamond \langle * \rangle p$ as required.

(⇐) suppose that $\mathcal{X}_f$ is not topologically transitive. Then there exist non-empty $U, V \in \tau$ for which $f^n(u) \notin V$ for all $u \in U$ and $n \geq 0$. We define the following valuation ν on $\mathcal{X}_f$:

$$\nu(p) = V.$$

Then clearly for all $v \in V$ we have $\langle \mathcal{X}_f, \nu \rangle, v \models \Box p$ and in particular $\langle \mathcal{X}_f, \nu \rangle, u \models \exists \Box p$ for some $u \in U$. By assumption, there is no point in U that reaches V and thus $\langle \mathcal{X}_f, \nu \rangle, u \not\models \Diamond \langle * \rangle p$. It follows that $\mathcal{X}_f \not\models \exists \Box p \to \Diamond \langle * \rangle p$, as required. □

It is also a well known result that continuity is expressible in our language.

Lemma 3.0.2. A dense-in-itself topological system with a function $\mathcal{X}_f = \langle X, \tau, f \rangle$ is continuous iff $\mathcal{X}_f \models \bigcirc \Diamond p \to \Diamond \bigcirc p$.

Proof. This is a known result. Its proof can be found for example in [1]. □

Corollary 3.0.3. *A dense-in-itself topological space $\langle X, \tau \rangle$ with a map $f : X \to X$ is a topologically transitive dynamic topological system iff it validates*

$$\bigcirc \Diamond p \to \Diamond \bigcirc p \;\; \textit{and} \;\; \exists \Box p \to \Diamond \langle * \rangle p.$$

At this point, we make the following conjecture.

Conjecture 3.0.1. *The property of a dense set of periodic points is not expressible in the language $\mathcal{L}^{\bigcirc[*]}_{\Box\forall}$.*

As mentioned before, it is known that there is no finite axiomatisation for the logic of all dynamic topological systems under the trimodal topo-temporal language. Therefore, if we want to construct a logic for all topologically transitive dynamic topological systems, then we cannot use the normal deterministic setting. In Chapter 3.2 we will use the non-deterministic setting from Chapter 2 and show how completeness becomes possible as we can drop the temporal operator $\bigcirc$. Prior to that, we examine the connection between p-morphic and quasiconjugate dynamic topological systems in the next section.

3.1 p-morphism as strong quasiconjugacy

The truth preserving maps of modal formulas between structures are often called *p-morphisms.* In this section we show the connection between quasiconjugacies and p-morphisms between dynamic topological systems.

Let $\mathcal{X}_f = \langle X, \tau, f \rangle$ and $\mathcal{Y}_g = \langle Y, \upsilon, g \rangle$ be two dynamic topological systems. A map $\pi : \mathcal{X}_f \to \mathcal{Y}_g$ is a *dynamic p-morphism* if the following conditions are satisfied:

1. π is open and continuous map from X onto Y;

2. $\pi(f(x)) = g(\pi(x))$;

In case $\mathcal{M}_f = \langle X, \tau, f, V\rangle$ and $\mathcal{N}_g = \langle Y, \upsilon, g, V'\rangle$ are instead two dynamic topological models, we say that π is a dynamic p-morphism between $\mathcal{X}_f$ and $\mathcal{Y}_g$ if in addition the following condition is satisfied:

3. $V(p) = \pi^{-1}(V'(p))$, for all $p \in \mathsf{PV}$.

Lemma 3.1.1. Let $\mathcal{M}_f = \langle X, \tau, f, V\rangle$ and $\mathcal{N}_g = \langle Y, \upsilon, g, V'\rangle$ be two dynamic topological models and let $\pi : \mathcal{M}_f \to \mathcal{N}_g$ is a dynamic p-morphism. Then for all $\varphi \in \mathcal{L}_{\Box\forall}^{\bigcirc[*]}$ we have:

$$V(\varphi) = \pi^{-1}(V'(\varphi)).$$

Proof. We prove this by induction on the complexity of φ. The base case for the atomic formulas and the induction steps for the boolean connectives are routine. Moreover, the induction step for the universal operator is trivial by surjectivity. We consider the remaining induction steps of the operators $\Box, [*]$ and $\bigcirc$.

- $\varphi := \Box\psi$

$$\begin{aligned} V(\Box\psi) &= Int(V(\psi)) \\ &= Int(\pi^{-1}(V'(\psi))) \quad \textit{by the IH} \\ &= \pi^{-1}(Int(V'(\psi))) \quad \textit{by condition } 1 \\ &= \pi^{-1}(V'(\Box\psi)). \end{aligned}$$

- $\varphi := [*]\psi$

$$\begin{aligned} x \in V([*]\psi) &\iff x \in \bigcap_{n\geq 0} f^n(V(\psi)) \\ &\iff \bigcup_{n\geq 0} f^n(x) \subseteq V(\psi) \\ &\iff \bigcup_{n\geq 0} f^n(x) \subseteq \pi^{-1}(V'(\psi)) \;\; \textit{by the IH} \\ &\iff \pi(\bigcup_{n\geq 0} f^n(x)) \subseteq V'(\psi) \\ &\iff \bigcup_{n\geq 0} g^n(\pi(x)) \subseteq V'(\psi) \qquad \textit{by condition } 2 \\ &\iff \pi(x) \in V'([*]\psi) \\ &\iff x \in \pi^{-1}(V'([*]\psi)). \end{aligned}$$

- $\varphi := \bigcirc\psi$

$$\begin{aligned} x \in V(\bigcirc\psi) &\iff f(x) \in V(\psi) \\ &\iff f(x) \in \pi^{-1}(V'(\psi)) \quad \textit{by the IH} \\ &\iff \pi(f(x)) \in V'(\psi) \\ &\iff g(\pi(x)) \in V'(\psi) \qquad \textit{by condition } 2 \\ &\iff \pi(x) \in V'(\bigcirc\psi) \\ &\iff x \in \pi^{-1}(V'(\bigcirc\psi)). \end{aligned}$$

□

Theorem 3.1.2. *Let $\mathcal{X}_f = \langle X, \tau, f\rangle$ and $\mathcal{Y}_g = \langle Y, \upsilon, g\rangle$ be two dynamic topological systems. Suppose $\pi : \mathcal{X}_f \to \mathcal{Y}_g$ is a dynamic p-morphism. Then $\mathcal{X}_f \models \varphi$ implies $\mathcal{Y}_g \models \varphi$ for any $\varphi \in \mathcal{L}^{\bigcirc[*]}_{\Box\forall}$.*

Proof. By contraposition we prove that $\mathcal{Y}_g \not\models \varphi$ implies $\mathcal{X}_f \not\models \varphi$. Suppose $\mathcal{Y}_g \not\models \varphi$, then by definition there exists V' such that $\langle \mathcal{Y}_g, V'\rangle \not\models \varphi$ and so $V'(\varphi) \neq Y$. We fix $V(p) = \pi^{-1}(V'(p))$ for each propositional variable p. Since π is a dynamic p-morphism, then by Lemma 3.1.1 we get $V(\varphi) = \pi^{-1}(V'(\varphi))$. Since π is surjective, then $V(\varphi) \neq X$. By definition it follows that $\mathcal{X}_f \not\models \varphi$. □

Let $\mathcal{X}_f$ and $\mathcal{Y}_g$ be dynamic topological systems and $\pi : \mathcal{X}_f \to \mathcal{Y}_g$ a commuting map with respect to f and g. Then if π is surjective and $\mathcal{Y}_g$ is an interior image of $\mathcal{X}_f$, then $\mathcal{X}_f$ has dense image in $\mathcal{Y}_g$. By Theorem 3.1.2 we derive the following:

Corollary 3.1.3. *Let $\mathcal{X}_f$ be a dynamic topological system and S some property expressible by $\varphi \in \mathcal{L}^{\bigcirc[*]}_{\Box\forall}$. If the property S is preserved under quasiconjugacy, then φ is preserved under dynamic p-morphism.*

Note that the opposite direction does not follow. We provide the following example in order to illustrate this.

Definition 3.1.4. (*extremally disconnectedness*) A topological space $\langle X, \tau\rangle$ is extremally disconnected if $Cl(U) \in \tau$, for all $U \in \tau$.

Proposition 3.1.5. Extremally disconnectedness is preserved under dynamic p-morphism but is not preserved under quasiconjugacy.

Proof. Let $\varphi := \Diamond\Box p \to \Box\Diamond\, p$. It is straightforward to show that $\mathcal{X}_f \models \varphi$ iff $\mathcal{X}_f$ is extremally disconnected:

$$\begin{aligned} \mathcal{X}_f \models \Diamond\Box p \to \Box\Diamond\, p &\iff Cl(Int(S)) \subseteq Int(Cl(S)), \text{ for all } S \subseteq X \\ &\iff Cl(Int(S)) = Int(Cl(Int(S))), \text{ for all } S \subseteq X \\ &\iff Cl(U) \in \tau, \text{ for all } U \in \tau \\ &\iff \mathcal{X}_f \text{ is extremally disconnected} \end{aligned}$$

Clearly, φ is preserved under dynamic p-morphism. However, φ is not preserved under quasiconjugacy. Consider the two dynamic topological systems $\mathcal{X}_f = \langle X, \tau, f\rangle$ and $\mathcal{Y}_g = \langle Y, \upsilon, g\rangle$, where

- $X = \{a, b, c\}$ and $Y = \{a', b', c'\}$;
- $\tau = \wp(X)$ and $\nu = \{\varnothing, \{a'\}, \{b'\}, Y\}$;
- $f(x) = c$ and $g(x') = c'$, for all $x \in \{a, b, c\}$ and $x' \in \{a', b', c'\}$.

Moreover we defined the map $\pi : \mathcal{X}_f \to \mathcal{Y}_g$ as

$$\pi(x) = x', \text{ for all } x \in \{a, b, c\}.$$

Since $\mathcal{X}_f$ is discrete, then it is also extremally disconnected. Note that π commutes with respect to f and g, and is image-dense since it is surjective. However, since $Cl(a') = \{a', c'\}$ and $\{a', c'\} \notin \upsilon$, then $\mathcal{Y}_g$ is not extremally disconnected. It follows that the property of extremally disconnectedness is not preserved under quasiconjugacy. $\square$

We showed that dynamic p-morphisms are but a generalisation of quasiconjugacies and we demonstrated why p-morphisms are stronger. An interesting question would be which fragment of $\mathcal{L}_{\Box\forall}^{\bigcirc[*]}$ is preserved under quasiconjugacy. From the example above it is rather clear that quasiconjugacies cannot deal with spatial operations very well, yet temporal operations seem to work adequately. It is fair to assume that such a fragment will include the operators $\bigcirc$ and $[*]$.

As finite axiomatisation with respect to all dynamic topological systems for the trimodal language is impossible, we have a few options in case we want to establish completeness for chaotic systems. We could consider a non-deterministic dynamical system where instead of a function we have a relation. We could consider fragments of $\mathcal{L}_{\Box\forall}^{\bigcirc[*]}$, or we could consider completeness in respect to a smaller class of spaces. In the next section we will use a mixture of the first two options. We will discuss the third option in chapter 4.

3.2 The logic ND4TT

In this section we provide a a complete axiomatisation for the logic of all topologically transitive non-deterministic dynamic topological systems with a transitive relation. In order to work with only one modality, we will assume $R = R^*$, and thus have R play both the role of the continuous transition relation and of its transitive closure. Accordingly, we only consider the fragment $\mathcal{L}_{\Box\forall}^{[*]}$.

Further, we make the following revision to our c-semantics in order to accommodate non-deterministic dynamic topological systems. We define a non-deterministic dynamic topological model as $\mathcal{M}_R = \langle X, \tau, R, \nu\rangle$, where $\langle X, \tau, R\rangle$ is a non-deterministic dynamic topological system and $\nu : \mathsf{PV} \to \wp(X)$ is a valuation function. Given a non-deterministic dynamic topological model $\mathcal{M}_R = \langle X, \tau, R, \nu\rangle$, the truth condition of $[*]$ replaces the original truth condition 6 and is defined as follows:

6'. $\mathcal{M}_R, x \models [*]\varphi \iff \forall y \in X$, if xRy then $\mathcal{M}_R, y \models \varphi$.

As was conjectured before, the property of a dense set of periodic points is non-definable in our language. This conjecture also applies to the non-deterministic definition of a dense set of periodic points. On the other hand, topological transitivity is definable in our language as in the deterministic case.

Lemma 3.2.1. A dense-in-itself NDTS $\mathcal{X}_R = \langle X, \tau, R\rangle$ is topologically transitive iff $\mathcal{X}_R \models \exists\Box p \to \Diamond\langle *\rangle p$.

Proof. This follows similarly to the proof of Lemma 3.0.1 for topological transitivity in a dynamic topological system. $\square$

We show that the property of continuity in a Kripke frame is definable in our language. Continuity on a Kripke frame is interpreted in the usual sense where the open sets are the topologically upward-closed sets. We call a bimodal Kripke frame $\mathfrak{F}_R = \langle W, T, R\rangle$, a *weak dynamic Kripke frame*. If R is continuous with respect to T, we call it a *dynamic Kripke frame* .

Lemma 3.2.2. A weak dynamic Kripke frame $\mathfrak{F}_R = \langle W, T, R\rangle$ with a transitive relation R is continuous iff $\mathfrak{F}_R \models \langle *\rangle\Box p \to \Box\langle *\rangle p$.

Proof. ($\Rightarrow$) Suppose $\mathfrak{F}_R$ is continuous and $w \models \langle *\rangle\Box p$ for some $w \in W$. Then there exists some $v \in W$ with wRv and for all $u \in W$ such that vTu we have $u \models p$. Note that v and all u such that vTu constitute a T-upward-closed set. We denote it by U. Since R is continuous, then $R^{-1}(U)$ is also T-upward-closed. In particular for any $v' \in W$ such that wTv' we have that $v'Ru$ for some $u \in U$. It follows that $w \models \Box\langle *\rangle p$.

($\Leftarrow$) Suppose $\mathfrak{F}_R$ is not continuous. Then there is some T-upward-closed set U such that $R^{-1}(U)$ is not a T-upward-closed set. We set the following valuation:

$$V(p) = U.$$

Now since $R^{-1}(U)$ is not open, then there exists some $u \in R^{-1}(U)$ such that $u' \notin U$ for some $u' \in T(u)$. We clearly have $u \models \langle *\rangle\Box p$. Moreover, by the transitivity of R we have $u' \not\models \langle *\rangle p$ and hence $u \not\models \Box\langle *\rangle p$, as required. $\square$

Note that the property of continuity on Kripke frames is topological in nature and is in fact equivalent to the purely relational property of *confluence*. We say that a Kripke frame $\langle W, T, R\rangle$ is confluent if whenever wTw' and wRv, then there is v' such that $w'Rv'$ and vTv'.

Lemma 3.2.3. A weak dynamic Kripke frame $\mathfrak{F}_R = \langle W, T, R\rangle$ with a transitive relation R is confluent iff $\mathfrak{F}_R \models \langle *\rangle\Box p \to \Box\langle *\rangle p$.

Proof. ($\Rightarrow$) Suppose $\mathfrak{F}_R$ is confluent and $w \models \langle *\rangle\Box p$. So there exists $v \in W$ such that wRv and for all v' such that vTv' we have $v' \models p$. Let w' be such that wTw'. By confluence, there exists some z such that both vTz and $w'Rz$ hold. But since v' is some arbitrary successor of v and w' is some arbitrary successor of w, then $w \models \Box\langle *\rangle p$.

($\Leftarrow$) Suppose $\mathfrak{F}_R$ is not confluent. Then there is wTw' and wRv, but there exists no v' such that $w'Rv'$ and vTv'. We set the following valuation:

$$V(p) = \{v\} \cup \{v' : vTv'\}.$$

Then clearly $w \models \langle * \rangle \Box p$. However, by the transitivity of R and since wTw' while $w' \not R\, v'$ for any v' such that vTv', then $w \not\models \Box \langle * \rangle p$. $\Box$

Corollary 3.2.4. *Continuity and confluence are equivalent for the class of weak dynamic Kripke frames with a transitive relation R.*

Finally, we show the validity of $\langle * \rangle \Box p \to \Box \langle * \rangle p$ on all continuous topological spaces.

Lemma 3.2.5. The axiom $\langle * \rangle \Box p \to \Box \langle * \rangle p$ is valid on all non-deterministic dynamic topological systems.

Proof. Let $\mathcal{X}_f = \langle X, \tau, R \rangle$ be a non-deterministic dynamic topological system. Suppose R is continuous and $\langle \mathcal{X}_R, \nu \rangle, x \models \langle * \rangle \Box p$ for some $x \in X$ and some valuation ν. Then either $\exists U \in \tau$ with $x \in U$ and $U \subseteq \nu(p)$ and then trivially $\langle \mathcal{X}_R, \nu \rangle, x \models \Box \langle * \rangle p$. Otherwise, there exists $y \in R^k(x)$, for some $k \geq 1$, and $\exists V \in \tau$ with $y \in V$ such that $V \subseteq \nu(p)$. Since R is continuous then $R^{-n}(V)$ is an open set for all $n \geq 1$. In particular, it holds when $n = k$ which implies $x \in R^{-k}(V) = U$, for some $U \in \tau$. Moreover, for all $u \in U$ we have $R^k(u) \subseteq V \subseteq \nu(p)$. It follows that $x \models \Box \langle * \rangle p$, as required. $\Box$

Note that none of the properties above require the use of the 'next' operator $\bigcirc$, and as mentioned, our system only requires the fragment $\mathcal{L}^{[*]}_{\Box\forall}$ of the trimodal language. This fact simplifies the problem of completeness significantly as we do not have to deal with the difficulty that functionality usually provides. In particular, this will allow us to easily obtain a completeness result. We start by presenting the axioms and inference rules of the logic **ND4TT**.

The Logic **ND4TT** consists of the following axioms and inference rules.

- (Taut) All propositional tautologies.
- Axioms for $\Box$:
    1. (K) $\Box(p \to q) \to (\Box p \to \Box q)$
    2. (T) $\Box p \to p$
    3. ($\mathsf{4}$) $\Box p \to \Box\Box p$
- Axioms for $[*]$:
    1. ($\mathsf{K}_{[*]}$) $[*](p \to q) \to ([*]p \to [*]q)$
    2. ($\mathsf{4}_{[*]}$) $[*]p \to [*][*]p$
    3. ($\mathsf{D}_{[*]}$) $[*]p \to \langle * \rangle p$

- Axioms for $\forall$:
  1. ($\mathsf{K}_\forall$) $\forall(p \to q) \to (\forall p \to \forall q)$
  2. ($\mathsf{T}_\forall$) $\forall p \to p$
  3. ($\mathsf{4}_\forall$) $\forall p \to \forall\forall p$
  4. ($\mathsf{B}_\forall$) $p \to \forall\exists p$
  5. ($\mathsf{Com}_\forall$) $\forall p \to (\Box p \wedge [*]p)$
- (TT) $\exists\Box p \to \Diamond\langle * \rangle p$
- ($\mathsf{Con}_{\langle * \rangle}$) $\langle * \rangle \Box p \to \Box\langle * \rangle p$
- Rules:
  1. (MP) Modus ponens
  2. (Sub) Substitution
  3. ($\mathsf{N}_\Box$) Necessitation for $\Box$
  4. ($\mathsf{N}_{[*]}$) Necessitation for $[*]$

ND4TT Kripke frames consist of two relations: The $\Box$-relation is a reflexive-transitive relation and it represents the spatial component of the frame. The $[*]$-relation is a transitive-serial relation that is also continuous in respect to the $\Box$-relation. It represents the temporal component of the frame. Seriality is essential to us as it manages to capture the important idea of deterministic chaos where each point has a defined future.

Note that all of the axioms in **ND4TT** are Sahlqvist. We show the first-order correspondences of the axioms $\mathsf{Con}_{\langle * \rangle}$ and TT. Let T denote the $\Box$-relation and R denote the $[*]$-relation.

Proposition 3.2.6. The first-order correspondence of $\mathsf{Con}_{\langle * \rangle}$ is

$$\forall x \forall y \forall z((Txy \wedge Rxz) \to \exists w(Tzw \wedge Ryw)).$$

The first-order correspondence of TT is

$$\forall x \forall y \exists z(xTz \wedge \exists w(Tyw \wedge Rzw)).$$

Proof. This follows from the Sahlqvist algorithm for correspondence [8]. $\Box$

According to Sahlqvist's theorem, a normal modal logic axiomatised by Sahlqvist axioms will have a canonical model whose frame validates these axioms [8]. Since the universal modality is axiomatised by **S5**, this means that the canonical model will be of the form $\mathfrak{M} = (W, R_1, \ldots, R_n, E)$, where E is an equivalence relation used to interpret the universal modality. However, the intended interpretation of the universal modality is as the total relation $W \times W$ and there is no guarantee that E is total.

In order to overcome this difficulty, we choose a world $w \in W$ such that $\mathfrak{M}, w \models \varphi$, for each formula $\varphi \in \mathcal{L}^{[*]}_{\Box\forall}$ that we wish to satisfy. Then, we look at the submodel $\mathfrak{M}_w$ of the canonical model generated by φ. We claim that the restriction of E is a total relation on the submodel $\mathfrak{M}_w$. We prove this by showing that the axiom $\mathsf{Com}_\forall$ implies that the E-cluster of w is already closed under each relation R_i. It follows that the model $\mathfrak{M}_w$ is in fact just the restriction of the canonical model to the E-cluster of w, and since E is total on this cluster, the model $\mathfrak{M}_w$ has the desired form. This can be found for example in [37].

Since every Sahlqvist formula is canonical for the property it defines [8], and **ND4TT** is a normal logic, then we get the following consequence:

Corollary 3.2.7. **ND4TT** *is sound and complete with respect to the class of topologically transitive dynamic Kripke frames with a transitive-serial* $[*]$*-relation.*

Note that similarly, we can show that **ND**, which excludes the axioms TT and $4_{[*]}$, is sound and complete with respect to the class of all weak dynamic Kripke frames with a serial $[*]$-relation.

We can now show that **ND4TT** is topologically sound and complete.

Theorem 3.2.8. (*topological soundness and completeness*) **ND4TT** *is sound and complete with respect to the class of all topologically transitive non-deterministic dynamic topological systems with a transitive relation.*

Proof. For soundness we only need to show that TT and $\mathsf{Cont}_{\langle * \rangle}$ are valid on the class of all dynamic topological systems. This follows from Lemma 3.2.1 and Lemma 3.2.5 respectively. Since **ND4TT** is a language above **S4**, then by Proposition 1.3.3 it immediately follows that **ND4TT** is topologically complete. $\Box$

This completeness result applies to the topological closure semantics. Recall that we have a special interest in d-completeness of dynamic topological systems. In order to start discussing such systems, we will work our way up starting with the most fundamental d-logic and with one special logic above it. That will be our primary concern in the next chapter.

Chapter 4

Derivative Logics of Dynamic Topological Systems

In this chapter we will discuss the finite model property and d-completeness results for dynamic topological logics. Specifically, we will prove the finite model property, soundness and completeness of the d-logic **wK4C** with respect to all dynamic topological systems and for the d-logic **GLC** with respect to all dynamic topological systems based on a scattered space. Recall that by d-completeness we refer to completeness with respect to the topological semantics with the Cantor derivative interpretation of $\Diamond$. Using this interpretation we can express properties that we cannot express under the closure interpretation of $\Diamond$, such as a space being dense-in-itself. This type of systems have not yet been studied in relation to dynamic topological systems and we will therefore start with the d-logic **wK4C** of all dynamic topological systems. Before we do that, we will need to show Kripke completeness of the logic **wK4C** which defines the class of *weakly-transitive* and *weakly-monotonic* Kripke frames. These properties are defined as follows:

Definition 4.0.1. (*weak transitivity*) A binary relation R is weakly transitive if $wRvRu$ implies wRu or $w = u$.

Definition 4.0.2. (*monotonicity and weak monotonicity*) A function f is monotonic if wRv implies $f(w)Rf(v)$ and weakly monotonic if wRv implies $f(w)Rf(v)$ or $f(w) = f(v)$.

Any other d-logic for dynamic topological systems will be above **wK4C**. From this point onward, we restrict ourselves to the fragment $\mathcal{L}^{\bigcirc}_{\Box}$. We will start by introducing the topological d-semantics of our logic with the Cantor derivative interpretation.

Definition 4.0.3. (*d-semantics*) Given a dynamic topological system $\mathcal{X}_f = \langle X, \tau, f \rangle$, we define a valuation ν from the set PV of propositional variables to $\wp(X)$. A pair $\mathcal{M}_f = \langle \mathcal{X}_f, \nu \rangle$ is called a model of $\mathcal{X}_f$. Given a model $\mathcal{M}_f$ and a point $x \in X$, we define the d-satisfaction relation $\models_d$ by induction on the complexity of φ:

1. $x \models_d p \iff x \in \nu(p)$;
2. $x \models_d \neg\varphi \iff x \not\models_d \varphi$;
3. $x \models_d \varphi \wedge \psi \iff x \models_d \varphi$ and $x \models_d \psi$;
4. $x \models_d \Box\varphi \iff \exists U \in \tau$ s.t. $x \in U$ and $\forall y \in U \backslash \{x\}(y \models_d \varphi)$,
   and therefore dually:
   $x \models_d \Diamond\varphi \iff \forall U \in \tau$, if $x \in U$ then $\exists y \in U \backslash \{x\}(y \models_d \varphi)$;
5. $x \models_d \bigcirc\varphi \iff f(x) \models \varphi$.

This will be the semantics used in order to prove the topological d-completeness of the systems **wK4C** and **GLC**. In particular, we will provide a method for proving completeness of any logic above **wK4C**. We have a special interest in the logic **GLC** that is motivated by prior work of Fernández-Duque [18]. In his paper, he proved that there exists a complete axiomatisation for the language $\mathcal{L}_{\Box}^{\bigcirc[*]}$. However, it necessitates the addition of the *tangled closure* to our language and moreover this axiomatisation is not finite. The tangled closure is defined as follows:

Definition 4.0.4. (*tangled closure*) Let $\langle X, \tau \rangle$ be a topological space and $\mathcal{S} \subseteq \wp(X)$. Given $A \subseteq X$, we say that $\mathcal{S}$ is tangled in A if for all $S \in \mathcal{S}$

$$d(S \cap A) = A.$$

We define the *tangled closure* of $\mathcal{S}$ as

$$\mathcal{S}^* := \bigcup \mathcal{A},$$

where $\mathcal{A}$ is the set of all $A \subseteq X$ such that $\mathcal{S}$ is tangled in A.

The addition of this operator results in non-finite axiomatisation of DTL. We will show, that for a specific class of spaces, namely *scattered spaces*, this addition is redundant.

Definition 4.0.5. (*scattered space*) A topological space $\langle X, \tau \rangle$ is called scattered if for every $S \subseteq X$

$$S \subseteq d(S) \text{ implies } S = \varnothing.$$

This is equivalent to the more common definition of scattered space where a topological space is called scattered if it does not contain nonempty dense-in-itself subsets. That is because $S = \varnothing$ implies that there exists an isolated point $y \in S \backslash d(S)$.

Proposition 4.0.6. [9] Let $\mathfrak{F} = \langle W, R \rangle$ be a **K4** Kripke frame with the upset topology. Then the topological space $\langle X, \tau \rangle$ induced by the Alexandroff topology of $\mathfrak{F}$ is a scattered space if and only if Löb's axiom $\Box(\Box p \to p) \to \Box p$ is valid on $\mathfrak{F}$ if and only if R is converse well-founded, i.e. there is no infinite sequence

$$a_0 R a_1 R a_2 R \ldots$$

In case $\mathfrak{F}$ is a finite **K4** frame, then $\langle X, \tau \rangle$ is scattered if and only if R is irreflexive. The class of transitive, converse well-founded frames is named **GL** after Gödel and Löb. The logic **GLC** includes a weakly monotonic function as in **wK4C**.

In order to show completeness of **GLC** with respect to all dynamic topological systems based on a scattered space, we will need to consider d-completeness similarly to **wK4C**. The result of d-completeness for **GL** is a well-known result by the works of Simmons [38] and Esakia [15]. In case of d-completeness we will need to revise the definition of the tangled closure to a *tangled derivative*. We denote its corresponding operator by $\diamond\!\!\!\diamond$ and define its operation as follows:

Definition 4.0.7. (*tangled derivative*) Let $\langle X, \tau \rangle$ be a topological space and $\{\varphi_1, \ldots, \varphi_n\}$ a set of formulas. We denote by $[\![\varphi]\!] := \{x : x \models \varphi\}$ the truth set of φ. Suppose $x \in X$, then

$$x \models \diamond\!\!\!\diamond\{\varphi_1, \ldots \varphi_n\} \iff \exists S \subseteq X \text{ s.t. } x \in S \text{ and } S \subseteq d(S \cap [\![\varphi_i]\!]), \text{ for all } i \leq n.$$

The logic **GLC*** is an extension of **GLC** that includes the axioms for the temporal operator $[*]$. Unlike the case of DTL that requires the tangled operator, in the case of **GLC***, we will be able to avoid this and have the regular operator $\Diamond$ alone. This is shown by the following theorem.

Theorem 4.0.8. *Let $\mathcal{X} = \langle X, \tau \rangle$ be a scattered space and $\{\varphi_1, \ldots, \varphi_n\}$ a set of formulas. Then*

$$\diamond\!\!\!\diamond\{\varphi_1, \ldots, \varphi_n\} \equiv \bot.$$

Proof. Suppose $x \models \diamond\!\!\!\diamond\{\varphi_1, \ldots, \varphi_n\}$ for some $x \in X$. Then by definition there exists $S \subseteq X$ s.t. $x \in S$ and $S \subseteq d(S \cap [\![\varphi_i]\!])$, for all $i \leq n$. It follows that $S \subseteq d(S)$. Since $\mathcal{X}$ is scattered, then $S = \varnothing$ in contradiction. It follows that $x \not\models \diamond\!\!\!\diamond\{\varphi_1, \ldots, \varphi_n\}$ for all $x \in X$. $\Box$

We can therefore achieve a neat completeness result for **GLC*** with finite axiomatisation. However, this is beyond the scope of this thesis. In the next section we establish the first steps of d-completeness and finite model property results for dynamic topological systems. In particular, we show the existence of the finite model property, soundness and completeness of the d-logic **wK4C** by a proof technique that could be applied to any logic above **wK4C**. We use **GLC** as a case study of such logic. This is an essential step for achieving a full completeness result for **GLC***. We will embed the proof of **GLC** in the proof of **wK4C** and thus prove them simultaneously.

4.1 Finite Model Property and d-Completeness

Consider the logic **wK4C** that consists of the following axioms and derivation rules:

- (Taut) All propositional tautologies.
- Axioms for $\Box$:
  1. ($\mathsf{K}_\Box$) $\Box(p \to q) \to (\Box p \to \Box q)$
  2. ($\mathsf{w4}$) $(p \land \Box p) \to \Box\Box p$
- Axioms for $\bigcirc$:
  1. ($\mathsf{Or}_\bigcirc$) $\bigcirc(p \lor q) \equiv (\bigcirc p \lor \bigcirc q)$
  2. ($\mathsf{Neg}_\bigcirc$) $\bigcirc\neg p \equiv \neg\bigcirc p$
- ($\mathsf{Cont_d}$) $(\bigcirc\Box p \land \bigcirc p) \to \Box\bigcirc p$
- Inference rules:
  1. (MP) Modus ponens
  2. (Sub) Substitution
  3. ($\mathsf{N}_\Box$) Necessitation for $\Box$
  4. ($\mathsf{N}_\bigcirc$) Necessitation for $\bigcirc$

For the logic **GLC** we replace w4 with the axiom

$$\mathsf{L\ddot{o}b} := \Box(\Box p \to p) \to \Box p.$$

As the following lemmas show, the axioms w4 and $\mathsf{Cont_d}$ define the class of weakly-transitive and weakly-monotonic frames respectively.

Lemma 4.1.1. w4 defines the class of weakly-transitive Kripke frames.

Proof. The proof of this is well known and can be found in [15]. $\Box$

Lemma 4.1.2. $\mathsf{Cont_d}$ defines the class of weakly-monotonic Kripke frames.

Proof. Let $\mathfrak{F}_f = \langle W, R, f\rangle$ be a dynamic Kripke frame.

($\Rightarrow$) Suppose for contradiction that $\langle\mathfrak{F}_f, V\rangle, w \not\models (\bigcirc\Box p \land \bigcirc p) \to \Box\bigcirc p$ for some $w \in W$ and a valuation V. Then for all v such that $f(w)Rv$ we have $\langle\mathfrak{F}_f, V\rangle, v \models p$ and also $\langle\mathfrak{F}_f, V\rangle, f(w) \models p$. Moreover, there exists u such that wRu and $\langle\mathfrak{F}_f, V\rangle, f(u) \not\models p$. But since f is weakly-monotonic, then $f(w)Rf(u)$ or $f(w) = f(u)$. If $f(w)Rf(u)$, then we have a contradiction since $\langle\mathfrak{F}_f, V\rangle, v \models p$ for all v such that $f(w)Rv$. If $f(w) = f(u)$ then we have a contradiction once again since $\langle\mathfrak{F}_f, V\rangle, f(w) \models p$.

($\Leftarrow$) Suppose $\mathfrak{F}_f$ is not weakly-monotonic. Then there exists $w, v \in W$ such that wRv, $f(w) \neq f(v)$ and $f(w) \not R f(v)$. Consider the following valuation V:

$$V(p) = \{x : f(w)Rx\} \cup \{f(w)\}.$$

By this valuation, since $f(w) \not R f(v)$ and $f(w) \neq (v)$, then $\langle \mathfrak{F}_f, V\rangle, f(v) \not\models p$. Then wRv implies $\langle \mathfrak{F}_f, V\rangle, w \not\models \Box\bigcirc p$. Next, since $\langle \mathfrak{F}_f, V\rangle, f(w) \models p$ then clearly $\langle \mathfrak{F}_f, V\rangle, w \models \bigcirc p$ and by the definition of V also $\langle \mathfrak{F}_f, V\rangle, w \models \bigcirc\Box p$. It follows that $\langle \mathfrak{F}_f, V\rangle, w \not\models (\bigcirc\Box p \wedge \bigcirc p) \to \Box\bigcirc p$, as required. $\Box$

4.1.1 The Canonical Model

Let Λ be any normal logic and let R^+ be a weakly-transitive relation. We show that if $\varphi \notin \Lambda$, then there is a finite dynamic Kripke model $\mathfrak{M} = \langle W, R^+, g, V\rangle$ such that $\mathfrak{M} \not\models \varphi$. We will start with a few definitions.

A *maximal consistent set* (MCS) w is a set of formulas that is Λ-consistent, i.e. $w \nvdash_\Lambda \bot$, and any set of formulas that properly contains it is Λ-inconsistent.

Let $\mathfrak{M} = \langle W, R^+, g, V\rangle$ be the canonical model, where:

1. W is the set of all Λ-MCSs;
2. wR^+v iff for all formulas φ if $\Box\varphi \in w$, then $\varphi \in v$;
3. $g(w) = \{\varphi : \bigcirc\varphi \in w\}$;
4. $V(p) = \{w : p \in w\}$.

Let Λ, Λ' be normal logics. We say that Λ extends Λ' if all the axioms and rules of Λ' are derivable in Λ.

Lemma 4.1.3. If Λ extends **wK4C**, then the canonical model for Λ is a **wK4C** model. If Λ extends **K4C**, then the canonical model of Λ is a **K4C** model.

Proof. Suppose that Λ extends **wK4C**. We prove that g is weakly monotonic. Suppose wR^+v and $g(w) \neq g(v)$. Since $g(w) \neq g(v)$, there exists φ such that $\varphi \in g(w)$ and $\varphi \notin g(v)$. We consider an arbitrary $\Box\psi \in g(w)$ then clearly $\Box(\psi \vee \varphi) \in g(w)$ and $(\psi \vee \varphi) \in g(w)$. In particular, $\bigcirc\Box(\psi \vee \varphi) \in w$ and $\bigcirc(\psi \vee \varphi) \in w$. Since $(\bigcirc\Box p \wedge \bigcirc p) \to \Box\bigcirc p \in$ **wK4C**, then $\Box\bigcirc(\varphi \vee \psi) \in w$. Since wR^+v, then $\bigcirc(\varphi \vee \psi) \in v$ and $(\varphi \vee \psi) \in g(v)$. Since $\varphi \notin g(v)$, then $\psi \in g(v)$ and because $\Box\psi$ is arbitrary, then $g(w)R^+g(v)$, as required. Hence g is weakly monotonic.

We prove that R^+ is weakly transitive. Suppose wR^+vR^+u and $w \neq u$. From $w \neq u$ it follows that there exists φ such that $\varphi \in w$ and $\varphi \notin u$. We consider an arbitrary $\Box\psi \in w$, then clearly $\Box(\psi \vee \varphi) \in w$ and $(\psi \vee \varphi) \in w$. Since $(p \wedge \Box p) \to \Box\Box p \in$ **wK4C**, then $\Box\Box(\psi \vee \varphi) \in w$. Then wR^+vR^+u implies $(\psi \vee \varphi) \in u$ and as $\varphi \notin u$ we have $\psi \in u$. Since $\Box\psi$ is arbitrary, then wR^+u holds, as required.

It follows that g is weakly monotonic and R^+ is weakly transitive and hence $\mathfrak{M}$ is a **wK4C** model.

Suppose that Λ extends **K4C**, then weak monotonicity holds as before since **K4C** extends **wK4C**. Therefore, we only need to prove that R^+ is transitive. Suppose wR^+vR^+u. We consider an arbitrary $\Box\psi \in w$. Since $\Box p \to \Box\Box p \in$ **wK4C**, then $\Box\Box\psi \in w$. Then, wR^+vR^+u implies $\psi \in u$ and since $\Box\psi$ is arbitrary, then wR^+u, as required.

It follows that g is weakly monotonic and R^+ is transitive and thus $\mathfrak{M}$ is a **K4C** model. ☐

It is a well-known result that the transitivity axiom $\Box p \to \Box\Box p$ is derivable in **GL** (see [39]). Therefore, **GLC** extends the system **K4C**.

Lemma 4.1.4. (*existence lemma*) For any normal modal logic Λ and any point $w \in W$, if $\Diamond\varphi \in w$ then there exists a point $v \in W$ such that wRv and $\varphi \in v$.

Proof. The proof for this lemma could be found for example in [8]. ☐

Lemma 4.1.5. (*truth lemma*) For every $w \in W$ and every formula φ in our language

$$w \in V(\varphi) \text{ iff } \varphi \in w.$$

Proof. This follows by standard argument using the definition of R^+ and the existence lemma. ☐

Corollary 4.1.6. **wK4C** *is sound and complete with respect to the class of all weakly-monotonic weakly-transitive Kripke frames.* **K4C** *is sound and complete with respect to the class of all weakly-monotonic transitive Kripke frames.*

We will add a few additions to our structure. We define a R^+-cluster $C(w)$ for each point $w \in W$ as

$$C(w) = \{w\} \cup \{v : wR^+vR^+w\}.$$

Definition 4.1.7. (*φ-maximal consistent set*) A set w is said to be φ-maximal consistent set if w is a MCS, $\varphi \in w$ and whenever wR^+v and $\varphi \in v$, it follows that $v \in C(w)$. We call φ-MCSs simply φ-maximal sets.

Lemma 4.1.8. (*Zorn's Lemma*) Let $(A, \leq)$ be a preordered set where A is non-empty. A chain is a set $\mathcal{C} \subseteq A$ whose elements are totally ordered by $\leq$. Suppose that every chain $\mathcal{C}$ has an upper bound in A. Then, A has a $\leq$-maximal element.

Lemma 4.1.9. If $\Diamond\varphi \in w$, then there is φ-maximal v so that wR^+v.

Proof. Let $\underline{R}^+$ be the reflexive closure of R^+. Suppose that $\mathcal{C}$ is an $\underline{R}^+$-chain in $A := R^+(w)$. We show that there is an upper bound of $\mathcal{C}$ that belongs to A. If $\mathcal{C}$ has a maximal element, then we are done. Suppose that $\mathcal{C}$ does not have a maximal element. Let Γ be the set

$$\{\varphi : \exists w \in \mathcal{C}(\Box\varphi \in w)\}.$$

Suppose $\varphi_1, \dots, \varphi_n$ is inconsistent and $\varphi_1, \dots, \varphi_n \in \Gamma$. Then there exists $w_0 \in \mathcal{C}$ such that $\Box\varphi_1 \in w_0$. Since w_0 is not maximal in $\mathcal{C}$ by assumption, we have $w_1 \in \mathcal{C}$ such that $w_0 R^+ w_1$ but $w_1 \not R^+ w_0$. Then if there is v such that $w_1 \underline{R}^+ v$, then $\varphi_1 \wedge \Box\varphi_1 \in v$ by weak transitivity of R^+. Continuing in that manner, we construct $w_1, \dots, w_n$ such that $\varphi_1 \wedge \Box\varphi_1, \dots, \varphi_j \wedge \Box\varphi_j \in w_j$ and in particular $\varphi_1 \wedge \Box\varphi_1, \dots, \varphi_n \wedge \Box\varphi_n \in w_n$. But since w_n is consistent, then so is $\varphi_1, \dots, \varphi_n$. Therefore, Γ is consistent.

By the Lindenbaum lemma we can extend Γ to be a MCS; we denote it by w^*. Suppose $\Box\varphi \in v \in \mathcal{C}$, then by definition $\varphi \in w^*$. It follows by definition that vR^+w^*. Since by our main assumption $\Diamond\varphi \in w$ then

$$\{v : \varphi \in v \text{ and } wR^+v\} \neq \varnothing.$$

Thus, by Lemma 4.1.8 we conclude that there is a φ-maximal world above w. $\Box$

Lemma 4.1.10. Let Φ be a finite set of formulas. There is an auxiliary relation R^- on the canonical model of Λ such that:

(i) $R^- \subset R^+$;

(ii) For each $w \in W$, the set $R^-(w)$ is finite;

(iii) If $\Diamond\varphi \in w \cap \Phi$, then there exists $v \in W$ with wR^-v and $\varphi \in v$;

(iv) If wR^-vR^-w then $R^-(w) \cup \{w\} = R^-(v) \cup \{v\}$;

(v) If Λ extends **GLC**, then R^- is irreflexive.

Proof. Let C be any cluster of points in W and define

$$\bigcup R^+(C) = \bigcup \{R^+(v) : v \in C\}.$$

We construct the weakly transitive relation R^- as follows. Using Lemma 4.1.9 we use the axiom of choice to choose a function that for each formula φ and each cluster C such that $\Diamond\varphi \in \bigcup C$, assigns a φ-maximal world $w(\varphi, C)$ such that $w(\varphi, C) \in \bigcup R^+(C)$. We choose a second point $w'(\varphi, C)$, possibly equal to $w(\varphi, C)$, such that

- if $\Diamond\varphi \in w(\varphi, C)$, then $w'(\varphi, C)$ is any φ-maximal point such that $w(\varphi, C) R^+ w'(\varphi, C)$;
- otherwise, $w'(\varphi, C) = w(\varphi, C)$.

Let φ be a formula and Φ be the set of subformulas of φ. We set uR_0^-v iff there exists $\psi \in \Phi$ such that $\Diamond\psi \in u$ and $v \in \{w(\psi, C(u)), w'(\psi, C(u))\}$. Let R^- be the weakly transitive closure of R_0^-.

It is clear that (i), (iii) and (iv) follow directly from the construction. We therefore only need to verify conditions (ii) and (v). First, we check that for each $w \in W$ the set $R^-(w)$ is finite. If wR^-v, then by the definition of weakly transitive closure it follows that either wR_0^-v and v is an immediate successor of w, or there exists u such that $wR_0^-uR^-v$ and $w \neq v$. Note that there are finitely many such u, since we fixed finitely many points in $R_0^-(w)$, namely, there are finitely many formulas in Φ, and for each formula there are at most two successors of u, and so w has finitely many successors.

Now, if uR^-v then there is a sequence

$$uR_0^-v_1R_0^- \dots R_0^-v_n = v.$$

Suppose that the sequence is minimal. If $n > 1$, we have $uR_0^-v_1R_0^-v_2$. Then by the weak transitivity of R_0^- either $u = v_2$ and then we can shorten the full sequence to $uR_0^-v_1R_0^-v_3 \dots R_0^-v$ or else $u \neq v_2$ and so $uR_0^-v_2$ since for any x, y we have that xR_0^-y depends only on $C(x)$ when $y \neq x$. Therefore, in that case we can shorten the sequence to $uR_0^-v_2 \dots R_0^-v$. By the minimality of the sequence, it follows that $n = 1$ and since $R_0^-(v)$ is finite then there are finitely many points v with uR^-v via $uR_0^-v_1R_0^- \dots R_0^-v$ $v_1 \in C(u)$ such that $v_1 \in C(u)$.

It remains to show that there are finitely many points v with uR^-v via $uR_0^-v_1R_0^- \dots R_0^-v$ such that $v_1 \notin C(u)$. We already know that there are finitely many options for v_1, so it suffices to show that given a fixed v , there are finitely many v with v_1R^-v. Given a world u consider the set

$$u^\Diamond = \{\varphi \in \Phi : \exists v(uR^+v, v \not\!R^+ u \text{ and } \varphi \in v\}.$$

We prove that for each v_1 we have $u^\Diamond \supsetneq v_1^\Diamond$ and since $u^\Diamond, v_1^\Diamond$ are finite then we can do an induction on $|v_1^\Diamond| < |u^\Diamond|$.

Since v_1 is φ-maximal for some φ such that $\Diamond\varphi \in u$, then $\varphi \in u^\Diamond \setminus v_1^\Diamond$ and hence $u^\Diamond \neq v_1^\Diamond$. We show that $u^\Diamond \supseteq v_1^\Diamond$. Since $uR_0^-v_1$ then in particular $v_1 \not\!R^+ u$ as $v_1 \notin C(u)$. Suppose $\exists v(v_1R^+v, v \not\!R^+ v_1$ and $\varphi \in v)$, then $\varphi \in v_1^\Diamond$. As before, $uR^+v_1R^+v$ and by weak transitivity either $u = v$ or uR^+v. If $u = v$ then vR^+v_1 in contradiction, and therefore uR^+v. Moreover, $v \not\!R^+ u$, for else either $v = v_1$ in contradiction since v_1R^+v and $v \not\!R^+ v_1$, or vR^+v_1 which again yields contradiction. It follows that $\varphi \in u^\Diamond$ and so $u^\Diamond \supseteq v_1^\Diamond$.

We have established that $u^\Diamond \supsetneq v_1^\Diamond$ and since $u^\Diamond, v_1^\Diamond$ are finite then $|u^\Diamond| > |v_1^\Diamond|$. We can apply the induction hypothesis to conclude that $R^-(v_1)$ is finite. Then the set

$$R^-(u) = R_0^-(u) \cup \bigcup\{R^-(v_1) : wR_0^-v_1 \not\!R_0^- w\}$$

is finite as it is a union of finitely many finite sets. So there are finitely many points v such that $v_1 \notin C(u)$, as required.

We conclude that for each $w \in W$, the set $R^-(w)$ is finite.

Next, we verify condition (v). Suppose that w is φ-maximal, then $\varphi \in w$ and let

$$w_\Box := \{\varphi : \Box\varphi \in w\}.$$

We prove that $\Box\neg\varphi \in w$. For the sake of contradiction suppose $\Diamond\varphi \in w$. First, note that $\Gamma := \{\varphi, \Box\neg\varphi\} \cup w_\Box \cup \Box w_\Box$ is consistent, for if it is not, then

$$\Box w_\Box \cup w_\Box \vdash_{\mathbf{GLC}} \Box\neg\varphi \to \neg\varphi.$$

It follows that there exist finitely many $\chi_1, \ldots, \chi_n \in w_\Box$, whose conjunction can be represented as a single formula χ by the closure under derivability of w, and

$$\vdash_{\mathbf{GLC}} (\Box\chi \wedge \chi) \to (\Box\neg\varphi \to \neg\varphi) \vdash_{\mathbf{GLC}} \Box(\Box\chi \wedge \chi) \to \Box(\Box\neg\varphi \to \neg\varphi).$$

Now, using the fact that

$$\Box p \vdash_{\mathbf{GLC}} \Box\Box p \wedge \Box p \vdash_{\mathbf{GLC}} \Box(\Box p \wedge p),$$

together with substitution and $\mathsf{L\ddot{o}b}$ axiom, we get

$$\Box\chi \vdash_{\mathbf{GLC}} \Box\neg\varphi.$$

But that means that w is inconsistent, in contradiction. We extend Γ to be a maximal consistent set v. Since $w_\Box \subset v$ then wR^+v and moreover $v \not\!R^+w$ for else $\neg\varphi \in w$ in contradiction. But that contradicts the fact that w is φ-maximal and therefore $\Box\neg\varphi \in w$ which implies that w is irreflexive. $\Box$

4.1.2 Gluons and Weak p-Morphisms

In this subsection we show that the logics **wK4C** and **GLC** have the finite model property by constructing finite models and a truth preserving maps from these models to our canonical model. We will start with a few definitions.

We denote by $C^-(w)$ the R^- cluster of w, i.e.

$$C^-(w) = \{w\} \cup \{v : wR^-vR^-w\}.$$

Definition 4.1.11. (*gluon*) We define a gluon with duration I as a model

$$\mathfrak{g} = \langle |\mathfrak{g}|, \prec, f, \nu \rangle,$$

where $|\mathfrak{g}| \neq \varnothing$ is a finite set of points, $\prec$ is a weakly transitive and weakly monotonic relation, $C^\prec(w)$ denotes the $\prec$ cluster of w, the reflexive closure of $\prec$ is denoted by $\preceq$, and

1. $|\mathfrak{g}| = \bigsqcup_{i \leq I} |\mathfrak{g}|_i$, where each $|\mathfrak{g}|_i$ is open and has a cluster-root, i.e. there is $C^-(w) \subseteq |\mathfrak{g}|_i$ such that $w \prec v$ for all $w \in C^-(w)$ and for all $v \in |\mathfrak{g}|_i$ where $v \neq w$. The cluster-root of $|\mathfrak{g}|_0$ is called the cluster-root of the gluon, or simply the root of the gluon;

2. $f : |\mathfrak{g}| \to |\mathfrak{g}|$ is a continuous function such that for all $i < I$, $f|\mathfrak{g}|_i \subseteq |\mathfrak{g}|_{i+1}$ and $f \restriction |\mathfrak{g}|_I$ is the identity map;

3. ν is an evaluation function assigning to each propositional variable a subset of $|\mathfrak{g}|$;

4. $x \prec y$ implies that $x, y \in |\mathfrak{g}|_i$ for some $i \leq I$;

5. If $x \prec f(y)$ then there exists $z \preceq y$ such that $f(z) \in C^\prec(x)$.

Let $dpt_{\bigcirc}(w) = k$ whenever $w \in |\mathfrak{g}|_i$ and $i + k = I$, where I is the duration of the gluon. We define a map from gluons to the canonical model as follows:

Definition 4.1.12. (*weak p-morphism*) A map $\pi : |\mathfrak{g}| \to W$ is called a *weak p-morphism* if for all $x \in |\mathfrak{g}|$ the following conditions are satisfied:

1. $x \in \nu(p) \iff p \in \pi(x)$;
2. If $x \in |\mathfrak{g}|_i$ for some $i < I$, then $g(\pi(x)) = \pi(f(x))$;
3. if $x \prec y$ then $\pi(x)R^+\pi(y)$;
4. If $\pi(x)R^-v$ for some $v \in W$, then there exists $y \in |\mathfrak{g}|$ such that

$$x \prec y \text{ and } v = \pi(y).$$

We now show that a weak p-morphism π preserves truth.

Lemma 4.1.13. (*truth lemma*) Suppose φ is a formula of $\bigcirc$-depth less than I. Then $\varphi \in \pi(x)$ iff $x \in \nu(\varphi)$.

Proof. We prove this by induction on the complexity of φ. For the base case, suppose $p \in \pi(x)$. Then by the definition of weak p-morphism we have $x \in \nu(p)$ as required. The other direction follows similarly. The boolean cases are routine and we therefore show only the temporal and spatial cases. To show both directions of the statement for the spatial case, we prove left-to-right direction for each dual:

- If $\Box\varphi \in \pi(x)$ then for all $v \in W$ such that $\pi(x)R^+v$ we have $\varphi \in v$. By the induction hypothesis it follows that $v' \in \nu(\varphi)$, where $\pi(v') = v$. Now suppose there is $z \in |\mathfrak{g}|$ such that $x \prec z$ and $z \notin \nu(\varphi)$. But by the definition of weak p-morphism if $x \prec z$ then $\pi(x)R^+\pi(z)$ and thus by the induction hypothesis $\varphi \in \pi(z)$ which yields $z \in \nu(\varphi)$ in contradiction. Hence $x \in \nu(\Box\varphi)$.

  If $\Diamond\varphi \in \pi(x)$ then there exists $v \in W$ such that $\pi(x)R^-v$ and $\varphi \in v$. By the definition of weak p-morphism there exists $v' \in |\mathfrak{g}|$ such that $x \prec v'$ and $v = \pi(v')$. By the induction hypothesis $v' \in \nu(\varphi)$ and thus $x \in \nu(\Diamond\varphi)$.

- If $\bigcirc\varphi \in \pi(x)$ then there exists $v \in W$ such that $g(\pi(x)) = v$ and $\varphi \in v$. By the definition of weak p-morphism we get $v = \pi(f(x))$ and from the induction hypothesis we get $f(x) \in \nu(\varphi)$. Hence definition $x \in \nu(\bigcirc\varphi)$.

  If $x \in \nu(\bigcirc\varphi)$ then there exists $y \in |\mathfrak{g}|$ such that $f(x) = y$ and $y \in \nu(\varphi)$. By the induction hypothesis it follows that $\varphi \in \pi(f(x))$ and by the definition of weak p-morphism we get $\pi(f(x)) = g(\pi(x))$. Therefore $\bigcirc\varphi \in \pi(x)$. $\Box$

Definition 4.1.14. (*gluonide*) Let $\vec{C}' = (C'_0, \ldots, C'_n)$ be a sequence of sets such that $C'_i \subseteq C(x_i)$ for some x_i in the canonical model $\mathfrak{M}$ and $g(C'_i) \subseteq C'_{i+1}$. Let $\vec{\mathfrak{a}} = \langle \mathfrak{a}^m \rangle_{m<N}$ be a sequence of gluons of duration $n+1$. We define the gluonide $\mathfrak{g} = \vec{C}' \oplus \vec{\mathfrak{a}}$ as follows:

- for any $i < n+1$
$$|\mathfrak{g}|_i = C'_i \cup \bigsqcup_{m<N} |\mathfrak{a}^m|_i;$$

- $x \prec y$ if either
  - $x, y \in C'_i$ and $x \neq y$ or $x = y$ and xR^-y;
  - $x \in C'_i$ and $y \in |\mathfrak{a}^m|_i$ for some m;
  - or $x, y \in |\mathfrak{a}^m|$ and $x \prec_{\mathfrak{a}^m} y$ for some m;

- $f(x) = \begin{cases} g(x) & , if \;\; x \in C'_i \\ f_{\mathfrak{a}^m}(x) & , if \;\; x \in |\mathfrak{a}^m| \end{cases}$;

- $\nu^{-1}(x) = \begin{cases} \{p : x \in V(p)\} & , if \;\; x \in C'_i \\ \nu^{-1}_{\mathfrak{a}^m}(x) & , if \;\; x \in |\mathfrak{a}^m| \end{cases}$.

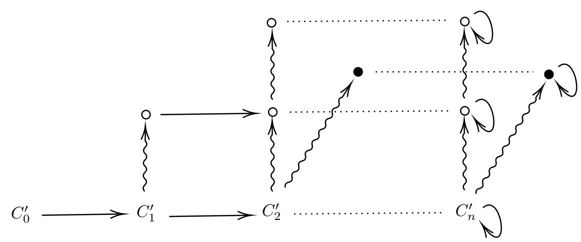

Figure 4.1: A gluonide. The squiggly arrows represent the $\prec$ relation while the straight arrows represent the function f. The $\bigcirc$-depth of this gluonide is $n+1$, while the $\prec$-depth is of degree of at most two.

Lemma 4.1.15. Let $\mathfrak{g} = \langle |\mathfrak{g}|, \prec, f, \nu \rangle$ be a gluonide. Then $\mathfrak{g}$ is a gluon. Moreover, if each $\mathfrak{a}^m$ is a **GLC** gluon and each C'_i is a singleton, then $\mathfrak{g}$ is a **GLC** model.

Proof. We prove that $\prec$ is weakly transitive. Suppose $x \prec y \prec z$ and $x \neq z$. If $x \in C'_i$ for some i, then by definition either $y \in C'_i$ or $y \in |\mathfrak{a}|_i$ for some $\mathfrak{a}$. In either case $z \in |\mathfrak{g}|_i$ and since $x \neq z$ then clearly $x \prec z$. If $x \in |\mathfrak{a}|_i$ then also $y, z \in |\mathfrak{a}|_i$ and by weak transitivity of $\prec_{\mathfrak{a}}$ we have $x \prec_{\mathfrak{a}} z$. By definition $x \prec z$.

We prove that f is weakly monotonic. Suppose $x \prec y$ and $f(x) \neq f(y)$. If $x, y \in C'_i$ then $f(x), f(y) \in C'_{i+1}$ and thus by definition $f(x) \prec f(y)$. If $x \in C'_i$ and $y \in |\mathfrak{a}|_i$ for some gluon $\mathfrak{a}$, then by the definition of f and the definition of a gluon we get $f(x) \in C'_{i+1}$ and $f(y) \in |\mathfrak{a}|_{i+1}$. Thus, $f(x) \prec f(y)$. Similarly, if $x, y \in |\mathfrak{a}|_i$ for some gluon $\mathfrak{a}$, then by the definition of a gluon and since $\prec_{\mathfrak{a}}$ is weakly-monotonic we get $f(x) \prec f(y)$. Hence f is weakly monotonic. The rest of the conditions easily follow from the definitions.

Finally, suppose that each $\mathfrak{a}^m$ is a **GLC** gluon and each C'_i is a singleton. As before, both weak transitivity and weak monotonicity follow. Moreover, since each C'_i is a singleton, and thus vacuously transitive, and R^- is irreflexive in addition to each point of each $\mathfrak{a}^m$ being irreflexive, then $\mathfrak{g}$ is both transitive and irreflexive. Therefore, $\mathfrak{g}$ is a **GLC** model. □

We define the notions of a *quotient set* and a *super weak p-morphism* that will be essential for the rest of the proof.

Definition 4.1.16. (*quotient set*) Let $\mathfrak{g}$ be a gluonide and fix some arbitrary $\hat{\pi} : |\mathfrak{g}| \to W$. We say that x is *at the* Λ-*bottom* for $\Lambda \in \{\mathbf{wK4C}, \mathbf{GLC}\}$ if

- $\hat{\pi}(x) \in C'_i$, for some i, where $\Lambda = \mathbf{wK4C}$;
- for all $y \prec x$, $\hat{\pi}(y) = \hat{\pi}(x)$, where $\Lambda = \mathbf{GLC}$.

We will refer to 'Λ-bottom' simply as 'bottom' when we want to refer to both **wK4C**-bottom and **GLC**-bottom.

We define $x \sim y$ if $x = y$ or x, y are at the bottom and $\hat{\pi}(x) = \hat{\pi}(y)$.

We call a map $\hat{\pi} : |\hat{\mathfrak{g}}| \to W$ *super weak p-morphism* if it fulfils conditions 1,2 and 4 of a weak p-morphism, and

$$x \hat{\prec} y \text{ implies } \hat{\pi}(x) R^+ \hat{\pi}(y) \text{ or } x, y \text{ are at the bottom.}$$

Let x_i be a point in W. We define the set $C'(x_i) \subseteq W$ recursively as follows:

$$C'(x_i) = \begin{cases} C^-(x_i), & if \;\; i = 0 \\ f(C'(x_{i-1})), & otherwise \end{cases} .$$

We denote by $\vec{C'}(x_0) = (C'(x_0), \ldots, C'(x_{n-1}))$ the cluster path of length n emanating at $C'(x_0)$. Note that for the case of **GLC** each $C'(x_i)$ is a singleton. For if this is not the case, then there exists v such that $v R^- x_i R^- v$, and thus $v R^+ v$. However, we already showed that any such v must be irreflexive with respect to R^+, because if v is φ-maximal then $\Diamond\varphi \notin v$.

We can now prove that there exists a gluon $\hat{\mathfrak{g}}$ and a super weak p-morphism $\hat{\pi} : |\mathfrak{g}| \to W$. Let $w \in W$. We prove this by induction on the temporal depth $dpt_{\bigcirc}(w)$ and with a secondary induction on $|w^{\Diamond}|$. Let $\vec{x}_0 = x_0, \ldots, x_{n-1}$ be the orbit of x_0. From the main induction hypothesis we get that for each v such that $x_i R^- v$, for some $i > 0$, there is a gluon $\mathfrak{g}_v$ and a weak p-morphism $\pi_v : |\mathfrak{g}_v| \to W$ that maps the root of $\mathfrak{g}_v$ to v. when $i = 0$ the same conclusion follows by the secondary induction hypothesis.

Accordingly, we define the gluon $\hat{\mathfrak{g}} = \langle |\hat{\mathfrak{g}}|, \hat{\prec}, \hat{f}, \hat{\nu} \rangle$ as

$$\hat{\mathfrak{g}} = (C'(x_0), \ldots, C'(x_{n-1})) \oplus \{\mathfrak{g}_v : x_i R^-_1 v, for\ any\ i\},$$

where R^-_1 is the strict R^- successor, i.e. $w R^-_1 v$ iff $w R^- v$ and $v \not{R}^- w$. Further, if $x_i R^- v$ for any i, then $|\mathfrak{g}_v|_j = \varnothing$ for any $j < i$.

Next we define a map $\hat{\pi} : |\hat{\mathfrak{g}}| \to W$ as

$$\hat{\pi}(x) = \begin{cases} x, & if \;\; x \in C'(x_i), \text{ for some } i \\ \hat{\pi}_v(x), & if \;\; x \in |\hat{\mathfrak{g}}_v| \end{cases}.$$

Since $\hat{g}$ is a gluonide, then by Lemma 4.1.15 it is a gluon. We prove that $\hat{\pi}$ is a super weak p-morphism.

Lemma 4.1.17. $\hat{\pi}$ is a super weak p-morphism.

Proof. Suppose $x \hat{\prec} y$. We check that either $\hat{\pi}(x)R^+\hat{\pi}(y)$ or x, y are at the bottom. There are two cases to consider:

1. Suppose $x \in |\hat{\mathfrak{g}}_{x'}|$ for some x' and $y \in |\hat{\mathfrak{g}}_{y'}|$ for some y'. Then by the definition of a gluon $x' = y'$. By the induction hypothesis, since $\hat{\pi}_{x'}$ is a weak p-morphism, then $\hat{\pi}(x)R^+\hat{\pi}(y)$.

2. Suppose $x \in C'(x_i)$ for some i and $\hat{\pi}(x) \not R^+\hat{\pi}(y)$. Then clearly x is at the **wK4C**-bottom. Moreover, if y is not at the bottom then $y \in |\hat{\mathfrak{g}}_{y'}|$ for some y'. Since $\hat{\pi}(x)R^-\hat{\pi}(y')$ by definition then $\hat{\pi}(x)R^+\hat{\pi}(y')$. If $y = y'$ we have a contradiction, so we suppose otherwise. If $y \neq y'$ then since by the induction hypothesis $\hat{\pi}_{y'}$ is a weak p-morphism, then $y' \hat{\prec} y$ implies $\hat{\pi}(y')R^+\hat{\pi}(y)$. By weak transitivity of R^+ we have $\hat{\pi}(x) \not R^+\hat{\pi}(y)$ implies $\hat{\pi}(x) = \hat{\pi}(y) = y \in C'(x_i)$ and so y is at the **wK4C**-bottom.

   It is clear that x is also at the **GLC**-bottom. Moreover it must be the case that $x \neq y$ since each C_i' is an irreflexive-singleton and a **GLC**-gluonide. But then y is not at the **GLC**-bottom. By the same reasoning as before we get $\hat{\pi}(x)R^+\hat{\pi}(y')R^+\hat{\pi}(y)$ and since $\hat{\pi}(x) \neq \hat{\pi}(y)$ then $\hat{\pi}(x)R^+\hat{\pi}(y)$.

Suppose $\hat{\pi}(x)R^-v$. We check that there exists y such that $x \prec y$ and $v = \hat{\pi}(y)$:

1. Suppose $x \in |\mathfrak{g}_{x'}|$ for some x'. Then $\hat{\pi}_{x'}(x)R^-v$ and since $\hat{\pi}_{x'}$ is a weak p-morphism by the induction hypothesis, then there is y such that $x \prec y$ and $v = \hat{\pi}_{x'}(y)$.

2. Suppose $x \in C'(x_i)$ for some i. Then $\hat{\pi}(x) = x$ and xR^-v. By definition of $\mathfrak{g}$, we observe that $v = \hat{\pi}(r)$ where r is the root of $\mathfrak{g}_v$. Thus also $x \prec r$.

Suppose $z = g(\hat{\pi}(x))$. We now prove that $g(\hat{\pi}(x)) = \hat{\pi}(\hat{f}(x))$.

1. Suppose $x \in |\mathfrak{g}_{x'}|$ for some x'. Since $\hat{\pi}_{x'}$ is a weak p-morphism by the induction hypothesis, then $\hat{\pi}(x) = \hat{\pi}_{x'}(x)$ and thus $g(\hat{\pi}_{x'}(x)) = \hat{\pi}_{x'}(\hat{f}(x))$ and so $z = \hat{\pi}_{x'}(\hat{f}(x))$ which yields $z = \hat{\pi}(\hat{f}(x))$.

2. Suppose $x \in C'(x_i)$ for some i. Then by definition $\hat{\pi}(x) = x$ and $z = g(x)$. By definition $\hat{\pi}(g(x)) = \hat{\pi}(\hat{f}(x)) = g(x)$ and so $z = \hat{\pi}(\hat{f}(x))$. ∎

We note the following observation.

Proposition 4.1.18. Let $x, y \in |\hat{\mathfrak{g}}|$. If x is at the Λ-bottom and $y \hat{\prec} x$, then y is at the Λ-bottom. Moreover, x is at the Λ-bottom iff $f(x)$ is at the Λ-bottom.

Proof. Suppose x is at the **wK4C**-bottom and $y \hat{\prec} x$. Then by definition $y \in C'_i$ and so $\hat{\pi}(y) \in C'_i$ and y is at the **wK4C**-bottom. Suppose x is at the **GLC**-bottom. Since $y \hat{\prec} x$ then $y \neq x$ since R^- is irreflexive. But then by definition $y \in C'_i$ in contradiction to the fact that C'_i is a singleton. The statement then follows trivially.

Next suppose that x is at the Λ-bottom. If $\Lambda = \mathbf{wK4C}$ then $C'_i(x) = x$ for some i and since $f(x) \in C'_{i+1}$ by definition, then $C'_{i+1}(f(x)) = f(x)$ and therefore $f(x)$ is at the bottom. The other direction follows similarly.

If $\Lambda = \mathbf{GLC}$ then x is the singleton C'_i for some i or else R^- is not irreflexive in contradiction. It follows that $f(x)$ is at the bottom for the same reason. The other direction follows similarly. □

Next we define the quotient construction of $\hat{\mathfrak{g}}$ that will be used for the rest of the proof.

Definition 4.1.19. (*quotient gluon*) Given $y \in |\hat{\mathfrak{g}}|$ we denote $[y] = \{z : z \sim y\}$. The *quotient gluon* $\mathfrak{g} = \{|\mathfrak{g}|, \prec, f, \nu\}$ of $\hat{\mathfrak{g}}$ and its respective morphism $\pi : |\mathfrak{g}| \to W$ are defined as follows, where $x, y \in |\mathfrak{g}|$:

(QG1) $|\mathfrak{g}| = \{[x] : x \in |\hat{\mathfrak{g}}|\}$;

(QG2) $[x] \prec [y]$ iff $x, y \in |\mathfrak{g}|_i$ for some i and one of the following conditions is satisfied:

- x, y are at the bottom and $\hat{\pi}(x) R^+ \hat{\pi}(y)$;
- x, y are at the bottom and $\hat{\pi}(x) \neq \hat{\pi}(y)$;
- x is at the bottom and y is not at the bottom;
- x, y are not at the bottom and $x \hat{\prec} y$.

(QG3) $f([x]) = [\hat{f}(x)]$;

(QG4) $\nu(p) = \{[x] : x \in \hat{\nu}(p)\}$;

(QG5) $\pi([x]) = \hat{\pi}(x)$.

Proposition 4.1.20. The maps f and π are well defined.

Proof. To show that π is well defined, suppose that for some $x, y \in |\hat{\mathfrak{g}}|_i$ we have $x \neq y$ and $[x] = [y]$. By definition of $\sim$ it follows that $\hat{\pi}(x) = \hat{\pi}(y)$.

to show that f is well defined, we prove that for each x, y such that $x \neq y$ and $[x] = [y]$ we have $[\hat{f}(x)] = [\hat{f}(y)]$. Since $\hat{\pi}$ is a super weak p-morphism, then $\hat{\pi}(\hat{f}(x)) = g(\hat{\pi}(x))$ and $\hat{\pi}(\hat{f}(y)) = g(\hat{\pi}(y))$. Since $[x] = [y]$ while $x \neq y$, then $g(\hat{\pi}(x)) = g(\hat{\pi}(y))$. It follows that $\hat{\pi}(\hat{f}(x)) = \hat{\pi}(\hat{f}(y))$.

Since x, y are at the bottom, then $\hat{f}(x), \hat{f}(y)$ are at the bottom. It follows that f is well defined. □

We now need to prove that $\mathfrak{g}$ is a gluon and that π is a weak p-morphism.

Lemma 4.1.21. The quotient gluon $\mathfrak{g}$ of $\hat{\mathfrak{g}}$ is a gluon. Moreover, if $\hat{\mathfrak{g}}$ is a **GLC** gluon, then so is $\mathfrak{g}$.

Proof. We check the nontrivial steps. To see that $\prec$ is weakly transitive, suppose that $[x] \prec [y] \prec [z]$ and $[x] \neq [z]$. By QG2 and Proposition 4.1.18 there are four cases to consider:

1. Suppose x, y, z are at the bottom and $\hat{\pi}(x)R^+\hat{\pi}(y)R^+\hat{\pi}(z)$. Then either $\hat{\pi}(x) = \hat{\pi}(z)$ and since x, z are at the bottom we get $[x] = [z]$ in contradiction, or $\hat{\pi}(x)R^+\hat{\pi}(z)$ and then since both x and z are at the bottom by QG2 we get $[x] \prec [z]$.

2. Suppose x, y, z are at the bottom, $\hat{\pi}(x) \neq \hat{\pi}(y)$ and $\hat{\pi}(y) \neq \hat{\pi}(z)$. Since x, z are at the bottom then by the definition of $\sim$, if $\hat{\pi}(x) = \hat{\pi}(z)$ then $[x] = [z]$, in contradiction to our assumption. Then $\hat{\pi}(x) \neq \hat{\pi}(z)$ and thus $[x] \prec [z]$.

3. Suppose x is at the bottom and z is not at the bottom. Then by QG2 we immediately get $[x] \prec [z]$.

4. Suppose x, y, z not at the bottom and $x \hat{\prec} y \hat{\prec} z$. Since $[x] \neq [z]$ then by definition $x \neq z$. By the weak transitivity of $\hat{\prec}$ it follows that $x \hat{\prec} z$ and by QG2 $[x] \prec [z]$.

We conclude that $\prec$ is weakly transitive.

To see that $\prec$ is weakly monotonic suppose $[x] \prec [y]$ and $f([x]) \neq f([y])$. It follows that $[\hat{f}(x)] \neq [\hat{f}(y)]$ and thus $\hat{f}(x) \neq \hat{f}(y)$. We prove $[\hat{f}(x)] \prec [\hat{f}(y)]$ by cases on QG2:

1. If x, y are at the bottom and $\hat{\pi}(x)R^+\hat{\pi}(y)$, then by weak monotonicity of R^+ we get $g(\hat{\pi}(x)) = g(\hat{\pi}(y))$ or $g(\hat{\pi}(x))R^+g(\hat{\pi}(y))$. Moreover, $\hat{f}(x), \hat{f}(y)$ are at the bottom by Proposition 4.1.18. Suppose $g(\hat{\pi}(x)) = g(\hat{\pi}(y))$ then $\hat{\pi}(\hat{f}(x)) = \hat{\pi}(\hat{f}(y))$. But then by the definition of $\sim$ we have $[\hat{f}(x)] = [\hat{f}(y)]$, in contradiction. If $g(\hat{\pi}(x))R^+g(\hat{\pi}(y))$ then $\hat{\pi}(\hat{f}(x))R^+\hat{\pi}(\hat{f}(y))$ and by QG2 it follows that $[\hat{f}(x)] \prec [\hat{f}(y)]$.

2. Suppose x, y are at the bottom and $\hat{\pi}(x) \neq \hat{\pi}(y)$. Then $\hat{f}(x), \hat{f}(y)$ are at the bottom by Proposition 4.1.18. Since $[\hat{f}(x)] \neq [\hat{f}(y)]$ and $\hat{f}(x) \neq \hat{f}(y)$, then $\hat{\pi}(\hat{f}(x)) \neq \hat{\pi}(\hat{f}(y))$ and thus $[\hat{f}(x)] \prec [\hat{f}(y)]$.

3. Suppose x is at the bottom and y is not at the bottom. Then by Proposition 4.1.18, $\hat{f}(x)$ is at the bottom and $\hat{f}(y)$ is not at the bottom. Then it follows by QG2 that $[\hat{f}(x)] \prec [\hat{f}(y)]$.

4. Suppose x, y are not at the bottom and $x \hat{\prec} y$. Then $\hat{f}(x), \hat{f}(y)$ are not at the bottom by Proposition 4.1.18. Since $\hat{f}(x) \neq \hat{f}(y)$ and $\hat{\prec}$ is weakly monotonic, then $\hat{f}(x) \hat{\prec} \hat{f}(y)$ and by QG2 $[\hat{f}(x)] \prec [\hat{f}(y)]$.

We conclude that f is weakly monotonic.

Finally, we show that if $[x] \prec f([y])$ there is $[z] \preceq [y]$ with $f([z]) \in C^{\prec}([x])$. Suppose $[x] \prec f([y])$. First assume that x is at the bottom. Then $x \in C'(x_i)$ for some i and so $x = \hat{\pi}(x) = \pi([x])$. This means that $[x]$ is a root and so $[x_{i-1}] \prec [y]$. Since $\hat{f}(x_{i-1}) \in C'(x_i)$, then by definition of f and since $f([x_{i-1}]) = [\hat{f}(x_{i-1})]$ we get that $f([x_{i-1}])$ is at the bottom . Therefore, $f([x_{i-1}]) \in C^{\prec}([x])$, as required.

Next, suppose x is not at the bottom. Then by QG2 we have $x \hat{\prec} \hat{f}(y)$. Since $\hat{\mathfrak{g}}$ is a gluon then there exists $z \hat{\preceq} y$ with $\hat{f}(z) \in C^{\hat{\prec}}(x)$. But then by QG2 we have $f([z]) \in C^{\hat{\prec}}([x])$. Moreover, y is not at the bottom for else $\hat{f}(y) \in C'(x_i)$ for some i, in contradiction. By QG2, it follows that $[z] \prec [y]$ and $[z]$ is the required witness.

The reset of the conditions immediately follow from the definition of quotient gluon and the fact that $\hat{\mathfrak{g}}$ is a gluon.

We now prove the last claim regarding **GLC**. Suppose $\hat{\mathfrak{g}}$ is a **GLC** gluon and $[x] \prec [y]$. If neither x, y are at the bottom then $x \hat{\prec} y$ and since $\hat{\mathfrak{g}}$ is a **GLC** gluon, then from the irreflexivity of $\hat{\mathfrak{g}}$ follows that $x \neq y$ and so $[x] \neq [y]$. If x is at the bottom and y is not at the bottom, then clearly $x \neq y$ and thus $[x] \neq [y]$. Finally, suppose that both x, y are at the bottom. Since $\hat{\mathfrak{g}}$ is a **GLC** gluon, then $x = y$ is an irreflexive singleton such that $C'(x) = C'(y)$. If $\hat{\pi}(x) R^+ \hat{\pi}(y)$ then $x R^+ x$. But we know that any such point must be irreflexive with respect to R^+, in contradiction. Moreover, since obviously $\hat{\pi}(x) = \hat{\pi}(y)$ then $[x] = [y]$ while $[x] \not\prec [y]$. So $\prec$ is irreflexive.

Finally, f is weakly-monotonic as before and it is easy to see that $\prec$ is transitive. Since $\prec$ is irreflexive and transitive, then $\mathfrak{g}$ is a **GLC** gluon. □

Lemma 4.1.22. $\pi : |\mathfrak{g}| \to W$ is a weak p-morphism.

Proof. We already showed that $\hat{\pi}$ is a super weak p-morphism and therefore we only need to show that $[x] \prec [y]$ implies $\pi([x]) R^+ \pi([y])$. Suppose $[x] \prec [y]$, then $x, y \in |\mathfrak{g}|_i$ for some i and there are four cases to consider:

1. Suppose both x, y are at the bottom and $\hat{\pi}(x) R^+ \hat{\pi}(y)$, then by definition of quotient gluon $\pi([x]) R^+ \pi([y])$.

2. Suppose x, y are at the bottom and $\hat{\pi}(x) \neq \hat{\pi}(y)$. Then clearly $x \neq y$ and thus $[x] \neq [y]$ by the definition of $\sim$. Also $\pi([x]) \neq \pi([y])$ by the definition of a quotient gluon. Since x, y are at the bottom then $x, y \in C'(x_i)$ and $\hat{\pi}(x) R^+ \hat{\pi}(y)$. By the definition of a quotient gluon $\pi([x]) R^+ \pi([y])$.

3. Suppose x is at the bottom and y is not at the bottom. Then $x \hat{\prec} y$ implies either $\hat{\pi}(x) R^+ \hat{\pi}(x)$, which implies $\pi([x]) R^+ \pi([y])$, or y is at the bottom in contradiction.

4. Suppose x, y are not at the bottom and $x \hat{\prec} y$. Then by similar reasoning to case 3, we get $\pi([x]) R^+ \pi([y])$.

We conclude that π is a weak p-morphism. □

Theorem 4.1.23. **wK4C** *and* **GLC** *have the finite model property and are Kripke complete.*

Proof. Let $\Lambda \in \{\mathbf{wK4C}, \mathbf{GLC}\}$ and suppose $\Lambda \not\vdash \varphi$. Then in the canonical model $\mathfrak{M} = \langle W, R^+, g, V\rangle$ there is $w \in W$ that refutes φ. Then, by the previous lemmas, there is a gluon $\mathfrak{g}$ and a weak p-morphism $\pi : |\mathfrak{g}| \to W$ such that $w = \pi(r)$, where r is a root of $\mathfrak{g}$. It follows that $\mathfrak{g}, r \not\models \varphi$. Recall that $\mathfrak{g}$ is a finite weakly-transitive and weakly-monotonic frame and in the case of **GLC** also irreflexive. Then **wK4C** is complete with respect to finite, weakly-transitive and weakly-monotonic Kripke frames and **GLC** is complete with respect to finite, irreflexive weakly-transitive and weakly-monotonic Kripke frames. $\square$

4.1.3 Topological d-Completeness

The topological d-completeness for **GLC** is almost immediate. For soundness, we will need to define the dual of the Cantor derivative $\hat{d}$, called the *co-derivative.*

Definition 4.1.24. (*co-derivative*) For each $S \subseteq X$ let $\hat{d}(S) := X \backslash d(X \backslash S)$ denote the co-derivative of S. Let $A, B \subseteq X$. The co-derivative as the following axioms:

1. $\hat{d}(X) = X$;
2. $x \in \hat{d}(A) \iff x \in \hat{d}(A \cup \{x\})$;
3. $A \cap \hat{d}(A) \subseteq \hat{d}(A \cap \hat{d}(A))$;
4. $\hat{d}(A \cap B) = \hat{d}(A) \cap \hat{d}(B)$.

Moreover, there is a close connection between the co-derivation and the interior of a set, namely $Int(A) = A \cap \hat{d}(A)$ for each $A \subseteq X$. This implies that $U \subseteq \hat{d}(U)$ for each open set U, but not necessarily $\hat{d}(U) \subseteq U$.

Lemma 4.1.25. Let $\mathcal{X}_f = \langle X, \tau, f\rangle$ be a dynamic topological system. Then $\mathcal{X}_f \models_d (\bigcirc\Box p \wedge \bigcirc p) \to \Box\bigcirc p$ iff f is continuous.

Proof. $\Rightarrow$ Let ν be some valuation on $\mathcal{X}_f$. First, note that:

$$\langle \mathcal{X}_f, \nu\rangle, X \models_d (\bigcirc\Box p \wedge \bigcirc p) \to \Box\bigcirc p \iff f^{-1}(\hat{d}(A)) \cap f^{-1}(A) \subseteq \hat{d}(f^{-1}(A)),$$

for each $A \subseteq X$. Suppose $f^{-1}(\hat{d}(A)) \cap f^{-1}(A) \subseteq \hat{d}(f^{-1}(A))$, then since for each $U \in \tau$ we have $U \subseteq \hat{d}(U)$ and $Int(A) \subseteq A$ for all $A \subseteq X$, then:

$$\begin{aligned} Int(f^{-1}(U)) \subseteq f^{-1}(U) &\subseteq f^{-1}(\hat{d}(U)) \cap f^{-1}(U) \\ &\subseteq \hat{d}(f^{-1}(U)) \cap f^{-1}(U) = Int(f^{-1}(U)). \end{aligned}$$

It follows that $f^{-1}(U) = Int(f^{-1}(U))$, hence f is continuous.

$\Leftarrow$ Suppose that f is continuous and for some point $x \in X$ and for some valuation ν, we have $\langle \mathcal{X}_f, \nu\rangle, x \models_d \bigcirc\Box p \wedge \bigcirc p$. Then, there exists $U \in \tau$ such that $f(x) \in U$ and for all $u \in U$ we have $u \models_d p$. Since f is continuous, then $f^{-1}(U) = V$ is an open set. In particular, since $f(x) \in U$ then $x \in V$, and for every $v \in V$ we have that $f(v) \in U$ and hence for all $v \in V$ we have $f(v) \models_d p$. It follows that there exists an open set of x, that is V, such that for all $v \in V$, we have $f(v) \models_d p$, and thus $\langle \mathcal{X}_f, \nu \rangle, x \models_d \Box \bigcirc p$ as required. $\square$

The last missing step for showing soundness and completeness is the equivalence between the relational and d-semantics for irreflexive-transitive dynamic Kripke frames.

Proposition 4.1.26. Suppose $\mathfrak{F}_g = \langle W, R, g, V \rangle$ is an irreflexive-transitive dynamic Kripke model and let τ be the upset topology on W. Then

$$x \models_d \varphi \iff x \models \varphi.$$

Proof. We only need to prove the inductive step for $\varphi := \Box\psi$ as all the other steps are routine.

($\Rightarrow$) If $x \models_d \Box\psi$ then there exists $U \in \tau$ such that $x \in U$ and $U \backslash \{x\} \models_d \psi$. By the induction hypothesis $U \backslash \{x\} \models \psi$. By definition, $x \models \Box\psi$ as $\uparrow(x) \backslash \{x\} \subseteq U$ and R is irreflexive.

($\Leftarrow$) If $x \models \Box\psi$ then as R is irreflexive and transitive then $\uparrow(x) \backslash \{x\} \models \psi$. Let $U := \uparrow x$. Then clearly U is open and by the induction hypothesis $U \backslash \{x\} \models_d \psi$. By definition it follows that $x \models_d \Box\psi$. $\square$

We can now prove the main theorem for **GLC**.

Theorem 4.1.27. **GLC** *is the d-logic of all dynamic topological systems based on scattered spaces.*

Proof. (*soundness*) It follows from Proposition 4.0.6 and Lemma 4.1.25.

(*Completeness*) Suppose $\nvdash_{\mathbf{GLC}} \varphi$. Then by Theorem 4.1.23 and Proposition 4.1.26 there is a dynamic topological system that d-refutes φ and thus **GLC** is topologically d-complete. $\square$

In order to prove topological d-completeness for **wK4C**, we first provide a few definitions and some generalisations of known results. We use similar constructions as in [6].

Definition 4.1.28. (*dynamic d-morphism*) A map π from a dynamic topological system $\mathcal{X}_f = \langle X, \tau, f \rangle$ to a **wK4C**-frame $\mathfrak{F}_g = \langle W, R, g \rangle$ is called a *dynamic d-morphism* if

1. π is i-discrete, i.e. $\pi^{-1}(w)$ is a discrete subspace of X for each irreflexive $w \in W$;
2. π is r-dense, i.e. $\pi^{-1}(w) \subseteq d_X \pi^{-1}(w)$ for each reflexive $w \in W$;
3. $\pi : \mathcal{X} \to \langle W, \tau_{\mathfrak{F}} \rangle$ is an interior map, where $\tau_{\mathfrak{F}}$ denotes the Alexandroff topology on $\mathfrak{F}$;

4. $\pi^{-1}(g^{-1}(w)) = f^{-1}(\pi^{-1}(w))$.

Theorem 4.1.29. *Let $\mathcal{X}_f = \langle X, \tau, f\rangle$ be a continuous dynamic topological system and $\mathfrak{F}_g = \langle W, R, g\rangle$ a **wK4C**-frame. Let $\pi : \mathcal{X}_f \to \mathfrak{F}_g$ be a map. Then π is a dynamic d-morphism iff $\pi^{-1}(R^{-1}(A)) = d_X(\pi^{-1}(A))$ and $\pi^{-1}(g^{-1}(A)) = f^{-1}(\pi^{-1}(A))$ for each $A \subseteq W$.*

Proof. Follows from Theorem 2.7 in [5]. □

Corollary 4.1.30. *If $\pi : \mathcal{X}_f \to \mathfrak{F}_g$ is a surjective dynamic d-morphism, then $L(\mathcal{X}_f) \subseteq L(\mathfrak{F}_g)$.*

Proof. Suppose $\varphi \notin L(\mathfrak{F}_g)$. Then there exists a valuation V on $\mathfrak{F}_g$ such that $V(\varphi) \neq W$. We set a new valuation ν on $\mathcal{X}_f$, such that for each propositional variable $p \in \mathsf{PV}$ we have $\nu(p) = \pi^{-1}(V(p))$. Since π is surjective and by Theorem 4.1.29 the inverse π^{-1} commutes with the derived set and temporal operators, it follows that $\nu(\varphi) = \pi^{-1}(V(\varphi)) \neq X$ and therefore $\varphi \notin L(\mathcal{X}_f)$. □

Definition 4.1.31. Let $\mathfrak{F}_g = \langle W, R, g\rangle$ be a **wK4C**-frame and let $\mathcal{X}_w = \langle X_w, \tau_w\rangle$ be a topological space indexed by $w \in W$. Let $X_\oplus = \bigsqcup_{w\in W} X_w$ be the disjoint union of all X_w. Further, for each $A \subseteq X_\oplus$ and $w \in W$, define $A_w = A \cap X_w$.

Finally, we denote by $\mathcal{X}_\oplus$ the topological space $\langle X_\oplus, \tau_\oplus\rangle$, where $U \in \tau_\oplus$ iff for all $w, v \in W$ we have:

1. $U_w \in \tau_w$;
2. If wRv, $w \neq v$ and $U_w \neq \varnothing$, then $U_v = X_v$.

Proposition 4.1.32. $\mathcal{X}_\oplus$ is a topological space.

Proof. The proof can be found in [6]. □

Theorem 4.1.33. **wK4C** *is the d-logic of all dynamic topological systems.*

Proof. (*Soundness*) We only need to verify that $(\bigcirc\Box p \wedge \bigcirc p) \to \Box\bigcirc p$ is d-valid on all continuous dynamic topological systems. This follows from the right-to-left direction of Lemma 4.1.25.

(*Completeness*) Let **wK4C** $\nvdash \varphi$. Then by Theorem 4.1.23 there exists a finite **wK4C**-frame $\mathfrak{F}_g = \langle W, R, g\rangle$ such that $\mathfrak{F}_g \not\models \varphi$. We now construct the dynamic topological system $\mathcal{X}_\oplus$ based on $\mathfrak{F}_g$. For a reflexive $w \in W$ let w' be a copy of w and let $X_w = \{w, w'\}$ denote the two-points trivial space where $\tau_w = \{\varnothing, X_w\}$. For irreflexive $w \in W$, let $X_w = \{w\}$ denote the singleton space. For each $w, v \in W$ such that $g(w) = v$, we construct a function $f : X_\oplus \to X_\oplus$ as

$$f(x) = g(w) \text{ whenever } x \in X_w.$$

It is easy to verify that f is continuous.

Let $\pi : X_\oplus \to W$ be a map sending any $x \in X_w$ to w. We show that π is a dynamic d-morphism:

(i) $\pi : X_\oplus \to \langle W, \tau_{\mathfrak{F}} \rangle$ is an interior map:

(*Open*) Suppose $U \in \tau_\oplus$. Then $\pi(U) = U'$ where $U' \subseteq W$. Suppose that for some $w, v \in W$ we have $w \in U'$, wRv but not $v \in U'$. Then $U' \notin \tau_{\mathfrak{F}}$. However, since $U_w \neq \varnothing$, then from Definition 4.1.31 we get $U_v = X_v$ and since $X_v = \pi^{-1}(v)$ and $X_v \subseteq U$, then it must be the case that $v \in U'$, in contradiction. It follows that $U' \in \tau_{\mathfrak{F}}$.

(*Continuous*) Suppose $U' \in \tau_{\mathfrak{F}}$, then U' is an upset of $\mathfrak{F}$ and $\pi^{-1}(U') = U$ where $U \subseteq X_\oplus$. If wRv for some $w, v \in U'$, and $w \neq v$ then $U_w \neq \varnothing$ since $\pi^{-1}(w) \in X_w \subseteq U$. By Definition 4.1.31 we get $U_v = X_v$ and since $\pi^{-1}(v) = X_v$ and $w, v \in W$ are arbitrary, by Definition 4.1.31 we get that $U \in \tau_\oplus$.

(ii) π is r-dense: We show that $\pi^{-1}(w)$ is dense in itself for any reflexive $w \in W$. Let $w \in W$ be reflexive. Then by definition $\pi^{-1}(w) = \{w, w'\} = X_w$. Since on a space X_w we have the trivial topology, then the points w and w' cannot be separated by an open set of τ_w. It follows that $\pi^{-1}(w)$ is dense-in-itself, i.e. $\pi^{-1}(w) \subseteq d_{X_\oplus}\pi^{-1}(w)$.

(iii) π is i-discrete: Suppose $w \in W$ is an irreflexive point. Then $\pi^{-1}(w) = X_w = \{w\}$. Clearly $X_w \cap d(X_w) = \varnothing$ since X_w is a singleton. It follows that $\pi^{-1}(w)$ is a discrete subspace of $X_\oplus$ for each irreflexive $w \in W$.

(iv) $\pi^{-1}(g^{-1}(w)) = f^{-1}(\pi^{-1}(w))$:

($\subseteq$) Suppose $x \in \pi^{-1}(g^{-1}(w))$, then there exists $v \in W$ such that $g(v) = w$ and $\pi(x) = v$. By the way we defined $\mathcal{X}_\oplus$, there exists a unique $y \in X_\oplus$ such that $f(x) = y$ and $\pi(y) = w$. It follows that $x \in f^{-1}(\pi^{-1}(w))$.

($\supseteq$) Suppose $x \in f^{-1}(\pi^{-1}(w))$, then there exists $y \in X$ such that $\pi(y) = w$ and $f(x) = y$. By the way we defined $\mathcal{X}_\oplus$, there exists a unique $v \in W$ such that $\pi(x) = v$ and $g(v) = w$. It follows that $x \in \pi^{-1}(g^{-1}(w))$.

We conclude that π is a dynamic d-morphism. Since π is clearly surjective and $\mathfrak{F} \not\models \varphi$, then by Corollary 4.1.30 we have $\mathcal{X}_\oplus \not\models_d \varphi$. Therefore, we provided a dynamic topological system that d-refutes φ. We conclude that the d-logic **wK4C** is topologically sound and complete with respect to the class of all dynamic topological systems. It is therefore the d-logic of all dynamic topological systems. $\square$

Chapter 5

Conclusions and Future Work

We have presented Devaney's interpretation of chaos, which consists of specific conditions that a dynamical system must fulfil in order to be considered chaotic. We provided some examples for such systems and showed that the original definition can turn redundant if we exclude some particular trivialities. We then provided an alternative non-deterministic interpretation to dynamical systems, which is easier to work with if one is interested in formal systems, as functionality is no longer necessary. Further investigation of dynamical systems as non-deterministic entities may be helpful in providing alternative notions for properties that belong exclusively to deterministic dynamical systems, such as chaos. This may open new doors to the discussion of mathematical chaos, and other similar properties from dynamical systems theory, in physics, computer science, philosophy and other research fields.

We have compared homomorphism between dynamical systems, called quasiconjugacies, and homomorphisms between dynamic topological systems, called dynamic p-morphisms. We investigated the logical correspondences and complete axiomatisation of the logic of all topologically transitive non-deterministic dynamic topological systems with a transitive temporal relation. There are several noticeable difficulties when considering dynamical systems and their logics. First of all, in order to express properties such as a dense set of periodic points, we will need to extend our language beyond what has been considered before in relation to dynamical systems. Accordingly, this may lead to issues that would be difficult to settle. This problem extends to the non-deterministic setting as well. For if we seek to express any properties that either distinguish between different paths or express the interconnection between the temporal and topological aspects of the system, e.g. dense set of periodic points, then we might need a stronger language in order to do so.

The chief endeavour of this thesis was the establishment of the first soundness and completeness results for dynamic topological d-logics. This broadens the research landscape for further investigations of more expressible methods for the formalisation of dynamical systems. Expressing certain properties, such as a space being dense-in-itself is impossible using topological c-semantics, yet possible with topological d-semantics. This setting may also allow for simplifications of other open problems and may render them solvable. For instance, we already mentioned in Chapter 4 that d-completeness for **GLC** with the 'henceforth' operator, or simply $\mathbf{GLC}^*$, should be possible without any changes to the original trimodal language and with a finite set of axioms. That provides a first immediate direction for future work. Another very natural path is to prove d-completeness with respect to interesting well-known spaces given they could be described by some set of axioms above **wK4C**. For example, we already showed that **GLC** is d-complete in respect to the class of scattered spaces.

We may also want to pursue the direction of Intuitionistic temporal logics. Boudou, Diéguez, Fernández-Duque and Romero [10] proposed four axiomatic systems for intuitionistic linear temporal logic and showed that each of these systems is sound with respect to a class of structures that are based either on dynamic topological systems or on Kripke frames. Their topological semantics offers an alternative interpretation for the 'henceforth' modality $[*]$, which is a natural intuitionistic adaptation of the traditional one. In a succeeding paper, Boudou, Diéguez and Fernández-Duque [11] provided a complete axiomatisation for the 'henceforth'-free fragment, albeit with its dual, and proved completeness for some well-known spaces. The problem of completeness for the language with the 'henceforth' operator remains open and provides yet another direction for future research. It might be possible to use the intuitionistic setting to prove completeness of other systems with axioms that express interesting properties of dynamical systems. Another noteworthy candidate for an alternative system is provided by *hybrid logic* in which the notions of a fixed-point and a periodic point are easily expressible [7]. Regardless, a complete axiomatisation for the class of all deterministic chaotic systems seems implausible without some major changes, either by adapting and extending the intuitionistic temporal framework or by extending our language and our semantics.

Whether we can fully express chaos in a dynamical system remains an open question. In the meantime, there is much to do to bring forward the logic of dynamic topological systems with the tools we do have and with the properties we can express. Since we can represent dynamical systems as Kripke frames, we can ask how will the properties of dynamical systems will look in Kripke frames and what can we learn from them. For instance, how will topological transitivity look in the setting of Kripke frames? For the sake of example, we will use an interesting class of Kripke frames called *crown frames*. Those frames were introduced by David Gabelaia et al. [20] who proved that the logic of all validities on planar polygons $\mathbf{PL}_2$ is sound complete with respect to the class of finite crown frames and thus has the finite model property. They did it by viewing each crown frame as an Alexandroff space and by showing that any crown frame is an interior image of the polygonal plane. Then, they proved that if a formula is satisfied on the polygonal plane, it is also satisfied on a crown frame. The crown frames are interesting structures as they represent

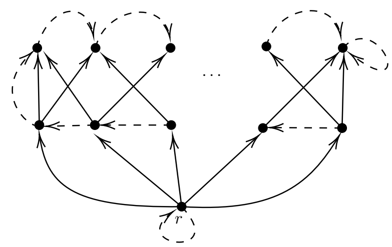

Figure 5.1: Topologically transitive topo-temporal crown frame.

convex polygons, which are finite intersections of half planes, on the Euclidean plane. If we add a function to such structures, we can easily see that topological transitivity could be obtained by a single path through the levels of the crown in addition to a f-reflexive root r. This is illustrated in Figure 5.1.

Being able to represent dynamic topological systems as graphs is also useful for investigations of possible applications. While the study of dynamical systems is varied and its applications could be found in almost any academic discipline in some form or another, in order to efficiently use and compute results concerning such systems we need to consider the computational complexity aspects of the problems they generate. Since we have the finite model property for both c-logics and d-logics of the restricted language $\mathcal{L}_{\square}^{\bigcirc}$, then the dynamic topological models of these logics could be represented as finite graphs. Unfortunately, as in most modal systems, the questions of model checking and satisfiability are highly intractable, as first noted by Ladner [27]. As a result, one may want to try and use approximations or special methods such as parameterised complexity in order to bound by some constants various properties of graphs representing dynamic topological systems. Although we will not be able to answer general satisfiability or model checking problems, we could still ask meaningful questions regarding those spaces. For instance, we might want to calculate whether a function between some subsets of cardinality k is topologically transitive or not. Using parameterised complexity, we might even be able to answer such questions effectively and in tractable time.

Bibliography

[1] Artemov, S., Davoren, J., and Nerode, A. (1997). *Modal Logics and Topological Semantics for Hybrid Systems.* Technical Report MSI 97-05, Cornell University. Available at http://web.cs.gc.cuny.edu/~sartemov/.

[2] Banks, J., Brooks, J., Cairns, G., Davis, G., and Stacey, P. (1992). *On Devaney's definition of chaos.* The American mathematical monthly, 99(4), 332-334.

[3] Baptista, M. S. (1998). *Cryptography with chaos.* Physics letters A, 240(1-2), 50-54.

[4] van Benthem, J., and Bezhanishvili, G. (2007). *Modal logics of space.* In Handbook of spatial logics (pp. 217-298). Springer, Dordrecht.

[5] Bezhanishvili, G., Esakia, L., Gabelaia, D. (2005). *Some results on modal axiomatization and definability for topological spaces.* Studia Logica 81(3), 325–355.

[6] Bezhanishvili, G., Esakia, L., and Gabelaia, D. (2009). *Spectral and T 0-spaces in d-semantics.* In International Tbilisi Symposium on Logic, Language, and Computation (pp. 16-29). Springer, Berlin, Heidelberg.

[7] Blackburn, P. (2000). *Representation, reasoning, and relational structures: a hybrid logic manifesto.* Logic Journal of the IGPL, 8(3), 339-365.

[8] Blackburn, P., De Rijke, M., and Venema, Y. (2002). *Modal logic: graph. Darst (Vol. 53).* Cambridge University Press.

[9] Boolos, G. (1984). *The logic of provability.* The American Mathematical Monthly, 91(8), 470-480.

[10] Boudou, J., Diéguez, M., Fernández-Duque, D., and Romero, F. (2019). *Axiomatic systems and topological semantics for intuitionistic temporal logic.* In European Conference on Logics in Artificial Intelligence (pp. 763-777). Springer, Cham.

[11] Boudou, J., Diéguez, M., and Fernández-Duque, D. (2019). *Complete Intuitionistic Temporal Logics in Topological Dynamics.* arXiv preprint arXiv:1910.00907.

[12] Brouwer, L. E. J. (1910). *On the structure of perfect sets of points.* In KNAW, Proceedings (Vol. 12, pp. 785-794).

[13] Degn, H., Holden, A. V., and Olsen, L. F. (Eds.). (2013). *Chaos in biological systems (Vol. 138).* Springer Science & Business Media.

[14] Devaney, R. (1989). *An Introduction to Chaotic Dynamical Systems.* Addison–Wesley.

[15] Esakia, L. (1981). *Diagonal constructions, Löb's formula and Cantor's scattered spaces.* in: Studies in Logic and Semantics, Metsniereba, Tbilisi, pp. 128–143. In Russian.

[16] Esakia, L. (2001). *Weak transitivity–restitution.* In Study in Logic, volume 8, pp. 244–254. Nauka. In Russian.

[17] Fernández-Duque, D. (2007). *Dynamic Topological Completeness for* $\mathbb{R}^2$. Logic Journal of the IGPL, 15(1), 77-107.

[18] Fernández-Duque, D. (2012). *A sound and complete axiomatization for dynamic topological logic.* Journal of Symbolic Logic, 77(3), 947-969.

[19] Fernández-Duque, D. (2014). *Non-finite axiomatizability of dynamic topological logic.* ACM Transactions on Computational Logic (TOCL), 15(1), 1-18.

[20] Gabelaia, D., Gogoladze, K., Jibladze, M., Kuznetsov, E., and Marx, M. (2018). *Modal logic of planar polygons.* arXiv preprint arXiv:1807.02868.

[21] Huang, W., Khilko, D., Kolyada, S., and Zhang, G. (2016). *Dynamical compactness and sensitivity.* Journal of Differential Equations, 260(9), 6800-6827.

[22] Kato, T. (1964). *Demicontinuity, hemicontinuity and monotonicity.* Bulletin of the American Mathematical Society, 70(4), 548-550.

[23] Kelsey, D. (1988). *The economics of chaos or the chaos of economics.* Oxford economic papers, 40(1), 1-31.

[24] Konev, B., Kontchakov, R., Tishovsky, D., Wolter, F., and Zakharyaschev, M. (2006a). *On Dynamic Topological and Metric Logics.* Studia Logica.

[25] Kremer, P., and Mints, G. (2007). *Dynamic topological logic.* In Handbook of spatial logics (pp. 565-606). Springer, Dordrecht.

[26] Kripke, S. (1963). *Semantical Analysis of Modal Logic I, Normal Propositional Calculi.* Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, 9:67–96.

[27] Ladner, R. E. (1977). *The computational complexity of provability in systems of modal propositional logic.* SIAM journal on computing, 6(3), 467-480.

[28] Layek, G. C. (2015). *An introduction to dynamical systems and chaos*. India: Springer.

[29] Lichtenberg, A. J., and Lieberman, M. A. (2013). *Regular and chaotic dynamics (Vol. 38)*. Springer Science & Business Media.

[30] Lorenz, E. (2000). *The butterfly effect*. World Scientific Series on Nonlinear Science Series A, 39, 91-94.

[31] McKinsey, J. C. C. and Tarski, A. (1944). *The Algebra of Topology*. Annals of Mathematics, 45:141–191.

[32] Mints, G. and Zhang, T. (2005). *Propositional Logic of Continuous Transformation in Cantor Space*. Archive for Mathematical Logic, 44(6):783-799.

[33] Poincaré, H. (1905). *Leçons de mécanique céleste: professées à la Sorbonne (Vol. 1)*. Gauthier-Villars.

[34] Poincaré, H. (1992). *New methods of celestial mechanics (Vol. 13)*. Springer Science & Business Media.

[35] Pnueli, A. (1977). *The temporal logic of programs*. In 18th Annual Symposium on Foundations of Computer Science (sfcs 1977) (pp. 46-57). IEEE.

[36] Popper, K. R. (1992). *The open universe: An argument for indeterminism (Vol. 2)*. Psychology Press.

[37] Shehtman, V. (1999). *≪Everywhere≫ and ≪here≫*. Journal of Applied Non-Classical Logics, 9(2-3), 369-379.

[38] Simmons, H. (1975). *Topological aspects of suitable theories*. Proc. Edinb. Math. Soc. 2 (19) 383–391.

[39] Smoryǹski, C. (1984). *Modal logic and self-reference*. In Handbook of philosophical logic (pp. 441-495). Springer, Dordrecht.