Studies in the Extension of Standard Modal Logic with an Infinite Modality.

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

We consider the modal logic ML^∞, the extension of standard modal logic where the modality $\Diamond^\infty$ is added to the signature. Interpreted using Kripke semantics, the $\Diamond^\infty$ modality captures the distinction between finite and infinite. We first provide a collection of results on the model theoretic aspects of this logic. Introducing an alternative definition of bisimulation, we establish a collection of invariance results as well as a characterization of ML^∞ in terms of this new notion of bisimulation. Furthermore we adapt the Hennessy-Milner property to the ML^∞ framework and characterize a collection of frames that enjoy this property.

In a second line of research we establish some positive results on the finite axiomatization of ML^∞. We introduce the ML^∞ logics K^∞ and $\mathrm{S5}^\infty$ and we show that they are, respectively, sound and weakly complete with respect to the class of Kripke frames and the class of equivalent Kripke frames.

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Finally, I would like to dedicate this thesis to my grandfather Rafael. Your passion on mathematics has been a source of inspiration all these years.

Contents

Introduction.

This master thesis introduces the logic ML^∞; the extension of standard modal logic (ML) where we boost the ML language by introducing the modality $\Diamond^\infty$, embodying the existence of infinitely many successors enjoying a certain property. The modal logic ML^∞ is interpreted on Kripke models with a unique accessibility relation, where a world w satisfies the formula $\Diamond^\infty\varphi$ if and only if it has infinitely many successors satisfying φ. Inspired by the extensive research that has occurred around ML, this master thesis aims to provide a first analysis of some model theoretic and axiomatic properties of the modal logic ML^∞.

Our motivation. Generalized quantifiers have been an active field of research in the logic community since they were first defined in the late 50's by Mostowski [Mos57]. The abstract model theoretic properties of cardinal quantifiers have been a prominent line of research in the past decades. In the 50's and 60's FO^∞, the extension of first-order logic with the cardinal quantifier $\exists^\infty$ embodying the finite/infinite distinction, was taken as an object of research in this field. However, it was soon noticed that FO^∞ lacks of important model theoretic properties such as compactness (see Section 1.1 of Chapter IV in [BF17]), Craig-Interpolation property (see [Mos68]), axiomatization and the Löwenheim-Skolem property (see Proposition 1.3.2 of Chapter IV in [BF17]). Therefore the logic community opted to substitute this logic for other cardinal logics that enjoy better model theoretic properties (see [Fuh65; Vau64; Kei70; BF17]). Recent research, however by Carreiro et al. [Car+18] has led to positive improvements on the model theoretic and complexity aspects of the monadic fragment of FO^∞. Thus, in this thesis we extend the object of research considered by Carreiro et al. to the monadic fragment of FO^∞ with an additional binary relation and provide positive results on the model theoretic and axiomatization properties of this fragment.

The interplay between generalized quantifiers and modal logic has been, to some extent, a minor area of research in both modal logic and generalized quantifier theory. However, the connections that can be drawn between these two theories has resulted in positive results. In particular we should highlight the celebrated results obtained by van der Hoek and de Rijke [VD93] on the expressive power of graded modal logic in connection to the generalized quantifiers expressible in first-order logic. In other respects, model theory has played a major role in modal logic, not only to answer model theoretic questions in the modal framework, but model theoretic techniques have been used to study the expressive properties of modal logics [HM85], to characterize modal logics [TBV07; De 95], or to study the interplay between modal logics and first and second order logic [Ben76; Ros97].

Related work. Whilst the literature devoted to the study of ML^{∞} is limited, the next paragraph provides an overview of the research surrounding this field:

It was in the early 80's when Emerson and Halper [EH86] first proposed an interpretation of the $\Diamond^{\infty}$ modality to the temporal logic framework. Influenced by a previous paper from Clarke and Emerson [EC80] where the infinite quantifier $\exists^{\infty}$ is introduced to the field of automata theory, Emerson and Halper introduced the infinite modality F^{∞}, capturing the existence of events that occur infinitely often. This modality is established in the language of the temporal logic CTL^{*}, an extension of propositional logic that is equipped with the modalities F ("sometimes"), X ("next time"), U ("until") as well as the previously mentioned modality F^{∞} ("infinitely often"). CTL^{*}, furthermore, was conceived primarily to unify the branch and linear interpretation of temporal logic discussed in [Lam80] into a unique language, and has been proven to enjoy both positive complexity and expressive properties.

In 2007, van Benthem et al. [TBV07] introduced the logic $\mathrm{ML}^{\bullet}$. The language of this logic is obtained by introducing a weaker version of the $\Diamond^{\infty}$ modality to the ML language, namely the $\bullet$ modality. This new modality is interpreted on Kripke models where a world satisfies the formula $\bullet\varphi$ if and only if it has infinitely many reflexive successors satisfying φ. In addition, van Benthem et al. show that $\mathrm{ML}^{\bullet}$ is a non elementary extension of ML that is not contained in first-order logic, but is still well behaved in model theoretic terms. Meaning that $\mathrm{ML}^{\bullet}$ satisfies the Löwenheim-Skolem Theorem (see Proposition 3.7 in [TBV07]), the Compactness Theorem (see Page 14 in [TBV07]), the Craig-Interpolation property (see Proposition 3.11 in [TBV07]) and it is finitely axiomatizable (see Proposition 3.9 in [TBV07]).

Our contribution. The contributions of this master thesis are of different flavours. In a first line of research we introduce an alternative notion of bisimulation, namely ML^{∞}-bisimulation, that is enhanced in such a way that is able to capture the infinite behaviour of $\Diamond^{\infty}$. Under this new definition we show that a significant amount of the preservation results are recovered. Moreover, we show that under this new definition of bisimulation the class of $\aleph_0^{\infty}$-saturated models (an adaptation of the concept of saturation) enjoy of the Hennessy-Milner property. Our contribution in the expressive power of ML^{∞} ends with an adaptation of the celebrated van Benthem characterization theorem on the ML^{∞}

framework.

On a second line of research we introduce the logics K^∞ and $S5^\infty$, the supplements of the well-known modal logics K and S5. In Chapter 6, we provide a finite axiomatization of these logics and show that they are weakly complete with respect to the class of Kripke models (for the K^∞ case) and with respect to the class of equivalence frames (for the $S5^\infty$ case).

In the following paragraphs the reader can find a more detailed breakdown of the analysis this thesis provides:

Chapter 2 deals with all the preliminary syntactic and semantic concepts. We introduce the language of ML^∞, an extension of standard modal logic with an additional modality $\Diamond^\infty$ embodying the finite/infinite distinction. We provide the standard Kripke-style semantics for this language, restricting them to a unique accessibility relation. A contribution of this chapter is the introduction of an alternative semantic for ML^∞ based on Kripke models equipped with two accessibility relations, where each relation captures the behaviour of each modality. We conclude this chapter by showing a method to convert the bimodal Kripke models to the standard Kripke models while still preserving the truth value of the ML^∞ formulas (see Lemma 2.29).

In Chapter 3 we tackle the bisimulation variance failure by providing a stronger definition of bisimulation that is able to capture the infinite behaviour of $\Diamond^\infty$. This is achieved by adapting the bisimulation game, allowing `Spoiler` to launch two types of challenges towards `Duplicator`. Each of the movements is able to capture, in game-theoretic terms, the semantic behaviour of the modalities $\Diamond$ and $\Diamond^\infty$. Finally, under this new bisimulation we are able to recover all the desired preservation results that hold in ML.

In Chapter 4 our study focuses on the research of the Hennessy-Milner property of ML^∞, i.e. the classes of frames for which the concept of ML^∞-bisimulation and ML^∞-equivalence coincide. Motivated by this topic and the research on the bisimulation invariance of ML^∞ that is discussed in Chapter 5, a major section of this chapter is devoted to introducing the reader to the model theoretic concepts of FO^∞; the extension of first-order logic with an additional cardinal quantifier $\exists^\infty$ embodying the finite/infinite distinction. In addition, we introduce the reader to the concepts of ω-types and κ^∞-saturation. Two concepts that are conceived as a natural extension of the model theoretic concepts of type and saturated models, but with the additional power that allows us to prove the Hennessy-Milner property for the class of $\aleph_0^\infty$-saturated models.

Building on the preservation results obtained in 3 and following the tradition of van Benthem's work [Ben76] on the relationship between ML and first-order logic, in Chapter 5 we will show a bisimulation invariance theorem for ML^∞. In particular, we show that the modal logic ML^∞ represents the ML^∞-bisimulation invariant fragment of FO^{∞}. The failure of the Compactness Theorem was an inevitable obstacle to follow van Benthem's original strategy to show the desired result. However, Rosen's technique [Ros97] on the bisimulation invariance result over finite structures is powerful enough that it could be applied to our framework and hence obtain the desired result of ML^{∞}-bisimulation invariance.

Motivated by the lack of axiomatization for FO^{∞} we conclude this master thesis by throwing some light onto this matter. We start Chapter 5 by introducing the reader to the logics K^{∞} and $S5^{\infty}$, two ML^{∞} logics that supplement the normal modal logics K and S5. A direct consequence of the failure of the Compactness Theorem is the failure of the usual strategy using Canonical Models to prove the completeness result. To overcome this complication we adapt the filtrated canonical models as in [FL79] combined with the model theoretic machinery described in Chapter 3 and thus showing that $S5^{\infty}$ is sound and weakly complete with respect to the class of Kripke frames equipped with an equivalence accessibility relation.

On the contrary, some additional problems arose when we tried to show completeness for the logic K^{∞}. This problem was resolved by introducing the unravelling technique of bi-modal Kripke to the model theoretic machinery developed to prove completeness for the logic $S5^{\infty}$. We conclude this chapter by showing that that K^{∞} is complete with respect to the class of Kripke frames.

Preliminaries.

This section is intended to introduce the reader to the necessary background information required to follow this thesis. First, we present the language of ML^∞, an extension of the language of standard modal logic where we add an additional modality to the modal signature. Second, we introduce two variants of the Kripke semantics on which ML^∞ is interpreted. On one hand we introduce the standard semantic utilizing Kripke models with a unique binary relation and we observe that under this semantic the compactness property is not satisfied. On the other hand we introduce an alternative interpretation of the ML^∞ language. This new interpretation is an adaptation, on the ML^∞ framework, of the alternative semantics proposed by van Benthem et al. [TBV07]. In particular, the alternative Kripke models that we propose are equipped with two accessibility relations where each relation captures the semantic behaviour of each modality. We conclude this chapter by providing a technique to transform Kripke structures with two accessibility relations to Kripke models with a unique accessibility relation while preserving the truth value of the ML^∞ formulas.

For the sake of simplicity we fix an arbitrary countably infinite set of propositional variables that is denoted by Φ and we denote by $\mathfrak{P}(X)$ the power set of X.

2.1 Syntax of ML^∞.

Definition 2.1. Let Φ be a collection of propositional variables. The collection of $\mathrm{ML}^\infty(\Phi)$-formulas (over Φ) is defined by the following grammar:

$$\varphi ::= \bot \mid p \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Diamond\varphi \mid \Diamond^\infty\varphi,$$

where p is a propositional variable in Φ. Apart from the well-known expressions $\varphi \to \psi, \varphi \vee \psi$ or $\Box\varphi$ we let $\Box^\infty\varphi$ to be $\neg\Diamond^\infty\neg\varphi$. Since we have fixed a set Φ of propositional variables, we will write ML^∞ instead of $\mathrm{ML}^\infty(\Phi)$. Any subset of ML^∞ is said to be an ML^∞-theory.

Definition 2.2. For any ML$^\infty$ formula φ we define its modal depth, noted by $\mathtt{md}(\varphi)$ as follows:

$$\begin{aligned}
&\mathtt{md}(p) := 0 \text{ for every } p \in \Phi \cup \{\bot\},\\
&\mathtt{md}(\neg\varphi) := \mathtt{md}(\varphi),\\
&\mathtt{md}(\varphi \wedge \psi) := \max\{\mathtt{md}(\varphi), \mathtt{md}(\psi)\},\\
&\mathtt{md}(\bigcirc\varphi) := \mathtt{md}(\varphi) + 1 \text{ where } \bigcirc \in \{\Diamond, \Diamond^\infty\}.
\end{aligned}$$

2.2 Kripke semantics.

As we mentioned at the beginning of this chapter, we will interpret the ML$^\infty$ language using Kripke models. In this section, thus, we introduce the Kripke semantics for the modal logic ML$^\infty$. In addition we present the well-known concepts of generated submodel, disjoint union and bounded morphism. We conclude this section by introducing the unravelling technique. A method to construct *new* Kripke models from *old* ones presenting particular properties that will be used later in connection with the blooming technique (see Lemma 2.29).

Definition 2.3. A Kripke frame is a tuple $\mathscr{F} = (W, R)$, where W, the universe of worlds, is a non-empty set and $R \subseteq W \times W$ is the accessibility relation. A Kripke model over a set of propositional variables Φ is a triple $\mathscr{M} = (W, R, V)$ where (W, R) is a Kripke frame equipped with a valuation function $V : \Phi \to \mathfrak{P}(W)$.

A pointed Kripke model denoted by $(\mathscr{M}, w)$ is a tuple where w is a world of the universe of $\mathscr{M}$. We denote by $R[w]$ the set of successors of w.

Definition 2.4. Let $\mathscr{M}$ be a Kripke frame. Given a set $W' \subseteq W$, the submodel of $\mathscr{M}$ induced by W', denoted by $\mathscr{M}|_{W'}$ is the triple (W', R', V') where $R' = R \cap (W' \times W')$ and $V'(p) = V(p) \cap W'$ for every $p \in \Phi$. A submodel $\mathscr{M}'$ of $\mathscr{M}$ is said to be a generated submodel if it is closed under the following rule:

$$\text{If } w \in W' \text{ and } wRv, \text{ then } v \in W'.$$

Moreover for any set $X \subseteq W$ we let $\mathscr{M}_X$ to be the smallest generated submodel of $\mathscr{M}$ containing X. If $\mathscr{M}$ is a Kripke model generated by a singleton $\{w\}$ we say that $\mathscr{M}$ is rooted at w.

Definition 2.5. Let $\mathscr{M} = (W, R, V)$ and $\mathscr{M}' = (W', R', V')$ be two Kripke models. A map $\rho : W \to W'$ is a bounded morphism if the following properties are satisfied:

- w and $\rho(w)$ satisfy the same propositional variables **(atomic)**
- If wRu, then $\rho(w)R'\rho(u)$, **(forth)**
- If $\rho(w)R'u'$ then there is some u such that $\rho(u) = u'$ and wRu. **(back)**

If, in addition ρ satisfies the following condition:

- If $X \subseteq R[x]$ is an infinite set, so is $\{\rho(x) \mid x \in X\}$, **(strongforth)**

we say that ρ is a strong bounded morphism. If there exists a surjective strong bounded morphism from $\mathscr{M}$ to $\mathscr{M}'$ we denote it by $\mathscr{M} \overset{s}{\twoheadrightarrow} \mathscr{M}'$.

Definition 2.6. Let $\mathscr{M}_i = (W_i, R_i, V_i)$ be a collection of Kripke models indexed by a set I. We define the disjoint union $\biguplus_{i \in I} \mathscr{M}_i = (W, R, V)$ to be the Kripke model where:

- $W := \biguplus\limits_{i \in I} W_i$ is the disjoint union of universes,
- $R := \biguplus\limits_{i \in I} R_i$ is the disjoint union of the accessibility relations,
- $V(p) := \biguplus\limits_{i \in I} V_i(p)$ for every $p \in \Phi$.

Definition 2.7. Let $(\mathscr{M}, w)$ be a pointed Kripke model. For, every natural number n we define the n-neighbourhood of w, denoted by $N_n(w)$, recursively as follows:

- $N_0(w) := \{w\}$,
- $N_{n+1} := \{v \in W \mid \text{there is a } u \in N_n(w)(uRv \text{ or } vRu \text{ or } v = u)\}$.

Definition 2.8. A Kripke model $\mathscr{M} = (W, R, V)$ is said to be a tree model if there exists a unique world $w \in W$ satisfying the following properties:

- $W = R^*[w]$,
- for every $t \in W \backslash \{w\}$ there exists a unique $t' \in W$ such that $t'Rt$,
- the accessibility relation R is acyclic, meaning that for every $t \in W(\neg tRt)$.

where R^* is the transitive and reflexive closure of R. If this is the case we say that $\mathscr{M}$ is rooted at w. Furthermore, $\mathscr{M}$ is an n-pseudotree for some natural number n if the submodel $\mathscr{M}|_{N_n(w)}$ is a tree rooted at w.

Definition 2.9. For any pointed Kripke model $(\mathscr{M}, w)$ we define the satisfaction relation $\Vdash$ recursively as follows:

$$\begin{aligned}
\mathscr{M}, w \Vdash p &\Longleftrightarrow_{\text{Def}} w \in V(p).\\
\mathscr{M}, w \Vdash \bot &\Longleftrightarrow_{\text{Def}} \text{ Never.}\\
\mathscr{M}, w \Vdash \neg\varphi &\Longleftrightarrow_{\text{Def}} \mathscr{M}, w \nVdash \varphi.\\
\mathscr{M}, w \Vdash \varphi \wedge \psi &\Longleftrightarrow_{\text{Def}} \mathscr{M}, w \Vdash \varphi \text{ and } \mathscr{M}, w \Vdash \psi.\\
\mathscr{M}, w \Vdash \Diamond\varphi &\Longleftrightarrow_{\text{Def}} \text{ there is a } v \in W(wRv \text{ and } \mathscr{M}, v \Vdash \varphi).\\
\mathscr{M}, w \Vdash \Diamond^{\infty}\varphi &\Longleftrightarrow_{\text{Def}} \text{ there are infinitely many } v \in W(wRv \text{ and } \mathscr{M}, v \Vdash \varphi).
\end{aligned}$$

If two pointed Kripke models $(\mathscr{M}, w)$ and $(\mathscr{M}', w')$ satisfy the same ML^{∞}-formulas we say that $(\mathscr{M}, w)$ and $(\mathscr{M}', w')$ are ML^{∞}-equivalent and we will denote it by $\mathscr{M}, w \equiv^{\infty} \mathscr{M}', w'$. Moreover we say that $(\mathscr{M}, w)$ and $(\mathscr{M}', w')$ are ML^{∞}_n-equivalent and denote it by $\mathscr{M}, w \equiv^{\infty}_n \mathscr{M}', w'$ if both pointed Kripke models satisfy the same ML^{∞}-formulas up to modal depth n.

***Remark* 2.10.** For any formula φ and any Kripke frame $\mathscr{F}$ we say that φ is valid in $\mathscr{F}$ and denote it by $\mathscr{F} \Vdash \varphi$ if for every pointed Kripke model $(\mathscr{M}, w)$ where $\mathscr{F}$ is the underlying frame of $\mathscr{M}$ the following holds: $\mathscr{M}, w \Vdash \varphi$. Moreover, if $\mathbb{F}$ is a collection of Kripke frames and $\Gamma, \{\varphi\}$ are two sets of ML^{∞} formulas, we denote by $\Gamma \Vdash_{\mathbb{F}} \varphi$ the following property:

If for every $\mathscr{F} \in \mathbb{F}$ and every $\psi \in \Gamma$: $\mathscr{F} \Vdash \psi$, then $\mathscr{F} \Vdash \varphi$ for every $\mathscr{F} \in \mathbb{F}$.

***Observation* 2.11.** As we mentioned in the beginning of this chapter, ML^{∞} is not a compact logic. Recall that a logic Λ is compact given that for every set $\Gamma \subseteq \Lambda$, if every finite subset A of Γ has a model that satisfies every formula in A, then there exists a model that satisfies every formula in Γ.

Now consider the following ML^{∞}-theory $T := \{\Diamond\psi_n \mid n \in \mathbb{N}\} \cup \{\neg\Diamond^{\infty}\top\}$ where $\psi_n := \bigwedge_{i<n} \neg p_i \wedge p_n$ and $p_0, ..., p_n$ are distinct propositional variables in Φ. It is not difficult to see that every finite subset of T has a finite Kripke model that satisfies every formula in it. However, every pointed Kripke model $(\mathscr{M}, w)$ that satisfies all the formulas in $\{\Diamond\psi_n \mid n \in \mathbb{N}\}$ must have infinitely many successors. But if this is the case, $(\mathscr{M}, w)$ does not satisfy the $\neg\Diamond^{\infty}\top$ formula. Therefore ML^{∞} is not a compact logic.

2.2.1 Unravelling a Kripke model

In this subsection we present the unravelling technique. This method has been widely studied in standard modal logic and allows us to transform *old* Kripke models to pseudotrees while still preserving the semantic behaviour of the *old* Kripke model. To achieve this goal we first introduce the reader to the concepts of path and family of a path. Then we give a formal definition of the unravelling of a Kripke model.

Definition 2.12. Let $\mathfrak{F} = (W, R)$ be a Kripke frame and let n be a natural number. An R-path of length n over W is a sequence $(w_0, w_1, ..., w_n)$ such that $w_i R w_{i+1}$ for every $i < n$. The collection of R-paths of length n over W starting at w will be denoted by $\mathrm{Path}_n(w)$.

***Remark* 2.13.** Let $\bar{a}$ be any R-path over W of length n. We denote by $\mathtt{last}(\bar{a})$ the last element $\bar{a}$. Moreover, for every element $w \in W$ we denote by $\bar{a} * w$ to be the sequence extending $\bar{a}$ where we add w to the extreme of $\bar{a}$. Finally, if $\bar{b}$ is an R-path of length k for some $k > n$ we say that $\bar{a}$ is the n-subpath of $\bar{b}$ if there exists some $w_{n+1}, ..., w_k \in W$ such that $\bar{b} = \bar{a} * w_{n+1} * ... * w_k$.

Definition 2.14. For any Kripke frame $\mathscr{F} = (W, R)$ rooted at $w \in W$ and any $n \in \mathbb{N}$ we define the set $S_n[w]$ to be:

$$S_n[w] := \bigcup_{i \leq n} \mathrm{Path}_i(w).$$

Definition 2.15. For any Kripke frame $\mathscr{F} = (W, R)$, any natural number $n \in \mathbb{N}$ and any R-path $\bar{a}$ of length n we define the family of $\bar{a}$, denoted by $\mathrm{Fam}(\bar{a})$ to be:

$$\mathrm{Fam}(\bar{a}) := \{(w, \bar{a}) \mid w \in W_{\mathtt{last}(\bar{a})}\}$$

where $W_{\mathtt{last}(\bar{a})}$ is the universe of the generated submodel $\mathscr{M}_{\{\mathtt{last}(\bar{a})\}}$. Moreover, we let $\pi_0 : \mathrm{Fam}(\bar{a}) \to W$ be the projection map such that for any $(w, \bar{a}) \in \mathrm{Fam}(\bar{a})$:

$$\pi_0((w, \bar{a})) := w.$$

The n-unravelling of a Kripke model $\mathscr{M}$ rooted at w is done in two steps. In the first step we construct the tree section of the *new* model by taking all the R-paths starting from w that have at most length n. We then equip this set with a binary relation in the following way: An R-path $\bar{a}$ is linked to an R-path $\bar{b}$ if $\bar{b} = \bar{a} * \mathtt{last}(b)$, meaning that $\bar{b}$ is the extension of $\bar{a}$ by adding an element at the end of the sequence. The reader might have noticed that under this relation, the tuple formed by the set of R-paths of length at most n and the binary relation described forms a tree:

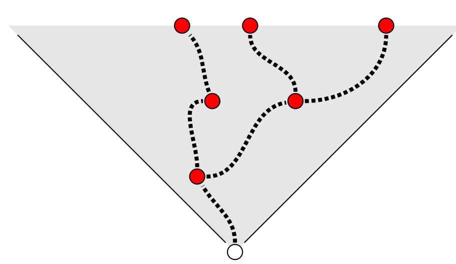


Figure 2.1: Figure representing the tree we described in the previous paragraph.

On the second step of the unravelling technique we now complete the tree. For every leaf $\bar{a}$ in the tree, we take a copy of its family and glue it to the model. We do so by preserving the relation of the *old* model in this newly glued copy of the family of $\bar{a}$, meaning that two worlds in the family of a leaf are linked by the *new* accessibility relation if and only if these worlds are linked by the *old* accessibility relation. Moreover, we link a leaf $\bar{a}$ to any element of its family if $\texttt{last}(\bar{a})$ is linked to such world in the *old* model:

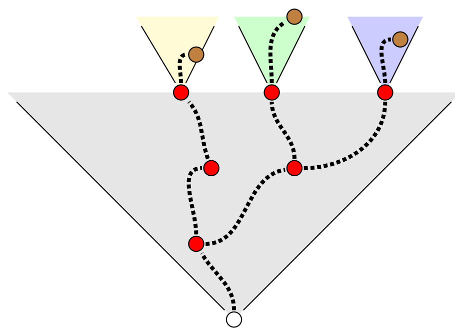


Figure 2.2: Figure representing the n-unravelling of a Kripke model

Definition 2.16. Let $\mathscr{M} = (W, R, V)$ be a Kripke frame rooted at $w \in W$. For any $n \in \omega$, we let $\mathfrak{W}_n[w]$ to be:

$$\mathfrak{W}_n[w] := S_n[w] \cup \bigcup_{\bar{a} \in \mathrm{Path}_n(w)} \mathrm{Fam}(\bar{a})$$

Moreover, we let $\chi_n : \mathfrak{W}_n[w] \to W$ to be the map such that for every $\alpha \in \mathfrak{W}_n$:

$$\chi_n(\alpha) := \begin{cases} \texttt{last}(\bar{a}) & \text{If } \alpha = \bar{a} \text{ for some } \bar{a} \in S_n[w], \\ \pi_0(\alpha) & \text{If } \alpha \in \mathrm{Fam}(\bar{b}) \text{ for some } \bar{b} \in \mathrm{Path}_n(w). \end{cases}$$

Finally, we equip the set $\mathfrak{W}_n[w]$ with an accessibility relation $\mathfrak{R}_n[w]$ and a valuation function $\mathfrak{V}_n[w]$ to define the Kripke model $\mathfrak{M}_n[w]$:

- For every $\alpha, \beta \in \mathfrak{W}_n[w]$, $\alpha \mathfrak{R}_n[w] \beta$ if one of the following properties is satisfied:

  i $\alpha = \bar{a} \in \mathrm{Path}_k(w)$, $\beta = \bar{b} \in \mathrm{Path}_{k+1}(w)$ for some $k < n$ and $\bar{a}$ is the k-subsequence of $\bar{b}$.

  ii $\alpha = \bar{a} \in \mathrm{Path}_n(w)$, $\beta \in \mathrm{Fam}(\bar{a})$ and $\chi_n(\bar{a}) R \chi_n(\beta)$, i.e. $\texttt{last}(\bar{a}) R \pi_0(\beta)$.

  iii $\alpha, \beta \in \mathrm{Fam}(\bar{c})$ for some $\bar{c} \in \mathrm{Path}_n(w)$ and $\chi_n(\bar{a}) R \chi_n(\beta)$, i.e. $\pi_0(\alpha) R \pi_0(\beta)$.

- $\mathfrak{V}_n[w](p) := \{\alpha \in \mathfrak{W}_n[w] \mid \chi_n(\alpha) \in V(p)\}$ for every $p \in \Phi$.

Example 2.17. Consider the following example of the unravelling of a Kripke model. The figure on the left represents a Kripke model $\mathscr{M}$ rooted at w (the black node). The figure on the middle represents the 1-unravelling of $\mathscr{M}$ around w and the third figure is the 2-unravelling of $\mathscr{M}$ around w. Moreover the different colors of the nodes represent the maps χ_1 and χ_2 i.e. the χ_1-image of any green node in the second figure is the green node in the first figure:

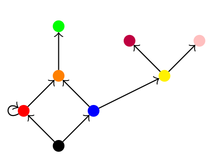


Figure 2.3: The Kripke model $\mathscr{M}$ rooted at w.

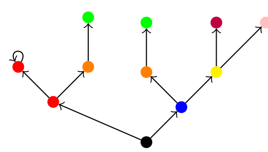


Figure 2.4: The Kripke model $\mathfrak{M}_1[w]$.

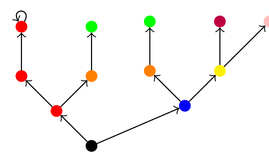


Figure 2.5: The Kripke model $\mathfrak{M}_2[w]$.

2.3 ML$^\infty$-Kripke semantics.

In this section we introduce an alternative semantic for the ML$^\infty$ language based on bimodal Kripke models that we will denote by ML$^\infty$-Kripke models. In contrast to the previous semantics, each accessibility relation of the ML$^\infty$-Kripke model describes the behaviour of one of the modalities. As in the previous section, we introduce the concept of bounded morphism and generated submodel to this framework. We conclude this section by introducing an adaptation of the unravelling technique.

Definition 2.18. A ML$^\infty$-Kripke frame is a triple $\mathcal{F} = (W, R, R^\infty)$ where the tuple (W, R) is a Kripke frame and $R^\infty \subseteq R$ is the infinite accessibility relation. A ML$^\infty$-Kripke model over a set of propositional variables Φ is a quadruple $\mathcal{M} = (W, R, R^\infty, V)$, where the triple (W, R, V) is a Kripke model over Φ and the triple (W, R, R^∞) is a ML$^\infty$-Kripke frame.

***Remark* 2.19.** For any ML$^\infty$-Kripke model $\mathcal{M} = (W, R, R^\infty, V)$ we let (W, R, V) be its underlying Kripke model and we will denote it by $\mathscr{M}$.

Definition 2.20. Let $\mathcal{M} = (W_0, R_0, R_0^\infty, V_0)$ and $\mathcal{M}' = (W_1, R_1, R_1^\infty, V_1)$ be two ML$^\infty$-Kripke models. $\mathcal{M}'$ is a ML$^\infty$-submodel of $\mathcal{M}$ if $\mathscr{M}_1$ is a submodel of $\mathscr{M}_0$ and $R_1^\infty = R_0^\infty \cap (W_1 \times W_1)$. Similarly $\mathcal{M}'$ is the ML$^\infty$-submodel of $\mathcal{M}$ generated by X if $\mathscr{M}_1$ is the submodel of $\mathscr{M}_0$ generated by X and $\mathcal{M}'$ is a ML$^\infty$-submodel of $\mathcal{M}$. As in the Kripke semantics, for every set $X \subseteq W_0$ we denote by $\mathcal{M}|_X$ the ML$^\infty$-submodel of $\mathcal{M}$ induced by X and we denote by $\mathcal{M}_X$ the smallest ML$^\infty$-submodel of $\mathcal{M}$ generated by X.

Furthermore, a map $\rho : W_0 \to W_1$ is said to be a ML$^\infty$-bounded morphism if ρ is a bounded morphism with respect to both accessibility relations, meaning the following properties are satisfied:

- w and $\rho(w)$ satisfy the same propositional variables, **(atom)**
- If wRu, then $\rho(w)R'\rho(u)$, **(R-forth)**
- If $wR^{\infty}u$, then $\rho(w){R'}^{\infty}\rho(u)$, **(R$^{\infty}$-forth)**
- If $\rho(w)R'u'$, then there is a u such that $\rho(u) = u'$ and wRu, **(R-back)**
- If $\rho(w){R'}^{\infty}u'$, then there is a u such that $\rho(u) = u'$ and $wR^{\infty}u$. **(R$^{\infty}$-back.)**

Moreover if there exists a surjective bounded morphism from $\mathcal{M}$ to $\mathcal{M}'$ we denote it by $\mathcal{M} \twoheadrightarrow \mathcal{M}'$.

Definition 2.21. For any pointed ML$^{\infty}$-Kripke model $(\mathcal{M}, w)$ we define the infinity satisfaction relation $\Vdash^{\infty}$ recursively as follows:

$$\begin{aligned}
\mathcal{M}, w \Vdash^{\infty} p &\Longleftrightarrow_{\text{Def}} w \in V(p).\\
\mathcal{M}, w \Vdash^{\infty} \bot &\Longleftrightarrow_{\text{Def}} \text{ Never.}\\
\mathcal{M}, w \Vdash^{\infty} \neg\varphi &\Longleftrightarrow_{\text{Def}} \mathcal{M}, w \nVdash \varphi.\\
\mathcal{M}, w \Vdash^{\infty} \varphi \wedge \psi &\Longleftrightarrow_{\text{Def}} \mathcal{M}, w \Vdash^{\infty} \varphi \text{ and } \mathcal{M}, w \Vdash^{\infty} \psi.\\
\mathcal{M}, w \Vdash^{\infty} \Diamond\varphi &\Longleftrightarrow_{\text{Def}} \text{ there is a } v \in W(wRv \text{ and } \mathcal{M}, v \Vdash^{\infty} \varphi).\\
\mathcal{M}, w \Vdash^{\infty} \Diamond^{\infty}\varphi &\Longleftrightarrow_{\text{Def}} \text{ there is a } v \in W(wR^{\infty}v \text{ and } \mathcal{M}, v \Vdash^{\infty} \varphi).
\end{aligned}$$

Moreover, two pointed ML$^{\infty}$-Kripke models $(\mathcal{M}, w)$ and $(\mathcal{M}', w')$ are ML$^{\infty}$-equivalent, denoted by $\mathcal{M}, w \equiv^{\infty} \mathcal{M}', w'$ if they satisfy the same ML$^{\infty}$ formulas. For every natural number n, we say that $(\mathcal{M}, w)$ and $(\mathcal{M}', w')$ are ML$_n^{\infty}$-equivalent, denoted by $\mathcal{M}, w \equiv_n^{\infty} \mathcal{M}', w'$, if $(\mathcal{M}, w)$ and $(\mathcal{M}', w')$ satisfy the same ML$^{\infty}$ formulas up to modal depth n.

2.3.1 Unravelling a ML$^{\infty}$-Kripke model

Definition 2.22. For any ML$^{\infty}$-Kripke model $\mathcal{M} = (W, R, R^{\infty}, V)$ rooted at $w \in W$ and any natural number n, we define its n-unravelling around w, denoted by $\mathfrak{M}_n[w]$ to be the tuple $(\mathfrak{W}_n[w], \mathfrak{R}_n[w], \mathfrak{R}_n^{\infty}[w], \mathfrak{V}_n[w])$, where:

- $(\mathfrak{W}_n[w], \mathfrak{R}_n[w], \mathfrak{V}_n[w])$ is the n-unravelling around w of $\mathscr{M}$.
- $\mathfrak{R}_n^{\infty}[w] := \{(\alpha, \beta) \in \mathfrak{W}_n[w] \times \mathfrak{W}_n[w] \mid \alpha\mathfrak{R}_n[w]\beta \text{ and } \chi_n(\alpha)R^{\infty}\chi_n(\beta)\}$

Proposition 2.23. Let $\mathfrak{M}_n[w]$ be the n-unravelling of a ML$^{\infty}$-Kripke model $\mathcal{M}$ rooted at $w \in W$. For every $\alpha, \beta \in S_n[w]$ and every $\gamma \in \mathfrak{W}_n[w]$ the following holds:

$$\text{If } \alpha\mathfrak{R}_n[w]\gamma \text{ and } \beta\mathfrak{R}_n^{\infty}[w]\gamma, \text{ then } \alpha = \beta.$$

Proof. Follows directly from Definition 2.22 and Definition 2.16 by making two distinctions. One where γ lies in $S_n[w]$ and another case where γ lies in $\text{Fam}(\bar{a})$ for some R-path $\bar{a}$ of length n. □

Proposition 2.24. Let $\mathcal{M} = (W, R, R^\infty, V)$ be a finite ML^∞-Kripke model rooted at $w \in W$. For every natural number $n \in \mathbb{N}$ its n-unravelling, $\mathfrak{M}_n[w]$, is also a finite ML^∞-Kripke model.

Proof. Follows from a combinatorial argument where the key observation is that at most $\sum_{i \leq n} |W|^n$ many elements were added to the pseudotree segment of the n-unravelling and at most $|W|$ many elements were added to each of the families. Therefore the set $\mathfrak{W}_n[w]$ has at most $(\sum_{i \leq n} |W|^n) \cdot (1 + |W|)$ many worlds. □

2.4 The blooming Technique

We conclude this chapter by introducing the blooming technique. This method aims to transform ML^∞-Kripke models to Kripke models but maintain the truth value of the ML^∞-formulas. Meaning that for every pointed ML^∞-Kripke $(\mathcal{M}, w)$, its Bloomed pointed Kripke model, $(\mathscr{M}, w)$ satisfies the same ML^∞-Kripke formulas. In the following paragraph the reader can find a brief introduction to the blooming technique:

First, we transform the universe W by substituting any world that lies in the image of the R^∞-function to countably infinite many copies of the same world. Second, we transform the accessibility relation R and link those worlds in the *new* universe if and only their correspondent in the *old* universe were linked by the accessibility relation R. Finally any world in the new model will satisfy $p \in \Phi$ if its original copy satisfies p in $\mathcal{M}$.

However, it was noted that the truth value of the ML^∞-formulas was not preserved throughout this method. We found out that when blooming an ML^∞-Kripke infinitely many successors could be added to a world, leading to misadjustments on the truth value of ML^∞-formulas (see Observation 2.27). In order to resolve this issue, we first observed that tree ML^∞-Kripke models did not suffer from these misadjustments, therefore combining the unravelling technique previously described with the blooming technique led us to our desired result.

Definition 2.25. Let $\mathcal{M} = (W, R, R^\infty, V)$ be a ML^∞-Kripke model. We define the non-empty set $\underline{W}$ as follows:

$\underline{W} := \{(v, n) \mid v \in \bigcup_{w \in W} R^\infty[w] \text{ and } n \in \mathbb{N}\} \cup \{(v, 0) \mid v \in W \setminus \bigcup_{w \in W} R^\infty[w]\}$.

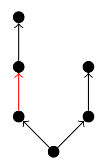
Moreover, let $\underline{\pi} : \underline{W} \to W$ be the surjective projection map such that:

$$\underline{\pi}((w, i)) := w \text{ for every } (w, i) \in \underline{W}.$$

Finally we equip the set $\underline{W}$ with a binary relation $\underline{R}$ and a valuation function $\underline{V} : \Phi \to \mathfrak{P}(\underline{W})$ to define the bloomed Kripke model $\underline{\mathscr{M}} = (\underline{W}, \underline{R}, \underline{V})$ of $\mathcal{M}$:

- $\underline{R} := \{((w, i), (v, j)) \in \underline{W}^2 \mid \underline{\pi}(w, i) R \underline{\pi}(v, j)\}$
- $\underline{V}(p) := \{(w, i) \in \underline{W} \mid \underline{\pi}(w, i) \in V(p)\}$ for every $p \in \Phi$.

Example 2.26. In the following diagram we provide a graphic example of Definition 2.25. On the left the reader can find a figure representing a ML^∞-Kripke model $\mathcal{M}$. The paths coloured in black represent those elements that are related by the accessibility relation R but not by the accessibility relation R^∞. On the contrary the arrows coloured in red represent those worlds that are linked by the accessibility relation R^∞. The figure of the left represents a ML^∞-Kripke model $\mathcal{M}$ and the figure located represents the Kripke model $\underline{\mathscr{M}}$, the bloomed Kripke model of $\mathcal{M}$:



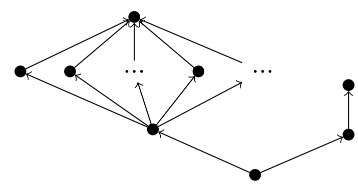


Figure 2.6: The ML^∞-Kripke model $\mathcal{M}$. Figure 2.7: The bloomed model $\underline{\mathscr{M}}$.

***Observation* 2.27.** As we stated in the beginning of this section, the blooming technique does not always preserve the theory of a pointed ML^∞-Kripke model. To show this, consider the following ML^∞-Kripke model $\mathcal{M} := (W, R, R^\infty, V)$, where $W := \{a, b, c, d\}$, $R := \{(a, b), (b, d), (a, c), (c, d)\}$ and $R^\infty := \{(c, d)\}$. Moreover, consider the simple case where Φ consists of a unique propositional variable p and let $V(p) := \{d\}$. The following figure represents the ML^∞-Kripke model, where the black arrow represents the R-relation and the red one represent the R^∞-relation:

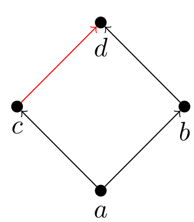


Figure 2.8: The ML^∞-Kripke model $\mathcal{M}$.

Note that under the ML^∞-semantics $\mathcal{M}, b \Vdash^\infty \Diamond p \wedge \neg \Diamond^\infty p$. If we now apply the blooming technique towards $\mathscr{M}$ we obtain the Kripke model $\underline{\mathscr{M}} := (\underline{W}, \underline{R}, \underline{V})$:

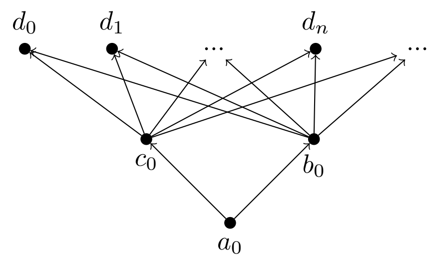


Figure 2.9: The Kripke model $\underline{\mathscr{M}}$.

Therefore $\underline{\mathscr{M}}, b_0 \Vdash \Diamond p \wedge \Diamond^{\infty} p$. Hence the theory of $(\mathscr{M}, b)$ is not preserved under the blooming technique.

However as we noted in the introduction of this section, if we combine the unravelling technique with the blooming technique we achieve our goal. Lemma 2.29 shows that if $\mathcal{M}$ is a finite n-pseudotree rooted at w, the theory $T_n := \{\varphi \in \mathrm{ML}^{\infty} \mid \mathtt{qd}(\varphi) \leq n \text{ and } \mathcal{M}, w\}$ is preserved under the blooming procedure.

Proposition 2.28. Let $\mathfrak{M}_n^{\infty}[w]$ be the n-unravelling of a finite ML^{∞}-Kripke model $\mathcal{M}$ rooted at $w \in W$. Moreover, let $\underline{\mathfrak{M}_n[w]}$ be the bloomed Kripke model of $\mathfrak{M}_n[w]$. Let $\alpha \in \underline{\mathfrak{W}_n[w]}$ be any world such that $\underline{\pi}(\alpha) \in \mathrm{Path}_k(w)$ for some $k < n$. If the set $\underline{\mathfrak{R}_n[w]}[\alpha]$ is infinite, then there exists an infinite set $X \subseteq \underline{\mathfrak{R}_n[w]}[\alpha]$ and some $b \in \mathfrak{W}_n[w]$ such that:

i $\underline{\pi}[X] = \{b\}$,

ii $\underline{\pi}(\alpha)\mathfrak{R}_n^{\infty}[w]b$.

Proof. By 2.24, the set $\mathfrak{W}_n[w]$ is finite. Therefore $\underline{\pi}(\alpha)$ has finitely many successors. By Definition 2.25, every $\beta \in \underline{\mathfrak{W}_n[w]}$ is an $\underline{\mathfrak{R}_n[w]}$-successor of α if and only if $\underline{\pi}(\beta)$ is an $\mathfrak{R}_n[w]$-successor of $\underline{\pi}(\alpha)$. Combining these facts with the pigeonhole principle we can find an infinite set $X \subseteq \underline{\mathfrak{R}_n[w]}[\alpha]$ and a unique $b \in \mathfrak{W}_n[w]$ such that $\underline{\pi}[X] = \{b\}$. Finally, it suffices to show that $\underline{\pi}(\alpha)\mathfrak{R}_n^{\infty}[w]b$.

By the Definition 2.25 there exists some $c \in \mathfrak{W}_n$ such that $c\mathfrak{R}_n^{\infty}[w]b$. Besides, by our assumption $\underline{\pi}(\alpha) \in \mathrm{Path}_k(w)$ for some $k < n$. Therefore, in view of Proposition 2.23, we conclude that $b = c$ and thus $\underline{\pi}(\alpha)\mathfrak{R}_n^{\infty}[w]b$. $\square$

Lemma 2.29. Let $\mathfrak{M}_n[w]$ be the n-unravelling of a finite ML^{∞}-Kripke model $\mathcal{M}$ rooted at w. For every $\alpha \in \underline{\mathfrak{W}_n[w]}$, if $\underline{\pi}(\alpha) \in \mathrm{Path}_k(w)$ for some $k \leq n$ and for every $\varphi \in \mathrm{ML}^{\infty}$ with $\mathtt{md}(\varphi) \leq n - k$ the following holds:

$$\underline{\mathfrak{M}_n[w]}, \alpha \Vdash \varphi \Longleftrightarrow \mathfrak{M}_n[w], \underline{\pi}(\alpha) \Vdash^{\infty} \varphi. \tag{2.1}$$

Proof. This proof follows by reverse induction on k, where the only non-trivial case involve the modalities. Let $\alpha \in \underline{\mathfrak{W}_n[w]}$ such that $\underline{\pi}(\alpha) \in \mathrm{Path}_k(w)$ for some $k < n$. Moreover, consider any φ with $\mathtt{md}(\varphi) = n - k$. Then consider the two possible cases:

- Suppose that $\varphi = \Diamond\psi$ with $\mathtt{md}(\psi) = n - (k+1)$. Then:

$$\begin{aligned}
\underline{\mathfrak{M}_n[w]}, \alpha \Vdash \Diamond\psi &\Longrightarrow \exists\beta \in \underline{\mathfrak{W}_n}(\alpha\underline{\mathfrak{R}_n[w]}\beta \wedge \underline{\mathfrak{M}_n[w]}, \beta \Vdash \psi) && (\Vdash \text{ definition})\\
&\Longrightarrow \exists\beta \in \underline{\mathfrak{W}_n[w]}(\underline{\pi}(\alpha)\mathfrak{R}_n[w]\underline{\pi}(\beta) \wedge \underline{\mathfrak{M}_n}, \beta \Vdash \psi) && (\text{Blooming technique})\\
&\Longrightarrow \exists\beta \in \underline{\mathfrak{W}_n[w]}(\underline{\pi}(\alpha)\mathfrak{R}_n[w]\underline{\pi}(\beta) \wedge \mathfrak{M}_n[w], \underline{\pi}(\beta) \Vdash^{\infty} \psi) && (\text{Inductive hypoyhesis})\\
&\Longrightarrow \exists b \in \mathfrak{W}_n[w](\underline{\pi}(\alpha)\mathfrak{R}_n[w]b \wedge \mathfrak{M}_n[w], b \Vdash^{\infty} \psi) && (\text{Equivalent formulation})\\
&\Longrightarrow \mathfrak{M}_n[w], \underline{\pi}(\alpha) \Vdash^{\infty} \Diamond\psi && (\Vdash^{\infty} \text{ definition})\\
\mathfrak{M}_n[w], \underline{\pi}(\alpha) \Vdash^{\infty} \Diamond\psi &\Longrightarrow \exists b \in Path_{k+1}(w)(\underline{\pi}(\alpha)\mathfrak{R}_n[w]b \wedge \mathfrak{M}_n[w], b \Vdash^{\infty} \psi) && (\Vdash \text{ definition})\\
&\Longrightarrow \exists\beta \in \underline{\mathfrak{W}_n}(\underline{\pi}(\beta) = b \wedge \underline{\pi}(\alpha)\mathfrak{R}_n[w]b \wedge \mathfrak{M}_n[w], b \Vdash^{\infty} \psi) && (\underline{\pi} \text{ is surjective})\\
&\Longrightarrow \exists\beta \in \underline{\mathfrak{W}_n}(\underline{\pi}(\alpha)\mathfrak{R}_n[w]\underline{\pi}(\beta) \wedge \mathfrak{M}_n[w], \underline{\pi}(\beta) \Vdash^{\infty} \psi) && (\text{Equivalent formulation})\\
&\Longrightarrow \exists\beta \in \underline{\mathfrak{W}_n}(\alpha\underline{\mathfrak{R}_n[w]}\beta \wedge \underline{\mathfrak{M}_n[w]}, \beta \Vdash \psi) && (\text{Inductive hypothesis})\\
&\Longrightarrow \underline{\mathfrak{M}_n[w]}, \alpha \Vdash \Diamond\psi && (\Vdash^{\infty} \text{ definition})
\end{aligned}$$

- Alternatively suppose that $\varphi = \Diamond^{\infty}\psi$ with $\mathtt{md}(\psi) = n - (k+1)$. Then:

$$\begin{aligned}
\underline{\mathfrak{M}_n[w]}, \alpha \Vdash \Diamond^{\infty}\psi &\Longrightarrow \exists^{\infty}\beta \in \underline{\mathfrak{W}_n[w]}(\alpha\underline{\mathfrak{R}_n[w]}\beta \wedge \underline{\mathfrak{M}_n[w]}, \beta \Vdash \psi) && (\Vdash \text{ definition})\\
&\Longrightarrow \exists b \in \mathfrak{W}_n[w](\underline{\pi}(\beta) = b \wedge \underline{\pi}(\alpha)\mathfrak{R}^{\infty}_n[w]b \wedge \mathfrak{M}_n[w], \beta \Vdash \psi) && (\text{Proposition 2.28})\\
&\Longrightarrow \exists b \in \mathfrak{W}_n[w](\underline{\pi}(\alpha)\mathfrak{R}_n[w]^{\infty}b \wedge \mathfrak{M}_n[w], b \Vdash^{\infty} \psi) && (\text{Inductive hypothesis})\\
&\Longrightarrow \mathfrak{M}_n[w], \underline{\pi}(\alpha) \Vdash^{\infty} \Diamond^{\infty}\psi && (\Vdash^{\infty} \text{ definition})\\
\mathfrak{M}_n[w], \underline{\pi}(\alpha) \Vdash^{\infty} \Diamond^{\infty}\psi &\Longrightarrow \exists b \in Path_{k+1}(w)(\underline{\pi}(\alpha)\mathfrak{R}^{\infty}_n[w]b \wedge \mathfrak{M}_n[w], b \Vdash^{\infty} \psi) && (\Vdash^{\infty} \text{ definition})\\
&\Longrightarrow \forall n \in \mathbb{N}(\underline{\pi}(\alpha)\mathfrak{R}_n[w]\underline{\pi}((b,n)) \wedge \mathfrak{M}_n[w], \underline{\pi}((b,n)) \Vdash^{\infty} \psi) && (\underline{\mathfrak{M}_n[w]} \text{ definition})\\
&\Longrightarrow \exists^{\infty}\beta \in \underline{\mathfrak{W}_n[w]}(\underline{\pi}(\alpha)\mathfrak{R}_n[w]\underline{\pi}(\beta) \wedge \mathfrak{M}_n[w], \underline{\pi}(\beta) \Vdash^{\infty} \psi) && (\text{Equivalent formulation})\\
&\Longrightarrow \exists^{\infty}\beta \in \underline{\mathfrak{W}_n[w]}(\alpha\underline{\mathfrak{R}_n[w]}\beta \wedge \underline{\mathfrak{M}_n[w]}, \beta \Vdash \psi) && (\text{Inductive hypothesis})\\
&\Longrightarrow \underline{\mathfrak{M}_n[w]}, \alpha \Vdash \Diamond^{\infty}\psi && (\Vdash \text{ definition})
\end{aligned}$$

$\square$

Bisimulation in ML^∞.

Just as the Kripke semantics of basic modal logic is invariant under bisimulation (see Theorem 3.3 in [TBV07]), in this chapter we observe that this condition does not hold for the Kripke semantics of ML^∞. In order to solve this issue and introduce a notion of bisimulation that recovers the invariant results, we present the notion of ML^∞-bisimulation. We do so by adapting the bisimulation game, a game-theoretical definition of bisimulation (see Section 3.1 in [GO07]), to the Kripke semantics and we allow `Spoiler` to launch two types of challenges towards `Duplicator`. Next we focus on the concept of bisimulation on the ML^∞-Kripke semantics that we proposed in the previous chapter. Since in these semantics the $\Diamond^\infty$ was interpreted in terms of the R^∞ accessibility relation we observe that a version of the general bisimulation game is strong enough to achieve our purpose. We conclude this section by showing that the satisfaction relation defined in Kripke semantics for the logic ML^∞ is invariant under ML^∞-bisimulation. In a parallel way we also prove that the ML^∞-Kripke semantics is invariant under bisimulation.

3.1 The ML^∞-bisimulation game.

***Observation* 3.1.** Firstly, we will show that the modal logic ML^∞ is not invariant under bisimulation. Recall that for two Kripke models $\mathscr{M} = (W, R, V)$ and $\mathscr{M}' = (W', R', V')$ a bisimulation is a binary relation $Z \subseteq W \times W'$ satisfying the following properties:

- If vZv' then v and v' satisfy the same propositional variables. **(atom)**
- If vZv' and vRu then there is some u' such that $v'R'u'$ and uZu'. **(forth)**
- If vZv' and $v'R'u'$ then there is some u such that vRu and uZu'. **(back)**

If two points $(v, v') \in W \times W'$ are linked by a bisimulation we say that are bisimilar and denote it by $\mathscr{M}, w \leftrightarroweq \mathscr{M}', w'$. On one hand consider the Kripke model $\mathscr{M} = (W, R, V)$ where $W := \{n \mid n \in \mathbb{N}\} \cup \{\omega\}$, $R := \{(\omega, n) \mid n \in \mathbb{N}\}$ and $V(p) := \{n \mid n \in \mathbb{N}\}$ for every $p \in \Phi$. On the other hand consider the Kripke model $\mathscr{M} = (W, R, V)$ where $W' := \{a, b\}$, $R' := \{(a, b)\}$ and $V'(p) := \{b\}$

for every $p \in \Phi$. Moreover, notice that the binary relation $Z := \{(n, b) \mid n \in \mathbb{N}\} \cup \{(\omega, a)\}$ is a bisimulation over these two structures:

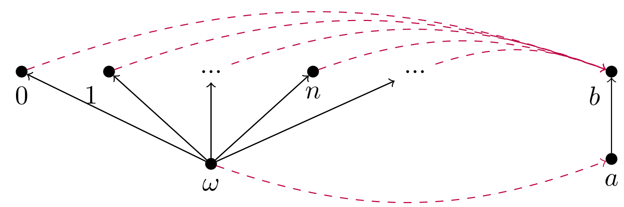


Figure 3.1: The bisimulation between $\mathscr{M}$ and $\mathscr{M}'$

However, notice that for every $p \in \Phi$ we have that $\mathscr{M}, \omega \Vdash \Diamond^{\infty} p$ whereas $\mathscr{M}', a \nVdash \Diamond^{\infty} p$. Hence in the ML^{∞} framework bisimulation does not imply ML^{∞}-equivalence.

Definition 3.2. The ML^{∞}- bisimulation game is played by two players, `Spoiler` (that is a male) and `Duplicator` (that is a female) over two Kripke models, $\mathscr{M}_0$ and $\mathscr{M}_1$. Moreover, each of the models is equipped with a pebble. Each round of the ML^{∞}-bisimulation game over $\mathscr{M}_0$ and $\mathscr{M}_1$ has a starting configuration $(w_0, w_1) \in W_0 \times W_1$ (the worlds that were pebbled in the previous round) and continues as follows:

`Spoiler` chooses one of the two structures i.e. $\mathscr{M}_i$, then he continues by making one of the two possible moves allowed:

F.O.M.: `Spoiler` moves the pebble located at w_i to a successor world $w_i^+ \in W_i$. `Duplicator` replies by moving the $\mathscr{M}_{-i}$-pebble from w_{-i} to a successor world w_{-i}^+.

S.O.M.: Alternatively, `Spoiler` chooses an infinite set X of successors of w_i, i.e. $X \subseteq \{v \in W_i \mid w_i R_i v\}$. `Duplicator` replies by choosing an infinite set Y of successors of w_{-i}, i.e. $Y \subseteq \{v \in W_{-i} \mid w_{-i} R_{-i} v\}$. Finally, `Spoiler` moves the $\mathscr{M}$-pebble from w_i to a world $w_i^+ \in Y$ and `Duplicator` responds by placing the $\mathscr{M}_i$-pebble from w_i to a world $w_i^+ \in X$.

`Duplicator` wins the game either if at some round each of the worlds constituting the initial configuration does not have any successors or if she can survive indefinitely. On the contrary, `Spoiler` wins the game if at any round the worlds constituting the outcome configuration do not satisfy the same propositional variables.

We denote the ML^{∞}-bisimulation game over $\mathscr{M}_0$ and $\mathscr{M}_1$ with initial configuration (w_0, w_1) by $\mathtt{Bis}^{\infty}(\mathscr{M}_0, \mathscr{M}_1)@(w_0, w_1)$.

***Remark* 3.3.** The following diagram gives a more graphic explanation of the second order movement we have just introduced:

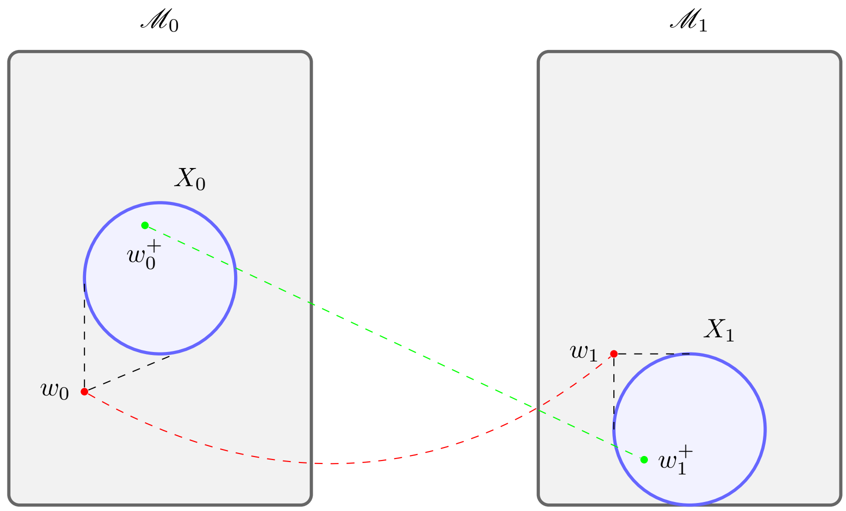


Figure 3.2: Second order movement.

In this case suppose that the round has started with the initial configuration (w_0, w_1) and suppose that `Spoiler` launches a second order challenge towards `Duplicator` by taking X_0, an infinite set of R_0-successors of w_0. `Duplicator` replies by choosing an infinite set X_1 of R_1-successors of w_1. `Spoiler` continues by choosing the world w_1^+ in X_1 and `Duplicator` answers by taking the element w_0^+ in X_0. `Spoiler` wins the round and the game if w_0^+ and w_1^+ are not atomically equivalent. On the contrary, if w_0^+ and w_1^+ satisfy the same propositional variables `Duplicator` survives the round.

***Remark* 3.4.** We say that `Duplicator` has a winning strategy in $\texttt{Bis}^\infty(\mathscr{M}_0, \mathscr{M}_1)$ with initial configuration (w_0, w_1) if she can respond to any challenge that `Spoiler` may launch at her.

Definition 3.5. Let $(\mathscr{M}, w)$ and $(\mathscr{M}', w')$ be two pointed Kripke models. We say that $(\mathscr{M}, w)$ is ML^∞-bisimilar to $\mathscr{M}, w'$ and denote it by $(\mathscr{M}, w \;\underline{\leftrightarrow}^\infty\; \mathscr{M}', w')$ if `Duplicator` has a winning strategy in $\texttt{Bis}^\infty(\mathscr{M}, \mathscr{M}')\ @\ (w, w')$.

Definition 3.6. Given two pointed Kripke models $(\mathscr{M}_0, w_0)$ and $(\mathscr{M}_1, w_1)$ and a natural number n we define the ML^∞-bisimulation game of length n over $\mathscr{M}_0$ and $\mathscr{M}_1$ with initial configuration (w_0, w_1), denoted by $\texttt{Bis}_n^\infty(\mathscr{M}_0, \mathscr{M}_1)@(w_0, w_1)$, to be the ML^∞-bisimulation game that terminates after n rounds.

Definition 3.7. Let n be a natural number and let $(\mathscr{M}, w)$ and $(\mathscr{M}', w')$ be two pointed Kripke models. We say that $\mathscr{M}, w$ is ML_n^∞-bisimilar to $\mathscr{M}, w'$ and denote it by $\mathscr{M}, w \;\underline{\leftrightarrow}_n^\infty\; \mathscr{M}', w'$ if `Duplicator` has a winning strategy in $\mathtt{Bis}_n^\infty(\mathscr{M}, \mathscr{M}')$ @ (w, w').

3.2 The bisimulation game over ML$^\infty$-Kripke models.

In this section we introduce the concept of bisimulation for the ML$^\infty$-Kripke semantics. As in the previous section we will adapt the standard bisimulation game taking into account the interpretation of the $\Diamond^\infty$ modality in this semantics. We do so by allowing `Spoiler` to make two kinds of moves, each of which is aims to capture the behaviour of each modality.

Definition 3.8. For any two pointed ML$^\infty$-Kripke models $(\mathcal{M}_0, w_0)$ and $(\mathcal{M}_1, w_1)$, the bisimulation game over $\mathcal{M}_0$ and $\mathcal{M}_1$ with initial configuration (w_0, w_1) (denoted by $\mathtt{Bis}(\mathcal{M}_0, \mathcal{M}_1)@(w_0, w_1)$) is played by two players `Spoiler` and `Duplicator`. As in the ML$^\infty$-bisimulation game, each structure has a pebble. A round of the ML$^\infty$-bisimulation with configuration (u_0, u_1) goes as follows:

`Spoiler` chooses one of the two structures, namely $\mathcal{M}_i$, and makes one of the permitted moves:

R-move: `Spoiler` moves the $\mathcal{M}_i$-pebble from u_i to an R_i-successor world u_i^+. Then `Duplicator` advances the $\mathcal{M}_{-i}$-pebble from u_{-i} to an R_{-i}-successor world u_{-i}^+.

R$^\infty$-move: Alternatively, `Spoiler` moves the $\mathcal{M}_i$-pebble from u_i to an R_i^∞-successor world u_i^+. `Duplicator` replies to the challenge by moving the $\mathcal{M}_{-i}$-pebble from u_{-i} to an R_{-i}^∞-successor u_{-i}^+.

`Spoiler` wins the game if at any round of the game `Duplicator` cannot reply to a challenge launched by `Spoiler` or if the outcome sequence is not atomically equivalent. On the contrary, `Duplicator` wins the game if she can effectively reply to any challenge that `Spoiler` launches at her.

***Remark* 3.9.** We would like to clarify some issues that might arise from Definition 3.8. The misunderstanding that can arise from this definition is concerned with the **R-move**. Note that since the accessibility relations R^∞ and R are related in the following way: $R^\infty \subseteq R$, the following situation can occur and is in fact a valid movement: `Spoiler` makes an **R-move** and relocates the $\mathcal{M}_i$-pebble from a world w_i to an R_i-successor w_i^+ but for which it is not the case that $w_i R_i^\infty w^+$. Then `Duplicator` replies by moving w_{-i} to an R_{-i}^∞-successor w_{-i}^+. If w_i^+ and w_{-i}^+ are atomically equivalent `Duplicator` survive this round of the game.

However the converse situation is not a valid movement. If `Spoiler` makes an $\mathbf{R}^\infty$**-movement** and relocates the $\mathscr{M}_i$-pebble from a world w_i to an R_i^∞-successor w_i^+, `Duplicator` must move the $\mathscr{M}_{-i}$-pebble from the world w_{-i} to an R_i^∞-successor. She is not allowed to move the world w_{-i} to an R_i-successor w_{-i}^+ for which it is not the case that $w_{-i} R_{-i}^\infty w_{-i}^+$.

Definition 3.10. Let $(\mathcal{M}, w)$ and $(\mathcal{M}', w')$ be two pointed ML$^\infty$-Kripke models. We say that $(\mathcal{M}, w)$ is bisimilar to $(\mathcal{M}', w')$ and denote it by $\mathscr{M}, w \leftrightarroweq \mathscr{M}', w'$ if `Duplicator` has a winning strategy in $\mathtt{Bis}(\mathcal{M}, \mathcal{M}')@(w, w')$.

3.3 Invariance results.

We conclude Chapter 3 by showing a significant amount of the preservation results that hold for ML.

Lemma 3.11. For any two pointed Kripke models $(\mathscr{M}, w)$ and $(\mathscr{M}', w')$:

$$\text{If } \mathscr{M}, w \ \leftrightarroweq^\infty \mathscr{M}', w', \text{ then } \mathscr{M}, w \equiv^\infty \mathscr{M}', w'.$$

Proof. As in theorem 2.20 in [BRV02], this is shown by induction on the ML$^\infty$-formulas. The fundamental observation relies on the connection between the capability of `Duplicator` to reply to every first (second) order move and the semantic behaviour of $\Diamond$ ($\Diamond^\infty$). $\square$

Lemma 3.12. For any two pointed ML$^\infty$-Kripke models $(\mathcal{M}, w)$ and $(\mathcal{M}', w')$:

$$\text{If } \mathcal{M}, w \leftrightarroweq \mathcal{M}', w', \text{ then } \mathcal{M}, w \equiv^\infty \mathcal{M}', w'.$$

Proof. See theorem 2.20 in [BRV02]. $\square$

Lemma 3.13. Let $(\mathscr{M}, w)$ and $(\mathscr{M}', w')$ be two pointed Kripke models, then the following properties are satisfied:

i If $\mathscr{M}, w \ \leftrightarroweq^\infty \mathscr{M}', w'$ then $\mathscr{M}, w \leftrightarroweq \mathscr{M}', w'$,

ii If $\mathscr{M}, w \leftrightarroweq_n^\infty \mathscr{M}', w'$ then $\mathscr{M}, w \leftrightarroweq_n \mathscr{M}', w'$,

for every $n \in \mathbb{N}$.

Proof. Note that if `Duplicator` has a winning strategy in $\mathtt{Bis}^\infty(\mathscr{M}, \mathscr{M}')@(w, w')$ ($\mathtt{Bis}_n^\infty(\mathscr{M}, \mathscr{M}')@(w, w')$) then she has a winning strategy in $\mathtt{Bis}(\mathscr{M}, \mathscr{M}')@(w, w')$ ($\mathtt{Bis}_n(\mathscr{M}, \mathscr{M}')@(w, w')$). In view of Proposition 28 in [GO07] we obtain our desired result. $\square$

Lemma 3.14. Let $\mathscr{M} = (W, R, V)$ and $\mathscr{M}' = (W', R', V')$ be two Kripke models and $w \in W$. Then the following properties are satisfied:

i $\mathscr{M}, w \;\underline{\leftrightarrow}^{\infty}\; \mathscr{M}', w$ if $\mathscr{M} = \mathscr{M}'_X$ for some $X \subseteq W'$ and $w \in W$.

ii $\mathscr{M}, w \;\underline{\leftrightarrow}^{\infty}\; \mathscr{M}', \rho(w)$ if $\rho : \mathscr{M} \to \mathscr{M}'$ is a strong bounded morphism.

iii $\mathscr{M}, w \;\underline{\leftrightarrow}^{\infty}\; \mathscr{M}', w$ if $\mathscr{M}' = \biguplus_{i \in I} \mathscr{M}_i$ is a disjoint union where $\mathscr{M} = \mathscr{M}_i$ for some $i \in I$ and $w \in W$.

Proof. (ii) follows immediately since the **(forth)** and **(back)** conditions present a method for `Duplicator` to reply to every first order challenge launched by `Spoiler`. Similarly, the previous two conditions combined with **(strongforth)** determine a strategy for `Duplicator` to reply to every second order challenge launched by `Spoiler`. (i) and (iii) follow since the maps $id : \mathscr{M} \to \mathscr{M}'$ is a strong bounded morphism in both cases. □

Lemma 3.15. Let $\mathcal{M} = (W, R, R^{\infty}, V)$ and $\mathcal{M}' = (W', R', R'^{\infty}, V')$ be two ML^{∞}-Kripke models and $w \in W$. Then the following properties hold:

i $\mathcal{M}, w \,\underline{\leftrightarrow}\, \mathcal{M}', \rho(w)$ if $\rho : W \to W'$ is a bounded morphism and $w \in W$.

ii $\mathscr{M}, w \,\underline{\leftrightarrow}\, \mathscr{M}', w$ if $\mathcal{M} = \mathcal{M}'_X$ for some $X \subsetneq W'$ and $w \in W$.

Proof. See Proposition 2.19 in [BRV02]. □

Proposition 3.16. Let $\mathcal{M} = (W, R, R^{\infty}, V)$ be a ML^{∞}-Kripke model rooted at $w \in W$. For every $n \in \mathbb{N}$ and every $a \in \mathfrak{W}_n[w] : \mathfrak{M}_n[w], a \;\underline{\leftrightarrow}^{\infty}\; \mathscr{M}, \chi_n(a)$.

Proof. Follows immediately from Lemma 3.15 part ii since the map χ_n is a surjective strong bounded morphism. □

Model theory of FO^∞

At the end of the previous chapter we proved the invariance result for the modal logic ML^∞. In other words we demonstrated that if two pointed Kripke models are ML^∞-bisimilar then they are ML^∞-equivalent. It is a well-known result from modal logic that bisimulation implies modal equivalence. However, the converse of this result does not always hold. Therefore a natural question on this matter arises: For which Kripke models does the notion of equivalence and bisimulation coincide? Positive results by Hennessy and Milner [HM85] on this question paved the concept of the Hennessy-Milner property:

Hennessy-Milner property: A collection $\mathbb{C}$ of pointed Kripke models has the Hennessy-Milner property if for every $(\mathscr{M}, w), (\mathscr{M}', w') \in \mathbb{C}$ the following holds:

$$(\mathscr{M}, w) \leftrightarroweq (\mathscr{M}', w') \text{ if and only if } (\mathscr{M}, w) \equiv (\mathscr{M}', w').$$

Therefore the core goal of this section is to study the Hennessy-Milner property on the modal logic ML^∞. To achieve such goal we first need to introduce the reader to predicate logic FO^∞, an extension of first order logic with an additional quantifier $\exists^\infty$. This new quantifier improves the expressive power of first order logic by manifesting the existence of infinitely many elements satisfying a certain formula.

This chapter is divided into two sections. In the first, we introduce the reader to the basic semantic and syntactic concepts of FO^∞. In addition we define ω-type, a generalization of the model theoretic concept of type and κ^∞-saturation, an adaptation of the model theoretic concept of saturation to the FO^∞ framework.

In the second section we adapt the definition of the Hennessy-Milner property to the ML^∞ framework and prove the main results of this chapter, namely that the class of image-finite Kripke models and the class of $\aleph_0$-saturated Kripke models enjoy the Hennessy-Milner property.

4.1 Basic concepts of $\mathbf{FO}^\infty$

As we previously mentioned, we first introduce all the basic semantic and syntactic concepts of FO^∞. Most of these definitions are adaptations of the standard model theoretic concepts that can be found in [Mar02].

Definition 4.1. A signature $\mathscr{L}$ is a set containing a possibly empty collection of constant symbols, a possibly empty collection of function symbols of finite arity and a possibly empty collection of relation symbols of finite arity.

Definition 4.2. Let $\mathscr{L}$ be a signature. The set of $\mathscr{L}$-terms, denoted by $\mathfrak{Term}(\mathscr{L})$ is defined by the following grammar:

$$t ::= c \mid x \mid f(t, ..., t),$$

where c is a constant symbol in $\mathscr{L}$, x is a variable and f is a n-ary function symbol in $\mathscr{L}$.

Definition 4.3. Let $\mathscr{L}$ be a signature. We define $\mathtt{FV} : \mathfrak{Term}(\mathscr{L}) \to \mathfrak{P}(\mathrm{Var})$ recursively as follows:

- $\mathtt{FV}(x) = \{x\}$ for every variable x,
- $\mathtt{FV}(c) = \emptyset$ for every constant symbol c,
- $\mathtt{FV}(f(t_1, ..., t_n)) = \bigcup_{i \leq n} \mathtt{FV}(t_i)$ for every $f(t_1, ..., t_n) \in \mathfrak{Term}(\mathscr{L})$.

Definition 4.4. The collection of atomic $\mathscr{L}$-formulas denoted by $\mathfrak{Atom}(\mathscr{L})$ is defined by the following grammar:

$$\alpha ::= \quad t_1 = t_2 \mid R(t_0, ..., t_n),$$

where $t_1, ..., t_n \in \mathfrak{Term}(\mathscr{L})$ and R is a n-ary relation symbol in $\mathscr{L}$. Moreover, the collection of $\mathscr{L}^\infty$-formulas, denoted by $\mathfrak{Form}^\infty(\mathscr{L})$ is defined by the following grammar:

$$\varphi ::= \alpha \mid \neg\varphi \mid \varphi \wedge \varphi \mid \exists x\varphi \mid \exists^\infty x\varphi,$$

where $\alpha \in \mathfrak{Atom}(\mathscr{L})$ and x is a variable. Moreover, we let $\mathfrak{Form}(\mathscr{L})$ to be the fragment of $\mathfrak{Form}^\infty(\mathscr{L})$ where $\exists^\infty$ does not occur.

Definition 4.5. For any formula φ in $\mathfrak{Form}^\infty(\mathscr{L})$ we define its quantifier depth, noted by $\mathtt{qd}(\varphi)$ as follows:

- $\mathtt{qd}(\alpha) = 0$ for every $\alpha \in \mathfrak{Atom}^\infty(\mathscr{L})$,
- $\mathtt{qd}(\neg\varphi) = \mathtt{qd}(\varphi)$,
- $\mathtt{qd}(\varphi \wedge \psi) = \max\{\mathtt{qd}(\varphi), \mathtt{qd}(\psi)\}$,
- $\mathtt{qd}(Qx\varphi) = \mathtt{qd}(\varphi) + 1$ where $Q \in \{\exists, \exists^\infty\}$.

Definition 4.6. We define $\mathtt{FV} : \mathfrak{Form}^\infty(\mathscr{L}) \to \mathfrak{P}(\mathrm{Var})$ recursively as follows:

- $\mathtt{FV}(R(t_1, ..., t_n)) := \bigcup_{0<i\leq n} \mathtt{FV}(t_i)$,
- $\mathtt{FV}(\neg\varphi) := \mathtt{FV}(\varphi)$,
- $\mathtt{FV}(\varphi \wedge \psi) := \mathtt{FV}(\varphi) \cap \mathtt{FV}(\psi)$,
- $\mathtt{FV}(Qx\varphi)) := \mathtt{FV}(\varphi)\backslash\{x\}$ where $Q \in \{\exists, \exists^\infty\}$.

Moreover, we say that a formula $\varphi \in \mathfrak{Form}^\infty(\mathscr{L})$ is a sentence if $\mathtt{FV}(\varphi) = \emptyset$. The collection of all sentences in $\mathfrak{Form}^\infty(\mathscr{L})$ will be denoted by $\mathfrak{Sent}^\infty(\mathscr{L})$.

Definition 4.7. An $\mathscr{L}$-structure M is a tuple $(\mathtt{dom}(M), (\cdot)^M)$, where:

- $\mathtt{dom}(M)$ is a non-empty set.
- $(\cdot)^M$ is an interpretation function on $\mathscr{L}$ satisfying the following properties:
  - i $c^M \in \mathtt{dom}(M)$ for every constant symbol c,
  - ii $f^M : \mathtt{dom}(M)^n \to \mathtt{dom}(M)$ for every n-ary function symbol f in $\mathscr{L}$,
  - iii $R^M \subseteq M^n$ for every n-ary relation symbol R in $\mathscr{L}$.

Definition 4.8. For any $\mathscr{L}$-structure M we define the satisfaction relation $\vDash$ by induction on the complexity of the formulas as follows:

$$\begin{aligned}
M \vDash (t = s) &\iff_{\mathrm{Def}} (s^M = t^M),\\
M \vDash R(t_0, ..., t_n) &\iff_{\mathrm{Def}} (t_0^M, ..., t_n^M) \in R^M,\\
M \vDash \varphi \wedge \psi &\iff_{\mathrm{Def}} M \vDash \varphi \text{ and } M \vDash \psi,\\
M \vDash \neg\varphi &\iff_{\mathrm{Def}} M \nvDash \varphi,\\
M \vDash \exists x\varphi(x) &\iff_{\mathrm{Def}} \text{ there exists some } m \in M \text{ such that } M \vDash \varphi(m),\\
M \vDash \exists^\infty x\varphi(x) &\iff_{\mathrm{Def}} \text{ there are infinitely } m \in M \text{ such that } M \vDash \varphi(m).
\end{aligned}$$

Definition 4.9. Two $\mathscr{L}$-structures M and N are FO$^\infty$-elementary equivalent, denoted by $M \equiv^\infty N$ given that for every $\varphi \in \mathfrak{Sent}^\infty(\mathscr{L})$:

$$M \vDash \varphi \iff N \vDash \varphi.$$

Moreover, for any $n \in \mathbb{N}$, we write $M \equiv^\infty_n N$ if and only if for every sentence $\varphi \in \mathfrak{Sent}^\infty_{\mathscr{L}}$ with $\mathtt{qd}(\psi) \leq n$, the following holds:

$$M \vDash \varphi \iff N \vDash \varphi.$$

Definition 4.10. Let φ and ψ be two $\mathfrak{Form}^\infty(\mathscr{L})$ formulas, we say that φ is equivalent to ψ up to logical equivalence if $M \vDash \varphi \leftrightarrow \psi$ for every structure M.

Definition 4.11. A map $\eta : \mathtt{dom}(M) \to \mathtt{dom}(N)$ between two $\mathscr{L}$-structures M and N is an $\mathscr{L}$-embedding if η is injective and the following properties are satisfied:

i $\eta(c^M) = c^N$ for every constant symbol c,

ii $\eta(f^M(t_1, ..., t_n)) = f^N(\eta(t_1), ..., \eta(t_n))$ for every n-ary function symbol $f \in \mathscr{L}$ and terms $t_1, ..., t_n \in \mathfrak{Term}(\mathscr{L})$,

iii $(t_1, ..., t_n) \in R^M$ if and only if $(\eta(t_0), ..., \eta(t_n)) \in R^N$ for every n-ary relation symbol $R \in \mathscr{L}$ and terms $t_1, ..., t_n \in \mathfrak{Term}(\mathscr{L})$.

A bijective $\mathscr{L}$-embedding is called an $\mathscr{L}$-isomorphism. In addition an $\mathscr{L}$-embedding η is said to be FO$^\infty$-elementary if for every $\varphi(x_0, ..., x_n) \in \mathfrak{Form}^\infty(\mathscr{L})$ and any $(m_0, ..., m_n) \in \mathtt{dom}(M)^{n+1}$:

$$M \vDash \varphi(m_0, ..., m_n) \iff N \vDash \varphi(\eta(m_0), ..., \eta(m_n)).$$

Definition 4.12. Let $\eta : \{m_0, ..., m_k\}\mathtt{dom}(M) \to \mathtt{dom}(N)$ be a partial map between two $\mathscr{L}$-structures M and N. We say that η is a local isomorphism if for every atomic formula $\alpha(x_0, ..., x_k)$:

$$M \vDash \alpha(m_0, ..., m_k) \iff N \vDash \alpha(\eta(m_0), ..., \eta(m_k)).$$

Definition 4.13. An $\mathscr{L}^\infty$-theory T is a subset of $\mathfrak{Form}^\infty(\mathscr{L})$. An $\mathscr{L}^\infty$-theory T is satisfiable if there exists an $\mathscr{L}$-structure M that makes true every $\varphi \in T$. Moreover, we say that T is finitely satisfiable if there exists a $\mathscr{L}$-structure M that satisfies every finite subset of T.

Definition 4.14. For any $\mathscr{L}$-structure M, we let $\mathtt{Th}^\infty(M)$ to be:

$$\mathtt{Th}^\infty(M) := \{\psi \in \mathfrak{Form}^\infty(\mathscr{L}) \mid M \vDash \psi\}$$

4.2 FO$^\infty$-Ehrenfeucht–Fraïssé games

We now introduce FO$^\infty$-Ehrenfeucht–Fraïssé game, a generalization of the well-known Ehrenfeucht–Fraïssé game to the FO$^\infty$ framework. Moreover, we define the finite version of the FO$^\infty$-Ehrenfeucht–Fraïssé game and we show that `Duplicator` has a winning strategy on the version of this game that ends after n rounds if and only if the structures on which `Spoiler` and `Duplicator` are playing are FO$^\infty_n$-equivalent.

Definition 4.15. Let M_0, M_1 be two $\mathscr{L}$-structures. The FO$^\infty$-Ehrenfeucht–Fraïssé game on M_0 and M_1, denoted by $\mathtt{EF}^\infty(M_0, M_1)$, is played by `Spoiler` (that is a male) and `Duplicator` (that is a female). The $(n+1)$-round of $\mathtt{EF}^\infty(M_0, M_1)$ has an initial configuration $(\bar{m}_0; \bar{m}_1)$ where $\bar{m}_0 \in \mathtt{dom}(M_0)^n$ and $\bar{m}_1 \in \mathtt{dom}(M_1)^n$. `Spoiler` starts by making a move that `Duplicator` replies immediately afterwards. `Spoiler` is allowed to perform two kinds of moves:

First order move: `Spoiler` chooses an arbitrary element of one of the two structures, i.e. $m_i \in \mathtt{dom}(M_i)$. `Duplicator` responds to this move by picking an element of the other structure $m_{-i} \in \mathtt{dom}(M_{-i})$.

Second order move: `Spoiler` chooses an infinite set of one of the structures $X_i \subseteq \mathtt{dom}(M_i)$. `Spoiler` responds by choosing an infinite subset of the domain of the other structure: $X_{-i} \subseteq \mathtt{dom}(M_{-i})$. Finally, `Spoiler` chooses an element $m_{-i} \in X_{-i}$ and `Duplicator` responds by choosing an element $m_i \in X_i$.

After a move is completed, the sequence $\bar{m}_i$ is extended by adding the selected element at the end of the sequence: $\bar{m}_i{}' = \bar{m}_i * m_i$ and the partial map $f_{n+1} : \bar{m}_0' \to \bar{m}_1'$ is defined, where:

$$f_{n+1}(m_{i,0}) = m_{i,1}.$$

`Spoiler` wins the game if at any of the rounds, the constructed map $f_n : \bar{m}_0 \to \bar{m}_1$ does not form a local isomorphism. On the contrary, `Duplicator` wins the game if she can survive every round of the FO$^\infty$-Ehrenfeucht–Fraïssé game.

Definition 4.16. Let M and N be two $\mathscr{L}$-structures. `Duplicator` has a winning strategy in $\mathtt{EF}^\infty(M, N)$ if `Duplicator` can effectively respond to any challenge launched by `Spoiler`.

Definition 4.17. Let M, N be two $\mathscr{L}$-structures and $n \in \mathbb{N}$. We let $\mathtt{EF}^\infty_n(M, N)$ be the FO$^\infty$-Ehrenfeucht–Fraïssé game that terminates after n rounds. Analogous to our previous definition, `Spoiler` wins the $\mathtt{EF}^\infty_n(M, N)$ game if at any of the rounds the partial map f_k is not a local isomorphism. On the contrary, `Duplicator` wins the FO$^\infty$-Ehrenfeucht–Fraïssé game of length n if she can survive the n rounds of the game.

Definition 4.18. Let M and N be two $\mathscr{L}$-structures. We write $M \cong^\infty_n N$ when `Duplicator` has a winning strategy in the $\mathtt{EF}^\infty_n(M, N)$ game. Moreover, for any $m \in \mathtt{dom}(M)$ and any $n \in \mathtt{dom}(N)$ we write $(M, m) \cong^\infty_n (N, n)$ whenever `Duplicator` has a winning strategy in the $\mathtt{EF}^\infty_n(M, N)$ with initial configuration $(m; n)$.

Lemma 4.19. Let $\mathscr{L}$ be a finite signature without function symbols. For every $n \in \omega$, the collection:

$$\mathfrak{Form}^\infty_n(\mathscr{L}) := \{\varphi \in \mathfrak{Form}^\infty(\mathscr{L}) \mid \mathtt{qd}(\varphi) \leq n\}$$

is finite up to logical equivalence.

Proof. Follows by a straightforward adaptation of Lemma 2.4.8 in [Mar02] to the FO$^\infty$ framework. □

Theorem 4.20. Let $\mathscr{L}$ be a finite signature without function symbols and let M, N be two $\mathscr{L}$-structures. The following are equivalent:

i $M \cong_n^\infty N$,

ii $M \equiv_n^\infty N$.

Proof. This is a generalization of Lemma 2.4.9 in [Mar02]. $\square$

4.3 Types and saturated models

In this section we introduce the concepts of ω-type and κ^∞-saturation. These are generalization of the well-known model theoretic sconcepts of type and saturation. Moreover, we conclude by showing that if an $\aleph_0^\infty$-saturated model finitely satisfies an ω-type, then it realizes such type.

Definition 4.21. Let M be an $\mathscr{L}$-structure. For every $X \subseteq \mathtt{dom}(A)$, we define the signature $\mathscr{L}_X$ to be the extension of $\mathscr{L}$, where we add a constant symbol for every element in X.

Definition 4.22. Let M be an $\mathscr{L}$-structure and let $X \subseteq \mathtt{dom}(M)$. Moreover, let $p \subseteq \mathfrak{Form}^\infty(\mathscr{L}_X)$ such that for every $\varphi \in p$:

$$\mathtt{FV}(\varphi) \subseteq \{x_1, ..., x_n\}.$$

We call p an n-type if $p \cup \mathtt{Th}_X(M)$ is satisfiable. We say that p is a complete type if for every $\varphi \in \mathfrak{Form}^\infty(\mathscr{L}_X)$ either $\varphi \in p$ or $\neg\varphi \in p$. Otherwise we say that p is a partial n-type. Finally, we let $\mathbb{S}_n^M(X)$ be the collection of all the complete n-types over X.

Definition 4.23. Let p be an n-type over X. For every $k < n$, we let the k-type $p|_k$ to be:

$$p|_k := \{\varphi \in p \mid \mathtt{FV}(\varphi) \subseteq \{x_1, ..., x_k\}\}$$

Definition 4.24. Let p be an n-type over X. We say that the $\mathscr{L}_X$-structure M satisfies p if there exists a tuple $(m_1, ..., m_n) \in \mathtt{dom}(M)^n$ such that:

$$M \vDash \varphi(m_1, ..., m_n) \text{ for every } \varphi(x_1, ..., x_n) \in p.$$

If not such tuple exists, we say that M omits the type p. Moreover, we say that p is finitely satisfied by M if for every finite subset $\Sigma \subseteq p$, there exists a tuple $(m_{\Sigma,1}, ..., m_{\Sigma,n}) \in \mathtt{dom}(M)^n$ that realizes Σ.

Definition 4.25. A set $p \subseteq \mathfrak{Form}^{\infty}(\mathscr{L}_X)$ is said to be an ω-type if there exists a sequence $(p_n)_{n\in\mathbb{N}}$ such that:

i p_n is an n-type for every $n \in \mathbb{N}$,

ii $p_n \subseteq p_{n+1}$ for every $n \in \mathbb{N}$,

iii $p = \bigcup_{n\in\mathbb{N}} p_n$.

The ω-type is finitely realized by the $\mathscr{L}$-structure M if every p_k is finitely realized by M. Moreover an ω-sequence $(m_n)_{n\in\mathbb{N}} \in \mathtt{dom}(M)^{\omega}$ realizes p if for every $k \in \omega$, the subsequence $(m_0, ..., m_k)$ realizes p_k.

Definition 4.26. Let κ be a cardinal and M be a $\mathscr{L}$-structure. We say that M is a κ^{∞}-saturated model if for every set $X \subseteq \mathtt{dom}(M)$ with $|X| < \kappa$ and every n-type p over X the following property is satisfied:

$$\text{If } (M, x)_{x\in X} \text{ finitely satisfies } p, \text{ then } (M, x)_{x\in X} \text{ satisfies } p.$$

Proposition 4.27. Let M be an $\aleph_0^{\infty}$-saturated $\mathscr{L}$-structure. Moreover, let p be a complete $(n+1)$-type. If p is finitely satisfied by M and there exists a tuple $\bar{a} \in \mathtt{dom}(M)^n$ that satisfies $p|_n$, then there exists some $a_{n+1} \in \mathtt{dom}(M)$ that satisfies $p(\bar{a}, x_{n+1})$ in $(M, \bar{a})$.

Proof. Since M is $\aleph_0^{\infty}$-saturated, it suffices to show that $p(\bar{a}, x_{n+1})$ is finitely realized by $(M, \bar{a})$. Thus, take any finite subset $\Sigma \subseteq p(\bar{x}, x_{n+1})$ and let:

$$\psi(\bar{x}) = \exists x_{n+1} \left(\bigwedge_{\sigma(\bar{x}, x_{n+1})\in\Sigma} \sigma(\bar{x}, x_{n+1}) \right)$$

Since $p(\bar{x}, x_{n+1})$ is a complete type, $\psi(\bar{x}) \in p(\bar{x}, x_{n+1})$, otherwise $\neg\psi(\bar{x}) \in p(\bar{x}, x_{n+1})$ and this will contradict the fact that M finitely realizes $p(\bar{x}, x_{n+1})$.

Since $\psi(\bar{x}) \in p(\bar{x}, x_{n+1})$ and $FV(\psi(\bar{x})) \subseteq \bar{x}$, we can infer that $\psi(\bar{x}) \in p(\bar{x})|_n$. By assumption $\bar{a}$ realizes $p(\bar{x})|_n$, hence:

$$M \vDash \psi(\bar{a}).$$

Therefore there is some $m \in \mathtt{dom}(M)$ that realizes Σ. Since the choice of Σ is arbitrary, we conclude that $(M, \bar{a})$ finitely realizes $p(\bar{a}, x_n)$. By the $\aleph_0^{\infty}$-saturated nature of M we can find some $a_{n+1} \in M$ that realizes $p(\bar{a}, x_{n+1})$. □

Proposition 4.28. Let $\mathscr{L}$ be a signature such that $|\mathscr{L}| \leq \aleph_0$ and let p be an ω-type that is finitely satisfied by an $\mathscr{L}$-structure M. Then p can be extended to a complete ω-type q that is finitely satisfied by M.

Proof. Note that by assumption $|\mathfrak{Form}^{\infty}(\mathscr{L})| = \aleph_0$. Let $\{\varphi_i \mid i \in \mathbb{N}\}$ be an enumeration of $\mathfrak{Form}^{\infty}(\mathscr{L})$. Now, we construct a sequence of ω-types $(q_i)_{i\in\omega}$ such that $q_0 \subseteq ... \subseteq q_i \subseteq q_{i+1} \subseteq ...$ as follows:

- $q_0 = p$
- $q_{i+1} = q_i \cup \{\varphi_i\}$ if M finitely satisfies $q_i \cup \{\varphi_i\}$ and we let $q_{i+1} = q_i \cup \{\neg\varphi_i\}$ otherwise.

We now show by induction on i that M finitely realizes q_i. Note that by our assumption M finitely realizes q_0. Now, suppose for sake of contradiction that $q_i \cup \{\varphi_i\}$ and $q_i \cup \{\neg\varphi_i\}$ are not finitely realized by M. Then, without loss of generality, we can find two finite subsets $\Sigma_0, \Sigma_1 \subseteq q_i$ such that for every $\sigma_0 \in \Sigma_0$ and $\sigma_1 \in \Sigma_1$:

$$\mathtt{FV}(\sigma_0) = \mathtt{FV}(\sigma_1) = \mathtt{FV}(\varphi_i) = \bar{x}$$

and moreover:

$$M \vDash \forall \bar{x}(\bigwedge_{\sigma_0 \in \Sigma_0} \sigma_0(\bar{x}) \rightarrow \neg\varphi_i(\bar{x}))$$
$$M \vDash \forall \bar{x}(\bigwedge_{\sigma_1 \in \Sigma_1} \sigma_1(\bar{x}) \rightarrow \varphi_i(\bar{x})).$$

Since Σ_0 and Σ_1 are finite, so is $\Sigma := \Sigma_0 \cup \Sigma_1$. Since by the inductive hypothesis q_i is finitely satisfiable,we can find some $\bar{m} \in \mathtt{dom}(M)^k$ such that:

$$M \vDash \bigwedge_{\sigma \in \Sigma} \sigma(\bar{m})$$

Leading us into a contradiction. Thus, M finitely satisfies $q_i \cup \{\varphi_i\}$ or $q_i \cup \{\neg\varphi_i\}$ and hence q_{i+1} is finitely satisfiable by M. Finally if we let $q = \bigcup_{i \in \mathbb{N}} q_i$, it is clearly a complete type and is, as we have shown, finitely satisfiable by M. $\square$

Proposition 4.29. Let M be a $\aleph_0^\infty$-saturated model. Moreover, suppose that p is a complete ω-type. If M finitely satisfies p, then there exists some $(m_n)_{n \in \mathbb{N}} \in \mathtt{dom}(M)^\omega$ that realizes p.

Proof. Let p be a complete ω-type and and let $(p_n)_{n \in \mathbb{N}}$ be the sequence of types as in Definition 4.25. We construct a sequence $(m_n)_{n \in \mathbb{N}}$ by induction on n such that for every $i \in \mathbb{N}$, $(m_1, ..., m_i)$ realizes p_i.

Since p is a complete type, so is p_1. Moreover, since M finitely realizes p, M also finitely satisfies p_1. By the $\aleph_0^\infty$-saturated nature of M, there exists some $m_1 \in M$ that realizes p_1. Now suppose that $(m_1, ..., m_n)$ realize p_n. Since M finitely realizes p, M finitely satisfies p_{n+1}. If we combine these facts with Proposition 4.27, we can find some m_{n+1} that realizes $p_{n+1}(m_1, ..., m_n, x_{n+1})$. Thus $(m_0, ..., m_{n+1})$ realizes p_{n+1}.

Therefore, by construction $(m_n)_{n \in \mathbb{N}}$ realizes p. $\square$

4.4 The Hennessy-Milner property in ML^{∞}.

We conclude Chapter 4 by proving the Hennessy-Milner property for the class of finite Kripke models and the class of $\aleph_0^{\infty}$-saturated Kripke models. To achieve this goal we first need to embed the Kripke semantics into FO^{∞}-semantics. Therefore the first part of this section will be devoted to give a clear introduction of how to achieve this. We then conclude this chapter by proving the results stated above.

Definition 4.30. Given a set of propositional variables Φ we define the signature $\mathscr{L}_\Phi := \{P \mid p \in \Phi\} \cup \{R\}$, where P is a unary relation and R is a binary relation. Therefore every Kripke model (over Φ) $\mathscr{M} = (W, R, V)$ is interpreted as an $\mathscr{L}_\Phi$-structure where:

- $\mathtt{dom}(\mathscr{M}) := W$,
- $P^{\mathscr{M}} := V(p)$ for every $p \in \Phi$,
- $R^{\mathscr{M}} := \{(w, v) \in \mathtt{dom}(\mathscr{M})^2 \mid wRv\}$

***Remark* 4.31.** In what remains of section we will fix the signature $\mathscr{L}_\Phi$ of an arbitrary but fixed set of propositional variables Φ. Therefore we let $\mathfrak{Form}^{\infty}$ to be $\mathfrak{Form}^{\infty}(\mathscr{L}_\Phi)$.

Definition 4.32. We define the standard translation $\mathtt{ST}_x : \mathrm{ML}^{\infty} \to \mathfrak{Form}^{\infty}$ recursively as follows:

- $\mathtt{ST}_x(p) := P(x)$ for every $p \in \Phi$,
- $\mathtt{ST}_x(\bot) := (x \neq x)$,
- $\mathtt{ST}_x(\neg\varphi) := \neg\mathtt{ST}_x(\varphi)$,
- $\mathtt{ST}_x(\varphi \wedge \hat{\psi}) := \mathtt{ST}_x(\varphi) \wedge \mathtt{ST}_x(\hat{\psi})$,
- $\mathtt{ST}_x(\Diamond\varphi) := \exists y(R(x, y) \wedge \mathtt{ST}_x(\varphi))$,
- $\mathtt{ST}_x(\Diamond^{\infty}\varphi) := \exists^{\infty} y(R(x, y) \wedge \mathtt{ST}_x(\varphi))$.

Theorem 4.33. For any pointed Kripke model $(\mathscr{M}, w)$ and any $\varphi \in \mathrm{ML}^{\infty}$ the following two conditions hold:

i $\mathscr{M}, w \Vdash \varphi \iff \mathscr{M} \vDash \mathtt{ST}_x(\varphi)[w]$,

ii $\mathscr{M} \Vdash \varphi \iff \mathscr{M} \vDash \forall x \mathtt{ST}_x(\varphi)$.

Proof. It can be easily proven by a simultaneous induction on the formula $\hat{\varphi}$. $\square$

Definition 4.34. A Kripke model $\mathscr{M} = (W, R, V)$ is image finite if for every $w \in W$ the set $R[w]$ is finite.

We now have enough information to prove the first of our results:

Theorem 4.35. Let $\mathbb{C}$ be the class of finite pointed Kripke models. Then for every $(\mathscr{M}, w), (\mathscr{M}', w') \in \mathbb{C}$ the following holds:

$$\mathscr{M}, w \;\underline{\leftrightarrow}^{\infty}\; \mathscr{M}', w' \text{ if and only if } \mathscr{M}, w \;\equiv^{\infty}\; \mathscr{M}', w'.$$

The rest of the section is concerned with the proof of our second result:

Proof. Since $(\mathscr{M}, w), (\mathscr{M}', w')$ are two finite-image Kripke models notice that the following holds: $\mathscr{M}, w \;\underline{\leftrightarrow}^{\infty}\; \mathscr{M}', w'$ if and only if $\mathscr{M}, w \;\underline{\leftrightarrow}\; \mathscr{M}', w'$ and $\mathscr{M}, w \equiv \mathscr{M}', w'$ if and only if $\mathscr{M}, w \;\equiv^{\infty}\; \mathscr{M}', w'$. In view of theorem 2.24 in [BRV02], we conclude that $\mathscr{M}, w \;\underline{\leftrightarrow}^{\infty}\; \mathscr{M}', w'$ if and only if $\mathscr{M}, w \;\equiv^{\infty}\; \mathscr{M}', w'$. □

Definition 4.36. Let $\mathscr{M}$ be a Kripke model. For any world $w \in W$ and any infinite set A of successors of w satisfying the same ML$^{\infty}$-formulas, i.e. $A \subseteq \{v \in W \mid wRv\}$ and any natural number $n \in \mathbb{N}$ we let:

$$\Sigma^A_n := \{\mathtt{ST}_{x_n}(\varphi) \mid \varphi \in \mathrm{ML}^{\infty} \text{ and } \mathscr{M}, a \Vdash \varphi\} \cup \{R(w, x_n)\}$$

Moreover, we let $\Sigma^A := \bigcup\limits_{n \in \omega} \Sigma^A_n \cup \{x_i \neq x_j \mid i \neq j\}$.

Proposition 4.37. Let $(\mathscr{M}_0, w_0)$ and $(\mathscr{M}_1, w_1)$ be two pointed $\aleph_0^{\infty}$-saturated Kripke models. Suppose that $\mathscr{M}_0, w_0 \;\equiv^{\infty}\; \mathscr{M}_1, w_1$. If $A_0 \subseteq R_0[w_0]$ is an infinite set of modally equivalent worlds, then the ω-type $\Sigma^A[w_1/w_0]$ is finitely satisfied by $(\mathscr{M}_1, w_1)$.

Proof. Firstly, define Γ to be:

$$\Gamma := \{\varphi \mid \varphi \in \mathrm{ML}^{\infty} \text{ such that } \mathscr{M}_0, a \Vdash \varphi \text{ for every } a \in A\}.$$

Claim 1: For every finite subset $\Delta \subseteq \Gamma$, the set:

$$Y := \{v \in W_1 \mid w_1 R v \text{ and } \mathscr{M}_1, v \Vdash \delta \text{ for every } \delta \in \Delta\}$$

is infinite.

Claim proof: Fix an arbitrary finite set $\Delta \subseteq \Gamma$ and let $\delta := \bigwedge_{\varphi \in \Delta} \varphi$. Note that every $a \in A$ makes δ true, therefore $\mathscr{M}_0, w_0 \Vdash \Diamond^{\infty}\delta$. Since by assumption $\mathscr{M}_0, w_0 \;\equiv^{\infty}\; \mathscr{M}_1, w_1$, we conclude that $\mathscr{M}_1, w_1 \Vdash \Diamond^{\infty}\delta$. Therefore Y is an infinite set. ■
Therefore every finite $\Delta \subseteq \Gamma$ is realized in $(\mathscr{M}_1, w_1)$. Combining Theorem 4.33 with the previous claim we infer that $\Sigma^A[w_1/w_0]$ is finitely satisfiable in $(\mathscr{M}_1, w_1)$. □

Corollary 4.38. Let $(\mathscr{M}_0, w_0)$ and $(\mathscr{M}_1, w_1)$ be two pointed $\aleph_0^\infty$-saturated Kripke models such that $\mathscr{M}_0, w_0 \ \equiv^\infty \ \mathscr{M}_1, w_1$. If $A \subseteq R_0[w_0]$ is an infinite set of modally equivalent worlds, then there exists an infinite set $B \subseteq R_1[w_1]$ of modally equivalent worlds such that for every $a \in A$ and $b \in B$:

$$\mathscr{M}_0, a \ \equiv^\infty \ \mathscr{M}_1, b.$$

Where R_0 and R_1 are the accessibility relations in $\mathscr{M}_1$ and $\mathscr{M}_2$ respectively.

Proof. In view of Proposition 4.37, $(\mathscr{M}_1, w_1)$ finitely realizes $\Sigma^A[w_1/w_0]$. Moreover, by Proposition 4.28, we can extend $\Sigma[w_1/w_0]$ to a complete ω-type Γ that is finitely realized by $(\mathscr{M}_1, w_1)$. Finally, in view of Proposition 4.29, we can find an ω-sequence $(b_n)_{n\in\mathbb{N}} \in (W_1)^\omega$ that realizes Γ. If we let $B = \{b_i \mid i \in \mathbb{N}\}$, it suffices to show that for every $a \in A$ and every $b \in B$, $\mathscr{M}_0, a \ \equiv^\infty \ \mathscr{M}_1, b$. To prove so fix an arbitrary $a \in A$ and an arbitrary $b \in B$. Note that the standard translation of every ML^∞-formula that a satisfies is in Σ, then by definition b satisfies such ML^∞-formula. To prove the converse suppose that a does not make an ML^∞ formula φ true, then a satisfies $\neg\varphi$. By our previous argument b will satisfy $\neg\varphi$. Hence we conclude that $\mathscr{M}_0, a \ \equiv^\infty \ \mathscr{M}_1, b$. $\square$

Theorem 4.39. Let $\mathscr{M}_0$ and $\mathscr{M}_1$ be two $\aleph_0^\infty$-saturated Kripke models. For every $w_0 \in W_0$ and $w_1 \in W_1$:

$$\text{If } \mathscr{M}_0, w_0 \ \equiv^\infty \ \mathscr{M}_1, w_1, \text{ then } \mathscr{M}_0, w_0 \ \leftrightarroweq^\infty \mathscr{M}_1, w_1.$$

Proof. We show that `Duplicator` has a winning strategy in the $\mathtt{Bis}_n^\infty(\mathscr{M}_0, \mathscr{M}_1)$ game with initial configuration (w_0, w_1). This is shown by induction on the round and we will discuss, without loss of generality, the n-round of the game that starts with configuration $(\mathscr{M}_0, u_0; \mathscr{M}_1, u_1)$. Moreover, we will assume that the pointed Kripke model $(\mathscr{M}_0, u_0)$ is modally equivalent to $(\mathscr{M}_1, u_1)$ and that `Spoiler` decides to make a move on $\mathscr{M}_0$. Then we can consider two cases:

- Firstly, suppose that `Spoiler` moves the $\mathscr{M}_0$-pebble from u_0 to an R-successor element u_0^+. Then we let:

  $$\Sigma := \{\mathtt{ST}_x(\varphi) \mid \varphi \in \mathrm{ML}^\infty \text{ and } \mathscr{M}_0, u_0^+ \Vdash \varphi\}$$

  Since $u_0 \ \equiv^\infty \ u_1$, the 1-type $\Gamma := \Sigma \cup \{u_1 R_1 x\}$ is finitely satisfiable in $(\mathscr{M}_1, u_1)$. Invoking the $\aleph_0^\infty$-saturated property of $\mathscr{M}_1$, we can find some u_1^+ that realizes Γ. Clearly $u_1 R_1 u_1^+$ and $u_0^+ \ \equiv^\infty \ u_1^+$. Therefore `Spoiler` survives to the first order challenge.

- Secondly, suppose that `Spoiler` selects an infinite set $A \subseteq R_0[u_0]$. Such set A can be split into λ many disjoint and modally equivalent sets for some cardinal λ. Therefore $A := \biguplus_{\alpha<\lambda} A_\alpha$ where $A_\alpha \subseteq A$ is a modally equivalent collection of worlds.

  Claim 1: We claim that for every $\alpha < \lambda$, there exists a $B_\alpha \subseteq R_1[u_1]$ such that:

i For every $a \in A_\alpha, b \in B_\alpha : \mathscr{M}_0, a \equiv^\infty \mathscr{M}_1, b$,

ii $|A_\alpha| < \aleph_0$ if and only if $|B_\alpha| < \aleph_0$.

Claim proof: Fix an arbitrary $\alpha < \lambda$. Firstly suppose that A_α is a finite set. Since A_α is a modally equivalent set, we fix an arbitrary $a \in A_\alpha$. By the argument discussed on the previous point we can find some $b \in R_1[u_1]$ such that $\mathscr{M}_0, a \equiv^\infty \mathscr{M}_1, b$. If we let $B_\alpha = \{b\}$, we get that $|B_\alpha| < \aleph_0$ and for any $a \in A : \mathscr{M}_0, a \equiv^\infty \mathscr{M}_1, b$.

Secondly, suppose that A_α is an infinite set. In view of Corollary 4.38, there exists a set $B_\alpha \subseteq R_1[u_1]$ such that $|B_\alpha| \geq \aleph_0$ and for every $a \in A_\alpha, b \in B_\alpha : \mathscr{M}_0, a \equiv^\infty \mathscr{M}_1, b$.

Then we get that for every $\alpha < \lambda$, there exists a set $B_\alpha \subseteq R_1[u_1]$ satisfying *i* and *ii*. ■

Now, if we let $B := \bigcup_{\alpha<\lambda} B_\alpha$ Note that since A is infinite, so is B. Finally, for every $b \in B$ on which `Spoiler` might place the $\mathscr{M}_1$-pebble, `Duplicator` can reply by advancing the $\mathscr{M}_0$-pebble from u_1 to some $a \in A$ such that $\mathscr{M}_0, a \equiv^\infty \mathscr{M}_1, b$. Hence we conclude that `Duplicator` has a winning strategy on the ML^∞-bisimulation game.

□

Corollary 4.40. Let $\mathscr{M}_0$ and $\mathscr{M}_1$ be two $\aleph_0^\infty$-saturated Kripke models. For every $w_0 \in W_0$ and $w_1 \in W_1$:

$$\mathscr{M}_0, w_0 \equiv^\infty \mathscr{M}_1, w_1 \text{ if and only if } \mathscr{M}_0, w_0 \;\underline{\leftrightarrow}^\infty\; \mathscr{M}_1, w_1.$$

Proof. The left-to-right direction of the proof follows from Theorem 4.39. The left-to-righ direction follows from Lemma 3.11. □

ML^∞-Bisimulation Invariance Theorem.

We conclude the research on the expressiveness of the modal logic ML^∞ by showing an adaptation of the celebrated van Benthem Characterization theorem [Ben76] to the modal logic ML^∞. We showed at the end of Chapter 4 that the modal logic ML^∞ is a fragment of the predicate logic FO^∞ (see Theorem 4.33). Therefore, it is only natural to research into this direction in order to provide a full characterisation of ML^∞.

We soon noticed that the original proof by van Benthem could not be adapted to ML^∞. The failure of the Compactness Theorem (see Observation 2.11) was an obstacle that could not be avoided, therefore a different approach was needed to fulfill our goal. We then noticed that Rosen developed an alternative technique to prove the bisimulation preservation theorem (see [Ros97]). Unlike van Benthem's approach, Rosen's strategy does not rely on the Compactness Theorem in any way. In this chapter we follow an adaptation of Rosen's original proof and Goranko's and Otto's approach in [GO07].

Before sharing all the details of the proof, we would like to take the opportunity to provide some insight of the strategy we follow to achieve our last goal. As we already mentioned, our strategy is based on Rosen's technique. Such technique was developed in the framework of modal logic with finite signature. This restriction allows us to draw an equivalence between the concept of bisimulation and logical equivalence (see Theorem 5.4): However, the key point of Rosen's technique resides in the link he drew between the model theory of modal logic and the model theory of first order logic. He first observed that first order formulas that are bisimilar invariant also enjoy the property of being n-local for some $n \in \mathbb{N}$ (see Corollary 5.12), meaning that the truth value of such formula is preserved under pointed structures where Duplicator has a winning strategy in the Ehrenfeucht–Fraïssé game of length n (see Definition 4.17). Moreover, he managed to quantify how long such Ehrenfeucht-Fraïssé game based on the quantifier depth of the formula. Finally, he recognized that two pointed structures that are n-elementary equivalent are also n-bisimilar. Combining these three observations, Rosen managed to prove that preservation under bisimulation implies preservation under sufficiently large finite bisimulation.

We notice that these three properties do not vary when we move from the ML framework to the ML^{∞} framework. In order to give a clear description on why these observations are preserved from ML to ML^{∞}, consider a tree-like Kripke model. First notice that the $\Diamond^{\infty}$ modality is only concerned with the *horizontal* behaviour of the Kripke model, meaning that it can capture how wide certain branches of the Kripke model is. However, Rosen's observations are uniquely based on the length of the tree, on what we might think to be the *vertical* properties of the Kripke model:

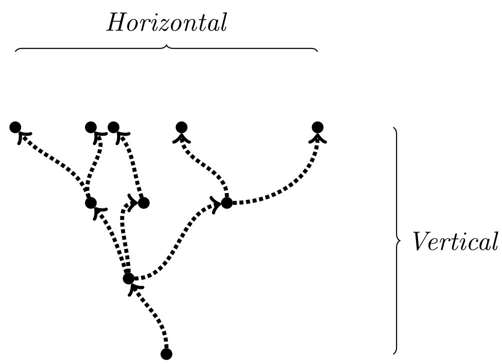


Figure 5.1: Figure representing the tree we described.

***Convention* 5.1.** In order to simplify the arguments of this chapter we will denote ML^{∞}-formulas by $\hat{\varphi}, \hat{\psi}, \hat{\chi}$ etc. On the contrary FO^{∞}-formulas will be denoted by φ, ψ, χ etc. Since we will be working with the language of ML^{∞} over a finite set of propositional variables, we fix an arbitrary finite set of propositional variables $\Phi := \{p_1, ..., p_n\}$. Moreover, in view of Remark 4.31 we will denote the set $\mathfrak{Form}^{\infty}(\mathscr{L}_{\Phi})$ by $\mathfrak{Form}^{\infty}$.

Proposition 5.2. For every $n \in \mathbb{N}$ the set ML^{∞}_n is finite up to logical equivalence.

Proof. In view of Proposition 4.33, we know that ML^{∞} is a fragment of $\mathfrak{Form}^{\infty}$. It is not difficult to check that for every $\hat{\varphi} \in \mathrm{ML}^{\infty}$: $\mathtt{md}(\hat{\varphi}) = \mathtt{qd}(\mathrm{ST}_x(\hat{\varphi}))$. Combining both results we conclude that for every $n \in \mathbb{N}$, ML^{∞}_n is a fragment of $\mathfrak{Form}^{\infty}_n$. It then follows from Lemma 4.19 that ML^{∞}_n is finite up to logical equivalence. $\square$

***Convention* 5.3.** We know by Proposition 5.2 that ML^{∞}_n is finite, up to logical equivalence, for every $n \in \mathbb{N}$. We then let $\{\hat{\varphi}_0, ..., \hat{\varphi}_n\}$ be an enumeration of the representatives of ML^{∞}_n.

Theorem 5.4. Let $(\mathscr{M}, w)$ and $(\mathscr{M}', w')$ be two pointed Kripke models. The following are equivalent:

i $\mathscr{M}, w \;\underline{\leftrightarrow}^{\infty}\; \mathscr{M}', w'$,

ii $\mathscr{M}, w \;\equiv^{\infty}\; \mathscr{M}', w'$.

Proof. The ($i \implies ii$) direction is shown by induction on the modal depth of the formulas. To prove the converse we first suppose that $\mathscr{M}, w \not\underline{\leftrightarrow}_n^{\infty} \mathscr{M}', w'$. Combining the winning strategy of `Spoiler` and Proposition 5.2 we can construct an ML$^{\infty}$ formula $\hat{\psi}$ with $\mathtt{md}(\hat{\psi}) \leq n$ such that $\mathscr{M}, w \Vdash \hat{\psi}$ and $\mathscr{M}', w' \nVdash \hat{\psi}$. □

Corollary 5.5. Let $\mathbb{C}$ be a collection of pointed Kripke models, let $\mathbb{C}'$ be a subcollection of $\mathbb{C}$ and $n \in \mathbb{N}$ be a natural number. The following are equivalent:

i For every $(\mathscr{M}', w') \in \mathbb{C}'$ and every $(\mathscr{M}, w) \in \mathbb{C}\backslash\mathbb{C}'$: $\mathscr{M}, w \not\underline{\leftrightarrow}_n^{\infty} \mathscr{M}', w'$.

ii There exists a formula $\hat{\psi} \in \mathrm{ML}_n^{\infty}$ such that $\mathbb{C}' = \{(\mathscr{M}, w) \mid \mathscr{M}, w \Vdash \hat{\psi}\}$.

Proof. Follows directly from Theorem 5.4. □

Lemma 5.6. Let $\mathscr{M}, \mathscr{M}'$ be two n-pseudotrees rooted at $w \in W$ and $w' \in W'$ respectively. If $\mathscr{M}, w \underline{\leftrightarrow}_n^{\infty} \mathscr{M}', w'$ then there exists two n-pesudotrees $\mathscr{N}$ and $\mathscr{N}'$ rooted at $v \in W_{\mathscr{N}}$ and $v' \in W_{\mathscr{N}'}$ such that:

i $\mathscr{M}, w \;\underline{\leftrightarrow}^{\infty}\; \mathscr{N}, v$,

ii $\mathscr{M}', w' \;\underline{\leftrightarrow}^{\infty}\; \mathscr{N}', v'$,

iii $\mathscr{N}|_{N_n(v)} = \mathscr{N}'|_{N_n(v')}$.

Proof. By assumption: $\mathscr{M}, w \underline{\leftrightarrow}_n^{\infty} \mathscr{M}', w'$. Therefore the two pointed Kripke models $(\mathscr{M}, w)$ and $(\mathscr{M}', w')$ are n-bisimilar in the standard modal logic sense i.e. $(\mathscr{M}, w) \underline{\leftrightarrow}_n (\mathscr{M}', w')$. In view of Definition 2.30 in [BRV02], we can find an $n+1$-sequence of binary relations $Z_0 \subseteq ... \subseteq Z_n \subseteq (N_n(w) \times N_n(w'))$ with the following properties (for $i+1 < n$):

(i) wZ_0w'.

(ii) If uZ_iu' then u and u' satisfy the same propositional variables.

(iii) If uZ_iu' and uRv, then there exists some v' with $u'R'v'$ and $vZ_{i+1}v'$.

(iv) If uZ_iu' and $u'R'v'$, then there exists some v with uRv and $vZ_{i+1}v'$.

Now, we define an $n+1$-sequence of sets $S_0 \subseteq ... \subseteq S_n \subseteq Z_n$ by induction:

- $S_0 := \{(w, w')\}$
- $S_{i+1} := S_i \cup \{(u, u') \in Z_n \mid \exists(v, v') \in N_i(vRu \text{ and } v'R'u')\}$.

The set S_n is a refinement of the n-bisimulation Z_n where we only take into account those tuples (u, u') in Z_n where the tuple formed from the predecesor worlds worlds $(u^-, u^{'-})$ also lie in Z_n.

Claim 1: The following two conditions hold in S_n:

- If $u \in N_n(w)$ then there exists some $u' \in N_n(w')$ such that uS_nu'.
- If $u' \in N_n(w')$ then there exists some $u \in N_n(w)$ such that uS_nu'.

Claim proof: Note that for every $u \in N_n(w)$ $(N_n(w'))$ there exists an R-path (R'-path) $w = w_0, ..., w_k$ of length at most n. Therefore in view of the observations (ii) and (iii) above the previous statement follows directly. ■

Claim 2: There exists a tree Kripke model $\mathscr{S}$ rooted at s such that:

- $\mathscr{M}, w \leftrightarroweq_n^\infty \mathscr{S}, s$,
- $\mathscr{M}', w' \leftrightarroweq_n^\infty \mathscr{S}, s$.

Claim proof: First, we equip the set S with an accessibility relation R_S and a valuation function V_S as follows:

- $(u, u')R_S(v, v') \Longleftrightarrow_{\text{Def}} uRv$ and $u'R'v'$,
- $(u, u') \in V_S(p) \Longleftrightarrow_{\text{Def}} u \in V(p)$ and $u' \in V'(p)$.

Thus, we define the model $\mathscr{S}$ to be the tuple (S_n, R_S, V_S). Note that since $\mathscr{M}$ and $\mathscr{M}'$ are two n-pseudotrees the binary relation R_S is acyclic and every element in W_Z has a unique predecessor. By the first claim we conclude that $S_n = R_S^*[(w, w')]$. Moreover, it follows from the first claim and the definitions of R_S and V_S that $\mathscr{M}, w \leftrightarroweq_n^\infty \mathscr{S}, (w, w')$ and $\mathscr{M}', w' \leftrightarroweq_n^\infty \mathscr{S}, (w, w')$. ■

Finally we will construct the Kripke models $\mathscr{N}$ and $\mathscr{N}'$ by extending $\mathscr{S}$ in two different ways. The strategy that we follow may remind the reader of the unravelling technique previously described. We extend $\mathscr{S}$ into $\mathscr{N}$ Kripke model by gluing a copy of the universe of the generate submodel $\mathscr{M}_u$ of every leaf (u, u') in S_n. Similarly we extend $\mathscr{S}$ to $\mathscr{N}'$ by attaching a copy of the universe of the generate submodel $\mathscr{M}'_{u'}$ for every leaf (u, u') in S_n.

Claim 3: There exists two n-pseudotrees $\mathscr{N}$ and $\mathscr{N}'$ rooted at z and z' such that:

i $\mathscr{M}, w \leftrightarroweq^\infty \mathscr{N}, z$,

ii $\mathscr{M}', w' \leftrightarroweq^\infty \mathscr{N}', z'$,

iii $\mathscr{N}|_{N_n(z)} = \mathscr{S} = \mathscr{N}'|_{N_n(z')}$.

Claim proof: As we already mentioned we will construct $\mathscr{N}$ and $\mathscr{N}'$ using a method that resembles the unravelling technique. Since the procedure is parallel in both cases we will only describe the method to construct $\mathscr{N}$ from $\mathscr{S}$. For sake of simplicity let $L_{\mathscr{S}}$ be the seat of leafs in $\mathscr{S}$. Now we define the set N to be:

$$N := S_n \cup \bigcup_{(l,l')\in L_{\mathscr{S}}} \{(u,(l,l')) \mid u \in W_l\}$$

where W_l is the universe of the submodel of $\mathscr{M}$ generated by the singleton $\{l\}$. Moreover, we let $\pi_0 : N \to W$ be the projection map, sending every tuple to its first element. We now equip the set N with a binary relation R_N and a valuation function V_N as follows:

- $(u,u')R_N(v,v')$ if one of the following situations is satisfied:
  - i If $(u,u'),(v,v') \in S_n$ and $(u,u')R_S(v,v')$.
  - ii If $(u,u') \in L_{\mathscr{S}}$, $v \in W_u$, $v' = (u,u')$ and uRv.
  - iii If $u' = v' = (x,x') \in L_{\mathscr{S}}$, $u,v \in W_x$ and uRv.
- $(u,u') \in V_N(p)$ if and only if $u \in V(p)$.

It can be easily verified that the projection map π_0 from $\mathscr{N} = (N,R_N,V_N)$ to $\mathscr{M}$ is a strong bounded morphism. Hence in view of Proposition 3.16 we can conclude that $\mathscr{N},(w,w') \leftrightarrows^{\infty} \mathscr{M},w$. Similarly, the n-neighbourhood of $\mathscr{N}$ is just S_n thus we infer that $\mathscr{N}|_{N_n(w,w')} = \mathscr{S}$. Obtaining our desired result ■

□

Example 5.7. To clarify the previous proof consider the following case. Let $\mathscr{M} = (W,R,V)$ and $\mathscr{M}' = (W',R',V')$ be the following Kripke models over $\Phi := \{p,q\}$:

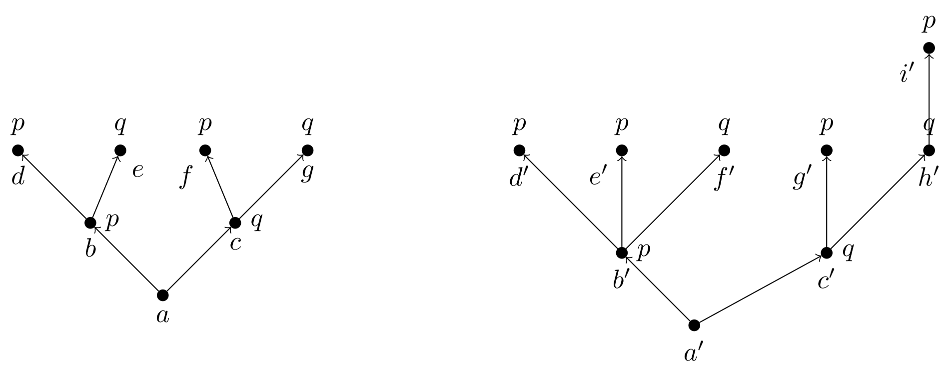


Figure 5.2: Caption

It is not difficult to see that $\mathscr{M},a \leftrightarrows_2^{\infty} \mathscr{M}',a'$. In fact the n-bisimulation relation $Z_2 \subseteq W \times W'$ can be represented by the following diagram:

(d, d') (d, e') (d, g') (e, f') (e, h') (f, d') (f, e') (f, g') (g, f') (g, h')

(b, b') (c, c')

(a, a')

Figure 5.3: The bisimulation Z

We then *clean* the set Z_2 to only take into account those tuples in Z_2 whose predecessors are a tuple in Z_2 as well. We denote such set to be S and we construct a Kripke model $\mathscr{S}$ in the following way:

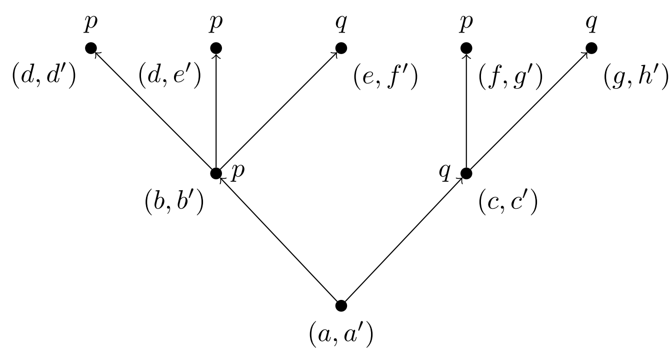


Figure 5.4: The *cleaned* Kripke model $\mathscr{S}$.

Finally we extend the Kripke model $\mathscr{Z}$ to the 2-peusotrees $\mathscr{N}$ and $\mathscr{N}'$. In the case of $\mathscr{N}$ we do not need to add any extra world because the projections of the leafs of $\mathscr{S}$ in $\mathscr{M}$ do not have any world that is not already present on $\mathscr{S}$. In the case of $\mathscr{N}'$, note that the projection of the tuple (g, h') in $\mathscr{M}'$ has a successor that has not been added to $\mathscr{S}$, i.e. the family of the projection of (g, h') in $\mathscr{M}'$ is a non-empty set. Hence we need to add this worlds to complete the Kripke model $\mathscr{N}'$:

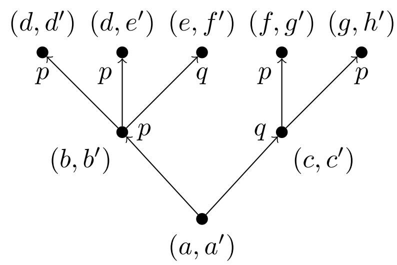


Figure 5.5: The 2-pseudotree $\mathscr{N}$

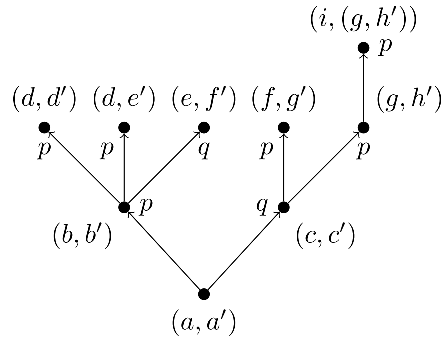


Figure 5.6: The 2-pseudotree $\mathscr{N}'$

Definition 5.8. Let $\varphi(x) \in \mathfrak{Form}^{\infty}$. We say that φ is ML$^{\infty}$-bisimulation invariant if for any two pointed Kripke models $(\mathscr{M}, w)$ and $(\mathscr{M}', w')$:

$$(\mathscr{M}, w \;\underline{\leftrightarrow}^{\infty}\; \mathscr{M}', w') \text{ implies that } (\mathscr{M} \vDash \varphi(w) \Longleftrightarrow \mathscr{M}' \vDash \varphi(w')).$$

Definition 5.9. Let $\varphi(x) \in \mathfrak{Form}^{\infty}$. We say that $\varphi(x)$ is FO$_n^{\infty}$-local if for any two pointed Kripke models $(\mathscr{M}, w)$ and $(\mathscr{M}', w')$ the following condition is satisfied:

$$\text{If } (\mathscr{M}|_{N_n(w)}, w \cong_n^{\infty} \mathscr{M}'|_{N_n(w')}, w'), \text{ then } (\mathscr{M} \vDash \varphi(w) \Longleftrightarrow \mathscr{M}' \vDash \varphi(w')).$$

Definition 5.10. Let $\mathscr{M}$ be a Kripke model. We define the Gaifman distance $\mathtt{gaif}(\cdot,\cdot) : W^2 \to \mathbb{N}$ as follows:

- $\mathtt{gaif}(w,w) = 0$ for every w,
- $\mathtt{gaif}(w,w') = 1$ given that $\mathscr{M} \vDash (wRw' \vee w'Rw)$,
- $\mathtt{gaif}(w,w') \leq n+1$ if and only if there is some v such that $\mathtt{gaif}(w,v) \leq n$ and $\mathtt{gaif}(v,w') = 1$,
- $\mathtt{gaif}(w,w') = n$ if and only if $\mathtt{gaif}(w,w') \geq n$ and $\mathtt{gaif}(w,w') \nleq n+1$.

Lemma 5.11. Let $\varphi(x) \in \mathfrak{Form}^{\infty}$ with $\mathtt{qd}(\varphi) = q$ be a ML$^{\infty}$-bisimilar invariant formula. Then for every Kripke model $\mathscr{M}$ rooted at $w \in W$ the following holds:

$$\mathscr{M} \vDash \varphi(w) \Longleftrightarrow \mathscr{N} \vDash \varphi(w),$$

where $\mathscr{N} = \mathscr{M}|_{N_n(w)}$ and $n = 2^q - 1$.

Proof. To prove this statement we will adapt the proof of Lemma 58 [GO07] to the ML$^{\infty}$ framework. Our first task then is to show the following statement:

$$\mathscr{M} \uplus \mathscr{O}, w \cong_q^\infty \mathscr{N} \uplus \mathscr{O}, w,$$

where $\mathscr{O}$ is the disjoint union of q-many copies of $\mathscr{M}$ and q-many copies of $\mathscr{N}$. In order to provide a clear proof we will introduce some notation that will make the argument easier to follow. Firstly, we will label the copies of $\mathscr{O}$ in the following way:

- $\mathscr{M} \uplus \mathscr{O} = \mathscr{M} \uplus \biguplus_{i \leq q} \mathscr{M}_i^{\mathscr{M}} \uplus \biguplus_{i \leq q} \mathscr{N}_i^{\mathscr{M}}$.
- $\mathscr{N} \uplus \mathscr{O} = \mathscr{N} \uplus \biguplus_{i \leq q} \mathscr{M}_i^{\mathscr{N}} \uplus \biguplus_{i \leq q} \mathscr{N}_i^{\mathscr{N}}$.

Moreover for each $X, Y \in \{\mathscr{M}, \mathscr{N}\}$ and $i \leq q$ we label the universe of the Kripke model X_i^Y in $\mathscr{O}$ as follows:

- $W_{X_i^Y} := \{(w, Y, i) \mid w \in W_X\}$.

Lastly, we define $\pi_{\mathscr{M}} : \mathscr{M} \uplus \mathscr{O} \to \mathscr{M} \uplus \mathscr{N}$ and $\pi_{\mathscr{N}} : \mathscr{N} \uplus \mathscr{O} \to \mathscr{M} \uplus \mathscr{N}$ to be the projection maps such that for every $X \in \{\mathscr{M}, \mathscr{N}\}$ and any $w \in X \uplus \mathscr{O}$:

$$\pi_X(w) := \begin{cases} w & \text{If } w \in X, \\ v & \text{If } w = (v, Y, i) \text{ for some } i \leq q \text{ and } Y \in \{\mathscr{M}, \mathscr{N}\}. \end{cases}$$

As previously mentioned, we will first show that $\mathscr{M} \uplus \mathscr{O}, w \cong_q^\infty \mathscr{N} \uplus \mathscr{O}, w$ by adapting the proof of Lemma 58 in [GO07]. To do so we will show that `Duplicator` has a winning strategy in $\mathtt{EF}_n^\infty(\mathscr{M}, \mathscr{N})$ with initial configuration (w, w).

Claim 1: `Duplicator` has a winning strategy in $\mathtt{EF}_n^\infty(\mathscr{M} \uplus \mathscr{O}, \mathscr{N} \uplus \mathscr{O})@(w, w)$.
Claim proof: We show by induction on the round number k that the partial map constructed from the sequences $(w = m_0, ..., m_k; w = n_0, ..., n_k)$ of the worlds that have been selected by `Spoiler` and `Duplicator` forms a local isomorphism satisfying the following conditions:

i $\mathtt{gaif}(m_i, m_j) \geq 2^{q-k} \iff \mathtt{gaif}(n_i, n_j) \geq 2^{q-k}$

ii If $\mathtt{gaif}(m_i, m_j) < 2^{q-k}$, then $\mathtt{gaif}(n_i, n_j) = \mathtt{gaif}(m_i, m_j)$.

iii If $\mathtt{gaif}(n_i, n_j) < 2^{q-k}$, then $\mathtt{gaif}(n_i, n_j) = \mathtt{gaif}(m_i, m_j)$.

Note that the base case follows immediately. Therefore we will only deal with the inductive case. Suppose that $(w = m_0, ..., m_{k-1}; w = n_0, ..., n_{k-1})$ is the ordered sequence containing the elements that `Spoiler` and `Duplicator` have chosen in the first k rounds of the game. Moreover suppose that conditions (i)-(iii) are satisfied by this sequence. Then the $k+1$-round of the game starts and `Spoiler` chooses one of the two movements that are allowed. For sake of simplicity we will assume that he decides to make a movement in the $\mathscr{M} \uplus \mathscr{O}$ structure.

If `Spoiler` makes a first order move, `Duplicator` must respect the critical distance of 2^{q-k}. Meaning that if the chosen world m_k by `Spoiler` lies within distance 2^{q-k} from an already pebbled element i.e. m_i, then `Duplicator` must respect that distance and choose the world n_k such that:

- $\mathtt{gaif}(m_i, m_k) = \mathtt{gaif}(n_i, n_k)$
- m_k and n_k satisfy the same atomic formulas.

If on the contrary the world m_k chosen lies further than the distance 2^{q-k} from every already pebbled element in $\mathscr{M} \uplus \mathscr{O}$, then `Spoiler` answers by selecting the world m_k of an isomorphic copy of $\mathscr{M}$ or $\mathscr{N}$ that has not been pebbled yet. For a further explanation of this strategy please see Lemma 58 in [GO07].

Now suppose that he makes a second order move and selects an infinite subset X of $\mathscr{M} \uplus \mathscr{O}$. For every $i < k$, we define the set X_i to be:

$$X_i := \{x \in X \mid \mathtt{gaif}(x, m_i) < 2^{q-k}\}.$$

Intuitively, X_i captures all the elements in X that lie within a distance 2^{q-k} of m_i. Therefore for each $i < k$ and each $x \in X_i$ we can find a unique copy of x, denoted by y_x, in the universe of $\mathscr{N} \uplus \mathscr{O}$ such that:

- $\pi_{\mathscr{M}}(x) = \pi_{\mathscr{M}}(y_x)$,
- $\mathtt{gaif}(m_i, x) = \mathtt{gaif}(n_i, y_x)$,

Since the element y_x is located at the same distance from n_i as x is from m_i and x and y_x satisfy the same propositional variables we infer that:

$$\mathscr{M} \vDash \alpha(m_0, ..., x) \iff \mathscr{N} \vDash \alpha(n_0, ..., y_x) \text{ for every atomic formula } \alpha.$$

We let Y_i be the collection of the worlds y_x:

$$Y_i := \{y \in W_{\mathscr{N} \uplus \mathscr{O}} \mid \exists! x \in X_i(\pi_{\mathscr{M}}(x) = \pi_{\mathscr{N}}(y) \text{ and } \mathtt{gaif}(m_i, x) = \mathtt{gaif}(n_i, y))\}.$$

Note that since we have respected the critical distance of 2^{q-k} with respect to the already pebbled elements the following conditions hold:

- $|X_i| = |Y_i|$
- $|X_i \cap X_j| = |Y_i \cap Y_j|$

for every $i, j < k$. Therefore there exists a 1-1 map between $\bigsqcup_{i<k} X_i$ and Y. Next we take the collection of worlds in X that lie further than a distance 2^{q-k} from any already pebbled element m_i, i.e. $X \setminus \bigcup_{i<k} X_i$ and partition it into two disjoint sets. On one hand we will collect all those worlds that are in an isomorphic copy of $\mathscr{M}$ and on the other hand we collect those worlds that lie in an isomorphic copy of $\mathscr{N}$:

- $X^{\mathscr{M}} := \{x \in X \setminus \bigcup_{i \in k} X_i \mid \pi_{\mathscr{M}}(x) \in W_{\mathscr{M}}\}$,

- $X^{\mathscr{N}} := \{x \in X \setminus \bigcup_{i \in k} X_i \mid \pi_{\mathscr{M}}(x) \in W_{\mathscr{N}}\}$

Since only $k-1$-many rounds have been completed so far, there are at least $q-(k-1)$ many isomorphic copies of $\mathscr{M}$ and $\mathscr{N}$ in $\mathscr{N} \uplus \mathscr{O}$ that have not been pebbled yet. We will assume, without loss of generality, that $\mathscr{M}_0^{\mathscr{N}}$ and $\mathscr{N}_0^{\mathscr{N}}$ are two isomorphic copies of $\mathscr{M}$ and $\mathscr{N}$ that have not been pebbled yet. Moreover, we define the functions:

- $\eta_{\mathscr{M}} : X^{\mathscr{M}} \to \mathscr{M}_0^{\mathscr{N}}$ where $\eta_{\mathscr{M}}(x) := (\pi_{\mathscr{M}}(x), \mathscr{M}, 0)$.
- $\eta_{\mathscr{N}} : X^{\mathscr{N}} \to \mathscr{N}_0^{\mathscr{N}}$ where $\eta_{\mathscr{N}}(x) := (\pi_{\mathscr{N}}(x), \mathscr{N}, 0)$.

Now we define $Y_{\mathscr{M}}$ to be image of the map $\eta_{\mathscr{M}}$. In a similar way we define $Y_{\mathscr{N}}$ to be image of the map $\eta_{\mathscr{N}}$:

- $Y^{\mathscr{M}} = \eta_{\mathscr{M}}[X^{\mathscr{M}}]$.
- $Y^{\mathscr{N}} = \eta_{\mathscr{N}}[X^{\mathscr{N}}]$.

Since every world x in $X^{\mathscr{M}} \cup X^{\mathscr{N}}$ cannot be reached from any already pebbled world m_i by the R accessibility relation R we conclude that:

$$\mathscr{M} \vDash \alpha(m_0, ..., x) \Longleftrightarrow \mathscr{N} \vDash \alpha(n_0, ..., \eta_Z(x))$$

for every $Z \in \{\mathscr{M}, \mathscr{N}\}$, $x \in X^Z$ and atomic formula α. Note that by construction for every element y in $Y^{\mathscr{M}}$ ($Y^{\mathscr{N}}$) the set $\eta_{\mathscr{M}}^{-1}(y)$ ($\eta_{\mathscr{N}}^{-1}(y)$) has at most q-many elements. Therefore $X^{\mathscr{M}}$ ($X^{\mathscr{N}}$).

Finally let $Y := \bigcup_{i \in k} Y_i \cup Y^{\mathscr{M}} \cup Y^{\mathscr{N}}$. Since X is infinite, at least $\bigcup_{i \in k} X_i, X^{\mathscr{M}}$ or $X^{\mathscr{N}}$ must be finite. Then by our previous argument $\bigcup_{i \in k} Y_i, Y^{\mathscr{M}}$ or $Y^{\mathscr{M}}$ is infinite, hence Y is an infinite set. Moreover for any $y \in Y$ that `Spoiler` chooses `Duplicator` can find a unique $x \in X$ such that the map $f_k : (m_0, ..., x) \to (n_0, ..., y)$ is a local isomorphism. Therefore `Duplicator` has a winning strategy in $\mathtt{EF}_n^\infty(\mathscr{M} \uplus \mathscr{O}, \mathscr{N} \uplus \mathscr{O})@(w, w)$. ■

In view of Theorem 4.20, we infer that $\mathscr{M} \uplus \mathscr{O} \equiv_q^\infty \mathscr{N} \uplus \mathscr{O}$. We now have all the necessary information to show our desired result:

$$\begin{aligned} \mathscr{M} \vDash \varphi(w) &\Longleftrightarrow \mathscr{M} \uplus \mathscr{O} \vDash \varphi(w) && \text{(Lemma 3.15 and Lemma 3.11)} \\ &\Longleftrightarrow \mathscr{N} \uplus \mathscr{O} \vDash \varphi(w) && (\mathscr{M} \uplus \mathscr{O} \equiv_q^\infty \mathscr{N} \uplus \mathscr{O}) \\ &\Longleftrightarrow \mathscr{N} \vDash \varphi(w) && \text{(Lemma 3.15 and Lemma 3.11)} \end{aligned}$$

□

Corollary 5.12. Let $\varphi(x) \in \mathfrak{Form}^\infty$ be a ML^∞-bisimilar invariant formula with $\mathtt{qd}(\varphi) = q$. Then $\varphi(x)$ is FO_n^∞-local for $n = 2^q - 1$.

Proof. Let $(\mathscr{M}, w)$ and $(\mathscr{M}', w')$ be two arbitrary pointed Kripke models such that $\mathscr{M}|_{N_n(w)}, w \cong_n^\infty \mathscr{M}'|_{N_n(w')}, w'$. Now, let $\mathscr{N} = \mathscr{M}_{\{w\}}$ and $\mathscr{N}' = \mathscr{M}'_{\{w'\}}$ be the generated submodels of $\mathscr{M}$ and $\mathscr{M}'$ respectively. Since $\mathscr{M}|_{N_n(w)}, w \cong_n^\infty \mathscr{M}'|_{N_n(w')}, w'$ it follows that $\mathscr{N}|_{N_n(w)}, w \cong_n^\infty \mathscr{N}'|_{N_n(w')}, w'$. Finally, consider the following derivation:

$$\begin{aligned}
\mathscr{M} \vDash \varphi(w) &\iff \mathscr{N} \vDash \varphi(w) && \text{(Lemma 3.15 and Lemma 3.11)}\\
&\iff \mathscr{N}|_{N_n(w)} \vDash \varphi(w) && \text{(Lemma 5.11)}\\
&\iff \mathscr{N}'|_{N_n(w')} \vDash \varphi(w') && (\text{Since } \mathscr{N}|_{N_n(w)}, w \cong_n^\infty \mathscr{N}'|_{N_n(w')}, w')\\
&\iff \mathscr{N}' \vDash \varphi(w') && \text{(Lemma 5.11)}\\
&\iff \mathscr{M}' \vDash \varphi(w'). && \text{(Lemma 3.15 and Lemma 3.11)}
\end{aligned}$$

Therefore $\varphi(x)$ is a FO_n^∞-local formula. □

Theorem 5.13. Let $\varphi(x) \in \mathfrak{Form}^\infty$. Then the following are equivalent:

i $\varphi(x)$ is ML^∞-bisimilar invariant.

ii $\varphi \equiv \mathtt{ST}_x(\hat{\varphi})$ for some $\hat{\varphi} \in \mathrm{ML}^\infty$.

Proof. The $(ii \implies i)$ direction falls directly from Proposition 4.33 and Lemma 3.11. Conversely suppose that (i) holds. Note that in view of Lemma 5.5 it suffices to show that $\varphi(x)$ is ML_n^∞-bisimulation invariant formula.

In view of corollary 5.12, $\varphi(x)$ is FO_n^∞-local. Now, consider any two Kripke pointed models $(\mathscr{M}, w)$ and $(\mathscr{M}', w')$ such that $\mathscr{M}, w \,\underline{\leftrightarrow}_n^\infty\, \mathscr{M}', w'$. In view of Lemma 3.15 $\mathscr{M}, w \,\underline{\leftrightarrow}^\infty\, \mathscr{N}, w$ and $\mathscr{M}', w' \,\underline{\leftrightarrow}^\infty\, \mathscr{N}', w'$, where $\mathscr{N}$ ($\mathscr{N}'$) is the submodel of $\mathscr{M}$ ($\mathscr{M}'$) generated by $\{w\}$ ($\{w'\}$).

Now let $\mathfrak{N}_n[w]$ and $\mathfrak{N}'_n[w']$ be the n-unravelling of $\mathscr{N}$ and $\mathscr{N}'$ along w and w' respectively. In view of Proposition 3.16 and Lemma 3.15 we conclude that $\mathfrak{N}_n[w], (w) \,\underline{\leftrightarrow}^\infty\, \mathscr{N}, w$ and $\mathfrak{N}'_n[w'], (w') \,\underline{\leftrightarrow}^\infty\, \mathscr{N}', w'$. Combining this result with our assumption we conclude that $\mathfrak{N}_n[w], (w) \,\underline{\leftrightarrow}_n^\infty\, \mathfrak{N}'_n[w'], (w')$. Finally, in view of Lemma 5.6, there are two n-pseudotrees $\mathfrak{U}$, $\mathfrak{U}'$ rooted at $\mathfrak{u} \in W_{\mathfrak{U}}$, $\mathfrak{u}' \in W_{\mathfrak{U}'}$ such that:

- $\mathfrak{U}, \mathfrak{u} \,\underline{\leftrightarrow}^\infty\, \mathfrak{N}_n[w], (w)$,
- $\mathfrak{U}', \mathfrak{u} \,\underline{\leftrightarrow}^\infty\, \mathfrak{N}'_n[w'], (w')$,
- $\mathfrak{U}|_{N_n(\mathfrak{u})} = \mathfrak{U}'|_{N_n(\mathfrak{u}')}$.

Since $\varphi(x)$ is an FO_n^∞-local formula by Corollary 5.12 and $\mathfrak{U}|_{N_n(w)} = \mathfrak{U}'|_{N_n(w)}$ implies that $\mathfrak{U}|_{N_n(w)} \cong_n^\infty \mathfrak{U}'|_{N_n(w)}$ we infer that:

$$\mathfrak{U} \vDash \varphi(\mathfrak{u}) \iff \mathfrak{U}' \vDash \varphi(\mathfrak{u}').$$

Therefore:

$$\mathscr{M} \vDash \varphi(w) \iff \mathscr{M} \vDash \varphi(w').$$

Hence $\varphi(x)$ is a ML^{∞}_{n}-bisimulation invariant formula. In view of Corollary 5.5 we can find some $\hat{\varphi} \in \mathrm{ML}^{\infty}$ such that $\varphi \equiv \mathtt{ST}_x(\hat{\varphi})$. $\square$

***Remark* 5.14.** In the following diagram we provide a more visual interpretation of the strategy we have followed to show the $(ii \implies i)$ direction of the previous theorem:

$$\begin{array}{ccccccccc}
\mathscr{M}, w & \underline{\leftrightarrow}^{\infty} & \mathscr{N}, w & \underline{\leftrightarrow}^{\infty} & \mathfrak{N}_n[w], (w) & \underline{\leftrightarrow}^{\infty} & \mathfrak{U}, \mathfrak{u} & \underline{\leftrightarrow}^{\infty}_{n} & \mathfrak{U}|_{N_n(\mathfrak{u})}, \mathfrak{u} \\
\\
\updownarrow^{\infty}_{n} & & \updownarrow^{\infty}_{n} & & \updownarrow^{\infty}_{n} & & \updownarrow^{\infty}_{n} & & \cong^{\infty}_{n} \\
\\
\mathscr{M}', w' & \underline{\leftrightarrow}^{\infty} & \mathscr{N}, w' & \underline{\leftrightarrow}^{\infty} & \mathfrak{N}'_n[w'], (w') & \underline{\leftrightarrow}^{\infty} & \mathfrak{U}', \mathfrak{u}' & \underline{\leftrightarrow}^{\infty} & \mathfrak{U}|_{N_n(\mathfrak{u})}, \mathfrak{u}'
\end{array}$$

Figure 5.7: Strategy

Soundness and completeness.

We conclude this master thesis by exploring the finite axiomatizability of the modal logic ML^∞. We start this chapter by introducing the concept of ML^∞-normal modal logic, a variation of the normal modal logic definition, where we take into account the interplay between the $\Diamond$ and the $\Diamond^\infty$ definition. After introducing the usual concepts of soundness and completeness.

We first noticed that a direct consequence of the failure of the Compactness Theorem for the modal logic ML^∞ (see Observation 2.11) was the impossibility to apply the usual strategy of Canonical Models to prove the completeness result. Therefore we adapt the finitary method developed by Fischer and Ladner [FL79] in the context of PDL to the ML^∞ framework. This method is based on the filtration of the Canonical models over a finite set of formulas. We conclude this section by proving the Truth Lemma over the ML^∞ semantics.

In the second section of this chapter we introduce the ML^∞-normal modal logic $\mathrm{S5}^\infty$, a supplement of the well-known modal logic S5 in the ML^∞ context. We first prove that under this fragment of ML^∞, the issue that is highlighted in Observation 2.27 does not occur. Therefore we show that combining the blooming technique with the Truth Lemma obtained for the ML^∞-semantics we can show that $\mathrm{S5}^\infty$ is weakly complete with respect to the class of pointed Kripke models whose accessibility relation R is an equivalence relation.

6.1 Basic concepts.

Definition 6.1. A set $\Lambda \subseteq \mathrm{ML}^\infty$ is a ML^∞-normal modal logic if it contains the following axioms:

All propositional tautologies.	**(Taut)**
$\Box(p \to q) \to (\Box p \to \Box q)$.	**(K)**
$\Box^\infty(p \to q) \to (\Box^\infty p \to \Box^\infty q)$.	**(K$^\infty$)**
$\Box p \leftrightarrow \neg\Diamond\neg p$.	**(Dual)**
$\Box^\infty p \leftrightarrow \neg\Diamond^\infty\neg p$.	**(Dual$^\infty$)**
$\Box p \to \Box^\infty p$.	**(Dist)**

Moreover it is closed under the following rules:

If $\varphi, \varphi \to \psi \in \Lambda$ then $\psi \in \Lambda$.	**(MP)**
If $\varphi \in \Lambda$ then $\varphi[\theta/p] \in \Lambda$ for every $p \in \Phi$ and $\theta \in \mathrm{ML}^\infty$.	**(Subs)**
If $\varphi \in \Lambda$ then $\Box\varphi \in \Lambda$.	**(N)**

We define the smallest ML^∞-normal modal logic to be the logic K^∞.

Definition 6.2. Let Λ be a ML^∞-normal modal logic and let $\varphi_0, ...\varphi_n, \psi$ be a collection of ML^∞-formulas. It is said that ψ is Λ-deducible from $\varphi_0, ...\varphi_n$ if:

$$(\bigwedge_{i \leq n} \varphi_i \to \psi) \in \Lambda.$$

Moreover for every (possibly infinite) set $\Gamma \subseteq \mathrm{ML}^\infty$ set. We say that ψ is Λ-deducible from Γ, denoted by $\Gamma \vdash_\Lambda \psi$ if there exists a finite set of formulas $\{\theta_0, ..., \theta_n\} \subseteq \Gamma$ such that ψ is Λ-deducible from $\theta_0, ..., \theta_n$. Furthermore Γ is Λ-consistent if $\Gamma \nvdash_\Lambda \bot$.

***Remark* 6.3.** For any $\Gamma, \Delta \in \mathfrak{P}(\mathrm{ML}^\infty)$ and any $\varphi_0, ..., \varphi_n, \psi \in \mathrm{ML}^\infty$, we will make use of the following notation:

- $\vdash_\Lambda \psi \iff \emptyset \vdash_\Lambda \psi$,
- $\Gamma \vdash_\Lambda \Delta \iff \Gamma \vdash_\Lambda \psi$ for every $\psi \in \Delta$,
- $\varphi_0, ..., \varphi_n \vdash \psi \iff \{\varphi_0, ..., \varphi_n\} \vdash_\Lambda \psi$,
- $\Gamma, \Delta \vdash_\Lambda \psi \iff \Gamma \cup \Delta \vdash_\Lambda \psi$.

Definition 6.4. Let $\mathbb{F}$ be a collection of Kripke frames. A ML^∞-normal modal logic Λ is sound with respect to $\mathbb{F}$ if:

$$\Lambda \subseteq \{\varphi \in \mathrm{ML}^\infty \mid \mathbb{F} \Vdash \varphi\}.$$

Definition 6.5. A ML^{∞}-normal modal logic Λ is strongly complete with respect to a class of frames $\mathbb{F}$ if for every $\Gamma \subseteq \mathrm{ML}^{\infty}$ and any $\varphi \in \mathrm{ML}^{\infty}$:

$$\text{If } \Gamma \Vdash_{\mathbb{F}} \varphi \text{ then } \Gamma \vdash_{\Lambda} \varphi.$$

Definition 6.6. A ML^{∞}-normal modal logic Λ is weakly complete with respect to a class of frames $\mathbb{F}$ if:

$$\{\varphi \in \mathrm{ML}^{\infty} \mid \mathbb{F} \Vdash \varphi\} \subseteq \Lambda.$$

Definition 6.7. For any ML^{∞}-formula φ we define $\sim\varphi$ to be θ in the case that φ is the negation of the ML^{∞}-formula θ. Otherwise we define it to be $\neg\varphi$. Moreover, for every set X of ML^{∞}-formulas, we let $\sim X$ to be the smallest set containing X that is closed under the following rule:

$$\text{If } \varphi \in X \text{ then } \sim\varphi \in X.$$

Remark **6.8.** By a straight-forward combinatorial argument it can be shown that for every finite set X of ML^{∞}-formulas, the set $\sim X$ is finite as well.

Definition 6.9. For any set $X \subseteq \mathrm{ML}^{\infty}$ we define the closure of X, denoted by $\mathtt{CL}(X)$ to be the smallest set containing X that is subformula closed and closed under the following rule:

$$\text{If } \Diamond^{\infty}\varphi \in \mathtt{CL}(X) \text{ then } \Diamond\varphi \in \mathtt{CL}(X).$$

Remark **6.10.** It can be easily verified by induction that the closure of every ML^{∞}-formula is finite. Similarly, the closure of every finite set is finite.

Definition 6.11. Let Λ be a ML^{∞}-normal modal logic and let $\Sigma \subseteq \mathrm{ML}^{\infty}$ be a set of formulas. A set $A \subseteq \sim\mathtt{CL}(\Sigma)$ is a Λ-atom over Σ if A is a maximal Λ-consistent set over $\sim\mathtt{CL}(\Sigma)$. We denote the collection of Λ-atom over Σ as $\mathrm{At}_{\Lambda}(\Sigma)$.

Proposition 6.12. For any atom $A \in \mathrm{At}_{\Lambda}(\Sigma)$ the following properties are satisfied:

i For every $\varphi, \varphi \to \psi \in \sim\mathtt{CL}(\Sigma)$: If $\varphi, \varphi \to \psi \in A$ then $\psi \in A$,

ii $(\Lambda \cap \sim\mathtt{CL}(\Sigma)) \subseteq A$,

iii For any $\varphi \in \sim\mathtt{CL}(\Sigma)$: Either $\varphi \in A$ or $\sim\varphi \in A$,

iv For any $\varphi \vee \psi \in \sim\mathtt{CL}(\Sigma)$: $(\varphi \vee \psi) \in A \iff \varphi \in A$ or $\psi \in A$.

Proof. Follows immediately by adapting the proof of Lemma 4.81 in [BRV02] to the framework of ML^{∞}. $\square$

Proposition 6.13. For any Λ-consistent formula $\varphi \in {\sim}\mathtt{CL}(\Sigma)$ there exists some $A \in \mathrm{At}_\Lambda(\Sigma)$ such that $\varphi \in A$.

Proof. Follows from an adaptation of Lemma 4.83 in [BRV02] to the ML^∞ framework. $\square$

Definition 6.14. For any ML^∞-normal modal logic Λ and any finite set of formulas $\Sigma \subseteq \mathrm{ML}^\infty$, we define the tuple $\mathcal{M}_{\Lambda,\Sigma} = (W_{\Lambda,\Sigma}, R_{\Lambda,\Sigma}, R^\infty_{\Lambda,\Sigma}, V_{\Lambda,\Sigma})$ where:

- $W_{\Lambda,\Sigma} := \mathrm{At}_\Lambda(\Sigma)$,
- $R_{\Lambda,\Sigma} := \{(A, B) \in W^2_{\Lambda,\Sigma} \mid (\hat{A} \wedge \Diamond \hat{B}) \text{ is } \Lambda\text{-consistent}\}$,
- $R^\infty_{\Lambda,\Sigma} := \{(A, B) \in W^2_{\Lambda,\Sigma} \mid (\hat{A} \wedge \Diamond^\infty \hat{B}) \text{ is } \Lambda\text{-consistent}\}$,
- $V_{\Lambda,\Sigma}(p) := \{A \in W_{\Lambda,\Sigma} \mid p \in A\}$ for any $p \in \Phi$.

By the **(Dist)** axiom in Definition 6.1 it is easy to see that $R^\infty_{\Lambda,\Sigma} \subseteq R_{\Lambda,\Sigma}$. Therefore for any ML^∞-normal modal logic Λ and any finite set $\Sigma \subseteq \mathrm{ML}^\infty$ $\mathcal{M}_{\Lambda,\Sigma}$ is an ML^∞-Kripke model.

Lemma 6.15. For any $A \in W_{\Lambda,\Sigma}$ and any $\varphi \in \mathrm{ML}^\infty$. If $\Diamond\varphi \in A$, then there exists some $B \in W_{\Lambda,\Sigma}$ such that $\varphi \in B$ and $AR_{\Lambda,\Sigma}B$.

Proof. Combining the adaptation of Lemma 4.86 in [BRV02] to the ML^∞ framework and Definition 6.1 we obtain the desired result. $\square$

Lemma 6.16. For any $A \in W_{\Lambda,\Sigma}$ and any $\varphi \in \mathrm{ML}^\infty$. If $\Diamond^\infty\varphi \in A$, then there exists some $B \in W_{\Lambda,\Sigma}$ such that $\varphi \in B$ and $AR^\infty_{\Lambda,\Sigma}B$.

Proof. By a straightforward adaptation of Lemma 6.15 to the $\Diamond^\infty$ modality. $\square$

Lemma 6.17. Let Λ be any normal modal logic and $\Sigma \subseteq \mathrm{ML}^\infty$ be a finite set. Then for any $A \in W_{\Lambda,\Sigma}$ and any $\varphi \in {\sim}\mathtt{CL}(\Sigma)$:

$$\varphi \in A \iff \mathcal{M}_{\Lambda,\Sigma}, A \Vdash^\infty \varphi.$$

Proof. Follows by a straightforward induction on the formula, where the only interesting cases are the ones concerned with the modalities. However these cases are a direct corollary of Lemma 6.15 and Lemma 6.16. $\square$

6.2 The ML$^\infty$-normal modal logic S5$^\infty$

Definition 6.18. The ML$^\infty$-normal modal logic $\mathbf{S5}^\infty$ is the smallest ML$^\infty$-normal modal logic containing the following axioms:

$$
\begin{aligned}
&p \to \Diamond p, && \textbf{(T)}\\
&\Diamond\Diamond p \to \Diamond p, && \textbf{(4)}\\
&\Diamond\Box p \to p, && \textbf{(B)}\\
&\Diamond^\infty p \to \Box\Diamond^\infty p. && \textbf{(VB)}
\end{aligned}
$$

Definition 6.19. We let S5$^\infty$ be the collection of all Kripke frames whose accessibility relation R is an equivalence relation, i.e. reflexive, transitive and symmetric.

Lemma 6.20. The ML$^\infty$-normal modal logic S5$^\infty$ is sound with respect to $\mathbb{S}5$.

Proof. The proof that every Kripke frames satisfies all the axioms of the ML$^\infty$-normal modal logic K$^\infty$ is given in Lemma 6.29. Therefore, in view of Page 193 in [BRV02] it only suffices to show that $\mathbb{S}5$ satisfies the **(VB)** axiom. However this can be easily checked since R is an equivalence relation. $\Box$

Proposition 6.21. Let $\Sigma \subseteq$ ML$^\infty$ be a finite set. For any atom $A \in W_{\mathrm{S5}^\infty,\Sigma}$ the following properties hold:

i For every $\Box\varphi \in {\sim}\mathtt{CL}(\Sigma)$: If $\Box\varphi \in A$, then $\varphi \in A$,

ii For every $\Diamond\Diamond\varphi \in {\sim}\mathtt{CL}(\Sigma)$: If $\Diamond\Diamond\varphi \in A$, then $\Diamond\varphi \in A$,

iii For every $\Diamond\Box\varphi \in {\sim}\mathtt{CL}(\Sigma)$: If $\Diamond\Box\varphi \in A$, then $\varphi \in A$,

iv For every $\Box\Diamond^\infty\varphi \in {\sim}\mathtt{CL}(\Sigma)$: If $\Diamond^\infty\varphi \in A$, then $\Box\Diamond^\infty\varphi \in A$.

Proof. (i-iv) is proved similarly. Therefore we will only focus on (iv). Suppose for sake of contradiction that $\Diamond^\infty\varphi \in A$ but $\Box\Diamond^\infty\varphi \notin A$. In view of Proposition 6.12 Part iii, we infer that ${\sim}\Box\Diamond^\infty\varphi \in A$. Therefore we obtain that:

$$\vdash_{\mathrm{S5}^\infty} \hat{A} \to {\sim}\Box\Diamond^\infty\varphi.$$

On the other hand, by the **(VB)** axiom we obtain that:

$$\vdash_{\mathrm{S5}^\infty} \hat{A} \to \Box\Diamond^\infty\varphi.$$

Leading us into a contradiction. Therefore $\Box\Diamond^\infty\varphi \in A$. $\Box$

Corollary 6.22. Let $\Sigma \subseteq \mathrm{ML}^\infty$ be a finite set and any $A, B, C \in W_{\mathrm{S5}^\infty,\Sigma}$ be any atoms. The following properties hold:

i $AR_{\mathrm{S5}^\infty,\Sigma}A$,

ii $AR_{\mathrm{S5}^\infty,\Sigma}B$ if and only if $BR_{\mathrm{S5}^\infty,\Sigma}A$,

iii If $AR_{\mathrm{S5}^\infty,\Sigma}B$ and $BR_{\mathrm{S5}^\infty,\Sigma}C$, then $AR_{\mathrm{S5}^\infty,\Sigma}C$,

iv If $AR_{\mathrm{S5}^\infty,\Sigma}B$ and $BR^\infty_{\mathrm{S5}^\infty,\Sigma}C$, then $AR^\infty_{\mathrm{S5}^\infty,\Sigma}C$.

Proof. (i-iv) follow automatically from clauses (i-iv) in Proposition 6.21. □

***Remark* 6.23.** Note that Corollary 6.22 (iv) is a weaker result that the one we obtained in Proposition 2.23. However as we see in the following Proposition, this condition is strong enough to avoid the node issue that we discussed in Observation 2.27.

Proposition 6.24. Let $\Sigma \subseteq \mathrm{ML}^\infty$ be a finite set and $A, B, C \in W_{\mathrm{S5}^\infty,\Sigma}$ be any three atoms. If $AR_{\mathrm{S5}^\infty,\Sigma}B$ and $CR^\infty_{\mathrm{S5}^\infty,\Sigma}B$, then $AR^\infty_{\mathrm{S5}^\infty,\Sigma}B$.

Proof. Suppose that $AR_{\mathrm{S5}^\infty,\Sigma}B$ and $CR^\infty_{\mathrm{S5}^\infty,\Sigma}B$. In view of Definition 6.14, $CR_{\mathrm{S5}^\infty,\Sigma}B$. Combining Corollary 6.22 (ii) and (iii) we infer that $AR_{\mathrm{S5}^\infty,\Sigma}C$. Since $AR_{\mathrm{S5}^\infty,\Sigma}C$ and $CR^\infty_{\mathrm{S5}^\infty,\Sigma}B$ and in view of Proposition 6.22 (iv) we conclude that $AR^\infty_{\mathrm{S5}^\infty,\Sigma}B$. □

Proposition 6.25. Let $\Sigma \subseteq \mathrm{ML}^\infty$ be a finite set and $(A, i) \in \underline{W_{\mathrm{S5}^\infty,\Sigma}}$ be a world in $\underline{\mathscr{M}_{\mathrm{S5}^\infty,\Sigma}}$. If there are infinitely many $(B, j) \in \underline{W_{\mathrm{S5}^\infty,\Sigma}}$ such that $(A, i)\underline{R_{\mathrm{S5}^\infty,\Sigma}}(B, j)$, then there exists an infinite set $X \subseteq \underline{R_{\mathrm{S5}^\infty,\Sigma}}[(A, i)]$ and some $C \in W_{\mathrm{S5}^\infty,\Sigma}$ such that:

i $\underline{\pi}[X] = \{C\}$.

ii $\underline{\pi}(A, i)R^\infty_{\mathrm{S5}^\infty,\Sigma}C$.

Proof. In view of Remark 6.8 and Remark 6.10 we infer that the set $R_{\mathrm{S5}^\infty,\Sigma}[\underline{\pi}(A, i)]$ is finite. By the assumption we know that the set $\underline{R_{\mathrm{S5}^\infty,\Sigma}}[(A, i)]$ is infinite and by Definition 2.25 we know $\underline{\pi}[\underline{R_{\mathrm{S5}^\infty,\Sigma}}[(A, i)]] \subseteq \underline{R_{\mathrm{S5}^\infty,\Sigma}}[(A, i)]$. Then there exists an infinite set $X \subseteq \underline{R_{\mathrm{S5}^\infty,\Sigma}}[(A, i)]$ and an element $C \in \underline{R_{\mathrm{S5}^\infty,\Sigma}}[(A, i)]$ such that $\underline{\pi}[X] = \{C\}$. Moreover note that this is only possible if C is the $R^\infty_{\mathrm{S5}^\infty,\Sigma}$ successor of some atom $D \in W_{\mathrm{S5}^\infty,\Sigma}$. However in view of Proposition 6.25 this implies that $\underline{\pi}(A, i)R^\infty_{\mathrm{S5}^\infty,\Sigma}C$. □

Lemma 6.26. For any finite set $\Sigma \subseteq \mathrm{ML}^\infty$, any $(A, i) \in \underline{W_{\mathrm{S5}^\infty,\Sigma}}$ and any $\varphi \in \mathrm{ML}^\infty$:

$$\mathcal{M}_{\mathrm{S5}^\infty,\Sigma}, A \Vdash^\infty \varphi \iff \underline{\mathscr{M}_{\mathrm{S5}^\infty,\Sigma}}, (A, i) \Vdash \varphi$$

Proof. This is shown by induction on the formula φ. The base case as well as the cases involved with the Boolean connectives can be easily proved by the definitions. The case that involve the $\Diamond$ modality follow directly from Lemma 6.15 and Definition 2.25. Therefore we will only discuss the $\Diamond^\infty$ case:

Suppose that $\mathcal{M}_{\mathrm{S5}^\infty,\Sigma}, A \Vdash^\infty \Diamond^\infty\varphi$. In view of Lemma 6.16 there exists some $B \in W_{\mathrm{S5}^\infty,\Sigma}$ such that $AR^\infty_{\mathrm{S5}^\infty,\Sigma}B$ and $\mathcal{M}_{\mathrm{S5}^\infty,\Sigma}, B \Vdash^\infty \varphi$. In view of the Inductive hypothesis, we conclude that for every $n \in \mathbb{N}$, $(A,i)\underline{R^\infty_{\mathrm{S5}^\infty,\Sigma}}(B,n)$ and $\underline{\mathcal{M}_{\mathrm{S5}^\infty,\Sigma}}, (B,n) \Vdash \varphi$. Hence we conclude that $\underline{\mathscr{M}_{\mathrm{S5}^\infty,\Sigma}}, (A,i) \Vdash \Diamond^\infty\varphi$.

Conversely suppose that $\underline{\mathscr{M}_{\mathrm{S5}^\infty,\Sigma}}, (A,i) \Vdash \Diamond^\infty\varphi$. By unfolding the semantics we get that there are infinitely many $(B,j) \in \underline{W_{\mathrm{S5}^\infty,\Sigma}}$ such that $(A,i)\underline{R_{\mathrm{S5}^\infty,\Sigma}}(B,j)$ and $\underline{\mathscr{M}_{\mathrm{S5}^\infty,\Sigma}}, (B,j) \Vdash \varphi$. Combining the Inductive hypothesis with Proposition 6.25 we can find some $C \in W_{\mathrm{S5}^\infty,\Sigma}$ such that $AR^\infty_{\mathrm{S5}^\infty,\Sigma}C$ and $\mathcal{M}_{\mathrm{S5}^\infty,\Sigma}, C \Vdash \varphi$. Hence $\mathcal{M}_{\mathrm{S5}^\infty,\Sigma}, A \Vdash^\infty \Diamond^\infty\varphi$. $\square$

Theorem 6.27. The ML^∞-normal modal logic $\mathbf{S5}^\infty$ is weakly complete with respect to $\mathbb{S}5$.

Proof. Consider any $\varphi \in \mathrm{ML}^\infty$ such that $\nvdash_{\mathrm{S5}^\infty} \varphi$. Therefore the ML^∞ formula $\neg\varphi$ is $\mathrm{S5}^\infty$-consistent. Let $\Sigma := \{\neg\varphi\}$. In view of Proposition 6.13 there exists some atom $A \in W_{\mathrm{S5}^\infty,\Sigma}$ such that $\neg\varphi \in A$. Moreover by Lemma 6.17 we conclude that $\mathcal{M}_{\mathrm{S5}^\infty,\Sigma}, A \Vdash^\infty \neg\varphi$. Finally, in view of Lemma 6.26 we conclude that $\underline{\mathscr{M}_{\mathrm{S5}^\infty,\Sigma}}, (A,i) \Vdash \neg\varphi$ for some $i \in \mathbb{N}$. Finally it just suffices to show that the accessibility relation $\underline{R_{\mathrm{S5}^\infty,\Sigma}}$ is reflexive, transitive and symmetric. However this follows directly from Corollary 6.22 and the fact that for any $(B,i),(C,j) \in \underline{W_{\mathrm{S5}^\infty,\Sigma}}$:

$$\underline{\pi}(B,i)R_{\mathrm{S5}^\infty,\Sigma}\underline{\pi}(C,j) \text{ if and only if } (B,i)\underline{R_{\mathrm{S5}^\infty,\Sigma}}(C,j).$$

$\square$

6.3 The ML^∞-normal modal logic $\mathbf{K}^\infty$

Definition 6.28. We let $\mathbb{K}$ to be the collection of all Kripke frames.

Lemma 6.29. The ML^∞-normal modal logic K^∞ is sound with respect to $\mathbb{K}$

Proof. In view of Page 193 in [BRV02] it only suffices to show that every Kripke frame validates the **(Dual)** and **($\mathbf{K}^\infty$)** axiom. However, it is not difficult to see that the $\Diamond^\infty$ modality distributes over the $\vee$ connective and that $\Box^\infty$ is the dual of the $\Diamond^\infty$ modality. $\square$

Theorem 6.30. The ML^∞-normal modal logic K^∞ is weakly complete with respect to $\mathbb{K}$.

Proof. Consider any $\varphi \in \mathrm{ML}^\infty$ such that $\nvdash_{\mathrm{K}^\infty} \varphi$. Let $\Sigma = {\sim}\mathtt{CL}(\{\neg\varphi\})$ and $n = qd(\neg\varphi)$. Combining Proposition 6.13 and Lemma 6.17 we can find an atom $A \in \mathrm{At}_{\mathrm{K}^\infty}(\Sigma)$ such that $\neg\varphi \in A$ and $\mathcal{M}_{\mathrm{K}^\infty,\Sigma}, A \Vdash^\infty \neg\varphi$. Secondly, let $\mathcal{N}$ be the weak ML^∞-Kripke submodel of $\mathcal{M}_{\mathrm{K}^\infty,\Sigma}$ generated by the singleton $\{A\}$. Combining the previous result with Lemma 3.15 and Lemma 3.12, we conclude that $\mathcal{N}, A \Vdash^\infty \neg\varphi$. The reader might have noticed that $\mathcal{N}$ is a ML^∞-Kripke model rooted at A. Following the unravelling technique described in Chapter 2, we let $\mathfrak{N}_n[A]$ be the n-unravelling of $\mathcal{N}$ around A. In view of Proposition 3.16 we infer that $\mathfrak{N}_n[A], (A) \Vdash^\infty \neg\varphi$. Since $\mathfrak{N}_n[A]$ is a finite model and $\underline{\pi}$ is a surjective map, we can find some $\alpha \in \underline{\mathfrak{W}_n[A]}$ such that $\underline{\pi}(\alpha) = (A)$. Finally by Lemma 2.29, we conclude that $\underline{\mathfrak{N}_n[A]}, \alpha \Vdash \neg\varphi$ and thus $\mathbb{K} \nVdash \varphi$. $\square$

Conclusion

In this thesis, we have introduced the modal logic ML^∞ and have provide a first line of research on the model theoretic and axiomatization aspects of this logic. The first four chapters of the thesis have been concerned with the model theoretic aspects of the modal logic ML^∞, while Chapter 6 has provided some contributions in the axiomatization of ML^∞.

In Chapter 2, we introduced two different Kripke semantics for the modal logic ML^∞. First, we defined the standard semantics based on Kripke models with a unique accessibility relation. Second, we introduced the ML^∞-Kripke models, an alternative version of the Kripke semantics for the modal logic ML^∞ based on structures equipped with two accessibility relations. Furthermore we introduced the blooming technique, a model theoretic method that allows us to transform ML^∞-Kripke models to standard Kripke models. In Chapter 3, we discussed the failure of the bisimulation invariance property and we proposed an alternative definition of bisimulation for the modal logic ML^∞. We concluded this chapter by recovering a significant amount of the preservation results. Chapter 4 provided an introduction to the basic syntactic and semantic concepts of FO^∞. In addition, we introduced the concept of κ^∞-saturation, a generalization of the well-known concept of saturation that arises in classical model theory. We concluded this chapter by proving, in the ML^∞ framework, that the class of $\aleph_0^\infty$-saturated Kripke models enjoys of the Hennessy-Milner property. In Chapter 5, we concluded our research on the model theoretic properties of ML^∞ by proving the bisimulation invariance result for the modal logic ML^∞. In particular we showed that ML^∞ is the fragment of FO^∞ that is invariant under ML^∞-bisimulation.

Chapter 6 provided an introduction to the concept of ML^∞-normal modal logic, and we introduced the ML^∞-normal modal logics K^∞ and $\mathrm{S5}^\infty$. Employing the blooming technique developed in Chapter 2, we proved that the logic $\mathrm{S5}^\infty$ is sound and weakly complete with respect to the class of equivalent Kripke models. In a second instance, we adapted the unravelling technique to the ML^∞-Kripke semantics and combining it with the blooming technique we showed that K^∞ is sound and weakly complete with respect to the class of Kripke frames.

Future work

As we mentioned in the introduction, this thesis provides a first analysis on the logic ML^{∞}. Some positive results concerning the model theoretic and axiomatizability properties of ML^{∞} have been provided throughout this thesis. Moreover the modal logic ML^{∞} opens up questions within this discipline and in relation to other areas of logic such as graded modal logic and model theory. In the following paragraphs we propose some directions in which the research could continue:

Correspondence. On Chapter 6 we show that the class of Kripke frames and the class of equivalence Kripke frames can be defined in ML^{∞}. A first line of research would be to expand these results and provide a definability of well-known Kripke frames (reflexive, reflexive and transitive etc.). Moreover, we believe that a question to address would be to adapt the important research done to the frame definability of standard modal logic (see [Ben93; GT75; Sah75; Fit73] to the ML^{∞} framework.

$\aleph_0^{\infty}$-saturation. In Chapter 4 we introduce $\aleph_0^{\infty}$-saturation, an extension of the classic model theoretic notion of saturation to the FO^{∞} framework. Unlike in the first order logic situation, in the FO^{∞} framework we cannot elementary embed every structure into a κ^{∞}-saturated structure. However it is still unknown if κ-saturated models can be elementary embedded into κ^{∞}-ones. Therefore we believe that a first line of research should be focused on the connections that can be drawn between κ^{∞}-saturation and κ-saturation.

In Theorem 4.39 we show that the class of $\aleph_0^{\infty}$-saturated Kripke models enjoy of the Hennessy-Milner property. However, as we stated on the previous paragraph, the existence of κ^{∞}-saturated Kripke models is still unknown. Therefore, we propose a second line of research that aims to show for which cardinal κ does the class of κ-saturated Kripke models satisfy the Hennessy-Milner property.

Alternative semantics. Algebraic semantics have been a prominent structure for interpreting logic since more than a century ago. This tradition led to an algebrization of modal logic by introducing the Boolean algebras with operators (BAO's). A prominent result on the algebraization of standard modal logic is the Jónsonn-Tarski Thoerem [JT51] that opened a door to study the axiomatization of modal logic from an algebraic perspective. Furthermore, topological semantics have been a more recent interpretation of logic. It was first observed by Stone [Sto36] the connections between algebraic and topological semantics of propositional logic. This result influenced the work of algebraists and topologists and was further exported onto other logics [Esa74; Gol74; Gol89].

We therefore propose a line of research in this direction, first by providing an algebraization of the modal logic ML^∞. Second by designing topological semantics for ML^∞ and in a later stage we propose to develop a duality relation between these two semantics. An important consideration that needs to be done is the failure of the Compactness Theorem in ML^∞. Recent techniques developed by Bezhanishvili and Henke [HB20] have allowed us to overcome the compactness failure of descriptive frames. In this sense, we believe that this technique could be applied to this framework to obtain a topological compactification of the modal logic ML^∞.

Lindström characterization. Characterizing logics based on their abstract model theoretic properties has been an important program within logic since Lindström first published the characterization of first order logic based on its abstract model theoretic properties [Lin69]. Since then, positive results have been obtained in the Lindström-style characterization of modal logics. First by de Rijke [De 95] when he showed that standard modal logic is the most expressive logic that enjoys the compactness property and the bisimulation invariance property. This result has been further improved by van Benthem et al. [TBV07] where they provide Lindström-style characterization for graded modal logic (see Theorem 3.14 in [TBV07]) and the binary guarded fragment (see Theorem 3.23 in [TBV07]). In this thesis we have shown that ML^∞ fails to satisfy the Compactness Theorem (see Observation 2.11) and is not invariant under bisimulation (see Observation 3.1). While we have not been able to provide a property that substitutes compactness we have shown that the modal logic ML^∞ is closed under ML^∞-bisimulation. Therefore we propose the following question: Can the modal logic ML^∞ be Lindström-style characterized in terms of the invariance under ML^∞-bisimulation?

Graded modal logic. Graded modal logic (GML) extends standard modal logic by introducing a series of modalities $\Diamond_n$ for every natural number n. These are modalities that enlarge the expressive power of standard modal logic by allowing the language to capture the number of finite successors that satisfy a certain property. However, the expressive power of this logic is stopped when we reach the infinity boundary. ML^∞ is more expressive than GML when we reach the infinity, but lacks of expressive power on the finite cases. Therefore we propose the modal logic GML^∞, the extension of GML where we add the $\Diamond^\infty$ modality to the language as an object of study, where interesting questions concerning the expressiveness, model theoretic properties can be considered.

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