DYNAMIC LOGICS FOR MODEL TRANSFORMATIONS

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

This thesis is concerned with internalizing model transformations as 'dynamic' modalities in modal logic. The model-theoretic operations discussed here are (strong) simulations, homomorphisms, and functional powersets. For each of these transformations, we will extend the basic modal language with a modality, which expresses truth along this transformation. We then provide a sound and complete axiomatization for the new logic. As a part of the axiomatization of the simulation modality, we also study the so-called validity modality, which expresses whether a formula is a validity in a given logic. It is shown how to obtain a sound and complete axiomatization for this logic of validity, given an axiomatization and refutation system for the underlying logic. Finally, we show how the preservation laws for (strong) simulations and homomorphisms can be derived syntactically inside the logic. To this end, we provide a new proof of Lyndon's positivity theorem in modal logic: a modal formula is preserved under homomorphisms iff it is equivalent to a positive existential formula.

Contents

1. Introduction

This thesis is concerned with internalizing several model transformations as modalities in modal logic. In each chapter, we will extend the basic modal language with a different modality, which expresses truth along a model-theoretic transformation. For each of these logics we investigate expressivity, axiomatizations, and preservation laws. In this chapter, we motivate the research in this thesis and put it in the context of related work. In addition, each chapter will contain a short section discussing the motivation for and the use of the particular modal language discussed in that chapter.

1.1 Motivation for the thesis

Preservation results have always been at the core of first-order model theory. A logical formula is preserved under a model-theoretic operation if whenever it holds in a model, it will also hold after applying the respective operation on the model. Preservation theorems syntactically define which formulas are preserved under the model-theoretic operation in question. For example, the Łos-Tarski preservation theorem states that a formula is preserved under submodels if and only if it is equivalent to a formula without existential quantifiers (Chang and Keisler, 1990). Other well-known operations for which preservation laws are known are unions of chains (the Chang-Łos-Suzko theorem) and surjective homomorphisms (Lyndon's theorem).

Since modal logic is a fragment of first-order logic, such preservation laws do also exist for modal logic. For example, a basic modal logic formula is preserved under submodels iff it is equivalent to a $\Box$-free formula, that is, a universal formula (De Rijke, 1993).

Modal logic also has a tradition of enriching the basic modal language with additional modalities, such as the universal modality or the difference operator. In the same way, it is possible to add a modality that expresses truth in a certain related model. More specifically, let R be a relation between pointed Kripke models. For example, we can take R to be the submodel relation or the homomorphism relation. We can then study the language $\mathcal{L}_{[R]}$ containing the modality $[R]\varphi$, whose semantics is given by

$$\mathfrak{M}, x \models [R]\varphi \text{ iff for all } \mathfrak{M}', x' \text{ such that} (\mathfrak{M}, x) R (\mathfrak{M}', x'), \text{ we have } \mathfrak{M}', x' \models \varphi$$

We say that the language $\mathcal{L}_{[R]}$ *internalizes* the relation R. One well-known example of such a language is Public Announcement Logic, or PAL for short (Plaza, 1989; Gerbrandy and Groeneveld, 1997). This logic is an extension of the basic modal language with an additional operator $[!\varphi]\psi$, which denotes that if we restrict the model only to worlds where φ holds, then ψ holds in this restricted model. PAL, together with some other examples, will be discussed in more detail in section 1.3 on related work. However, many of the important model-theoretic operations from first-order logic have not been internalized in modal logic yet.

In this thesis, we will consider some of these well-known operations on models and add them as modalities to the modal language. The operations that are investigated are simulations, homomorphisms, and functional powersets. For all these operations, a sound and complete axiomatization will be provided. Once we have added the modality to the language, the language can express properties of the model-theoretic relation in question. As mentioned above, preservation laws are important properties of a relation. Therefore, we use the axiomatization to provide a syntactic proof inside the language that all formulas with the respective property are indeed preserved. For example, chapter 5 is concerned with homomorphisms. In this chapter, it is shown that a formula is preserved under homomorphisms iff it

is logically equivalent to a positive existential formula. After axiomatizing the logic with the homomorphism modality, a syntactic proof is provided that all positive existential formulas are preserved under homomorphisms.

As mentioned above, three different modalities are being investigated. In chapter 5, we discuss a homomorphism modality. Homomorphisms occur all throughout mathematics and are a fundamental notion in model theory. Lyndon's theorem, which states that a formula is preserved under surjective homomorphisms iff it is equivalent to a positive formula (Lyndon et al., 1959), is, therefore, one of the three fundamental preservation results in model theory. However, an internalization of homomorphisms in modal logic has, as far as the author is aware, never been studied before. Therefore, it is logical to investigate this internalization and prove the preservation law in basic modal logic.

A key notion in modal logic is a bisimulation, which preserves all formulas in modal logic. A simulation is simply a bisimulation, where the back clause is omitted and only the truth of propositional variables is preserved. Homomorphisms are then simply functional simulations. Simulations have many applications, mostly in theoretical computer science. For example, they are used in process theory to state that one process *implements* another process. They are also used in database theory to describe that a database *conforms to* a database schema. Internalizing this modality allows us to express that a statement holds in all implementations of a process for example (Henzinger et al., 1995; Buneman et al., 1997). In theoretical computer science, two notions of simulations occur. Besides the simulations described above, it is often required that the simulation preserves the falsehood of the propositional variables as well. Such simulations will be called *strong* simulations. In chapter 4 both the logic of simulation as well as the logic of strong simulations are investigated and axiomatized.

However, to axiomatize the logic of simulations, the language must be able to express that a formula is valid on all models in a logic. Therefore, chapter 3 is devoted to an investigation of the validity operator, which expresses precisely this. Any axiomatization of a language that includes a modality that quantifies over model extensions will most likely require the validity operator as well. Hence, the logic of validity has the potential to be used in many different logics.

Finally, in chapter 6, an introduction to powerset models is given. Powerset models have applications in many different areas of logic, such as linguistics and philosophy. One field of application is inquisitive logic. The worlds in a powerset model are then called *information states* (Ciardelli, 2014) and used to model dialogues and questions. In addition, information states allow for more flexibility in the semantics. For example, there are at least three reasonable ways to define disjunction in powerset models (Aloni, 2016). Therefore, powerset operators have numerous applications in different branches of logic. In this chapter, only the functional powerset operator is axiomatized. Functional powerset models have been introduced as a new perspective on the possibility semantics in modal logic (Holliday, 2016; Van Benthem, 1999; Humberstone, 1981).

1.2 Contributions of this thesis

As described above, this thesis is concerned with languages that can express entailment along a relation. In this thesis, we will investigate the simulation relation, homomorphism relation, and the functional powerset relation. For the simulation relation, we first need to provide a sound and complete axiomatization for the validity operator: an operator that expresses that a formula is valid in all models of a given logic. Combining all this, the main contributions of the thesis are:

- A procedure of how to obtain a sound and complete axiomatization of the language with the validity operator given a sound and complete refutation system for any normal modal logic.
- A sound and complete axiomatization for the language with the simulation modality and a syntactic proof that all positive existential formulas are preserved under simulations. The same is done for the strong simulation modality.

- A proof that (surjective) homomorphisms preserve exactly the formulas that are logically equivalent to a positive existential formula.
- A sound and complete axiomatization for the language with a modality $f\varphi$, which denotes that φ holds after applying a homomorphism $\mathbf{f}$, which is specified in the semantics. Next, a new modality $[\mathcal{H}]$ is introduced, which quantifies over a set of admissible homomorphisms. The logic with this new modality is axiomatized and used to derive syntactically that all positive existential formulas are preserved.
- A sound and complete axiomatization for the functional powerset modality.

1.3 Related Work

Internalizing model-constructions as a modality in modal logic is not new. A well-known example is the aforementioned Public Announcement Logic. This logic contains the additional modality $[!\varphi]\psi$, which states that in the submodel defined by φ, ψ holds. More formally, let $\mathfrak{M} = (W, R, V)$ be a model. Then $\mathfrak{M}_\varphi = (W_\varphi, R_\varphi, V_\varphi)$ is given by $W_\varphi = \{w \in W \mid \mathfrak{M}, w \models \varphi\}$ and R_φ and V_φ are the restrictions of R, V to W_φ respectively. Then the definition of the public announcement operator is given by:

$$\mathfrak{M}, x \models [!\varphi]\psi \text{ iff } \mathfrak{M}, x \not\models \varphi \text{ or } \mathfrak{M}_\varphi, x \models \psi$$

PAL has numerous applications, mostly in Dynamic Epistemic Logic (DEL). In such logics, $\Box\varphi$ is read as "the agent knows that φ". In this case, $[!\varphi]\psi$ denotes that after a truthful public announcement of φ, ψ holds. For example, $[!p]Kp$ denotes that if I truthfully tell you that p holds, then you will know that p holds. Also, $[!p]p$ is a tautology, since all the announcements are truthful. Using this interpretation, PAL can be used to solve puzzles like the Muddy Children Puzzle or the Numbers Puzzle (Baltag and Renne, 2016). PAL has been axiomatized by reducing the language into basic modal logic and is therefore equally expressive as the basic modal logic.

Another popular operation that has been internalized is the product update. This operation was introduced as a generalization of PAL (Baltag et al., 1998; Baltag and Moss, 2004; Baltag and Renne, 2016). It can be used for modeling a wide range of informational actions, including private communications and lying. Such actions can be modeled by so-called *action models*. Given the model for the initial state, one can then perform the *product update operation* to obtain the model that depicts the state of matter after the informational action is performed. The corresponding logic is called EAL and a sound and complete axiomatization is given in Baltag et al. (1998) and Baltag and Moss (2004). See Van Ditmarsch et al. (2007) or Van Benthem (2011) for more information on DEL.

One other line of research that is closely related to the work in this thesis is the Logic of Abstraction (Baltag et al., 2017; Ilin, 2018). In this logic, the operation of taking quotients is internalized as a modality in the language. This modality then represents 'abstracting' away from unimportant facts in the possible world semantics. Closely related to this is the Logic of Questions (Van Benthem and Minică, 2012). This logic aims to model questions and issues, by introducing an issue relation to the model and an issue modality to the language.

A special case of the Logic of Abstraction is the Logic of Filtration, where the quotient is defined by a set of modal logic formulas. Let Σ be a finite set of modal formulas and $\mathfrak{M}_\Sigma$ be the filtration of a model $\mathfrak{M}$. Then we can define

$$\mathfrak{M}, x \models [\Sigma]\varphi \text{ iff } \mathfrak{M}_\Sigma, |x| \models \varphi$$

This modality in particular interesting in the context of this thesis, since filtrations have a well-known preservation law: the filtration theorem states that all modal formulas are preserved under filtrations. That is, $\models \varphi \leftrightarrow [\Sigma]\varphi$ for all $\varphi \in \Sigma$. The logic of filtration has been axiomatized using reduction laws into the basic modal language with the universal quantifier. In this way, it is possible to internalize the filtration theorem and obtain $\vdash \varphi \leftrightarrow [\Sigma]\varphi$ inside the logic (Ilin, 2018) for all $\varphi \in \Sigma$. This logic of

Filtration is further explored in Van Benthem and Bezhanishvili (2020). This piece gives an overview of existing work and directions of future research related to dynamic logics and filtration. In addition, the logic of filtration is compared and combined with other dynamic logics, such as PAL and a modal logic of issues.

1.4 Structure of the thesis

After introducing some preliminary notions in chapter 2, each chapter is concerned with a different modality. The validity operator is the topic of chapter 3. The sound and complete axiomatization of this logic of validity is then used in chapter 4 to axiomatize the logic of simulations. Next, we switch to the closely related notion of homomorphisms, which is the topic of chapter 5. Finally, chapter 6 is devoted to the (functional) powerset model. Each chapter starts with a short motivation for the modality in question together with an overview of related work.

2. Preliminaries

In this chapter, we will introduce and define the concepts needed in the rest of the thesis. This will mostly consist of introducing modal logic and the notation used in this text. See Blackburn et al. (2001) for a more extensive introduction.

2.1 Syntax and Semantics

We first fix a countably infinite set PROP of proposition letters. The proposition letters are denoted by $p, q, r, \ldots$.

Definition 2.1. Given a set PROP of propositional variables, the language $\mathcal{BML}$ (Basic Modal Language) is defined recursively by the following grammar:

$$\varphi := p \in \mathsf{PROP} \mid \bot \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid \Box\varphi$$

Furthermore, we define $\top = \neg\bot, (\varphi \vee \psi) = \neg(\neg\varphi \wedge \neg\psi), (\varphi \to \psi) = \neg\varphi \vee \psi$,$(\varphi \leftrightarrow \psi) = (\varphi \to \psi) \wedge (\psi \to \varphi)$ and $\Diamond = \neg\Box\neg$ as usual.

Now that we have defined the syntax of the language, let us turn to the semantics.

Definition 2.2. A *frame* $\mathfrak{F}$ is defined as a pair $\langle W, R\rangle$ where W is a set of worlds and $R \subseteq W \times W$ is a binary relation on W.

Definition 2.3. A *(Kripke) model* $\mathfrak{M}$ is defined as a pair $\langle W, R, V\rangle$ where $\langle W, R\rangle$ is a frame and $V : \mathsf{PROP} \to W$ a valuation.

Worlds are denoted by $w, v, \ldots$ and x, y, z. Given a model $\mathfrak{M} = \langle W, R, V\rangle$, we may write $x \in \mathfrak{M}$ to denote $x \in W$. Next, we define

$$R(x) = \{x \in W \mid (\exists y \in W)(xRy)\}.$$

Finally, sometimes we will abuse notation slightly and treat V as a function from W to PROP. Then $V(x)$ denotes the set of proposition letters that are hold at world x.

A pointed model is a model together with a point of evaluation. We can now define the satisfaction relation between the pointed models and the formulas in the basic modal language.

Definition 2.4. Given a model $\mathfrak{M} = \langle W, R, V\rangle$ and a world $x \in \mathfrak{M}$, The satisfaction relation $\models$ is defined inductively on the formulas of $\mathcal{BML}$ as follows

$$\begin{aligned}
\mathfrak{M}, x &\models p \text{ iff } w \in V(p)\\
\mathfrak{M}, x &\models \bot \text{ iff never}\\
\mathfrak{M}, x &\models \neg\varphi \text{ iff not } \mathfrak{M}, x \models \varphi\\
\mathfrak{M}, x &\models \varphi \wedge \psi \text{ iff } \mathfrak{M}, x \models \varphi \text{ and } \mathfrak{M}, x \models \psi\\
\mathfrak{M}, x &\models \Box\varphi \text{ iff for all } y \in R(x) \text{ we have } \mathfrak{M}, y \models \varphi
\end{aligned}$$

If we omit the model or world, we imply universal quantification over the missing item. For example, $\mathfrak{M} \models \varphi$ denotes that $\mathfrak{M}, x \models \varphi$ for all $x \in \mathfrak{M}$. Similarly $\models \varphi$ is true iff $\mathfrak{M} \models \varphi$ for all Kripke models

$\mathfrak{M}$. It is also convenient to extend the valuation from proposition letters to arbitrary formulas. We therefore we define

$$V(\varphi) = \{x \in W \mid \mathfrak{M}, x \models \varphi\}$$

Finally, we introduce the notion of logical consequence.

Definition 2.5. Let $\Phi \cup \{\varphi\}$ be a set of $\mathcal{BML}$-formulas and S a class of Kripke models. Then we write

$$\Phi \models_{\mathsf{S}} \varphi \text{ iff for all } \mathfrak{M} \in \mathsf{S} \text{ we have } \mathfrak{M} \models \Phi \text{ implies } \mathfrak{M} \models \varphi$$

2.2 Normal Modal Logics

Now that we have introduced the syntax and semantics of modal logics, we will now relate the two using the notion of a normal modal logic.

Definition 2.6. A normal modal logic L is a set of formulas that can be derived using the axioms and rules in table 2.1.

Table 2.1: Normal modal logic

1.	All propositional tautologies
2.	All substitution instances of axiom K: $\Box(p \to q) \to (\Box p \to \Box q)$
3.	*Modus ponens*: From φ and $\varphi \to \psi$, prove ψ
4.	*Generalization*: From φ, prove $\Box\varphi$

This definition is slightly different from Blackburn et al. (2001), in that is does not treat uniform substitution as a rule of normal modal logics. Since many of the logics considered in this thesis do not obey uniform substitution, we have chosen to include every substitution instance of the K-axiom instead. Given a logic L, we will write $\mathsf{L} \vdash \varphi$ (or $\vdash_{\mathsf{L}} \varphi$) if φ is provable from the axioms and rules of L. If a formula φ is provable from L, we sometimes write $\varphi \in \mathsf{L}$. Using this notation, we can also define a similar semantic notion.

Definition 2.7. A L-model $\mathfrak{M}$ is a model such that for every $\varphi \in \mathsf{L}$ we have $\mathfrak{M} \models \varphi$.

The smallest normal modal logic is called logic K, after logician Saul Kripke. This logic has the property that it is *sound* and *strongly complete* with respect to the class of Kripke models.

Definition 2.8. Let S be a class of models.
A logic L is *sound* for S if for any formula φ, we have that $\vdash_{\mathsf{L}} \varphi$ implies $\mathsf{S} \models \varphi$.
A logic is *complete* for S if for any formula φ, we have that $\mathsf{S} \models \varphi$ implies $\vdash_{\mathsf{L}} \varphi$.
A logic is *strongly complete* for S if for any set of formulas $\Gamma \cup \{\varphi\}$ we have that $\Gamma \models_{\mathsf{S}} \varphi$ implies $\Gamma \vdash_{\mathsf{L}} \varphi$.

Theorem 2.9 (Soundness and Completeness for K)**.** *The logic* K *is sound and strongly complete with respect to all Kripke models*

We can extend the minimal modal logic with axioms such as transitivity ($\Box\varphi \to \Box\Box\varphi$) or reflexivity ($\Box\varphi \to \varphi$). For many such logics, sound and complete axiomatizations have been given in the literature. For a more extensive introduction on the topic, see Blackburn et al. (2001).

2.3 Bisimulations

Every field in mathematics has comes with its own notion of equivalence. In modal logic, this notion is bisimulation.

Definition 2.10. Let $\mathfrak{M} = \langle W, R, V\rangle$ and $\mathfrak{M}' = \langle W', R', V'\rangle$ be models. A *bisimulation* Z is a non-empty relation such that

- **atoms:** If wZw', then $w \in V(p)$ iff $w' \in V'(p)$.
- **forth:** If wZw' and wRv, then there exists $v' \in W'$, such that vZv' and $w'R'v'$.
- **back:** If wZw' and $w'R'v'$, then there exists $v \in W$, such that vZv' and wRv'.

If there is a bisimulation Z from $\mathfrak{M}$ to $\mathfrak{M}'$ we write $\mathfrak{M} \leftrightarrows \mathfrak{M}'$. If $(w, w') \in Z$, then we write $\mathfrak{M}, w \leftrightarrows \mathfrak{M}', w'$.

Theorem 2.11. *Modal formulas are invariant under bisimulation. That is, for all pointed models $\mathfrak{M}, x$ and $\mathfrak{M}', x'$ such that $\mathfrak{M}, x \leftrightarrows \mathfrak{M}', x'$, we have that $\mathfrak{M}, x$ and $\mathfrak{M}', x'$ model the same modal formulas.*

3. Logic of Satisfiability

One of the most basic operators we can define is the satisfiability operator. This modality expresses that a formula is satisfiable on a set of models S. One example is where S consists only of the model of evaluation, in which the modality is the same as the well-known universal modality in modal logic. In this text, we will consider the case where S consists of all models of a given logic. The formal details of this operator will be given in the first section of this chapter. After that, we will introduce *refutation systems*, a tool needed for defining the sound and complete axiomatization in section 3.3

3.1 Introduction

3.1.1 Preliminaries

The modality in this chapter is meant to express that a formula is satisfiable on some model of a given logic L. Therefore, the interpretation of this modality depends on the underlying logic and we will denote the modality by $\exists^+_{\mathsf{L}}$. Formally, its semantics is as follows:

$$\mathfrak{M}, x \models \exists^+_{\mathsf{L}}\varphi \text{ iff there is an } \mathsf{L}\text{-model } \mathfrak{M}' \text{ and world } x' \in \mathfrak{M}' \text{ such that } \mathfrak{M}', x' \models \varphi$$

Dually to the satisfiability operator, we can also define the validity operator $\forall^+_{\mathsf{L}}$ as $\neg\exists^+_{\mathsf{L}}\neg$. It follows that $\forall^+_{\mathsf{L}}\varphi$ holds iff φ is valid on all models of the underlying logic. If we assume that the logic L is complete with respect to all L-models, this modality expresses whether φ is a theorem of the logic. For the rest of this chapter, we assume that this is the case, so the meanings of the words theorem and validity will coincide. Note, however, that an axiomatization of the base language L is not enough to obtain an axiomatization of $\mathcal{SL}_{\mathsf{L}}$, since we need to distinguish whether a formula is a theorem or not a theorem. Since an axiomatization of the base language only derives theorems, we require axioms to derive the non-theorems as well. Hence, we need an axiomatization of the non-theorems, which leads to the definition of so-called refutation systems in section 3.2.

3.1.2 Motivation and Related Work

The main motivation for this modality comes from the work done in chapter 4. In this chapter, we consider the simulation modality, which states that φ holds in all models that are similar to the model of evaluation. However, a similar model can contain more worlds than the original model. Therefore, as will be explained more in that chapter, axiomatizing this modality requires an expression for whether a formula is satisfiable at all. This is exactly the modality considered in this chapter. However, its use is not limited to simulations only. Any operation that would allow for extensions of the original model would likely include some sort of validity operator.

As far as the author knows, the validity operator has never been mentioned in scientific literature before. Of course, the work is closely related to the work in refutation systems. The achievement in this chapter is that a sound and complete axiomatization and a sound and complete refutation system are combined into a sound and complete axiomatization for the language $\mathcal{SL}_{\mathsf{L}}$.

3.2 Refutation systems

3.2.1 Introduction

Refutation systems have been introduced in Łukasiewicz (1957). In this book, Łukasiwiecz notes that most axiom systems produce a list of theorems, while the task of listing the non-theorems is hardly addressed. Therefore, he proposes an axiomatization of the non-theorems of propositional logic, by introducing the refutation symbol $\dashv$. Here we will give a short introduction into refutation systems, using the notation from Goranko (1994). In the rest of this section, we let L be some propositional (normal modal) logic.

Definition 3.1. A *refutation system* $\mathbf{R}$ is any set of axioms $\dashv\varphi$ and refutation rules of the form

$$\frac{\vdash \varphi_1, \ldots \vdash \varphi_k,\ \dashv\psi_1, \ldots\ \dashv\psi_n}{\dashv\psi}$$

Each logic L has a corresponding refutation system, namely the system that derives exactly all non-theorems. Similarly, each refutation system has a corresponding logic: the logic that derives all formulas that are not refuted. The following definitions make these notions more precise.

Definition 3.2. A refutation system for a logic L is a refutation system $\mathbf{R}$ in which $\vdash \varphi$ is interpreted as 'φ is provable in L' and $\dashv\varphi$ as 'φ is refutable in L'.

Definition 3.3. Given a refutation system $\mathbf{R}$ for L, an *inference in* $\mathbf{R}$ is a sequence of formulas $\varphi_1, \ldots \varphi_n$, in which each formula φ_k is either an axiom of $\mathbf{R}$ or the result of applying a rule in $\mathbf{R}$ to any of $\varphi_1, \ldots \varphi_{k-1}$. The last formula φ of such an inference is called L *-rejected*, written $\mathsf{L} \dashv_{\mathbf{R}} \varphi$. If $\mathbf{R}$ is fixed, we will drop the subscript and write $\mathsf{L} \dashv \varphi$.

Definition 3.4. A refutation system $\mathbf{R}$ for a logic L is *sound* if $\mathsf{L} \dashv \varphi$ implies $\mathsf{L} \nvdash \varphi$. A refutation system $\mathbf{R}$ is *Lukasiewicz-complete* (short: Ł-complete) if for every formula φ we have either $\mathsf{L} \vdash \varphi$ or $\mathsf{L} \dashv \varphi$.

So a sound refutation system rejects only non-theorems. A complete refutation system either proves or refutes every formula. Therefore, a sound and complete refutation proves all theorems and rejects all non-theorems for the given logic.

3.2.2 Two examples

In the book where Łukasiewicz introduced refutation systems (Łukasiewicz, 1957), he also provided a sound and Ł-complete refutation system for propositional logic. The system is called CPC^* and consists of one axiom and two rules:

Table 3.1: The refutation system CPC *

1.	$\dashv\bot$	
2.	*Reverse substitution*:	$\frac{\dashv\sigma(\varphi)}{\dashv\varphi}$ for any uniform substitution σ
3.	*Modus Tollens*:	$\frac{\vdash\varphi\rightarrow\psi,\ \dashv\psi}{\dashv\varphi}$

For example, we can show that the formula p is not a theorem in CPC, by using reverse substitution on $\dashv\bot$ to derive $\dashv p$. Later, this system was extended to construct refutation systems for several modal logics (Goranko, 1991, 1994; Skura, 1995, 2002). For example, a sound and Ł-complete refutation system for the minimal logic K is given in Goranko (1991):

Table 3.2: The refutation system for modal logic K

1.	The axioms and rules of CPC *
2.	$\dashv\Diamond\top$
3.	From $\dashv\lambda,\ \dashv\psi\vee\theta_1,\ldots,\ \dashv\psi\vee\theta_k$ (for λ $\Box$-free) Prove $\dashv\lambda\vee\Box\theta_1\vee\ldots\theta_k\vee\Diamond\psi$

In the same paper, he also presents several strategies for finding refutation systems for modal logics and the reader is referred to this paper for more information on refutation systems.

3.3 Axiomatizing the Logic of Satisfiability

Suppose we have a logic L and a refutation system $\mathbf{R}$ for L. How can we use this to obtain an axiomatization for the corresponding logic of satisfiability? In this section, we will provide an axiomatization for $\mathcal{SL}_\mathsf{L}$. First, we need to introduce a normal form for the language $\mathcal{SL}_\mathsf{L}$. Then we will introduce a so-called *combined system* for the logic of satisfiability. We will use these two notions to define the logic SL_L and proof soundness and completeness. In these sections, we will use $\forall^+$ as the primitive modality. Therefore, we define the following language:

Definition 3.5. The language $\mathcal{SL}_\mathsf{L}$ is given by the following syntax:

$$\varphi := p \mid \neg\varphi \mid \varphi\wedge\varphi \mid \Box\varphi \mid \forall^+_\mathsf{L}\varphi$$

where the semantics of the $\forall^+_\mathsf{L}$ operator is given by:

$$\mathfrak{M},x\models\forall^+_\mathsf{L}\varphi \text{ iff for every } \mathsf{L}\text{-model } \mathfrak{M}' \text{ and world } x'\in\mathfrak{M}', \text{ we have } \mathfrak{M}',x'\models\varphi$$

If the logic L is clear from context, we will sometimes drop the subscript L and write $\forall^+$ and $\exists^+$ respectively.

3.3.1 Satifiability Normal Form

Definition 3.6. The class of formulas in Satifiability Normal Form (SNF) is defined inductively as follows:

- Every $\forall^+$-free formula is an SNF-formula
- If λ is $\forall^+$-free and χ an SNF-formula, then $\lambda\vee\neg\forall^+\chi$ is an SNF-formula.
- If φ,ψ are SNF-formulas, then $\varphi\vee\forall^+\psi$ is an SNF-formula
- If φ,ψ are SNF-formulas, then $\varphi\wedge\psi$ is an SNF-formula

Note that the definition of the SNF has a close resemblance with the Normal Modal Form in Goranko (1994). In fact, the SNF is obtained from the NMF by replacing $\Box$ with $\forall^+$ and $\Diamond$ with $\neg\forall^+$. It is therefore also easy to see that every formula in $\mathcal{SL}_\mathsf{L}$ is logically equivalent to an SNF-formula. A formal proof of this is provided in section 3.3.5. However, the main property that makes SNF-formulas nice, is that they possess a form of the disjunction property. Here, and in the rest of this chapter, the satisfaction relation $\models$ ranges over the set of L-models.

Lemma 3.7. *Let λ be a $\forall^+$-free formula and $\chi,\psi_1,\ldots\psi_n$ SNF-formulas. Then*

$$\models\lambda\vee\neg\forall^+\chi\vee\bigvee_{i\leq n}\forall^+\psi_i \textit{ iff } \models\lambda \textit{ or } \not\models\chi \textit{ or } \models\psi_i \textit{ for some } i$$

Proof. Right-to-left is the easiest direction. First, if $\models \lambda$, then the statement is immediate. Secondly, if $\models \psi_i$ for some i, then $\models \forall^+\psi_i$, so the left-hand side follows. Finally, if $\not\models \chi$, then χ is not an L-validity, hence $\models \neg\forall^+\chi$. So, also in this case, the left-hand side follows. So if $\models \lambda$ or $\not\models \chi$ or $\models \psi_i$ for some i, then $\models \lambda \vee \neg\forall^+\chi \vee \bigvee_{i\leq n} \forall^+\psi_i$.

For the other direction, suppose $\not\models \lambda$ and $\models \chi$ and $\not\models \psi_i$. Then, by definition, $\models \forall^+\chi$ and $\models \neg\forall^+\psi_i$ for all i. Since $\not\models \lambda$, there is some model $\mathfrak{M}, x$ such that $\mathfrak{M}, x \not\models \lambda$. Then also $\mathfrak{M}, x \models \neg\lambda \wedge \forall^+\chi \wedge \bigwedge \neg\forall^+\psi_i$. So $\mathfrak{M}, x \not\models \lambda \vee \neg\forall^+\chi \vee \bigvee_{i\leq n} \forall^+\psi_i$. So we have $\not\models \lambda \vee \neg\forall^+\chi \vee \bigvee_{i\leq n} \forall^+\psi_i$ as required.

Note that the key reason why this lemma holds, is that if $\forall^+\varphi$ is true in some world in some model, then it is true in all worlds in all models. □

3.3.2 Combined system

Definition 3.8. A combined system consists of an axiom system L, a refutation system $\mathbf{R}$ and rules of the form

$$\frac{\vdash \varphi_1, \dots \vdash \varphi_k,\ \dashv\psi_1, \dots\ \dashv\psi_n}{\vdash \psi}$$

In other words, a combined system consists of a logic and a refutation system, with the only addition being that we can derive validities from refutation results. The definition of soundness and Ł-completeness are the same as for refutation systems. We will now create a combined system $\mathbf{SAT}_{\mathsf{L}}$.

Definition 3.9. Let L be a logic in a language $\mathcal{L}$ and let $\mathbf{R}$ be a sound and Ł-complete refutation system for L. The combined system $\mathbf{SAT}_{\mathsf{L}}$ is then given by

1. The axioms and rules of L.
2. The axioms and rules of $\mathbf{R}$
3. From $\vdash \varphi$, deduce $\vdash \forall^+\varphi$.
4. From $\dashv\varphi$, deduce $\vdash \neg\forall^+\varphi$.
5. From $\dashv\varphi$ and $\vdash \psi$, deduce $\dashv\varphi \vee \neg\forall^+\psi$.
6. From $\dashv\varphi$ and $\dashv\psi$, deduce $\dashv\varphi \vee \forall^+\psi$.

Proposition 3.10. *The combined system* $\mathbf{SAT}_{\mathsf{L}}$ *is sound.*

Proof. Clearly, all the axioms and rules of L and $\mathbf{R}$ are sound, by soundness of the logic and refutation system respectively. The next two rules follow immediately from the definition of $\forall^+$. Therefore we will only provide a proof of the last two rules.

Suppose that $\not\models \varphi$ and $\models \psi$. Then $\models \forall^+\psi$ and there is a model $\mathfrak{M}, x$ such that $\mathfrak{M}, x \not\models \varphi$. Then also $\mathfrak{M}, x \models \forall^+\psi$, so by double negation we obtain $\mathfrak{M}, x \not\models \neg\forall^+\psi$. Hence $\mathfrak{M}, x \not\models \varphi \vee \neg\forall^+\psi$. So $\not\models \varphi \vee \neg\forall^+\psi$.

Similarly, suppose $\not\models \varphi$ and $\not\models \psi$. Then $\models \neg\forall^+\psi$ and there is a model $\mathfrak{M}, x$ such that $\mathfrak{M}, x \models \neg\varphi$. Then $\mathfrak{M}, x \models \neg\varphi \wedge \neg\forall^+\psi$. Hence $\mathfrak{M}, x \not\models \varphi \vee \forall^+\psi$, so we conclude $\not\models \varphi \vee \forall^+\psi$. □

It turns out that this system is also 'Ł-complete' on the formulas in SNF.

Proposition 3.11. *For all SNF-formulas* φ *we have* $\mathbf{SAT}_{\mathsf{L}} \vdash \varphi$ *or* $\mathbf{SAT}_{\mathsf{L}} \dashv\varphi$.

Proof. We proceed by induction on the structure of the SNF-formulas. If a formula φ is $\forall^+$-free, then by Ł-completeness of $\mathbf{R}$, we have $\vdash \varphi$ or $\dashv\varphi$.

Next, suppose that we have a formula of the form $\lambda \vee \neg\forall^+\chi$ where λ is $\forall^+$-free and χ is an SNF-formula. By the induction hypothesis, we have $\vdash \lambda$ or $\dashv\lambda$ and $\vdash \chi$ or $\dashv\chi$. We consider the following three exhaustive cases:

- If $\vdash \lambda$, then we have by propositional logic that $\vdash \lambda \vee \neg\forall^+\psi$, as required.
- If $\dashv\chi$, then by rule 4 $\vdash \neg\forall^+\chi$. Then, by propositional logic, we have $\vdash \lambda \vee \neg\forall^+\chi$ as required.
- If $\dashv\lambda$ and $\vdash \chi$, then by rule 5 we have $\dashv\lambda \vee \neg\forall^+\chi$ as required.

Next, suppose that we have a formula of the form $\varphi \vee \forall^+\psi$, where φ and ψ are in SNF. By the induction hypothesis, we have $\vdash \varphi$ or $\dashv\varphi$ and $\vdash \psi$ or $\dashv\psi$. We distinguish the following 3 exhaustive cases:

- If $\vdash \varphi$, then we have by propositional logic that $\vdash \varphi \vee \forall^+\psi$, as required.
- If $\vdash \psi$, then by rule 3 we have $\vdash \forall^+\psi$. Then we have by propositional logic that $\vdash \varphi \vee \forall^+\psi$, as required.
- If $\dashv\varphi$ and $\dashv\psi$, then the conclusion follows immediately by rule 6.

If $\varphi = \psi \wedge \chi$, then by induction hypothesis, we have $\vdash \psi$ or $\dashv\psi$ and $\vdash \chi$ or $\dashv\chi$. If $\vdash \psi$ and $\vdash \chi$, then $\vdash \psi \wedge \chi$ by propositional logic. If $\dashv\psi$, then by $\vdash (\psi \wedge \chi) \to \psi$ and modus tollens, we have $\dashv\psi \wedge \chi$, as required. If $\dashv\chi$, we can apply the same reasoning to obtain $\dashv\psi \wedge \chi$. This last case completes the proof of the proposition. □

3.3.3 Axiomatization of $\mathcal{SL}_{\mathsf{L}}$

Now that we have defined a normal form and a combined system, we have all the tools needed to define the logic SL_{L} and show that it is sound and complete with respect to all L-models.

Definition 3.12. Let L be a logic in a language $\mathcal{L}$ and let $\mathbf{R}$ be a sound and Ł-complete refutation system for L. Let the logic SL_{L} consist of the following formulas:

1. All axioms and rules of L
2. $\vdash \forall^+(\varphi \to \psi) \to (\forall^+\varphi \to \forall^+\psi)$
3. $\vdash \Box\big(\lambda \vee \neg\forall^+\chi \vee \bigvee_{i\in n} \forall^+\psi_i\big) \leftrightarrow \big(\Box\lambda \vee \neg\forall^+\chi \vee \bigvee_{i\in n} \forall^+\psi_i\big)$ for $\lambda \in \mathcal{BML}$
4. For every axiom $\dashv\varphi$ in $\mathbf{SAT}_{\mathsf{L}}$, add the axiom $\vdash \neg\forall^+_{\mathsf{L}}\varphi$
5. For every rule of the form
$$\frac{\vdash \varphi_1, \ldots \vdash \varphi_k,\ \dashv\psi_1, \ldots\ \dashv\psi_n}{\dashv\psi}$$
in $\mathbf{SAT}_{\mathsf{L}}$, we add the following rule to SL_{L}.
$$\frac{\vdash \varphi_1, \ldots \vdash \varphi_k, \vdash \neg\forall^+_{\mathsf{L}}\psi_1, \ldots, \vdash \neg\forall^+_{\mathsf{L}}\psi_n}{\vdash \neg\forall^+_{\mathsf{L}}\psi}$$
6. For every rule of the form
$$\frac{\vdash \varphi_1, \ldots \vdash \varphi_k,\ \dashv\psi_1, \ldots\ \dashv\psi_n}{\vdash \psi}$$
in $\mathbf{SAT}_{\mathsf{L}}$, we add the following rule to SL_{L}
$$\frac{\vdash \varphi_1, \ldots \vdash \varphi_k, \vdash \neg\forall^+_{\mathsf{L}}\psi_1, \ldots, \vdash \neg\forall^+_{\mathsf{L}}\psi_n}{\vdash \psi}$$

The main goal of this section is to proof that SL_{L} is sound and complete with respect to all L-frames.

3.3.4 Soundness of SL_L

Theorem 3.13. *The logic* SL_L *is sound on all* L*-models.*

Proof. Clearly, all the axioms and rules of L are sound on all L-models. Next, if $\varphi \to \psi$ is a validity and φ is a validity, then ψ is a validity. The final three rules follow by noting that $\dashv\varphi$ iff $\models \neg\forall^+_\mathsf{L}\varphi$. Namely, by soundness and completeness of $\mathbf{R}$, we have that $\dashv\varphi$ iff φ is not a validity iff $\models \neg\forall^+_\mathsf{L}\varphi$. Therefore, only axiom 3 remains to be proven. However, using lemma 3.7 and the fact that validities are global, we have

$$\begin{aligned}
\mathfrak{M}, x \models \Box\big(\lambda \vee \forall^+\chi \vee \bigvee_{i\in n} \forall^+\psi_i\big) \text{ iff for all } y \in R(x) \text{ we have } \mathfrak{M}, y \models \lambda \vee \neg\forall^+\chi \vee \bigvee_{i\in n} \forall^+\psi_i \\
\text{iff for all } y \in R(x) \text{ we have } \mathfrak{M}, y \models \lambda \text{ or } \mathfrak{M}, y \models \neg\forall^+\chi \\
\quad \text{or } \mathfrak{M}, y \models \forall^+\psi_i \text{ for some } i \qquad \text{(lemma 3.7)} \\
\text{iff for all } y \in R(x) \text{ we have } \mathfrak{M}, y \models \lambda \text{ or } \mathfrak{M} \models \neg\forall^+\chi \\
\quad \text{or } \mathfrak{M} \models \forall^+\psi_i \text{ for some } i \\
\text{iff } \mathfrak{M}, x \models \Box\lambda \text{ or } \mathfrak{M} \models \neg\forall^+\chi \text{ or } \mathfrak{M} \models \forall^+\psi_i \text{ for some } i \\
\text{iff } \mathfrak{M}, x \models \Box\lambda \vee \neg\forall^+\chi \vee \forall^+\psi_i \text{ for some } i
\end{aligned}$$

□

3.3.5 Satifiability Normal Form Theorem

It turns out that every formula φ in $\mathcal{SL}_\mathsf{L}$, there is an equivalent formula ψ in SNF such that $\mathsf{SL}_\mathsf{L} \vdash \varphi \leftrightarrow \psi$. This will be proven syntactically. In the proofs below, some steps, such as application of modus ponens, are omitted for brevity.

Lemma 3.14 (Substitution of equivalences)**.** *If* $\mathsf{SL}_\mathsf{L} \vdash \psi \leftrightarrow \chi$*, then* $\mathsf{SL}_\mathsf{L} \vdash \varphi[\psi/p] \leftrightarrow \varphi[\chi/p]$*.*

Proof. The substitution of equivalences can be proven by induction on φ. Most cases are standard. For the case $\forall^+\varphi$, we have the following derivation:

$$\begin{aligned}
&\vdash \varphi[\psi/p] \leftrightarrow \varphi[\chi/p] && \text{(Induction Hypothesis)} \\
&\vdash \forall^+\big(\varphi[\psi/p] \leftrightarrow \varphi[\chi/p]\big) && \text{(Rule 3 in } \mathbf{SAT}_\mathsf{L}\text{)} \\
&\vdash \forall^+\varphi[\psi/p] \leftrightarrow \forall^+\varphi[\chi/p] && \text{(Rule 2 twice)}
\end{aligned}$$

Note that this lemma requires the underlying logic to satisfy substitution of equivalences. Since we restricted ourselves to normal modal logics, this does not pose a problem. □

Lemma 3.15. $\mathsf{SL}_\mathsf{L} \vdash \forall^+(\varphi \wedge \psi) \leftrightarrow (\forall^+\varphi \wedge \forall^+\psi)$

Proof. First we prove left-to-right:

$$\begin{aligned}
&\vdash \varphi \wedge \psi \to \varphi && \text{(propositional tautology)} \\
&\vdash \forall^+(\varphi \wedge \psi \to \varphi) && \text{(rule 3 in } \mathbf{SAT}_\mathsf{L}\text{)} \\
&\vdash \forall^+(\varphi \wedge \psi) \to \forall^+\varphi && \text{(axiom 2)}
\end{aligned}$$

The case for ψ is analogous. For the right-to-left direction, we have

$$\begin{aligned}
&\vdash \varphi \to (\psi \to \varphi \wedge \psi) && \text{(propositional tautology)}\\
&\vdash \forall^+(\varphi \to (\psi \to \varphi \wedge \psi)) && \text{(rule 3 in } \mathbf{SAT}_\mathsf{L})\\
&\vdash \forall^+\varphi \to (\forall^+\psi \to \forall^+(\varphi \wedge \psi)) && \text{(axiom 2 twice)}\\
&\vdash (\forall^+\varphi \wedge \forall^+\psi) \to \forall^+(\varphi \wedge \psi) && \text{(propositional logic)}
\end{aligned}$$

□

Proposition 3.16. *For every formula $\varphi \in \mathcal{SL}_\mathsf{L}$, there is an SNF-formula ψ such that $\mathsf{SL}_\mathsf{L} \vdash \varphi \leftrightarrow \psi$.*

Proof. We prove this lemma by induction on the structure of the formula in $\mathcal{SL}_\mathsf{L}$.

- Every formula consisting of a propositional variable is a $\forall^+$-free formula, so already in SNF.
- Suppose we have a formula of the form $\neg\varphi$. By the induction hypothesis, φ is equivalent to a formula SNF. So
$$\mathcal{SL}_\mathsf{L} \vdash \neg\varphi \leftrightarrow \neg\left(\bigwedge\left(\lambda \vee \neg\forall^+\chi \vee \forall^+\psi_1 \vee \ldots \forall^+\psi_n\right)\right)$$
Then, by distributivity of $\neg$, we have
$$\mathcal{SL}_\mathsf{L} \vdash \varphi \leftrightarrow \left(\bigvee\left(\neg\lambda \wedge \forall^+\chi \wedge \neg\forall^+\psi_1 \wedge \ldots \wedge \neg\forall^+\psi_n\right)\right)$$
Then using distributivity of $\vee$ over $\wedge$ and using lemma 3.15 to group terms of $\neg\forall^+$, it is easy to see that we get a conjunction of SNF formulas. Hence $\neg\varphi$ is equivalent to an SNF-formula.
- Suppose we have a formula of the form $\varphi \wedge \psi$. By the induction hypothesis, there are SNF-formulas φ' and ψ' such that $\mathsf{SL}_\mathsf{L} \vdash \varphi \leftrightarrow \varphi'$ and $\mathsf{SL}_\mathsf{L} \vdash \psi \leftrightarrow \psi'$. Then by substitution of equivalences we have $\mathsf{SL}_\mathsf{L} \vdash (\varphi \wedge \psi) \leftrightarrow (\varphi' \wedge \psi')$. Since $\varphi' \wedge \psi'$ is an SNF-formula, this completes the step in the induction.
- $\Box\varphi$: By the induction hypothesis, φ is equivalent to a formula in SNF. Therefore
$$\begin{aligned}
&\mathsf{SL}_\mathsf{L} \vdash \varphi \leftrightarrow \bigwedge\left(\lambda \vee \neg\forall^+\chi \vee \forall^+\psi_1 \vee \ldots \forall^+\psi_n\right) && \text{(IH)}\\
&\mathsf{SL}_\mathsf{L} \vdash \Box\varphi \leftrightarrow \Box\bigwedge\left(\lambda \vee \neg\forall^+\chi \vee \forall^+\psi_1 \vee \ldots \forall^+\psi_n\right) && \text{(Modal Logic)}\\
&\mathsf{SL}_\mathsf{L} \vdash \Box\varphi \leftrightarrow \bigwedge\Box\left(\lambda \vee \forall^+\psi_1 \vee \ldots \forall^+\psi_n \vee \exists^+\chi\right) && \text{(axiom K)}\\
&\mathsf{SL}_\mathsf{L} \vdash \Box\varphi \leftrightarrow \bigwedge\left(\Box\lambda \vee \neg\forall^+\chi \vee \forall^+\psi_1 \vee \ldots \forall^+\psi_n\right) && \text{(Axiom 3)}
\end{aligned}$$
The last line shows that $\Box\varphi$ is equivalent to a formula in SNF.
- Suppose we have a formula of the form $\forall^+\varphi$. By the induction hypothesis φ is equivalent to an SNF formula φ'. Then by substitution of equivalents we have $\mathsf{SL}_\mathsf{L} \vdash \forall^+\varphi \leftrightarrow \forall^+\varphi'$. Since $\forall^+$ applied to an SNF formula is itself an SNF formula, we are done.

□

3.3.6 Completeness of SL_L

We are now finally ready to prove completeness of SL_L. For this, we first show a correspondence between the combined system $\mathbf{SAT}_\mathsf{L}$ and our logic SL_L.

Lemma 3.17. *If $\mathbf{SAT}_\mathsf{L} \vdash \varphi$, then $\mathsf{SL}_\mathsf{L} \vdash \varphi$ and if $\mathbf{SAT}_\mathsf{L} \dashv\varphi$ then $\mathsf{SL}_\mathsf{L} \vdash \neg\forall^+\varphi$*

Proof. We proceed by induction on the derivation in $\mathbf{SAT}_{\mathsf{L}}$. If $\vdash \varphi$ is an axiom in L, then we are done immediately, since L is included in SL_{L}. Similarly, if $\dashv\varphi$ is an axiom in $\mathbf{SAT}_{\mathsf{L}}$, then $\neg\forall^+\varphi$ is an axiom in SL_{L} and the result is also immediate. Also if $\vdash \varphi$ is the result of applying a rule from L, then by the induction hypothesis and the fact that L is included in SL_{L} we have $\vdash \varphi$. Therefore, suppose that φ is the result of applying a rule of the form

$$\frac{\vdash \varphi_1, \ldots \vdash \varphi_k,\ \dashv\psi_1, \ldots\ \dashv\psi_n}{\dashv\varphi}$$

then, by the induction hypothesis, we have $\vdash \varphi_1, \ldots \vdash \varphi_k, \vdash \neg\forall^+_{\mathsf{L}}\psi_1, \ldots, \vdash \neg\forall^+_{\mathsf{L}}\psi_n$ and since we added

$$\frac{\vdash \varphi_1, \ldots \vdash \varphi_k, \vdash \neg\forall^+_{\mathsf{L}}\psi_1, \ldots, \vdash \neg\forall^+\psi_n}{\vdash \neg\forall^+\psi}$$

as a rule to SL_{L}, we deduce that $\mathsf{SL}_{\mathsf{L}} \vdash \neg\forall^+\varphi$. The case where we have rule ending in $\vdash \varphi$ is analogous. □

Lemma 3.18. *For all SNF-formulas φ we have $\models \varphi$ implies $\vdash \varphi$*

Proof. Let φ be an SNF-formula. By proposition 3.11, it follows that either $\mathbf{SAT}_{\mathsf{L}} \vdash \varphi$ or $\mathbf{SAT}_{\mathsf{L}} \dashv\varphi$. By proposition 3.10, we know that $\mathbf{SAT}_{\mathsf{L}}$ is sound. Therefore, if $\models \varphi$, it follows that $\mathbf{SAT}_{\mathsf{L}}$ does not refute φ. Therefore, we know that $\mathbf{SAT}_{\mathsf{L}} \vdash \varphi$. Then it follows by lemma 3.17 that $\mathsf{SL}_{\mathsf{L}} \vdash \varphi$. □

Theorem 3.19. *The logic SL_{L} is complete with respect to all L-models.*

Proof. Suppose that φ is valid. By proposition 3.16, there exists an SNF-formula ψ such that $\mathsf{SL}_{\mathsf{L}} \vdash \varphi \leftrightarrow \psi$. Since SL_{L} is sound, we have $\models \psi$. We can then apply lemma 3.18 to see that we also have $\mathsf{SL}_{\mathsf{L}} \vdash \psi$. Since $\mathsf{SL}_{\mathsf{L}} \vdash \varphi \leftrightarrow \psi$, and SL_{L} contains modus ponens, we have $\mathsf{SL}_{\mathsf{L}} \vdash \varphi$. □

3.4 Some axiomatizations

In this section, we will use the results from the previous section to provide some sound and complete axiomatization for several (modal) logics. We will start with an axiomatization of the classical propositional calculus. The refutation system has been given in table 3.1. We simply take the axioms and rules of SL_{L}, except all the rules involving $\Box$. This leads to the following system:

Table 3.3: The logic $\mathsf{SL}_{\mathsf{CPC}}$

1.	All the axioms and rules of CPC
2.	$\forall^+(\varphi \to \psi) \to (\forall^+\varphi \to \forall^+\psi)$
3.	$\neg\forall^+\bot$
4.	From $\neg\forall^+\sigma(\varphi)$ infer $\neg\forall^+\varphi$ for any uniform substitution σ
5.	From $\varphi \to \psi$ and $\neg\forall^+\psi$, infer $\neg\forall^+\varphi$
6.	From φ, infer $\forall^+\varphi$
7.	From $\neg\forall^+\varphi$ and ψ, infer $\neg\forall^+(\varphi \vee \neg\forall^+\psi)$
8.	From $\neg\forall^+\varphi$ and $\neg\forall^+\psi$, infer $\neg\forall^+(\varphi \vee \forall^+\psi)$

For the logic K, we obtain the following axiomatization:

Table 3.4: The logic $\mathsf{SL_K}$

1.	All the axioms and rules of $\mathsf{SL_{CPC}}$
2.	$\Box(\lambda \vee \neg\forall^+\chi \vee \bigvee_{i\in n}\forall^+\psi_i) \leftrightarrow (\Box\lambda \vee \neg\forall^+\chi \vee \bigvee_{i\in n}\forall^+\psi_i)$ for $\lambda \in \mathcal{BML}$
3.	$\neg\forall^+\Diamond\top$
4.	From $\neg\forall^+\lambda, \neg\forall^+(\psi \vee \theta_1), \ldots, \neg\forall^+(\psi \vee \theta_k)$ infer $\neg\forall^+(\lambda \vee \Box\theta_1 \vee \ldots \theta_k \vee \Diamond\psi)$ for λ $\Box$-free

Using the refutation systems in, for example, Goranko (1991) it is easy to construct axiomatizations for the logic of satisfiability for KW, T or $\mathsf{S4.Grz}$. A refutation system for $\mathsf{S4}$ is provided in Skura (1995).

It can be argued that some of the axioms and rules in $\mathsf{SL_K}$ are complex and opaque. Also, formulas like $\varphi \to \forall^+_{\mathsf{L}}\varphi$ and $\Box\forall^+_{\mathsf{L}}\varphi \leftrightarrow \forall^+_{\mathsf{L}}\varphi$ which might be expected in an axiomatization, are derivable formulas. This suggests that a cleaner, more transparent axiomatization might be possible. This is left as a direction for future work on this topic.

3.5 Conclusion

In this chapter, we consider the language $\mathcal{SL}_{\mathsf{L}}$. This language consists of basic modal logic, extended with an operator $\forall^+_{\mathsf{L}}\varphi$, expressing that the formula φ is valid on all L-models. It is shown how an axiomatization and refutation system can be turned into a sound and complete axiomatization for this new language. For this purpose, we defined the notion of a *combined system*, which merges the axiomatization and the refutation system into one system. Based on this, we were able to find an axiomatization $\mathcal{SL}_{\mathsf{L}}$ for any propositional (normal modal) logic L. In the end, concrete axiomatizations were given for classical propositional logic and K.

The operator $\forall^+_{\mathsf{L}}$ fits right into the broader scheme of this thesis since it is an instance of the more general type of operators we are discussing. Namely, we can consider an operator $\forall^+_{\mathsf{S}}$, which has, given a set of pointed models S, the following semantics:

$$\mathfrak{M}, x \models \forall^+_S\varphi \text{ iff for all } (\mathfrak{M}', x') \in S \text{ we have } \mathfrak{M}', x', \models \varphi$$

In this chapter, we picked S to be the set of pointed L models. However, as mentioned in the introduction, many different options are available. In the next chapters, we consider different sets S and in all future chapters, S will depend on the model of evaluation. We start by considering the case where, given a model $\mathfrak{M}, x$, we choose S to be the set of all simulations of $\mathfrak{M}, x$.

4. Logic of Simulations

In this chapter, we will study the logic of simulations. After a short introduction to simulations and their applications to logic, we will axiomatize two logics: the first one is concerned with *strong simulations* and the latter one with 'regular' simulations. In the end, both logics can be used to prove their respective preservation law inside the language.

4.1 Introduction

4.1.1 Preliminaries

A simulation is a relation between (Kripke) models that preserves the structure of the original model. That is, it preserves the truth of variables and any relation in the original model has a corresponding relation in the destination model.

Definition 4.1. Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be models. A simulation $Z \subseteq W \times W'$ is a non-empty relation such that

- If wZw' and $w \in V(p)$ then $w' \in V'(p)$.
- If wZw' and wRv, then there exists $v' \in W'$, such that vZv' and $w'R'v'$.

If there is a simulation Z from $\mathfrak{M}$ to $\mathfrak{M}'$, then we say that $\mathfrak{M}'$ is *similar* to $\mathfrak{M}$ and write $Z : \mathfrak{M} \rightrightarrows \mathfrak{M}'$. We will write $\mathfrak{M}, w \rightrightarrows \mathfrak{M}', w'$ when there is a $Z : \mathfrak{M} \rightrightarrows \mathfrak{M}'$ such that wZw'. In this case, we call $\mathfrak{M}, w'$ similar to $\mathfrak{M}, w$.

The definition above is used in modal logic (e.g. see Blackburn et al. (2001) or De Rijke (1993)) and in database theory (e.g. see Buneman et al. (1997)). However, in process algebra, a slightly stronger notion of a simulation is prevalent as well. For example, Alur et al. (1998) and Henzinger et al. (1995) require similar worlds have satisfy exactly the same proposition letters. To distinguish the two kinds of relations, we name the latter *strong simulations*.

Definition 4.2. Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be models. A *strong simulation* Z is simulation with the property: if wZw', then $w \in V(p)$ iff $w' \in V'(p)$. If there is a strong simulation from w in $\mathfrak{M}$ to w' in $\mathfrak{M}'$, we write $\mathfrak{M}, w \rightrightarrows \mathfrak{M}', w'$.

4.1.2 Motivation and Related Work

Simulations have mainly been studied in the field of theoretical computer science. Within this field, there are at least two branches where the use of simulations is prevalent: process theory and database theory (Blackburn et al., 2001).

In process theory, the elements of Kripke models are seen as *states*, and the relation represents *transitions* between the states. In this context, if $\mathfrak{M} \rightrightarrows \mathfrak{M}'$, it means that every transition in $\mathfrak{M}$ can also be performed in $\mathfrak{M}'$. It is said that $\mathfrak{M}$ *refines* or *implements* $\mathfrak{M}'$, since $\mathfrak{M}$ has fewer options for transitions than $\mathfrak{M}$. This leads to the dual notion of simulations, called *refinement*. The corresponding logic, refinement logic, has been axiomatized in Bozzelli et al. (2014) and inspired the axiomatization for strong simulations. Whereas refinements correspond to the notion of implementations of abstract programs, simulations correspond to the verification of concrete programs. In this context, $\mathfrak{M}$ is the

concrete program of which we need to make sure that it implements the abstract program $\mathfrak{M}'$ (Woodcock and Davies, 1996). This involves quantification over the simulations of $\mathfrak{M}$, which corresponds to the modality we introduced above. One final use of simulations in process theory is given in Henzinger et al. (1995). In process theory, it is often desirable to reduce the size of automata by taking quotients with respect to some similarity relation. In the paper, it is argued that simulations are in many cases the appropriate abstraction for computer-aided verification.

In abstract database theory, simulations are used the other way around. The elements in a Kripke model can be seen as databases containing objects that have relations between them (Buneman et al., 1997). Then, if $\mathfrak{M} \rightrightarrows \mathfrak{M}'$, the database $\mathfrak{M}$ cannot have more relations than the database $\mathfrak{M}'$. That is, the *database schema* $\mathfrak{M}'$ *constraints* the database $\mathfrak{M}$ or, put differently, the database $\mathfrak{M}$ *conforms to* database schema $\mathfrak{M}'$. Therefore, given a database $\mathfrak{M}$, the modality introduced above can be used to express that there exists a database schema $\mathfrak{M}'$ that has a certain definable property and to which $\mathfrak{M}$ conforms.

One final possible application of simulation quantifiers lies in game theory. In this semantics, worlds represent the states of a game and the relation depicts the possible moves the player can make at a state. In this context, one model $\mathfrak{M}$ simulates another model $\mathfrak{M}'$ if all transitions and states of model $\mathfrak{M}$ are also present in model $\mathfrak{M}'$. However, model $\mathfrak{M}'$ can contain more states and/or transitions. For example, suppose we restrict ourselves to surjective simulations. Then the simulation modality defined above will check whether a formula still holds when a player can do forbidden moves. Not restricting to surjective simulations represents the situation where a player can do forbidden moves and end up in forbidden states. Such a modality can be used to reason about cheating players and which statement still hold even if the player does a forbidden move.

As mentioned above, the refinement modality has been introduced and axiomatized in Bozzelli et al. (2014). In this paper, they use the so-called Cover Logic to axiomatize their Refinement Modal Logic and an extension to μ-calculus. Even though the extension to μ-calculus is not performed in this thesis, the axiomatization of the logic with the strong simulation operator is also axiomatized using Cover Logic, thereby closely following Bozzelli et al. (2014). This would probably allow for a similar extension of simulation logic to μ-calculus.

There is one other paper that considers the simulation relation as a modality (Allwein et al., 2013). In this article, simulations are considered in a category of general frames. They axiomatize this logic in more abstract categorical terms. This approach is, however, completely different from the reduction style axiomatizations that are provided in this chapter, which are more transparent than the more general categorical approach.

4.2 Preservation laws

Before stating the axiomatizations for the logics defined above, we will now first prove some properties of the simulation modality. One goal of the axiomatization is to internalize the preservation law for simulations. A proof of the preservation law for simulations is given in Blackburn et al. (2001), theorem 2.78. In this section, we will only prove the preservation law for strong simulations, since this proof is only a minor adaption of the preservation proof for simulations in Blackburn et al. (2001).

We will now first define the class of formulas that are preserved under (strong) simulations respectively.

Definition 4.3. A formula in $\mathcal{BML}$ is existential iff it has been built up using only (negated) proposition letters, $\vee, \wedge$, and $\Diamond$.

Definition 4.4. A formula in $\mathcal{BML}$ is positive existential iff it has been built up using only proposition letters, $\vee, \wedge$, and $\Diamond$.

Theorem 4.5. *A formula φ is preserved under simulations iff it is equivalent to an positive-existential formula.*

Proof. See Theorem 2.7.8 in Blackburn et al. (2001). □

Theorem 4.6. *A formula φ is preserved under strong simulations iff it is equivalent to an existential formula.*

Proof. The proof that existential formulas are preserved under strong simulation follows by an easy induction proof. Here we will prove that if a formula φ is preserved under strong simulations then it is equivalent to an existential formula. The proof will be analogous to the proof that positive existential formulas are preserved under simulations as presented in Blackburn et al. (2001).

Let φ be a formula that is preserved under strong simulations. Then consider the set

$$PEC(\varphi) = \{\psi \mid \psi \text{ is existential and } \varphi \models \psi\}$$

It now remains to show that $PEC(\varphi) \models \varphi$. Namely, if this holds, then by compactness there is a finite subset Ψ of $PEC(\varphi)$ such that $X \models \varphi$. Let $\bigwedge \Psi$ be the conjunction of all formulas in Ψ. Then $\bigwedge \Psi \models \varphi$ and since φ models every formula in X, it follows that also $\varphi \models \bigwedge \Psi$. Hence $\bigwedge \Psi$ is logically equivalent to Ψ and $\bigwedge \Psi$ is an existential formula.

Let $\mathfrak{M}, x$ be a model such that $\mathfrak{M}, x \models PEC(\varphi)$. We must show that $\mathfrak{M}, x \models \varphi$. For this purpose, consider the set

$$\Gamma = \{\neg\psi \mid \psi \text{ is existential and } \mathfrak{M}, x \not\models \psi\}$$

Then $\{\varphi\} \cup \Gamma$ is consistent. Namely, suppose otherwise. Then there are formulas $\neg\psi_1, \ldots, \neg\psi_n \in \Gamma$ such that $\varphi \models \psi_1 \vee \ldots \vee \psi_n$. Since each ψ_i is existential, $\psi_1 \vee \ldots \vee \psi_n$ is existential and hence $\psi_1 \vee \ldots \vee \psi_n \in PEC(\varphi)$. Hence $\mathfrak{M}, x \models \psi_1 \vee \ldots \vee \psi_n$. Hence, there is a ψ_i such that $\mathfrak{M}, x \models \psi_i$. However, since $\neg\psi_i \in \Gamma$, we also have $\mathfrak{M}, x \not\models \psi_i$. This is our required contradiction.

So we conclude that $\{\varphi\} \cup \Gamma$ is consistent. By completeness, $\{\varphi\} \cup \Gamma$ is then also satisfiable, so there is a model $\mathfrak{N}, w$ such that $\mathfrak{N}, w \models \varphi \wedge \bigwedge \Gamma$. Then, for every existential formula ψ, if $\mathfrak{N}, w \models \psi$, then $\mathfrak{M}, x \models \psi$. Since all formulas are preserved under ultrafilter extensions, we get that for all existential ψ we have $\mathfrak{ue}\ \mathfrak{N}, \pi_w \models \psi$ implies $\mathfrak{ue}\ \mathfrak{M}, \pi_x \models \psi$, where π_w denotes the principal ultrafilter generated at w. It remains to show that there is a strong simulation from $\mathfrak{ue}\ \mathfrak{N}, \pi_w$ to $\mathfrak{ue}\ \mathfrak{M}, \pi_x$. Once we established this, we have that $\mathfrak{N}, w \models \varphi$, so $\mathfrak{ue}\ \mathfrak{N}, \pi_w \models \varphi$. Since φ is preserved under strong simulations, $\mathfrak{ue}\ \mathfrak{M}, \pi_x \models \varphi$. Then $\mathfrak{M}, x \models \varphi$, which is what we needed to show.

We will show that the following relation is a simulation between $\mathfrak{ue}\ \mathfrak{M}$ and $\mathfrak{ue}\ \mathfrak{N}$:

$$Z = \{(\pi_w, \pi_x) \mid \text{ for all existential } \psi : \mathfrak{ue}\ \mathfrak{N}, \pi_w \models \psi \text{ implies } \mathfrak{ue}\ \mathfrak{M}, \pi_x \models \psi\}$$

Clearly, $\mathfrak{ue}\ \mathfrak{N}, \pi_w Z \mathfrak{ue}\ \mathfrak{M}, \pi_x$. Since p and $\neg p$ are existential formulas for all proposition letters p, it follows that $\mathfrak{ue}\ \mathfrak{N}, \pi_w$ and $\mathfrak{ue}\ \mathfrak{M}, \pi_x$ satisfy the same proposition letters. Next, take any successor π_v of π_w in $\mathfrak{ue}\ \mathfrak{N}$. We then set

$$\Delta = \{\psi \mid \psi \text{ is existential and } \mathfrak{ue}\ \mathfrak{N}, \pi_v \models \psi\}$$

Take any finite subset X of Δ. Since every formula in X is existential, $\Diamond \bigwedge X$ is an existential. Also since $\mathfrak{ue}\ \mathfrak{N}, \pi_v \models \bigwedge X$, it follows that $\mathfrak{ue}\ \mathfrak{N}, \pi_w \models \Diamond \bigwedge X$. Hence also $\mathfrak{ue}\ \mathfrak{M}, \pi_x \models \Diamond \bigwedge X$. So X is satisfiable on a successor of π_x. Since ultrafilter extensions are m-saturated and X was an arbitrary finite subset, it follows that there is some π_y such that $\mathfrak{ue}\ \mathfrak{M}, \pi_y \models \Delta$. Hence, if $\pi_w Z \pi_x$, then for every successor π_v of π_w in $\mathfrak{ue}\ \mathfrak{N}$, there is a successor π_y of π_x in $\mathfrak{ue}\ \mathfrak{M}$ such that $\pi_v Z \pi_y$. Hence Z is a strong simulation. □

4.3 Bisimulation invariance

A second property that will be crucial in the soundness proofs for the reduction axioms, is the bisimulation invariance of the new modalities. We will first show that simulations are transitive, from which the bisimulation invariance follows as a corollary.

Lemma 4.7. *Let $\mathfrak{M}_i = (W_i, R_i, V_i)$ be models for $i = 1, 2, 3$. If there is a (strong) simulation Z_1 from $\mathfrak{M}_1$ to $\mathfrak{M}_2$ and a (strong) simulation Z_2 from $\mathfrak{M}_2$ to $\mathfrak{M}_3$, then there is a (strong) simulation Z from $\mathfrak{M}_1$ to $\mathfrak{M}_3$.*

Proof. We will only prove the case for regular simulations. The case for strong simulations follows analogously.

Take

$$Z = \{(x_1, x_3) \in W_1 \times W_3 \mid (\exists x_2 \in W_2)(x_1 Z_1 x_2 \wedge x_2 Z_2 x_3)\}$$

We will now show that Z is a simulation. Suppose $x_1 Z x_3$. Then there exists an x_2 such that $x_1 Z_1 x_2$ and $x_2 Z_2 x_3$. Therefore, since Z_1 and Z_2 are simulations, if $x_1 \in V_1(p)$, then $x_2 \in V_2(p)$, so $x_3 \in V_3(p)$. So the first condition of a simulation is satisfied. Next, if there is a $y \in W_1$ such that $x_1 R_1 y_1$, then there is a $y_2 \in W_2$ such that $x_2 R_2 y_2$ and $y_1 Z_1 y_2$. Again, then there must be a $y_3 \in W_3$ such that $x_3 R y_3$ and $y_2 Z_2 y_3$. Hence by the definition of Z, $y_1 Z y_3$. So Z also satisfies the second condition of a simulation and we conclude that Z is a simulation between $\mathfrak{M}_1$ and $\mathfrak{M}_3$. □

Proposition 4.8. *The modality $[\rightarrow]$ and $[\overrightarrow{=}]$ are invariant under bisimulation.*

Proof. Again, we only prove the case for $[\rightarrow]$, the case for $[\overrightarrow{=}]$ follows analogously.

Let $\mathfrak{M}, x \models [\rightarrow]\varphi$ and $\mathfrak{M}, x \leftrightarroweq \mathfrak{M}', x'$. Since obviously every bisimulation is a simulation, there is a simulation from $\mathfrak{M}, x$ to $\mathfrak{M}', x'$. Hence, by lemma 4.7, for every model $\mathfrak{M}'', x''$ that is similar to $\mathfrak{M}, x$, there exists a simulation from $\mathfrak{M}, x$ to $\mathfrak{M}'', x''$. Hence $\mathfrak{M}'', x'' \models \varphi$. Since $\mathfrak{M}''$ was picked arbitrarily, it follows that $\mathfrak{M}', x \models [\rightarrow]\varphi$. □

4.4 Axiomatizing the Logic of Strong Simulations

In this section, we will axiomatize the logic of strong simulations. The language of this logic consists of the language $\mathcal{SL}_{\mathsf{K}}$ (the logic of satisfiability for basic modal logic K) and a new modality that quantifies over the similar models.

Definition 4.9. The syntax of the language $\mathcal{L}_{\forall^+\overrightarrow{=}}$ is given by

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \forall^+_{\mathsf{K}}\varphi \mid [\overrightarrow{=}]\varphi$$

where the semantics of $\forall^+_{\mathsf{K}}$ and $[\overrightarrow{=}]$ is given by

$$\mathfrak{M}, x \models \forall^+_{\mathsf{K}}\varphi \text{ iff for all } \mathsf{K}\text{-models } \mathfrak{M}' \text{ and } w \in \mathfrak{M}', \text{ we have } \mathfrak{M}', w \models \varphi$$
$$\mathfrak{M}, w \models [\overrightarrow{=}]\varphi \text{ iff for all } \mathfrak{M}', w', \text{ such that } \mathfrak{M}, w \overrightarrow{=} \mathfrak{M}', w' \text{ we have } \mathfrak{M}', w' \models \varphi$$

However, for the axiomatization of the logic of strong simulations we will also make use of cover operator, an alternative primitive operator for modal logic (Bílková et al., 2008). However, in this

thesis, we will introduce it as an abbreviation. Given a finite set Φ of $\mathcal{L}_{\forall^+\overrightarrow{\rightleftharpoons}}$-formulas, we will write

$$\nabla\Phi \text{ abbreviates } \Box \bigvee_{\varphi\in\Phi} \varphi \wedge \bigwedge_{\varphi\in\Phi} \Diamond\varphi$$
$$\nabla^+\Phi \text{ abbreviates } \forall^+_{\mathsf{K}} \bigvee_{\varphi\in\Phi} \varphi \wedge \bigwedge_{\varphi\in\Phi} \exists^+_{\mathsf{K}}\varphi$$

By writing out the definitions, it is easy to see that we can also express the $\Box$ and $\Diamond$ modalities in terms of the cover modality ∇.

$$\Box\varphi \text{ iff } \nabla\emptyset \vee \nabla\{\varphi\}$$
$$\Diamond\varphi \text{ iff } \nabla\{\varphi, \top\}$$

Similarly, $\forall^+_{\mathsf{K}}$ and $\exists^+_{\mathsf{K}}$ can also be expressed in terms of the ∇^+ operator:

$$\forall^+_{\mathsf{K}}\varphi \text{ iff } \nabla^+_{\mathsf{K}}\{\varphi\}$$
$$\exists^+_{\mathsf{K}}\varphi \text{ iff } \nabla^+_{\mathsf{K}}\{\varphi, \top\}$$

Also, conjunction of two cover modalities is again a cover modality and similarly for negation: (Bílková et al., 2008)

$$\nabla\Phi \wedge \nabla\Psi \text{ iff } \nabla\left(\bigcup_{\varphi\in\Phi}\left(\varphi \wedge \bigvee\Psi\right) \cup \bigcup_{\psi\in\Psi}\left(\psi \wedge \bigvee\Phi\right)\right)$$
$$\neg\nabla\Phi \text{ iff } \nabla\{\bigwedge_{\varphi\in\Phi} \neg\varphi, \top\} \vee \bigvee_{\varphi\in\Phi} \nabla\{\neg\varphi\} \vee \nabla\emptyset$$

By writing out the definitions, it can be seen that the identities above also hold for ∇^+.

Using these new operators, we will introduce a new class of formulas on the $[\overrightarrow{\rightleftharpoons}]$-free fragment of $\mathcal{L}_{\forall^+\overrightarrow{\rightleftharpoons}}$. This class of formulas will be called the *disjunctive $\mathcal{SL}_{\mathsf{K}}$-formulas*. We will then show that all formulas in $\mathcal{SL}_{\mathsf{K}}$ are equivalent to a disjunctive $\mathcal{SL}_{\mathsf{K}}$-formula.

Definition 4.10. A disjunctive $\mathcal{SL}_{\mathsf{K}}$-formula is a formula that can be produced by the following grammar:

$$\varphi := \varphi \vee \varphi \mid \varphi_0 \wedge \nabla^+_{\mathsf{K}}\{\varphi, \ldots, \varphi\} \mid \varphi_0 \wedge \nabla\{\varphi, \ldots, \varphi\} \wedge \nabla^+_{\mathsf{K}}\{\varphi, \ldots, \varphi\}$$

where φ_0 is a formula in propositional logic.

Proposition 4.11. *Every formula in $\mathcal{SL}_{\mathsf{K}}$ is logically equivalent to a disjunctive $\mathcal{SL}_{\mathsf{K}}$-formula.*

Proof. The proof goes by induction on the complexity of the formula φ. Previously, it was shown that every formula in $\mathcal{BML}$ is equivalent to a formula of the form $\varphi \vee \varphi$ or $\varphi_0 \wedge \nabla\{\varphi, \ldots, \varphi\}$ (Hales et al., 2012). Therefore, the lemma holds for the case where φ is $\forall^+_{\mathsf{K}}$-free.

Now suppose that $\varphi = \neg\chi$. By the induction hypothesis, ψ is equivalent to a disjunctive $\mathcal{SL}_{\mathsf{K}}$-formula. Here we will only prove the most complicated case, where χ is equivalent to

$$\bigvee_i \varphi_{0i} \wedge \nabla\Phi_i \wedge \nabla^+_{\mathsf{K}}\Psi_i$$

Using distributive laws and the observations above, we then have the following equivalences:

$$\begin{aligned}
\neg\chi \text{ iff } & \neg\left(\bigvee_i \varphi_{0i} \wedge \nabla\Phi_i \wedge \nabla^+_{\mathsf{K}}\Psi_i\right)\\
\text{iff } & \bigwedge_i \left(\neg\varphi_{0i} \vee \neg\nabla\Phi_i \vee \neg\nabla^+_{\mathsf{K}}\Psi_i\right)\\
\text{iff } & \bigwedge_i \left[\neg\varphi_{0i} \vee \left(\nabla\{\bigwedge_{\varphi\in\Phi}\neg\varphi, \top\} \vee \bigvee_{\varphi\in\Phi}\nabla\{\neg\varphi\} \vee \nabla\emptyset\right) \vee \left(\nabla^+_{\mathsf{K}}\{\bigwedge_{\psi\in\Psi}\neg\psi, \top\} \vee \bigvee_{\psi\in\Psi}\nabla^+_{\mathsf{K}}\{\neg\psi\}\right)\right]
\end{aligned}$$

We can then use the distributive laws to rewrite the last line into a disjunction of conjunctions. We can then use the identities above to group conjunctions of ∇ and ∇^+. Since each disjunct will then be a disjunctive formula, the resulting formula is a disjunctive formula.

Next, suppose that we have a formula of the form $\varphi \wedge \psi$. If either of the two cases is a disjunction, we can apply the distributive laws to obtain a disjunction of two formulas of lower complexity. By the induction hypothesis, both formulas are equivalent to a disjunctive $\mathcal{SL}_{\mathsf{K}}$-formula, so the conclusion follows. Now consider the case where where $\varphi = \varphi_0 \wedge \nabla\Phi_1 \wedge \nabla^+_{\mathsf{K}}\Phi_2$ and $\psi = \psi_0 \wedge \nabla\Psi_1 \wedge \nabla^+_{\mathsf{K}}\Psi_2$. By noting the equivalences above, it is easy to see that this is equivalent to

$$(\varphi_0\wedge\psi_0)\wedge\nabla\left(\bigcup_{\varphi\in\Phi_1}\left(\varphi\wedge\bigvee\Psi_1)\right)\cup\bigcup_{\psi\in\Psi_1}\left(\psi\wedge\bigvee\Phi_1\right)\right)\wedge\nabla^+_{\mathsf{K}}\left(\bigcup_{\varphi\in\Phi_2}\left(\varphi\wedge\bigvee\Psi_2)\right)\cup\bigcup_{\psi\in\Psi_2}\left(\psi\wedge\bigvee\Phi_2\right)\right)$$

By the induction hypothesis, it follows that this is equivalent to a disjunctive formula. The case where φ or ψ is equivalent to $\varphi_0 \wedge \nabla^+_{\mathsf{K}}\Phi_2$ is analogous.

Finally, the cases for disjunction, ∇, and ∇^+ follow immediately by the induction hypothesis. □

4.4.1 The axiomatization

In chapter 3 we provided an axiomatization for $\mathcal{SL}_{\mathsf{K}}$. The resulting logic was called SL_{K}. Using the axioms and rules in table 4.1, we can rewrite every $\mathcal{L}_{\forall^+\overrightarrow{\rightleftharpoons}}$-formula into a logically equivalent formula in $\mathcal{SL}_{\mathsf{K}}$. Completeness of this system will then follow from the completeness of SL_{K}. Before we prove this in more detail, we will first prove the soundness of the axioms and rules in $\mathsf{StrongSimLog}_{\nabla}$.

Table 4.1: The logic $\mathsf{StrongSimLog}_{\nabla}$

1.	Axioms and rules of SL_{K}
2.	$[\overrightarrow{\rightleftharpoons}]\varphi \leftrightarrow \neg\langle\overrightarrow{\rightleftharpoons}\rangle\neg\varphi$
3.	$[\overrightarrow{\rightleftharpoons}]p \leftrightarrow p$
4.	$[\overrightarrow{\rightleftharpoons}]\neg p \leftrightarrow \neg p$
5.	$[\overrightarrow{\rightleftharpoons}](\varphi \to \psi) \to ([\overrightarrow{\rightleftharpoons}]\varphi \to [\overrightarrow{\rightleftharpoons}]\psi)$
6.	$\langle\overrightarrow{\rightleftharpoons}\rangle\nabla\Phi \leftrightarrow \Box\bigvee_{\varphi\in\Phi}\langle\overrightarrow{\rightleftharpoons}\rangle\varphi \wedge \bigwedge_{\varphi\in\Phi}\exists^+_{\mathsf{K}}\varphi$
7.	$[\overrightarrow{\rightleftharpoons}]\nabla^+_{\mathsf{K}}\Phi \leftrightarrow \nabla^+_{\mathsf{K}}\Phi$
8.	From φ, deduce $[\overrightarrow{\rightleftharpoons}]\varphi$

4.4.2 Soundness

Proposition 4.12. *The logic* $\mathsf{StrongSimLog}_{\nabla}$ *is sound*

Proof. It was shown in chapter 3 that all axioms and rules of SL_{L} are sound on all L-models. Also, axiom 2-5 and 8 are immediate from the definition of the simulation. Next, axiom 7 follows from the

fact that the truth of the validity operator is independent of the model of evaluation. Here we will only prove the axiom involving the cover modality: $\langle\rightrightarrows\rangle\nabla\Phi \leftrightarrow \Box\bigvee_{\varphi\in\Phi}\langle\rightrightarrows\rangle\varphi \wedge \bigwedge_{\varphi\in\Phi}\exists^+_{\mathsf{K}}\varphi$

Suppose that $\mathfrak{M}, x \models \langle\rightrightarrows\rangle\nabla\Phi$. Then there is a simulation $\mathfrak{M}', x'$ such that $\mathfrak{M}', x' \models \nabla\Phi$. Now, take any $y \in R(x)$. Then there is a $y' \in R'(x')$ such that yZy'. Since $\mathfrak{M}', x' \models \nabla\Phi$, there is some $\varphi_0 \in \Phi$ such that $\mathfrak{M}', y' \models \varphi_0$. Hence $\mathfrak{M}, y \models \langle\rightrightarrows\rangle\varphi_0$. So for all $y \in R(x)$ we have $\mathfrak{M}, y \models \bigvee_{\varphi\in\Phi}\langle\rightrightarrows\rangle\varphi$. From this, we get that $\mathfrak{M}, x \models \Box\bigvee\langle\rightrightarrows\rangle\varphi$. Also, since $\mathfrak{M}', x' \models \nabla\Phi$, there is for every $\varphi \in \Phi$ a $y' \in R'(x')$ such that $\mathfrak{M}', y' \models \varphi$. Hence, every $\varphi \in \Phi$ is satisfiable. Therefore $\mathfrak{M}, x \models \Box\bigvee_{\varphi\in\Phi}\langle\rightrightarrows\rangle\varphi \wedge \bigwedge_{\varphi\in\Phi}\exists^+_{\mathsf{K}}\varphi$.

Next, consider any model $\mathfrak{M}, x$ such that $\mathfrak{M}, x \models \Box\bigvee_{\varphi\in\Phi}\langle\rightrightarrows\rangle\varphi\wedge\bigwedge_{\varphi\in\Phi}\exists^+_{\mathsf{K}}\varphi$. Without loss of generality, we can assume that $\mathfrak{M}$ is tree-like. For every $y \in R(y)$, let $\mathfrak{M}_y$ denote the point-generated model at y. By the assumption, for every y there is a $\varphi_y \in \Phi$ such that $\mathfrak{M}_y, y \models \langle\rightrightarrows\rangle\varphi_y$. Hence there is a model $\mathfrak{M}'_y, y'$ such that $\mathfrak{M}_y, y \rightrightarrows \mathfrak{M}'_y, y'$ and $\mathfrak{M}'_y, y' \models \varphi_y$. Also, for every φ, there a model $\mathfrak{M}_\varphi, x_\varphi$ such that $\mathfrak{M}_\varphi, x_\varphi \models \varphi$. Then consider the model $\mathfrak{M}' = \{x\} \cup \bigcup_{y\in R(y)}\mathfrak{M}'_y \cup \bigcup_{\varphi\in\Phi}\mathfrak{M}_\varphi$, where xRy' for all $y \in R(x)$ and xRx_φ for all $\varphi \in \Phi$. Also $V'(x) = V(x)$, the valuation of x in $\mathfrak{M}$. Then clearly $\mathfrak{M}', x \models \nabla\varphi$. Also, for all successors $y \in R(x)$ there is a successor $y' \in R'(x)$ such that $\mathfrak{M}, y \rightrightarrows \mathfrak{M}', y'$. Hence $\mathfrak{M}, x \rightrightarrows \mathfrak{M}', x$, so we conclude $\mathfrak{M}, x \models \langle\rightrightarrows\rangle\nabla\varphi$. □

4.4.3 Completeness

The completeness proof goes by the usual reduction method. We will show that for every formula φ in $\mathcal{L}_{\forall^+\rightrightarrows}$, there is a formula $\psi \in \mathcal{SL}_{\mathsf{K}}$ such that $\vdash_{\mathsf{StrongSimLog}_\nabla} \varphi \leftrightarrow \psi$. This will be done by pushing the $\langle\rightrightarrows\rangle$ modality inside, until it reaches either a proposition letter or a $\forall^+_{\mathsf{K}}$-operator in which case the $\langle\rightrightarrows\rangle$ modality disappears by axiom 3, 4 or 7 respectively. The only problem is that we do not have a rule to let $\langle\rightrightarrows\rangle$ distribute over disjunction. This is where we will make use of the disjunctive $\mathcal{SL}_{\mathsf{K}}$ formulas, as shown in lemma 4.18. Before we can prove completeness this way, we must first prove some auxiliary lemmas.

Lemma 4.13. $\mathsf{StrongSimLog}_\nabla$ *proves substitution of equivalences*

Proof. The proof goes by induction as usual, with the case for $[\rightrightarrows]$ following by axiom 5 and rule 8. □

Lemma 4.14. $\vdash [\rightrightarrows](\varphi \wedge \psi) \leftrightarrow ([\rightrightarrows]\varphi \wedge [\rightrightarrows]\psi)$ *and* $\vdash \langle\rightrightarrows\rangle(\varphi \vee \psi) \leftrightarrow (\langle\rightrightarrows\rangle\varphi \vee \langle\rightrightarrows\rangle\psi)$

Proof. As usual, for example, analogous to the proof of lemma 3.15. □

Lemma 4.15. $\vdash ([\rightrightarrows]\varphi \vee [\rightrightarrows]\psi) \to [\rightrightarrows](\varphi \vee \psi)$ *and* $\vdash \langle\rightrightarrows\rangle(\varphi \wedge \psi) \to (\langle\rightrightarrows\rangle\varphi \wedge \langle\rightrightarrows\rangle\psi)$

Proof. Here we will only prove the first statement. The second one follows by contraposition.

$\vdash \varphi \to (\varphi \vee \psi)$	(propositional tautology)
$\vdash [\rightrightarrows](\varphi \to (\varphi \vee \psi)$	(rule 8)
$\vdash [\rightrightarrows]\varphi \to [\rightrightarrows](\varphi \vee \psi)$	(axiom 5)
$\vdash [\rightrightarrows]\psi \to [\rightrightarrows](\varphi \vee \psi)$	(same as for φ)
$\vdash ([\rightrightarrows]\varphi \vee [\rightrightarrows]\psi) \to [\rightrightarrows](\varphi \vee \psi)$	(propositional logic)

□

Lemma 4.16. *For all propositional* φ*, we have* $\vdash [\rightrightarrows]\varphi \leftrightarrow \varphi$ *and* $\vdash \langle\rightrightarrows\rangle\varphi \leftrightarrow \varphi$.

Proof. We will only prove $\vdash [\rightrightarrows]\varphi \leftrightarrow \varphi$, the case for $\langle\rightrightarrows\rangle$ is analogously by using the duals of the axioms. Without loss of generality, we assume that φ is in conjunctive normal form. We then proceed by induction on the structure of φ. If φ is a literal, then the lemma follows immediately by axiom 1 and 2.

Now suppose $\varphi = l_1 \vee \ldots \vee l_k$, for literals l_i. The right-to-left direction is as follows:

$$\begin{aligned}
\vdash [\rightrightarrows]\varphi &\to [\rightrightarrows](l_1 \vee l_2 \vee \ldots \vee l_k) && \text{(Definition)}\\
&\to [\rightrightarrows]\big(\neg l_1 \to (\neg l_2 \to (\ldots \to (\neg l_{k-1} \to l_k)\ldots)\big) && \text{(propositional logic)}\\
&\to [\rightrightarrows]\neg l_1 \to ([\rightrightarrows]\neg l_2 \to (\ldots \to ([\rightrightarrows]\neg l_{k-1} \to [\rightrightarrows] l_k)\ldots) && \text{(axiom 5)}\\
&\to \neg l_1 \to (\neg l_2 \to \ldots \to (\neg l_{k-1} \to l_k)\ldots) && \text{(induction hypothesis)}\\
&\to l_1 \vee l_2 \vee \ldots \vee l_k && \text{(propositional logic)}\\
&\to \varphi && \text{(definition)}
\end{aligned}$$

The other direction follows by reversing the arrows and using lemma 4.15.

If φ is a conjunction, then we simply apply lemma 4.14 and we are done. This case completes the proof by induction. □

Lemma 4.17. $\vdash \langle\rightrightarrows\rangle(\varphi \wedge \nabla^+_{\mathsf{K}}\Psi) \leftrightarrow (\varphi \wedge \nabla^+_{\mathsf{K}}\Psi)$ *for all propositional* φ.

Proof. Left-to-right is the easiest direction:

$$\begin{aligned}
&\vdash \langle\rightrightarrows\rangle(\varphi \wedge \nabla^+_{\mathsf{K}}\Psi) \to (\langle\rightrightarrows\rangle\varphi \wedge \langle\rightrightarrows\rangle\nabla^+_{\mathsf{K}}\Psi) && \text{(lemma 4.15)}\\
&\vdash \langle\rightrightarrows\rangle(\varphi \wedge \nabla^+_{\mathsf{K}}\Psi) \to (\varphi \wedge \langle\rightrightarrows\rangle\nabla^+_{\mathsf{K}}\Psi) && \text{(lemma 4.16)}\\
&\vdash \langle\rightrightarrows\rangle(\varphi \wedge \nabla^+_{\mathsf{K}}\Psi) \to (\varphi \wedge \nabla^+_{\mathsf{K}}\Psi) && \text{(axiom 7)}
\end{aligned}$$

The derivation for the right-to-left direction is done using the contrapositive.

$$\begin{aligned}
&\vdash \neg(\varphi \wedge \nabla^+_{\mathsf{K}}\Psi) \to (\nabla^+_{\mathsf{K}}\Psi \to \neg\varphi) && \text{(tautology)}\\
&\vdash [\rightrightarrows]\big(\neg(\varphi \wedge \nabla^+_{\mathsf{K}}\Psi) \to (\nabla^+_{\mathsf{K}}\Psi \to \neg\varphi)\big) && \text{(axiom 8)}\\
&\vdash [\rightrightarrows]\neg(\varphi \wedge \nabla^+_{\mathsf{K}}\Psi) \to ([\rightrightarrows]\nabla^+_{\mathsf{K}}\Psi \to [\rightrightarrows]\neg\varphi) && \text{(axiom 5)}\\
&\vdash [\rightrightarrows]\neg(\varphi \wedge \nabla^+_{\mathsf{K}}\Psi) \to ([\rightrightarrows]\nabla^+_{\mathsf{K}}\Psi \to \neg\varphi) && \text{(lemma 4.16)}\\
&\vdash [\rightrightarrows]\neg(\varphi \wedge \nabla^+_{\mathsf{K}}\Psi) \to (\nabla^+_{\mathsf{K}}\Psi \to \neg\varphi) && \text{(axiom 7)}\\
&\vdash (\varphi \wedge \nabla^+_{\mathsf{K}}\Psi) \to \langle\rightrightarrows\rangle(\varphi \wedge \nabla^+_{\mathsf{K}}\Psi) && \text{(propositional logic)}
\end{aligned}$$

□

Lemma 4.18. $\vdash \langle\rightrightarrows\rangle(\varphi \wedge \nabla^+_{\mathsf{K}}\Psi \wedge \nabla X) \leftrightarrow (\varphi \wedge \nabla^+_{\mathsf{K}}\Psi \wedge \langle\rightrightarrows\rangle\nabla X)$ *for all propositional* φ.

Proof. The proof of this lemma is very similar to the proof of lemma 4.17. Therefore, we will only proof the right-to-left direction. Again, we will prove the contrapositive, namely

$$[\rightrightarrows]\neg(\varphi \wedge \nabla^+_{\mathsf{K}}\Psi \wedge \nabla X) \to (\neg\varphi \vee \neg\nabla^+_{\mathsf{K}}\Psi \vee [\rightrightarrows]\neg\nabla X)$$

. This derivation is as follows:

$$\vdash \neg(\varphi \wedge \nabla^+_{\mathsf{K}}\Psi \wedge \nabla X) \to \big((\varphi \wedge \nabla^+_{\mathsf{K}}\Psi) \to \neg\nabla X\big) \qquad \text{(propositional tautology)}$$
$$\vdash [\rightrightarrows]\Big(\neg(\varphi \wedge \nabla^+_{\mathsf{K}}\Psi \wedge \nabla X) \to \big((\varphi \wedge \nabla^+_{\mathsf{K}}\Psi) \to \neg\nabla X\big)\Big) \qquad \text{(rule 8)}$$
$$\vdash [\rightrightarrows]\neg(\varphi \wedge \nabla^+_{\mathsf{K}}\Psi \wedge \nabla X) \to \big([\rightrightarrows](\varphi \wedge \nabla^+_{\mathsf{K}}\Psi) \to [\rightrightarrows]\neg\nabla X\big) \qquad \text{(rule 5)}$$
$$\vdash [\rightrightarrows]\neg(\varphi \wedge \nabla^+_{\mathsf{K}}\Psi \wedge \nabla X) \to \big(([\rightrightarrows]\varphi \wedge [\rightrightarrows]\nabla^+_{\mathsf{K}}\Psi) \to [\rightrightarrows]\neg\nabla X\big) \qquad \text{(lemma 4.14)}$$
$$\vdash [\rightrightarrows]\neg(\varphi \wedge \nabla^+_{\mathsf{K}}\Psi \wedge \nabla X) \to \big(([\rightrightarrows]\varphi \wedge \nabla^+_{\mathsf{K}}\Psi) \to [\rightrightarrows]\neg\nabla X\big) \qquad \text{(axiom 7)}$$
$$\vdash [\rightrightarrows]\neg(\varphi \wedge \nabla^+_{\mathsf{K}}\Psi \wedge \nabla X) \to \big((\varphi \wedge \nabla^+_{\mathsf{K}}\Psi) \to [\rightrightarrows]\neg\nabla X\big) \qquad \text{(lemma 4.16)}$$
$$\vdash [\rightrightarrows]\neg(\varphi \wedge \forall^+_{\mathsf{K}}\psi \wedge \chi) \to (\neg\varphi \vee \neg\nabla^+_{\mathsf{K}}\Psi \vee [\rightrightarrows]\neg\nabla X)) \qquad \text{(propositional logic)}$$

□

Proposition 4.19. *For every formula $\varphi \in \mathcal{L}_{\forall^+\rightrightarrows}$ there is a formula $\psi \in \mathcal{SL}_{\mathsf{K}}$ such that $\vdash_{\mathsf{StrongSimLog}_\nabla} \varphi \leftrightarrow \psi$.*

Proof. Take any $\varphi \in \mathcal{L}_{\forall^+\rightrightarrows}$. We will prove the proposition by induction on the number of occurrences of $\langle\rightrightarrows\rangle$. For the base case, if φ is $\langle\rightrightarrows\rangle$-free, then the statement is obvious. Therefore, suppose that φ has $n+1$ occurrences of $[\rightrightarrows]$. Take any subformula of the form $\langle\rightrightarrows\rangle\varphi_0$, with $\varphi_0 \in \mathcal{SL}_{\mathsf{K}}$. Since our language is finite, such a subformula exists. We prove by induction on the structure of φ_0 that there is a formula ψ_0 such that $\vdash_{\mathsf{StrongSimLog}_\nabla} \langle\rightrightarrows\rangle\varphi_0 \leftrightarrow \psi_0$, where $\psi_0 \in \mathcal{SL}_{\mathsf{K}}$. Then replacing $\langle\rightrightarrows\rangle\varphi_0$ by ψ_0 in φ results in a formula ψ such that $\vdash_{\mathsf{StrongSimLog}_\nabla} \varphi \leftrightarrow \psi$ and ψ has n occurrences of $\langle\rightrightarrows\rangle$. Hence we have proven the induction step.

So it remains to prove that ψ_0 exists. By proposition 4.11, every formula in $\mathcal{SL}_{\mathsf{K}}$ is logically equivalent to a disjunctive $\mathcal{SL}_{\mathsf{K}}$-formula. Therefore, we can without loss of generality assume that φ_0 is a disjunctive $\mathcal{SL}_{\mathsf{K}}$-formulas. First, suppose that $\varphi_0 = \varphi_1 \wedge \nabla^+_{\mathsf{K}}\Psi \wedge \nabla X$. Then we have

$$\vdash \varphi_0 \leftrightarrow \varphi_1 \wedge \nabla^+_{\mathsf{K}}\Psi \wedge \nabla X$$
$$\vdash \langle\rightrightarrows\rangle\varphi_0 \leftrightarrow \langle\rightrightarrows\rangle(\varphi_1 \wedge \nabla^+_{\mathsf{K}}\Psi \wedge \nabla X) \qquad \text{(axiom 5 and 8)}$$
$$\vdash \langle\rightrightarrows\rangle\varphi_0 \leftrightarrow (\varphi_1 \wedge \nabla^+_{\mathsf{K}}\Psi \wedge \langle\rightrightarrows\rangle\nabla X) \qquad \text{(lemma 4.18)}$$
$$\vdash \langle\rightrightarrows\rangle\varphi_0 \leftrightarrow \Big(\varphi_1 \wedge \nabla^+_{\mathsf{K}}\Psi \wedge \Box \bigvee_{\chi\in X} \langle\rightrightarrows\rangle\chi \wedge \bigwedge_{\chi\in X} \exists^+_{\mathsf{K}}\chi\Big) \qquad \text{(axiom 6)}$$

Then we can apply the induction hypothesis, to obtain a formula $\psi_0 \in \mathcal{SL}_{\mathsf{K}}$ such that $\vdash \langle\rightrightarrows\rangle\varphi_0 \leftrightarrow \psi_0$.

Next, suppose that φ_0 is of the form $\varphi_1 \wedge \nabla^+_{\mathsf{K}}\Psi$. Then we have the following:

$$\vdash \varphi_0 \leftrightarrow \varphi_1 \wedge \nabla^+_{\mathsf{K}}\Psi$$
$$\vdash \langle\rightrightarrows\rangle\varphi_0 \leftrightarrow \langle\rightrightarrows\rangle(\varphi_1 \wedge \nabla^+_{\mathsf{K}}\Psi) \qquad \text{(axiom 5 and 8)}$$
$$\vdash \langle\rightrightarrows\rangle\varphi_0 \leftrightarrow (\varphi_1 \wedge \nabla^+_{\mathsf{K}}\Psi) \qquad \text{(lemma 4.17)}$$

Finally, for the induction step, suppose that $\varphi = \varphi_1 \vee \varphi_2$. Then by lemma 4.14, it follows that

$$\vdash \langle\rightrightarrows\rangle\varphi \leftrightarrow \big(\langle\rightrightarrows\rangle\varphi_1 \vee \langle\rightrightarrows\rangle\varphi_2\big)$$

Then we can apply the induction hypothesis, to obtain a formula $\psi_0 \in \mathcal{SL}_{\mathsf{K}}$ such that $\vdash \langle\rightrightarrows\rangle\varphi_0 \leftrightarrow \psi_0$. Hence the proposition holds. □

Theorem 4.20. *The logic $\mathsf{StrongSimLog}_\nabla$ is complete for the logic $\mathcal{L}_{\forall^+\rightrightarrows}$: $\models \varphi$ implies $\vdash \varphi$ for all $\varphi \in \mathcal{L}_{\forall^+\rightrightarrows}$.*

Proof. This argument goes by the usual reduction argument. Let φ be any formula in $\mathcal{L}_{\forall^+\rightrightarrows}$ such that $\models \varphi$. By proposition 4.19, there is a $\psi \in \mathcal{SL}_{\mathsf{K}}$ such that $\vdash_{\mathsf{StrongSimLog}_\nabla} \varphi \leftrightarrow \psi$. Since $\mathsf{StrongSimLog}_\nabla$ is sound, we have $\models \psi$. It follows by completeness of SL_{K} for $\mathcal{SL}_{\mathsf{K}}$ that $\vdash_{\mathsf{SL}_{\mathsf{K}}} \psi$. Since all axioms of SL_{K} are part of $\mathsf{StrongSimLog}_\nabla$, we have $\vdash_{\mathsf{StrongSimLog}_\nabla} \psi$. Combining $\vdash_{\mathsf{StrongSimLog}_\nabla} \varphi \leftrightarrow \psi$ and $\vdash_{\mathsf{StrongSimLog}_\nabla} \psi$, we find that $\vdash_{\mathsf{StrongSimLog}_\nabla} \varphi$. Hence $\mathsf{StrongSimLog}_\nabla$ is complete for $\mathcal{L}_{\forall^+\rightrightarrows}$. □

4.4.4 Internalization

Now that we have a sound and complete axiom system for $[\rightrightarrows]$, we can prove the preservation law for strong simulations inside the logic. However, because the modality does not distribute over arbitrary disjunctions, we need to work with a normal form for the existential formulas.

Lemma 4.21. *Every existential formula is logically equivalent to a formula produced by the following grammar:*

$$\varphi := \varphi_0 \mid \varphi \wedge \varphi \mid \varphi_0 \vee \Diamond\varphi$$

where φ_0 is a formula in propositional logic.

Proof. We proceed by induction on the complexity of the positive existential formulas.

- In propositional case follows immediately by the first clause of the grammar.
- If we have a formula of the form $\varphi \wedge \psi$, then the statement holds by the second clause of the grammar and the induction hypothesis.
- Now suppose that we have a formula of the form $\varphi \vee \psi$. By the induction hypothesis, the following cases are exhaustive:
  - φ and ψ are propositional. Then $\varphi \vee \psi$ is propositional, so the lemma holds by the first clause of the grammar.
  - $\varphi = \varphi_1 \wedge \varphi_2$ or $\psi = \psi_1 \wedge \psi_2$. If $\varphi = \varphi_1 \wedge \varphi_2$, then $\varphi \vee \psi$ is logically equivalent to $(\varphi_1 \vee \psi) \wedge (\varphi_2 \vee \psi)$. By the induction hypothesis, both disjuncts are logically equivalent to a formula produced by the grammar, so the conclusion follows by the second clause of the grammar. The case for $\psi = \psi_1 \vee \psi_2$ is analogous.
  - φ is propositional and ψ is logically equivalent to $\psi_0 \vee \Diamond\psi_1$, where ψ_0 is propsositional (or the other way around). Then $\varphi \vee \psi$ is equivalent to $(\varphi \vee \psi_0) \vee \Diamond\psi_1$, so it can be produced by the third clause of the grammar.
  - φ and ψ are logically equivalent to $\varphi_0 \vee \Diamond\varphi_1$ and $\psi_0 \wedge \Diamond\psi_1$ respectively. Then $\varphi \vee \psi$ is logically equivalent to $(\varphi_0 \vee \psi_0) \vee \Diamond(\varphi_1 \vee \psi_1)$, so the conclusion follows by the third clause of the grammar.
- If we have a formula of the form $\Diamond\varphi$ the conclusion follows by the induction hypothesis and the third clause of the grammar (setting $\varphi = \top$).

□

Proposition 4.22. $\vdash \varphi \leftrightarrow [\rightrightarrows]\varphi$ *for all existential* φ.

Proof. We proceed by induction on the structure of φ in the form specified in lemma 4.21. The base case, where φ is propositional, is a direct consequence of lemma 4.16, where we proved that $\vdash [\rightrightarrows]\varphi \leftrightarrow \varphi$ for all propositional φ.

Next, suppose that $\varphi = \varphi_1 \wedge \varphi_2$. Then we have

$$\begin{aligned} \vdash (\varphi_1 \wedge \varphi_2) &\leftrightarrow ([\rightrightarrows]\varphi_1 \wedge [\rightrightarrows]\varphi_2) && \text{(Induction hypothesis)} \\ &\leftrightarrow [\rightrightarrows](\varphi_1 \wedge \varphi_2) && \text{(Lemma 4.14)} \end{aligned}$$

Finally, suppose that $\varphi = \varphi_0 \vee \Diamond\varphi_1$. Then we have

$$\begin{aligned} \vdash [\rightrightarrows](\varphi_0 \vee \Diamond\varphi_1) &\leftrightarrow \varphi_0 \vee [\rightrightarrows]\Diamond\varphi_1 && \text{(Lemma 4.15)} \\ &\leftrightarrow \varphi_0 \vee [\rightrightarrows]\nabla\{\varphi_1, \top\} \\ &\leftrightarrow \varphi_0 \vee \Diamond[\rightrightarrows]\varphi_1 \wedge \Diamond[\rightrightarrows]\top \wedge \forall^+_{\mathsf{K}}(\varphi \vee \top) && \text{(Dual of axiom 6)} \\ &\leftrightarrow \varphi_0 \vee \Diamond([\rightrightarrows]\varphi_1 \wedge [\rightrightarrows]\top) \wedge \forall^+_{\mathsf{K}}(\varphi \vee \top) \\ &\leftrightarrow \varphi_0 \vee \Diamond[\rightrightarrows]\varphi \\ &\leftrightarrow \varphi_0 \vee \Diamond\varphi && \text{(Induction Hypothesis)} \end{aligned}$$

□

4.5 Axiomatizing the Logic of Simulations

In the previous section, we axiomatized the logic of *strong* simulations. In the logic of regular simulations, the axiomatization becomes slightly more complicated, since we do not have $\models \varphi \leftrightarrow [\rightrightarrows]\varphi$ for all propositional φ anymore. For this reason, reduction axioms with cover modalities become way more complicated. That is the reason why we do not use those in this section. Instead, we can introduce a special normal form to axiomatize the logic of simulations. This time, we use the notion of BMNF formulas, which are closely related to the Normal Modal Form in Goranko (1991).

4.5.1 Basic Normal Modal Form

The Normal Modal Form (NMF in short) is introduced in Goranko (1991), as a tool for proving completeness of refutation systems. In our axiomatizations, we need a slightly stronger notion, which we call the Basic Normal Modal Form (BNFM for short).

Definition 4.23. A class of formulas in Normal Modal Form is defined inductively by the following rules:

1. All $\Box$-free formulas are in NMF.
2. If λ is propositional and $\psi, \theta_1, \ldots, \theta_k$ are in NMF, then $\lambda \vee \Diamond\psi$ and $\lambda \vee \Box\theta_1 \vee \ldots \vee \Box\theta_k \vee \Diamond\psi$ and $\lambda \vee \Box\theta_1 \vee \ldots \vee \Box\theta_k$ are in NMF.
3. If φ and ψ are in NMF, then $\varphi \wedge \psi$ is in NMF.

We will now define the BNFM, which is basically the NFM, except that we require the formulas to be in conjunctive normal form. In addition, we also need a clause for the validity operator, since it will be part of the axiomatization.

Definition 4.24. A formula $\varphi = \bigvee \varphi_i$ is a *Basic Normal Disjunction* (BND for short) iff (1) there is at most one i such that φ_i is of the form $\Diamond\psi$ and (2) no φ_i has conjunction as main connective.

Definition 4.25. The class of formulas in *Basic Normal Modal Form* (BNMF) is defined inductively:

1. p and $\neg p$ are in BNMF for all proposition letters p.
2. If φ is in BNMF, then $\Box\varphi, \Diamond\varphi, \forall^+_{\mathsf{K}}\varphi$ and $\exists^+_{\mathsf{K}}\varphi$ are in BNMF.
3. If φ is a BND and each disjunct of φ is a BNMF formula, then φ is a BNMF.

4. If $\varphi_1, \ldots, \varphi_n$ are BNMF formulas, then $\varphi_1 \wedge \ldots \wedge \varphi_n$ is a BNMF formula.

Equivalently, a formula is in BNMF form if it is in conjunctive normal form, except that the diamonds are pushed outwards instead of inwards. For example, $\Diamond q \vee \Diamond\Box p \vee \neg p$ is in disjunctive normal form, while it is not in BNMF, since the disjunction contains two terms of the form $\Diamond\varphi$. However, the equivalent formula $\Diamond(q \vee \Box p) \vee \neg p$ is in BNMF. Similarly, $(p \wedge q) \vee \Diamond\psi$ is in NMF, but not in BNMF. However, $(p \vee \Diamond\psi) \wedge (q \vee \Diamond\psi)$ is in BNMF. Indeed, as expected for a normal form, it turns out that every formula in $\mathcal{SL}_{\mathsf{K}}$ is equivalent to a formula in BNMF. For this, we need the following lemma.

Lemma 4.26. *Let $\varphi = \varphi_0 \vee \ldots \vee \varphi_k$. Then φ is logically equivalent to a conjunction of Basic Normal Disjunctions.*

Proof. Suppose φ is not already an BND. Then there are two disjuncts of the form $\Diamond\psi$, and/or there is at least one φ_i that has conjunction as main connective. We can then apply the following procedure:

1. First, replace every pair of disjuncts of the form $\Diamond\psi_1$ and $\Diamond\psi_2$, by $\Diamond(\psi_1 \vee \psi_2)$. Do this until at most 1 disjunct of the form $\Diamond\psi$ is left.
2. Suppose we have a disjunct $\varphi_i = \bigwedge_j \varphi_{ij}$. Then, by distributivity, φ is logically equivalent to $\bigwedge_j(\varphi_{ij} \vee \varphi_1 \vee \ldots \vee \varphi_{i-1} \vee \varphi_{i+1} \vee \ldots \vee \varphi_k)$. Repeatedly apply step this to all conjuncts until φ is in conjunction normal form.

By the procedure it is now clear that the resulting formula will be a conjunction of BNDs. □

Proposition 4.27. *Every formula φ in $\mathcal{SL}_{\mathsf{K}}$ has a logically equivalent formula in BNMF.*

Proof. All cases are immediate, except disjunction. Suppose $\varphi = \bigvee \varphi_i$. By the induction hypothesis, we can without loss of generality assume that all φ_i are in BNMF. By lemma 4.26, φ is equivalent to a conjunction of BNDs. Since each disjunct is in BNMF, it follows by the second item in the definition of BNMFs that φ is equivalent to a formula in BNMF. This completes the proof. □

4.5.2 The axiomatization

Given this new normal form, we can define our reduction system for the simulation modality in table 4.2. The soundness and completeness proofs are similar to the ones in the previous section. Note that almost all the axioms of SimLog are also sound for $[\rightrightarrows]$, except for axiom 4.

Table 4.2: The logic SimLog

1.	Axioms and rules of SL_{K}
2.	$[\rightrightarrows]\varphi \leftrightarrow \neg\langle\rightrightarrows\rangle\neg\varphi$
3.	$[\rightrightarrows]p \leftrightarrow p$
4.	$[\rightrightarrows]\neg p \leftrightarrow \bot$
5.	$[\rightrightarrows](\varphi \to \psi) \to ([\rightrightarrows]\varphi \to [\rightrightarrows]\psi)$
6.	$[\rightrightarrows](\bigvee \varphi_i) \leftrightarrow \bigvee_{\varphi_i}[\rightrightarrows]\varphi_i \vee \forall^+_{\mathsf{K}} \bigvee \varphi_i$ for every BND $\bigvee \varphi_i$
7.	$[\rightrightarrows]\Box\varphi \leftrightarrow \forall^+_{\mathsf{K}}\varphi$
8.	$[\rightrightarrows]\Diamond\varphi \leftrightarrow \Diamond[\rightrightarrows]\varphi$
9.	$[\rightrightarrows]\forall^+_{\mathsf{K}}\varphi \leftrightarrow \forall^+_{\mathsf{K}}\varphi$
9.	$[\rightrightarrows]\exists^+_{\mathsf{K}}\varphi \leftrightarrow \exists^+_{\mathsf{K}}\varphi$
10.	From φ, deduce $[\rightrightarrows]\varphi$

Lemma 4.28. *The following are derivable in* SimLog*:*

1. *Substitution of equivalences:* $\vdash_{\mathsf{SimLog}} [\rightrightarrows](\varphi \wedge \psi) \leftrightarrow ([\rightrightarrows]\varphi \wedge [\rightrightarrows]\psi)$

2. $[\rightrightarrows]$ *distributes over conjunction:* $\vdash_{\mathsf{SimLog}} [\rightrightarrows](\varphi \wedge \psi) \leftrightarrow ([\rightrightarrows]\varphi \wedge [\rightrightarrows]\psi)$

Proof. As usual using axiom 5 and rule 11. □

4.5.3 Soundness

The soundness proof for axiom 6 is quite long, so we treat this as a separate lemma. Then we will prove the soundness of the other axioms.

Lemma 4.29. *For every BND* $\varphi = \bigvee \varphi_i$*, we have*

$$\models [\rightrightarrows] \bigvee \varphi_i \leftrightarrow (\bigvee [\rightrightarrows]\varphi_i \vee \forall^+_{\mathsf{K}} \bigvee \varphi_i)$$

Proof. The right-to-left direction is trivial. Therefore we will only prove the left-to-right direction. Take any model $\mathfrak{M}, x \models [\rightrightarrows] \bigvee \varphi_i$ and suppose that $\bigvee \varphi_i$ is *not* a validity. For contradiction, suppose that $\mathfrak{M}, x \not\models \bigvee [\rightrightarrows]\varphi_i$. Without loss of generality, we can assume that $\mathfrak{M}$ is tree-like, using tree unraveling.

We will now construct an extension $\mathfrak{M}'$ of $\mathfrak{M}$ such that $\mathfrak{M}', x \not\models \bigvee \varphi_i$. Since every extension is a simulation of the original model, this contradicts our initial assumption that every models similar to $\mathfrak{M}$ satisfies $\bigvee \varphi_i$. Let $\varphi = \varphi_1 \vee \ldots \vee \varphi_k$. We now build $\mathfrak{M}'$ in stages: $\mathfrak{M}' = \mathfrak{M}_k$. During the construction, we will ensure that each model $\mathfrak{M}_i$ has the property that $\mathfrak{M}_i \not\models \bigvee_{j<i} \varphi_j$. Then $\mathfrak{M}' \not\models \bigvee \varphi_i$ as required. We set $\mathfrak{M}_0 = \mathfrak{M}$ and build $\mathfrak{M}_i$ as follows:

- If $\varphi_i = p$, then $\mathfrak{M}, x \not\models p$. Namely, otherwise we would have by the definition of a simulation that $\mathfrak{M}, x \models [\rightrightarrows]p$, which implies that $\mathfrak{M}, x \models \bigvee_i [\rightrightarrows]\varphi_i$. This contradicts our assumption though. Therefore, let $\mathfrak{M}_i = \mathfrak{M}_{i-1}$ and we conclude by the induction hypothesis that $\mathfrak{M}_i \not\models \bigvee_{j \leq i} \varphi_j$.
- Suppose $\varphi_i = \neg p$. Then let $\mathfrak{M}_i = (W_{i-1}, R_{i-1}, V_{i-1} \cup \{(p, x)\})$. Clearly $\mathfrak{M}_i, x \not\models \varphi_i$. Also note that the only formula that could become true at x after this change, would be p (since $\mathfrak{M}$ is irreflexive). However, if p is a disjunct in φ, then φ contains $p \vee \neg p$ and φ would be equivalent to $\top$, which is not the case. Therefore, for every disjunct $j \leq i$, if $\mathfrak{M}_{i-1}, x \not\models \varphi_j$, then $\mathfrak{M}_i, x \not\models \varphi_j$.
- Suppose $\varphi_i = \Box\psi$. Let $\mathfrak{N}, w$ be a model of $\neg\psi$ such that for the disjunct of φ that is of the form $\Diamond\chi$, we have $\mathfrak{N}, x \not\models \chi$. If this is impossible, then $\psi \vee \chi$ is a validity. Then $\Box\psi \vee \Diamond\chi$ is a validity, so φ is a validity, which we assumed not to be the case. Then we set $\mathfrak{M}_i$ as the disjoint union of $\mathfrak{M}_{i-1}$ and $\mathfrak{N}$, with xR_iw. Clearly, $\mathfrak{M}_i, x \not\models \Box\psi$. Also, the only type of formulas that could become true are of the form $\Diamond\chi$, but since $\mathfrak{N}, w \not\models \chi$, these will not become true by switching from $\mathfrak{M}_{i-1}$ to $\mathfrak{M}_i$. Therefore, for every disjunct φ_j of φ, if $\mathfrak{M}_{i-1}, x \not\models \varphi_j$, then $\mathfrak{M}_i, x \not\models \varphi_j$.
- Next, suppose that $\varphi_i = \Diamond\psi$. Let $y_1, y_2 \ldots$ be the successors of x that satisfy ψ. Take any such y_j. Then $\mathfrak{M}, y_j \not\models [\rightrightarrows]\psi$. Namely, otherwise we would have $\mathfrak{M}, x \models \Diamond[\rightrightarrows]\psi$, which, as will be shown in proposition 4.30 point 8, is equivalent to $\mathfrak{M}, x \models [\rightrightarrows]\Diamond\psi$. However, then $\mathfrak{M}, x \models [\rightrightarrows]\varphi_i$ and also $\mathfrak{M}, x \models \bigvee [\rightrightarrows]\varphi_i$, which is a contradiction. So there is some $\mathfrak{M}'_{y_j}$ that is similar to $\mathfrak{M}_{y_j}$ (the generated submodel of $\mathfrak{M}$ at y_j). Let $\mathfrak{M}_i$ be $\mathfrak{M}_{i-1}$ where each $\mathfrak{M}_{y_j}$ is replaced by $\mathfrak{M}_{y_j}$. Then $\mathfrak{M}_i, x \not\models \Diamond\psi$. Also, no other disjuncts become true by this operation, since there are no other diamonds in the disjunction and all boxes were false already by other successors of x.
- Finally, suppose that $\varphi_i = \forall^+_{\mathsf{K}}\psi$. If $\mathfrak{M}, x \models \forall^+_{\mathsf{K}}\psi$, then $\mathfrak{M}, x \models [\rightrightarrows]\forall^+_{\mathsf{K}}\psi$ (see proposition 4.30, item 9). However, then $\mathfrak{M}, x \models [\rightrightarrows]\varphi_i$, so also $\mathfrak{M}, x \models \bigvee [\rightrightarrows]\varphi_i$, which contradicts our assumption. Hence we set $\mathfrak{M}_i = \mathfrak{M}_{i-1}$ and conclude that $\mathfrak{M}_i \not\models \bigvee_{j<i} \varphi_j$.
- The case where $\varphi_i = \exists^+_{\mathsf{K}}\psi$ is analogous to the case where $\varphi_i = \forall^+_{\mathsf{K}}\psi$.

□

Proposition 4.30. *The logic* SimLog *is sound on all Kripke models.*

Proof. First note that $[\rightrightarrows]\varphi \to \varphi$, since the identity is a simulation. Therefore, we will only prove the other direction for the axioms.

1. Modal logic K is sound on all models.
2. By definition.
3. $p \to [\rightrightarrows]p$, since simulations preserve truth of propositional variables.
4. Suppose $\mathfrak{M}, x \models [\rightrightarrows]\neg p$. Then $\mathfrak{M}, x \models \neg p$. Consider $\mathfrak{M}'$ which is identical to $\mathfrak{M}$ except that $x \in V'(p)$. Then $\mathfrak{M}'$ is a simulation of $\mathfrak{M}$ and $\mathfrak{M}', x \not\models \neg p$. Hence $\mathfrak{M}, x \not\models [\rightrightarrows]\neg p$, which is a contradiction.
5. We have

$$\begin{aligned}
\mathfrak{M}, x \models [\rightrightarrows](\varphi \wedge \psi) \text{ iff for every } \mathfrak{M}', x' \rightrightarrows \mathfrak{M}, x, \text{ we have } \mathfrak{M}', x' &\models \varphi \wedge \psi \\
\text{iff for every } \mathfrak{M}', x' \rightrightarrows \mathfrak{M}, x, \text{ we have } \mathfrak{M}', x' &\models \varphi \text{ and } \mathfrak{M}', x' \models \psi \\
\text{iff for every } \mathfrak{M}', x' \rightrightarrows \mathfrak{M}, x, \text{ we have } \mathfrak{M}', x' &\models \varphi \\
\text{and for every } \mathfrak{M}', x' \rightrightarrows \mathfrak{M}, x, \text{ we have } \mathfrak{M}', x' &\models \psi \\
\text{iff } \mathfrak{M}, x \models [\rightrightarrows]\varphi \text{ and } \mathfrak{M}, x &\models [\rightrightarrows]\psi \\
\text{iff } \mathfrak{M}, x \models [\rightrightarrows]\varphi \wedge [\rightrightarrows]\psi &
\end{aligned}$$

6. See lemma 4.29.
7. It is obvious that if φ is a validity, then $\Box\varphi$ is a validity, and also $[\rightrightarrows]\Box\varphi$ is a validity. Therefore we are left with the left-to-right-direction, which we will prove by contraposition. Suppose that $\mathfrak{M}, x \not\models \forall^+_{\mathsf{K}}\varphi$. Then there is some model $\mathfrak{N}, w$, such that $\mathfrak{N}, w \not\models \varphi$. Consider the model $\mathfrak{M}' = (W', R', V')$ that is the disjoint union of $\mathfrak{M}$ and $\mathfrak{N}$, with the additional relation $xR'w$. Since every extension of a model is a simulation, $\mathfrak{M}'$ is a simulation of $\mathfrak{M}$. Clearly, also $\mathfrak{M}', x \not\models \Box\varphi$, so we have $\mathfrak{M}, x \not\models [\rightrightarrows]\Box\varphi$, as required.
8. Suppose that $\mathfrak{M}, x \models \Diamond[\rightrightarrows]\varphi$. Then there is a $y \in R(x)$ such that $\mathfrak{M}, y \models [\rightrightarrows]\varphi$. Take any $\mathfrak{M}', x'$ such that there is a simulation $Z : \mathfrak{M}, x \rightrightarrows \mathfrak{M}', x'$. Then there is a $y \in \mathfrak{M}'$ such that $x'R'y'$ and yZy'. Since $\mathfrak{M}, y \models [\rightrightarrows]\varphi$, we have $\mathfrak{M}', y' \models \varphi$. Hence $\mathfrak{M}, x \models \Diamond\varphi$. Since $\mathfrak{M}'$ was an arbitrary similar model, we conclude that $\mathfrak{M}, x \models [\rightrightarrows]\Diamond\varphi$.

   For the other direction, suppose that $\mathfrak{M}, x \not\models \Diamond[\rightrightarrows]\varphi$. If x has no successors, then by the identity simulation we conclude that $\mathfrak{M}, x \not\models \Diamond[\rightrightarrows]\varphi$. Next, suppose that x does have successors $y_1, y_2 \ldots$. For each y_i there is a model $\mathfrak{M}_i$ such that $\mathfrak{M}_{y_i}, y_i \rightrightarrows \mathfrak{M}_i, y_i$ and $\mathfrak{M}_i, y_i \not\models \varphi$. Consider the model $\mathfrak{M}'$ where each $\mathfrak{M}_{y_i}$ is replaced by $\mathfrak{M}_i$. This model is similar and $\mathfrak{M}', x \not\models \Diamond\varphi$. Hence $\mathfrak{M}, x \not\models [\rightrightarrows]\Diamond\varphi$.
9. The soundness of this axiom follows immediately from the fact that the truth of $\forall^+_{\mathsf{K}}\varphi$ is independent from the model: If $\mathfrak{M}, x \models \forall^+_{\mathsf{K}}\varphi$, then for all simulations $\mathfrak{M}', x'$ we have $\mathfrak{M}', x' \models \forall^+_{\mathsf{K}}\varphi$. Hence $\mathfrak{M}, x \models [\rightrightarrows]\forall^+_{\mathsf{K}}\varphi$.
10. This axiom is very similar to the previous axiom: if $\mathfrak{M}, x \models \exists^+_{\mathsf{K}}\varphi$, then for all simulations $\mathfrak{M}', x'$ we have $\mathfrak{M}', x' \models \exists^+_{\mathsf{K}}\varphi$. Hence $\mathfrak{M}, x \models [\rightrightarrows]\exists^+_{\mathsf{K}}\varphi$.
11. If φ holds in all models, then, in particular, it holds in all simulations of all models.

□

4.5.4 Completeness

Proposition 4.31. *For every $\varphi \in \mathcal{L}_{\underline{\rightarrow}}$, there is a $\psi \in \mathcal{SL}_{\mathsf{K}}$ such that $\vdash_{\mathsf{SimLog}} \varphi \leftrightarrow \psi$.*

Proof. We prove this by induction on the number of occurrences of $[\underline{\rightarrow}]$ in the formula φ. If $[\underline{\rightarrow}]$ does not occur in φ, then $\varphi \in \mathcal{SL}_{\mathsf{K}}$, so the proposition follows immediately by $\psi = \varphi$.

Now suppose that there are $n+1$ occurrences of $[\underline{\rightarrow}]$ in φ. Then there is a subformula in φ of the form $[\underline{\rightarrow}]\varphi_0$, where φ_0 is $[\underline{\rightarrow}]$-free. We will now prove that there is a formula ψ_0 in $\mathcal{SL}_{\mathsf{K}}$ such that $\vdash_{\mathsf{SimLog}} [\underline{\rightarrow}]\varphi_0 \leftrightarrow \psi_0$. Once we have proven this, we replace $[\underline{\rightarrow}]\varphi_0$ in φ by ψ_0. By substitution of equivalences, this formula is provably equivalent to φ. Since this new formula contains one fewer occurrence of $[\underline{\rightarrow}]$, it follows by the induction hypothesis that it is equivalent to a formula in $\mathcal{SL}_{\mathsf{K}}$. Hence φ is provably equivalent to a formula in $\mathcal{SL}_{\mathsf{K}}$.

So it remains to show that there exists a ψ_0 such that $\vdash_{\mathsf{SimLog}} [\underline{\rightarrow}]\varphi_0 \leftrightarrow \psi_0$. We do this by induction on the structure of φ_0. By proposition 4.27 and completeness of SL_{K}, we can without loss of generality assume that φ_0 is in BNMF.

- Suppose $\varphi_0 = p$. Then by axiom 3, we have $\vdash [\underline{\rightarrow}]p \leftrightarrow p$.
- Suppose $\varphi_0 = \neg p$. Then by axiom 4, we have $\vdash [\underline{\rightarrow}]\neg p \leftrightarrow \bot$.
- Suppose $\varphi_0 = \forall^+_{\mathsf{K}}\varphi_1$. Then by axiom 9, we have $\vdash [\underline{\rightarrow}]\forall^+_{\mathsf{K}}\varphi_1 \leftrightarrow \forall^+_{\mathsf{K}}\varphi_1$.
- Suppose $\varphi_0 = \exists^+_{\mathsf{K}}\varphi_1$. Then by axiom 10, we have $\vdash [\underline{\rightarrow}]\exists^+_{\mathsf{K}}\varphi_1 \leftrightarrow \exists^+_{\mathsf{K}}\varphi_1$.
- Suppose $\varphi_0 = \Box\varphi_1$. Then by axiom 7, we have $\vdash [\underline{\rightarrow}]\Box\varphi_1 \leftrightarrow \forall^+_{\mathsf{K}}\varphi_1$.
- Suppose $\varphi_0 = \Diamond\varphi_1$. Then we have

$$\begin{aligned}
&\vdash [\underline{\rightarrow}]\varphi_1 \leftrightarrow \psi_1 && \text{(Induction hypothesis)}\\
&\vdash \Diamond[\underline{\rightarrow}]\varphi_1 \leftrightarrow \Diamond\psi_1 && \text{(Modal logic)}\\
&\vdash [\underline{\rightarrow}]\Diamond\varphi_1 \leftrightarrow \Diamond\psi_1 && \text{(Axiom 8)}\\
&\vdash [\underline{\rightarrow}]\varphi_0 \leftrightarrow \Diamond\psi_1 && \text{(Definition)}
\end{aligned}$$

- Suppose $\varphi_0 = \varphi_1 \wedge \varphi_2$. Then it follows directly from lemma 4.28 and the induction hypothesis.
- Suppose $\varphi_0 = \bigvee \varphi_i$ is a Basic Normal Disjunction. This case follows immediately by axiom 6.

□

Theorem 4.32. *The logic* SimLog *is complete with respect to all Kripke models: $\models \varphi$ implies $\vdash \varphi$ for all $\varphi \in \mathcal{L}_{\underline{\rightarrow}}$.*

Proof. Suppose $\models \varphi$. By proposition 4.31, there is a $\psi \in \mathcal{SL}_{\mathsf{K}}$, such that $\vdash_{\mathsf{SimLog}} \varphi \leftrightarrow \psi$. Therefore, by soundness of SimLog, we have $\models \varphi \leftrightarrow \psi$ and $\models \psi$. Then, by completeness of SL_{K}, we find that $\vdash_{\mathsf{SL}_{\mathsf{K}}} \psi$. Since SL_{K} is included in SimLog, we get $\vdash_{\mathsf{SimLog}} \psi$. Combining this with $\vdash_{\mathsf{SimLog}} \varphi \leftrightarrow \psi$, it follows that $\vdash_{\mathsf{SimLog}} \varphi$, as required. □

4.5.5 Internalization

Just as for the strong simulation logic, we can use our axiom system to provide a syntactic proof that all positive existential formulas are preserved under simulations.

Lemma 4.33. $\vdash_{\mathsf{SimLog}} \varphi \leftrightarrow [\underline{\rightarrow}]\varphi$ *for all positive propositional formulas.*

Proof. Let φ be a positive formula in propositional logic. Without loss of generality, we can assume that the positive formula is equivalent to a positive formula in conjunction normal form. Then we have

$$\begin{aligned}
\vdash \varphi &\leftrightarrow \bigwedge\bigvee p_i \\
&\leftrightarrow \bigwedge\bigvee [\rightrightarrows] p_i && \text{(Axiom 3)} \\
&\leftrightarrow \bigwedge [\rightrightarrows] \bigvee p_i && \text{(Axiom 6)} \\
&\leftrightarrow [\rightrightarrows] \bigwedge\bigvee p_i && \text{(Lemma 4.28)} \\
&\leftrightarrow [\rightrightarrows]\varphi
\end{aligned}$$

□

Proposition 4.34. $\vdash_{\mathsf{SimLog}} \varphi \leftrightarrow [\rightrightarrows]\varphi$ *for all positive existential* φ

Proof. By slightly altering lemma 4.21, we find that all positive existential formulas are equivalent to a formula that can be produced by the following grammar:

$$\varphi = \varphi_0 \mid \varphi \wedge \varphi \mid \varphi_0 \vee \Diamond\varphi$$

where φ_0 is a positive propositional formula. We now prove the lemma by induction on the grammar above.

The base case where we have a positive propositional follows immediately from lemma 4.33. The conjunction case also follows immediately from lemma 4.28 and the induction hypothesis. Finally, for the case where we have a formula of the form $\varphi_0 \vee \Diamond\varphi_1$, note that

$$\begin{aligned}
\vdash \varphi_0 \vee \Diamond\varphi_1 &\leftrightarrow [\rightrightarrows]\varphi \vee \Diamond[\rightrightarrows]\varphi_1 && \text{(Induction hypothesis)} \\
&\leftrightarrow [\rightrightarrows]\varphi \vee [\rightrightarrows]\Diamond\varphi && \text{(Axiom 8)} \\
&\leftrightarrow [\rightrightarrows](\varphi \vee \Diamond\varphi) && \text{(Axiom 6)}
\end{aligned}$$

□

4.6 Conclusion

The main topic of this chapter was the basic modal language with an additional modality for quantifying over (strong) simulations. Simulations have many applications, ranging from process theory to abstract database theory and game theory. In this chapter, two sound and complete axiomatizations are provided for both the simulation modality as well as the strong simulation modality. Besides the pointers for further research mentioned at the end of this thesis (see chapter 7), further research in this topic could for example focus on extending the logic of strong simulations to μ-calculus in line with Bozzelli et al. (2014). Another possible extension could consist of extending the logic to a multi-agent setting. This would consist of altering and adding a few axioms, but probably not be too complex.

5. Logic of Homomorphisms

5.1 Introduction

Homomorphisms are structure-preserving maps and exist in virtually every branch of mathematics. In the case of modal logic, they are functions between models (or frames) that preserve the relations and truth of propositional variables. In this sense, they are very similar to simulations, with the only difference that homomorphisms are functions, whereas simulations are arbitrary relations. In fact, as will be shown in this chapter, two models have simulation between them iff there is a homomorphism on bisimilar models. Defining simulations in this way also leads to an alternative proof of the preservation law for homomorphisms in modal logic.

The rest of this chapter is focused on proving the soundness and completeness of two axiomatizations concerning homomorphisms. The first logic axiomatizes the basic model language enhanced with a modality that expresses that a formula φ holds on a given homomorphic model. The second axiomatization builds on this and is concerned with quantifying over homomorphisms. It studies a modality that expresses that a formula φ holds after applying any homomorphism from a set of homomorphisms. The main goal is to provide a proof inside the logic that all positive existential formulas are preserved under homomorphisms in both logics. Before all that, we will first establish the necessary notions and provide an overview of related work.

5.1.1 Preliminaries

Definition 5.1. Let $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ be two models. A *homomorphism* is a function $f : W \to W'$ such that

1. For every $w \in \mathfrak{M}$, if $w \in V(p)$ then $f(w) \in V'(p)$.
2. For every $w, v \in \mathfrak{M}$, if Rwv, then $R'f(w)f(v)$.

Homomorphisms are usually denoted with $f, g, h \ldots$. Given a model $\mathfrak{M}$, we call $\mathfrak{M}'$ a $\mathfrak{M}$-homomorphic model iff there is a homomorphism between $\mathfrak{M}$ and $\mathfrak{M}'$. If we have specified a homomorphism $f : \mathfrak{M} \to \mathfrak{M}'$, then $\mathfrak{M}'$ is also called the f-homomorphic model of $\mathfrak{M}$.

Definition 5.2. A formula φ is preserved under (surjective) homomorphisms iff whenever $f : \mathfrak{M} \to \mathfrak{M}'$ is a (surjective) homomorphism and $w \in \mathfrak{M}$, then $\mathfrak{M}, w \models \varphi$ implies $\mathfrak{M}', f(w) \models \varphi$.

5.1.2 Motivation and Related Work

Lyndon's theorem is one of the three fundamental preservation laws in classical model theory (Rossman, 2008). There are two versions of the theorem. The first one states that a formula in first-order logic is preserved under surjective homomorphisms if and only if it is equivalent to a positive formula. The second version states that a formula is preserved under arbitrary homomorphisms if and only if it is equivalent to a positive existential formula. Preservation results have always been at the core of model theory, especially around the sixties of the previous century (De Rijke, 1993). Therefore, it is interesting to consider a logical language that can talk about homomorphisms and thereby prove the preservation law inside the language. A modal analog of Lyndon's theorem was considered in De Rijke (1993) (theorem 6.6.16). However, there seems to be a mistake in the statement of the theorem. It states that a formula is preserved under surjective homomorphisms iff it is positive. However, $\Box p$ is

positive, while it is not preserved under homomorphisms. As will be shown later in this chapter, it turns out that the positive existential formulas are preserved under homomorphisms.

Besides internalizing the preservation law, there are are more reasons for considering this modality. Many of the examples that motivated the study of the simulation relation can, with a slight change, also apply to homomorphisms. Here we will highlight a few connections to other, related logics.

First of all, there is a connection between the Logic of Homomorphisms and the Logic of Abstraction, as in Ilin (2018). The Logic of Abstraction equips Kripke models with an equivalence relation Q over the world in the model. It then introduces the notion of a quotient model $\mathfrak{M}_Q$ and a modality $[Q]\varphi$, which states that φ holds in the quotient model. Given a model $\mathfrak{M} = (W, R, V, Q)$, the quotient model is defined as $\mathfrak{M}_Q = (W_Q, R_Q, V_Q, \mathsf{id})$, where

$$\begin{aligned} W_Q &= \{|w| \mid w \in W\} \\ |w| R_Q |v| &\text{ iff there is } w' \in |w| \text{ and } v' \in |v| \text{ such that } wRv \\ V(p) &= \{|w| \mid w \in V(p)\} \end{aligned}$$

With these relations, it follows that the map $f : \mathfrak{M} \to \mathfrak{M}_Q$ given by $w \mapsto |w|$ is a homomorphism. However, not every homomorphism is a map to a quotient model. Therefore, the logic of homomorphisms is weaker than the logic of abstraction, in the sense that it can prove fewer theorems. However, given the axiomatization of the logic of abstraction in Ilin (2018), it would be interesting to consider a modality that quantifies over equivalence relations, similar to the quantification over homomorphisms considered in this chapter.

The proof methods in this paper are inspired by results in other papers. The axiomatization of the homomorphism quantifier logic was inspired by the axiomatization of Arbitrary Public Announcement Logic (Balbiani et al., 2007, 2008) and Dynamic Logic for Learning Theory (Vargas Sandoval, 2020). Both papers use the technique of necessity forms (which originally goes back to Goldblatt (1982)) and witnessed theories. The notion of a so-called 'Admissible set of homomorphisms' is also used in Vargas Sandoval (2020), even though it is not mentioned explicitly. The application to homomorphisms, the definition of the admissible sets, and the internalization of the preservation law have never been investigated before.

5.2 Preservation results

As mentioned in the introduction, Lyndon's theorem states that a formula in first-order logic is preserved under surjective homomorphisms iff it is equivalent to a positive formula (Lyndon et al., 1959). Later, it was shown that formula is preserved under arbitrary homomorphisms iff it equivalent to a positive existential formula (Rossman, 2008). However, by the locality of the basic modal language, the preservation laws for surjective and non-surjective homomorphisms will be the same.

Proposition 5.3. *In the basic modal language, homomorphisms and surjective homomorphisms preserve exactly the same formulas.*

Proof. It is clear that all formulas that are preserved under homomorphisms are also preserved under surjective homomorphisms. For the other direction, suppose that there is some formula φ that is preserved under surjective homomorphisms. Take any homomorphism $f : \mathfrak{M} \to \mathfrak{M}'$. Then consider the homomorphism $g : \mathfrak{M} \sqcup \mathfrak{M}' \to \mathfrak{M}'$ given by

$$g(x) = \begin{cases} x & \text{if } x \in \mathfrak{M}' \\ f(x) & \text{if } x \in \mathfrak{M} \end{cases}$$

Clearly, g is a surjective homomorphism. Therefore, suppose that $\mathfrak{M}, x \models \varphi$ for some $x \in \mathfrak{M}$. Since formulas in the basic modal language are preserved under disjoint union, it follows that $\mathfrak{M} \sqcup \mathfrak{M}', x \models \varphi$.

Since φ is preserved under surjective homomorphisms, it follows $\mathfrak{M}', g(x) \models \varphi$. Since $x \in \mathfrak{M}$, it follows by definition of g that $\mathfrak{M}', f(x) \models \varphi$. Hence φ is preserved under f. Since f was an arbitrary homomorphism, it follows that φ is preserved under all homomorphisms. Hence the proposition follows. □

Note that the fact that the basic modal language is unable to distinguish between a surjective and a non-surjective homomorphism follows from the locality of modal logic. In the following, everything we prove for surjective homomorphisms does also hold for arbitrary homomorphisms.

The preservation law in this section, is based on the following relation between simulations and homomorphisms.

Proposition 5.4. *Take any two pointed models $\mathfrak{M}, x$ and $\mathfrak{N}, w$. Then the following are equivalent:*

1. *There is a simulation $Z : \mathfrak{M}, x \rightrightarrows \mathfrak{N}, w$.*
2. *There are $\mathfrak{M}', x' \leftrightarrows \mathfrak{M}, x$ and $\mathfrak{N}', w' \leftrightarrows \mathfrak{N}, w$ and a homomorphism $f : \mathfrak{M}' \to \mathfrak{N}'$ such that $f(x') = w'$.*

The proof of this proposition is split into two lemmas, each covering one of the two direction of the equality. First, we will prove the harder direction of the two: that every simulation can be turned into a homomorphism on bisimilar models.

Lemma 5.5. *Take any two models $\mathfrak{M}, x$ and $\mathfrak{N}, w$ such that there is a simulation Z from $\mathfrak{M}, x$ to $\mathfrak{N}, w$. Then there exists $\mathfrak{M}', x' \leftrightarrows \mathfrak{M}, x$ and $\mathfrak{N}', w' \leftrightarrows \mathfrak{N}, w$ such that there exists a homomorphism $f : \mathfrak{M} \to \mathfrak{N}$ such that $f(x') = w'$.*

Proof. Let us look at the construction of $\mathfrak{M}'$. For any point $y \in \mathfrak{M}$ we denote the generated submodel of $\mathfrak{M}$ at y by $\mathfrak{M}_y$. Let $\mathfrak{N} = \mathfrak{N}_w$ and $\mathfrak{M}'' = \mathfrak{M}_x$. This way, the simulation is defined on all elements in $\mathfrak{M}''$. We now build $\mathfrak{M}'$ and f at the same time by induction on the depth of the worlds in $\mathfrak{M}''$. We start at the point $x \in \mathfrak{M}$.

Let $Z_x = \{v \mid xZv\} = \{v_1, v_2, \ldots\}$. Now add $|Z_x|$ many copies of $\mathfrak{M}_x$ to $\mathfrak{M}''$, and label the roots $x_1, x_2, \ldots$. Then set $f(x_i) = v_i$. Next, take any point y_n of depth n and suppose that f is defined on the predecessor y_{n-1}. We denote $f(y_{n-1}) = v_{n-1}$. By the simulation, there is a point v_n such that $v_{n-1}Rv_n$ and y_nZv_n. If there are multiple, say k, such worlds, we do the same as before and make k copies of $\mathfrak{M}_{v_n}$ and map the roots of each copy to one of the v_ns. In this way, in the end, f is functional.

Call the resulting model $\mathfrak{M}'$. Since generated submodels andZ disjoint unions preserve bisimilarity, we have $\mathfrak{M}', x \leftrightarrows \mathfrak{M}'', x \leftrightarrows \mathfrak{M}, x$. Now, we will show that f is actually a homomorphism from $\mathfrak{M}'$ to $\mathfrak{N}'$. Clearly, f is functional by its construction. Next, take any $y' \in \mathfrak{M}'$. Then, by construction, there is an $y \in \mathfrak{M}$ such that $\mathfrak{M}', y' \leftrightarrows \mathfrak{M}, y$ and $yZf(y')$. Therefore if $y' \in V'(p)$ then $y \in V(p)$, and, since simulations preserve truth of proposition letters, we have $f(y)$ models p in $\mathfrak{N}'$. So the first condition of simulations is satisfied. Now, take any $x', y' \in \mathfrak{M}'$ such that $x'R'y'$. Then there are $x, y \in \mathfrak{M}$ such that $\mathfrak{M}', x' \leftrightarrows \mathfrak{M}, y$ and $\mathfrak{M}', y' \leftrightarrows \mathfrak{M}, y$ and xRy and $xZf(x')$ and $yZf(y')$. Since we restricted to generated submodels, it follows that $f(x')R'f(y')$ in $\mathfrak{N}'$ and the forth condition for the homomorphism is satisfied. Therefore, it follows that f is surjective homormorphism. □

Figure 5.1 below illustrates the construction in the proof of lemma 5.6 for a specific $\mathfrak{M}, \mathfrak{N}$. On the left, we see a simulation between two models, on the right we see $\mathfrak{M}'$ and $\mathfrak{N}'$ with the surjective homomorphism between them.

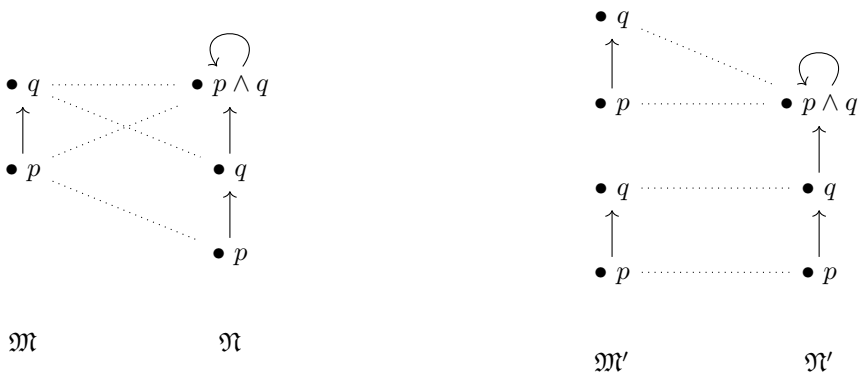

Figure 5.1: An illustration of the construction in lemma 5.5

Lemma 5.6. *Take any two models $\mathfrak{M}, x$ and $\mathfrak{N}, w$. If there exists $\mathfrak{M}', x' \leftrightarrows \mathfrak{M}, x$ and $\mathfrak{N}', w' \leftrightarrows \mathfrak{N}, w$ such that there exists a homomorphism $f : \mathfrak{M} \to \mathfrak{N}$ such that $f(x') = w'$, then there is a simulation Z from $\mathfrak{M}, x$ to $\mathfrak{N}, w$*

Proof. Let B_1 be the bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$ and B_2 the bisimulation between $\mathfrak{N}$ and $\mathfrak{N}'$. For every world $x \in \mathfrak{M}$, we let

$$xZw \text{ iff } (\exists x' \in \mathfrak{M}')(\exists w' \in \mathfrak{N}')(xB_1x' \wedge w' = f(x') \wedge wB_2w')$$

We will now show that Z is a simulation from $\mathfrak{M}$ to $\mathfrak{N}$. First suppose that xZw and $\mathfrak{M}, x \models p$. Then $\mathfrak{M}', x' \models p$ and since homomorphisms preserve the truth of proposition letters, we have $\mathfrak{N}', w' \models p$. Then by bisimilarity, $\mathfrak{N}, w \models p$, so Z preserves the truth of proposition letters.

Next, suppose xZw and $xR_{\mathfrak{M}}y$. Then there are $x' \in \mathfrak{M}'$ and $w' \in \mathfrak{N}'$ such that xB_1x' and $f(x') = w'$ and wB_2w'. By the conditions on simulations, there is a $y' \in \mathfrak{M}'$ such that $x'R_{\mathfrak{M}'}y'$. By the forth condition of homomorphisms, there is a $v' \in \mathfrak{N}'$ such that $w'R_{\mathfrak{N}'}v$ and $v' = f(y')$. However, since B_2 is a bisimulation, then we also know there is a $v \in \mathfrak{N}$ such that $wR_{\mathfrak{N}}v$ and vB_2v'. In addition we have yZv as required. Hence Z satisfies the forth condition and it follows that Z is a simulation. □

In the previous chapter, it was shown that a formula is preserved under simulations iff it is equivalent to a positive existential formula. Recall that a positive existential formula is a formula that does not contain $\Box$ and negation. Given proposition 5.4, it follows that exactly the same class of formulas is preserved under (surjective) homomorphisms.

Theorem 5.7 (Lyndon Theorem)**.** *A basic modal formula is preserved under (surjective) homomorphisms iff it is equivalent to a positive existential modal logic formula.*[1]

Proof. The proof that positive existential formulas are preserved under homomorphisms is a standard induction proof. Here, we will only proof the hard direction. Suppose that φ is *not* equivalent to a positive existential formula. Then, by theorem 4.5, there are models $\mathfrak{M}, \mathfrak{N}$, with worlds $x \in \mathfrak{M}$ and $w \in \mathfrak{N}$ and a simulation Z from $\mathfrak{M}, x$ to $\mathfrak{N}, w$, such that $\mathfrak{M}, x \models \varphi$ and $\mathfrak{N}, w \not\models \varphi$. By lemma 5.5, there are $\mathfrak{M}', x' \leftrightarrows \mathfrak{M}, x$ and $\mathfrak{N}', w' \leftrightarrows \mathfrak{N}, w$ such that there is a homomorphism $f : \mathfrak{M}' \to \mathfrak{N}'$ and $f(x') = w'$. By bisimilarity, we have $\mathfrak{M}', x' \models \varphi$, while $\mathfrak{N}', w' \not\models \varphi$. Hence φ is not preserved under f, so φ is not preserved under homomorphisms. This completes the proof. Figure 5.2 provides a visual overview of the proof. □

[1] Compare with De Rijke (1993), which states that all positive formulas are preserved.

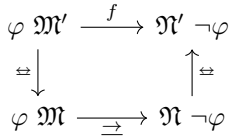

Figure 5.2: A visual representation of the proof of theorem 5.7

5.3 Axiomatizing the Logic of Homomorphisms

Given the characterization of simulations as homomorphisms on bisimilar models, one might wonder whether the axiomatization of the logic of homomorphisms in chapter 4 can also be used for homomorphisms. Unfortunately, the answer is negative. It turns out that the axiom involving disjunction (axiom 6) does not hold for homomorphisms. This axiom states that the simulation modalities distributes over certain disjunctions. However, consider a model $\mathfrak{M}$, consisting of a single reflexive world x and no propositional variables hold at x. Then for any homomorphism $f : \mathfrak{M} \to \mathfrak{M}_f$, we have that $\mathfrak{M}_f, f(x)$ will be a reflexive world, and therefore $\mathfrak{M}_f, f(x) \models \neg p \vee \Diamond p$. However, it is easy to construct homomorphisms $f : \mathfrak{M} \to \mathfrak{M}_f$ such that $\mathfrak{M}_f, x \not\models \neg p$ or where $\mathfrak{M}_f, x \not\models \Diamond p$. Hence, the homomorphism modality does not distribute over basic normal disjunctions. The underlying reason is that the homomorphism modality is not bisimulation invariant, unlike the simulation modality.

Therefore, we need a different approach. As a first step, we consider the homomorphism modality. Given a model $\mathfrak{M}$ and a homomorphism $f : \mathfrak{M} \to \mathfrak{M}'$, this modality expresses whether a formula holds at $\mathfrak{M}', f(x)$. Therefore we first enrich the syntax with the additional modality f.

Definition 5.8. Let $\mathcal{L}_f$ denote the language $\mathcal{BML}$ with an additional operator f:

$$\varphi = p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid f\varphi$$

In the semantics, we must also specify which homomorphism the operator f denotes. To this end, we specify a new type of model: the **f**-models.

Definition 5.9. An **f**-model is a tuple $\langle \mathfrak{M}, \mathbf{f} \rangle$, where $\mathfrak{M}$ is a Kripke model and $\mathbf{f} : \mathfrak{M} \to \mathfrak{M}$ an homomorphism.

Given an **f**-model $\mathfrak{M}$, we can define the semantics of the operator f as follows:

$$\mathfrak{M}, x \models f\varphi \text{ iff } \mathfrak{M}, \mathbf{f}(x) \models \varphi$$

Note. At first sight, the definition of **f**-models might seem very restrictive, since we require the homomorphism **f** to go from $\mathfrak{M}$ to $\mathfrak{M}$, instead of allowing arbitrary models as the range of **f**. However, this requirement does not lead to any loss of expressivity. Namely, suppose we have a homomorphism f_1 between **f**-model $\mathfrak{M}_1$ and **f**-model $\mathfrak{M}_2$. The model $\mathfrak{M}_2$ then also comes with a homomorphism $f_2 : \mathfrak{M}_2 \to \mathfrak{M}_3$ and so on. Then consider the model $\mathfrak{M} = \mathfrak{M}_1 \sqcup \mathfrak{M}_2 \sqcup \mathfrak{M}_3 \ldots$, i.e. the disjoint union of the models $\mathfrak{M}_1, \mathfrak{M}_2, \ldots$. Then we can construct a homomorphism $\mathbf{f}'$ from $\mathfrak{M}$ to $\mathfrak{M}$ that is logically equivalent to **f** since all modal formulas are preserved under disjoint union. Therefore, we can without loss of generality assume that **f** is a homomorphism from $\mathfrak{M}$ to $\mathfrak{M}$.

5.3.1 Axiomatization

In this subsection, we will provide a sound and complete axiomatization for the language $\mathcal{L}_f$. The axiomatization is given in table 5.1. Immediately below this axiomatization, we prove that some useful formulas are derivable in the logic Hom_f. After that, we proceed by proving soundness and

completeness. We end this section by showing that $\vdash_{\mathsf{Hom}_f} \varphi \to f\varphi$ for all positive existential φ, as expected.

Table 5.1: The logic Hom_f

1.	Axioms and rules of the basic modal logic K
2.	$f\bot \to \bot$
3.	$p \to fp$
4.	$\neg f\varphi \to f\neg\varphi$
5.	$f(\varphi \to \psi) \to (f\varphi \to f\psi)$
6.	$\Diamond f\varphi \to f\Diamond\varphi$
7.	From $\vdash \varphi$, prove $\vdash f\varphi$

Lemma 5.10. *The following are provable in the logic* Hom_f*:*

1. *If* $\vdash \varphi_1 \leftrightarrow \varphi_2$ *then* $\vdash \varphi[p\backslash\varphi_1] \leftrightarrow \varphi[p\backslash\varphi_2]$ *(Substitution of equivalences)*
2. $\vdash f(\varphi \wedge \psi) \leftrightarrow (f\varphi \wedge f\psi)$ *and* $\vdash (f\varphi \vee f\psi) \to f(\varphi \vee \psi)$
3. $\vdash f\neg\varphi \leftrightarrow \neg f\varphi$

Proof. All are straightforward. The first follows from axiom 5 and f-generalization (rule 7). The proof of the second is analogous to the proof for $\Box$ in K. For the last one, right-to-left follows immediately by axiom 4. The derivation for left-to-right is as follows (some steps in propositional logic are omitted for brevity):

$$\begin{aligned}
&\vdash \varphi \to (\neg\varphi \to \bot) && \text{(tautology)}\\
&\vdash f\big(\varphi \to (\neg\varphi \to \bot)\big) && \text{(rule 7)}\\
&\vdash f\varphi \to (f\neg\varphi \to f\bot) && \text{(axiom 5 twice)}\\
&\vdash f\varphi \to (f\neg\varphi \to \bot) && \text{(axiom 2)}\\
&\vdash f\varphi \to \neg f\neg\varphi && \text{(propositional logic)}\\
&\vdash f\neg\varphi \to \neg f\varphi && \text{(contrapositive)}
\end{aligned}$$

□

5.3.2 Soundness

Proposition 5.11. *The system* Hom_f *is sound on all models.*

Proof. Clearly all axioms and rules of K are sound. Axiom 2 is immediate. Axiom 3 follows by the fact that homomorphisms preserve the truth of proposition letters. Axiom 4, 5 and 6 follow by the functionality and the forth-condition of homomorphisms respectively. Finally, rule 7 is also obvious from the semantics. Therefore, we will only prove axiom 4 as an example.

$$\begin{aligned}
\mathfrak{M}, x \models \neg f\varphi &\text{ iff not } \mathfrak{M}, x \models f\varphi\\
&\text{iff not } \mathfrak{M}, \mathbf{f}(x) \models \varphi\\
&\text{iff } \mathfrak{M}, \mathbf{f}(x) \models \neg\varphi\\
&\text{iff } \mathfrak{M}, x \models f\neg\varphi
\end{aligned}$$

□

5.3.3 Completeness

Let us now turn to completeness. The completeness proof goes via the usual canonical model method. We will first show that every consistent set of formulas is contained in a maximally consistent set of formulas. Then we will define a canonical **f**-model, consisting of all maximally consistent sets and a particular homomorphism on this model. Then we will show that each maximally consistent set is satisfied at its own world in the canonical model. For this, we must first define what it means for a set of formulas to be a (maximal) theory.

Definition 5.12.
A Hom_f*-theory* is a Hom_f-consistent set of formulas.
A Hom_f*-maximal theory* is a theory Γ such that for all $\Gamma' \supset \Gamma$, Γ' is Hom_f-inconsistent.

When the logic is clear, we often drop the prefix and simply talk about a *theory* and maximal theory. The following properties of maximal theories will be useful in the proof.

Lemma 5.13. *Let Γ be an Hom_f-maximal theory. Then:*

1. *Γ is closed under Modus Ponens;*
2. *$\mathsf{Hom}_f \subseteq \Gamma$;*
3. *For every formula φ, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$;*
4. *For all φ, ψ, we have $\varphi \vee \psi \in \Gamma$, iff $\varphi \in \Gamma$ or $\psi \in \Gamma$.*

Proof. As usual. The lemma is an obvious extension of proposition 4.16 in Blackburn et al. (2001). □

Now that we have defined the necessary notions, we will work towards proving Lindenbaum's lemma, which states that every theory can be extended to a maximal theory. For this, we first prove several small lemmas.

Lemma 5.14. *If Γ is a theory and $\Gamma \nvdash \neg\varphi$, then $\Gamma \cup \{\varphi\}$ is a theory.*

Proof. Let Γ be any theory such that $\Gamma \nvdash \neg\varphi$. Now suppose that $\Gamma \cup \{\varphi\}$ is inconsistent. Then $\Gamma \vdash \neg\varphi$, which we assumed not to be the case. Hence $\Gamma \cup \{\varphi\}$ is consistent. □

Lemma 5.15. *If $\Gamma_1 \subseteq \Gamma_2 \subseteq \Gamma_3 \ldots$ are theories, then $\bigcup \Gamma_i$ is a theory.*

Proof. Suppose $\bigcup \Gamma_i$ is inconsistent. Then there are formulas $\varphi_1, \ldots, \varphi_n \in \Gamma$ such that $\vdash \varphi_1 \wedge \ldots \wedge \varphi_n \rightarrow \bot$. Then there are $k_1, \ldots, k_n$ such that $\varphi_i \in \Gamma_{k_i}$. Then let $m = \max\{k_1, \ldots, k_n\}$, i.e. the maximum of the k_is. Then $\varphi_1, \ldots, \varphi_n \in \Gamma_m$. Since a theory is closed under modus ponens, it follows that Γ_m is inconsistent. However, this contradicts that Γ_m is a theory. Hence we conclude that $\bigcup \Gamma_i$ is a theory. □

Lemma 5.16 (Lindenbaum Lemma)**.** *Let Γ be a theory. Then there is a $\Gamma' \supseteq \Gamma$ such that Γ' is a maximal theory.*

Proof. The proof follows the standard proof of Lindenbaum's theory for basic modal logic. Let $\varphi_1, \varphi_2, \ldots$ be an enumeration of the formulas in $\mathcal{L}_f$. We will construct an increasing chain $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$, such that for each $i \geq 0$, Γ_i is a theory. First, we set $\Gamma_0 = \Gamma$. We then define Γ_{i+1} as follows:

$$\Gamma_{i+1} = \begin{cases} \Gamma_i \cup \{\varphi_i\} & \text{if } \Gamma_i \nvdash \neg\varphi_i \\ \Gamma_i & \text{otherwise} \end{cases}$$

By lemma 5.14, Γ_{i+1} is consistent whenever Γ_i is. Now, let $\Gamma' = \bigcup \Gamma_i$. The claim is that Γ' is a maximal theory. We know that Γ' is a theory by lemma 5.15, so it remains to be proven that Γ' is maximal.

We show maximality by proving that for every formula φ, either $\varphi \in \Gamma'$ or $\neg\varphi \in \Gamma'$. If this is the case, any proper extension of Γ' will be inconsistent, hence Γ' is maximal. Therefore, take any formula φ_i in the enumeration. If $\Gamma_i \cup \{\varphi_i\}$ is consistent, then $\varphi_i \in \Gamma_{i+1} \subseteq \Gamma'$, so we are done. Now suppose $\Gamma_i \cup \{\varphi_i\}$ is inconsistent. Either $\neg\varphi_i \in \Gamma$ or $\Gamma \vdash \neg\varphi_i$. In the first case, we are done since $\Gamma_i \subseteq \Gamma'$. In the latter case, we have $\varphi_j = \neg\varphi_i$ for some $j > i$. Since $\Gamma_i \vdash \varphi_j$ and $\Gamma_i \subseteq \Gamma_j$, we have $\Gamma_j \vdash \varphi_j$. Hence $\Gamma_j \cup \{\varphi_j\}$ is consistent, so $\varphi_j \in \Gamma_{j+1} \subseteq \Gamma'$. Hence either $\varphi_i \in \Gamma'$ or $\neg\varphi_i \in \Gamma'$. □

The canonical model

We are now almost ready to define the canonical **f**-model. The model will consist of all maximal theories, as usual. However, we also need to define the homomorphism **f**. In the end, we want to have for any maximal theory Γ and any formula φ that $\mathfrak{M}, \Gamma \models \varphi$ iff $\varphi \in \Gamma$. In particular, we want for all ψ that $f\psi \in \Gamma$ iff $\mathfrak{M}, \mathbf{f}(\Gamma) \models \psi$. This naturally leads to the following definition.

Definition 5.17. For every set of formulas Γ, we define $\Gamma^f = \{\varphi \mid f\varphi \in \Gamma\}$

Lemma 5.18. *If Γ is a maximal theory, then Γ^f is a maximal theory.*

Proof. First, we show that Γ^f is consistent. For contradiction, suppose that Γ^f is not consistent. Then there are $\varphi_1, \ldots \varphi_n \in \Gamma^f$, such that $\vdash (\varphi_1 \wedge \ldots \wedge \varphi_n) \to \bot$. Then also:

$$\begin{aligned}
&\vdash \varphi_1 \to (\ldots \to (\varphi_n \to \bot)\ldots) && \text{(propositional logic)}\\
&\vdash f\big(\varphi_1 \to (\ldots \to (\varphi_n \to \bot)\ldots)\big) && \text{(Rule 7)}\\
&\vdash f\varphi_1 \to (\ldots \to (f\varphi_n \to f\bot)\ldots) && \text{(axiom 5)}\\
&\vdash f\varphi_1 \to (\ldots \to (f\varphi_n \to \bot)\ldots) && \text{(axiom 2)}\\
&\vdash (f\varphi_1 \wedge \ldots \wedge f\varphi_n) \to \bot && \text{(propositional logic)}
\end{aligned}$$

Since $\varphi_1, \ldots, \varphi_n \in \Gamma^f$, we have by definition that $f\varphi_1, \ldots f\varphi_n \in \Gamma$. Hence, by modus ponens, Γ is inconsistent, which is a contradiction. Hence Γ is consistent.

Next we show that Γ^f is maximal. Take any formula φ. By maximality of Γ, either $f\varphi \in \Gamma$ or $\neg f\varphi \in \Gamma$. If $f\varphi \in \Gamma$, then $\varphi \in \Gamma^f$. If $\neg f\varphi \in \Gamma$, then by axiom 4, we have $f\neg\varphi \in \Gamma$. Hence $\neg\varphi \in \Gamma$. So in either case, $\varphi \in \Gamma^f$ or $\neg\varphi \in \Gamma^f$, so Γ^f is maximal. □

Definition 5.19. Let the canonical **f**-model be $\mathfrak{M}_c = \langle W_c, R_c, V_c, \mathbf{f}_c\rangle$, where

$$\begin{aligned}
W_c &= \{\Gamma \mid \Gamma \text{ is a maximally consistent set}\}\\
R_c &= \{(\Gamma, \Delta) \mid \varphi \in \Delta \text{ implies } \Diamond\varphi \in \Gamma\}\\
V_c(p) &= \{\Gamma \mid p \in \Gamma\}\\
\mathbf{f}_c(\Gamma) &= \Gamma^f
\end{aligned}$$

This canonical **f**-model is very similar to the usual canonical model for basic modal logic, except that we have specified the additional homomorphism $\mathbf{f}_c$. This also means that results like the existence lemma and most cases of the truth lemma hold for the canonical **f**-model above. However, we must first prove that $\mathfrak{M}_c$ actually is an **f**-model. Once we have shown this, we can prove the Truth lemma. The completeness proof follows from the Truth lemma as usual.

Lemma 5.20. *The canonical model $\mathfrak{M}_c$ is well-defined. That is, $\mathfrak{M}_c$ is a valid* **f***-model.*

Proof. Here we need to show that $\mathbf{f}$ is a homomorphism. It is clear that $\mathbf{f}$ is a function. Therefore, suppose that $\mathfrak{M}_c, \Gamma \models p$. Then $\Gamma \in V_c(p)$, so by definition $p \in \Gamma$. Since $p \to fp$ is an axiom, we have $fp \in \Gamma$. Hence $p \in \Gamma^f$ and by definition of the valuation we then also have $\mathfrak{M}_c, \Gamma^f \models p$. Hence $\mathbf{f}$ preserves the truth of propositional variables.

Next, suppose that $\Gamma R_c \Delta$. We must now show that $\Gamma^f R_c \Delta^f$. That is, we must show that if $\varphi \in \Delta^f$ implies $\Diamond\varphi \in \Gamma^f$. Therefore, suppose $\varphi \in \Delta^f$. Then $f\varphi \in \Delta$. By definition of R_c, it follows that $\Diamond f\varphi \in \Gamma$. Since $\Diamond f\varphi \to f\Diamond\varphi$ is an axiom, we also have $f\Diamond\varphi \in \Gamma$. Hence, by definition, $\Diamond\varphi \in \Gamma^f$. So $\mathbf{f}$ satisfies the forth-condition as well, so we can conclude that $\mathbf{f}$ is a homomorphism. □

Lemma 5.21 (Truth lemma)**.** $\mathfrak{M}_c, \Gamma \models \varphi$ *iff* $\varphi \in \Gamma$

Proof. We proceed by induction on the structure of the formula φ. Almost all cases are identical to the proof of the truth lemma for the basic modal logic K. For a detailed proof, see lemma 4.21 in Blackburn et al. (2001) or the proof of lemma 5.46 later in this chapter. Here, we only prove the case where φ is of the form $f\psi$. The induction hypothesis states that for all $\Gamma \in \mathfrak{M}_c$, we have $\psi \in \Gamma$ iff $\mathfrak{M}_c, \Gamma \models \psi$.

$$\begin{aligned}
\mathfrak{M}_c, \Gamma \models f\psi &\iff \mathfrak{M}_c, \mathbf{f}(\Gamma) \models \psi \\
&\iff \mathfrak{M}_c, \Gamma^f \models \psi && \text{(Definition of } \mathbf{f}\text{)} \\
&\iff \psi \in \Gamma^f && \text{(Induction hypothesis)} \\
&\iff f\psi \in \Gamma && \text{(Definition of } \Gamma^f\text{)}
\end{aligned}$$

□

Corollary 5.22. *The system* Hom_f *is strongly complete: if* $\Phi \models \varphi$*, then* $\Phi \vdash \varphi$*.*

Proof. Suppose that $\Phi \nvdash \varphi$. Then the set $\Phi \cup \{\neg\varphi\}$ is consistent, so by lemma 5.16, there is some maximal theory Γ such that $\Phi \cup \{\varphi\} \in \Gamma$. By the truth lemma, it follows that $\mathfrak{M}_c, \Gamma \models \Gamma$. Hence Γ is satisfiable and so is $\Phi \cup \{\neg\varphi\}$. Hence $\Phi \not\models \varphi$. □

5.3.4 Internalization of Lyndon's theorem

As mentioned in the beginning of this chapter, Lyndon's theorem states that a formula is preserved under (surjective) homomorphisms iff it is equivalent to a positive existential formula. Or, expressed in the language of this section:

$$\models \varphi \to f\varphi \text{ iff } \varphi \text{ is positive existential}$$

Of course, by completeness it immediately follows that we can now 'prove' Lyndon's theorem inside the logic:

$$\vdash_{\mathsf{Hom}_f} \varphi \to f\varphi \text{ iff } \varphi \text{ is positive-existential}$$

By induction on the structure of the positive existential formulas, it is straightforward to construct proofs that derive the above implication, as shown in the following lemma.

Lemma 5.23. $\vdash \varphi \to f\varphi$ *for all positive existential* φ

Proof. We proceed by induction on the structure of the formula φ.

- Suppose $\varphi = p$. Then the conclusion is immediate by axiom 3.

- Suppose $\varphi = \varphi_1 \wedge \varphi_2$. Then it follows immediately by the induction hypothesis, lemma 5.10 and some propositional logic:

$$\begin{aligned}
&\vdash \varphi_1 \to f\varphi_1 && \text{(Induction hypothesis)}\\
&\vdash \varphi_2 \to f\varphi_2 && \text{(Induction hypothesis)}\\
&\vdash (\varphi_1 \wedge \varphi_2) \to (f\varphi_1 \wedge f\varphi_2) && \text{(Propositional logic)}\\
&\vdash (\varphi_1 \wedge \varphi_2) \to f(\varphi_1 \wedge \varphi_2) && \text{(Lemma 5.10)}
\end{aligned}$$

- The case where $\varphi = \varphi_1 \vee \varphi_2$ is identical to the case for conjunction.
- Suppose $\varphi = \Diamond\psi$. Then we have

$$\begin{aligned}
&\vdash \psi \to f\psi && \text{(Induction hypothesis)}\\
&\vdash \neg f\psi \to \neg\psi && \text{(Contraposition)}\\
&\vdash \Box(\neg f\psi \to \neg\psi) && \text{($\Box$-generalization)}\\
&\vdash \Box\neg f\psi \to \Box\neg\psi && \text{(Axiom K)}\\
&\vdash \Diamond\psi \to \Diamond f\psi && \text{(Contraposition)}\\
&\vdash \Diamond\psi \to f\Diamond\psi && \text{(Axiom 6)}
\end{aligned}$$

□

To internalize Lyndon's theorem one step further, we can introduce a modality that states that a formula φ holds in all models of an admissible set of homomorphic images. This is exactly the logic we consider in the following section.

5.4 Axiomatizing the logic of Homomorphism Quantifiers

In the previous section, a modality was introduced to represent the truth of a formula in a given homomorphic model. However, a Logic of Homomorphisms would ideally consist of a modality that states "for every homomorphism **f**, φ holds in the **f**-homomorphic model". Such an unbounded quantification would be hard to axiomatize. In this section, we therefore consider the language with a modality that quantifies over a set of homomorphisms. This approach is similar to general models in second-order logic: instead of quantifying over all relations and functions, the quantification happens over a specified subset of them (Henkin, 1950). In the following, we introduce the notion of such admissible sets, axiomatize the logic, and prove Lyndon's theorem inside this logic.

5.4.1 Preliminaries

Definition 5.24. Let the language $\mathcal{L}_{[\mathcal{H}]}$ be given by

$$\varphi = p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid f\varphi \mid [\mathcal{H}]\varphi$$

where $p \in \mathsf{PROP}$ and $f \in \mathsf{PROP}_{\mathsf{Hom}}$, a countably infinite set of homomorphism letters. Given a model $\mathfrak{M}$, let $\mathsf{Hom}_{\mathfrak{M}}$ denote the set of homomorphisms from $\mathfrak{M}$ to $\mathfrak{M}$.

Definition 5.25. Let $\mathfrak{M}$ be a Kripke model. An admissible set of homomorphisms $\mathcal{A}$ for $\mathfrak{M}$ is a set of homomorphisms from $\mathfrak{M}$ to $\mathfrak{M}$ such that:

1. The identity homomorphism is in $\mathcal{A}$
2. For all $f, g \in \mathcal{A}$, the composition $f \circ g$ is in $\mathcal{A}$.

That is, $\mathcal{A}$ is a monoid, with function composition as operation. For example, the set of all homomorphisms is clearly an admissible set. Also, the set consisting of only the identity homomorphism is also an admissible set. Suppose we have a set X consisting of homomorphisms. The admissible set generated by X consists of all homomorphisms in X, together with the identity homomorphism and is closed under function composition. From this it follows that for any set X, there exists an admissible set $\mathcal{A}_X$, such that $X \subseteq \mathcal{A}_X$. We now use the notion of the admissible set in the definition of H-models.

Definition 5.26. An H-model is a pair $\langle \mathfrak{M}, \mathcal{A}, H \rangle$, where $\mathfrak{M}$ is a Kripke model, $\mathcal{A}$ is an admissible set of homomorphisms for $\mathfrak{M}$ and $H : \mathsf{PROP}_{\mathsf{Hom}} \to \mathcal{A}$ is a valuation.

Given an H-model $\mathfrak{M}$, the semantics of the f and $[\mathcal{H}]$ modalities are given by

$$\mathfrak{M}, x \models f\varphi \text{ iff } \mathfrak{M}, H(f)(x) \models \varphi$$
$$\mathfrak{M}, x \models [\mathcal{H}]\varphi \text{ iff for all homomorphisms } \mathbf{f} \text{ in } \mathcal{A}, \text{ we have } \mathfrak{M}, \mathbf{f}(x) \models \varphi$$

As an example, consider the model $\mathfrak{M}$ in figure 5.3. The model consists of the natural numbers, the successor relation and each even number has an extra world it is related to. Then the function $\mathbf{f}_n : \mathfrak{N} \to \mathfrak{N}$ given by $i \mapsto i + 2n$ is a homomorphism for all $n \in \mathfrak{N}$. In addition $\mathcal{A} = \{\mathbf{f}_0, \mathbf{f}_1, \mathbf{f}_2, \ldots\}$ is an admissible set: $\mathbf{f}_0$ is the identity homomorphism and $\mathbf{f}_i \circ \mathbf{f}_j = \mathbf{f}_{i+j}$. Similarly $\mathcal{A}_e = \{\mathbf{f}_0, \mathbf{f}_2, \mathbf{f}_4, \ldots\}$ is an admissible set. However, $\mathcal{A}_o = \{\mathbf{f}_1, \mathbf{f}_3, \mathbf{f}_5, \ldots\}$ is not admissible, since the identity homomorphism is not in $\mathcal{A}_o$ and $\mathbf{f}_1 \circ \mathbf{f}_1 = \mathbf{f}_2 \notin \mathcal{A}_o$. Finally, the admissible set generated by $\{\mathbf{f}_2\}$ is precisely $\mathcal{A}_e$.

In this example, we can set $H(f_i) = \mathbf{f}_i$. Then $\langle M, \mathcal{A}, H \rangle$ is an H-model.

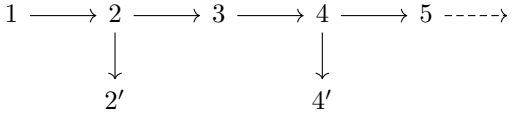

Figure 5.3: An example of an H-model

5.4.2 Axiomatization

Before we can state the axiomatization, we first need to introduce the notion of a necessity form as in Balbiani et al. (2007) and Vargas Sandoval (2020). A necessity form $[s]$ contains a unique occurrence of the symbol $\sharp$. Given such a necessecity form $[s]$ and a formula $\psi \in \mathcal{L}_{[\mathcal{H}]}$, we let $[s]\psi$ denote the formula obtained by replacing the occurrence of $\sharp$ in $[s]$ with ψ. Similarly as for modalities, we let $\langle s \rangle$ denote $\neg[s]\neg\psi$ and call them possibility forms.

Definition 5.27. The set of *necessity forms* is defined inductively as follows:

- $\sharp$ is a necessity form,
- If φ is a formula and $[s]$ a necessity form, then $\varphi \to [s]$ is a necessity form,
- If $[s]$ is a necessity form, then $f[s]$ is a necessity form.
- If $[s]$ is a necessity form, then $\Box[s]$ is a necessity form.

For example, $[s] = f_1(\varphi \to f_2\sharp)$ is a necessity form and $[s]\psi$ becomes $f_1(\varphi \to f_2\psi)$. Using the necessity forms, we can define the axiomatization for the language with homomorphism quantifiers. The axiomatization is given in table 5.2.

Table 5.2: The logic Hom

1.	Axioms and rules of the basic modal logic K
2.	$f\bot \to \bot$
3.	$p \to fp$
4.	$\neg f\varphi \to f\neg\varphi$
5.	$f(\varphi \to \psi) \to (f\varphi \to f\psi)$
6.	$\Diamond f\varphi \to f\Diamond\varphi$
7.	From φ, infer $f\varphi$
8.	$[\mathcal{H}]\varphi \to f\varphi$
9.	$[\mathcal{H}]\varphi \to \varphi$
10.	$[\mathcal{H}]\varphi \to [\mathcal{H}][\mathcal{H}]\varphi$
11.	From $[s]f\psi$, infer $[s][\mathcal{H}]\psi$ for $f \notin [s], \psi$

The first seven axioms are the same as in the logic Hom_f. These rules axiomatize the behaviour of the homomorphism letters. Axioms 8 to 11 are added to axiomatize the $[\mathcal{H}]$-modality. First, axiom 8 allows us to go from the $[\mathcal{H}]$-operator to the f-operator, thereby requiring that the admissible set is non-empty. Axiom 9 and 10 follow from the restrictions for the admissible set: axiom 9 states that the identity homomorphism is in $\mathcal{A}$ and axiom 10 states that the composition of two homomorphisms is again a homomorphism. Restricting or expanding the admissible set would result in removing or adding more axioms of this kind respectively. Finally, axiom 11 is interesting. It is inspired by the axiomatization of Arbitrary Public Announcement Logic in Balbiani et al. (2007) and the Dynamic Logic of Learning Theory in Vargas Sandoval (2020). The technique of using necessity forms goes back to Goldblatt in Goldblatt (1982). The idea behind the rule is that whenever we can prove for a fresh homomorphism f that $f\varphi$ holds, then φ holds for all homomorphism. In some sense, it is the reverse of axiom 8. The use of necessity forms is required to ensure that every consistent set can be 'witnessed', as will become clear later.

Before we continue with proving the soundness of the system, we will first state and prove a lemma for later use.

Lemma 5.28. *The following are provable in the logic* Hom*:*

1. $\vdash [\mathcal{H}]\varphi \leftrightarrow [\mathcal{H}][\mathcal{H}]\varphi$
2. *From* $\vdash \varphi$ *infer* $\vdash [\mathcal{H}]\varphi$
3. $\vdash [\mathcal{H}](\varphi \wedge \psi) \leftrightarrow ([\mathcal{H}]\varphi \wedge [\mathcal{H}]\psi)$ *and* $\vdash ([\mathcal{H}]\varphi \vee [\mathcal{H}]\psi) \to [\mathcal{H}](\varphi \vee \psi)$

Proof. We prove the statements one-by-one:

1. Left-to-right follows immediately from axiom 10 and right-to-left follows from axiom 9.
2. If $\vdash \varphi$, then by f-generalization, we have $\vdash f\varphi$. Since the formula φ is finite, we can without loss of generality assume that $f \notin \varphi$. Hence by rule 11 we get $\vdash [\mathcal{H}]\varphi$.
3. Analogous to the proof that $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ implies $\Box(\varphi \wedge \psi) \leftrightarrow (\Box\varphi \wedge \Box\psi)$ and $(\Box\varphi \vee \Box\psi) \to \Box(\varphi \vee \psi)$.

□

5.4.3 Soundness

We will first prove the soundness of the logic Hom. The soundness proof for $[\mathcal{H}]$-generalization requires the following two lemmas. The first is a version of the coincidence lemma for homomorphism letters and the latter proves the soundness of rule 11.

Lemma 5.29. *Let $\mathfrak{M}, x$ and $\mathfrak{M}', x$ be H-models, such that $\mathfrak{M}$ and $\mathfrak{M}'$ differ only by the valuation of some homomorphism letter f. Then for all formulas φ such that $f \notin \varphi$, we have $\mathfrak{M}, x \models \varphi$ iff $\mathfrak{M}', x \models \varphi$.*

Proof. We proceed by induction on the structure of the formula φ. If $\varphi = p$, a proposition letter, the lemma follows immediately, since the valuation for proposition letters in $\mathfrak{M}$ and $\mathfrak{M}'$ is identical. The Boolean cases and $\Box$-case are straightforward.

Suppose $\varphi = f_0\psi$ and $\mathfrak{M}, x \models f_0\psi$. Then $\mathfrak{M}, H(f_0)(x) \models \psi$. Since $f \notin \varphi$, we get $f \neq f_0$ and therefore $H(f_0) = H'(f_0)$. This, in combination with the induction hypothesis, gives $\mathfrak{M}', H'(f_0)(x) \models \psi$. So it follows that $\mathfrak{M}', x \models f_0\psi$. The other direction is analogous.

Suppose $\varphi = [\mathcal{H}]\psi$ and $\mathfrak{M}, x \models [\mathcal{H}]\psi$. Then for all $\mathbf{f} \in \mathcal{A}$, we get $\mathfrak{M}, \mathbf{f}(x) \models \psi$. Hence, by the induction hypothesis, $\mathfrak{M}', \mathbf{f}(x) \models \psi$ for all $\mathbf{f} \in \mathcal{A}'$. Since $\mathcal{A} = \mathcal{A}'$, it follows that $\mathfrak{M}', x \models [\mathcal{H}]\psi$. Again, the other direction is analogous. □

Lemma 5.30. *Let $\mathfrak{M}, x$ be an H-model. If $\mathfrak{M}, x \not\models [s][\mathcal{H}]\psi$, then there is an $\mathfrak{M}', x'$, such that $\mathfrak{M}', x' \not\models [s]f\psi$ for some $f \notin [s], \psi$.*

Proof. We proceed by induction over the structure of the necessity form $[s]$. Let $\mathfrak{M} = \langle W, R, V, \mathcal{A}, H\rangle$. The induction hypothesis states that if $\mathfrak{M}, x \not\models [s][\mathcal{H}]\psi$, then there is an $\mathfrak{M}' = \langle W, R, V, \mathcal{A}, H'\rangle$, such that (1) $\mathfrak{M}', x \not\models [s]f\psi$ for some $f \notin [s], \psi$, and (2) H' and H only differ at the valuation of f.

Suppose $[s] = \sharp$, so $[s][\mathcal{H}]\psi$ is $[\mathcal{H}]\psi$. Let $\mathfrak{M}, x \not\models [\mathcal{H}]\psi$, where $\mathfrak{M} = \langle W, R, V, \mathcal{A}, H\rangle$. Then there is some homomorphism $\mathbf{f} \in \mathcal{A}$, such that $\mathfrak{M}, \mathbf{f}(x) \not\models \psi$. Now take any homomorphism letter $f \notin \psi$. Then we set $\mathfrak{M}' = \langle W, R, V, \mathcal{A}, H'\rangle$, where $H' = H$ except that $H(f) = \mathbf{f}$. Since $f \notin \psi$, it follows by lemma 5.29 that $\mathfrak{M}', \mathbf{f}(x) \not\models \psi$, so $\mathfrak{M}', x \not\models f\psi$. Equivalently, $\mathfrak{M}', x \not\models [s]f\psi$.

Suppose $[s] = f_0[s']$. Then $[s][\mathcal{H}]\psi = f_0[s'][\mathcal{H}]\psi$. Let $\mathfrak{M} = \langle W, R, V, \mathcal{A}, H\rangle$ and suppose $\mathfrak{M}, x \not\models f_0[s'][\mathcal{H}]\psi$. Then $\mathfrak{M}, H(f_0)(x) \not\models [s'][\mathcal{H}]\psi$. Hence by the induction hypothesis, there is a model $\mathfrak{M}' = \langle W, R, V, \mathcal{A}, H'\rangle$ such that $\mathfrak{M}', H'(f_0)(x) \not\models [s']f\psi$ for some $f \notin [s'], \psi$. Without loss of generality, we can assume that $f \neq f_0$. It then follows that $\mathfrak{M}', x \not\models f_0[s']f\psi$, as required.

Suppose $[s][\mathcal{H}]\psi = \varphi \to [s'][\mathcal{H}]\psi$. Again let $\mathfrak{M}, x \not\models \varphi \to [s']\langle\mathcal{H}\rangle\psi$. Then $\mathfrak{M}, x \models \varphi$ and $\mathfrak{M}, x \not\models [s']\langle\mathcal{H}\rangle\psi$. By the induction hypothesis, there is an $\mathfrak{M}'$ such that $\mathfrak{M}', x \not\models [s']f\psi$ for some $f \notin [s'], \psi$. By the induction hypothesis, the only difference between $\mathfrak{M}$ and $\mathfrak{M}'$ is the valuation of $H(f)$. Without loss of generality, we can assume that $f \notin \varphi$, so by lemma 5.29, it follows that $\mathfrak{M}', x \models \varphi$. Therefore it follows that $\mathfrak{M}', x \models \varphi \wedge \neg[s']f\psi$. This is equivalent to $\mathfrak{M}', x \not\models \varphi \to [s']f\psi$, which is what we need to prove.

Suppose $[s] = \Box[s']$. Let $\mathfrak{M}, x \not\models \Box[s']\langle\mathcal{H}\rangle\psi$. Then there is a $y \in \mathfrak{M}$ such that xRy and $\mathfrak{M}, y \not\models [s']\langle\mathcal{H}\rangle\psi$. By the induction hypothesis, there is some $\mathfrak{M}'$ such that $\mathfrak{M}', y \not\models [s']f\psi$ for some $f \notin [s], \psi$. Since $\mathfrak{M}'$ differs some $\mathfrak{M}$ only by the homomorphism valuation at f and $f \notin [s], \psi$, it follows that $\mathfrak{M}', x \not\models \Box[s']f\psi$, so $\mathfrak{M}', x \not\models [s]f\psi$. □

Proposition 5.31. *The logic* Hom *is sound on all models*

Proof. The first 6 axioms and f-generalization are sound by proposition 5.11. Axiom 8 is immediate from the definition. Axiom 9 holds since the identity homomorphism is in all admissible sets of homomorphisms. Similarly, axiom 10 holds since the admissible sets of homomorphisms are closed under function composition. Therefore we only prove rule 11. Suppose $\not\models [s][\mathcal{H}]\psi$. Then there is some model $\mathfrak{M}, x$ such that $\mathfrak{M}, x \not\models [s][\mathcal{H}]\psi$. By lemma 5.30, there is a model $\mathfrak{M}', x'$ such that $\mathfrak{M}', x' \not\models [s]f\psi$ for some $f \notin [s], \psi$. Hence $\not\models [s]f\psi$, as required. □

5.4.4 Completeness

Now that we have established the soundness of the axiom system, let us turn to completeness. The completeness proof follows the standard canonical method, only this time we work with so-called *witnessed* maximal theories. We will now first introduce witnessed theories and prove Lindenbaum's lemma for witnessed theories.

Definition 5.32.
A Hom-*theory* is a Hom-consistent set of formulas.
A *maximal* Hom-*theory* is a Hom-theory Γ that is not a proper subset any other Hom-theory.
A $\mathsf{PROP}_{\mathsf{Hom}}$-*witnessed theory* is a theory Γ such that if $\langle s\rangle\langle\mathcal{H}\rangle\psi$ is consistent with Γ, then there is some homomorpism letter $f \in \mathsf{PROP}_{\mathsf{Hom}}$ such that $f \notin \langle s\rangle, \psi$ and $\langle s\rangle f\psi$ is consistent with Γ.
A *maximal, witnessed theory* is a witnesed theory Γ is not a proper subset of any witnessed theory.

When the set $\mathsf{PROP}_{\mathsf{Hom}}$ is fixed, we often talk simply about a witnessed theory and omit the prefix $\mathsf{PROP}_{\mathsf{Hom}}$. Also, similarly as before, we will usually drop the prefix Hom, when no confusion can arise.

Lemma 5.33. *If Γ is a witnessed theory and $\Gamma \nvdash \neg\varphi$, then $\Gamma \cup \{\varphi\}$ is a witnessed theory.*

Proof. We know that $\Gamma\cup\{\varphi\}$ is a theory by lemma 5.14. It remains to show that $\Gamma\cup\{\varphi\}$ is witnessed. Suppose for contradiction that $\Gamma \cup \{\varphi\}$ is not witnessed. Then there is a possibility form $\langle s\rangle$ and a formula ψ such that $\langle s\rangle\langle\mathcal{H}\rangle\psi$ is consistent with $\Gamma\cup\{\varphi\}$ but for all homomorphism letters $f \notin \langle s\rangle, \psi$, we have that $\langle s\rangle f\psi$ is not consistent with $\Gamma \cup \{\varphi\}$. Hence $\Gamma \cup \{\varphi\} \vdash \neg\langle s\rangle f\psi$ for all $f \notin \langle s\rangle, \psi$. Using the definition of the possibility form and axiom 4, it follows that $\Gamma\cup\{\varphi\} \vdash [s]f\neg\psi$. Hence $\Gamma \vdash \varphi \to [s]f\neg\psi$. Now, let $[s'] = \varphi \to [s]$ be a necessity form. Then $\Gamma \vdash [s']f\neg\psi$ for all $f \notin \langle s\rangle, \psi$. By axiom 4, it follows that $\Gamma \vdash \neg\langle s'\rangle f\psi$. Now, since Γ contains witnesses and $\langle s'\rangle f\psi$ is inconsistent with Γ for all f, it follows that $\Gamma \vdash \neg\langle s'\rangle\langle\mathcal{H}\rangle\psi$. Using axiom 4 again, we obtain $\Gamma \vdash [s'][\mathcal{H}]\psi$. Hence, by definition, $\Gamma \vdash \varphi \to [s][\mathcal{H}]\neg\psi$ and so $\Gamma \cup \{\varphi\} \vdash \neg\langle s\rangle\langle\mathcal{H}\rangle\psi$. However, we assumed that $\langle s\rangle\langle\mathcal{H}\rangle\psi$ is consistent with $\Gamma \cup \{\varphi\}$, so we have a contradiction. Hence we conclude that $\Gamma \cup \{\varphi\}$ contains witnesses. □

Lemma 5.34 (Lindenbaum's Lemma)**.** *Every witnessed theory Γ can be extended to a maximal witnessed theory Γ'.*

Proof. The proof is similar to the proof of the Lindenbaum lemma in the previous section (lemma 5.16), but now we need to make sure that the resulting set is witnessed. Therefore we change the definition of Γ_{i+1} slightly. Again, we will construct a increasing chain $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$, such that for each $i \geq 0$, Γ_i is a witnessed theory. First, we set $\Gamma_0 = \Gamma$.

Now assume that Γ_i is a witnessed theory. If $\Gamma_i\cup\{\varphi_i\}$ is inconsistent, then set $\Gamma_{i+1} = \Gamma_i$. Now suppose that $\Gamma_i \cup \{\varphi_i\}$ is consistent. If φ_i is of the form $\langle s\rangle\langle\mathcal{H}\rangle\psi$, then, since Γ_i is witnessed, there is a homomorphism letter $f \notin \langle s\rangle, \psi$ such that $\Gamma \cup \{\langle s\rangle f\psi\}$ is consistent. Then set $\Gamma_i = \Gamma \cup \{\langle s\rangle\langle\mathcal{H}\rangle\psi, \langle s\rangle f\psi\}$. Finally, if φ_i is not of the form $\langle s\rangle\langle\mathcal{H}\rangle\psi$, then we set $\Gamma_{i+1} = \Gamma \cup \{\varphi\}$. By lemma 5.33, Γ_{i+1} is a witnessed theory.

The proof of maximality is identical to the proof of maximality in lemma 5.16. So we will focus on showing that Γ' is witnessed. Take any formula $\varphi_i = \langle s\rangle\langle\mathcal{H}\rangle\psi$ that is consistent with Γ'. By maximality of Γ', it follows that $\langle s\rangle\langle\mathcal{H}\rangle\psi \in \Gamma'$. Hence $\langle s\rangle\langle\mathcal{H}\rangle\psi$ was added at stage i. However, then $\langle s\rangle f\psi$ was also added at stage i, for some $f \notin \langle s\rangle, \psi$ such that $\langle s\rangle f\psi$ is consistent with Γ_i. Hence $\langle s\rangle f\psi \in \Gamma'$, so $\langle s\rangle f\psi$ is consistent with Γ'. Hence Γ' is witnessed. □

Now, we have proven Lindenbaum's lemma for witnessed theories. However, this version requires a witnessed theory to begin with! Therefore, we need another lemma, which allows us to form witnessed theories from arbitrary theories. This is taken care of by the Extension lemma.

Lemma 5.35 (Extension lemma)**.** *Let P be a set of homomorphism letters and P' a countably infinite set of 'fresh' homomorphism letters, that is, $P \cap P' = \emptyset$. For every P-theory Γ, there is a $P \cup P'$-witnessed theory $\Gamma' \supseteq \Gamma$.*

Proof. Take any theory Γ. Let $f'_0, f'_1, \ldots$ be an enumeration of the homomorphism letters in P'. Next, let $\varphi_0, \varphi_1, \ldots$ be an enumeration of all formulas of the form $\langle s\rangle\langle\mathcal{H}\rangle\psi$ for $\psi \in \mathcal{L}_{[\mathcal{H}]}$. We now define an increasing chain $\Gamma_0 \subseteq \Gamma_1 \subseteq \ldots$ of P'-theories. First, we set $\Gamma_0 = \Gamma$. We then define Γ_{i+1} as follows:

$$\Gamma_{i+1} = \begin{cases} \Gamma_i \cup \{\langle s\rangle f'_i\psi\} & \text{for } \varphi_i = \langle s\rangle\langle\mathcal{H}\rangle\psi \text{ and } \Gamma_i \cup \{\varphi_i\} \text{ consistent} \\ \Gamma_i & \text{otherwise} \end{cases}$$

Since f_i is fresh, Γ_{i+1} is consistent if Γ_i is consistent. Since $\Gamma_0 = \Gamma$ is consistent, it follows by construction that every Γ_i is consistent. Then, by lemma 5.15, $\bigcup\Gamma_i$ is also a theory. By construction, $\bigcup\Gamma_i$ is $P \cup P'$-witnessed. □

Corollary 5.36. *Let P be the set of homomorphism letters and P' a countably infinite set of 'fresh' homomorphism letters. Then every P-theory Γ can be extended to a maximal, P'-witnessed theory.*

Proof. Immediate from the Extension Lemma and Lindenbaum's lemma. □

Canonical model

Now we have all the tools to build our canonical model. The definition of the model will be very similar to the canonical model for Hom_f, with the main difference being that the states in the model are the maximal, *witnessed* theories, instead of the all maximal theories. The steps towards the truth lemma, are similar to the ones in the previous section.

Definition 5.37. For every set of formulas Γ, we define $\Gamma^f = \{\varphi \mid f\varphi \in \Gamma\}$

Lemma 5.38. *If Γ is a maximally witnessed theory, then Γ^f is a maximally witnessed theory*

Proof. By lemma 5.18, Γ^f is a maximal theory.

To show it is witnessed, take any formula of the form $\langle s\rangle\langle\mathcal{H}\rangle\psi \in \Gamma^f$. Then $f\langle s\rangle\langle\mathcal{H}\rangle\psi \in \Gamma$. Since $f\langle s\rangle$ is a possibility form and Γ is witnessed, there is an f_0 such that $f\langle s\rangle f_0\psi \in \Gamma$, where $f_0 \notin f\langle s\rangle, \psi$. Hence $\langle s\rangle f_0\psi \in \Gamma^f$, so we conclude that Γ^f is witnessed. □

Definition 5.39. Let the canonical H-model be $\mathfrak{M}_c = \langle W_c, R_c, V_c, \mathcal{A}_c, H_c\rangle$, where

$$\begin{aligned} W_c &= \{\Gamma \mid \Gamma \text{ is a maximal witnessed theory}\} \\ R_c &= \{(\Gamma, \Delta) \mid \varphi \in \Delta \text{ implies } \Diamond\varphi \in \Gamma\} \\ V_c(p) &= \{\Gamma \mid p \in \Gamma\} \\ H_c(f) &= \{(\Gamma, \Gamma^f) \mid \Gamma \in W_c\} \\ \mathcal{A}_c &= \text{The admissible set generated by } \{H_c(f) \mid f \in \mathsf{PROP}_{\mathsf{Hom}}\} \end{aligned}$$

Lemma 5.40. *The relation $H(f) : \mathfrak{M}_c \to \mathfrak{M}_c$ is a homomorphism.*

Proof. It is clear that $H_c(f)$ is a function. Therefore, suppose that $\mathfrak{M}_c, \Gamma \models p$. Then $\Gamma \in V_c(p)$, so by definition $p \in \Gamma$. Since $p \to fp$ is an axiom, we have $fp \in \Gamma$. Hence $p \in \Gamma$ and by definition of the valuation we then also have $\mathfrak{M}_c, \Gamma^f \models p$. Hence truth of propositional variables is preserved under $H_c(f)$ for all f.

Next, suppose that $\Gamma R_c \Delta$. We must now show that $\Gamma^f R_c \Delta^f$. That is, we must show that if $\varphi \in \Delta^f$ implies $\Diamond\varphi \in \Gamma^f$. Therefore, suppose $\varphi \in \Delta^f$. Then $f\varphi \in \Delta$. By definition of R_c, it follows that

$\Diamond f\varphi \in \Gamma$. Since $\Diamond f\varphi \to f\Diamond\varphi$ is an axiom, we also have $f\Diamond\varphi \in \Gamma$. Hence, by definition, $\Diamond\varphi \in \Gamma^f$. So it follows that $H_c(f)$ is a homomorphism. □

Corollary 5.41. *The canonical model $\mathfrak{M}_c$ is well-defined. That is, $\mathfrak{M}_c$ is a valid H-model.*

Now that we have shown that $\mathfrak{M}_c$ is a valid H-model, we will show the Existence lemma for $\mathfrak{M}_c$ in the usual way.

Lemma 5.42. *If Γ is a maximal, witnessed theory, then $\Delta = \{\varphi \mid \Box\varphi \in \Gamma\}$ is a witnessed theory.*

Proof. First we show that if Γ is a theory, then Δ is a theory. We do this by proving the contrapositive. Suppose Δ is inconsistent. Then there are $\varphi_1, \ldots, \varphi_n \in \Delta$ such that $\varphi_1, \wedge \ldots \wedge \varphi_n \to \bot$. Then, by a simple derivation, we obtain that $\Box\varphi_1, \wedge \ldots \wedge \Box\varphi_n \to \bot$. Since $\Box\varphi_1, \ldots, \Box\varphi_n \in \Gamma$, it follows that Γ is inconsistent, which is what we needed to show.

Next, we show that Δ is witnessed, by proving the contrapositive. So suppose $\Delta \vdash \neg\langle s\rangle f\psi$ for all $f \notin \langle s\rangle, \psi$. By axiom 4, it follows that $\Delta \vdash [s]f\neg\psi$ all these f. Hence, there are $\varphi_1, \ldots, \varphi_n \in \Delta$ such that $\varphi_1, \wedge \ldots \wedge \varphi_n \to [s]f\neg\psi$. We then also have $\Box\varphi_1, \wedge \ldots \wedge \Box\varphi_n \to \Box[s]f\neg\psi$. Since $\Box\varphi_1, \ldots, \Box\varphi_n \in \Gamma$, we get that $\Gamma \vdash \Box[s]f\neg\psi$. Note that $[s'] = \Box[s]$ is a necessity form. Then we have

$$\begin{aligned}
&\Gamma \vdash [s']f\neg\psi && \text{(for all } f\text{)}\\
&\Gamma \vdash \neg\langle s'\rangle f\psi && \text{(Rewriting)}\\
&\Gamma \vdash \neg\langle s'\rangle\langle\mathcal{H}\rangle\psi && \text{(Γ contains witnesses)}\\
&\Gamma \vdash \Box[s][\mathcal{H}]\neg\psi && \text{(Rewriting and def. } [s']\text{)}
\end{aligned}$$

Hence, by maximality of Γ and the definition of Δ, we get $[s][\mathcal{H}]\neg\psi \in \Delta$. So $\Delta \vdash \neg\langle s\rangle\langle\mathcal{H}\rangle\psi$, which implies that $\langle s\rangle\langle\mathcal{H}\rangle\psi$ is inconsistent with Δ and which is exactly what we needed to prove. □

Lemma 5.43. *Let Γ, Δ be maximal, witnessed theories. Then $\Gamma R_c\Delta$ iff for all φ, if $\Box\varphi \in \Gamma$, then $\varphi \in \Delta$.*

Proof. Suppose $\Gamma R_c\Delta$ and let φ be any formula such that $\Box\varphi \in \Gamma$. If $\varphi \notin \Delta$, then by maximality of Δ, we have $\neg\varphi \in \Delta$. Then by definition of R_c, we get $\Diamond\neg\varphi \in \Delta$ and by the duality, $\neg\Box\varphi \in \Gamma$, which is a contradiction. Hence we conclude that $\varphi \in \Delta$.

For the other direction, suppose that for all φ, if $\Box\varphi \in \Gamma$ then $\varphi \in \Delta$. Now, suppose $\varphi \in \Delta$ and suppose for contradiction that $\neg\Diamond\varphi \in \Gamma$. Then $\Box\neg\varphi \in \Gamma$, so by our assumption, $\neg\varphi \in \Delta$, which is a contradiction. Hence $\neg\Diamond\varphi \notin \Gamma$, so by maximality of Γ we conclude that $\Diamond\varphi \in \Gamma$, as required. □

Lemma 5.44 (Existence Lemma for $\Diamond$)**.** *If $\Diamond\varphi \in \Gamma$ in $\mathfrak{M}_c$, then there is a $\Delta \in \mathfrak{M}_c$ such that $\Gamma R_c\Delta$ and $\varphi \in \Delta$*

Proof. Take $\Diamond\varphi \in \Gamma$. Let $X = \{\varphi\} \cup \{\psi \mid \Box\psi \in \Gamma\}$. By lemma 5.42, $\{\psi \mid \Box\psi \in \Gamma\}$ is a witnessed theory, so by lemma 5.33, it follows that X is a witnessed theory. Hence, by Lindenbaum's lemma (lemma 5.34), there is a maximal witnessed theory $\Delta \supseteq X$. Since $\Box\psi \in \Gamma$ implies $\psi \in \Delta$, it follows that $\Gamma R_c\Delta$. By definition, $\varphi \in \Delta$, so we have constructed the required Δ. □

Now that we have proven the existence lemma for $\Diamond$ (lemma 5.44) and f (lemma 5.38), we are almost ready to prove the truth lemma and the completeness theorem. For clarity in the proof of the truth lemma, we first prove a lemma connecting the homomorphisms in $\mathcal{A}_c$ with the homomorphism letters in the syntax.

Lemma 5.45. *Take any $\mathbf{f} \in \mathcal{A}_c$. Then there are $f_1, \ldots f_n \in \mathsf{PROP}_{\mathit{Hom}}$ (for $n \geq 0$) such that for all φ we have $\mathfrak{M}_c, \mathbf{f}(\Gamma) \models \varphi$ iff $\mathfrak{M}, \Gamma \models f_n \ldots f_1\varphi$.*

Proof. If $\mathbf{f} \in \mathcal{A}_c$, then, by the definition of $\mathcal{A}_c$, there are $f_1, \ldots f_n$ (for $n \geq 0$) such that $\mathbf{f} = H(f_1) \circ \ldots \circ H(f_n)$. We now prove the lemma by induction on n. If $n = 0$, then $\mathbf{f}$ is the identity, so for all φ

$$\mathfrak{M}_c, \mathbf{f}(\Gamma) \models \varphi \iff \mathfrak{M}_c, \Gamma \models \varphi$$

Now suppose that $n = k + 1$. Then for all φ

$$\begin{aligned} \mathfrak{M}_c, \mathbf{f}(\Gamma) \models \varphi &\iff \mathfrak{M}_c, H(f_{k+1}) \circ H(f_k) \circ \ldots \circ H(f_1)(\Gamma) \models \varphi \\ &\iff \mathfrak{M}_c, H(f_k) \circ \ldots \circ H(f_1)(\Gamma) \models f_{k+1}\varphi \\ &\iff \mathfrak{M}_c, \Gamma \models f_1 \ldots f_k f_{k+1}\varphi && \text{(induction hypothesis)} \end{aligned}$$

□

Lemma 5.46 (Truth Lemma)**.** $\mathfrak{M}_c, \Gamma \models \varphi$ *iff* $\varphi \in \Gamma$

Proof. We proceed by induction over the complexity of φ. The induction hypothesis states that for all $\Gamma \in \mathfrak{M}_c$, we have $\varphi \in \Gamma$ iff $\mathfrak{M}_c, \Gamma \models \varphi$.

- Suppose $\varphi = p$. Then the conclusion follows by observing the following equivalences:

$$\mathfrak{M}_c, \Gamma \models p \iff \Gamma \in V(p) \iff p \in \Gamma$$

- Suppose $\varphi = \neg\psi$. Then the conclusion follows by observing the following equivalences:

$$\mathfrak{M}_c, \Gamma \models \neg\psi \iff \text{not } \mathfrak{M}_c, \Gamma \models \psi \iff \text{not } \psi \in \Gamma \iff \neg\psi \in \Gamma$$

  where the last equivalence follows from the fact that Γ is a maximal theory.

- Suppose $\varphi = \psi_1 \wedge \psi_2$. Then the conclusion follows by observing the following equivalences:

$$\begin{aligned} \mathfrak{M}_c, \Gamma \models \psi_1 \wedge \psi_2 &\iff \mathfrak{M}_c, \Gamma \models \psi_1 \text{ and } \mathfrak{M}_c, \Gamma \models \psi_2 \\ &\iff \psi_1 \in \Gamma \text{ and } \psi_2 \in \Gamma && \text{(Induction hypothesis)} \\ &\iff \psi_1 \wedge \psi_2 \in \Gamma && (\Gamma \text{ is a maximal theory}) \end{aligned}$$

- Suppose $\varphi = \Diamond\psi$. Then the conclusion follows by observing the following equivalences:

$$\begin{aligned} \mathfrak{M}_c, \Gamma \models \Diamond\psi &\iff \text{there is a } \Delta \text{ such that } \Gamma R_c \Delta \text{ and } \mathfrak{M}_c, \Delta \models \psi \\ &\iff \text{there is a } \Delta \text{ such that } \Gamma R_c \Delta \text{ and } \psi \in \Delta && \text{(Induction hypothesis)} \\ &\iff \Diamond\psi \in \Gamma \end{aligned}$$

  On the last line, the left-to-right direction follows from the definition of R_c. The other direction follows directly from the existence lemma (lemma 5.44).

- Suppose $\varphi = f\psi$. Then the conclusion follows by observing the following equivalences:

$$\begin{aligned} \mathfrak{M}_c, \Gamma \models f\psi &\iff \mathfrak{M}_c, H(f)(\Gamma) \models \psi \\ &\iff \mathfrak{M}_c, \Gamma^f \models \psi && \text{(Definition of } H) \\ &\iff \psi \in \Gamma^f && \text{(Induction hypothesis)} \\ &\iff f\psi \in \Gamma && \text{(Definition of } \Gamma^f) \end{aligned}$$

- Suppose $\varphi = \langle \mathcal{H} \rangle \psi$. In this case, we prove both direction separately.

  First, suppose that $\mathfrak{M}_c, \Gamma \models \psi$. Then, by lemma 5.45, there are $f_1, \dots f_n$ such that $\mathfrak{M}_c, \Gamma \models f_1 \dots f_n \varphi$. We can now apply the induction hypothesis and obtain that $f_1 \dots f_n \varphi \in \Gamma$. Next, we can apply the contrapositive of axiom 8 multiple times and we get that $\langle \mathcal{H} \rangle \dots \langle \mathcal{H} \rangle \varphi \in \Gamma$. Then by axiom 10 (transitivity), we obtain $\langle \mathcal{H} \rangle \psi \in \Gamma$.

  For the other direction, suppose that $\langle \mathcal{H} \rangle \psi \in \Gamma$. Then, since Γ is witnessed and maximal, there is an $f \notin \psi$ such that $f\psi \in \Gamma$. Hence, by the induction hypothesis, $\mathfrak{M}, \Gamma \models f\psi$. By the semantics, this means that $\mathfrak{M}, H(f)(\Gamma) \models \varphi$. Since we picked $\mathcal{A}_c$ such that $H(f) \in \mathcal{A}_c$, this implies that $\mathfrak{M}, x \models \langle \mathcal{H} \rangle \psi$.

□

Theorem 5.47. *The logic* Hom *is complete: if* $\models \varphi$*, then* $\vdash \varphi$*.*

Proof. Let P_φ be the set of homomorphism letters in φ. Now suppose that $\nvdash \varphi$. Then $\{\neg\varphi\}$ is a P_φ-theory. We can apply the extension lemma (corollary 5.36) and conclude that there is some maximal $\mathsf{PROP}_{\mathsf{Hom}}$-witnessed theory Γ such that $\neg\varphi \in \Gamma$. By the truth lemma (lemma 5.46), it follows that $\mathfrak{M}_c, \Gamma \models \neg\varphi$. Hence $\neg\varphi$ is satisfiable and so $\not\models \varphi$. □

5.4.5 Internalization of Lyndon's theorem

Using the new semantics, we can internalize Lyndon's theorem as

$$\models \varphi \to [\mathcal{H}]\varphi \text{ iff } \varphi \text{ is positive existential}$$

By completeness, it follows that $\vdash \varphi \to [\mathcal{H}]\varphi$ holds iff φ is positive existential. However, it is also easy to obtain a syntactic proof that φ is preserved whenever φ is positive existential. Since Hom contains all axioms of Hom_f, we can apply lemma 5.23 to get proofs for $\vdash \varphi \to f\varphi$. By the fact that $\varphi \to \sharp$ is a necessity form, rule 11 immediately gives the desired result.

5.5 Conclusion

In this chapter, we discussed the role of homomorphisms in modal logic. First, we provided a proof that a formula is preserved under homomorphisms iff it is positive existential. Next, a sound and complete axiomatization for the homomorphism quantifiers was given. This axiomatization has then been used to prove syntactically that all positive existential formulas are preserved. The axiomatization makes use of an admissible set of homomorphisms. Clearly, an interesting point for future research would be to axiomatize the logic where $[\mathcal{H}]$ quantifies over *all* homomorphisms. This will most likely require a different technique than used in this chapter since the number of homomorphisms is uncountable.

However, the proof technique used here can be applied in many different situations. On a general level, the steps were the following:

1. Axiomatize the logic for the single modality;
2. Add the closure rules for the admissible set and the rule involving the necessity forms;
3. Show that every formula is satisfied at a world in the canonical model.

Most of the proof steps will be identical to the ones in this chapter. Suggestions for other modalities that can likely be axiomatized using this technique include submodels, extensions, or filtrations. For submodels, the admissible set could consist of the topology, in which case the modality is very similar to the one in Vargas Sandoval (2020). Filtrations would be interesting, both from a mathematical as a philosophical point of view. The logic of finite filtrations has been axiomatized in a PDL-style manner

in Ilin (2018). Since the number of finite filtrations is countable, the use of admissible sets would not be needed.

6. Powerset Models

6.1 Introduction

The relation considered in this chapter is the one where a model is mapped to its powerset. Powerset models have various uses in philosophy, mathematics, and linguistics. We will first define what powerset models are and give some examples of where they are used in logic. We then consider one powerset lifting in particular: the functional powerset. This logic is then axiomatized using reduction axioms.

6.1.1 Preliminaries

Definition 6.1. Let $\mathfrak{M} = (W, R, V)$ be a model. The powerset model is a model $\mathfrak{M}^{\mathcal{P}} = (W^{\mathcal{P}}, R^{\mathcal{P}}, V^{\mathcal{P}})$ where $W^{\mathcal{P}} = \mathcal{P}(W) \backslash \{\emptyset\}$.

There are several ways to lift the relation R to $R^{\mathcal{P}}$ and the valuation V to $V^{\mathcal{P}}$. In the next section, we will explore one particular lifting in more detail. Given a relation $R^{\mathcal{P}}$ and valuation $V^{\mathcal{P}}$, we can define a modality $[\mathcal{P}]$ with the semantics:

$$\mathfrak{M}, x \models [\mathcal{P}]\varphi \text{ iff } \mathfrak{M}^{\mathcal{P}}, \{x\} \models \varphi$$

6.1.2 Motivation and Related Work

Powerset models occur in several fields of logic. They allow formulas to be evaluated at a set of worlds, instead of a single world. These sets of worlds are often called *information states* (Ciardelli, 2014). Using states leads to more flexibility in choosing the semantics. For example, there are many different ways to define disjunction on powerset models, each with their application (Aloni, 2016). Choosing the valuation lifting in certain ways allows for states where neither p nor $\neg p$ is supported. Powerset models are closely related to the *team semantics* in propositional and modal logic (Lück, 2016). In team semantics, formulas are not evaluated at a single world in the model, but at a subset of the worlds. This semantics has been later been developed into inquisitive logic, which aims to model not just statements in logic, but also dialogues and questions (Ciardelli, 2014).

In this chapter, we will axiomatize the Functional Powerset Logic. Functional powersets have been introduced in Holliday (2016) and Van Benthem (1999) as a new perspective on the possibility semantics by Humberstone (1981). In the following, we show that the functional powerset models as defined later in this chapter, do not add expressivity with respect to their original Kripke model: for every formula $\varphi \in \mathcal{BML}$, there is a $\psi \in \mathcal{BML}$ such that φ holds on the function powerset iff ψ holds on the underlying Kripke model.

6.2 Axiomatizing the Logic of Functional Powersets

The lifting we discuss is the functional powerset lifting and was introduced in Holliday (2016) and Van Benthem et al. (2016). It is defined as $\mathfrak{M}_f^{\mathcal{P}} = (W^{\mathcal{P}}, R_f^{\mathcal{P}}, V_f^{\mathcal{P}})$, where:

$$\begin{aligned} W^{\mathcal{P}} &= \mathcal{P}(W) \backslash \{\emptyset\} \\ R_f^{\mathcal{P}} &= \{(X, Y) \mid R[X] = Y\} \text{ with } R[X] = \{y \mid (\exists x \in X) xRy\} \\ V_f^{\mathcal{P}}(p) &= \{X \in W^{\mathcal{P}} \mid X \subseteq V(p)\} \end{aligned}$$

Using this definition, the powerset model $\mathfrak{M}_f^{\mathcal{P}}$ is called the *functional powerset possibilization* of $\mathfrak{M}$. Indeed, the powerset model $\mathfrak{M}^{\mathcal{P}}$ under this lifting is functional. Namely, a successor of a world $X \in \mathfrak{M}^{\mathcal{P}}$ is a set that contains exactly the successors of the elements in X. Since this set is unique, the world X will have at most one successor. Note that worlds in $\mathfrak{M}$ may not have any successors. Therefore, since $\emptyset \notin \mathfrak{M}_f^{\mathcal{P}}$, there might be worlds in $\mathfrak{M}_f^{\mathcal{P}}$ that do not have any successors. Generally, we say that the *n-successor* in $\mathfrak{M}_f^{\mathcal{P}}$ is the set of *n-successors* in $\mathfrak{M}$.

Definition 6.2. Let $\mathfrak{M} = (W, R, V)$ and $x \in W$. A world y is an *n-successor* of x if there exists $y_1, \dots y_{n-1}$ such that $xRy_1R \dots Ry_{n-1}Ry$

Let us now also introduce a modality $[\mathcal{FP}]$, which allows for reasoning about the functional powerset model. Here we choose to evaluate the formula in the powerset model at the singleton of the world (see below). Evaluating the formula at different worlds in the powerset model are left for future research.

Definition 6.3. Let the language $\mathcal{L}_{[\mathcal{FP}]}$ be $\mathcal{BML}$ with the extra modality $[\mathcal{FP}]$ whose semantics is given by

$$\mathfrak{M}, w \models [\mathcal{FP}]\varphi \text{ iff } \mathfrak{M}_f^{\mathcal{P}}, \{w\} \models \varphi$$

6.2.1 Preservation of Propositional Formulas

Just as before, we can define the notion of being preserved, which is independent of the lifting of the relation and valuation. Then, we can show that under the valuation of the functional powerset model, all propositional formulas are preserved.

Definition 6.4. A formula is preserved under functional powersets if $\mathfrak{M}, w \models \varphi$ implies $\mathfrak{M}_f^{\mathcal{P}}, \{w\} \models \varphi$.

Proposition 6.5. *All propositional formulas are preserved under powersets.*

Proof. We proceed by induction over the formulas. We will prove that $\models \varphi \leftrightarrow [\mathcal{FP}]\varphi$ for all propositional φ. First consider the base case for propositional variables.

$$\mathfrak{M}, x \models [\mathcal{FP}]p \text{ iff } \mathfrak{M}_f^{\mathcal{P}}, \{x\} \models p \text{ iff } \{x\} \subseteq V(p) \text{ iff } x \in V(p) \text{ iff } \mathfrak{M}, x \models p$$

Then for negation we have

$$\begin{aligned} \mathfrak{M}, x \models [\mathcal{FP}]\neg\varphi &\text{ iff } \mathfrak{M}_f^{\mathcal{P}}, \{x\} \models \neg\varphi \\ &\text{iff } \mathfrak{M}_f^{\mathcal{P}}, \{x\} \not\models \varphi \\ &\text{iff } \mathfrak{M}, x \not\models [\mathcal{FP}]\varphi \\ &\text{iff } \mathfrak{M}, x \not\models \varphi && \text{(induction hypothesis)} \\ &\text{iff } \mathfrak{M}, x \models \neg\varphi \end{aligned}$$

Then, finally, for conjunction we have

$$\begin{aligned} \mathfrak{M}, x \models [\mathcal{FP}](\varphi \wedge \psi) &\text{ iff } \mathfrak{M}_f^{\mathcal{P}}, \{x\} \models \varphi \text{ and } \mathfrak{M}^{\mathcal{P}}, \{x\} \models \psi \\ &\text{iff } \mathfrak{M}, x \models [\mathcal{FP}]\varphi \text{ and } \mathfrak{M}, x \models [\mathcal{FP}]\psi \\ &\text{iff } \mathfrak{M}, x \models \varphi \text{ and } \mathfrak{M}, x \models \psi && \text{(induction hypothesis)} \\ &\text{iff } \mathfrak{M}, x \models \varphi \wedge \psi \end{aligned}$$

Hence all propositional formulas are preserved under functional powersets. □

Note that in the proof of proposition 6.5 we also see that $[\mathcal{FP}]$ distributes over conjunction and disjunction.

6.2.2 Axiomatization

In this section, we will give an axiomatization of the logic of functional powersets by means of reduction axioms. The required axioms are given in table 6.1 and the logic is called FPL.

Table 6.1: The logic FPL

1.	Axioms and rules of the basic modal logic K		
2.	$[\mathcal{FP}]\Box^n p$	$\leftrightarrow$	$\Box^n p$
3.	$[\mathcal{FP}]\Box^n \neg p$	$\leftrightarrow$	$\Diamond^n \neg p \vee \Box^n \bot$
4.	$[\mathcal{FP}]\Box^n(\varphi \vee \psi)$	$\leftrightarrow$	$[\mathcal{FP}]\Box^n \varphi \vee [\mathcal{FP}]\Box^n \psi$
5.	$[\mathcal{FP}]\Box^n(\varphi \wedge \psi)$	$\leftrightarrow$	$[\mathcal{FP}]\Box^n \varphi \wedge [\mathcal{FP}]\Box^n \psi$
6.	$[\mathcal{FP}]\Box^n \Diamond \varphi$	$\leftrightarrow$	$([\mathcal{FP}]\Box^{n+1} \varphi \wedge \Diamond^{n+1} \top) \vee \Box^n \bot$
7.	$[\mathcal{FP}](\varphi \to \psi)$	$\to$	$([\mathcal{FP}]\varphi \to [\mathcal{FP}]\psi)$
8.	From φ infer $[\mathcal{FP}]\varphi$		

Before we prove the soundness of FPL, we first prove an auxiliary lemma.

Lemma 6.6. *For all $\varphi \in \mathcal{L}_{[\mathcal{FP}]}$, we have $\models [\mathcal{FP}]\Box^n \Diamond^{m+1} \varphi \leftrightarrow \big([\mathcal{FP}]\Box^{n+m+1} \varphi \wedge \Diamond^{n+m+1} \top\big) \vee \Box^n \bot$*

Proof. Let $\mathfrak{M}$ be any model and $x \in \mathfrak{M}$. Then $\mathfrak{M}, x \models [\mathcal{FP}]\Box^n \Diamond^{m+1} \varphi$ iff $\mathfrak{M}_f^{\mathcal{P}}, \{x\} \models \Box^n \Diamond^{m+1} \varphi$. Now assume $\mathfrak{M}_f^{\mathcal{P}}, \{x\} \not\models \Box^n \bot$. Since $\mathfrak{M}_f^{\mathcal{P}}$ is functional, this is implies $\mathfrak{M}_f^{\mathcal{P}}, \{x\} \models \Box^{n+m+1} \varphi$ and $\mathfrak{M}_f^{\mathcal{P}}, \{x\} \models \Diamond^{n+m+1} \top$. Therefore, either $\mathfrak{M}_f^{\mathcal{P}}, \{x\} \models \Box^n \bot$ or $\mathfrak{M}_f^{\mathcal{P}}, \{x\} \models \Box^{n+m+1} \varphi \wedge \Diamond^{n+m+1} \top$. Hence $\mathfrak{M}, x \models \big([\mathcal{FP}]\Box^{n+m+1} \varphi \wedge \Diamond^{n+m+1} \top\big) \vee \Box^n \bot$. So the left-hand side implies the right-hand side.

Now suppose $\mathfrak{M}, x \models \big([\mathcal{FP}]\Box^{n+m+1} \varphi \wedge \Diamond^{n+m+1} \top\big) \vee \Box^n \bot$. If $\mathfrak{M}, x \models \Box^n \bot$, then $\mathfrak{M}_f^{\mathcal{P}}, \{x\} \models \Box^n \bot$, since there are no n-successor of x. Therefore $\mathfrak{M}_f^{\mathcal{P}} \models \Box^n \Diamond^{m+1} \varphi$ and $\mathfrak{M}, x \models [\mathcal{FP}]\Box^n \Diamond^{m+1} \varphi$. Now, suppose that $\mathfrak{M}, x \models [\mathcal{FP}]\Box^{n+m+1} \varphi \wedge \Diamond^{n+m+1} \top$. Then, $\{x\}$ has an $(n+m+1)$-successor. Since $\mathfrak{M}_f^{\mathcal{P}}, \{x\} \models \Box^{n+m+1} \varphi$, every $(n+m+1)$-successor satisfies φ. This implies that $\mathfrak{M}_f^{\mathcal{P}}, \{x\} \models \Box^n \Diamond^{m+1} \varphi$. So we conclude that $\mathfrak{M}, x \models [\mathcal{FP}]\Box^n \Diamond^{m+1} \varphi$ and that the right-hand side implies the left-hand side. This completes the proof. □

Proposition 6.7. *The axioms of* FPL *are sound on all Kripke models.*

Proof. The modal logic K is sound on all models. Also axiom 7 and rule 8 are immediate. Below we will prove the other axioms. Fix any model $\mathfrak{M}$ and a world x and recall our definition of n-successor in definition 6.2. Let us start by proving the soundness of axiom 2.

$$
\begin{aligned}
\mathfrak{M}, x \models [\mathcal{FP}]\Box^n p \text{ iff } & \mathfrak{M}_f^{\mathcal{P}}, \{x\} \models \Box^n p \\
& \text{iff the } n\text{-successor of } \{x\} \text{ satisfies } p \\
& \text{iff the } n\text{-successor of } \{x\} \text{ is a subset of } V(p) \\
& \text{iff every } n\text{-successor of } x \text{ is in } V(p) \\
& \text{iff } \mathfrak{M}, x \models \Box^n p
\end{aligned}
$$

Next, we prove the soundness of axiom 3.

$$
\begin{aligned}
\mathfrak{M}, x \models [\mathcal{FP}]\Box^n \neg p \text{ iff } & \mathfrak{M}_f^{\mathcal{P}}, \{x\} \models \Box^n \neg p \\
& \text{iff every } n\text{-successor of } \{x\} \text{ satisfies } \neg p \\
& \text{iff every } n\text{-successor of } \{x\} \text{ is a not subset of } V(p) \\
& \text{iff there is an } n\text{-successor of } x \text{ that is not in } V(p) \text{ or } x \text{ has no } n\text{-successor} \\
& \text{iff } \mathfrak{M}, x \models \Diamond^n p \vee \Box^n \bot
\end{aligned}
$$

The soundness proof of axiom 4 is slightly more involved. Recall that in the proof of proposition 6.5 we also see that $[\mathcal{FP}]$ always distributes over conjunction and disjunction.

$$\begin{aligned}
\models [\mathcal{FP}]\Box^n(\psi \vee \chi) \text{ iff } & \models [\mathcal{FP}]\Diamond^n(\psi \vee \chi) \vee \Box^n\bot && \text{(lemma 6.6)}\\
\text{iff } & \models [\mathcal{FP}]\Diamond^n\psi \vee [\mathcal{FP}]\Diamond^n\chi \vee \Box^n\bot && \text{(distributivity)}\\
\text{iff } & \models \Big(\big([\mathcal{FP}]\Box^n\psi \wedge \Diamond^n\top\big) \vee \big([\mathcal{FP}]\Box^n\chi \wedge \Diamond^n\top\big)\Big) \vee \Box^n\bot && \text{(lemma 6.6)}\\
\text{iff } & \models \Big(\big([\mathcal{FP}]\Box^n\psi \vee [\mathcal{FP}]\Box^n\chi\big) \wedge \Diamond^n\top\Big) \vee \Box^n\bot && \text{(lemma 6.6)}\\
\text{iff } & \models \big([\mathcal{FP}]\Box^n\psi \vee [\mathcal{FP}]\Box^n\chi \vee \Box^n\bot\big) \wedge (\Diamond^n\top \vee \Box^n\bot) && \text{(rewriting)}\\
\text{iff } & \models [\mathcal{FP}]\Box^n\psi \vee [\mathcal{FP}]\Box^n\chi && \text{(rewriting)}
\end{aligned}$$

The soundness of axiom 5 follows almost directly from the distributivity of $[\mathcal{FP}]$ over $\Box$.

$$\begin{aligned}
\models [\mathcal{FP}]\Box^n(\psi \wedge \chi) \text{ iff } & \models [\mathcal{FP}](\Box^n\psi \wedge \Box^n\chi)\\
\text{iff } & \models [\mathcal{FP}]\Box^n\psi \wedge [\mathcal{FP}]\Box^n\chi && \text{(distributivity)}
\end{aligned}$$

Finally, axiom 6 follows immediately from lemma 6.6, setting $m = 0$. □

Completeness follows by a standard reduction argument.

Lemma 6.8. FPL *satisfies substitution of equivalences*

Proof. As usual using axiom 7 and rule 8. □

Proposition 6.9. *For every $\varphi \in \mathcal{L}_{[\mathcal{FP}]}$, there is a $\psi \in \mathcal{BML}$ such that $\vdash_{\mathsf{FPL}} \varphi \leftrightarrow \psi$.*

Proof. We proceed by induction on the number of occurrences of $[\mathcal{FP}]$ in φ. If φ is $[\mathcal{FP}]$-free, the proposition vacuously holds. Therefore, suppose that φ contains $n+1$ occurrences of $[\mathcal{FP}]$. Then there is a subformula $[\mathcal{FP}]\varphi_0$ in φ, such that φ_0 is a $\mathcal{BML}$ formula. We will show that there is a formula ψ_0, such that $\vdash [\mathcal{FP}]\varphi_0 \leftrightarrow \chi_0$. Let χ be the formula φ, where $[\mathcal{FP}]\varphi_0$ is replaced by χ_0. By substitution of equivalences (lemma 6.8), it follows that $\vdash \varphi \leftrightarrow \chi$. The formula χ has n occurrences of $[\mathcal{FP}]$, so we can use the induction hypothesis, to find a formula $\psi \in \mathcal{BML}$ such that $\vdash \chi \leftrightarrow \psi$. Hence $\vdash \varphi \leftrightarrow \chi$, so the proposition follows.

It remains to show that such a formula χ_0 exists. We proceed by induction on the complexity of the formula φ_0. Since the basic modal logic K is part of the axiomatization, we can without loss of generality assume that φ_0 is in negation normal form. That is, all negation signs occur in front of a propositional variable. We then have the following cases:

Suppose that $\varphi_0 = [\mathcal{FP}]\Box^n p$. Then, by axiom 2, it follows that $\vdash \varphi_0 \leftrightarrow \Box^n p$ and we are done.

Suppose that $\varphi_0 = [\mathcal{FP}]\Box^n\neg p$. Then, by axiom 3, it follows that $\vdash \varphi_0 \leftrightarrow \Diamond^n\neg p \vee \Box^n\bot$ and we are done.

Suppose that $\varphi_0 = [\mathcal{FP}]\Box^n(\varphi_1 \vee \varphi_2)$. Then, by axiom 4, we have $\vdash \varphi_0 \leftrightarrow \big([\mathcal{FP}]\Box^n\varphi_1 \vee [\mathcal{FP}]\Box^n\varphi_2\big)$. Hence, by the induction hypothesis, there are $\chi_1, \chi_2 \in \mathcal{BML}$ such that $\vdash [\mathcal{FP}]\Box^n\varphi_1 \leftrightarrow \chi_1$ and $\vdash [\mathcal{FP}]\Box^n\varphi_2 \leftrightarrow \chi_2$. Hence $\vdash \varphi_0 \leftrightarrow \chi_1 \vee \chi_2$, so the conclusion follows by setting $\chi = \chi_1 \vee \chi_2$.

The case where $\varphi_0 = [\mathcal{FP}]\Box^n(\varphi_1 \wedge \varphi_2)$ is analogous to the case for disjunction, using axiom 5.

Finally, suppose that $\varphi_0 = [\mathcal{FP}]\Box^n\Diamond\varphi_1$. By the induction hypothesis, there is a formula χ_1 such that $\vdash \chi_1 \leftrightarrow [\mathcal{FP}]\Box^{n+1}\varphi_1$. Then we have

$$\begin{aligned}
& \vdash \varphi_0 \leftrightarrow ([\mathcal{FP}]\Box^{n+1}\varphi \wedge \Diamond^{n+1}\top) \vee \Box^n\bot && \text{(axiom 6)}\\
& \vdash \varphi_0 \leftrightarrow (\chi_1 \wedge \Diamond^{n+1}\top) \vee \Box^n\bot
\end{aligned}$$

Then the conclusion follows by setting $\chi = (\chi_1 \wedge \Diamond^{n+1}\top) \vee \Box^n\bot$. □

Theorem 6.10. *The axiom schema* FPL *is sound and complete with respect to all Kripke models.*

Proof. The soundness proof is given in proposition 6.7. Therefore, we are left with completeness. Let $\varphi \in \mathcal{L}_{[\mathcal{FP}]}$ be valid. From proposition 6.9, it follows that there is a formula $\psi \in \mathcal{BML}$ such that $\vdash \varphi \leftrightarrow \psi$. Since FPL is sound, it follows that $\models \varphi \leftrightarrow \psi$. Hence ψ is also valid. By the completeness of modal logic K, we then have that $\vdash \psi$. Since $\vdash \varphi \leftrightarrow \psi$, we conclude that $\vdash \varphi$, as required. □

6.3 Other Liftings

There are many other liftings that could also be considered. Attempts to find an axiomatization have been unsuccessful so far, so these are left for future research. Also, the preservation theorems for the liftings are still open questions. Some possible lifings for $R^{\mathcal{P}}$ and $V^{\mathcal{P}}$ include are:

$$R^{\mathcal{P}} = \{(X,Y) \mid (\forall x \in X)(\exists y \in Y)(xRy)\}$$
$$R^{\mathcal{P}} = \{(X,Y) \mid (\forall x \in X)(\forall y \in Y)(xRy)\}$$
$$R^{\mathcal{P}} = \{(X,Y) \mid (\forall y \in Y)(\exists x \in X)(xRy)\}$$
$$R^{\mathcal{P}} = \{(X,Y) \mid (\forall x \in X)(\exists y \in Y)(xRy) \wedge (\forall y \in Y)(\exists x \in X)(xRy)\}$$

$$V^{\mathcal{P}}(p) = \{X \mid (\exists x \in X)(x \in V(p))\}$$
$$V^{\mathcal{P}}(p) = \{X \mid V(p) \subseteq X\}$$

6.4 Conclusion

In this section, we provided a sound and complete axiomatization for the function powerset operator. As the name suggests, the accessibility relation in the functional powerset is functional. Therefore, $\Box$ distributes over conjunction in the functional powerset. This makes it possible to state clear reduction axioms for the formulas in $\mathcal{L}_{[\mathcal{FP}]}$. The preservation law for the functional powerset lifting is unknown and left as an open question. Similarly, the many other possible liftings for the relation and the valuation are also left for future research.

7. Conclusion

This thesis has been concerned with dynamic logics for several model constructions. In the field of Dynamic Epistemic Logic, modalities can often be viewed as model transformers or actions. Building on this tradition, this thesis adds several model-theoretic constructions as modalities to the basic modal language, thereby allowing the language to express properties of related models. After an introduction of the field and the technical preliminaries, each chapter is devoted to the study of a specific modality.

In chapter 3, the main topic is the validity/satisfiability operator. This operator, denoted $\forall^+_{\mathsf{L}}$, expresses that a formula holds in all models belonging to a certain logic L. It studies the basic modal language together with this operator $\forall^+_{\mathsf{L}}$. However, to be able to find a sound and complete axiom system for this language $\mathcal{SL}_{\mathsf{L}}$, we must also be able to derive all formulas of the form $\neg\forall^+_{\mathsf{L}}\varphi$. That is, we must be able to derive all non-theorems. For this, we use the notion of refutation systems and combined systems. It is shown how, given a sound and Ł-complete refutation system for a logic L, one can obtain a sound and complete axiomatization for the corresponding Logic of Satisfiability SL_{L}.

The main application of the logic of satisfiability is in chapter 4, where we study simulations. Simulations are widely used in several branches of theoretical computer science as a form of equivalence between trace equivalence and bisimulations. Simulations are studied, as well as strong simulations, which also preserve the falsehood of proposition letters. The main achievement in this chapter is the axiomatization of the logic for simulations and strong simulations, by the means of reduction into the language $\mathcal{SL}_{\mathsf{K}}$. Together with the results of chapter 1, this leads to a sound and complete axiom system for (strong) simulations. At the end, it is shown that we can syntactically derive that all positive existential formulas are preserved under simulations inside the new logic.

In chapter 5, we move from simulations to homomorphisms. The first result in this chapter is that there exists a simulation between two models if and only if there exists a homomorphism on bisimilar models This immediately shows that the preservation laws for simulations and homomorphisms are the same, thereby providing an alternative proof for Lyndon's theorem in modal logic. After this, a new modality $[\mathcal{H}]$ is introduced that quantifies over all homomorphisms in a given admissible set. The language with this new modality is axiomatized using a technique with witnessed theories, also used in Arbitrary Public Announcement Logic (Balbiani et al., 2007) and Dynamic Logic for Learning Theory (Vargas Sandoval, 2020). The same technique can easily be adapted for use for other dynamic modalities, such as submodels or quotients.

Finally, in chapter 6, we consider the powerset models. Powerset models provide a flexible semantics, which is useful in both mathematics, as well as philosophy, and linguistics. After a general introduction on powerset models, we focus on a particular instance: the functional powerset model. It is shown that the functional powerset model does not add any expressivity compared to the regular Kripke models. For this, we define reduction rules for the functional powerset modality into the basic modal language.

Directions for Future Research

The final section of this thesis is devoted to some directions for future research. At the end of each chapter, there have been directions for future research directly related to the specific modality. Here we will focus on some points that apply to all modalities in question.

A first, and obvious, direction for future research consists of investigating new operations. Especially in the field of the powerset operators, there are many interesting and useful modalities to internalize into the language. However, axiomatizing these modalities is not as easy as the functional powerset

operator in chapter 6. Other research in this field could focus on finding preservation laws for these operators, something that has not yet been studied as far as the author is aware.

Instead of introducing new modalities, one could also focus on the preservation laws for the existing modalities. The preservation laws discussed in this thesis were all based on the basic modal logic. However, the basic language with the additional modality comes with its own, new preservation law. For example, it would be interesting to study which formulas in $\mathcal{L}_{[\mathcal{H}]}$ are preserved under homomorphisms between the H-models. In addition, preservation is a special case of 'entailment along a relation'[1] (Barwise and Van Benthem, 1999). We say that φ entails ψ along a relation R if in all models $\mathfrak{M}, x$ and $\mathfrak{N}, w$ such that $\mathfrak{M}, xR\mathfrak{N}, w$, we have $\mathfrak{M} \models \varphi$ implies $\mathfrak{N} \models \psi$. Preservation is then the case where $\varphi = \psi$. The results in this thesis could be used as a basis for studying entailment along simulations, homomorphisms or functional powersets.

In addition, the logics in this thesis are only able to prove the 'easy' direction of the preservation laws syntactically: that all formulas of a certain form are preserved. Of course, the other direction follows by completeness. To prove the other direction syntactically, one would need a refutation system to show that all for all other formulas, the preservation is *not* derivable. Further research could therefore focus on finding refutation systems for the logics in this thesis. Finding such systems would provide an alternative proof of the preservation law in question.

Finally, all the modalities were studied in the single-agent setting. Dynamic Epistemic Logic often deals with multi-agent environments by adding more accessibility relations, one for each agent. Therefore, in the context of DEL, it would be interesting to extend the results in this thesis to a multi-agent setting.

[1] This observation was pointed out to me by Van Benthem in a conversation.

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