The paradoxicality of Curry

MSc Thesis *(Afstudeerscriptie)*

written by

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under the supervision of **Thomas Schindler**, and submitted to the Examinations Board in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam.*

Date of the public defense:
July 11, 2025

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Abstract

Curry's paradox is a peculiar paradox for at least two reasons: it is largely left out by common definitions of *paradoxicality* and it challenges paraconsistent solutions to the paradoxes because it can persist in a paraconsistent system. In this thesis, we study the dynamics of this paradox by adapting Kripke's definition of paradoxicality (Kripke, 1975) and a definition of paradoxicality introduced by Hsiung (2024) such that they include more of Curry's paradox. We moreover explore the possibility of a notion of paradoxicality that can distinguish Curry's paradox from other paradoxes like the Liar, by embedding the definition in a paraconsistent logic. The result is an overview of different formal characterisations of paradoxicality, that capture Curry's paradox and its relatives to various degrees.

Acknowledgements

This thesis would not have existed without the help of a number of people – I want to thank some of them here.

First and foremost, I want to thank Thomas Schindler for being a wonderful supervisor. His supervision struck the perfect balance between freedom and support – giving me complete freedom in choosing my research topic, but always guiding me to make sure that the project would result in a coherent piece of work. Thank you for your interest in this research, your spot-on questions and remarks, and believing (more so than I) that this thesis could be finished in time.

Many thanks go out to the committee members Luca Incurvati and Giorgio Sbardolini for their time reading this thesis and for their inspiring questions and engagement in fruitful discussions. Many thanks, too, to Balder ten Cate for chairing the defence and for his insights about my work, even without having read the thesis.

On the more emotional side of the spectrum, I want to thank my mother for her endless patience, her listening ear, and for giving me a space to work and rest. I would like to thank Josef and Michael for being great friends throughout the Master of Logic, and for lighting up my time at the ILLC.

I want to mention someone whose influence has been more indirect, but no less important: Graham Priest, whose work on logical paradoxes has ignited my fascination with logic – the interest in paraconsistent logic that shines through this thesis is owed to him.

Finally, I want to thank my father, whose bright mind has always inspired me, and who would have been so very proud.

Contents

1. Introduction

> *[Paradoxes] are associated with crises in thought and with revolutionary advances. To grapple with them is not merely to engage in an intellectual game, but is to come to grips with key issues.*
>
> – Sainsbury (2009, p. 1)

Ever since the ancient Greek prophet Epimenides, the Cretan, introduced the paradox of the Liar by uttering the words 'Every Cretan always lies', thinkers have been fascinated by paradox. In modern times, too, paradox is an important subject within the research areas of logic and philosophy. Research on the notion of *paradoxicality* abounds in modern literature on philosophical logic. Much of this is still centered around the Liar (in its simpler form: 'This sentence is false') and variations on the Liar, such as Yablo's paradox, a version of the Liar containing infinitely many sentences. In this thesis, we focus on a different paradox: Curry's paradox.

Curry's paradox is of the form 'If this sentence is true, then the moon is made out of blue cheese.' It is named after its discoverer Curry (1942),[1] and arises in both truth theory and set theory. We focus on the truth-theoretic variant, which can be formally expressed as follows:

$$(\mathcal{C}) \qquad T\ulcorner\mathcal{C}\urcorner \to B,$$

where T is a truth predicate, $\ulcorner - \urcorner$ is a naming device, and B is an arbitrary sentence. Given only minimal conditions on the background logic, Curry's sentence allows us to derive the sentence B, trivialising the logic. Thus, Curry's paradox has the same disastrous effect on a classical logical system as other famous self-referential paradoxes, such as the Liar. In other respects, however, Curry's paradox is different. Curry's paradox, for instance, causes great trouble for paraconsistent solutions to the paradoxes. Such solutions work by rejecting the rule of *explosion*: $A \wedge \neg A \models B$. This allows them to accommodate for the Liar, but not – in general – for Curry. For Curry's paradox, unlike the Liar, does not involve negation: it involves implication. This makes it notoriously difficult to deal with; Weber has called it 'the most diabolical of all the paradoxes' (Weber, 2021, p. 24).

Another remarkable feature of Curry is that it is not *one* paradox, but a family of paradoxes: one for each sentence B. And Curry's paradox behaves differently depending on the status of B: if B is a contradiction, then Curry's sentence behaves like the Liar – it is equivalent its own negation. If B is a tautology, on the other hand, then Curry's sentence is a tautology too. If B is neither a tautology nor a contradiction, then the status of Curry's sentence varies, depending on the interpretation of B. Common existing notions of paradoxicality, such as the ones put forward by Kripke (1975), Cook (2004), Hsiung (2024), and Priest (1995), do not capture the complete family of Curry's paradox: they capture only the instances with a contradictory consequent – those that behave like the Liar.

[1] Though Löb (1955) independently came up with a derivation of the paradox that is closer to the one most commonly known today.

But there is more to Curry than the Liar – even if its consequent B is a tautology, Curry's sentence seems paradoxical. Not because of the *fact* that we can derive B, but because of the *way* in which B is derived: it is not right that its truth is due to the existence of Curry's paradox. As Pleitz (2020, p. 194) has put it, concerning the example 'If this sentence is true then snow is white', which allows us to infer that snow is white:

> we want to protest, 'sure, snow is white – but not solely in virtue of some facts that concern sentences!'

Thus, Curry's paradox challenges traditional notions of paradoxicality, and pushes negation-oriented solutions to the paradoxes to look beyond just negation. The aim of this thesis is to study the dynamics of this interesting paradox more explicitly than has been done so far. Our starting point is the question: 'what makes Curry's sentence paradoxical?' Based on different informal answers to this question, we propose different formal definitions of 'Curry-paradoxicality'. Some of these are based on existing notions of paradoxicality; we will be considering Kripke's 1975 influential definition and a more recent definition introduced by Hsiung (2020, 2024). Other notions will be built up from scratch. The result is an overview of different formal definitions of 'Curry-paradoxicality', each of which can be seen as a formalisation of an informal view of Curry's paradox.

1.1. Overview of the thesis

Chapter 2 is an informal chapter, in which we propose different perspectives on Curry's paradox. We call these perspectives 'intuitions', and divide them into two groups: one group is framed in terms of entailment, and one is framed in terms of models.

In Chapter 3, we introduce the necessary preliminaries for the formal definitions that follow in subsequent chapters. We succinctly introduce the language of Peano arithmetic and prove two distinct methods of self-reference, which we will both be using: strong and weak diagonalisation.

In Chapters 4 and 5, we take two existing notions of paradoxicality as starting points: those introduced by Kripke (1975) and Hsiung (2020), both of which are framed in terms of models. We adapt these notions in order formalise the intuitions involving models that we introduce in Chapter 2. In Chapter 4, we first introduce Kripke's theory of truth, and Kripke's corresponding notion of paradoxicality. We then consider the behaviour of Curry in this theory, and see how this naturally gives rise to a definition that captures more instances of Curry's sentence (in Section 4.2). We will see that this definition does not, however, capture Curry's paradox in case its consequent is a tautology. To include these instances as well, we introduce a second notion (Section 4.3). Chapter 5 is a shorter chapter, in which we aim to show that the method we employed in Section 4.2 extends to Hsiung's definition of paradoxicality too.

In Chapter 6, we formalise the intuitions that are framed in terms of entailment; we introduce a notion of paradoxicality that is framed in terms of entailment, and adapt this notion as we go along to accommodate for the different intuitions. Among other things, we will employ a paraconsistent background logic in order to capture the crucial difference between Curry and the Liar in paraconsistent systems (Section 6.4).

2. Perspectives on Curry

In this chapter, we introduce different ways of thinking about Curry. As we have seen, Curry's paradox is unique among the self-referential paradoxes in that it is troublesome for paraconsistent logics and that it is not fully recognised as paradoxical on standard definitions of paradoxicality. What makes this so? By introducing different characterisations of *Curry-paradoxical* sentences, we propose different perspectives on this question. These perspectives will come back in the more technical chapters that follow, where we will see how these intuitions can be formalised.

2.1. Flavours of Curry

Before we outline the different informal views on Curry's paradox, let us have a closer look at the paradox itself. We will denote Curry's sentence with consequent B by $\mathcal{C}_B$:

$$(\mathcal{C}_B) \qquad T\ulcorner\mathcal{C}_B\urcorner \to B.$$

We mentioned in the introduction that this sentence entails B to be true, given minimal requirements on the background logic. Let us go through the argument informally. Suppose for the moment that $\mathcal{C}_B$ is true. Then, by definition of $\mathcal{C}_B$, the formula $T\ulcorner\mathcal{C}_B\urcorner \to B$ is true. So, since we assumed that $\mathcal{C}_B$ is true, we have B by modus ponens. We just showed that if $\mathcal{C}_B$ is true, then B. That is, we showed that $T\ulcorner\mathcal{C}_B\urcorner \to B$ is true! So $\mathcal{C}_B$ is true, and we just saw that this means that B is true as well. One thing to note here is that $\mathcal{C}_B$ does not only entail the truth of B, but also of itself: it is a sentence whose very existence proves its own truth.

As we saw in the introduction as well, Curry's paradox is not one paradox, but a family of paradoxes, whose members differ substantially. *Tautological Curry* is the mildest form of Curry: it is the instance whose consequent is $\top$.

$$(\mathcal{C}_\top) \qquad T\ulcorner\mathcal{C}_\bot\urcorner \to \top.$$

From the point of view of logical models, this sentence is quite unproblematic: in many logics, it is simply equivalent to $\neg T\ulcorner\mathcal{C}_\top\urcorner \vee \top$, which is in turn equivalent to $\top$. So, as far as logic is concerned, tautological Curry is just a tautology. Any instance of Curry with its consequent *equivalent* to $\top$, i.e. a tautological consequent, of course behaves the same. In the sequel, we will often be sloppy and use 'tautological Curry' to refer to any instance of Curry with a tautological consequent.

Contradictory Curry is the most severe form of Curry. It is the instance whose consequent is (equivalent to) $\bot$:

$$(\mathcal{C}_\bot) \qquad T\ulcorner\mathcal{C}_\bot\urcorner \to \bot.$$

In classical logic, and many other logics, this sentence is equivalent to $\neg T\ulcorner\mathcal{C}_\bot\urcorner$: it behaves like the Liar.[1] In between tautological Curry and contradictory Curry are those instances

[1] The requirement here is *negation equivalence* $\neg A \leftrightarrow (A \to \bot)$. This is certainly not an exclusively classical property: intuitionistic negation, for instance, is defined in exactly this way.

of Curry whose consequent is neither contradictory or tautological. These instances are not equivalent to $\top$, and not to a sentence expressing its own falsity either. They are, in a way, the most interesting instances of Curry's paradox: their behaviour is not set in stone, but varies depending on the interpretation of their consequents. If their consequent is interpreted as true, they behave like tautological Curry; while if it is interpreted as false, they behave like contradictory Curry.

The takeaway is that Curry is not *one* paradox, but rather a family of sentences, some of which are more obviously paradoxical than others. Among the different intuitive notions that follow, some will capture tautological Curry and/or contradictory Curry, while others will not.

2.2. First perspective: entailment

On the first perspective, a paradoxical sentence is a sentence whose existence logically entails something that is unacceptable. This is close to the standard definition of paradox that was proposed by Sainsbury, according to which a paradox is

> an apparently unacceptable conclusion derived by apparently acceptable reasoning from apparently acceptable premises. (Sainsbury, 2009, p. 1)

The 'apparently unacceptable conclusion' can take on various forms. As we saw in the introduction, in the context of logical paradoxes, the conclusion is often a plain contradiction of the form 'A and not-A'. As Priest has put it:

> The paradoxes are all arguments starting with apparently analytic principles concerning truth, membership, etc., and proceeding via apparently valid reasoning to a conclusion of the form 'α and not-α'. (Priest, 2006, p. 9)

This would give us the following characterisation of paradoxicality:

Intuition 0. A paradoxical sentence is a sentence whose existence entails a contradiction.

The Liar is paradoxical in this sense; but Curry is not, unless its consequent is contradictory. As we saw in the introduction, other instances of Curry's paradox seem paradoxical too – so the above notion is not satisfactory. One alternative may be the following, which captures more of Curry's paradox:

Intuition 1. A Curry-paradoxical sentence is a sentence whose existence entails the truth of a sentence that is not a tautology.

Curry's sentence is paradoxical in this sense as long as its consequent is not a tautology. We will turn to a characterisation that captures Curry even in case its consequent is a tautology in a moment – but first we sharpen the above intuition.

Intuition 1 captures both Curry and the Liar, while, as we saw in the introduction, there a crucial difference between any instance of Curry and the Liar: there is no explosion involved in Curry's paradox – even in the case of contradictory Curry, the consequent is entailed directly, without appealing to the principle of explosion. Curry moreover includes its entailed sentence as a subformula, while the Liar does not. In these two senses, Curry stands in a more direct relationship to its entailed sentence than the Liar. This invites a sharpening of the above intuition, such that it captures only Curry and not the Liar:

Intuition 2. A Curry-paradoxical sentence is a sentence whose existence entails the truth of a sentence that is not a tautology *by means of a direct argument.*

The notion of a 'direct' argument can be explicated in the two ways we mentioned before; the first and perhaps most obvious option is to use the fact that the Liar relies on explosion, while Curry does not:

Intuition 2a. A Curry-paradoxical sentence is a sentence whose existence entails the truth of a sentence that is not a tautology *by means of an argument that does not use explosion.*

This means, in particular, that Curry's paradox can persist in logics that are *paraconsistent*, i.e. logics that do not obey the rule of explosion, while the Liar cannot. We will use this fact to formalise Intuition 2a using a paraconsistent framework (in Section 6.4).

The second option is to note that Curry contains the sentence that it forces to be true as its consequent, while the Liar does not:

Intuition 2b. A Curry-paradoxical sentence is a sentence whose existence entails the truth of one of its *subformulas* B, where B is not a tautology.

We will formalise this intuition in Section 6.3.

2.2.1. Curry's paradoxical argument

All of the above intuitions capture Curry's paradox in case its consequent is not a tautology. However, as we noted in the introduction, even if Curry's consequent is a tautology – say, '$0 = 0$' – it still can be seen as paradoxical. Not because it is unacceptable to be able to derive that 0 equals 0, but because the argument that establishes it is unacceptable. The tautological conclusion is thus not unacceptable by itself, but it is unacceptable as a conclusion of the argument in question. As Pleitz has put it:

> [T]hat snow is white is surely unacceptable as the conclusion of the Curry argument that starts from considering the sentence 'If this sentence is true, then snow is white', because it clearly cannot be warranted by a bunch of non-empirical premises alone. (Pleitz, 2015, p. 8, fn 8)

Pleitz subsequently proposes that we broaden our notion of an 'unacceptable conclusion' by 'think[ing] of a conclusion as unacceptable either if it is false or if it intuitively cannot be warranted by the premises of the argument it is the conclusion of' (Pleitz, 2015, p. 8, fn 8). It is not immediately clear how this last part should be made more precise. What we seem to be dealing with here is the question of whether a (formal) argument is intuitively unproblematic, and how to characterise problematic arguments formally. This is not an easy question, and moreover one that goes beyond the scope of this thesis, which takes a model-theoretic approach and therefore does not lend itself to the analysis of arguments or proofs. An answer might be found in approaches built on proof theory, such as Tennant's proof-theoretic criterion of paradoxicality (Tennant, 1982, 1995).[2] We will not go into these technicalities here. Having said this, we still propose one intuition intended to capture the paradoxicality of Curry's argument.

The idea is the following: what is strange about the argument involved in Curry's paradox is that it is independent of the conclusion it proves. That is: from the existence of Curry's sentence with consequent B we can prove B, and exactly the same argument – modulo replacing B by A – establishes A from the existence of Curry's sentence with consequent A. This thus means that Curry's family allows us to derive any sentence whatsoever, leading to triviality. The proposal is that the fact that the argument is independent of the conclusion is to blame for this, and that this is what makes Curry's sentence paradoxical – even in case the consequent is a tautology. For the fact that the argument is independent of the tautology in question tells us that *the same* argument could establish something that is *not* a tautology too, and this cannot be right. There must therefore be something fishy about the argument.

To be sure: what is problematic is not the fact that the existence of some sentence A entails another sentence B – for instance, this is always the case if B is a tautology. What is problematic is that the argument has nothing at all to do with B. In the innocent case, in which we derive a tautology from the existence of an arbitrary sentence, the argument in question will appeal to the nature of the tautology: if we wish to prove $0 = 0$, we appeal to the definition of $=$ in our logic, and the fact that the same logical symbol occurs on both sides. By contrast, in the derivation of $0 = 0$ from Curry's sentence with consequent $0 = 0$, we do not care about the meaning of $0 = 0$ at all. This is what is odd, and what we mean by saying that Curry's argument is *independent* of its consequent.[3]

Intuition 3. A Curry-paradoxical sentence is a sentence A whose existence entails the truth of a sentence B by means of an argument that is independent of B.

Where the argument which derives B from the existence of A is *independent* from B if replacing B by an arbitrary sentence C in both A and the argument gives us a valid argument with the conclusion C.

Like Intuition 1, this intuition includes both Curry's paradox and the Liar: from the existence of the Liar we can derive any arbitrary sentence through the rule of explosion, and this argument is clearly independent of the sentence in question – since explosion is blind the conclusion in question. Unique about Curry is the fact that it entails its consequent by means of an argument that is independent of this consequent yet does not use explosion. It

[2]This criterion is based on the observation that the known paradoxes (including Curry's paradox) exhibit proofs that cannot be brought into normal form – reduction sequences would enter a loop or be nonterminating in another way. This is thought to be the determining factor: 'The test was to see whether the (dis)proof in question could be brought into normal form, by means of allowable reduction procedures. *If it could not*, then one would be dealing with a genuine paradox.' (Tennant, 2024, p. 935)

[3]Arguments that are independent of their conclusions are, of course, not problematic in general. They are in fact very common. For instance, the argument that establishes A from the premise $A \wedge B$ is independent of A, in the sense that replacing A by C in both premise, argument, and conclusion would result in a correct argument. The argument thus does not care about the meaning of A. This is not problematic – because A is part of the premise. Here, however, we are talking about arguments with as sole premise the *existence* of a certain sentence. And this changes the game. For suppose that the existence of a sentence A entails B by means of an argument independent of B; then replacing B by something unacceptable, say $\perp$, in premise, argument and conclusion would give us some sentence whose existence entails $\perp$ – and this is clearly unacceptable.

moreover contains the sentence it entails as a subformula. In other words, we have the same sharpenings of Intuition 3 that we encountered for Intuition 1:

> **Intuition 4.** A Curry-paradoxical sentence is a sentence A whose existence entails the truth of a sentence B by means of a *direct* argument that is independent of B.

We can put this in terms of the lack of explosion:

> **Intuition 4a.** A Curry-paradoxical sentence is a sentence A whose existence entails the truth of a sentence B by means of an argument that is independent of B *and that does not use explosion.*

This characterisation captures every instance of Curry's paradox, but not the Liar. In a similar vein, we might restrict Intuition 3 by demanding that the entailed sentence B is a subformula of the Curry-paradoxical sentence A:

> **Intuition 4b.** A Curry-paradoxical sentence is a sentence A whose existence entails the truth of one of its *subformulas* B by means of an argument that is independent of B.

The intuitions related to entailment will be formalised in Chapter 6. (Intuition 1 in Section 6.2, Intuition 2a in Section 6.4, and Intuition 2b in Section 6.3.)

2.3. Second perspective: models

The previous four intuitions were framed in terms of entailment. There is, however, another natural way to view Curry's paradox, that arises when we consider the behaviour of Curry's paradox in the context of existing definitions of paradoxicality. This will be our starting point in Chapter 4 and Chapter 5. The resulting notion of Curry-paradoxicality frames Curry in terms of its behaviour across models, rather than entailment.

We assume an existing model-theoretic definition of paradoxicality which does not necessarily include Curry's sentence, but does include the Liar. Given such a definition, we can look at the behaviour of the instances of Curry's sentence in logical models. As we saw in Section 2.1, contradictory Curry behaves just like the Liar in any logical model. Since any notion of paradoxicality worth its salt should capture the Liar, this instance of Curry is likely recognised as paradoxical by our chosen definition. The idea now is to view any other instance of Curry as a variation of contradictory Curry: it would behave like the Liar, if its consequent were false. This means that it is *potentially* paradoxical: it is paradoxical on some interpretations of its consequent. Assuming that the base definition of paradoxicality that we are working with is framed in terms of models, this means that Curry's sentence is paradoxical in at least some models:

> **Intuition 5a.** A Curry-paradoxical sentence is a sentence that behaves paradoxically in some models.

This intuition does *not* capture tautological Curry: we saw that tautological Curry is – classically – equivalent to $\top$, and thus behaves like a tautology in any model. It moreover captures both contradictory Curry and the Liar: we cannot distinguish Curry from the Liar on this approach, because both are paradoxical in all models. We can, however, look at a restricted class of sentences, which includes only those instances of Curry's sentence whose consequent is neither a contradiction nor a tautology:

> **Intuition 5b.** A Curry-paradoxical sentence is a sentence that behaves paradoxically in some models, but not in others.

Since the Liar and contradictory Curry are paradoxical in any model, this set includes only the instances of Curry with a consequent that is true in some models, but false in others. It therefore is quite a restricted notion of Curry-paradoxicality; but it has the advantage that it uniquely characterises Curry's paradox. It moreover highlights an aspect that seems to be unique to Curry's paradox: its changeability across logical models.

The previous two intuitions do not capture tautological Curry; yet tautological Curry, too, can be seen as a variation on the undoubtedly paradoxical contradictory Curry. Indeed, if the consequent of tautological Curry *were* false, it would behave like the Liar; however, since the consequent in question is $\top$, this is never the case. What we seem to be dealing with here is the *logical form* of tautological Curry, as opposed to its de facto behaviour. Indeed, every instance of Curry has the same logical form – in particular, the logical form of tautological Curry and contradictory Curry are the same. Since contradictory Curry is clearly paradoxical, we might infer that tautological Curry is too – because it shares the same logical form. This gives rise to the following characterisation:

> **Intuition 6.** A Curry-paradoxical sentence is a sentence whose logical form is paradoxical.

This same suggestion was made by Oms (2023) who proposed the following characterisation in the context of Curry's paradox and the Sorites paradox: 'A paradox is an apparently valid argument whose logical form can be used to derive an apparently false conclusion from apparently true premises.'[4] (Oms, 2023, p. 217)

Intuition 5a is formalised based on Kripke's notion of paradoxicality in Chapter 4, and based on a notion of paradoxicality introduced by Hsiung (2020) in Chapter 5. Both formalisations in principle allow for a formalisation of Intuition 5b as well; but we especially pay attention to this intuition in Chapter 4 on Kripke's definition. Formalising Intuition 6 is less straightforward, but we propose a potential formalisation in terms of Kripke's definition in Section 4.3.

[4] In fact, this proposal was only Oms' first attempt at defining an appropriate notion of paradoxicality that captures both Curry and the Sorites paradox, and one that he comes to reject later. His reasons to reject it, however, are related to the Sorites paradox rather than Curry.

Below is an overview of the intuitions and where their formalisations can be found.

Intuition	Section
Intuition 0	Section 6.2
Intuition 1	Section 6.2
Intuition 2a	Section 6.4
Intuition 2b	Section 6.3
Intuition 3	Section 6.5
Intuition 5a	Section 4.2
Intuition 5b	Section 4.2 and Chapter 5
Intuition 6	Section 4.3

3. Preliminaries

This chapter introduces the preliminaries that are necessary to formally express Curry's paradox. In particular, we fix the language of Peano arithmetic and consider the difference between weak and strong diagonalisation. We conclude with an overview of some standard paradoxes that will figure throughout the thesis.

3.1. Self-reference and diagonalisation

We define the necessary ingredients to formally express Curry's paradox and the other paradoxical sentences (see Halbach (2011)). The first ingredient we need is a method of naming sentences: a function $\ulcorner - \urcorner$ which takes a sentence and outputs a name. The second is a method of self-reference, which allows a sentence to contain its own name. And the final ingredient is a truth predicate. The first two ingredients are provided for in the language and theory of Peano arithmetic, and the last one is accommodated for by adding a truth predicate to the language.

We start from an arithmetical language without a truth predicate, a *ground language*. We will be working either in the first-order language of Peano arithmetic, or an expansion of this language that we specify in a moment. The first-order language of Peano arithmetic is the language $\mathcal{L}_{\mathsf{PA}}$ which includes a countably infinite set of variables, the quantifiers $\vee$ and $\neg$, the quantifier $\exists$, and the identity symbol $=$. All other connectives and the universal quantifier are defined in terms of these in the usual way. The signature of the language consists of the constant $\bar{0}$, the successor function symbol S, and two binary operators $+$ and $\times$. The *expanded* language of Peano arithmetic $\mathcal{L}_{\mathsf{PA}^+}$ is obtained from $\mathcal{L}_{\mathsf{PA}}$ by adding a function symbol for a certain primitive recursive function that will be specified in a moment.

For naming and self-reference, we assume a Gödel numbering, which assigns a natural number to every formula in the language $\mathcal{L}_{\mathsf{PA}}$ or $\mathcal{L}_{\mathsf{PA}^+}$. We denote the Gödel number of a formula A by $\#A$. We denote the numeral of a natural number n by $\bar{n}$, which is obtained by applying the successor function n times to the constant $\bar{0}$. We denote the numeral of the Gödel number $\#A$ by $\ulcorner A \urcorner$. We will also call $\ulcorner A \urcorner$ the *standard name* of A.

The theory PA is the usual axiomatic theory of Peano arithmetic. The theory PA^+ additionally contains the defining axiom for the primitive recursive function symbol that is included in the language $\mathcal{L}_{\mathsf{PA}^+}$.

Given a ground $\mathcal{L}_{\mathsf{PA}}$ or $\mathcal{L}_{\mathsf{PA}^+}$, we obtain the full language by adding a truth predicate T. We denote the resulting language by $\mathcal{L}^+_{\mathsf{PA}}$ and $\mathcal{L}^+_{\mathsf{PA}^+}$, respectively. When it is clear which of $\mathcal{L}_{\mathsf{PA}}$ and $\mathcal{L}_{\mathsf{PA}^+}$ we are considering, we will simply denote the resulting full language by $\mathcal{L}^+$.

For any first-order language L, the set of its formulas is defined inductively as usual, and we denote this set by $\mathrm{Form}(L)$.

When the background logic is classical first-order logic, the theory PA proves the Diagonal Lemma, which is the usual means of self-reference (see Boolos (1993, pp. 53–54)):

Lemma 3.1.1 (Diagonal Lemma)**.** Let $A(y)$ be a formula in $\mathcal{L}_{\mathsf{PA}}$. Then there exists a sentence G of $\mathcal{L}_{\mathsf{PA}}$ such that:

$$\mathsf{PA} \models G \leftrightarrow A(\ulcorner G \urcorner).$$

Proof. Consider the primitive recursive function f defined as follows: $f(n) = \#A(\bar{n})$, if $\#A(x) = n$, and $f(n) = 0$ otherwise. Since f is primitive recursive, it is represented in PA by some formula $f^\circ(x, y)$. That is, f° is such that $\mathsf{PA} \models f^\circ(\bar{n}, \bar{m})$ if and only if $f(n) = m$. Now let $A(x)$ be any formula. Consider the formula

$$B(y) = \exists x(f^\circ(y, x) \wedge A(x)).$$

And define G to be $B(\ulcorner B \urcorner)$. Then $f(\#B) = \#B(\ulcorner B \urcorner) = \#G$. Hence $\mathsf{PA} \models f^\circ(\ulcorner B \urcorner, \ulcorner G \urcorner)$. By definition of G, we have

$$\mathsf{PA} \models G \leftrightarrow \exists x(f^\circ(\ulcorner B \urcorner, x) \wedge A(x))$$

By the fact that $\mathsf{PA} \models f^\circ(\ulcorner B \urcorner, \ulcorner G \urcorner)$, it follows that

$$\mathsf{PA} \models G \leftrightarrow A(\ulcorner G \urcorner).$$

□

There is, however, a sense in which the self-reference obtained by the Diagonal Lemma is not *truly* self-referential: the sentence G does not refer to itself explicitly. Rather, it *describes* itself by mentioning a certain property which happens to be possessed only by itself. It thus refers to itself by definite description. For most purposes, this is enough: we have a sentence that is provably equivalent to a sentence containing its own name.

In non-classical logics, however, the equivalence given by the Diagonal Lemma may break down – for instance, if the theory of the biconditional is weakened, as we will see in Section 4.1.1. It is then useful to work in the expanded language $\mathcal{L}_{\mathsf{PA}^+}$, which we take to contain a function symbol $\mathbf{s}$ for the primitive recursive function s, defined as follows:

$$s(m, n) = \begin{cases} \ulcorner A(\bar{n}/x) \urcorner, & \text{if } m = \#A \\ 0, & \text{otherwise.} \end{cases}$$

In this expanded language, we can prove a stronger form of self-reference: one that *is* literal, in the sense that we obtain a sentence containing (a term provably equal to) its own name. It is given by the Strong Diagonal Lemma, which is originally due to Jeroslow (1973). We follow the proof of Schindler (2015, p. 20).

Lemma 3.1.2 (Strong Diagonal Lemma)**.** Let $A(x)$ be any formula in $\mathcal{L}_{\mathsf{PA}^+}$. Then there exists a term t of $\mathcal{L}_{\mathsf{PA}^+}$ such that:

$$\mathsf{PA}^+ \models t = \ulcorner A(t) \urcorner.$$

In particular, the sentence $G = A(t)$ satisfies:

$$\mathsf{PA}^+ \models G \leftrightarrow A(\ulcorner G \urcorner).$$

Proof. Define the term t by $t = \mathbf{s}(\ulcorner A(\mathbf{s}(x,x))\urcorner, \ulcorner A(\mathbf{s}(x,x))\urcorner)$. Then note that, by definition of s (and the fact that PA^+ contains the defining axiom for $\mathbf{s}$),

$$\mathsf{PA}^+ \models t = \ulcorner A(\mathbf{s}(\ulcorner A(\mathbf{s}(x,x))\urcorner, \ulcorner A(\mathbf{s}(x,x))\urcorner))\urcorner.$$

And thus, by definition of t,

$$\mathsf{PA}^+ \models t = \ulcorner A(t)\urcorner.$$

□

The proof of this lemma still goes through in languages richer than $\mathcal{L}_{\mathsf{PA}^+}$, even if we assume a weaker logic for the non-arithmetical part of the language. In particular, we may extend $\mathcal{L}_{\mathsf{PA}^+}$ with a truth predicate and use a weaker non-classical logic to govern the portion of the language including truth predicate.

To see why, simply note that the above proof takes place entirely in the arithmetical part of the language, *even if* the formula $A(x)$ is non-arithmetical. The term $t = \mathbf{s}(\ulcorner A(\mathbf{s}(x,x))\urcorner, \ulcorner A(\mathbf{s}(x,x))\urcorner$ only contains the standard name of A, which simply refers to a natural number. Since we assumed classical logic for the arithmetical part, this means that the proof goes through unchanged. This will be very important to us in Chapter 4 and Section 6.4, when we will be needing a method of self-reference for non-classical logics.

The second benefit of using strong rather than weak diagonalisation is more philosophical: strong diagonalisation gives us a sentence which naturally corresponds to how we think of the paradoxes in natural language. We expect the Liar to be of the form $\neg T(l)$, with as its only subsentences itself and $T(l)$. The Strong Diagonal Lemma gives us such a sentence. The regular Diagonal Lemma, by contrast, gives us a sentence of the form

$$\exists x(f^\circ(\ulcorner B\urcorner, x) \wedge \neg T(x)),$$

where B is in turn the formula $\exists x(f^\circ(y,x) \wedge \neg T(x))$. This is a rather complicated construction, which is quite far from how we tend to think of paradoxes informally. The Strong Diagonal Lemma thus has the advantage of capturing our intuitive picture of paradoxical sentences. This will be especially relevant when we will be considering subsentences of the paradoxical sentences in Sections 4.3 and 6.3: the Strong Diagonal Lemma guarantees that the paradoxical sentences have the subsentences we expect them to have, and thus allow for a straightforward formalisation of the subsentence-based intuitions we saw in Chapter 2.

Remark 3.1.3. It should be noted that, besides the formal and philosophical benefits that are relevant here, the choice of strong diagonalisation over weak diagonalisation has important formal consequences too; for instance, certain truth theoretic axioms that are consistent over $\mathcal{L}_{\mathsf{PA}}$ become inconsistent over the expanded language $\mathcal{L}_{\mathsf{PA}^+}$; see Heck (2007), Schindler (2015).

3.2. Paradoxes

Given a particular choice of diagonalisation – weak or strong – we use the following notation to succinctly define the self-referential sentence G that it yields:

(G) $\qquad A(\ulcorner G\urcorner).$

That is, this defines the sentence G such that $\mathsf{PA} \models G \leftrightarrow A(\ulcorner G \urcorner)$ (weak diagonalisation) or the sentence $G = A(t)$ such that $\mathsf{PA}^+ \models t = A(t)$ (strong diagonalisation).

Both diagonal lemmas carry over immediately if we enrich the (corresponding) language of Peano arithmetic by adding a truth predicate. This gives us the means to express our classic Liar sentence:

$$(\lambda) \qquad \neg T \ulcorner \lambda \urcorner.$$

And, for any sentence B in the ground language, a Curry sentence $\mathcal{C}_B$ such that

$$(\mathcal{C}_B) \qquad T \ulcorner \mathcal{C}_B \urcorner \to B.$$

These two paradoxes will be the main players of this thesis, with an emphasis on the latter. But we will be interested in non-self-referential generalisations of these paradoxes as well: we are talking about multi-sentence versions. Such paradoxes are interesting because they are no longer explicitly self-referential, and they raise questions about the structure of paradoxes.

In order to express them, we need multi-sentence generalisations of the diagonal lemmas. Both the Diagonal Lemma and the Strong Diagonal Lemma generalise to versions in which multiple sentences refer to each other:

Lemma 3.2.1 (Diagonal Lemma, multi-sentence version)**.** Let n be a natural number. For each $i \leq n$, let $A_i(y_1, \ldots, y_n)$ be a formula of $\mathcal{L}^+_{\mathsf{PA}}$. Then there exist sentences $G_0, \ldots, G_n$ of $\mathcal{L}_{\mathsf{PA}}$ such that, for each $i \leq n$:

$$\mathsf{PA} \models G_i \leftrightarrow A_i(\ulcorner G_0 \urcorner, \ldots, \ulcorner G_n \urcorner).$$

Lemma 3.2.2 (Strong Diagonal Lemma, multi-sentence version)**.** Let n be a natural number. For each $i \leq n$, let $A_i(y_1, \ldots, y_n)$ be a formula of $\mathcal{L}^+_{\mathsf{PA}^+}$. Then there exist terms $t_0, \ldots, t_n$ of $\mathcal{L}_{\mathsf{PA}^+}$ such that, for each $i \leq n$:

$$\mathsf{PA}^+ \models t_i = \ulcorner A_i(t_1, \ldots, t_n) \urcorner.$$

In particular, there are formulas $G_0, \ldots, G_n$, defined by $G_i = A_i(t_1, \ldots, t_n)$, such that

$$\mathsf{PA}^+ \models G_i \leftrightarrow A_i(\ulcorner G_0 \urcorner, \ldots, \ulcorner G_n \urcorner).$$

The Card Liar might be the simplest non-self-referential paradox. It consists of two sentences that refer to each other:

$$(\lambda_1) \qquad T \ulcorner \lambda_2 \urcorner$$
$$(\lambda_2) \qquad \neg T \ulcorner \lambda_1 \urcorner.$$

The Card Liar is no longer explicitly self-referential, but still quite close: its reference pattern is a loop. This raises the question: is circularity a requirement for paradox? Yablo (1985) has argued to the contrary, proposing a version of the Liar that is not – prima facie – circular, and contains *infinitely* many sentences. It is the set of sentences $\{S_n : n \in \omega\}$ such that:

$$(S_0) \qquad (\forall m > 0)(\neg T\ulcorner S_{\dot{m}}\urcorner)$$
$$(S_1) \qquad (\forall m > 1)(\neg T\ulcorner S_{\dot{m}}\urcorner)$$
$$(S_2) \qquad (\forall m > 2)(\neg T\ulcorner S_{\dot{m}}\urcorner)$$
$$\vdots$$

Explicitly, S_n is defined as

$$(S_n) \qquad (\forall m > n)(\neg T\ulcorner S_{\dot{m}}\urcorner).$$

Here $\dot{m}$ is a device that allows us to quantify over m inside the name of S_m; see Picollo (2012) for details.

Yablo's paradox and the other multi-sentence versions of the Liar have inspired much research on the general shape of paradoxes.[1] Of particular interest for us are 'Curried' versions of the Card Liar and Yablo's paradox. The former is a straightforward adaptation of the Card Liar:

Definition 3.2.3 (Card Curry)**.** *Card Curry* is the set of sentences $\{\mathcal{C}_{B1}, \mathcal{C}_{B2}\}$, defined as follows:

$$(\mathcal{C}_{B1}) \qquad T\ulcorner \mathcal{C}_{B2}\urcorner$$
$$(\mathcal{C}_{B2}) \qquad T\ulcorner \mathcal{C}_{B1}\urcorner \to B$$

A Curried version of Yablo's paradox was introduced by Cook (2009), under the name *Yablurry*:

Definition 3.2.4 (Yablurry (Cook, 2009, p. 617))**.** *Yablurry* is the set of sentences $\{S_n : n \in \omega\}$ defined as follows:

$$(S_0) \qquad (\forall m > 0)(T\ulcorner S_{\dot{m}}\urcorner \to B)$$
$$(S_1) \qquad (\forall m > 1)(T\ulcorner S_{\dot{m}}\urcorner \to B)$$
$$(S_2) \qquad (\forall m > 2)(T\ulcorner S_{\dot{m}}\urcorner \to B)$$
$$\vdots$$

Explicitly, S_n is defined as

$$(S_n) \qquad (\forall m > n)(T\ulcorner S_{\dot{m}}\urcorner \to B).$$

These two multi-sentence versions of Curry's paradox will recur throughout the coming chapters.

[1] See e.g. Beringer and Schindler (2017), Cook (2006).

4. Curry in Kripke's theory of truth

In this chapter, we look at the behaviour of Curry's sentence in Kripke's theory of truth. Our main goal is to define a notion of paradoxicality based on Kripke's definition that is tailored to Curry's paradox. We will be formalising Intuitions 5b and 5a and Intuition 6.

The chapter is structured as follows. In the first section, we set the stage by introducing Kripke's theory of truth. In the second section, we examine the behaviour of Curry's paradox in Kripke's theory and formalise Intuitions 5b and 5a. In the third section, we formalise Intuition 6. We conclude with an evaluation and comparison of the different definitions.

4.1. Kripke's fixed point definition

In this section, we work towards Kripke's definition of paradoxicality by reviewing those parts of Kripke's theory of truth that we need. We follow Kripke's seminal paper (Kripke, 1975).

Kripke's theory of truth is based on a simple idea: the concept of truth is to be understood in stages. Suppose we speak a language that does not include the concept of truth; let us call this the *ground language*. At stage 0, we do not know yet what 'truth' is. It is thus impossible for us to understand sentences involving the concept of truth, such as 'The sentence "Snow is white" is true.' However, we may advance in our understanding of truth by accepting the following rule:

> we are entitled to assert (or deny) of any sentence that it is true precisely under the circum- stances when we can assert (or deny) the sentence itself. (Kripke, 1975, p. 701)

We thus come to understand what it means to attribute truth to a sentence as 'Snow is white.' Indeed, we come to learn that the sentence 'Snow is white' is true. Having done so, we move on to the next stage, at which we learn, by following the rule, that 'The sentence "Snow is white" is true' is true. Continuing this process, we are able to assert the truth of more and more complex sentences involving the notion of truth. This process is reflected in Kripke's *fixed point construction* that we will see in a moment (Definition and Theorem 4.1.13). The trick is that, if we pursue this into the transfinite ordinals, the process will come to a halt: there will be a stage at which all sentences involving truth of which we *can* know whether they are true or false, we will know. This is the fixed point. Will this include every sentence? Undoubtedly not: only those sentences whose truth is grounded in facts that do *not* involve truth will obtain a truth value. As Kripke has put it:

> There is no reason to suppose that all statements involving 'true' will become decided in this way, but most will. Indeed, our suggestion is that the "grounded" sentences can be characterized as those which eventually get a truth value in this process. (Kripke, 1975, p. 701)

Paradoxical sentences are examples of such *ungrounded* sentences: the truth of a sentence such as the Liar, which asserts its own falsity, can evidently not be reduced to the truth of sentences in the ground language. This, then, is how Kripke's theory of truth gives rise to a definition of paradoxicality. In what follows, we will see how this is done formally; we first consider the necessary technical preliminaries, and then turn to the definitions of fixed points and paradoxicality in Kripke's theory.

4.1.1. Technical preliminaries

Kripke's theory of truth starts from a ground language that is rich enough to express elementary arithmetic. We use the expanded language of arithmetic $\mathcal{L}_{\mathsf{PA}^+}$ described in Chapter 3 as our ground language – this will give us the Strong Diagonal Lemma. We will simply denote it by $\mathcal{L}$ throughout this chapter. The full language $\mathcal{L}^+$ is obtained from $\mathcal{L}$ by adding a truth predicate T.

We will be considering classical first-order models $\mathcal{M} = (M, I)$ for the ground language $\mathcal{L}$ that make the theory PA^+ true. We will call such models *ground models*.

Definition 4.1.1 (Classical model for $\mathcal{L}$)**.** A classical model for the language $\mathcal{L}$ is a pair $\mathcal{M} = (M, I)$ where M is the domain of $\mathcal{M}$ and I is an interpretation function such that:

1. For each function symbol f of arity n in $\mathcal{L}$, $I(f)$ is a function on M of the same arity;
2. For each predicate symbol P of arity n in $\mathcal{L}$, $I(P)$ is a relation on M of the same arity.

We will also denote $I(f)$ and $I(P)$ by $f^{\mathcal{M}}$ and $P^{\mathcal{M}}$, respectively. The interpretation function induces an interpretation of each closed term t, which we will denote by $t^{\mathcal{M}}$.

Truth in a classical model is defined as usual in classical first-order logic:

Definition 4.1.2 (Truth in a classical model)**.** *Truth of a sentence A in a classical model* $\mathcal{M} = (D, I)$ for $\mathcal{L}$, denoted $\mathcal{M} \models A$, is recursively defined as follows:

$$\begin{array}{lll} \mathcal{M} \models P(t_1, \ldots, t_n) & \iff & (t_1^{\mathcal{M}}, \ldots, t_n^{\mathcal{M}}) \in P^{\mathcal{M}} \\ \mathcal{M} \models t_1 = t_2 & \iff & t_1^{\mathcal{M}} = t_2^{\mathcal{M}} \\ \mathcal{M} \models A \vee B & \iff & \mathcal{M} \models A \text{ or } \mathcal{M} \models B \\ \mathcal{M} \models \neg A & \iff & \mathcal{M} \not\models A \\ \mathcal{M} \models \exists x (A(x)) & \iff & \text{there exists some } d \in D \text{ such that } \mathcal{M} \models A(\mathbf{d}). \end{array}$$

If Σ is a set of sentences, we write $\mathcal{M} \models \Sigma$ if $\mathcal{M} \models A$ for all $A \in \Sigma$.

Let us stress that a ground model is not just any model for the ground language, but one that makes the theory PA^+ true:

Definition 4.1.3 (Ground model)**.** A *ground model* is a classical model $\mathcal{M}$ for $\mathcal{L}$ such that $\mathcal{M} \models \mathsf{PA}^+$.

For the full language $\mathcal{L}^+$, we use non-classical models that allow for truth value *gaps*: sentences that are evaluated as neither true nor false. This will allow certain sentences involving the truth predicate – notably, the paradoxical sentences – to lack a truth value. We denote these non-classical models for $\mathcal{L}^+$ by $\mathcal{M}_{(T^+, T^-)}$, given by ground model $\mathcal{M}$ together with a partial interpretation (T^+, T^-) of the truth predicate:

Definition 4.1.4 (Partial interpretation of the truth predicate)**.** A *partial interpretation of* T is a pair (T^+, T^-) consisting of an extension $T^+ \subset M$ and an anti-extension $T^- \subset M$ such that:

1. All members of T^+ are (interpretations of Gödel numbers of) sentences of $\mathcal{L}^+$,
2. The intersection of T^+ and T^- is empty.

Truth or falsity of a formula in a model $\mathcal{M}_{(T^+,T^-)}$ is determined by following one of three possible valuation schemas: the strong Kleene truth conditions, the weak Kleene truth conditions, or van Fraassen's supervaluationism schema. Each of these logics gives rise to a different theory, and each of these theories has its own advantages and disadvantages.

In all of the coming definitions, we tacitly enrich the language with a constant symbol $\mathbf{d}$ for each element d in the domain of a model $\mathcal{M} = (M, I)$.

Definition 4.1.5 (Strong Kleene truth conditions)**.** For a model $\mathcal{M} = (M, I)$ and a partial interpretation (T^+, T^-) of T, we define recursively when a sentence is true ($\models^+$) or false ($\models^-$) in $\mathcal{M}_{(T^+,T^-)}$ according to the strong Kleene (SK) conditions as follows:

$$\begin{aligned}
\mathcal{M}_{(T^+,T^-)} \models^+ P(t_1, \ldots, t_n) &\iff \mathcal{M} \models P(t_1, \ldots, t_n), \text{ if } P \neq T \\
\mathcal{M}_{(T^+,T^-)} \models^- P(t_1, \ldots, t_n) &\iff \mathcal{M} \not\models P(t_1, \ldots, t_n), \text{ if } P \neq T \\
\mathcal{M}_{(T^+,T^-)} \models^+ t_1 = t_2 &\iff t_1^{\mathcal{M}} = t_2^{\mathcal{M}} \\
\mathcal{M}_{(T^+,T^-)} \models^- t_1 = t_2 &\iff t_1^{\mathcal{M}} \neq t_2^{\mathcal{M}} \\
\mathcal{M}_{(T^+,T^-)} \models^+ T(t) &\iff I(t) \in T^+ \\
\mathcal{M}_{(T^+,T^-)} \models^- T(t) &\iff I(t) \in T^- \\
\mathcal{M}_{(T^+,T^-)} \models^+ A \vee B &\iff \mathcal{M}_{(T^+,T^-)} \models^+ A \text{ or } \mathcal{M}_{(T^+,T^-)} \models^+ B \\
\mathcal{M}_{(T^+,T^-)} \models^- A \vee B &\iff \mathcal{M}_{(T^+,T^-)} \models^- A \text{ and } \mathcal{M}_{(T^+,T^-)} \models^- B \\
\mathcal{M}_{(T^+,T^-)} \models^+ \neg A &\iff \mathcal{M}_{(T^+,T^-)} \models^- A \\
\mathcal{M}_{(T^+,T^-)} \models^- \neg A &\iff \mathcal{M}_{(T^+,T^-)} \models^+ A \\
\mathcal{M}_{(T^+,T^-)} \models^+ \exists x(A(x)) &\iff \text{there exists some } d \in M \text{ such that } \mathcal{M}_{(T^+,T^-)} \models^+ A(\mathbf{d}) \\
\mathcal{M}_{(T^+,T^-)} \models^- \exists x(A(x)) &\iff \text{for all } d \in M : \mathcal{M}_{(T^+,T^-)} \models^- A(\mathbf{d})
\end{aligned}$$

Here and throughout the thesis, the connectives $\wedge$ and $\rightarrow$ and $\leftrightarrow$ are defined as usual in terms of $\neg$ and $\vee$, and $\forall$ is defined as usual in terms of $\neg$ and $\exists$. We take $\top$ to be defined as $\top \overset{\text{def}}{=} \bar{0} = \bar{0}$, and $\bot$ as $\bot \overset{\text{def}}{=} \neg\top$.

If A is true in $\mathcal{M}_{(T^+,T^-)}$ or false in $\mathcal{M}_{(T^+,T^-)}$, we say that A *has a truth value in* $\mathcal{M}_{(T^+,T^-)}$.

Definition 4.1.6 (Weak Kleene truth conditions)**.** For a model $\mathcal{M} = (M, I)$ and a partial interpretation (T^+, T^-) of T, we define recursively when a sentence is true ($\models^+$) or false

($\models^-$) in $\mathcal{M}_{(T^+,T^-)}$ according to the weak Kleene (WK) definitions as follows:

$$
\begin{aligned}
\mathcal{M}_{(T^+,T^-)} &\models^+ P(t_1,\ldots,t_n) &&\iff \mathcal{M} \models P(t_1,\ldots,t_n), \text{ if } P \neq T \\
\mathcal{M}_{(T^+,T^-)} &\models^- P(t_1,\ldots,t_n) &&\iff \mathcal{M} \not\models P(t_1,\ldots,t_n), \text{ if } P \neq T \\
\mathcal{M}_{(T^+,T^-)} &\models^+ t_1 = t_2 &&\iff t_1^{\mathcal{M}} = t_2^{\mathcal{M}} \\
\mathcal{M}_{(T^+,T^-)} &\models^- t_1 = t_2 &&\iff t_1^{\mathcal{M}} \neq t_2^{\mathcal{M}} \\
\mathcal{M}_{(T^+,T^-)} &\models^+ T(t) &&\iff I(t) \in T^+ \\
\mathcal{M}_{(T^+,T^-)} &\models^- T(t) &&\iff I(t) \in T^- \\
\mathcal{M}_{(T^+,T^-)} &\models^+ A \vee B &&\iff \mathcal{M}_{(T^+,T^-)} \models^+ A \text{ or } \mathcal{M}_{(T^+,T^-)} \models^+ B, \\
& && \quad \text{and both } A \text{ and } B \text{ have a truth value in } \mathcal{M}_{(T^+,T^-)} \\
\mathcal{M}_{(T^+,T^-)} &\models^- A \vee B &&\iff \mathcal{M}_{(T^+,T^-)} \models^- A \text{ and } \mathcal{M}_{(T^+,T^-)} \models^- B, \\
& && \quad \text{and both } A \text{ and } B \text{ have a truth value in } \mathcal{M}_{(T^+,T^-)} \\
\mathcal{M}_{(T^+,T^-)} &\models^+ \neg A &&\iff \mathcal{M}_{(T^+,T^-)} \models^- A \\
\mathcal{M}_{(T^+,T^-)} &\models^- \neg A &&\iff \mathcal{M}_{(T^+,T^-)} \models^+ A \\
\mathcal{M}_{(T^+,T^-)} &\models^+ \exists x(A(x)) &&\iff \text{there exists some } d \in M \text{ such that } \mathcal{M}_{(T^+,T^-)} \models^+ A(\mathbf{d}), \\
& && \quad \text{and } A(\mathbf{e}) \text{ has a truth value in } \mathcal{M}_{(T^+,T^-)} \text{ for every } e \in M \\
\mathcal{M}_{(T^+,T^-)} &\models^- \exists x(A(x)) &&\iff \text{for all } d \in M : \mathcal{M}_{(T^+,T^-)} \models^- A(\mathbf{d})
\end{aligned}
$$

For the supervaluationism conditions, there are multiple options; we consider the simplest one, following Field (2009, Chapter 10.1) and Kremer and Urquhart (2008, Chapter 2).[1]

Definition 4.1.7 (Supervaluationism truth conditions)**.** A *classical model* $\mathcal{M}_X$ *for the language* $\mathcal{L}^+$ is a ground model $\mathcal{M} = (D, I)$ for $\mathcal{L}$ together with a classical interpretation $X \subseteq D$ of the truth predicate. Truth in $\mathcal{M}_X$, denoted $\mathcal{M}_X \models A$, is determined as usual in classical first-order logic.

If (T^+, T^-) is a partial interpretation of the truth predicate, we say that X is a *precisification of* (T^+, T^-) if $T^+ \subseteq X$ and $T^- \subseteq D \setminus X$. Truth ($\models^+$) and falsity ($\models^-$) in a partial model $\mathcal{M}_{(T^+,T^-)}$ are defined according to the supervaluationism (SV) truth conditions as follows:

$$
\begin{aligned}
\mathcal{M}_{(T^+,T^-)} \models^+ A &\iff \mathcal{M}_X \models A \text{ for every precisification } X \text{ of } (T^+, T^-) \\
\mathcal{M}_{(T^+,T^-)} \models^- A &\iff \mathcal{M}_X \not\models A \text{ for every precisification } X \text{ of } (T^+, T^-).
\end{aligned}
$$

Definition 4.1.8 (Logical consequence)**.** For any logic $L \in \{\mathsf{SK}, \mathsf{WK}, \mathsf{SV}\}$, we define logical consequence for L as follows. Let Σ be a set of sentences and let A be a sentence.

$$
\Sigma \models_L A \text{ iff, for every partial model } \mathcal{M}_{(T^+,T^-)} \text{ for } \mathcal{L}^+\text{: if } \mathcal{M}_{(T^+,T^-)} \models^+ B \text{ for all } B \in \Sigma \\
\text{then } \mathcal{M}_{(T^+,T^-)} \models^+ A \text{ according to the valuation scheme of } L.
$$

[1] The more involved variants of supervaluationism conditions place additional restrictions on the precisifications X; see e.g. (Field, 2009, Chapter 11.2).

For our method of self-reference, we employ the Strong Diagonal Lemma. The regular Diagonal Lemma does not literally go through: in classical logic, this gives us, for any formula A, a sentence G such that

$$\mathsf{PA} \models G \leftrightarrow A(\ulcorner G \urcorner).$$

However, in the current setting it is possible that both G and $A(\ulcorner G \urcorner)$ have no truth value, in which case the biconditional $G \leftrightarrow A(\ulcorner G \urcorner)$ has no truth value either.

The Strong Diagonal Lemma, on the other hand, goes through: as we mentioned in Section 3.1, it holds even if we use a non-classical logic to govern the truth predicate, so long as the logic governing the arithmetical part is classical.

Lemma 4.1.9 (Strong Diagonal Lemma)**.** Let n be a natural number. For each $i \leq n$, let $A_i(y_1, \ldots, y_n)$ be a formula of $\mathcal{L}^+$. Then there exist terms $t_0, \ldots, t_n$ of $\mathcal{L}_{\mathsf{PA}}$ such that, for each $i \leq n$:

$$\mathsf{PA}^+ \models_L t_i = \ulcorner A_i(t_1, \ldots, t_n) \urcorner,$$

for any logic $L \in \{\mathsf{SK}, \mathsf{WK}, \mathsf{SV}\}$.

Proof. See Lemma 3.1.2 and the subsequent remark. □

This gives us formulas $G_0, \ldots, G_n$, defined by $G_i = A_i(t_1, \ldots, t_n)$, such that, for any model $\mathcal{M}_{(T^+,T^-)}$ for PA, we have the following:

$$\mathcal{M}_{(T^+,T^-)} \models^+ G_i \iff \mathcal{M}_{(T^+,T^-)} \models^+ A_i(\ulcorner G_0 \urcorner, \ldots, \ulcorner G_n \urcorner)$$
$$\mathcal{M}_{(T^+,T^-)} \models^- G_i \iff \mathcal{M}_{(T^+,T^-)} \models^- A_i(\ulcorner G_0 \urcorner, \ldots, \ulcorner G_n \urcorner),$$

on each of the strong Kleene, weak Kleene, and supervaluationism schemes. As a shorthand for the above definition, we will usually simply denote each of the formulas G_i by

$$(G_i) \qquad A_i(\ulcorner G_0 \urcorner, \ldots, \ulcorner G_n \urcorner).$$

We say that the formula G_i is *diagonally defined* by the formulas $A_1, \ldots, A_n$.

To shorten notation, we introduce the following, slightly unconventional definition of equivalence of sentences:

Definition 4.1.10 (Equivalence)**.** We say that sentences A and B are *equivalent*, written $A \equiv B$, if they have the same truth value in every model:

$$\mathcal{M}_{(T^+,T^-)} \models^+ A \iff \mathcal{M}_{(T^+,T^-)} \models^+ B$$
$$\mathcal{M}_{(T^+,T^-)} \models^- A \iff \mathcal{M}_{(T^+,T^-)} \models^- B.$$

Remark 4.1.11. Equivalence $A \equiv B$ is commonly defined as $\mathcal{M}_{(T^+,T^-)} \models^+ A \leftrightarrow B$ for every model $\mathcal{M}_{(T^+,T^-)}$. On our definition, this is *not* the case: if both A and B are without a truth value, then so is $A \leftrightarrow B$. Our definition is thus not put in terms of the biconditional of the object language, but rather in terms of sameness of truth value.

The Strong Diagonal Lemma thus gives us sentences G_i such that $G_i \equiv A(\ulcorner G_0 \urcorner, \ldots, \ulcorner G_n \urcorner)$. We moreover know that each sentence G_i is truly referring to the sentences $G_0, \ldots, G_n$ in the sense that G_i contains terms t_k such that $\mathsf{PA}^+ \models t_k = \ulcorner G_k \urcorner$.

As we saw in Section 3.2, the Strong Diagonal Lemma allows us to express the paradoxes that we will be studying. To obtain the Liar, applying the Strong Diagonal Lemma to the formula $\neg T(x)$ gives us a term l such that

$$\mathsf{PA}^+ \models l = \ulcorner \neg T(l) \urcorner.$$

The Liar sentence is then the sentence $\lambda := \neg T(l)$, and we have $\lambda \equiv \neg T \ulcorner \lambda \urcorner$. Similarly, Curry's sentence is obtained by applying the Strong Diagonal Lemma to the formula $T(x) \to B$. Curry's sentence is then defined as $\mathcal{C}_B := T(c) \to B$, where c is the term such that

$$\mathsf{PA}^+ \models c = \ulcorner T(c) \to B \urcorner.$$

To obtain multi-sentence paradoxes, we use multiple defining formulas A_i: for instance, the Card Liar is the set of sentences $\{\lambda_1, \lambda_2\}$ defined as

(λ_1) $$T \ulcorner \lambda_2 \urcorner$$

(λ_2) $$\neg T \ulcorner \lambda_1 \urcorner.$$

4.1.2. Kripke's fixed point definition

The definition of paradoxicality that we are after is framed in terms of the *fixed points* that are central to Kripke's theory:

Definition 4.1.12. A partial interpretation (T^+, T^-) of T is a *fixed point* of the ground model $\mathcal{M}$ if

$$\begin{aligned} T^+ &= \{\ulcorner A \urcorner^{\mathcal{M}} : A \in \text{Form}(\mathcal{L}^+) \text{ and } \mathcal{M}_{(T^+,T^-)} \models^+ A\}, \\ T^- &= \{d \in M : d \neq \ulcorner B \urcorner^{\mathcal{M}} \text{ for every sentence } B \in \mathcal{L}^+\} \cup \\ &\quad \{\ulcorner A \urcorner^{\mathcal{M}} : A \in \text{Form}(\mathcal{L}^+) \text{ and } \mathcal{M}_{(T^+,T^-)} \models^- A\}. \end{aligned}$$

If (T^+, T^-) is a fixed point, we will refer to the corresponding model $\mathcal{M}_{(T^+,T^-)}$ as a fixed point as well.

If (T^+, T^-) is a fixed point, we have $\mathcal{M}_{(T^+,T^-)} \models^+ T \ulcorner A \urcorner$ iff $\mathcal{M}_{(T^+,T^-)} \models^+ A$ and $\mathcal{M}_{(T^+,T^-)} \models^- T \ulcorner A \urcorner$ iff $\mathcal{M}_{(T^+,T^-)} \models^- A$. That is, the language contains its own truth predicate, obeying a kind of T-schema: we have $T \ulcorner A \urcorner \equiv A$. Note however that we do *not* have $T \ulcorner A \urcorner \leftrightarrow A$: it may be that neither A nor $T \ulcorner A \urcorner$ has a truth value, in which case $T \ulcorner A \urcorner \leftrightarrow A$ does not have a truth value either.

At the heart of Kripke's theory is the fact that there exists a fixed point for every ground model; a *least* fixed point, in particular. The construction of the least fixed point will be of use later on, so we describe it here.

Definition and Theorem 4.1.13 (Strong Kleene, weak Kleene, supervaluationism (Kripke, 1975, pp. 704–705))**.** Let $\mathcal{M}$ be any ground model. The *least fixed point construction* for $\mathcal{M}$ is the sequence $((T^+_\alpha, T^-_\alpha))_{\alpha \in \text{Ord}}$ of partial interpretations of the truth predicate defined recursively as follows:

For $\alpha = 0$, we define $(T^+_0, T^-_0) = (\varnothing, \varnothing)$.

For a successor ordinal $\alpha = \beta + 1$, we define

$$T^+_\alpha = \{\ulcorner A \urcorner : A \in \text{Form}(\mathcal{L}^+) \text{ and } \mathcal{M}_{(T^+_\beta, T^-_\beta)} \models^+ A\}$$
$$T^-_\alpha = \{n \in M : n \text{ is not the Gödel number of a sentence of } \mathcal{L}^+\} \cup \{\ulcorner A \urcorner : A \in \text{Form}(\mathcal{L}^+) \text{ and } \mathcal{M}_{(T^+_\beta, T^-_\beta)} \models^- A\}.$$

For a limit ordinal λ, we define

$$T^+_\lambda = \bigcup \{T^+_\beta : \beta < \lambda\}$$
$$T^-_\lambda = \bigcup \{T^-_\beta : \beta < \lambda\}.$$

Then there exists an ordinal β such that $(T^+_\beta, T^-_\beta) = (T^+_{\beta+1}, T^-_{\beta+1})$. This is a fixed point of $\mathcal{M}$, and it contains any other fixed point of $\mathcal{M}$. We call it the *least fixed point* of $\mathcal{M}$.

Proof. To show that there exists β such that $(T^+_\beta, T^-_\beta) = (T^+_{\beta+1}, T^-_{\beta+1})$, note that the extensions and antiextensions keep growing as we travel up the ordinals: it is easy to check that – in all three logics – if $\alpha < \beta$, then $T^+_\alpha \subseteq T^+_\beta$ and $T^-_\alpha \subseteq T^-_\beta$. But they cannot keep growing forever: $\text{Form}(\mathcal{L}^+)$ is a set of fixed cardinality κ, so there must be some ordinal $\beta \leq \kappa$ such that $(T^+_\beta, T^-_\beta) = (T^+_{\beta+1}, T^-_{\beta+1})$.

By the very definition of $T^+_{\beta+1}$ and $T^-_{\beta+1}$, we have that (T^+_β, T^-_β) is a fixed point of $\mathcal{M}$. To show that (T^+_β, T^-_β) is a least fixed point of $\mathcal{M}$, let (T^+, T^-) be any fixed point of $\mathcal{M}$. It can be shown by induction on α that $T^+_\alpha \subseteq T^+$ and $T^-_\alpha \subseteq T^-$ for any α; it then follows that $(T^+_\beta, T^-_\beta) \subseteq (T^+, T^-)$. □

The above construction can be repeated to obtain other fixed points by starting from non-empty T^+_0 and T^-_0. This allows us to create fixed points in which more sentences have a truth value: some sentences have no truth value in the least fixed point of a model $\mathcal{M}$, but they do have a truth value in some other fixed points. A typical example of such a sentence is the *truth teller*, which asserts of itself that it is true:

$$(\mathcal{T}) \qquad T \ulcorner \mathcal{T} \urcorner.$$

It is straightforward to check by induction that $\mathcal{T}$ does not have a truth value in the least fixed point of any model $\mathcal{M}$. However, there are fixed points in which the truth teller does have a truth value: if we repeat the construction by starting from $T^+_0 = \{\mathcal{T}\}$ and $T^-_0 = \varnothing$, then we obtain a fixed point of $\mathcal{M}$ in which the it is true. Alternatively, defining $T^+_0 = \varnothing$ and $T^-_0 = \{\mathcal{T}\}$ gives us a fixed point in which it is false. Thus, the truth teller has a truth value in some, but not all fixed points.

Proposition 4.1.14 (Strong Kleene, weak Kleene, supervaluationism)**.** Let $\mathcal{M}$ be any ground model. The truth teller $\mathcal{T}$ does not have a truth value in the least fixed point of $\mathcal{M}$, but there exists a fixed point of $\mathcal{M}$ in which $\mathcal{T}$ does have a truth value.

Sentences such as $\mathcal{T}$ stand in contrast to sentences that have no truth value in any fixed point, such as the Liar and the Card Liar. It follows from the definition of negation in strong Kleene, weak Kleene, and supervaluationism, that it is impossible to assign a truth value to any of these sentences in any fixed point.

Proposition 4.1.15 (Strong Kleene, weak Kleene, supervaluationism)**.** Let $\mathcal{M}$ be any ground model. The Liar sentence λ and both sentences λ_1 and λ_2 of the Card Liar do not have a truth value in any fixed point of $\mathcal{M}$.

Proof. Suppose (T^+, T^-) is a fixed point of $\mathcal{M}$. Then (in any of the three logics), $\mathcal{M}_{(T^+,T^-)} \models^+ \lambda$ iff $\mathcal{M}_{(T^+,T^-)} \models^+ T\ulcorner\lambda\urcorner$, by definition of a fixed point. However, by definition of λ, we have $\mathcal{M}_{(T^+,T^-)} \models^+ \lambda$ iff $\mathcal{M}_{(T^+,T^-)} \models^+ \neg T\ulcorner\lambda\urcorner$. This is in turn equivalent to $\mathcal{M}_{(T^+,T^-)} \models^- \lambda$ in all three logics. So we obtain that $\mathcal{M}_{(T^+,T^-)} \models^+ \lambda$ iff $\mathcal{M}_{(T^+,T^-)} \models^- \lambda$, which means that λ does not have a truth value in $\mathcal{M}_{(T^+,T^-)}$.

For the Card Liar, similar reasoning gives us the following equivalences: $\mathcal{M}_{(T^+,T^-)} \models^+ \lambda_1$ iff $\mathcal{M}_{(T^+,T^-)} \models^+ T\ulcorner\lambda_2\urcorner$ iff $\mathcal{M}_{(T^+,T^-)} \models^+ \lambda_2$ iff $\mathcal{M}_{(T^+,T^-)} \models^+ \neg T\ulcorner\lambda_1\urcorner$ iff $\mathcal{M}_{(T^+,T^-)} \models^- \lambda_1$. So $\mathcal{M}_{(T^+,T^-)} \models^+ \lambda_1$ iff $\mathcal{M}_{(T^+,T^-)} \models^- \lambda_1$, telling us that λ_1 does not have a truth value in $\mathcal{M}_{(T^+,T^-)}$. An analogous argument shows us that λ_2 does not have a truth value in $\mathcal{M}_{(T^+,T^-)}$ either. □

The difference between the truth teller and the Liar can be thought of as follows: The truth value of a sentence like the truth teller cannot be determined, but it can still be assigned a truth value coherently. The Liar, by contrast, cannot coherently be assigned any truth value at all. This finally brings us to Kripke's definition of paradoxicality and the closely related definition of *groundedness*:

Definition 4.1.16 (Adaptation of (Kripke, 1975, p. 694))**.** A sentence is *grounded in* $\mathcal{M}$ if it has a truth value in every fixed point of $\mathcal{M}$. Otherwise, it is *ungrounded in* $\mathcal{M}$.

The Liar and the truth teller are both examples of ungrounded sentences, for any ground model $\mathcal{M}$.

Definition 4.1.17 (Adaptation of (Kripke, 1975, p. 708))**.** A sentence is *(Kripke-)paradoxical in a ground model* $\mathcal{M}$ if it has no truth value in any fixed point of $\mathcal{M}$. Otherwise, it is *unparadoxical* in $\mathcal{M}$.

What Propositions 4.1.14 and 4.1.15 have told us is that the Liar and both sentences in the Card Liar are paradoxical (in any ground model), while the truth teller is ungrounded but unparadoxical (in any ground model).

Remark 4.1.18. As we will see in Chapter 5, not all definitions of paradoxicality recognize multi-sentence paradoxes, such as the Card Liar, as paradoxical. It can be seen of one of the strengths of Kripke's framework that it does account for such multi-sentence paradoxes.

Remark 4.1.19. The truth teller, the Liar, and the Card Liar have the same status across all three logics; but note that it may in general depend on which logic is used (strong Kleene, weak Kleene, or supervaluationism) whether a sentence has a truth value in a model, and thus whether a sentence is grounded or paradoxical in a model. In other words, the notions of groundedness and paradoxicality are logic-dependent. In the next section, we will see some examples of sentences whose groundedness and/or paradoxicality depends on which logic is used (in particular: the (un)groundedness of Curry's sentence in Propositions 4.2.5 and 4.2.6, and the (un)paradoxicality of some composite sentences in Propositions 4.2.11 and 4.2.13).

Kripke fixes the ground model $\mathcal{M}$ at the outset and is not explicit about the fact that his notions of groundedness and paradoxicality depend on $\mathcal{M}$, and this is the custom in current work too. That is not unreasonable: for most sentences that are typically considered, their paradoxicality and groundedness do not depend on $\mathcal{M}$. We saw that the truth teller is ungrounded and unparadoxical in any $\mathcal{M}$, and the Liar and Card Liar are paradoxical regardless of the ground model. But for some sentences, the choice of ground model matters – they are paradoxical in some ground models, but not others. Curry's sentence is an example of such a sentence.

4.2. Curry's sentence in Kripke's theory

In this section, we see how Kripke's definition of paradoxicality naturally gives rise to two characterisations of Curry-paradoxical sentences, in the form of what we call *local* paradoxicality and *local-but-not-global* paradoxicality. The former definition can be seen as a formalisation of Intuition 5a: 'A Curry-paradoxical sentence is a sentence that behaves paradoxically in some models.' The latter is a more restrictive notion that can be seen as a formalisation of Intuition 5b: 'A Curry-paradoxical sentence is a sentence that behaves paradoxically in some models, but not in others.'

The main question we aim to answer in this section is: how do our proposed formalisations play out formally? Do they capture the sentences that we would like to capture, or do they have unwanted consequences? To answer these questions, we will be studying the set of locally (but not globally) paradoxical sentences, and collect examples of its members.

4.2.1. Capturing Curry

Some sentences are not like the Liar in that they are paradoxical in every ground model, but they still are paradoxical in *some* ground models. They are sentences such as Curry's. In this section, we will be considering Curry's sentence with its consequent B in the ground language:

Definition 4.2.1. Let $B \in \mathcal{L}$. Curry's sentence is the sentence

$$(\mathcal{C}_B) \qquad T\ulcorner\mathcal{C}_B\urcorner \to B.$$

By definition of $\to$, we have $\mathcal{C}_B \equiv \neg T\ulcorner\mathcal{C}_B\urcorner \vee B$.

To distinguish sentences such as the Liar from sentences such as Curry's, we free Kripke's definition of paradoxicality from its dependence on a ground model. This gives rise to the notions of *local* and *global* paradoxicality:

Definition 4.2.2. A sentence A of $\mathcal{L}^+$ is *globally (Kripke-)paradoxical* if A is paradoxical in every ground model. It is *globally (Kripke-)unparadoxical* if it is unparadoxical in every ground model.

Definition 4.2.3. A sentence A of $\mathcal{L}^+$ is *locally (Kripke-)paradoxical* if there exists some ground model $\mathcal{M}$ such that A is paradoxical in $\mathcal{M}$.

Sentences like the Liar are globally paradoxical – and, a fortiori, also locally paradoxical.

Proposition 4.2.4. The Liar sentence λ is globally paradoxical.

Proof. Follows from Proposition 4.1.15. □

Sentences such as Curry's – except tautological Curry – are locally paradoxical, but not necessarily globally so. The notion of local paradoxicality corresponds to Intuition 5a that we saw in Chapter 2:

Intuition 5a. A Curry-paradoxical sentence is a sentence that behaves paradoxically in some models.

We also proposed a more restrictive characterisation – not with the goal of capturing all of Curry's paradox, but with the aim of distinguishing what makes Curry special – according to which a Curry-paradoxical sentence is paradoxical in some models, but not in others:

Intuition 5b. A Curry-paradoxical sentence is a sentence that behaves paradoxically in some models, but not in others.

This intuition does not capture contradictory Curry: contradictory Curry is paradoxical in every model. But it does capture a unique feature of Curry's paradox: some of its instances behave differently depending on the interpretation of their consequent. Formally, this intuition allows to be captured by the notion of *local-but-not-global* paradoxicality: a sentence is locally but not globally paradoxical precisely when there exists some ground model in which it is paradoxical, and another in which it is unparadoxical.

In what follows, we examine how local paradoxicality and local-but-not-global paradoxicality fare as a formalisation of the above intuitions. We first verify that Curry is indeed locally paradoxical so long as its consequent is not tautological, and locally but not globally paradoxical in case its consequent is neither a contradiction nor a tautology. Curry's sentence behaves slightly differently in weak Kleene than in the logics strong Kleene and supervaluationism; we consider these two cases separately.

Proposition 4.2.5 (Strong Kleene, supervaluationism)**.** Let $\mathcal{M}$ be any ground model. If $\mathcal{M} \models B$, then Curry's sentence $\mathcal{C}_B$ is grounded (and hence unparadoxical) in $\mathcal{M}$. If $\mathcal{M} \not\models B$, then $\mathcal{C}_B$ is paradoxical in $\mathcal{M}$.

Proof. Strong Kleene. Let $\mathcal{M}_{(T^+,T^-)}$ be a fixed point of $\mathcal{M}$. We show that Curry's sentence $\mathcal{C}_B$ has a truth value in $\mathcal{M}_{(T^+,T^-)}$ iff $\mathcal{M} \models B$. In order to do so, we spell out the truth conditions of $\mathcal{C}_B$ in $\mathcal{M}_{(T^+,T^-)}$. The strong Kleene truth conditions tell us that $\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_B$ iff $\mathcal{M}_{(T^+,T^-)} \models^- T\ulcorner\mathcal{C}_B\urcorner$ or $\mathcal{M}_{(T^+,T^-)} \models^+ B$. Since $\mathcal{M}_{(T^+,T^-)}$ is a fixed point, $\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_B$ implies that $\mathcal{M}_{(T^+,T^-)} \models^+ T\ulcorner\mathcal{C}_B\urcorner$. So the above equivalence reduces to: $\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_B$ iff $\mathcal{M}_{(T^+,T^-)} \models^+ B$ iff $\mathcal{M} \models B$.

Similarly, we have $\mathcal{M}_{(T^+,T^-)} \models^- \mathcal{C}_B$ iff $\mathcal{M}_{(T^+,T^-)} \models^+ T\ulcorner\mathcal{C}_B\urcorner$ and $\mathcal{M}_{(T^+,T^-)} \models^- B$. Again, since $\mathcal{M}_{(T^+,T^-)}$ is a fixed point, $\mathcal{M}_{(T^+,T^-)} \models^- \mathcal{C}_B$ implies that $\mathcal{M}_{(T^+,T^-)} \models^- T\ulcorner\mathcal{C}_B\urcorner$. This gives us that $\mathcal{M}_{(T^+,T^-)} \not\models^+ T\ulcorner\mathcal{C}_B\urcorner$, so it is in fact impossible that $\mathcal{M}_{(T^+,T^-)} \models^- \mathcal{C}_B$.

This tells us: if $\mathcal{M} \models B$, then $\mathcal{C}_B$ is true in every fixed point of $\mathcal{M}$. Hence if $\mathcal{M} \models B$, then $\mathcal{C}_B$ is grounded in $\mathcal{M}$. If $\mathcal{M} \not\models B$, however, $\mathcal{C}_B$ does not receive a truth value in any fixed point of $\mathcal{M}$, so $\mathcal{C}_B$ is paradoxical in $\mathcal{M}$.

Supervaluationism. As for strong Kleene, we show that if $\mathcal{C}_B$ is true in a fixed point of $\mathcal{M}$, then $\mathcal{M} \models B$; and it is impossible that $\mathcal{C}_B$ is false in a fixed point of $\mathcal{M}$. According to the supervaluationism conditions, we have: $\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_B$ iff for any precisification X of (T^+, T^-) we have $\mathcal{M}_X \models T\ulcorner\mathcal{C}_B\urcorner \to B$, ie. $\mathcal{M}_X \not\models T\ulcorner\mathcal{C}_B\urcorner$ or $\mathcal{M}_X \models B$. Since (T^+, T^-) is a fixed point, we have: $\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_B$ implies $\ulcorner\mathcal{C}_B\urcorner^{\mathcal{M}} \in T^+$. This in turn implies that $\ulcorner\mathcal{C}_B\urcorner^{\mathcal{M}} \in X$, ie. $\mathcal{M}_X \models T\ulcorner\mathcal{C}_B\urcorner$, for any precisification X of (T^+, T^-). So we obtain that for any fixed point (T^+, T^-): $\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_B$ iff $\mathcal{M} \models B$.

Similar reasoning gives us that $\mathcal{M}_{(T^+,T^-)} \models^- \mathcal{C}_B$ is impossible: we have $\mathcal{M}_{(T^+,T^-)} \models^- \mathcal{C}_B$ iff for any precisification X of (T^+, T^-) we have $\mathcal{M}_X \not\models T\ulcorner\mathcal{C}_B\urcorner \to B$, ie. $\mathcal{M}_X \models T\ulcorner\mathcal{C}_B\urcorner$ and $\mathcal{M}_X \not\models B$. But $\mathcal{M}_X \not\models T\ulcorner\mathcal{C}_B\urcorner$ is impossible, since (T^+, T^-) is a fixed point and $\mathcal{M}_{(T^+,T^-)} \models^- \mathcal{C}_B$.

Our claims now follow: if $\mathcal{M} \models B$, then $\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_B$ for any fixed point (T^+, T^-) of $\mathcal{M}$, so $\mathcal{C}_B$ is grounded in $\mathcal{M}$. If $\mathcal{M} \not\models B$, then $\mathcal{M}_{(T^+,T^-)} \not\models^+ \mathcal{C}_B$ and $\mathcal{M}_{(T^+,T^-)} \not\models^- \mathcal{C}_B$, so $\mathcal{C}_B$ receives no truth value in any fixed point $\mathcal{M}_{(T^+,T^-)}$; hence $\mathcal{C}_B$ is paradoxical in $\mathcal{M}$. □

Proposition 4.2.6 (Weak Kleene)**.** If $\mathcal{M} \models B$, then Curry's sentence $\mathcal{C}_B$ is ungrounded but unparadoxical in $\mathcal{M}$. If $\mathcal{M} \not\models B$, then $\mathcal{C}_B$ is paradoxical in $\mathcal{M}$.

Proof. We first spell out what it means for $\mathcal{C}_B$ to be true or false in a fixed point $\mathcal{M}_{(T^+,T^-)}$ of $\mathcal{M}$. For exactly the same reason as in the proof of the previous proposition, it is impossible that $\mathcal{M}_{(T^+,T^-)} \models^- \mathcal{C}_B$, since the falsity condition for $\mathcal{C}_B$ in a model is the same in strong and weak Kleene.

For $\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_B$, the weak Kleene truth conditions give us:

$$\begin{aligned} \mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_B \quad \iff \quad & \mathcal{M}_{(T^+,T^-)} \models^+ T\ulcorner\mathcal{C}_B\urcorner \text{ and } \mathcal{M}_{(T^+,T^-)} \models^+ B \\ & \text{or } \mathcal{M}_{(T^+,T^-)} \models^- T\ulcorner\mathcal{C}_B\urcorner \text{ and } \mathcal{M}_{(T^+,T^-)} \models^+ B \\ & \text{or } \mathcal{M}_{(T^+,T^-)} \models^- T\ulcorner\mathcal{C}_B\urcorner \text{ and } \mathcal{M}_{(T^+,T^-)} \models^- B. \end{aligned}$$

The last two options are impossible, since $\mathcal{M}_{(T^+,T^-)} \models^+ T\ulcorner\mathcal{C}_B\urcorner$ by the fact that (T^+, T^-) is a fixed point. We obtain:

$$\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_B \iff \mathcal{M}_{(T^+,T^-)} \models^+ T\ulcorner\mathcal{C}_B\urcorner \text{ and } \mathcal{M} \models B.$$

This is impossible if $\mathcal{M} \not\models B$; hence, if $\mathcal{M} \not\models B$, then $\mathcal{C}_B$ does not receive a truth value in any fixed point of $\mathcal{M}$, ie. $\mathcal{C}_B$ is paradoxical in $\mathcal{M}$.

Now suppose that $\mathcal{M} \models B$. Then we get truth-teller-like behaviour: for a fixed point (T^+, T^-), we have $\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_B$ iff $\mathcal{M}_{(T^+,T^-)} \models^+ T\ulcorner\mathcal{C}_B\urcorner$. On the other hand, as before, it is impossible that $\mathcal{M}_{(T^+,T^-)} \models^- \mathcal{C}_B$.

A straightforward induction shows that $\mathcal{C}_B$ does not have a truth value in the least fixed point of $\mathcal{M}$. Thus $\mathcal{C}_B$ is ungrounded in $\mathcal{M}$. However, like the truth teller, it has a truth value in some fixed points of $\mathcal{M}$: the least fixed point construction starting from $(T_0^+, T_0^-) = (\{\ulcorner \mathcal{C}_B \urcorner^{\mathcal{M}}\}, \varnothing)$ instead of $(\varnothing, \varnothing)$ gives us a fixed point of $\mathcal{M}$ in which $\mathcal{C}_B$ is true. Thus, $\mathcal{C}_B$ is ungrounded but not paradoxical. □

The previous propositions give us the following characterisation of the paradoxicality of Curry in Kripke's fixed point theory, put in terms of the status of its consequent:

Proposition 4.2.7 (Strong Kleene, weak Kleene, supervaluationism)**.** If $B \not\equiv \top$ and $B \not\equiv \bot$, then Curry's sentence $\mathcal{C}_B$ is locally paradoxical but not globally paradoxical. If $B \equiv \top$, then $\mathcal{C}_B$ is globally unparadoxical. If $B \equiv \bot$, then $\mathcal{C}_B$ is globally paradoxical.

Proof. If $B \not\equiv \top$ and $B \not\equiv \bot$, then there exists some ground model $\mathcal{M}$ such that $\mathcal{M} \not\models B$ and some ground model $\mathcal{M}'$ such that $\mathcal{M}' \models B$. So by Propositions 4.2.5 and 4.2.6, we have that $\mathcal{C}_B$ is paradoxical in $\mathcal{M}$ but not paradoxical in $\mathcal{M}'$ in strong Kleene, weak Kleene, and supervaluationism. This means that $\mathcal{C}_B$ is locally paradoxical but not globally paradoxical.

On the other hand, if $B \equiv \top$, then $\mathcal{M} \models B$ for every ground model $\mathcal{M}$, so by Propositions 4.2.5 and 4.2.6, $\mathcal{C}_B$ is not paradoxical in any ground model $\mathcal{M}$. Hence $\mathcal{C}_B$ is globally unparadoxical. If $B \equiv \bot$, then $\mathcal{M} \not\models B$ for every ground model $\mathcal{M}$, so (by Propositions 4.2.5 and 4.2.6) $\mathcal{C}_B$ is paradoxical in every ground model $\mathcal{M}$ – hence $\mathcal{C}_B$ is globally paradoxical. □

Thus, our formalisations of Intuitions 5b and 5a seem to do what we want them to do: they capture some instances of Curry's paradox because they are paradoxical in some ground model (but not in others). This is a generalisation over Kripke's notion of paradoxicality, which considers only one ground model. However, we should put a technical sidenote: since we are only considering ground models for PA^+, our definition only makes a difference for instances of Curry with consequents that are *independent of* PA^+. We are thus working in quite a small margin. Still, we hope that our work demonstrates a certain *method* for characterising more of Curry's paradox as paradoxical, and a certain mechanism underlying Curry's paradox. The same method might be applied in an adapted context, in which its impact might be bigger – for instance, if we adopt a different (artificial) method of self-reference[2], or if we add a non-arithmetical part to the language.

4.2.2. Multi-sentence Curry

Having seen that Curry's sentence is locally (but not globally) paradoxical depending on the status of B, we examine what other sentences are locally (but not globally) paradoxical. We first turn to multi-sentence variants of Curry, which we would like to fall under our definition.

Recall the definitions of Card Curry and Yablurry:

Definition 4.2.8 (Card Curry)**.** *Card Curry* is the set of sentences $\{\mathcal{C}_{B1}, \mathcal{C}_{B2}\}$, defined as follows:

$(\mathcal{C}_{B1})$ $\quad T\ulcorner \mathcal{C}_{B2} \urcorner$

$(\mathcal{C}_{B2})$ $\quad T\ulcorner \mathcal{C}_{B1} \urcorner \to B$

[2] As in Cook, 2004.

Definition 4.2.9 (Yablurry (Cook, 2009, p. 617))**.** *Yablurry* is the set of sentences $\{S_n : n \in \omega\}$ defined as follows:

(S_0) $\quad (\forall m > 0)(T\ulcorner S_{\dot{m}} \urcorner \to B)$

(S_1) $\quad (\forall m > 1)(T\ulcorner S_{\dot{m}} \urcorner \to B)$

(S_2) $\quad (\forall m > 2)(T\ulcorner S_{\dot{m}} \urcorner \to B)$

$\vdots$

Explicitly, S_n is defined as

(S_n) $\quad (\forall m > n)(T\ulcorner S_{\dot{m}} \urcorner \to B),$

where $\dot{m}$ is a device that allows us to quantify over m inside the name of S_m, as in Section 3.2.

Both these paradoxes behave as we would like them to with respect to local(-but-not-global) paradoxicality; in all three logics, all sentences of Card Curry and Yablurry are locally (but not globally) paradoxical for an appropriate consequent B:

Proposition 4.2.10 (Strong Kleene, weak Kleene, supervaluationism)**.** Let B be a sentence in the ground language. If $B \not\equiv \top$, then the following sentences are locally paradoxical. If additionally $B \not\equiv \bot$, then they are locally but not globally paradoxical.

1. Both sentences $\mathcal{C}_{B1}$ and $\mathcal{C}_{B2}$ of Card Curry.
2. Every sentence S_n of Yablurry.

Proof. 1. Suppose that $B \not\equiv \top$ and let $\mathcal{M}$ be such $\mathcal{M} \not\models B$. Then we have that $\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_{B1}$ iff $\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_{B2}$, since (T^+, T^-) is a fixed point. And, in all three logics, $\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_{B2}$ iff $\mathcal{M}_{(T^+,T^-)} \models^- T\ulcorner \mathcal{C}_{B1}$ iff $\mathcal{M}_{(T^+,T^-)} \models^- \mathcal{C}_{B1}$ since $\mathcal{M} \not\models B$. So $\mathcal{C}_{B1}$ is true in $\mathcal{M}_{(T^+,T^-)}$ iff $\mathcal{C}_{B1}$ is false in $\mathcal{M}_{(T^+,T^-)}$; we conclude that $\mathcal{C}_{B1}$ has no truth value in $\mathcal{M}_{(T^+,T^-)}$. It follows that $\mathcal{C}_{B2}$ has no truth value in $\mathcal{M}_{(T^+,T^-)}$ either. So both $\mathcal{C}_{B1}$ and $\mathcal{C}_{B2}$ are paradoxical in $\mathcal{M}$, ie. they are locally paradoxical.

   Now suppose that $B \not\equiv \bot$ and let $\mathcal{M}'$ be such that $\mathcal{M}' \models B$. One can then check that $\mathcal{C}_{B2}$ is unparadoxical in $\mathcal{M}'$; the argument is the same as in Propositions 4.2.5 and 4.2.6. So there is some fixed point (T^+, T^-) in which $\mathcal{C}_{B2}$ has a truth value. It then follows that $\mathcal{C}_{B1}$ is unparadoxical as well: we have $\mathcal{M}'_{(T^+,T^-)} \models^+ T\ulcorner \mathcal{C}_{B2} \urcorner$ since (T^+, T^-) is a fixed point. This shows that both $\mathcal{C}_{B1}$ and $\mathcal{C}_{B2}$ are not globally paradoxical.

2. Let $\mathcal{M}$ be such that $\mathcal{M} \not\models B$. Let (T^+, T^-) be any fixed point of $\mathcal{M}$. One can then check that $\mathcal{M}_{(T^+,T^-)} \models^+ T\ulcorner S_m \urcorner \to B$ iff $\mathcal{M}_{(T^+,T^-)} \models^- T\ulcorner S_m \urcorner$, for any $m \in \omega$. The argument then proceeds as in the standard Yablo's paradox.

   If $B \not\equiv \bot$, let $\mathcal{M}'$ be such that $\mathcal{M}' \models B$. Then it is again straightforward to verify that any sentence S_n in Yablurry is unparadoxical in $\mathcal{M}'$.

□

This is a positive result: It is one of the strengths of Kripke's definition of paradoxicality that it recognises multiple-sentence versions of the Liar, like the Card Liar and Yablo's paradox, as paradoxical. It might therefore be demanded of a definition of Curry-paradoxicality within this framework that it captures the Curried versions of the Card Liar and Yablo's paradox. What the previous proposition shows is that the notion of local-but-not-global paradoxicality meets this demand.

4.2.3. Relatives to Curry: strong Kleene and supervaluationism

What other sentences are locally but not globally paradoxical depends on the underlying logic. In Strong Kleene and supervaluationism, we can build locally but not globally paradoxical sentences from globally paradoxical sentences, such as the Liar:

Proposition 4.2.11 (Strong Kleene, supervaluationism)**.** Let S be any globally paradoxical sentence and let A and B be sentences in the ground language. If $A \not\equiv \bot$ and $B \not\equiv \top$, then the following sentences are locally paradoxical. If additionally $A \not\equiv \top$ and $B \not\equiv \bot$, then they are locally but not globally paradoxical.

1. $S \wedge A$,
2. $S \vee B$,
3. $S \to B$.

Proof. Let us consider the first example. In strong Kleene we have, for any fixed point $\mathcal{M}_{(T^+,T^-)}$: $\mathcal{M}_{(T^+,T^-)} \models^+ S \wedge A$ iff $\mathcal{M}_{(T^+,T^-)} \models^+ S$ and $\mathcal{M}_{(T^+,T^-)} \models^+ A$. Since S is globally paradoxical, S has no truth value in $\mathcal{M}_{(T^+,T^-)}$, so this is impossible. On the other hand, $S \wedge A$ is false in $\mathcal{M}_{(T^+,T^-)}$ iff $\mathcal{M}_{(T^+,T^-)} \models^- S$ or $\mathcal{M}_{(T^+,T^-)} \models^- A$. Again, it is impossible that S is false in $\mathcal{M}_{(T^+,T^-)}$, so it must be that $\mathcal{M}_{(T^+,T^-)} \models^- A$, ie. $\mathcal{M} \not\models B$. We see that $S \wedge B$ has a truth value in a fixed point of $\mathcal{M}$ iff $\mathcal{M} \not\models B$; so $S \wedge A$ is paradoxical in $\mathcal{M}$ iff $\mathcal{M} \models A$. Since $A \not\equiv \bot$, there exists some ground model $\mathcal{M}$ such that $\mathcal{M} \models A$. In this model, $S \wedge A$ is paradoxical. This means that $S \wedge B$ is locally paradoxical. If moreover $A \not\equiv \top$, then there are other models in which $S \wedge A$ is paradoxical: for then there exists some ground model $\mathcal{M}'$ such that $\mathcal{M}' \not\models A$. Hence $S \wedge A$ is locally but not globally paradoxical.

In supervaluationism, we have that $\mathcal{M}_{(T^+,T^-)} \models^+ S \wedge A$ iff, for every precisification X of (T^+, T^-): $\mathcal{M}_X \models S$ and $\mathcal{M}_X \models A$. Since S is globally paradoxical, it is impossible that $\mathcal{M}_X \models S$ for every precisification X of (T^+, T^-), since this would imply that $\mathcal{M}_{(T^+,T^-)} \models^+ S$. On the other hand, have $\mathcal{M}_{(T^+,T^-)} \models^- S \wedge A$ iff, for every precisification X of (T^+, T^-): $\mathcal{M}_X \models S$ or $\mathcal{M}_X \models A$. Now if $\mathcal{M} \models B$, this reduces to: $\mathcal{M}_X \models S$ for every precisification X of (T^+, T^-), which is again impossible since S is globally paradoxical. So if $S \wedge B$ has a truth value in $\mathcal{M}_{(T^+,T^-)}$, then it must be that $\mathcal{M} \not\models A$, in which case we get $\mathcal{M}_{(T^+,T^-)} \models^- S \wedge A$; as in strong Kleene, we have that $S \wedge A$ is paradoxical in $\mathcal{M}$ iff $\mathcal{M} \models A$. This means that there exist ground models in which $S \wedge A$ is paradoxical precisely when $A \not\equiv \bot$. If $A \not\equiv \top$, there also exist ground models in which $S \wedge A$ is unparadoxical – so $S \wedge A$ is locally but not globally paradoxical.

In a similar manner, one can check that examples 2 and 3 are paradoxical in a model $\mathcal{M}$ iff $\mathcal{M} \not\models B$. □

Time for a moment of reflection: is it desirable that the sentences above belong to the class of intuitively Curry-paradoxical sentences? This might not be immediately obvious: a sentence such as "The Liar is true or I am Santa Claus" ($\lambda \vee B$) certainly does not have the same ring as "If this sentence is true, then I am Santa Claus" ($\mathcal{C}_B$). What the above considerations show is simply is that these two sentences formally behave in a very similar way – at least in Kripke's setting and in the logics under consideration. Indeed, they both are paradoxical in some models, but not in others – and so they match Intuition 5a.

In particular, like Curry, all three sentences above force A or B to take on a certain truth value. We go through the example $\lambda \vee B$ informally. Either $\lambda \vee B$ is true or it is false. If it is true, then either λ or B is true. If λ is true, then λ is false by definition of λ; so, since we assumed that $\lambda \vee B$ is true, it must be that B is true. If $\lambda \vee B$ is false, then both λ and B are false. So λ is true, which means that $\lambda \vee B$ is true – and we just saw that this means that B is true. Either way, B is true. We have avoided reasoning by contradiction, in order to highlight the similarity between this argument and the standard argument of Curry's paradox (Chapter 2) as much as possible. And indeed, though the arguments are different, we hope that the similarities are clear as well. Either way, this illustrates that these sentences share a crucial power with Curry: to force another sentence (or its negation) to be true.

What about weak Kleene? The previous examples are all globally paradoxical in weak Kleene, simply because any sentence containing a globally paradoxical sentence is again globally paradoxical:

Proposition 4.2.12 (Weak Kleene)**.** If S is paradoxical in $\mathcal{M}$, then so is any sentence containing S as a subformula.

Proof. It can be shown by induction on A that, for any sentence A and any model $\mathcal{M}_{(T^+,T^-)}$, A has a truth value in $\mathcal{M}_{(T^+,T^-)}$ under the weak Kleene truth conditions iff every subformula of A has a truth value in $\mathcal{M}_{(T^+,T^-)}$ under the weak Kleene truth conditions.

Now let S be paradoxical in $\mathcal{M}$ and let A be any sentence containing S. Let (T^+, T^-) be any fixed point of $\mathcal{M}$. If A has a truth value in $\mathcal{M}_{(T^+,T^-)}$, then, by the above, S has a truth value in $\mathcal{M}_{(T^+,T^-)}$. This contradicts the assumption that S is paradoxical in $\mathcal{M}$. We conclude that A does not have a truth value in any fixed point of $\mathcal{M}$; so A is paradoxical in $\mathcal{M}$. □

Proposition 4.2.13 (Weak Kleene)**.** The sentences in Proposition 4.2.11 are all globally paradoxical in weak Kleene.

Proof. Follows from the previous proposition. □

In general, we can generate locally but not globally paradoxical sentences given a single globally paradoxical sentence by appending to it any amount of sentences in the ground language, so long as they meet a strong independence requirement.

Definition 4.2.14 (Joint independence)**.** A set of sentences $\{B_1, \ldots, B_n\}$ is *jointly independent* iff any combination of truth values of these sentences is witnessed by some model; that is: for every function $f : \{B_1, \ldots, B_n\} \to \{0, 1\}$ there exists a ground model $\mathcal{M}_f$ such that $\mathcal{M}_f \models B_i$ iff $f(B_i) = 1$.

Proposition 4.2.15 (Strong Kleene, supervaluationism)**.** Let S be any globally paradoxical sentence and let $\{B_1, \ldots, B_n\} \subseteq \text{Form}(\mathcal{L})$ be a jointly independent set of sentences in the ground language, for some $n \geq 1$. Furthermore assume that $B_i \neq \top$ and $B_i \neq \bot$ for every i. Then the following sentence is locally but not globally paradoxical, for any choice of connectives $\circ_i \in \{\wedge, \vee, \to\}$ and $\sim_i \in \{\neg, -\}$:

$$\sim_n (\cdots \sim_2 (\sim_1 (S \circ_1 B_1) \circ_2 B_2) \cdots \circ_n B_n).$$

Here "−" indicates the *absence* of a negation, ie. is to be ignored.

Proof. We prove the result by induction on n. For $n = 1$, Proposition 4.2.11 shows that $S \circ_1 B_1$ is locally but not globally paradoxical. It is moreover straightforward to verify the following: if A is locally but not globally paradoxical, then so is $\neg A$. This follows from the fact that $\mathcal{M}_{(T^+,T^-)} \models^+ \neg A$ iff $\mathcal{M}_{(T^+,T^-)} \models^- A$ and $\mathcal{M}_{(T^+,T^-)} \models^- \neg A$ iff $\mathcal{M}_{(T^+,T^-)} \models^+ A$ in both strong Kleene and supervaluationism. Thus, $\sim_1 (S \circ_1 B_1)$ is locally but not globally paradoxical for any choice of $\sim_1$ and $\circ_1$.

For the induction step, suppose that $\Phi_n = \sim_n (\cdots \sim_2 (\sim_1 (S \circ_1 B_1) \circ_2 B_2) \cdots \circ_n B_n)$ is locally but not globally paradoxical. Let B_{n+1} be such that $\{B_1, \ldots, B_{n+1}\}$ is jointly independent and let $\circ_i \in \{\wedge, \vee, \rightarrow\}$ and $\sim_i \in \{\neg, -\}$. We show that

$$(\Phi_{n+1}) \qquad \sim_{n+1} (\sim_n (\cdots \sim_2 (\sim_1 (S \circ_1 B_1) \circ_2 B_2) \cdots \circ_n B_n) \circ_{n+1} B_{n+1})$$

is locally but not globally paradoxical. As before, we may disregard $\sim_{n+1}$ since the negation of a locally but not globally paradoxical sentence is again locally but not globally paradoxical. We consider the case that $\circ_i = \wedge$; the other cases are analogous. Let $\mathcal{M}_1$ be such that Φ_n is paradoxical in $\mathcal{M}_1$. Since $\{B_1, \ldots, B_{n+1}\}$ is jointly independent, there exists a model $\mathcal{M}'_1$ such that

$$\mathcal{M}'_1 \models B_1 \text{ iff } \begin{cases} \mathcal{M}_1 \models B_i, & \text{if } i \leq n \\ \text{always}, & \text{if } i = n+1. \end{cases}$$

One can check by induction that then, Φ_n is still paradoxical in $\mathcal{M}'_1$ in both strong Kleene and supervaluationism. This fact will give us that Φ_{n+1} is paradoxical in $\mathcal{M}'_1$ as well. Let (T^+, T^-) be any fixed point of $\mathcal{M}'_1$. In strong Kleene, we have $\mathcal{M}'_{1(T^+,T^-)} \models^+ \Phi_{n+1}$ iff $\mathcal{M}'_{1(T^+,T^-)} \models^+ \Phi_n$ and $\mathcal{M}'_1 \models B_{n+1}$. Since Φ_n has no truth value in $\mathcal{M}'_{1(T^+,T^-)}$, we obtain that $\mathcal{M}'_{1(T^+,T^-)} \not\models^+ \Phi_{n+1}$. Similarly, $\mathcal{M}'_{1(T^+,T^-)} \models^- \Phi_{n+1}$ iff $\mathcal{M}'_{1(T^+,T^-)} \models^+ \Phi_n$ and $\mathcal{M}'_1 \models B_{n+1}$. Since $\mathcal{M}'_1 \not\models B_{n+1}$, we obtain that $\mathcal{M}'_{1(T^+,T^-)} \not\models^- \Phi_{n+1}$. So we conclude that Φ_{n+1} does not have a truth value in any fixed point of $\mathcal{M}'_1$.

In supervaluationism, we have $\mathcal{M}'_{1(T^+,T^-)} \models^+ \Phi_{n+1}$ iff $\mathcal{M}'_{1X} \models \Phi_n$ and $\mathcal{M}'_1 \models B_{n+1}$ for any precisification X of (T^+, T^-). It is straightforward to verify that $\mathcal{M}'_{1X} \models \Phi_n$ iff $\mathcal{M}_{1X} \models \Phi_n$. Since Φ_n does not have a truth value in $\mathcal{M}'_{1(T^+,T^-)}$, there exists some precisification Y of (T^+, T^-) such that $\mathcal{M}'_{1X} \not\models \Phi_n$. So $\mathcal{M}'_{1(T^+,T^-)} \not\models^+ \Phi_{n+1}$. Similarly, we have $\mathcal{M}'_{1(T^+,T^-)} \models^- \Phi_{n+1}$ iff $\mathcal{M}'_{1X} \not\models \Phi_n \wedge B_{n+1}$; but $\mathcal{M}'_1 \models B_{n+1}$, so this is never the case. We conclude that Φ_{n+1} does not have a truth value in any fixed point of $\mathcal{M}'_1$. This means that Φ_{n+1} is locally paradoxical.

Now let $\mathcal{M}_2$ be such that Φ_n is not paradoxical in $\mathcal{M}_2$. Then it is straightforward to verify that Φ_{n+1} has a truth value in every fixed point of $\mathcal{M}_2$ as well. So Φ_n is not globally paradoxical. □

Examples of the previous proposition are sentences such as $(S \vee A) \wedge B$ and $\neg(S \wedge A) \vee B$: these are locally but not globally paradoxical for globally paradoxical S, so long as A and B are independent and not equivalent to $\top$ or $\bot$.

To see why the independence requirement is needed, we give some examples of sentences that are *not* locally paradoxical:

Proposition 4.2.16 (Strong Kleene, weak Kleene, supervaluationism)**.** Let S be any sentence in $\mathcal{L}^+$ and let B and C be sentences in the ground language. The following sentences are not locally paradoxical:

1. $(S \wedge B) \wedge \neg B$
2. $(S \vee B) \vee \neg B$
3. $(S \wedge B) \vee B$
4. $(S \vee B) \wedge B$
5. $(S \wedge (B \wedge \neg B)) \vee C$
6. $(S \vee (B \vee \neg B)) \wedge C$

Proof. Straightforward to verify by unfolding the truth conditions of these sentences in the three logics. $\square$

4.2.4. Relatives to Curry: weak Kleene

Propositions 4.2.11 and 4.2.15 have provided us with examples of sentences that behave like Curry in strong Kleene and supervaluationism. However, these are all globally paradoxical in weak Kleene. We can still arrive at sentences that are locally but not globally paradoxical in all three logics – they are sentences that are closer to Curry's paradox than the examples we saw before. Like Curry's paradox, they assert their own falsity depending on the behaviour of some sentence in the ground language.

Proposition 4.2.17 (Strong Kleene, weak Kleene, supervaluationism)**.** Let A and B be sentences in the ground language $\mathcal{L}$. If $A \not\equiv \bot$ and $B, C \not\equiv \top$, then the following sentences are locally paradoxical. If furthermore $A \not\equiv \top$ and $B, C \not\equiv \bot$, then they are locally but not globally paradoxical.

1. The sentence

$$(\mathcal{D}) \qquad \neg T\ulcorner\mathcal{D}\urcorner \wedge A,$$

2. The sentence

$$(\mathcal{E}) \qquad (T\ulcorner\mathcal{E}\urcorner \vee B) \to C.$$

3. The sentence

$$(\mathcal{F}) \qquad (T\ulcorner\mathcal{F}\urcorner \wedge A) \to B.$$

Proof. The arguments are similar to those we saw for Curry's paradox in Propositions 4.2.5 and 4.2.6. For $\mathcal{D}$, a ground model $\mathcal{M}$ such that $\mathcal{M} \models A$ witnesses that $\mathcal{D}$ is not globally paradoxical, while a model $\mathcal{M}'$ such that $\mathcal{M} \not\models A$ witnesses that $\mathcal{D}$ is locally paradoxical. The sentence $\mathcal{E}$ is paradoxical in $\mathcal{M}$ if $\mathcal{M} \not\models B$ and $\mathcal{M} \not\models C$, while $\mathcal{E}$ is unparadoxical in $\mathcal{M}$ if either $\mathcal{M} \models B$ or $\mathcal{M} \models C$. For $\mathcal{F}$, we have that $\mathcal{F}$ is paradoxical in $\mathcal{M}$ if and only if $\mathcal{M} \models A$ and $\mathcal{M} \not\models B$.

It is worth noting that $\mathcal{F}$ is simply equivalent to an instance of Curry's sentence: we have $\mathcal{F} \equiv T\ulcorner\mathcal{F}\urcorner \to (\neg A \vee B)$. $\square$

Remark 4.2.18. In the above proposition, we cannot take a sentence of the form

$$(\mathcal{H}) \qquad \neg T\ulcorner\mathcal{H}\urcorner \to A,$$

since $\neg T\ulcorner\mathcal{H}\urcorner \to B \equiv T\ulcorner\mathcal{H}\urcorner \vee B$: the negation disappears and we get unparadoxical truth-teller like behaviour.

The above examples are again instances of a more general pattern, similar to our generalisation of the locally but not globally paradoxical sentences in strong Kleene and supervaluationism of Proposition 4.2.15. The difference is that the sentence must refer to itself rather than to an external paradoxical sentence: it must assert its own falsity modulo the truth or falsity of some formulas in the ground language. We must moreover be careful with negation and the implication connective (since it is defined in terms of negation, as we saw in Remark 4.2.18), since these may cancel the fact that the sentence expresses its own falsity.

Proposition 4.2.19 (Strong Kleene, weak Kleene, supervaluationism)**.** Let $\{B_1, \ldots, B_n\} \subseteq \mathrm{Form}(\mathcal{L})$ be a jointly independent set of sentences in the ground language, for some $n \geq 1$. Furthermore assume that $B_i \neq \top$ and $B_i \neq \bot$ for every i. Then the following sentence is locally but not globally paradoxical, for any choice of connectives $\circ_i \in \{\wedge, \vee\}$:

$$(S) \qquad (\ldots((\neg T\ulcorner S\urcorner \circ_1 B_1) \circ_2 B_2) \cdots \circ_n B_n).$$

Proof. The proof is similar to the proof of Proposition 4.2.15. The proof is by induction on n, and the base case that $n = 1$ is given by the local-but-not-global paradoxicality of Curry's sentence $\mathcal{C}_B$ and the sentence $\mathcal{D}$ of Proposition 4.2.17. □

Note that the sentences $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ of Proposition 4.2.17 are all equivalent to sentences of the form displayed in Proposition 4.2.19. The latter proposition thus really is a generalisation of the former.

We thus have two groups of sentences that are similar to Curry: those that are locally (but not globally) paradoxical in the logics strong Kleene and supervaluationism, and those that are locally (but not globally) paradoxical in all three logics, including weak Kleene. We did not see any examples of sentences that are locally (but not globally) paradoxical in weak Kleene but not in strong Kleene or supervaluationism. In other words: so far, it seems like the notion of weak Kleene local(-but-not-global) paradoxicality is contained in that for strong Kleene and supervaluationism. Proving such an inclusion, however, is far from straightforward, since the fixed points of the three logics are not necessarily comparable.[3]

4.2.5. Summary

We proposed the notion of local paradoxicality as a formalisation of Intuition 5a, and the notion of local-but-not-global paradoxicality as a formalisation of Intuition 5b. Both definitions are very literal translations of the corresponding informal characterisations, so in this respect

[3]What we would like is an implication of the following form: 'if S has no truth value in any fixed point in weak Kleene, then S has no truth value in any fixed point in strong Kleene or supervaluationism.' However, Kremer (2009) has shown that the relation between various fixed points of the three logics is rather complex. For instance, the greatest intrinsic fixed points of weak Kleene, strong Kleene, and supervaluationism are incomparable (Kremer, 2009, p. 372). This makes an easy proof of such an implication unlikely.

they are successful. The main question was: how do these definitions play out formally? We saw that Curry's paradox is captured by both definitions in the manner that we expected; we moreover saw that multi-sentence variants of Curry's sentence are captured as well. In the final subsection, we encountered other relatives to Curry's paradox that fit the definitions as well. Which sentences are captured depends on the background logic: as far as our examples are concerned, the weak Kleene logic results in a smaller class of locally (but not globally) paradoxical sentences than the other two logics.

We encountered two groups of sentences that are similar to Curry – in the sense that they are locally (but not globally) paradoxical as well. The first group we encountered consists of sentences of the form $S \circ B$, for some globally paradoxical sentence S (such as the Liar) and some connective $\circ$. These are locally (but not globally) paradoxical in strong Kleene and supervaluationism, but not in weak Kleene. The second group consists of sentences that are, like Curry, self-referential; they are sentences such as $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$ of Proposition 4.2.17. These are locally (but not globally) paradoxical in all three logics.

Our results may be summarised as in the table of Figure 4.1.

	$\mathcal{C}$	$\mathcal{D}$	$\mathcal{E}$	$\mathcal{F}$	$\lambda \circ B$	Card Curry	Yablurry
Locally, not globally paradoxical (WK)	$*$	$*$	$*$	$*$	$\times$	$*$	$*$
Locally, not globally paradoxical (SK, SV)	$*$	$*$	$*$	$*$	$*$	$*$	$*$

($\times$) not captured
($*$) captured, under the condition that $A \not\equiv \bot$, $B \not\equiv \top$, and $C \not\equiv \top$.

Figure 4.1: Overview of Curry-paradoxical sentences on the definition of local-but-not-global Kripke-paradoxicality.

4.3. Capturing all of Curry

In this section, we again take Kripke's definition as a starting point. This time the aim is to formalise Intuition 6: 'A Curry-paradoxical sentence is a sentence whose logical form is paradoxical.' The corresponding definition should capture tautological Curry, unlike the previous notion of local (but not global) paradoxicality.

4.3.1. Tautological Curry and logical form

We saw that the previous notion of local paradoxicality captures Curry's sentence just in case its consequent is not logically equivalent to $\top$ – in particular, it does not capture tautological Curry: the instance of Curry's sentence whose consequent is $\top$. This matches our informal understanding of Intuitions 5b and 5a; as we saw in Section 2.3, neither of these intuitions recognises tautological Curry as paradoxical. In the same section, we proposed the following intuition that does include tautological Curry:

Intuition 6. A Curry-paradoxical sentence is a sentence whose logical form is paradoxical.

This intuition does not translate as straightforwardly to the framework at hand as the previous intuitions Intuitions 5b and 5a. Still, we propose one possible formalisation in the current setting; it is a valuable exercise to attempt to find a form of Kripke-paradoxicality that includes tautological Curry as well. What is at play here is the fact that tautological Curry behaves like a tautology, yet it has the same logical form as other instances of Curry. We propose one way of cashing this out formally into a notion of *Curry-type* sentences, which includes tautological Curry.

Let us first be a bit more precise about the sense in which tautological Curry 'behaves like a tautology'. In strong Kleene and supervaluationism, tautological Curry is, literally, tautological in the sense of being true in every model:

Proposition 4.3.1 (Strong Kleene, supervaluationism)**.** Tautological Curry is tautological, ie. $\mathcal{C}_\top \equiv \top$.

Proof. Let $\mathcal{M}_{(T^+,T^-)}$ be a model for $\mathcal{L}^+$. In strong Kleene, $\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_\top$ iff $\mathcal{M}_{(T^+,T^-)} \models^- T\ulcorner\mathcal{C}_\top\urcorner$ or $\mathcal{M}_{(T^+,T^-)} \models^+ \top$. The latter is always the case, so $\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_\top$.

In supervaluationism, $\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_\top$ iff, for every precisification X of (T^+,T^-): $\mathcal{M}_X \not\models T\ulcorner\mathcal{C}_\top\urcorner$ or $\mathcal{M}_X \models \top$. Again, since the latter is always the case, we have $\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_\top$. □

This means that tautological Curry has a truth value in every fixed point of every ground model; for this reason, the local-but-not-global paradoxicality of the previous section cannot capture tautological Curry. In weak Kleene, however, the situation is different: tautological Curry is *not* tautological, because it receives no truth value at all in some models. Still, tautological Curry is never false:

Proposition 4.3.2 (Weak Kleene)**.** Let $\mathcal{M}_{(T^+,T^-)}$ be any model for $\mathcal{L}^+$. Then, in weak Kleene, either $\mathcal{C}_\top$ is true in $\mathcal{M}_{(T^+,T^-)}$, or $\mathcal{C}_\top$ has no truth value in $\mathcal{M}_{(T^+,T^-)}$.

Proof. Let $\mathcal{M}_{(T^+,T^-)}$ be a model for $\mathcal{L}^+$. In weak Kleene, $\mathcal{M}_{(T^+,T^-)} \models^- \mathcal{C}_\top$ iff $\mathcal{M}_{(T^+,T^-)} \models^+ T\ulcorner\mathcal{C}_\top\urcorner$ and $\mathcal{M}_{(T^+,T^-)} \models^- \top$. The latter is impossible, so $\mathcal{M}_{(T^+,T^-)} \not\models^- \mathcal{C}_\top$. We conclude that $\mathcal{C}_\top$ is either true in $\mathcal{M}_{(T^+,T^-)}$ or has no truth value in $\mathcal{M}_{(T^+,T^-)}$. □

Still, the notion of local-but-not-global-paradoxicality cannot capture tautological Curry in weak Kleene either, because any ground model $\mathcal{M}$ has a fixed point in which it is true; we can obtain one by repeating the least fixed point construction from $(T_0^+, T_0^-) = (\{\ulcorner\mathcal{C}_B\urcorner^{\mathcal{M}}\}, \varnothing)$ (as mentioned in Proposition 4.2.6).

To capture tautological Curry, we need to look beyond the behaviour of this particular instance, and consider its logical form instead. One way to do so is by allowing the *replacement* of one of its subsentences by another sentence (of the same kind); in this way, the logical form is not altered, but its behaviour in logical models may change. In particular, it may become (globally) paradoxical. Indeed, if we take tautological Curry $\mathcal{C}_\top$ and replace its consequent $\top$ by $\bot$, we obtain contradictory Curry – which is a globally paradoxical sentence. That is the intuition that the following definition tries to capture:

Definition 4.3.3. For any two sentences A and S any subformula E of A, the sentence $A[S/E]$ is the sentence obtained from A by replacing every occurrence of E by S.

Definition 4.3.4 (Basic Curry-type sentences, first attempt)**.** A sentence S of $\mathcal{L}^+$ is a *basic Curry-type* sentence if there exists some subformula $E \in \text{Form}(\mathcal{L})$ of S such that:

1. $S[\bot/E]$ is globally paradoxical,
2. $S[\top/E]$ is globally unparadoxical.

The second clause is added in order to capture only those sentences that behave very much like Curry: note that replacing the consequent of any instance of Curry by $\top$ gives us a globally unparadoxical sentence. The definition expresses that the paradoxicality of S depends on the truth value of the subformula E: if E is evaluated as false, the sentence is paradoxical, and if it is true, then the sentence is unparadoxical. It thus captures those sentences whose logical form has a strong resemblance to that of Curry – it is for this reason that we call them Curry-type sentences.

While the above definition captures the above description intuitively, it has a technical problem: it does not take into account the fact that the name of S changes when one or more of its subsentences is replaced by $\top$ or $\bot$. This means that if S was self-referential, its altered version $S[\Delta_1/B_1]\ldots[\Delta_n/B_n]$ may no longer be. Consider, for instance, tautological Curry $\mathcal{C}_\top \equiv T\ulcorner\mathcal{C}_\top\urcorner \to \top$. Replacing $\top$ by $\bot$ gives us

$$\mathcal{C}_\top[\bot/\top] \equiv T\ulcorner\mathcal{C}_\top\urcorner \to \bot,$$

while the desired sentence is of the form $\mathcal{C}_\top{}^* \equiv T\ulcorner\mathcal{C}_\top{}^*\urcorner \to \bot$.

Given this notation, we can introduce a more sophisticated notion of basic Curry-type sentences, which solves the problem we encountered:

> **Definition 4.3.5** (Basic Curry-type sentences)**.** Let the sentence S in $\mathcal{L}^+$ be diagonally defined by the formulas $A_1(y_1,\ldots,y_n),\ldots,A_n(y_1,\ldots,y_n)$. Then S is *basic Curry-type* if there exists a subformula $E \in \text{Form}(\mathcal{L})$ of S satisfying
>
> 1. $S^\bot_E$ is globally paradoxical,
> 2. $S^\top_E$ is globally unparadoxical.
>
> Where, for any formula D and any subformula $E \in \text{Form}(\mathcal{L})$ of S, we let S^D_E denote the sentence that is diagonally defined by the formulas
>
> $$A_1(y_1,\ldots,y_n)[D/E],\ldots,A_n(y_1,\ldots,y_n)[D/E].$$

Example 4.3.6. Consider Curry's sentence $\mathcal{C}_B$ that is diagonally defined by

$$A(x) = T(x) \to B.$$

Then $S^\top_B$ is diagonally defined by

$$A(x)[\top/B] = T(x) \to \top,$$

and $S_B^\bot$ is diagonally defined by

$$A(x)[\bot/B] = T(x) \to \bot.$$

Thus, $S_B^\top$ is the sentence which has the same truth value as $T(\ulcorner S_B^\top \urcorner) \to \top$ in ground model, and $S_B^\bot$ has the same truth value as $T(\ulcorner S_B^\bot \urcorner) \to \bot$ in every ground model.

Remark 4.3.7. In the above definition, we restrict our attention to subformulas in the ground language. This matches the fact that we are only considering instances of Curry that have their consequent in the ground language. It also aligns with our previous notion of local-but-not global paradoxicality, which is framed in terms of ground models. Moreover, it helps avoid problems with sentences such as the following:

$$\lambda \vee \neg\lambda.$$

This sentence should not be classified as Curry-type: its logical form is not paradoxical. However, it would be if we consider subsentences in the full language $\mathcal{L}^+$: We can take $\lambda \vee \neg\lambda$ to simply be (vacuously) diagonally defined by the formula

$$A(x) = (\lambda \vee \neg\lambda) \wedge x = x,$$

and consider its subformula $\neg\lambda$. Replacing this subformula with $\bot$ gives us a globally paradoxical sentence, and replacing it with $\top$ gives us a globally unparadoxical sentence. Hence the sentence $\lambda \vee \neg\lambda$ meets the definition of basic Curry-type sentences. Restricting our attention to subformulas in the ground language resolves this problem.

Remark 4.3.8. One might wonder whether it is necessary to include the second clause in the above definition; the answer is yes, if we want to capture precisely those sentences whose paradoxicality depends on the truth or falsity of one of its subformulas. The requirement is motivated by sentences such as the following:

$$S := (\lambda \wedge E) \vee (\lambda \wedge \neg E).$$

Note that both $S[\bot/E] \equiv \bot \vee \lambda \equiv \lambda$ and $S[\top/E] \equiv \lambda \vee \bot \equiv \lambda$ are globally paradoxical. Thus, while S satisfies the first clause, it does not satisfy the second; it is paradoxical regardless of the status of E. Including the second clause means that we rule out such sentences.

Remark 4.3.9. The above definition is rather syntactical at first sight – and it has to be, because tautological Curry cannot be distinguished from $\top$ semantically in strong Kleene and supervaluationism. But note that, in effect, replacing part of the formula in question with $\top$ or $\bot$ is no more than a tool to control the interaction between the models and the formula: to ensure that the subformula in question always receives value 1 or 0, respectively.

With the notion of basic Curry-type sentences, we have a definition intended to capture those sentences whose logical form behaves similarly to that of Curry. Let us verify that Curry's sentence is indeed basic Curry-type:

Proposition 4.3.10 (Strong Kleene, weak Kleene, supervaluationism)**.** Every instance of Curry's sentence is basic Curry-type, including tautological Curry; ie. $\mathcal{C}_B$ is basic Curry-type for every B.

Proof. We have seen in Proposition 4.2.7 that $\mathcal{C}_B$ is globally paradoxical when $B \equiv \bot$, while $\mathcal{C}_B$ is globally unparadoxical if $B \equiv \top$. It is straightforward to check that this means that $\mathcal{C}_B^{\bot}$ is globally paradoxical, while $\mathcal{C}_B^{\top}$ is globally unparadoxical. □

In what follows, we consider other sentences that fit the mould of basic Curry-type sentences, as well as a more general notion of Curry-type sentences.

4.3.2. Relatives to Curry

If basic Curry-type sentences are those that force one of their subformulas to be true, then we might be interested in a more general notion – sentences that force more than one of their subformulas either to be true or to be false. This is the notion of *Curry-type* sentences, in which we allow the replacement of more than one subformula, and by either $\top$ or $\bot$. To state it concisely, we first introduce some notation:

Definition 4.3.11. Let the sentence S in $\mathcal{L}^+$ be diagonally defined by the formulas $A_1(y_1, \ldots, y_n), \ldots, A_n(y_1, \ldots, y_n)$. Let $\mathcal{B} = \{B_1, \ldots, B_m\} \subseteq \text{Form}(\mathcal{L})$ be a set of subformulas of S in $\mathcal{L}$, and let $f : \{B_1, \ldots, B_n\} \to \{\top, \bot\}$ be a function. Then we let $S_{\mathcal{B}}^f$ denote the sentence that is obtained from S by replacing every occurrence of B_i by $f(B_i)$ in the defining formulas $A_1(y_1, \ldots, y_n), \ldots, A_n(y_1, \ldots, y_n)$. I.e., $S_{\mathcal{B}}^f$ is diagonally defined by the following formulas, for $i \leq n$:

$$A_i(y_1, \ldots, y_n)[f(B_1)/B_1] \ldots [f(B_n)/B_n].$$

Example 4.3.12. Consider the sentence $\mathcal{E} \equiv (T\ulcorner\mathcal{E}\urcorner \vee A) \to B$ of Proposition 4.2.6 that is diagonally defined by

$$D(x) = (T(x) \wedge A) \to B.$$

Let $f(A) = A'$ and $f(B) = B'$. Then $\mathcal{E}_{\{A,B\}}^f$ is diagonally defined by

$$((T(x) \wedge A) \to B)[A/A'][B/B'] = (T(x) \wedge A') \to B',$$

so that $\mathcal{E}_{\{A,B\}}^f \equiv T\ulcorner\mathcal{E}_{\{A,B\}}^f\urcorner$.

Given this notation, Curry-type sentences may be defined as follows:

Definition 4.3.13 (Curry-type sentences)**.** Let S in $\mathcal{L}^+$ be diagonally defined by formulas $A_1(y_1, \ldots, y_n), \ldots, A_n(y_1, \ldots, y_n)$. Then S is *Curry-type (for $\mathcal{B}$)* if there exists a set of subformulas $\mathcal{B} = \{B_1, \ldots, B_n\} \subseteq \text{Form}(\mathcal{L})$, for some $n \geq 1$, together with a function $f : \{B_1, \ldots, B_n\} \to \{\top, \bot\}$ such that

1. $S_{\mathcal{B}}^f$ is globally paradoxical,
2. $S_{\mathcal{B}}^{f^*}$ is globally unparadoxical,

where f^* is defined by: $f^*(B_i) = \top$ iff $f(B_i) = \bot$, and $f^* = \bot$ iff $f(B_i) = \top$.

The notion of Curry-type sentences is closely related to the previous notion of globally but not locally paradoxical sentences:

Proposition 4.3.14. If S is Curry-type for $\{B_1, \ldots, B_n\}$, and $B_i \not\equiv \bot$ and $B_i \not\equiv \bot$ for every i, then S is locally paradoxical but not globally paradoxical.

Proof. Let S be as described. Let $A_1(y_1, \ldots, y_n), \ldots, A_n(y_1, \ldots, y_n)$ be the formulas that diagonally define S and let $f : \{B_1, \ldots, B_n\} \to \{\top, \bot\}$ be as in Definition 4.3.13.

We use the following fact: suppose that $\mathcal{M} \models B_i$ iff $f(B_i) = \top$ and $\mathcal{M} \not\models B_i$ iff $f(B_i) = \bot$. Then $\mathcal{M}_{(T^+,T^-)} \models^+ S$ iff $\mathcal{M}_{(T^+,T^-)} \models^+ S^f_{\mathcal{B}}$.

With this in mind, define the ground model $\mathcal{M}_1$ such that $\mathcal{M}_1 \models B_i$ iff $f(B_i) = \top$ and $\mathcal{M}_1 \not\models B_i$ iff $f(B_i) = \bot$. Since each $B_i \not\equiv \bot$ and $B_i \not\equiv \top$, there exists such a ground model. Then, by the aforementioned fact, S has a truth value in a fixed point of $\mathcal{M}_1$ iff $S^f_{\mathcal{B}}$ does. By assumption and Definition 4.3.13, $S^f_{\mathcal{B}}$ is globally paradoxical, so in particular $S^f_{\mathcal{B}}$ has no truth value in any fixed point of $\mathcal{M}_1$. Hence, by the mentioned fact, S has no truth value in any fixed point of $\mathcal{M}_1$. This tells us that S is locally paradoxical.

Similarly, show that S is not globally paradoxical, we define $\mathcal{M}_2$ such that S has a truth value in a fixed point of $\mathcal{M}_2$ iff $S^{f^*}_{\mathcal{B}}$ does. To achieve this, let $\mathcal{M}_2$ be such that $\mathcal{M}_2 \models B_i$ iff $f(B_i) = \bot$ and $\mathcal{M}_2 \not\models B_i$ iff $f(B_i) = \top$. Since $S^{f^*}_{\mathcal{B}}$ is globally unparadoxical, it has a truth value in some fixed point of $\mathcal{M}_2$; so S is unparadoxical in $\mathcal{M}$ as well. This tells us that S is not globally paradoxical. □

This means that many Curry-type sentences are in fact locally but not globally paradoxical. But not all are; crucially, tautological Curry – which is a globally unparadoxical sentence – is (basic) Curry-type.

Some examples of locally but not globally paradoxical sentences for strong Kleene and supervaluationism that we saw in Proposition 4.2.11 are basic Curry-type:

Proposition 4.3.15 (Strong Kleene, supervaluationism)**.** Let S be any globally paradoxical sentence and let B be a sentence in $\mathcal{L}$. The following sentences are basic Curry-type sentences:

1. $S \vee B$,
2. $S \to B$.

Note that unlike in Proposition 4.2.11, B may be equivalent to $\top$ or $\bot$. The other example in Proposition 4.2.11 and all examples in Proposition 4.2.17 are not basic Curry-type because of their syntactic shape, but they are still closely related: they are equivalent to basic Curry-type sentences.

Proposition 4.3.16 (Strong Kleene, supervaluationism)**.** Let S be any globally paradoxical sentence and let B be any sentence in $\mathcal{L}^+$. The following sentence is not basic Curry-type, but is logically equivalent to a basic Curry-type sentence:

$$S \wedge B.$$

Proof. Note that $S \wedge B \equiv \neg(\neg S \vee \neg B)$; take $E = \neg B$. □

Proposition 4.3.17 (Strong Kleene, weak Kleene, supervaluationism)**.** The following sentences are not basic Curry-type, but are logically equivalent to basic Curry-type sentences.

1. The sentence

$$(\mathcal{D}) \qquad \neg T\ulcorner\mathcal{D}\urcorner \wedge B,$$

2. The sentence

$$(\mathcal{F}) \qquad (T\ulcorner\mathcal{F}\urcorner \wedge A) \to B.$$

Proof. 1. Note that $\mathcal{D} \equiv \neg(T\ulcorner\mathcal{D}\urcorner \vee \neg B)$; consider the subsentence $E = \neg B$. Then this latter sentence is basic Curry-type for E.

2. Note that $\mathcal{F}$ is simply equivalent to an instance of Curry's sentence: in all three logics, we have

$$\mathcal{F} \equiv \neg T\ulcorner\mathcal{F}\urcorner \vee (\neg A \vee B) \equiv T\ulcorner\mathcal{F}\urcorner \to (\neg A \vee B).$$

Thus $\mathcal{F}$ is equivalent to a basic Curry-type sentence by taking $E = (\neg A \vee B)$.

□

As is straightforward to check, the above sentences are all examples of Curry-type sentences; we conjecture[4] that it is the case in general that any sentence equivalent to a basic Curry-type sentence is Curry-type.

Conjecture 4.3.18. If a sentence A is equivalent to a basic Curry-type sentence, then A is Curry-type.

We conjecture that the converse is not true:

Conjecture 4.3.19. Not every Curry-type sentence is equivalent to a basic Curry-type sentence; the sentence

$$(\mathcal{E}) \qquad (T\ulcorner\mathcal{E}\urcorner \vee A) \to B.$$

is Curry-type but not basic Curry-type.

Explanation. This sentence is globally paradoxical if $B \equiv \bot$ and $A = \bot$, and globally unparadoxical if either $B = \top$ or $A = \top$. It can therefore be checked that it is Curry-type for $\{A, B\}$. To show that it is basic Curry-type, we would like to rewrite it to a sentence containing the subsentence $A \vee B$, and such that A and B do not occur 'by themselves' outside of this subsentence. This does not seem possible; note for instance that the following rewriting process is not helpful, since it still contains an isolated instance of B:

$$\begin{aligned}\mathcal{E} &\equiv \neg(T\ulcorner\mathcal{E}\urcorner \vee A) \vee B\\ &\equiv (\neg T\ulcorner\mathcal{E}\urcorner \wedge \neg A) \vee B\\ &\equiv (\neg T\ulcorner\mathcal{E}\urcorner \vee \neg B) \wedge (\neg A \vee \neg B).\end{aligned}$$

□

[4]This conjecture should really have been a proposition; it is only due to limitations of time that the proof was not worked out.

4.3.3. Multi-sentence Curry

We saw examples of locally but not globally paradoxical sentences that are Curry-type. But not all locally but not globally paradoxical sentences are Curry-type: in particular, the definition of Curry-type sentences does not carry over nicely to multi-sentence Curry paradoxes.

Proposition 4.3.20 (Strong Kleene, weak Kleene, supervaluationism)**.** Of the sentences $\mathcal{C}_{B1}$ and $\mathcal{C}_{B2}$ in Card Curry, only $\mathcal{C}_{B2}$ is Curry-type.

Proof. Recall that $\mathcal{C}_{B1} \equiv T\ulcorner\mathcal{C}_{B2}\urcorner$ and $\mathcal{C}_{B2} \equiv T\ulcorner\mathcal{C}_{B1}\urcorner \to B$. It is immediate that $\mathcal{C}_{B1}$ is not Curry-type: its only subformula is itself. The sentence $\mathcal{C}_{B2}$ is Curry-type, as one can check by replacing B by $\bot$ or $\top$. □

Corollary 4.3.21. There exist locally but not globally paradoxical sentences that are not Curry-type.

So, though many locally but not globally paradoxical sentences are Curry-type, not all of them are. That is, the set of Curry-type sentences does not contain the set of locally but not globally paradoxical sentences. The converse does not hold either: we saw that tautological Curry is Curry-type but not locally paradoxical. In this sense, the two notions are incomparable.

It can be seen as a weakness that the notion of Curry-type sentences does not include multi-sentence versions of Curry; as we argued in the previous section, one of the strengths of Kripke's definition of paradoxicality is its capacity to capture multi-sentence paradoxes, so it is unfortunate that this does not carry over to the definition of Curry-type sentences. It is, however, not entirely surprising: the definition of Curry-type sentences only considers subformulas of the sentence in question, which causes it to capture only single-sentence paradoxes. A more general definition, which allows us to replace formulas across the entire language with $\top$ or $\bot$, might be fuel for further research.

4.3.4. Summary

Our aim was to find a definition of Curry-paradoxicality based on Kripke's notion of paradoxicality that captures *all* instances of Curry, not just those with a non-tautological consequent. We took Intuition 6, according to which the paradoxicality of a sentence depends on its logical form, as a starting point. The result is the notion of *Curry-type* sentences, which indeed succeeds in capturing tautological Curry.

Curry-type sentences come in two flavours: basic Curry-type and Curry-type. The notion of basic Curry-type sentences is quite restrictive, and captures only sentences that are very similar to Curry's paradox in that they force one certain subsentence to be true. The general Curry-type sentences, on the other hand, may also force a set of numerous subsentences to take on a certain combination of truth values. We conjectured that the set of Curry-type sentences is not simply the set of basic Curry-type sentences closed under logical equivalence; this was witnessed by the sentence $\mathcal{E}$.

A major weakness of our definition of Curry-type is that multi-sentence Curry paradoxes are not included. Since multi-sentence Curry paradoxes seem to have a paradoxical logical form just as much as Curry's paradox itself, this points to a defect of our definition of Curry-type as a formalisation of Intuition 6.

An overview of the behaviour of different sentences on the definitions we saw in this section can be found in Figure 4.2 in the next section.

4.4. Summary and evaluation

In this chapter, we have studied the interaction between Kripke's popular definition of paradoxicality and Curry's paradox, and have aimed to adapt the definition to make it more Curry-friendly. We introduced two candidate notions of 'Curry-paradoxicality', based on two intuitions that we found in Chapter 2. The notion of local Kripke-paradoxicality arises naturally from Kripke's definition of paradoxicality, by the simple observation that Curry's sentence is paradoxical in some ground models, but not in others. This allowed us to distinguish Curry's sentence from the Liar, which is paradoxical in all ground models. By considering locally-but-not-globally paradoxical sentences, we obtained a set of sentences which includes some instances of Curry but not the Liar. These definitions can be seen as formalisations of Intuitions 5b and 5a: A Curry-paradoxical sentence is a sentence that behaves paradoxically in some models (but not in others).

The strength of these definitions is that they keep Kripke's framework intact: it is a natural tweak of the original definition that keeps all the benefits of the original definition. This manifests as the fact that multi-sentence Curry paradoxes are captured by the notion of local(-but-not-global) paradoxicality, just like Kripke's original definition captures multi-sentence Liar paradoxes. The weakness of the definition is that it is somewhat limited in the range of Curry sentences that it captures: it captures only those instances of Curry whose consequent is not equivalent to either $\top$ or $\bot$. Since we are working exclusively with models of the theory PA^+, this means that it in fact only captures those instances whose consequent is independent of PA^+.

In order to address the issue of capturing more instances of Curry, we introduced the notion of *Curry-type* sentences, which is based on Intuition 6: A Curry-paradoxical sentence is a sentence whose logical form is paradoxical. This definition includes all instances of Curry, including tautological Curry. Its weakness, however, lies in the fact that it disturbs Kripke's construction – to determine whether a given sentence is Curry-type, we replace its subsentences by $\top$ or $\bot$, but we do not do the same in other sentences. This means that multi-sentence paradoxes are no longer properly captured. Since multi-sentence Curry paradoxes seem to have a paradoxical logical form as well, this is undesirable. A remedy may be found in allowing the replacement of certain sentences by $\top$ or $\bot$ across the entire language – or framing the notion of Curry-type sentences in terms of *sets* of sentences, as we will see in the definition of paradoxicality that we will see in the next chapter. This remains food for further thought.

Across all definitions, our main find was the existence of two groups of 'Curry-like' sentences: sentences such as $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$, and sentences such as $\lambda \circ B$, for some connective $\circ$. The former are closer to Curry's paradox in the sense that they, like Curry, are self-referential. The latter form a novel class of sentences that are recognised as Curry-paradoxical (on all our formal definitions) in strong Kleene and supervaluationism. In weak Kleene, the situation is more complex: $\lambda \circ B$ is globally paradoxical in weak Kleene for *any* B – including $B = \top$

(see Proposition 4.2.13). This means that the sentence comes out as locally paradoxical, but not as basic Curry-type nor Curry-type.

The table of Figure 4.2 gives an overview of of our main results in this chapter.

	$\mathcal{C}$	$\mathcal{D}$	$\mathcal{E}$	$\mathcal{F}$	$\lambda \circ B$	$\mathcal{C}_{B1}$	$\mathcal{C}_{B2}$
Locally Kripke-paradoxical (WK)	$*$	$*$	$*$	$*$	✓	$*$	$*$
Locally Kripke-paradoxical (SK, SV)	$*$	$*$	$*$	$*$	$*$	$*$	$*$
Equivalent to basic Curry-type (WK)	✓	✓	×	✓	×	×	✓
Equivalent to basic Curry-type (SK, SV)	✓	✓	×	✓	✓	×	✓
Curry-type (WK)	✓	✓	✓	✓	×	×	✓
Curry-type (SK, SV)	✓	✓	✓	✓	✓	×	✓

(✓) captured
(×) not captured
($*$) captured, under the condition that $A \not\equiv \bot$, $B \not\equiv \top$, and $C \not\equiv \top$.

Figure 4.2: Overview of Curry-paradoxical sentences on the different definitions of Curry-paradoxicality we saw in this chapter.

($\mathcal{C}_B$)	$T\ulcorner\mathcal{C}\urcorner \to B$
($\mathcal{D}$)	$\neg T\ulcorner\mathcal{D}\urcorner \wedge A$
($\mathcal{E}$)	$(T\ulcorner\mathcal{E}\urcorner \vee B) \to C$
($\mathcal{F}$)	$(T\ulcorner\mathcal{F}\urcorner \wedge A) \to B$
($\mathcal{C}_{B1}$)	$T\ulcorner\mathcal{C}_{B2}\urcorner$
($\mathcal{C}_{B2}$)	$T\ulcorner\mathcal{C}_{B1}\urcorner \to B$

Figure 4.3: Definitions of the sentences in Figure 4.2.

5. The 'folk' notion of paradoxicality

In this chapter, we consider the a definition of paradoxicality introduced by Hsiung (2020). We show that the method of local paradoxicality that we used in the previous chapter on Kripke's notion applies to this definition as well; this shows that our strategy is more generally applicable than just to the case of Kripke's definition. In the first section, we introduce Hsiung's definition. In the second section, we adapt it to capture Curry's paradox.

5.1. Hsiung's definition

The notion of paradoxicality that we consider here was introduced in the present form by Hsiung (2020). Other definitions that are in essence the same can be found in numerous sources. Hsiung calls it the 'folk notion of paradoxicality', because it has been used so widely to (informally) verify whether a given sentence is paradoxical. The notion is different than Kripke's notion because it is framed in terms of *sets* of sentences. This means that it gives rise not only to a notion of paradoxicality, but also potentially to a notion of *paradox*: when does a set of sentences constitute a paradox? It is in this context that it was used by Hsiung (2024). We do not go into the question of definitions of paradox (as opposed to paradoxicality) here, but the work that we do in this chapter may lay the basis for a more Curry-friendly definition of paradox along the lines of Hsiung's work.

The background logic of this definition is classical first-order logic; unlike in the previous chapter, we will not be using non-classical interpretations of the truth predicate here. We have the choice of working in the language $\mathcal{L}_{\mathsf{PA}}$ of Peano Arithmetic or the *expanded* language of $\mathcal{L}_{\mathsf{PA}^+}$ – as we saw in Chapter 3, the former gives rise to the Diagonal Lemma, and the latter to the Strong Diagonal Lemma. In other chapters, we work in $\mathcal{L}_{\mathsf{PA}^+}$ because we employ non-classical logics (that require a stronger form of self-reference) and because we want to consider subsentences of the self-referential sentences (which is more natural on a stronger form of self-reference). Here, however, we work in classical logic and will not be considering subsentences – there is therefore no need to work in the expanded language. Hsiung moreover formulates his definition in the language $\mathcal{L}_{\mathsf{PA}}$, so by working in $\mathcal{L}_{\mathsf{PA}}$ we remain close to his approach. It should be noted, however, that the self-referential sentences we consider here are strictly speaking different from those that we consider in the other chapters: they are obtained from the Diagonal Lemma rather than the Strong Diagonal Lemma. This means that they are not *explicitly* self-referential; for an explanation of the difference, see Section 3.1.

So we take as our ground language the first-order language of Peano Arithmetic $\mathcal{L}_{\mathsf{PA}}$, denoted simply by $\mathcal{L}$ throughout the chapter. We extend this language with a truth predicate T to obtain the full language $\mathcal{L}^+$. A *ground model* is a model $\mathcal{M} = (D, I)$ for the ground language such that $\mathcal{M} \models \mathsf{PA}$, and a model $\mathcal{M}_X$ for $\mathcal{L}^+$ is given by a ground model $\mathcal{M}$ together with an interpretation X of the truth predicate.

Hsiung fixes a ground model $\mathcal{M}$ at the outset, and defines paradoxicality relative to this model. In light of our aim of capturing Curry's paradox, we give an adapted formulation, in which the dependence on the ground model is made explicit:

Definition 5.1.1 (Folk-paradoxicality)**.** (Adapted from (Hsiung, 2024, p. 2549)). A set of sentences Σ is *folk-paradoxical in a ground model* $\mathcal{M}$ if there exists no interpretation X of the truth predicate such that $\mathcal{M}_X \models A \leftrightarrow T\ulcorner A \urcorner$ for all $A \in \Sigma$.

If the singleton $\{A\}$ is folk-paradoxical, we say that A is folk-paradoxical.

On this definition, a sentence is paradoxical in a model precisely when it asserts its own falsity in that model. To put it crudely, the Liar is the only paradoxical sentence.

Proposition 5.1.2. (Adapted from (Hsiung, 2024, p. 2550)) A sentence A is folk-paradoxical in a ground model $\mathcal{M}$ if and only if $\mathcal{M}_X \models A \leftrightarrow \neg T\ulcorner A \urcorner$ for every X.

Proof. Suppose A is paradoxical; then $\mathcal{M}_X \not\models A \leftrightarrow T\ulcorner A \urcorner$ for every X. But (by definition of $\rightarrow$) this means that $\mathcal{M}_X \models A \leftrightarrow \neg T\ulcorner A \urcorner$. $\square$

For Curry's sentence $\mathcal{C}_B$, this means that $\mathcal{C}_B$ is paradoxical in some ground models, but not in others, depending on the behaviour of B in the ground model in question.

Proposition 5.1.3. Let B be any sentence in $\mathcal{L}^+$. Curry's sentence $\mathcal{C}_B$ is locally folk-paradoxical in a ground model $\mathcal{M}$ if and only if $\mathcal{M}_X \not\models B$ for every X. In particular, if B is a sentence in the ground language, then $\mathcal{C}_B$ is locally folk-paradoxical in $\mathcal{M}$ if and only if $\mathcal{M} \not\models B$.

Proof. By Proposition 5.1.2, $\mathcal{C}_B$ is folk-paradoxical in $\mathcal{M}$ if and only if $\mathcal{M}_X \models \mathcal{C}_B \leftrightarrow \neg T\ulcorner \mathcal{C}_B \urcorner$ for all X. We claim that $\mathcal{M}_X \models \mathcal{C}_B \leftrightarrow \neg T\ulcorner \mathcal{C}_B \urcorner$ iff $\mathcal{M}_X \not\models B$. To see why, note that

$$\begin{aligned}\mathcal{M}_X \models \mathcal{C}_B \leftrightarrow \neg T\ulcorner \mathcal{C}_B \urcorner &\iff \mathcal{M}_X \models (T\ulcorner \mathcal{C}_B \urcorner \rightarrow B) \leftrightarrow \neg T\ulcorner \mathcal{C}_B \urcorner \\ &\iff \mathcal{M}_X \not\models B.\end{aligned}$$

Hence $\mathcal{C}_B$ is locally folk-paradoxical in $\mathcal{M}$ if and only if $\mathcal{M}_X \not\models B$ for every X. If B is a sentence in the ground language, then $\mathcal{M}_X \not\models B$ for all X if and only if $\mathcal{M} \not\models B$. $\square$

Remark 5.1.4. Hsiung (2024, p. 2550) notes that Curry's sentence $\mathcal{C}_B$ is folk-paradoxical if and only if $B \equiv \bot$. This seems to be a different situation than the one described above. However, the two can be reconciled. The apparent difference is due to the fact that Hsiung fixes a ground model $\mathcal{M}$ at the start, and defines both logical equivalence and paradoxicality in terms of this particular model. Curry's sentence is then paradoxical if and only if $\mathcal{M}_X \not\models B$ for every X. At the same time, $B \equiv \bot$ precisely when $\mathcal{M}_X \models B \leftrightarrow \bot$ for every X. Thus, Curry's sentence is paradoxical (in $\mathcal{M}$) precisely when $B \equiv \bot$ (in $\mathcal{M}$) – despite appearances, this matches our account.

5.2. Capturing Curry: local folk-paradoxicality

We formalise Intuition 5a for folk-paradoxicality by introducing local folk-paradoxicality. Recall Intuition 5a:

Intuition 5a. A Curry-paradoxical sentence is a sentence that behaves paradoxically in some models.

In Chapter 4, we formalised both Intuition 5a and Intuition 5b; the latter was obtained by demanding that a sentence is locally, but not globally paradoxical. Here, we restrict our attention to Intuition 5a and local paradoxicality for simplicity; but everything that we do here can be generalised to local-but-not-global paradoxicality in the manner that one would expect.

5.2.1. Local folk-paradoxicality

The definition of local folk-paradoxicality is analogous to the definition of local Kripke-paradoxicality:

Definition 5.2.1. A set of sentences Σ is *locally folk-paradoxical* if there exists some ground model $\mathcal{M}$ such that Σ is folk-paradoxical in $\mathcal{M}$.

Since a sentence is locally paradoxical if it is paradoxical in some model, it prefers some ground models and rules out others. It therefore it forces some sentences to be true on pain of avoiding paradoxicality – namely, those sentences that are true in all ground models in which it is not paradoxical.

We have the following adaptation of the characterisation of folk-paradoxical sentences that we saw in Proposition 5.1.2:

Proposition 5.2.2. A sentence A is locally folk-paradoxical iff there exists some model $\mathcal{M}$ such that, for any interpretation X of the truth predicate, we have: $\mathcal{M}_X \models A \leftrightarrow \neg T\ulcorner A \urcorner$.

Proof. Immediate by Definition 5.2.1 and Proposition 5.1.2. □

Like on the notion of local Kripke-paradoxicality, the instances of Curry that are locally folk-paradoxical are precisely those whose consequent B is not tautological, so long as B is a sentence in the ground language. Unlike on local Kripke-paradoxicality, we can now also easily characterise the instances of Curry that are captured in the more general case that B contains the truth predicate:

Proposition 5.2.3. Let B be any sentence in $\mathcal{L}^+$. Curry's sentence $\mathcal{C}_B$ is locally folk-paradoxical if and only if there exists some ground model $\mathcal{M}$ such that $\mathcal{M}_X \not\models B$ for every X. In particular, if B is a sentence in the ground language, then $\mathcal{C}_B$ is locally folk-paradoxical if and only if $\mathsf{PA} \not\models B$.

Proof. By Proposition 5.2.2, $\mathcal{C}_B$ is locally folk-paradoxical if and only if there exists some ground model $\mathcal{M}$ such that $\mathcal{M}_X \models \mathcal{C}_B \leftrightarrow \neg T\ulcorner \mathcal{C}_B \urcorner$ for all X. We claim that $\mathcal{M}_X \not\models \mathcal{C}_B \leftrightarrow T\ulcorner \mathcal{C}_B \urcorner$ iff $\mathcal{M}_X \not\models B$. To see why, note that

$$\begin{aligned} \mathcal{M}_X \models \mathcal{C}_B \leftrightarrow \neg T\ulcorner \mathcal{C}_B \urcorner &\iff \mathcal{M}_X \models (T\ulcorner \mathcal{C}_B \urcorner \to B) \leftrightarrow \neg T\ulcorner \mathcal{C}_B \urcorner \\ &\iff \mathcal{M}_X \not\models B. \end{aligned}$$

It follows that $\mathcal{C}_B$ is locally folk-paradoxical iff there exists some ground model $\mathcal{M}$ such that $\mathcal{M}_X \not\models B$ for all X. If B is a sentence in the ground language, then $\mathcal{M}_X \not\models B$ for all X if and only if $\mathcal{M} \not\models B$. So then $\mathcal{C}_B$ is locally folk-paradoxical if and only if there exists some ground model $\mathcal{M}$ such that $\mathcal{M} \not\models B$. □

Example 5.2.4. For an example of a locally folk-paradoxical instance of Curry's sentence with a consequent containing the truth predicate, consider the sentence

$$(\mathcal{C}) \qquad T\ulcorner\mathcal{C}\urcorner \to (T(\bar{0}) \wedge \neg T(\bar{0})).$$

How does this notion compare to the notion of local Kripke-paradoxicality? Recall that different logics gave us different flavours of Kripke-paradoxicality; the set of local Kripke-paradoxical sentences includes more sentences in the logics strong Kleene and supervaluationism, while the weak Kleene local Kripke-paradoxical sentences constitute a smaller class. The sentences $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$ that we saw to be locally Kripke-paradoxical on all three logics (Proposition 4.2.17) are locally folk-paradoxical too:

Proposition 5.2.5. Let A, B and C be sentences in the ground language $\mathcal{L}$ such that $A \not\equiv \bot$, $B \not\equiv \top$ and $C \not\equiv \top$. Then the following sentences are locally folk-paradoxical:

1. The sentence
$$\mathcal{D} \equiv \neg T\ulcorner\mathcal{D}\urcorner \wedge A.$$
2. The sentence
$$\mathcal{E} \equiv (T\ulcorner\mathcal{E}\urcorner \vee B) \to C.$$
3. The sentence
$$\mathcal{F} \equiv (T\ulcorner\mathcal{F}\urcorner \wedge A) \to B.$$

Proof. For $\mathcal{D}$, let $\mathcal{M}$ be a ground model such that $\mathcal{M} \models A$. Let X be any interpretation of the truth predicate for $\mathcal{M}$. Then $\mathcal{M}_X \models \neg T\ulcorner\mathcal{D}\urcorner \wedge A$ if and only if $\mathcal{M}_X \models \neg T\ulcorner\mathcal{D}\urcorner$. Hence $\mathcal{M}_X \models \mathcal{D} \leftrightarrow \neg T\ulcorner\mathcal{D}\urcorner$. By Proposition 5.2.2, $\mathcal{D}$ is locally folk-paradoxical.

Similarly, for $\mathcal{E}$ let $\mathcal{M}'$ be a ground model such that $\mathcal{M}' \not\models B$ and $\mathcal{M}' \not\models C$. Then we have that $\mathcal{M}'_X \models (T\ulcorner\mathcal{E}\urcorner \vee B) \to C$ if and only if $\mathcal{M}'_X \models \neg T\ulcorner\mathcal{E}\urcorner$, for any X. It follows that $\mathcal{E}$ is locally folk-paradoxical.

Finally, $\mathcal{F}$ is simply equivalent to an instance of Curry's paradox: we have $\mathcal{F} \equiv \neg T\ulcorner\mathcal{F}\urcorner \vee (\neg A \vee B)$. This means that a ground model $\mathcal{M}''$ such that $\mathcal{M}'' \models A$ and $\mathcal{M}'' \not\models B$ will witness that $\mathcal{F}$ is locally folk-paradoxical. □

In Proposition 4.2.15, we saw that combining a globally Kripke-paradoxical sentence – such as the Liar – with any other sentence B gives us a locally Kripke-paradoxical sentence in the logics strong Kleene and supervaluationism (so long as $B \not\equiv \top$). This same trick does not go through for local folk-paradoxicality:

Proposition 5.2.6. Let A be any sentence. Then the following sentences are *not* locally folk-paradoxical:

1. $\lambda \wedge A$

2. $\lambda \vee A$

3. $\lambda \to A$.

Proof. We consider only the first case; the other proofs are similar. Let $\mathcal{M}$ be any ground model; we find an interpretation X of the truth predicate for $\mathcal{M}$ such that $\mathcal{M}_X \models (\lambda \wedge A) \leftrightarrow T\ulcorner\lambda \wedge A\urcorner$. Note that $\mathcal{M}_X \models \lambda \wedge A$ iff $\mathcal{M}_X \models \neg T\ulcorner\lambda\urcorner$ and $\mathcal{M} \models A$. On the other hand, $\mathcal{M}_X \models T\ulcorner\lambda \wedge A\urcorner$ iff $\ulcorner\lambda \wedge A\urcorner \in X$. Defining $X = \{\lambda\}$ gives us that $\mathcal{M}_X \models T\ulcorner\lambda\urcorner$ and thus $\mathcal{M}_X \not\models \lambda \wedge A$. At the same time, $\mathcal{M}_X \not\models T\ulcorner\lambda \wedge A\urcorner$. We obtain $\mathcal{M}_X \models (\lambda \wedge A) \leftrightarrow T\ulcorner\lambda \wedge A\urcorner$. □

The key to the previous proposition is the fact that folk-paradoxicality is framed in terms of sets of sentences, and does not take any sentences into account beyond those in the set in question. For this reason, the sentence $\lambda \wedge A$ does not come out as paradoxical: it is perfectly possible to interpret the truth predicate coherently for this sentence alone. What is not possible is to interpret the truth predicate coherently for this sentence *and* simultaneously for the sentence λ. That is, the singleton $\{\lambda \wedge A\}$ is not locally folk-paradoxical, but the set $\{\lambda \wedge A,\ \lambda\}$ *is* locally folk-paradoxical.

Proposition 5.2.7. Let A be any sentence. The following sets are locally folk-paradoxical:

1. $\{\lambda \wedge A, \lambda\}$
2. $\{\lambda \vee A, \lambda\}$
3. $\{\lambda \to A, \lambda\}$.

Proof. Immediate by the fact that λ is locally folk-paradoxical, and is a member of each set. □

This brings us to the matter of multi-sentence paradoxes.

5.2.2. Multi-sentence paradoxes

Multi-sentence paradoxes such as the Card Liar and Yablo's paradox are captured by the notion of folk-paradoxicality. However, one needs to be careful here: as Proposition 5.1.2 shows, the single sentences, viewed in isolation, are *not* paradoxical – they are not equivalent to their own negation. When viewed as sets, however, the paradoxes *are* paradoxical. But, as Hsiung points out, this is a notion of paradoxicality, not of paradox; a set of sentences containing *more* than just the paradox is still paradoxical.

Proposition 5.2.8. The sentences λ_1 and λ_2 in the Card Liar are not folk-paradoxical paradoxical in any ground model. But, if $\{\lambda_1, \lambda_2\} \subseteq \Sigma$, then Σ is paradoxical in every ground model.

As we saw in the previous chapter (e.g. Remark 4.1.18), it is a strength of Kripke's definition of paradoxicality that all other sentences are always – implicitly – taken into consideration when determining the paradoxicality of a single sentence. For this reason, there is no need to consider sets of sentences. This makes Kripke's notion more sophisticated than the notion considered here. However, there is something to be said in favour of a notion involving sets of sentences too: such a notion makes explicit how paradoxical sentences depend on one another. If a certain paradoxical set of sentences Σ ceases to be paradoxical when a certain sentence A is removed from it, this means that the sentence A is part of the paradox expressed by Σ. In other words, a set-based notion of paradoxicality, such as the one we consider here, can give rise to a notion of *paradox*: when does a certain set of sentences constitute a paradox? Hsiung (2024) uses the notion of folk-paradoxicality to develop a definition of 'paradox'.

Analogously, the adapted notion of local folk-paradoxicality could serve as a basis to develop a more Curry-friendly definition of 'paradox'. To start investigating whether this would be feasible, we consider the behaviour of multi-sentence Curry paradoxes in our adapted framework. Recall that Card Curry (Definition 4.2.8) and Yablurry (Definition 4.2.9) are Curried versions of the Card Liar and Yablo's paradox, respectively.

Proposition 5.2.9. Let B be a sentence in the ground language such that $B \not\equiv \top$. Any set containing both sentences $\mathcal{C}_{B1}$ and $\mathcal{C}_{B2}$ of the Card Liar is locally folk-paradoxical, i.e. if $\{\mathcal{C}_{B1}, \mathcal{C}_{B2}\} \subseteq \Sigma$, then Σ is locally-folk paradoxical.

Proof. Let $\mathcal{M}$ be a ground model such that $\mathcal{M} \not\models B$. Let Σ be such that $\{\mathcal{C}_{B1}, \mathcal{C}_{B2}\} \subseteq \Sigma$. Suppose for contradiction that there exists some interpretation X of the truth predicate such that $\mathcal{M}_X \models S \leftrightarrow T\ulcorner S \urcorner$ for all $S \in \Sigma$. So $\mathcal{M}_X \models \mathcal{C}_{B1} \leftrightarrow T\ulcorner \mathcal{C}_{B1} \urcorner$ and $\mathcal{M}_X \models \mathcal{C}_{B2} \leftrightarrow T\ulcorner \mathcal{C}_{B2} \urcorner$. □

Proposition 5.2.10. Let B be a sentence in the ground language such that $B \not\equiv \top$. Any set containing all sentences S_n of Yablurry is locally folk-paradoxical, i.e. if $\{S_n : n \in \omega\} \subseteq \Sigma$, then Σ is locally-folk paradoxical.

Proof. Let $\mathcal{M}$ be a ground model such that $\mathcal{M} \not\models B$, and let Σ be such that $\{S_n : n \in \omega\} \subseteq \Sigma$. If $\mathcal{M}_X \models S \leftrightarrow T\ulcorner S \urcorner$ for all $S \in \Sigma$, then $\mathcal{M}_X \models S_n \leftrightarrow T\ulcorner S_n \urcorner$ for every n. Let $m \in \omega$ be arbitrary. By definition of S_m and the Diagonal Lemma, we have $\mathcal{M}_X \models S_m$ if and only if $\mathcal{M}_X \models (\forall l > m)(T\ulcorner S_m \urcorner \to B)$. Since $\mathcal{M} \not\models B$, this reduces to: $\mathcal{M}_X \models S_m$ if and only if $\mathcal{M}_X \models (\forall l > m)(\neg T\ulcorner S_l \urcorner)$. Since $\mathcal{M}_X \models S_n \leftrightarrow T\ulcorner S_n \urcorner$ for all $n \in \omega$, we obtain: $\mathcal{M}_X \models S_m$ if and only if $\mathcal{M}_X \models (\forall l > m)\neg S_l$. The argument is finalised as in the standard Yablo's paradox. □

5.3. Summary and evaluation

In this short chapter, we showed that the method of 'localising' a model-dependent definition of paradoxicality can be generalised to Hsiung's 'folk' notion of paradoxicality. The result is a definition of paradoxicality that captures more instances of Curry's paradox. Hsiung's notion is framed in terms of *sets* of sentences, which means that multi-sentence paradoxes are only captured if all of the necessary sentences are included. This makes the notion apt for defining a notion of *paradox*, and our work might thus contribute to a more Curry-tailored notion of paradox.

Like in the chapter on Kripke's theory, our definition of *local folk-paradoxicality* formalises Intuition 5a: A Curry-paradoxical sentence is a sentence that behaves paradoxically in some models. The definition behaves as expected; like that of local Kripke-paradoxicality, but with the properties of folk-paradoxicality rather than Kripke-paradoxicality. In particular, multi-sentence Curry paradoxes are only captured if all of the relevant sentences are included. Interestingly, we found that sentences of the form $\lambda \circ B$ are, in a sense, multi-sentence paradoxes too: they are only captured when B is included explicitly – the singleton $\lambda \circ B$ is not paradoxical. The sentences $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$ that we saw are single-sentence paradoxes, so they are captured.

Like local Kripke-paradoxicality, local folk-paradoxicality captures only those instances of Curry's paradox with a non-tautological consequent. Further research might lie in developing a notion of Curry-type sentences (as defined in Section 4.3) for the current framework; this notion captures tautological Curry, but we saw that it struggles with multi-sentence paradoxes. That might be resolved in the setting of folk-paradoxicality, where all sentences of a multi-sentence paradox are always included explicitly – this makes it easier to perform substitutions across multi-sentence paradoxes.

	$\mathcal{C}$	$\mathcal{D}$	$\mathcal{E}$	$\mathcal{F}$	$\lambda \circ B$	$\{\lambda \circ B, \lambda\}$	$\{\mathcal{C}_{B1}, \mathcal{C}_{B2}\}$	$\{S_n : n \in \omega\}$
Locally folk-paradoxical	$*$	$*$	$*$	$*$	$\times$	$*$	$*$	$*$

($\times$) not captured
($*$) captured, under the condition that $A \not\equiv \bot$, $B \not\equiv \top$, and $C \not\equiv \top$.

Figure 5.1: Overview of Curry-paradoxical sentences on the definition of local folk-paradoxicality.

6. Paradoxicality in terms of entailment

In this chapter, we formalise the intuitions that were discussed in Section 2.2, according to which Curry-paradoxicality is framed in terms of entailment. The first section sets the stage by introducing a definition of paradoxicality that is framed in terms of entailment. In the subsequent sections, we adapt this definition to include Curry's paradox in line with different intuitive notions; below is an overview.

Section	Intuition
Section 6.2	Intuitions 0 and 1
Section 6.3	Intuition 2b
Section 6.4	Intuition 2a
Section 6.5	Intuition 3

6.1. Classical paradoxicality

In this section, we propose what we call *classical paradoxicality*; this is a notion of paradoxicality framed in terms of entailment that does not capture Curry's paradox, but sets the stage for coming notions that do. It is a formalisation of Intuition 0 that we saw in Chapter 2:

> **Intuition 0.** A paradoxical sentence is a sentence whose existence entails a contradiction.

6.1.1. Preliminaries

Like in Chapter 4, our ground language $\mathcal{L}$ is the expanded first-order language of arithmetic $\mathcal{L}_{\mathsf{PA}^+}$, as defined in Chapter 3. We work in this expanded language in order to have access to the Strong Diagonal Lemma, which will facilitate the straightforward expression of genuine self-referential sentences, even in weak non-classical logics. This is helpful when talking about their subsentences in Section 6.3, and when we will considering paraconsistent background logics in Section 6.4. The extended language $\mathcal{L}^+$ is obtained from $\mathcal{L}$ by adding a truth predicate T.

The set of terms of a language L is defined as usual based on the function symbols, constants and variables. The *closed terms* are those terms that do not contain any variables. The set of formulas of a language L is defined as usual from the terms and predicate symbols of the language, and denoted by $\text{Form}(L)$.

To formalise Intuition 0, we work in classical logic. We therefore consider classical models for the ground language $\mathcal{L}$ and the full language $\mathcal{L}^+$. As in the previous chapters, we restrict our attention to models of the ground language that make the theory PA^+ (as defined in Chapter 3) true – we need this in order to prove the Strong Diagonal Lemma.

Definition 6.1.1 (Classical model for $\mathcal{L}$ and $\mathcal{L}^+$)**.** A classical model for $\mathcal{L}$ is a pair $\mathcal{M} = (D, I)$ where M is the domain of $\mathcal{M}$ and I is an interpretation function such that:

1. For each function symbol f of arity n in L, $I(f)$ is a function on M of the same arity;
2. For each predicate symbol P of arity n in L, $I(P)$ is a relation on M of the same arity.

The interpretation function induces an interpretation of each closed term t, which we will denote by $t^{\mathcal{M}}$.

A classical model $\mathcal{M}_X$ for $\mathcal{L}^+$ consists of a classical model $\mathcal{M} = (D, I)$ for $\mathcal{L}$ together with an interpretation of the truth predicate $X \subseteq D$. We denote the induced interpretation of a closed term t in $\mathcal{M}_X$ by $t^{\mathcal{M}_X}$.

If $\mathcal{M} = (D, I)$ is a model for $\mathcal{L}$ and $X \subseteq D$, the model $\mathcal{M}_X$ is the model for $\mathcal{L}^+$ in which the interpretation of T is given by X.

Truth in a model for $\mathcal{L}$ is defined as usual in classical first order logic; truth in a model for $\mathcal{L}^+$ is defined the same, with no special restrictions on the interpretation of the truth predicate.

As in the previous chapters, we will be restricting our attention to *ground models* that make the theory PA^+ true:

Definition 6.1.2 (Ground model)**.** A *ground model* is a model $\mathcal{M}$ for $\mathcal{L}$ such that $\mathcal{M} \models \mathsf{PA}^+$.

And as in the previous chapters, logical equivalence of sentences is defined with respect to the class of models that make PA^+ true. That is, $A \equiv B$ if and only if, for every ground model $\mathcal{M}$ and every interpretation of the truth predicate X: $\mathcal{M}_X \models A \leftrightarrow B$.

6.1.2. Classical paradoxicality

The formalisation of Intuition 0 we propose is framed in terms of what we call T*-models* (for a sentence S): these are models that make the T-schema true for a particular sentence S.

Definition 6.1.3 (T-model for S)**.** Let S be a sentence in $\mathcal{L}^+$. A model $\mathcal{M}_X$ for $\mathcal{L}^+$ is a T*-model for* S if $\mathcal{M}$ is a ground model and X is such that (the name of) S satisfies the T-schema in $\mathcal{M}_X$, i.e.:

$$\mathcal{M}_X \models T\ulcorner S \urcorner \iff \mathcal{M}_X \models S.$$

This notion allows us to formalise what we mean by the *existence* of a sentence in Intuition 0. We do not translate this literally by adding new sentences to the language – it is easiest to keep the language fixed throughout, and assume that all sentences are already available. However, since paradoxical sentences always involve the truth predicate, we may frame the existence of such a sentence in terms of the truth predicate. We propose the following: we translate the 'existence' of a sentence by the demand that the T-schema holds for that sentence. Since the T-schema ensures that the truth predicate means what we expect it to mean, this is not unreasonable: demanding that the T-schema holds will ensure that the sentence expresses what we expect it to express – that is, the sentence exists *and has its intended meaning.*

Based on this idea, we may translate the fact that the existence of a sentence S entails a contradiction as follows: there exist no models in which S satisfies the T-schema, i.e. there are no T-models for S.

Definition 6.1.4 (Classical paradoxicality)**.** A sentence S in $\mathcal{L}^+$ is *classically paradoxical* if there exists no T-model for S.

Remark 6.1.5. This definition can be generalised to sets of sentences Σ, analogous to the folk-paradoxicality that we saw in the Definition 5.1.1. For simplicity, we only consider single sentences here; this means that we will not be able to capture multi-sentence paradoxes.

If we look at a paradoxical sentence like the Liar, it has no T-models:

Proposition 6.1.6. The Liar sentence λ is classically paradoxical.

Proof. Let $\mathcal{M}$ be a T-model for λ such that $\mathcal{M} \models \mathsf{PA}$. Then $\mathcal{M} \models \lambda$ iff $\mathcal{M} \models \neg T\ulcorner\lambda\urcorner$ iff $\mathcal{M} \not\models T\ulcorner\lambda\urcorner$ iff $\mathcal{M} \not\models \lambda$. This is impossible; so there is no T-model for λ. □

Thus, if we demand that the Liar falls under the T-schema, the existence of the Liar entails a contradiction – every model that makes the Liar true makes $\perp$ true, since there are no such models. This is how Definition 6.1.4 formalises Intuition 0.

It might not come as a surprise that this definition is closely related to our our previous notion of folk-paradoxicality (Definition 5.1.1), which was based on the T-schema in a classical setting as well. It turns out that it *coincides* with folk-paradoxicality, when we take folk-paradoxicality to range over every ground model. There just is one point where we need to be careful: the notion of folk-paradoxicality was defined using the language $\mathcal{L}_{\mathsf{PA}}$, while we are using the language $\mathcal{L}_{\mathsf{PA}^+}$ here. This means that the equivalence does not hold for the definition of classical paradoxicality as we defined it here, but rather for the corresponding definition framed in the language $\mathcal{L}_{\mathsf{PA}}$:[1]

Proposition 6.1.7. Let the definition of classical paradoxicality be defined for the language $\mathcal{L}_{\mathsf{PA}}$ instead of $\mathcal{L}_{\mathsf{PA}^+}$.[2] Then a sentence S is classically paradoxical (viewed in $\mathcal{L}_{\mathsf{PA}}$) if and only if it is folk-paradoxical in every ground model.

Proof. Note that S is classically paradoxical if and only if, for every ground model $\mathcal{M}$ and every X, we have: $\mathcal{M}_X \not\models T\ulcorner S\urcorner \leftrightarrow S$.

Recall that a sentence S is folk-paradoxical in a ground model $\mathcal{M}$ if and only if $\mathcal{M}_X \not\models T\ulcorner S\urcorner \leftrightarrow S$ for every X. This means that S is folk-paradoxical in *every* ground model if and only if $\mathcal{M}_X \not\models T\ulcorner S\urcorner \leftrightarrow S$ for every ground model $\mathcal{M}$ and every X. Thus, the two notions coincide. □

We have thus not introduced any novel characterisation of paradoxicality in this section; what we do have, however, is a definition that allows to be adapted to capture the other intuitions that are framed in terms of entailment.

[1] We *can* frame our definition in the language $\mathcal{L}_{\mathsf{PA}}$ as well; we are working in classical logic, so the (weak) Diagonal Lemma goes through. Our preference for framing it in the expanded language $\mathcal{L}_{\mathsf{PA}^+}$ (which gives us the Strong Diagonal Lemma) is only due to our consideration of subsentences of self-referential sentences in Section 6.3, and the general philosophical advantages of having truly self-referential sentences.

[2] That is: a T-model for S is a model $\mathcal{M}_X$ *for the language* $\mathcal{L}_{\mathsf{PA}}$ such that $\mathcal{M}_X$ makes the T-schema true for S, and S is classically paradoxical if there exists no T-model for S.

6.2. Capturing Curry

Building on the previous definition of classical paradoxicality, we introduce *contingent-paradoxicality*, which intends to capture Intuition 1:

> **Intuition 1.** A Curry-paradoxical sentence is a sentence whose existence entails the truth of a sentence that is not a tautology.

We compare our definition to the previous definition of local folk-paradoxicality and look at what relatives of Curry are captured.

6.2.1. Contingent-paradoxicality

Intuition 1 readily allows to be formalised in the current framework as follows:

> **Definition 6.2.1** (Classical contingent-paradoxicality)**.** A sentence S in $\mathcal{L}^+$ is *(classically) contingent-paradoxical (for A)* if there exists some sentence A in $\mathcal{L}$ such that $A \not\equiv \top$, but $\mathcal{M}_X \models A$ for every model $\mathcal{M}_X$ which is a T-model for S.

In other words, in every model (for PA^+) that makes the T-schema true for S, the sentence A is true – despite the fact that A is not a logical consequence of PA^+. This means that, if we demand that the T-schema holds for S, then the very presence of this sentence entails A. Since A is not a tautology, this is problematic. We call this notion 'contingent-paradoxical'.[3]

The restriction to sentences A in the ground language is necessary; if we drop this restriction, then *every* sentence S is contingent-paradoxical by simply taking $A = T\ulcorner S \urcorner \leftrightarrow S$. Clearly, $T\ulcorner S \urcorner \leftrightarrow S$ is true in every T-model for S, but it is not a consequence of PA^+. Restriction to sentences in the ground language assures us that we have no problems of this kind.[4]

Contingent-paradoxicality clearly subsumes classical paradoxicality: if there are no T-models for S at all, then the condition for contingent-paradoxicality trivially holds.

Proposition 6.2.2. If a sentence is classically paradoxical, then it is contingent-paradoxical.

As expected, Curry's sentence is contingent-paradoxical precisely when its consequent is not a tautology:

Proposition 6.2.3. Let B be any sentence in $\mathcal{L}$. Curry's sentence $\mathcal{C}_B$ is contingent-paradoxical for B if and only if $B \not\equiv \top$.

[3] The word 'contingent' is only to be interpreted loosely here; A may be a contradiction, which would not normally be seen as a contingent sentence. We still choose this terminology because distinguishing part of this definition is the case where A is not a contradiction: if A is a contradiction, then S is simply classically paradoxical.

[4] This measure might seem drastic, and one might wonder if it would suffice, for instance, to demand only that A does not contain $T\ulcorner S \urcorner$ as a subsentence. It does not: consider for instance $S = \top$. Then $\mathcal{M} \models T\ulcorner S \urcorner$ for every T-model $\mathcal{M}$ for S. So, for every T-model for S, we have $\mathcal{M} \models \exists x(T(x))$. Note that $\exists x(T(x))$ does not contain $T\ulcorner S \urcorner$. So, since $\mathsf{PA}^+ \not\models \exists x(T(x))$, this means that the sentence $\top$ would come out as contingent-paradoxical – which is not what we want.

Proof. First suppose that $\mathcal{C}_B$ is contingent-paradoxical for B. Then $B \not\equiv \top$ by definition of contingent-paradoxicality, so this direction is immediate.

Conversely, suppose that $B \not\equiv \top$. Let $\mathcal{M}$ be a T-model for $\mathcal{C}_B$ such that $\mathcal{M} \models \mathsf{PA}^+$. By definition of $\mathcal{C}_B$ and the fact that $\mathcal{M} \models \mathsf{PA}^+$, we have $\mathcal{M} \models \mathcal{C}_B$ iff $\mathcal{M} \models T\ulcorner\mathcal{C}_B\urcorner \to B$. Since $\mathcal{M}$ is a T-model, this reduces to: $\mathcal{M} \models \mathcal{C}_B$ iff $\mathcal{M} \models \mathcal{C}_B \to B$. Now if $\mathcal{M} \not\models B$, this in turn reduces to: $\mathcal{M} \models \mathcal{C}_B$ iff $\mathcal{M} \not\models \mathcal{C}_B$, which is impossible; so $\mathcal{M} \models B$. Hence B is true in every model for PA^+ that is a T-model for $\mathcal{C}_B$. Since $B \not\equiv \top$, this means that $\mathcal{C}_B$ is contingent-paradoxical for B. □

Classical paradoxicality coincides with folk-paradoxicality in every ground model; does contingent-paradoxicality then coincide with local folk-paradoxicality? The answer is no: the notion of contingent-paradoxicality is less general, because we restricted our attention to sentences A in the ground language. Contingent-paradoxicality does imply local folk-paradoxicality, so we have a strict inclusion of the set of contingent-paradoxical sentences in that of local folk-paradoxical sentences.

Like in Proposition 6.1.7, we need to be careful about the language we are working in: in the previous chapter, we worked in the language $\mathcal{L}_{\mathsf{PA}}$ rather than $\mathcal{L}_{\mathsf{PA}^+}$. The two notions can only properly be compared if we phrase them in the same language; so for the following proposition, we assume a definition of contingent-paradoxicality for $\mathcal{L}_{\mathsf{PA}}$ rather than $\mathcal{L}_{\mathsf{PA}^+}$.

Proposition 6.2.4. Let the definition of contingent-paradoxicality be defined for the language $\mathcal{L}_{\mathsf{PA}}$ instead of $\mathcal{L}_{\mathsf{PA}^+}$. If a sentence S is contingent-paradoxical (viewed in $\mathcal{L}_{\mathsf{PA}}$), then S is locally folk-paradoxical.

Proof. If S is contingent-paradoxical for the sentence A in $\mathcal{L}_{\mathsf{PA}}$, then there exists some ground model $\mathcal{M}_0$ for $\mathcal{L}_{\mathsf{PA}}$ such that $\mathcal{M}_0 \not\models A$. Moreover, $\mathcal{M}_X \models A$ for every model $\mathcal{M}_X$ for $\mathcal{L}^+_{\mathsf{PA}}$ such that $\mathcal{M}_X$ is a T-model for S. Then S is folk-paradoxical in $\mathcal{M}_0$: Suppose that there exists some extension $\mathcal{M}_{0_X}$ of $\mathcal{M}_0$ to a model for the language $\mathcal{L}^+$ such that $\mathcal{M}_{0_X} \models S \leftrightarrow T\ulcorner S\urcorner$. Then $\mathcal{M}_{0_X}$ is a T-model for S, and $\mathcal{M}_{0_X} \not\models A$. This contradicts the fact that A is true in every T-model for S. So S is folk-paradoxical in $\mathcal{M}_0$. □

Conjecture 6.2.5. Not every locally folk-paradoxical sentence is contingent-paradoxical: the sentence

$$(\mathcal{C}) \qquad T\ulcorner\mathcal{C}\urcorner \to (T(\bar{0}) \wedge \neg T(\bar{0}))$$

is locally folk-paradoxical (see Example 5.2.4) but we conjecture that it is not contingent-paradoxical, because the sentence $T(\bar{0}) \wedge \neg T(\bar{0})$ is not a sentence in the ground language.

6.2.2. Relatives to Curry

We saw that not every locally paradoxical sentence is contingent-paradoxical; the two notions do not coincide. However, with respect to the sentences that have been the recurring examples of Curry-paradoxical sentences throughout this thesis – the sentences $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ and those of the form $\lambda \circ B$ – contingent-paradoxicality behaves just like local folk-paradoxicality.

Proposition 6.2.6. Let A and B be sentences in the ground language $\mathcal{L}$. If $A \not\equiv \bot$ and $B, C \not\equiv \top$, then the following sentences are contingent-paradoxical.

1. The sentence

   (𝒟) $$\neg T\ulcorner\mathcal{D}\urcorner \wedge A$$

   is contingent-paradoxical for $\neg A$,

2. The sentence

   (ℰ) $$(T\ulcorner\mathcal{E}\urcorner \vee B) \to C$$

   is contingent-paradoxical for $B \vee C$,

3. The sentence

   (ℱ) $$(T\ulcorner\mathcal{F}\urcorner \wedge A) \to B$$

   is contingent-paradoxical for $\neg A \vee B$.

Proof. Similar to the proof of the contingent-paradoxicality of Curry (Proposition 6.2.3). □

The fact that sentences of the form $\lambda \circ B$ are not contingent-paradoxical simply follows from the fact that they are not locally folk-paradoxical:

Proposition 6.2.7. Let A be any sentence. Then the following sentences are *not* contingent-paradoxical:

1. $\lambda \wedge A$
2. $\lambda \vee A$
3. $\lambda \to A$.

Proof. By Proposition 5.2.7, we know that these sentences are not locally folk-paradoxical. Proposition 6.2.4 then tells us that they are not contingent-paradoxical either. □

6.2.3. Summary

The notion of contingent-paradoxicality formalises Intuition 1: A Curry-paradoxical sentence is a sentence whose existence entails the truth of a sentence that is not a tautology. We saw that this notion is closely related to local folk-paradoxicality – it seems that they coincide, spare exceptions involving the discrepancy between the ground language and the full language for the notion of contingent-paradoxicality. This tells us that Intuition 1 and Intuition 5a – according to which a Curry-paradoxical sentence is paradoxical in some models – can coincide on certain formalisations. This hinges, of course, on the definition of paradoxicality that is used as a basis in Intuition 5a. What we have shown here is that, if we use Hsiung's definition of paradoxicality as this basis, then Intuition 5a aligns naturally with Intuition 1 on the current formalisation.

The overview of contingent-paradoxical sentences below is identical to that of the locally folk-paradoxical sentences (Figure 5.1) – except that we did not consider *sets* of sentences. As we noted in Remark 6.1.5, the current framework can be extended straightforwardly to a set-based definition à la Hsiung: this might be of interest for further research.

	$\mathcal{C}$	$\mathcal{D}$	$\mathcal{E}$	$\mathcal{F}$	$\lambda \circ B$
Contingent-paradoxical	$*$	$*$	$*$	$*$	$\times$

($\times$) not captured
($*$) captured, under the condition that $A \not\equiv \bot$, $B \not\equiv \top$, and $C \not\equiv \top$.

Figure 6.1: Overview of Curry-paradoxical sentences on the definition of contingent-paradoxicality.

6.3. The subsentence restriction

In this section, we briefly consider a formalisation of Intuition 2b, which is straightforwardly implemented in the current framework. Intuition 2b was designed to capture a distinction between Curry's paradox and the Liar in terms of subsentences:

Intuition 2b. A Curry-paradoxical sentence is a sentence whose existence entails the truth of one of its *subformulas* B, where B is not a tautology.

We formalise it by restricting Definition 6.2.1 such that A has to be a subsentence of S:

Definition 6.3.1 (Sub-contingent-paradoxicality)**.** A sentence S in $\mathcal{L}^+$ is *sub-contingent-paradoxical (for A)* if there exists some subsentence $A \in \text{Form}(\mathcal{L})$ of S such that $A \not\equiv \top$, but $\mathcal{M}_X \models A$ for every model $\mathcal{M}_X$ which is a T-model for S.

It is immediate that Curry's sentence $\mathcal{C}_B$ is sub-contingent-paradoxical: it is contingent-paradoxical for its subsentence B. The other contingent-paradoxical sentences $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ are *not* sub-contingent-paradoxical:

Proposition 6.3.2. Let A and B be arbitrary. The sentences $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$ of Proposition 6.2.6 are not sub-contingent-paradoxical.

Proof. The proof proceeds by verifying that these sentences are not contingent-paradoxical for any of their subsentences. □

The sentences $\mathcal{D}$ and $\mathcal{F}$, however, are equivalent to sub-contingent-paradoxical sentences:

Proposition 6.3.3. Let A and B be sentences in the ground language such that $A \not\equiv \bot$ and $B \not\equiv \top$. Then the sentences $\mathcal{D}$ and $\mathcal{F}$ are equivalent to sub-contingent-paradoxical sentences.

Proof. Note that $\mathcal{D} \equiv \neg(T\ulcorner\mathcal{D}\urcorner \vee \neg A)$ and $\mathcal{F} \equiv \neg T\ulcorner\mathcal{F}\urcorner \vee (\neg A \vee B)$. We saw that $\mathcal{D}$ is contingent-paradoxical for $\neg A$, and that $\mathcal{F}$ is contingent-paradoxical for $\neg A \vee B$ (Proposition 6.2.6). The result follows. □

While we conjecture that $\mathcal{E}$ is *not* (just like we conjectured that $\mathcal{E}$ is not equivalent to a basic Curry-type sentence in Conjecture 4.3.19):

Conjecture 6.3.4. The sentence $\mathcal{E}$ is not equivalent to a sub-contingent-paradoxical sentence.

Explanation. Note that $\mathcal{E}$ is contingent-paradoxical for $B \vee C$, but we conjecture – like in Conjecture 4.3.19 – that it is not possible to rewrite $\mathcal{E}$ to a sentence containing $B \vee C$ as a subsentence, without any isolated occurrences of B or C. This would mean that $\mathcal{E}$ is not sub-contingent-paradoxical. □

Finally, the Liar is not sub-contingent-paradoxical, simply because the Liar sentence $\lambda = \neg T(l)$ has no subsentences in the ground language – and we demand that the subsentence A is in the ground language.

Thus, the notion of sub-contingent-paradoxicality gives us a formalisation of Intuition 2b that succeeds in capturing Curry but not the Liar. But, similar to the notion of basic Curry-type sentences (Definition 4.3.5), it is rather limited in what relatives to Curry's paradox it captures. This is, of course, not a bad thing *per se*; but to distinguish sentences like Curry's from sentences like $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ – and sentences like $\mathcal{E}$ from sentences like $\mathcal{D}$ and $\mathcal{F}$ – only because they do not contain the sentence they entail as an *exact* subsentence seems rather superficial. The important fact seems to be that each of the sentences $\mathcal{C}_B$, $\mathcal{D}$, $\mathcal{E}$, and $\mathcal{F}$ contains complete information about the sentence they entail – in the sense of containing all parts of the sentence as subsentences. The current definition might thus benefit for a generalisation to *multiple* subsentences as well as negated subsentences, just like we generalised basic Curry-type sentences to obtain Curry-type sentences.

	$\mathcal{C}$	$\mathcal{D}$	$\mathcal{E}$	$\mathcal{F}$	$\lambda \circ B$
Sub-contingent-paradoxical	$*$	$\equiv$	$\times$	$\equiv$	$\times$

($\times$) not captured
($*$) captured, under the condition that $B \not\equiv \top$.
($\equiv$) equivalent to a sentence that is captured, under the condition that $A \not\equiv \bot$ and $B \not\equiv \top$.

Figure 6.2: Overview of Curry-paradoxical sentences on the definition of sub-contingent-paradoxicality.

6.4. Curry in a paraconsistent logic

In this section, we study Curry's paradox in a paraconsistent context, formalising Intuition 2a:

Intuition 2a. A Curry-paradoxical sentence is a sentence whose existence entails the truth of a sentence that is not a tautology *by means of an argument that does not use explosion.*

Our aim in doing so is to distinguish Curry from the Liar (see Section 2.2). In this section, we will find out whether our formalisation succeeds. We will moreover see that our approach gives us a systematic way to compare the behaviour of Curry's paradox in a paraconsistent logic with its behaviour in classical logic.

We will first consider a paraconsistent variant on classical paradoxicality; like classical paradoxicality, this notion does not capture Curry. We then adapt our paraconsistent notion of paradoxicality to capture non-tautological instances of Curry's paradox.

6.4.1. Paraconsistent paradoxicality

We will be working in the paraconsistent logic $\mathsf{LP}^{\rightarrow}$, which is obtained from Priest's popular paraconsistent Logic of Paradox LP (Priest, 2006) by adding a simple conditional that obeys modus ponens (see e.g. Hazen and Pelletier (2019, Section 6.1)). This logic accommodates for the Liar without exploding, while the existence of Curry's sentence leads to triviality. This allows us to distinguish the two in this logic.

Definition 6.4.1 ($\mathsf{LP}^{\rightarrow}$-model for $\mathcal{L}^+$)**.** An $\mathsf{LP}^{\rightarrow}$-model $\mathcal{M}_{(T^+,T^-)}$ for $\mathcal{L}^+$ consists of a classical ground model $\mathcal{M} = (M, I)$ for $\mathcal{L}$ together with a *glutty interpretation* (T^+, T^-) of the truth predicate, consisting of an extension $T^+ \subseteq M$ and an anti-extension $T^- \subseteq M$ such that $T^+ \cup T^- = M$.

Definition 6.4.2 ($\mathsf{LP}^{\rightarrow}$ truth conditions)**.** For a model $\mathcal{M} = (M, I)$ and a glutty interpretation (T^+, T^-) of the truth predicate, we recursively define when a sentence is true ($\models^+$) or false ($\models^-$) in $\mathcal{M}_{(T^+,T^-)}$ according to the $\mathsf{LP}^{\rightarrow}$ conditions as follows:

$$
\begin{aligned}
\mathcal{M}_{(T^+,T^-)} \models^+ P(t_1,\dots,t_n) &\iff \mathcal{M} \models P(t_1,\dots,t_n),\ \text{if } P \neq T \\
\mathcal{M}_{(T^+,T^-)} \models^- P(t_1,\dots,t_n) &\iff \mathcal{M} \not\models P(t_1,\dots,t_n),\ \text{if } P \neq T \\
\mathcal{M}_{(T^+,T^-)} \models^+ t_1 = t_2 &\iff t_1^{\mathcal{M}} = t_2^{\mathcal{M}} \\
\mathcal{M}_{(T^+,T^-)} \models^- t_1 = t_2 &\iff t_1^{\mathcal{M}} \neq t_2^{\mathcal{M}} \\
\mathcal{M}_{(T^+,T^-)} \models^+ T(t) &\iff I(t) \in T^+ \\
\mathcal{M}_{(T^+,T^-)} \models^- T(t) &\iff I(t) \in T^- \\
\mathcal{M}_{(T^+,T^-)} \models^+ A \vee B &\iff \mathcal{M}_{(T^+,T^-)} \models^+ A \text{ or } \mathcal{M}_{(T^+,T^-)} \models^+ B \\
\mathcal{M}_{(T^+,T^-)} \models^- A \vee B &\iff \mathcal{M}_{(T^+,T^-)} \models^- A \text{ and } \mathcal{M}_{(T^+,T^-)} \models^- B \\
\mathcal{M}_{(T^+,T^-)} \models^+ \neg A &\iff \mathcal{M}_{(T^+,T^-)} \models^- A \\
\mathcal{M}_{(T^+,T^-)} \models^- \neg A &\iff \mathcal{M}_{(T^+,T^-)} \models^+ A \\
\mathcal{M}_{(T^+,T^-)} \models^+ A \rightarrow B &\iff \text{if } \mathcal{M}_{(T^+,T^-)} \models^+ A,\ \text{then } \mathcal{M}_{(T^+,T^-)} \models^+ B \\
\mathcal{M}_{(T^+,T^-)} \models^- A \rightarrow B &\iff \mathcal{M}_{(T^+,T^-)} \models^- A \text{ and } \mathcal{M}_{(T^+,T^-)} \models^+ B \\
\mathcal{M}_{(T^+,T^-)} \models^+ \exists x(A(x)) &\iff \text{there exists some } d \in M \text{ such that } \mathcal{M}_{(T^+,T^-)} \models^+ A(\mathbf{d}) \\
\mathcal{M}_{(T^+,T^-)} \models^- \exists x(A(x)) &\iff \text{for all } d \in M : \mathcal{M}_{(T^+,T^-)} \models^- A(\mathbf{d})
\end{aligned}
$$

As we did in Chapter 4, we adopt a notion of logical equivalence framed in terms of sameness of truth values. We say that sentences A and B are *equivalent*, written $A \equiv B$, if they have the same truth value in every $\mathsf{LP}^{\rightarrow}$-model:

$$\mathcal{M}_{(T^+,T^-)} \models^+ A \iff \mathcal{M}_{(T^+,T^-)} \models^+ B$$
$$\mathcal{M}_{(T^+,T^-)} \models^- A \iff \mathcal{M}_{(T^+,T^-)} \models^- B.$$

Remark 6.4.3. Unlike in the case of the gappy logics of Kripke's construction (weak Kleene, strong Kleene, and supervaluationism), we can express this equivalence in the logic as well: A and B have the same truth value in a model $\mathcal{M}_{(T^+,T^-)}$ precisely when $\mathcal{M}_{(T^+,T^-)} \models (A \leftrightarrow B) \wedge (\neg A \leftrightarrow \neg B)$. Thus, $A \equiv B$ if and only if $\models_{\mathsf{LP}^{\rightarrow}} (A \leftrightarrow B) \wedge (\neg A \leftrightarrow \neg B)$.

As we saw in Section 3.1, the Strong Diagonal Lemma extends to logics that are weaker than classical, so long as classical logic is used for the arithmetical part of the language. This gives us access to the Strong Diagonal Lemma in the current setting.

The stage is now set to generalise our definition of classical paradoxicality (Definition 6.1.4) to the paraconsistent setting of $\mathsf{LP}^{\rightarrow}$. We first adapt our definition of a T-model (Definition 6.1.3) to glutty models:

Definition 6.4.4 (T-model for S in an $\mathsf{LP}^{\rightarrow}$-model)**.** Let S be a sentence in $\mathcal{L}^+$. An $\mathsf{LP}^{\rightarrow}$-model $\mathcal{M}_{(T^+,T^-)}$ for $\mathcal{L}^+$ is *a T-model for S* if the following holds:

$$\mathcal{M}_{(T^+,T^-)} \models^+ T\ulcorner S \urcorner \iff \mathcal{M}_{(T^+,T^-)} \models^+ S.$$

This yields the following paraconsistent variant on classical paradoxicality:

Definition 6.4.5 (Paradoxicality in $\mathsf{LP}^{\rightarrow}$)**.** A sentence S in $\mathcal{L}^+$ is *paradoxical in* $\mathsf{LP}^{\rightarrow}$ if there exists no $\mathsf{LP}^{\rightarrow}$-model for $\mathcal{L}^+$ which is a T-model for S.

The Liar sentence is not paradoxical in this sense, because paraconsistent logics can deal with the Liar:

Proposition 6.4.6. The Liar sentence λ is not paradoxical in $\mathsf{LP}^{\rightarrow}$.

Proof. Let $\mathbb{N}$ be the intended model of PA^+. Now consider the $\mathsf{LP}^{\rightarrow}$-model $\mathbb{N}_{(T^+,T^-)}$ given by $T^+ = T^- = \{\#\lambda\}$. Then $\mathbb{N}_{(T^+,T^-)} \models^+ T\ulcorner\lambda\urcorner$, since $\ulcorner\lambda\urcorner^{\mathbb{N}} \in T^+$. At the same time, $\mathbb{N}_{(T^+,T^-)} \models^- T\ulcorner\lambda\urcorner$; so $\mathbb{N}_{(T^+,T^-)} \models^+ \neg T\ulcorner\lambda\urcorner$. By definition of λ, this means that $\mathbb{N}_{(T^+,T^-)} \models^+ \lambda$. Thus, $\mathbb{N}_{(T^+,T^-)} \models^+ T\ulcorner\lambda\urcorner$ and $\mathbb{N}_{(T^+,T^-)} \models^+ \lambda$, so $\mathbb{N}_{(T^+,T^-)}$ is a T-model for λ. □

The logic $\mathsf{LP}^{\rightarrow}$, however, cannot handle Curry's paradox:

Proposition 6.4.7. Contradictory Curry $\mathcal{C}_\perp$ is paradoxical in $\mathsf{LP}^{\rightarrow}$.

Proof. Let $\mathcal{M}_{(T^+,T^-)}$ be any glutty model for $\mathcal{L}^+$, and assume the $\mathsf{LP}^{\rightarrow}$ valuation scheme; we show that $\mathcal{M}_{(T^+,T^-)}$ is not a T-model for $\mathcal{C}_\perp$. Note that $\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_\perp$ if and only if $\mathcal{M}_{(T^+,T^-)} \models^+ T\ulcorner\mathcal{C}_\perp\urcorner \rightarrow \perp$, by definition of $\mathcal{C}_\perp$. Now by definition of $\rightarrow$ in $\mathsf{LP}^{\rightarrow}$, we have:

$$\mathcal{M}_{(T^+,T^-)} \models^+ T\ulcorner\mathcal{C}_\perp\urcorner \rightarrow \perp \text{ iff } \mathcal{M}_{(T^+,T^-)} \not\models^+ T\ulcorner\mathcal{C}_\perp\urcorner.$$

We thus have: $\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_\perp$ if and only if $\mathcal{M}_{(T^+,T^-)} \not\models^+ \mathcal{C}_\perp$. This shows that $\mathcal{M}_{(T^+,T^-)}$ is not a T-model for $\mathcal{C}_\perp$. □

It is worth noting that the fact that contradictory Curry is paradoxical in $\mathsf{LP}^{\rightarrow}$ relies on the fact that Curry uses the implication connective, and that the implication connective is *not* the material conditional. Indeed, *material Curry* $\neg T\ulcorner\mathcal{C}\urcorner \vee \bot$ is not paradoxical in $\mathsf{LP}^{\rightarrow}$:

Proposition 6.4.8. The contradictory instance of *material Curry*

$$(\mathcal{C}_{\bot}{}^{m}) \qquad \neg T\ulcorner\mathcal{C}_{\bot}{}^{m}\urcorner \vee \bot$$

is not paradoxical in $\mathsf{LP}^{\rightarrow}$.

Proof. Note that $\mathcal{C}_{\bot}{}^{m} \equiv \neg T\ulcorner\mathcal{C}_{\bot}{}^{m}\urcorner$; we thus have a proof similar to the case of the Liar (Proposition 6.4.6). □

In fact, we conjecture that it is *impossible* to find a paradoxical sentence in $\mathsf{LP}^{\rightarrow}$ that does not use the implication connective:

Conjecture 6.4.9. If a sentence S in $\mathcal{L}^{+}$ does not contain $\rightarrow$, then S is not paradoxical in $\mathsf{LP}^{\rightarrow}$.

This gives us the means to differentiate at least one of the instances of Curry – contradictory Curry – from the Liar, in a very natural manner. However, as on classical paradoxicality, all other instances of Curry are left out:

Proposition 6.4.10. Let B be any sentence in the ground language. Curry's sentence $\mathcal{C}_B$ is paradoxical in $\mathsf{LP}^{\rightarrow}$ if and only if $B \equiv \bot$.

Proof. If $B \equiv \bot$, then the proof that $\mathcal{C}_B$ is paradoxical is the same as in Proposition 6.4.7. For the other direction, suppose $B \not\equiv \bot$: so there exists some ground model $\mathcal{M}$ such that $\mathcal{M} \models^{+} B$. Then $\mathcal{M}$ can serve as the basis for a T-model for $\mathcal{C}_B$: define (T^{+}, T^{-}) by $T^{+} = \{\ulcorner\mathcal{C}_B\urcorner^{\mathcal{M}}\}$ and $T^{-} = \varnothing$. Then $\mathcal{M}_{(T^{+},T^{-})} \models^{+} T\ulcorner\mathcal{C}_B\urcorner$. Moreover, by definition of $\mathcal{C}_B$, we have $\mathcal{M}_{(T^{+},T^{-})} \models^{+} \mathcal{C}_B$ iff $\mathcal{M}_{(T^{+},T^{-})} \models^{+} T\ulcorner\mathcal{C}_B\urcorner \rightarrow B$. By definition of $\rightarrow$ in $\mathsf{LP}^{\rightarrow}$ and the fact that $\mathcal{M}_{(T^{+},T^{-})} \models^{+} B$, we have $\mathcal{M}_{(T^{+},T^{-})} \models^{+} \mathcal{C}_B$. Since $\mathcal{M}_{(T^{+},T^{-})} \models^{+} \mathcal{C}_B$ and $\mathcal{M}_{(T^{+},T^{-})} \models^{+} T\ulcorner\mathcal{C}_B\urcorner$, we conclude that $\mathcal{M}_{(T^{+},T^{-})}$ is a T-model for $\mathcal{C}_B$. □

6.4.2. Capturing Curry, paraconsistently

To capture instances of Curry's paradox whose consequent is not contradictory, we adapt the definition of contingent-paradoxicality to the current paraconsistent framework. This gives us a formalisation of Intuition 2a, which was the goal of this section. We examine how this formalisation plays out; our aim was to find a definition that captures non-tautological instances of Curry, yet recognises the difference between Curry's paradox and the Liar – we will see that it meets this goal.

> **Definition 6.4.11** (Contingent-paradoxicality in $\mathsf{LP}^{\rightarrow}$)**.** A sentence S in $\mathcal{L}^{+}$ is *contingent-paradoxical (for A) in* $\mathsf{LP}^{\rightarrow}$ if there exists some sentence A in $\mathcal{L}$ such that $A \not\equiv \top$, but $\mathcal{M}_{(T^{+},T^{-})} \models^{+} A$ for every $\mathsf{LP}^{\rightarrow}$-model $\mathcal{M}_{(T^{+},T^{-})}$ that is a T-model for S.

Proposition 6.4.12. Let B be any sentence in the ground language. Curry's sentence $\mathcal{C}_B$ is contingent-paradoxical in $\mathsf{LP}^{\rightarrow}$ if and only if $B \not\equiv \top$.

Proof. For left-to-right, suppose that $\mathcal{C}_B$ is contingent-paradoxical in LP; so there exists some sentence A in the ground language such that $A \not\equiv \top$, but $\mathcal{M}_{(T^+,T^-)} \models^+ A$ for every $\mathsf{LP}^{\rightarrow}$-model $\mathcal{M}_{(T^+,T^-)}$ which is a T-model for S. Let $\mathcal{M}_0$ be the ground model witnessing that $A \not\equiv \top$. Now assume for contradiction that $B \equiv \top$. Then, by a similar argument as in the proof of Proposition 6.4.10, we can define a T-model $\mathcal{M}_{0_{(T^+,T^-)}}$ for $\mathcal{C}_B$ with $\mathcal{M}_0$ as its ground model. Hence $\mathcal{M}_{0_{(T^+,T^-)}} \not\models A$. So we have a T-model for $\mathcal{C}_B$ in which A is not true; this means that $\mathcal{C}_B$ is not contingent-paradoxical for A.

For the right-to-left direction, it suffices to show that, if $\mathcal{M}_{(T^+,T^-)}$ is a T-model for $\mathcal{C}_B$, then $\mathcal{M}_{(T^+,T^-)} \models B$. Suppose for contradiction that that $\mathcal{M}_{(T^+,T^-)}$ is a T-model for $\mathcal{C}_B$ such that $\mathcal{M}_{(T^+,T^-)} \not\models^+ B$. Then, by definition of $\rightarrow$ in $\mathsf{LP}^{\rightarrow}$, we have $\mathcal{M}_{(T^+,T^-)} \models^+ \mathcal{C}_B$ if and only if $\mathcal{M}_{(T^+,T^-)} \not\models^+ T\ulcorner\mathcal{C}_B\urcorner$. Hence $\mathcal{M}_{(T^+,T^-)}$ is not a T-model for $\mathcal{C}_B$. □

Unlike the previous definitions of Curry-paradoxicality, this definition does not include a large class of sentences that are 'like Curry'; most examples that we considered were variants on material Curry. Whereas here, as we saw in Proposition 6.4.8, the paradoxicality of Curry crucially depends on its use of the non-material implication connective. The same holds for contingent-paradoxicality:

Proposition 6.4.13. Let B be any sentence in $\mathcal{L}^+$. Then material Curry

$$(\mathcal{C}_B{}^m) \qquad \neg T\ulcorner\mathcal{C}_B{}^m\urcorner \vee B$$

is not contingent-paradoxical in $\mathsf{LP}^{\rightarrow}$.

Proof. Let A be any sentence in the ground language such that $A \not\equiv \top$. We show that $\mathcal{C}_B{}^m$ is not contingent-paradoxical for A. Since $A \not\equiv \top$, there exists some ground model $\mathcal{M}$ such that $\mathcal{M} \not\models A$. No matter the ground model, defining $T^+ = T^- = \{\ulcorner\mathcal{C}_B{}^m\urcorner^{\mathcal{M}}\}$ gives us a T-model for $\mathcal{C}_B{}^m$, as is straightforward to check. In particular, $\mathcal{M}_{(T^+,T^-)}$ is a T-model for $\mathcal{C}_B{}^m$, and $\mathcal{M}_{(T^+,T^-)} \not\models^+ A$ since $\mathcal{M} \not\models A$. □

Similarly, the sentence $\mathcal{D}$, which does not employ the implication connective, is not contingent-paradoxical in $\mathsf{LP}^{\rightarrow}$:

Proposition 6.4.14. Let A be any sentence in $\mathcal{L}^+$. Then the sentence

$$(\mathcal{D}) \qquad \neg T\ulcorner\mathcal{D}\urcorner \wedge A$$

is not contingent-paradoxical in $\mathsf{LP}^{\rightarrow}$.

Proof. Similar to the proof of Proposition 6.4.13. □

Indeed – like for plain paradoxicality in $\mathsf{LP}^{\rightarrow}$ – we conjecture that only sentences containing the implication connective are paradoxical in $\mathsf{LP}^{\rightarrow}$; the implication is to blame for all evil.

Conjecture 6.4.15. If a sentence S in $\mathcal{L}^+$ does not contain $\rightarrow$, then S is not contingent-paradoxical in $\mathsf{LP}^{\rightarrow}$.

Having said this, we can still find some variants on Curry's paradox that are contingent-paradoxical in $\mathsf{LP}^{\rightarrow}$. Here are two examples that we encountered in previous chapters as well:

Proposition 6.4.16. Let A, B and C be sentences in $\mathcal{L}$ such that $A \not\equiv \bot$, $B \not\equiv \top$, and $C \not\equiv \top$. Then the following sentences are contingent-paradoxical in $\mathsf{LP}^{\to}$:

1. The sentence

   $$(\mathcal{E}) \qquad (T\ulcorner\mathcal{E}\urcorner \vee B) \to C.$$

2. The sentence

   $$(\mathcal{F}) \qquad (T\ulcorner\mathcal{F}\urcorner \wedge A) \to B.$$

Proof. Straightforward to check by an argument similar to the right-to-left direction of Proposition 6.4.12. □

6.4.3. Summary

We proposed the definition of contingent-paradoxicality in the paraconsistent logic $\mathsf{LP}^{\to}$ to formalise Intuition 2a. The result is a definition of paradoxicality that captures non-tautological instances of Curry's paradox, but does not capture the Liar – this was our aim, so our definition is successful in this respect. It also highlights the differences between classical logic and paraconsistent logic in their treatment of the logical paradoxes. What we seem to have here is a device that allows us to characterise which sentences are 'problematic' in a given logic, and this invites for a generalisation to other logics – paraconsistent or otherwise. One might consider paracomplete logics such as the logics weak Kleene, strong Kleene, and supervaluationism that we saw in Chapter 4, or weaker paraconsistent logics such as the logic LP. A study and comparison of definitions of paradoxicality embedded in different logics would highlight the changeablity of the notion of paradoxicality across different logics, and provoke the question: is paradoxicality a logic dependent-notion?

For now, we conclude with a summary of the results of this section, in terms of the status of some sentences on the two definitions of paradoxicality that we introduced.

	$\mathcal{C}$	$\mathcal{C}^m$	$\mathcal{D}$	$\mathcal{E}$	$\mathcal{F}$	$\lambda \circ B$
Paradoxical in $\mathsf{LP}^{\to}$	$**$	$\times$	$\times$	$\times$	$\times$	$\times$
Contingent-paradoxical in $\mathsf{LP}^{\to}$	$*$	$\times$	$\times$	$*$	$*$	$\times$

($\times$) not captured
($*$) captured, under the condition that $A \not\equiv \bot$, $B \not\equiv \top$, and $C \not\equiv \top$
($**$) captured, under the condition that $B \equiv \bot$

Figure 6.3: Overview of Curry-paradoxical sentences on the definition of contingent-paradoxicality.

6.5. Independent arguments

Before we conclude this chapter, we briefly consider Intuition 3 that we saw in Chapter 2:

Intuition 3. A Curry-paradoxical sentence is a sentence A whose existence entails the truth of a sentence B by means of an argument that is independent of B.

Where the argument which derives B from the existence of A is *independent* from B if replacing B by an arbitrary sentence C in both A and the argument gives us a valid argument with the conclusion C.

This characterisation is framed in terms of entailment, and therefore invites for a formalisation in the setting of this chapter. Due to constraints of time and space, we will not discuss this proposal in as much detail as the previous definitions. Still, we would like to outline a possible formalisation and point to the prospects for further research.

As we noted in Chapter 2, this intuition is framed in terms of *arguments*, and the model-theoretic approach we take here is not well-suited to talk about arguments, proofs, or reasoning. However, we might still try to approach the intuition by model-theoretic means. The idea is the following: the intuition demands that replacing B by arbitrary C in both A and the argument yields the conclusion C. Thus, the existence of the sentence $A[C/B]$ entails the truth of C. While we cannot talk about the argument involved in the present framework, we *can* formalise the fact that the existence of $A[C/B]$ entails the truth of C. This gives us a notion that is less strict, since it does not demand that the argument involved is identical modulo replacing B by C. However, if we range over *all* possible sentences C, it seems likely that the argument in question is indeed the same in every case – though this is no more than a speculation at this point, and requires further research.

The formal definition in question is the following:

Definition 6.5.1 (Independent-paradoxicality)**.** Let the sentence S in $\mathcal{L}^+$ be diagonally defined by the formulas $A_1(y_1, \dots, y_n), \dots, A_n(y_1, \dots, y_n)$. Then S is *independent-paradoxical* if there exists some subsentence $B \in \mathrm{Form}(\mathcal{L})$ of S such that, for any C in $\mathcal{L}$, we have: $\mathcal{M}_X \models C$ for every model $\mathcal{M}_X$ which is a T-model for S^C_B.

As in Definition 4.3.5, S^C_B denotes the sentence that is diagonally defined by the formulas

$$A_1(y_1, \dots, y_n)[C/B], \dots, A_n(y_1, \dots, y_n)[C/B].$$

The need to talk about sentences of the form S^C_B, rather than simply $S[C/B]$, is due to problems of (self-)reference that we encountered earlier in Definition 4.3.13.

In other words, demanding that the T-schema holds for the sentence S with B appropriately replaced by C yields the truth of C. Thus, the existence of the sentence S with B replaced by C entails C, which gives us a partial formalisation of Intuition 3.

Let us verify that Curry's sentence is indeed independent-paradoxical:

Proposition 6.5.2. Every instance of Curry's sentence with a consequent in the ground-language is independent-paradoxical, i.e. $\mathcal{C}_B$ is independent-paradoxical for every sentence B in $\mathcal{L}$.

Proof. Let C be an arbitrary sentence in $\mathcal{L}$. We show that $\mathcal{C}_B$ is independent-paradoxical for B. We denote $\mathcal{C}_B$ by $\mathcal{C}$ to avoid cluttered notation. We show that, for any T-model $\mathcal{M}_X$ for $\mathcal{C}_B^C$, we have $\mathcal{M}_X \models C$. So let $\mathcal{M}_X$ be a T-model for $\mathcal{C}_B^C$. Then $\mathcal{M}_X \models T\ulcorner\mathcal{C}_B^C\urcorner$ if and only if $\mathcal{M}_X \models \mathcal{C}_B^C$. But, by definition of $\mathcal{C}_B^C$, we have

$$\mathcal{M}_X \models \mathcal{C}_B^C \iff \mathcal{M}_X \models T\ulcorner\mathcal{C}_B^C\urcorner \to C.$$

It thus follows that $\mathcal{M}_X \models C$, as desired. □

This formal definition thus succeeds in capturing at least Curry's paradox, including tautological Curry. It does so by allowing the replacement of a certain subsentence by another sentence. In this sense, it places emphasis of the *logical form* of the sentence. Indeed, there is a strong parallel between this definition and the definition of Curry-type sentences that we saw in Definitions 4.3.5 and 4.3.13.

6.6. Summary and evaluation

In this chapter, we attempted to formalise each of the intuitions framed in terms of entailment that we saw in Section 2.2. The orthodox Intuition 0 was our starting point, which we formalised as *classical paradoxicality*. This was only stage setting, since this definition does not capture Curry. We then generalised classical paradoxicality to obtain a formalisation of Intuition 1, in the form of *contingent-paradoxicality*. Most of the chapter centered around this notion: in the two subsequent sections, we introduced adapted versions of contingent-paradoxicality to capture Intuitions 2b and 2a – designed to distinguish Curry's paradox from the Liar. In the final section, we briefly considered the prospects of capturing Intuition 3 in this framework – an intuition designed to capture the paradoxicality of the *argument* involved in Curry's paradox.

We saw that the central notion of contingent-paradoxicality (Section 6.2) is closely related to local folk-paradoxicality; every contingent-paradoxical sentence is locally folk-paradoxical, and the converse holds for many key examples as well. This points to a potential correspondence between Intuition 1 and Intuition 5a.

Sub-contingent-paradoxicality (Section 6.3) was proposed as a formalisation of Intuition 2b, which highlights the fact that Curry's sentence contains the formula that it forces to be true as a subformula. We obtained a definition that captures Curry's paradox, but not the Liar – which was our aim. We saw, however, that the definition is restrictive, capturing only sentences that are like Curry's paradox in that they contain the sentence they entail as an *exact* subsentence – sentences like $\mathcal{E}$ are not captured. This seems to me like it draws a rather superficial distinction between different members of the Curry family, so this might not be a desirable property.

The notion of *contingent-paradoxicality in* $\mathsf{LP}^{\to}$ (Section 6.4) offers a more profound characterisation of the difference between Curry's sentence and the Liar by employing a paraconsistent background logic. We use the logic $\mathsf{LP}^{\to}$, in which Curry's paradox is problematic but the Liar is not. This results in a definition that captures Curry's paradox but not the Liar, and does not make superficial distinctions based on syntactic shape. It moreover gave us a means to form a clear picture of the different behaviour of paradoxical sentences across logics

– classical versus paraconsistent. We saw that the paradoxicality of Curry in paraconsistent logics hinges on its use of the implication connective. This is well-known, but the definition we presented gave us one framework in which to express this formally. The attractive thing about this framework is that it can be extended readily to other non-classical logics, which might be the basis for a unified overview of paradoxical sentences across different logics.

In the final section, we proposed *independent-paradoxicality* as a potential partial formalisation of Intuition 3. This intuition characterises Curry by means of its paradoxical argument, which we argued to be independent of its conclusion in Section 2.2. We hold that this is difficult to faithfully represent in the present model-theoretic framework, but we have considered whether there might be a (partial) way out. We have aimed to argue that our formal strategy captures an aspect of the intuition, but how the definition works out formally remains open for further research.

The table in Figure 6.4 gives an overview of the behaviour of our example sentences in the definitions we proposed.

	$\mathcal{C}$	$\mathcal{D}$	$\mathcal{E}$	$\mathcal{F}$	$\lambda \circ B$
Contingent-paradoxical	$*$	$*$	$*$	$*$	$\times$
Sub-contingent-paradoxical	$*$	$\equiv$	$\times$	$\equiv$	$\times$
Contingent-paradoxical in $\mathsf{LP}^{\rightarrow}$	$*$	$\times$	$*$	$*$	$\times$

($\times$) not captured
($*$) captured, under the condition that $A \not\equiv \bot$, $B \not\equiv \top$ and $C \not\equiv \top$
($\equiv$) equivalent to a sentence that is captured, under the condition that $A \not\equiv \bot$ and $B \not\equiv \top$.

Figure 6.4: Overview of Curry-paradoxical sentences on the different definitions of contingent-paradoxicality.

7. Conclusion and outlook

In this thesis, we have aimed to investigate the paradoxicality of Curry by asking the question: 'what makes Curry's sentence paradoxical?' and taking different answers to this question as starting points for formal definitions of paradoxicality. The result is an overview of formal definitions of paradoxicality that each correspond to a certain perspective on Curry's paradox. We have aimed to investigate how these definitions play out formally, to answer the question: what are the formal upshots of the proposed views on Curry?

In Chapter 2, we introduced the different perspectives on Curry's paradox that figured throughout the thesis, which we called intuitions. They were divided into two groups: those that are framed in terms of models, and those that are framed in terms of entailment. In both groups, there are two key themes: the question of distinguishing Curry's paradox from the Liar and the question of capturing all instances of Curry, including tautological Curry.

In Chapter 3, we discussed the necessary preliminaries for the formal chapters to come. We payed special attention to the distinction between strong and weak diagonalisation, and motivated our choice to use strong diagonalisation in most of the chapters.

In Chapters 4 to 6, we formalised the intuitions proposed in Chapter 2. Chapters 4 and 5 gave us two different formalisations of the intuitions that were framed in terms of models, based on the existing definitions of paradoxicality introduced by Kripke and Hsiung. In Chapter 6, we formalised the intuitions involving entailment.

In Chapter 4, we saw that Curry's paradox is only captured by Kripke's definition of paradoxicality in case its consequent is false in a chosen ground model; in order to capture more instances of Curry's paradox, we defined the notions of *local paradoxicality* and *local but not global paradoxicality*. We saw that these notions capture Curry's paradox, so long as its consequent is not a consequence of the theory PA^+. In order to moreover account for tautological Curry, we introduced the notion of *Curry-type* sentences, which succeeded to capture tautological Curry but missed out on the benefits of Kripke's theory regarding multi-sentence paradoxes: multi-sentence versions of Curry were no longer included.

The main objective of Chapter 5 was to show that the strategy of defining a notion of local paradoxicality extends to other definitions of paradoxicality that are framed in terms of models – such as Hsiung's notion of folk-paradoxicality. This notion is framed in terms of *sets* of sentences, which causes some differences in the treatment of multi-sentence paradoxes. We introduced the notion of *local folk-paradoxicality*, which behaved as we expected on the basis of the results in Kripke's chapter.

In the final Chapter 6, we formalised the four different intuitions that were framed in terms of entailment. The central notion was *contingent-paradoxicality*, which we saw was closely related to local folk-paradoxicality. In order to differentiate Curry from the Liar on this definition, we proposed two adaptations: one that was framed in terms of subsentences, and one that uses a paraconsistent background logic. The definition that uses a paraconsistent logic seemed promising because it highlights a deeper difference between Curry's paradox and the Liar than just syntactic shape, and because it provides a general framework that can be adapted to compare the behaviour of paradoxical sentences across logics. We concluded with a definition that attempted to capture the paradoxicality of Curry's *argument*, but we hold that this is difficult in the current model-theoretic setting.

7.1. Outlook

Several potential avenues for further research present themselves. The first and perhaps most natural option is to consider other influential model-theoretic definitions of paradoxicality and consider whether any of our strategies (especially the ones of the first two chapters) carry over. This would give us a more broad insight into the nature of Curry's paradox, and how its paradoxicality depends on different starting points. One relevant such definition of paradoxicality, which seems to allow for a straightforward application of our strategy, is given by Gupta and Belnap's *revision sequences* (Gupta & Belnap, 1993).

A more radically different approach lies in proof-theoretic approaches to Curry's paradox, such as the one proposed by Tennant (1982, 1995). As we have argued in Section 2.2 and again in Section 6.5, the paradoxicality of Curry's argument is not easy to catch in a model-theoretic environment. A proof-theoretic approach highlights the structure of the involved proofs, which brings us closer to arguments. The question would be whether such an approach would, indeed, be able to formalise our Intuition 3 – or perhaps it would show us how the intuition can be improved.

There are also some more specific open questions that we encountered throughout the thesis.

In Chapter 5, we met the opportunity for a notion of folk-paradoxicality that captures tautological Curry by adopting the same strategy that we used when defining Curry-type sentences in Section 4.3. We argued that the setting of folk-paradoxicality might even be better suited for such a definition, since it would not suffer the problems with multi-sentence paradoxes that we encountered for the Curry-type sentences.

We moreover noted that all definitions of contingent-paradoxicality (Sections 6.2 to 6.4) extend naturally to a framework that considers sets of sentences rather than individual sentences, just like Hsiung's definition of paradoxicality. This would give us the opportunity to develop a notion of *paradox* based based on our notion of contingent-paradoxicality, as in the work of Hsiung (2024).

Another point of interest is the definition of paradoxicality in a paraconsistent setting that we introduced (Section 6.4). As we mentioned, it would be worth looking into a generalisation of this definition to different background logics; for instance, to the paracomplete logics weak Kleene, strong Kleene, and supervaluationism, or other paraconsistent logics such as LP.

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