Union-splittings, the Axiomatization Problem, and the Rule Dichotomy Property in Modal Logic

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written by

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Abstract

This thesis studies logical properties in lattices of modal logics, focusing on union-splittings, the axiomatization problem, and the rule dichotomy property. We use semantic approaches to investigate these topics by working with the theory of stable canonical rules and formulas. Under the modal duality, we apply universal algebra to modal algebras and combinatorial methods to modal spaces.

We reformulate and extend the theory of stable canonical rules and formulas by introducing the notion of definable filtration, which will be the semantic foundation for much of the thesis. Building on this, a new combinatorial method, the Refinement Construction, is developed to prove the finite model property for a large class of logics and rule systems, generalizing the finite model property of union-splittings in NExtK, $\mathsf{K4}$-stable logics, and stable rule systems.

We then give a semantic characterization of union-splittings in the lattice NExtK and show that both being a union-splitting and a splitting are decidable in NExtK. This yields two more decidable properties in NExtK, namely, being a decidable formula and having a decidable axiomatization problem. These results answer the open questions [WZ07, Problem 2] and [CZ97, Problem 17.3] in the affirmative.

Finally, we study admissibility and the rule dichotomy property in the weak transitive logic $\mathsf{wK4}$ and the basic modal logic K. We refine the notion of rule dichotomy property, and show that stable canonical rules have the rule dichotomy property over $\mathsf{wK4}$ but fail over K. The latter supports Jeřábek's remark that the rule dichotomy property is a very strong property and thus is likely to fail for many logics [Jeř09].

The last chapter applies descriptive set theory to study the cardinality of sets of logics without assuming the Continuum Hypothesis and resolves the open questions [JL18, Question 6.4 (ii)] and [BBM25, Section 8 (1)].

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Chapter 1

Introduction

Modal logics are obtained by adding *modal operators* to classical logic. They are very expressive, yet often still reasonably easy to handle. While modal logics originated in the study of logical reasoning regarding necessity and possibility, different interpretations of the modal operators enable the modeling of notions and reasoning in various areas, such as: *provability logic*, modeling provability in mathematics; *epistemic logic*, modeling knowledge and belief; *tense logic*, modeling the flow of time; and *propositional dynamic logic*, modeling the behavior of computer programs. Additionally, modal operators allow translations from many non-classical logics to modal logics, including *intuitionistic logic* and *quantum logic*. As a result, studying modal logics often yields valuable insights and results across various applied fields and other logical systems. For more details, see [BdV01] and [CZ97] and the references therein. In this thesis, we will exclusively focus on modal logics that involve a single modal operator.

Modal logic, as a branch of mathematical logic, is not merely a collection of individual logical systems. Since the 1960s, the focus has shifted from examining various concrete modal logics independently to studying classes of modal logics, usually lattices of extensions of a fixed base logic. We denote K as the least modal logic and NExtK as the lattice of extensions of K, namely all normal modal logics. A significant turning point is the result proved by Jankov [Jan68] that there are continuum many superintuitionistic (and therefore modal) logics. Consequently, the primary research question has become understanding the structure of these lattices of modal logics and achieving general results on the logical properties of these systems. Such studies offer an abstract mathematical understanding of modal operators in formal systems and make individual results into corollaries. Some notable results in this area include:

- Bull [Bul66] and Fine [Fin71] showed that all extensions of $\mathsf{S4.3}$ have the fmp, are finitely axiomatizable, and thus decidable.
- Maksimova [Mak79] showed that there are exactly 8 superintuitionistic logics that have the Craig interpolation property (this result is for superintuitionistic logics but has the same spirit).
- The Blok-Esakia theorem [Blo76; Esa76]: there is a lattice isomorphism between the lattice of all superintuitionistic logics and the lattice of all normal extensions of the modal logic Grz.
- Blok's dichotomy theorem [Blo78]: the degree of Kripke incompleteness of a modal logic in NExtK can only be 1 or $2^{\aleph_0}$, and a logic has the degree of Kripke incompleteness 1 iff it is a union-splitting in NExtK. (A logic L is a *splitting* in NExtK if there is a logic L' such that (L, L')

is a *splitting pair* of NExtK in lattice theoretic sense; a logic L is a *union-splitting* in NExtK if it is a join of splittings in NExtK.)

This thesis aligns with this direction of research and studies the logical properties of modal logics.

We will work with both *algebraic semantics* and *relational semantics* for modal logic. On the algebraic semantics, we use *modal algebras* and apply notions and results from universal algebra. In particular, we will exploit the one-to-one correspondence between modal logics and varieties of modal algebras. On the relational semantics, we use *modal spaces*, a generalization of *Kripke frames* by equipping them with a topology. These relational structures can be visually represented, and thus are convenient for combinatorial proofs and constructing counterexamples. The two semantics are combined by the *Jónsson–Tarski duality* based on [JT51]. It states that the category of modal algebras and homomorphisms and the category of modal spaces and p-morphisms are dually equivalent. Consequently, we can move freely between the two realms, using the most appropriate semantics for our purposes. For a comprehensive overview of the historical development of modal logic, primarily from a mathematical perspective, we refer to [Gol03].

One of the most powerful tools in the study of lattices of logics is *characteristic formulas*. These formulas are defined from finite structures (e.g., finite modal algebras or finite Kripke frames) in a way that their validity has a semantic characterization. This semantic characterization allows us to reformulate questions about logics into purely semantic arguments when working with logics that are axiomatized by characteristic formulas, which gives us more control over them. The pioneering work by Jankov [Jan63; Jan68] and [de 68] introduced the first type of characteristic formulas, now referred to as *Jankov-de Jongh formulas*. The modal logic counterpart, known as *Jankov-Fine formulas*, was introduced by Fine [Fin74b]. A generalization to n-transitive modal logics was constructed by Rautenberg [Rau80]. Subsequent developments in this area include *subframe formulas*, *cofinal subframe formulas*, and *canonical formulas* (see, e.g., [CZ97, Chapter 9]). A milestone result regarding canonical formulas is that they axiomatize all transitive logics [Zak92], which means that we can semantically study all logics over the transitive modal logic K4.

The idea of generalizing characteristic formulas to inference rules was first introduced by Jeřábek [Jeř09], who extended canonical formulas to *canonical rules*. Both canonical rules and canonical formulas rely on *selective filtration* (see, e.g., [CZ97, Section 5.5]). Bezhanishvili et al. [BBI16], motivated by the *filtration* method (see, e.g., [BdV01, Section 2.3] and [CZ97, Section 5.3]), introduced *stable canonical rules* and *stable canonical formulas*. While selective filtration is mostly used for transitive logics such as K4 and S4, standard filtration is also constructed for non-transitive logics such as K. So, aiming to study all normal modal logics, in particular non-transitive ones, we will work with stable canonical rules and formulas in this thesis.

As we will observe in Chapter 3, an essential point needed for the theory of stable canonical rules and formulas is *definable filtration*, a generalization of the standard filtration introduced in [KSZ20]. We generalize the axiomatization result via stable canonical rules by allowing the base rule system to be any one that admits definable filtration. Furthermore, using Gabbay's filtration [Gab72], we generalize the stable canonical formulas to pre-transitive logics of the form $\mathsf{K} + \Diamond^{m+1}p \to \Diamond p$ and generalize the axiomatization result via stable canonical formulas from K4 to the pre-transitive logics. Through our proof, we observe that the axiomatization result can be further strengthened by considering a stronger type of formulas, which we call *m-stable canonical formulas*. These notions and results will be used in the subsequent chapters.

A modal logic (resp. rule system) has the *finite model property* (*fmp* for short) if, for any formula (resp. rule) that is not provable, there exists a finite countermodel. The finite model property is one of the most important and well-studied properties of modal logics (see, e.g., [BdV01] and [CZ97]). A crucial step toward Blok's dichotomy theorem, which is interesting in itself, is that all union-splittings in NExtK have the fmp. In Chapter 4, we significantly generalize this result by introducing a novel combinatorial method on modal spaces, which we call the *Refinement Construction*. This method is built on top of the theory of stable canonical rules and formulas and gives an explicit construction of finite countermodels. Using the Refinement Construction, we prove the fmp for a large class of logics and rule systems, which implies the fmp of union-splittings in NExtK [Blo78], $\mathsf{K4}$-stable logics [BBI18], and stable rule systems [BBI16].

In addition to the question of whether a logic has a particular property, it is also natural to ask if it is decidable to determine whether a logic has that property. This question has been extensively studied (see [WZ07] for a comprehensive survey), but in NExtK most known results are negative; almost all logical properties that one could consider were proved to be undecidable, except for the consistency and the coincidence with K. In contrast, we show in Chapter 5 that the property of being a union-splitting is decidable in NExtK. The key to our proof is a semantic characterization of union-splittings. From the characterization, it also follows the decidability of being a splitting. These results answer the open question [WZ07, Problem 2] affirmatively, adding two properties to the decidable properties in NExtK. Moreover, we observe that there is a somewhat mysterious connection between union-splittings and decidability. A formula φ is a *decidable formula* if it is decidable, given a formula ψ, whether $\varphi \in \mathsf{K}+\psi$. The *axiomatization problem* for a logic L is the problem of, given a formula ψ, deciding whether $L = \mathsf{K}+\psi$. It turns out that a formula φ is a decidable formula in NExtK iff the axiomatization problem for $\mathsf{K}+\varphi$ is decidable iff $\mathsf{K}+\varphi$ is a union-splitting in NExtK or the inconsistent logic. Consequently, our result implies the decidability of being a decidable formula and having a decidable axiomatization problem, and therefore positively answers another open question [CZ97, Problem 17.3].

The study of *admissible rules* is related to both decidability and logical properties. Admissible rules in a logic are valid inferences in that logic. The decidability of admissibility in a logic is then a natural strengthening of the decidability of the logic. Friedman [Fri75] posed the question of whether the admissibility of a given inference rule in IPC is decidable. Rybakov showed that this is the case for IPC and a large class of transitive modal and superintuitionistic logics (see [Ryb97] for a comprehensive overview and references). However, the decidability of admissibility in K is a long-standing open question (e.g., [CZ97, Problem 16.4]). Recently, Jeřábek [Jeř09] introduced a new method to establish the decidability of admissibility. This method involves proving the *rule dichotomy property over L* for a class of rules, that is, every rule in the class is either admissible or equivalent to an assumption-free rule. While Jeřábek used canonical rules, which are less effective in the non-transitive setting, our approach in Chapter 6 focuses on stable canonical rules, as they do axiomatize all rules over K. We show that stable canonical rules have the rule dichotomy property over $\mathsf{wK4}$, but not over K. This partially confirms Jeřábek's conjecture that many logics may lack the rule dichotomy property. Unfortunately, it also suggests that the method may not be very effective for K. Finally, we provide sufficient conditions for both admissibility and inadmissibility in K, which highlight the combinatorial complexity involved.

Studying the logical properties of lattices of modal logics from an abstract perspective often involves counting logics with or without certain properties. In Chapter 7, we take a slightly different point of view and study the cardinality of classes of modal logics without assuming the *Continuum Hypothesis*.

We introduce the idea of applying descriptive set theory, specifically, the theory of *Borel* sets, to logic and universal algebra through coding. We can naturally encode formulas/identities as natural numbers and logics/equational theories as real numbers. Then, by characterizing the arithmetical hierarchy to which the set of reals corresponding to the set of logics/equational theories belongs, we show that the cardinality of any subvariety lattice, the cardinality of any interval, and the degree of fmp of any logic is either $\leq \aleph_0$ or $2^{\aleph_0}$. This answers the questions [JL18, Question 6.4 (ii)] and [BBM25, Section 8 (1)] in the positive. Chapter 7 is based on a joint work with Juan P. Aguilera and Nick Bezhanishvili.

Finally, we summarize the main contributions of this thesis:

- We reformulate the theory of stable canonical rules in a more general setting where the base logic admits definable filtration (Chapter 3);
- We give an algebraic proof of Gabbay's filtration and generalize the theory of stable canonical formulas to pre-transitive logics $\mathsf{K} + \Diamond^{m+1}p \to \Diamond p$ (Chapter 3);
- We introduce m-stable canonical formulas, a stronger notion of stable canonical formulas, and show that they also axiomatize all logics extending the pre-trantive logics (Chapter 3);
- We generalize the fmp results of union-splitting in NExtK, $\mathsf{K4}$-stable logics, and stable rule systems, in a unified way by introducing a new combinatorial method, called *Refinement Construction* (Chapter 4);
- We observe that in a stable canonical rule or formula defined from a modal algebra of finite height, the closed domain essentially does not increase the expressivity (Chapter 4);
- We show the decidability of being a union-splitting and a splitting in NExtK by providing a semantic characterization of union-splittings (Chapter 5);
- We show as a result that having a decidable axiomatization problem and being a decidable formula is also decidable (Chapter 5);
- We redefine the rule dichotomy property (over a logic) so that it makes sense in a broader context (Chapter 6);
- We show that stable canonical rules have the rule dichotomy over $\mathsf{wK4}$ but not over K (Chapter 6);
- We provide sufficient conditions for a stable canonical rule to be (in)admissible in K and discuss some examples, including a full characterization of the admissibility for stable rules (Chapter 6);
- We show (in ZFC) that any interval of varieties and thus any interval of modal logics has the cardinality either $\leq \aleph_0$ or $2^{\aleph_0}$ (Chapter 7);
- We show (in ZFC) that the degree of fmp of any logic is either $\leq \aleph_0$ or $2^{\aleph_0}$ (Chapter 7).

Chapter 2

Preliminaries

In this chapter, we recall notions and results that we will use in the thesis. Notations will be fixed along the way.

2.1 Universal algebra

We recall basic notions and results from universal algebra, which serves as the foundation of algebraic semantics for logics. We refer to [Ber11] and [BS81] for details.

Algebras

Definition 2.1. A *language* or *similarity type* of algebras is a set F of function symbols such that each $f \in F$ is assigned a non-negative integer n, called the *arity* of f. A function symbol $f \in F$ with arity n is called an *n-ary function symbol*, and a 0-ary function symbol is also called a *constant symbol*.

Definition 2.2. Let F be a language. An algebra $\mathfrak{A}$ of type F is a pair $(A, F^{\mathfrak{A}})$ where A is a non-empty set and $F^{\mathfrak{A}}$ is a set of *functions* or *operations* on A such that for each n-ary function symbol $f \in F$, there is a corresponding n-arry function $f^{\mathfrak{A}} : A^n \to A$ in $F^{\mathfrak{A}}$.

Definition 2.3. Let $\mathfrak{A} = (A, F^{\mathfrak{A}})$ and $\mathfrak{B} = (B, F^{\mathfrak{B}})$ be algebras of type F. A map $h : A \to B$ is called a *homomorphism* if for any n-ary function symbol $f \in F$,

$$h(f^{\mathfrak{A}}(a_1, \ldots, a_n)) = f^{\mathfrak{B}}(h(a_1), \ldots, h(a_n)) \text{ for all } a_0, \ldots, a_n \in A.$$

An injective homomorphism is called an *embedding*. A bijective homomorphism is called an *isomorphism*.

We often denote $h : \mathfrak{A} \to \mathfrak{B}$, $h : \mathfrak{A} \hookrightarrow \mathfrak{B}$, and $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$, to emphasize that h is a homomorphism, an embedding, and a surjective homomorphism, respectively.

Definition 2.4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be algebras of the same type.

1. $\mathfrak{A}$ is a *subalgebra* of $\mathfrak{B}$ if $A \subseteq B$ and the inclusion map $i : A \to B$ is a homomorphism.

2. $\mathfrak{B}$ is a *homomorphic image* of $\mathfrak{A}$ if there is a surjective homomorphism $f : \mathfrak{A} \to \mathfrak{B}$.

3. $\mathfrak{A}$ is *isomorphic* to $\mathfrak{B}$ if there is an isomorphism $h : \mathfrak{A} \to \mathfrak{B}$.

For any $X \subseteq A$, there is a least subalgebra of $\mathfrak{A}$ containing X, which we call the subalgebra of $\mathfrak{A}$ *generated* by X. A subalgebra $\mathfrak{A}'$ of $\mathfrak{A}$ is called *finitely generated* if it is generated by a finite subset of A.

Definition 2.5. Let $\{\mathfrak{A}_i\}_{i \in I}$ be a family of algebras where each $\mathfrak{A}_i = (A_i, F^{\mathfrak{A}_i})$ is an algebra of type F. The *direct product* or *product* of $\{\mathfrak{A}_i\}_{i \in I}$ is the algebra $\Pi_{i \in I} \mathfrak{A}_i = (\Pi_{i \in I} A_i, F^{\Pi_{i \in I} \mathfrak{A}_i})$ where

1. $\Pi_{i \in I} A_i$ is the cartesian product of $\{A_i\}_{i \in I}$, and
2. for each n-ary function symbol $f \in F$,

$$f^{\Pi_{i \in I} \mathfrak{A}_i}(\alpha_1, \ldots, \alpha_n)(i) = f^{\mathfrak{A}_i}(\alpha_1(i), \ldots, \alpha_n(i)) \text{ for all } \alpha_1, \ldots, \alpha_n \in \Pi_{i \in I} A_i.$$

The notions above induce the following operations on a class $\mathcal{K}$ of algebras:

- $\mathsf{H}(\mathcal{K}) = \{\mathfrak{B} : \mathfrak{B}$ is a homomorphic image of some $\mathfrak{A} \in \mathcal{K}\}$,
- $\mathsf{S}(\mathcal{K}) = \{\mathfrak{B} : \mathfrak{B}$ is a subalgebra of some $\mathfrak{A} \in \mathcal{K}\}$,
- $\mathsf{I}(\mathcal{K}) = \{\mathfrak{B} : \mathfrak{B}$ is isomorphic to some $\mathfrak{A} \in \mathcal{K}\}$,
- $\mathsf{P}(\mathcal{K}) = \{\mathfrak{B} : \mathfrak{B}$ is a product of a family of algebras in $\mathcal{K}\}$,
- $\mathsf{P}_U(\mathcal{K}) = \{\mathfrak{B} : \mathfrak{B}$ is an ultraproduct of a family of algebras in $\mathcal{K}\}$.

Definition 2.6. Let $\mathcal{K}$ be a class of algebras.

1. $\mathcal{K}$ is called a *variety* if it is closed under H, S, and P. We denote by $\mathcal{V}(\mathcal{K})$ the least variety containing $\mathcal{K}$.
2. $\mathcal{K}$ is called a *universal class* if it is closed under I, S, and P_U. We denote by $\mathcal{U}(\mathcal{K})$ the least universal class containing $\mathcal{K}$.

The following characterization of varieties and universal classes has been obtained by Tarski (see [BS81, Chapter 2, Theorem 9.5] and [BS81, Chapter 5, Theorem 2.20]).

Theorem 2.7. *Let $\mathcal{K}$ be a class of algebras. Then,*

1. $\mathcal{V}(\mathcal{K}) = \mathsf{HSP}(\mathcal{K})$,
2. $\mathcal{U}(\mathcal{K}) = \mathsf{ISP}_U(\mathcal{K})$.

Definition 2.8. An algebra $\mathfrak{A}$ is called *locally finite* if every finitely generated subalgebra of $\mathfrak{A}$ is finite. A class $\mathcal{K}$ of algebras is called *locally finite* if every algebra in $\mathcal{K}$ is locally finite.

Syntax and semantics

Definition 2.9. Let F be a language and X be a set. Elements of X are called *variables.*

1. *Terms* of type F over X are defined recursively by
   (a) Each variable $x \in X$ and each constant symbol $f \in F$ are terms,
   (b) If $f \in F$ is an n-ary function symbol and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term.

2. An *identity* of type F over X is an expression of the form

$$s \approx t$$

for some terms s and t.

Definition 2.10. A *valuation* over a set X of variables on an algebra $\mathfrak{A} = (A, F^{\mathfrak{A}})$ is a function $V : X \to A$. Then, V naturally generalizes to a function from terms to A. We say that $\mathfrak{A}$ *satisfies* an identity $s \approx t$ under V, written $\mathfrak{A}, V \models s \approx t$, if $V(s) = V(t)$ in $\mathfrak{A}$. We say that $\mathfrak{A}$ *validates* $s \approx t$, written $\mathfrak{A} \models s \approx t$, if $\mathfrak{A}$ satisfies $s \approx t$ for any valuation on $\mathfrak{A}$. Moreover, for a set Σ of identities, we write $\mathfrak{A} \models \Sigma$ if $\mathfrak{A} \models \varphi$ for all $\varphi \in \Sigma$; for a class $\mathcal{K}$ of algebras, we write $\mathcal{K} \models \varphi$ if $\mathfrak{A} \models \varphi$ for all $\mathfrak{A} \in \mathcal{K}$.

An algebra $\mathfrak{A}$ can also be seen as a first-order structure. Followng first-oder model theory, for a frist-order sentence ψ, we write $\mathfrak{A} \models \psi$ if $\mathfrak{A}$ satisfies ψ.

For a set Σ of identities, let $\mathcal{V}(\Sigma) = \{\mathfrak{A} : \mathfrak{A} \models \Sigma\}$. For a class $\mathcal{K}$ of algebras, let $\mathsf{Th}(\mathcal{K}) = \{\varphi : \mathcal{K} \models \varphi\}$. The following characterization, due to Birkhoff, bridges between the syntax and the semantics of universal algebra.

Theorem 2.11. *Let $\mathcal{K}$ be a class of algebras.*

1. *$\mathcal{K}$ is a variety iff $\mathcal{K} = \mathcal{V}(\Sigma)$ for a set Σ of identities,*

2. *$\mathcal{K}$ is a universal class iff $\mathcal{K} = \{\mathfrak{A} : \mathfrak{A} \models \Gamma\}$ for a set Γ of first-order universal sentences.*

We call a set Σ of idenenties an *equational theory* if $\Sigma = \mathsf{Th}(\mathcal{K})$ for a variety $\mathcal{K}$. It is well-known in universal algebra that varieties and equational theories respectively form a complete lattice, and operations $\mathsf{Th}(-)$ and $\mathcal{V}(-)$ are dual isomorphisms between these two lattices that are inverse of each other.

We have a proof system-like characterization of equational theories (see, e.g., [BS81, Chapter 2, Definition 14.16 and Theorem 14.17]).

Definition 2.12.

- For a term t and an identity $s \approx s'$, an identity $t \approx t'$ is a *replacement* instance of t and $s \approx s'$ if t' is the result of replacing an occurrence of s in t by s'.

- For an identity $s \approx s'$ and a tuple of terms $(t_1, \ldots, t_n)$, the *substitution* instance of $s \approx s'$ and $(t_1, \ldots, t_n)$ is the resulting identity by simultaneously replacing every occurrence of each variable x_i in $s \approx s'$ by t_i.

Theorem 2.13. *Let Σ be a set of identities. Then $\mathsf{Th}(\Sigma)$, the least equational theory containing Σ, is the least set of identities containing Σ such that:*

1. $s \approx s \in \mathsf{Th}(\Sigma)$ *for* $s \in \mathsf{T}$,

2. $s \approx t \in \mathsf{Th}(\Sigma) \Rightarrow t \approx s \in \mathsf{Th}(\Sigma)$,

3. $s \approx t,\ t \approx u \in \mathsf{Th}(\Sigma) \Rightarrow s \approx u \in \mathsf{Th}(\Sigma)$,

4. $\mathsf{Th}(\Sigma)$ *is closed under replacement,*

5. $\mathsf{Th}(\Sigma)$ *is closed under substitution.*

Congruences and subdirectly irreducible algebras

Definition 2.14. Let $\mathfrak{A} = (A, F^{\mathfrak{A}})$ be an algebra of type F. A binary relation $\theta \subseteq A \times A$ is a *congruence* on $\mathfrak{A}$ if θ is an equivalence relation (i.e., reflexive, transitive, and symmetric and for each n-ary function symbol $f \in F$,

$$(f^{\mathfrak{A}}(a_1, \dots, a_n), f^{\mathfrak{A}}(b_1, \dots, b_n)) \in \theta \text{ for all } (a_1, b_1), \dots, (a_n, b_n) \in \theta.$$

Every algebra $\mathfrak{A}$ has the least congruence $\Delta = \{(a, a) : a \in \mathfrak{A}\}$ and the greatest congruence $A \times A$. In fact, congruences on $\mathfrak{A}$ form a lattice, called the *congruence lattice* of $\mathfrak{A}$.

Definition 2.15. Let $\mathfrak{A} = (A, F^{\mathfrak{A}})$ be an algebra of type F and θ be a congruence on $\mathfrak{A}$. Then the quotient of $\mathfrak{A}$ by θ is the algebra $\mathfrak{A}/\theta = (A/\theta, F^{\mathfrak{A}/\theta})$ where

1. A/θ is the quotient set of A by θ, and
2. for each n-ary function symbol $f \in F$,

$$f^{\mathfrak{A}/\theta}(a_1/\theta, \dots, a_n/\theta) = (f^{\mathfrak{A}}(a_1, \dots, a_n))/\theta \text{ for all } a_1, \dots, a_n \in A.$$

For each quotient $\mathfrak{A}/\theta$, the projection map from $\mathfrak{A}$ to $\mathfrak{A}/\theta$ is a surjective homomorphism. Conversely, each surjective homomorphism from $\mathfrak{A}$ induces a congruence on $\mathfrak{A}$. Thus, there is a one-to-one correspondence between congruences on $\mathfrak{A}$ and homomorphic images of $\mathfrak{A}$ (up to isomorphism).

Definition 2.16. Let $\{\mathfrak{A}_i\}_{i \in I}$ be a family of algebras and $p_i : \Pi_{i \in I}\mathfrak{A}_i \to \mathfrak{A}_i$ be the projection map for each $i \in I$.

1. An algebra $\mathfrak{B}$ is called a *sudirect product* of $\{\mathfrak{A}_i\}_{i \in I}$ if $\mathfrak{B}$ is a subalgebra of $\Pi_{i \in I}\mathfrak{A}_i$ such that $p_i{\restriction}B : \mathfrak{B} \to \mathfrak{A}_i$ is surjective.
2. An embedding $h : \mathfrak{B} \hookrightarrow \Pi_{i \in I}\mathfrak{A}_i$ is called a *subdirect embedding* if the image $h[\mathfrak{B}]$ is a subdirect product of $\{\mathfrak{A}_i\}_{i \in I}$.

Definition 2.17. An algebra $\mathfrak{A}$ is called *subdirectly irreducible* (*s.i.* for short) if for any subdirect embedding $h : \mathfrak{A} \hookrightarrow \Pi_{i \in I}\mathfrak{A}_i$, there is an $i \in I$ such that $p_i \circ h : \mathfrak{A} \to \mathfrak{A}_i$ is an isomorphism.

Subdirectly irreducible algebras can be characterized by congruences (see, e.g., [BS81, Chapter 2, Theorem 8.4])

Theorem 2.18. *An algebra $\mathfrak{A}$ is subdirectly irreducible iff there is a second least congruence on $\mathfrak{A}$, that is, there is a congruence θ on $\mathfrak{A}$ such that $\Delta \subsetneq \theta$ and $\theta \subseteq \theta'$ for any congruence $\theta' \neq \Delta$ on $\mathfrak{A}$.*

Given a class $\mathcal{K}$ of algebras, we write $\mathcal{K}_{\mathrm{si}}$ for the set of subdirectly irreducible members of $\mathcal{K}$. Subdirectly irreducible algebras are particularly important in universal algebra because of the following result by Birkhoff (see, e.g., [BS81, Chapter 2, Theorem 8.6]). Intuitively, subdirectly irreducible algebras serve as building blocks of all algebras.

Theorem 2.19.

1. *Every algebra $\mathfrak{A}$ is isomorphic to a subdirect product of subdirectly irreducible algebras that are homomorphic images of $\mathfrak{A}$.*

2. For any variety $\mathcal{V}$, it holds that $\mathcal{V} = \mathcal{V}(\mathcal{V}_{\mathrm{si}})$.

Definition 2.20. A variety $\mathcal{V}$ is called *congruence-distributive* if the congruence lattice of every algebra $\mathfrak{A} \in \mathcal{V}$ is a distributive lattice.

The following results about congruence-distributive varieties are consequences of the Jónsson's Lemma [Jón67] (see also [Ber11, Section 5.2] and [BS81, Section 4.6]).

Theorem 2.21. *Let $\mathcal{K}$ be a class of algebras such that $\mathcal{V}(\mathcal{K})$ is a congruence-distributive variety. Then $\mathcal{V}(\mathcal{K})_{\mathrm{si}} \subseteq \mathsf{HSP}_U(\mathcal{K})$.*

Corollary 2.22. *Let $\mathcal{K}$ be a finite set of finite algebras such that $\mathcal{V}(\mathcal{K})$ is a congruence-distributive variety. Then $\mathcal{V}(\mathcal{K})_{\mathrm{si}} \subseteq \mathsf{HS}(\mathcal{K})$.*

2.2 Lattice theory

In this section, we recall the theory of *lattices*. Lattices, on the one hand, serve as a base structure of various algebras used in algebraic semantics. On the other hand, lattice theory is useful in studying a class of logics as they often form a lattice. We refer to [Ber11] for details.

Definition 2.23. A pair (X, R) of a set X and a binary relation $R \subseteq X \times X$ is called a *partially ordered set* (*poset*) if R is:

1. reflexive: $\forall x \in X\ (xRx)$,
2. transitive: $\forall x, y, z \in X\ (xRy \wedge yRz \rightarrow xRz)$, and
3. antisymmetric: $\forall x, y \in X\ (xRy \wedge yRx \rightarrow x = y)$.

If (X, R) is a poset, the relation R is called a *partial order* on X and often denoted by $\leq$.

Definition 2.24. Let $(X, \leq)$ be a poset and $Y \subseteq X$. An element $x \in X$ is called a *lower bound* of Y if $x \leq y$ for all $y \in Y$. The element $x \in X$ is called a *greatest lower bound* or *infimum* of Y if x is a lower bound of Y and $x' \leq x$ for any lower bound $x' \in X$ of Y. The notions *upper bound* and *least upper bound* (or *supremum*) are defined dually.

An infimum and a supremum of Y are unique if they exist; they are denoted as $\bigwedge Y$ and $\bigvee Y$ respectively.

Definition 2.25. A poset $(L, \leq)$ is called a *lattice* if any $\{x, y\} \subseteq L$ has an infimum and a supremum.

Lattices also have an algebraic definition.

Definition 2.26. A *lattice* is an algebra $(L, \wedge, \vee)$ with two binary operations $\wedge$ (called *meet*) and $\vee$ (called *join*) validating the following identities:

(Associativity) $\quad x \wedge (y \wedge z) \approx (x \wedge y) \wedge z, \quad x \vee (y \vee z) \approx (x \vee y) \vee z,$

(Idempotence) $\quad x \wedge x \approx x, \quad x \vee x \approx x,$

(Commutativity) $\quad x \wedge y \approx y \wedge x, \quad x \vee y \approx y \vee x,$

(Absorption) $x \wedge (x \vee y) \approx x, \quad x \vee (x \wedge y) \approx x.$

These two definitions are equivalent. A lattice $(L, \leq)$ in Definition 2.25 induces a lattice $(L, \wedge, \vee)$ in Definition 2.26 by defining $x \wedge y = \bigwedge\{x, y\}$ and $x \vee y = \vee\{x, y\}$. Conversely, a lattice $(L, \wedge, \vee)$ in Definition 2.26 induces a lattice $(L, \leq)$ in Definition 2.25 by defining $x \leq y$ iff $x \wedge y = x$ (or, equivalently, $x \vee y = y$). Moreover, these two constructions are inverse to each other.

We often simply write L for a lattice $(L, \wedge, \vee)$.

Definition 2.27. A lattice L is called *distributive* if L validates the identities

$$(x \wedge y) \vee z \approx (x \vee z) \wedge (y \vee z) \text{ and } (x \vee y) \wedge z \approx (x \wedge z) \vee (y \wedge z).$$

In fact, for lattices, the two identities above are equivalent to each other.

Definition 2.28. An algebra $(L, \wedge, \vee, 0, 1)$ is a *bounded lattice* if $(L, \wedge, \vee)$ is a lattice and the identities $x \wedge 0 \approx 0$ and $x \vee 1 \approx 1$ hold.

Definition 2.29. A lattice L is called *complete* if the infimum $\wedge X$ and the supremum $\bigvee X$ exist for every subset $X \subseteq L$.

Note that a complete lattice is always bounded, that is, it has the greatest element and the least element. Moreover, a lattice is complete iff the infimum exists for any subset iff the supremum exists for any subset.

Definition 2.30. Let L be a bounded lattice. A subset $F \subseteq L$ is called a *filter* on L if the following hold:

1. If $x \in F$ and $x \leq y$, then $y \in F$,
2. If $x, y \in F$, then $x \wedge y \in F$.

The filter F is called *trivial* if $F = \{1\}$. The filter F is called *proper* if $F \neq L$.

The following notions and results will be useful in the study of lattices of modal logics.

Definition 2.31. A *closure operator* on a set A is a function $C : \mathcal{P}(A) \to \mathcal{P}(A)$ such that for any $X, Y \subseteq A$,

(increasing) $X \subseteq C(X)$,

(idempotent) $C(C(X)) = C(X)$,

(monotone) $X \subseteq Y \implies C(X) \subseteq C(Y)$.

For a closure operator C on A, a set $X \subseteq A$ is called *C-closed* if $C(X) = X$.

Definition 2.32. An closure operator C on a set A is called *algebraic* if for any $X \subseteq A$,

$$C(X) = \bigcup\{C(Y) : Y \text{ is a finite subset of } X\}.$$

Definition 2.33. Let L be a complete lattice. An element $x \in L$ is called *compact* if for any $X \subseteq L$,

$$x \leq \bigvee X \implies x \leq \bigvee Y \text{ for some finite } Y \subseteq X.$$

The lattice L is called *algebraic* if every element is a join of compact elements.

An algebraic closure operator induces an algebraic lattice (see, e.g., [Ber11, Theorem 2.30])

Theorem 2.34. *Let C be an algebraic closure operator on a set A. Then C-closed sets with the subset relation form an algebraic lattice where compact elements are $C(X)$ for finite sets $X \subseteq A$.*

Definition 2.35. Let L be a complete lattice. A *splitting pair* of L is a pair (x, y) of elements of L such that $x \not\leq y$ and for any $z \in L$, either $x \leq z$ or $z \leq y$. If (x, y) is a splitting pair of L, we say that x *splits* L and y is a splitting in L.

Note that if x splits L, then there is a unique $y \in L$ such that (x, y) is a splitting pair, which we denote L/x. Conversely, a splitting y also uniquely determines x.

2.3 Modal algebras and modal duality

In this section, we recall modal algebras used in algebra semantics for modal logics. The materials are collected from [BdV01, Chapter 5], [CZ97, Chapter 7 and 8], [Kra99, Chapter 2], and [Ven07]. Note that modal algebras are a type of algebra, so results from universal algebras apply.

Modal algebras

Definition 2.36. An algebra $\mathfrak{A} = (A, \wedge, \vee, \to, 0, 1)$ is a *Boolean algebra* if $(A, \wedge, \vee, 0, 1)$ is a bounded lattice and the identities $x \wedge (x \to 0) \approx 0$ and $x \vee (x \to 0) \approx 1$.

We abbreviate $x \to 0$ as $\neg x$. Since a Boolean algebra is also a lattice, all the notions for lattices apply. In particular, a filter F on a Boolean algebra $\mathfrak{A}$ is called an *ultrafilter* if it is proper and for any $a \in A$, $a \in F$ or $\neg a \in F$.

Definition 2.37. An algebra $\mathfrak{A} = (A, \wedge, \vee, \to, 0, 1, \Diamond)$ is a *modal algebra* if $(A, \wedge, \vee, \to, 0, 1)$ is a Boolean algebra and the identities $\Diamond 0 \approx 0$ and $\Diamond(x \vee y) \approx \Diamond x \vee \Diamond y$ hold.

We abbreviate $\neg\Diamond\neg x$ as $\Box x$. Then the above two identities are equivalent to $\Box 1 = 1$ and $\Box(x \wedge y) = \Box x \wedge \Box y$. Also, we often write a modal algebra as a pair $(A, \Diamond)$ where A is its base Boolean algebra and $\Diamond$ is the additional operation.

Definition 2.38. Let $\mathfrak{A}$ be a modal algebra. A subset $F \subseteq A$ is called a *modal filter* on $\mathfrak{A}$ if F is a filter on A and $a \in F$ implies $\Box a \in F$.

There is a one-to-one correspondence between modal filters and congruences, and thus a one-to-one correspondence between modal filters and homomorphic images.

Theorem 2.39. *Let $\mathfrak{A}$ be a modal algebra. The lattice of modal filters on $\mathfrak{A}$ is dually isomorphic to the lattice of congruences on $\mathfrak{A}$.*

Modal algebras are congruence-distributive, so the consequences of Jónsson's Lemma apply. The following characterization of s.i. modal algebras is from Rautenberg [Rau80]. Note that an opremum may not be unique.

Proposition 2.40. *A modal algebra $\mathfrak{A}$ is s.i. iff $\mathfrak{A}$ has an* opremum*, that is, an element $c \in \mathfrak{A}$ such that $c \neq 1$ and for any $a \neq 1$, there is $n \in \omega$ such that $\Box^{\leq n} a \leq c$.*

Modal duality

Duality theory is useful in studying lattice-based algebraic structures. Some well-known dualities include: *Priestley duality* for bounded distributive lattices and Priestley spaces, *Esakia duality* for Heyting algebras and Esakia spaces, and *Stone duality* for Boolean algebras and Stone spaces. In the rest of this section, we recall the *modal duality* between modal algebras and modal spaces (see, e.g., [BdV01, Chapter 5]). This duality is also known as Jósson-Tarski duality as it is based on Jónsson-Tarski representation theorem [JT51].

Let X be a set and $R \subseteq X \times X$ be a binary relation on X. We use the following notations. For $U \subseteq X$, $R[U]$ and $R^{-1}[U]$ denote the direct image and the inverse image of U under R respectively. Let R^n denote the n times composition of R, where R^0 is the identity. For $\alpha \leq \omega$, let $R^{<\alpha}[U] = \bigcup_{n<\alpha} R^n[U]$ and $R^{\leq\alpha}[U] = \bigcup_{n\leq\alpha} R^n[U]$. In addition, if $U = \{x\}$, we will write $R[x]$ instead of $R[\{x\}]$, and similarly for other cases. A subset $U \subseteq X$ is called an *upset* of (X, R) if $R^{\leq 1}[U] = U$; *downsets* are defined dually.

Definition 2.41. A *Stone space* is a compact Hausdorff totally disconnected topological space.

Definition 2.42. A *modal space* $\mathfrak{X}$ is a pair (X, R) where X is a Stone space and $R \subseteq X \times X$ is a binary relation on X satisfying:

1. $R[x]$ is closed for each $x \in X$,
2. For any clopen set $U \subseteq X$, the set $R^{-1}[U]$ is also clopen.

Definition 2.43. Let $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, Q)$ be modal spaces. A continuous map $f : X \to Y$ is called a *p-morphism* if the following conditions hold:

(forth) For any $x, x' \in X$, xRx' implies $f(x)Qf(x')$,

(back) For any $x \in X$, if $f(x)Qy'$ for some $y' \in Y$, then there is an $x' \in X$ such that xRx' and $f(x') = y'$.

Definition 2.44. Let $\mathfrak{X} = (X, R)$ be a modal space. A point $x \in X$ is called a *topo-root* of $\mathfrak{X}$ if the closure of $R^{<\omega}[x]$ is X. The modal space $\mathfrak{X}$ is called *topo-rooted* if the set of topo-roots of $\mathfrak{X}$ has a non-empty interior.

Theorem 2.45. *The category of modal algebras and homomorphisms is dually equivalent to the category of modal spaces and p-morphisms.*

We sketch the construction of the dual equivalence functors, which is built on top of *Stone duality* (see e.g. [BdV01, Section 5.4]). Given a modal algebra $\mathfrak{A}$, the dual modal space of $\mathfrak{A}$ is $\mathfrak{A}_* = (A_*, R)$ where

1. A_* is the dual Stone space of A: A_* as a set is the set of all ultrafilters on A, and the topology is generated by the base $\{\beta(a) : a \in A\}$, where the function $\beta : A \to \mathcal{P}(A_*)$ is such that $\beta(a) = \{x \in A_* : a \in x\}$,
2. R is defined by
$$xRy \text{ iff for any } a \in A,\ a \in y \text{ implies } \Diamond a \in x.$$

Given a homomorphism $h : \mathfrak{A} \to \mathfrak{B}$, the dual p-morphism of h is $h_* : \mathfrak{B}_* \to \mathfrak{A}_*; x \to h^{-1}[x]$. Conversely, given a modal space $\mathfrak{X} = (X, R)$, the dual modal algebra of $\mathfrak{X}$ is $\mathfrak{X}^* = (X^*, \diamond)$ where

1. X^* is the Boolean algebra of clopen subsets of X,
2. $\diamond a = R{-}1[a]$ for $a \in A$.

Given a p-morphism $f : \mathfrak{X} \to \mathfrak{Y}$, the dual homomorphism is $f^* : \mathfrak{Y}^* \to \mathfrak{X}^*; a \to f^{-1}[a]$.

The following table summarizes the correspondence between notions for modal algebras and modal spaces under this dual equivalence.

modal algebras	**modal spaces**
homomorphic images	closed upsets
subalgebras	p-morphic images
subdirectly irreducible	topo-rooted

Kripke frames and general frames

Another commonly used semantics of modal logic is *Kripke frames*, which resemble modal spaces but lack the topological structure.

Definition 2.46. A *Kripke frame* is a pair (X, R) of a set X and a binary relation $R \subseteq X \times X$.

Since finite Stone spaces are discrete, finite Kripke frames coincide with finite modal spaces.

A *subframe* of a Kripke frame is a model-theoretic substructure. A *generated subframe* of a Kripke frame is an upset. *P-morphisms* between Kripke frames are defined the same as for modal spaces, except that the continuity is dropped.

Modal logics are not complete with respect to classes of Kripke frames in general. One can overcome this by adding extra structures on Kripke frames.

Definition 2.47. A *general frame* or *frame* is a tuple $(X, R, \mathcal{A})$ where (X, R) is a Kripke frame and $\mathcal{A}$ is a non-empty family of subsets of X that is closed under Boolean operations $(\cap, \cup, -^c)$ and R^{-1}. Elements of $\mathcal{A}$ are called *admissible sets*.

Definition 2.48. A general frame $(X, R, \mathcal{A})$ is called *descriptive* if it is:

differenciated: for any distinct $x, y \in X$, there is some $A \in \mathcal{A}$ such that $x \in A$ and $y \notin A$,

tight: for any $x, y \in X$ such that $x \not{R} y$, there is some $A \in \mathcal{A}$ such that $y \in A$ and $x \notin R^{-1}[A]$,

compact: for any $\mathcal{A}' \subseteq \mathcal{A}$, if any finite intersection of elements of $\mathcal{A}'$ is non-empty, then $\bigcap \mathcal{A}'$ is non-empty.

Descriptive frames are equivalent to modal spaces in the following sense. For a descriptive frame $(X, R, \mathcal{A})$, if we equip X with the topology generated by the base $\mathcal{A}$, then (X, R) becomes a modal space. For a modal space (X, R), letting $\mathcal{A}$ be the set of all clopen subsets of X, we obtain a descriptive frame $(X, R, \mathcal{A})$. Moreover, these two operations are the inverse of each other.

2.4 Modal logics

In this section, we recall basic notions and results for modal logics. We refer to [BdV01] and [CZ97] for details.

Syntax and semantics

We recall the syntax of modal logics. *Formulas* are defined by the following syntax, where Prop is a countable set of propositional variables. Let Fml be the set of all formulas.

$$\varphi ::= p \mid \bot \mid \varphi \wedge \psi \mid \varphi \to \psi \mid \Box\varphi, \quad p \in \mathsf{Prop}$$

We will use the following abbreviation: $\neg\varphi$ for $\varphi \to \bot$, $\top$ for $\neg\bot$, $\varphi \vee \psi$ for $\neg(\neg\varphi \wedge \neg\psi)$, $\varphi \leftrightarrow \psi$ for $(\varphi \to \psi) \wedge (\psi \to \varphi)$, and $\Diamond\varphi$ for $\neg\Box\neg\varphi$.

A *substitution* is a function $\sigma : \mathsf{Prop} \to \mathsf{Fml}$. For a substitution σ and a formula φ, the *substitution instance* of φ under σ is the resulting formula by simultaneously replacing each occurrence of each propositional variable p by $\sigma(p)$.

A *normal modal logic* L is a set of formulas that contains all the propositional tautologies and the K axiom $\Box(p \to q) \to \Box p \to \Box q$, and is closed under:

modus ponens: from formulas φ and $\varphi \to \psi$, to deduce ψ,

necessitation: from a formula φ, to deduce $\Box\varphi$

uniform substitution: froma a formula φ, to deduce a substitution instance of φ.

We will work solely with normal modal logics in this thesis, so we simply call them *logics*.

A logic L is called a *normal extension* or simply an *extension* of another logic L_0 if $L_0 \subseteq L$. For any logic L_0, the extensions of L_0 form a complete lattice, denoted $\mathsf{NExt}L_0$, where the order is the subset relation $\subseteq$. For a set Σ of formulas, let $\mathsf{Log}(\Sigma)$ be the least logic containing Σ. The meet in $\mathsf{NExt}L_0$ is $\cap$ and the join of two logics L and L' is given by $L + L' := \mathsf{Log}(L \cup L')$. For a logic L, a formula φ, and a set Σ of formulas, let $L + \varphi = \mathsf{Log}(L \cup \{\varphi\})$ and $L + \Sigma = \mathsf{Log}(L \cup \Sigma)$. When the base logic L_0 is fixed and $L = L_0 + \Sigma$, we call elements of Σ *axioms* of L and say that L is *axiomatized* by Σ (over L_0).

We write K for the least normal modal logic, $\mathsf{K4}$ for the transitive modal logic $\mathsf{K} + \Box p \to \Box\Box p$, $\mathsf{wK4}$ for the weak transitive modal logic $\mathsf{K} + \Box p \wedge p \to \Box\Box p$, and $\mathsf{S4}$ for the logic $\mathsf{K4} + \Box p \to p$.

We recall the semantics of modal logic with respect to modal algebras, modal spaces, Kripke frames, and general frames.

Definition 2.49. Let $\mathfrak{A}$ be a modal algebra. A *valuation* on $\mathfrak{A}$ is a function $V : \mathsf{Prop} \to A$. This naturally generalizes to $V : \mathsf{Fml} \to A$. We say that $\mathfrak{A}$ *satisfies* a formula φ under the valuation V, written $\mathfrak{A}, V \models \varphi$, if $V(\varphi) = 1$. We say that $\mathfrak{A}$ *validates* φ, written $\mathfrak{A} \models \varphi$, if $\mathfrak{A}, V \models \varphi$ for any valuation V on $\mathfrak{A}$.

Definition 2.50. A logic L is said to be *complete* with respect to a class $\mathcal{K}$ of modal algebras if for any formula φ,

$$\varphi \in L \text{ iff } \mathcal{K} \models \varphi \text{ (i.e., } \mathfrak{A} \models \varphi \text{ for all } \mathfrak{A} \in \mathcal{K}).$$

For a modal algebra $\mathfrak{A}$, let $\mathsf{Log}(\mathfrak{A}) = \{\varphi : \mathfrak{A} \models \varphi\}$. Similarly, for a class $\mathcal{K}$ of modal algebras, let $\mathsf{Log}(\mathcal{K}) = \{\varphi : \mathcal{K} \models \varphi\}$. For a logic L, let $\mathcal{V}(L) = \{\mathfrak{A} : \mathfrak{A} \models L\}$. The following is known as the *completeness theorem* for algebraic semantics.

Theorem 2.51. *Every logic L is complete with respect to the class $\mathcal{V}(L)$. Moreover, the operations $\mathsf{Log}(-)$ and $\mathcal{V}(-)$ are dual isomorphisms between the lattice of varieties of modal algebras and the lattice of modal logics that are the inverse of each other.*

In addition, there is a one-to-one correspondence between logics and equational theories for modal algebras.

The semantics with respect to modal spaces can be obtained by dualizing the one with respect to modal algebras, but we give an explicit account.

Definition 2.52. Let $\mathfrak{X} = (X, R)$ be a modal space.

1. A *valuation* on $\mathfrak{X}$ is a function $V : \mathsf{Prop} \to \mathcal{P}(X)$ such that every value of V is clopen in $\mathfrak{X}$. This generalizes to $V : \mathsf{Fml} \to A$ as follows: Boolean connectives are interpreted by Boolean operations on $\mathcal{P}(X)$, and $V(\Diamond\varphi) = R^{-1}[V(\varphi)]$.

2. We say that a point $x \in X$ *satisfies* a formula φ under a valuation V, written $\mathfrak{X}, V, x \models \varphi$, if $x \in V(\varphi)$.

3. We say that φ is *universally true* in $\mathfrak{X}$ under a valuation V, written $\mathfrak{X}, V \models \varphi$, if $\mathfrak{X}, V, x \models \varphi$ for all $x \in X$.

4. We say that $\mathfrak{X}$ *validates* φ, written $\mathfrak{X} \models \varphi$, if $\mathfrak{X}, V \models \varphi$ for any valuation V on $\mathfrak{X}$.

The semantics with respect to Kripke frames and general frames are similar except that: for Kripke frames, a valuation is a function $V : \mathsf{Prop} \to \mathcal{P}(X)$; for general frames, a valuation is a function $V : \mathsf{Prop} \to \mathcal{A}$. The operation $\mathsf{Log}(-)$ is defined similarly for modal spaces, Kripke frames, general frames, and classes of them. By duality, modal logics are also complete with respect to modal spaces and descriptive frames. However, there are modal logics that are not complete with any class of Kripke frames. The following is a useful completeness result (see, e.g., [CZ97, Corollary 3.19]).

Theorem 2.53. K *is sound and complete with respect to the class of finite rooted irreflexive intransitive trees (viewed as finite Kripke frames).*

Logical properties

We recall various logical properties and special classes of logics. We state them for modal logics, while they often apply to other types of logics as well.

Definition 2.54. Let L_0 and L be logics.

1. L is *Kripke complete* iff it is complete with respect to a class of Kripke frames.

2. L has the *finite model property* (*fmp* for short) iff it is complete with respect to a class of finite modal algebras iff it is complete with respect to a class of finite modal spaces (i.e., finite Kripke frames).

3. L is *tabular* iff $L = \mathsf{Log}(\mathfrak{A})$ for some finite modal algebra iff $L = \mathsf{Log}(\mathfrak{X})$ for some finite modal space (i.e., finite Kripke frames).

4. L is *finitely axiomatizable* over L_0 iff $L = L_0 + \Sigma$ for a finite set Σ.

5. L is *decidable* iff there is an algorithm that, given a formula φ, to decide if $\varphi \in L$.

6. L is a *splitting logic* in $\mathsf{NExt}L_0$ or an L_0-*splitting* iff it is a lattice-theoretic splitting of the lattice $\mathsf{NExt}L_0$.

7. L is a *union-splitting logic* in $\mathsf{NExt}L_0$ or an L_0*-union-splitting* iff it is the join of a set of splitting logics in $\mathsf{NExt}L_0$.

Some logical properties can be rephrased in terms of varieties of modal algebras. For example, a logic L has the finite modal property iff the variety $\mathcal{V}(L)$ is generated by a class of finite algebras.

Harrop's theorem [Har58] (see also [CZ97, Theorem 16.13]) is one of the most well-known criteria for decidability.

Theorem 2.55. *Every finitely axiomatizable logic with the fmp is decidable.*

The following notion was introduced by Fine [Fin74a] to measure the extent to which a logic is Kripke incomplete.

Definition 2.56. Let L_0 be a logic. For a logic $L \in \mathsf{NExt}L_0$, the *degree of Kripke incompleteness* of L in $\mathsf{NExt}L_0$ is the cardinal

$$|\{L' \in \mathsf{NExt}L_0 : \mathsf{KF}(L') = \mathsf{KF}(L)\}|,$$

where $\mathsf{KF}(L)$ is the class of Kripke frames validating L.

Blok [Blo78] showed the connection between degrees of Kripke incompleteness and union-splittings, and fully characterized the degrees of Kripke incompleteness in NExtK: surprisingly, only 1 and $2^{\aleph_0}$ are realized as the degree of Kripke incompleteness of some logic L in NExtK (see also [CZ97, Section 10.5]).

Theorem 2.57. *Let L_0 be a logic with the fmp. For any union-splitting L in $\mathsf{NExt}L_0$, if L is Kripke complete, then L has degree of Kripke incompleteness 1 in $\mathsf{NExt}L_0$.*

Theorem 2.58 (Blok's dichotomy theorem)**.** *Let L be a modal logic. If L is a K-union-sptting, then it has degree of Kripke incompleteness 1 in NExtK; otherwise it has degree of Kripke incompleteness $2^{\aleph_0}$ in NExtK*

2.5 Modal multi-conclusion rules

In this section, we recall the notions and properties of *modal multi-conclusion rules.* We refer to [Kra07] and [Jeř09] for details.

Definition 2.59. A *modal multi-conclusion rule* ρ is an expression of the form

$$\frac{\Gamma}{\Delta}$$

or Γ/Δ where Γ and Δ are finite sets of formulas. If Δ is a singleton, ρ is called *single-conclusion.* If $\Gamma = \emptyset$, ρ is called *assumption-free.*

We will work solely with modal multi-conclusion rules in this thesis, so we simply call them *rules.* A single-conclusion assumption-free rule $/\varphi$ can be identified with the formula φ.

Definition 2.60. A *normal modal multi-conclusion consequence relation* or *normal modal multi-conclusion rule system* is a set $\mathcal{S}$ of rules satisfying:

1. $\varphi/\varphi \in \mathcal{S}$,

2. $\varphi, \varphi \to \psi / \psi \in \mathcal{S}$,

3. $\varphi / \Box\varphi \in \mathcal{S}$,

4. $/\varphi \in \mathcal{S}$ for all $\varphi \in \mathsf{K}$,

5. $\Gamma/\Delta \in \mathcal{S}$ implies $\Gamma, \Gamma'/\Delta, \Delta' \in \mathcal{S}$ (*weakening*),

6. $\Gamma/\Delta, \varphi \in \mathcal{S}$ and $\Gamma, \varphi/\Delta \in \mathcal{S}$ implies $\Gamma/\Delta \in \mathcal{S}$ (*cut*),

7. $\Gamma/\Delta \in \mathcal{S}$ implies $\sigma(\Gamma)/\sigma(\Delta) \in \mathcal{S}$ for any substitution σ (*substitution*).

We simply call normal modal multi-conclusion consequence relations *rule systems*. Similar to logics, for any rule system $\mathcal{S}_0$, the extensions of $\mathcal{S}_0$ form a complete lattice, denoted $\mathsf{NExt}\mathcal{S}_0$. Splitting rule systems and union-splitting rule systems are defined analogously to logics (Definition 2.54). For a set $\mathcal{R}$ of rules, let $\mathcal{S}_0 + \mathcal{R}$ be the least rule system containing $\mathcal{S}_0 \cup \mathcal{R}$. When $\mathcal{S} = \mathcal{S}_0 + \mathcal{R}$, we call elements of $\mathcal{R}$ *axioms* of $\mathcal{S}$ and say that $\mathcal{S}$ is *axiomatized* by $\mathcal{R}$ over $\mathcal{S}_0$. For a logic L, let $\mathcal{S}_L$ be the least rule system containing L.

Definition 2.61. A modal algebra $\mathfrak{A}$ *validates* a rule Γ/Δ, written $\mathfrak{A} \models \Gamma/\Delta$, if for any valuation V on $\mathfrak{A}$, $V(\gamma) = 1$ for all $\gamma \in \Gamma$ implies $V(\delta) = 1$ for some $\delta \in \Delta$.

The semantics with respect to modal spaces can be obtained by duality, but we provide an explicit account.

Definition 2.62. A modal space $\mathfrak{X}$ *validates* a rule Γ/Δ, written $\mathfrak{X} \models \Gamma/\Delta$, if for any valuation V on $\mathfrak{X}$, $V(\gamma) = X$ for all $\gamma \in \Gamma$ implies $V(\delta) = X$ for some $\delta \in \Delta$.

Since each formula corresponds to an identity for modal algebras, each rule corresponds to a universal sentence. Thus, for a rule system $\mathcal{S}$, $\mathcal{U}(\mathcal{S}) := \{\mathfrak{A} : \mathfrak{A} \models \mathcal{S}\}$ is a universal class. Conversely, for a universal class $\mathcal{U}$, $\mathcal{S}(\mathcal{U}) := \{\rho : \mathcal{U} \models \rho\}$ is a rule system. Moreover, the operations $\mathcal{U}$ and $\mathcal{S}$ are dual isomorphisms between the $\mathsf{NExt}\mathcal{S}_\mathsf{K}$ and the lattice of all universal classes.

For a logic L, let $\Sigma(L)$ be the least rule system containing $\{/\varphi : \varphi \in L\}$, that is, $\Sigma(L) = \mathcal{S}_\mathsf{K} + \{/\varphi : \varphi \in L\}$. Note that $\Sigma(L) = \mathcal{S}_L$. For a rule system $\mathcal{S}$, let $\Lambda(\mathcal{S}) = \{\varphi : /\varphi \in \mathcal{S}\}$. Then, $\Sigma : \mathsf{NExtK} \to \mathsf{NExt}\mathcal{S}\mathsf{K}$ and $\Lambda : \mathsf{NExt}\mathcal{S}_\mathsf{K} \to \mathsf{NExtK}$ are order-preserving map. Since, dually, we have $\mathcal{V}(\mathcal{U}(\mathcal{V})) = \mathcal{V}$ for a variety $\mathcal{V}$ and $\mathcal{U} \subseteq \mathcal{U}(\mathcal{V}(\mathcal{U}))$ for a universal class $\mathcal{U}$, it follows that $\Lambda(\Sigma(L)) = L$ for a logic L and $\Sigma(\Lambda(\mathcal{S})) \subseteq \mathcal{S}$ for a rule system $\mathcal{S}$.

The finite model property generalizes straightforwardly to rule systems. A rule system $\mathcal{S}$ has the *finite model property* (*fmp*) if for any rule $\rho \notin \mathcal{S}$, there is a finite modal algebra $\mathfrak{A}$ such that $\mathfrak{A} \models \mathcal{S}$ and $\mathfrak{A} \not\models \rho$. This is equivalent to the universal class $\mathcal{U}(\mathcal{S})$ being generated by a class of finite algebras. The fmp is preserved by operations Σ and Λ.

Proposition 2.63. *If a rule system $\mathcal{S}$ has the fmp, then the logic $\Lambda(\mathcal{S})$ has the fmp.*

Proof. If a rule system $\mathcal{S}$ has the fmp, then $\mathcal{U}(\mathcal{S}) = \mathcal{U}(\mathcal{K})$ for some class $\mathcal{K}$ of finite algebras, so $\mathcal{V}(\mathcal{U}(\mathcal{S})) = \mathcal{V}(\mathcal{U}(\mathsf{K})) = \mathcal{V}(\mathsf{K})$, which implies that the logic $\Lambda(\mathcal{S})$ has the fmp. $\Box$

2.6 Descriptive set theory

In this section, we recall the basics of descriptive set theory, particularly the notions and results about *Borel sets*, which will be used in Chapter 7. We work in ZFC and do not assume the Continuum Hypothesis. We refer to [Kan08] for details.

Recall that *Cantor space* is 2^ω endowed with the topology generated by the clopen basis $\{B_s : s \in 2^{<\omega}\}$, where $B_s = \{x \in 2^\omega : s \subseteq x\}$. Elements of Cantor space are often called *reals* because they can be identified with standard real numbers. There are two types of measures on the complexity of subsets of Cantor space: *Borel hierarchy* and *arithmetical hierarchy*.

Definition 2.64. The *Borel hierarchy* is defined as follows: for $A \subseteq 2^\omega$ and $1 < \alpha < \omega_1$,

$$\begin{aligned}
&A \in \boldsymbol{\Sigma}^0_1 \text{ iff } A \text{ is open,}\\
&A \in \boldsymbol{\Pi}^0_1 \text{ iff } A \text{ is closed,}\\
&A \in \boldsymbol{\Sigma}^0_\alpha \text{ iff } A \text{ is a countable union of sets in } \bigcup_{0<\beta<\alpha} \boldsymbol{\Pi}^0_\beta,\\
&A \in \boldsymbol{\Pi}^0_\alpha \text{ iff } A \text{ is a countable intersection of sets in } \bigcup_{0<\beta<\alpha} \boldsymbol{\Sigma}^0_\beta,\\
&A \text{ is } \textit{Borel} \text{ iff } A \in \bigcup_{0<\alpha<\omega_1} \boldsymbol{\Sigma}^0_\alpha = \bigcup_{0<\alpha<\omega_1} \boldsymbol{\Pi}^0_\alpha.
\end{aligned}$$

Definition 2.65. The *arithmetical hierarchy* is defined as follows: for $A \subseteq 2^\omega$ and $n > 0$,

$$\begin{aligned}
&A \in \Sigma^0_n \text{ iff } A \text{ is defined by } \exists m_1 \forall m_2 \cdots Q m_n R \text{ for some recursive } R \subseteq \omega^n \times 2^\omega,\\
&A \in \Pi^0_n \text{ iff } A \text{ is defined by } \forall m_1 \exists m_2 \cdots Q m_n R \text{ for some recursive } R \subseteq \omega^n \times 2^\omega.
\end{aligned}$$

For $a \in 2^\omega$, by allowing the use of a as a parameter in R, we obtain the relativized notions: $\Sigma^0_n(a)$ and $\Pi^0_n(a)$.

The following proposition characterizes the relation between the two hierarchies (see, e.g., [Kan08, Proposition 12.6]).

Proposition 2.66. *Let $A \subseteq 2^\omega$ and $n > 0$.*

1. *$A \in \boldsymbol{\Sigma}^0_n$ iff $A \in \Sigma^0_n(a)$ for some $a \in 2^\omega$.*
2. *$A \in \boldsymbol{\Pi}^0_n$ iff $A \in \Pi^0_n(a)$ for some $a \in 2^\omega$*

In short, if a set $A \subseteq 2^\omega$ is $\Sigma^0_n(a)$ or $\Pi^0_n(a)$ for some $a \in 2^\omega$, then A is Borel.

The *perfect set property* dates back to Cantor's effort to establish the Continuum Hypothesis. Cantor [Can84] showed that any perfect set has the cardinality $2^{\aleph_0}$ (see also [Kan08, Proposition 11.3]).

Definition 2.67. A set $A \subseteq 2^\omega$ is *perfect* if it is nonempty, closed, and has no isolated points. The set A has the *perfect set property* if it is either countable or contains a perfect subset.

Proposition 2.68. *Every perfect set of reals has the cardinality $2^{\aleph_0}$.*

It is a classical result in descriptive set theory that Borel sets have the perfect set property (see, e.g., [Kan08, Theorem 12.2]).

Theorem 2.69. *Every Borel set has the perfect set property. Consequently, every Borel set has the cardinality either* $\leq \aleph_0$ *or* $2^{\aleph_0}$.

We will exploit this fact to study the cardinality of sets of equational theories and logics in Chapter 7.

Chapter 3

Stable Canonical Rules and Formulas

One of the most powerful tools in the study of the lattice of modal logics and superintuitionistic logics is *characteristic formulas*. Characteristic formulas, also called *algebra-based formulas* or *frame-based formulas*, refer to variations of formulas that are defined from finite algebras or finite relational structures (e.g., finite Kripke frames) so that their validity has a semantic characterisation. Thus, logics axiomatized by characteristic formulas are often easier to analyse.

The first type of characteristic formulas, *Jankov formulas* or *Jankov-de Jongh formulas*, was introduced and studied by Jankov [Jan63] and independently by de Jongh [de 68] for superintuitionistic logics. Jankov formulas were used to construct continuum many superintuitionistic logics [Jan63]. Their modal logic analogue, *Fine formulas* or *Jankov-Fine formulas*, were introduced by Fine [Fin74b]. A generalization to n-transitive modal logics was constructed by Rautenberg [Rau80]. Further development includes *subframe formulas*, *cofinal subframe formulas*, and *canonical formulas* (see, e.g., [CZ97, Chapter 9]). A remarkable feature of canonical formulas is that they axiomatize all transitive logics [Zak92]. See also [Bez08] for a historical overview and a unified framework to address frame-based formulas.

Recently, Jeřábek [Jeř09] generalized the idea of canonical formulas to inference rules and defined *canonical rules*. They are based on *selective filtration* (see, e.g., [CZ97, Section 5.5]) and axiomatize all rule systems over $\mathsf{K4}$. Bezhanishvili et al. [BBI16], motivated by the filtration method (see, e.g., [BdV01, Section 2.3] and [CZ97, Section 5.3]), defined *stable canonical rules* and showed that all rule systems can be axiomatized by stable canonical rules over $\mathcal{S}_{\mathsf{K}}$. They also introduced *stable canonical formulas* for $\mathsf{K4}$ as an alternative to canonical formulas.

In this chapter, we review the theory of stable canonical rules and formulas, which will be used throughout the thesis. We recall *definable filtrations* from [KSZ20] and generalize the axiomatization result via stable canonical rules by allowing the base rule system to be any one that admits definable filtration. Moreover, using Gabbay's filtration [Gab72], we generalize the stable canonical formulas to pre-transitive logics of the form $\mathsf{K} + \Diamond^{m+1}p \to \Diamond p$. We also define *$m$-stable canonical formulas* and show that they provide an alternative axiomatization result for pre-transitive logics.

3.1 Stable homomorphisms and the closed domain condition

Stable homomorphisms and the closed domain condition for modal algebras and modal spaces are introduced in [BBI16], generalizing that for Heyting algebras and Priestley spaces introduced in [BB17]. Intuitively, a stable homomorphism h does not preserve the modal operator (so it is not a modal algebra homomorphism), but it does satisfy the inequality $\Diamond h(a) \leq h(\Diamond a)$; the closed domain condition indicates for which elements a the equality $\Diamond h(a) = h(\Diamond a)$ should hold. Dually, for modal spaces and continuous maps, a stable map f is order-preserving, or in other words, f satisfies the forth condition in the definition of p-morphisms; the closed domain condition requires f to meet the back condition, but only partially. Stable homomorphisms and the closed domain condition will be used in the semantic characterization of the validity of stable canonical rules and formulas.

The definitions and facts in this section are from [BBI16, Section 3].

Definition 3.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be modal algebras. A Boolean homomorphism $h : A \to B$ is a *stable homomorphism* if $\Diamond h(a) \leq h(\Diamond a)$ for all $a \in A$.

Definition 3.2. Let $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, Q)$ be modal spaces. A continuous map $f : X \to Y$ is a *stable map* if xRy implies $f(x)Qf(y)$ for all $x, y \in X$.

The two definitions are dual to each other.

Proposition 3.3. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be modal algebras and $h : A \to B$ be a Boolean homomorphism. Then $h : A \to B$ is stable iff $h_* : \mathfrak{B}_* \to \mathfrak{A}_*$ is stable.*

Definition 3.4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be modal algebras and $h : A \to B$ be a stable homomorphism. For $a \in A$, we say that h satisfies the *closed domain condition (CDC) for a* if $h(\Diamond a) = \Diamond h(a)$. For $D \subseteq A$, we say that h satisfies the *closed domain condition (CDC) for D* if h satisfies CDC for all $a \in D$.

Note that for a stable homomorphism h, $h(\Diamond a) = \Diamond h(a)$ is equivalent to $h(\Diamond a) \leq \Diamond h(a)$ as the other inequality is guaranteed by being stable.

Definition 3.5. Let $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, Q)$ be modal spaces and $f : X \to Y$ be a stable map. For a clopen subset $D \subseteq Y$, we say that f satisfies the *closed domain condition (CDC) for D* if

$$Q[f(x)] \cap D \neq \emptyset \Rightarrow f(R[x]) \cap D \neq \emptyset.$$

For a set $\mathcal{D}$ of clopen subsets of Y, we say that $f : X \to Y$ satisfies *the closed domain condition (CDC) for $\mathcal{D}$* if f satisfies CDC for all $D \in \mathcal{D}$.

Again, these definitions are dual to each other.

Proposition 3.6. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be modal algebras and $h : A \to B$ be a stable homomorphism. For any $a \in A$,*

h satisfies CDC for a iff h_* satisfies CDC for $\beta(a)$.

For any $D \subseteq A$,

h satisfies CDC for D iff h_* satisfies CDC for $\beta[D]$.

It follows directly from the definition that a stable homomorphism $h : A \to B$ satisfying CDC for A is a modal algebra homomorphism, and a stable map $f : X \to Y$ satisfying CDC for $\mathcal{P}(Y)$ is a p-morphism.

We will primarily focus on stable embeddings between modal algebras and their dual surjective stable maps between modal spaces. Therefore, we introduce abbreviated notation.

Notation 3.7. We write $h : \mathfrak{A} \hookrightarrow_D \mathfrak{B}$ if h is a stable embedding satisfying CDC for D and $\mathfrak{A} \hookrightarrow_D \mathfrak{B}$ if there is such an h. We write $f : \mathfrak{X} \twoheadrightarrow_{\mathcal{D}} \mathfrak{Y}$ if f is a surjective stable map satisfying CDC for $\mathcal{D}$ and $\mathfrak{X} \twoheadrightarrow_{\mathcal{D}} \mathfrak{Y}$ if there is such an f.

3.2 Definable filtrations

The *filtration* method was first used in its algebraic form by McKinsey [McK41], and the frame-thoeretic approach was later developed by Lemmon and Scott [Lem77]. It has been the most powerful tool to establish the finite modal property for modal logics. We refer to [BdV01, Definition 2.36] and [CZ97, Section 5.3] for the standard filtration for Kripke frames and models, and to [BBI16, Section 4] for an algebraic account.

For our purposes, we will use a slightly generalized version of filtrations, *definable filtration*, introduced in [KSZ20] for Kripke frames. The idea is to encompass filtrations where one extends the subformula-closed set of formulas before taking the quotient. We first define definable filtrations in a frame-theoretic way for modal spaces.

Definition 3.8. Let $\mathfrak{X} = (X, R)$ be a modal space, V be a valuation on $\mathfrak{X}$, Θ be a finite subformula-closed set of formulas, and Θ' be a finite subformula-closed set of formulas containing Θ. A *definable filtration of* $(\mathfrak{X}, V)$ *for* Θ *through* Θ' is a modal space $\mathfrak{X}' = (X', R')$ with a valuation V' such that:

1. $X' = X/{\sim_{\Theta'}}$, where

$$x \sim_{\Theta'} y \text{ iff } (\mathfrak{X}, V, x \models \varphi \iff \mathfrak{X}, V, y \models \varphi \text{ for all } \varphi \in \Theta'),$$

2. $V'(p) = \{[x]_{\Theta'} : x \in V(p)\}$ for $p \in \Theta'$ and $V'(p) = \emptyset$ for $p \notin \Theta'$,

3. xRy implies $[x]_{\Theta'} R' [y]_{\Theta'}$,

4. if $[x]_{\Theta'} R' [y]_{\Theta'}$ then ($y \models \varphi$ implies $x \models \Diamond\varphi$ for $\Diamond\varphi \in \Theta$).

We also call $\mathfrak{X}'$ a *definable filtration of* $\mathfrak{X}$ *for* Θ *through* Θ'.

We drop the subscript Θ' when it is clear from the context.

Remark 3.9. The only difference between a standard filtration and a definable filtration is that in a definable filtration, one can use a finer equivalence relation to obtain the quotient X'. For a finite subformula-closed set Θ of formulas, a definable filtration for Θ through Θ is just a standard filtration through Θ. The idea of using a different set of formulas to define X' appears earlier in [Gab72].

One could go further and generalize item (1) to allow any equivalence relations as long as the quotient X' is finite. The Filtration Lemma (Lemma 3.13) still holds, and the fmp can be proved. This more generalized filtration was used earlier in [She87]. Recently, [KS25] used it to prove the fmp for pre-transitive analogues of $\mathsf{wK4}$. However, we only allow equivalence relations induced by a set of formulas because by doing so, we guarantee that the projection $X \twoheadrightarrow X'$ is a continuous map, which allows an algebraic presentation as in Definition 3.10. This will be crucial for developing the theory of stable canonical rules and formulas in the subsequent sections.

We refer to [vB23] for an overview and discussion of the filtration method. They also define *weak filtration*, which is sufficient to prove the Filtration Lemma (Lemma 3.13).

Now we define definable filtrations for modal algebras.

Definition 3.10. Let $\mathfrak{A} = (A, \Diamond)$ be a modal algebra, V be a valuation on $\mathfrak{X}$, Θ be a finite subformula-closed set of formulas, and Θ' be a finite subformula-closed set of formulas containing Θ. A *definable filtration of* $(\mathfrak{A}, V)$ *for* Θ *through* Θ' is a modal algebra $\mathfrak{A}' = (A', \Diamond')$ with a valuation V' such that:

1. A' is the Boolean subalgebra of A generated by $V[\Theta'] \subseteq A$,

2. $V'(p) = V(p)$ for $p \in \Theta'$ and $V'(p) = 0$ for $p \notin \Theta'$,

3. The inclusion $\mathfrak{A}' \hookrightarrow \mathfrak{A}$ is a stable homomorphism satisfying CDC for D, where

$$D = \{V(\varphi) : \Diamond\varphi \in \Theta\}.$$

We also call $\mathfrak{A}'$ a *definable filtration of* $\mathfrak{A}$ *for* Θ *through* Θ'.

It is straightforward to verify that the two definitions of definable filtrations are dual to each other, generalizing the proof for standard filtrations in [BBI16, Theorem 4.2]. We will mostly use algebraic filtrations throughout the chapter, while one can always reformulate it using the language of modal spaces.

Proposition 3.11. *Let $\mathfrak{A} = (A, \Diamond)$ be a modal algebra with the dual space $\mathfrak{X} = (X, R)$. For a valuation V on $\mathfrak{A}$, a finite subformula-closed set Θ of formulas, and a finite subformula-closed set Θ' of formulas containing Θ, let A' be the Boolean subalgebra of A generated by $V[\Theta'] \subseteq A$ and $D = \{V(\varphi) : \Diamond\varphi \in \Theta\}$. For a modal operator $\Diamond'$ on A', the following two conditions are equivalent:*

1. *The inclusion $(A', \Diamond') \hookrightarrow (A, \Diamond)$ is a stable homomorphism satisfying (CDC) for D,*

2. *Viewing V as a valuation on $\mathfrak{X}$, there is a definable filtration $(\mathfrak{X}', V')$ of $(\mathfrak{X}, V)$ for Θ through Θ' such that R' is the dual of $\Diamond'$.*

Example 3.12 ([BBI16])**.**

1. Recall that the *least filtration* and the *greatest filtration* are standard filtrations defined frame-theoretically by

$$[x]R^l[y] \text{ iff } (x \sim x' \text{ and } y \sim y' \text{ and } x'Ry', \text{ for some } x', y' \in X)$$

and

$$[x]R^g[y] \text{ iff } (y \models \varphi \text{ implies } x \models \Diamond\varphi, \text{ for all } \Diamond\varphi \in \Theta)$$

respectively. The algebraic constructions of them are

$$\Diamond^l a = \bigwedge\{b \in A' : \Diamond a \leq b\}$$

and

$$\Diamond^g a = \bigwedge\{\Diamond b : a \leq b \text{ and } b \in D^\vee\}$$

respectively, where $D^\vee$ is the $(\vee, 0)$-subsemilattice of $\mathfrak{A}'$ generated by D.

2. Recall that the *Lemmon filtration*, also called the transitive filtration (see, e.g., [BdV01, Section 2.3] or [CZ97, Section 5.3]) is a standard filtration defined frame-theoretically by

$$[x]R^L[y] \text{ iff } (y \models \Diamond^{\leq 1}\varphi \text{ implies } x \models \Diamond\varphi, \text{ for all } \Diamond\varphi \in \Theta),$$

where $\Diamond^{\leq 1}a = a \vee \Diamond a$. The algebraic construction of the Lemmon filtration is

$$\Diamond^L a = \bigwedge\{\Diamond b : \Diamond a \leq \Diamond b \text{ and } \Diamond^{\leq 1}a \leq \Diamond^{\leq 1}b \text{ and } b \in D^\vee\}.$$

The importance of the filtration method is illustrated in the following lemma.

Lemma 3.13 (Filtration Lemma). *Let $(\mathfrak{A}', V')$ be a definable filtration of $(\mathfrak{A}, V)$ for Θ through Θ'. Then $V(\varphi) = V'(\varphi)$ for all $\varphi \in \Theta$.*

Proof. By a routine induction on the complexity of φ. Use the fact that $\Theta \subseteq \Theta'$ for the base case, that A' is a Boolean subalgebra of A for the Boolean cases, and that the inclusion is stable and satisfies CDC for D for the $\Diamond$ case. □

Using the Filtration Lemma, we can automatically deduce the fmp for logics and rule systems that "admit definable filtration."

Definition 3.14.

1. A class $\mathcal{C}$ of modal algebras *admits definable filtration* if for any finite subformula-closed set Θ of formulas, there is a finite subformula-closed set Θ' containing Θ such that, for any modal algebra $\mathfrak{A} \in \mathcal{C}$ and any valuation V on $\mathfrak{A}$, there is a definable filtration $(\mathfrak{A}', V')$ of $(\mathfrak{A}, V)$ for Θ through Θ' such that $\mathfrak{A}' \in \mathcal{C}$.
2. A modal logic L *admits definable filtration* if the variety $\mathcal{V}(L)$ admits definable filtration.
3. A rule system $\mathcal{S}$ *admits definable filtration* if the universal class $\mathcal{U}(\mathcal{S})$ admits definable filtration.

Note that a logic L admits definable filtration iff the rule system $\mathcal{S}_L$ admits definable filtration because they correspond to the same class of modal algebras.

Remark 3.15. The notion of admitting filtration has been used in the literature with various meanings and strengths. An explicit definition of *admitting filtration in the weak sense* and *admitting filtration in the strong sense* is provided in [BBI18], where the relation of the two definitions is also discussed. A similar discussion can also be found in [KSZ20]. We define it as Definition 3.14 because this is just enough to prove Theorem 3.24 and Corollary 3.25, though it is stronger than what is needed to prove Proposition 3.16.

Proposition 3.16. *If a logic L admits definable filtration, then it has the fmp. If a rule system $\mathcal{S}$ admits filtration, then it has the fmp.*

Proof. We only show the statement for rule systems, and the case for logics can be obtained similarly. Let $\mathcal{S}$ be a rule system that admits definable filtration. For any rule $\rho = \Gamma/\Delta \notin \mathcal{S}$, there is an $\mathcal{S}$-algebra $\mathfrak{A}$ such that $\mathfrak{A} \not\models \rho$, witnessed by some valuation V on $\mathfrak{A}$. By the assumption that $\mathcal{S}$ admits definable filtration, since $\mathfrak{A} \in \mathcal{U}(\mathcal{S})$, there is a finite subformula-closed set Θ containing $\mathsf{Sub}(\Gamma \cup \Delta)$ and a filtration $(\mathfrak{A}', V')$ of $(\mathfrak{A}, V)$ for $\Gamma \cup \Delta$ through Θ such that $\mathfrak{A}' \in \mathcal{U}(\mathcal{S})$. By Lemma 3.13, $\mathfrak{A}, V \not\models \Gamma/\Delta$ implies $\mathfrak{A}', V' \not\models \Gamma/\Delta$. Thus, $\mathfrak{A}'$ is a finite $\mathcal{S}$-algebra that refutes ρ. Hence, $\mathcal{S}$ has the fmp. □

Many logics are known to admit standard filtration and thus admit definable filtration. For example, K, T, and D admit the least and the greatest filtration. For transitive logics, $\mathsf{K4}$ and $\mathsf{S4}$ admit the Lemmon filtration.

In the rest of this section, we give an algebraic account of the fact that a certain type of pre-transitive logics admits definable filtration. Recall that pre-transitive logics $\mathsf{K4}^{\mathsf{m}}_{\mathsf{n}}$ are logics axomatized by $\Diamond^m p \to \Diamond^n p$ (or equivalently, $\Box^n p \to \Box^m p$) over K. The logic $\mathsf{K4}^{\mathsf{m}}_{\mathsf{n}}$ defines the condition

$$\forall x \forall y\, (xR^m y \to xR^n y),$$

where R^k is the k-time composition of R, for modal spaces. Note that $\mathsf{K4}_1^2$ is the transitive logic $\mathsf{K4}$.

We present a definable filtration construction for $\mathsf{K4}_1^{\mathsf{m+1}}$ ($m \geq 1$), meaning that when applied to a $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra, the filtrated algebra is also a $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra. This filtration will be used to develop the theory of stable canonical formulas for pre-transitive logics $\mathsf{K4}_1^{\mathsf{m+1}}$ in Section 3.4. The construction is the algebraic dual of the frame-theoretic filtration presented in the proof of [Gab72, Theorem 8].

Lemma 3.17. *Let $\mathfrak{A} = (A, \Diamond)$ be a $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra, V be a valuation on $\mathfrak{A}$, and Θ be a finite subformula-closed set of formulas. Let $\Theta' = \mathsf{Sub}(\Theta \cup \{\Diamond^m\varphi : \varphi \in \Theta\})$ and A' be the Boolean subalgebra of A generated by $V[\Theta']$. Define the modal oparators $\Diamond_0$ and $\Diamond_1$ on A' by*

$$\Diamond_0 a = \bigwedge\{b \in A' : \Diamond a \leq b\} \text{ and } \Diamond_1 a = \bigvee\{\Diamond_0^{km+1} a : k \in \omega\}.$$

Then, $\mathfrak{A}' = (A', \Diamond_1)$ is a definable filtration of $\mathfrak{A}$ for Θ through Θ' and $\mathfrak{A}' \models \mathsf{K4}_1^{\mathsf{m+1}}$.

Proof. Note that Θ' is a finite subformula-closed set, so A' is finite since Boolean algebras are locally finite. For each $a \in A'$, since A' is finite, there exists $k_a \in \omega$ such that $\{\Diamond_0^{km+1}a : k \leq k_a\} = \{\Diamond_0^{km+1}a : k \in \omega\}$. Let $K = \max\{k_a : a \in A'\}$. Then, $\Diamond_1 a = \bigvee\{\Diamond_0^{km+1}a : k \leq K\}$ for any $a \in A'$, so $\Diamond_1$ is always a finite join and well-defined. In fact, $\Diamond_1 a = \bigvee\{\Diamond_0^{km+1}a : k \leq K'\}$ for any $K' \geq K$.

Let $D' = \{V(\varphi) : \Diamond\varphi \in \Theta'\}$. We know from Example 3.12 that $(A', \Diamond_0)$ is the least filtration of $\mathfrak{A}$ through Θ', so $i : (A', \Diamond_0) \hookrightarrow_{D'} (A, \Diamond)$, where i is the inclusion map. Since $\Diamond_0 0 = 0$, we have $\Diamond_1 0 = 0$. Since A' is closed under finite joins and $\Diamond_0$ preserves them, we have

$$\begin{aligned}
\Diamond_1 a \vee \Diamond_1 b &= \bigvee\{\Diamond_0^{km+1}a : k \leq K\} \vee \bigvee\{\Diamond_0^{km+1}b : k \leq K\} \\
&= \bigvee\{\Diamond_0^{km+1}a \vee \Diamond_0^{km+1}b : k \leq K\} \\
&= \bigvee\{\Diamond_0^{km+1}(a \vee b) : k \leq K\} \\
&= \Diamond_1(a \vee b).
\end{aligned}$$

So, $\Diamond_1$ preserves 0 and $\vee$, hence $(A', \Diamond_1)$ is a modal algebra. Next, we show by induction that $\Diamond_1^l a = \bigvee\{\Diamond_0^{km+l}a : k \leq K\}$ for $l \geq 1$. This holds for $l = 1$ by the definition of $\Diamond_1$. Assuming that it holds for l, we have

$$\begin{aligned}
\Diamond_1^{l+1} a &= \Diamond_1 \Diamond_1^l a \\
&= \bigvee\{\Diamond_0^{km+1}\Diamond_1^l a : k \leq K\} \\
&= \bigvee\{\Diamond_0^{km+1}\bigvee\{\Diamond_0^{k'm+l}a : k' \leq K\} : k \leq K\} \\
&= \bigvee\{\bigvee\{\Diamond_0^{km+1}\Diamond_0^{k'm+l}a : k' \leq K\} : k \leq K\} \\
&= \bigvee\{\Diamond_0^{(k+k')m+l+1}a : k, k' \leq K\} \\
&= \bigvee\{\Diamond_0^{km+l+1}a : k \leq 2K\} \\
&= \bigvee\{\Diamond_0^{km+l+1}a : k \leq K\},
\end{aligned}$$

showing that the statement also holds for $l + 1$. So, $\Diamond_1^l a = \bigvee\{\Diamond_0^{km+l}a : k \leq K\}$ holds for $l \geq 1$, and we obtain that for any $a \in A'$,

$$\begin{aligned}\Diamond_1^{m+1} a &= \bigvee\{\Diamond_0^{km+m+1} a : k \leq K\} \\ &= \bigvee\{\Diamond_0^{(k+1)m+1} a : k \leq K\} \\ &\leq \bigvee\{\Diamond_0^{km+1} a : k \leq K+1\} \\ &= \bigvee\{\Diamond_0^{km+1} a : k \leq K\} \\ &= \Diamond_1 a.\end{aligned}$$

Thus, $(A', \Diamond_1)$ is a $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra.

Let $D = \{V(\varphi) : \Diamond\varphi \in \Theta\}$. It remains to show that $i' : (A', \Diamond_1) \hookrightarrow_D (A, \Diamond)$, where $i' = i$ as a map. Note that i' is a Boolean embedding. Since i is stable and $\Diamond_0 a \leq \Diamond_1 a$ by definition, we have $\Diamond i'(a) = \Diamond i(a) \leq i(\Diamond a) = i'(\Diamond_0 a) \leq i'(\Diamond_1 a)$. So, i' is stable.

Let $d \in D$. Then, $d = V(\varphi)$ for some $\Diamond\varphi \in \Theta$, so $\Diamond^{m+1}\varphi, \dots, \Diamond\varphi \in \Theta'$ and $\Diamond^m d, \dots, d \in D'$. Moreover, since $\Diamond_0$ is the least filtration, we have $\Diamond^m d = \Diamond_0^m d, \dots, \Diamond d = \Diamond_0 d$ (see, e.g., [BBI16, Lemma 4.5]). Since i satisfies CDC for D' and $d \in D'$, we have $i'(\Diamond_0 d) \leq \Diamond i'(d)$. Assume that $i'(\Diamond_0^{m'} d) \leq \Diamond^{m'} i'(d)$ for some $1 \leq m' \leq m$. Then $\Diamond i'(\Diamond_0^{m'} d) \leq \Diamond^{m'+1} i'(d)$. Again since i satisfies CDC for D' and $\Diamond_0^{m'} d = \Diamond^{m'} d \in D'$, we have $i'(\Diamond_0^{m'+1} d) \leq \Diamond i'(\Diamond_0^{m'} d)$, thus $i'(\Diamond_0^{m'+1} d) \leq \Diamond^{m'+1} i'(d)$. Inductively, we obtain $i'(\Diamond_0^{m+1} d) \leq \Diamond^{m+1} i'(d)$. Morover, since $(A, \Diamond)$ is a $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra and i is stable, we have $\Diamond^{m+1} i'(d) \leq \Diamond i'(d) \leq i'(\Diamond_0 d)$. So, $i'(\Diamond_0^{m+1} d) \leq i'(\Diamond_0 d)$, and $\Diamond_0^{m+1} d \leq \Diamond_0 d$ since i' is a Boolean embedding. Since $\Diamond_0$ is monotone, we inductively obtain $\Diamond_0^{km+1} d \leq \Diamond_0 d$ for all $k \geq 1$. So,

$$i'(\Diamond_1 d) = i'(\bigvee\{\Diamond_0^{km+1} d : k \leq K\}) \leq i'(\Diamond_0 d) \leq \Diamond i'(d).$$

Thus, i' satisfies CDC for D. Hence, $i' : (A', \Diamond_1) \hookrightarrow_D (A, \Diamond)$ and $(A', \Diamond_1)$ is a definable filtration of $(A, \Diamond)$ for Θ through Θ'. □

Theorem 3.18. *For any $m \geq 1$, the logic $\mathsf{K4}_1^{\mathsf{m+1}}$ admits definable filtration.*

Proof. This follows immediately from Lemma 3.17. Note that Θ' defined in Lemma 3.17 does not depend on $\mathfrak{A}$. □

Corollary 3.19 ([Gab72])**.** *For any $m \geq 1$, the logic $\mathsf{K4}_1^{\mathsf{m+1}}$ has the fmp.*

Proof. This follows from Theorem 3.18 and Proposition 3.16. □

3.3 Stable canonical rules

Stable canonical rules are introduced in [BBI16] as an alternative to canonical rules, the theory of which is developed in [Jeř09]. Both of them are generalizations of characteristic formulas to multi-conclusion rules. They are defined from finite modal algebras or finite modal spaces, and their validity has purely semantic characterizations. While canonical rules, generalizing Zakharyaschev's canonical formulas (see, e.g., [CZ97, Chapter 9]), use selective filtration (see, e.g., [CZ97, Section 5.5]), stable canonical rules use the standard filtration. A special feature of stable canonical rules is that, contrary to canonical rules, any rule can be axiomatized by stable canonical rules over the least normal modal rule system $\mathcal{S}_{\mathsf{K}}$ [BBI16]. We see in this section that the key component for the axiomatization result via stable canonical rules is definable filtration, and we generalize the result to any rule system that admits definable filtration.

The basic idea of stable canonical rules, like characteristic formulas as well as canonical rules, is very similar to *diagrams*, widely used in model theory: to encode the structure of finite algebras or finite spaces (frames), but only partially. That is why subalgebras and homomorphisms (or p-morphisms and clopen upsets for modal spaces) are not enough to capture the validity of stable canonical rules, and we have to consider stable homomorphisms and the closed domain condition.

Definition 3.20. Let $\mathfrak{A}$ be a finite modal algebra and $D \subseteq A$. The *stable canonical rule* $\rho(\mathfrak{A}, D)$ associated to $\mathfrak{A}$ and D is the rule Γ/Δ, where:

$$\begin{aligned}\Gamma = &\{p_{a \vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \cup \\ &\{\neg p_a \leftrightarrow \neg p_a : a \in A\} \cup \\ &\{\Diamond p_a \to p_{\Diamond a} : a \in A\} \cup \\ &\{p_{\Diamond a} \to \Diamond p_a : a \in D\},\end{aligned}$$

and

$$\Delta = \{p_a : a \in A, a \neq 1\}.$$

Stable canonical rules can also be defined directly via finite modal spaces (i.e., finite Kripke frames).

Definition 3.21. Let $\mathfrak{X} = (X, R)$ be a finite modal space and $\mathcal{D} \subseteq \mathcal{P}(X)$. Define the *stable canonical rule* $\rho(\mathfrak{X}, \mathcal{D})$ associated to $\mathfrak{X}$ and $\mathcal{D}$ as the rule Γ/Δ, where:

$$\begin{aligned}\Gamma = &\{\bigvee\{p_x : x \in X\}\} \cup \\ &\{p_x \to \neg p_y : x, y \in X, x \neq y\} \cup \\ &\{p_x \to \neg\Diamond p_y : x, y \in X, x \not R y\} \cup \\ &\{p_x \to \bigvee\{\Diamond p_y : y \in D\} : x \in X, D \in \mathcal{D}, x \in R^{-1}[D]\},\end{aligned}$$

and

$$\Delta = \{\neg p_x : x \in X\}.$$

We will work primarily with modal algebras, but one can obtain the dual results by either using the duality or working directly with Definition 3.21.

Below is the semantic characterization of the validity of stable canonical rules, which we will use freely throughout the thesis.

Theorem 3.22 ([BBI16])**.** *Let $\mathfrak{A}$ be a finite modal algebra, $D \subseteq A$, and $\mathfrak{B}$ be a modal algebra. Then*

$$\mathfrak{B} \not\models \rho(\mathfrak{A}, D) \textit{ iff } \mathfrak{A} \hookrightarrow_D \mathfrak{B}.$$

For convenience, we provide the dual presentation of this fact. For a finite modal space $\mathcal{F}$, $\mathcal{D} \subseteq \mathcal{P}(F)$, and a modal space $\mathfrak{X}$, we have

$$\mathfrak{X} \not\models \rho(\mathcal{F}, \mathcal{D}) \text{ iff } \mathfrak{X} \twoheadrightarrow_{\mathcal{D}} \mathcal{F}.$$

Before proving the axiomatization result, we remark that one may restrict oneself to stable canonical rules $\rho(\mathfrak{A}, D)$ where D is a $(\vee, 0)$-subsemilattice of A. We do not need this fact to show the following results, but it will simplify things, for example, when we count stable canonical rules in Example 6.37.

Proposition 3.23. *For any finite modal algebra $\mathfrak{A}$ and any $D \subseteq A$, there is a $(\vee, 0)$-subsemilattice D' of A such that, for any modal algebra $\mathfrak{B}$,*

$$\mathfrak{B} \models \rho(\mathfrak{A}, D) \textit{ iff } \mathfrak{B} \models \rho(\mathfrak{A}, D').$$

Proof. Let $\mathfrak{A}$ be a finite modal algebra and $D \subseteq A$. Let D' be the $(\vee, 0)$-subsemilattice of $\mathfrak{A}$ generated by D. For any modal algebra $\mathfrak{B}$, if $\mathfrak{B} \not\models \rho(\mathfrak{A}, D')$, then $\mathfrak{A} \hookrightarrow_{D'} \mathfrak{B}$, which implies $\mathfrak{A} \hookrightarrow_D \mathfrak{B}$ since $D \subseteq D'$, hence $\mathfrak{B} \not\models \rho(\mathfrak{A}, D)$. Conversely, assume that $\mathfrak{B} \not\models \rho(\mathfrak{A}, D)$, then there is some stable homomorphism $h : \mathfrak{A} \hookrightarrow_D \mathfrak{B}$. For any $d' \in D'$, since D' is a $(\vee, 0)$-subsemilattice of A generated by D, $d' = d_0 \vee \cdots \vee d_n$ such that each $d_i = 0$ or $d_i \in D$. For $0 \in \mathfrak{A}$, we have $h(\Diamond 0) = h(0) = 0 = \Diamond 0 = \Diamond h(0)$. For $d \in D$, we have $h(\Diamond d) = \Diamond h(d)$ since h satisfies CDC for D. Thus,

$$\begin{aligned} h(\Diamond d') &= h(\Diamond(d_0 \vee \cdots \vee d_n)) \\ &= h((\Diamond d_0) \vee \cdots \vee (\Diamond d_n)) \\ &= h(\Diamond d_0) \vee \cdots \vee h(\Diamond d_n) \\ &= \Diamond h(d_0) \vee \cdots \vee \Diamond h(d_n) \\ &= \Diamond(h(d_0) \vee \cdots \vee h(d_n)) \\ &= \Diamond h(d_0 \vee \cdots \vee d_n) = \Diamond h(d'). \end{aligned}$$

Therefore, h satisfies CDC for D'. It follows that $\mathfrak{A} \hookrightarrow_{D'} \mathfrak{B}$, namely, $\mathfrak{B} \not\models \rho(\mathfrak{A}, D')$. Hence, $\mathfrak{B} \models \rho(\mathfrak{A}, D)$ iff $\mathfrak{B} \models \rho(\mathfrak{A}, D')$. ☐

Now we prove the main theorems of this section, generalizing [BBI16, Theorem 5.5 and Theorem 5.6] by extending the base rule system from $\mathcal{S}_\mathsf{K}$ to any one that admits definable filtration. We already extracted the key feature of $\mathcal{S}_\mathsf{K}$ used in the original proofs, namely that it admits definable filtration; the proofs below are essentially the same as the original ones. The main idea is to construct finite *refutation patterns* for a given rule and represent them by stable canonical rules.

Theorem 3.24. *Let $\mathcal{S}$ be a rule system that admits definable filtration. For any rule ρ, there exist stable canonical rules $\rho(\mathfrak{A}_1, D_1), \ldots, \rho(\mathfrak{A}_n, D_n)$ where each $\mathfrak{A}_i$ is a finite $\mathcal{S}$-algebra and $D_i \subseteq A_i$, such that for any $\mathcal{S}$-algebra $\mathfrak{B}$,*

$$\mathfrak{B} \models \rho \textit{ iff } \mathfrak{B} \models \rho(\mathfrak{A}_1, D_1), \ldots, \rho(\mathfrak{A}_n, D_n),$$

Proof. Let $\rho = \Gamma/\Delta$ be a rule. If $\rho \in \mathcal{S}_L$, then we take $n = 0$. Assume that $\rho \notin \mathcal{S}_L$ and let $\Theta = \mathsf{Sub}(\Gamma \cup \Delta)$. By the assumption that $\mathcal{S}$ admits definable filtration, there is a finite subformula-closed set Θ' containing Θ such that, for any $\mathcal{S}$-algebra $\mathfrak{B}$ and any valuation V on $\mathfrak{B}$, there is a definable filtration $(\mathfrak{B}', V')$ of $(\mathfrak{B}, V)$ for Θ through Θ' such that $\mathfrak{B}' \models \mathcal{S}$. Let $m = |\Theta'|$. Since Boolean algebras are locally finite, up to isomorphism, there are finitely many tuples $(\mathfrak{A}, V, D)$ satisfying the following conditions:

1. $\mathfrak{A}$ is a finite $\mathcal{S}$-algebra based on an at most m-generated Boolean algebra and $\mathfrak{A} \not\models \rho$,

2. V is a valuation on $\mathfrak{A}$ such that $\mathfrak{A}, V \not\models \rho$ and $V(p) = 0$ for $p \notin \Theta'$,

3. $D = \{V(\psi) : \Diamond\psi \in \Theta\}$.

Let $(\mathfrak{A}_1, V_1, D_1), \ldots, (\mathfrak{A}_n, V_n, D_n)$ be an enumaration of such tuples. We show that for any $\mathcal{S}$-algebra $\mathfrak{B}$, $\mathfrak{B} \models \rho$ iff $\mathfrak{B} \models \rho(\mathfrak{A}_1, D_1), \ldots, \rho(\mathfrak{A}_n, D_n)$.

Suppose that $\mathfrak{B} \not\models \rho(\mathfrak{A}_i, D_i)$ for some $1 \leq i \leq n$. Then there is a stable embedding $h : \mathfrak{A}_i \hookrightarrow_{D_i} \mathfrak{B}$. Define a valuation V on $\mathfrak{B}$ by $V(p) = h \circ V_i(p)$. Since h satisfies CDC for D_i, we have $V(\varphi) = h \circ V_i(\varphi)$ for all $\varphi \in \Theta$. Therefore, since $V_i(\gamma) = 1$ for all $\gamma \in \Gamma$ and $V_i(\delta) \neq 1$ for all $\delta \in \Delta$, it follows that $V(\gamma) = 1$ for all $\gamma \in \Gamma$ and $V(\delta) \neq 1$ for all $\delta \in \Delta$, namely, $\mathfrak{B}, V \not\models \rho$.

Conversely, suppose that $\mathfrak{B} \not\models \rho$. Let V be a valuation on $\mathfrak{B}$ such that $\mathfrak{B}, V \not\models \rho$ and $V(p) = 0$ for $p \notin \Theta'$. Then, there is a definable filtration $(\mathfrak{B}', V')$ of $(\mathfrak{B}, V)$ for Θ through Θ' such that $\mathfrak{B}' \models \mathcal{S}$. By the definition of definable filtrations, B' is a Boolean subalgebra of B generated by $V[\Theta']$, so B' as a Boolean algebra is at most m-generated. By Lemma 3.13, since $\Gamma \cup \Delta \subseteq \Theta$ and $\mathfrak{B}, V \not\models \rho$, we obtain $\mathfrak{B}', V' \not\models \rho$. Let $D = \{V(\psi) : \Diamond\psi \in \Theta\}$. Then, the tuple $(\mathfrak{B}', V', D)$ is identical to $(\mathfrak{A}_i, V_i, D_i)$ for some $1 \leq i \leq n$. Since $\mathfrak{B}' \hookrightarrow_D \mathfrak{B}$ by the definition of definable filtration, we have $\mathfrak{A}_i \hookrightarrow_{D_i} \mathfrak{B}$, namely, $\mathfrak{B} \not\models \rho(\mathfrak{A}_i, D_i)$. Therefore, we conclude that $\mathfrak{B} \models \rho$ iff $\mathfrak{B} \models \rho(\mathfrak{A}_1, D_1), \ldots, \rho(\mathfrak{A}_n, D_n)$. □

It follows directly from Theorem 3.24 that any formula φ is also semantically equivalent to finitely many stable canonical rules over any rule system that admits definable filtration, by identifying φ with the rule $/\varphi$. Thus, we obtain the following axiomatization result. Recall that for a logic L, $\mathcal{S}_L$ is the rule system $\Sigma(L)$.

Corollary 3.25.

1. *Let $\mathcal{S}$ be a rule system that admits definable filtration. Any rule system $\mathcal{S}' \supseteq \mathcal{S}$ is axiomatizable over $\mathcal{S}$ by stable canonical rules. Moreover, if $\mathcal{S}'$ is finitely axiomatizable over $\mathcal{S}$, then $\mathcal{S}'$ is axiomatizable over $\mathcal{S}$ by finitely many stable canonical rules.*

2. *Let L be a logic that admits definable filtration. Any logic $L' \supseteq L$ is axiomatizable over $\mathcal{S}_L$ by stable canonical rules. Moreover, if L' is finitely axiomatizable over L, then L' is axiomatizable over $\mathcal{S}_L$ by finitely many stable canonical rules.*

Proof.

1. Let $\mathcal{S}$ be a rule system that admits definable filtration. For any rule system $\mathcal{S}' \supseteq \mathcal{S}$, $\mathcal{S}' = \mathcal{S} + \{\rho_i : i \in I\}$ for a set $\{\rho_i : i \in I\}$ of rules. By Theorem 3.24, each rule ρ_i is semantically equivalent to a finite set of stable canonical rules $\{\rho(\mathfrak{A}_{ij}, D_{ij}) : 1 \leq j \leq n_i\}$ for $\mathcal{S}$-algebras. So, for any $\mathcal{S}$-algebra $\mathfrak{B}$, $\mathfrak{B} \models \mathcal{S}'$ iff $\mathfrak{B} \models \rho(\mathfrak{A}_{ij}, D_{ij})$ for all $i \in I$ and $1 \leq j \leq n_i$. Thus, $\mathcal{S}' = \mathcal{S} + \{\rho(\mathfrak{A}_{ij}, D_{ij}) : i \in I, 1 \leq j \leq n_i\}$. Moreover, if $\mathcal{S}'$ is finitely axiomatizable over $\mathcal{S}$, then we can choose I to be finite, hence the set $\{\rho(\mathfrak{A}_{ij}, D_{ij}) : i \in I, 1 \leq j \leq n_i\}$ is also finite.

2. Let L be a logic that admits definable filtration. For any logic $L' \supseteq L$, $\mathcal{S}_{L'} = \mathcal{S}_L + \{/\varphi_i : i \in I\}$ for a set $\{\varphi_i : i \in I\}$ of formulas. By (1), there is a set $\{\rho(\mathfrak{A}_j, D_j) : j \in J\}$ of stable canonical rules such that $\mathcal{S}_{L'} = \mathcal{S}_L + \{\rho(\mathfrak{A}_j, D_j) : j \in J\}$. Thus, $L' = \Lambda(\mathcal{S}_{L'}) = \Lambda(\mathcal{S}_L + \{\rho(\mathfrak{A}_j, D_j) : j \in J\})$. Moreover, if L' is finitely axiomatizable over L, then we can choose I to be finite, so that J is also finite by (1).

□

As we mentioned, many logics such as K, T, D, $\mathsf{K4}$, and $\mathsf{S4}$ admit definable filtration; we also showed that pre-transitive logics $\mathsf{K4}_1^{\mathsf{m}+1}$ $(m \geq 1)$ admit definable filtration (Theorem 3.18). Thus, their corresponding rule systems admit definable filtration, and Corollary 3.25 applies to $\mathcal{S}_K$, $\mathcal{S}_{\mathsf{T}}$, $\mathcal{S}_{\mathsf{D}}$, $\mathcal{S}_{\mathsf{K4}}$, $\mathcal{S}_{\mathsf{S4}}$, and $\Sigma(\mathsf{K4}_1^{\mathsf{m}+1})$ for $m \geq 1$.

Remark 3.26. If the base rule system $\mathcal{S}$ is decidable, then the result of Corollary 3.25 is computable; that is, given a finite axiomatization of $\mathcal{S}'$ over $\mathcal{S}$, a finite set of stable canonical rules that axiomatize $\mathcal{S}'$ over $\mathcal{S}$ can be computed. This can be shown by observing that if $\mathcal{S}$ is decidable, then in the proof of Theorem 3.24, the enumeration of tuples $(\mathfrak{A}, V, D)$ is computable, so the finite set $\rho(\mathfrak{A}_1, D_1), \ldots, \rho(\mathfrak{A}_n, D_n)$ of stable canonical rules that is equivalent to the given rule ρ is also computable. Since a finitely axiomatizable rule system with the fmp is decidable, this is the case for all the rule systems mentioned above.

Finally, we briefly discuss two extreme types of stable canonical rules, namely stable canonical rules $\rho(\mathfrak{A}, D)$ with $D = \emptyset$ and those with $D = A$. We refer to [BBI16, Section 7] for details.

Definition 3.27. A stable canonical rule $\rho(\mathfrak{A}, \emptyset)$ is called a *stable rule.* A stable canonical rule $\rho(\mathfrak{A}, A)$ is called a *Jankov rule.*

The name Jankov rules comes from the analogy to Jankov formulas (see Definition 3.41 and the discussion after). It follows immediately from Theorem 3.22 that for any modal algebra $\mathfrak{B}$,

$$\mathfrak{B} \not\models \rho(\mathfrak{A}, \emptyset) \text{ iff } \mathfrak{A} \hookrightarrow_{\emptyset} \mathfrak{B}, \text{ i.e., there is a satble embedding from } \mathfrak{A} \text{ to } \mathfrak{B}$$

and

$$\mathfrak{B} \not\models \rho(\mathfrak{A}, A) \text{ iff } \mathfrak{A} \hookrightarrow_{A} \mathfrak{B}, \text{ i.e., } \mathfrak{A} \text{ is (isomorphic to) a subalgera of } \mathfrak{B}.$$

It turns out that stable rules and Jankov rules respectively axiomatize certain classes of rule systems with special features.

Definition 3.28. A rule system $\mathcal{S}$ is *stable* if the universal class $\mathcal{U}(\mathcal{S})$ is closed under stable subalgebras; that is, for any modal algebras $\mathfrak{A}$ and $\mathfrak{B}$, if $\mathfrak{A} \hookrightarrow_{\emptyset} \mathfrak{B}$ and $\mathfrak{B} \in \mathcal{U}(\mathcal{S})$, then $\mathfrak{A} \in \mathcal{U}(\mathcal{S})$.

It follows directly from the definition that stable rule systems admit definable filtration.

Proposition 3.29. *Any stable rule system admits definable filtration.*

Proof. Let $\mathcal{S}$ be a stable rule system. For any finite subformula-closed set Θ of formulas, $\mathcal{S}$-algebra $\mathfrak{A}$, and valuation V on $\mathfrak{A}$, let $(\mathfrak{A}', V')$ be the least filtration (Example 3.12) of $(\mathfrak{A}, V)$ through Θ. Then $(\mathfrak{A}', V')$ is a definable filtration of $(\mathfrak{A}, V)$ for Θ through Θ. Thus, $\mathfrak{A}'$ is a stable subalgebra of $\mathfrak{A}$, so $\mathfrak{A}' \models \mathcal{S}$ since $\mathcal{S}$ is stable and $\mathfrak{A}' \models \mathcal{S}$. Therefore, $\mathcal{S}$ admits definable filtration. □

Theorem 3.30 ([BBI16]). *A rule system $\mathcal{S}$ is stable iff it is axiomatizable over $\mathcal{S}_{\mathsf{K}}$ by stable rules.*

We can also define stability for logics in a similar manner.

Definition 3.31. Let M be a logic.

1. A class $\mathcal{K}$ of M-agebras is M-*stable* if for any M-algebra $\mathfrak{A}$ and any $\mathfrak{B} \in \mathcal{K}$, if $\mathfrak{A} \hookrightarrow_{\emptyset} \mathfrak{B}$, then $\mathfrak{A} \in \mathcal{K}$. The class $\mathcal{K}$ is *finitely* M-*stable* if for any finite M-algebra $\mathfrak{A}$ and any $\mathfrak{B} \in \mathcal{K}$, if $\mathfrak{A} \hookrightarrow_{\emptyset} \mathfrak{B}$, then $\mathfrak{A} \in \mathcal{K}$.

2. A logic $L \supseteq \mathsf{M}$ is M-*stable* if $\mathcal{V}(L)$ is generated by an M-stable class. A K-stable logic is simply called a *stable logic*.

The notion of M-stable logics is studied in [BBI18] (see also [Ili18] for a comprehensive account).

Theorem 3.32 ([Jeř09]). *Let $\mathcal{S}$ be a rule system.*

1. *$\mathcal{S}$ is splitting in $\mathsf{NExt}\mathcal{S}_{\mathsf{K}}$ iff $\mathcal{S}$ is axiomatizable by a Jankov rule.*
2. *$\mathcal{S}$ is a union-splitting in $\mathsf{NExt}\mathcal{S}_{\mathsf{K}}$ iff $\mathcal{S}$ is axiomatizable by Jankov rules.*

An algebraic proof of this result can be found in [BBI16, Theorem 7.10].

3.4 Stable canonical formulas for pre-transitive logics

One of the main reasons that we had to move to rules when defining stable canonical rules is that there is a hidden universal quantifier in the frame-theoretic definition of the validity of rules. This allows stable canonical rules to have better control over the structure of modal algebras and enables us to prove the semantic characterization Theorem 3.22. However, if the *master modality* (see, e.g., [BdV01, Section 6.5]) is "definable" in the base logic, then we can turn stable canonical rules into formulas. This idea is realized in [BBI16, Section 6] for $\mathsf{K4}$, which developed the theory of *stable canonical formula* for $\mathsf{K4}$ as an alternative to Zakharyaschev's canonical formulas.

In this section, we develop the theory of stable canonical formulas for pre-transitive logics $\mathsf{K4}_1^{\mathsf{m+1}}$, generalizing the stable canonical formulas for $\mathsf{K4}$. We already showed that these logics admit definable filtration (Theorem 3.18). As we can observe in Definition 3.35, the master modality is also definable in $\mathsf{K4}_1^{\mathsf{m+1}}$ in a similar manner as in $\mathsf{K4}$.

As we are dealing with logics in this section, it is useful to work with s.i. modal algebras. We recall the following lemma from [BB11, Lemma 6.4], which is proved as a corollary of Venema's characterization [Ven04] of s.i. modal algebras. Recall that $\Box^m\varphi$ is an abbreviation of $\Box\cdots\Box\varphi$ with m many $\Box$ and $\Box^{\leq m}$ is an abbreviation of $\varphi \wedge \cdots \wedge \Box^m\varphi$; similarly for elements of modal algebras.

Lemma 3.33. *Let $\mathfrak{A}$ be a finite modal algebra and $\mathfrak{B}$ be a s.i. modal algebra. If $\mathfrak{A}$ is a stable subalgebra of $\mathfrak{B}$, i.e., $\mathfrak{A} \hookrightarrow_\emptyset \mathfrak{B}$, then $\mathfrak{A}$ is also s.i.*

We first show how to construct finite refutation patterns for modal formulas. The proof is similar to Theorem 3.24.

Theorem 3.34. *For any formula φ, there exist pairs $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ such that each $\mathfrak{A}_i = (A_i, \Diamond_i)$ is a finite s.i. $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra, $D_i \subseteq A_i$, and for any s.i. modal algebra $\mathfrak{B} = (B, \Diamond)$, the following conditions are equivalent:*

(1) $\mathfrak{B} \not\models \varphi$.

(2) There is $1 \leq i \leq n$ and a stable embedding $h : \mathfrak{A}_i \hookrightarrow_{D_i} \mathfrak{B}$.

(3) There is a s.i. homomorphic image $\mathcal{C} = (C, \Diamond)$ of $\mathfrak{B}$, $1 \leq i \leq n$, and a stable embedding $h : \mathfrak{A}_i \hookrightarrow_{D_i} \mathcal{C}$.

Proof. If $\varphi \in \mathsf{K4}_1^{\mathsf{m+1}}$, then let $n = 0$. Assume that $\varphi \notin \mathsf{K4}_1^{\mathsf{m+1}}$ and let $\Theta = \mathsf{Sub}(\varphi)$. Since $\mathsf{K4}_1^{\mathsf{m+1}}$ admits definable filtration by Theorem 3.18, there is a finite subformula-closed set Θ' containing Θ such that, for any $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra $\mathfrak{B}$ and any valuation V on $\mathfrak{B}$, there is a definable filtration $(\mathfrak{B}', V')$ of $(\mathfrak{B}, V)$ for Θ through Θ' such that $\mathfrak{B}' \models \mathsf{K4}_1^{\mathsf{m+1}}$. Let $m = |\Theta'|$. Since Boolean algebras are locally finite, up to isomorphism, there are finitely many tuples $(\mathfrak{A}, V, D)$ satisfying the following conditions:

1. $\mathfrak{A}$ is a finite s.i. $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra based on an at most m-generated Boolean algebra and $\mathfrak{A} \not\models \varphi$,

2. V is a valuation on $\mathfrak{A}$ such that $\mathfrak{A}, V \not\models \varphi$ and $V(p) = 0$ for $p \notin \Theta'$,

3. $D = \{V(\psi) : \Diamond\psi \in \Theta\}$.

Let $(\mathfrak{A}_1, V_1, D_1), \ldots, (\mathfrak{A}_n, V_n, D_n)$ be an enumaration of such tuples. We show that $(\mathfrak{A}_1, D_1), \ldots, (\mathfrak{A}_n, D_n)$ is the desired pairs. Let $\mathfrak{B}$ be a s.i. $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra.

(1) $\Rightarrow$ (2). Suppose that $\mathfrak{B} \not\models \varphi$. Let V be a valuation on $\mathfrak{B}$ such that $\mathfrak{B}, V \not\models \varphi$. Then there is a definable filtration $(\mathfrak{B}', V')$ of (B, V) for Θ through Θ' such that $\mathfrak{B}' \models \mathsf{K4}_1^{\mathsf{m+1}}$. By the definition of definable filtration, $\mathfrak{B}'$ is a stable subalgebra of $\mathfrak{B}$, so $\mathfrak{B}'$ is also s.i. by Lemma 3.33. Then the same argument as in the proof of Theorem 3.24 shows that the tuple $(\mathfrak{B}', V', D)$ is identical to $(\mathfrak{A}_i, V_i, D_i)$ for some $1 \leq i \leq n$. Since $\mathfrak{B}' \hookrightarrow_D \mathfrak{B}$ by the definition of definable filtration, we conclude $\mathfrak{A}_i \hookrightarrow_{D_i} \mathfrak{B}$.

(2) $\Rightarrow$ (3). This is obvious by taking $\mathcal{C} = \mathfrak{B}$.

(3) $\Rightarrow$ (1). Suppose that there is a s.i. homomorphic image $\mathcal{C}$ of $\mathfrak{B}$, $1 \leq i \leq n$, and a stable embedding $h : \mathfrak{A}_i \hookrightarrow_{D_i} \mathcal{C}$. Let V_i be valuation on $\mathfrak{A}_i$ such that $\mathfrak{A}_i, V_i \not\models \varphi$. Define a valuation V on $\mathcal{C}$ by $V(p) = h(V_i(p))$. The same argument as in the proof of Theorem 3.24 shows that $\mathcal{C} \not\models \varphi$. Thus, $\mathfrak{B} \not\models \varphi$ since $\mathcal{C}$ is a homomorphic image of $\mathfrak{B}$.

$\square$

Now we define *stable canonical formulas* for pre-transitive logics $\mathsf{K4}_1^{\mathsf{m+1}}$, which capture the semantic condition (3) in the theorem above.

Definition 3.35. Let $\mathfrak{A}$ be a finite s.i. $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra and $D \subseteq A$. Let $\rho(\mathfrak{A}, D) = \Gamma/\Delta$ be the stable canonical rule defined in Definition 3.20. We define the *stable canonical formula* $\gamma^m(\mathfrak{A}, D)$ as

$$\begin{aligned}\gamma^m(A, D) &= \bigwedge\{\Box^{\leq m}\gamma : \gamma \in \Gamma\} \to \bigvee\{\Box^{\leq m}\delta : \delta \in \Delta\} \\ &= \Box^{\leq m} \bigwedge \Gamma \to \bigvee\{\Box^{\leq m}\delta : \delta \in \Delta\}.\end{aligned}$$

We will write $\gamma(\mathfrak{A}, D)$ for $\gamma^1(\mathfrak{A}, D)$; this notation is consistent with [BBI16]. The following lemma is a straightforward generalization of [BB11, Lemma 4.1] to pre-transitive logics $\mathsf{K4}_1^{\mathsf{m+1}}$.

Lemma 3.36. *Let $\mathfrak{A}$ be a $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra and $a, b \in \mathfrak{A}$ such that $\Box^{\leq m}a \not\leq b$. Then there exists a s.i. $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra $\mathfrak{B}$ and a surjective homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ such that $f(\Box^{\leq m}a) = 1$ and $f(b) \neq 1$.*

Proof. Let $\mathfrak{A}$ be a $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra and $a, b \in \mathfrak{A}$ such that $\Box^{\leq m}a \not\leq b$. Let $F = {\uparrow}\Box^{\leq m}a$. Then F is a filter on $\mathfrak{A}$. If $c \in F$, i.e., $c \geq \Box^{\leq m}a$ for some $c \in \mathfrak{A}$, then since $\Box a \leq \Box^{m+1}a$ by $\mathfrak{A} \models \mathsf{K4}_1^{\mathsf{m+1}}$, we have

$$\Box c \geq \Box(\Box^{\leq m}a) = \Box a \wedge \cdots \wedge \Box^{m+1}a \geq \Box a \wedge \cdots \wedge \Box^m a \geq \Box^{\leq m}a,$$

thus $\Box c \in F$. So, F is a $\Box$-filter. By Zorn's Lemma, there is a $\Box$-filter M such that $\Box^{\leq m}a \in M$ and $b \notin M$, and M is maximal in this sense.

Let $B = A/\sim$, where $a \sim b$ iff $a \leftrightarrow b \in M$ for $a, b \in A$. Since M is a $\Box$-filter, operations on $\mathfrak{A}$ induce operations on B and turn it into a modal algebra $\mathfrak{B}$. Let $f : A \to B$ be the projection map. It is straightforward to verify that f is a surjective modal algebra homomorphism. Moreover, $f(\Box^{\leq m}a) = 1$ and $f(b) \neq 1$ since $\Box^{\leq m}a \in M$ and $b \notin M$.

By the correspondence between $\Box$-filters of $\mathfrak{B}$ and $\Box$-filters of $\mathfrak{A}$ containing M, if F' is a $\Box$-filter on $\mathfrak{B}$ such that $\{1\} \subsetneq F'$, then its corresponding $\Box$-filter on $\mathfrak{A}$ contains b by the maximality of M, so $f(b) \in F'$. It follows that $\Box^{\leq m}f(b) \in F'$. So, $\uparrow \Box^{\leq m}f(b)$ is the smallest $\Box$-filter of $\mathfrak{B}$ properly containing $\{1\}$. Therefore, B is s.i. ☐

Theorem 3.37. *Let $\mathfrak{A}$ be a finite s.i. $\mathsf{K4}_1^{m+1}$-algebra and $D \subseteq A$. Then, for any $\mathsf{K4}_1^{m+1}$-algebra $\mathfrak{B}$,*

$$\mathfrak{B} \not\models \gamma^m(\mathfrak{A}, D) \text{ iff there is a s.i. homomorphic image } \mathcal{C} \text{ of } \mathfrak{B} \text{ such that } \mathfrak{A} \hookrightarrow_D \mathcal{C}.$$

Proof. Suppose that there is a s.i. homomorphic image $\mathcal{C}$ of $\mathfrak{B}$ and a stable embedding $h : \mathfrak{A} \hookrightarrow_D \mathcal{C}$. Define a valuation V_A on $\mathfrak{A}$ by $V_A(p_a) = a$. It follows from the definition of Γ and Δ (Definition 3.20) that $V_A(\gamma) = 1$ for all $\gamma \in \Gamma$ and $V_A(\delta) \neq 1$ for all $\delta \in \Delta$. Thus, $V_A(\Box^{\leq m} \bigwedge \Gamma) = 1$ and $V_A(\Box^{\leq m}\delta) = \Box^{\leq m}V_A(\delta) \neq 1$ for all $\delta \in \Delta$. Since $\mathfrak{A}$ is s.i., $\mathfrak{A}$ has an opremum c by Proposition 2.40. So, for each $\delta \in \Delta$, there is $n \in \omega$ such that $\Box^n(\Box^{\leq m}V_A(\delta)) \leq c$. Since $\mathfrak{A} \models \mathsf{K4}_1^{m+1}$, for any $a \in \mathfrak{A}$ and $k \in \omega$, $\Box^{k'}a \leq \Box^k$ for some $0 \leq k' \leq m$. Thus, we have $\Box^{\leq m}V_A(\delta) \leq \Box^n(\Box^{\leq m}V_A(\delta)) \leq c$. Hence, $\bigvee \Box^{\leq m}V_A(\delta) \leq c$, which implies $\mathfrak{A} \not\models \gamma^m(\mathfrak{A}, D)$. Next, define a valuation V_C on $\mathcal{C}$ by $V_C(p_a) = h(V_A(p_a))$. The same argument as in the proof of Theorem 3.24 shows that $V_C(\gamma) = 1$ for all $\gamma \in \Gamma$ and $V_C(\delta) \neq 1$ for all $\delta \in \Delta$. Thus, $V_C(\Box^{\leq m} \bigwedge \Gamma) = 1$ and $V_C(\Box^{\leq m}\delta) = \Box^{\leq m}V_C(\delta) \neq 1$ for all $\delta \in \Delta$. Since $\mathcal{C}$ is a homomorphic image of $\mathfrak{B}$ and $\mathfrak{B} \models \mathsf{K4}_1^{m+1}$, $\mathcal{C} \models \mathsf{K4}_1^{m+1}$ as well. Since $\mathcal{C}$ is also s.i., applying the same argument as for $\mathfrak{A}$, we obtain that $V_C(\Box^{\leq m} \bigwedge \Gamma) = 1$ and $\bigvee \Box^{\leq m}V_C(\delta)$ is below or equal to the opremum of $\mathcal{C}$, so $\mathcal{C} \not\models \gamma^m(\mathfrak{A}, D)$. It follows that $\mathfrak{B} \not\models \gamma^m(\mathfrak{A}, D)$ since $\mathcal{C}$ is a homomorphic image of $\mathfrak{B}$.

Conversely, suppose that $\mathfrak{B} \not\models \gamma^m(\mathfrak{A}, D)$. Then there is a valuation V_B on $\mathfrak{B}$ such that $\Box^{\leq m}V_B(\bigwedge \Gamma) \not\leq V_B(\bigvee \Box^{\leq m}\delta)$. By Lemma 3.36, there is a s.i. homomorphic image $\mathcal{C}$ of $\mathfrak{B}$ and a valuation V_C on $\mathcal{C}$ such that $V_C(\Box^{\leq m} \bigwedge \Gamma) = 1$ and $V_C(\bigvee \Box^{\leq m}\delta) \neq 1$. Define a map $h : \mathfrak{A} \to \mathcal{C}$ by $h(a) = V_C(p_a)$. Unfolding the definition of Γ and Δ (Definition 3.20), it is straightforward to verify that h is a stable embedding satisfying CDC for D. ☐

Combining Theorem 3.34 and Theorem 3.37, we obtain the following corollary, which is a version of Theorem 3.24 for stable canonical formulas for pre-transitive logics.

Corollary 3.38. *For any formula φ, there exist stable canonical formulas $\gamma^m(\mathfrak{A}_1, D_1), \ldots, \gamma^m(\mathfrak{A}_n, D_n)$ where each $\mathfrak{A}_i$ is a finite s.i. $\mathsf{K4}_1^{m+1}$-algebra and $D_i \subseteq A_i$, such that for any s.i. modal algebra $\mathfrak{B}$,*

$$\mathfrak{B} \models \varphi \text{ iff } \mathfrak{B} \models \bigwedge\{\gamma^m(\mathfrak{A}_i, D_i) : 1 \leq i \leq n\}.$$

Proof. This follows directly from Theorem 3.34 and Theorem 3.37. ☐

Now we arrive at the axiomatization result for logics above $\mathsf{K4}_1^{m+1}$, generalizing the result for logics above $\mathsf{K4}$ ([BBI16, Theorem 6.10]).

Theorem 3.39. *Let $m \geq 1$. Any logic $L \supseteq \mathsf{K4}_1^{m+1}$ is axiomatizable over $\mathsf{K4}_1^{m+1}$ by stable canonical formulas. Moreover, if L is finitely axiomatizable over $\mathsf{K4}_1^{m+1}$, then L is axiomatizable over $\mathsf{K4}_1^{m+1}$ by finitely many stable canonical formulas.*

Proof. Let L be a logic containing $\mathsf{K4}_1^{\mathsf{m+1}}$. Then $L = \mathsf{K4}_1^{\mathsf{m+1}} + \{\varphi_i : i \in I\}$ for a set $\{\varphi_i : i \in I\}$ of formulas. By Corollary 3.38, for each formula φ_i, there exists a finite set of stable canonical formulas $\{\gamma^m(\mathfrak{A}_{ij}, D_{ij}) : 1 \leq j \leq n_i\}$ such that for any s.i. $\mathsf{K4}_1^{\mathsf{m+1}}$-algebras $\mathfrak{B}$, $\mathfrak{B} \models \varphi$ iff $\mathfrak{B} \models \{\gamma^m(\mathfrak{A}_{ij}, D_{ij}) : 1 \leq j \leq n_i\}$. So, for any s.i. $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra $\mathfrak{B}$, $\mathfrak{B} \models L$ iff $\mathfrak{B} \models \gamma^m(\mathfrak{A}_{ij}, D_{ij})$ for all $i \in I$ and $1 \leq j \leq n_i$. Since, by Theorem 2.19, every logic is determined by the class of its s.i. modal algebras, it follows that $L = \mathsf{K4}_1^{\mathsf{m+1}} + \{\gamma^m(\mathfrak{A}_{ij}, D_{ij}) : i \in I, 1 \leq j \leq n_i\}$. Moreover, if L is finitely axiomatizable over $\mathsf{K4}_1^{\mathsf{m+1}}$, then we can choose I to be finite, hence the set $\{\gamma^m(\mathfrak{A}_{ij}, D_{ij}) : i \in I, 1 \leq j \leq n_i\}$ is also finite. □

Remark 3.40. We have a computability result similar to Remark 3.26. Since the logic $\mathsf{K4}_1^{\mathsf{m+1}}$ is finitely axiomatizable and has the fmp (Corollary 3.19), it is decidable. Thus, the enumeration in the proof of Theorem 3.34 is computable. Therefore, the result of Theorem 3.39 is computable; that is, given a finite axiomatization of L over $\mathsf{K4}_1^{\mathsf{m+1}}$, a finite set of stable canonical formulas that axiomatize L over $\mathsf{K4}_1^{\mathsf{m+1}}$ can be computed.

As in Definition 3.27, stable canonical formulas also induce two extreme types of formulas.

Definition 3.41. A stable canonical formula $\gamma^m(\mathfrak{A}, \emptyset)$ is called a *stable formula*. A stable canonical formula $\gamma^m(\mathfrak{A}, A)$ is called a *Jankov formula*.

It follows immediately from Theorem 3.37 that for any $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra $\mathfrak{B}$,

$$\mathfrak{B} \not\models \gamma^m(\mathfrak{A}, \emptyset) \text{ iff } \mathfrak{A} \text{ is a stable subalgebra of a s.i. homomorphic image of } \mathfrak{B}$$

and

$$\mathfrak{B} \not\models \gamma^m(\mathfrak{A}, D) \text{ iff } \mathfrak{A} \text{ is a subalgebra of a s.i. homomorphic image of } \mathfrak{B}.$$

The relation of stable formulas for $\mathsf{K4}$ ($\mathsf{S4}$) and $\mathsf{K4}$($\mathsf{S4}$)-stable logics is studied in [BBI18] (see also [Ili18] for a comprehensive account).

The terminology Jankov formulas is motivated by the fact that, by Theorem 3.37, a stable canonical formula $\gamma^m(\mathfrak{A}, A)$ has the same semantic characterization as the original *Jankov formula* [Jan63] for Heyting algebras. Finally, we describe a characterization of splitting logics and union-splitting logics in $\mathsf{NExtK4}_1^{\mathsf{m+1}}$ via Jankov formulas.

Theorem 3.42. *Let $m \geq 1$.*

1. *A logic $L \in \mathsf{NExtK4}_1^{\mathsf{m+1}}$ is a splitting logic in $\mathsf{NExtK4}_1^{\mathsf{m+1}}$ iff $L = \mathsf{K4}_1^{\mathsf{m+1}} + \gamma^m(\mathfrak{A}, A)$ for some finite s.i. $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra $\mathfrak{A}$.*

2. *A logic $L \in \mathsf{NExtK4}_1^{\mathsf{m+1}}$ is a union-splitting logic in $\mathsf{NExtK4}_1^{\mathsf{m+1}}$ iff $L = \mathsf{K4}_1^{\mathsf{m+1}} + \{\gamma^m(\mathfrak{A}_i, A_i) : i \in I\}$ for a set $\{\mathfrak{A}_i : i \in I\}$ finite s.i. $\mathsf{K4}_1^{\mathsf{m+1}}$-algebras.*

Proof. (2) is an immediate consequence of (1). To show (1), first suppose that $L = \mathsf{K4}_1^{\mathsf{m+1}} + \gamma^m(\mathfrak{A}, A)$ for some finite s.i. $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra $\mathfrak{A}$. We show that $(L, \mathsf{Log}(\mathfrak{A}))$ is a splitting pair in $\mathsf{NExtK4}_1^{\mathsf{m+1}}$. For any logic $L' \in \mathsf{NExtK4}_1^{\mathsf{m+1}}$ such that $L \not\subseteq L'$, there is a modal algebra $\mathfrak{B}$ such that $\mathfrak{B} \models L'$ and $\mathfrak{B} \not\models L$. Thus, $\mathfrak{B} \not\models \gamma^m(\mathfrak{A}, A)$, namely, $\mathfrak{A}$ is a subalgebra of a homomorphic image of $\mathfrak{B}$. So, $L' \subseteq \mathsf{Log}(\mathfrak{B}) \subseteq \mathsf{Log}(\mathfrak{A})$. Hence, $(L, \mathsf{Log}(\mathfrak{A}))$ is a splitting pair in $\mathsf{NExtK4}_1^{\mathsf{m+1}}$.

Conversely, suppose that L is a splitting logic in $\mathsf{NExtK4}_1^{\mathsf{m+1}}$. Then by Theorem 3.43 below, since $\mathsf{NExtK4}_1^{\mathsf{m+1}}$ has the fmp (Corollary 3.19), $(L, \mathsf{Log}(\mathfrak{A}))$ is a splitting pair in $\mathsf{NExtK4}_1^{\mathsf{m+1}}$ for some finite s.i. $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra $\mathfrak{A}$. By the argument above,$(\mathsf{K4}_1^{\mathsf{m+1}} + \gamma^m(\mathfrak{A}, A), \mathsf{Log}(\mathfrak{A}))$ is a splitting pair in $\mathsf{NExtK4}_1^{\mathsf{m+1}}$. It follows that $L = \mathsf{K4}_1^{\mathsf{m+1}} + \gamma^m(\mathfrak{A}, A)$. □

A similar characterization of (union-)splitting logics in $\mathsf{NExtwK4}$ can be found in [BB12]. Note that both of these characterizations follow from the combination of the following two general results by McKenzie [McK72] and by Rautenberg [Rau80]; the proof of the latter uses a version of characteristic formulas.

Theorem 3.43 ([McK72]). *Let L be a logic with the fmp. If a logic L' splits $\mathsf{NExt}L$, then $L' = \mathsf{Log}(\mathfrak{A})$ for some finite s.i. L-algebra $\mathfrak{A}$.*

Theorem 3.44 ([Rau80]). *Let L be a logic containing the logic $\mathsf{K} + p \wedge \cdots \wedge \Box^m p \to \Box^{m+1} p$. Then for any finite s.i. L-algebra $\mathfrak{A}$, the logic $\mathsf{Log}(\mathfrak{A})$ splits $\mathsf{NExt}L$.*

3.5 The m-closed domain condition and m-stable canonical formulas

In this section, we consider a variation of the closed domain condition and stable canonical formulas for pre-transitive logics. The intuitive idea is to preserve the modality not only one step but also up to m steps. This leads to an alternative axiomatization result as Theorem 3.39 for logics above pre-transitive logics $\mathsf{K4}_1^{\mathsf{m+1}}$. However, note that the notions and results in this section will not be used in other parts of the thesis.

Definition 3.45. Let $\mathfrak{A}$ and $\mathfrak{B}$ be modal algebras and $h : A \to B$ be a stable homomorphism. For $a \in A$, we say that h satisfies the *m-closed domain condition (m-CDC) for a* if $h(\Diamond^k a) = \Diamond^k h(a)$ for all $1 \leq k \leq m$. For $D \subseteq A$, we say that h satisfies the *m-closed domain condition (m-CDC) for D* if h satisfies m-CDC for all $a \in D$.

Similar to the case of CDC, for stable homomorphisms, $h(\Diamond^k a) = \Diamond^k h(a)$ holds iff $h(\Diamond^k a) \leq \Diamond^k h(a)$ holds. We also provide the frame-theoretic version of the definition and show that the two definitions are dual to each other.

Definition 3.46. Let $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, Q)$ be modal spaces and $f : X \to Y$ be a stable map. For a clopen subset $D \subseteq Y$, we say that f satisfies the *m-closed domain condition (m-CDC) for D* if for all $1 \leq k \leq m$,

$$Q^k[f(x)] \cap D \neq \emptyset \Rightarrow f(R^k[x]) \cap D \neq \emptyset.$$

For a set $\mathcal{D}$ of clopen subsets of Y, we say that $f : X \to Y$ satisfies the *m-closed domain condition (m-CDC) for $\mathcal{D}$* if f satisfies m-CDC for all $D \in \mathcal{D}$.

Proposition 3.47. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be modal algebras and $h : A \to B$ be a stable homomorphism. For any $a \in A$,*

h satisfies m-CDC for a iff h_* satisfies m-CDC for $\beta(a)$.

For any $D \subseteq A$,

h satisfies m-CDC for D iff h_* satisfies m-CDC for $\beta[D]$.

Proof. Let $\mathfrak{A}$ and $\mathfrak{B}$ be modal algebras with dual spaces $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, Q)$ and $h : A \to B$ be a stable homomorphism. Let $a \in A$. It follows from the duality that

$$\begin{aligned}
h \text{ satisfies } m\text{-CDC for } a &\text{ iff } h(\Diamond^k a) \leq \Diamond^k h(a) \text{ for all } 1 \leq k \leq m \\
&\text{ iff } h_*^{-1} \Diamond^k \beta(a) \subseteq \Diamond^k h_*^{-1} \beta(a) \text{ for all } 1 \leq k \leq m \\
&\text{ iff } Q^k[h_*(x)] \cap \beta(a) \neq \emptyset \Rightarrow h_*(R^k[x]) \cap \beta(a) \neq \emptyset \text{ for all } 1 \leq k \leq m \\
&\text{ iff } h_* \text{ satisfies } m\text{-CDC for } \beta(a).
\end{aligned}$$

The second statement follows directly from the first. □

Notation 3.48. We write $h : \mathfrak{A} \hookrightarrow_D^m \mathfrak{B}$ if h is a stable embedding satisfying m-CDC for D and $\mathfrak{A} \hookrightarrow_D^m \mathfrak{B}$ if there is such an h. We write $f : \mathfrak{X} \twoheadrightarrow_{\mathcal{D}}^m \mathfrak{Y}$ if f is a surjective stable map satisfying m-CDC for $\mathcal{D}$ and $\mathfrak{X} \twoheadrightarrow_{\mathcal{D}}^m \mathfrak{Y}$ if there is such an f.

Now we define *m-stable canonical formulas* for pre-transitive logics $\mathsf{K4}_1^{\mathsf{m+1}}$ ($m \geq 1$). The only difference with standard stable canonical formulas is that in Γ, the part using the closed domain D is generalized so that it captures m-CDC.

Definition 3.49. Let $\mathfrak{A}$ be a finite s.i. $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra and $D \subseteq A$. We define the *m-stable canonical formula* $\gamma_+^m(\mathfrak{A}, D)$ as

$$\begin{aligned}
\gamma_+^m(A, D) &= \bigwedge\{\Box^{\leq m}\gamma : \gamma \in \Gamma\} \to \bigvee\{\Box^{\leq m}\delta : \delta \in \Delta\} \\
&= \Box^{\leq m} \bigwedge \Gamma \to \bigvee\{\Box^{\leq m}\delta : \delta \in \Delta\},
\end{aligned}$$

where

$$\begin{aligned}
\Gamma =& \{p_{a \vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \cup \\
& \{\neg p_a \leftrightarrow \neg p_a : a \in A\} \cup \\
& \{\Diamond p_a \to p_{\Diamond a} : a \in A\} \cup \\
& \{p_{\Diamond^k a} \to \Diamond^k p_a : a \in D, 1 \leq k \leq m\},
\end{aligned}$$

and

$$\Delta = \{p_a : a \in A, a \neq 1\}.$$

Remark 3.50. As stable canonical rules, m-stable canonical formulas can also be defined directly from finite modal spaces (i.e., finite Kripke frames). The basic idea is the same as Definition 3.49, but we use Γ and Δ in Definition 3.21 instead and change the last clause in Γ to

$$\{p_x \to \bigvee\{\Diamond^k p_y : y \in D\} : x \in X, D \in \mathcal{D}, x \in (R^{-1})^k[D], 1 \leq k \leq m, \}$$

in light of Definition 3.46.

Remark 3.51. In general, m-CDC is stronger than the standard CDC. For $m = 1$, where the base logic is $\mathsf{K4}$, the m-CDC and m-stable canonical formulas reduce to the standard CDC and stable canonical formulas. It is clear that m-stable canonical formulas induce the same stable formulas (up to equivalence) as stable canonical formulas. Moreover, since a modal homomorphism always satisfies m-CDC for any D, they also induce the same Jankov formulas (up to equivalence).

Following the proof of Theorem 3.37, it is straightforward to verify the semantic condition of the validity of m-stable canonical formulas.

Theorem 3.52. *Let $\mathfrak{A}$ be a finite s.i. $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra and $D \subseteq A$. Then, for any $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra $\mathfrak{B}$,*

$$\mathfrak{B} \not\models \gamma_+^m(\mathfrak{A}, D) \text{ iff there is a s.i. homomorphic image } \mathcal{C} \text{ of } \mathfrak{B} \text{ such that } \mathfrak{A} \hookrightarrow_D^m \mathcal{C}.$$

Now we show that any logic above $\mathsf{K4}_1^{\mathsf{m+1}}$ can be axiomatized by m-stable canonical formulas, thus providing an alternative to Theorem 3.39. Recall that in the case of stable canonical formulas, the essential idea was to construct finite refutation patterns as done in Theorem 3.34. The construction in turn essentially depends on the definable filtration developed in Lemma 3.17. Thus, to adapt the whole proof for stable canonical formulas to m-stable canonical formulas, it suffices to observe that the definable filtration in Lemma 3.17 in fact induces a stable homomorphism satisfying m-CDC.

Lemma 3.53. *Let $\mathfrak{A} = (A, \Diamond)$ be a $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra, V be a valuation on $\mathfrak{A}$, Θ be a finite subformula-closed set of formulas, and $\Theta' = \mathsf{Sub}(\Theta \cup \{\Diamond^m\varphi : \varphi \in \Theta\})$. Let $\Diamond_0$ and $\Diamond_1$ be the modal operators on A' defined in Lemma 3.17. Then, the inclusion $i' : \mathfrak{A}' = (A', \Diamond_1) \hookrightarrow_D \mathfrak{A}$ satisfies m-CDC for D, where $D = \{V(\varphi) : \Diamond\varphi \in \Theta\}$.*

Proof. Let $D' = \{V(\varphi) : \Diamond\varphi \in \Theta'\}$. Let $d \in D$. Then, as we saw in the proof of Lemma 3.17, $\Diamond^m d = \Diamond_0^m d, \dots, \Diamond d = \Diamond_0 d, d \in D'$. Since the inclusion $i : (A', \Diamond_0) \hookrightarrow'_D \mathfrak{A}$ is the same map as i', we have $i'(\Diamond_0^l d) \leq \Diamond^l i'(d)$ for all $1 \leq l \leq m$. Also, recall from the proof of Lemma 3.17 that $\Diamond_1^l a = \bigvee\{\Diamond_0^{km+l} a : k \leq K\}$ for $l \geq 1$ and $a \in A'$ and $\Diamond_0^{km+1} d \leq \Diamond_0 d$ for all $k \geq 1$. Thus, for any $1 \leq l \leq m$ we have

$$i'(\Diamond_1^l d) = i'(\bigvee\{\Diamond_0^{km+l} d : k \leq K\}) \leq i'(\{\Diamond_0^l d) \leq \Diamond^l i'(d).$$

Hence, we conclude that i' satisfies m-CDC for D. □

Theorem 3.54. *For any formula φ, there exist stable canonical formulas $\gamma_+^m(\mathfrak{A}_1, D_1), \dots, \gamma_+^m(\mathfrak{A}_n, D_n)$ where each $\mathfrak{A}_i$ is a finite s.i. $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra and $D_i \subseteq A_i$, such that for any s.i. modal algebra $\mathfrak{B}$,*

$$\mathfrak{B} \models \varphi \text{ iff } \mathfrak{B} \models \bigwedge\{\gamma_+^m(\mathfrak{A}_i, D_i) : 1 \leq i \leq n\}.$$

Proof. Lemma 3.53 shows that the condition (3) in Theorem 3.34 can be strengthened such that the stable embedding $h : \mathfrak{A}_i \hookrightarrow_{D_i} \mathcal{C}$ satisfies m-CDC for D_i. Thus, combined with Theorem 3.52, the statement follows. □

Corollary 3.55. *Let $m \geq 1$. Any logic $L \supseteq \mathsf{K4}_1^{\mathsf{m+1}}$ is axiomatizable over $\mathsf{K4}_1^{\mathsf{m+1}}$ by m-stable canonical formulas. Moreover, if L is finitely axiomatizable over $\mathsf{K4}_1^{\mathsf{m+1}}$, then L is axiomatizable over $\mathsf{K4}_1^{\mathsf{m+1}}$ by finitely many m-stable canonical formulas.*

Proof. This follows from Theorem 3.54 by a similar argument as in the proof of Theorem 3.39. □

Remark 3.56. The above results can be seen as an improvement of Corollary 3.38 in the sense that every m-stable canonical formula is equivalent over $\mathsf{K4}_1^{\mathsf{m+1}}$ to a stable canonical formula, but not vice versa. It is easy to see from the semantic characterizations Theorems 3.37 and 3.52 that, any

m-stable canonical formula $\gamma_+^m(\mathfrak{A}, D)$ is equivalent to the stable canonical formula $\gamma^m(\mathfrak{A}, D')$, where $D' = \{\Diamond^{k-1} d : d \in D, 1 \leq k \leq m\}$. The following example shows that the converse does not hold.

We construct a counterexample using finite modal spaces. Let $\mathfrak{X}$ be the finite rooted $\mathsf{K4}^4_1$-space depicted as below and $D = \{d\}$.

$\mathfrak{X}$

Assume for a contradiction that the stable canonical formula $\gamma^m(\mathfrak{X}, \{D\})$ is equivalent to some m-stable canonical formula $\gamma_+^m(\mathfrak{X}', \mathcal{D}')$. Then $\mathfrak{X} \not\models \gamma_+^m(\mathfrak{X}', \mathcal{D}')$ and $\mathfrak{X}' \not\models \gamma^m(\mathfrak{X}, \{D\})$. By the dual of Theorem 3.37 and Theorem 3.52, one can verify that $\mathfrak{X}' = \mathfrak{X}$. Let $\mathfrak{Y}$ and $\mathfrak{Y}'$ be rooted $\mathsf{K4}^4_1$-spaces depicted as below.

$\mathfrak{Y}$ $\mathfrak{Y}'$

Note that there is only one stable map from $\mathfrak{Y}$ or $\mathfrak{Y}'$ to $\mathfrak{X}$. It is easy to see that $\mathfrak{Y} \twoheadrightarrow_D \mathfrak{X}$ and $\mathfrak{Y}' \not\twoheadrightarrow_D \mathfrak{X}$, thus $\mathfrak{Y} \not\models \gamma^m(\mathfrak{X}, \{D\})$ and $\mathfrak{Y}' \models \gamma^m(\mathfrak{X}, \{D\})$. If $d \in D'$ for some $D' \in \mathcal{D}'$, then $\mathfrak{Y} \not\twoheadrightarrow^m_{\mathcal{D}'} \mathfrak{X}$, so $\mathfrak{Y} \models \gamma_+^m(\mathfrak{X}', \mathcal{D}')$. If there is no $D' \in \mathcal{D}'$ such that $d \in D'$, then $\mathfrak{Y}' \twoheadrightarrow^m_{\mathcal{D}'} \mathfrak{X}$, so $\mathfrak{Y}' \not\models \gamma_+^m(\mathfrak{X}', \mathcal{D}')$. This contradicts that $\gamma^m(\mathfrak{X}, \{D\})$ and $\gamma_+^m(\mathfrak{X}', \mathcal{D}')$ are equivalent.

3.6 Splitting formulas

Although the master modality is not fully definable in K, if a stable canonical rule is defined from a finite s.i. modal algebra (or dually, a finite rooted cycle-free modal space), then we can mimic the master modality and turn the rule into a formula. We call the formulas obtained in this way *splitting formulas* because they axiomatize union-splitting logics in NExtK. This type of characteristic formulas is studied in [BBI16, Section 7] under the name stable canonical formulas. We summarize the results while leaving out the proofs.

Definition 3.57. Let $\mathfrak{A}$ be a finite s.i. modal algebra of height $\leq n$ and $D \subseteq A$. Let $\rho(\mathfrak{A}, D) = \Gamma/\Delta$ be the stable canonical rule defined in Definition 3.20. We define the *splitting formula* $\epsilon(\mathfrak{A}, D)$ as

$$\begin{aligned}\epsilon(\mathfrak{A}, D) &= (\Box^{n+1}\bot \wedge \bigwedge\{\Box^{\leq n}\gamma : \gamma \in \Gamma\}) \to \bigvee\{\Box^{\leq n}\delta : \delta \in \Delta\} \\ &= (\Box^{n+1}\bot \wedge \Box^{\leq n} \bigwedge \Gamma) \to \bigvee\{\Box^{\leq n}\delta : \delta \in \Delta\}.\end{aligned}$$

Splitting formulas have exactly the same validity characterization as stable canonical formulas, except that here we consider all modal algebras. Thus, they are semantically equivalent and axiomatize the same logics over pre-transitive logics.

Theorem 3.58 ([BBI16])**.** *Let $\mathfrak{A}$ be a finite s.i. modal algebra of finite height and $D \subseteq A$. Then, for any modal algebra $\mathfrak{B}$,*

$$\mathfrak{B} \not\models \epsilon(\mathfrak{A}, D) \text{ iff there is a s.i. homomorphic image } \mathcal{C} \text{ of } \mathfrak{B} \text{ such that } \mathfrak{A} \hookrightarrow_D \mathcal{C}.$$

Corollary 3.59. *Let $m \geq 1$ and L be a logic containing $\mathsf{K4}_1^{m+1}$. Then, for any finite s.i. $\mathsf{K4}_1^{m+1}$-algebra $\mathfrak{A}$ of finite height and $D \subseteq A$, we have*

$$L + \epsilon(\mathfrak{A}, D) = L + \gamma^m(\mathfrak{A}, D)$$

Proof. By Theorem 3.37 and Theorem 3.58, for any $\mathsf{K4}_1^{m+1}$-algebra $\mathfrak{B}$, we have $\mathfrak{B} \models \epsilon(\mathfrak{A}, D)$ iff $\mathfrak{B} \models \gamma^m(\mathfrak{A}, D)$. This holds for all L-modal algebras because $L \supseteq \mathsf{K4}_1^{m+1}$. Thus, the logics $L + \epsilon(\mathfrak{A}, D)$ and $L + \gamma^m(\mathfrak{A}, D)$ correspond to the same variety, and hence they are the same logic. □

Similar to stable canonical rules and formulas, we can consider special splitting formulas of the form $\epsilon(\mathfrak{A}, A)$. We also call them *Jankov formula* in light of Corollary 3.59. In fact, this form of Jankov formulas appeared early in the study of union-splitting in NExtK (see, e.g., [CZ97, Section 10.5]).

Contrary to pre-transitive logics, where every finite s.i. modal algebra splits the lattice $\mathsf{NExtK4}_1^{m+1}$, only finite s.i. modal algebras of finite height split the lattice NExtK. Moreover, we will see in Theorem 4.27 that in fact any splitting formula is equivalent to a set of Jankov formulas, thus all splitting formulas axiomatize union-splittings. The following theorem is proved by Blok [Blo78], and an alternative proof can be found in [BBI16].

Theorem 3.60 ([Blo78])**.** *Let L be a logic.*

1. *L is a splitting in NExtK iff L is axiomatizable by a Jankov formula of finite s.i. modal algebras of finite height.*
2. *L is a union-splitting in NExtK iff L is axiomatizable by Jankov formulas of finite s.i. modal algebras of finite height.*

3.7 Summary

We reviewed and generalized the theory of stable canonical rules and stable canonical formulas. We identified definable filtration, a generalization of filtration, as the key property for the axiomatization results. If $\mathcal{S}$ is a rule system that admits definable filtration, then any rule system in $\mathsf{NExt}\mathcal{S}$ is axiomatizable by stable canonical rules over $\mathcal{S}$. Another factor that the master modality is definable is needed to define stable canonical formulas. We provided a dual presentation of Gabbay's filtration for pre-transitive logics $\mathsf{K4}_1^{m+1} = \mathsf{K} + \Diamond^{m+1}p \to \Diamond p$ $(m \geq 1)$ and defined stable canonical formulas for them. Similar to the case of stable canonical rules, we showed that any extension of $\mathsf{K4}_1^{m+1}$ is axiomatizable by stable canonical formulas over $\mathsf{K4}_1^{m+1}$. Moreover, we defined the m-closed domain condition and m-stable canonical formulas, strengthening the closed domain condition and stable canonical formulas for $\mathsf{K4}_1^{m+1}$. Up to logical equivalence, m-stable canonical formulas form a proper subset of stable canonical formulas. We showed that any extension of $\mathsf{K4}_1^{\mathsf{m}+1}$ is axiomatizable by m-stable canonical formulas over $\mathsf{K4}_1^{\mathsf{m}+1}$.

The aforementioned two points, i.e., admitting definable filtration and being able to define the master modality, are also the obstacles when trying to further generalize the theory of stable canonical formulas. Given the success with the logics $\mathsf{K4}_1^{\mathsf{m}+1}$, one might expect a similar result for weakly transitive logics and other pre-transitive logics. However, even though $\mathsf{wK4}$ has the fmp [BEG11], it is unknown whether it admits definable filtration (see, e.g., [KS25, Section 5]). The fmp of pre-transitive logics in general, such as $\mathsf{K4}_2^3$, is a long-standing open problem (e.g., [CZ97, Problem 11.2]), let alone definable filtration. Stable canonical rules work well for $\mathcal{S}_{\mathsf{K}}$ because rules have the master modality built into their semantics. Since the master modality is not definable in K, it seems quite challenging to define stable canonical formulas for K in a meaningful way. We leave it open to generalize stable canonical formulas to non-transitive logics other than $\mathsf{K4}_1^{\mathsf{m}+1}$.

Chapter 4

The Finite Model Property via the Refinement Construction

The *finite model property* (*fmp* for short) is one of the most important and thus well-studied properties of modal logics. It implies Kripke completeness; a finitely axiomatizable logic with the fmp is decidable. Filtration, as we saw in the last chapter, has been used to show the fmp for individual logics. Earlier fmp results for classes of logics include the following. Bull [Bul66] and Fine [Fin71] showed that every extension of $\mathsf{S4.3}$ has the fmp. Segerberg [Seg71] showed that every extension of $\mathsf{K4}$ of finite depth has the fmp. Later, the interests shifted to the fmp for classes of logics with certain semantic properties. *Subframe logics* in $\mathsf{NExtK4}$ have been defined and studied by Fine [Fin74b; Fin85], and Zakharyaschev [Zak92; Zak96] generalized them to *cofinal subframe logics*; all transitive subframe logics and cofinal subframe logics have the fmp. Moreover, Zakharyaschev [Zak97] showed the fmp for a large class of extensions of cofinal subframe logics, which implies the fmp of extensions of $\mathsf{K4}$ with modal reduction principle and the fmp of extensions of $\mathsf{S4}$ with a formula of one variable (see [CZ97, Chapter 11] for an overview). Blok [Blo78] showed that all union-splittings in NExtK have the fmp. Recently, Bezhanishvili et al. [BBI18] introduced *stable logics* and *stable rule systems* and showed that they have the fmp.

In this chapter, we introduce a novel combinatorial method, which we call the *Refinement Construction*, to establish the fmp for a broad class of logics and rule systems. This method relies on the theory of stable canonical formulas and stable canonical rules we discussed in the last chapter. It turns out that the construction works well with stable formulas/rules and stable canonical formulas/rules defined from finite modal algebras of finite height and splitting formulas. Thus, our fmp results at the same time generalize the fmp of union-splittings in NExtK and the fmp of stable logics/rule systems.

4.1 Refinement construction

In this section, we focus on the combinatorics of modal spaces. We will prove the main lemma (Lemma 4.6) by introducing a new construction, *Refinement Construction*, for modal spaces. The usefulness and applications of the main lemma will become clear in the subsequent sections.

We begin by introducing a measure that will be used in our inductive construction. For a modal space $\mathfrak{X}$ and $x \in X$, we define rank(x), the *rank of x (in $\mathfrak{X}$)*, to be the length of the longest path starting at x: formally, $\mathrm{rank}(x) = \sup\{n \in \omega : \mathfrak{X}, x \not\models \Box^{n-1}\bot\}$. For example, $\mathrm{rank}(x) = 1$ iff x is a dead end, and $\mathrm{rank}(x) = \omega$ iff there is an arbitrarily long path starting at x. It is easy to see that the rank $< \kappa$ ($\kappa \leq \omega$) part of a modal space is an upset, and the rank is non-decreasing under stable maps.

Lemma 4.1. *Let $\mathfrak{X} = (X, R)$ be a modal space and $\kappa \leq \omega$. Then the rank $< \kappa$ part of $\mathfrak{X}$ is an upset of $\mathfrak{X}$.*

Proof. Let $x, y \in \mathfrak{X}$ such that xRy and $\text{rank}(x) < \kappa$. For any finite path starting at y, adding x at the beginning results in a longer path starting at x, so $\text{rank}(y) < \text{rank}(x) < \kappa$. Thus, the rank $< \kappa$ part of $\mathfrak{X}$ is an upset of $\mathfrak{X}$. □

Lemma 4.2. *Let $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, Q)$ be modal spaces and $f : \mathfrak{X} \to \mathfrak{Y}$ be a stable map. Then, for any $x \in \mathfrak{X}$, $\text{rank}(x) \leq \text{rank}(f(x))$.*

Proof. Let $x \in \mathfrak{X}$ and $n = \text{rank}(x)$. Then there is a path $x = x_1 R \cdots R x_n$ of length n in $\mathfrak{X}$. Since f is stable, $f(x) = f(x_1) Q \cdots Q f(x_n)$ is a path of length n in $\mathfrak{Y}$, so $\text{rank}(f(x)) \geq n$. □

A related notion, *cycle-free*, has been widely used in the literature. A *cycle* in a modal space is a finite path of length ≥ 2 with the same initial point and the terminal point. A modal space $\mathfrak{X}$ is *cycle-free* if there is no cycle in $\mathfrak{X}$. For finite modal spaces, being cycle-free admits a useful characterization via the rank function.

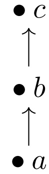
Proposition 4.3. *Let $\mathfrak{X} = (X, R)$ be a finite modal space. The following are equivalent.*

1. *$\mathfrak{X}$ is cycle-free,*
2. $\text{rank}(x) < \omega$ *for all $x \in \mathfrak{X}$,*
3. *there is some $n < \omega$ such that* $\text{rank}(x) < n$ *for all $x \in \mathfrak{X}$.*

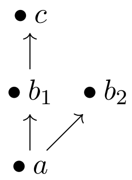
Proof. If $\mathfrak{X}$ has a cycle, then any point in the cycle has rank ω, so (2) $\Rightarrow$ (1) follows. If $\text{rank}(x) = \omega$ for some $x \in \mathfrak{X}$, then there is no $n < \omega$ such that $\text{rank}(x) < n$, so (3) $\Rightarrow$ (2) follows. Finally, suppose that there is no $n < \omega$ such that $\text{rank}(x) < n$ for all $x \in \mathfrak{X}$. So, there is some $x \in \mathfrak{X}$ such that $\text{rank}(x) > |X|$. Then there is a path of length $> |X|$ (starting at x) in $\mathfrak{X}$. Such a path must contain two identical points, which implies that $\mathfrak{X}$ has a cycle. This shows (1) $\Rightarrow$ (3). □

Before diving into the proof of the main lemma, we present a simple example of the Refinement Construction. This example illustrates how the construction works and the motivation behind it.

Example 4.4. Let $\mathcal{F}_1$ be the following modal space.

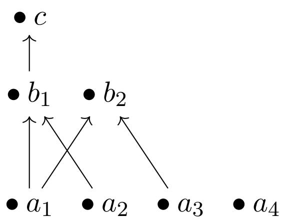


Let $\mathfrak{X} = (X, R)$ be a modal space and $f_1 : \mathfrak{X} \twoheadrightarrow_{\emptyset} \mathcal{F}_1$ be a stable map. All points in $f_1^{-1}(c)$ are dead ends. A point in $f_1^{-1}(b)$ can either see some point in $f_1^{-1}(c)$ or be a dead end. However, this distinction is lost when moving to $\mathcal{F}_1$ by f_1. We can rescue this information by letting f_1 factor through the following modal space $\mathcal{F}_2$.



Let $f_2 : \mathfrak{X} \twoheadrightarrow_\emptyset \mathcal{F}_2$ be a map behaving the same way as f_1 except for that f_2 sends points in $f_1^{-1}(b)$ that see some point in $f_1^{-1}(c)$ to b_1 and that are dead ends to b_2. One can verify that f_2 is continuous, surjective (assuming that in $f_1^{-1}(b)$ there are points that see some point in $f_1^{-1}(c)$ and that are dead ends), and stable. Also, there is a canonical stable map $g_2 : \mathcal{F}_2 \twoheadrightarrow_\emptyset \mathcal{F}_1$ identifying b_1 and b_2 as b. Indeed, we have $f_1 = g_2 \circ f_2$. Moreover, all points in $f_2^{-1}(b_1)$ see some points in $f_2^{-1}(c)$ and all points in $f_2^{-1}(b_2)$ are dead ends. In this sense, we recover the information that was lost when applying f_1 directly.

Next, we proceed with $f_2^{-1}(a)$ in a similar manner. In principal, there are eight possibilities for points in $f_2^{-1}(a)$, depending on whether they see some point in $f_2^{-1}(c)$, $f_2^{-1}(b_1)$, and $f_2^{-1}(b_2)$. But since f_2 is stable, a point in $f_2^{-1}(a)$ cannot see any point in $f_2^{-1}(c)$. So, we may consider the following modal space $\mathcal{F}_3$ and construct a similar factorization of f_2 as $g_3 \circ f_3$.



The finite modal space $\mathcal{F}_3$ and the maps $f_3 : \mathfrak{X} \twoheadrightarrow_\emptyset \mathcal{F}_3$ and $g_3 : \mathcal{F}_3 \twoheadrightarrow_\emptyset \mathcal{F}_1$ are the results of the Refinement Construction applied to the map $f_1 : \mathfrak{X} \twoheadrightarrow_\emptyset \mathcal{F}_1$. One can verify from the construction that f_3 is in fact a p-morphism.

There may be two concerns about this example of the Refinement Construction. First, not all possibilities are realized in a preimage. For example, there might be no dead end in $f_2^{-1}(a)$, so f_3 might not be surjective. This issue can be overcome by adding points only when the corresponding possibility is realized in the preimage. One may also wonder what happens if we take the closed domain $\mathcal{D}$ into account. However, as we will see in the proof of the main lemma, the CDC works perfectly with this construction.

Moreover, it should be clear from this example that the Refinement Construction only works for the cycle-free part, in particular, the irreflexive part, of a modal space. Indeed, if a is reflexive, then points in $f_2^{-1}(a)$ can see each other, and there is no appropriate way to define the relation on $\mathcal{F}_3$. This concludes our example.

Definition 4.5. Let $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, Q)$ be modal spaces and $f : \mathfrak{X} \to \mathfrak{Y}$ be a stable map. We call f *a p-morphism for* $x \in \mathfrak{X}$ if, for any $y' \in Y$ such that $f(x)Qy'$, then there exists some $x' \in X$ such that xRx' and $f(x') = y'$.

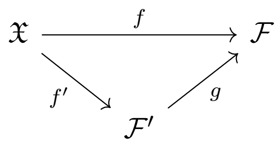
So, a stable map $f : \mathfrak{X} \to \mathfrak{Y}$ is a p-morphism iff for any $x \in X$, the map f is a p-morphism for x.

Now we are ready to introduce the *Refinement Construction*. The output of the construction is summarized in the following lemma, whose proof will be given along the way.

Lemma 4.6 (Refinement Lemma). *Let $\mathfrak{X}$ be a modal space, $\mathcal{F}$ be a finite modal space, $\mathcal{D} \subseteq \mathcal{P}(F)$, and $f : \mathfrak{X} \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$ be a stable map. Then, the Refinement Construction applied to $f : \mathfrak{X} \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$ produces*

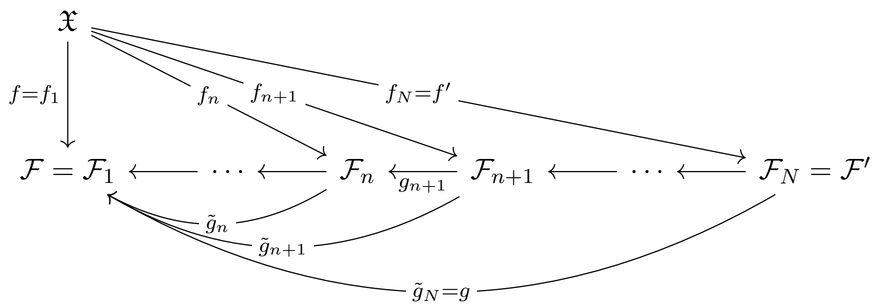
1. *a finite modal space $\mathcal{F}'$,*
2. *a stable map $f' : \mathfrak{X} \twoheadrightarrow_{\emptyset} \mathcal{F}'$ such that f' is a p-morphism for $x \in \mathfrak{X}$ with* $\mathrm{rank}(f'(x)) < \omega$,
3. *a stable map $g : \mathcal{F}' \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$ such that g is the identity between the rank ω part of $\mathcal{F}'$ and that of $\mathcal{F}$,*

such that the following diagram commutes.



Refinement Construction

Let $\mathfrak{X}$ be a modal space, $\mathcal{F}$ be a finite modal space, $\mathcal{D} \subseteq \mathcal{P}(F)$, and $f : \mathfrak{X} \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$ be a stable map. Let $\mathcal{F}_1 = \mathcal{F}$, $f_1 = f$, and $\tilde{g}_1$ be the identity map on $\mathcal{F}$. Suppose that we have constructed a finite modal space $\mathcal{F}_n = (F_n, Q_n)$, a stable map $f_n : \mathfrak{X} \twoheadrightarrow_{\emptyset} \mathcal{F}_n$ such that f_n is a p-morphism for $x \in \mathfrak{X}$ with $\mathrm{rank}(f_n(x)) \leq n$, and a stable map $\tilde{g}_n : \mathcal{F}_n \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$ such that $\tilde{g}_n$ is the identity between the rank $> n$ part of $\mathcal{F}_n$ and that of $\mathcal{F}$, and $\tilde{g}_n \circ f_n = f$. These hold for $n = 1$: in particular, note that for any $x \in \mathfrak{X}$ such that $\mathrm{rank}(f_1(x)) \leq 1$, f_1 is a p-morphism for x since $f_1(x)$ is a dead end. We construct $\mathcal{F}_{n+1}$, f_{n+1}, and $\tilde{g}_{n+1}$ as follows.



Let $C = \{c_0, \ldots, c_{m-1}\}$ be the set of points in $\mathcal{F}_n$ with rank $\leq n$. Let $V = \{v \in F_n : \mathrm{rank}(v) = n+1\}$. Fix a $v \in V$. Divide the clopen set $U^v := f_n^{-1}(v)$ by boolean combinations of the clopen sets $\{R^{-1}[f_n^{-1}(c)] : c \in C\}$:

1. $U^v = \coprod_{s \in 2^m} U^v_s$, where each U^v_s is a (possibly empty) clopen set in $\mathfrak{X}$,
2. for any $s \in 2^m$ and $x \in U^v_s$, $x \in R^{-1}[f_n^{-1}(c_i)] \iff s(i) = 1$.

Let $S^v = \{s \in 2^m : U^v_s \neq \emptyset\}$. Take a set of fresh points $W = \{w^v_s : v \in V, s \in S^v\}$. Let $\mathcal{F}_{n+1} = (F_{n+1}, Q_{n+1})$ the finite modal space where $F_{n+1} = (F_n \setminus V) \cup W$ and

1. $pQ_{n+1}q \iff pQ_nq$ for $p, q \notin W$,
2. $pQ_{n+1}w^v_s \iff pQ_nv$ for $p \notin W$,
3. $w^v_sQ_{n+1}q \iff q = c_i \wedge s(i) = 1$ for $q \notin W$,

4. $Q_{n+1} \cap W \times W = \emptyset$.

Every point in $F_{n+1} \setminus W$ has the same rank in $\mathcal{F}_{n+1}$ as in $\mathcal{F}_n$, and every point in W has rank $\leq n+1$ in $\mathcal{F}_{n+1}$. The rank $> n+1$ part of $\mathcal{F}_{n+1}$ is isomorphic to that of $\mathcal{F}_n$, and therefore to that of $\mathcal{F}$. For the sake of simplicity, we will use the same letter to denote a point in $F_{n+1} \setminus W$ and its counterpart in $F_n \setminus V$, and when the point has rank $> n+1$, also its counterpart in F.

Let $U = \bigcup_{v \in V} U^v$. Define $f_{n+1} : X \to F_{n+1}$ as follows:

1. $f_{n+1}(x) = w_s^v \quad$ if $x \in U_s^v$ for some $v \in V$ and $s \in S^v$,

2. $f_{n+1}(x) = f_n(x) \quad$ if $x \notin U$.

Then, for any $x \in X$, $f_{n+1}(x) \in W$ iff $x \in U$ iff $f_{n+1}(x) \neq f_n(x)$.

Claim 4.7. f_{n+1} is continuous and surjective.

Proof. For each $b \in F_{n+1} \setminus W$, $f_{n+1}^{-1}(b) = f_n^{-1}(b)$, which is clopen and nonempty since f_n is continuous and surjective. For each $w_s^v \in W$, $f_{n+1}^{-1}(w_s^v) = U_s^v$, which is clopen by definition and nonempty since $s \in S^v$. Thus, f_{n+1} is continuous and surjective. □

Claim 4.8. f_{n+1} is stable.

Proof. Assume that xRy in $\mathfrak{X}$. If $x, y \notin U$, then $f_{n+1}(x)Q_{n+1}f_{n+1}(y)$ since $f_n(x)Q_nf_n(y)$, $f_{n+1}(x) = f_n(x)$, and $f_{n+1}(y) = f_n(y)$. If $y \in U_s^v$ and $x \notin U$, then $f_n(x)Q_nv$ since $f_n(y) = v$, so $f_{n+1}(x)Q_{n+1}w_s^v$ by the definition of Q_{n+1} and $f_{n+1}(x) = f_n(x)$. If $y \notin U$ and $x \in U_s^v$, then $vQ_nf_n(y)$ since $f(x) = v$, so $\text{rank}((f_n(y)) < \text{rank}(v) = n+1$; it follows that $f_n(y) = c_i$ for some i, and $s(i) = 1$ since $x \in R^{-1}[f_n^{-1}(c_i)]$, hence $w_s^vQ_{n+1}c_i$ by the definition of Q_{n+1}. If $x, y \in U$, then $\text{rank}(f_n(x)) = \text{rank}(f_n(y)) = n+1$ by the definition of U and V, which contradicts $f_n(x)Q_nf_n(y)$ by f_n being stalbe, so this case cannot happen. So, in all possible cases, we verified that $f_{n+1}(x)Q_{n+1}f_{n+1}(y)$, hence f_{n+1} is stable. □

Claim 4.9. f_{n+1} is a p-morphism for $x \in \mathfrak{X}$ with $\text{rank}(f_{n+1}(x)) \leq n+1$.

Proof. Let $x \in X$ such that $\text{rank}(f_{n+1}(x)) \leq n+1$ and $q \in F_{n+1}$ such that $f_{n+1}(x)Q_{n+1}q$. If $f_{n+1}(x) \notin W$, then $f_{n+1}(x) = f_n(x)$ and $\text{rank}(f_{n+1}(x)) \leq n$ in both $\mathcal{F}_{n+1}$ and $\mathcal{F}_n$. So, f_n is a p-morphism for x, hence there is a $y \in X$ such that xRy and $f_n(y) = q$. Since $f_n(x)Q_nq$, $\text{rank}(q) \leq n$ in $\mathcal{F}_n$, so $f_{n+1}(y) = f_n(y) = q$. If $f_{n+1}(x) \in W$, then $f_{n+1}(x) = w_s^v$ for some $v \in V$ and $s \in S^v$, and $q = c_i$ for some i such that $s(i) = 1$ by the definition of Q_{n+1}. So, $x \in U_s^v$, which by definition imlies $x \in R^{-1}[f_n^{-1}(c_i)]$, hence $x \in R^{-1}[f_{n+1}^{-1}(c_i)]$. Therefore, f_{n+1} is a p-morphism for $x \in \mathfrak{X}$ with $\text{rank}(f_{n+1}(x)) \leq n+1$. □

Define $g_{n+1} : \mathcal{F}_{n+1} \to \mathcal{F}_n$ as follows:

1. $g_{n+1}(w_s^v) = v$ for $w_s^v \in W$,

2. $g_{n+1}(w) = w$ for $w \notin W$.

Let $\tilde{g}_{n+1} = \tilde{g}_n \circ g_{n+1} : \mathcal{F}_{n+1} \to \mathcal{F}$. Since $g_{n+1} \circ f_{n+1} = f_n$ and $\tilde{g}_n \circ f_n = f$, we have $\tilde{g}_{n+1} \circ f_{n+1} = f$. Also, since g_{n+1} is the identity on the rank $> n+1$ part of $\mathcal{F}_{n+1}$, and this part is isomorphic to the corresponding part of $\mathcal{F}_n$, the assumption that $\tilde{g}_n$ is the identity between the rank $> n$ part of $\mathcal{F}_n$ and that of $\mathcal{F}$ implies that $\tilde{g}_{n+1}$ is the identity between the the rank $> n+1$ part of $\mathcal{F}_{n+1}$ and that of $\mathcal{F}$.

Claim 4.10. $\tilde{g}_{n+1}$ is continuous and surjective.

Proof. Continuity is clear since $\mathcal{F}_{n+1}$ is finite, hence discrete. For the subjectivity, given that $\tilde{g}_n$ is surjective, it suffices to show that g_{n+1} is surjective, which is clear since $U^v = f_n^{-1}(v)$ is nonempty by the subjectivity of f_n for every $v \in V$. □

Claim 4.11. $\tilde{g}_{n+1}$ is stable.

Proof. Given that $\tilde{g}_n$ is stable, it suffices to show that g_{n+1} is stable. Assume that $pQ_{n+1}q$ in F_{n+1}. If $p, q \notin W$, then $g_{n+1}(p) = p$, $g_{n+1}(q) = q$, and pQ_nq by the definition of Q_{n+1}. If $p \notin W$ and $q = w_s^v$, then $g_{n+1}(q) = v$ and pQ_nv by definition of Q_{n+1}. If $p = w_s^v$ and $q \notin W$, then $q = c_i$ for some i and $s(i) = 1$ by the definition of Q_{n+1}, so there exists some (in fact, for any) $x \in U_s^v$ such that $x \in R^{-1}[f_n^{-1}(c_i)]$, and vQ_nc_i by the stability of f_n. If $p, q \in W$, then $\neg pQ_{n+1}q$ by the definition of Q_{n+1}, so this case cannot happen. So, in all possible cases, we have verified that $g_{n+1}(p)Q_ng_{n+1}(q)$, therefore g_{n+1}, and hence $\tilde{g}_{n+1}$ is stable. □

Claim 4.12. $\tilde{g}_{n+1}$ satisfies CDC for $\mathcal{D}$.

Proof. Let $D \in \mathcal{D}$ and $d \in D$. Assume that pQd in $\mathcal{F}$ and $\tilde{g}_{n+1}(p') = p$. Let $x \in f_{n+1}^{-1}(p')$, which exists since f_{n+1} is surjective. Since $f(x) = \tilde{g}_{n+1}(f_{n+1}(x)) = \tilde{g}_{n+1}(p') = p$ and f satisfies CDC for D, there exists some $y \in X$ such that xRy and $f(y) \in D$. So, $p'Q_{n+1}f_{n+1}(y)$ by the stability of f_{n+1}, and $\tilde{g}_{n+1}(f_{n+1}(y)) = f(y) \in D$. Therefore, $\tilde{g}_{n+1}$ satisfies CDC for $\mathcal{D}$. □

Since $\mathcal{F}$ is finite, there is an upperbound on the rank of all points in $\mathcal{F}$ of finite rank, say, N. Inductively applying the above construction, we obtain a finite modal space $\mathcal{F}_N = (F_N, Q_N)$, a stable map $f_N : \mathfrak{X} \twoheadrightarrow_\emptyset \mathcal{F}_N$ such that f_N is a p-morphism for $x \in \mathfrak{X}$ with $\mathrm{rank}(f_N(x)) \leq N$, and a stable map $\tilde{g}_N : \mathcal{F}_N \twoheadrightarrow_\mathcal{D} \mathcal{F}$ such that $\tilde{g}_N$ is the identity between the rank $> N$ part of $\mathcal{F}_N$ and that of $\mathcal{F}$. By the definition of N, for any $x \in \mathfrak{X}$, if $\mathrm{rank}(f_N(x)) < \omega$ then $\mathrm{rank}(f_N(x)) \leq N$. So, it follows that f_N is a p-morphism for $x \in \mathfrak{X}$ with $\mathrm{rank}(f_N(x)) < \omega$. Moreover, since, by the construction, the rank $> N$ part of $\mathcal{F}_N$ is isomorphic to that of $\mathcal{F}$, $\tilde{g}_N$ is the identity between the rank ω part of $\mathcal{F}_N$ and that of $\mathcal{F}$. The tuple $(\mathcal{F}_N, f_N, \tilde{g}_n)$ is the result of the construction, and we conclude the proof of Lemma 4.6 by taking $\mathcal{F}' = \mathcal{F}_N$, $f' = f_N$, and $g = \tilde{g}_N$.

Remark 4.13. The idea of modifying the cycle-free part of a modal space and leaving the rest as they are is inspired by the direct proof of Corollary 4.20 given in [CZ97, Theorem 10.54], which was originally proved by Blok [Blo78].

4.2 The finite model property of union-splitting logics

In this section, we apply the Refinement Construction (Lemma 4.6) to establish the fmp for union-splittings in NExtK and their relatives, thus generalizing Blok's fmp result in [Blo78].

First, we relate finite cycle-free modal spaces to their algebraic counterpart, namely, finite modal algebras of finite height. Let $\mathcal{FH}$ be the class of modal algebras *of finite height*, that is, $\mathcal{FH} = \{\mathfrak{A} : \exists n \in \omega(\mathfrak{A} \models \Box^n \bot)\}$, and $\mathcal{FH}_{\mathrm{fin}}$ be the class of finite algebras in $\mathcal{FH}$.

Proposition 4.14. *Let $\mathfrak{A}$ be a finite modal algebra and $\mathfrak{X}$ be its dual space. Then, $\mathfrak{A}$ is of finite height iff $\mathfrak{X}$ is cycle-free. Therefore, $\mathcal{FH}_{\mathrm{fin}}$ is the class of modal algebras dual to finite cycle-free modal spaces.*

Proof. Let $\mathfrak{A}$ be a finite modal algebra and $\mathfrak{X}$ be its dual space. Then,

$$\begin{aligned}\mathfrak{A} \text{ is of finite height} &\iff \text{there is some } n \in \omega \text{ such that } \mathfrak{A} \models \Box^n \bot \\ &\iff \text{there is some } n \in \omega \text{ such that } \mathfrak{X} \models \Box^n \bot \\ &\iff \text{there is some } n \in \omega \text{ such that rank}(x) < n+2 \text{ for all } x \in \mathfrak{X} \\ &\iff \mathfrak{X} \text{ is cycle free (by Proposition 4.3).}\end{aligned}$$

Thus, a finite modal algebra is of finite height iff its dual modal space is cycle-free. □

Moreover, with the dual approach, we can easily show that the property of having finite height is reflected by stable subalgebras.

Proposition 4.15. *Let $\mathfrak{A}$ be a modal algebra and $\mathfrak{B}$ be a modal algebra of finite height such that $\mathfrak{B} \hookrightarrow_\emptyset \mathfrak{A}$. Then $\mathfrak{A}$ is also of finite height. Therefore, $\mathcal{FH}$ reflects stable subalgebras.*

Proof. Let $\mathfrak{X} = (X, R)$ and $\mathfrak{Y} = (Y, Q)$ be the dual spaces of $\mathfrak{A}$ and $\mathfrak{B}$ respectively. Then there is some $n \in \omega$ such that $\mathfrak{Y} \models \Box^n \bot$ and $\mathfrak{X} \twoheadrightarrow_\emptyset \mathfrak{Y}$. Let $f : \mathfrak{X} \twoheadrightarrow_\emptyset \mathfrak{Y}$. Suppose, for a contradiction, that $\mathfrak{A}$ is not of finite height. Then, in particular, $\mathfrak{X} \not\models \Box^n \bot$, so there is a path $x_0 R x_1 R \cdots R x_n$ in $\mathfrak{X}$. Since f is stable, $f(x_0) Q f(x_1) Q \cdots Q f(x_n)$ is a path in $\mathfrak{Y}$, so $\mathfrak{Y}, f(x_0) \not\models \Box^n \bot$, which is a contradiction. Thus, $\mathfrak{A}$ is of finite height. □

Now we proceed to prove the two main theorems of this section. Both of them have a similar form: "a base rule system or logic + stable rules or formulas + splitting formulas" has the fmp. In general, the Refinement Construction does not preserve properties of modal spaces due to its combinatorial nature. But when it does, it will produce a finite countermodel that is useful for showing the fmp.

Recall that $\Sigma(L) = \mathcal{S}_\mathsf{K} + \{/\varphi : \varphi \in L\}$ for a logic L and $\Lambda(\mathcal{S}) = \{\varphi : /\varphi \in \mathcal{S}\}$ for a rule system $\mathcal{S}$.

Theorem 4.16. *Let $\{\epsilon(\mathfrak{A}_i, D_i) : i \in I\}$ be a set of splitting formulas and $\mathcal{S}$ be a rule system that admits definable filtration. Suppose that the Refinement Construction preserves $\mathcal{S}$, that is, if the modal spaces $\mathfrak{X}$ and $\mathcal{F}$ validate $\mathcal{S}$, then the modal space $\mathcal{F}'$ obtained in Lemma 4.6 also validates $\mathcal{S}$. Then, for any set of stable rules $\{\rho(\mathfrak{A}_j, \emptyset) : j \in J\}$, the rule system*

$$\mathcal{S} + \{\rho(\mathfrak{A}_j, \emptyset) : j \in J\} + \Sigma(\{\epsilon(\mathfrak{A}_i, D_i) : i \in I\})$$

has the fmp. Consequently, the logic

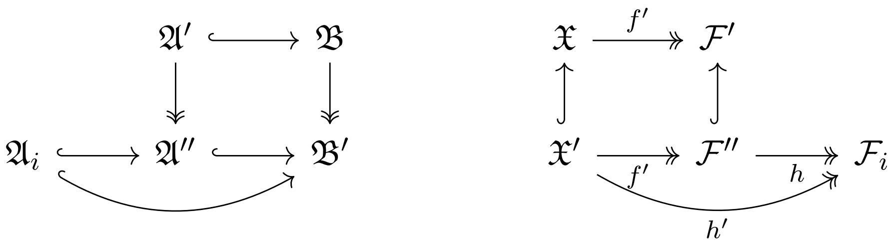
$$\Lambda(\mathcal{S} + \{\rho(\mathfrak{A}_j, \emptyset) : j \in J\} + \Sigma(\{\epsilon(\mathfrak{A}_i, D_i) : i \in I\}))$$

has the fmp.

Proof. The latter statement follows from the former by Proposition 2.63. Let $\mathcal{S}' = \mathcal{S} + \{\rho(\mathfrak{A}_j, \emptyset) : j \in J\} + \Sigma(\{\epsilon(\mathfrak{A}_i, D_i) : i \in I\}$. Let $\rho \notin \mathcal{S}'$. Then there is a modal algebra $\mathfrak{B}$ such that $\mathfrak{B} \models \mathcal{S}$, $\mathfrak{B} \models \rho(\mathfrak{A}_j, \emptyset)$ for all $j \in J$, $\mathfrak{B} \models \epsilon(\mathfrak{A}_i, D_i)$ for all $i \in I$, and $\mathfrak{B} \not\models \rho$. By Theorem 3.24, for $\mathcal{S}$-algebras, ρ is equivalent to a set of stable canonical rules $\{\rho(\mathfrak{A}_k, D_k) : 1 \leq k \leq n\}$ where each $\mathfrak{A}_k$ is a finite $\mathcal{S}-$algebra and $D_k \subseteq A_k$, so $\mathfrak{B} \not\models \rho(\mathfrak{A}_k, D_k)$ for some $1 \leq k \leq n$, namely $\mathfrak{A}_k \hookrightarrow_{D_k} \mathfrak{B}$.

Let $\mathfrak{X} = (X, R)$ and $\mathcal{F} = (F, Q)$ be the dual space of $\mathfrak{B}$ and $\mathfrak{A}_k$ respectively, and $\mathcal{D}_k = \beta[D_k]$. Then there is a stable map $f : \mathfrak{X} \twoheadrightarrow_{\mathcal{D}_k} \mathcal{F}$. Applying the Refinement Construction (Lemma 4.6), we obtain a finite modal space $\mathcal{F}'$, a stable map $f' : \mathfrak{X} \twoheadrightarrow_{\emptyset} \mathcal{F}'$ such that f' is a p-morphism for $x \in \mathfrak{X}$ with $\text{rank}(f'(x)) < \omega$, and a stable map $g : \mathcal{F}' \twoheadrightarrow_{\mathcal{D}_k} \mathcal{F}$ such that g is the identity between the rank ω part of $\mathcal{F}'$ and that of $\mathcal{F}$, and $f = g \circ f'$. Let $\mathfrak{A}'$ be the dual algebra of $\mathcal{F}'$. Since by our assumption the Refinement Construction preserves $\mathcal{S}$, we have $\mathcal{F}' \models \mathcal{S}$, so $\mathfrak{A}'$ is a finite $\mathcal{S}$-algebra. If $\mathfrak{A}' \not\models \rho(\mathfrak{A}_j, \emptyset)$ for some $j \in J$, then $\mathfrak{A}_j \hookrightarrow_{\emptyset} \mathfrak{A}'$. But since $\mathfrak{X} \twoheadrightarrow_{\emptyset} \mathcal{F}'$, i.e., $\mathfrak{A}' \hookrightarrow_{\emptyset} \mathfrak{B}$, we would obtain $\mathfrak{A}_j \hookrightarrow_{\emptyset} \mathfrak{B}$, i.e., $\mathfrak{B} \not\models \rho(\mathfrak{A}_j, \emptyset)$, which is a contradiction. Moreover, since $\mathcal{F}' \twoheadrightarrow_{\mathcal{D}_k} \mathcal{F}$, dually we have $\mathfrak{A}_k \hookrightarrow_{D_k} \mathfrak{A}'$, so $\mathfrak{A}' \not\models \rho(\mathfrak{A}_k, D_k)$, hence $\mathfrak{A}' \not\models \rho$.

It remains to show that $\mathfrak{A}' \models \epsilon(\mathfrak{A}_i, D_i)$ for all $i \in I$. Assume for a contradiction that $\mathfrak{A}' \not\models \epsilon(\mathfrak{A}_i, D_i)$ for some $i \in I$. Then $\mathfrak{A}_i \hookrightarrow_{D_i} \mathfrak{A}''$ for a s.i. homomorphic image $\mathfrak{A}''$ of $\mathfrak{A}'$. Dually, this means that there is a rooted closed upset $\mathcal{F}'' \subseteq \mathcal{F}'$ such that $\mathcal{F}'' \twoheadrightarrow_{\mathcal{D}_i} \mathcal{F}_i$ ($\mathcal{D}_i = \beta[D_i]$), where $\mathcal{F}''$ is dual to $\mathfrak{A}''$ and $\mathcal{F}_i$ to $\mathfrak{A}_i$. Since $\mathfrak{A}_i$ is of finite height, $\mathfrak{A}''$ is so by Proposition 4.15, namely, $\mathcal{F}''$ is cycle-free by Proposition 4.14, which by Proposition 4.3 implies that any point in $\mathcal{F}'$ has rank $\leq N$ for some N. Let s_0 be the root of $\mathcal{F}''$ and $x_0 \in f'^{-1}(s_0)$. Then $\text{rank}(s_0) \leq N$ and $\text{rank}(x_0) \leq N$ since f' is stable. Let $X' = R^{<\omega}[x_0] = R^{\leq N}[x_0]$ and $\mathfrak{X}'$ be X' with the topology and relation induced by $\mathfrak{X}$. Then $\mathfrak{X}'$ is a closed upset of $\mathfrak{X}$, and $\mathfrak{X}'$ is topo-rooted because x_0 is a topo-root of $\mathfrak{X}'$ and $\{x_0\} = f'^{-1}(s_0) \cap X'$ is clopen in $\mathfrak{X}'$.



Recall that $f' : \mathfrak{X}' \twoheadrightarrow_{\emptyset} \mathcal{F}''$ is surjective and stable. Since $f' : \mathfrak{X} \twoheadrightarrow_{\emptyset} \mathcal{F}'$ is a p-morphism for $x \in \mathfrak{X}'$ with $\text{rank}(f'(x)) < \omega$ and $\mathcal{F}''$ is cycle-free, $f' \restriction \mathfrak{X}' : \mathfrak{X}' \twoheadrightarrow \mathcal{F}''$ is a surjective p-morphism. Let $h : \mathcal{F}'' \twoheadrightarrow_{\mathcal{D}_i} \mathcal{F}_i$ and $h' = h \circ f' \restriction \mathfrak{X}'$. Then, we have $h' : \mathfrak{X}' \twoheadrightarrow_{\mathcal{D}'} \mathcal{F}_i$. Thus, $\mathfrak{X}'$ is a topo-rooted closed upset of $\mathfrak{X}$ such that $\mathfrak{X}' \twoheadrightarrow_{\mathcal{D}'} \mathcal{F}_i$. Dually, if we let $\mathfrak{B}'$ be the dual of $\mathfrak{X}'$, then this means that $\mathfrak{B}'$ is a s.i. homomorphic image of $\mathfrak{B}$ such that $\mathfrak{A}_i \hookrightarrow_{D_i} \mathfrak{B}'$, namely, $\mathfrak{B} \not\models \epsilon(\mathfrak{A}_i, D_i)$, which is a contradiction.

Therefore, $\mathfrak{A}'$ is a finite modal algebra such that $\mathfrak{A}' \models \mathcal{S}'$ and $\mathfrak{A}' \not\models \rho$, and we conclude that the rule system $\mathcal{S}'$ has the fmp. □

Corollary 4.17. *For any stable rule system $\mathcal{S}$, the rule system*

$$\mathcal{S} + \Sigma(\{\epsilon(\mathfrak{A}_i, D_i) : i \in I\})$$

has the fmp. Consequently, the logic

$$\Lambda(\mathcal{S} + \Sigma(\{\epsilon(\mathfrak{A}_i, D_i) : i \in I\}))$$

has the fmp.

Proof. $\mathcal{S}_{\mathsf{K}}$ admits (definable) filtration and is clearly preserved by the Refinement Construction. Thus, the former statement follows from Theorems 3.30 and 4.16. The latter statement follows from the former by Proposition 2.63. □

Dropping stable rules in Theorem 4.16, we obtain a general fmp result for logics.

Lemma 4.18. *For any logics L and L',*

$$\Lambda(\Sigma(L) + \Sigma(L')) = L + L'.$$

Proof. Let $\mathcal{V}$ and $\mathcal{V}'$ be the corresponding varieties of L and L' respectively. Then $\Sigma(L)$ and $\Sigma(L')$ also correspond to $\mathcal{V}$ and $\mathcal{V}'$. Thus, since $\mathcal{V} \cap \mathcal{V}'$ is a variety and hence a universal class, $\Sigma(L) + \Sigma(L')$ corresponds to $\mathcal{V} \cap \mathcal{V}'$, which also corresponds to $\Lambda(\Sigma(L) + \Sigma(L'))$. Since $L + L'$ corresponds to $\mathcal{V} \cap \mathcal{V}'$, we conclude $\Lambda(\Sigma(L) + \Sigma(L')) = L + L'$. □

Theorem 4.19. *Let $\{\epsilon(\mathfrak{A}_i, D_i) : i \in I\}$ be a set of splitting formulas and L be a logic that admits definable filtration. Suppose that the Refinement Construction preserves L, that is, if the modal spaces $\mathfrak{X}$ and $\mathcal{F}$ validate L, then the modal space $\mathcal{F}'$ obtained in Lemma 4.6 also validates L. Then, the logic*

$$L + \{\epsilon(\mathfrak{A}_i, D_i) : i \in I\}$$

has the fmp.

Proof. Let L be a logic that admits definable filtration. Then the rule system $\mathcal{S}_L = \Sigma(L)$ also admits definable filtration because they correspond to the same class of modal algebras. So, by Theorem 4.16, the logic $\Lambda(\Sigma(L) + \Sigma(\{\epsilon(\mathfrak{A}_i, D_i) : i \in I\}))$ has the fmp. By Lemma 4.18, we have

$$\Lambda(\Sigma(L) + \Sigma(\{\epsilon(\mathfrak{A}_i, D_i) : i \in I\})) = L + \{\epsilon(\mathfrak{A}_i, D_i) : i \in I\},$$

where the statement follows. □

This immediately implies Blok's fmp result [Blo78] (see also [CZ97, Theorem 10.54]).

Corollary 4.20 ([Blo78]). *Every union-splitting in* NExtK *has the fmp.*

Proof. K admits (definable) filtration and is clearly preserved by the Refinement Construction. Thus, the statement follows from Theorems 3.60 and 4.19. □

Corollary 4.21. *Every finitely axiomatizable union-splitting in* NExtK *is decidable.*

In practice, it is not always easy to verify whether a logic is preserved by the Refinement Construction. We show in the following lemma that this is the case for pre-transitive logics $\mathsf{K4}_1^{\mathsf{m+1}}$ ($m \geq 1$) considered in Section 3.2. Recall that pre-transitive logics $\mathsf{K4}_1^{\mathsf{m+1}}$ are logics of the form $\mathsf{K} + \Diamond^{m+1}p \to \Diamond p$. They define the condition $\forall x \forall y\, (xR^{m+1}y \to xRy)$ for modal spaces. Note that $\mathsf{K4}_1^2$ is the transitive logic $\mathsf{K4}$.

Lemma 4.22. *Let $m \geq 1$. The Refinement Construction preserves $\mathsf{K4}_1^{\mathsf{m+1}}$, that is, if the modal spaces $\mathfrak{X}$ and $\mathcal{F}$ validate $\mathsf{K4}_1^{\mathsf{m+1}}$, then the modal space $\mathcal{F}'$ obtained in Lemma 4.6 also validates $\mathsf{K4}_1^{\mathsf{m+1}}$.*

Proof. Let $\mathfrak{X} = (X, R)$ be a $\mathsf{K4}_1^{\mathsf{m+1}}$-space and $\mathcal{F} = (F, Q)$ be a finite $\mathsf{K4}_1^{\mathsf{m+1}}$-space. Applying the Refinement Construction (Lemma 4.6), we obtain a finite modal space $\mathcal{F}' = (F', Q')$, a stable map $f' : \mathfrak{X} \twoheadrightarrow_{\emptyset} \mathcal{F}'$ such that f' is a p-morphism for $x \in \mathfrak{X}$ with $\mathrm{rank}(f'(x)) < \omega$, and a stable map $g : \mathcal{F}' \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$ such that g is the identity between the rank ω part of $\mathcal{F}'$ and that of $\mathcal{F}$, and $f = g \circ f'$. Let $s_0, \ldots, s_m \in \mathcal{F}'$ be such that $s_{i-1}Q's_i$ for each $1 \leq i \leq m$. We show that $s_0 Q' s_m$.

If $\mathrm{rank}(s_0) < \omega$, then $\mathrm{rank}(s_i) < \omega$ for all $0 \leq i \leq m$. Since $f' : \mathfrak{X} \twoheadrightarrow_{\emptyset} \mathcal{F}'$ is a p-morphism for $x \in \mathfrak{X}$ with $\mathrm{rank}(f'(x)) < \omega$, we obtain a chain $x_0 R \cdots R x_m$ in $\mathfrak{X}$ such that $f'(x_i) = s_i$ for each $0 \leq i \leq m$. Then $x_0 R x_m$ since $\mathfrak{X} \models \mathsf{K4}_1^{\mathsf{m+1}}$ by assumption, which implies $s_0 Q' s_m$ since f' is stable.

Suppose that $\mathrm{rank}(s_0) = \omega$. Since $g : \mathcal{F}' \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$ is stable, we have $g(s_0)Q \cdots Qg(s_m)$ in $\mathcal{F}$. Then $g(s_0)Qg(s_m)$ since $\mathcal{F} \models \mathsf{K4}_1^{\mathsf{m+1}}$ by assumption. Again since g is stable, $\mathrm{rank}(g(s_0)) = \omega$. Reviewing the Refinement Construction, either $\mathrm{rank}(g(s_m)) = \omega$ so that s_m remains the same as $g(s_m)$ or $\mathrm{rank}(g(s_m)) < \omega$ so that s_m is added with $g(s_m)$ replaced, we must have $s_0Q's_m$ by $g(s_0)Qg(s_m)$.

Thus, in both cases we have $s_0Q's_m$. Hence, $\mathcal{F}'$ validates $\mathsf{K4}_1^{\mathsf{m+1}}$. □

The following characterization of $\mathsf{K4}_1^{\mathsf{m+1}}$-stable logics is a straightforward generalization of the series of characterizations of $\mathsf{K4}$-stable logics obtained in [BBI18].

Lemma 4.23. *Let $m \geq 1$. A logic $L \supseteq \mathsf{K4}_1^{\mathsf{m+1}}$ is $\mathsf{K4}_1^{\mathsf{m+1}}$-stable iff the class $\mathcal{V}(L)_{\mathrm{si}}$ is finitely $\mathsf{K4}_1^{\mathsf{m+1}}$-stable. Moreover, each $\mathsf{K4}_1^{\mathsf{m+1}}$-stable logic is axiomatizable by stable formulas over $\mathsf{K4}_1^{\mathsf{m+1}}$.*

Proof. We only sketch the proof. Since $\mathsf{K4}_1^{\mathsf{m+1}}$ admits definable filtration, following the proof of [BBI18, Theorem 3.8], we can show that $\mathcal{V}(L)$ is generated by a $\mathsf{K4}_1^{\mathsf{m+1}}$-stable class $\mathcal{K}$ of finite $\mathsf{K4}_1^{\mathsf{m+1}}$-algebras. Then, [BBI18, Lemma 4.5] also holds for $\mathsf{K4}_1^{\mathsf{m+1}}$ because for any finite $\mathsf{K4}_1^{\mathsf{m+1}}$-space $\mathcal{F}$, adding an extra point to $\mathcal{F}$ so that it sees every point in $\mathcal{F}$ and itself results in another $\mathsf{K4}_1^{\mathsf{m+1}}$-sapce. Finally, following the proof of [BBI18, Theorem 4.7] and using Lemma 3.33, we obtain the statement. □

Now we show our second main theorem. Recall that a logic $L \supseteq \mathsf{K4}_1^{\mathsf{m+1}}$ is $\mathsf{K4}_1^{\mathsf{m+1}}$-stable if the variety $\mathcal{V}(L)$ is generated by a $\mathsf{K4}_1^{\mathsf{m+1}}$-stable class.

Theorem 4.24. *Let $m \geq 1$ and L be a $\mathsf{K4}_1^{\mathsf{m+1}}$-stable logic. For any set of splitting formulas $\{\epsilon(\mathfrak{A}_i, D_i) : i \in I\}$, the logic*

$$L + \{\epsilon(\mathfrak{A}_i, D_i) : i \in I\}$$

has the fmp. Consequently, for any set of stable canonical formulas $\{\gamma^m(\mathfrak{A}_j, D_j) : j \in J\}$ where each $\mathfrak{A}_j$ is a finite s.i. $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra of finite height, the logic

$$L + \{\gamma^m(\mathfrak{A}_j, D_j) : j \in J\}$$

has the fmp.

Proof. The latter statement follows from the former with Corollary 3.59. By Lemma 4.23, there is a set of stable formulas $\{\gamma^m(\mathfrak{A}_j, \emptyset) : j \in J\}$ such that $L = \mathsf{K4}_1^{\mathsf{m+1}} + \{\gamma^m(\mathfrak{A}_j, \emptyset) : j \in J\}$. Let $L' = L + \{\epsilon(\mathfrak{A}_i, D_i) : i \in I\}$. Let $\varphi \notin L'$. Then there is a s.i. modal algebra $\mathfrak{B}$ such that $\mathfrak{B} \models \mathsf{K4}_1^{\mathsf{m+1}}$, $\mathfrak{B} \models \gamma^m(\mathfrak{A}_j, \emptyset)$ for all $j \in J$, $\mathfrak{B} \models \epsilon(\mathfrak{A}_i, D_i)$ for all $i \in I$, and $\mathfrak{B} \not\models \varphi$. Since $\mathsf{K4}_1^{\mathsf{m+1}}$ admits definable filtration and $\mathsf{K4}_1^{\mathsf{m+1}}$ and $\Sigma(\mathsf{K4}_1^{\mathsf{m+1}})$ correspond to the same class of modal algebras, by Theorem 3.24, for $\mathsf{K4}_1^{\mathsf{m+1}}$-algebras, φ is equivalent to a set of stable canonical rules $\{\rho(\mathfrak{A}_k, D_k) : 1 \leq k \leq n\}$ where each $\mathfrak{A}_k$ is a finite $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra and $D_k \subseteq A_k$. So, $\mathfrak{B} \not\models \rho(\mathfrak{A}_k, D_k)$ for some $1 \leq k \leq n$, namely $\mathfrak{A}_k \hookrightarrow_{D_k} \mathfrak{B}$.

Let $\mathfrak{X} = (X, R)$ and $\mathcal{F} = (F, Q)$ be the dual space of $\mathfrak{B}$ and $\mathfrak{A}_k$ respectively, and $\mathcal{D}_k = \beta[D_k]$. Then there is a stable map $f : \mathfrak{X} \twoheadrightarrow_{\mathcal{D}_k} \mathcal{F}$. Applying the Refinement Construction (Lemma 4.6), we obtain a finite modal space $\mathcal{F}'$, a stable map $f' : \mathfrak{X} \twoheadrightarrow_{\emptyset} \mathcal{F}'$ such that f' is a p-morphism for $x \in \mathfrak{X}$ with $\mathrm{rank}(f'(x)) < \omega$, and a stable map $g : \mathcal{F}' \twoheadrightarrow_{\mathcal{D}_k} \mathcal{F}$ such that g is the identity between the rank ω part of $\mathcal{F}'$ and that of $\mathcal{F}$, and $f = g \circ f'$. Let $\mathfrak{A}'$ be the dual algebra of $\mathcal{F}'$. Since the Refinement Construction preserves $\mathsf{K4}_1^{\mathsf{m+1}}$ by Lemma 4.22, the same argument in the proof of Theorem 4.16 shows that $\mathfrak{A}'$ is a finite $\mathsf{K4}_1^{\mathsf{m+1}}$-algebra, $\mathfrak{A}' \models \epsilon(\mathfrak{A}_i, D_i)$ for all $i \in I$, and $\mathfrak{A}' \not\models \varphi$.

It remains to show that $\mathfrak{A}' \models \gamma^m(\mathfrak{A}_j, \emptyset)$ for all $j \in J$. It suffices to show that $\mathfrak{A}' \models L$. Note that $\mathfrak{B}$ is s.i. and $\mathfrak{A}'$ is a finite stable subalgebra of $\mathfrak{B}$ since $\mathfrak{X} \twoheadrightarrow_{\emptyset} \mathcal{F}'$. Thus, since L is $\mathsf{K4}_1^{\mathsf{m+1}}$-stable, $\mathfrak{A} \models L$ by Lemma 4.23. Therefore, $\mathfrak{A}'$ is a finite modal algebra such that $\mathfrak{A}' \models L'$ and $\mathfrak{A}' \not\models \varphi$, and we conclude that the logic L' has the fmp. □

Remark 4.25. The series of fmp results so far can be compared with the general fmp result regarding *cofinal subframe logics* and *canonical formulas* in [CZ97, Theorem 11.55]. These results have interesting consequences that every extension of $\mathsf{S4}$ (or IPC) with finitely many axioms in one variable has the fmp [CZ97, Theorem 11.58 and Corollary 11.59]. It remains open if our results lead to such a concrete fmp result.

Another naturally arising question from our results is which logics or rule systems admit definable filtration and are preserved by the Refinement Construction. A reasonable sufficient condition will make the results more useful. We leave this as an open question.

Question 4.26. *Is there a non-trivial class of logics or rule systems that admit definable filtration and are preserved by the Refinement Construction, other than the pre-transitive logics* $\mathsf{K4}_1^{\mathsf{m+1}}$ $(m \geq 1)$ *we have discussed?*

The proof strategy we have used in this section yields another result about splitting formulas and stable canonical formulas defined from finite s.i. modal algebras of finite height. Essentially, it implies that the closed domain condition collapses to p-morphisms when working with finite s.i. modal algebras of finite height.

Theorem 4.27. *Let* $\mathfrak{A}$ *be a finite s.i. modal algebra of finite height and* $D \subseteq A$*. Then, the logic* $L = \mathsf{K} + \epsilon(\mathfrak{A}, D)$ *is a union-splitting in* NExtK*.*

Proof. Let

$$L' = \mathsf{K} + \{\epsilon(\mathfrak{A}_i, A_i) : \mathfrak{A} \text{ is a finite s.i. modal algebra of finite height such that } \mathfrak{A} \hookrightarrow_D \mathfrak{A}_i\}.$$

It suffices to show $L = L'$ since L' is a union-splitting in NExtK by Theorem 3.60. Let $\mathfrak{B}$ be a modal algebra.

Suppose that $\mathfrak{B} \not\models L'$. Then $\mathfrak{B} \not\models \epsilon(\mathfrak{A}_i, A_i)$ for some $i \in I$. So, $\mathfrak{A}_i$ is a subalgebra of a s.i. homomorphic image $\mathfrak{B}'$ of $\mathfrak{B}$. Since $\mathfrak{A} \hookrightarrow_D \mathfrak{A}_i$, we have $\mathfrak{A} \hookrightarrow_D \mathfrak{B}'$, which implies $\mathfrak{B} \not\models \epsilon(\mathfrak{A}, D)$, hence $\mathfrak{B} \not\models L$.

Conversely, suppose that $\mathfrak{B} \not\models L$. Then $\mathfrak{B} \not\models \epsilon(\mathfrak{A}, D)$, namely, $\mathfrak{A} \hookrightarrow_D \mathfrak{B}'$ for a s.i. homomorphic image $\mathfrak{B}'$ of $\mathfrak{B}$. Let $\mathfrak{X}'$ and $\mathcal{F}$ be the dual space of $\mathfrak{B}'$ and $\mathfrak{A}$ respectively, and $\mathcal{D} = \beta[D]$. Then $\mathfrak{X}'$ is a topo-rooted closed upset of $\mathfrak{X}$ such that $\mathfrak{X}' \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$. Let $f : \mathfrak{X}' \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$. Since $\mathfrak{A}$ is a finite modal algebra of finite height, $\mathcal{F}$ is finite and cycle-free by Proposition 4.14, so there is some $N \in \omega$ such that any point in $\mathcal{F}$ has rank $\leq N$ by Proposition 4.3. Since $f : \mathfrak{X}' \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$ is stable, any point in $\mathfrak{X}'$ has rank $\leq N$. $\mathfrak{X}'$ is topo-rooted since $\mathfrak{B}'$ is s.i. If x is a topo-root of $\mathfrak{X}'$, then $\mathfrak{X}'$ is the closure of $R^N[x]$, which implies $X' = R_X^n[x]$ since $R^N[x]$ is closed in $\mathfrak{X}$. Thus, a topo-root of $\mathfrak{X}'$ is a root of $\mathfrak{X}'$. So, $\mathfrak{X}'$ is rooted.

Applying the Refinement Construction (Lemma 4.6) to $f : \mathfrak{X}' \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$, we obtain a finite modal space $\mathcal{F}'$, a stable map $f' : \mathfrak{X}' \twoheadrightarrow_{\emptyset} \mathcal{F}'$ such that f' is a p-morphism for $x \in \mathfrak{X}'$ with $\mathrm{rank}(f'(x)) < \omega$, and a stable map $g : \mathcal{F}' \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$ such that $f = g \circ f'$. Since $\mathcal{F}$ is cycle-free, $\mathcal{F}'$ is also cycle-free. Thus, $f' : \mathfrak{X} \twoheadrightarrow \mathcal{F}'$ is a p-morphism. Since $\mathfrak{X}'$ is rooted and f' is surjective and stable, $\mathcal{F}'$ is also rooted. Let

$\mathfrak{A}'$ be the dual algebra of $\mathcal{F}'$. Then $\mathfrak{A}'$ is a finite s.i. modal algebra of finite height such that $\mathfrak{A} \hookrightarrow_D \mathfrak{A}'$ and $\mathfrak{A}' \hookrightarrow \mathfrak{B}'$. Thus, $\mathfrak{A}'$ coincides with $\mathfrak{A}_i$ for some $i \in I$ and $\mathfrak{B}' \not\models \epsilon(\mathfrak{A}', A')$. Then $\mathfrak{B} \not\models \epsilon(\mathfrak{A}', A')$ since $\mathfrak{B}'$ is a homomorphic image of $\mathfrak{B}$. So, $\mathfrak{B} \not\models L'$, and therefore, $L = L'$. □

Corollary 4.28. *For any set of splitting formulas* $\{\epsilon(\mathfrak{A}_i, D_i) : i \in I\}$*, the logic*

$$\mathsf{K} + \{\epsilon(\mathfrak{A}_i, D_i) : i \in I\}$$

is a union-splitting in NExtK*.*

Theorem 4.29. *For any set of stable canonical formulas* $\{\gamma^m(\mathfrak{A}_i, D_i) : i \in I\}$ *where each* $\mathfrak{A}_i$ *is a finite s.i.* $\mathsf{K4}_1^{\mathsf{m+1}}$*-algebra of finite height, the logic*

$$\mathsf{K4}_1^{\mathsf{m+1}} + \{\gamma^m(\mathfrak{A}_i, D_i) : i \in I\}$$

is a union-splitting in $\mathsf{NExtK4}_1^{\mathsf{m+1}}$*.*

Proof. Since the Refinement Construction preserves $\mathsf{K4}_1^{\mathsf{m+1}}$ by Lemma 4.22, the statement follows from the same proof as in Theorem 4.27. □

In terms of the lattice of pre-transitive logics, our fmp results also have implications on the degree of Kripke incompleteness.

Corollary 4.30. *Every union-splitting in* $\mathsf{NExtK4}_1^{\mathsf{m+1}}$ *split by a set of finite s.i.* $\mathsf{K4}_1^{\mathsf{m+1}}$*-algebras of finite height has the degree of Kripke incompleteness 1 in* $\mathsf{NExtK4}_1^{\mathsf{m+1}}$*.*

Proof. This follows directly from Theorem 4.24 and Theorem 2.57. □

Determining the degree of Kripke incompleteness in the lattice $\mathsf{NExtK4}$ is a long-standing open question (e.g., [CZ97, Problem 10.5]). It is even unknown whether all union-splittings in $\mathsf{NExtK4}$ are Kripke complete, let alone what their degree of Kripke incompleteness is. In this respect, Corollary 4.30 identifies a subclass of union-splittings in $\mathsf{NExtK4}$ that have the fmp and thus is Kripke complete.

4.3 The finite model property of union-splitting rule systems

In this section, we apply the Refinement Construction to study the fmp for rule systems. The proofs will be quite similar to the ones in the last section, and are even simpler because stable canonical rules have a simpler semantic characterization than stable canonical formulas, so we will only present sketches. The following fmp result is a rule system analogue of Theorem 4.16.

Theorem 4.31. *Let* $\{\rho(\mathfrak{A}_i, D_i) : i \in I\}$ *be a set of stable canonical rules where each* $\mathfrak{A}_i$ *is a finite modal algebra of finite height, and* $\mathcal{S}$ *be a rule system that admits definable filtration. Suppose that the Refinement Construction preserves* $\mathcal{S}$*, that is, if the modal spaces* $\mathfrak{X}$ *and* $\mathcal{F}$ *validate* $\mathcal{S}$*, then the modal space* $\mathcal{F}'$ *obtained in Lemma 4.6 also validates* $\mathcal{S}$*. Then, for any set of stable rules* $\{\rho(\mathfrak{A}_j, \emptyset) : j \in J\}$*, the rule system*

$$\mathcal{S} + \{\rho(\mathfrak{A}_j, \emptyset) : j \in J\} + \{\rho(A_i, \mathcal{D}_i) : i \in I\}$$

has the fmp.

Proof. This is proved in a similar manner to the proof of Theorem 4.16 by using Theorem 3.24. □

Corollary 4.32. *For any stable rule system $\mathcal{S}$, the rule system*

$$\mathcal{S} + \{\rho(A_i, \mathcal{D}_i) : i \in I\}$$

has the fmp.

Proof. $\mathcal{S}_\mathsf{K}$ admits (definable) filtration and is clearly preserved by the Refinement Construction. So, the statement follows directly from Theorem 4.31 and Theorem 3.30. □

Similar to Theorem 4.27, we also obtain an observation about the closed domain in stable canonical rules defined from finite modal algebras of finite height.

Theorem 4.33. *Let $\mathfrak{A}$ be a finite modal algebra of finite height and $D \subseteq A$. Then, the rule system $\mathcal{S} = \mathcal{S}_\mathsf{K} + \rho(\mathfrak{A}, D)$ is a union-splitting in $\mathsf{NExt}\mathcal{S}_\mathsf{K}$.*

Proof. Let

$$\mathcal{S}' = \mathcal{S}_\mathsf{K} + \{\rho(\mathfrak{A}_i, A_i) : \mathfrak{A}_i \text{ is a finite modal algebra of finite height such that } \mathfrak{A} \hookrightarrow_D \mathfrak{A}_i\}.$$

It suffices to show $\mathcal{S} = \mathcal{S}'$ since $\mathcal{S}'$ is a union-splitting in $\mathsf{NExt}\mathcal{S}_\mathsf{K}$ by Theorem 3.32. This is proved by a similar argument as in the proof of Theorem 4.27. □

Corollary 4.34. *For any set of stable canonical rules $\{\rho(\mathfrak{A}_i, D_i) : i \in I\}$ where each $\mathfrak{A}_i$ is a finite modal algebra of finite height, the rule system*

$$\mathcal{S}_\mathsf{K} + \{\rho(\mathfrak{A}_i, D_i) : i \in I\}$$

is a union-splitting in $\mathsf{NExt}\mathcal{S}_\mathsf{K}$.

This subclass of union-splittings in $\mathsf{NExt}\mathcal{S}_\mathsf{K}$ has a neat characterization. It turns out that they are the lower part of union-splittings in $\mathsf{NExt}\mathcal{S}_\mathsf{K}$. A similar idea for K-union-splittings will appear later in Theorem 5.5. Recall that $\mathcal{FH}$ is the class of modal algebras of finite height. Let $\mathcal{C}$ be the complement of $\mathcal{FH}$, and $\mathcal{C}_{\text{fin}}$ be the class of finite members of $\mathcal{C}$.

Theorem 4.35. *For any rule system $\mathcal{S}$, the following are equivalent:*

1. $\mathcal{C} \subseteq \mathcal{U}(\mathcal{S})$,
2. $\mathcal{C}_{\text{fin}} \subseteq \mathcal{U}(\mathcal{S})$,
3. *$\mathcal{S}$ is axiomatized over $\mathcal{S}_\mathsf{K}$ by Jankov rules of finite modal algebras of finite height,*
4. *$\mathcal{S}$ is axiomatized over $\mathcal{S}_\mathsf{K}$ by stable canonical rules of finite modal algebras of finite height,*
5. $\mathcal{S} \subseteq \mathcal{S}_\mathsf{K} + \{\rho(\mathfrak{A}, A) : \mathfrak{A} \in \mathcal{FH}_{\text{fin}}\}$.

Proof. (1) ⇒ (2): This is clear.

(2) ⇒ (3): Suppose that $\mathcal{C}_{\text{fin}} \subseteq \mathcal{U}(\mathcal{S})$. Let

$$\mathcal{S}' = \mathcal{S}_\mathsf{K} + \{\rho(\mathfrak{A}, A) : \mathfrak{A} \text{ is a finite modal algebra such that} \mathfrak{A} \not\models \mathcal{S}\}.$$

Since $\mathcal{C}_{\text{fin}} \subseteq \mathcal{U}(\mathcal{S})$, $\mathcal{S}'$ is axiomatized over $\mathcal{S}_{\mathsf{K}}$ by Jankov rules of finite modal algebras of finite height, thus $\mathcal{S}'$ has the fmp by Theorem 4.31. It suffices to show that $\mathcal{S} = \mathcal{S}'$.

If $\mathcal{S} \not\subseteq \mathcal{S}'$, then by the fmp of $\mathcal{S}'$, there is a finite modal algebra $\mathfrak{A}$ such that $\mathfrak{A} \models \mathcal{S}'$ and $\mathfrak{A} \not\models \mathcal{S}$, which contradicts the definition of $\mathcal{S}'$ since $\mathfrak{A} \not\models \rho(\mathfrak{A}, A)$. So, $\mathcal{S} \subseteq \mathcal{S}'$. Convsersely, if $\mathcal{S}' \not\subseteq \mathcal{S}$, then there is a modal algebra $\mathfrak{B}$ such that $\mathfrak{B} \models \mathcal{S}$ and $\mathfrak{B} \not\models \rho(\mathfrak{A}, A)$ for some finite modal algebra $\mathfrak{A} \not\models \mathcal{S}$. So, $\mathfrak{A} \hookrightarrow \mathfrak{B}$, and since $\mathcal{U}(\mathcal{S})$ is closed under subalgebras, $\mathfrak{A} \in \mathcal{U}(\mathcal{S})$, which contradicts $\mathfrak{A} \not\models \mathcal{S}$. So, $\mathcal{S}' \subseteq \mathcal{S}$.

(3) $\Rightarrow$ (1): For any $\mathfrak{B} \in \mathcal{C}$ and any finite modal algebra $\mathfrak{A}$ of finite height, since being of finite height reflects stable subalgebras by Proposition 4.15, $\mathfrak{A} \not\hookrightarrow_{\emptyset} \mathfrak{B}$, so $\mathfrak{B} \models \rho(\mathfrak{A}, A)$. So, if $\mathcal{S}$ is axiomatized over $\mathcal{S}_{\mathsf{K}}$ by Jankov rules of finite algebras of finite height, then $\mathcal{C} \models \mathcal{S}$, i.e., $\mathcal{C} \subseteq \mathcal{U}(\mathcal{S})$.

(3) $\Leftrightarrow$ (4): This follows from Corollary 4.34.

(3) $\Rightarrow$ (5): This is clear.

(5) $\Rightarrow$ (2): This follows from a similar argument as in the case (3) $\Rightarrow$ (1). ☐

Similar to the case of $\mathsf{NExtK4}$ and contrary to the case of NExtK, it is unknown if all union-splittings in $\mathsf{NExt}\mathcal{S}_{\mathsf{K}}$ have the fmp. This is expected since all finite algebras split $\mathsf{NExt}\mathcal{S}_{\mathsf{K}}$.

Question 4.36. *Does every union-splitting in* $\mathsf{NExt}\mathcal{S}_{\mathsf{K}}$ *have the fmp?*

4.4 Summary

We introduced the Refinement Construction as a method for showing the fmp for logics and rule systems. Given a modal space $\mathfrak{X}$, a finite modal space $\mathcal{F}$, and a stable map $f : \mathfrak{X} \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$ where $\mathcal{D} \subseteq \mathcal{P}(\mathcal{F})$, the construction makes the map f factor through a finite modal space $\mathcal{F}'$, which will be the desired finite countermodel. We summarize the logics and rule systems that were shown in this chapter to have the fmp as follows:

- A logic that admits definable filtration and is preserved by the Refinement Construction + splitting formulas.
- A $\mathsf{K4}_1^{\mathsf{m+1}}$-stable logic + splitting formulas (or equivalently, stable canonical formulas defined from finite s.i. $\mathsf{K4}_1^{\mathsf{m+1}}$-algebras of finite height).
- A rule system that admits definable filtration and is preserved by the Refinement Construction + stable rules + splitting formulas.
- A stable rule system + splitting formulas.
- A rule system that admits definable filtration and is preserved by the Refinement Construction + stable rules + stable canonical rules defined from finite modal algebras of finite height.
- A stable rule system + stable canonical rules defined from finite modal algebras of finite height.

Note that by Proposition 2.63, for a rule system $\mathcal{S}$ in the above list, the logic $\Lambda(\mathcal{S})$ also has the fmp. These fmp results imply the fmp of union-splittings in NExtK [Blo78], $\mathsf{K4}$-stable logics [BBI18], and stable rule systems [BBI16]. Moreover, it follows that every union-splitting in $\mathsf{NExtK4}_1^{\mathsf{m+1}}$ split by a set of finite s.i. $\mathsf{K4}_1^{\mathsf{m+1}}$-algebras of finite height has the degree of Kripke incompleteness 1 in $\mathsf{NExtK4}_1^{\mathsf{m+1}}$.

As a side result, we observed that for splitting formulas and stable canonical formulas defined from finite s.i. $\mathsf{K4}_1^{\mathsf{m+1}}$-algebras of finite height, the closed domain condition collapses to p-morphisms. Thus, logics of the form "K + splitting formulas" are union-splittings in NExtK and logics of the form "$\mathsf{K4}_1^{m+1}$ + stable canonical formulas defined from finite s.i. $\mathsf{K4}_1^{m+1}$-algebras of finite height" are union-splittings in $\mathsf{NExtK4}_1^{m+1}$.

Further applications of the Refinement Construction are open. A characterization or sufficient condition of logics or rule systems preserved by the Refinement Construction will lead to more fmp results. Although the above list contains many logics and rule systems, they are presented in a rather technical way. These results are in line with Zakharyaschev's fmp result [CZ97, Theorem 11.55] proved using canonical formulas, while it has the consequence that every extension of $\mathsf{S4}$ with finitely many axioms in one variable has the fmp [CZ97, Theorem 11.58]. We leave it for further research whether our results and technique can be used to obtain such a concrete fmp result.

Chapter 5

Decidability of Logical Properties

In this chapter, we study the decidability of logical properties. Intuitively, our question is: is there an algorithm that, given a modal logic, decides whether it has a specific property? There are different ways to formulate this question, depending on how logics are encoded as input. Note that an input must be a finite object, so it is certainly not possible to take all of the continuum many logics into account. The most general possible formulation is to consider all recursively axiomatizable logics, encoded by recursive functions that enumerate them. However, Kuznetsov showed that this only leads to triviality, similar to Rice's Theorem for partial recursive functions. Kuznetsov left the result unpublished, but one can find a proof in [CZ97, Section 17.1]. The result also holds for $\mathsf{NExtK4}$, $\mathsf{NExtS4}$, and other lattices of normal modal logics.

Theorem 5.1 (Kuznetsov)**.** *Let P be a non-trivial property of recursively axiomatizable logics, that is, there are a recursively axiomatizable logic that has P and a recursively axiomatizable logic that does not have P. Then it is undecidable whether a recursively axiomatizable logic in* NExtK *has P.*

So, we will confine ourselves to finitely axiomatizable logics. Since most logics we encounter in practice are finitely axiomatizable, this is not a serious drawback. A finitely axiomatizable logic will be encoded by a finite subformula-closed set of formulas axiomatizing the logic, or equivalently, a single formula axiomatizing the logic. Now we can formulate our question as follows.

Definition 5.2. Let L_0 be a modal logic. A logical property P is *decidable* in $\mathsf{NExt}L_0$ iff the set $\{\varphi : L_0 + \varphi \text{ has } P\}$ is decidable.

We will identify a property P with the set of logics having P and write $L \in P$ if L has P.

5.1 Undecidable logical properties

Many logical properties have been shown to be undecidable, see [CZ97, Section 17.6] for a historical overview. The proof of [WZ07, Theorem 9], based on Chagrov's method and the proof of Theorem 2.58, provides a very general scheme to establish the undecidability of logical properties, which can be summarized as the following theorem. Note that although the statements there were written for specific logical properties, their proof in fact shows more.

Theorem 5.3. *Let P be a logical property. If there is a logic $L \in P$ that is not a* K*-union-splitting, and any logic in P is Kripke complete, then P is undecidable in* NExtK*.*

Sketch. We briefly sketch the proof idea of [WZ07, Theorem 9]. Recall that $\mathsf{KF}(L)$ is the class of Kripke frames validating L.

We want to reduce an undecidable problem Q ([CZ97, Theorem 16.3]) about Minsky machines, also called counter machines with two tapes (see, e.g., [CZ97, Section 16.1]). Fix an arbitrary logic L that is not a K-union-splitting. Then we can construct a reduction f, given an input x of Q, computing a finitely axiomatizable logic $f(x)$ such that, for any input x of Q, $x \in Q$ implies $f(x) = L$ and $x \notin Q$ implies $f(x) \neq L$ and $\mathsf{KF}(f(x)) = \mathsf{KF}(L)$. So, $x \in Q$ implies $f(x) \in P$ since $L \in P$, and $x \notin Q$ implies $f(x) \notin P$ since any logic in P is Kripke complete. Thus, this reduction shows that P is undecidable. □

Theorem 5.3 shows that almost all interesting logical properties are undecidable in NExtK, including Kripke completeness, finite model property, tabularity, first-order definability, to name a few. One can also show that decidability is undecidable by adjusting this proof. For the details, we refer to [WZ07, Theorem 9] and the subsequent discussion. We add one more to this family.

Corollary 5.4. *Being a stable logic is undecidable in* NExtK.

Proof. All stable logics have the fmp, and all extensions of $\mathsf{S5}$ are stable [BBI18]. Since $\mathsf{KD} = \mathsf{K} + \Diamond\top$ is the largest K-union-splitting, any extension of $\mathsf{S5}$ witnesses that there is a stable logic that is not a K-union-splitting. It follows from Theorem 5.3 that being a stable logic is undecidable. □

On the other hand, we point out three limitations of this scheme. First, the construction of the reduction in the proof of Theorem 5.3 relies heavily on the proof of Blok's dichotomy Theorem 2.58, which only works for NExtK. The other two limitations are when a property P does not satisfy the assumptions of the theorem. For example, since there are Kripke incomplete subframe logics over K [Wol93], Theorem 5.3 does not apply to subframeness. Moreover, there is certainly no K-union-splitting that is not a K-union-splitting, so it neither applies to union-splittings. It turns out that being a union-splitting is decidable in NExtK.

5.2 Decidability of being a K-union-splitting logic

Given the generality of Theorem 5.3, it might seem that all meaningful logical properties would be undecidable in NExtK. Indeed, it was pointed out in [WZ07] that "we know only two interesting decidable properties of finitely axiomatizable logics in NExtK: consistency and coincidence with K," which are more or less trivial. However, in this section, we show that being a union-splitting is decidable in NExtK, answering the open question [WZ07, Problem 2] in the affirmative.

We first give a semantic characterization of K-union-splittings. The idea is similar to that of Theorem 4.35. Recall that $\mathcal{FH}$ is the class of modal algebras of finite height, that is, $\mathcal{FH} = \{\mathfrak{A} : \exists n \in \omega(\mathfrak{A} \models \Box^n\bot)\}$.

Theorem 5.5. *For any modal logic L, the following are equivalent:*

1. *L is a K-union-splitting.*
2. *L is axiomatized over K by Jankov formulas of finite s.i. modal algebras of finite height,*
3. *L is axiomatized over K by splitting formulas of finite s.i. modal algebras of finite height,*
4. *For any modal algebra $\mathfrak{A}$, $\mathsf{H}(\mathfrak{A})_{\mathrm{si}} \cap \mathcal{FH} \subseteq \mathcal{V}(L)$ implies $\mathfrak{A} \in \mathcal{V}(L)$,*

5. *For any finite modal algebra* $\mathfrak{A}$*,* $\mathsf{H}(\mathfrak{A})_{\text{fsi}} \cap \mathcal{FH} \subseteq \mathcal{V}(L)$ *implies* $\mathfrak{A} \in \mathcal{V}(L)$*,*

Proof. (1) ⇔ (2) ⇔ (3): this follows from Theorem 3.60 and Corollary 4.28.

(2) ⇒ (4): let $\mathfrak{A}$ be a modal algebra such that $\mathfrak{A} \not\models L$. By (2), $\mathfrak{A} \not\models \epsilon(\mathfrak{B}, \mathfrak{B})$ for some finite s.i. modal algebra $\mathfrak{B}$ of finite height such that $\epsilon(\mathfrak{B}, \mathfrak{B}) \in L$. So, $\mathfrak{B} \hookrightarrow \mathfrak{A}'$ for a s.i. homomorphic image $\mathfrak{A}'$ of $\mathfrak{A}$. Since $\mathfrak{B}$ is of finite height, it follows from Proposition 4.15 that $\mathfrak{A}'$ is also of finite height. Thus, $\mathfrak{A}' \in \mathsf{H}(\mathfrak{A})_{\text{si}} \cap \mathcal{FH}$. Moreover, since $\mathfrak{B} \hookrightarrow \mathfrak{A}'$, $\mathfrak{A}' \not\models \epsilon(\mathfrak{B}, \mathfrak{B})$, hence $\mathfrak{A}' \not\models L$. So, $\mathsf{H}(\mathfrak{A})_{\text{si}} \cap \mathcal{FH} \not\subseteq \mathcal{V}(L)$.

(4) ⇒ (5): this is clear because for any finite modal algebra $\mathfrak{A}$, $\mathsf{H}(\mathfrak{A})_{\text{fsi}} = \mathsf{H}(\mathfrak{A})_{\text{si}}$.

(5) ⇒ (2): let $L' = \mathsf{K} + \{\epsilon(\mathfrak{B}, \mathfrak{B}) : \mathfrak{B} \in \mathcal{FH}_{\text{fsi}}, \mathfrak{B} \not\models L\}$. It suffices to show that $L = L'$.

We know from Corollary 4.20 that L' has the fmp. If $L \not\subseteq L'$, then by the fmp of L', there is a finite modal algebra $\mathfrak{A}$ such that $\mathfrak{A} \models L'$ and $\mathfrak{A} \not\models L$. So, by (5), there is a $\mathfrak{B} \in \mathsf{H}(\mathfrak{A})_{\text{fsi}} \cap \mathcal{FH}$ such that $\mathfrak{B} \not\models L$. Since $\epsilon(\mathfrak{B}, \mathfrak{B}) \in L'$ by the definition of L' and $\mathfrak{B} \not\models \epsilon(\mathfrak{B}, \mathfrak{B})$, $\mathfrak{B} \not\models L'$. Since H preserves validity, we have $\mathfrak{A} \not\models L'$, which is a contradiction. Thus, $L \subseteq L'$.

If $L' \not\subseteq L$, then there is a modal algebra $\mathfrak{A}$ such that $\mathfrak{A} \models L$ and $\mathfrak{A} \not\models L'$. By the definition of L', $\mathfrak{A} \not\models \epsilon(\mathfrak{B}, \mathfrak{B})$ for some $\mathfrak{B} \in \mathcal{FH}_{\text{fsi}}$ such that $\mathfrak{B} \not\models L$. So, $\mathfrak{B} \hookrightarrow \mathfrak{A}'$ for a s.i. homomorphic image $\mathfrak{A}'$ of $\mathfrak{A}$. Since H and $\mathcal{S}$ preserve validity, $\mathfrak{A} \not\models L$, which is a contradiction. Thus, $L' \subseteq L$, and therefore, $L = L'$. □

As an immediate consequence, we see that KD is the largest K-union-splitting.

Corollary 5.6. KD *is the largest* K*-union-splitting.*

Proof. For any $\mathfrak{A} \in \mathcal{FH}_{\text{fsi}}$, the dual space of $\mathfrak{A}$ is cycle-free and refutes $\mathsf{D} = \Diamond\top$, so $\mathfrak{A} \notin \mathcal{V}(L)$. Thus, the condition (5) in Theorem 5.5 is trivially satisfied, hence KD is a K-union-splitting.

On the other hand, let L be a K-union-splitting. For any finite D-algebra $\mathfrak{A}$, since the dual space of $\mathfrak{A}$ is serial, there is no cycle-free upset of $\mathfrak{A}$, so $\mathsf{H}(\mathfrak{A})_{\text{fsi}} \cap \mathcal{FH} = \emptyset$, thus $\mathfrak{A} \in \mathcal{V}(L)$ by (5). It follows that $\mathcal{V}(\mathsf{KD}) \subseteq \mathcal{V}(L)$, namely, $L \subseteq \mathsf{KD}$. So, KD is the largest K-union-splitting. □

Moreover, we can use the characterization to decide whether a concrete logic is a K-union-splitting or not.

Example 5.7.

1. $\mathsf{K} + \Box\Diamond\top$ is not a K-union-splitting. Let $\mathfrak{X}$ be the modal space $\circ\!\to\!\bullet$ and $\mathfrak{A}$ be its dual algebra. Clearly, $\mathfrak{A} \not\models \Box\Diamond\top$. On the other hand, since the only cycle-free upset of $\mathfrak{X}$ is the irreflexive singleton, which validates $\Box\Diamond\top$, it holds that $\mathsf{H}(\mathfrak{A})_{\text{fsi}} \cap \mathcal{FH} \subseteq \mathcal{V}(\mathsf{K} + \Box\Diamond\top)$.

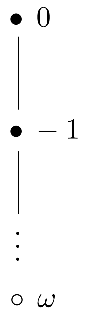
2. $\mathsf{K} + \Box\Diamond\top \vee \Diamond\Diamond\top$ is a K-union-splitting. Let $\varphi = \mathsf{K} + \Box\Diamond\top \vee \Diamond\Diamond\top$. Let $\mathfrak{A}$ be a finite modal algebra with the dual space $\mathfrak{X}$ such that $\mathfrak{A} \not\models \varphi$. Then there is a rooted upset $\mathfrak{X}' \subseteq \mathfrak{X}$ where φ is refuted at the root. It follows that $\mathfrak{X}'$ has depth 2, so $\mathfrak{X}'$ is cycle-free. Let $\mathfrak{A}'$ be the dual algebra of $\mathfrak{X}'$. Then $\mathfrak{A}' \in \mathsf{H}(\mathfrak{A})_{\text{fsi}} \cap \mathcal{FH}$ and $\mathfrak{A}' \notin \mathcal{V}(\mathsf{K} + \varphi)$, hence $\mathsf{H}(\mathfrak{A})_{\text{fsi}} \cap \mathcal{FH} \not\subseteq \mathcal{V}(\mathsf{K} + \varphi)$.

The condition (5) in Theorem 5.5 is special in its finitary nature. It should be seen in contrast to subframe logics over K4 and K4-stable logics. The following examples show that:

- there is a logic $L \in \mathsf{NExtK4}$ such that the class of finite rooted L-spaces is closed under subframes, while there is an L-frame $\mathcal{F}$ and a finite subframe $\mathcal{F}'$ of $\mathcal{F}$ such that $\mathcal{F}' \not\models L$, so L is not a subframe logic,

- there is a logic $L \in \mathsf{NExtK4}$ such that the class of finite rooted L-spaces is closed under stable images, while there is an L-frame $\mathcal{F}$ and a finite stable image $\mathcal{F}'$ of $\mathcal{F}$ such that $\mathcal{F}' \not\models L$, so L is not a $\mathsf{K4}$-stable logic.

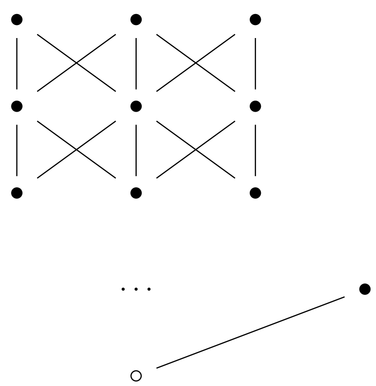
Example 5.8. Let $\mathcal{F}$ be the Kripke frame of negative integers with the order $<$ and a reflexive root ω at the bottom.



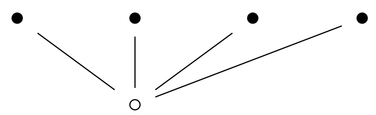
$\mathcal{F}$

For any finite rooted frame $\mathcal{G} \models \mathsf{Log}\mathcal{F}$, since $\mathcal{G} \not\models \gamma(\mathcal{G}, \mathcal{P}(\mathcal{G}))$, $\gamma(\mathcal{G}, \mathcal{P}(\mathcal{G})) \notin \mathsf{Log}\mathcal{F}$, namely $\mathcal{F} \not\models \gamma(\mathcal{G}, \mathcal{P}(\mathcal{G}))$, so $\mathcal{G}$ is a p-morphic image of a generated subframe of $\mathcal{F}$. Note that a p-morphism cannot identify two irreflexive points in $\mathcal{F}$. So, since $\mathcal{G}$ is finite, $\mathcal{G}$ must be a finite irreflexive chain. Thus, the class of finite rooted $\mathsf{Log}\mathcal{F}$-spaces is the class of finite irreflexive chains, which is closed under subframes. However, $\mathcal{F}$ has the single reflexive point as its subframe, which refutes $\mathsf{Log}\mathcal{F}$.

Example 5.9. Let $\mathcal{F}$ and $\mathcal{F}'$ be the $\mathsf{K4}$-s shown below. These frames are inspired by the proof of [CZ97, Proposition 9.50]



$\mathcal{F}$



$\mathcal{F}'$

Let $\mathcal{C}$ be the class of all finite rooted frames of width ≤ 3 and $L = \mathsf{Log}(\mathcal{C} \cup \{\mathcal{F}\})$. For any finite rooted frame $\mathcal{G} \models L$, since $\mathcal{G} \not\models \gamma(\mathcal{G}, \mathcal{P}(\mathcal{G}))$, $\gamma(\mathcal{G}, \mathcal{P}(\mathcal{G})) \notin L$, so $\mathcal{C} \cup \{\mathcal{F}\} \not\models \gamma(\mathcal{G}, \mathcal{P}(\mathcal{G}))$. Then $\mathcal{G}$ is a p-morphic image of a generated subframe of some frame in $\mathcal{C}$ or $\mathcal{F}$. Since p-morphisms do not increase width, in both cases $\mathcal{G}$ has width ≤ 3. For any stable image $\mathcal{G}'$ of $\mathcal{G}$, since stable maps do not increase width, $\mathcal{G}'$ has width ≤ 3, so $\mathcal{G}' \in \mathcal{C}$ and $\mathcal{G}' \models L$. However, $\mathcal{F}'$ is a finite stable image of $\mathcal{F}$, and $\mathcal{F}' \not\models L$ since it has width 4.

Now we turn to the decidability of union-splittings in NExtK.

Lemma 5.10. *Let L_0 be a modal logic. If $L_0 + \varphi = L_0 + \{\psi_i : i \in I\}$, then there is a finite subset $I' \subseteq I$ such that $L_0 + \varphi = L_0 + \{\psi_i : i \in I'\}$.*

Proof. Let Φ be a set of formulas. Since modal logics have finitary proof systems, for any $\varphi \in L_0 + \Phi$, $\varphi \in L_0 + \Phi'$ for some finite subset $\Phi' \subseteq \Phi$. It follows that the closure operator $\Phi \mapsto L_0 + \Phi$ on the set of formulas is algebraic. So, by Theorem 2.34, $\mathsf{NExt}L_0$ is an algebraic lattice, and finitely axiomatizable logics are exactly compact elements in $\mathsf{NExt}L_0$.

If $L_0 + \varphi = L_0 + \{\psi_i : i \in I\}$, then by the definition of compact elements, there is a finite subset $I' \subseteq I$ such that $L_0 + \varphi \subseteq L_0 + \{\psi_i : i \in I'\}$. Since $L_0 + \{\psi_i : i \in I'\} \subseteq L_0 + \{\psi_i : i \in I\} = L_0 + \varphi$, we obtain $L_0 + \varphi = L_0 + \{\psi_i : i \in I'\}$. □

Theorem 5.11. *Being a union-splitting is decidable in* NExtK*. That is, it is decidable, given a formula* φ*, whether the logic* $\mathsf{K} + \varphi$ *is a union-splitting in* NExtK*.*

Proof. First, we show that the union-splitting problem is Σ^0_1. By Lemma 5.10, $\mathsf{K} + \varphi$ is a union-splitting iff there is a finite set $\{\mathfrak{A}_i : i \leq n\}$ of finite s.i. modal algebras of finite height such that $\mathsf{K} + \varphi = \mathsf{K} + \{\epsilon(\mathfrak{A}_i, \mathfrak{A}_i) : i \leq n\}$. Dually, this means that $\mathsf{K} + \varphi = \mathsf{K} + \{\epsilon(\mathfrak{X}_i, \mathcal{P}(\mathfrak{X}_i)) : i \leq n\}$ for a finite set $\{\mathfrak{X}_i : i \leq n\}$ of finite rooted cycle-free modal spaces. Since it is decidable whether a finite modal space is rooted and cycle-free, finite rooted cycle-free modal spaces and finite sets of them can be effectively enumerated. Moreover, since a finitely axiomatized logic is recursively enumerable, the problem whether $\mathsf{K} + \varphi = \mathsf{K} + \{\epsilon(\mathfrak{X}_i, \mathcal{P}(\mathfrak{X}_i)) : i \leq n\}$ is Σ^0_1. As Σ^0_1 is closed under existential quantification, it follows that the union-splitting problem is Σ^0_1.

Next, we show that the union-splitting problem is Π^0_1. By Theorem 5.5 and the duality, $\mathsf{K} + \varphi$ is not a union-splitting iff there is a finite modal space $\mathfrak{X}$ such that all rooted cycle-free upsets of $\mathfrak{X}$ validate φ and $\mathfrak{X} \not\models \varphi$. A finite modal space $\mathfrak{X}$ only has finitely many rooted cycle-free upsets, which can be effectively computed. So, whether $\mathsf{K} + \varphi$ is not a union-splitting is Σ^0_1, hence the union-splitting problem is Π^0_1.

Thus, the union-splitting problem is both Σ^0_1 and Π^0_1, hence decidable. □

We can also provide an intuitive description of an algorithm that decides K-union-splittings as follows. We start enumerating all finite sets $\{\epsilon(\mathfrak{X}_i, \mathcal{P}(\mathfrak{X}_i)) : i \leq n\}$ where $\mathfrak{X}_i$'s are finite rooted cycle-free modal spaces. During the enumeration, for each enumerated finite set, we start verifying whether $\mathsf{K} + \varphi = \mathsf{K} + \{\epsilon(\mathfrak{X}_i, \mathcal{P}(\mathfrak{X}_i)) : i \leq n\}$ holds. If $K + \varphi$ is a union-splitting, then eventually the enumeration will find a finite set $\{\epsilon(\mathfrak{X}_i, \mathcal{P}(\mathfrak{X}_i)) : i \leq n\}$ that axiomatizes $\mathsf{K} + \varphi$ and the identification verification halts. (One might be concerned about "nested" computation here, but this is fine because we have a computable bijection from ω to $\omega \times \omega$, which is the main reason that Σ^0_1 is closed under existential quantification.) Simultaneously, we start enumerating all finite rooted cycle-free modal spaces. For each of them, we compute all its rooted cycle-free upsets and check if it witnesses that $K + \varphi$ breaks the condition (5) in Theorem 5.5. If $K + \varphi$ is not a union-splitting, we will eventually find such a witness. Combining these two, we can decide whether $K + \varphi$ is a union-splitting.

This algorithm is constructive, in the sense that if $\mathsf{K} + \varphi$ is a union-splitting, then it can output a finite set $\{\epsilon(\mathfrak{X}_i, \mathcal{P}(\mathfrak{X}_i)) : i \leq n\}$ that axiomatizes $\mathsf{K} + \varphi$. From this, we can show the decidability of splittings in NExtK.

Theorem 5.12. *Being a splitting is decidable in* NExtK*. That is, it is decidable, given a formula* φ*, whether the logic* $\mathsf{K} + \varphi$ *is a splitting in* NExtK*.*

Proof. First, we run the algorithm from Theorem 5.11. If $\mathsf{K} + \varphi$ is not a union-splitting, then the algorithm outputs false, and we are done because $\mathsf{K} + \varphi$ is not a splitting. Suppose that $\mathsf{K} + \varphi$ is a union-splitting. Then the algorithm can output an axiomatization $\mathsf{K} + \varphi = \mathsf{K} + \{\epsilon(\mathfrak{A}_i, \mathfrak{A}_i) : i \leq n\}$, where each $\mathfrak{A}_i$ is a finite s.i. modal algebra of finite height. If $n = 0$, then $\mathsf{K} + \varphi = \mathsf{K}$ is not a splitting. So, we may assume $n \geq 1$.

If $\mathsf{K} + \varphi$ is a splitting, then $\mathsf{K} + \varphi = \mathsf{K} + \epsilon(\mathfrak{B}, \mathfrak{B})$ for some finite s.i. modal algebra $\mathfrak{B}$ of finite height. For each $\mathfrak{A}_i$, since $\mathfrak{A}_i \not\models \epsilon(\mathfrak{A}_i, \mathfrak{A}_i)$, we have $\mathfrak{A}_i \not\models \epsilon(\mathfrak{B}, \mathfrak{B})$, so there is a homomorphic image $\mathfrak{A}'_i$ of $\mathfrak{A}_i$ such that $\mathfrak{B} \hookrightarrow \mathfrak{A}'_i$, thus $|B| \leq |A_i|$. Let $m = \max\{|\mathfrak{A}_i| : i \leq n\}$. Then $|B| \leq m$. Thus, $\mathsf{K} + \varphi$ is a splitting iff $\mathsf{K} + \{\epsilon(\mathfrak{A}_i, \mathfrak{A}_i) : i \leq n\} = \mathsf{K} + \epsilon(\mathfrak{B}, \mathfrak{B})$ for some finite s.i. modal algebra $\mathfrak{B}$ of finite height such that $|B| \leq m$.

There are only finitely many such $\mathfrak{B}$, and we can effectively enumerate all of them. Also, given a $\mathfrak{B}$, whether $\mathsf{K} + \{\epsilon(\mathfrak{A}_i, \mathfrak{A}_i) : i \leq n\} = \mathsf{K} + \epsilon(\mathfrak{B}, \mathfrak{B})$ holds is decidable because both logics are decidable by Corollary 4.21. Thus, it is decidable whether $\mathsf{K} + \varphi$ is a splitting. □

Another application of this constructive algorithm is on the axiomatization problem for union-splittings. Given a modal logic L_0 and a formula φ, the *axiomatization problem* for $L_0 + \varphi$ is, given a formula ψ, to decide whether $L_0 + \psi = L_0 + \varphi$. In other words, the axiomatization problem for $L_0 + \varphi$ is decidable iff the property "$L_0 + (-) = L_0 + \varphi$" is decidable in $\mathsf{NExt}L_0$.

Lemma 5.13. *Let $\mathsf{K}+\varphi$ be a K-union-splitting. Then the axiomatization problem for $\mathsf{K}+\varphi$ is decidable.*

Proof. Let $\mathsf{K} + \varphi$ be a K-union-splitting. Applying the algorithm from Theorem 5.11, we obtain an axiomatization $\mathsf{K} + \varphi = \mathsf{K} + \{\epsilon(\mathfrak{A}_i, \mathfrak{A}_i) : i \leq n\}$, where each $\mathfrak{A}_i$ is a finite s.i. modal algebra of finite height. If $n = 0$, then $\mathsf{K} + \varphi = \mathsf{K}$, and the decidability of the axiomatization problem for $\mathsf{K} + \varphi$ follows from the decidability of K. So, we may assume $n \geq 1$.

Given a formula ψ, $\mathsf{K} + \psi = \mathsf{K} + \varphi$ iff $\psi \in \mathsf{K} + \{\epsilon(\mathfrak{A}_i, \mathfrak{A}_i) : i \leq n\}$ and $\epsilon(\mathfrak{A}_i, \mathfrak{A}_i) \in \mathsf{K} + \psi$ for $i \leq n$. Whether $\psi \in \mathsf{K} + \{\epsilon(\mathfrak{A}_i, \mathfrak{A}_i) : i \leq n\}$ holds is decidable by Corollary 4.21. To decide whether $\epsilon(\mathfrak{A}_i, \mathfrak{A}_i) \in \mathsf{K} + \psi$ holds, note that if $\epsilon(\mathfrak{A}_i, \mathfrak{A}_i) \in \mathsf{K} + \psi$ then $\mathfrak{A}_i \not\models \psi$ since $\mathfrak{A}_i \not\models \epsilon(\mathfrak{A}_i, \mathfrak{A}_i)$, and if $\mathfrak{A}_i \not\models \psi$, then $\mathsf{K} + \psi \not\subseteq \mathsf{Log}\mathfrak{A}_i$, so $\mathsf{K} + \epsilon(\mathfrak{A}_i, \mathfrak{A}_i) \subseteq \mathsf{K} + \psi$ since $(\mathsf{K} + \epsilon(\mathfrak{A}_i, \mathfrak{A}_i), \mathsf{Log}\mathfrak{A}_i)$ is a splitting pair, and thus $\epsilon(\mathfrak{A}_i, \mathfrak{A}_i) \in \mathsf{K} + \psi$. So, it suffices to check whether $\mathfrak{A}_i \not\models \psi$ holds, which is decidable. Hence, it is decidable whether $\mathsf{K} + \psi = \mathsf{K} + \varphi$. □

Theorem 5.14. *The axiomatization problem for a consistent logic $\mathsf{K} + \varphi$ is decidable iff $\mathsf{K} + \varphi$ is a union-splitting in* NExtK.

Proof. The left-to-right direction can be proved by a similar method used in the proof of Theorem 5.3, combining Chagrov's method and the proof of Theorem 2.58; see [WZ07, Theorem 7] for details. □

Remark 5.15. Theorem 5.14 is already proved in [WZ07, Theorem 7], but unlike their proof, our proof is constructive, in the sense that there is an algorithm that, given a formula φ such that $\mathsf{K}+\varphi$ is a K-union-splitting, outputs an algorithm that decides the axiomatization problem for $\mathsf{K} + \varphi$. Essentially, Lemma 5.13 says that the process of obtaining a finite axiomatization via Jankov formulas for a given K-union-splitting $\mathsf{K} + \varphi$ is computable.

A formula φ is called a *(un)decidable formula* in $\mathsf{NExt}L$ if it is (un)decidable, given a formula ψ, whether $\varphi \in L + \psi$ (see, e.g., [CZ97, Section 16.4]). From the proof of Lemma 5.13, we can observe that if $\mathsf{K} + \varphi$ is a K-union-splitting, then φ is a decidable formula in NExtK: we first compute an axiomatization $\mathsf{K} + \varphi = \mathsf{K} + \{\epsilon(\mathfrak{A}_i, \mathfrak{A}_i) : i \leq n\}$, then $\varphi \in \mathsf{K} + \psi$ iff $\mathfrak{A}_i \not\models \psi$ for all i. Moreover, the proof of [WZ07, Theorem 7] in fact established the converse: if $\mathsf{K} + \varphi$ is not a K-union-splitting, then φ is an undecidable formula in NExtK. Hence, we obtain another characterization of K-union-splittings.

Theorem 5.16. *φ is a consistent decidable formula in NExtK iff $\mathsf{K} + \varphi$ is a union-splitting in NExtK.*

Thus, we also obtain a somewhat mysterious equivalence between decidable axiomatization problems and decidable formulas.

Corollary 5.17. *φ is a (un)decidable formula in NExtK iff the axiomatization problem for $\mathsf{K} + \varphi$ is (un)decidable.*

Proof. For a consistent formula φ, the equivalence follows from Theorem 5.14 and Theorem 5.16. For an inconsistent formula φ, the equivalence holds because $\varphi \in \mathsf{K} + \psi$ iff $\mathsf{K} + \varphi = \mathsf{K} + \psi$ (= the inconsistent logic). □

Finally, we see three immediate consequences of Theorem 5.11.

Corollary 5.18. *Blok's dichotomy is decidable. In other words, it is decidable whether $\mathsf{K} + \varphi$ has degree of Kripke incompleteness 1 or $2^{\aleph_0}$ in NExtK.*

Proof. This follows from Theorem 2.58 and Theorem 5.11. □

Corollary 5.19. *It is decidable whether the axiomatization problem for $\mathsf{K} + \varphi$ is decidable.*

Proof. This follows from Theorem 5.14, Theorem 5.11, and the fact that consistency is decidable (φ is consistent iff $\circ \models \varphi$ or $\bullet \models \varphi$). □

Corollary 5.20. *It is decidable whether φ is a (un)decidable formula in NExtK.*

Proof. This follows from Theorem 5.16, Theorem 5.11, and the fact that consistency is decidable. □

This answers [CZ97, Problem 17.3] for NExtK.

5.3 Summary

We showed that the properties of being a union-splitting and being a splitting are decidable in NExtK by providing a semantic characterization of K-union-splittings. These answer the open question [WZ07, Problem 2] affirmatively. Moreover, we observed that a formula φ is a decidable formula in NExtK iff the axiomatization problem for $\mathsf{K} + \varphi$ is decidable iff $\mathsf{K} + \varphi$ is a union-splitting in NExtK or the inconsistent logic. Consequently, the decidability of being a union-splitting implies the decidability of being a decidable formula and having a decidable axiomatization problem. This answers another open question [CZ97, Problem 17.3] for NExtK in the affirmative. We leave it open whether these three properties are decidable in other lattices of modal logics such as $\mathsf{NExtK4}$ and $\mathsf{NExtS4}$.

Chapter 6

Decidability of Admissibility and the Rule Dichotomy Property

Admissible rules in a logic can be regarded as valid inferences in that logic. The decidability of admissibility in a logic is then a natural strengthening of the decidability of a logic. Friedman [Fri75] asked whether the admissibility of a given inference rule in IPC is decidable. Rybakov showed that this is the case for IPC and a large class of transitive modal and superintuitionistic logics (see [Ryb97] for a comprehensive overview and references). However, the decidability of admissibility in K is a long-standing open question (e.g., [CZ97, Problem 16.4]).

Recently, Jeřábek [Jeř09] introduced a new method to construct admissible bases and establish the decidability of admissibility. Since the admissibility in a decidable logic is Π^0_1 by definition, the existence of a Σ^0_1 admissible base implies the decidability of admissibility. Given a logic L, the method consists of two parts:

1. find a class of rules that (effectively) axiomatizes all rules over L,

2. provide an admissible base for those selected rules over L.

The latter is done concurrently by proving the *rule dichotomy property over L* (Definition 6.6) for the class of rules.

Jeřábek [Jeř09] employed canonical rules, a generalization of Zakharyaschev's canonical formulas [Zak92], to study admissibility in IPC and several transitive modal logics. An admissible base for these logics was constructed and the decidability of admissibility was established. However, canonical rules are less effective in the non-transitive setting as they do not axiomatize all rules over K. In this chapter, we study the possibility of generalizing this method with stable canonical rules to non-transitive modal logics wK4 and K. Note that an alternative proof of the existence of an admissible base and the decidability of admissibility in K4, S4, and IPC via stable canonical rules was presented in [Bez+16].

The weak transitive logic $\mathsf{wK4} = \mathsf{K} + \Diamond\Diamond p \to \Diamond p \vee p$ characterizes the derived set operator in topological spaces (see, e.g., [Esa04]). Moreover, studying wK4 is a first step of the generalization from the transitive setting to the non-transitive one, as it is weaker than K4 but still has clear similarity with K4. We show in Section 6.2 that stable canonical rules enjoy the rule dichotomy property over wK4, following the proof for K4 presented in [Bez+16].

However, as we will see in Section 6.3, stable canonical rules do not have the rule dichotomy property over K. This aligns with Jeřábek's remark that the rule dichotomy property is a very strong property and thus is likely to fail for many logics [Jeř09]. Finally, in Section 6.4, we provide some sufficient conditions on the admissibility and inadmissibility in K. On the one hand, these conditions yield a full characterization of admissibility for stable rules in K. On the other hand, they illustrate the combinatorial difficulty of the admissibility in K.

6.1 Admissibility and the rule dichotomy property

Definition 6.1. Let L be a modal logic. A rule Γ/Δ is *admissible in L* or *L-admissible* if for any substitution σ,

$$\forall\gamma \in \Gamma\ (\sigma\gamma \in L) \text{ implies } \exists\delta \in \Delta\ (\sigma\delta \in L).$$

Admissible rules are sometimes informally described as rules that, when added to the base logic, do not produce new theorems. Though Definition 6.1 implies this informal description, the converse does not hold in general. We show that the converse holds for modal logics with the *modal disjunction property* (see, e.g., [CZ97, Chapter 15]). See [Iem16] and [Met12] for more discussion on the two formalizations of admissibility. Recall that $\Lambda(\mathcal{S}) = \{\varphi : /\varphi \in \mathcal{S}\}$ for a rule system $\mathcal{S}$.

Proposition 6.2. *Let L be a modal logic. For any L-admissible rule ρ,*

$$\Lambda(\mathcal{S}_L + \rho) = L.$$

Proof. Let ρ be an L-admissible rule. From the characterization of admissibility in [Met12], we obtain that $\mathsf{HSP_U}(\mathcal{V}(L)) = \mathsf{HSP_U}(\mathcal{V}(L) \cap \mathcal{U}(\rho))$. Clearly, the lefthand side is equal to $\mathcal{V}(L)$. Also, since $\mathcal{V}(L) = \mathcal{U}(\mathcal{S}_L)$, the righthand side is equal to $\mathsf{HSP_U}(\mathcal{U}(\mathcal{S}_L + \rho))$. Applying $\mathcal{V}$ to both, we obtain $\mathcal{V}(L) = \mathcal{V}(\mathcal{U}(\mathcal{S}_L + \rho))$, namely, $L = \Lambda(\mathcal{S}_L + \rho)$. □

Definition 6.3. A modal logic L has the *modal disjunction property* if for any formulas $\varphi_1, \ldots, \varphi_n$,

$$\Box\varphi_1 \vee \cdots \vee \Box\varphi_n \in L \text{ implies } \varphi_i \in L \text{ for some } 1 \leq i \leq n.$$

Proposition 6.4. *Let L be a modal logic with the modal disjunction property. Then for any rule ρ,*

$$\Lambda(\mathcal{S}_L + \rho) = L \textit{ implies } \rho \textit{ is } L\textit{-adimissible.}$$

Proof. Let $\rho = \Gamma/\Delta$ be a rule such that $\Lambda(\mathcal{S}_L + \rho) = L$. Let σ be a substitution such that $\sigma\gamma \in L$ for all $\gamma \in \Gamma$, namely, $/\sigma\gamma \in \mathcal{S}_L$ for all $\gamma \in \Gamma$. Let $\mathcal{S} = \mathcal{S}_L + \rho$ and $\Delta = \{\delta_1, \ldots, \delta_m\}$. Then we have:

$$\begin{aligned}
&\Gamma/\Delta \in \mathcal{S} \\
\Longrightarrow\ &\sigma\Gamma/\sigma\Delta \in \mathcal{S} \\
\Longrightarrow\ &/\sigma\Delta \in \mathcal{S} \quad \text{(by applying the cut rule with each rule } /\sigma\gamma) \\
\Longrightarrow\ &/\Box\sigma\delta_1, \ldots, \Box\sigma\delta_m \in \mathcal{S} \quad \text{(by applying the cut rule with necessitation rules)} \\
\Longrightarrow\ &/\Box\sigma\delta_1 \vee \cdots \vee \Box\sigma\delta_m \in \mathcal{S} \\
\Longrightarrow\ &\Box\sigma\delta_1 \vee \cdots \vee \Box\sigma\delta_m \in \Lambda(\mathcal{S}) = L \\
\Longrightarrow\ &\sigma\delta_i \in L \text{ for some } 1 \leq i \leq n \quad \text{(by the modal disjunction property of } L).
\end{aligned}$$

Thus, ρ is L-adimissible. □

We recall some notions of rules and rule systems.

Definition 6.5. Let L be a logic.

1. Two rules ρ and ρ' are *equivalent over L* or *L-equivalent* if they derive each other over L, in other words, $\mathcal{S}_L + \rho = \mathcal{S}_L + \rho'$; similar for sets of rules.

2. A rule ρ is *axiomatized over L* by a class $\mathcal{R}$ of rules if ρ is L-equivalent to a set of rules $\mathcal{R}' \subseteq \mathcal{R}$.

3. An *admissible base for $\mathcal{R}$ over L* is a class of L-admissible rules that axiomatizes all L-admissible rules in $\mathcal{R}$. An admissible base for all rules over L is simply called an *admissible base over L.*

Definition 6.6. Let L be a modal logic and $\mathcal{R}$ be a class of rules.

1. $\mathcal{R}$ has the *rule dichotomy property over L* if every rule in $\mathcal{R}$ is either L-admissilbe or L-equivalent to an assumption-free rule.

2. L has the *rule dichotomy property* if every rule is L-equivalent to a set of rules which are either L-admissible or assumption-free.

Remark 6.7. The rule dichotomy property was first introduced in [Jeř09], where a transitive modal logic L was said to have the *strong rule dichotomy property* if the class of canonical rules has the rule dichotomy property over L. We modified the definition so that it covers other classes of rules. A transitive modal logic L has the strong rule dichotomy property in Jeřábek's terminology iff canonical rules have the rule dichotomy property over L in our terminology.

It is clear from the definition that if there is a class of rules that axiomatizes all rules over L and has the rule dichotomy property over L, then L has the rule dichotomy property.

It was shown in [Jeř09] that canonical rules have the rule dichotomy property over several transitive modal logics and IPC, in the process constructing an admissible base over these logics and proving the decidability of admissibility of them. But the rule dichotomy property is not only a byproduct of the decidability of admissibility and thus a strong sign of the decidability of admissibility, but also has consequences on the admissibility in the extensions.

Theorem 6.8 ([Jeř09])**.** *If L has the rule dichotomy property and L' is an extension of L, then L' has an admissible base consisting of rules both L- and L'-admissible*

However, canonical rules do not axiomatize all rules over K. For example, D, viewed as the rule $/\Diamond\top$ cannot be axiomatized by canonical rules over K [Jeř09, Example 6.3]. So, they are less effective in the non-transitive setting. On the other hand, stable canonical rules do axiomatize all rules over K [BBI16], and are applied to IPC, K4, and S4 in [Bez+16] to construct an admissible base and prove the rule dichotomy property. The following theorems summarize the known results about the rule dichotomy property.

Theorem 6.9 ([Jeř09])**.** *The class of canonical rules has the rule dichotomy property over* IPC*,* K4*,* GL*,* S4*,* K4.3*,* S4.3*, and* GL.3*. Therefore, these logics have the rule dichotomy property.*

Theorem 6.10 ([Bez+16])**.** *The class of stable canonical rules has the rule dichotomy property over* IPC*,* K4*, and* S4*. Therefore, these logics have the rule dichotomy property.*

6.2 The rule dichotomy property over wK4

In this section, we prove that stable canonical rules have the rule dichotomy property over wK4. The proof largely follows the proof in [Bez+16] for K4, fixing some ambiguous points and minor issues there. However, note that this does not immediately imply the decidability of admissibility in wK4; see the discussion following Corollary 6.18. We will work mostly with modal spaces and exploit combinatorics on them.

We will establish the rule dichotomy property over wK4 for stable canonical rules, construct an explicit admissible base for them over wK4, and provide a decidable characterization of admissibility for them all at once. We begin with a lemma providing a sufficient condition for a stable canonical rule to be wK4-equivalent to an assumption-free rule. Recall that $\Box^{\leq 1}\varphi = \varphi \wedge \Box\varphi$.

Lemma 6.11. *Let $\rho(\mathcal{F}, \mathcal{D})$ be a stable canonical rule over* wK4*, that is, $\mathcal{F}$ is a finite* wK4*-space. Suppose that for any* wK4*-space $\mathfrak{X}$, any clopen upset $X' \subseteq X$, and any stable map $f' : \mathfrak{X}' \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$, there is a stable map $f : \mathfrak{X} \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$. Then $\rho(\mathcal{F}, \mathcal{D})$ is equivalent to an assumption-free rule over* wK4.

Proof. Let $\rho = \rho(\mathcal{F}, \mathcal{D})$ be rule

$$\frac{\gamma}{\delta_1, \cdots, \delta_n}$$

and ρ' be the rule

$$\frac{}{\Box^{\leq 1}\gamma \to \delta_1, \cdots, \Box^{\leq 1}\gamma \to \delta_n}$$

We show that ρ and ρ' are wK4-equivalent. It is clear that ρ' derives ρ: for any wK4-space $\mathfrak{X} \models \rho'$ and any valuation V on $\mathfrak{X}$, if $\mathfrak{X}, V \models \gamma$, then $\mathfrak{X}, V \models \Box^{\leq 1}\gamma$ and $\mathfrak{X}, V \models \Box^{\leq 1}\gamma \to \delta_i$ for some $1 \leq i \leq n$, so $\mathfrak{X}, V \models \delta_i$ for some $1 \leq i \leq n$, thus $\mathfrak{X} \models \rho$.

For the other direction, let $\mathfrak{X}$ be a wK4-space that refutes ρ'. Then there is a valuation V on $\mathfrak{X}$ such that $\mathfrak{X}, V \not\models \Box^{\leq 1}\gamma$ for all $1 \leq i \leq n$. Let $\mathfrak{X}'$ be the subspace of $\mathfrak{X}$ with the underlying set $X' = V[\Box^{\leq 1}\gamma]$. Since $\mathfrak{X} \models$ wK4, X' is a clopen upset of X, and $\mathfrak{X}'$ is a wK4-space. So, $\mathfrak{X}', V \models \Box^{\leq 1}\gamma$ implies $\mathfrak{X}', V \models \gamma$. For each $1 \leq i \leq n$, since $\mathfrak{X}, V \not\models \Box^{\leq 1}\gamma \to \delta_i$, there is some $x \in X'$ such that $\mathfrak{X}, V, x \not\models \delta_i$, hence $\mathfrak{X}', V, x \not\models \delta_i$. So, $\mathfrak{X}' \not\models \rho$. Then there is a stable map $f' : \mathfrak{X}' \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$. By the assumption, there is a stable map $f : \mathfrak{X} \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$, so $\mathfrak{X} \not\models \rho$. Hence, ρ derives ρ', and they are equivalent over wK4. ☐

Next, we introduce a class of rules that will be shown to be an admissible base for stable canonical rules over wK4. For $l, m, n \in \omega$, let $S_n^{l,m}$ and T_n^m be the following rules, where we follow the convention $\bigwedge \emptyset = \top$ and $\bigvee \emptyset = \bot$. The intuition behind these formulas will be made clear in the proof of (3) $\Rightarrow$ (2) in Theorem 6.13.

$$S_n^{l,m} \quad \frac{[\bigwedge_{i=1}^{l}(\Box v_i \to v_i) \wedge \bigwedge_{i=1}^{m} \Box(r_i \to \Box(r_i \vee \Box^{\leq 1}q))] \to \bigvee_{i=1}^{n} \Box p_i}{\Box^{\leq 1}q \to p_1, \cdots, \Box^{\leq 1}q \to p_n}$$

$$T_n^m \quad \frac{\bigwedge_{i=1}^{m}(\Diamond r_i \to \Diamond(r_i \wedge \Box^{\leq 1}q)) \to \bigvee_{i=1}^{n} \Box p_i}{\Box^{\leq 1}q \to p_1, \cdots, \Box^{\leq 1}q \to p_n}$$

Before going into the main theorem, we need to define an auxiliary property for stable canonical rules. This is not only a technical tool for the proof, but also a decidable criterion for a stable canonical rule to be admissible in wK4. Recall that $\uparrow^{\leq 1}X = X \cup \uparrow X$ and $\downarrow^{\leq 1}X = X \cup \downarrow X$.

Definition 6.12. A stable canonical rule $\rho(\mathcal{F}, \mathcal{D})$ is called *trivial* if for any $S \subseteq F$, there are $s^\circ, s^\bullet \in F$ such that:

1. $S \subseteq {\uparrow} s^\circ$ and s° is reflexive,
2. for any $D \in \mathcal{D}$, if $D \cap {\uparrow} s^\circ \neq \emptyset$ then $D \cap (\{s^\circ\} \cup {\uparrow}^{\leq 1} S) \neq \emptyset$,
3. $S \subseteq {\uparrow} s^\bullet$,
4. for any $D \in \mathcal{D}$, if $D \cap {\uparrow} s^\bullet \neq \emptyset$ then $D \cap {\uparrow}^{\leq 1} S \neq \emptyset$.

Note that $s^\bullet$ does not have to be irreflexive; s° and $x^\bullet$ can even coincide.

Theorem 6.13. *Let $\mathcal{F}$ be a finite rooted* wK4*-space and $\mathcal{D} \subseteq \mathcal{P}(\mathcal{F})$. The following are equivalent:*

1. *$\rho(\mathcal{F}, \mathcal{D})$ is admissible in* wK4*.*
2. *$\rho(\mathcal{F}, \mathcal{D})$ is derivable from $\{S_n^{l,m}, T_n^m : l, m, n \in \omega\}$.*
3. *$\rho(\mathcal{F}, \mathcal{D})$ is not trivial.*
4. *$\rho(\mathcal{F}, \mathcal{D})$ is not equivalent to an assumption-free rule over* wK4*.*

Proof. (2) $\Rightarrow$ (1). It suffices to show that $S_n^{l,m}$ and T_n^m are admissible for all $l, m, n \in \omega$.

Suppose that $n = 0$. Let σ be a substitution. Let $\mathcal{M}$ be the Kripke model with a single reflexive point x and the empty valuation. Then $\mathcal{M}, x \not\models [\bigwedge_{i=1}^{l} (\Box \sigma v_i \to \sigma v_i) \wedge \bigwedge_{i=1}^{m} \Box(\sigma r_i \to \Box(\sigma r_i \vee \Box^{\leq 1} \sigma q))] \to \bot$, which thus is not a theorem of wK4. Using the Kripke model with a single irreflexive point and the empty valuation, the same argument shows that $\bigwedge_{i=1}^{m} (\Diamond \sigma r_i \to \Diamond(\sigma r_i \wedge \Box^{\leq 1} \sigma q)) \to \bot \notin \mathsf{wK4}$. So, $S_0^{l,m}$ and T_0^m are admissible.

Suppose that $n > 0$. Let σ be a substitution such that $\Box^{\leq 1} \sigma q \to \sigma p_i \notin \mathsf{wK4}$ for all $1 \leq i \leq n$. Then, by the finite model property of wK4 [BEG11], for each i, there is a finite rooted Kripke model $\mathcal{M}_i$ of wK4 with a root x_i such that $\mathcal{M}_i, x_i \models \Box^{\leq 1} \sigma q$ and $\mathcal{M}_i, x_i \not\models \sigma p_i$. By the weak transitivity, $\mathcal{M}_i \models \Box^{\leq 1} \sigma q$.

Let $\mathcal{M}$ be the Kripke model consisting of the disjoint union of $\mathcal{M}_1, \ldots, \mathcal{M}_n$ with an extra reflexive root x, where all propositional variables are false. Then $\mathcal{M}, x \models \bigwedge_{i=1}^{l} (\Box \sigma v_i \to \sigma v_i)$ since x is reflexive, $\mathcal{M}, x \models \bigwedge_{i=1}^{m} \Box(\sigma r_i \to \Box(\sigma r_i \vee \Box^{\leq 1} \sigma q))$ since $\mathcal{M}_i \models \Box^{\leq 1} \sigma q$ for all $1 \leq i \leq n$, and $\mathcal{M}, x \not\models \Box \sigma p_i$ since x sees x_i for all $1 \leq i \leq n$. So, $\mathcal{M}, x \not\models [\bigwedge_{i=1}^{l} (\Box \sigma v_i \to \sigma v_i) \wedge \bigwedge_{i=1}^{m} \Box(\sigma r_i \to \Box(\sigma r_i \vee \Box^{\leq 1} \sigma q))] \to \bigvee_{i=1}^{n} \Box \sigma p_i$, which thus is not a theorem of wK4. Using the Kripke model consisting of the disjoint union of $\mathcal{M}_1, \ldots, \mathcal{M}_n$ with an extra irreflexive root x, the same argument shows that $\bigwedge_{i=1}^{m} (\Diamond \sigma r_i \to \Diamond(\sigma r_i \wedge \Box^{\leq 1} \sigma q)) \to \bigvee_{i=1}^{n} \Box \sigma p_i \notin \mathsf{wK4}$. So, $S_n^{l,m}$ and T_n^m are admissible for $n > 0$.

(3) $\Rightarrow$ (2). Suppose that $\rho(\mathcal{F}, \mathcal{D})$ is not derivable from $\{S_n^{l,m}, T_n^m : l, m, n \in \omega\}$. Then there is a wK4-space $\mathfrak{X} = (X, R)$ validating $\{S_n^{l,m}, T_n^m : l, m, n \in \omega\}$ and refuting $\rho(\mathcal{F}, \mathcal{D})$. So, there is a stable map $f : \mathfrak{X} \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$. We show that $\rho(\mathcal{F}, \mathcal{D})$ is trivial. Let $S \subseteq F$.

We work in the modal algebra $\mathfrak{X}^*$ dual to $\mathfrak{X}$. For the sake of readability, we use propositional variables and formulas to also denote the corresponding elements in $\mathfrak{X}^*$ and the corresponding clopen subsets of $\mathfrak{X}$, with a specified valuation. Let $C = \{v_1, \ldots, v_l\}$ be a finite set of clopen subsets of $\mathfrak{X}$. Let $p_s = X \setminus f^{-1}(s)$ for $s \in S$, $q = f^{-1}[{\uparrow}^{\leq 1} S]$, and $r_a = f^{-1}(a)$ for $a \in F$. p_s, q, r_a are all clopen since f is continuous. Since f is stable and $\mathcal{F}$ is weakly transitive, we have $q = \Box^{\leq 1} q$, so $\Box^{\leq 1} q \not\subseteq p_s$ for all $s \in S$. So, the conclusion of $S_n^{l,m}$ is falsified in $\mathfrak{X}$, thus the assumption is also falsified. It follows that the set

$\{\Box v \to v : v \in X^*\} \cup \{\bigcap_{a\in F} \Box(r_a \to \Box(r_a \cup \Box^{\leq 1} q)) \cap \bigcap_{s\in S} \Diamond\neg p_s\}$ has the finite intersection property. Since X is compact, there is some x in all of these clopens.

Let $s^\circ = f(x)$. Since $x \in \Box v \to v$ for all $v \in X^*$, x is reflexive, so s° is reflexive. Since $x \in \bigcap_{s\in S} \Diamond\neg p_s$, for any $s \in S$, xRy for some $y \in f^{-1}(s)$, so $S \subseteq {\uparrow} s^\circ$. Since $x \in \bigcap_{a\in F} \Box(r_a \to \Box(r_a \cup \Box^{\leq 1} q))$ and x is reflexive, $x \in r_{s^\circ} \to \Box(r_{s^\circ} \cup \Box^{\leq 1} q)$, so $x \in \Box(r_{s^\circ} \cup \Box^{\leq 1} q)$ by $f(x) = s^\circ$. If $D \cap {\uparrow} s^\circ \neq \emptyset$ for some $D \in \mathcal{D}$, then since f satisfies CDC for D, there is some y such that xRy and $f(y) \in D$. Then $y \in r_{s^\circ} \cup \Box^{\leq 1} q$, so $f(y) \in \{s^\circ\} \cup {\uparrow}^{\leq 1} S$. Hence, $D \cap (\{s^\circ\} \cup {\uparrow}^{\leq 1} S) \neq \emptyset$.

Again, let $p_s = X \setminus f^{-1}(s)$ for $s \in S$, $q = f^{-1}[{\uparrow}^{\leq 1} S]$, and $r_a = f^{-1}(a)$ for $a \in F$. Then, the conclusion of T^m_n is falsified in X, so the assumption is also falsified. It follows that there is some $x' \in \bigcap_{a\in F}(\Diamond r_a \to \Diamond(r_a \cap \Box^{\leq 1} q)) \cap \bigcap_{s\in S} \Diamond\neg p_s$. Let $s^\bullet = f(x')$. Since $x' \in \bigcap_{s\in S} \Diamond\neg p_s$ for $s \in S$, $S \subseteq {\uparrow} s^\bullet$. If $D \cap {\uparrow} s^\bullet \neq \emptyset$ for some $D \in \mathcal{D}$, then since f satisfies CDC for D, $x' \in \Diamond r_d$ for some $d \in D$, so $x \in \Diamond(r_d \cap \Box^{\leq 1} q)$, which implies $D \cap {\uparrow}^{\leq 1} S \neq \emptyset$. Hence, $\rho(\mathcal{F}, \mathcal{D})$ is trivial.

(4) ⇒ (3). Assume that $\rho(\mathcal{F}, \mathcal{D})$ is trivial. We use Lemma 6.11 to show that $\rho(\mathcal{F}, \mathcal{D})$ is equivalent to an assumption-free rule. Let $\mathfrak{X} = (X, R)$ be a $\mathsf{wK4}$-space, $Y \subseteq X$ be a clopen upset, and $f : Y \twoheadrightarrow_{\mathcal{D}} F$ be a stable map satisfying CDC for $\mathcal{D}$. Let $f^0 = f$ and $Y^0 = Y$. Given a clopen upset $Y^k \subseteq X$ and a stable map $f^k : Y^k \twoheadrightarrow_{\mathcal{D}} F$, we extend f^k to f^{k+1} as follows.

For $x \in X$, let $f^k_x = f^k[Y^k \cap {\uparrow} x]$. For $S \subseteq F$, let $Y^k_S = \{x \in X \setminus Y^k : f^k_x = S\}$. Take a minimal $S \subseteq F$ such that $Y^k_S \neq \emptyset$. Such an S always exists because F is finite.

Case (I): S has a reflexive root $s \in S$. Define

$$f^{k+1}(x) = \begin{cases} f^k(x) & \text{if } x \in Y^k, \\ s & \text{if } x \in Y^k_S. \end{cases}$$

Let $Y^{k+1} = \mathrm{dom} f = Y^k \cup Y^k_S$.

Claim 6.14. Y^{k+1} is a clopen upset and f^{k+1} is stable and continuous.

Proof. Since for any $x \in X$, $x \in \Box(Y^k \to (f^k)^{-1}[S]) \setminus Y^k$ iff $f^k_x = f^k[Y^k \cap {\uparrow} x] \subseteq S$ and $x \notin Y^k$, $Y^k_S = \Box(Y^k \to (f^k)^{-1}[S]) \setminus Y^k$ by the minimality of S. So, Y^k_S and Y^{k+1} are clopen. If $x \in Y^k_S$ and xRy for some $y \notin Y^k$, then ${\uparrow} y \subseteq {\uparrow} x \cup \{x\}$ by the weak transitivity, so $f^k_y \subseteq f^k_x = S$, hence $f^k_y = S$ and $y \in Y^k_S$ by the minimality of S. So, $Y^{k+1} \subseteq X$ is an upset. It follows that f^{k+1} is stable (since s is a reflexive root of S) and continuous. □

Claim 6.15. f^{k+1} satisfies CDC for $\mathcal{D}$.

Proof. It suffices to verify CDC for $x \in Y^k_S$ since f^k satisfies CDC. Suppose $D \cap {\uparrow} f^{k+1}(x) \neq \emptyset$ for some $D \in \mathcal{D}$ and $x \in Y^k_S$. Since $f^k_x = S$, there is a $y \in Y^k \cap {\uparrow} x$ such that $f^k(y) = f^{k+1}(y) = s = f^{k+1}(x)$, so $D \cap {\uparrow} f^k(y) \neq \emptyset$. Since f^k satisfies CDC, there is a $z \in Y^k$ such that yRz and $f(z) \in D$. We have $xRyRz$. Since $x \notin Y^k$ and $z \in Y^k$, $x \neq z$, so xRz by the weak transitivity. This shows that $D \cap f^{k+1}[{\uparrow} x] \neq \emptyset$. □

Case (II): S does not have a reflexive root. Then $s^\bullet \notin S$ since $S \subseteq {\uparrow} s^\bullet$. Let $Y^{k\bullet}_S = Y^k_S \setminus {\downarrow} Y^k_S$ and $Y^{k\circ}_S = Y^k_S \setminus {\downarrow}^{\leq 1} Y^{k\bullet}_S$. $Y^{k\bullet}_S$ consists of irreflexive R-maximal points in Y^k_S. All R-maximal points in $Y^{k\circ}_S$ are reflexive: if $x \in Y^{k\circ}_S$ is maximal in $Y^{k\circ}_S$ but not maximal in Y^k_S, then xRy for some $y \in {\downarrow}^{\leq 1} Y^{k\bullet}_S$, so $x \in {\downarrow}^{\leq 1} Y^{k\bullet}_S$ by the weak transitivity, which contradicts $x \in Y^{k\circ}_S$; if $x \in Y^{k\circ}_S$ is maximal in Y^k_S, then it is reflexive since $x \notin Y^{k\bullet}_S$.

Define:

$$f^{k+1}(x) = \begin{cases} f^k(x) & \text{if } x \in Y^k, \\ s^\bullet & \text{if } x \in Y_S^{k\bullet}, \\ s^\circ & \text{if } x \in Y_S^{k\circ}. \end{cases}$$

Let $Y^{k+1} = \text{dom} f = Y^k \cup Y_S^{k\bullet} \cup Y_S^{k\circ}$.

Claim 6.16. Y^{k+1} is a clopen upset and f^{k+1} is stable and continuous.

Proof. Since Y_S^k is clopen as shown in Case (I), $Y_S^{k\bullet}$, $Y_S^{k\circ}$, and Y^{k+1} are clopen. So, f^{k+1} is conitnous. Suppose that xRy. If $x \in Y^k$, then $y \in Y^k$ since Y^k is an upset. If $x \in Y_S^{k\bullet}$, then $y \in Y^k \cup Y_S^k$ since $Y^k \cup Y_S^k$ is an upset as shown in Case (I), and $y \in Y^k$ since x is irreflexive and R-maximal in Y_S^k. If $x \in Y_S^{k\circ}$, then again $y \in Y^k \cup Y_S^k$. If furhter $y \in {\downarrow} Y_S^{k\bullet}$, then yRz for some $z \in Y_S^{k\bullet}$, and xRz by the weak transitivity since $x \notin Y_S^{k\bullet}$, which contradicts $x \notin {\downarrow} Y_S^{k\bullet}$. Also, $y \notin Y_S^{k\bullet}$ since $x \notin {\downarrow} Y_S^{k\bullet}$. So, $y \in Y^k \cup Y_S^{k\circ}$. It follows that $Y^{k+1} \subseteq X$ is an upset and f^{k+1} is stable (since s° is reflexive). □

Claim 6.17. f^{k+1} satisfies CDC for $\mathcal{D}$.

Proof. It suffices to verify CDC for $x \in Y_S^{k\bullet}$ and $x \in Y_S^{k\circ}$ since f^k satisfies CDC. Suppose that $D \cap {\uparrow} f^{k+1}(x) \neq \emptyset$ for some $D \in \mathcal{D}$ and $x \in Y_S^{k\bullet}$. Then $D \cap {\uparrow}^{\leq 1} S \neq \emptyset$ since $f^{k+1}(x) = s^\bullet$. If $D \cap S \neq \emptyset$, then $D \cap f^k[Y^k \cap {\uparrow} x] \neq \emptyset$ since $f_x^k = S$, so $D \cap f^{k+1}[{\uparrow} x] \neq \emptyset$. If $D \cap {\uparrow} S \neq \emptyset$, then there is some $s \in S$ such that $D \cap {\uparrow} s \neq \emptyset$. Since $f_x^k = S$, there is a $y \in Y^k \cap {\uparrow} x$ such that $f^k(y) = s$. Since f^k satisfies CDC for D, there is a $z \in Y^k$ such that yRz and $f^k(z) \in D$. We have $xRyRz$. Since $x \notin Y^k$ and $z \in Y^k$, $x \neq z$, so xRz by the weak transitivity. This shows that $D \cap f^{k+1}[{\uparrow} x] \neq \emptyset$.

Suppose that $D \cap {\uparrow} f^{k+1}(x) \neq \emptyset$ for some $D \in \mathcal{D}$ and $x \in Y_S^{k\circ}$. Then $D \cap (\{s^\circ\} \cup {\uparrow}^{\leq 1} S) \neq \emptyset$ since $f^{k+1}(x) = s^\circ$. If $s^\circ \in D$, then $D \cap f^{k+1}[{\uparrow} x] \neq \emptyset$ since all R-maximal points in $Y_S^{k\circ}$ are reflexive. If $D \cap {\uparrow}^{\leq 1} S \neq \emptyset$, then $D \cap f^{k+1}[{\uparrow} x] \neq \emptyset$ follows as the above case of $Y_S^{k\bullet}$. □

It remains to show that the construction halts in finitely many steps. Suppose that $f^0, \ldots, f^n$ are constructed as above and $S \subseteq F$ is used to construct f^k for some $0 < k \leq n$. We show that S cannot be used to construct f^{n+1}. Note that $Y^0 \subseteq \cdots \subseteq Y^n$ by the construction.

Suppose that S occurs as one of the candidates to construct f^{n+1} for a contradiction. Then $S = f_x^n$ for some $x \in X \setminus Y^n$, so $x \notin Y^k$. If f^k is constructed by Case (I), then $Y^k = Y^{k-1} \cup Y_S^{k-1}$, so $x \notin Y^{k-1}$ and $x \notin Y_S^{k-1}$, which implies $S \neq f_x^{k-1}$. By the minimality of S, $f_x^{k-1} \setminus S \neq \emptyset$, so $f_x^n \setminus S \neq \emptyset$ since $f^{k-1} \subseteq f^n$, which contradicts $S = f_x^n$. If f^k is constructed by Case (II), then $Y^k = Y^{k-1} \cup Y_S^{k-1\bullet} \cup Y_S^{k-1\circ}$. If $x \notin Y_S^{k-1}$, then we again obtain a contradiction with $S = f_x^n$. If $x \in Y_S^{k-1}$, then $x \in {\downarrow} Y_S^{k-1\bullet}$ since $x \notin Y_S^{k-1\bullet} \cup Y_S^{k-1\circ}$, so xRy for some $y \in Y_S^{k-1\bullet}$, which contradicts $S = f_x^n$ since $f^n(y) = f^k(y) = s^\bullet \notin S$.

So, since F is finite, there are only finitely many S to use in the construction, so the construction halts in finitely many steps. In the end, we obtain a clopen upset $\bar{Y} \subseteq X$ and a stable map $\bar{f} : \bar{Y} \twoheadrightarrow_{\mathcal{D}} F$. Since the construction cannot be applied to $\bar{f}$, for any $S \subseteq F$ there is no $x \in X \setminus \bar{Y}$ such that $\bar{f}_x = S$, which implies that $X = \bar{Y}$. Thus, $\bar{f} : X \twoheadrightarrow_{\mathcal{D}} F$. Hence, by Lemma 6.11, $\rho(\mathcal{F}, \mathcal{D})$ is equivalent to an assumption-free rule.

(1) ⇒ (4). Suppose that $\rho(\mathcal{F}, \mathcal{D})$ is admissible and equivalent to an assumption-free rule $/\Delta$ for a contradiction. Then one of the formulas in Δ is a theorem of wK4, so $/\Delta$ is valid on all wK4-sapces. However, this contradicts $\mathcal{F} \not\models \rho(\mathcal{F}, \mathcal{D})$.

□

Corollary 6.18.

1. *Stable canonical rules have the rule dichotomy property over* wK4*,*
2. *The set* $\{S_n^{l,m}, T_n^m : l, m, n \in \omega\}$ *forms an admissible base for stable canonical rules over* wK4*,*
3. *It is decidable whether a stable canonical rule is admissible in* wK4*.*

However, Corollary 6.18 does not immediately imply the decidability of admissibility in wK4, as it remains open whether all rules can be axiomatized by stable canonical rules over wK4 (note that this was the first part of Jeřábek's method as described in the introduction). In particular, we lack a filtration for wK4 (cf. Section 3.2). Although the finite model property of wK4 was proved in [BEG11] and recently in [KS25], their filtrations are non-standard, and it is unclear whether they are continuous.

Corollary 6.19. *If* wK4 *admits filtration, then the admissibility is decidable in* wK4*.*

The following problem is also asked in [KS25].

Question 6.20. *Does* wK4 *admit filtration?*

Taking another look at the rules $S_n^{l,m}$ and T_n^m, one may notice that these rules are exactly the rules forming the admissible base for K4 constructed in [Bez+16]. This is not a coincidence, given the following fact, which aligns with the idea in Theorem 6.8.

Proposition 6.21. *Let $\mathcal{R}$ be a class of rules that has the rule dichotomy property over L, L' be an extension of L, and $\mathcal{R}' \subseteq \mathcal{R}$. If $\mathcal{A}$ is an admissible base for $\mathcal{R}$ over L, then $\mathcal{A}' = \{\rho \in \mathcal{A} :$ ρ is L'-admissible$\}$ is an admissible base for $\mathcal{R}'$ over L'.*

Proof. Let $\mathcal{A}$ be an admissible base for $\mathcal{R}$ over L. For any L'-admissible rule $\rho \in \mathcal{R}'$, ρ is either L-admissible or L-equivalent to an assumption-free rule by the rule dichotomy property over L. If ρ is L-admissible, then $\mathcal{S}_L + \rho = \mathcal{S}_L + \mathcal{A}_0$ for some $\mathcal{A}_0 \subseteq \mathcal{A}$, and $\mathcal{S}_{L'} + \rho = \mathcal{S}_{L'} + \mathcal{A}_0$. Since ρ is L'-admissible, rules in $\mathcal{A}_0$ are L'-admissible, so $\mathcal{A}_0 \subseteq \mathcal{A}'$. Thus, ρ is axiomatized by $\mathcal{A}'$. If ρ is L-equivalent to an assumption-free rule, then ρ is L'-equivalent to an assumption-free rule ρ', and since ρ is L'-admissible, ρ' is so, so one of the conclusion of ρ' is a theorem in L', thus ρ' is derivable in L', hence ρ is so. This means that ρ is trivially axiomatized by $\mathcal{A}'$. So, $\mathcal{A}'$ is an admissible base for $\mathcal{R}'$ over L'. □

Corollary 6.22 ([Bez+16])**.**

1. *The set* $\{S_n^{l,m}, T_n^m : l, m, n \in \omega\}$ *forms an admissible base for stable canonical rules over* K4*,*
2. *The set* $\{S_n^{l,m}, T_n^m : l, m, n \in \omega\}$ *forms an admissible base in* K4*,*
3. *The admissibility is decidable in* K4*.*

Proof. Stable canonical rules over K4 are stable canonical rules over wK4. So, by Corollary 6.18 and Proposition 6.21, the rules $S_n^{l,m}$ and T_n^m that are K4-admissible form an admissible base for stable canonical rules over K4. As done in the proof of Theorem 6.13 (2) ⇒ (1), it can be verified that these rules are K4-admissilbe, where (1) follows. It was shown in [Bez+16] that stable canonical rules axiomatize all rules over K4, which together with (1) implies (2). Finally, it is clear that the set $\{S_n^{l,m}, T_n^m : l, m, n \in \omega\}$ is recursively enumerable. So, as we mentioned at the beginning of this chapter, (3) follows. □

6.3 The rule dichotomy property over K

Contrary to $\mathsf{wK4}$ and many transitive modal logics, we show that the rule dichotomy property over K fails for stable canonical rules. In fact, we will construct infinitely many stable rules (that is, stable canonical rules with $\mathcal{D} = \emptyset$) that are neither K-admissible nor K-equivalent to an assumption-free rule. So, even though stable canonical rules axiomatize all rules over K (Theorem 3.24), we cannot prove a similar result as Theorem 6.13 for K.

It is well-known that K has the modal disjunction property (see, e.g., [CZ97, Theorem 3.72]). So, by Propositions 6.2 and 6.4, a rule ρ is admissible in K iff $\Lambda(\mathcal{S}_{\mathsf{K}} + \rho) = \mathsf{K}$. We will take the latter as our working definition of K-admissibility in this section.

For $n \in \omega$, let $\mathcal{F}_n$ be the following modal space. Points in the circle form a cluster; they all see u and are not seen by u.

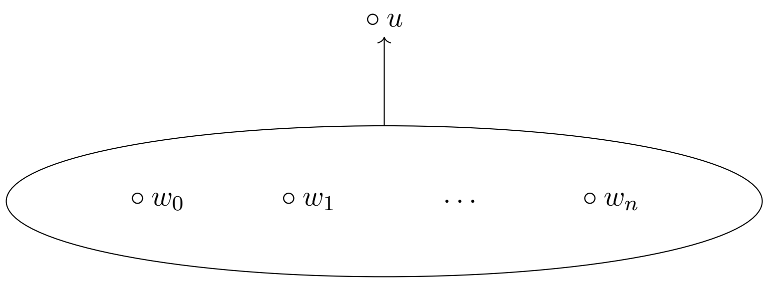


Figure 6.1: $\mathcal{F}_n$

Lemma 6.23. *For any $n \in \omega$, $\rho(\mathcal{F}_n, \emptyset)$ is inadmissible in* K.

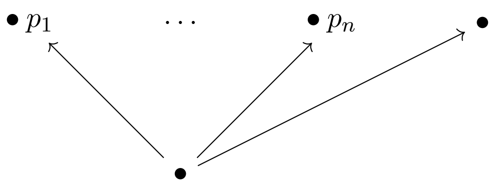
Proof. Let $n \in \omega$. We show that there is a formula φ such that $\varphi \in \Lambda(\mathcal{S}_{\mathsf{K}} + \rho(\mathcal{F}_n, \emptyset))$ but $\varphi \notin \mathsf{K}$. For $0 \le i \le n$, let

$$\psi_i = p_i \wedge \bigwedge\{\neg p_j : 0 \le j \le n, i \ne j\}$$

and

$$\varphi = \bigwedge\{\Diamond\psi_i : 0 \le i \le n\} \to \Box(\Diamond\top \vee \bigvee\{p_i : 0 \le i \le n\}).$$

As φ is refuted at the root in the following model, $\varphi \notin \mathsf{K}$.



Assume that $\varphi \notin \Lambda(\mathcal{S}_{\mathsf{K}} + \rho(\mathcal{F}_n, \emptyset))$ for a contradiction. Then, $\mathcal{V}(\mathcal{U}(\mathcal{S}_{\mathsf{K}} + \rho(\mathcal{F}_n, \emptyset))) \not\models \varphi$, so $\mathcal{U}(\mathcal{S}_{\mathsf{K}} + \rho(\mathcal{F}_n, \emptyset)) \not\models \varphi$. So, there is a modal space $\mathfrak{X}$ such that $\mathfrak{X} \models \rho(\mathcal{F}_n, \emptyset)$ and $\mathfrak{X} \not\models \varphi$. Let

$$\psi = \bigwedge\{\neg p_i : 0 \le i \le n\}.$$

Let V be a valuation on $\mathfrak{X}$ such that $\mathfrak{X}, V \not\models \varphi$. Define $f : \mathfrak{X} \to \mathcal{F}_n$ by $f(x) = w_i$ if $x \in V(\psi_i)$ for $0 \le i \le n$, $f(x) = u$ if $x \in V(\Box\bot \wedge \psi)$, and $f(x) = w_0$ otherwise. As $\{V(\psi_i) : 0 \le i \le n\} \cup \{V(\Box\bot \wedge \psi)\}$ is a pairwise disjoint family of non-empty clopen subsets of $\mathfrak{X}$, f is well-defined, continuous, and surjective. Since all points in $f^{-1}(u) = V(\Box\bot \wedge \psi)$ are dead ends, f is stable. So, $f : \mathfrak{X} \twoheadrightarrow_{\emptyset} \mathcal{F}_n$.

Thus, $\mathfrak{X} \not\models \rho(\mathcal{F}_n, \emptyset)$, which is a contradiction. Therefore, $\varphi \in \Lambda(\mathcal{S}_{\mathsf{K}} + \rho(\mathcal{F}_n, \emptyset)) \setminus \mathsf{K}$, and $\rho(\mathcal{F}_n, \emptyset)$ is inadmissible in K. □

Remark 6.24. Lemma 6.23 can also be derived as an immediate consequence of Theorem 6.33 proved in the next section.

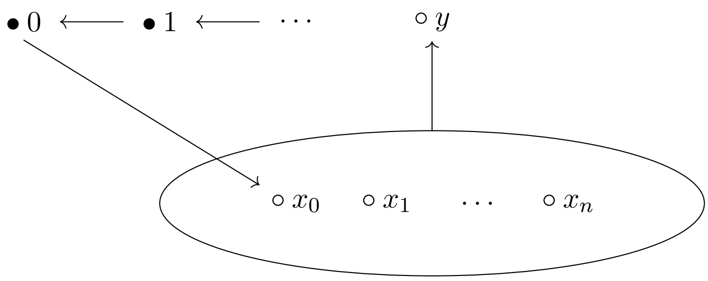
To show that $\rho(\mathcal{F}_n, \emptyset)$ is not K-equivalent to an assumption-free rule, we use the following fact.

Proposition 6.25 ([Jeř09])**.** *For any rule system $\mathcal{S}$, the following are equivalent.*

1. *Validity of $\mathcal{S}$ is preserved by generated subframes (i.e., upsets),*
2. *$\mathcal{S}$ is axiomatized by assumption-free rules over K.*

Lemma 6.26. *For any $n \in \omega$, $\mathcal{S}_{\mathsf{K}} + \rho(\mathcal{F}_n, \emptyset)$ is not axiomatized by assumption-free rules over K. In particular, $\rho(\mathcal{F}_n, \emptyset)$ is not K-equivalent to any assumption-free rule.*

Proof. Let $n \in \omega$. Let $\mathfrak{X} = (X, R)$ be the following modal space, where $U \subseteq X$ is clopen iff $U \cap (\omega \cup \{y\})$ is finite without y or cofinite with y. Note that R is not transitive, y only sees y, and 0 only sees x_0.



$\mathcal{F}_n$ is (isomorphic to) a closed upset of $\mathfrak{X}$. We show that $\mathfrak{X} \models \rho(\mathcal{F}_n, \emptyset)$ and $\mathcal{F}_n \not\models \rho(\mathcal{F}_n, \emptyset)$. The latter is clear. Suppose that there is a stable map $f : \mathfrak{X} \twoheadrightarrow_{\emptyset} \mathcal{F}_n$ for a contradiction. Since u has no proper successor, if $f(x) = u$ and xRy then $f(y) = u$. If $f(0) = u$, then $f(x_0) = u$. If $f(x_i) = u$ for some $0 \leq i \leq n$, then $f(x_i) = u$ for all $0 \leq i \leq n$ and $f(y) = u$. If $f(y) = u$, then, since $f^{-1}(y)$ is clopen, there is some $k \in \omega$ such that $f(l) = u$ for all $l \geq k$. If $f(k) = u$ for some $k > 0$, then $f(k-1) = u$. Since $f^{-1}(u) \neq \emptyset$, this argument implies that $f^{-1}(u) = X$, which is a contradiction. So, $X \not\twoheadrightarrow_{\emptyset} F$ and $X \models \rho(\mathcal{F}_n, \emptyset)$.

This shows that the validity of $\rho(\mathcal{F}_n, \emptyset)$ is not preserved by upsets, so by Proposition 6.25, $\rho(\mathcal{F}_n, \emptyset)$ is not axiomatized by assumption-free rules over K. □

Theorem 6.27. *There are infinitely many stable (canonical) rules that are neither K-admissible nor K-equivalent to an assumption-free rule. Therefore, stable canonical rules do not have the rule dichotomy property over K.*

Proof. This follows directly from Lemma 6.23 and Lemma 6.26. □

Jeřábek remarked in [Jeř09] that "the rule dichotomy is a very strong property which is unlikely to hold for a substantial class of logics." Since there are only countably many stable canonical rules, Theorem 6.27 shows that the rule dichotomy property over K fails for stable canonical rules as badly as possible. This is further evidence to the fact that K may lack the rule dichotomy property. At least, we can conclude that it is impossible to prove Theorem 6.13 for K, thus we cannot apply Jeřábek's method of establishing the decidability of admissibility with stable canonical rules to K, even though, unlike canonical rules, stable canonical rules do axiomatize all rules over K.

However, the fact that stable canonical rules do not have the rule dichotomy over K does not imply that K does not have the rule dichotomy property. Specifically, it does not exclude the possibility that a stable canonical rule, though itself is neither K-admissible nor K-equivalent to an assumption-free rule, is K-equivalent to a set of rules such that each of them is either K-admissible or K-equivalent to an assumption-free rule. It remains open whether one can design a class of rules that axiomatizes all rules over K (which canonical rules fail to do) and has the rule dichotomy property over K (which stable canonical rules fail to do), and thus prove the rule dichotomy property for K. The existence of such a class of rules would lead to an affirmative solution to the decidability of admissibility in K.

Question 6.28. *Is there a class of rules that axiomatizes all rules over* K *and has the rule dichotomy property over* K*? Does* K *have the rule dichotomy property?*

6.4 Decidable sufficient conditions for (in)admissibility in K

Moving away from Jeřábek's method of proving the decidability of admissibility in K via the rule dichotomy property, in this section, we try to study the admissibility of stable canonical rules directly by working with combinatorics on modal spaces. We present some combinatorial sufficient conditions for stable canonical rules to be K-admissible or K-inadmissible.

We will use the following lemma as our main strategy to obtain sufficient conditions for being K-admissible.

Lemma 6.29. *Let $\mathcal{F}$ be a finite modal space and $\mathcal{D} \subseteq \mathcal{P}(F)$. Suppose that for any finite rooted irreflexive tree $\mathfrak{X}$, there is a modal space $\mathfrak{X}'$ such that $\mathfrak{X}$ is a closed upset of $\mathfrak{X}'$ and $\mathfrak{X}' \not\twoheadrightarrow_{\mathcal{D}} \mathcal{F}$. Then, $\rho(\mathcal{F}, \mathcal{D})$ is admissible in* K.

Proof. Let $\mathfrak{X}$ be an arbitrary finite rooted irreflexive tree, and $\mathfrak{X}'$ be given as in the assumption. Since $\mathfrak{X}' \not\twoheadrightarrow_{\mathcal{D}} \mathcal{F}$, we have $\mathfrak{X}' \models \rho(\mathcal{F}, \mathcal{D})$, in particular, $\mathfrak{X}' \models \Lambda(\mathcal{S}_{\mathsf{K}} + \rho(\mathcal{F}, \mathcal{D}))$. So, since the validity of formulas is preserved by closed upsets, $\mathfrak{X} \models \Lambda(\mathcal{S}_{\mathsf{K}} + \rho(\mathcal{F}, \mathcal{D}))$. This holds for any finite rooted irreflexive tree. Therefore, since K is sound and complete with respect to finite rooted irreflexive trees (Theorem 2.53), it follows that $\mathsf{K} = \Lambda(\mathcal{S}_{\mathsf{K}} + \rho(\mathcal{F}, \mathcal{D}))$, namely, $\rho(\mathcal{F}, \mathcal{D})$ is admissible in K. □

For a modal space $\mathfrak{X} = (X, R)$, a point $x \in X$ is called a *sharp root* of $\mathfrak{X}$ if xRy for all $y \in X$. A sharp root must be reflexive.

Lemma 6.30. *Let $\mathcal{F} = (F, Q)$ be a finite modal space and $\mathcal{D} \subseteq \mathcal{P}(F)$. If $\mathcal{F}$ has no sharp root r such that either $\forall D \in \mathcal{D}(D \neq \emptyset \to r \in D)$ or $\exists w \in F(w \neq r \wedge wQw \wedge wQr)$, then $\rho(\mathcal{F}, \mathcal{D})$ is admissible in* K.

Proof. Assume that F has no sharp root r such that either $\forall D \in \mathcal{D}(D \neq \emptyset \to r \in D)$ or $\exists w \in F(w \neq r \wedge wQw \wedge wQr)$. Let $\mathfrak{X} = (X, R)$ be an arbitrary finite rooted irreflexive tree. Let $\mathfrak{X}' = (X \cup \{x_0, x_1\}, R')$ be the finite modal space where $R' = R \cup \{(x_0, x) : x \in X'\} \cup \{(x_1, x_0)\} \cup \{(x_1, x_1)\}$. Then, $\mathfrak{X}$ is a closed upset of $\mathfrak{X}'$.

Case (I): $\mathcal{F}$ has no sharp root. Then, it is clear that there is no stable map $\mathfrak{X}' \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$, since $\mathfrak{X}'$ has a sharp root x_0.

Case (II): $\mathcal{F}$ has sharp roots and for any sharp root r, $\exists D \in \mathcal{D}(D \neq \emptyset \wedge r \notin D)$ and $\forall w \in F(w \neq r \wedge wQw \to \neg wQr)$. Suppose for a contradiction that there exists a stable map $f : \mathfrak{X}' \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$. Then $f(x_0)$ is a sharp root in $\mathcal{F}$. By the assumption, there is some $D \in \mathcal{D}$ such that $D \neq \emptyset$ and $f(x_0) \notin D$. Let $w \in D$. If $f(x_1) = f(x_0)$, then $f(x_1)Qw$, which contradicts CDC for D since x_1 only sees x_0 and x_1 and $f(x_1) = f(x_0) \notin D$. So, $f(x_1) \neq f(x_0)$. Then, since $f(x_1)Qf(x_1)$ by $x_1R'x_1$, we have $\neg f(x_1)Qf(x_0)$ by the assumption, which contradicts f being stable since $x_1R'x_0$. Thus, there is no stable map $\mathfrak{X}' \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$.

In both cases, we have shown that $\mathfrak{X}' \not\twoheadrightarrow_{\mathcal{D}} \mathcal{F}$, so it follows from Lemma 6.29 that $\rho(\mathcal{F}, \mathcal{D})$ is admissible in K. □

Lemma 6.31. *Let $\mathcal{F} = (F, Q)$ be a finite modal space and $\mathcal{D} \subseteq \mathcal{P}(F)$. If there is some $\mathcal{D}' \subseteq \mathcal{D}$ and $d \in \bigcup \mathcal{D}'$ such that there is no path in $\bigcup \mathcal{D}'$ from d to a maximal irreflexive point in $\bigcup \mathcal{D}'$, then $\rho(\mathcal{F}, \mathcal{D})$ is admissible in K.*

Proof. Let $\mathcal{D}' \subseteq \mathcal{D}$ and $d \in \bigcup \mathcal{D}'$ such that there is no path in $\bigcup \mathcal{D}'$ from d to a maximal irreflexive point in $\bigcup \mathcal{D}'$. Let $\mathfrak{X} = (X, R)$ be an arbitrary finite rooted irreflexive tree. Suppose that there is a stable map $f : \mathfrak{X} \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$ for a contradiction.

Let $d_0 = d$ and $x_0 \in f^{-1}(d_0)$. Assume that we obtained a path $d_0Q \cdots Qd_n$ in $\mathcal{F}$ and a path $x_0R \cdots Rx_n$ in $\mathfrak{X}$ such that $d_i \in \bigcup \mathcal{D}'$ and $f(x_i) = d_i$ for all $0 \leq i \leq n$. By our assumption, d_n is not a maximal irreflexive point in $\bigcup \mathcal{D}'$, so there is some $e \in \bigcup \mathcal{D}'$ such that d_nQe. Let $E \in \mathcal{D}'$ be such that $e \in E$. Since f satisfies CDC for E, there is some $x_{n+1} \in X$ such that $f(x_{n+1}) \in E$. Let $d_{n+1} = f(x_{n+1})$, so $d_{n+1} \in \bigcup \mathcal{D}'$. Thus, we obtain a path $d_0Q \cdots Qd_{n+1}$ in $\mathcal{F}$ and a path $x_0R \cdots Rx_{n+1}$ in $\mathfrak{X}$ such that $d_i \in \bigcup \mathcal{D}'$ and $f(x_i) = d_i$ for all $0 \leq i \leq n+1$. Repeating the construction, it follows that there is an infinite path in $\mathfrak{X}$, which contradicts the assumption that $\mathfrak{X}$ is a finite rooted irreflexive tree. So, $f : \mathfrak{X} \not\twoheadrightarrow_{\mathcal{D}} \mathcal{F}$, and therefore, $\rho(\mathcal{F}, \mathcal{D})$ is admissible in K by Lemma 6.29. □

Summarizing Lemma 6.30 and Lemma 6.31, we obtain the following sufficient condition for K-admissibility.

Theorem 6.32. *Let $\mathcal{F} = (F, Q)$ be a finite modal space and $\mathcal{D} \subseteq \mathcal{P}(F)$. If one of the following conditions is* not *satisfied, then $\rho(\mathcal{F}, \mathcal{D})$ is K-admissible:*

1. *$\mathcal{F}$ has a sharp root r such that $\forall D \in \mathcal{D}(D \neq \emptyset \to r \in D)$ or $\exists w \in F(w \neq r \wedge wQw \wedge wQr)$,*
2. *For any $\mathcal{D}' \subseteq \mathcal{D}$ and $d \in \bigcup \mathcal{D}'$, there is a path in $\bigcup \mathcal{D}'$ from d to a maximal irreflexive point in $\bigcup \mathcal{D}'$.*

Now we turn to the inadmissibility. A set $\mathcal{D} \subseteq \mathcal{P}(F)$ is called *trivial* if $\mathcal{D} = \emptyset$ or $\mathcal{D} = \{\emptyset\}$. If $\mathcal{D}$ is trivial, then any stable map satisfies CDC for $\mathcal{D}$.

Theorem 6.33. *Let $\mathcal{F} = (F, Q)$ be a finite modal space with a sharp root r and $\mathcal{D} \subseteq \mathcal{P}(F)$ be trivial. Then, $\rho(\mathcal{F}, \mathcal{D})$ is inadmissible in K.*

Proof. Let $n = |F| - 1$ and $F = \{r, w_1, \ldots, w_n\}$. If $n = 0$, then $\mathcal{F}$ consists of a single reflexive point since $\mathcal{F}$ has a sharp root, so $\mathfrak{X} \twoheadrightarrow_{\mathcal{D}} \mathcal{F}$ for any modal space $\mathfrak{X}$, thus $\mathcal{U}(\mathcal{S}_{\mathsf{K}} + \rho(\mathcal{F}, \mathcal{D})) = \emptyset$. Then, $\mathcal{V}(\mathcal{U}(\mathcal{S}_{\mathsf{K}} + \rho(\mathcal{F}, \mathcal{D}))) = \emptyset$, which means that $\Lambda(\mathcal{S}_{\mathsf{K}} + \rho(\mathcal{F}, \mathcal{D}))$ is the inconsistent logic, hence $\rho(\mathcal{F}, \mathcal{D})$ is inadmissible in K.

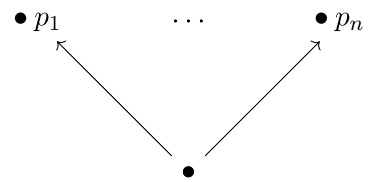
Assume that $n > 0$. For each $1 \leq i \leq n$, let

$$\psi_i = p_i \wedge \bigwedge\{\neg p_j : 1 \leq j \leq n, i \neq j\}$$

and

$$\varphi = \neg(\bigwedge\{\Diamond(\psi_i \wedge \Box\bot) : 1 \leq i \leq n\}).$$

We show that $\varphi \in \Lambda(\mathcal{S}_\mathsf{K} + \rho(\mathcal{F}, \mathcal{D}))$ and $\varphi \notin \mathsf{K}$. As φ is refuted in the following model, $\varphi \notin \mathsf{K}$.



Suppose that $\varphi \notin \Lambda(\mathcal{S}_\mathsf{K} + \rho(\mathcal{F}, \mathcal{D}))$ for a contradiction. Then, $\mathcal{V}(\mathcal{U}(\mathcal{S}_\mathsf{K} + \rho(\mathcal{F}, \mathcal{D}))) \not\models \varphi$, so $\mathcal{U}(\mathcal{S}_\mathsf{K} + \rho(\mathcal{F}, \mathcal{D})) \not\models \varphi$. So, there is a modal space $\mathfrak{X} = (X, R)$ such that $\mathfrak{X} \models \rho(\mathcal{F}, \mathcal{D})$ and $\mathfrak{X} \not\models \varphi$. Let V be a valuation on $\mathfrak{X}$ such that $\mathfrak{X}, V \not\models \varphi$.

Define $f : \mathfrak{X} \to \mathcal{F}$ by $f(x) = w_i$ if $x \in V(\psi_i \wedge \Box\bot)$ for $1 \leq i \leq n$, and $f(x) = r$ otherwise. Since φ is refuted at some point in $\mathfrak{X}$, $V(\psi_i \wedge \Box\bot) \neq \emptyset$ for all $1 \leq i \leq n$ and $V(\Diamond\top) \neq \emptyset$. Then, $\{V(\psi_i \wedge \Box\bot) : 1 \leq i \leq n\} \cup \{V(\Diamond\top)\}$ is a pairwise disjoint family of non-empty clopen subsets of $\mathfrak{X}$, so f is well-defined, continuous, and surjective. If $x \in X$ has a successor, then $x \not\models \Box\bot$, so $f(x) = r$. Thus, f is stable since r is a sharp root of F. So, $X \twoheadrightarrow_\mathcal{D} F$, naemly, $X \not\models \rho(\mathcal{F}, \mathcal{D})$, which is a contradiction. Therefore, $\varphi \in \Lambda(\mathcal{S}_\mathsf{K} + \rho(\mathcal{F}, \mathcal{D})) \setminus \mathsf{K}$, and $\rho(\mathcal{F}, \mathcal{D})$ is inadmissible in K. □

Remark 6.34. Note that both conditions in Theorem 6.32 and Theorem 6.33 are decidable. In particular, the second condition in Theorem 6.32 is decidable because in a finite modal space $\mathcal{F}$, there is a path from x to y iff there is a path of length $\leq |F|$ from x to y.

Finally, as an application, we consider some concrete examples.

Corollary 6.35. *A stable rule $\rho(\mathcal{F}, \emptyset)$ is K-admissible iff $\mathcal{F}$ has no sharp root.*

Example 6.36. An explicit axiomatization was presented for several logics and rule systems in [BBI16, Section 8].

1. Let **Rooted** be the class of all finite rooted modal spaces. Then $\mathcal{S}(\textbf{Rooted}) = \mathcal{S}_\mathsf{K} + \rho(\,) + \rho(\circ\,\circ) + \rho(\circ\!\rightarrow\!\circ\!\leftarrow\!\circ)$. The three rules are all K-admissible since none of the corresponding spaces has a sharp root. It follows that $\Lambda(\mathcal{S}(\textbf{Rooted})) = \mathsf{K}$. This confirms that K is complete with respect to **Rooted**.

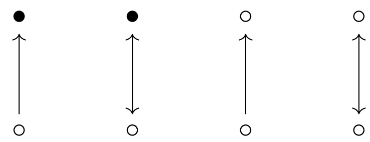
2. $\mathsf{KD} = \Lambda(\mathcal{S}_\mathsf{K} + \rho(\bullet) + \rho(\circ\!\rightarrow\!\bullet))$. Since $\mathsf{K} \subsetneq \mathsf{KD}$, at least one of the two rules must be K-inadmissible. Corollary 6.35 tell us that $\rho(\bullet)$ is K-admissbile and $\rho(\circ\!\rightarrow\!\bullet)$ is not.

3. $\mathsf{KT} = \Lambda(\mathcal{S}_\mathsf{K} + \rho(\bullet) + \rho(\circlearrowleft\!\circ\!\rightarrow\!\bullet))$. Similarly, it follows from Corollary 6.35 that $\rho(\bullet)$ is K-admissbile and $\rho(\circlearrowleft\!\circ\!\rightarrow\!\bullet)$ is not.

Example 6.37. We clarify the (in)admissibility in K for stable canonical rules $\rho(\mathcal{F}, \mathcal{D})$ such that $|F| = 1$ and $|F| = 2$. Proposition 3.23 dually states that, up to equivalence, we can work with $\mathcal{D}$ that contains $\emptyset$ and is closed under unions. For the sake of simplicity, we will only count and present stable canonical rules with such $\mathcal{D}$'s.

1. Among stable canonical rules $\rho(\mathcal{F}, \mathcal{D})$ such that $|F| = 1$, up to equivalence, there is only one rule that is K-inadmissible. Theorem 6.33 implies that $\rho(\circ, \{\emptyset\})$ is K-inadmissible, and Theorem 6.32 implies that all the others are K-admissible.

2. Among stable canonical rules $\rho(\mathcal{F}, \mathcal{D})$ such that $|F| = 2$, Theorem 6.33 implies that $\rho(\mathcal{F}, \{\emptyset\})$ is K-inadmissible for the following four $\mathcal{F}$'s.

   Theorem 6.32 implies that all the others are K-admissible, except for $\rho = \rho(\mathcal{F}, \{\emptyset, F\})$, where $\mathcal{F}$ is the leftmost one in the above figure.

   We can manually verify that ρ is K-inadmissible. Let $\varphi = \neg\Diamond\Box\bot$. It is clear that $\varphi \notin \mathsf{K}$. If $\varphi \notin \Lambda(\mathcal{S}_\mathsf{K} + \rho)$, then, as done in the proof of Theorem 6.33, there is a modal space $\mathfrak{X}$ such that $\mathfrak{X} \models \rho$ and $\mathfrak{X} \not\models \varphi$. Then, $\mathfrak{X}$ contains points that are dead ends and that are not, so $\mathfrak{X} \twoheadrightarrow_{\{\emptyset, F\}} \mathcal{F}$ by mapping dead ends to the irreflexive point and the other points to the reflexive root. However, this contradicts that $\mathfrak{X} \models \rho = \rho(\mathcal{F}, \{\emptyset, F\})$. Thus, $\rho \in \Lambda(\mathcal{S}_\mathsf{K} + \rho) \setminus \mathsf{K}$, which implies that ρ is K-inadmissible.

   Hence, we conclude that, up to equivalence, there are exactly five stable canonical rules $\rho(\mathcal{F}, \mathcal{D})$ such that $|F| = 2$ that are K-inadmissible.

6.5 Summary

We studied the rule dichotomy property and the decidability of admissibility for non-transitive logics K4 and K. We showed that stable canonical rules have the rule dichotomy property over wK4 by generalizing the proof for K4 in [Bez+16]. However, this does not immediately yield the decidability of admissibility in wK4, as it is unknown whether stable canonical rules axiomatize all rule systems over wK4. In addition, we showed that stable canonical rules do not have the rule dichotomy property over K. This partially confirms Jeřábek's conjecture that many logics may lack the rule dichotomy property. As the rule dichotomy property is a crucial step in Jeřábek's method, unfortunately, this also suggests that the method may be less effective for K. Finally, we provided sufficient conditions for both admissibility and inadmissibility in K and discussed some examples. In particular, we obtained a full characterization of K-admissibility for stable rules.

In the following table, we summarize our results and the known results on the rule dichotomy property and the decidability of admissibility for different logics L and classes $\mathcal{R}$ of rules. We say the $\mathcal{R}$ is *complete* over L if $\mathcal{R}$ axiomatizes all rule systems over L. We abbreviate the rule dichotomy property as *rdp*.

	$\mathcal{R}$ is complete over L	$\mathcal{R}$ has the rdp over L	L has the rdp	admissibility is decidable in L
$L = \mathsf{K4}$, $\mathcal{R} = \{$canonical rules$\}$	✓	✓	✓	✓
$L = \mathsf{K4}$, $\mathcal{R} = \{$stable canonical rules$\}$	✓	✓	✓	✓
$L = \mathsf{wK4}$, $\mathcal{R} = \{$canonical rules$\}$	?	?	?	?
$L = \mathsf{wK4}$, $\mathcal{R} = \{$stable canonical rules$\}$	?	✓	?	?
$L = \mathsf{K}$, $\mathcal{R} = \{$canonical rules$\}$	×	✓	?	?
$L = \mathsf{K}$, $\mathcal{R} = \{$stable canonical rules$\}$	✓	×	?	?

We leave it open if stable canonical rules axiomatize all rule systems over wK4. If this is the case, then we obtain the rule dichotomy property and the decidability of admissibility in wK4 for free. As wK4 shares more similarities with K4 than K, canonical rules could be more effective for wK4 than stable canonical rules, though we have not discussed this approach. We leave it for further research to apply Jeřábek's method with canonical rules to wK4, aiming to fill in the third row in the table.

Chapter 7

Cardinality of Sets of Logics

In this short chapter, we study the cardinality of sets of logics. This chapter is based on a joint work with Juan P. Aguilera and Nick Bezhanishvili.

Counting logics has been an interesting question in the study of lattices of logics. Since we assume the set of propositional variables to be countable, there are at most $2^{\aleph_0}$ many logics, being sets of formulas. Jankov [Jan68] showed that there are indeed $2^{\aleph_0}$ many distinct superintuitionistic logics, which implies that there are also $2^{\aleph_0}$ many modal logics. Many results on logical properties are about the number of logics with or without a certain logical property. Blok's dichotomy result (Theorem 2.58) can also be seen in this regard.

The current study is motivated by the following question posed recently in [BBM25].

Question 7.1 ([BBM25])**.** *Does every superintuitionistic and transitive modal logic have the* degree of fmp *(Definition 7.8) either* $\leq \aleph_0$ *or* $2^{\aleph_0}$*?*

This question can be reformulated as the following, where the interval between two logics L_0 and L_1 is the set

$$[L_0, L_1] = \{L : L_0 \subseteq L \subseteq L_1\}.$$

Question 7.2 ([BBM25])**.** *Does every interval of superintuitionistic or transitive modal logics have the cardinality either* $\leq \aleph_0$ *or* $2^{\aleph_0}$*?*

We applied descriptive set theory via coding and showed that this is the case. While the proof was written for logics, we noticed that it works in a much more general setting. Later, George Metcalfe, Niels Vooijs, and Simon Santchi pointed out that the proof works in the setting of varieties. In particular, the following open question in [JL18] was communicated to us by Niels Vooijs.

Question 7.3 ([JL18, Question 6.4 (ii)])**.** *Does the subvariety lattice of every variety have the cardinality either* $\leq \aleph_0$ *or* $2^{\aleph_0}$*?*

We answer all three questions in the positive; the latter two are addressed in Section 7.1 and the first one is addressed in Section 7.2, generalized to all modal logics. Throughout the chapter, we work in ZFC and do not assume the Continuum Hypothesis.

7.1 Cardinality of intervals of varieties and equational theories

We first illustrate the main idea behind the proofs: applying descriptive set theory to logic/universal algebra via *coding*. We assume our algebraic language and the set of variables to be countable. So, there are only countably many terms and identities, and they can be effectively coded by natural numbers. Let T be the set of all terms and Id be the set of all identities. We will identify T and Id with their corresponding subsets of ω.

Equational theories, viewed as sets of identities, thus correspond to subsets of ω, or elements of the *Cantor space* 2^ω, which are in turn often identified with real numbers. Explicitly, an equational theory Φ corresponds to an element $A \in 2^\omega$, i.e., an infinite 0-1 sequence, such that $A(i) = 1$ iff the identity with the code i belongs to Φ. So, instead of varieties or equational theories, we can count sets of subsets of ω, in other words, sets of reals. With this correspondence, it is no surprise that *descriptive set theory* can be used to study the cardinality of sets of equational theories/varieties, as it is exactly the theory about sets of reals. In particular, descriptive set theory is good at handling cardinalities between $\aleph_0$ and $2^{\aleph_0}$; any explicit construction of an uncountable family of varieties or equational theories would result in a family of the cardinality $2^{\aleph_0}$.

Recall the definition of the *arithmetical hierarchy*, the *Borel hierarchy*, and *Borel* sets from Section 2.6. All of our cardinality results will be obtained in the same scheme. Given a set of logics/equational theories, we show that it is Borel by finding an arithmetical hierarchy it belongs to, and then apply the fact that every Borel set has the cardinality $\leq \aleph_0$ or $2^{\aleph_0}$ (Theorem 2.69).

We start with intervals of varieties and equational theories. Let Φ_0 and Φ_1 be equational theories. Recall that the *interval* between Φ_0 and Φ_1 is the set

$$[\Phi_1, \Phi_2] = \{\Phi : \Phi \text{ is an equational theory such that } \Phi_1 \subseteq \Phi \subseteq \Phi_2\}.$$

An interval in this sense may not be linear. Since each equational theory corresponds to a real, we can view an interval as a set of reals. Thus, it is meaningful to talk about the arithmetical hierarchy or Borel hierarchy of an interval. Recall that for $A, B \subseteq \omega$, the *join* $A \oplus B \subseteq \omega$ is the set

$$\{2n : n \in A\} \cup \{2n+1 : n \in B\}.$$

Intuitively, using the parameter $A \oplus B$ amounts to using the parameters A and B.

Lemma 7.4. *The set* $[\Phi_0, \Phi_1]$ *is* $\Pi^0_1(\Phi_0 \oplus \Phi_1)$.

Proof. A set $\Phi \subseteq \omega$ is in $[\Phi_0, \Phi_1]$ iff each element in Φ indeed codes an identity, Φ is an equational theory, and $\Phi_1 \subseteq \Phi \subseteq \Phi_2$. Recall that a set of identities is an equational theory iff it satisfies the five conditions in Theorem 2.13. Thus, $\Phi \in [\Phi_0, \Phi_1]$ iff it satisfies all the following conditions:

1. $\Phi \subseteq \mathsf{Id}$,
2. $s \approx s \in \Phi$ for $s \in \mathsf{T}$,
3. $s \approx t \in \Phi \Rightarrow t \approx s \in \Phi$,
4. $s \approx t,\ t \approx u \in \Phi \Rightarrow s \approx u \in \Phi$,
5. Φ is closed under replacement,
6. Φ is closed under substitution,
7. $\Phi_0 \subseteq \Phi \subseteq \Phi_1$.

We show that all these conditions can be expressed by Π^0_1 formulas, with the parameters Φ_0 and Φ_1 used in the last one. We only address items (6) and (7). The others can be verified in a similar manner. It follows from Definition 2.12 that the ternary relation "φ' is a replacement instance of t and φ" is recursive. Thus, using a recursive predicate Rep, item (6) can be expressed by the Π^0_1 formula:

$$\forall i \forall j \forall k \ (j \in \mathsf{T} \wedge k \in \Phi \wedge \mathsf{Rep}(i,j,k) \rightarrow i \in \Phi).$$

Item (7) can be expressed by the Π^0_1 formula with parameters Φ_0 and Φ_1:

$$\forall i \ [(i \in \Phi_0 \rightarrow i \in \Phi) \wedge (i \in \Phi \rightarrow i \in \Phi_1)].$$

Thus, the set $[\Phi_0, \Phi_1]$ can be defined by a Π^0_1 formula with parameters Φ_0 and Φ_1, hence it is $\Pi^0_1(\Phi_0 \oplus \Phi_1)$. □

Theorem 7.5. *For any equational theories Φ_0 and Φ_1, the interval $[\Phi_0, \Phi_1]$ has the cardinality $\leq \aleph_0$ or $2^{\aleph_0}$. Dually, for any varieties $\mathcal{V}_0$ and $\mathcal{V}_1$, the interval $[\mathcal{V}_0, \mathcal{V}_1]$ has the cardinality $\leq \aleph_0$ or $2^{\aleph_0}$.*

Proof. By Lemma 7.4 and Proposition 2.66, the interval $[\Phi_0, \Phi_1]$ is $\mathbf{\Pi}^0_1$, so it is Borel. Thus, it has the cardinality $\leq \aleph_0$ or $2^{\aleph_0}$ by Theorem 2.69. The dual statement follows from the correspondence between equational theories and varieties. □

Corollary 7.6. *Every euqtional theory has $\leq \aleph_0$ or $2^{\aleph_0}$ many extensions. Dually, every variety has $\leq \aleph_0$ or $2^{\aleph_0}$ many subvarieties.*

Proof. This follows from Theorem 7.5 by considering the contradictory equational theory or the trivial variety. □

This answers the open question [JL18, Question 6.4 (ii)] in the affirmative.

Remark 7.7. The cardinality results were formulated for equational theories and varieties, so they apply to a wide range of logics, including modal logics, superintuitionistic logics, and substructural logics. Moreover, it should be clear from the proofs that the same idea works for even general settings, such as *quasivarieties* and *universal classes.*

7.2 Cardinality of the degrees of the finite model property

Next, we present a particular application of our main idea to logics and study the *degrees of the finite model property* (*degrees of fmp* for short). The degrees of fmp were introduced in [BBM25] as a modified version of the degrees of Kripke incompleteness (Definition 2.56) using finite Kripke frames. We first recall the definition for modal logics; the case of superintuitionistic logics is analogous. Let L_0 be a logic.

Definition 7.8. Let FFr be the set of all finite Kripke frames. For $L \in \mathsf{NExt}L_0$, let

$$\mathsf{FFr}(L) = \{\mathcal{F} \in \mathsf{FFr} : \mathcal{F} \vDash L\},$$

and the *fmp span* of L (in $\mathsf{NExt}L_0$) be the set

$$\mathrm{fmp}_{L_0}(L) = \{L' \in \mathsf{NExt}L_0 : \mathsf{FFr}(L') = \mathsf{FFr}(L)\}.$$

The *degree of fmp* of L (in $\mathsf{NExt}L_0$) is the cardinality of the set $\mathrm{fmp}_{L_0}(L)$.

The relation $\mathsf{FFr}(L) = \mathsf{FFr}(L')$ induces an equivalence relation on the lattice $\mathsf{NExt}L_0$, and the fmp span of L refers to the equivalence class that L belongs to. Thus, intuitively, the degree of fmp of L measures to what extent L cannot be distinguished from other logics by the means of finite Kripke frames. The following antidichotomy theorems were proved in [BBM25], which states that every cardinal $0 < \kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$ is realized as the degree of fmp in superintuitionistic logics and transitive modal logics. This makes a clear contrast with Blok's dichotomy theorem about the degrees of Kripke incompleteness (Theorem 2.58).

Theorem 7.9 ([BBM25])**.** *For each cardinal $0 < \kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, there is a superintuitionistic logic that has the degree of fmp κ.*

Theorem 7.10 ([BBM25])**.** *Let $L_0 \subseteq \mathsf{Grz}$ be a modal logic with the fmp such that Grz is a union-splitting in $\mathsf{NExt}L_0$. For each nonzero cardinal $0 < \kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, there is a normal extension L of L_0 that has the degree of fmp κ in $\mathsf{NExt}L_0$.*

Using the same idea introduced at the beginning of the last section, we apply descriptive set theory and show that every cardinal $\aleph_0 < \kappa < 2^{\aleph_0}$ *cannot* be realized as the degree of fmp of any modal logic. Let L_0 be a logic.

Lemma 7.11. *For any logic $L \in \mathsf{NExt}L_0$, the set $\mathrm{fmp}_{L_0}(L)$ is $\Pi^0_2(L_0 \oplus L)$.*

Proof. By Lemma 7.4, there is a Π^0_1 formula α with the parameter L_0 that defines $\mathsf{NExt}L_0$.

A finite Kripke frame is a finite set with a binary relation. Given a finite Kripke frame $\mathcal{F}$ and a formula φ, it is decidable whether $\mathcal{F} \models \varphi$ by considering all possible valuations on $\mathcal{F}$ of propositional variables occurring in φ. So, finite Kripke frames (up to iso) can be recursively coded by natural numbers such that: the validity relation

$$\begin{aligned}\mathsf{Val}(f, i) \text{ iff } & f \text{ is the code of a finite Kripke frame } \mathcal{F} \text{ and} \\ & i \text{ is the code of a formula } \varphi \text{ and } \mathcal{F} \models \varphi\end{aligned}$$

is recursive.

For any $L \in \mathsf{NExt}L_0$ and $L' \subseteq \omega$, we have $L' \in \mathrm{fmp}_{L_0}(L)$ iff L' satisfies α and the formula

$$\beta = \forall f[f \in \mathsf{FFr} \to [(\forall i \in L' \ \mathsf{Val}(f, i)) \leftrightarrow (\forall j \in L \ \mathsf{Val}(f, j))]],$$

which is readily verified to be a Π^0_2 formula with the parameter L. Thus, the set $\mathrm{fmp}_{L_0}(L)$ is defined by $\alpha \wedge \beta$, which is a Π^0_2 formula with the parameters L_0 and L. Hence, the set $\mathrm{fmp}_{L_0}(L)$ is $\Pi^0_2(L_0 \oplus L)$. □

Theorem 7.12. *For any logic L_0 and $L \in \mathsf{NExt}L_0$, the degree of fmp of L in $\mathsf{NExt}L_0$ is either $\leq \aleph_0$ or $2^{\aleph_0}$.*

Proof. This follows from Lemma 7.11 analogously to the proof of Theorem 7.5. □

Remark 7.13. It is straightforward to modify the proofs of Lemma 7.11 and Theorem 7.12 and prove similar results for superintuitionistic logics.

This shows that we can obtain complete antidichotomy results without using the Continuum Hypothesis, answering the question posed in [BBM25, Section 8 (1)].

Remark 7.14. Such a characterization in terms of the arithmetical hierarchy can be obtained for various logical properties, viewed as sets of logics. For example, it can be shown that for any logic L_0, the set of tabular logics in $\mathsf{NExt}L_0$ is $\Delta^0_2(L_0)$. However, this has no non-trivial implication on the cardinality since there are only countably many tabular logics in total. We leave it as future research to obtain meaningful results about logical properties other than cardinality from their characterizations in the arithmetical hierarchy.

Remark 7.15. Another limitation of this method is that it heavily relies on coding. So, it works only if everything involved (e.g., identities, formulas, finite Kripke frames) can be effectively coded. For example, this is not the case for Kripke frames, which form a proper class. It is unknown how to apply this method to count Kripke complete logics or the degree of Kripke incompleteness.

7.3 Summary

We studied the cardinality of sets of equational theories and logics without the Continuum Hypothesis by applying descriptive set theory via coding. The main idea is to code identities and formulas by natural numbers and equational theories and logics by real numbers. Then, sets of equational theories and sets of logics correspond to sets of reals. Characterizing the arithmetical hierarchy they belong to, we exploited the well-known fact that Borel sets have the perfect set property and thus the cardinality either $\leq \aleph_0$ or $2^{\aleph_0}$. We showed that every interval of equational theories has the cardinality $\leq \aleph_0$ or $2^{\aleph_0}$. Thus, it holds for every interval of varieties and interval of logics as well. This affirmatively answers the open question [JL18, Question 6.4 (ii)]. The proof does not rely on anything peculiar to equational theories or logics, and it works in a more general setting, such as quasi-equational theories and rule systems. We also showed that the degree of fmp for any modal logic is either $\leq \aleph_0$ or $2^{\aleph_0}$ by coding finite Kripke frames in addition to the syntax. This answers a recently posted question [BBM25, Section 8 (1)] in the positive.

The perfect set property is more than having the cardinality $\leq \aleph_0$ or $2^{\aleph_0}$. We leave it for future research if the method can be applied to solve another open question that appeared in [JL18]. Note that if this were true, then it would imply that every subvariety lattice has the cardinality $\leq \aleph_0$ or $2^{\aleph_0}$ as continuum many equational theories can be constructed from an independent system. We conjecture this does not hold.

Question 7.16 ([JL18, Question 6.4 (i)])**.** *Does every variety with uncountably many subvarieties have an independent system extending its equational theory?*

Moreover, it is not clear what implications the characterization in the arithmetical hierarchy or Borel hierarchy may have for studying logical properties beyond the cardinality argument we presented. For example, if a logical property is shown to be Borel, analytic, or precisely in some complexity class, what conclusion can we draw about that property?

Chapter 8

Conclusion and Future Work

In this thesis, we started by generalizing the theory of stable canonical rules and formulas and studied several topics of lattices of modal logics. We established the finite model property for a large class of modal logics, including union-splittings and their relatives. We showed the decidability of being a union-splitting, which implies the decidability of being a decidable formula and having a decidable axiomatization problem. We saw that stable canonical rules have the rule dichotomy property over $\mathsf{wK4}$ but not over K. Finally, we applied descriptive set theory to study the cardinality of intervals of logics without assuming the Continuum Hypothesis. Many open questions and further research have been discussed along the way. We summarize them as well as some general directions for future work as follows.

Stable canonical rules and formulas

A main open question in this regard is to construct (definable) filtrations. Though the fmp of $\mathsf{wK4}$ was proved in [BEG11] and a variation of filtration was constructed recently in [KS25], there is no known definable filtration for $\mathsf{wK4}$ (see, e.g., [KS25, Section 5]). The fmp of pre-transitive logics $\mathsf{K4}_\mathsf{n}^\mathsf{m}$ in general is a long-standing open question (e.g., [CZ97, Problem 11.2]).

Another interesting direction for future work is to construct variations of stable canonical rules and formulas, or other types of characterization rules and formulas, and study their implications. We introduced m-stable canonical formulas for the pre-transitive logics $\mathsf{K4}_1^{\mathsf{m}+1}$, but we did not really utilize them. We leave it open what can be shown via m-stable canonical formulas beyond our results via stable canonical formulas.

The finite modal property and the Refinement Construction

Further applications of the Refinement Construction are open. We verified by hand that the pre-transitive logics $\mathsf{K4}_1^{\mathsf{m}+1}$ are preserved by the Refinement Construction, but we do not have a systematic way to determine whether a logic is preserved or not. We leave it for future research to identify other (classes of) logics that are preserved by the construction and obtain a sufficient condition to be preserved. Such results will lead to more fmp results. Moreover, contrary to Zakharyaschev's result that every extension of $\mathsf{S4}$ with finitely many axioms in one variable has the fmp [CZ97, Theorem 11.58], all of our results have a technical presentation. It would be very interesting if such a concrete fmp result could be drawn from the fmp results we presented or the Refinement Construction.

Decidability of being a (union-)splitting

The decidability of being a union-splitting or a splitting is unknown for many important lattices of modal logics, such as NExtK4 and NExtS4. We refer to [WZ07] for more discussion of results and questions on the decidability of logical properties for modal logics.

Another related area is the degree of Kripke incompleteness. We remark that, while Blok's dichotomy theorem solves it for NExtK, determining the degree of Kripke incompleteness in NExtK4, NExtS4, or the lattice of all superintuitionistic logics is a long-standing open question (see, e.g., [CZ97, Problem 10.5]). It is even open whether all (union-)splittings in these lattices are Kripke complete.

Admissibility and the rule dichotomy property

The rule dichotomy property and the decidability of admissibility remain open for wK4 and K. As for wK4, canonical rules could be more effective than stable canonical rules, since wK4 is more similar to K4 than K. We leave it for further research whether Jeřábek's method with canonical rules applies to wK4. If we want to apply the method to K, it seems that a new type of characteristic rules is needed.

A more general research direction is to use stable canonical rules to study other proof-theoretical notions. For example, stable rules were shown to have the *bounded proof property* [BG14].

Cardinality of sets of logics

A question in [JL18] that we did not address is whether every variety with uncountably many subvarieties has an independent system extending its equational theory. The idea of coding identities by natural numbers and equational theories by real numbers seems also useful to this question, and our conjecture is that this does not hold.

While we characterized the arithmetical hierarchy to which intervals of logics and sets of logics sharing the same class of finite Kripke frames as a given logic belong, it is not clear what implications we can draw from such characterizations other than the cardinality results we presented. For example, if the set of reals corresponding to a logical property is shown to be Borel, analytic, or precisely in some complexity class, what conclusion can we draw about that property?

Finally, in this thesis, we constrained ourselves to the simplest type of modal logics, namely, modal logics with a single modal operator. It is a natural question to ask if our methods can be applied to other types of modal logics with richer languages, such as tense logic, propositional dynamic logic, or more ambitiously, modal mu-calculus. One can also consider other non-classical logics. The main challenge would be to construct a definable filtration for the logic in question, which is a crucial step toward an axiomatization result.

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