

Tense Information Logic

MSc Thesis (*Afstudeerscriptie*)

written by

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ABSTRACT

This thesis studies *tense information logic* (TIL), an extension of modal information logic (MIL). MIL was introduced by van Benthem [2] to model information flow using possible worlds semantics by adding a binary modality $\langle \text{sup} \rangle$ to the language of propositional logic, interpreted via the supremum of two states; TIL adds a second binary modality $\langle \text{inf} \rangle$ interpreted via the infimum of two states.

We give a sound and complete axiomatization of TIL on posets, extending Knudstorp's [17] axiomatization of MIL. As a corollary, we obtain completeness of TIL on preorders. We also show that TIL has the finite model property with respect to a generalized class of structures, thereby establishing its decidability.

Beyond completeness and decidability, we develop a Stone–Jónsson–Tarski duality for TIL, show that interpreting the modalities via minimal and maximal bounds leaves the logic unchanged, and construct two translations between weak positive logic (WPL) and an extended version of TIL containing Kleene star-like versions of the $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$ modalities.

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1 INTRODUCTION

Modal information logic (MIL), introduced by van Benthem [2], models information flow using possible-worlds semantics of modal logic by interpreting worlds as information states and introducing additional modalities. Because informational inclusion should at least form a partial order, the natural semantic framework is that of posets, whose suprema provide a notion of informational addition (or ‘merging’). Recently, Knudstorp [17] axiomatized MIL with a supremum modality on poset frames.

The notion of binary addition through suprema in posets has an obvious downward dual: infima. This motivates the introduction of a second modality, interpreted as the information two states share; that is, if x is the infimum of y and z , then x is the most informative state that contains only the information that y and z have in common, or alternatively, x is the most informative state such that y and z both refine x .

Given this motivation for the introduction of an infimum operator, the following question arises: Can the axiomatization of MIL be extended to a version that includes both modalities? Concretely, the added infimum operator will have the following semantics:

$$\mathfrak{M}, x \Vdash \langle \text{inf} \rangle \varphi \psi \quad \text{iff} \quad \text{there exist } y, z \text{ such that } \mathfrak{M}, y \Vdash \varphi, \mathfrak{M}, z \Vdash \psi \text{ and } x = \text{inf}\{y, z\}.$$

This operator complements the $\langle \text{sup} \rangle$ operator, which has the following semantics:

$$\mathfrak{M}, x \Vdash \langle \text{sup} \rangle \varphi \psi \quad \text{iff} \quad \text{there exist } y, z \text{ such that } \mathfrak{M}, y \Vdash \varphi, \mathfrak{M}, z \Vdash \psi \text{ and } x = \text{sup}\{y, z\}.$$

We call this extended logic *tense information logic* (TIL), by analogy with tense logic [10], whose future and past-looking modalities are definable in the language of TIL. By axiomatizing TIL, we answer another question already posed by van Benthem [3]: Are there interesting general axioms that link $\langle \text{sup} \rangle \varphi \psi$ to $\langle \text{inf} \rangle \varphi \psi$? The first goal of this thesis is to find a complete axiomatization of TIL on poset frames, thereby addressing both questions simultaneously.

To arrive at a complete axiomatization of TIL, we first define its semantics over poset frames. We then apply the step-by-step method introduced by Burgess [11] (and also followed by Knudstorp [17]), which for a given consistent set of formulas constructs a poset model witnessing its satisfiability. This approach recursively repairs so-called ‘defects’ and ‘labels’ points of a subset of our frame with maximally consistent sets for which we prove a truth lemma. We then show that this result extends to TIL on the class of preorder frames.

Secondly, we prove that TIL is decidable, thus resolving an open problem raised in [4]. Since TIL lacks the finite model property with respect to posets, we instead consider a generalized class of structures. Since the axiomatization is also complete with respect to this class, we prove that TIL has the finite model property (FMP) by showing that every countermodel in this class can be transformed into a finite countermodel. This proof follows the approach outlined in Theorem 3.9 of [17], adapting it to the new setting.

Beyond completeness and decidability, we interpret the axioms of TIL algebraically on boolean algebras with two binary operators and, dually, on modal Stone spaces with

two ternary relations. We work out a Stone–Jónsson–Tarski dual equivalence between the respective categories and show that it restricts to the subcategories validating TIL.

Replacing the supremum and infimum semantics of $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$ with semantics based on minimal upper bounds and maximal lower bounds yields the logic of incomparable fusions and common information: two information states may have, for example, several minimal fusions that are pairwise incomparable. We prove that the same axiomatization is sound and complete for this semantics as well; hence both semantics induce the same logic (the same set of validities). The argument follows the representation method of [18], adapted to the setting with two modalities.

In [23, 24], Wang and Wang view lattices as relational structures based on partial orders and study the modal logics over them. Using the same language as TIL, they aim to capture the structure of lattices with this language. In lattices, every pair of elements has a unique supremum and infimum (as opposed to posets where suprema and infima do not necessarily exist), a property that is impossible to express in the language of TIL [24, Theorem 25]. They therefore add nominals to the language to strengthen the expressive power and give a complete axiomatization of TIL on lattices in a hybrid language. In comparison, the completeness proof for TIL presented in this thesis does not include nominals, and is over posets instead of lattices.

This work shows that there is an incentive to interpret TIL on lattices. In the literature, TIL on lattices is discussed alongside weak positive logic (WPL) [7]: positive modal logic (the negation- and implication-free fragment of classical propositional logic) that is not necessarily distributive. Interestingly, the TIL operator $\langle \text{inf} \rangle$ behaves much like disjunction in WPL. This suggests that a translation between the two logics is possible.

In order to extend the proposed translation in [7] to a full and faithful translation, it turns out that the modalities are not sufficient. This motivates introducing two additional binary modalities: Kleene star-like variants of $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$ which, instead of taking suprema and infima over pairs of worlds, take the join / meet of a any non-empty finite set of witnesses.

Having extended the language, there are many interesting directions to study. In this thesis, we return to poset semantics for these Kleene star-like supremum and infimum modalities and investigate how TIL (with and without these extra modalities) can be used as a modal lens on posets. We show that several natural poset operations are definable, while much is still unexplored.

The main contributions of this thesis are:

- We give a sound and complete axiomatization of TIL on poset frames and show that this result transfers to preorder frames.
- We prove that TIL is decidable by showing that it has the finite model property with respect to a generalized class of frames.
- We derive a Stone–Jónsson–Tarski duality for TIL between boolean algebras with two binary operators and modal Stone spaces with two ternary relations.
- By interpreting the modalities in terms of minimal upper bounds and maximal lower bounds rather than suprema and infima, we obtain the logic of incomparable fusions and common information. For this logic, we prove that the same axiomatization is sound and complete with respect to posets via a representation argument.

- We give a full and faithful translation of the $\perp$ -free fragment of WPL into an extension of TIL with two Kleene star-like modalities.
- We give a full and faithful translation of WPL (including $\perp$), relative to a fixed consequence pair $\varphi \trianglelefteq \psi$ into the same extension of TIL.
- We show that the extended logic enables us to define the supremum and infimum closure operators on posets.

1.1 GUIDE TO SECTIONS

The structure of the thesis follows the order of the results mentioned above. We start by providing the preliminaries in Chapter 2 and show that TIL on posets lacks the finite model property. Chapter 3 introduces an axiomatization for TIL and proves that it is sound and complete with respect to poset frames. Using this axiomatization, we prove in Chapter 4 that TIL has the finite model property with respect to a generalized class of frames, thus establishing its decidability.

In Chapter 5, we work out a Stone–Jónsson–Tarski duality for TIL and in Chapter 6, we show that changing the semantics of the modal operators to the minimal upper bound and maximal lower bound interpretation does not change the logic. Lastly, we study the relation between WPL and TIL in Chapter 7, provide two translations between these logics and show how TIL can be used as a modal lens on posets.

2 PRELIMINARIES

In this chapter, we set up the formal framework of TIL: we define the language, present the semantics, and thereby lay the groundwork for the rest of this thesis. We then show that the logic fails to have the finite model property and outline how we will nevertheless prove its decidability.

2.1 LANGUAGE AND SEMANTICS

Most of the notions in this chapter extend the definitions given in [17]. To capture infima alongside suprema, we expand the base language $\mathcal{L}_M$ of MIL [17, Definition 1.1] with an additional binary modality.

Definition 2.1.1. Given a countable set of propositional letters $\mathbf{P}$, we define *the language $\mathcal{L}_T$ of tense information logic* using two binary modalities $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$ by the following BNF grammar:

$$\varphi ::= p \mid \perp \mid \neg\varphi \mid \varphi \wedge \psi \mid \langle \text{sup} \rangle \varphi \psi \mid \langle \text{inf} \rangle \varphi \psi.$$

We denote by $\mathcal{L}_M$ the $\langle \text{inf} \rangle$ -free fragment of $\mathcal{L}_T$.

Definition 2.1.2. A (Kripke) *poset model* for $\mathcal{L}_T$ is a triple $\mathfrak{M} = (W, \leq, V)$, where W is a set, $\leq$ is a partial order, and V is a valuation $V : \mathbf{P} \rightarrow \mathcal{P}(W)$.

To interpret the binary modalities $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$, we recall the standard order-theoretic notions of supremum (join) and infimum (meet).

Definition 2.1.3 (Supremum and infimum). Let $\mathfrak{F} = \langle W, \leq \rangle$ be a poset and $S \subseteq W$ a non-empty subset.

- An element $u \in W$ is an *upper bound* of S if $s \leq u$ for every $s \in S$. A *supremum* (or *least upper bound*) of S , written $\sup S$, is an upper bound u such that whenever v is an upper bound of S we have $u \leq v$.
- An element $l \in W$ is a *lower bound* of S if $l \leq s$ for every $s \in S$. An *infimum* (or *greatest lower bound*) of S , written $\inf S$, is a lower bound l such that whenever m is a lower bound of S we have $m \leq l$.

For the two-element set $S = \{y, z\}$ we write $\sup\{y, z\}$ (respectively $\inf\{y, z\}$) and call it the *join* (respectively *meet*) of y and z .

Remark 2.1.1. Since a partial order is antisymmetric, infima and suprema are unique in posets if they exist. However, note that they do not necessarily exist.

Definition 2.1.4 (Semantics). The *interpretation* of a formula $\varphi \in \mathcal{L}_T$ at a state $x \in W$ is defined recursively as follows:

$$\begin{aligned}
& \mathfrak{M}, x \not\models \perp, \\
& \mathfrak{M}, x \models p \quad \text{iff} \quad x \in V(p), \\
& \mathfrak{M}, x \models \neg\varphi \quad \text{iff} \quad \mathfrak{M}, x \not\models \varphi, \\
& \mathfrak{M}, x \models \varphi \wedge \psi \quad \text{iff} \quad \mathfrak{M}, x \models \varphi \text{ and } \mathfrak{M}, x \models \psi, \\
& \mathfrak{M}, x \models \langle \sup \rangle \varphi \psi \quad \text{iff} \quad \text{there exist } y, z \in W \text{ s.t. } \mathfrak{M}, y \models \varphi, \mathfrak{M}, z \models \psi \text{ and } x = \sup\{y, z\}, \\
& \mathfrak{M}, x \models \langle \inf \rangle \varphi \psi \quad \text{iff} \quad \text{there exist } y, z \in W \text{ s.t. } \mathfrak{M}, y \models \varphi, \mathfrak{M}, z \models \psi \text{ and } x = \inf\{y, z\}.
\end{aligned}$$

Definition 2.1.5 (Dual box modalities). For $\varphi, \psi \in \mathcal{L}_T$, define the duals of the binary diamonds as

$$[\inf] \varphi \psi := \neg \langle \inf \rangle \neg \varphi \neg \psi, \quad [\sup] \varphi \psi := \neg \langle \sup \rangle \neg \varphi \neg \psi.$$

By Definition 2.1.4, their truth conditions are:

$$\begin{aligned}
\mathfrak{M}, x \models [\inf] \varphi \psi & \quad \text{iff} \quad \text{for every } y, z \in W (x = \inf\{y, z\} \Rightarrow \mathfrak{M}, y \models \varphi \text{ or } \mathfrak{M}, z \models \psi), \\
\mathfrak{M}, x \models [\sup] \varphi \psi & \quad \text{iff} \quad \text{for every } y, z \in W (x = \sup\{y, z\} \Rightarrow \mathfrak{M}, y \models \varphi \text{ or } \mathfrak{M}, z \models \psi).
\end{aligned}$$

Definition 2.1.6. With these semantics at hand we are able to define the standard *past / future looking diamond / box* of temporal logic [10] in the following way:

$$\begin{aligned}
P \varphi &:= \langle \sup \rangle \varphi \top \quad \text{past looking diamond}, \\
F \varphi &:= \langle \inf \rangle \varphi \top \quad \text{future looking diamond}, \\
H \varphi &:= \neg \langle \sup \rangle \neg \varphi \top \quad \text{past looking box}, \\
G \varphi &:= \neg \langle \inf \rangle \neg \varphi \top \quad \text{future looking box}.
\end{aligned}$$

Recall the semantics of the above-mentioned temporal modalities:

$$\begin{aligned}
\mathfrak{M}, x \models P \varphi & \quad \text{iff} \quad \text{there exists } y \in W \text{ such that } y \leq x \text{ and } \mathfrak{M}, y \models \varphi, \\
\mathfrak{M}, x \models F \varphi & \quad \text{iff} \quad \text{there exists } y \in W \text{ such that } x \leq y \text{ and } \mathfrak{M}, y \models \varphi, \\
\mathfrak{M}, x \models H \varphi & \quad \text{iff} \quad \text{for every } y \in W \text{ such that } y \leq x \text{ it holds that } \mathfrak{M}, y \models \varphi, \\
\mathfrak{M}, x \models G \varphi & \quad \text{iff} \quad \text{for every } y \in W \text{ such that } x \leq y \text{ it holds that } \mathfrak{M}, y \models \varphi.
\end{aligned}$$

Remark 2.1.2. We show that these semantics correspond to the definitions of P, F, H and G of Definition 2.1.6.

- Assume $\mathfrak{M}, x \models \langle \inf \rangle \varphi \top$. By the semantics of $\langle \inf \rangle$ this holds iff there are $y, z \in W$ such that $\mathfrak{M}, y \models \varphi$, $\mathfrak{M}, z \models \top$ and $x = \inf\{y, z\}$. Since $\mathfrak{M}, x \models \top$ always holds, it follows that $\mathfrak{M}, x \models \langle \inf \rangle \varphi \top$ iff there is $y \in W$ such that $\mathfrak{M}, y \models \varphi$ and $x = \inf\{y, x\}$. Because $x = \inf\{y, x\}$ is equivalent to $x \leq y$ we get the desired result.
- The case of $\langle \sup \rangle \varphi \top$ is shown in a similar way, using the fact that $x = \sup\{y, x\}$ is equivalent to $y \leq x$.
- Since H is the dual of P, it follows that $H\varphi$ is defined as $\neg P \neg \varphi$, which coincides with the above definition. To get a better feeling for the semantics of the modalities, we still show it directly:

Assume $\mathfrak{M}, x \Vdash \neg\langle\text{sup}\rangle\neg\varphi\top$. By the semantics of $\langle\text{sup}\rangle$ this holds iff there are no $y, z \in W$ such that $\mathfrak{M}, y \Vdash \neg\varphi$, $\mathfrak{M}, z \Vdash \top$ and $x = \text{sup}\{y, z\}$. Since we always have $\mathfrak{M}, x \Vdash \top$, it follows that $\mathfrak{M}, x \Vdash \neg\langle\text{sup}\rangle\neg\varphi\top$ iff there is no $y \in W$ such that $\mathfrak{M}, y \Vdash \neg\varphi$ and $x = \text{sup}\{y, x\}$. Because $x = \text{sup}\{y, x\}$ is equivalent to $y \leq x$, $\mathfrak{M}, x \Vdash \neg\langle\text{sup}\rangle\neg\varphi\top$ holds iff $\mathfrak{M}, y \Vdash \varphi$ for every $y \leq x$.

- The case of $\neg\langle\text{inf}\rangle\neg\varphi\top$ is shown in a similar way.

Based on these semantics, we define the logic *TIL*. For completeness, we repeat the definition of *MIL* as presented in [17]:

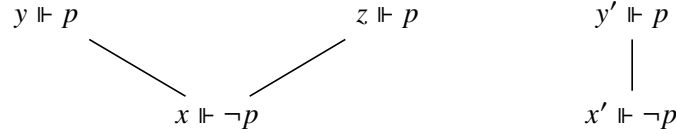
Definition 2.1.7. Tense information logic on posets is denoted *TIL* and defined as the set of all $\mathcal{L}_T$ -validities on poset frames; that is

$$\text{TIL} := \{ \varphi \in \mathcal{L}_T \mid \text{for every poset model } \mathfrak{M} = (W, \leq, V) \text{ and every } x \in W : \mathfrak{M}, x \Vdash \varphi \}.$$

MIL is defined as the set of all validities in the $\langle\text{inf}\rangle$ -free fragment of $\mathcal{L}_T$ on poset frames; that is,

$$\text{MIL} := \{ \varphi \in \mathcal{L}_M \mid \text{for every poset model } \mathfrak{M} = (W, \leq, V) \text{ and every } x \in W : \mathfrak{M}, x \Vdash \varphi \}.$$

Remark 2.1.3. It should be clear that the expressive power of *TIL* strictly extends *MIL*, but for completeness we give an example that shows this. Consider the following two models:



Then $x \Vdash \langle\text{inf}\rangle pp$, since $x = \text{inf}\{y, z\}$, $y \Vdash p$ and $z \Vdash p$ while $x' \not\Vdash \langle\text{inf}\rangle pp$, since x' is not the infimum of two p worlds. But, as an induction readily shows, for every $\varphi \in \mathcal{L}_M : x \Vdash \varphi$ iff $x' \Vdash \varphi$.

2.2 FINITE MODEL PROPERTY

Knudstorp [17] shows that *MIL* lacks the finite model property (FMP) with respect to preorder frames. Because *TIL* strictly extends *MIL* and since every poset is also a preorder, the failure of the FMP carries over to *TIL* on poset frames as an immediate corollary. For completeness (to give a specific example involving both modalities), we still provide a counterexample in the language $\mathcal{L}_T$ that shows that *TIL* does not have the FMP with respect to posets.

Proposition 2.2.1. *TIL* does not have the FMP with respect to poset frames.

Proof. Consider the following formula:

$$\chi_\infty := G(p \rightarrow F\neg p) \wedge G(\neg p \rightarrow Fp) \wedge H(p \rightarrow P\neg p) \wedge H(\neg p \rightarrow Pp).$$

We first show that χ_∞ is satisfiable on an infinite poset model. Let $\mathfrak{M} = (W, \leq, V)$ be defined as follows:

1. $W = \mathbb{Z}$,
2. $\leq$ is the standard less than or equal relation on the integers,
3. $V : \mathbf{P} \rightarrow \mathcal{P}(W)$ is defined as $V(p) = \{z \in \mathbb{Z} : z \text{ is even}\}$.

Fix an arbitrary $z \in \mathbb{Z}$.

- *If z is even.* Then p holds at z and $\neg p$ holds at $z \pm 1$. Hence $F \neg p$ and $P \neg p$ hold at z , so the first and third implications in χ_∞ are satisfied; the other two implications are vacuously true. It follows that

$$\mathfrak{M}, z \models p \rightarrow F \neg p \wedge \neg p \rightarrow F p \wedge p \rightarrow P \neg p \wedge \neg p \rightarrow P p.$$

- *If z is odd.* The situation is symmetric, interchanging p and $\neg p$.

Thus, $\mathfrak{M}, z \models \chi_\infty$ for every $z \in \mathbb{Z}$.

On the other hand, assume $\mathfrak{M} = (W, \leq, V)$ is a finite poset model. It follows that the model has a minimal element m_1 [8]. Either $m_1 \models p$ or $m_1 \models \neg p$.

- If $m_1 \models p$, then $m_1 \not\models p \rightarrow P \neg p$ because there is no $w \in W$ such that $w \leq m_1$ and $w \models \neg p$, since m_1 is minimal.
- If $m_1 \models \neg p$, the clause $\neg p \rightarrow P p$ fails by the same reasoning.

Hence $m_1 \not\models p \rightarrow P \neg p$ or $m_1 \not\models \neg p \rightarrow P p$. This holds for any minimal element $m \in W$. Since for every $y \in W$ there exists a minimal element $m \in W$ such that $m \leq y$, it follows that no world satisfies both $H(p \rightarrow P \neg p)$ and $H(\neg p \rightarrow P p)$. Thus, no world satisfies χ_∞ , so $\mathfrak{M}$ does not satisfy χ_∞ . We conclude that no finite poset model satisfies χ_∞ .

Because χ_∞ is satisfiable on posets frames but only on infinite ones, *TIL* lacks the finite model property with respect to posets. $\square$

Remark 2.2.2. Note that χ_∞ is satisfiable on a finite preorder frame. Let $\mathfrak{M} = (W, \leq, V)$ be the preorder model that consists of two worlds x_1, x_2 such that $x_1 \leq x_2$ and $x_2 \leq x_1$. If $V(p) = x_1$ it follows that both worlds satisfy χ_∞ .

It follows that χ_∞ only shows that *TIL* on posets does not have the FMP, while the lack of the FMP for *TIL* on preorders is presented in [17, Proposition 1.7].

To still prove decidability of *TIL*, we will follow the same route as [17]. In the chapters that follow we will:

- present a sound and complete axiomatization of *TIL*; and
- prove that *TIL* has the FMP with respect to a generalized class of structures.

3 SOUNDNESS AND COMPLETENESS OF TIL

In this chapter, we present one of the thesis' main results: a sound and complete axiom system for *TIL* over poset frames, which we subsequently use in the next chapter to prove decidability.

Section 3.1 introduces a proof-theoretic description of *TIL* and establishes its soundness with respect to poset semantics. Section 3.2 explains why the usual canonical model construction fails and proves some auxiliary lemmas. In Section 3.3 we introduce the step-by-step method, which will be used to nevertheless prove completeness. Section 3.4 contains the actual strong completeness proof of *TIL* on posets.

3.1 AXIOMATIZATION AND SOUNDNESS

To study modal information logics on posets in a symmetric way, we extended the language by including the infimum operator. However, it is not immediately clear which axioms link the two modalities (see [3]). Surprisingly, it turns out that the relation between $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$ can be fully captured by standard temporal axioms. To show this, we present the following logic.

Definition 3.1.1. Let **TIL** be the least normal modal logic in the language $\mathcal{L}_T$ containing all propositional tautologies, instances of the K-axiom for $[\text{inf}]$ and $[\text{sup}]$ ¹ and the following axioms:

- (Re.) $(p \wedge q \rightarrow \langle \text{sup} \rangle pq) \wedge (p \wedge q \rightarrow \langle \text{inf} \rangle pq),$
- (4) $(\text{PP } p \rightarrow \text{P } p) \wedge (\text{FF } p \rightarrow \text{F } p),$
- (Co.) $(\langle \text{sup} \rangle pq \rightarrow \langle \text{sup} \rangle qp) \wedge (\langle \text{inf} \rangle pq \rightarrow \langle \text{inf} \rangle qp),$
- (Dk1) $(p \wedge \langle \text{sup} \rangle qr) \rightarrow \langle \text{sup} \rangle pq,$
- (Dk2) $(p \wedge \langle \text{inf} \rangle qr) \rightarrow \langle \text{inf} \rangle pq,$
- (Sy.) $(p \rightarrow \text{GP } p) \wedge (p \rightarrow \text{HF } p),$

and closed under modus ponens, uniform substitution and generalization (i.e., if $\vdash \varphi$ then $\vdash [\text{inf}] \varphi \psi$ (and thus $\vdash [\text{inf}] \psi \varphi$) for all $\varphi, \psi \in \mathcal{L}_T$ and similarly for $[\text{sup}]$).

For each axiom in the axiomatization of MIL from [17] (namely (Re.), (4), (Co.) and (Dk)), we added the same axiom for the new $\langle \text{inf} \rangle$ -operator. The only truly new axiom is the standard temporal axiom (Sy.) [14, 20]. That this axiom is to be included is expected, since the temporal operators P, F, G and H are definable. On the other hand, what may be unexpected is that this is the *only* axiom that links $\langle \text{sup} \rangle$ to $\langle \text{inf} \rangle$ that we need to add to the axiomatization of MIL to obtain an axiomatization of TIL. We will proceed to show that this axiom system is strongly complete for *TIL* and thus that the single axiom (Sy.) already suffices to syntactically link $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$.

¹ For each operator we include both coordinate-wise schemes: $[\text{sup}](p \rightarrow q)r \rightarrow ([\text{sup}]pr \rightarrow [\text{sup}]qr),$ $[\text{sup}]r(p \rightarrow q) \rightarrow ([\text{sup}]rp \rightarrow [\text{sup}]rq)$ and analogously for $[\text{inf}]$.

First off, the axiomatization is readily seen to be sound by checking that *TIL* is a normal modal logic validating all the axioms of **TIL**:

Theorem 3.1.1 (Soundness). **TIL** $\subseteq$ *TIL*.

To familiarize the reader with the framework and provide some hands-on intuition, we include parts of the soundness proof.

Proof. Let $\mathfrak{M} = (W, \leq, V)$ be a poset model. We must verify that every propositional tautology, the normal modal schemata *K* and the axioms of **TIL** are valid in $\mathfrak{M}$, and, further, that validity on poset models is preserved under modus ponens and generalization:

- (4) This follows directly from transitivity of $\leq$. Assume $\mathfrak{M}, x \Vdash \text{PP} p$, so there are $y, z \in W$ such that $y \leq x$, $z \leq y$ and $\mathfrak{M}, y \Vdash \text{P} p$ and $\mathfrak{M}, z \Vdash p$. By transitivity, we get $z \leq x$ and so $\mathfrak{M}, x \Vdash \text{P} p$. The second part is shown symmetrically.
- (Dk2) Assume $\mathfrak{M}, x \Vdash p \wedge \langle \text{inf} \rangle qr$, so $\mathfrak{M}, x \Vdash p$ and there are y, z such that $x = \text{inf}\{y, z\}$, $\mathfrak{M}, y \Vdash q$ and $\mathfrak{M}, z \Vdash r$. Since $x = \text{inf}\{y, z\}$ implies $x = \text{inf}\{x, y\}$, it follows that $\mathfrak{M}, x \Vdash \langle \text{inf} \rangle pq$.
- (Sy.) Assume $\mathfrak{M}, x \Vdash p$. Now, let y be arbitrary such that $x \leq y$. Since $\mathfrak{M}, x \Vdash p$ it follows that $\mathfrak{M}, y \Vdash \text{P} p$, so since y was an arbitrary element greater than x it follows that $\mathfrak{M}, x \Vdash \text{GP} p$. On the other hand, take any y such that $y \leq x$; then $\mathfrak{M}, y \Vdash \text{F} p$, so $\mathfrak{M}, x \Vdash \text{HF} p$.
- (K) We show one instance:

$$([\text{sup}](p \rightarrow q)r) \rightarrow ([\text{sup}]pr \rightarrow [\text{sup}]qr).$$

Assume $\mathfrak{M}, x \Vdash [\text{sup}](p \rightarrow q)r$, so for every y, z such that $x = \text{sup}\{y, z\}$ we have $\mathfrak{M}, y \Vdash p \rightarrow q$ or $\mathfrak{M}, z \Vdash r$. Now assume $\mathfrak{M}, x \Vdash [\text{sup}]pr$, so for every u, v such that $x = \text{sup}\{u, v\}$ we have $\mathfrak{M}, u \Vdash p$ or $\mathfrak{M}, v \Vdash r$. We need to show $\mathfrak{M}, x \Vdash [\text{sup}]qr$. So let y, z be arbitrary such that $x = \text{sup}\{y, z\}$. Since $x = \text{sup}\{y, z\}$, we have $\mathfrak{M}, y \Vdash p \rightarrow q$ or $\mathfrak{M}, z \Vdash r$. In case of the latter, we are done, so assume $\mathfrak{M}, y \Vdash p \rightarrow q$. Then $\mathfrak{M}, x \Vdash [\text{sup}]pr$ and $x = \text{sup}\{y, z\}$ imply $\mathfrak{M}, y \Vdash p$ or $\mathfrak{M}, z \Vdash r$, and again, in case of the latter, we are done. If $\mathfrak{M}, y \Vdash p$, then $\mathfrak{M}, y \Vdash p \rightarrow q$ implies $\mathfrak{M}, y \Vdash q$. Hence $\mathfrak{M}, x \Vdash [\text{sup}]qr$.

Lastly, we check whether the rule of generalization preserves validity. We again show one instance: Assume $\Vdash \varphi$. We must show that for any $\psi \in \mathcal{L}_T : \Vdash [\text{inf}]\varphi\psi$. So let $\mathfrak{M}$ be an arbitrary poset model and $x \in W$. Let y, z be arbitrary so that $x = \text{inf}\{y, z\}$. We have $\mathfrak{M}, y \Vdash \varphi$, so $\mathfrak{M}, x \Vdash [\text{inf}]\varphi\psi$. $\square$

3.2 CANONICAL MODEL AND AUXILIARY LEMMAS

To prove completeness, we will use a similar approach to the one presented in [17, Section 2]. To avoid redundancy, we will omit some proofs that are identical (or nearly so), instead focusing on the aspects that provide insight into the relation between $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$ in our extended system.

We begin by constructing the canonical model. However, this canonical model lacks the properties required to establish the necessary truth lemma for the completeness proof; in particular, the underlying frame is not a poset frame where the canonical relations refer to

the supremum and infimum induced by the ordering those relations induce. We therefore make use of the step-by-step method (see [10]).

Definition 3.2.1. Let $W_{\mathbf{TIL}}$ be the set that contains all maximally consistent $\mathbf{TIL}$ -sets. Let C_{sup} and C_{inf} be the two induced ternary relations of the canonical $\mathbf{TIL}$ -frame:

$$C_{\text{sup}} \Gamma \Delta \Theta \iff \forall \delta \in \Delta, \theta \in \Theta (\langle \text{sup} \rangle \delta \theta \in \Gamma),$$

$$C_{\text{inf}} \Gamma \Delta \Theta \iff \forall \delta \in \Delta, \theta \in \Theta (\langle \text{inf} \rangle \delta \theta \in \Gamma).$$

We define the following binary relation on the canonical frame:

$$\leq_{\text{pre}} := \{(\Delta, \Gamma) \in W_{\mathbf{TIL}} \times W_{\mathbf{TIL}} \mid C_{\text{sup}} \Gamma \Gamma \Delta\}.$$

It may appear bizarre that our definition of $\leq_{\text{pre}}$ only depends on C_{sup} , and indeed we will need that it admits an equivalent characterization in terms of C_{inf} :

$$\leq_{\text{pre}} := \{(\Delta, \Gamma) \in W_{\mathbf{TIL}} \times W_{\mathbf{TIL}} \mid C_{\text{inf}} \Delta \Delta \Gamma\}. \quad (1)$$

To show this, we rely on two auxiliary lemmas (Lemma 3.2.1(1) and (2)). The equivalence itself is given by Lemma 3.2.1(3), while Lemma 3.2.1(4), (5) and (6) are needed for later proofs.

Lemma 3.2.1. 1. $p \rightarrow P p$ and $p \rightarrow F p$ are derivable in $\mathbf{TIL}$.

$$2. \forall \Gamma, \Delta, \Theta \in W_{\mathbf{TIL}} : (C_{\text{sup}} \Gamma \Delta \Theta \text{ iff } C_{\text{sup}} \Gamma \Theta \Delta) \text{ and } (C_{\text{inf}} \Gamma \Delta \Theta \text{ iff } C_{\text{inf}} \Gamma \Theta \Delta).$$

$$3. \forall \Gamma, \Delta \in W_{\mathbf{TIL}} : C_{\text{sup}} \Gamma \Gamma \Delta \text{ iff } C_{\text{inf}} \Delta \Delta \Gamma.$$

$$4. \forall \Gamma, \Delta \in W_{\mathbf{TIL}} : \Delta \leq_{\text{pre}} \Gamma \text{ iff } (\forall \delta \in \Delta : P \delta \in \Gamma \text{ and } \forall \gamma \in \Gamma : F \gamma \in \Delta).$$

$$5. \leq_{\text{pre}} \text{ is a preorder.}$$

$$6. \forall \Gamma, \Delta, \Theta \in W_{\mathbf{TIL}} : (C_{\text{sup}} \Gamma \Delta \Theta \text{ only if } \Delta \leq_{\text{pre}} \Gamma \Theta \leq_{\text{pre}} \Gamma) \text{ and } (C_{\text{inf}} \Gamma \Delta \Theta \text{ only if } \Gamma \leq_{\text{pre}} \Delta, \Gamma \leq_{\text{pre}} \Theta).$$

We explicitly present the proof of Lemma 3.2.1(3), as the proofs of the other statements are analogous to the proofs of Observation 2.5 and Lemma 2.6 in [17]. In each case, the version involving the $\langle \text{inf} \rangle$ operator is proven in the same way as the version involving $\langle \text{sup} \rangle$.

Proof of Lemma 3.2.1(3). Throughout, we use the standard properties of $\mathbf{TIL}$ -maximally consistent sets (MCSs), which hold because $\mathbf{TIL}$ is a normal modal logic.

Assume $C_{\text{sup}} \Gamma \Gamma \Delta$. To show that $C_{\text{inf}} \Delta \Delta \Gamma$, let $\delta \in \Delta$ and $\gamma \in \Gamma$ be arbitrary. (Sy.) and uniform substitution entail

$$\gamma \rightarrow \neg \langle \text{sup} \rangle (\neg \langle \text{inf} \rangle \gamma \top) \top \in \Gamma.$$

Since $\gamma \in \Gamma$, modus ponens yields

$$\neg \langle \text{sup} \rangle (\neg \langle \text{inf} \rangle \gamma \top) \top \in \Gamma.$$

Consistency then forces

$$\langle \text{sup} \rangle (\neg \langle \text{inf} \rangle \gamma \top) \top \notin \Gamma.$$

By Lemma 3.2.1 (2), $C_{\text{sup}} \Gamma \Gamma \Delta$ implies $C_{\text{sup}} \Gamma \Delta \Gamma$, so it follows from

$$\langle \text{sup} \rangle (\neg \langle \text{inf} \rangle \gamma \top) \top \notin \Gamma$$

that

$$\neg \langle \text{inf} \rangle \gamma \top \notin \Delta \text{ or } \top \notin \Gamma.$$

But $\top \in \Gamma$, so $\neg \langle \text{inf} \rangle \gamma \top \notin \Delta$, hence $\langle \text{inf} \rangle \gamma \top \in \Delta$ by maximality of Δ . Since $\delta \in \Delta$, an application of US and (Dk2) yields $\langle \text{inf} \rangle \delta \gamma \in \Delta$. Hence $C_{\text{inf}} \Delta \Delta \Gamma$.

The other direction is proven similarly. $\square$

In case C_{sup} and C_{inf} would correspond to the infimum and supremum relation with respect to $\leq_{\text{pre}}$, and if $\leq_{\text{pre}}$ was a partial order instead of merely a preorder, we would have completeness in our pocket. The next examples show that neither are the case. We first show that $\leq_{\text{pre}}$ is not antisymmetric:

Example 3.2.2. Consider the following model:

1. $W = \mathbb{Z}$,
2. $\leq$ is the standard less than or equal relation on the integers,
3. $V : \mathbf{P} \rightarrow \mathcal{P}(W)$ is defined as $V(p) = \{z \in \mathbb{Z} : z \text{ is even}\}$.

For every $i \in \mathbb{Z}$ put

$$\Delta_i := \{\varphi \in \mathcal{L}_T \mid i \Vdash \varphi, i \text{ is even}\}, \quad \Gamma_i := \{\varphi \in \mathcal{L}_T \mid i \Vdash \varphi, i \text{ is odd}\}.$$

It is easily seen that

$$\cdots = \Delta_{-2} = \Delta_0 = \Delta_2 = \cdots, \tag{2}$$

and

$$\cdots = \Gamma_{-3} = \Gamma_{-1} = \Gamma_1 = \cdots. \tag{3}$$

Our goal is to show

$$\Delta_0 \leq_{\text{pre}} \Gamma_1 \text{ and } \Gamma_1 \leq_{\text{pre}} \Delta_0,$$

while $\Delta_0 \neq \Gamma_1$, thus showing that $\leq_{\text{pre}}$ is not anti-symmetric.

We have $0 \Vdash p$ and $1 \Vdash \neg p$, hence $p \in \Delta_0$ and $\neg p \in \Gamma_1$, so $\Delta_0 \neq \Gamma_1$.

Now take $\delta \in \Delta_0$, i.e. $0 \Vdash \delta$. Because $-1 \leq 0 \leq 1$,

$$-1 \Vdash F\delta \text{ and } 1 \Vdash P\delta,$$

so by (3) $F\delta, P\delta \in \Gamma_1$.

Conversely, for $\gamma \in \Gamma_1$ (so $1 \Vdash \gamma$) we have $0 \leq 1 \leq 2$, hence

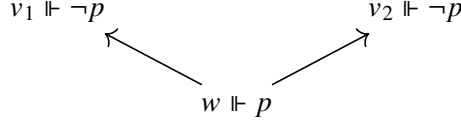
$$0 \Vdash F\gamma \text{ and } 2 \Vdash P\gamma.$$

It follows from (2) that $F\gamma, P\gamma \in \Delta_0$.

Lemma 3.2.1(4) yields $\Delta_0 \leq_{\text{pre}} \Gamma_1$ and $\Gamma_1 \leq_{\text{pre}} \Delta_0$.

Secondly, we give an example showing that C_{inf} is not the infimum relation of $\leq_{\text{pre}}$.

Example 3.2.3. Suppose we have MCSs Γ and Δ for which it holds that $C_{\text{inf}} \Gamma \Delta \Delta$. If C_{sup} and C_{inf} matched the supremum and infimum relations induced by $\leq_{\text{pre}}$, we would have $\Gamma \leq_{\text{pre}} \Delta$ and $\Delta \leq_{\text{pre}} \Gamma$. We show that $C_{\text{inf}} \Gamma \Delta \Delta$ can hold without $\Delta \leq_{\text{pre}} \Gamma$ being true. Consider the following model:



And let p be the only propositional variable that the states satisfy / falsify. We define the following MCSs:

$$\Gamma := \{ \varphi \in \mathcal{L}_T \mid \mathfrak{M}, w \Vdash \varphi \}$$

$$\Delta := \{ \varphi \in \mathcal{L}_T \mid \mathfrak{M}, v_1 \Vdash \varphi \} = \{ \varphi \in \mathcal{L}_T \mid \mathfrak{M}, v_2 \Vdash \varphi \}$$

We see that $p \in \Gamma$, but $p \notin \Delta$, so by Lemma 3.2.1(4) we get that $\Delta \not\leq_{\text{pre}} \Gamma$. On the other hand, since $w = \inf\{v_1, v_2\}$ we see that $C_{\text{inf}} \Gamma \Delta \Delta$ holds.

3.3 REPAIR LEMMAS

To prove that syntactic consistency implies satisfiability (and hence completeness), we use the *step-by-step defect–repair construction* to build a model $\mathfrak{M}$ whose worlds are labeled by a function l , assigning each world a MCS. The objective is then to ensure that the truth lemma for labeled worlds

$$\mathfrak{M}, x \Vdash \varphi \iff \varphi \in l(x)$$

holds.

Concretely, we start with a single world labeled by our initial MCS Γ , so that if the truth lemma holds, then we have our satisfying model. For the truth lemma to hold, a world's label $l(x)$ can dictate that it should satisfy some formula involving $\langle \text{sup} \rangle$ or $\langle \text{inf} \rangle$; for example, if $\langle \text{sup} \rangle \varphi \psi \in l(x)$, then x should satisfy $\langle \text{sup} \rangle \varphi \psi$.

If the model under construction does not yet provide points witnessing this (so $x \not\Vdash \langle \text{sup} \rangle \varphi \psi$), we call this shortfall a *defect*. These defects (Definitions 3.3.2, 3.3.3, 3.3.4 and 3.3.5) are *repaired* in stages. In case of the defect we just described, it is resolved by adding two fresh worlds y and z , such that $x = \text{sup}\{y, z\}$, $\varphi \in l(y)$ and $\psi \in l(z)$.

That the procedures we describe actually resolve the defects is proven in their respective repair lemmas (see Lemma 3.3.1 and 3.3.2). Iterating this process until no defects remain yields our full model.

We will mostly use the outline presented in Section 4.6 of [10] and Section 2.2 of [17]. We now turn to the formal definition of the tuples $(l, \leq)$ that at each stage determine the ‘approximating frame’:

Definition 3.3.1. Let W be a countable set, and $\mathbb{P}$ the set of all tuples $(l, \leq)$ such that:

1. l is a partial function from W to the set of all MCSs, W_{TIL} .
2. $\text{dom}(l)$ is finite.
3. $\leq$ is a partial order on $\text{dom}(l)$, and the identity relation on $W \setminus \text{dom}(l)$.
4. If $y \leq x$ then $l(y) \leq_{\text{pre}} l(x)$ (whenever $x, y \in \text{dom}(l)$).

There are four different types of defects that can occur. In addition to the two defects described in Definitions 2.8 and 2.9 of [17], there are now two additional defects involving the new $\langle \text{inf} \rangle$ operator. To be complete, we define all possible defects that can occur:

Definition 3.3.2 ($\langle \text{sup} \rangle$ -defect). Let $(l, \leq) \in \mathbb{P}$. Then a pair $(\langle \text{sup} \rangle \varphi \psi, x)$ is a $\langle \text{sup} \rangle$ -defect (of $(l, \leq)$) iff

- (i) $x \in \text{dom}(l)$
- (ii) $\langle \text{sup} \rangle \varphi \psi \in l(x)$, and
- (iii) there are no $y, z \in \text{dom}(l)$ such that:

$$\begin{aligned} \varphi \in l(y), \quad C_{\text{sup}} l(x) l(y) l(z), \quad \uparrow y = \uparrow x \cup \{y\} \cup (\uparrow y \cap \{w \mid \uparrow w \cap \uparrow x = \emptyset\}), \\ \psi \in l(z), \quad x = \text{sup}\{y, z\}, \quad \uparrow z = \uparrow x \cup \{z\} \cup (\uparrow z \cap \{w \mid \uparrow w \cap \uparrow x = \emptyset\}), \end{aligned}$$

where $\uparrow w := \{v \mid w \leq v\}$.

Definition 3.3.3 ($\langle \text{inf} \rangle$ -defect). Let $(l, \leq) \in \mathbb{P}$. Then a pair $(\langle \text{inf} \rangle \varphi \psi, x)$ denotes an $\langle \text{inf} \rangle$ -defect (of $(l, \leq)$) iff

- (i) $x \in \text{dom}(l)$
- (ii) $\langle \text{inf} \rangle \varphi \psi \in l(x)$, and
- (iii) there are no $y, z \in \text{dom}(l)$ such that:

$$\begin{aligned} \varphi \in l(y), \quad C_{\text{inf}} l(x) l(y) l(z), \quad \downarrow y = \downarrow x \cup \{y\} \cup (\downarrow y \cap \{w \mid \downarrow w \cap \downarrow x = \emptyset\}), \\ \psi \in l(z), \quad x = \text{inf}\{y, z\}, \quad \downarrow z = \downarrow x \cup \{z\} \cup (\downarrow z \cap \{w \mid \downarrow w \cap \downarrow x = \emptyset\}), \end{aligned}$$

where $\downarrow w := \{v \mid v \leq w\}$.

Definition 3.3.4 ($\neg \langle \text{sup} \rangle$ -defect). Let $(l, \leq) \in \mathbb{P}$. Then a quadruple $(\neg \langle \text{sup} \rangle \varphi \psi, x, y, z)$ denotes a $\neg \langle \text{sup} \rangle$ -defect (of $(l, \leq)$) iff:

$$\begin{aligned} x \in \text{dom}(l), \quad x = \text{sup}\{y, z\}, \quad \neg \langle \text{sup} \rangle \varphi \psi \in l(x), \\ \varphi \in l(y), \quad \psi \in l(z). \end{aligned}$$

Definition 3.3.5 ($\neg \langle \text{inf} \rangle$ -defect). Let $(l, \leq) \in \mathbb{P}$. Then a quadruple $(\neg \langle \text{inf} \rangle \varphi \psi, x, y, z)$ denotes a $\neg \langle \text{inf} \rangle$ -defect (of $(l, \leq)$) iff:

$$\begin{aligned} x \in \text{dom}(l), \quad x = \text{inf}\{y, z\}, \quad \neg \langle \text{inf} \rangle \varphi \psi \in l(x), \\ \varphi \in l(y), \quad \psi \in l(z). \end{aligned}$$

[17] faces the same defects, but constrained to $(\neg) \langle \text{sup} \rangle$ -defects. The question is whether a similar approach to resolving the defects in this setting would also work. It is not difficult to see that in the case of the $\langle \text{sup} \rangle$ - and $\langle \text{inf} \rangle$ -defects, the same solution as presented in Lemma 2.11 of [17] can be applied. For the sake of completeness, we discuss how an $\langle \text{inf} \rangle$ -defect is repaired as an illustrative case.

Assume $(\langle \text{inf} \rangle \varphi \psi, x)$ denotes an $\langle \text{inf} \rangle$ -defect of $(l, \leq) \in \mathbb{P}$. Then $\langle \text{inf} \rangle \varphi \psi \in l(x)$, but no $y, z \in \text{dom}(l)$ satisfy the conditions in clause (iii) of definition 3.3.3, i.e. there are no y, z

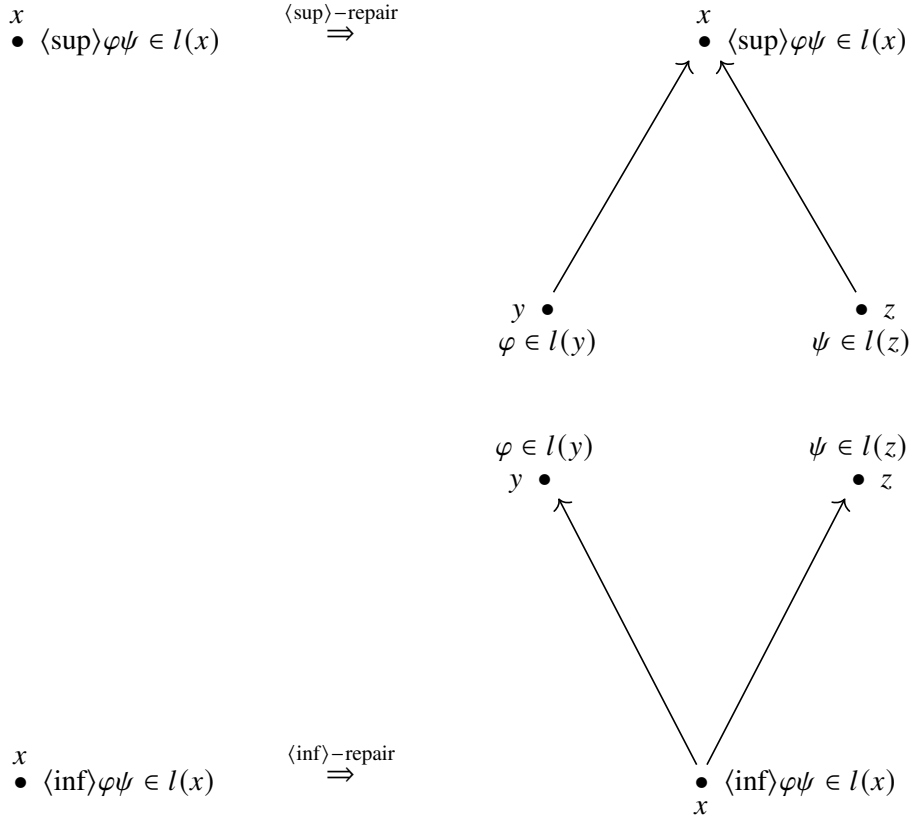
such that $x = \inf\{y, z\}$, $\varphi \in l(y)$ and $\psi \in l(z)$. Since $l(x)$ is a MCS of the canonical model, the existence lemma guarantees that there are MCSs Δ and Θ such that $C_{\inf} l(x) \Delta \Theta$, $\varphi \in \Delta$ and $\psi \in \Theta$. Take fresh, distinct $y, z \in W$, place them directly above x and extend l to l' by adding $l'(y) = \Delta$, $l'(z) = \Theta$. We will show in Lemma 3.3.1 that the resulting pair $(l', \leq')$ is still in $\mathbb{P}$ and that the defect is indeed resolved.

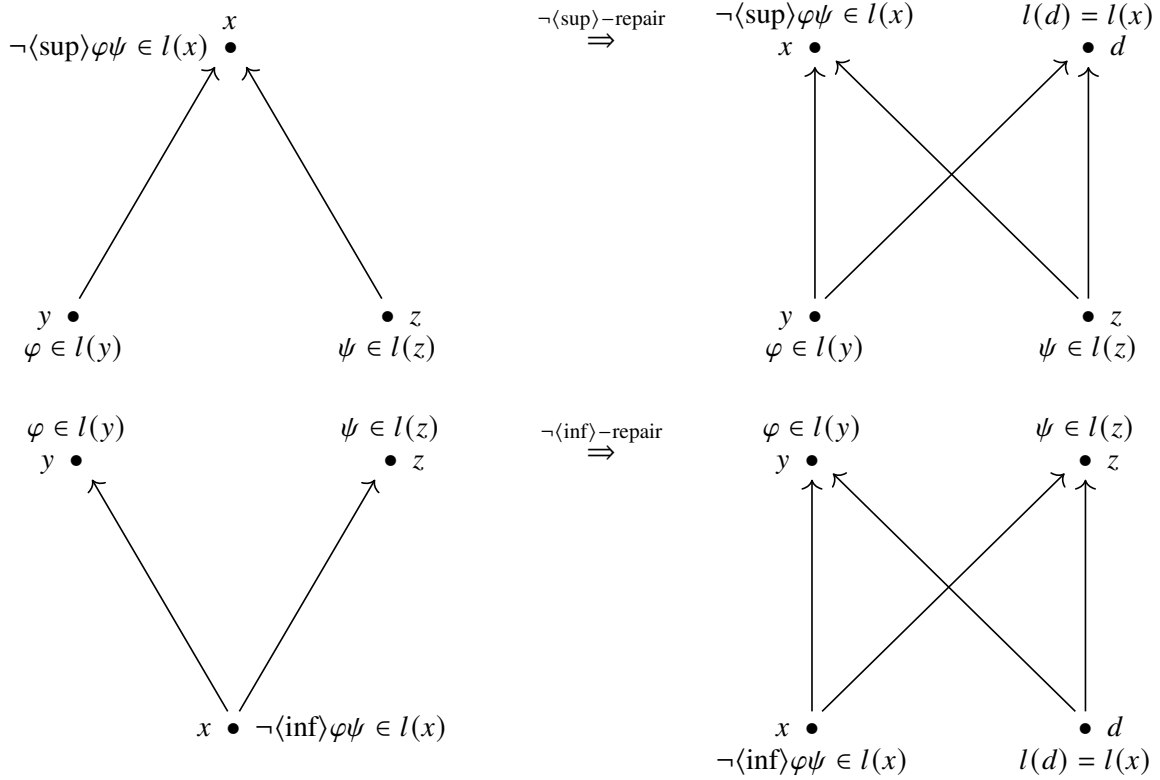
Lemma 2.12 of [17] repairs $\neg\langle\text{sup}\rangle$ -defects by adding dummy states without labels. This tactic fails here: since the $\langle\text{sup}\rangle$ and $\langle\text{inf}\rangle$ modalities act in opposite directions, every world in the model becomes reachable by looking ‘up’ and ‘down’ with the $\langle\text{inf}\rangle$ and $\langle\text{sup}\rangle$ operators. It thus matters with which MCS the new points get labeled. To resolve this, we duplicate the label of the point that constitutes the defect we are resolving.

To sketch how the updated $\neg\langle\text{inf}\rangle$ - and $\neg\langle\text{sup}\rangle$ -repair lemmas work, we briefly discuss the former.

Assume $(\neg\langle\text{inf}\rangle\varphi\psi, x, y, z)$ constitutes a $\neg\langle\text{inf}\rangle$ -defect. Then $\neg\langle\text{inf}\rangle\varphi\psi \in l(x)$ (so the label of x dictates that it does not satisfy $\langle\text{inf}\rangle\varphi\psi$), but at the same time there are y, z such that $\varphi \in l(y)$, $\psi \in l(z)$ and $x = \inf\{y, z\}$ contradicting this. To resolve this, we modify the model such that x is no longer the infimum of y and z by adding a duplicate of x that is incomparable with x itself. That the resulting tuple is still in $\mathbb{P}$ is proven formally in lemma 3.3.2.

We provide illustrations of the four repair lemmas to give an intuition:





We will only work out the $\langle \text{inf} \rangle$ - and $\neg \langle \text{inf} \rangle$ -repair lemmas as explanatory cases. The $\langle \text{sup} \rangle$ - and $\neg \langle \text{sup} \rangle$ -repair lemmas are in fact mirrored versions of these ones. For a worked out version of the $\langle \text{sup} \rangle$ -repair lemma we refer to Lemma 2.11 of [17].

Lemma 3.3.1 ($\langle \text{inf} \rangle$ -repair). Let $(\langle \text{inf} \rangle \varphi \psi, x)$ be an $\langle \text{inf} \rangle$ -defect of $(l, \leq)$. Then we can resolve this defect by extending $(l, \leq)$ to $(l', \leq')$ in the following way:

Take distinct $y, z \in W \setminus \text{dom}(l)$ and let:

$$\begin{aligned} l' &:= l \cup \{(y, \Gamma), (z, \Delta)\}, & \leq' &:= \leq \cup \{(u, y), (u, z) \mid u \leq x\}, \\ \varphi &\in \Gamma, \psi \in \Delta, & C_{\text{inf}} l(x) &\Gamma \Delta. \end{aligned}$$

Then y, z witness that $(\langle \text{inf} \rangle \varphi \psi, x)$ does not form an $\langle \text{inf} \rangle$ -defect anymore.

Proof. Take fresh, distinct $y, z \in W$ and map them to Γ and Δ respectively, which we know exist due to the existence lemma:

If $\langle \text{inf} \rangle \varphi \psi \in \Theta$, then there exist Γ, Δ such that $\varphi \in \Gamma, \psi \in \Delta$ and $C_{\text{inf}} \Theta \Gamma \Delta$.

We must show that $(l', \leq') \in \mathbb{P}$ and that the defect is resolved. That the defect is resolved follows clearly from how we define $(l', \leq')$. Also conditions 1.-3. clearly hold for $(l', \leq')$. The only condition we check is 4:

- The inequalities we added are of the form $u \leq y$ and $u \leq z$ for $u \leq x$, so we can focus on the subset $\{(u, y), (u, z) \mid u \leq x\} \subseteq \leq'$ and the two inequalities $y \leq' y$ and $z \leq' z$.
($y \leq' y$) Since $\leq_{\text{pre}}$ is a preorder, it is in particular reflexive. It follows that $l'(y) \leq_{\text{pre}} l'(y)$.

- ($x \leq' y$) We know that $C_{\text{inf}} l'(x) l'(y) l'(z)$. We have proven in Lemma 3.2.1 that $C_{\text{inf}} \Gamma \Delta \Theta$ only if $\Gamma \leq_{\text{pre}} \Delta, \Gamma \leq_{\text{pre}} \Theta$, so it follows that $l'(x) \leq_{\text{pre}} l'(y)$.
- ($u \leq' y$) The only cases left are elements $u < x$. Since $\leq_{\text{pre}}$ is a preorder, it is in particular transitive. So $l'(u) \leq_{\text{pre}} l'(x)$ combined with the previous result (namely $l'(x) \leq_{\text{pre}} l'(y)$) yields $l'(u) \leq_{\text{pre}} l'(y)$.

The same reasoning can be repeated for z . $\square$

Lemma 3.3.2 ($\neg\langle\text{inf}\rangle$ repair lemma). Let $(\neg\langle\text{inf}\rangle\varphi\psi, x, y, z)$ be a $\neg\langle\text{inf}\rangle$ -defect of $(l, \leq) \in \mathbb{P}$. Then we can resolve this defect by extending $(l, \leq)$ to $(l', \leq') \in \mathbb{P}$ in the following way:

Take $d \in W \setminus \text{dom}(l)$ and let:

$$l' := l \cup \{(d, l(x))\}, \quad \leq' := \leq \cup \{(d, u), (d, v) \mid y \leq u, z \leq v\}.$$

So we get $x \neq \inf\{y, z\}$.

Proof. Take a fresh $d \in W$ and map it to $l(x)$. We must show that $(l', \leq') \in \mathbb{P}$ and that the defect is resolved.

- It is not difficult to show that $(l', \leq') \in \mathbb{P}$, the only step we highlight is showing that $y \leq' x$ implies $l(y) \leq_{\text{pre}} l(x)$ since this is where we see that the new way of labeling (different from what is done in [17]) plays a role. We again only check the subset $\{(d, u), (d, v) : y \leq u, z \leq v\} \subseteq \leq'$ and the inequality $d \leq' d$. We go through all the cases:

($d \leq' d$) $l(d) \leq_{\text{pre}} l(d)$ follows from reflexivity of $\leq_{\text{pre}}$.

($d \leq' y$) Since $d \leq' y$, we should show that $l(d) = l(x) \leq_{\text{pre}} l(y)$. We know that $(l, \leq) \in \mathbb{P}$. So since we assumed $x = \inf_{\leq}\{y, z\}$, we have $x \leq y$, so $l(d) = l(x) \leq_{\text{pre}} l(y)$. We can replace y by z and repeat the same reasoning.

($d \leq' u$) If $y < u$ or $z < u$ we get $l(d) \leq_{\text{pre}} l(u)$ by transitivity of $\leq_{\text{pre}}$.

- We show that the $\neg\langle\text{inf}\rangle$ defect is resolved by showing that $d \not\leq' x$ (while d is a lower bound of $\{y, z\}$), which contradicts that $x = \inf_{\leq}\{y, z\}$. Assume $d \leq' x$, then by definition of $\leq'$ this could only be the case if $x = y, x = z, x > y$ or $x > z$. The latter two cases are impossible by antisymmetry of the ordering ($\leq$ is a partial order, not merely a preorder). For the former two cases, assume without loss of generality that $x = y$. Then since $x \leq z$, we get $l(x) \leq_{\text{pre}} l(z)$. So (1) $C_{\text{sup}} l(z) l(z) l(x)$ and (2) $C_{\text{inf}} l(x) l(x) l(z)$. By (2) we get $C_{\text{inf}} l(x) l(y) l(z)$ since $l(y) = l(x)$, but then we cannot have had a $\neg\langle\text{inf}\rangle$ defect, so we derived a contradiction. $\square$

3.4 COMPLETENESS PROOF

Theorem 3.4.1 (Completeness). **TIL** is strongly complete with respect to **TIL**.

We want to show that if Γ^* is a **TIL**-consistent set, then we can find a model defined on a poset-frame such that there is $x \in \mathfrak{M}$ for which $\mathfrak{M}, x \models \gamma$ for every $\gamma \in \Gamma^*$. The labeled models defined by $(l, \leq)$ will approximate such a model better at each step. We now explain how this approximation works.

First note that we can extend Γ^* to a **TIL**-MCS Γ . Let W be an arbitrary countable set and let $x \in W$ be arbitrary. Define $l_0 := \{(x, \Gamma)\}$ and let $\leq_0$ be the identity relation on W . Then $(l_0, \leq_0)$ satisfies all the conditions of Definition 3.3.1.

It is possible to enumerate all the potential $\langle \text{sup} \rangle$ -, $\langle \text{inf} \rangle$ -, $\neg \langle \text{sup} \rangle$ - and $\neg \langle \text{inf} \rangle$ -defects since the defects are defined as finite tuples, $\mathcal{L}_T$ is a countable language and W is countable. We construct a sequence

$$(l_0, \leq_0), \dots, (l_n, \leq_n), \dots$$

with $l_n \subseteq l_{n+1}$ and $\leq_n \subseteq \leq_{n+1}$, by constructing from $(l_n, \leq_n)$ the next element in the sequence $(l_{n+1}, \leq_{n+1})$ by taking the least tuple in our enumeration constituting a defect for $(l_n, \leq_n)$, and applying the corresponding repair lemma to it.

Let

$$(l_\omega, \leq_\omega) := \left(\bigcup_{n \in \mathbb{N}} l_n, \bigcup_{n \in \mathbb{N}} \leq_n \right),$$

and let $V(p) := \{x \in \text{dom}(l_\omega) \mid p \in l_\omega(x)\}$. To prove completeness, we want to prove the following:

Lemma 3.4.2 (Truth Lemma for labeled points).

$$\forall x \in \text{dom}(l_\omega), \forall \varphi \in \mathcal{L}_T : (W, \leq_\omega, V), x \Vdash \varphi \Leftrightarrow \varphi \in l_\omega(x)$$

Proof. We prove the lemma by induction on the complexity of φ . The base case and induction steps for $\neg, \wedge$ are routine arguments. We therefore focus on the cases of $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$:

$\Rightarrow$: Assume $(W, \leq_\omega, V), x \Vdash \langle \text{sup} \rangle \varphi \psi$. By definition, there exist $y, z \in W$ such that

$$(W, \leq_\omega, V), y \Vdash \varphi, \quad (W, \leq_\omega, V), z \Vdash \psi \quad \text{and} \quad x = \sup_{\omega} \{y, z\}.$$

We know that $y, z \in \text{dom}(l_\omega)$, since if not, then as $\leq_\omega$ is the identity relation on $W \setminus \text{dom}(l_\omega)$, either $x = y$ and / or $x = z$, contradicting $x \in \text{dom}(l_\omega)$. By the induction hypothesis, it follows that $\varphi \in l_\omega(y)$ and $\psi \in l_\omega(z)$. To conclude that $\langle \text{sup} \rangle \varphi \psi \in l_\omega(x)$, it would suffice that $(l_\omega, \leq_\omega)$ does not contain any $\neg \langle \text{sup} \rangle$ -defects, which we will address later.

$\Leftarrow$: Assume now that $\langle \text{sup} \rangle \varphi \psi \in l_\omega(x)$. Then, again provided that $(l_\omega, \leq_\omega)$ does not contain any $\langle \text{sup} \rangle$ -defects, there exist $y, z \in \text{dom}(l_\omega)$ such that

$$x = \sup_{\omega} \{y, z\}, \quad \varphi \in l_\omega(y) \quad \text{and} \quad \psi \in l_\omega(z).$$

By the induction hypothesis, this implies that $(W, \leq_\omega, V), y \Vdash \varphi$ and $(W, \leq_\omega, V), z \Vdash \psi$, so by definition $(W, \leq_\omega, V), x \Vdash \langle \text{sup} \rangle \varphi \psi$.

The case of $\langle \text{inf} \rangle$ is proven similarly. □

As pointed out in the proof, once we show that $(l_\omega, \leq_\omega)$ does not contain any defects, the truth lemma holds, and completeness follows. The argument to show this follows the proof of Theorem 2.13 from [17], where necessary adjustments have to be made to handle the $\langle \text{inf} \rangle$ -operator. For the sake of completeness, we include most parts of the proof here.

Lemma 3.4.3. $(l_\omega, \leq_\omega)$ does not have any $\langle \text{sup} \rangle$, $\neg \langle \text{sup} \rangle$, $\langle \text{inf} \rangle$ or $\neg \langle \text{inf} \rangle$ defect.

To prove this statement we first need some auxiliary lemmas.

Lemma 3.4.4. Let $n \in \omega$. If

$$\uparrow_n^2 y = \uparrow_n x \cup \{y\} \cup (\uparrow_n y \cap \{w \mid \uparrow_n w \cap \uparrow_n x = \emptyset\}),$$

then the same equality holds for every $m \geq n$, hence in particular

$$\uparrow_\omega y = \uparrow_\omega x \cup \{y\} \cup (\uparrow_\omega y \cap \{w \mid \uparrow_\omega w \cap \uparrow_\omega x = \emptyset\}).$$

Proof. We argue by induction on $m \geq n$.

Base case: Holds by assumption.

Induction step: Assume the equality is true at stage m ; we show it remains true at stage $m + 1$, i.e. we show

$$\uparrow_{m+1} y = \uparrow_{m+1} x \cup \{y\} \cup (\uparrow_{m+1} y \cap \{w \mid \uparrow_{m+1} w \cap \uparrow_{m+1} x = \emptyset\}).$$

We consider each possible repair in turn and verify that, if that repair is carried out at stage m , the equality is still true afterwards.

$\langle \text{sup} \rangle$ -repair: The fresh points are only below existing points, so the equality still holds.

$\langle \text{inf} \rangle$ -repair: Let $(\langle \text{inf} \rangle \varphi \psi, c)$ be the defect that is resolved at stage m and let a, b be the two fresh points. We discuss the three possible positions of c relative to x and y .

(i) Suppose $c \geq_m y$ but $c \not\leq_m x$. Then since

$$\leq_{m+1} = \leq_m \cup \{(u, a), (u, b) \mid u \leq c\},$$

$a, b \in \uparrow_{m+1} y$, $a, b \notin \uparrow_{m+1} x$ and, because $a, b \not\leq_m x$, we also have

$$a, b \in (\uparrow_{m+1} y \cap \{w \mid \uparrow_{m+1} w \cap \uparrow_{m+1} x = \emptyset\}).$$

(ii) Suppose $c \geq_m x$. Then $a, b \in \uparrow_{m+1} x \subseteq \uparrow_{m+1} y$ so the equality holds.

(iii) Suppose $c \not\leq_m y$. Then $a, b \notin \uparrow_{m+1} y$ and

$$a, b \notin \uparrow_{m+1} x \cup \{y\} \cup (\uparrow_{m+1} y \cap \{w \mid \uparrow_{m+1} w \cap \uparrow_{m+1} x = \emptyset\}).$$

$\neg \langle \text{sup} \rangle$ -repair: Let $(\neg \langle \text{sup} \rangle \psi \psi', a, b, c)$ be the defect that is repaired, introducing one fresh point d . Since $a \geq_m b, c$ and b and c are incomparable (for assume they were comparable, then $l(c) \geq_{\text{pre}} l(b)$ or $l(b) \geq_{\text{pre}} l(c)$ would imply that we cannot have had a $\neg \langle \text{sup} \rangle$ defect), the only new inequalities are of the form $u \leq_{m+1} d$ for $u \leq_m b$ and $u \leq_m c$. We go through all the possible cases:

- If $a \leq_m y$ or y is unrelated to a, b, c , then $d \notin \uparrow_{m+1} y$ and $d \notin \uparrow_{m+1} x$, so the equality still holds at stage $m + 1$.
- If $y = b$, $x = a$, then $d \in \uparrow_{m+1} y$ and $d \notin \uparrow_{m+1} x$. So since $\uparrow_{m+1} d = d$, it follows that $d \in \uparrow_{m+1} y \cap \{w \mid \uparrow_{m+1} w \cap \uparrow_{m+1} x = \emptyset\}$ (case where $x = a$ and $y = c$ is symmetric).

² $\uparrow_n y$ is defined as $\{x : y \leq_n x\}$

- If b and $/$ or c are greater than or equal to x under $\leq_m$, it follows that $d \in \uparrow_{m+1} x \subseteq \uparrow_{m+1} y$, so the equality still holds.
- If b and $/$ or c are greater than or equal to y under $\leq_m$ and unrelated to x , then $d \in \uparrow_{m+1} y$ and $d \notin \uparrow_{m+1} x$, so again since $\uparrow_{m+1} d = d$ it follows that $d \in \uparrow_{m+1} y \cap \{w \mid \uparrow_{m+1} w \cap \uparrow_{m+1} x = \emptyset\}$ which implies that the equality still holds.
- If $x = a$ and b and c are unrelated to y then d is also unrelated to x and y , so the equality still holds.

We see that in all cases the equality still holds after the repair step.

$\neg(\text{inf})$ -repair: The single fresh point d is only below existing points, but not above any, so the equality still holds.

In every repair, the equality is preserved; therefore it holds at stage $m + 1$, completing the induction. $\square$

Lemma 3.4.5. Let $n \in \omega$. If

$$\downarrow_n y = \downarrow_n x \cup \{y\} \cup (\downarrow_n y \cap \{w \mid \downarrow_n w \cap \downarrow_n x = \emptyset\}),$$

then the same equality holds for every $m \geq n$, hence in particular

$$\downarrow_\omega y = \downarrow_\omega x \cup \{y\} \cup (\downarrow_\omega y \cap \{w \mid \downarrow_\omega w \cap \downarrow_\omega x = \emptyset\}).$$

Proof. Symmetric proof to the previous lemma. $\square$

We now prove auxiliary lemmas stating that once a defect is repaired, it remains repaired in every subsequent stage of the construction. We start with $\langle \text{sup} \rangle$ defects.

Lemma 3.4.6. Let $x \in \text{dom}(l_n)$ and $\langle \text{sup} \rangle \varphi \psi \in l_n(x)$. If $(\langle \text{sup} \rangle \varphi \psi, x)$ does not constitute a defect for $(l_n, \leq_n)$, then $(\langle \text{sup} \rangle \varphi \psi, x)$ does not constitute a defect for $(l_m, \leq_m)$ for any $m \geq n$, thus showing that it does not constitute a defect in the limit $(l_\omega, \leq_\omega)$.

Proof. Because $(\langle \text{sup} \rangle \varphi \psi, x)$ does not constitute a defect at stage n , there exist witnesses y, z with $l_n(y) = \Gamma$ and $l_n(z) = \Delta$, such that:

$$\begin{aligned} \varphi \in \Gamma, \quad C_{\text{sup}} l_n(x) \Gamma \Delta, \quad \uparrow_n y &= \uparrow_n x \cup \{y\} \cup (\uparrow_n y \cap \{w \mid \uparrow_n w \cap \uparrow_n x = \emptyset\}), \\ \psi \in \Delta, \quad x &= \sup_{\leq_n} \{y, z\}, \quad \uparrow_n z = \uparrow_n x \cup \{z\} \cup (\uparrow_n z \cap \{w \mid \uparrow_n w \cap \uparrow_n x = \emptyset\}). \end{aligned}$$

Now let $m \geq n$. We show that y, z also witness that $(\langle \text{sup} \rangle \varphi \psi, x)$ is not a defect for $(l_m, \leq_m)$.

The previous lemma ensures that

$$\uparrow_m y = \uparrow_m x \cup \{y\} \cup (\uparrow_m y \cap \{w \mid \uparrow_m w \cap \uparrow_m x = \emptyset\})$$

and

$$\uparrow_m z = \uparrow_m x \cup \{z\} \cup (\uparrow_m z \cap \{w \mid \uparrow_m w \cap \uparrow_m x = \emptyset\})$$

for every $m \geq n$.

Since l_m extends l_n , the only thing that remains to show is that x stays the supremum of $\{y, z\}$ throughout the construction; that is, for every $m \geq n$ we still have $x = \sup_m \{y, z\}$. We proceed by induction on m :

Base case: The claim holds for $m = n$ by assumption.

Induction step: Assume $x = \sup_m \{y, z\}$. We consider each possible repair that could be carried out at stage $m + 1$ and verify that it leaves the ordering of x, y, z unchanged, hence $x = \sup_{m+1} \{y, z\}$.

The cases of $\langle \text{sup} \rangle$ -repair and $\neg \langle \text{sup} \rangle$ -repair are covered in Theorem 2.13 of [17], we therefore only cover the two remaining repairs:

$\langle \text{inf} \rangle$ -repair: Assume $(l_{m+1}, \leq_{m+1})$ was obtained by repairing an $\langle \text{inf} \rangle$ -defect for some world s by adding fresh states y_s and z_s to the domain of l_m and extending the inequality relation as described in the repair lemma

$$\leq_{m+1} = \leq_m \cup \{(u, y_s), (u, z_s) : u \leq s\} \cup \{(z_s, z_s), (y_s, y_s)\}.$$

By induction hypothesis and the definition of $\leq_{m+1}$, the only candidates that could now lie above $\{y, z\}$ but not above x under $\leq_{m+1}$ are y_s and z_s . Suppose $y, z \leq_{m+1} y_s$. Then $y, z \leq_m s$, hence $x \leq_m s$ by the induction hypothesis and therefore $x \leq_{m+1} y_s$ by definition of $\leq_{m+1}$. The same reasoning applies to z_s , so x is still the least upper bound of $\{y, z\}$.

$\neg \langle \text{inf} \rangle$ -repair: Assume $(l_{m+1}, \leq_{m+1})$ was obtained by repairing a $\neg \langle \text{inf} \rangle$ -defect for worlds s, y_s, z_s by introducing a fresh state d_s such that $l_{m+1}(d_s) = l_{m+1}(s)$ and

$$\leq_{m+1} = \leq_m \cup \{(d_s, u), (d_s, v) : y_s \leq_m u, z_s \leq_m v\} \cup \{(d_s, d_s)\}.$$

Any new upper bound of $\{y, z\}$ would have to be d_s itself. But d_s is below any of the old worlds, not above them, so $d_s \not\leq_{m+1} y$ and $d_s \not\leq_{m+1} z$, so the supremum of $\{y, z\}$ is still the same as before.

Thus x is still the supremum of $\{y, z\}$ at stage $m + 1$, completing the induction.

For every $m \geq n$ the same worlds y and z witness that $(\langle \text{sup} \rangle \varphi \psi, x)$ does not constitute a defect for $(l_m, \leq_m)$. So in particular it does not constitute a defect for $(l_\omega, \leq_\omega)$. $\square$

Lemma 3.4.7. Let $x \in \text{dom}(l_n)$ and $\langle \text{inf} \rangle \varphi \psi \in l_n(x)$. If $(\langle \text{inf} \rangle \varphi \psi, x)$ does not constitute a defect for $(l_n, \leq_n)$, then $(\langle \text{inf} \rangle \varphi \psi, x)$ does not constitute a defect for $(l_m, \leq_m)$ for any $m \geq n$, thus showing that it does not constitute a defect in the limit $(l_\omega, \leq_\omega)$.

Proof. The proof of this lemma is symmetric to the proof of the previous lemma. $\square$

It remains to prove that any $\neg \langle \text{sup} \rangle$ - or $\neg \langle \text{inf} \rangle$ -defect that is repaired, remains repaired in subsequent stages of the construction. For that we need an auxiliary lemma.

Lemma 3.4.8. Let $n \in \omega$ and $a, b \in \text{dom}(l_n)$. If $b \not\leq_n a$ then for every $m \geq n : b \not\leq_m a$; hence showing $b \not\leq_\omega a$.

Proof. We show by induction that for every $m \geq n$ we also have $b \not\leq_m a$.

Base case: The claim holds by assumption.

Induction case: Assume $b \not\leq_m a$. If $(l_{m+1}, \leq_{m+1})$ was obtained from $(l_m, \leq_m)$ by carrying out a $\langle \text{sup} \rangle$ - or $\neg \langle \text{sup} \rangle$ -repair, the claim is proven in Theorem 2.13 from [17], we therefore only cover the two remaining cases:

$\langle \text{inf} \rangle$ -repair: Assume $(l_{m+1}, \leq_{m+1})$ is obtained from $(l_m, \leq_m)$ by repairing an $\langle \text{inf} \rangle$ -defect at a world x , introducing two fresh worlds y, z . This gives

$$\leq_{m+1} = \leq_m \cup \{(u, y), (u, z) \mid u \leq_m x\}.$$

It follows that every $u \leq_m x$ now has the two additional successors y, z under the updated ordering $\leq_{m+1}$. Since $b \not\leq_m a$, the relation $b \leq_{m+1} a$ could only hold if $a = y$ or $a = z$; but that is impossible, because y and z are fresh.

$\neg \langle \text{inf} \rangle$ -repair: If $(l_{m+1}, \leq_{m+1})$ is obtained from $(l_m, \leq_m)$ by repairing a $\neg \langle \text{inf} \rangle$ -defect for x, y, z , one fresh state d is added and

$$\leq_{m+1} = \leq_m \cup \{(d, u), (d, v) \mid y \leq_m u, z \leq_m v\}$$

Thus every element that was above y or z at stage m , has a new predecessor d at stage $m + 1$. No other inequalities are introduced. Because $b \not\leq_m a$, the only way to get $b \leq_{m+1} a$ would be to have $b = d$; but that is impossible because d is fresh. Hence $b \not\leq_{m+1} a$ as required.

Hence $b \not\leq_m a$ for every $m \geq n$. In particular, if there exists an $n \in \omega$ with $b \not\leq_n a$, then $b \not\leq_\omega a$. $\square$

With this lemma at hand, we can prove the two last auxiliary lemmas:

Lemma 3.4.9. If $(\neg \langle \text{sup} \rangle \varphi \psi, x, y, z)$ does not constitute a defect for $(l_n, \leq_n)$ for which $x, y, z \in \text{dom}(l_n)$ and $\neg \langle \text{sup} \rangle \varphi \psi \in l_n(x)$, $\varphi \in l_n(y)$, $\psi \in l_n(z)$, then $(\neg \langle \text{sup} \rangle \varphi \psi, x, y, z)$ does not constitute a defect for $(l_m, \leq_m)$ for any $m \geq n$, thus not for $(l_\omega, \leq_\omega)$.

Proof. $x \neq \text{sup}_n \{y, z\}$ holds if either x is not an upper bound of $\{y, z\}$ (i.e. $y \not\leq_n x$ and / or $z \not\leq_n x$), or it is an upper bound but there is $u \in W$ such that $u \geq_n y, z$ and $x \not\leq_n u$. In case of the former, it follows from Lemma 3.4.8 that $y \not\leq_m x$ and / or $z \not\leq_m x$. In case of the latter; $\leq_n \subseteq \leq_m$ implies $u \geq_m y, z$ and Lemma 3.4.8 yields $x \not\leq_m u$. $\square$

Lemma 3.4.10. If $(\neg \langle \text{inf} \rangle \varphi \psi, x, y, z)$ does not constitute a defect for $(l_n, \leq_n)$ for which $x, y, z \in \text{dom}(l_n)$ and $\neg \langle \text{inf} \rangle \varphi \psi \in l_n(x)$, $\varphi \in l_n(y)$, $\psi \in l_n(z)$, then $(\neg \langle \text{inf} \rangle \varphi \psi, x, y, z)$ does not constitute a defect for $(l_m, \leq_m)$ for any $m \geq n$, thus not for $(l_\omega, \leq_\omega)$.

Proof. This lemma is proven symmetrically to the previous lemma. $\square$

Using what we proved so far, we can derive the following lemma:

Lemma 3.4.11. If a tuple constitutes a defect at some stage n , but not at a later stage $m > n$, then it never constitutes a defect at any stage $k \geq m$.

Proof. Every tuple occurs in our enumeration of potential defects, so if a tuple is a defect at stage n , it is repaired no later than at stage $n + i + 1$, where i is its position in the list. Once the repair has been completed, 3.4.6, 3.4.7, 3.4.9 and 3.4.10 guarantee that the defect is still repaired at all subsequent stages. $\square$

We now prove that $(l_\omega, \leq_\omega)$ does not contain any defects, thereby closing the final gap in the proof of the truth lemma for labeled points 3.4.2.

Proof of Lemma 3.4.3. In [17, Theorem 2.13], the $\langle \text{sup} \rangle$ and $\neg \langle \text{sup} \rangle$ cases are treated in detail. The arguments for the $\langle \text{inf} \rangle$ and $\neg \langle \text{inf} \rangle$ cases are analogous, but for the sake of completeness we include them here:

$\langle \text{inf} \rangle$ -defect: Assume $(\langle \text{inf} \rangle \varphi \psi, x)$ is the i -th in the enumeration of potential defects, with $x \in \text{dom}(l_\omega)$ and $\langle \text{inf} \rangle \varphi \psi \in l_\omega(x)$. Then there exists $n \in \omega$ such that $x \in \text{dom}(l_n)$ and $\langle \text{inf} \rangle \varphi \psi \in l_n(x)$. By definition of the construction process we have that for any $m \geq n$ it is the case that $\text{dom}(l_n) \subseteq \text{dom}(l_m)$, so $x \in \text{dom}(l_m)$ and $\langle \text{inf} \rangle \varphi \psi \in l_m(x)$ since the definition of l_m extends l_n . There are two possibilities:

- If $(\langle \text{inf} \rangle \varphi \psi, x)$ does not constitute a defect for $(l_n, \leq_n)$, then by Lemma 3.4.7, it does not constitute a defect for $(l_m, \leq_m)$ for any $m \geq n$ and thus not for $(l_\omega, \leq_\omega)$.
- If $(\langle \text{inf} \rangle \varphi \psi, x)$ does constitute a defect for $(l_n, \leq_n)$ the construction process repairs the i -th tuple no later than at stage $n + i + 1$. Lemma 3.4.7 then guarantees that the repair is permanent, so the tuple is not a defect in any $(l_m, \leq_m)$ for $m \geq n + i + 1$, and in particular not in the limit $(l_\omega, \leq_\omega)$.

$\neg \langle \text{inf} \rangle$ -defect: Assume for contradiction that $(\neg \langle \text{inf} \rangle \varphi \psi, x, y, z)$ constitutes a defect for $(l_\omega, \leq_\omega)$ and that it is the i -th defect in our enumeration. This means that

$$\neg \langle \text{inf} \rangle \varphi \psi \in l_\omega(x), \varphi \in l_\omega(y), \psi \in l_\omega(z) \text{ and } x = \inf_\omega \{y, z\}.$$

It follows that there is $n \in \omega$ such that

$$\neg \langle \text{inf} \rangle \varphi \psi \in l_n(x), \varphi \in l_n(y), \psi \in l_n(z) \text{ and } x = \inf_n \{y, z\}.$$

This means that $(\neg \langle \text{inf} \rangle \varphi \psi, x, y, z)$ constitutes a defect for $(l_n, \leq_n)$. Since $(\neg \langle \text{inf} \rangle \varphi \psi, x, y, z)$ is the i -th defect in our enumeration, the defect is repaired no later than at stage $n + i + 1$. From Lemma 3.3.2 it follows that $x \neq \inf_{n+i+1} \{y, z\}$. From $x = \inf_n \{y, z\}$ it follows that $x \leq_n y, z$, so since $\leq_n \subseteq \leq_{n+i+1}$ we get that $x \leq_{n+i+1} y, z$. Since x is not the infimum at stage $n + i + 1$, there exists $a \in \text{dom}(l_{n+i+1})$ with $a \leq_{n+i+1} y, z$ and $a \not\leq_{n+i+1} x$. Lemma 3.4.8 guarantees that this inequality still holds at the limit stage: $a \not\leq_\omega x$. Because $\leq_{n+i+1} \subseteq \leq_\omega$, we also have $a \leq_\omega y, z$, contradicting $x = \inf_\omega \{y, z\}$. Hence $(\neg \langle \text{inf} \rangle \varphi \psi, x, y, z)$ cannot constitute a defect for $(l_\omega, \leq_\omega)$. $\square$

Thus, the completeness of **TIL** is now fully established.

3.5 SOUNDNESS AND COMPLETENESS OF TIL ON PREORDERS

Having established completeness of TIL on poset frames, a direct corollary is completeness with respect to another class of frames, namely preorders. $\mathfrak{M} = (W, \leq, V)$ is a preorder

model if $\leq$ is a preorder on W . In this case, suprema and infima need not be unique, but can come in clusters.

Definition 3.5.1 (Supremum and infimum in preorders). Let $\mathfrak{F} := \langle W, \leq \rangle$ be a preorder and $y, z \in W$.

- An element $x \in W$ is a *quasi-supremum* of $\{y, z\}$ (write $x \in \sup\{y, z\}$) iff x is an upper bound of $\{y, z\}$ and $x \leq w$ for all upper bounds w of $\{y, z\}$.
- An element $x \in W$ is a *quasi-infimum* of $\{y, z\}$ (write $x \in \inf\{y, z\}$) iff x is a lower bound of $\{y, z\}$ and $x \leq w$ for all lower bounds w of $\{y, z\}$.

The semantics of the modal operators are then defined as:

Definition 3.5.2.

$\mathfrak{M}, x \Vdash \langle \sup \rangle \varphi \psi$ iff there exist $y, z \in W$ s.t. $\mathfrak{M}, y \Vdash \varphi$, $\mathfrak{M}, z \Vdash \psi$, and $x \in \sup\{y, z\}$,

$\mathfrak{M}, x \Vdash \langle \inf \rangle \varphi \psi$ iff there exist $y, z \in W$ s.t. $\mathfrak{M}, y \Vdash \varphi$, $\mathfrak{M}, z \Vdash \psi$, and $x \in \inf\{y, z\}$.

With these new semantics at hand, we can define TIL on preorders.

Definition 3.5.3.

$$TIL_{\text{pre}} := \{\varphi \in \mathcal{L}_T \mid \text{for every preorder model } \mathfrak{M} = (W, \leq, V) \text{ and every } x \in W : \\ \mathfrak{M}, x \Vdash \varphi\}$$

It is not difficult to show that the axiomatization we gave is also sound and complete with respect to TIL_{pre} , as the following theorem shows:

Theorem 3.5.1.

$$TIL_{\text{pre}} = TIL = \mathbf{TIL}$$

Proof. $TIL_{\text{pre}} \subseteq TIL$ follows from the fact that every poset frame is also a preorder frame. Since the soundness proof of $\mathbf{TIL}$ on poset frames carries over to preorders without significant changes, we also derive that $\mathbf{TIL} \subseteq TIL_{\text{pre}}$, which concludes the proof. $\square$

This result provides yet another example of how modal information logics cannot differentiate preorders from posets (see also [16, 18]).

4 DECIDABILITY OF TIL

Following the outline of [17, Section 3], we prove decidability of *TIL*. Since the logic does not have the finite model property (FMP) with respect to neither preorders nor posets (see section 2.2 and [16, Section 1.2]), we show that it does have the FMP with respect to a generalized class of frames. Again, we highlight the adjustments that have to be made to the proofs in [17], to make them work in the new setting with two modal operators: $\langle \text{inf} \rangle$ and $\langle \text{sup} \rangle$.

In Section 4.1 we introduce the generalized frame class on which **TIL** will be interpreted and derive first-order correspondents for all its axioms. Section 4.2 proves that **TIL** has the FMP with respect to this class and works out the filtration argument that yields decidability.

4.1 GENERALIZED FRAMES AND FIRST-ORDER CORRESPONDENTS

As we have already mentioned, the canonical ternary relations C_{sup} and C_{inf} do not necessarily match the order-theoretic supremum and infimum determined by the preorder they induce. We investigate whether $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$ can be reinterpreted via frame-correspondence techniques on the class of all tuples

$$(W, C_{\text{sup}}, C_{\text{inf}}),$$

where W is a set and C_{sup} and C_{inf} are arbitrary ternary relations on W .

Since the axioms of **TIL** are Sahlqvist, each has a first-order equivalent on these tuples $(W, C_{\text{sup}}, C_{\text{inf}})$ [21]. Concretely, for every axiom Ax in our axiomatization of **TIL**, we denote by $\text{FO}Ax$ its first-order correspondent, i.e. a first-order sentence in the signature $\{C_{\text{sup}}, C_{\text{inf}}\}$ such that

$$(W, C_{\text{sup}}, C_{\text{inf}}) \models Ax \iff (W, C_{\text{sup}}, C_{\text{inf}}) \models^3 \text{FO}Ax$$

Building on the suprema-only correspondences of Lemma 3.1 of [17], we add matching infimum clauses and introduce a new one for the (Sy.)-axiom.

Definition 4.1.1 (First-order frame conditions for TIL).

$$(\text{FORe}) := \forall x (C_{\text{sup}} x x x \wedge C_{\text{inf}} x x x)$$

$$(\text{FO4}') := \forall x, y, z, u, v \left([C_{\text{sup}} x y z \wedge C_{\text{sup}} y u v \rightarrow \exists w C_{\text{sup}} x u w] \right. \\ \left. \wedge [C_{\text{inf}} x y z \wedge C_{\text{inf}} y u v \rightarrow \exists w C_{\text{inf}} x u w] \right)$$

$$(\text{FOCo}) := \forall x, y, z \left([C_{\text{sup}} x y z \rightarrow C_{\text{sup}} x z y] \wedge [C_{\text{inf}} x y z \rightarrow C_{\text{inf}} x z y] \right)$$

$$(\text{FODk1}) := \forall x, y, z (C_{\text{sup}} x y z \rightarrow C_{\text{sup}} x x y)$$

$$(\text{FODk2}) := \forall x, y, z (C_{\text{inf}} x y z \rightarrow C_{\text{inf}} x x y)$$

3 In what follows, ' $\models$ ' is used for frame validity of FO-sentences.

$$(\text{FOSy}') := \forall x, y \left([\exists z C_{\text{sup}} x y z \rightarrow \exists u C_{\text{inf}} y x u] \wedge [\exists z C_{\text{inf}} x y z \rightarrow \exists u C_{\text{sup}} y x u] \right)$$

In addition to (FO4') and (FOSy'), we introduce first-order sentences (FO4) and (FOSy) that, while equivalent only modulo the other axioms, have a simpler form and are easier to apply in arguments.

$$(\text{FO4}) := \forall w, v, u \left([C_{\text{sup}} w w v \wedge C_{\text{sup}} v v u \rightarrow C_{\text{sup}} w w u] \right. \\ \left. \wedge [C_{\text{inf}} w w v \wedge C_{\text{inf}} v v u \rightarrow C_{\text{inf}} w w u] \right)$$

$$(\text{FOSy}) := \forall w, v \left([C_{\text{sup}} w w v \rightarrow C_{\text{inf}} v v w] \wedge [C_{\text{inf}} w w v \rightarrow C_{\text{sup}} v v w] \right)$$

Using standard frame correspondence proofs, we can show the following.

Theorem 4.1.1 (Sahlqvist correspondence for TIL).

$$(W, C_{\text{sup}}, C_{\text{inf}}) \models \mathbf{TIL} \iff (W, C_{\text{sup}}, C_{\text{inf}}) \models (\text{FORe}) \wedge (\text{FO4}) \wedge \\ (\text{FOCo}) \wedge (\text{FODk1}) \wedge (\text{FODk2}) \wedge (\text{FOSy}).$$

For a proof see [10, Theorem 3.54].

Lastly, we use the first-order correspondents of the axioms of **TIL** to define the class of tuples $(W, C_{\text{sup}}, C_{\text{inf}})$ that satisfy them. We call this class $\tilde{\mathcal{C}}$:

Definition 4.1.2.

$$\tilde{\mathcal{C}} := \{ (W, C_{\text{sup}}, C_{\text{inf}}) \mid (W, C_{\text{sup}}, C_{\text{inf}}) \models (\text{FORe}) \wedge (\text{FO4}) \wedge \\ (\text{FOCo}) \wedge (\text{FODk1}) \wedge (\text{FODk2}) \wedge (\text{FOSy}) \}$$

Any tuple that is an element of this class is called a $\tilde{\mathcal{C}}$ -frame.

4.2 FINITE MODEL PROPERTY AND DECIDABILITY

In order to prove that **TIL** is decidable, we first show that it has the finite model property. Specifically, we will show that if a formula φ is not derivable in **TIL**, then there exists a finite $\tilde{\mathcal{C}}$ -model that falsifies it. To do so, we need the following definitions:

Definition 4.2.1.

$$\tilde{\mathcal{C}}_F := \{ (W, C_{\text{sup}}, C_{\text{inf}}) \in \tilde{\mathcal{C}} \mid W \text{ finite} \}, \\ \text{Log}(\tilde{\mathcal{C}}) := \{ \varphi \in \mathcal{L}_T \mid (W, C_{\text{sup}}, C_{\text{inf}}) \models \varphi \text{ for every } (W, C_{\text{sup}}, C_{\text{inf}}) \in \tilde{\mathcal{C}} \}, \\ \text{Log}(\tilde{\mathcal{C}}_F) := \{ \varphi \in \mathcal{L}_T \mid (W, C_{\text{sup}}, C_{\text{inf}}) \models \varphi \text{ for every } (W, C_{\text{sup}}, C_{\text{inf}}) \in \tilde{\mathcal{C}}_F \}.$$

From the fact that poset frames are special cases of $\tilde{\mathcal{C}}$ -frames together with Theorem 4.1.1, we derive soundness and strong completeness of **TIL** with respect to the class $\tilde{\mathcal{C}}$, so in particular $\mathbf{TIL} = \text{Log}(\tilde{\mathcal{C}})$.

Theorem 4.2.1. **TIL** has the finite model property, that is, $\mathbf{TIL} = \text{Log}(\tilde{\mathcal{C}}_F)$.

Proof. We show $\text{Log}(\tilde{C}_F) \subseteq \text{Log}(\tilde{C})$ by contraposition. Assuming $\chi \notin \text{Log}(\tilde{C})$, we know that there is a $\tilde{C}$ -model $(W, C_{\text{sup}}, C_{\text{inf}}, V)$ based on a $\tilde{C}$ -frame such that $(W, C_{\text{sup}}, C_{\text{inf}}, V) \not\models \chi$. We turn $(W, C_{\text{sup}}, C_{\text{inf}}, V)$ into a finite model that refutes χ by a filtration argument.

The first step is to further extend the notion of a set of formulas being subformula closed. We introduce $\langle \text{inf} \rangle$ versions of the clauses (Com) and (S-P) already present in Definition 3.7 of [17] and we add a new (Symm) clause that keeps the set closed when $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$ are swapped:

Definition 4.2.2. A set of $\mathcal{L}_T$ formulas Σ is $\tilde{C}$ -closed if:

- It is subformula closed (Sub)
- If $\langle \text{sup} \rangle \varphi \psi \in \Sigma$ then $\langle \text{sup} \rangle \psi \varphi \in \Sigma$ and if $\langle \text{inf} \rangle \varphi \psi \in \Sigma$ then $\langle \text{inf} \rangle \psi \varphi \in \Sigma$ (Com)
- If $\langle \text{sup} \rangle \varphi \psi \in \Sigma$ then $P \varphi \in \Sigma$ (S-P)
- If $\langle \text{inf} \rangle \varphi \psi \in \Sigma$ then $F \varphi \in \Sigma$ (S-F)
- If $\langle \text{sup} \rangle \varphi \psi \in \Sigma$ then $\langle \text{inf} \rangle \varphi \psi \in \Sigma$ and if $\langle \text{inf} \rangle \varphi \psi \in \Sigma$ then $\langle \text{sup} \rangle \varphi \psi \in \Sigma$ (Symm)

Take $\Sigma' = \{\chi\}$ and let Σ be the least extension of Σ' such that it is $\tilde{C}$ -closed. We claim without proving that Σ is finite.

We construct a finite model out of $(W, C_{\text{sup}}, C_{\text{inf}}, V)$ that also falsifies χ . Define an equivalence relation $\sim_\Sigma$ on W as:

$$w \sim_\Sigma v \iff \forall \varphi \in \Sigma \left((W, C_{\text{inf}}, C_{\text{sup}}, V), w \models \varphi \iff (W, C_{\text{inf}}, C_{\text{sup}}, V), v \models \varphi \right).$$

Let $W_\Sigma := \{|w|_\Sigma : w \in W\}$ be the set of states. To define the relations $C_{\text{sup}}^\Sigma |w|_\Sigma |v|_\Sigma |u|_\Sigma$ and $C_{\text{inf}}^\Sigma |w|_\Sigma |v|_\Sigma |u|_\Sigma$, consider the following. We ultimately want to show that the quotient frame $(W_\Sigma, C_{\text{sup}}^\Sigma, C_{\text{inf}}^\Sigma)$ lies in the class $\tilde{C}_F$, so C_{sup}^Σ and C_{inf}^Σ must satisfy all first-order correspondents listed above. Reflexivity (FORe) will follow once we show that $C_{\text{sup}} w v u$ implies $C_{\text{sup}}^\Sigma |w|_\Sigma |v|_\Sigma |u|_\Sigma$ and $C_{\text{inf}} w v u$ implies $C_{\text{inf}}^\Sigma |w|_\Sigma |v|_\Sigma |u|_\Sigma$. Because the quotient map $x \mapsto |x|_\Sigma$ is surjective, this property lifts reflexivity to the quotient frame without further work.

Every other frame axiom is an implication. Take (FOSy) as an example:

$$C_{\text{sup}} w w v \rightarrow C_{\text{inf}} v v w$$

is equivalent to

$$\neg C_{\text{sup}} w w v \vee C_{\text{inf}} v v w.$$

If the implication is true because $\neg C_{\text{sup}} w w v$ is true, we cannot derive $\neg C_{\text{sup}}^\Sigma |w|_\Sigma |w|_\Sigma |v|_\Sigma$ from the fact that $C_{\text{sup}} w w v$ implies $C_{\text{sup}}^\Sigma |w|_\Sigma |w|_\Sigma |v|_\Sigma$. The definition of $\tilde{C}$ -closedness and the filtration relations C_{sup}^Σ and C_{inf}^Σ are therefore designed to preserve the implications in the other axioms.

Thus, define

$$C_{\text{sup}}^\Sigma |w|_\Sigma |v|_\Sigma |u|_\Sigma \iff \forall \langle \text{sup} \rangle \varphi \psi \in \Sigma :$$

- (a) $(v \models \varphi, u \models \psi) \Rightarrow w \models \langle \text{sup} \rangle \varphi \psi$
- (b) $(v \models P \varphi, u \models P \psi) \Rightarrow w \models P \varphi \wedge P \psi$
- (c) $(w \models \varphi, v \models \psi) \Rightarrow v \models \langle \text{inf} \rangle \varphi \psi$
- (d) $(w \models \varphi, u \models \psi) \Rightarrow u \models \langle \text{inf} \rangle \varphi \psi$
- (e) $w \models F \varphi \Rightarrow (v \models F \varphi \text{ and } u \models F \varphi)$

$$C_{\text{inf}}^\Sigma |w||v||u| \Leftrightarrow \forall \langle \text{inf} \rangle \varphi \psi \in \Sigma :$$

- (a) $(v \Vdash \varphi, u \Vdash \psi) \Rightarrow w \Vdash \langle \text{inf} \rangle \varphi \psi$
- (b) $(v \Vdash F\varphi, u \Vdash F\psi) \Rightarrow w \Vdash F\varphi \wedge F\psi$
- (c) $(w \Vdash \varphi, v \Vdash \psi) \Rightarrow v \Vdash \langle \text{sup} \rangle \varphi \psi$
- (d) $(w \Vdash \varphi, u \Vdash \psi) \Rightarrow u \Vdash \langle \text{sup} \rangle \varphi \psi$
- (e) $w \Vdash P\varphi \Rightarrow (v \Vdash P\varphi \text{ and } u \Vdash P\varphi)$

and $V_\Sigma(p) = \{x|_\Sigma \in W_\Sigma : x \in V(p)\}$ for all $p \in \Sigma$.

Thus, if we can show that

$$(W_\Sigma, C_{\text{sup}}^\Sigma, C_{\text{inf}}^\Sigma) \in \tilde{C}_F,$$

and that

$$(W_\Sigma, C_{\text{sup}}^\Sigma, C_{\text{inf}}^\Sigma, V_\Sigma)$$

is a filtration of $(W, C_{\text{sup}}, C_{\text{inf}}, V)$ through Σ (that is, for every $\varphi \in \Sigma$ and every $x \in W$,

$$(W_\Sigma, C_{\text{sup}}^\Sigma, C_{\text{inf}}^\Sigma, V_\Sigma), |x|_\Sigma \Vdash \varphi \iff (W, C_{\text{sup}}, C_{\text{inf}}, V), x \Vdash \varphi)$$

then the proof is complete.

Proposition 4.2.2. $(W_\Sigma, C_{\text{sup}}^\Sigma, C_{\text{inf}}^\Sigma) \in \tilde{C}_F$

Proof. We show that $(W_\Sigma, C_{\text{sup}}^\Sigma, C_{\text{inf}}^\Sigma)$ satisfies all the frame correspondences:

- $(W_\Sigma, C_{\text{sup}}^\Sigma, C_{\text{inf}}^\Sigma) \models (\text{FORe})$ follows from $(W, C_{\text{sup}}, C_{\text{inf}}) \models (\text{Re})$.
- $(W_\Sigma, C_{\text{sup}}^\Sigma, C_{\text{inf}}^\Sigma) \models (\text{FOCo})$ follows from (Com)-closure of Σ and $(W, C_{\text{sup}}, C_{\text{inf}}) \models (\text{Co})$.
- For $(W_\Sigma, C_{\text{sup}}^\Sigma, C_{\text{inf}}^\Sigma) \models (\text{FODk1})$, assume $C_{\text{sup}}|w||v||u|$ and let $\langle \text{sup} \rangle \varphi \psi \in \Sigma$ be arbitrary. We show that all clauses hold:

- (a) Assume $w \Vdash \varphi$ and $v \Vdash \psi$. From (Com)- and (S-P)-closure of Σ it follows that

$$\langle \text{sup} \rangle \psi \top = P\psi \in \Sigma.$$

$(W, C_{\text{sup}}, C_{\text{inf}}) \models (\text{Re})$ implies

$$u \Vdash \langle \text{sup} \rangle \top \top = P\top \text{ and } v \Vdash \langle \text{sup} \rangle \psi \top = P\psi.$$

Then from $C_{\text{sup}}|w||v||u|$ (b) we get $w \Vdash P\psi \wedge P\top$, hence $w \Vdash P\psi = \langle \text{sup} \rangle \psi \top$. From $(W, C_{\text{sup}}, C_{\text{inf}}) \models (\text{Dk1})$ and $w \Vdash \varphi$ it follows that $w \Vdash \langle \text{sup} \rangle \varphi \psi$.

- (b) Assume $w \Vdash P\varphi$ and $v \Vdash P\psi$. Again, since $u \Vdash P\top$, it follows from $v \Vdash P\psi$, (Com)- and (S-P)-closure of Σ (which implies $P\psi \in \Sigma$) and from $C_{\text{sup}}|w||v||u|$ (b) that $w \Vdash P\psi \wedge P\top$. Thus $w \Vdash P\psi$, so $w \Vdash P\varphi \wedge P\psi$.
- (c) Assume $w \Vdash \varphi$, $w \Vdash \psi$, then $w \Vdash \langle \text{inf} \rangle \varphi \psi$ follows directly from $(W, C_{\text{sup}}, C_{\text{inf}}) \models (\text{Re})$.
- (d) Assume $w \Vdash \varphi$ and $v \Vdash \psi$. $C_{\text{sup}}|w||v||u|$ (c) yields $v \Vdash \langle \text{inf} \rangle \varphi \psi$.

- (e) Assume $w \Vdash F\varphi$. Then $C_{\text{sup}}|w||v||u|(e)$ implies $v \Vdash F\varphi$ and $u \Vdash F\varphi$, so in particular $w \Vdash F\varphi$ and $v \Vdash F\varphi$.
- $(W_\Sigma, C_{\text{sup}}^\Sigma, C_{\text{inf}}^\Sigma) \models (\text{FODk2})$ is shown in the same way as $(W, C_{\text{sup}}, C_{\text{inf}}) \models (\text{FODk1})$.
- For $(W_\Sigma, C_{\text{sup}}^\Sigma, C_{\text{inf}}^\Sigma) \models (\text{FO4})$, assume $C_{\text{sup}}|w||w||v|$ and $C_{\text{sup}}|v||v||u|$ and let $\langle \text{sup} \rangle \varphi \psi \in \Sigma$ be arbitrary. We show that all clauses hold:
 - (a) Assume $w \Vdash \varphi$ and $u \Vdash \psi$. By (Com)- and (S-P)-closure of Σ we get

$$\langle \text{sup} \rangle \psi \top = P\psi \in \Sigma.$$

From

$$C_{\text{sup}}|v||v||u|(b), u \Vdash P\psi \text{ and } v \Vdash P\top$$

it follows that $v \Vdash P\psi$. Then

$$C_{\text{sup}}|w||w||v|(b), w \Vdash P\top \text{ and } v \Vdash P\psi$$

yield $w \Vdash P\psi$. Since $(W, C_{\text{sup}}, C_{\text{inf}}) \models (\text{Dk1})$, it follows from $w \Vdash \langle \text{sup} \rangle \psi \top$ and $w \Vdash \varphi$ that $w \Vdash \langle \text{sup} \rangle \varphi \psi$.

- (b) Assume $w \Vdash P\varphi$ and $u \Vdash P\psi$. Again, (Com)- and (S-P)-closure of Σ imply $P\psi \in \Sigma$. So again from (Com)-closure we get $\langle \text{sup} \rangle \top \psi \in \Sigma$. Since

$$v \Vdash \langle \text{sup} \rangle \top \top, u \Vdash \langle \text{sup} \rangle \psi \top,$$

it follows from $C_{\text{sup}}|v||v||u|(b)$ that $v \Vdash P\psi$.

By applying the exact same reasoning again but with $C_{\text{sup}}|w||w||v|$, we get that $w \Vdash P\psi$ since $v \Vdash P\psi$. From $w \Vdash P\varphi$ we derive that $w \Vdash P\varphi \wedge P\psi$.

- (c) Assume $w \Vdash \varphi$ and $w \Vdash \psi$. Then $w \Vdash \langle \text{inf} \rangle \varphi \psi$ follows from $(W, C_{\text{sup}}, C_{\text{inf}}) \models (\text{Re})$.
- (d) Assume $w \Vdash \varphi$ and $u \Vdash \psi$. (Symm) and (S-F)-closure imply $F\varphi \in \Sigma$. Since $(W, C_{\text{sup}}, C_{\text{inf}}) \models (\text{Re})$ and $w \Vdash \varphi$ imply $w \Vdash F\varphi$, it follows from

$$C_{\text{sup}}|w||w||v|(e) \text{ and } w \Vdash F\varphi$$

that $v \Vdash F\varphi$. We apply the same reasoning using

$$v \Vdash F\varphi \text{ and } C_{\text{sup}}|v||v||u|(e)$$

to get $u \Vdash F\varphi$. Then $u \Vdash \psi$ and $(W, C_{\text{sup}}, C_{\text{inf}}) \models (\text{Dk2})$ yield $u \Vdash \langle \text{inf} \rangle \psi \varphi$. Since $(W, C_{\text{sup}}, C_{\text{inf}}) \models (\text{Co.})$, it follows that $u \Vdash \langle \text{inf} \rangle \varphi \psi$.

- (e) Assume $w \Vdash F\varphi$. We already showed in (d) that this implies $u \Vdash F\varphi$.
- For $(W_\Sigma, C_{\text{sup}}^\Sigma, C_{\text{inf}}^\Sigma) \models (\text{FOSy})$, assume $C_{\text{sup}}|w||w||v|$ and let $\langle \text{inf} \rangle \varphi \psi \in \Sigma$ be arbitrary. Then by (Symm)-closure of Σ we have that $\langle \text{sup} \rangle \varphi \psi \in \Sigma$. We show that all clauses hold:
 - (a) Assume $v \Vdash \varphi$ and $w \Vdash \psi$. By (Com)-closure of Σ , it follows that $\langle \text{sup} \rangle \psi \varphi \in \Sigma$.

$$C_{\text{sup}}|w||w||v|(d) \text{ and } (W, C_{\text{sup}}, C_{\text{inf}}) \models (\text{Co.})$$

yield $v \Vdash \langle \text{inf} \rangle \varphi \psi$

- (b) Assume $v \Vdash F \varphi$ and $w \Vdash F \psi$. (Com)- and (S-F)-closure of Σ imply $F \psi \in \Sigma$.
From

$$C_{\text{sup}}|w||w||v|(e) \text{ and } w \Vdash F \psi$$

it follows that $v \Vdash F \psi$. Hence $v \Vdash F \varphi \wedge F \psi$.

- (c) Assume $v \Vdash \varphi$ and $v \Vdash \psi$. Then $v \Vdash \langle \text{sup} \rangle \varphi \psi$ follows from $(W, C_{\text{sup}}, C_{\text{inf}}) \Vdash$ (Re).
(d) Assume $v \Vdash \varphi$ and $w \Vdash \psi$. It follows from (S-P)-closure of Σ that $P \varphi \in \Sigma$.

$$C_{\text{sup}}|w||w||v|(a), w \Vdash \top \text{ and } v \Vdash \varphi$$

yield $w \Vdash \langle \text{sup} \rangle \top \varphi$. Since $(W, C_{\text{sup}}, C_{\text{inf}}) \Vdash$ (Co.) it follows that $w \Vdash \langle \text{sup} \rangle \varphi \top$.
Then

$$(W, C_{\text{sup}}, C_{\text{inf}}) \Vdash (\text{Dk1}), (W, C_{\text{sup}}, C_{\text{inf}}) \Vdash (\text{Co.}) \text{ and } w \Vdash \psi$$

imply $w \Vdash \langle \text{sup} \rangle \varphi \psi$.

- (e) Assume $v \Vdash P \varphi$. From (S-P)- and (Com)-closure it follows that $\langle \text{sup} \rangle \top \varphi \in \Sigma$.
Since

$$w \Vdash \langle \text{sup} \rangle \top \top \text{ and } v \Vdash \langle \text{sup} \rangle \varphi \top$$

it follows from $C_{\text{sup}}|w||w||v|(b)$ that $w \Vdash P \varphi$. □

Proposition 4.2.3. $(W_{\Sigma}, C_{\text{sup}}^{\Sigma}, C_{\text{inf}}^{\Sigma}, V_{\Sigma})$ is a filtration of $(W, C_{\text{sup}}, C_{\text{inf}}, V)$ through Σ .

Proof. There are four things we need to show:

1. $C_{\text{sup}}^{\Sigma}|w||v||u|$ implies that for every $\langle \text{sup} \rangle \varphi \psi \in \Sigma$ we have that $v \Vdash \varphi$ and $u \Vdash \psi$ imply $w \Vdash \langle \text{sup} \rangle \varphi \psi$.
2. $C_{\text{inf}}^{\Sigma}|w||v||u|$ implies that for every $\langle \text{inf} \rangle \varphi \psi \in \Sigma$ we have that $v \Vdash \varphi$ and $u \Vdash \psi$ imply $w \Vdash \langle \text{inf} \rangle \varphi \psi$.
3. $C_{\text{sup}} w v u$ implies $C_{\text{sup}}^{\Sigma}|w||v||u|$.
4. $C_{\text{inf}} w v u$ implies $C_{\text{inf}}^{\Sigma}|w||v||u|$.

1. and 2. follow by definition of C_{sup}^{Σ} and C_{inf}^{Σ} .

To prove 3., we first make the following observations:

$$(W, C_{\text{sup}}, C_{\text{inf}}), w \Vdash P \varphi \iff \exists v \in W : C_{\text{sup}} w w v \wedge (W, C_{\text{sup}}, C_{\text{inf}}), v \Vdash \varphi,$$

$$(W, C_{\text{sup}}, C_{\text{inf}}), w \Vdash F \varphi \iff \exists v \in W : C_{\text{inf}} w w v \wedge (W, C_{\text{sup}}, C_{\text{inf}}), v \Vdash \varphi.$$

Assume $C_{\text{sup}} w v u$ and let $\langle \text{sup} \rangle \varphi \psi \in \Sigma$ be arbitrary. We show that all clauses of $C_{\text{sup}}^{\Sigma}|w||v||u|$ hold:

- (a) This clause follows by definition.
(b) Assume $v \Vdash P \varphi$ and $u \Vdash P \psi$. From $(W, C_{\text{sup}}, C_{\text{inf}}) \models (\text{FODk1}) \wedge (\text{FOCo})$ it follows that $C_{\text{sup}} w w v$ and $C_{\text{sup}} w w u$. Thus, by our previous observation and by our assumptions, $w \Vdash P P \varphi \wedge P P \psi$. Then $(W, C_{\text{sup}}, C_{\text{inf}}) \Vdash (4)$, implies $w \Vdash P \varphi \wedge P \psi$.

- (c) Assume $w \Vdash \varphi$ and $v \Vdash \psi$. From $(W, C_{\text{sup}}, C_{\text{inf}}) \models (\text{FODk1}) \wedge (\text{FOSy})$ it follows that $C_{\text{inf}} v v w$, which implies $v \Vdash \langle \text{inf} \rangle \psi \varphi$. $(W, C_{\text{sup}}, C_{\text{inf}}) \models (\text{Co})$ yields $v \Vdash \langle \text{inf} \rangle \varphi \psi$.
- (d) This clause is proven in a similar fashion as (c), but additionally relying on $(W, C_{\text{sup}}, C_{\text{inf}}) \models (\text{FOCo})$.
- (e) Assume $w \Vdash \text{F} \varphi$. From $(W, C_{\text{sup}}, C_{\text{inf}}) \models \text{FODk1} \wedge \text{FOSy}$, it follows that $C_{\text{inf}} v v w$. By our observation, we have $v \Vdash \text{FF} \varphi$. Since $(W, C_{\text{sup}}, C_{\text{inf}}) \models (4)$, it follows that $v \Vdash \text{F} \varphi$. The result for u is obtained in a similar way, with an additional application of $(W, C_{\text{sup}}, C_{\text{inf}}) \models \text{FOCo}$.

This finalizes the proof of 3.; 4. is proven similarly. □

This finishes the proof, showing that $\mathbf{TIL} = \text{Log}(\tilde{C}_F)$. □

We conclude that

$$TIL = \mathbf{TIL} = \text{Log}(\tilde{C}_F).$$

Since $\mathbf{TIL}$ is a finitely axiomatizable normal modal logic that has the FMP, we conclude that TIL is decidable (see Theorem 6.15 in [10]). Note that from $TIL = TIL_{\text{pre}}$, we immediately get the same result for TIL_{pre} .

5 DUALITY

Having worked out a sound and complete axiomatization of TIL on posets, we now want to see how the same logic looks from an algebraic and topological point of view. The aim of this chapter is to work out the Stone–Jónsson–Tarski duality between the categories

2B-BAO (Boolean algebras with two binary operators)

and

2T-MS (Modal spaces with two ternary relations),

and to check that this duality carries over unchanged when we restrict to the objects that validate the axioms of TIL.

We assume familiarity with the usual categorical notions, that is, categories, functors, and natural transformations. Readers who are not familiar with these concepts can consult Awodey [1] and Mac Lane [19].

Sections 5.1 and 5.2 fix notation and recall a few less standard categorical facts that we will need. Section 5.3 sets up the duality itself, shows how it restricts to the TIL-valid subcategories, and ends with some brief comments.

5.1 ALGEBRAIC NOTIONS

We begin on the algebraic side by introducing the objects and morphisms that form the category **2B-BAO**. Each object is a boolean algebra that interprets two binary modal operator symbols, $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$.

Definition 5.1.1 (Boolean algebra with two binary operators). Let $\sigma = \{\langle \text{sup} \rangle, \langle \text{inf} \rangle\}$ be a modal similarity type. A *boolean algebra with two binary operators* is an algebra

$$\mathfrak{A} = (A, \vee, \neg, \perp, \langle \text{sup} \rangle, \langle \text{inf} \rangle)$$

such that

- $(A, \vee, \neg, \perp)$ is a boolean algebra,
- $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$ are operations of arity 2 satisfying
 - *normality*: $\langle \text{sup} \rangle(\perp, b) = \langle \text{sup} \rangle(a, \perp) = \perp$ and $\langle \text{inf} \rangle(\perp, b) = \langle \text{inf} \rangle(a, \perp) = \perp$,
 - *additivity* in both arguments, i.e. $\langle \text{sup} \rangle(a_1 \vee a_2, b) = \langle \text{sup} \rangle(a_1, b) \vee \langle \text{sup} \rangle(a_2, b)$ and $\langle \text{sup} \rangle(a, b_1 \vee b_2) = \langle \text{sup} \rangle(a, b_1) \vee \langle \text{sup} \rangle(a, b_2)$ and likewise for $\langle \text{inf} \rangle$.

Definition 5.1.2 (BAO-homomorphism). Let $\mathfrak{A} = (A, \vee, \neg, \perp, \langle \text{sup} \rangle, \langle \text{inf} \rangle)$ and $\mathfrak{A}' = (A', \vee', \neg', \perp', \langle \text{sup} \rangle', \langle \text{inf} \rangle')$ be two BAOs of the same similarity type. Then $g : A \rightarrow A'$ is a *BAO-homomorphism* if the following conditions hold:

- g is a *boolean homomorphism* (i.e. a homomorphism from $(A, \vee, \neg, \perp)$ to $(A', \vee', \neg', \perp')$),

- it preserves the modal operators: for every $a, b \in A$

$$g(\langle \sup \rangle(a, b)) = \langle \sup \rangle'(g(a), g(b))$$

and similarly for $\langle \inf \rangle$.

We are now ready to define the category we are interested in.

Definition 5.1.3 (The category **2B-BAO**). **2B-BAO**, the *category of boolean algebras with two binary operators*, is the category whose objects are BAOs over the similarity type $\{\langle \sup \rangle, \langle \inf \rangle\}$ and whose morphisms are BAO-homomorphisms.

Given a BAO $\mathfrak{A}$, an *algebraic valuation* is a map $V : \mathbf{P} \rightarrow A$ that assigns propositional variables to elements of the algebra. Let $Ter(\mathbf{P})_\sigma$ be the σ -term algebra over the same variable set. Every valuation V extends uniquely to a BAO-homomorphism

$$\tilde{V} : Ter(\mathbf{P})_\sigma \rightarrow A,$$

as follows:

$$\begin{aligned} \tilde{V}(p) &= V(p) \quad \text{for every } p \in \mathbf{P} \\ \tilde{V}(\perp) &= \perp \\ \tilde{V}(\varphi \vee \psi) &= \tilde{V}(\varphi) \vee \tilde{V}(\psi) \\ \tilde{V}(\neg \varphi) &= \neg \tilde{V}(\varphi) \\ \tilde{V}(\langle \sup \rangle \varphi \psi) &= \langle \sup \rangle \tilde{V}(\varphi) \tilde{V}(\psi) \\ \tilde{V}(\langle \inf \rangle \varphi \psi) &= \langle \inf \rangle \tilde{V}(\varphi) \tilde{V}(\psi). \end{aligned}$$

Write $\top := \neg \perp$. We say that $\mathfrak{A} \models_{BAO} \varphi \approx \top$ iff $\tilde{V}(\varphi) = \top$ for every algebraic valuation V . Likewise, $\mathfrak{A} \models_{BAO} \varphi \leq \psi$ iff $\tilde{V}(\varphi) \leq \tilde{V}(\psi)$ for all V , where the partial order $\leq$ is defined by $a \leq b$ iff $a \vee b = b$.

5.2 TOPOLOGICAL NOTIONS

We continue by introducing the objects and morphisms that form the category of modal spaces with two ternary relations (**2T-MS**). We first recall some topological notions we will use.

Definition 5.2.1. Let (X, τ) be a topological space. $\text{Clop}(X)$ is the set of clopen subsets of X , i.e. the subsets U of X such that $U \in \tau$ and $X \setminus U \in \tau$.

$\text{Clop}(X)$ is a Boolean algebra under union, complement, and with $\emptyset, X$ as 0, 1.

Definition 5.2.2 (Stone space). Let (X, τ) be a topological space. We say that (X, τ) is a Stone space if it is compact, Hausdorff and totally disconnected or, equivalently, a compact, Hausdorff space with $\text{Clop}(X)$ as a basis.⁴

Definition 5.2.3 (Modal space). Let R_1 and R_2 be ternary relations, then (X, τ, R_1, R_2) is a *modal space* iff

⁴ For a proof of this equivalence see [15].

- (X, τ) is a Stone space,
- R_i ($i = 1, 2$) satisfy
 - *point-closedness*: $R_i[(y, z)] := \{x \in X \mid R_i x y z\}$ is closed for every $(y, z) \in X^2$,
 - $U_1, U_2 \in \text{Clop}(X)$ implies

$$m_{R_i}(U_1, U_2) := \{x \in X \mid \exists u_1 \in U_1, u_2 \in U_2 \text{ s.t. } R_i x u_1 u_2\} \in \text{Clop}(X).$$

Definition 5.2.4 (Continuous bounded morphism). Let (X, τ, R_1, R_2) and (X', τ', R'_1, R'_2) be two modal spaces with two ternary relations. Then $\alpha : X \rightarrow X'$ is a *continuous bounded morphism* if the following holds:

- $U \in \text{Clop}(X')$ implies $\alpha^{-1}(U) \in \text{Clop}(X)$,
- R_i ($i = 1, 2$) satisfy
 - $R_i x x_1 x_2$ implies $R'_i \alpha(x) \alpha(x_1) \alpha(x_2)$,
 - If $R'_i \alpha(x) x'_1 x'_2$, then there exist $x_1, x_2 \in X$ such that $R_i x x_1 x_2$ and $\alpha(x_1) = x'_1, \alpha(x_2) = x'_2$.

Definition 5.2.5 (The category **2T-MS**). **2T-MS**, the *category of modal spaces with two ternary relations* R_1 and R_2 , is the category whose objects are modal spaces (X, τ, R_1, R_2) and whose morphisms are continuous bounded morphisms.

A *modal-space valuation* on (X, τ, R_1, R_2) is a map $V : \mathbf{P} \rightarrow \text{Clop}(X)$. Truth is defined as in ordinary Kripke semantics:

$$\begin{aligned} (X, \tau, R_1, R_2, V), x \Vdash_{MS} p & \text{ iff } x \in V(p), \\ (X, \tau, R_1, R_2, V), x \Vdash_{MS} \neg\varphi & \text{ iff } (X, \tau, R_1, R_2, V), x \not\Vdash_{MS} \varphi, \\ (X, \tau, R_1, R_2, V), x \Vdash_{MS} \varphi \wedge \psi & \text{ iff } (X, \tau, R_1, R_2, V), x \Vdash_{MS} \varphi \text{ and } \\ & (X, \tau, R_1, R_2, V), x \Vdash_{MS} \psi, \\ (X, \tau, R_1, R_2, V), x \Vdash_{MS} \langle \sup \rangle \varphi \psi & \text{ iff } \exists y \in \llbracket \varphi \rrbracket_V, \exists z \in \llbracket \psi \rrbracket_V \text{ such that } R_1 x y z, \\ (X, \tau, R_1, R_2, V), x \Vdash_{MS} \langle \inf \rangle \varphi \psi & \text{ iff } \exists y \in \llbracket \varphi \rrbracket_V, \exists z \in \llbracket \psi \rrbracket_V \text{ such that } R_2 x y z, \end{aligned}$$

where $\llbracket \varphi \rrbracket_V := \{x \in X \mid (X, \tau, R_1, R_2, V), x \Vdash_{MS} \varphi\}$.

Since $\text{Clop}(X)$ is closed under the boolean operations and, by Definition 5.2.3, under m_{R_i} , it follows that $\llbracket \varphi \rrbracket_V \in \text{Clop}(X)$ for all $\varphi \in \mathcal{L}_T$. We write $(X, \tau, R_1, R_2) \Vdash_{MS} \varphi$ iff $\llbracket \varphi \rrbracket_V = X$ for every valuation $V : \mathbf{P} \rightarrow \text{Clop}(X)$.

5.3 THE STONE-JÓNSSON-TARSKI DUALITY

With the algebraic and topological groundwork in place, we can now state the Stone-Jónsson-Tarski duality that links the categories **2B-BAO** and **2T-MS**. We define contravariant functors $F : \mathbf{2B-BAO} \rightarrow \mathbf{2T-MS}$ and $G : \mathbf{2T-MS} \rightarrow \mathbf{2B-BAO}$, and we show that they form a dual equivalence.

On objects, the functor $F : \mathbf{2B-BAO} \rightarrow \mathbf{2T-MS}$ sends a boolean algebra with operators $\mathfrak{A} = (A, \vee, \neg, \perp, \langle \sup \rangle, \langle \inf \rangle)$ to its Stone dual $X_A = \{U \subseteq A : U \text{ is an ultrafilter on } A\}$, with the following topological basis of clopens:

$$\{\widehat{a} \mid a \in A\}, \quad \widehat{a} := \{U \in X_A \mid a \in U\}.$$

The relations $R_{\langle \sup \rangle}$ and $R_{\langle \inf \rangle}$ on X_A are defined as follows:

$$R_{\langle \sup \rangle}GHK \Leftrightarrow \langle \sup \rangle ab \in G \text{ for all } a \in H, b \in K,$$

$$R_{\langle \inf \rangle}GHK \Leftrightarrow \langle \inf \rangle ab \in G \text{ for all } a \in H, b \in K.$$

On morphisms it acts as follows: let $g : A \rightarrow A'$ be a BAO-homomorphism, then we define:

$$\begin{aligned} F(g) : X_{A'} &\longrightarrow X_A \\ V &\longmapsto \{a \in A \mid g(a) \in V\}. \end{aligned}$$

Now let $G : \mathbf{2T-MS} \rightarrow \mathbf{2B-BAO}$ be the functor that on objects sends a modal space (X, τ, R_1, R_2) to the boolean algebra with operators $(\text{Clop}(X), \cup, \setminus, \emptyset, m_{R_1}, m_{R_2})$ where for $U, V \in \text{Clop}(X)$ we set

$$m_{R_1}(U, V) = \{w \in X : \exists u \in U, v \in V \text{ s.t. } R_1 wuv\},$$

$$m_{R_2}(U, V) = \{w \in X : \exists u \in U, v \in V \text{ s.t. } R_2 wuv\}.$$

If $\alpha : X \rightarrow X'$ is a continuous bounded morphism. Then we define:

$$\begin{aligned} G(\alpha) : \text{Clop}(X') &\longrightarrow \text{Clop}(X) \\ U &\longmapsto \{x \in X \mid \alpha(x) \in U\}. \end{aligned}$$

Theorem 5.3.1 (Duality theorem). The functors F and G constitute a dual equivalence between the categories $\mathbf{2T-MS}$ and $\mathbf{2B-BAO}$.

Proof. As shown in Theorem 5.76 and Propositions 5.79-5.80 of [10], F and G are well-defined functors between $\mathbf{2T-MS}$ and $\mathbf{2B-BAO}$. Theorem 5.28 in [22] asserts that F and G establish a dual equivalence between $\mathbf{2T-MS}$ and $\mathbf{2B-BAO}$. $\square$

5.4 RESTRICTING THE FUNCTORS

We are interested in those BAOs and modal spaces that are structures that *validate the axioms of TIL*. We will consider the appropriate subcategories of the ones we just defined. We recall the categorical notion of a (full) subcategory:

Definition 5.4.1. Let C be a category and let $ob(C)$ be the collections of objects of C . For any two objects C_1 and C_2 in $ob(C)$ we denote with $C(C_1, C_2)$ the collection of morphisms between C_1 and C_2 .

We say that $\mathcal{U}$ is a *subcategory* of C if $ob(\mathcal{U}) \subseteq ob(C)$ and for each $U_1, U_2 \in ob(\mathcal{U})$ $\mathcal{U}(U_1, U_2) \subseteq C(U_1, U_2)$ such that $\mathcal{U}$ is closed under composition and identity.

$\mathcal{U}$ is a *full subcategory* if $\mathcal{U}(U_1, U_2) = C(U_1, U_2)$ for all $U_1, U_2 \in ob(\mathcal{U})$.

On the algebraic side, defining the appropriate subcategory amounts to requiring that the two binary operations $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$ satisfy equational versions of the axioms of TIL.

Definition 5.4.2 (TIL-BAO). Let **TIL-BAO** be the full subcategory of **2B-BAO** consisting of all BAOs $\mathfrak{A}$ (we will also call them TIL-algebras from now on) such that $\mathfrak{A} \models_{BAO} \varphi \approx \top$ for all axioms φ of TIL as defined in Definition 3.1.1. I.e. for each such φ and every valuation $V : \mathbf{P} \rightarrow A$ we have $\check{V}(\varphi) = \top$. In the axioms of TIL $\alpha \rightarrow \beta$ abbreviates $\neg\alpha \vee \beta$.

On the topological side we give a frame-style definition of the subcategory that matches the first-order correspondents of TIL we spelled out in Section 4.1.⁵

Definition 5.4.3 (TIL-MS, frame-style). Let **TIL-MS** be the full subcategory of **2T-MS** consisting of all modal spaces (we will also call them TIL-spaces from now on) $(X, \tau, R_{\text{sup}}, R_{\text{inf}})$ such that the ternary relations R_{sup} and R_{inf} satisfy the following conditions:

- R_{sup} and R_{inf} are reflexive⁶, commutative in the second and third argument and transitive⁷.
- R_{sup} and R_{inf} satisfy (FODK1) and (FODK2): for all $x, y, z \in X$ it holds that $R_{\text{sup}}xyz \Rightarrow R_{\text{sup}}xxy$ and $R_{\text{inf}}xyz \Rightarrow R_{\text{inf}}xxy$.
- R_{sup} and R_{inf} are tense duals, that is for all $x, y \in X$: $R_{\text{sup}}xxy \Leftrightarrow R_{\text{inf}}yyx$.

Since the TIL axioms are Sahlqvist, they are d-persistent (Theorem 5.91 of [10]): if $(X, \tau, R_1, R_2) \models_{MS} \varphi$, then the underlying frame $(X, R_1, R_2) \models \varphi$, where $\models_{MS}$ quantifies over $V : \mathbf{P} \rightarrow \text{Clop}(X)$ and $\models$ over $V' : \mathbf{P} \rightarrow \mathcal{P}(X)$.

Combining this with the Sahlqvist frame correspondence result for the axioms of TIL (see 4.1.1) the frame-style Definition 5.4.3 is equivalent to the following definition:

Definition 5.4.4 (TIL-MS, axiom-validity). Let **TIL-MS** be the full subcategory of **2T-MS** consisting of all modal spaces (X, τ, R_1, R_2) such that $(X, \tau, R_1, R_2) \models_{MS} \varphi$ for all axioms φ of TIL as defined in Definition 3.1.1.

From now on we use the two representations interchangeably.

We check whether the dual equivalence between **2B-BAO** and **2T-MS** restricts to an equivalence between **TIL-BAO** and **TIL-MS**. First, we check that the restricted functors

$$F' := F \upharpoonright_{\mathbf{TIL-BAO}}, \quad G' := G \upharpoonright_{\mathbf{TIL-MS}}$$

are well-defined.

Theorem 5.4.1. For every BAO $\mathfrak{A}$, modal space (X, τ, R_1, R_2) , and formula $\varphi \in \mathcal{L}_T$, with F and G as defined above,

$$\mathfrak{A} \models_{BAO} \varphi \approx \top \iff F(\mathfrak{A}) \models_{MS} \varphi, \tag{4a}$$

$$(X, \tau, R_1, R_2) \models_{MS} \varphi \iff G(X, \tau, R_1, R_2) \models_{BAO} \varphi \approx \top. \tag{4b}$$

⁵ From now on we will use R_{sup} and R_{inf} because we are considering TIL-spaces. They correspond to C_{sup} and C_{inf} used in the mentioned section.

⁶ In this case reflexivity means that $R_{\text{sup}}xxx$ and $R_{\text{inf}}xxx$ hold for all $x \in X$ in accordance with (FORe).

⁷ Transitivity is defined on the induced order by R_{sup} and R_{inf} , that is for all $x, y \in X$: $x \leq y$ iff $R_{\text{sup}}yyx$ iff $R_{\text{inf}}xxy$ (see 3.2.1 for a proof that this equivalence holds). Transitivity then says that for any $x, y, z \in X$ if $x \leq y$ and $y \leq z$, then $x \leq z$.

Proof. For a proof of this theorem we refer to [10, Proposition 5.24, Theorem 5.75]. $\square$

It follows directly from Theorem 5.4.1 and the fact that **TIL-BAO** and **TIL-MS** are full subcategories that the restricted functors F' and G' are well-defined.

Since F and G establish a dual equivalence between **2B-BAO** and **2T-MS**, there are natural isomorphisms $\eta : 1_{\mathbf{2B-BAO}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow 1_{\mathbf{2T-MS}}$ for which the triangle identities hold (see page 85 of [19]). By the previous reasoning, it follows that if $\mathfrak{A}$ is in **TIL-BAO**, then $GF(\mathfrak{A})$ is in **TIL-BAO**. Thus the component $\eta_{\mathfrak{A}} : \mathfrak{A} \Rightarrow GF(\mathfrak{A})$ is a morphism inside **TIL-BAO**. We can therefore restrict η object-wise to get a natural isomorphism

$$\eta' : 1_{\mathbf{TIL-BAO}} \Rightarrow G'F'.$$

The same reasoning can be given to restrict ϵ to a natural isomorphism ϵ' between $F'G'$ and $1_{\mathbf{TIL-MS}}$.

Since the components of η' and ϵ' are unchanged, the triangle identities still hold. We conclude the following.

Theorem 5.4.2. F' and G' constitute a dual equivalence between **TIL-BAO** and **TIL-MS**

5.5 COROLLARIES

Having established these duality results, we give two applications. First we show that we can lift the decidability results of Chapter 4 to the variety **TIL-BAO**, then we prove that every normal extension L of **TIL** is sound and complete with respect to a class of **TIL**-spaces.

Corollary 5.5.1 (FMP and decidability lift to the variety **TIL-BAO**). Since **TIL** has the finite model property (Chapter 4), any non-theorem φ has a finite **TIL**-countermodel. This countermodel can be turned into a finite modal space $X \in \mathbf{TIL-MS}$ falsifying φ . Then $G(X) \in \mathbf{TIL-BAO}$ and by Theorem 5.4.1 (4b), $G(X)$ also falsifies $\varphi \approx \top$. Since by construction $G(X)$ is also finite, it follows that the equational theory

$$\text{Log}(\mathbf{TIL-BAO}) := \{ \varphi \in \mathcal{L}_T \mid \mathfrak{A} \Vdash_{\mathbf{BAO}} \varphi \approx \top \text{ for all } \mathfrak{A} \in \mathbf{TIL-BAO} \}$$

is decidable.

Definition 5.5.1 (Normal extensions of **TIL**). Let **TIL** be the normal modal logic of Definition 3.1.1. A logic L is a *normal extension* of **TIL** iff $\mathbf{TIL} \subseteq L$ and L is closed under modus ponens, uniform substitution, and the generalization rules for $[\text{sup}]$ and $[\text{inf}]$ ⁸.

Definition 5.5.2. For a normal extension L of **TIL**, define the following full subcategories of **TIL-BAO** and **TIL-MS** respectively:

$$\mathbf{TIL-BAO}_L := \{ \mathfrak{A} \in \mathbf{TIL-BAO} \mid \mathfrak{A} \Vdash_{\mathbf{BAO}} \varphi \approx \top \text{ for all } \varphi \in L \},$$

$$\mathbf{TIL-MS}_L := \{ (X, \tau, R_{\text{sup}}, R_{\text{inf}}) \in \mathbf{TIL-MS} \mid (X, \tau, R_{\text{sup}}, R_{\text{inf}}) \Vdash_{\mathbf{MS}} \varphi \text{ for all } \varphi \in L \}.$$

Theorem 5.5.2 (Soundness and completeness). For each normal extension L of **TIL**, L is sound and complete with respect to the class of modal spaces $\mathbf{TIL-MS}_L$, that is

$$L = \text{Log}(\mathbf{TIL-MS}_L),$$

⁸ To briefly recall the definition of generalization (see Definition 3.1.1): if $\vdash \varphi$ then $\vdash [\text{inf}] \varphi \psi$ for all $\varphi, \psi \in \mathcal{L}_T$ and similarly for $[\text{sup}]$.

where

$$\text{Log}(\mathbf{TIL-MS}_L) := \left\{ \varphi \in \mathcal{L}_T \mid (X, \tau, R_{\text{sup}}, R_{\text{inf}}) \Vdash_{MS} \varphi \text{ for all } (X, \tau, R_{\text{sup}}, R_{\text{inf}}) \in \mathbf{TIL-MS}_L \right\}$$

Proof. Soundness follows immediately from the definition of $\mathbf{TIL-MS}_L$.

For completeness we rely on algebraic completeness [10, Theorem 5.27]. It follows that for all $\varphi \in \mathcal{L}_T$:

$$\varphi \in L \iff \varphi \in \text{Log}(\mathbf{TIL-BAO}_L),$$

where

$$\text{Log}(\mathbf{TIL-BAO}_L) := \left\{ \varphi \in \mathcal{L}_T \mid \mathfrak{A} \Vdash_{BAO} \varphi \approx \top \text{ for all } \mathfrak{A} \in \mathbf{TIL-BAO}_L \right\}.$$

Applying Theorem 5.4.1 we derive

$$\mathfrak{A} \Vdash_{BAO} \varphi \approx \top \iff F(\mathfrak{A}) \Vdash_{MS} \varphi,$$

and similarly for G . It follows that $\text{Log}(\mathbf{TIL-BAO}_L) = \text{Log}(\mathbf{TIL-MS}_L)$, hence

$$L = \text{Log}(\mathbf{TIL-MS}_L). \quad \square$$

Summarizing, we have worked out the Stone–Jónsson–Tarski duality between BAOs with two binary operators and modal spaces with two ternary relations. We showed that this duality can be restricted to a duality between TIL-spaces and TIL-algebras and used this correspondence to show that the variety of TIL-algebras is decidable and that every normal extension L of TIL is sound and complete with respect to a class of TIL-spaces.

6 TIL OF MINIMAL UPPER BOUNDS AND MAXIMAL LOWER BOUNDS

There are several natural extensions of TIL worth exploring. Knudstorp studies many interesting extensions of MIL in [16]. A natural question is whether the techniques developed there can be applied in the present setting.

One alteration of TIL that can be studied is tense information logic of minimal upper bounds and maximal lower bounds (as opposed to least upper bound and greatest lower bound). Taking inspiration from [18], it can be proven that changing the interpretation of the modalities this way is, in fact, indistinguishable for the modal language. This chapter will be devoted to proving that result.

6.1 DEFINITIONS

We first give a formal definition of minimal upper bound and maximal lower bound.

Definition 6.1.1. Let $\mathfrak{F} = \langle W, \leq \rangle$ be a poset and $y, z \in W$.

- An element $x \in W$ is a *minimal upper bound* of $\{y, z\}$ if $y \leq x$ and $z \leq x$ (so x is an upper bound) and there is no upper bound strictly below x , i.e. for any $w \in W$ that is also an upper bound of $\{y, z\}$, it holds that $w \not\prec x$. We write $x \in \text{mub}\{y, z\}$.
- An element $x \in W$ is a *maximal lower bound* of $\{y, z\}$ if $x \leq y$ and $x \leq z$ (so x is a lower bound) and there is no lower bound strictly above x , i.e. for any $w \in W$ that is also a lower bound of $\{y, z\}$, it holds that $x \not\prec w$. We write $x \in \text{mlb}\{y, z\}$.

Having these definitions at hand, we are ready to give alternative semantics for the modalities $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$ on posets. To distinguish this new semantics from the original one, we will write $\Vdash_M$ for the alternative interpretation:

$$\begin{aligned} \mathfrak{M}, x \Vdash_M \langle \text{sup} \rangle \varphi \psi & \quad \text{iff} \quad \exists y, z \in W \text{ s.t. } \mathfrak{M}, y \Vdash_M \varphi, \mathfrak{M}, z \Vdash_M \psi, x \in \text{mub}\{y, z\}, \\ \mathfrak{M}, x \Vdash_M \langle \text{inf} \rangle \varphi \psi & \quad \text{iff} \quad \exists y, z \in W \text{ s.t. } \mathfrak{M}, y \Vdash_M \varphi, \mathfrak{M}, z \Vdash_M \psi, x \in \text{mlb}\{y, z\}. \end{aligned}$$

Just like before, we can define the past / future looking diamond / box in the same way. For example, the past looking diamond P is still definable as $\langle \text{sup} \rangle \varphi \top$, with the following semantics:

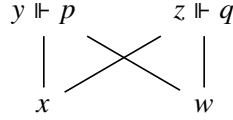
$$\mathfrak{M}, x \Vdash_M \langle \text{sup} \rangle \varphi \top \quad \text{iff} \quad \exists y, z \in W \text{ s.t. } \mathfrak{M}, y \Vdash_M \varphi, \mathfrak{M}, z \Vdash_M \top, x \in \text{mub}\{y, z\}.$$

Since $\mathfrak{M}, x \Vdash_M \top$ always holds, it follows that $\mathfrak{M}, x \Vdash_M \langle \text{sup} \rangle \varphi \top$ if there is $y \in W$ such that $\mathfrak{M}, y \Vdash_M \varphi$ and $x \in \text{mub}\{y, x\}$. Since $x \in \text{mub}\{y, x\}$ iff $y \leq x$, this corresponds to the semantics of P , just as we showed in Remark 2.1.2.

Definition 6.1.2. The modal logic of minimal upper bounds and maximal lower bounds is denoted $TIL_{\text{min-max}}$ and defined as the set of all $\mathcal{L}_T$ validities on poset frames:

$$TIL_{\min\text{-max}} := \{ \varphi \in \mathcal{L}_T \mid \text{for every poset model } \mathfrak{M} = (W, \leq, V) \text{ and every } x \in W, \\ \mathfrak{M}, x \Vdash_M \varphi \}.$$

At first glance, it seems likely that changing the interpretation of the modalities will change the logic. Consider, for example, the following model:



In this model the state x fails $\langle \text{inf} \rangle pq$ in the *least / greatest* (standard) sense because $x \neq \inf\{y, z\}$, while it satisfies the same formula in the *minimal / maximal* sense because $x \in \text{mlb}\{y, z\}$.

Surprisingly, we will show that $TIL = TIL_{\min\text{-max}}$, and hence (because $\mathbf{TIL} = TIL$) the axiomatization presented in Section 3.1 is also sound and complete for $TIL_{\min\text{-max}}$. One inclusion is not difficult to show:

Theorem 6.1.1. $TIL \subseteq TIL_{\min\text{-max}}$.

Proof. This comes down to showing $\mathbf{TIL} \subseteq TIL_{\min\text{-max}}$, that is, we should check that $TIL_{\min\text{-max}}$ is a normal modal logic that validates all the axioms of $\mathbf{TIL}$. Fortunately, the proof of Theorem 3.1.1 applies in this setting. We show this for the (Dk2) axiom:

(Dk2) Assume $\mathfrak{M}, x \Vdash_M p \wedge \langle \text{inf} \rangle qr$, so $\mathfrak{M}, x \Vdash_M p$ and there are y, z such that $x \in \text{mlb}\{y, z\}$ and $\mathfrak{M}, y \Vdash_M q$ and $\mathfrak{M}, z \Vdash_M r$. Since $x \in \text{mlb}\{x, y\}$, it follows that $\mathfrak{M}, x \Vdash_M \langle \text{inf} \rangle pq$. $\square$

6.2 PROOF STRATEGY

To show the converse inclusion $TIL_{\min\text{-max}} \subseteq TIL$ we follow the representation approach of [18]. We argue contrapositively:

$$\varphi \notin TIL \Rightarrow \varphi \notin TIL_{\min\text{-max}}.$$

Let $\varphi \in \mathcal{L}_T$ and assume $\varphi \notin TIL$. Then there exists a poset model $\mathfrak{M} = (W, \leq, V)$ and $x \in W$ with $\mathfrak{M}, x \not\Vdash \varphi$. We construct an extension

$$\mathfrak{M}' = (W', \leq', V') \quad \text{with} \quad W \subseteq W', \leq \subseteq \leq', V \subseteq V',$$

that still refutes φ , but now under the *minimal / maximal* semantics; that is, $\mathfrak{M}', x \not\Vdash_M \varphi$.

The difficulty lies with formulas containing $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$, as the example model we gave above shows. We therefore want to define $\mathfrak{M}'$ so that the least upper bound / greatest lower bound interpretation of $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$ (denoted by $\Vdash$) coincides with the minimal / maximal interpretation (denoted by $\Vdash_M$); that is,

$$\forall x \in W', \forall \varphi \in \mathcal{L}_T : \mathfrak{M}', x \Vdash \varphi \Leftrightarrow \mathfrak{M}', x \Vdash_M \varphi \quad (5)$$

One direction of (5) is immediate:

$$x = \text{sup}\{y, z\} \Rightarrow x \in \text{mub}\{y, z\}$$

(and dually for infima). For the reverse implication, we must ensure that

$$x \in \text{mub}\{y, z\} \Rightarrow x = \sup\{y, z\}, \quad (6)$$

$$x \in \text{mlb}\{y, z\} \Rightarrow x = \inf\{y, z\}, \quad (7)$$

for all $x, y, z \in W'$. The main part of this chapter is devoted to constructing such an extension for which (6) and (7) hold.

Next, the original worlds must preserve truth in the sup/inf interpretation:

$$\mathfrak{M}, x \Vdash \varphi \iff \mathfrak{M}', x \Vdash \varphi. \quad (8)$$

Putting everything together, the refutation in the supremum / infimum semantics then also holds under the minimal / maximal semantics:

$$\mathfrak{M}, x \not\models \varphi \xrightarrow{(8)} \mathfrak{M}', x \not\models \varphi \xrightarrow{(5)} \mathfrak{M}', x \not\models_M \varphi. \quad (9)$$

Hence $\varphi \notin TIL_{\min\text{-max}}$, thus completing the contraposition.

6.2.1 Constructing the extending frame

As mentioned above, the difficult part of this construction is defining an extension of $\mathfrak{M}$ such that (6) and (7) hold. The issue is when a world x is

- a minimal upper bound of y, z without being their supremum, or
- a maximal lower bound of y, z without being their infimum.

Because we must make sure that (8) holds, it does not work to turn such an x into the supremum / infimum of the two worlds in the new model. Instead, we make sure that it is no longer the minimal / maximal bound in the extension $\mathfrak{M}'$.

For every triple (x, y, z) that falls under one of the cases above, we add a new chain of bounds below or above x :

- **Upper-bound case:** add an infinite descending chain

$$x > x_0 > x_1 > \dots$$

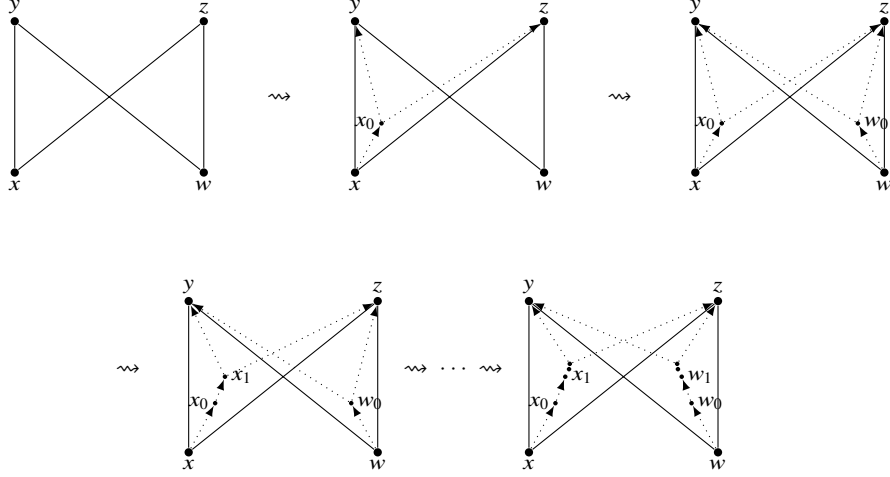
with $x_i > y, z$ for all i .

- **Lower-bound case:** add an infinite ascending chain

$$x < x_0 < x_1 < \dots$$

with $x_i < y, z$ for all i .

This process is carried out step-by-step and is depicted below for the case in which x is a maximal lower bound of the set $\{y, z\}$.



The case in which x is a minimal upper bound is shown in Section 3 of [18]: it is the mirror image of the construction above.

After this process, x is no longer minimal (or maximal) and hence the antecedent of (6) (and dually of (7)) never holds for such an x , so the implication is vacuously true.

We add these ascending and descending chains for every witness to the failure of (6) or (7) (possibly iterating the construction infinitely many times) to produce the required extension $\mathfrak{M}'$.

As required by (8), all *old* worlds $u \in W$ must keep their truth values under the original semantics after the extension. Because x will see all its copies, we must control the formulas those copies satisfy and make sure that they satisfy the exact same formulas as x itself. Only making x and its copies satisfy the same propositional variables is not enough: if $\mathfrak{M}, x \models \langle \sup \rangle \varphi \psi$ is witnessed by $u, v \leq x$, then since x_0 is *not* the supremum of $\{u, v\}$ in the picture above, x_0 would not satisfy the same formulas as x .

We solve this by copying not only x itself, but also its entire downset:

$$\downarrow x = \{u \in W : u \leq x\}.$$

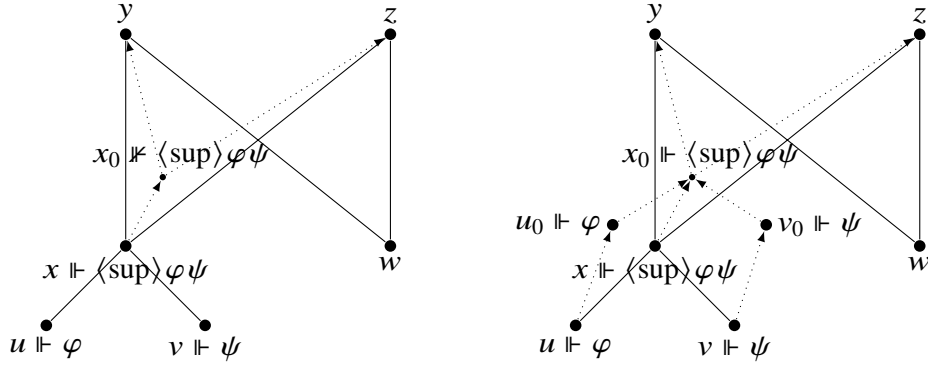
A symmetric issue arises for infima: if $\mathfrak{M}, x \models \langle \inf \rangle \varphi \psi$, then every copy of x must again be the greatest lower bound of the same elements as x itself. Therefore we also copy x 's upset:

$$\uparrow x = \{u \in W : u \geq x\}.$$

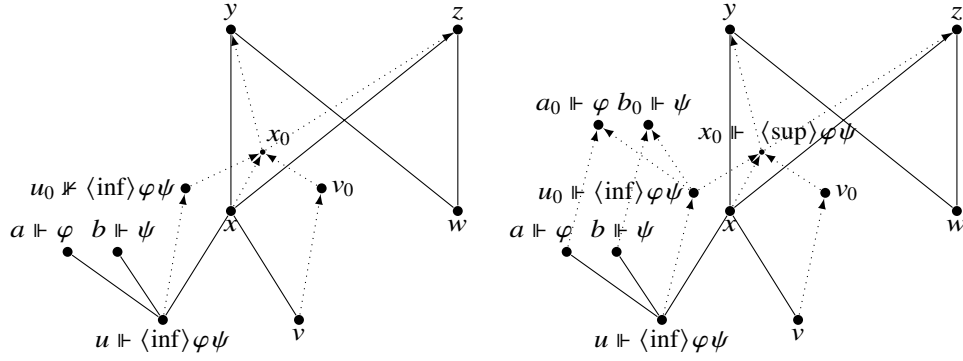
For each new x_n we thus duplicate its entire context $\uparrow x \cup \downarrow x$, extend $\leq$ in the obvious way, and lift the valuation:

$$V'(p) := V(p) \cup \{w_n : w \in V(p)\}.$$

The diagram below illustrates one step of this copying process.

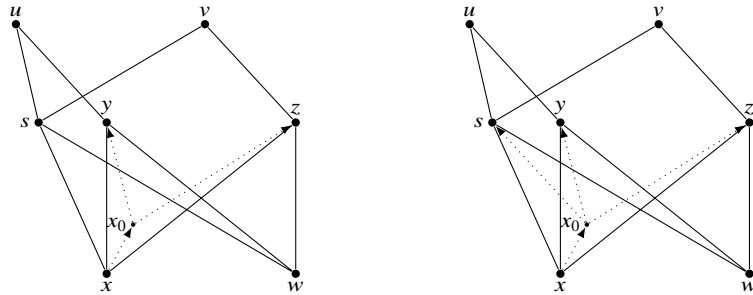


It turns out that this is still too naive. If u in our model above satisfies $\langle \text{inf} \rangle \varphi \psi$, witnessed by $a, b \geq u$, then u_0 would not satisfy the same formula. With P and F definable in our language, we can reach every world in the same ‘comparability component’, i.e. every state connected to the current one by a finite zigzag of $\leq$ - and $\geq$ -steps. We therefore copy the entire model at each step⁹:



Copying the entire set of worlds W is almost enough, but one final hurdle remains. Suppose $u \in \uparrow y$ and $v \in \uparrow z$ have infimum s in W . By adding the new lower bound x_0 under y, z , we also put x_0 below u, v . Unless $x_0 \leq s$, s is no longer the infimum of $\{u, v\}$. This is shown in the picture left below.

To solve this problem, we also connect x_0 (and thus any point in its downset by transitivity) to s . In general, every new lower bound of $\{y, z\}$ must be seen by all points in the least upset containing $\{y, z\}$ closed under binary infima. Dually, every added upper bound of a set $\{y', z'\}$ must be seen by all points in the least downset containing $\{y', z'\}$ closed under binary suprema.



⁹ Strictly speaking it would be enough to only copy the ‘comparability component’ x is part of, but for notational convenience we copy the whole model.

6.3 COMPLETENESS PROOF

In the previous section we sketched the idea behind the completeness proof. We now provide the technical details, beginning with two lemmas that remove a minimal (respectively maximal) upper (respectively lower) bound that is not a supremum (respectively infimum), which we will call a defect. In the end, this guarantees (6) and (7). Concretely, from a given frame $\mathfrak{F} = (W, \leq)$ we build a frame $\mathfrak{F}' = (W', \leq')$ in which the defect is repaired.

Definition 6.3.1. Let $(W, \leq)$ be a poset. A triple $(x, y, z) \in W^3$ is a *max-defect* iff

$$x \in \text{mub}\{y, z\} \quad \text{but} \quad x \neq \sup\{y, z\}.$$

Definition 6.3.2. Let $(W, \leq)$ be a poset. A triple $(x, y, z) \in W^3$ constitutes a *min-defect* iff

$$x \in \text{mlb}\{y, z\} \quad \text{but} \quad x \neq \inf\{y, z\}.$$

Our construction makes $\mathfrak{F}$ a p-morphic image of $\mathfrak{F}'$, thereby preserving validity under the sup/inf semantics.

Definition 6.3.3. Let $(W, \leq)$ and $(W', \leq')$ be posets, and let $f : W' \rightarrow W$ be a function. We say that f is a *sup / inf p-morphism* if the following hold:

- (sup-forth) if $x' = \sup'\{y', z'\}$, then $f(x') = \sup\{f(y'), f(z')\}$;
- (inf-forth) if $x' = \inf'\{y', z'\}$, then $f(x') = \inf\{f(y'), f(z')\}$;
- (sup-back) if $f(x') = \sup\{y, z\}$, then there exist $y', z' \in W'$ with $f(y') = y$, $f(z') = z$, and $x' = \sup'\{y', z'\}$;
- (inf-back) if $f(x') = \inf\{y, z\}$, then there exist $y', z' \in W'$ with $f(y') = y$, $f(z') = z$, and $x' = \inf'\{y', z'\}$.

The following lemmas make precise the method by which we repair the two possible defects identified above.

Lemma 6.3.1. Let $(W, \leq)$ be a poset and $(x, y, z) \in W^3$ such that (x, y, z) constitutes a max-defect. Then there exists a poset $(W', \leq')$ and a p-morphism $f : W' \rightarrow W$ such that:

- (1) $W \subseteq W'$ and $|W'| \leq \max\{\omega, |W|\}$,
- (2) $\leq' \cap (W \times W) = \leq$,
- (3) $f \upharpoonright_W = \text{id}_W$,
- (4) for all $u, v, w \in W$,

$$u = \sup\{v, w\} \Rightarrow u = \sup'\{v, w\}, \quad u = \inf\{v, w\} \Rightarrow u = \inf'\{v, w\},^{10}$$

- (5) $x \notin \text{mub}'\{y, z\}$.

¹⁰ Since $W \subseteq W'$, each element of W is also present in W' . Thus, when we write $u = \sup'\{v, w\}$ (respectively $u = \inf'\{v, w\}$), we mean that the elements $u, v, w \in W$, considered as elements in W' , still form the supremum (infimum) of $\{v, w\}$ with respect to the extended order $\leq'$.

Lemma 6.3.2. Let $(W, \leq)$ be a poset and $(x, y, z) \in W^3$ such that (x, y, z) constitutes a min-defect. Then there exists a poset $(W', \leq')$ and a p-morphism $f : W' \rightarrow W$ such that:

- (1) $W \subseteq W'$ and $|W'| \leq \max\{\omega, |W|\}$,
- (2) $\leq' \cap (W \times W) = \leq$,
- (3) $f \upharpoonright_W = \text{id}_W$,
- (4) for all $u, v, w \in W$,

$$u = \sup\{v, w\} \Rightarrow u = \sup'\{v, w\}, \quad u = \inf\{v, w\} \Rightarrow u = \inf'\{v, w\},$$

- (5) $x \notin \text{mlb}'\{y, z\}$.

We now prove Lemma 6.3.2. Lemma 6.3.1 is proven symmetrically. The proof adapts the strategy of Lemma 5.6 in [18] to our present setting, which includes the operator $\langle \inf \rangle$ and a modified extension construction.

Proof of Lemma 6.3.2. Let

$$W' := W \sqcup W = \{(v, 0), (v, 1) \mid v \in W\}, \quad f : W' \rightarrow W, \quad f(v, i) = v.$$

For $y, z \in W$ let $A(y, z)$ be the least upset containing $\{y, z\}$ and closed under binary infima:

$$A(y, z) := \bigcup_{n < \omega} A_n(y, z),$$

where

$$A_0(y, z) := \uparrow y \cup \uparrow z, \quad A_{n+1}(y, z) := \uparrow \left(A_n(y, z) \cup \{\inf\{b_n, c_n\} \mid b_n, c_n \in A_n(y, z)\} \right).$$

Define the extended order $\leq'$ on W' by

$$(v, i) \leq' (w, j) \iff i = 0 \text{ and } v \leq w, \tag{10a}$$

$$\text{or } i = j = 1 \text{ and } v \leq w, \tag{10b}$$

$$\text{or } i = 1, j = 0, w \in A(y, z), v \in \downarrow x. \tag{10c}$$

We prove three claims:

Claim 6.3.3. $(W', \leq')$ is a poset.

Claim 6.3.4. Conditions (1)–(5) of the lemma hold.

Claim 6.3.5. f is an onto sup / inf p-morphism.

We first prove two auxiliary lemmas that will be needed to prove these claims.

Lemma 6.3.6. For every $s \in A(y, z)$ we have $x < s$.

Proof. Because $x \in \text{mlb}\{y, z\}$ but $x \neq \inf\{y, z\}$, there exists $w \in W$ such that $w \leq y, z$ but $x \not\leq w$ and $w \not\leq x$. We prove by induction on n that

$$A_n(y, z) \subsetneq \uparrow x \text{ and } A_n(y, z) \subsetneq \uparrow w.$$

Base case: From $x \leq y, z$, $w \leq y, z$, $x \not\leq w$ and $w \not\leq x$ it follows that $x < y, z$ and $w < y, z$. Hence

$$A_0(y, z) = \uparrow y \cup \uparrow z \subseteq \uparrow x \quad \text{and} \quad A_0(y, z) = \uparrow y \cup \uparrow z \subseteq \uparrow w.$$

Inductive case: Assume the claim for $A_n(y, z)$. Let $b_n, c_n \in A_n(y, z)$ be such that $\inf\{b_n, c_n\}$ exists. By induction hypothesis

$$b_n, c_n \in A_n(y, z) \subseteq \uparrow x \quad \text{and} \quad b_n, c_n \in A_n(y, z) \subseteq \uparrow w$$

so $x \leq b_n, c_n$ and $w \leq b_n, c_n$. It follows that $x \leq \inf\{b_n, c_n\}$ and $w \leq \inf\{b_n, c_n\}$. On the other hand, since $x \not\leq w$ and $w \not\leq x$, it follows that the inequalities are strict. We conclude that

$$A_{n+1}(y, z) \subseteq \uparrow w \quad \text{and} \quad A_{n+1}(y, z) \subseteq \uparrow x.$$

Thus

$$A(y, z) = \bigcup_{n < \omega} A_n(y, z) \subseteq \uparrow x,$$

which implies $x < s$ for every $s \in A(y, z)$. □

Lemma 6.3.7. The map f is order-preserving.

Proof. Assume $(v, i) \leq' (w, j)$. We show $v = f(v, i) \leq w = f(w, j)$ by cases:

- $i = 0$. By clause 10a of the definition of $\leq'$ we have $v \leq w$.
- $i = j = 1$. Clause 10b implies $v \leq w$.
- $i = 1, j = 0$ with $w \in A(y, z)$ and $v \in \downarrow x$. Lemma 6.3.6 gives $x < w$, and $v \leq x$ by assumption, so $v \leq w$. □

Proof of Claim 6.3.3.

Reflexivity For any $(v, i) \in W'$ the corresponding clause (10a if $i = 0$ or 10b if $i = 1$) gives $(v, i) \leq' (v, i)$.

Transitivity: Suppose $(v, i) \leq' (w, j)$ and $(w, j) \leq' (u, k)$. Lemma 6.3.7 gives us $v \leq w \leq u$, so since $\leq$ is transitive we get $v \leq u$.

- If $i = 0$ or $i = k = 1$, $(v, i) \leq' (u, k)$ follows from clause 10a or 10b.
- If $i = 1$ and $k = 0$, we must show $u \in A(y, z)$ and $v \in \downarrow x$.
 1. If $j = 0$ then it follows from $(v, 1) \leq' (w, 0)$, that $w \in A(y, z)$ and $v \in \downarrow x$. From $w \leq u$ we get $u \in A(y, z)$, since it is an upset.
 2. If $j = 1$ then $(w, 1) \leq' (u, 0)$ implies $u \in A(y, z)$ and $w \in \downarrow x$. Because $v \leq w$, also $v \in \downarrow x$, so $(v, 1) \leq' (u, 0)$.

Antisymmetry: Assume $(v, i) \leq' (w, j)$ and $(w, j) \leq' (v, i)$. If $i = j$, Lemma 6.3.7 gives $v \leq w$, hence $v = w$ by antisymmetry of $\leq$.

If $i = 0$ and $j = 1$, it follows from $(w, 1) \leq' (v, 0)$ that $v \in A(y, z)$ and $w \in \downarrow x$. By Lemma 6.3.6 we get that $w \leq x < v$. But then we derived a contradiction because $(v, 0) \leq' (w, 1)$ implies that $v \leq w$ by Lemma 6.3.7. □

Proof of Claim 6.3.4. If we identify the set $\{(v, 0) : v \in W\} \subseteq W'$ with W , condition (1)–(3) of Lemma 6.3.2 follow immediately.

- (4) Assume $u = \sup\{v, w\}$. We need to show that $(u, 0) = \sup'\{(v, 0), (w, 0)\}$. From $v, w \leq u$, clause 10a gives $(v, 0), (w, 0) \leq' (u, 0)$. If also $(v, 0), (w, 0) \leq' (s, i)$ for some $(s, i) \in W'$, Lemma 6.3.7 yields $v, w \leq s$, hence $u \leq s$, since $u = \sup\{v, w\}$. Clause 10a then gives $(u, 0) \leq' (s, i)$. Thus $(u, 0) = \sup'\{(v, 0), (w, 0)\}$.

Now assume $u = \inf\{v, w\}$. Then from $u \leq v, w$, it follows by 10a that $(u, 0) \leq' (v, 0), (w, 0)$. Let $(s, i) \in W'$ be arbitrary such that $(s, i) \leq' (v, 0), (w, 0)$. By Lemma 6.3.7 it follows that $s \leq v, w$, so since $u = \inf\{v, w\}$ we get $s \leq u$. We need to show that $(s, i) \leq' (u, 0)$.

- If $i = 0$, then by 10a it follows that $(s, 0) \leq' (u, 0)$.
- If $i = 1$, we need to show that $s \in \downarrow x$ and $u \in A(y, z)$. $(s, 1) \leq' (v, 0), (w, 0)$ implies $s \in \downarrow x$ and $v, w \in A(y, z)$. Since $A(y, z)$ is closed under binary infima it follows that $\inf\{v, w\} = u \in A(y, z)$.

- (5) By construction $(x, 0) \leq' (x, 1)$, and because $y, z \in A(y, z)$ we have $(x, 1) \leq' (y, 0), (z, 0)$. Hence $(x, 0) \notin \text{mlb}'\{(y, 0), (z, 0)\}$. $\square$

Proof of Claim 6.3.5. f is clearly onto, so it remains to verify the four p-morphism conditions hold.

$\langle \text{sup} \rangle$ -forth Suppose $(u, i) = \sup'\{(v, j), (w, k)\}$. Then $(v, j), (w, k) \leq' (u, i)$, so Lemma 6.3.7 gives $v, w \leq u$. Let $a \in W$ satisfy $v, w \leq a$. By the definition of $\leq'$ we have $(v, j), (w, k) \leq' (a, 1)$, hence $(u, i) = \sup'\{(v, j), (w, k)\}$ implies $(u, i) \leq' (a, 1)$, and another application of Lemma 6.3.7 yields $u \leq a$.

$\langle \text{inf} \rangle$ -forth Suppose $(u, i) = \inf'\{(v, j), (w, k)\}$. From $(u, i) \leq' (v, j), (w, k)$, we get $u \leq v, w$ by Lemma 6.3.7. If $a \leq v, w$, then $(a, 0) \leq' (v, j), (w, k)$, so since $(u, i) = \inf'\{(v, j), (w, k)\}$ it follows that $(a, 0) \leq' (u, i)$ and therefore $a \leq u$. Thus $u = \inf\{v, w\}$.

$\langle \text{sup} \rangle$ -back Suppose $f(u, i) = \sup\{v, w\}$.

– If $i = 0$, then $(v, 0), (w, 0) \leq' (u, 0)$. If $(v, 0), (w, 0) \leq' (a, l)$ then Lemma 6.3.7 gives $v, w \leq a$, hence $u \leq a$ and clause 10a yields $(u, 0) \leq' (a, l)$.

– If $i = 1$ we get $(v, 1), (w, 1) \leq (u, 1)$. If $(v, 1), (w, 1) \leq' (a, l)$, then $l = 1$ implies $v, w \leq a$. So since $u = \sup\{v, w\}$ we get that $u \leq a$, hence $(u, 1) \leq' (a, 1)$.

If $l = 0$, then $(v, 1), (w, 1) \leq' (a, 0)$ implies $a \in A(y, z)$ and $v, w \in \downarrow x$. Since $u = \sup\{v, w\}$, and $v, w \leq x$, it follows that $u \leq x$, so $(u, 1) \leq' (a, 0)$.

$\langle \text{inf} \rangle$ -back Suppose $f(u, i) = \inf\{v, w\}$.

– If $i = 0$, then $(u, 0) \leq' (v, 0), (w, 0)$. If $(a, l) \leq' (v, 0), (w, 0)$, we get $a \leq v, w$ by an application of Lemma 6.3.7. So from $u = \inf\{v, w\}$ it follows that $a \leq u$. If $l = 0$, clause 10a gives $(a, 0) \leq (u, 0)$.

If $l = 1$, then $(a, 1) \leq' (v, 0), (w, 0)$ implies $a \in \downarrow x$ and $v, w \in A(y, z)$. Because $u = \inf\{v, w\}$ we have $u \in A(y, z)$, so $(a, 1) \leq' (u, 0)$.

– Assume $i = 1$, then $(u, 1) \leq' (v, 1), (w, 1)$. If $(a, l) \leq' (v, 1), (w, 1)$, then $a \leq v, w$, so $a \leq u$ and $(a, l) \leq' (u, 1)$. $\square$

This finalizes the proof of Lemma 6.3.2. $\square$

To prove that

$$TIL \supset TIL_{\min\text{-max}}$$

we need one more lemma showing that we can repair all max- and min-defects that a given poset may contain.

Lemma 6.3.8. Let $(W, \leq)$ be a poset. Then $(W, \leq)$ is the p-morphic image of a poset $(W', \leq')$ that contains no max- or min-defects.

Proof. Fix a countable set U disjoint from W . For convenience we assume that W is countable (the uncountable case can be handled by a standard transfinite recursion).

Enumerate all triples $(x, y, z) \in (W \cup U)^3$ and set $(W, \leq) = (W_0, \leq_0)$. Given $(W_n, \leq_n)$, let (x', y', z') be the least triple in the enumeration that is still a max- or min-defect in $(W_n, \leq_n)$.

- If it is a max-defect, apply Lemma 6.3.1 to obtain $(W_{n+1}, \leq_{n+1})$ such that $x' \notin \text{mub}_{\leq_{n+1}}\{y', z'\}$.
- If it is a min-defect, apply Lemma 6.3.2 to obtain $(W_{n+1}, \leq_{n+1})$ such that $x' \notin \text{mlb}_{\leq_{n+1}}\{y', z'\}$.

Clause (1) of the cited lemmas ensures that $|W_{n+1}| \leq \max\{\omega, |W_n|\}$, so new points can be chosen from U . Each application produces a p-morphism

$$f_{n+1} : (W_{n+1}, \leq_{n+1}) \rightarrow (W_n, \leq_n)$$

that acts as the identity on W_n (condition (3) in the cited lemmas). Since compositions of p-morphisms are p-morphisms, we can define

$$f_{n+1}^* := f_n^* \circ f_{n+1} : (W_{n+1}, \leq_{n+1}) \rightarrow (W_0, \leq_0) = (W, \leq).$$

Finally, set

$$(W_\omega, \leq_\omega) := \left(\bigcup_{n \in \mathbb{N}} W_n, \bigcup_{n \in \mathbb{N}} \leq_n \right)$$

and

$$f_\omega^* := \bigcup_{n \in \mathbb{N}} f_n^* : (W_\omega, \leq_\omega) \rightarrow (W_0, \leq_0) = (W, \leq).$$

We reason as follows:

1. $(W_\omega, \leq_\omega)$ is a poset. Each $\leq_{n+1}$ extends $\leq_n$ and agrees with it on W_n by clause (2) of Lemmas 6.3.1 and 6.3.2, so the union $\leq_\omega$ is again a partial order on W_ω .
2. f_ω^* is a surjective p-morphism. We first show that f_ω^* is a function. Let $x \in W_\omega$ and let n be least such that $x \in W_n$. For any $m \geq n$ f_m^* restricts to the identity on all W_k 's such that $k < m$, so $f_m^*(x) = f_{m-1}^*(x) = \dots = f_n^*(x)$. Thus the union $\bigcup_m f_m^*$ assigns a unique value to x , since f_n^* does, showing that f_ω^* is a well-defined function.

f_ω^* is onto since $f_1 \upharpoonright W = \text{Id}_W$ and $W \subseteq W_\omega$. Moreover, it follows that f_ω^* satisfies both $\langle \text{sup} \rangle$ - and $\langle \text{inf} \rangle$ -forth from condition (2) of Lemmas 6.3.1 and 6.3.2 and the fact that each f_n^* satisfies both forth conditions. For the back conditions we use that each f_n^* satisfies the back conditions, together with condition (3) of the before mentioned lemmas.

3. *No defects remain.* Every potential defect appears in the enumeration and is repaired at a certain stage in the recursive process. Once a defect is repaired, clause (2) and (5) of Lemmas 6.3.1 and 6.3.2 ensure that it stays repaired. $\square$

Theorem 6.3.9.

$$TIL = TIL_{\min\text{-max}}$$

Proof. The inclusion $TIL \subseteq TIL_{\min\text{-max}}$ was proven in Theorem 6.1.1. For the converse assume $\varphi \notin TIL$. Then there exists a poset model $\mathfrak{M} = (W, \leq, V)$ such that $\mathfrak{M} \not\models \varphi$.

By Lemma 6.3.8 there is a poset $(W', \leq')$ without defects and a surjective p-morphism

$$f : (W', \leq') \rightarrow (W, \leq).$$

Put

$$V'(p) := \{ w' \in W' \mid f(w') \in V(p) \}, \quad \mathfrak{M}' = (W', \leq', V').$$

Because p-morphisms preserve truth, $\mathfrak{M}' \not\models \varphi$. Since $(W', \leq')$ contains no max- or min-defects, the ordinary sup / inf semantics and the max / min semantics coincide on $\mathfrak{M}'$, so also $\mathfrak{M}' \not\models_M \varphi$. Hence $\varphi \notin TIL_{\min\text{-max}}$, showing $TIL_{\min\text{-max}} \subseteq TIL$. $\square$

Thus the logic of minimal upper bounds ('incomparable fusions') and maximal lower bounds ('incomparable common information') is the same as standard TIL under the supremum / infimum semantics.

7 TRANSLATING WPL TO TIL

So far the language of TIL has been interpreted on posets and preorders, structures in which suprema and infima need not exist. A natural next step (already explored by Wang and Wang in [23, 24]) is to move to lattices, where any two elements possess both a supremum and an infimum. In this chapter we follow this direction too.

Translations between logics are a way to reveal connections, bridging the worlds between different logics and this way enabling techniques and results to be imported and exported. Their importance was recently emphasized in a LIRA seminar by van Benthem [6]. Classic examples include Gödel’s embedding of intuitionistic logic into S4 [13] (see also [12] for a textbook proof) and the standard translation of modal formulas into first-order logic [9] and, an example more related to the subject of this thesis, the translation of truth maker logic into modal information logic [5]. Here we study the relationship between TIL (over lattices) and *weak positive logic* (WPL). The latter has the same language as positive logic (namely the negation- and implication-free fragment of classical propositional logic), but it does not, in general, satisfy the distributivity axiom.

Building on the partial translation sketched in [7], we provide a full, faithful translation from the $\perp$ -free fragment of WPL into an extended version of TIL and discuss the additional expressivity gained through this enrichment.

7.1 WPL AND TIL ON LATTICES

We begin by spelling out tense information logic over lattices.

7.1.1 TIL on lattices

Definition 7.1.1. A *lattice model* for the language $\mathcal{L}_T$ is a poset (X, R) in which for every pair of elements $x, y \in X$ both a supremum (alternatively called *join*) and an infimum (alternatively called *meet*) exist. We write

$$x \vee y := \sup\{x, y\}, \quad x \wedge y := \inf\{x, y\}.$$

Equivalently, a lattice can be described as a triple $(X, \wedge, \vee)$, where $\wedge, \vee : X^2 \rightarrow X$ are operations satisfying commutativity, associativity, idempotence, and absorption. A lattice is *bounded* if there are elements $0, 1 \in X$ such that $0 \leq a$ and $a \leq 1$ for all $a \in X$. We then write $(X, 0, 1, \wedge, \vee)$.

Every lattice $(X, 0, 1, \wedge, \vee)$ induces a partial order $\leq$ defined by

$$x \leq y \iff x \wedge y = x \quad (\text{equivalently, } x \vee y = y),$$

which is also the way to show that the two definitions are equivalent.

A notion we will use extensively in this chapter is that of filters on lattices.

Definition 7.1.2. A *filter* on a lattice $(X, 0, 1, \wedge, \vee)$ is a subset $F \subseteq X$ that is

- non-empty,
- upward closed: $x \in F$ and $x \leq y$ imply $y \in F$,
- closed under meet: $x, y \in F$ implies $x \wedge y \in F$.

The semantics for TIL on lattices is identical to the semantics of TIL on posets, as presented in Definition 2.1.4. For notational convenience, we write

$$\mathfrak{M}, x \Vdash_L \varphi$$

to mean that the formula $\varphi \in \mathcal{L}_T$ is true at the world $x \in X$ of the lattice model $\mathfrak{M} = (X, 1, 0, \wedge, \vee, V)$.

Definition 7.1.3 (TIL on lattices).

$$TIL_{\text{lat}} := \{ \varphi \in \mathcal{L}_T \mid \text{for every lattice model } \mathfrak{M} = (X, 1, 0, \wedge, \vee, V) \text{ and every } x \in X, \mathfrak{M}, x \Vdash_L \varphi \}$$

As mentioned before, Wang and Wang [23, 24] give a finite axiomatization of TIL over lattices for a hybrid language with supremum and infimum operators as well as nominals. Since the class of lattices is first-order definable [23, Definition 7], and the formulas of TIL_{lat} can be translated into first-order logic through the standard translation [10, Definition 2.45], it follows that TIL_{lat} is recursively enumerable [10, Lemma 6.32], hence axiomatisable.

What remains open is an axiomatization in our *non-hybrid* language $\mathcal{L}_T$: Wang and Wang show that uniqueness of joins / meets is not definable in the non-hybrid language [24, Theorem 25], which motivates their move to the hybrid setting. We therefore state the following open problem.

Open Problem 7.1.1 (Axiomatise non-hybrid TIL over lattices). Find a sound and complete axiomatization of TIL over lattices without hybrid features. Moreover, can we find a *finite* axiomatization or is it not finitely axiomatizable?

7.1.2 WPL

As mentioned before, WPL is the name for positive logic that is not necessarily distributive. [7] studies this logic and it is from this paper that we take the following definitions and results.

Definition 7.1.4. Let $\mathcal{L}_W$ be the language of weak positive logic, generated by the grammar

$$\varphi ::= p \mid \top \mid \perp \mid \varphi \wedge \psi \mid \varphi \vee \psi$$

Logics based on $\mathcal{L}_W$ are defined as sets of consequence pairs, i.e. expressions of the form $\varphi \sqsubseteq \psi$ where φ and ψ are formulas in $\mathcal{L}_W$.

Definition 7.1.5. Let **WPL** be the smallest set of consequence pairs containing the following axioms

$$\begin{array}{lll}
p \leq \top & \perp \leq p & \\
p \leq p & \frac{p \leq q \quad q \leq r}{p \leq r} & \\
p \wedge q \leq p & p \wedge q \leq q & \frac{r \leq p \quad r \leq q}{r \leq p \wedge q} \\
p \leq p \vee q & q \leq p \vee q & \frac{p \leq r \quad q \leq r}{p \vee q \leq r}
\end{array}$$

and closed under uniform substitution.

Algebraically, this is the positive fragment of bounded lattice logic: the connectives are interpreted as the lattice operations. The calculus above is sound and complete for the algebraic (lattice) semantics [7, Thm. 3.5].

The frame semantics for this logic are defined as follows:

Definition 7.1.6. An *L-model* is a bounded lattice $(X, 1, 0, \wedge, \vee)$ with a valuation

$$V : \mathbf{P} \longrightarrow \mathcal{F}(X, 1, 0, \wedge, \vee)^{11},$$

that assigns to each propositional variable $p \in \mathbf{P}$ a filter of $(X, 1, 0, \wedge, \vee)$.

Definition 7.1.7. The interpretation of an $\mathcal{L}_W$ -formula φ at a state x in an L-model $\mathfrak{M} = (X, 1, 0, \wedge, \vee, V)$ (which we will denote by $\Vdash_W$ to distinguish it from the semantics of TIL) is defined recursively as follows:

$$\begin{array}{ll}
\mathfrak{M}, x \Vdash_W \top & \text{always} \\
\mathfrak{M}, x \Vdash_W \perp & \text{iff } x = 1 \\
\mathfrak{M}, x \Vdash_W p & \text{iff } x \in V(p) \\
\mathfrak{M}, x \Vdash_W \varphi \wedge \psi & \text{iff } \mathfrak{M}, x \Vdash_W \varphi \text{ and } \mathfrak{M}, x \Vdash_W \psi \\
\mathfrak{M}, x \Vdash_W \varphi \vee \psi & \text{iff } \exists y, z \in X \text{ s.t. } \mathfrak{M}, y \Vdash_W \varphi, \mathfrak{M}, z \Vdash_W \psi \text{ and } y \wedge z \leq x
\end{array}$$

With these semantics at hand we can define weak positive logic on lattices:

Definition 7.1.8.

$$WPL_{\text{lat}} := \left\{ \varphi \in \mathcal{L}_W \mid \text{for every L-model } \mathfrak{M} = (X, 1, 0, \wedge, \vee, V) \text{ and every } x \in X : \right.$$

$$\left. \mathfrak{M}, x \Vdash_W \varphi \right\}$$

Soundness and completeness of **WPL** for the frame semantics follow from [7, Thm. 3.22].

¹¹ $\mathcal{F}(X, 1, 0, \wedge, \vee)$ denotes the set of all filters on the lattice $(X, 1, 0, \wedge, \vee)$.

7.2 TRANSLATING WPL TO TIL

Bezhanishvili et al. sketch in [7] a partial translation

$$T : \mathcal{L}_W \rightarrow \mathcal{L}_T.$$

The translation they propose is motivated by the close match between the semantics of disjunction in WPL and the semantics of the $\langle \text{inf} \rangle$ -operator in TIL. The only difference is that, for $\varphi \vee \psi$ to hold at x in an L-model, it suffices that the meet of the worlds that witness φ and ψ lies *below* x . Since the past-looking diamond is definable in TIL, let

$$T(\varphi \vee \psi) := P(\langle \text{inf} \rangle T(\varphi) T(\psi)).$$

This section will be devoted to defining a suitable translation function T and proving that for every consequence pair $\varphi \trianglelefteq \psi$:

$$\varphi \Vdash_W \psi \iff T(\varphi) \Vdash_T T(\psi). \quad (11)$$

To that end we first establish the following lemma by structural induction over φ :

$$(L, \rho V), x \Vdash_W \varphi \iff (L, V), x \Vdash_T T(\varphi) \quad (12)$$

for all lattice models (L, V) ¹², all $x \in L$, and all $\varphi \in \mathcal{L}_W$, where

$$\rho : \left(\mathbf{P} \xrightarrow{V} \mathcal{P}(X) \right) \mapsto \left(\mathbf{P} \xrightarrow{\rho V} \mathcal{F}(X, 1, 0, \wedge, \vee) \right). \quad (13)$$

is a function that sends a TIL subset-valuation to a WPL filter-valuation.

Concretely, we want to find a schema $\chi(a)$ such that

$$\rho V(p) := \{ x \in X \mid (L, V), x \Vdash_T \chi(p) \} \in \mathcal{F}(L) \quad \text{for all propositional } p.$$

Defining T on propositional variables by $T(p) := \chi(p)$ then yields the base case of (12):

$$(L, \rho V), x \Vdash_W p \iff x \in \rho V(p) \iff (L, V), x \Vdash_T \chi(p) \iff (L, V), x \Vdash_T T(p).$$

In accordance with the definition of filters, we must have $\rho V(p)$ closed under *finite* meets. However, $\mathcal{L}_T$ can only speak about a *binary* meet witness via $\langle \text{inf} \rangle$. To capture arbitrary finite meets inside the logic, we therefore enrich the language with two modalities:

$$\begin{aligned} \mathfrak{M}, x \Vdash_T \langle \text{inf}^* \rangle \varphi \psi \quad \text{iff} \quad & \exists y_1, \dots, y_n, z_1, \dots, z_m \text{ for } m, n \geq 1 \text{ such that} \\ & \forall y_i : y_i \Vdash_T \varphi, \forall z_j : z_j \Vdash_T \psi, \\ & \text{and } x = \text{inf}\{y_1, \dots, y_n, z_1, \dots, z_m\}. \end{aligned}$$

$$\begin{aligned} \mathfrak{M}, x \Vdash_T \langle \text{sup}^* \rangle \varphi \psi \quad \text{iff} \quad & \exists y_1, \dots, y_n, z_1, \dots, z_m \text{ for } m, n \geq 1 \text{ such that} \\ & \forall y_i : y_i \Vdash_T \varphi, \forall z_j : z_j \Vdash_T \psi, \\ & \text{and } x = \text{sup}\{y_1, \dots, y_n, z_1, \dots, z_m\}. \end{aligned}$$

¹² Here $L = (X, 1, 0, \wedge, \vee)$

We write $\mathcal{L}_{T^*}$ for this extension of $\mathcal{L}_T$.

Remark 7.2.1. For proving (12), it would actually suffice to add a *unary* finite-meet operator

$$\mathfrak{M}, x \Vdash_T \langle i \rangle \varphi \quad \text{iff} \quad \exists y_1, \dots, y_n \text{ for } n \geq 1 \text{ such that } \forall y_i : y_i \Vdash_T \varphi \\ \text{and } x = \inf\{y_1, \dots, y_n\}.$$

Also the “finite-join” operator $\langle \sup^* \rangle$ is not required. We nevertheless introduce both binary operators because

- the unary versions are immediate special cases:

$$\mathfrak{M}, x \Vdash_T \langle i \rangle \varphi \quad \Longleftrightarrow \quad \mathfrak{M}, x \Vdash_T \langle \inf^* \rangle \varphi \varphi,$$

and

- we want to keep studying the structures we interpret the language on (later we will also interpret it on posets again) in a symmetric way.

First, we show that these operators are not definable in $\mathcal{L}_T$.

Proposition 7.2.2. $\mathcal{L}_{T^*}$ is strictly more expressive on lattices than $\mathcal{L}_T$.

Proof. Define recursively a function $B: \mathcal{L}_T \rightarrow \mathbb{N}$ measuring an *upper bound* for the number of distinct witnesses a formula ever needs in any model.

Definition 7.2.1.

$$\begin{aligned} B(p) &= 1, \\ B(\neg \varphi) &= B(\varphi), \\ B(\varphi \wedge \psi) &= B(\varphi) + B(\psi), \\ B(\langle \sup \rangle(\varphi, \psi)) &= B(\varphi) + B(\psi), \\ B(\langle \inf \rangle(\varphi, \psi)) &= B(\varphi) + B(\psi). \end{aligned}$$

Assume, for contradiction, that the operator $\langle \inf^* \rangle$ is definable in $\mathcal{L}_T$. This means that there exists a schema $\beta(p, q) \in \mathcal{L}_T$, such that

$$\mathfrak{M}, x \Vdash_T \langle \inf^* \rangle \varphi \psi \quad \Longleftrightarrow \quad \mathfrak{M}, x \Vdash_T \beta(\varphi, \psi) \tag{14}$$

holds for all models $\mathfrak{M}, x \in \mathfrak{M}$ and $\varphi, \psi \in \mathcal{L}_{T^*}$. Let $n := B(\beta(p, p))$ and consider the lattice

$$X := \mathcal{P}(\{0, \dots, n\}), \quad \vee := \cup, \quad \wedge := \cap, \quad 0 := \emptyset, \quad 1 := \{0, \dots, n\},$$

together with the valuation

$$V(p) := \{\{1, \dots, n\}, \{0, 2, \dots, n\}, \dots, \{0, \dots, n-1\}\}.$$

At the point $0 = \emptyset$ we have $\mathfrak{M}, 0 \Vdash_T \langle \inf^* \rangle pp$, witnessed by the set

$$\{1, \dots, n\}, \{0, 2, \dots, n\}, \dots, \{0, \dots, n-1\}$$

of size $n+1$. No set of witnesses of size less than or equal to n suffices. Because $B(\beta(p, p)) = n$, the formula $\beta(p, p)$ cannot be satisfied at 0, contradicting (14). Hence $\langle \inf^* \rangle$ (and, by a symmetrical argument $\langle \sup^* \rangle$) is not expressible in $\mathcal{L}_T$. $\square$

Definition 7.2.2. Given a lattice model $\mathfrak{M} = (L, V)$, let

$$\rho V(p) := \{x \in X \mid \mathfrak{M}, x \Vdash_T P \langle \inf^* \rangle pp \vee \neg F P p\}$$

Claim 7.2.3. For every valuation V and propositional variable p , the set $\rho V(p)$ is a filter on L .

Proof. Let $\mathfrak{M} = (L, V)$.

(Non-emptiness) If $V(p) \neq \emptyset$, then $\mathfrak{M}, x \Vdash_T p$ implies $\mathfrak{M}, x \Vdash_T \langle \inf^* \rangle pp$, so $\mathfrak{M}, x \Vdash_T P \langle \inf^* \rangle pp$, thus $x \in \rho V(p)$ and $\rho V(p) \neq \emptyset$.

If $V(p) = \emptyset$, then $\mathfrak{M}, x \not\Vdash_T p$ for every $x \in X$. It follows that $\mathfrak{M}, 1 \Vdash_T \neg F P p$, so $1 \in \rho V(p)$, thus $\rho V(p) \neq \emptyset$.

(Upward-closedness) Assume $x \Vdash_T P \langle \inf^* \rangle pp \vee \neg F P p$ and $y \geq x$. Then $x \Vdash_T P \langle \inf^* \rangle pp$ implies there is $u \leq x$ such that $u \Vdash_T \langle \inf^* \rangle pp$. By transitivity it follows that $u \leq y$, so $y \Vdash_T P \langle \inf^* \rangle pp$.

If on the other hand $x \Vdash_T \neg F P p$ this implies $V(p) = \emptyset$, thus $\rho V(p) = X$. It follows that $y \in \rho V(p)$.

(Closed under meet) Assume $x \Vdash_T P \langle \inf^* \rangle pp$ and $y \Vdash_T P \langle \inf^* \rangle pp$. It follows that there exists $u \leq x$ such that $u \Vdash_T \langle \inf^* \rangle pp$ and $v \leq y$ such that $v \Vdash_T \langle \inf^* \rangle pp$, thus there are $y_1, \dots, y_n$ and $z_1, \dots, z_m$ such that $y_i \Vdash_T p$, $z_i \Vdash_T p$, $u = \inf\{y_1, \dots, y_n\}$ and $v = \inf\{z_1, \dots, z_m\}$. Then $u \wedge v = \inf\{y_1, \dots, y_n, z_1, \dots, z_m\}$, so $u \wedge v \Vdash_T \langle \inf^* \rangle pp$. From $u \leq x$ and $v \leq y$ it follows that $u \wedge v \leq x \wedge y$, so $x \wedge y \Vdash_T P \langle \inf^* \rangle pp$.

If $x \Vdash_T \neg F P p$ or $y \Vdash_T \neg F P p$, we get that $V(p) = \emptyset$, so $\rho V(p) = X$, so $x \wedge y \in \rho V(p)$. $\square$

Remark 7.2.4. Definition 7.2.2 is not the only way to map subset-valuations to filter-valuations (e.g. the trivial choice $\rho V(p) = X = \{x \mid \mathfrak{M}, x \Vdash_T \top\}$ also yields a filter). That we use Definition 7.2.2 is motivated as follows:

First of all, note that if $V(p)$ is a filter (thus non-empty), the first part of the disjunction (i.e. $P \langle \inf^* \rangle pp$) guarantees that $\rho V(p) = V(p)$. More generally, if $V(p)$ is non-empty, then $\{x \in X \mid \mathfrak{M}, x \Vdash_T P \langle \inf^* \rangle pp\}$ is the least filter containing $V(p)$:

Upward-closedness and closure under meet follow from similar reasoning as in the proof of Claim 7.2.3. Moreover, let H be a filter containing $V(p)$ and let $x \in X$ be such that $\mathfrak{M}, x \Vdash_T P \langle \inf^* \rangle pp$. If $\mathfrak{M}, x \Vdash_T P \langle \inf^* \rangle pp$, it follows that there is a world $u \leq x$ such that $\mathfrak{M}, u \Vdash_T \langle \inf^* \rangle pp$. Then $\mathfrak{M}, u \Vdash_T \langle \inf^* \rangle pp$ iff there are $u_1, \dots, u_n$ such that $u_i \Vdash_T p$ and $u = \sup\{u_1, \dots, u_n\}$. Since $V(p) \subseteq H$, it follows that $u_1, \dots, u_n \in H$. By closure under meet and upward closedness, it follows that $x \in H$, so $\{x \in X \mid \mathfrak{M}, x \Vdash_T P \langle \inf^* \rangle pp\} \subseteq H$.

The second part of the conjunct in the definition of $T(p)$ (i.e. $\neg F P p$) is motivated to guarantee that $\rho V(p)$ is never empty, which is required for it to be a filter. Assume $V(p) = \emptyset$, this implies that there is also no $x \in X$ such that $\mathfrak{M}, x \Vdash_T P \langle \inf^* \rangle pp$, so without the second part we would get that $\rho V(p) = \emptyset$. But since $V(p) = \emptyset$, and since $\mathfrak{M}$ is a model based on a lattice, it follows that for all $x \in X$ it holds that $\mathfrak{M}, x \Vdash_T \neg F P p$, so $\rho V(p) = X$. Thus $\rho V(p)$ is non-empty.

To give a full translation between $\mathcal{L}_W$ and $\mathcal{L}_T$, every formula needs to be translated. We already reasoned how disjunction and propositional variables should be translated. It turns

out that it is not evident how to translate the falsum symbol of $\mathcal{L}_W$. Since $\mathfrak{M}, x \Vdash_W \perp \iff x = 1$, a translation into TIL requires a formula $\varphi_1 \in \mathcal{L}_T$ such that

$$\mathfrak{M}, x \Vdash_T \varphi_1 \iff x = 1. \quad (15)$$

Such a formula does not exist, which we will show by a bisimulation argument.

Definition 7.2.3 (Bisimulation). Let $\mathfrak{M}$ and $\mathfrak{M}'$ be two lattice models. A non-empty binary relation $Z \subseteq X \times X'$ is a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$ iff

- xZx' implies that x and x' satisfy the same propositional formulas, and
- Z satisfies the back- and forth-conditions as stipulated in 6.3.3.

Proposition 7.2.5. There is no formula $\varphi_1 \in \mathcal{L}_T$ satisfying (15).

Proof. Consider the following two lattice models of TIL:



Set $V(p) := \{1\}$ and $V'(p) := \{1', w'\}$. The relation $Z := \{(1, 1'), (1, w'), (0, 0')\}$ is a bisimulation as defined in 7.2.3. It follows that 1 and w' satisfy the same $\mathcal{L}_T$ formulas. But 1 is the top element and w' is not, so the top element is undefinable in $\mathcal{L}_T$. Hence (15) is impossible. $\square$

This makes defining a complete translation difficult. We could add a constant for $\perp$ and shift to hybrid logic, but we will not explore this direction in this thesis. Instead, we will restrict the translation to the falsum-free fragment of $\mathcal{L}_W$, which we denote by $\mathcal{L}_W^-$.

Definition 7.2.4. Define $T : \mathcal{L}_W^- \rightarrow \mathcal{L}_T$ by

$$\begin{aligned} T(p) &= P(\langle \text{inf}^* \rangle p) \vee \neg \text{FP } p, \\ T(\top) &= \neg \perp, \\ T(\varphi \wedge \psi) &= T(\varphi) \wedge T(\psi), \\ T(\varphi \vee \psi) &= P(\langle \text{inf} \rangle T(\varphi) T(\psi)). \end{aligned}$$

We will prove that if we leave $\perp$ out of the picture, we have a full and faithful translation. This is a corollary of the following lemma, which captures (12).

Lemma 7.2.6. Let $T : \mathcal{L}_W^- \rightarrow \mathcal{L}_T$ be as defined above, then for all $\varphi \in \mathcal{L}_W^-$ and all lattice models $\mathfrak{M} = (L, V)$ (where L denotes a lattice $(X, 1, 0, \wedge, \vee)$):

$$(L, \rho V), x \Vdash_W \varphi \iff (L, V), x \Vdash_T T(\varphi)$$

Proof. We give a proof by induction on the complexity of φ .

Base case: $(L, \rho V), x \Vdash_W p$ holds iff

$$x \in \rho V(p) = \{x \in X \mid \mathfrak{M}, x \Vdash_T P(\langle \text{inf}^* \rangle p p) \vee \neg \text{FP } p\}.$$

Thus $x \in \rho V(p)$ iff $\mathfrak{M}, x \Vdash_T T(p)$, which is what we needed to show.

The case $\varphi := \top$ follows directly from the semantics and the definition of $\top$.

Inductive case: The case where $\varphi := \psi_1 \wedge \psi_2$ follows directly by induction hypothesis.

Assume $\varphi := \psi_1 \vee \psi_2$. Then $(L, \rho V), x \Vdash_W \psi_1 \vee \psi_2$ implies

$$\exists y, z \in X \text{ such that } (L, \rho V), y \Vdash_W \psi_1, (L, \rho V), z \Vdash_W \psi_2 \text{ and } y \wedge z \leq x.$$

It follows by induction hypothesis that $(L, V), y \Vdash_T T(\psi_1)$ and $(L, V), z \Vdash_T T(\psi_2)$, so since $y \wedge z \leq x$ it follows that

$$(L, V), x \Vdash_T P(\langle \text{inf} \rangle T(\psi_1) T(\psi_2)) = T(\psi_1 \vee \psi_2)$$

The converse direction follows in a similar way. $\square$

Theorem 7.2.7. For all consequence pairs $\varphi \leq \psi$ with $\varphi, \psi \in \mathcal{L}_W^-$:

$$\varphi \Vdash_W \psi \iff T(\varphi) \Vdash_T T(\psi)$$

Proof. Let $\varphi \leq \psi$ be an arbitrary consequence pair with $\varphi, \psi \in \mathcal{L}_W^-$ and assume $\varphi \Vdash_W \psi$. Let $\mathfrak{M} = (L, V)$ be an arbitrary lattice model and let $x \in X$ be arbitrary. Assume $\mathfrak{M}, x \Vdash_T T(\varphi)$. Then ρV is a filter valuation on L and by an application of Lemma 7.2.6 it follows that $(L, \rho V), x \Vdash_W \varphi$, so by assumption $(L, \rho V), x \Vdash_W \psi$. Applying Lemma 7.2.6 again yields $(L, V), x \Vdash_T T(\psi)$.

Conversely, assume $T(\varphi) \Vdash_T T(\psi)$, let $\mathfrak{M} = (L, V)$ be an arbitrary L-model, let $x \in X$ be arbitrary and assume $(L, V), x \Vdash_W \varphi$. Then (L, V) is also a lattice model and $\rho V = V$ since V is already a filter valuation. By an application of Lemma 7.2.6 it follows that $(L, V), x \Vdash_T T(\varphi)$, so by assumption $(L, V), x \Vdash_T T(\psi)$. Applying Lemma 7.2.6 again yields $(L, \rho V), x \Vdash_W \psi$ and since $\rho V = V$ we get the desired result. $\square$

7.3 TRANSLATING WPL TO TIL WITH $\perp$

Even though we showed that the top element is not definable in the language of TIL, we can still give a full translation of WPL (including $\perp$) into TIL by making the translation relative to a consequence pair. To this end, let $\varphi \leq \psi$ be an arbitrary consequence pair with $\varphi, \psi \in \mathcal{L}_W$. We define a translation function $T_{\varphi, \psi} : \mathcal{L}_W \rightarrow \mathcal{L}_T$ as follows.

Definition 7.3.1. For propositional letters $p, \top, \wedge$ and $\vee$, $T_{\varphi, \psi}$ is defined in the same way as T of Definition 7.2.4. Additionally, let $q \in \mathbf{P}$ be such that $q \notin \text{Prop}(\varphi, \psi)$, we define

$$T_{\varphi, \psi}(\perp) = \bigwedge_{p \in \text{Prop}(\varphi, \psi) \cup \{q\}} P \langle \text{inf}^* \rangle (\neg P p \rightarrow \neg \text{FP } p, \neg P p \rightarrow \neg \text{FP } p),$$

where $\text{Prop}(\varphi, \psi)$ is the set of all propositional letters that occur in φ and ψ .

We want to show

$$T_{\varphi,\psi}(\varphi) \Vdash_T T_{\varphi,\psi}(\psi) \iff \varphi \Vdash_W \psi.$$

For the right-to-left direction we argue by contraposition. Suppose there is a lattice model (L, V) and $x \in L$ such that $(L, V), x \Vdash_T T_{\varphi,\psi}(\varphi)$ yet $(L, V), x \not\Vdash_T T_{\varphi,\psi}(\psi)$. We now construct an L-model from (L, V) as follows. Because $\perp$ is true only at the top element of an L-model, we collapse all points of L that satisfy $T_{\varphi,\psi}(\perp)$ into a single class. Concretely, let $\sim$ be the equivalence relation that identifies exactly the points satisfying $T_{\varphi,\psi}(\perp)$ and leaves all other points distinct, and write $|y|$ for the $\sim$ -class of $y \in L$. On the set $\{|y| \mid y \in L\}$ we then define meet and join operations $\wedge_\sim, \vee_\sim$ (see below) and verify that they make this set into a lattice whose top element is the class of the $T_{\varphi,\psi}(\perp)$ -points. Defining an appropriate filter valuation on this lattice gives us an L-model in which the $\sim$ -class of x satisfies φ and refutes ψ , as required.

Let (L, V) be a lattice model. Define the following equivalence relation on L :

$$x \sim y \iff \begin{cases} x \Vdash T_{\varphi,\psi}(\perp) \text{ and } y \Vdash T_{\varphi,\psi}(\perp), \\ \text{or} \\ x \not\Vdash T_{\varphi,\psi}(\perp) \text{ and } x = y. \end{cases}$$

On the set $\{|x| \mid x \in L\}$ we define the following meet and join operations and claim that it defines a lattice:

- $|x| \vee_\sim |y| = |x \vee y|$
- $|x| \wedge_\sim |y| = \begin{cases} |x \wedge y|, & \text{if } |x| \neq |1| \neq |y|, \\ |x|, & \text{if } |y| = |1|, \\ |y|, & \text{if } |x| = |1|. \end{cases}$

The join is well-defined since $\llbracket T_{\varphi,\psi}(\perp) \rrbracket$ is an upset and the meet is well-defined by definition.

Claim 7.3.1. $(L/\sim, \wedge_\sim, \vee_\sim)$ is a lattice with $|1|$ as top element and $|0|$ as bottom element.

Proof. We first show that $|1|$ is the top element and $|0|$ is the bottom element:

$$\begin{aligned} |x| \vee_\sim |1| &= |1|, & |x| \vee_\sim |0| &= |x| && \text{by definition of } \vee_\sim, \\ |x| \wedge_\sim |1| &= |x|, & |x| \wedge_\sim |0| &= |0| && \text{by definition of } \wedge_\sim. \end{aligned}$$

Showing that the lattice laws hold is straightforward. The only more involved cases are the two absorption rules:

- $|x| = |x| \vee_\sim (|x| \wedge_\sim |y|)$
 - holds if $|x| \neq |1| \neq |y|$;
 - if $|x| = |1|$, then $|x| \wedge_\sim |y| = |y|$ and $|1| \vee_\sim |y| = |1| = |x|$;
 - if $|y| = |1|$, then $|x| \wedge_\sim |y| = |x|$ and $|x| \vee_\sim |x| = |x|$.

- $|x| = |x| \wedge_{\sim} (|x| \vee_{\sim} |y|)$
 - holds if $|x| \neq |1| \neq |y|$;
 - if $|x| = |1|$, then $|x| \wedge_{\sim} (|y| \vee_{\sim} |x|) = |y| \vee_{\sim} |x| = |x|$;
 - if $|y| = |1|$, then $|x| \vee_{\sim} |y| = |y|$ and $|x| \wedge_{\sim} |y| = |x|$. $\square$

Denote by $\text{Sub}(\varphi, \psi)$ the set of subformulas of φ and ψ . We will need the following auxiliary lemma for later proofs.

Lemma 7.3.2. Let $\chi \in \text{Sub}(\varphi, \psi)$ be arbitrary. Then

$$y \Vdash_T T_{\varphi, \psi}(\perp) \Rightarrow y \Vdash_T T_{\varphi, \psi}(\chi)$$

Proof. We give a proof by structural induction over χ .

Base case: The case of $\chi := \perp$ is trivial.

If $\chi := \top$, then $T_{\varphi, \psi}(\top) = \neg\perp$ and the statement follows by the semantics of $\perp$ in TIL.

For $\chi := r$ we need to show that

$$y \Vdash_T \bigwedge_{p \in \text{Prop}(\varphi, \psi) \cup \{q\}} P \langle \text{inf}^* \rangle (\neg P p \rightarrow \neg F P p, \neg P p \rightarrow \neg F P p) \Rightarrow y \Vdash_T P \langle \text{inf}^* \rangle r r \vee \neg F P r.$$

Since $r \in \text{Prop}(\varphi, \psi)$, we can assume that $y \Vdash_T P \langle \text{inf}^* \rangle (\neg P r \rightarrow \neg F P r, \neg P r \rightarrow \neg F P r)$. If $V(r) = \emptyset$ it follows that $y \Vdash_T \neg F P r$, so $y \Vdash_T T_{\varphi, \psi}(r)$. If $V(r) \neq \emptyset$ we reason as follows. $y \Vdash_T P \langle \text{inf}^* \rangle (\neg P r \rightarrow \neg F P r, \neg P r \rightarrow \neg F P r)$ implies that there exist $y_1, \dots, y_n \in \llbracket \neg P r \rightarrow \neg F P r \rrbracket$ such that $\text{inf}\{y_1, \dots, y_n\} \leq y$. If there exist $i \in \{1, \dots, n\}$ such that $y_i \Vdash_T \neg P r$, it follows that $y_i \Vdash_T \neg F P r$, so since L is a lattice it follows that $V(r) = \emptyset$, which is in contradiction with our assumption.

It follows that $y_i \Vdash_T P r$ for all $i \in \{1, \dots, n\}$. This implies that there exist $y'_1, \dots, y'_n$ such that $y'_i \leq y_i$ and $y'_i \Vdash_T r$ for all $i \in \{1, \dots, n\}$. Since $\text{inf}\{y'_1, \dots, y'_n\} \leq \text{inf}\{y_1, \dots, y_n\} \leq y$ it follows that $y \Vdash_T T_{\varphi, \psi}(r)$.

Inductive cases: The case where $\chi := \chi_1 \wedge \chi_2$ follows from the induction hypothesis.

Assume $\chi := \chi_1 \vee \chi_2$. Since $y \Vdash_T T_{\varphi, \psi}(\perp)$, it follows by the induction hypothesis that $y \Vdash_T T_{\varphi, \psi}(\chi_1)$ and $y \Vdash_T T_{\varphi, \psi}(\chi_2)$. Since $y \leq y$ and $y = \text{inf}\{y, y\}$ it follows that $y \Vdash_T P \langle \text{inf} \rangle T_{\varphi, \psi}(\chi_1) T_{\varphi, \psi}(\chi_2) = T_{\varphi, \psi}(\chi_1 \vee \chi_2)$. $\square$

We now consider the lattice $(L/\sim, \wedge_{\sim}, \vee_{\sim}, |1|, |0|)$ as a model of WPL. We define the following valuation on $(L/\sim, \wedge_{\sim}, \vee_{\sim}, |1|, |0|)$:

$$\rho' V(p) = \{ |x| \in L/\sim \mid (L, V), x \Vdash_T T_{\varphi, \psi}(p) \} \quad (16)$$

Remark 7.3.3. Note that the definition of $\rho' V$ might not be well-defined if $V(r) \neq \emptyset$ for any $r \notin \text{Prop}(\varphi, \psi)$. We prove that in that case we can define an alternative valuation V' that evaluates all subformulas of φ and ψ the same as V , which is what we need for our purposes.

Proposition 7.3.4. Let L be a lattice and let V and V' be subset valuations on L . If V and V' agree on all propositional letters in φ , then for any subformula χ of φ and any $x \in L$ it holds that

$$(L, V), x \Vdash_T \chi \iff (L, V'), x \Vdash_T \chi$$

Proof. By structural induction over χ . □

We argue that the definition of (16) is a well-defined filter valuation. For any $p \in \text{Prop}(\varphi, \psi)$ it follows from Lemma 7.3.2 that for all $x \in \llbracket T_{\varphi, \psi}(\perp) \rrbracket$ and for all $p \in \text{Prop}(\varphi, \psi)$ it is the case that $x \Vdash T_{\varphi, \psi}(p)$.

For any $p \notin \text{Prop}(\varphi, \psi)$, we may assume by Remark 7.3.3 that $V(p) = \emptyset$. This implies that $(L, V), x \Vdash \neg \text{FP } p$, so $(L, V), x \Vdash T_{\varphi, \psi}(p)$ for all $x \in L$. It follows that $x, y \in \llbracket T_{\varphi, \psi}(\perp) \rrbracket$ implies that

$$(L, V), x \Vdash T_{\varphi, \psi}(p) \iff (L, V), y \Vdash T_{\varphi, \psi}(p)$$

for any $p \in \mathbf{P}$.

It follows from similar reasoning as in Claim 7.2.3 that $\rho' V$ is a filter valuation.

We can now prove the following lemma.

Lemma 7.3.5. For all $\chi \in \text{Sub}(\varphi, \psi)$

$$(L, V), y \Vdash_T T_{\varphi, \psi}(\chi) \iff (L/\sim, \rho' V), |y| \Vdash_W \chi$$

Proof. We give a proof by induction on the complexity of χ .

Base case: The case $\chi := \top$ follows by the semantics.

If $\chi := \perp$, then $(L, V), y \Vdash_T T_{\varphi, \psi}(\perp)$ iff $|y| = |1|$ iff $(L/\sim, \rho' V), |y| \Vdash_W \perp$ by the semantics of WPL.

For $\chi := p$ the claim follows by definition of $\rho' V$.

Inductive case: The case where $\chi := \chi_1 \wedge \chi_2$ follows from the induction hypothesis.

Assume $\chi := \chi_1 \vee \chi_2$. For the left-to-right direction, assume $(L, V), y \Vdash_T T_{\varphi, \psi}(\chi_1 \vee \chi_2)$. Since $T_{\varphi, \psi}(\chi_1 \vee \chi_2) = \mathbf{P} \langle \inf \rangle T_{\varphi, \psi}(\chi_1) T_{\varphi, \psi}(\chi_2)$ this means there exist y_1 and y_2 such that

$$(L, V), y_1 \Vdash_T T_{\varphi, \psi}(\chi_1), (L, V), y_2 \Vdash_T T_{\varphi, \psi}(\chi_2) \text{ and } y_1 \wedge y_2 \leq y.$$

By induction hypothesis we get that

$$(L/\sim, \rho' V), |y_1| \Vdash_W \chi_1 \text{ and } (L/\sim, \rho' V), |y_2| \Vdash_W \chi_2.$$

The claim would follow if we showed that $|y_1| \wedge_{\sim} |y_2| \leq_{\sim}^{13} |y|$, i.e. $(|y_1| \wedge_{\sim} |y_2|) \wedge_{\sim} |y| = |y_1| \wedge_{\sim} |y_2|$. We distinguish cases:

- If $|y_1| \neq |1| \neq |y_2|$, then $|y_1| \wedge_{\sim} |y_2| = |y_1 \wedge y_2|$. Assume $|y| = |1|$, then $|y_1 \wedge y_2| \wedge_{\sim} |1| = |y_1 \wedge y_2|$.

If $|y| \neq |1|$, then $|y_1 \wedge y_2| \wedge_{\sim} |y| = |(y_1 \wedge y_2) \wedge y| = |y_1 \wedge y_2|$ since $y_1 \wedge y_2 \leq y$.

- Assume $|y_1| = |1|$ and $|y_2| \neq |1|$, then $|y_1| \wedge_{\sim} |y_2| = |y_2|$. Assume $|y| = |1|$, then $(|y_1| \wedge_{\sim} |y_2|) \wedge_{\sim} |y| = |y_2| \wedge_{\sim} |y| = |y_2|$.

Assume $|y| \neq |1|$. We get that $(|y_1| \wedge_{\sim} |y_2|) \wedge_{\sim} |y| = |y_2| \wedge_{\sim} |y| = |y_2 \wedge y|$. Since $|y_2 \wedge y| \vee_{\sim} |y| = |(y_2 \wedge y) \vee y| = |y|$ by definition of $\vee_{\sim}$ and absorption, it follows that $|y_2 \wedge y| \leq_{\sim} |y|$, which is what we needed to show.

- The case where $|y_1| \neq 1$ and $|y_2| = |1|$ is analogous.

¹³ Where $\leq_{\sim}$ is the ordering induced by $\wedge_{\sim}$ or equivalently by $\vee_{\sim}$

- Lastly, assume $|y_1| = |1| = |y_2|$. Since $\llbracket T_{\varphi,\psi}(\perp) \rrbracket$ is a filter, it follows that $y \in \llbracket T_{\varphi,\psi}(\perp) \rrbracket$, so $|y| = 1$ and the claim follows.

For the right-to-left direction, assume $(L/\sim, \rho' V), |y| \Vdash_W \chi_1 \vee \chi_2$. This implies there are $|y_1|, |y_2|$ such that

$$(L/\sim, \rho' V), |y_1| \Vdash_W \chi_1, (L/\sim, \rho' V), |y_2| \Vdash_W \chi_2 \text{ and } |y_1| \wedge_{\sim} |y_2| \leq_{\sim} |y|.$$

By induction hypothesis it follows that

$$(L, V), y_1 \Vdash_T T_{\varphi,\psi}(\chi_1) \text{ and } (L, V), y_2 \Vdash_T T_{\varphi,\psi}(\chi_2).$$

We need to show that $y_1 \wedge y_2 \leq y$. We distinguish cases:

- If $|y| = |1|$, then $(L, V), y \Vdash T_{\varphi,\psi}(\perp)$, so by Lemma 7.3.2 we get that $(L, V), y \Vdash T_{\varphi,\psi}(\chi_1 \vee \chi_2)$.
- If $|y| \neq |1|$, $|y_1| \neq |1| \neq |y_2|$, then since $|y_1| \wedge_{\sim} |y_2| \leq_{\sim} |y|$, it follows that $(|y_1| \wedge |y_2|) \wedge |y| = |(y_1 \wedge y_2) \wedge y| = |y_1 \wedge y_2|$, so as $|y_1 \wedge y_2|$ is a singleton, it follows that $(y_1 \wedge y_2) = y$, so we are done.
- If $|y| \neq |1| \neq |y_1|, |y_2| = |1|$, then $|y_1| \wedge_{\sim} |y_2| = |y_1|$, thus from

$$|y_1| = |y_1| \wedge_{\sim} |y_2| = (|y_1| \wedge_{\sim} |y_2|) \wedge_{\sim} |y| = |y_1| \wedge_{\sim} |y|,$$

it follows that $y_1 \wedge y = y_1$, so $y_1 \leq y$. We conclude that $y_1 \wedge y_2 \leq y$. $\square$

With all the groundwork in place we can prove the main theorem of this section.

Theorem 7.3.6. Let $\varphi \leq \psi$ be an arbitrary consequence pair with $\varphi, \psi \in \mathcal{L}_W$. Let $T_{\varphi,\psi}$ be as in Definition 7.3.1, then

$$T_{\varphi,\psi}(\varphi) \Vdash_T T_{\varphi,\psi}(\psi) \iff \varphi \Vdash_W \psi.$$

Proof. $\Rightarrow$ We show that

$$\varphi \not\Vdash_W \psi \implies T_{\varphi,\psi}(\varphi) \not\Vdash_T T_{\varphi,\psi}(\psi). \quad (17)$$

By assumption there exists a lattice $L, x \in L$ and a filter valuation V on L such that

$$(L, V), x \Vdash_W \varphi \text{ yet } (L, V), x \not\Vdash_W \psi.$$

Consider the L-model (L, V') , where $V'(p) = V(p)$ for $p \neq q$ and $V'(q) = 1$. As $q \notin \text{Prop}(\phi, \psi)$, it still holds that $(L, V'), x \Vdash_W \varphi$ and $(L, V'), x \not\Vdash_W \psi$.

Now consider (L, V') as a TIL-model. We prove by structural induction over χ that for all $y \in L$ and $\chi \in \text{Sub}(\varphi, \psi)$ it holds that

$$(L, V'), y \Vdash_W \chi \iff (L, V'), y \Vdash_T T_{\phi,\psi}(\chi).$$

Base case: The case where $\chi := \top$ follows by the semantics.

Assume $\chi := \perp$, then

$$(L, V'), y \Vdash_W \perp \iff y = 1.$$

On the other hand we claim that

$$(L, V'), y \Vdash_T T_{\varphi, \psi}(\perp) \iff y = 1.$$

Since $V'(q) = \{1\}$, 1 is the only element of L such that $(L, V'), 1 \Vdash_T \neg P q \rightarrow \neg F P q$. For let $z \in L$ be such that $z \neq 1$. Then $z < 1$, so $(L, V'), z \Vdash_T \neg P q$, while $(L, V'), z \Vdash_T F P q$. It follows that 1 is the only element such that

$$(L, V'), 1 \Vdash_T \bigwedge_{p \in \text{Prop}(\varphi, \psi) \cup \{q\}} P \langle \text{inf}^* \rangle (\neg P p \rightarrow \neg F P p, \neg P p \rightarrow \neg F P p) = T_{\varphi, \psi}(\perp).$$

Assume $\chi := p$. Then $(L, V'), y \Vdash_W p$ iff $(L, V'), y \Vdash_T p$, since both models have the same valuation and $(L, V'), y \Vdash_T p$ iff $(L, V'), y \Vdash_T T(p)$ since V' is a filter valuation.

Inductive case: The case where $\chi := \chi_1 \wedge \chi_2$ follows from the induction hypothesis.

Assume $\chi := \chi_1 \vee \chi_2$, then $(L, V'), y \Vdash_W \chi_1 \vee \chi_2$ iff there are $y_1, y_2 \in L$ such that $y_1 \wedge y_2 \leq y$, $(L, V'), y_1 \Vdash_W \chi_1$ and $(L, V'), y_2 \Vdash_W \chi_2$. By induction hypothesis it follows that $(L, V'), y_1 \Vdash_T T_{\varphi, \psi}(\chi_1)$ and $(L, V'), y_2 \Vdash_T T_{\varphi, \psi}(\chi_2)$, so since $y_1 \wedge y_2 \leq y$ it follows that

$$(L, V'), y \Vdash_T P \langle \text{inf} \rangle T_{\varphi, \psi}(\chi_1) T_{\varphi, \psi}(\chi_2) = T_{\varphi, \psi}(\chi_1 \vee \chi_2).$$

The other direction is proven similarly, which finishes the induction proof.

We conclude that (17) holds which finishes this direction of the proof.

$\Leftarrow$ We show that

$$T_{\varphi, \psi}(\varphi) \not\Vdash_T T_{\varphi, \psi}(\psi) \implies \varphi \not\Vdash_W \psi. \quad (18)$$

By assumption there exists a lattice L , $x \in L$ and subset valuation V on L such that

$$(L, V), x \Vdash_T T_{\varphi, \psi}(\varphi) \quad \text{yet} \quad (L, V), x \not\Vdash_T T_{\varphi, \psi}(\psi).$$

It follows from Lemma 7.3.5 that $(L/\sim, \rho' V), |x| \Vdash_W \varphi$ and $(L/\sim, \rho' V), |x| \not\Vdash_W \psi$, which proves (18). $\square$

We conclude that even though the top element of a lattice is not definable in the language of TIL, we can nevertheless define a translation of WPL into TIL that is relative to a consequence pair.

7.4 BACK TO POSETS

In order to give a full and faithful translation of the bottom-free fragment of WPL into TIL on lattices, we extended the language of TIL with two binary operators. These operators can also be interpreted on posets.

We therefore switch our attention back to posets and explore what the consequences are of interpreting the $*$ modalities on posets.

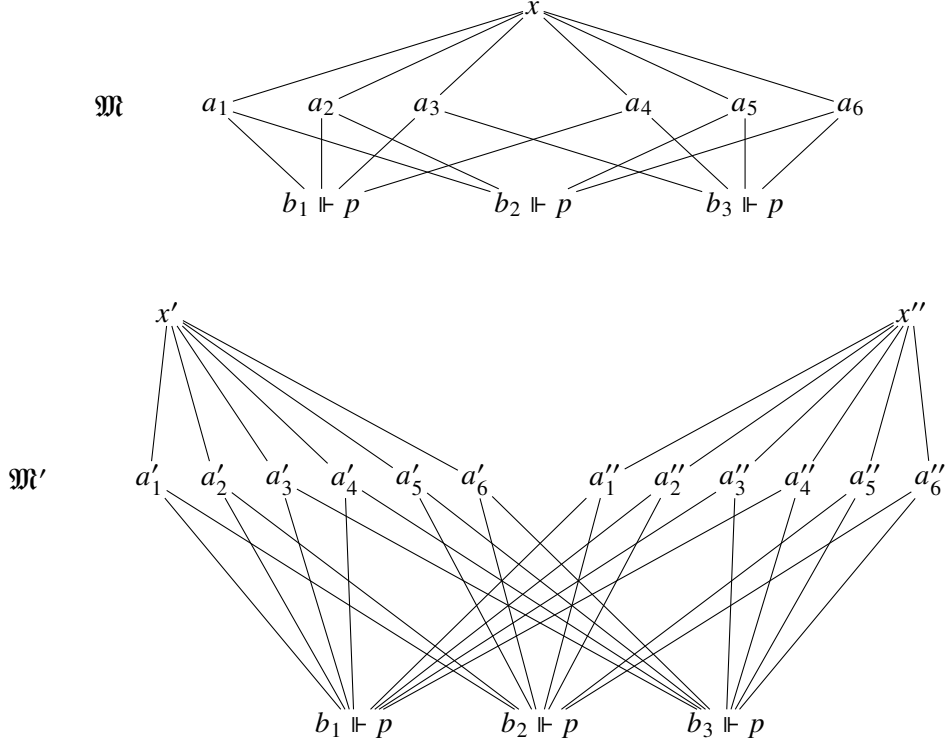
Definition 7.4.1.

$$\begin{aligned} \text{TIL}^* = \{ \varphi \in \mathcal{L}_T^* \mid & \text{for every poset model } \mathfrak{M} = (W, \leq, V) \text{ and every } x \in W, \\ & \mathfrak{M}, x \Vdash \varphi \} \end{aligned}$$

Even though the proof of Proposition 7.2.2 lifts to posets, we give an alternative proof that exploits the fact that not all binary suprema exist in posets.

Proposition 7.4.1. $\mathcal{L}_{T^*}$ is strictly more expressive on posets than $\mathcal{L}_T$.

Proof. Consider the following two poset models where the worlds satisfy all and only the proposition letters shown.



We define a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$ as defined in 7.2.3 as follows:

$$\begin{aligned} x &Z x', x'' \\ a_i &Z a'_i, a''_i \quad (1 \leq i \leq 6) \\ b_i &Z b'_i \quad (1 \leq i \leq 3) \end{aligned}$$

Since

$$\mathfrak{M}, x \Vdash \langle \text{sup}^* \rangle pp,$$

but

$$\mathfrak{M}', x' \not\Vdash \langle \text{sup}^* \rangle pp,$$

it follows that $\langle \text{sup}^* \rangle$ is not definable in $\mathcal{L}_T$. $\square$

7.4.1 Modal operations on posets

In this section we explore how TIL can be viewed as a modal lens on partial orders. With the basic modalities already present in $\mathcal{L}_T$ we can define four natural operations on posets, two of which go beyond the operations we can define with just the forward- and backward-looking operators of (tensed) S4.

Fix a poset model $\mathfrak{M} = (W, \leq, V)$. For every $\varphi \in \mathcal{L}_T^*$ we write

$$\llbracket \varphi \rrbracket^{\mathfrak{M}} := \{x \in W \mid \mathfrak{M}, x \models \varphi\}$$

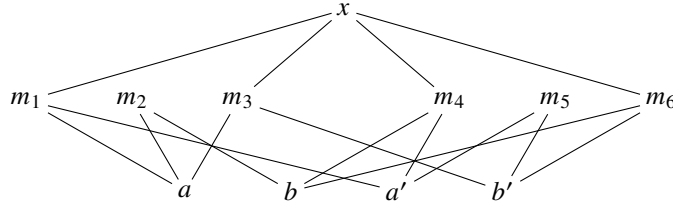
for the *truth set* (or *extension*) of φ .

Definition 7.4.2. For $A, B \subseteq W$ define

$$\begin{aligned} \downarrow A &= \{x \in W \mid x \leq a \text{ for some } a \in A\}, \\ \uparrow A &= \{x \in W \mid a \leq x \text{ for some } a \in A\}, \\ \sup(A, B) &= \{\sup\{a, b\} \mid a \in A, b \in B\}, \\ \inf(A, B) &= \{\inf\{a, b\} \mid a \in A, b \in B\}. \end{aligned}$$

Note that even if A and B are closed under suprema, $\sup(A, B)$ need not be, which is shown by the following example:

Example 7.4.2. Consider the following poset.



The subsets $A = \{a, b, m_2\}$ and $B = \{a', b', m_5\}$ are closed under suprema since $\sup\{a, b\} = m_2$ and $\sup\{a', b'\} = m_5$. On the other hand,

$$\sup(A, B) = \text{bigl}\{m_1, m_2, m_3, m_4, m_5, m_6\}$$

is not suprema closed since $\sup\{m_1, m_3\} = x \notin \sup\{A, B\}$.

Lemma 7.4.3. Let $\mathfrak{M} = (W, \leq, V)$ be a poset model. For any formulas $\varphi, \psi \in \mathcal{L}_T^*$ we have

$$\begin{aligned} \downarrow \llbracket \varphi \rrbracket^{\mathfrak{M}} &= \llbracket \text{F } \varphi \rrbracket^{\mathfrak{M}}, \\ \uparrow \llbracket \varphi \rrbracket^{\mathfrak{M}} &= \llbracket \text{P } \varphi \rrbracket^{\mathfrak{M}}, \\ \sup(\llbracket \varphi \rrbracket^{\mathfrak{M}}, \llbracket \psi \rrbracket^{\mathfrak{M}}) &= \llbracket \langle \sup \rangle \varphi \psi \rrbracket^{\mathfrak{M}}, \\ \inf(\llbracket \varphi \rrbracket^{\mathfrak{M}}, \llbracket \psi \rrbracket^{\mathfrak{M}}) &= \llbracket \langle \inf \rangle \varphi \psi \rrbracket^{\mathfrak{M}}. \end{aligned}$$

Proof. It is easy to check that the definitions are correct. As an example we verify two:

- $x \in \downarrow \llbracket \varphi \rrbracket^{\mathfrak{M}}$ iff there is $y \in \llbracket \varphi \rrbracket^{\mathfrak{M}}$ such that $x \leq y$, which by the semantics of $\text{F } \varphi$ is exactly the case if $x \models \text{F } \varphi$, so in other words $x \in \llbracket \text{F } \varphi \rrbracket^{\mathfrak{M}}$.
- $x \in \inf(\llbracket \varphi \rrbracket^{\mathfrak{M}}, \llbracket \psi \rrbracket^{\mathfrak{M}})$ iff there are $y \in \llbracket \varphi \rrbracket^{\mathfrak{M}}, z \in \llbracket \psi \rrbracket^{\mathfrak{M}}$ such that $x = \inf\{y, z\}$. By the semantics of $\langle \inf \rangle$ this corresponds exactly to $x \models \langle \inf \rangle \varphi \psi$, so $x \in \llbracket \langle \inf \rangle \varphi \psi \rrbracket^{\mathfrak{M}}$. $\square$

With the new modalities, we can define the *finite-supremum closure* operator. Formally, given a set X and a subset $A \subseteq X$, this operation is defined as follows:

$$\begin{aligned} \text{Cl}_{\text{fin}} : \mathcal{P}(X) &\rightarrow \mathcal{P}(X) \\ A &\mapsto \{\sup\{S\} : \emptyset \neq S \subseteq A \text{ such that } S \text{ is finite}\} \end{aligned}$$

We prove the following.

Proposition 7.4.4. Let $\varphi, \psi \in \mathcal{L}_T^*$ be arbitrary, then

$$\text{Cl}_{\text{fin}}(\llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket) = \llbracket \langle \text{sup}^* \rangle \varphi \psi \vee \langle \text{sup}^* \rangle \varphi \varphi \vee \langle \text{sup}^* \rangle \psi \psi \rrbracket$$

Proof. To prove the left to right inclusion, let $x \in \text{Cl}_{\text{fin}}(\llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket)$ be arbitrary, then there is finite non-empty $S \subseteq \llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket$ such that $x = \sup\{S\}$. Assume $S \cap \llbracket \varphi \rrbracket \neq \emptyset$ and $S \cap \llbracket \psi \rrbracket \neq \emptyset$, then there exist $y_1, \dots, y_n \in \llbracket \varphi \rrbracket$ (i.e. $\mathfrak{M}, y_i \Vdash \varphi$) and $z_1, \dots, z_m \in \llbracket \psi \rrbracket$ (i.e. $\mathfrak{M}, z_i \Vdash \psi$) such that $x = \sup\{y_1, \dots, y_n, z_1, \dots, z_m\}$. It follows that $x \Vdash \langle \text{sup}^* \rangle \varphi \psi$. In case $S \cap \llbracket \varphi \rrbracket = \emptyset$ it follows that $x \Vdash \langle \text{sup}^* \rangle \psi \psi$ and $S \cap \llbracket \psi \rrbracket = \emptyset$ yields $x \Vdash \langle \text{sup}^* \rangle \varphi \varphi$.

For the right to left inclusion, let $x \in \llbracket \langle \text{sup}^* \rangle \varphi \psi \vee \langle \text{sup}^* \rangle \varphi \varphi \vee \langle \text{sup}^* \rangle \psi \psi \rrbracket$ be arbitrary. Assume $\mathfrak{M}, x \Vdash \langle \text{sup}^* \rangle \varphi \psi$. Then there exists $y_1, \dots, y_n \in \llbracket \varphi \rrbracket$ and $z_1, \dots, z_m \in \llbracket \psi \rrbracket$ such that $x = \sup\{y_1, \dots, y_n, z_1, \dots, z_m\}$. It follows that $x \in \text{Cl}_{\text{fin}}(\llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket)$. If $x \Vdash \langle \text{sup}^* \rangle \varphi \varphi$, we get that there exist $y_1, \dots, y_n \in \llbracket \varphi \rrbracket$ such that $x = \sup\{y_1, \dots, y_n\}$, so $x \in \text{Cl}_{\text{fin}}(\llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket)$. The case where $x \Vdash \langle \text{sup}^* \rangle \psi \psi$ is handled similarly. $\square$

In conclusion, the extension $\mathcal{L}_T^*$ lets us define several natural operations on posets directly at the formula level. There are still many interesting directions to explore from here, some of which are discussed in the next section.

8 CONCLUSION AND FURTHER RESEARCH

In this thesis we study tense information logic, an extension of modal information logic with a second modality: the infimum operator.

Following the outline and proofs of [16], we presented an axiomatization of TIL on posets that only links $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$ through a standard axiom of temporal logic, thereby answering an open question posed by van Benthem in [4]. We extended the completeness proof for MIL in [16] using the step-by-step method and showed that this result extends to TIL on preorder frames as well. We proved that TIL enjoys the FMP with respect to a generalized class of frames, thereby establishing decidability of the logic and answering yet another question posed in [3].

Next, we studied tense information logic from an algebraic and topological perspective. We worked out the Stone–Jónsson–Tarski duality between the category of Boolean algebras with two binary operators (**2B-BAO**) and the category of modal Stone spaces with two ternary relations (**2T-MS**) and checked that this duality carries over unchanged when we restrict to the objects that validate the axioms of TIL.

Continuing our study of TIL, we examined whether the logic remains sound, complete and decidable when we change the semantics of the modal operators $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$: instead of interpreting them via the least upper bound and greatest lower bound, we define them with respect to minimal upper bounds and maximal lower bounds. Applying a modified version of the representation method of [18], we proved that no modal principle is valid under one interpretation but not the other. In particular, we showed that TIL and its variant based on minimal / maximal bounds can be axiomatized by, and are sound and complete with respect to, the same set of axioms. Tense information logic with minimal / maximal bounds shares the informational interpretation of TIL. However, instead of uniqueness, multiple incomparable bounds and thus merges and states containing most common information can exist.

Connecting TIL with other modal logics, the last chapter worked out a translation of the $\perp$ -free fragment of weak positive logic into tense information logic interpreted on lattices, starting from the translation suggested in [7]. To make this translation work, we extended TIL with two binary $*$ -modalities that take witness sets of any finite size in order to define meet-closure of filters in the logic. Moreover, by defining a translation relative to a consequence pair $\varphi \trianglelefteq \psi$ and using a fresh propositional letter $q \notin \text{Prop}(\varphi, \psi)$ in the translation of $\perp$, we obtained a full and faithful translation of WPL including $\perp$ into the extended version of TIL.

Extending the logic with these modalities turned out to have interesting implications. If we interpret the extended language on posets, the supremum and infimum closure operators are definable, thus going beyond the operators that are definable with only the past-/future-looking modalities of (tensed) S4. There seem to be many interesting directions one could go from here, of which some will be discussed in the following section.

8.1 FURTHER RESEARCH

During the investigation of TIL, multiple open problems and directions for further research presented themselves. We briefly discuss those that seem most interesting.

8.1.1 Informational implication

In [16], Knudstorp studies different alterations of modal information logic, one of which is MIL of minimal upper bounds. In Chapter 6 we lifted this result to the setting of TIL with two modalities.

Another extension that is studied in [16] is an enrichment of the language with the ‘informational implication’. Since the $\langle \text{sup} \rangle$ and $\langle \text{inf} \rangle$ operators behave in many ways like a ‘meet’ operator, it is natural to consider their residuals. We therefore enrich the language (denoting the result with $\mathcal{L}_{\sup T}$) with the *informational implications* $\supset_{\sup}$ and $\supset_{\inf}$, interpreted on posets as follows:

$$\begin{aligned} \mathfrak{M}, x \Vdash \varphi \supset_{\sup} \psi & \text{ iff } \forall u, v \in W, \text{ if } \mathfrak{M}, u \Vdash \varphi \text{ and } v = \sup\{u, x\} \text{ then } \mathfrak{M}, v \Vdash \psi, \\ \mathfrak{M}, x \Vdash \varphi \supset_{\inf} \psi & \text{ iff } \forall u, v \in W, \text{ if } \mathfrak{M}, u \Vdash \varphi \text{ and } v = \inf\{u, x\} \text{ then } \mathfrak{M}, v \Vdash \psi. \end{aligned}$$

Definition 8.1.1. Let $TIL_{\sup}$ be the logic of all $\mathcal{L}_{\sup T}$ validities on poset frames, that is

$$TIL_{\sup} = \{ \varphi \in \mathcal{L}_{\sup T} \mid \text{for every poset model } \mathfrak{M} = (W, \leq, V) \text{ and every } x \in W, \mathfrak{M}, x \Vdash \varphi \}.$$

In [17], Knudstorp presents an axiomatization of MIL extended with $\supset_{\sup}$ and uses bulldozing and representation theorems to prove soundness and completeness. The same approach seems to apply without problems to $TIL_{\sup}$, but one preliminary result turns out to be difficult to show: does introducing residuals increase expressive power? At this moment it remains an open problem whether $\supset_{\sup}$ and $\supset_{\inf}$ are definable in the original language.

8.1.2 Axiomatization of TIL^*

In Chapter 7 we extended TIL with two additional binary modalities that capture infima and suprema of sets of states of any finite size. The new operators quantify over finite but unbounded sets of witnesses. This raises the possibility that TIL^* is not compact (compare for example PDL’s non-compactness proof [10]). Whether or not the logic is compact is an open question. A further direction of research is to prove or refute that the logic is compact and, depending on the result, choose a strategy for finding an axiomatization and prove that it is sound and complete.

8.1.3 Operations on posets

In Section 7.4.1 we studied how TIL can be used as a modal lens on posets. We showed that the supremum and infimum closure operators are definable in the extended language $\mathcal{L}_*$.

Several other operations on posets might also be definable. One natural candidate is the *filter-closure* operation. Given a poset $(W, \leq)$, let

$$Fi : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$$

send a set $A \subseteq W$ to the least filter containing A . A first attempt to define this operator in $\mathcal{L}_*$ looks as follows:

$$Fi(\llbracket \varphi \rrbracket^{\mathfrak{M}} \cup \llbracket \psi \rrbracket^{\mathfrak{M}}) := \llbracket P \langle \text{inf}^* \rangle \varphi \psi \vee P \langle \text{inf}^* \rangle \varphi \varphi \vee P \langle \text{inf}^* \rangle \psi \psi \rrbracket^{\mathfrak{M}}.$$

Unfortunately, this definition does not apply in the current setting, since in posets suprema / infima do not necessarily exist. It motivates the introduction of a *filter modality*. Determining its exact definition and expressive power is an open problem and an interesting direction for further work.

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