HMS-Duality for Residuated Lattices

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

In this thesis we present a novel topological duality for not-necessarily-distributive residuated lattice ordered groupoids by modifying a recent duality for bounded lattices established by Bezhanishvili et al. (2024). Our duality establishes a natural connection between the algebraic semantics of substructural logics and the operational frame semantics originating in the work of Ono and Komori (1985), Humberstone (1987), and Došen (1989). This allow us to the further generalize the original completeness theorems for the operational semantics and to gain insight into the success of canonical model style proofs that were utilized. In particular we adapt a notion of persistence from Bezhanishvili et al. (2024) and show that the canonical model style proofs in Ono and Komori (1985), Humberstone (1987), and Došen (1989) can be explained by an analysis given in terms of algebraic completeness, topological duality, and the salient notion of persistence. We also explore the duality in its own right and obtain topological representations of the lattice of congruences and products of residuated lattices.

Contents

Chapter 1

Introduction

In the semantics of modal logic, topological dualities between categories of modal algebras and categories of modal spaces have played a central role in the development of the field [8, 10]. On one hand, these dualities have pragmatic purpose; they unify the available algebraic and model theoretic methods used to address questions about modal logics and their semantics. On the other hand, dualities have an explanatory purpose since they offer alternative perspectives on, and clarify why, the methods of one semantics or another are successful.

A paradigm case of the sort of insight duality can provide is the clarity it casts on canonical model completeness proofs with respect to Kripke frames. It is known that the canonical model of a normal modal logic comes with a natural modal space topology and that this modal space is the dual of Lindenbaum algebra of the same logic. This fact can then be coupled with what is sometimes called *d-persistence*, which is the property that if a formula is valid in a modal space, then it is also valid in the underlying Kripke frame of that modal space (see [8, 10]). Canonical model style Kripke completeness proofs can then be understood in virtue of three distinct steps: algebraic completeness, duality, and d-persistence. For more details we refer the reader to [8] and [10] for a proof of the Sahlqvist Completeness Theorem using topological duality - one of the most celebrated results in modal logic.

For non-classical logics, and in particular logics with algebraic semantics given in terms of residuated lattices, topological dualities have received varying degrees of attention. These dualities are often obtained by extending a duality for the lattice reducts of the algebras in question. For example, dualities for distributive residuated lattices tend to build on Priestley duality [11, 21]. A very successful instance of this strategy is the duality between Heyting algebras and Esakia Spaces [16]. Another more general method has been to modify Priestley spaces with a ternary relation [36]. For not-necessarily-distributive residuated lattices, dualities are obtained by building on dualities for bounded lattices. An illustrative example is Allwein and Dunn's [2] extension of Urquhart's topological representation of lattices to a representation of various residuated algebras [35]. Another are the dualities given in terms of canonical extensions [20, 15]. These are early examples, but a great number of dualities for various classes of residuated lattices could be listed, many building on various dualities for bounded lattices [5]. Another notable representation for residuated lattices is provided by residuated frames [17].

In just the same way that topological duality theory has clarified the Kripke semantics of modal logic, Esakia duality and dualities for other residuated lattices have offered the same sort of insight into the Kripke semantics of intuitionistic logic, Routley-Meyer semantics for relevance logics, and the semantics of various other substructural logics [36]. Despite these developments, a notable family of statebased semantics developed most maturely by in the work of Ono and Komori [32], Humberstone [23], and Došen [13], and often referred to as the operational semantics for substructural logics, have received no attention from the perspective of topological duality theory. Despite there not being a true topological duality, results in [32] and [37] suggest its possibility. Ono and Komori used the frames of their semantics to obtain embedding theorems for integral commutative residuated lattices in [32]. And more recently Weiss characterized a means of turn so-called Dunn-monoids into a kind of operational frame and vice versa [37]. This being said, neither of these results characterize a full duality and actually only partially characterize a dual-adjunction between a category of algebras and a class of frames for which no notion of morphism has previously been developed.

At their core, the semantics of Ono and Komori, Humberstone, and Došen, henceforth the OKHD-semantics, are a state based semantics similar to the standard Kripke semantics of modal or intuitionistic logic. So sentences are evaluated at states and proposition are construed as a regions of a structured set of those states. However, unlike Kripke semantics, the OKHD-semantics trades relational structure for operational structure and employs the idea of an algebra of states. The most general formulation of this idea that salient to this thesis appeared in [13], where the frames considered are pointed semilattice ordered groupoids with the special property that the groupoid operation distributes over the semilattice operation. The key insight that prompted the development of the these semantics was that disjunction could be interpreted as an intensional connective i.e. one whose satisfaction at state is determined by facts about other states of the model the and the way that each of these states are related. With this insight it became possible to prove "Kripke Style" completeness theorems for non-distributive logics.

In this thesis we present a novel topological duality for not-necessarily-distributive residuated lattice ordered groupoids by modifying a recent duality for bounded lattices established by Bezhanishvili et al. in [6]. In much the same way that the various dualities mentioned above link algebraic semantics to model theoretic semantics, our duality establishes a natural connection between the algebraic semantics of substructural logics and the operational frame semantics originating in the work of Ono and Komori [32], Humberstone [23], and Došen [13]. This connection will allow us to the further generalize the completeness theorems in [32, 23, 13] and to gain insight into the success of canonical model style proofs present in these papers. In particular we adapt the notion of Π_1-persistence from [6] and show that the canonical model style proofs in [32, 23, 13] can be explained by an analysis similar to the one given for canonical modal logics above.

Chapter 2 reminds the reader of preliminary notions regarding lattice expansions, residuated lattice ordered groupoids or $r\ell$-groupoids, and provides an introduction to L-spaces developed by the authors of [6].

Chapter 3 expands the theory of L-spaces by developing a general set of tools that ground the technical developments of the chapters to come. In particular, we provide a characterization of products and coproducts of L-spaces. Specifically, we obtain the some what counterintuitive result that products of L-spaces with a special notion of inclusion have the universal property of coproducts in the category of L-spaces(Theorem 3.1.10). We then this to obtain a topological representation of monotone lattice expansion (Theorem 3.2.7) and, more generally, a duality between the category of lattices with monotone operations as morphisms and the category of L-spaces with special continuous relations (Theorem 3.2.11). This chapter ends with some simple results characterzing the dual relations of join and meet preserving operations and provides a new representation of Modal Lattices and a discussion of how we can recover the representation of modal lattices presented by the authors of [6].

The importance of this chapter is twofold. From the conceptual point of view it provides a very general common framework for understanding both the results from later chapters and the results for the original work on L-spaces and modal lattices developed by the authors of [6]. At the technical level, the duality theorem proved in Theorem 3.2.7 is key to the Π_1-preservation results proved in the following chapter.

Chapter 4 uses the results of Chapter 3 to generalize the representation of completions of modal lattices by Bezhanishvili et al. [6] to all monotone lattice expansions. In particular, we show that the filter, ideal, and Π_1-completion of a monotone lattice expansion is representable by families of filters in the L-space that is dual to that lattice expansion (Theorem 4.2.3). We then use this representation to obtain a general result on the preservation of identities through the Π_1-completion (Theorem 4.3.5). This result generalizes the persistence results for modal lattices in [6] and will be applied to the case of $r\ell$-groupoids in Chapter 6.

Chapter 5 makes the move from the general representation theory of Chapter 3 and Chapter 4 to two categories of spaces that will occupy our attention for the rest of the thesis. In particular we introduce NRL-spaces and RML-spaces and morphisms for both types of spaces. RML-spaces are modifications of ternary relational instances of the spaces used to represent lattice expansions in Chapter 3. NRL-spaces are essentially topological versions of the frames used by the OKHD-semantics. We show this connection in detail by showing that the topology free reduct of an NRL-spaces is an OKHD-frame (Proposition 5.2.11). We then show that the category of RML-spaces and NRL-spaces are equivalent (Theorem 5.3.1). The chapter ends by showing the operations that witness the equivalence between RML-spaces and NRL-spaces do not generalize to an equivalence between the category of OKHD-frames and the category of what we call RML-frames.

This chapter is primarily of conceptual importance since it connects the general theory of L-spaces with additional continuous relations from Chapters 3 and 4 to the frames underlying the OKHD-semantics and the spacial objects we study the rest of the thesis.

Chapter 6 introduces our duality theory for $r\ell$-groupoids. We demonstrate that the category of NRL-spaces is dually equivalent to the category of $r\ell$-groupoids (Theorem 6.1.10). As an immediate corollary we also obtain duality between the category of RML-spaces and $r\ell$-groupoids. We then restrict the NRL-space duality to obtain duality for residuated lattices, FL-algebras, and Involutive residuated lattices and provide a number of other explicit correspondences between certain algebraic identities and topological properties. In the last section we obtain a representation of the Π_1-completion of an $r\ell$-groupoid and use the preservation theorem from Chapter 4 to characterize some classes of $r\ell$-groupoid that are closed under the Π_1-completion.

Chapter 7 reviews basic substructural logic and the OKHD-semantics and then uses the dualities and Π_1-persistence results obtained in Chapter 6 to provide both topological and frame based completeness for a wide range of substructural logics. In particular, we show every substructural logic extending a very minimal base logic that we call the Non-associative Positive Full Lambek Calculus or $\mathbf{NFL}^+$ is complete with with respect to a new topological semantics given in terms of NRL and RML-spaces (Theorem 7.4.3 and Theorem 7.4.4). We then adapt the notion of Π_1-persistence from Bezhanishvili et al. [6] and show that every logic axiomatized by sequents in specific signature is complete with respect to a class of OKHD-frames (Theorem 7.4.8) and a class of RML-frames (Theorem 7.4.9). These results generalize completeness proofs of Ono and Komori [32], Humberstone [23], and Došen [13] and also the completeness results reported recently in [37]. The chapter ends with of a discussion of how our duality bears the canonical model style completeness proofs of Ono and Komori [32], Humberstone [23], and Došen [13].

Chapter 8 extends the Theory of NRL-space duality in two directions and provides logical applications for each. While the logical results are either not novel or adapt existing results, they come with novel proofs that exploit the topological semantics given in terms of NRL-spaces. The first development for the theory of NRL-spaces is a representation of congruences of a residuated lattice in the dual space of that residuated lattice (Theorem 8.1.4). We use this representation to characterize the subspaces of NRL-spaces (Proposition 8.1.8 and Theorem 8.1.12). The logical application of the latter developments is a new proof of the Parameterize Local Deduction Theorem for the Positive Full Lambek calculus $\mathbf{FL}^+$ using the topological semantics of Chapter 7. The second development to NRL-space duality is a characterization of products and coproducts of NRL-spaces and a representation theorem for products of $r\ell$-groupoids. The logical application provided by these results is a characterization of when a substructural logic extending $\mathbf{NFL}^+$ has the disjunction property.

The contributions made by this thesis are summarized by the following list:

- A topological representation theorem for monotone lattice expansions.
- A characterization of a large class of identities that are preserved through the Π_1-completion of a lattice expansion.
- A novel topological duality for $r\ell$-Groupoids, Residuated Lattices, and FL-algebras.
- A characterization of classes of $r\ell$-Groupoids that are closed under the Π_1-completion.
- Topological semantics for substructural logics.
- A duality theoretic explication of the OKHD-semantics of substructural logics with logical applications.
- A representation of the lattice of congruences of residuated lattices and a characterization of subspaces of NRL-spaces.
- A novel proof of the Parameterized Local Deduction Theorem using topological semantics.
- A characterization of coproducts of NRL-spaces and a topological representation of products of $r\ell$-Groupoids.
- A characterization of substructural logic with the disjunction property using topological semantics.

Chapter 2

Preliminaries

In this chapter we introduce the algebraic that are and topological structures that are the focus of this thesis. In particular, Section 2.1 reviews the definitions of semilattices, lattices, and lattice expansions and then $r\ell$-groupoids, residuated lattices, and FL-algebras. In section 2.2 and we point to the topological notions we use and we review the L-space duality developed by Bezhanishvili et al. in [6].

We note that through out the preliminaries and thesis we assume familiarity with basic notions from catgeory theory such as the notions of an opposite category, a (contravariant) functors between categories, adjunctions, (dual) isomorphisms, products, coproducts, and concrete categories. We recommend [3] for a refresher.

2.1 Algebra

In this section we introduce the algebras we study in this thesis. We assume familiarity with the basic concepts of universal algebra including the notions of homomorphisms subalgebras, products, congruences, and free algebras. We also assume fundamental results like the isomorphism theorems. Finally, while knowledge of clones is not necessary, they will be remarked on and will be used to explain the Π_1-preservation result we obtain in Chapter 4. For introductory material on any of the topics listed above, see either the classic [9] or the more recent [4].

2.1.1 Lattices and Lattice Expansions

We begin by defining semilattices and lattices and then discuss lattice expansions. We provide the definition for completeness but assume basic properties without mention throughout the thesis.

Definition 2.1.1. *A $*$-semilattice is an algebra $\mathbf{S} = (S, *, e)$ where $*$ is associative, commutative, and idempotent and $a * e = a$ for all $a \in S$.*

We can always define two possible orders on a semilattice $\mathbf{S} = (S, *, e)$. When we define the order by $a \leq b$ iff $a * b = a$ we refer to $*$ as meet and e as top and call $\mathbf{S}$ a meet semilattice. If we define the order by $a \leq b$ iff $a * b = b$, then we refer to $*$ as join and e as bottom and call $\mathbf{S}$ a join semilattice.

Definition 2.1.2. *A (bounded) lattice is an algebra $\mathbf{L} = (L, \wedge, \vee, \top, \bot)$ where both $(L, \wedge, \top)$ and $(L, \vee, \bot)$ are semilattices and the absorbtion laws hold:*

$$a = a \wedge (b \vee a) \qquad\qquad a = a \vee (b \wedge a).$$

The standard order on the lattice $\mathbf{L}$ is defined $a \leq b$ iff $a \wedge b = a$ iff $a \vee b = b$.

We say an n-ary operation $f : \mathbf{L}_1 \times ... \times \mathbf{L}_n \to \mathbf{K}$ between lattices is monotone if it preserves the defined order in each coordinate.

Definition 2.1.3. *Let $f : \mathbf{L}_1 \times ... \times \mathbf{L}_n \to \mathbf{K}$ be n-ary operation between lattices. f is monotone if for any $\vec{a}, \vec{b} \in L_1 \times ... \times L_n$, if for each $i \leq n$: $\vec{a}(i) \leq \vec{b}(i)$, then $f(\vec{a}) \leq f(\vec{b})$.*

For example, if f is binary, then if $a \leq c$ and $b \leq d$, then $f(a, b) \leq f(c, d)$. We now define lattice expansions and monotone lattice expansions.

Definition 2.1.4. *(Lattice Expansions) A lattice expansion $(\mathbf{L}, F)$ is a lattice $\mathbf{L}$ with an additional family of operations F. We say that $(\mathbf{L}, F)$ is monotone if for each n-ary operation $f \in F$ is monotone.*

Homomorphism for semilattice, lattices, and lattice expansions are defined in the usual way. We denote the category of lattices with lattice homomorphisms by **Lat**.

Semilattice and Lattice as Ordered Sets. It is well known that lattice and semilattices have ordere theoretic definitions. From the perspective of order theory, a *meet semilattice* is a partial order with maximal element $\top$ where every two elements a, b have a greatest lower bound, for which we write $a \wedge b$. A *complete semilattice* is partial order with greatest lowerbounds $\bigwedge T$ for all subsets T. A (bounded) lattice is a bounded partial ordered with a least upped bound and greatest lower bound for each pair of elements, which we denote by $a \vee b$ and $a \wedge b$ respectively. A *complete lattice* is a partial order with least upper bounds and greatest lower bounds for all subsets. Every complete semilattice is a complete lattice. An element of a complete c lattice $\mathbf{C}$ is called *compact* if for each $T \subseteq C$, if $c \leq \bigvee T$, then there is a finite $T_0 \subseteq T$ such that $c \leq \bigvee T_0$. A *algebraic lattice* is a complete lattice lattice where every element is the join of compact elements.

Upsets, Filters, and Ideals. For a subset $U \subseteq P$ where P is a partial order, $\uparrow_P(U) = \{x \in P \mid \exists y \in U(y \leq x)\}$. When $U = \{x\}$ for some $x \in P$, we just write $\uparrow_P(x)$. We will drop the subscript P if the where the operation is being calculated is clear. An upset is subset U of a partial order such that $\uparrow U = U$. A *filter* of a meet semilattice is a non-empty upward closed subset that is also closed under meets. Given a subset $T \subseteq S$ of a semilattice, $[T) = \uparrow\{a_1 \wedge ... \wedge a_n \mid a_1, ...a_n \in T\}$ is the filter generated by T. A filter of the form $\uparrow x$ is called *principal.* For any given meet semilattice $\mathbf{S}$ we will denote the collection of all filters of S by $\mathcal{F}i(S)$. Dually, an *ideal* of a join semilattice is a non-empty downward closed subset that is also closed under joins. For a join semilattice $\mathbf{S}$, the set of all ideals is $\mathcal{I}d(\mathbf{S})$.

2.1.2 Residuated Lattice Expansions

We we define $e\ell$-groupoids, resisduated lattices, and FL-algebras. For a thorough introduction to residuated structures see [24] or [19]. We note that we only consider bounded alegbras here, which is not standard. So when we say residuated lattice, we mean a bounded residuated lattice.

Definition 2.1.5. *A pointed residuated lattice ordered groupoid or simply pointed $r\ell$-groupoid $\mathbf{G} = (G, \wedge, \vee, \top, \bot, \cdot, \backslash, /, e)$ an algebra where $(G, \wedge, \vee, \top, \bot)$ is a lattice, $\cdot : G \times G \to G$ is a binary monotone operation, e is a designated element, and $\cdot$, $\backslash$, and $/$ jointly satisfy the residual law:*

$$b \leq a \backslash c \iff a \cdot b \leq c \iff a \leq c/b$$

Note that pointed $r\ell$-groupoids have an equational definition and so form a variety (see Lemma 2.3 in [24]).

Example 2.1.6. *Let $(S, \cdot, e)$ be a pointed groupoid. Then $(\mathcal{P}(S), \cap, \cup, S, \emptyset, \circ, \subseteq, \backslash, /)$ is a pointed $r\ell$-groupoid such that $A \circ B = \{a \cdot b \mid a \in A \ \& \ b \in B\}$, $A \backslash B = \{b \in S \mid \forall a \in A(a \cdot b \in B\}$, and $A/B = \{b \in S \mid \forall b \in B,\ a \cdot b \in A\}$ for each $A, B \subseteq S$.*

We say an $r\ell$-groupoid is *unital* if e is a identity for $\cdot$ i.e. if $e \cdot a = a$ and $a \cdot e = a$. An $r\ell$-gropoid is integral if $e = \top$. An $r\ell$-groupoid is said to be associative or commutative if $\cdot$ satisfies the associative or commutative laws, respectively. A *residuated lattice* is a unital $r\ell$-groupoid where $\cdot$ is associative. If $\cdot$ is commutative, then for all a, b, $a \backslash b = b/a$, so we denote the application of either of these operations to two elements of the algebra by $'a \to b'$.

Note that in any $r\ell$-groupoid we have that $\cdot$ distributes over $\vee$.

$$a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c) \text{ and } (b \vee c) \cdot a = (b \cdot a) \vee (c \cdot a)$$

We also always have that $a \cdot \bot = \bot = \bot \cdot a$ and $\bot \backslash a = \top = a/\bot$.

By adding an additional constant f to the signature of a residuated lattice, we obtain what is called an FL-algebra. An *FL-algebra* $\mathbf{L} = (L, \wedge, \vee, \top, \bot, \cdot, \backslash, /, e, f)$ is a residuated lattice with the additional designated element f. These algebras are the used to provide the algebraic semantics of the Full Lambek Calculus. Finally, an *Involutive Residuated Lattice* is an Fl-algebra $\mathbf{L} = (L, \wedge, \vee, \top, \bot, \cdot, \backslash, /, e, f)$ with the property that both of the following identities hold:

$$f/(a\backslash f) = a \qquad\qquad (f/a)\backslash f = a.$$

We now list some useful properties or $r\ell$-groupoids.

Proposition 2.1.7. *In any $r\ell$-groupoid $\mathbf{G}$, if $X, Y \subseteq G$ and $\bigwedge Y$ and $\bigvee X$ exist, then:*

$$\bigvee X \backslash y = \bigwedge_{x \in X} (x \backslash y) \qquad\qquad y/\bigvee X = \bigwedge_{x \in X} (y/x)$$
$$x \backslash \bigwedge Y = \bigwedge_{y \in Y} (x \backslash y) \qquad\qquad \bigwedge Y/x = \bigwedge_{y \in Y} (y/x).$$

We also have the following.

Proposition 2.1.8. *In any associative $r\ell$-groupoid the following identities and their mirror images hold:*

(1) $(a\backslash b) \cdot z \leq x\backslash(y \cdot z)$,
(2) $a\backslash b \leq (c \cdot a)\backslash(c \cdot b)$,
(3) $(a\backslash b) \cdot (b\backslash c) \leq a\backslash c$,
(4) $(a \cdot b)\backslash c \leq b\backslash(a\backslash c)$, *and*
(5) $a\backslash(c/b) = (a\backslash c)/b$.

The properties mentioned above and more are discussed in detail in [24] and [19].

For any of the signature we consider, a homomorphism is a structure preserving function in the usual sense (see [9] or [4]). For future reference, we denote the category of pointed $r\ell$-groupoids with their homomorphisms by **RLG**. The category of residuated lattices will be **RL** and the category of FL-algebras will be **FLAlg**.

2.2 Topologies and L-spaces

We also assume familiarity with basic definitions from topology such as the notions of a basis, a subbase, subspaces, products, and compactness. For an introduction to the topology necessary for this thesis we recommend [11]. For a general introduction to topology see [29]. For completeness we include the definition of topological spaces and continuous functions.

Definition 2.2.1. *A topological space* $\mathbf{X} = (X, \tau)$ *is a pair consisting of set* X *and a family* $\tau \subseteq \mathcal{P}(X)$ *where:*

(1) $X, \emptyset \in \tau$,

(2) if $S \subseteq \tau$, *then* $\bigcup S \in \tau$, *and*

(3) if $U, V \in \tau$, *then* $U \cap V \in \tau$.

Element of τ are called the open sets of $\mathbf{X}$ and complements of elements of τ are called closed. If $U \in \tau$ and $X - U \in \tau$, then U is said to be clopen.

Morphisms between topologies are called continuous functions. For a function $f : X \to Y$ between sets, we define $f^{-1}[U] = \{x \in X \mid fx \in U\}$ for $U \subseteq Y$ and $f[V] = \{fx \in Y \mid x \in V\}$ for $V \subseteq X$.

Definition 2.2.2. *Let* $\mathbf{X} = (X, \tau)$ *and* $\mathbf{Y} = (Y, \sigma)$ *be topological spaces. A continuous function* $f : \mathbf{X} \to \mathbf{Y}$ *between topological spaces is function* $f : X \to Y$ *such that for each* $U \in \sigma$, $f^{-1}[U] \in \tau$.

2.2.1 L-spaces and Topological Duality for Bounded Lattices

We now introduce L-spaces and the duality between the category of L-spaces and the category of lattics developed by Bezhanishvili et al. [6]. We also recommend the [5] for a general review of duality theory for lattices and the connection between L-space duality and other prominent dualities for lattices. This section is fundamental to the entire thesis and we will often refer back to this section in the proofs we provide in the chapters to come. We refer the reader to [6] and [5] for proofs.

Definition 2.2.3. *An L-space* $X = (X, \curlywedge, 1, \tau)$ *is a compact 0-dimensional semilattice ordered topological space that satisfies the HMS-separation and two addition constraints:*

(1) $\forall x, y \in X$, *if* $x \not\leq y$, *then there is a clopen filter* U *such that* $x \in U$ *and* $y \notin U$.

(2) ($\triangledown$-closure) If U, V *are clopen filters, then* $U \triangledown V := \{z \in X \mid \exists x, y (x \in U \;\&\; y \in V \;\&\; x \curlywedge y \leq z)\}$ *is clopen as well,*

(3) $\{1\}$ *is clopen.*

A topological spaces (X, τ) is 0-dimensional if τ has a basis of clopens. Condition one is referred to as the *HMS-separation axioms.* We note that $U \triangledown V$ is the least filter containing the union of filters U and V. An HMS-space is an L-space without the requirement that the clopen filters be closed under $\triangledown$. Another important piece of notation is that $\mathcal{F}i_{clp}(\mathbf{X})$ is used to denote the set of all clopen filters of X. We also write $\mathcal{F}i(\mathbf{X})$ for the set of all filters of X, $\mathcal{F}i_k(\mathbf{X})$ for the closed filters, and $\mathcal{F}i_o(\mathbf{X})$ for the open filters of an L-space.

Morphism between L-spaces are continuous semilattice homomorphisms with a special back condition.

Definition 2.2.4. *(L-space morphism)*

An L-space morphism $f : X \to Y$ *is a continuous semi-lattice homomorphism that satisfies the following two constraints:*

i) for all $x \in X$, $fx = 1_Y$ iff $x = 1_X$

ii) for all $x', y' \in Y$ and $z \in X$, If $x' \curlywedge_Y y' \leq fz$, then there are $x, y \in X$ such that $x \curlywedge_X y \leq z$ and $x' \leq fx$ and $y' \leq fy$.

The category of L-spaces together with L-space morphisms is denoted **LSp**. The dual equivalence of **Lat** and **LSp** is derived from the followingpropositions. Lemma 2.2.5 encodes the operations of transforming an L-space into a corresponding lattice and lattice into a corresponding L-space.

Lemma 2.2.5. *(Between L-spaces and Lattices)*

(1) Let $\mathbf{X}$ be an L-space, then $\mathbf{L_X} = (\mathcal{F}i_{clp}(X), \cap, \triangledown, X, \{1\})$ is lattice, and

(2) If $L = (L, \wedge, \vee, \top, \bot)$ is a bounded lattice, then $\mathbf{X_L} = (\mathcal{F}i(L), \cap, L, \tau_L)$ in an L-space where τ_L is generated by the subbase $\{\phi(a) \mid a \in L\} \cup \{X - \phi(a) \mid a \in L\}$ and $\phi(a) = \{x \in \mathcal{F}i(\mathbf{L}) \mid a \in x\}$.

A proof can be found in [6], particularly in Theorem 2.14. We note that $\phi(a) = \uparrow_{X_G}(\uparrow_G(a))$. In general we use will always use $\mathbf{L_X}$ to denote the lattice clopen filters of an L-space $\mathbf{X}$ and $\mathbf{X_L}$ to denote the L-spaces on the filters of a lattice $\mathbf{L}$. It is worth noting that while the move from lattices to L-spaces is similar to the way a Stone space or Priestly space is constructed from it's corresponding dual algebra, it is also importantly different. Unlike the construction of a Stone space or Priestly space, rather than topologizing the collection of prime filters, the collection of all filters is topologized with a suitable subbase.

At the level of morphisms, the action of the inverse of an L-space morphism on clopen filters is a lattice homomorphism. Similarly, the inverse of a lattice homomorphism's action on its filters is a L-space morphism. The following lemma records this precisely.

Lemma 2.2.6. *(Bewteen Morphisms)*

(1) If $f : \mathbf{X} \to \mathbf{Y}$ is a L-space morphism, then $f^{-1} : \mathbf{L_Y} \to \{L_X\}$ is a lattice homomorphism, and

(2) if $f : L \to K$ is a lattice homomorphism, then $f^{-1} : \mathbf{X_K} \to \mathbf{X_L}$ is an L-space morphism.

Proofs can be found in [6] in Proposition 2.5 and 2.6. By combining Lemma 2.2.5 and Lemma 2.2.6, we define two functors **LSp** $\to$ **Lat** and **Lat** $\to$ **LSp** defined by the operations implicit in these lemmas.

$$\mathbf{X} \longmapsto \mathbf{L_X} \qquad\qquad \mathbf{L} \longmapsto \mathbf{X_L}$$
$$f \longmapsto f^{-1} \qquad\qquad g \longmapsto g^{-1}$$

These functors witness not only a dual adjunction between the categories **Lat** and **LSp**, but also a full dual equivalence. The following lemma provides us with the means to recognize this.

Lemma 2.2.7. *(The Units are Isomorphisms)*

(1) If L is a lattice, then $\phi_L : L \to \mathbf{L_{X_L}})$ is a lattice isomorphism,

(2) If $\mathbf{X}$ is an L-space, then $\eta_X : X \to \mathbf{X_{L_X}}$ is an L-space homeomorphism where $\eta_X(x) = \{U \in \mathcal{F}i_{clp}(X) \mid x \in U\}$.

We can now state the theorem that the category **Lat** of lattices is dual to the category of **LSp** of L-spaces.

Theorem 2.2.8. *The category* **Lat** *of lattices is dually isomorphic to the category of* **LSp** *of L-spaces.*

The above theorem grounds all of the developments of this thesis.

Finally, we state a few useful lemmas regarding properties of closed and clopen filters. We use these properties repeatedly throughout the thesis and sometimes without reference. A particularly useful fact is the following. We make heavy use of the fact that (i) and (ii) are equivalent.

Lemma 2.2.9. *Let $X = (X, \curlywedge, 1, \tau)$ be an L-space and U be a filter of X. Then the following conditions are equivalent: i) U is closed, ii) U is principle, and iii) U is the interesection of clopen filters.*

We do not provide a proof here. A proof can be found at Lemma 2.8 in [6].

Another useful lemma allows us to separate points in the space from closed downsets using clopen filters.

Lemma 2.2.10. *Let X be an L-space. If D is a closed downset and $y \notin D$, then there is a clopen filter containing y which is disjoint from D.*

Proof. The complement $X - D$ of D is an open upset and can therefore be written as a union $\bigcup_{i \in I} U_i$ of clopen filters $\{U_i\}_{i \in I}$. Since $y \notin D$, this means then that $y \in \bigcup_{i \in I} U_i$ and so there is some $i \in I$ such that $y \in U_i$. And since $X - D = \bigcup_{i \in I} U_i$, it means that $U_i \cap D = \emptyset$.

□

We now have a simple reformulation of compactness.

Lemma 2.2.11. *Let V be an open set and $\{U_i \mid i \in I\}$ be a family of closed sets of a compact topological space $\mathbf{X}$. If $\bigcap\{U_i \mid i \in I\} \subseteq V$, then there is a finite $J \subseteq I$ such that $\bigcap\{U_i \mid i \in J\} \subseteq V$.*

The last lemma we employ relates of the semilattice structure of an L-space to the topology of that L-space in a useful way.

Lemma 2.2.12. *Let $\mathbf{X}$ be an L-space and U be a clopen filter of X. If $x \curlywedge y \in U$, then there are clopen filters V and U such that $V \triangledown W \subseteq U$ and $x \in V$ and $y \in W$.*

Proof. Let U be a clopen filter of $\mathbf{X}$ and suppose that $x \curlywedge y \in U$. Then $\uparrow x \curlywedge y = \uparrow x \triangle \uparrow y \subseteq U$. However, by Lemma 2.2.9, $\uparrow x = \bigcap\{V' \in \mathcal{F}i_{clp}(\mathbf{X}) \mid x \in V'\}$ and $\uparrow y = \bigcap\{W' \in \mathcal{F}i_{clp}(\mathbf{X}) \mid y \in W'\}$. Since$\uparrow x \triangle \uparrow y \subseteq U$, we have that both $\bigcap\{V' \in \mathcal{F}i_{clp}(\mathbf{X}) \mid x \in V'\} \subseteq U$ and $\bigcap\{W' \in \mathcal{F}i_{clp}(\mathbf{X}) \mid y \in W'\} \subseteq U$. But then by compactness, there are clopen filters V and W such that $x \in V$ and $y \in W$ and $V \triangledown W \subseteq U$. □

Then last thing we would like to remark on is that the semilattice that underlies an L-space also always complete. This fact is proved in both [6] in Lemma and [5].

Lemma 2.2.13. *Let $X = (X, \curlywedge, 1, \tau)$ be an L-space then for all $S \subseteq X$, the meet of S exists in X.*

This conclude the preliminaries.

Chapter 3

Products of L-Spaces and the Representation of Monotone Operations

This chapter expands the theory of L-spaces originating in [6] by demonstrating a representation of arbitrary n-ary monotone operations between lattices as $n+1$-ary relations between L-spaces. The point of such a general result is to provide the setting in which we obtain our representation and duality results for $r\ell$-groupoids and residuated lattices in Chapter 6 and ultimately to derive the OKHD-semantics discussed in the introduction and in Chapter 7. In addition, the results reported in this chapter are analogous to the general representation of monotone operations obtained by Moshier and Jipsen in [27, 28] to study duality for lattice expansions. So in addition to generalizing aspects of the representation of modal lattices in [6] to apply to $r\ell$-groupoids and the semantics of subtructural logics, this chapter shows that theory of L-spaces is equally well suited for the study of lattice expansions and that these studies can be carried out in a style familiar to Jónsson and Tarski's famous representation of boolean algebras with operators [25].

In more detail, this chapter is structured as follows. We begin in Section 3.1 with a characterization of products and coproducts in the category of L-spaces (Theorem 3.1.10). This phrasing is a bit misleading since it will turn out that products of L-spaces actually play the role of coproducts. This will be discussed more thoroughly below. These results then naturally leads to a concrete representation of products of lattices (Theorem 3.1.11). In Section 3.2, we use the latter developments to obtain a representation of $n+1$-ary monotone operations between lattices by special continuous relations among L-spaces that we call filter continuous relations (Theorem 3.2.6). As an immediate corollary we obtain a representation theorem for all lattice expansions. We then demonstrate a duality between the category of lattices with monotone operations as morphisms and a category of L-spaces with filter continuous relations between them (Theorem 3.2.11). Finally, Section 3.3 observes some correspondences between properties of the filter continuous relations and their corresponding operations on the algebras of filters of L-spaces. In particular we characterize the filter continuous relations the correspond to meet and join preserving operations and then show how we can recover L-space morphisms from these properties. Notably, the representation of meet preserving operations is peculiar and contrasts the one given in [6]. To make this difference clear we remark on how to represent to modal lattices and how we could potentially recover the representation given by the authors of [6].

3.1 Duality for Products

We begin with the characterization of products and coproducts of L-spaces. Perhaps counterintuitively, we show that products of L-spaces and a special notion of inclusion into the product, have the universal property of coproducts. We then show how this characterization can be used to give a concrete topological representation of products of lattices. The next section will employ these results as a tool for obtaining the representation of monotone operations. In Chapter 8, we will extend these results to obtain characterizations of coproducts of the dual spaces of $r\ell$-groupoids and then a representation theorem for products of $r\ell$-groupoids.

3.1.1 (Co)Products of L-Spaces

One way of forming coproducts of L-spaces is by freely generating meets on the disjoint union of a (finite) family of L-spaces and then equipping this structure with an appropriate topology. However, it turns out that this free construction corresponds to simply taking the direct product of the underlying operational structures and equipping that structure with the product topology. Coproducts of L-spaces are then simply cartesian products at the object level. While this may appear counterintuitive, the role of a cartesian product as a coproduct can be explained in virtue of the semi-lattice structure underlying a given L-space.

As an example, let us consider the direct product $X \times Y$ of two semi-lattices, X and Y. There are inclusion homomorphisms $\gamma_X : X \to X \times Y$ and $\gamma_Y : Y \to X \times Y$ for both semi-lattices such that $\gamma_X(x) = (x, 1_Y)$ and $\gamma_Y(y) = (1_X, y)$. The images of these maps form copies of X and Y inside $X \times Y$. This corresponds roughly to the disjoint union of the X and Y after being quotiented so as to agree on what the top element is. What would be freely generated meets are then identified with the meets of pairs coming from $\gamma_X[X]$ and $\gamma_Y[Y]$. For example, given $\gamma_X(x) \in \gamma_X[X]$ and $\gamma_Y(y) \in \gamma_Y[Y]$, the meet of these two objects $\gamma_X(x) \curlywedge \gamma_Y(y) = (x, y)$.

In the remainder of this section we define products of L-spaces and show that the category of L-spaces is closed under finte products. We then demonstrate that products of L-spaces also play the role of coproducts in the category of L-spaces. While this is enough to conclude that the duals of products coincide with products of their duals spaces, we given a concrete description of the homeomorphism between the dual of a products of the lattice expansions we have been entertaining ourselves with and the products of duals of these algebras. It is important to note that our duality results for products only apply to finite products.

Let us now define direct products of semilattices In this definition we consider arbitrary products but we will restrict attention to finite products once we consider topologies on these structures.

Definition 3.1.1. *(Direct Products of Semilattices)*
Let $\{\mathbf{X}_i = (X_i, \curlywedge_i, 1_i)\}_{i \in I}$ *be a family semilattices.*

The direct product $\prod_{i \in I} \mathbf{X}_i = (\prod_{i \in I} X_i, \wedge, 1)$ *is defined such that:*
1) $\prod_{i \in I} X_i = \{\alpha : I \to \bigcup_{i \in I} X_i \mid \forall i \in I(\alpha(i) \in X_i)\}$,
2) $\curlywedge$ *is defined such that for* $i \in I$*:* $\alpha \wedge \beta$ *is the function such that* $(\alpha \wedge \beta)(i) = \alpha(i) \wedge_i \beta(i)$ *, and*
3) the top 1 *is defined as the function* $1(i) = 1_i$.
For each $i \in I$, p_i *is the projection map where* $p_i(\alpha) = \alpha(i)$.
In addition, for each i *we define the inclusion map* $\gamma_i : X_i \to X$ *so that* $\gamma_i(x)$ *is the function such that* $(\gamma_i(x))(i) = x$ *and for all* $j \neq i$, $(\gamma_i(x))(j) = 1_j$.

When it is convenient we will conflate $\prod_{i\in I} X_i$ with the cartesian product of sets X_i and think of the operations and constants as being defined pointwise. The following lemma states that each the inclusions maps γ_i are essentially injective L-space morphisms without necessarily being continuous since we are not yet considering topologies. We omit proof since the $\gamma_i[X]$ is easily see to be a copy of X in the product.

Lemma 3.1.2. *Each inclusion map $\gamma_i : X_i \to \prod_{i\in I} X_i$ is an injective semilattice homomorphisms that satisfies the following back condition: If $\alpha \curlywedge \beta \leq \gamma_i z$, then there are $x, y \in X$ such that $x \curlywedge_X y \leq z$ and $\alpha \leq fx$ and $\beta \leq fy$.*

In addition to essentially being L-space morhisms, the inclusion $\gamma_j : X_i \to \prod_{i\in I} X_i$ enjoys the property of being right adjoint to the projection $p_j : \prod_{i\in I} X_i \to X_j$. We again omit proofs in this case since the argument is quite straight forward.

Lemma 3.1.3. *(Adjunction Property) Let $\{\mathbf{X}_i\}_{i\in I}$ be a family of semilattices. Let $X = \prod_{i\in I} X_i$. Then for all i, then projection map p_i is left adjoint to the inclusion map γ_i:*

$$p_i(\alpha) \leq x \iff \alpha \leq \gamma_i(x).$$

A simple but useful property of the previous lemma is the following.

Lemma 3.1.4. *Let $\{\mathbf{X}_i\}_{i\in I}$ be a family of semilattices. Let $X = \prod_{i\in I} X_i$ and suppose that $S \subseteq X$ is upward closed. Then $\gamma_i^{-1}[S] = p_i[S]$.*

Proof. Let $S \subseteq X$ be upward closed. Let $x \in \gamma_i^{-1}[S]$. Then $\gamma_i(x) \in S$. $p_i(\gamma_i(x)) \in p_i[S]$. But since $x = p_i(\gamma_i(x))$ (see definitionnitions), we have that $x \in p_i[S]$. For the other inclusion if $x \in p_i[S]$, there is some $\alpha \in S$ such that $p_i(\alpha) = x$. By the adjunction property, it follows that $\alpha \leq \gamma_i(x)$. Therefore, since S is upward closed, we have that $\gamma_i(x) \in S$ and thus that $x \in \gamma_i^{-1}[s]$. We can therefore conclude that $\gamma_i^{-1}[S] = p_i[S]$. □

We now reach the final lemma regarding products of semilattices which we will make extensive use of. We show that products commute with the operation of taking filters.

Lemma 3.1.5. *Let $\{X_i\}_{i\in I}$ be a family of semilattices.*
(1) Suppose that $x \in \mathcal{F}i(\prod_{i\in I} X_i)$, then $p_i[x] \in \mathcal{F}i(X_i)$,
(2) Suppose that for each $i \in I$ we have some $x_i \in \mathcal{F}i(X_i)$, then $\Pi_{i\in I} x_i \in \mathcal{F}i(\prod_{i\in I} X_i)$, and
3) if I is finite, then $\mathcal{F}i(\prod X_i)$ and $\prod \mathcal{F}i(X_i)$ are in bijective correspondence.

Proof. For (1), suppose that $x \in \mathcal{F}i(\prod_{i\in I} X_i)$. clearly, $1_i \in p_i[x]$. For meet closure, let $a, b \in p_i[x]$. Then there are α and β in x such that $\alpha(i) = a$ and $\beta(i) = b$. Since x is filter we have that $\alpha \wedge \beta \in x$. Therefore $a \wedge b = \alpha(i) \wedge \beta(i) = (\alpha \wedge \beta)(i) \in p_i[x]$. For upward closure, let $a \in p_i[x]$ and $a \leq b$. If $a \in p_i[x]$, there is some $\alpha \in x$ such that $\alpha(i) = a$. If follows that the function $\beta \in \prod S_i$ defined such that $\beta(j) = \alpha(j)$ for all $j \neq i$ and $\beta(i) = b$ is such that $\alpha \leq \beta$. Therefore, $\beta \in x$ and so $b \in p_i[x]$. We conclude that $p_i[x]$ is a filter.

For (2), suppose that for each $i \in$ we have some $x_i \in \mathcal{F}i(X_i)$. define $x = \Pi_{i\in I} x_i$. Surely $1_{\prod S_i} \in x$. Now let $\alpha, \beta \in x$. For all i, $(\alpha \wedge \beta)(i) = \alpha(i) \wedge \beta(i) \in x_i$. Therefore $\alpha \wedge \beta \in x$. For upward closure, suppose that $\alpha \in x$ and $\alpha \leq \beta$. Then for all $i \in I$, $\alpha(i) \in x_i$ and $\alpha(i) \leq \beta(i)$. So for all $i \in I$, $\beta(i) \in x_i$. So $\beta \in x$, as desired. We conclude that x is a filter.

Finally, for (3) recall that we consider I to be finite. Also beware we conflate the definition of products as containing choice functions and the definition of products as containing sequences. Then we define the maps $\lambda : \mathcal{F}i(\prod X_i) \to \prod \mathcal{F}i(X_i)$ and $\lambda^* : \prod \mathcal{F}i(X_i) \to \mathcal{F}i(\prod X_i)$ such that

$$\lambda(x) = \langle p_i[x] \rangle_{i \in I} \qquad \lambda^*(\langle x_i \rangle_{i \in I}) = \Pi x_i.$$

λ and λ^* are well defined in virtue of parts (1) and (2) of this lemma. We need to show that λ and λ^* are inverses of one another and therefore need to show $\lambda^*(\lambda(x)) = x$ and $\lambda(\lambda^*(\langle x_i \rangle_{i \in I})) = \langle x_i \rangle_{i \in I}$.

For the identity $\lambda(\lambda^*(\langle x_i \rangle_{i \in I})) = \langle x_i \rangle_{i \in I}$, we note that $\lambda(\lambda^*(\langle x_i \rangle_{i \in I})) = \langle p_i[\Pi x_i] \rangle_{i \in I} = \langle x_i \rangle_{i \in I}$ because for each $i \in I$, $x_i = p_i[\Pi x_i]$.

For the identity $\lambda^*(\lambda(x)) = x$, we note that the inclusion $x \subseteq \lambda^*(\lambda(x))$ is essentially by definition. For the other inclusion, let $\alpha \in \lambda^*(\lambda(x))$ for some filter $x \in \mathcal{F}i(\prod X_i)$. If $\alpha \in \lambda^*(\lambda(x)) = \prod p_i[x]$, then $\alpha(i) \in p_i[x]$ for each i. However, if $\alpha(i) \in p_i[x]$ for each $i \in I$, then there is some $\beta^i \in x$ such that $\beta^i(i) = \alpha(i)$ for each $i \in I$. So since $\beta^i \leq \gamma_i(\alpha(i))$ for each $i \in I$ and x is upward closed, $\gamma(\alpha(i)) \in x$ for each $i \in I$. However, because I is finite and $\gamma_i(\alpha(i)) \in x$ for each $i \in I$, the fact that x is closed under meets implies that $\bigwedge \gamma_i(\alpha(i)) \in x$. However as $\alpha = \bigwedge \gamma_i(\alpha(i))$, we may conclude that $\alpha \in x$ and thus that $\alpha \in \lambda^*(\lambda(x)) \subseteq x$. $\square$

With each of the previous lemma available, let us now formally define the coproducts of L-spaces.

Definition 3.1.6. *(Products of L-spaces.) Let $\{(X_i, \tau_i)_i\}_{i \in I}$ be a family of L-spaces.*

The Product $\prod_{i \in I} \mathbf{X}_i = (X, \tau)$ is defined such that $X = \prod_{i \in I} X_i$ and τ is the product topology on X.

We now state a useful fact that states that the product topology is generated by a subbase consisting of products of subassic elements of the topologies on the factors.

Lemma 3.1.7. *Let $\{(X_i, \tau_i)_i\}_{i \in I}$ be a family of L-spaces. Then*

$$\begin{aligned} \mathcal{S}_P := \{\textstyle\prod_{i \in I} U_i \mid U_i \in \mathcal{F}i_{clp}(X_i) \;\&\; |\{i \mid U_i \neq X_i\}| < \aleph_0\} \\ \cup \{X - (\textstyle\prod_{i \in I} U_i) \mid U_i \in \mathcal{F}i_{clp}(X_i) \;\&\; |\{i \mid U_i \neq \emptyset\}| < \aleph_0\} \end{aligned}$$

is a subbase for the product topology on $\{(X_i, \tau_i)_i\}_{i \in I}$.

Proof. In general fact from topology is that given a collection of topologies $\{\mathbf{Y}_i\}_{i \in I}$ where for each $i \in I$, we have a subbase $\mathcal{S}_i$ for $\mathbf{Y}_i$, a the collection defined by subsituting $\mathcal{S}_i$ for $\mathcal{F}i_{clp}(\mathbf{X}_i)$ in the definition of $\mathcal{S}_P$ is a subbase for the product topology. Since $\mathcal{F}i_{clp}(\mathbf{X}_i) \cup \{X - U \mid U \in \mathcal{F}i_{clp}(\mathbf{X}_i)\}$ is a subbase for each L-space $\mathbf{X}_i$, we are ensured that $\mathcal{S}_P$ is a subbase for the product of L-spaces. $\square$

We now restrict attention to finite products and obtain our first result of this section. In particular we show that the the category of L-spaces is closed under finite products.

Proposition 3.1.8. *A finite product of L-spaces is an L-space.*

Proof. Let $\{(X_i, \tau_i)_i\}_{i \leq n}$ be a finite family of L-spaces and let $\prod_{i \leq n} \mathbf{X}_i = (X, \tau)$ be the product topology.

Given the previous Lemma, which essentially asserts that the product topology has a basis of clopens, and the fact that products preserve compactness, we need to check that HMS-separation holds and that the clopen filters of the product topology are closed $\triangledown$.

(HMS-separation) Suppose that $\alpha \not\leq \beta$. Then there is some i such that $\alpha(i) \not\leq \beta(i)$. Then by HMS-separation in X_i, there is a clopen filter U in X_i such that $\alpha(i) \in U$ and $\beta(i) \notin U$. It follows that $\alpha \in \prod_{j \leq n} V_j$ but $\beta \notin \prod_{j \leq n} V_j$, where $\prod_{j \leq n} V_j$ is such that $V_i = U$ and for all $j \neq i$, $V_j = X_j$. It is also a ready consequence that $\prod_{j \leq n} V_j$ is a clopen filter (See Lemma 3.1.7 and Lemma 3.1.5).

(Closure of Clopen Filters under $\triangledown$) A straight forward argument shows that $\prod_{i \leq n} U_i \triangledown \prod_{i \leq n} V_i = \prod_{i \leq n}(U_i \triangledown V_i)$. Since $\prod_{i \leq n}(U_i \triangledown V_i)$ is clopen in the product topology, so is $\prod_{i \leq n} U_i \triangledown \prod_{i \leq n} V_i = U \triangledown V$.

(1 is clopen) The last thing we need to check is that 1 is clopen. This is ensured again by the subbase lemma and the fact that $\{1\} = \prod \{1_{\mathbf{X}_i}\}$. $\square$

Lemma 3.1.9. *$\{(X, \tau_i)_i\}_{i \in I}$ be a family of L-spaces. Let $X = \prod_{i \in I} \mathbf{X}_i$ be the product. Then for all $i \in I$, $\gamma_i : X_i \to X$ is an L-space morphism.*

We now show that the inclusion morphisms a L-space morphisms (See Definition 2.2.4 for a reminder).

Proof. In virtue of Lemma 3.1.2, we just need to check that γ_i is continuous. However, it then suffices to check that if U is a clopen filter of $X = \prod_{i \in I} \mathbf{X}_i$, then $\gamma_i^{-1}[U]$ is a clopen filter of X_i. However, in virtue of Lemma 3.1.4, we have that $\gamma_i^{-1}[U] = p_i[U]$. So since U necessarily of the form $\prod_{j \in I} V_j$ for a choice of $V_j \subseteq X_j$ for each $j \in I$, we have that $\gamma_i^{-1}[U] = p_i[U] = p_i[\prod_{j \in I} V_j] = V_i$, which is a clopen filter of X_i. $\square$

We now arrive at one of the main results of this section. We show that the products of L-spaces also play the role of coproducts in the category of L-spaces. This ambiguity is what will later allow use to given our topological representation of products of lattices.

Theorem 3.1.10. *(Universal Property of Coproducts) Let $\{(X, \tau_i)_i\}_{i \leq n}$ and Y be a finite collection of L-spaces. Suppose that for each $i \leq n$, there is an L-space morphism $g_i : X_i \to Y$. Then there exists a map $g : \prod_{i \leq n} X_i \to Y$ that uniquely satisfies $g(\gamma_i(x)) = g_i(x)$ for all $x \in X_i$ and $i \leq n$.*

Proof. Let $\{(X, \tau_i)_i\}_{i \leq n}$ and Y be $(r)\ell G$-spaces. Suppose that for each $i \leq n$, there is some L-space morphism $g_i : X_i \to Y$. Recall that $(r)\ell G$-spaces have all meets (Lemma 2.2.13). We define $g : \prod_{i \leq n} X_i \to Y$ such that:

$$g(\alpha) = \bigcurlywedge_{i \leq n} g_i(\alpha(i)).$$

In virtue of this definition it is immediate that $g(\gamma_i(x)) = g_i(x)$. For uniqueness with respect to this property, we recall also that L-space morphisms preserve meets. Let $h : \prod_{i \leq n} X_i \to Y$ be a map such that $h(\gamma_i(x)) = g_i(x)$ for all $x \in X_i$ and $i \in I$. Let $\alpha \in \prod_{i \leq n} X_i$. We claim $h(\alpha) = g(\alpha)$ and therefore that $h = g$. Note that for all $i \leq n$, $g_i(\alpha(i)) = h(\gamma(\alpha(i)))$ by assumption. So we have:

$$g(\alpha) = \bigcurlywedge g_i(\alpha(i)) = \bigcurlywedge h(\gamma_i(\alpha(i)))) = h(\bigcurlywedge \gamma_i(\alpha(i))) = h(\alpha).$$

We must now check the various conditions for g an L-space morphism (see Definition 2.2.4).

For meet preservation:

$$\begin{aligned}
g(\alpha \curlywedge \beta) &= \bigcurlywedge_{i \leq n} g_i(\alpha(i) \curlywedge \beta(i)) \\
&= \bigcurlywedge_{i \leq n} g_i(\alpha(i)) \curlywedge g_i(\beta(i))) \\
&= \bigcurlywedge_{i \leq n} g_i(\alpha(i)) \curlywedge \bigcurlywedge_{i \leq n} g_i(\beta(i))) \\
&= g(\alpha) \curlywedge g(\beta).
\end{aligned}$$

For the back condition, suppose that $x \curlywedge y \leq g(\alpha) = \bigcurlywedge_{i \leq n} g_i(\alpha(i))$. We need to show that there are β and γ such that $\beta \curlywedge \gamma \leq \alpha$ and $x \leq g(\beta)$ and $y \leq g(\gamma)$. If $x \curlywedge y \leq \bigcurlywedge_{i \leq n} g_i(\alpha(i))$, then for each $i \leq n$, $x \curlywedge y \leq g_i(\alpha(i))$. By the back condition for each $i \leq n$, for each $i \leq n$ there are x_i and y_i such that $x_i \curlywedge y_i \leq \alpha(i)$ and $x \leq g_i(x_i)$ and $y \leq g_i(y_i)$. Define β and γ such that $\beta(i) = x_i$ and $\gamma(i) = y_i$ for each $i \leq n$. Then $\beta \curlywedge \gamma \leq \alpha$ and $x \leq g(\beta)$ and $y \leq g(\gamma)$, as desired.

Finally, we need to check that our map g is continuous. Given that each g_i is continuous, it is sufficient to show that for each clopen filter U in Y, then $\prod_{i \leq n} g_i^{-1}[U] = g^{-1}[U]$. The following chain equivalences proves this.

$$\begin{aligned}
\alpha \in \prod_{i \leq n} g_i^{-1}[U] &\iff \forall i \leq n : \; g_i(\alpha(i)) \in U \\
&\iff g(\alpha) = \bigcurlywedge_{i \leq n} g_i(\alpha(i)) \in U \\
&\iff \alpha \in g^{-1}[U].
\end{aligned}$$

The first and last equivalence are by definition. The second equivalence follows from U being a filter and so being closed under meets. Having shown that g meets the requirements to be an L-space morphism in Definition 2.2.4, we conclude our proof. $\square$

We have just shown the possibly counterintuitive result that products of L-spaces play the role of coproducts in the category of L-spaces. In the next section we exploit this fact to provide a representation of products of lattices.

3.1.2 A Representation for Products of Lattices

We now give a concrete representation of products of lattice in virtue of products of their dual L-spaces. This representation may be counter intuitive. But in light of the discussion at the beginning of the previous section, the results of the previous section, and particularly the demonstration that products of L-spaces with their inclusion maps have the universal property of coproducts (Theorem 3.1.10), this result becomes slightly more natural.

Theorem 3.1.11. *Let $\{\mathbf{L}_i\}_{i \leq n}$ be a finite family of lattices. Then:*

$$\mathbf{X}_{\prod_{i \leq n} L_i} \cong \prod_{i \leq n} X_{L_i}$$

Proof. Recall the bijection $\lambda : \mathbf{X}_{\prod_{i \leq n} L_i} \to \prod_{i \leq n} X_{L_i}$ from part 3 of Lemma 3.1.5. λ was defined so that $\lambda(x) = \langle p_i[x] \rangle_{i \in I}$. In order to demonstrate the claim, it is sufficient for us to show that λ is an L-space morphism (see definition 2.2.4).

For meet preservation, it is sufficient to show that for each $i \in I$, $p_i[x \cap y] = p_i[x] \cap p_i[y]$ because then $\lambda(x \cap y) = \langle p_i[x \cap y] \rangle_{i \in I} = \langle p_i[x] \cap p_i[y] \rangle_{i \in I} = \langle p_i[x] \rangle_{i \in I} \curlywedge \langle p_i[y] \rangle_{i \in I} = \lambda(x) \curlywedge \lambda(y)$. So let $a \in p_i[x \cap y]$. then there is some $\alpha \in x \cap y$ such that $\alpha(i) = a$. But then $a \in p_i[x_i] \cap p_i[y]$. For the other inclusion, suppose that $a \in p_i[x - \cap p_i[y]$. Then there are $\alpha \in x$ and $\beta \in y$ such that $a = \alpha(i) = \beta(i)$. But then $\alpha \leq \gamma_i(a)$ and $\beta \leq \gamma_i(a)$. Therefore, since both x and y are upward closed we have that $\gamma_i(a) \in x \cap y$. There we obtain $a \in p_i[x \cap y]$, as desired.

For the back condition associated with $\curlywedge$, let $x \curlywedge y \leq \lambda(z)$ where $x = \langle x_i \rangle_{i \in I}$ and $y = \langle y_i \rangle_{i \in I}$. Note the fact that λ preserves meets and is bijective implies that λ's inverse λ^* does too. So we have that $\lambda^* x \curlywedge \lambda^* y = \lambda(x \curlywedge y) \leq z$. Thus we have shown that that λ satisfies the back condition.

Finally, for continuity, let U be a clopen filter of $\prod_{i \leq n} X_{L_i}$. We must show that $\lambda^*[U]$ is clopen in $\mathbf{X}_{\prod_{i \leq n} L_i}$. For notations sake, let us have that $\mathbf{L} = \prod_{i \leq n} \mathbf{L}_i$. If U is a clopen filter of $\prod_{i \leq n} \mathbf{X}_{\mathbf{L_i}}$, then there are $a_1, ..., a_n$ such that $U = \prod_{i \leq n} \phi_{\mathbf{L}_i}(a_i)$ (see Lemma 3.1.7). We claim that $\lambda^*[U] = \phi_{\mathbf{L}}(\langle a_i \rangle_{i \leq n}$. So let $\lambda^*(\langle x_i \rangle_{i \leq n}) \in \lambda^*[U]$. Then since $U = \prod_{i \leq n} \phi_{\mathbf{L}_i}(a_i)$, for each $i \leq n$ $x_n \in \phi_{\mathbf{L}_i}(a_i)$. Therefore, $\langle a_i \rangle_{i \leq n} \in \lambda^*(\langle x_i \rangle_{i \leq n})$ and so $\lambda^*(\langle x_i \rangle_{i \leq n}) \in \phi_{\mathbf{L}}(\langle a_i \rangle_{i \leq n})$. For the other direction, let $x \in \phi_{\mathbf{L}}(\langle a_i \rangle_{i \leq n})$. Then $p_i[x] \in \phi_{\mathbf{L}_i}(a_i)$ for each $i \leq n$. This then implies that $\lambda(x) \in \prod_{i \leq n} \phi_{\mathbf{L}_i}(a_i)$ and so since $x = \lambda^*(\lambda(x))$ we can conclude that $x \in \lambda^*[U]$, as desired. We therefore have shown that λ is continuous and so meets all of the requirements of being an L-space morphism. $\square$

Corollary 3.1.12. *Let $\{\mathbf{L}_i\}_{i \leq n}$ be a finite family of lattices, then: $\phi_{L_1}(a_1) \times ... \times \phi_{L_n}(a_n) \cong \phi_{\prod L_{i \leq n}}((a_1, .., a_n))$.*

Proof. Corollary of 3.1.11, in particular the case of showing λ is continuous, and the fact that by duality the salient lattices of clopen sets are isomorphic. $\square$

We now move to the representation of monotone operations, which will make essentail use of the representation of products we have just given.

3.2 The General Representation for Monotone Operations

In [27, 28] and [22], which provide spectral style dualities for lattices and posets respectively, representations of monotonic functions between lattices are given. In this section we extend the L-space duality in a similar way by showing that every monotone n-ary operation between lattices can be represented as an $n+1$-ary relation between the duals of those lattices. We then obtain a representation theorem for all monotone lattice expansions and then generalize the L-spaces duality. Some of the results presented in this section are direct generalizations of those reported in [6] and do not provide significant theoretical advancements. This being said, generalizing to the case of n-ary operations will set the stage for chapters to come and provide the means of proving a general result on what sorts of identities can be preserved through the Π_1-completion.

We begin in Section 3.2.1 by defining *filter continuous relations.* In Section 3.2.2 we will show in Theorem 3.2.6 that every operation between lattices is represented uniquely by some filter continuous relation. This leads naturally to a representation theorem for all monotone lattice expansions in Corollary 3.2.7. We end this section with some remarks on the relation of this representation result to the dualities in [6]. Finally, in Section 3.2.3 will show in Theorem 3.2.11 that the category of lattices with monotone operations is dual to the category of the L-spaces with filter continuous relations as morphisms. These results will be key to those regarding completions and persistence in the following sections of the chapter.

3.2.1 Filter Continuous Relations

Definition 3.2.1. *Let $\{X_i = (X_i, \curlywedge_i, 1_i, \tau_i)\}_{i\leq n} \cup \{Y = (Y, \curlywedge, 1, \tau)\}$ be a family of L-spaces with $n \in \mathbb{N}$. Suppose that $R \subseteq Y \times X_1 \times .. \times X_n$.*

We say that R is filter continuous if it satisfies the following conditions:

(Clopen compatible) $R(y, x_1, .., x_n)$ iff for all in and $U_i \in \mathcal{F}i_{clp}(X_i)$: If $x_i \in U_i$, then $y \in F[U_1, ..., U_n]\{y \in Y \mid \exists x_1 \in U_1 ... \exists x_n \in U_n(R(y, x_1, .., x_n)\}$, and

(Clopen-continuous) If $U_i \in \mathcal{F}i_{clp}(X_i)$ for each $i \leq n$, then $F_R[U_1, .., U_n]$ is in $\mathcal{F}i_{clp}(Y)$.

Note that in the case that R is binary, $F_R[U] = R^{-1}[U]$.

We will now provide two lemmas we make use in the following pages. The first, Lemma 3.2.2, provides some useful properties of filter continuous relations. The second, Lemma 3.2.3, will made use of in two important places. The first is to show that the composition of filter continuous relations is a filter continuous relation in Lemma 3.2.8. The second is to show that the various completions we are interested in when applied to monotone lattice expansions can be represented in the dual space of the lattice being completed (Lemma 4.2.2 and Theorem 4.2.3).

Lemma 3.2.2. *(Properties of Filter Continuity)*

Let $\{X_i = (X_i, \curlywedge_i, 1_i, \tau_i)\}_{i\leq n} \cup \{Y = (Y, \curlywedge, 1, \tau)\}$ be a family of L-spaces with $n \in \mathbb{N}$. Suppose that $R \subseteq Y \times X_1 \times .. \times X_n$.

If R is clopen-compatible, then it satisfies each of the following properties:

i) (Order Compatibility) If $R(y, x_1, .., x_n)$, $y \leq y'$, and $x'_i \leq x_i$ for each $i \leq n$, we have $R(y', x'_1, .., x'_n)$,

ii) ($\curlywedge$-Compatibility) If $R(y_j, x_{j1}, ..x_{jn})$ for all $j \in J$, then $R(\bigcurlywedge y_j, \bigcurlywedge x_{j1}, .., \bigcurlywedge x_{jn})$.

iii) (Point Closed) For all $y \in Y$, $R[y] = \{(x_1, .., x_n) \mid R(y, x_1, .., x_n)\}$ is closed in the product topology $P = (\prod_{i\leq n} X_i, \curlywedge_P, 1_P, \tau_P)$.

iv) (Boundedness) For all $i \leq n$ and $x_i \in X_i$, $R(1, x_1, .., x_n)$.

Proof. We prove each claim in turn:

(Order Compatiblity) Suppose that $R(y, x_1, .., x_n)$, $y \leq y'$, and $x'_i \leq x_i$ for each $i \leq n$. Suppose now that for $U_i \in \mathcal{F}i_{clp}(X_i)$ we have $x'_i \in U_i$. Then clearly by upward closure of the U_i's, we have that $x_i \in U_i$ for each $i \leq n$. Therefore, by clopen compatibility and the assumption that $R(y, x_1, .., x_n)$, we have $y \in F_R[U_1, .., U_n]$ and thus by upward closure again, which is guaranteed by the fact that $F_R[U_1, .., U_n]$ is a filter, we have that $y' \in F_R[U_1, .., U_n]$. This then implies again by clopen compatibility that $R(y', x'_1, .., x'_n)$, as desired.

($\curlywedge$-Compatibility) An analogous argument holds but relies on the fact that clopen filters are closed under arbitrary meets (which follows from the fact that closed filters are principal. See Lemma 2.2.9).

(Point-Closure) We claim that

$$\{(x_1,..,x_n) \mid R(y,x_1,...,x_n)\} = \bigcap\{X^n - (\prod_{i\leq n} U_i) \mid U_i \in \mathcal{F}i_{clp}(X_i) \ \&\ y \notin F_R[U_1,..,U_n]\}.$$

($\subseteq$) suppose that $(x_1,..,x_n) \in \{(x'_1,..,x'_n) \mid R(y,x'_1,...,x'_n)\}$ and $y \notin F_R[U_1,..,U_n]$ for some clopen filters $U_i \in \mathcal{F}i_{clp}(X_i)$. Then $R(y,x_1,..,x_n)$. So by clopen-compatibility of R, we have that there is some $i \leq n$ such that $x_i \notin U_i$. It follows that then that $(x_1,..,x_n) \in X^n - (\prod_{i\leq n} U_i)$, as desired.

($\supseteq$) We reason contrapositively. Suppose that it is *not* the case that $(x_1,..,x_n) \in \{(x'_1,..,x'_n) \mid R(y,x'_1,...,x'_n)\}$ and therefore that not $R(y,x_1,..,x_n)$. It follows by clopen compatability that for each $i \leq n$ there is some $U_i \in \mathcal{F}i_{clp}(X_i)$ such that $x_i \in U_i$ for each $i \leq n$ but $y \notin F_R[U_1,..,U_n]$. It follows then that $(x_1,..,x_n) \notin \bigcap\{X^n - (\prod_{i\leq n} U_i) \mid U_i \in \mathcal{F}i_{clp}(X_i) \ \&\ y \notin F_R[U_1,..,U_n]\}$.

Finally we are ensured that $\bigcap\{X^n - (\prod_{i\leq n} U_i) \mid U_i \in \mathcal{F}i_{clp}(X_i) \ \&\ y \notin F_R[U_1,..,U_n]\}$ is a closed set in the product topology because $(\prod_{i\leq n} U_i)$ is clopen in the product topology by Lemma 3.1.7.

(Boundedness) By clopen continuity, for all $i \leq n$ and all $x_i \in X_i$ and $U_i \in \mathcal{F}i_{clp}(X_i)$ such that $x_i \in U_i$, we have that $1 \in F[U_1,..,U_n]$. Therefore by clopen compatibility, we have that $R(1,x_1,..,x_n)$. □

Our second lemma shows how F_R extends from the clopen filters to the closed filters and filters of an L-space. This lemma is essentially the same as Lemma 4.18 in [6].

Lemma 3.2.3. *Let $R \subseteq X \times Y_1 \times ... \times Y_n$ be a filter-continuous relation between L-spaces $\mathbf{X}$ and $\mathbf{Y}_i$ for $i \leq n$. Then:*

1) For all closed filter $C_1,..,C_n$ of $\mathbf{Y}$: $F_R[C_1,..,C_n] = \bigcap\{F_R[U_1,..,U_n] \mid U_i \in \mathcal{F}i_{clp}(\mathbf{Y}_i) \ \&\ C_i \subseteq U_i\}$,

2) For all filters $U_1,..,U_n$ of $\mathbf{Y}$: $F_R[U_1,..,U_n] = \bigvee\{F_R[C_1,..,C_n] \mid C_i \in \mathcal{F}i_k(\mathbf{Y}_i) \ \&\ C_i \subseteq U_i\}$.

Proof. The proof of this lemma is a direct generalization of the proof of Lemma 4.18 in [6].

For (1), let $C_1,..,C_n$ be closed filters of $\mathbf{Y}$. The inclusion $F_R[C_1,..,C_n] \subseteq \bigcap\{F_R[U_1,..,U_n] \mid U_i \in \mathcal{F}i_{clp}(\mathbf{Y}_i) \ \&\ C_i \subseteq U_i\}$ is immediate from monotonicity of F_R. For the other inclusions suppose that $x \notin F_R[C_1,..,C_n]$. Then $R[x] \cap (C_1 \times ... \times C_n) = \emptyset$. Since $R[x]$ is a closed downset, by lemmas 2.2.9 and 2.2.10 we obtain that there is some clopen filter $U_1 \times ... \times U_n$ where $U_i \in \mathcal{F}i_{clp}(\mathbf{Y}_i)$ (See Lemma 3.1.7 for subbase products.) and $C_1 \times ... \times C_n \subseteq U_1 \times ... \times U_n$ and $R[x] \cap (U_1 \times ... \times U_n) = \emptyset$. Therefore, we have that $x \notin F_R[U_1,..,U_n]$ for some $U_1,..,U_n$ such that for $i \leq n$, $U_i \in \mathcal{F}i_{clp}(\mathbf{Y}_i)$ and $C_i \subseteq U_i$. And thus we arrive at the conclusion that $x \notin \bigcap\{F_R[U_1,..,U_n] \mid U_i \in \mathcal{F}i_{clp}(\mathbf{Y}_i) \ \&\ C_i \subseteq U_i\}$.

For (2), The inclusion $\bigvee\{F_R[C_1,..,C_n] \mid C_i \in \mathcal{F}i_k(\mathbf{Y}_i) \ \&\ C_i \subseteq U_i\} \subseteq F_R[U_1,..,U_n]$ is immediate by monotonicity of F_R. For the other inclusion, let $x \in R_R[U_1,..,U_n]$. Then there are $y_i \in U_i$ such that $R(x,y_1,..,y_n)$. By Lemma 2.2.9, $\uparrow y_i$ is closed for each $i \leq n$. Since $R(x,y_1,..,y_n)$, it follows that $x \in F_R[\uparrow y_1,..,\uparrow y_n]$. Therefore, the preceding two sentences implt that $x \in \bigvee\{F_R[C_1,..,C_n] \mid C_i \in \mathcal{F}i_k(\mathbf{Y}_i) \ \&\ C_i \subseteq U_i\}$ as desired. □

With Lemmas 3.2.2 and 3.2.3 in hand, we turn to the representation of monotone operations by filter-continuous relations and then the duality result showing the category of lattice with monotone operations is dual to the category of the L-spaces with filter continuos relations among them.

3.2.2 The Representation of Monotone Operations

We will show in Theorem 3.2.6 that for every monotone operation $f : \mathbf{L}_1 \times ... \times \mathbf{L}_n \to \mathbf{K}$ between lattices, there is a unique filter continuous relation R_f among some L-spaces so that the following diagram commutes.

$$\begin{array}{ccc} L_1 \times ... \times \mathbf{L}_n & \xleftrightarrow{\phi_{L_1 \times ... \times L_n}} & \mathbf{L}_{\mathbf{X}_{\mathbf{L_1}}} \times ... \times \mathbf{L}_{\mathbf{X}_{\mathbf{L_n}}} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle F_{R_f}} \\ \mathbf{K} & \xleftrightarrow[\phi_K]{} & \mathbf{K}_{\mathbf{X}_\mathbf{K}} \end{array}$$

So if $f : \mathbf{L}_1 \times ... \times \mathbf{L}_n \to K$ is an n-ary operation between bounded lattices. We can canonically define a relation $R_f \subseteq \mathbf{X_K} \times \mathbf{X_{L_1}} \times ... \times \mathbf{X_{L_n}}$ between the filters of K and the L_1 up through L_n as follows.

For all $y \in X_K$ and $x_i \in X_{LIi}$, $R_f(y, x_1, .., x_n)$ iff if $a_1 \in x_1, .., a_n \in x_n$, then $f(a_1, .., a_n) \in y$.

Lemma 3.2.4. *(Filter Continuity)*

Let $f : \mathbf{L}_1 \times ... \times \mathbf{L}_n \to K$ be an n-ary operation between bounded lattices. Then the relation $R_f \subseteq \mathbf{X_K} \times \mathbf{X_{L_1}} \times ... \times \mathbf{X_{L_n}}$ is filter continuous.

Proof. We demonstrate the two conditions of filter continuity as follows:

(Clopen Compatability) The direction from left to right follows in virtue of the definition of $F_{R_f}[\cdot]$. For the direction from right to left we must rely on the equivalence between $a \in x$ and $x \in \phi(a)$.

(Clopen Continuous) We show that $F_{R_f}[\phi_{L_1}(a_1), .., \phi_{L_n}(a_n)] = \phi_K(f(a_1, .., a_n)$, which via the isomorphisms ϕ_K between K and $\mathcal{F}i_{clp}(X_K)$ and ϕ_{L_i} between L_i and $\mathcal{F}i_{clp}(X_i)$ for $i \leq n$, respectively, will establish the desired result.

($\subseteq$) Suppose that $y \in F_{R_f}[\phi_{L_1}(a_1), .., \phi_{L_n}(a_n)]$. Then for each $i \leq n$, there is some $x_i \in \phi_{K_i}(a_i)$ such that $R_f(y, x_1, .., x_n)$. However, if $x_i \in \phi_{K_i}(a_i)$, then $a_i \in x_i$ for each $i \leq n$. Therefore by the definitionnition of R_f, $f(a_1, .., a_n) \in y$. This in turn implies that $y \in \phi(f(a_1, ..a_n))$, as desired.

($\supseteq$) Suppose that $y \in \phi_K(f(a_1, .., a_n))$, then $f(a_1, .., a_n) \in y$. Note that for each $i \leq n$, $\uparrow(a_i) \in \phi(a_i)$. fro each $i \leq n$ suppose that there is a b_i such that $a_i \leq b_i$. Then $f(a_1, .., a_n) \leq f(b_1, .., b_n)$ by monotonicity and so $f(b_1, .., b_n) \in y$. Therefore, $R_f(y, \uparrow(a_1), .., \uparrow(a_n))$, which in turn implies $y \in F_{R_f}[\phi_{L_1}(a_1), .., \phi_{L_n}(a_n)]$. ☐

We provide a supporting lemma which will allow us to show the uniqueness of R_f with respect to the property expressed by the diagram above.

Lemma 3.2.5. *Suppose that $S, R \subseteq \mathbf{X_K} \times \mathbf{X_{L_1}} \times ... \times \mathbf{X_{L_n}}$ are filter continuous relations, then $S \subseteq R$ iff for all $a_1 \in L_1$,.., and $a_n \in L_n$, $F_S[\phi(a_1), .., \phi(a_n)] \subseteq F_R[\phi(a_1), .., \phi(a_n)]$.*

Proof. For the direction from left to right, the argument is straightforward. From right to left, we reason contrapositively. Suppose that $S \not\subseteq R$, then is a sequence $(y, x_1, .., x_n) \in S$ where $(y, x_1, .., x_n) \notin R$. This means that $(x_1, .., x_n) \in S[y]$ but $(x_1, .., x_n) \notin R[y]$. However, $R[y]$ is a closed downset in virtue of Lemma 3.2.2 (in particular point closedness and order compatibility) and $R[y]$ excludes $(x_1, .., x_n)$. So by Lemma 2.2.10, there is a clopen filter $U = \phi_{L_1}(a_1) \times, .., \phi_{L_n}(a_n)$ of $\prod_{i \leq n} X_{L_1}$ such that $(x_1, .., x_n \in U$ and $R[x] \cap U = \emptyset$. But then $y \in F_S[\phi_{L_1}(a_1), ., \phi_{L_n}(a_n)]$ but $x \notin F_R[\phi_{L_1}(a_1), ., \phi_{L_n}(a_n)]$. Therefore $F_S[\phi(a_1), .., \phi(a_n)] \not\subseteq F_R[\phi(a_1), .., \phi(a_n)]$. ☐

The above proof is accomplished in terms of the method used by Gehrke and van Gool to represent join preserving functions in in [21]. We are now in a position to conclude the remain result of this section. The preceding lemmas allow us to show that every monotone lattice operation corresponds uniquely to a filter continuous relation between relevant L-spaces.

Theorem 3.2.6. *(Representation of Monotone Maps) Let $f : \mathbf{L}_1 \times ... \times \mathbf{L}_n \to \mathbf{K}$ be a monotone lattice map. Then R_f is the unique n+1-ary filter-continuous relation on $\mathbf{X_K} \times \mathbf{X_{L_1}} \times ... \times \mathbf{X_{L_n}}$ such that $\phi_K(f(a_1, .., a_n)) = F_R[\phi_{L_1}(a), .., \phi_{L_n}(a_n)]$.*

Proof. In virtue of the lemma 3.2.4 we only need to show that that R_f *uniquely* satisfies $F_R[\phi_{L_1}(a), .., \phi_{L_n}(a_n)] = \phi_K(f(a_1, .., a_n))$. However the previous Lemma 3.2.5 implies this rather directly. Suppose that $S \subseteq X_K \times X_{L_1} \times .. \times X_{L_n}$ and suppose that $F_S[\phi_{L_1}(a_1), .., \phi_{L_n}(a_n)] = \phi_K(f(a_1, .., a_n)$. Then $F_S[\phi_{L_1}(a_1), .., \phi_{L_n}(a_n)] = \phi_K(f(a_1, .., a_n) = F_R[\phi_{L_1}(a_1), .., \phi_{L_n}(a_n)]$. Then by lemma 3.2.5, $S = R_h$. ☐

One corollary of the representation of monotone maps given in the previous theorem is that every monotone lattice expansion $(L, \{f_i\}_{i\in I})$ is isomorphic to $(\mathcal{F}i_{clp}(\mathbf{X_L}), \{F_{R_{f_i}}\}_{i\in I})$.

Corollary 3.2.7. *(Representation for Monotone Lattice Expansions) Every monotone lattice expansion $(L, \{f_i\}_{i\in I})$ is isomorphic to the clopen filters of an L-space $\mathbf{X} = (X, \{R_i\}_{i\in I})$ equipped with a family of filter continuous relations $\{R_i\}_{i\in I}$.*

Proof. Note that the lattice isomorphism $\phi : L \to \mathcal{F}i_{clp}(\mathbf{X_L})$ is also a homomorphism with respect to each operation f_i. In particular, in virtue of Theorem 3.2.6, we have $\phi_K(f(a_1, .., a_n)) = F_R[\phi_{L_1}(a), .., \phi_{L_n}(a_n)]$. ☐

We conclude this section. In the next we prove a generalization of L-space duality.

3.2.3 Duality for the Category of Lattices with Monotone Operations

A more general consequence of Theorem 3.2.6 is that the category $\mathbf{Lat}(\mathbb{O})$ of lattices with monotone operations as morphisms is dual to the category $\mathbf{LSp}(\mathbb{FC})$ of L-spaces with filter-continuous relations as morphisms. This is a generalization of the L-space duality of Bezhanishvili et al. [6] and we will informally discuss how to recover L-space duality in the following section.

To demonstrate that $\mathbf{LSp}(\mathbb{FC})$ is indeed a category, we must identify a means of composing filter continuous relations. To do this, we will utilize the fact that each $n+1$-ary filter-continuous relation is equivalent to binary one. In particular, given an n+1-ary filter-continuous relation $R \subseteq Y \times X_1 \times .. \times X_n$ we define the corresponding binary filter continuous relations $R^b \subseteq Y \times (X_1 \times .. \times X_n)$ (notice the brackets) in the natural way:

$$R^b(y, x) \iff R(y, x(1), .., x(n)).$$

The fact the R^b is filter continuous follows from Theorem 3.1.11, which tells us that the operations mapping lattices to L-spaces and L-spaces to lattices commute with products. This ambiguity between $n+1$-ary relations and binary relations will allow us treat composition of morphisms in $\mathbf{LSp}(\mathbb{FC})$ as ordinary relation composition. So given filter continuous relations $R \subseteq X \times Y$ and $S \subseteq Y \times X$: we define the composition $R * S$ of R and S in the usual way:

$$R * S = \{(x,y) \in X \times Z \mid \exists y \in Y((x,y) \in R \ \&\ (y,z) \in S))\}.$$

Finally, it is worth noting that given an n-ary monotone operation $f : L_1 \times ... \times L_n \to K$ we can think of the relation $R_f :\subseteq X_K \times X_{L_1} \times ... \times X_{L_n}$ and its binary counterpart $R^b_f \subseteq X_K \times (X_{L_1} \times ... \times X_{L_n})$ as being equivalent to the binary relation $R'_f \subseteq X_K \times X_{L_1 \times ... \times L_n}$. In essence this boils down to the following diagram commuting.

$$\begin{tikzcd}
\mathbf{X_K} \arrow[rr, leftrightarrow, "id"] \arrow[d, rightsquigarrow, "R^b_f"'] & & \mathbf{X_K} \arrow[d, rightsquigarrow, "R'_f"] \\
\mathbf{X_{L_1}} \times ... \times \mathbf{X_{L_n}} \arrow[rr, leftrightarrow, "\cong"] & & \mathbf{X_{(L_1 \times ... \times L_n)}}
\end{tikzcd}$$

With all of these considerations in mind, let us now prove the final lemmas needed to show that the categories $\mathbf{Lat}(\mathbb{O})$ and $\mathbf{LSp}(\mathbb{FC})$ are dually equivalent. In particular, we will check that really forms $\mathbf{LSp}(\mathbb{FC})$ a category in Lemma 3.2.10. To obtain this fact, we will first prove Lemma 3.2.8, which guarantees that the composition of filter continuous relations are filter continuous. We will then show how to turn L-space morphisms into filter continuous relations in Lemma 3.2.9. This allows us to characterize the identity morphisms in $\mathbf{LSp}(\mathbb{FC})$ and leads to the fact that $\mathbf{LSp}$ is a subcategory of $\mathbf{LSp}(\mathbb{FC})$.

Lemma 3.2.8. *Let* $\mathbf{X}$, $\mathbf{Y}$, *and* $\mathbf{Z}$ *be L-spaces and* $R \subseteq X \times Y$ *and* $S \subseteq Y \times Z$ *be filter continuous relations. Then:*

1) for all filters $U \in \mathcal{F}i(\mathbf{Z})$, $F_{R*S}[U] = F_R * F_S[U]$, *and*

2) $R * S$ *is a filter continuous relation.*

Proof. Let $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ be filter continuous relations.

For (1), Let U be a filter of $\mathbf{Z}$. Let $x \in F_{R*S}[U]$. Then there is some $z \in U$ such that $(x,z) \in R * S$. But then there is a $y \in Y$ such that $(x,y) \in R$ and $(y,z) \in S$. But then $y \in F_S[U]$ and $x \in F_R[F_S[U]]$. So $F_{R*S}[U] \subseteq F_R[F_S[U]]$. On the other hand, let $x \in F_R[F_S[U]]$. Then there is some $y \in F_S[U]$ such that $(x,y) \in R$. But if $y \in F_S[U]$, there is some $z \in U$ such that $(y,z) \in S$. It then follows that $(x,z) \in R * S$ and $x \in F_{R*S}[U]$.

For (2), we need to show both condition of Definition 3.2.1 hold of $R * S$.

(Clopen Compatibility) One direction of this condition is by definition. For the nontrivial direction, suppose that for all $V \in \mathcal{F}i_{clp}(\mathbf{Z})$, if $z \in V$, then $z \in F_{R*S}[V]$. We need to show that $(x,z) \in R * S$. Let us define $y = \curlywedge(F_S[\uparrow z])$.

Claim 1: $(y,z) \in S$. To show this, Let $U \in \mathcal{F}i_{clp}(\mathbf{Z})$ and suppose that $z \in U$. Since $\uparrow z \subseteq U$, we have that $F_S[\uparrow z] \subseteq F_S[U]$. It then follows from the fact that U is closed, and therefore principal (Lemma 2.2.9), that $y = \curlywedge(F_S[\uparrow z]) \in F_s[U]$. Therefore, by (Clopen Compatibility) of S, $(y,z) \in S$.

Claim 2: $(x,y) \in R$. To show this we will show that for all $U \in \mathcal{F}i_{clp}(\mathbf{Y})$, if $y \in U$, then $x \in F_R[U]$ and use clopen compatibility of R. So let $U \in \mathcal{F}i_{clp}(\mathbf{Y})$ and suppose that $y \in U$. Since $\uparrow z$ is closed, by Lemma 3.2.3, $F_S[\uparrow z] = \bigcap\{F_S[V] \mid V \in \mathcal{F}i_{clp}(\mathbf{Z}) \ \&\ z \in V\}$. Therefore, by Lemma 2.2.9 we have $\uparrow y = F_S[\uparrow z]$ and by the assumption that $y \in U$, we have $\bigcap\{F_S[V] \mid V \in \mathcal{F}i_{clp}(\mathbf{Z}) \ \&\ z \in V\} \subseteq U$. But then because U is clopen, by compactness we obtain that there are $V_1, .., V_n$ such that $z \in \bigcap_{i \leq n} V_i$ for each $i \leq n$ and $\bigcap_{i \leq n} F_S[V_i] \subseteq U$. Now by our assumption that for all $V \in \mathcal{F}i_{clp}(\mathbf{Z})$, if $z \in V$, then $z \in F_{R*S}[V]$ and the fact that $z \in \bigcap_{i \leq n} V_i$, we have that $x \in F_{R*S}[\bigcap_{i \leq n} V_i]$. But from (1) we then have that $x \in F_R[F_S[\bigcap_{i\leq n} V_i]]$. So from the monotonicity of F_R and F_S we have:

$$F_R[F_S[\bigcap_{i\leq n} V_i]] \subseteq F_R[\bigcap_{i\leq n} F_S V_i]] \subseteq F_R[U].$$

We may therefore conclude that $x \in F_R[U]$. So by generalizing on U, we obtain from clopen compatibility of R that $(x, y) \in R$, as desired.

It then follows from (Claim 1) and (Claim 2) that $(x, z) \in R * S$, as we wished to show. We therefore conclude that $R * S$ is clopen compatible.

(Clopen Continuity) Let $U\mathcal{F}i_{clp}(\mathbf{Z})$. By the clopen continuity of R and S, we have that $F_R * F_S[U] \in \mathcal{F}i_{clp}(\mathbf{X})$. By (1) $F_{R*S}[U] = F_R * F_S[U]$, so $F_{R*S}[U]$ is a clopen filter. Thus $R * S$ is clopen continuous. □

Let us now show that each L-space morphism can be turned into a filter continuous relation.

Lemma 3.2.9. *Let $f : X \to Y$ be an L-space morphism, then*

$$R^f := \bigcup\{\uparrow x \times \downarrow fx \mid x \in X\}$$

is a filter continuous relation.

Proof. We will first show that for all filters U of $\mathbf{Y}$, $F_{R^f}[U] = f^{-1}[U]$. Let U be a filter of $\mathbf{Y}$. For the inclusion $F_{R^f}[U] \subseteq f^{-1}[U]$, let $x \in F_{R^f}[U]$. Then there is some $y \in U$ such that $(x, y) \in R^f$. If $(x, y) \in R^f$, then by definition of ^{f} there is some $z \in X$ such that $z \leq x$ and $y \leq fz$. So since f is an L-space morphism, and is therefore monotone, $y \leq fz \leq fx$. But then since $y \in U$ and U is upward closed, $fx \in U$ and so $x \in f^{-1}[U]$, as desired. For the other inclusion $f^{-1}[U] \subseteq F_{R^f}[U]$, let $x \in f^{-1}[U]$. Then $(x, fx) \in R^f$ and $fx \in U$. Therefore $x \in F_{R^f}[U]$.

We now show that R^f is filter continuous. We have two conditions to check (See Definition 3.2.4). For the condition of clopen-continuity, whenever U is a clopen filter of $\mathbf{Y}$, $F_{R^f}[U] = f^{-1}[U]$ is clopen since f is a L-space morphism. For the condition of clopen-compatibility, we must check the non trivial direction. So suppose that for all clopen filters U of Y, if $y \in U$, then $x \in F_{R^f}[U]$. We must show that $(x, y) \in R^f$. By the hypothesis that for all clopen filters U of Y, if $y \in U$, then $x \in F_{R^f}[U]$, we obtain that $x \in \bigcap\{F_{R^f}[V] \mid V \in \mathcal{F}i_{clp}(\mathbf{Y})\ \&\ y \in V\}$. But by the fact that $\uparrow y$ is closed (Lemma 2.2.9), we obtain from Lemma 3.2.3 that $F_{R^f}[\uparrow y] = \bigcap\{F_{R^f}[V] \mid V \in \mathcal{F}i_{clp}(\mathbf{Y})\ \&\ y \in V\}$. So $x \in F_{R^f}[\uparrow y]$. But above we showed that for all filters V, $F_{R^f}[V] = f^{-1}[V]$, and so in particular that $F_{R^f}[\uparrow y] = f^{-1}[\uparrow y]$. So $x \in f^{-1}[\uparrow y]$ implying that $y \leq fx$. However, if $y \leq fx$, then $(x, y) \in \uparrow x \times \downarrow fx \subseteq R^f$, as desired.

We can therefore conclude that R^f is filter continuous. □

Lemma 3.2.10. **LSp**$(\mathbb{FC})$ *is a category.*

Proof. By Lemma 3.2.8 we are ensured that the composition of filter continuous relations are filter continuous. Therefore the morphisms of $\mathbf{LSp}(\mathbb{FC})$ are closed under composition. Therefore, all that is left to do is to confirm the existence of identity morphisms. Using Lemma 3.2.9, which gives a recipe for transforming L-space morphisms into filter continous relations, we have the following definition. Let $\mathbf{Y}$ be an L-space, then $R^{id}_{\mathbf{Y}} \subseteq Y \times Y$ is defined:

$$R^{id}_{\mathbf{Y}} = \bigcup\{\uparrow y \times \downarrow y \mid y \in Y\}.$$

By Lemma 3.2.9, $R^{id}_{\mathbf{X}}$ is filter continuous since $id : \mathbf{Y} \to \mathbf{Y}$ is an L-space morphism. We now check that for any filter continuous relations $R \subseteq X \times Y$ and $R' \subseteq Y \times Z$, we have that:

$$R * R^{id}_{\mathbf{Y}} = R \qquad R^{id}_{\mathbf{Y}} * R' = R'.$$

For showing $R * R^{id}_{\mathbf{Y}} = R$, note that $R \subseteq R * R^{id}_{\mathbf{Y}}$ is immediate since for any $(x, y) \in R$, $(y, y) \in R^{id}_{\mathbf{Y}}$. For the other inclusion, let $(x, z) \in R * R^{id}_{\mathbf{Y}}$. Then there is some $z' \in Y$ such that $(x, z') \in R$ and $(z', z) \in R^{id}_{\mathbf{Y}}$. However, if $(z', z) \in R^{id}_{\mathbf{Y}}$, then there is some $y \in Y$ such that $z \leq y \leq z'$. But since R is filter continuous and therefore is order compatibility (See Lemma 3.2.2), $(x, z') \in R$ and $z \leq y \leq z'$ imply that $(x, z) \in R$. Therefore, We may conclude that $R * R^{id}_{\mathbf{Y}} = R$.

Showing that $R^{id}_{\mathbf{Y}} * R' = R'$ holds follows by similar argument. □

We now arrive at the main result of the section. We show that the category of lattices and monotone operations is dual to the category of L-spaces with filter continuous relations. This Theorem lays the foundations for some of the results we will see through out the thesis and generalizes the L-space duality of [6]. The key application of this result will be to demonstrating a preservation theorem for the Π_1-completion in Chapter 4.

Theorem 3.2.11. *The categories* $\mathbf{Lat}(\mathbb{O})$ *and* $\mathbf{LSp}(\mathbb{FC})$ *are dually equivalent.*

Proof. The object part of our duality follows from the object part of the L-space duality (Theorem **??**). The duality at the level of morphisms holds because the categories $\mathbf{Lat}(\mathbb{O})$ and $\mathbf{LSp}(\mathbb{FC})$ are both concrete and because Lemma 3.2.6 guarantees that for each monotone operation $f : L_1, \times ... \times L_n \to K$, R_f is the unique $n+1$-ary filter continuous relation such that $\phi_K(f(a_1, .., a_n)) = F_R[\phi_{L_1}(a), .., \phi_{L_n}(a_n)]$. □

3.3 Some Correspondences for Filter Continuous Relations

We end this chapter with some observations regarding the representation of operations that preserve meets or joins. This will allow us to make the relationship between L-space morphisms and filter continuous relations more precise. We then discuss how to obtain the representation theorem for modal lattices in [6] from the representation theory developed here.

3.3.1 Preservation of Joins and Meets

In general, an operation $f : \mathbf{L}_1 \times ... \times \mathbf{L}_n \to \mathbf{K}$ corresponding to a filter continuous relation $R \subseteq X_K \times X_{L_1} \times ... \times X_{L_n}$ will preserve joins at some coordinate $i \leq n$ if and only if we have:

(Reflection) If $R(y, ..x_{i-1}, z \curlywedge z', x_{i+1}, ..x_n)$, then there are $t, t' \in X_{L_i}$ such that $t \curlywedge t' \leq y$ and $R(t, ..x_{i-1}, z, x_{i+1}, ..x_n)$ and $R(t', ..x_{i-1}, z', x_{i+1}, ..x_n)$.

We call this property reflection. Let us state and show this correspondence more precisely in the case when R is binary. Recall that $[T) = \uparrow\{a_1 \wedge ... \wedge a_n \mid a_1, .., a_n \in T\}$ denotes the filter generated by a subset T of a semilattice.

Proposition 3.3.1. *For all lattices* $\mathbf{K}$ *and* $\mathbf{L}$ *and monotone functions* $f : \mathbf{L} \to \mathbf{K}$ *for all* $a, b \in L$, $f(a \vee b) = f(a) \vee f(b)$ *if and only if for all* $x \in X_K$ *and* $y', z' \in X_L$:

(Reflection) If $R_f(x, y' \curlywedge z')$, then there are $y, z \in X_K$ such that $y \curlywedge z \leq x$ and $R_f(y, y')$ and $R_f(z, z')$.

Proof. Suppose that for all $a, b \in L$, $f(a \vee b) = f(a) \vee f(b)$. Now suppose that $R_f(x, y' \curlywedge z')$. Let us define $y := [f[y'])$ and $z := [f[z'])$. It is clear that $R_f(y, y')$ and $R_f(z, z')$. Let us now show that $y \curlywedge z \leq x$. So suppose that $c \in [fy') \cap [fz')$. Then there are $a_1, .., a_n \in y'$ and $b_1, .., b_m \in z'$ such that $\bigwedge_{i \leq n} fa_i \leq c$ and $\bigwedge_{i \leq m} fb_i \leq c$. Since f is monotone, it follows that both $f(\bigwedge_{i \leq n} a_i) \leq \bigwedge_{i \leq n} fa_i$ and $f(\bigwedge_{i \leq m} b_i) \leq \bigwedge_{i \leq m} fb_i$. We therefore obtain from join preservation:

$$f(\bigwedge_{i \leq n} a_i \vee \bigwedge_{i \leq m} b_i) = f(\bigwedge_{i \leq n} a_i) \vee f(\bigwedge_{i \leq m} b_i) \leq \bigwedge_{i \leq n} fa_i \vee \bigwedge_{i \leq m} fb_i \leq c.$$

But we have that $\bigwedge_{i \leq n} a_i \vee \bigwedge_{i \leq m} b_i \in y' \cap z'$ and thus that $f(\bigwedge_{i \leq n} a_i \vee \bigwedge_{i \leq m} b_i) \in f[y' \cap z']$. We therefore can conclude that $c \in [f[y' \cap z'])$. Now, from the supposition that $R(x, y' \curlywedge z')$, we also have $f[y' \cap z'] \subseteq x$ by definition of R_f and of $\curlywedge$. It then follows from the fact that x is filter that $[f(y' \curlywedge z']) \subseteq x$. So because $c \in [f(y' \curlywedge z'])$, we also have that $c \in x$ and thus that $[fy') \cap [fz') \subseteq x$. We may therefore conclude that $y \curlywedge z \leq x$.

For the other direction, suppose that for all $x \in X_K$ and $y', z' \in X_L$: If $R_f(x, y' \curlywedge z')$, then there are $y, z \in X_K$ such that $y \curlywedge z \leq x$ and $R_f(y, y')$ and $R_f(z, z')$. Let us first show that $F_{R_f}[U \triangledown V] = F_{R_f}[U] \triangledown F_{R_f}[V]$. The inclusion $F_{R_f}[U] \triangledown F_{R_f}[V] \subseteq F_{R_f}[U \triangledown V]$ follows by monotonicity and the fact that $\triangledown$ is join. So let $x \in F_{R_f}[U \triangledown V]$. Then there is some $x' \in U \triangledown V$ such that $R(x, x')$. If $x' \in U \triangledown V$, then there are $y' \in U$ and $z' \in V$ such that $y' \curlywedge z' \leq x'$. By Lemma 3.2.2, $R(x, y' \curlywedge z')$. So by assumption we have that there are filters y, z such that $R(y, y')$ and $R(z, z')$ and $y \curlywedge z \leq x$. But then $y \in F_{R_f}[U]$ and $z \in F_{R_f}[V]$ and $x \in F_{R_f}[U] \triangledown F_{R_f}[V]$, as desired.

Now since $\phi_{\mathbf{L}}$ and $\phi_{\mathbf{K}}$ are both isomorphism and $F_{R_f} * \phi_{\mathbf{L}} = \phi_{\mathbf{K}} * f$, we conclude that $f(a \vee b) = fa \vee fb$. $\square$

The more general case follows from a very similar argument. Let us now consider meet preserving operations.

An odd feature of the representation we have provided in this chapter is that the dual relation of *all* monotone operations are defined in a uniform way, even those operations that preserve meets. In the case of a unary operation f, the binary relation R_f on the relevant dual spaces was defined by

$$R_f(x, y) \text{ iff } f[y] \subseteq x.$$

While in the case of join preserving operations this is standard fare, the fact the meet preserving operations are represented this way too is unusual. To clarify this situation, let us provide the condition guaranteeing when the operation F_R associated with some filter continuous relations R preserves meets in some coordinate.

In the most general case, a operation $f : \mathbf{L}_1 \times ... \times \mathbf{L}_n \to \mathbf{K}$ corresponding to a filter continuous relation $R \subseteq X_K \times X_{L_1} \times ... \times X_{L_n}$ preserves meets at some coordinate $i \leq n$ if and only if we have:

(Idealization) If $R_f(y, ..x_{i-1}, x, x_{i+1}, ..x_n)$ and $R_f(y, ..x_{i-1}, x', x_{i+1}, ..x_n)$, then there is some $w \in X_i$ such that $x \leq w$ and $x' \leq w$ and $R_f(y, ..x_{i-1}, w, x_{i+1}, ..x_n)$.

We call this property idealization because the set $\{z \in X_i \mid R_f(y, ..x_{i-1}, z, x_{i+1}, ..x_n)\}$ forms an ideal of X_i.

Let us state and show this correspondence more precisely in the case when R is binary.

Proposition 3.3.2. *For all lattices* $\mathbf{K}$ *and* $\mathbf{L}$ *and monotone functions* $f : \mathbf{L} \to \mathbf{K}$ *for all* $a, b \in L$, $f(a \wedge b) = f(a) \wedge f(b)$ *if and only if for all* $x \in X_K$ *and* $y, z \in X_L$,

(Idealization) If $R(x, y)$ *and* $R(x, z)$*, then there is some* $w \in X_L$ *such that* $y \leq w$ *and* $z \leq w$ *and* $R(x, w)$.

Proof. Suppose that for all $a, b \in \mathbf{L}$, $f(a \wedge b) = fa \wedge fb$. Let $R(x, y)$ and $R(x, z)$. Defined $w := [y \cup z)$. We show that $R(x, w)$. So let $a \in [y \cup z)$. we will show that $fa \in x$. If $a \in [y \cup z)$, then because y and z are both filters, we find $b \in y$ and $c \in z$ such that $b \wedge c \leq a$. But because $b \in y$ and $c \in z$ and $R(x, y)$ and $R(x, z)$, we have that $fb \in x$ and $fc \in x$ and so $fb \wedge fb \in x$. So since $b \wedge c \leq a$, we obtain $fb \wedge fc = f(b \wedge c) \leq fa$. Therefore, $fa \in x$, as desired. We can therefore conclude that $R(x, w)$.

Suppose that for all $x \in X_K$ and $y, z \in X_L$, if $R(x, y)$ and $R(x, z)$, then there is some $w \in X_L$ such that $y \leq w$ and $z \leq w$ and $R(x, w)$. We show that $F_{R_F}[U \cap V] = F_{R_F}[U] \cap F_{R_F}[V]$. The inclusion $F_{R_F}[U \cap V] \subseteq F_{R_F}[U] \cap F_{R_F}[V]$ is by monotonicity and the fact that $\cap$ is meet. For the other inclusion, let $x \in F_{R_F}[U] \cap F_{R_F}[V]$. The there are $y \in U$ and $z \in V$ such that $R(x, y)$ and $R(x, z)$. By our assumption, there is some $w \in X_L$ such that $y \leq w$ and $z \leq w$ and $R(x, w)$. So $w \in U \cap V$ and $x \in F_{R_f}[U \cap V]$.

Since $\phi_{\mathbf{L}}$ and $\phi_{\mathbf{K}}$ are both isomorphism and $F_{R_f} * \phi_{\mathbf{L}} = \phi_{\mathbf{K}} * f$, we conclude that $f(a \wedge b) = fa \wedge fb$. □

With Propositions 3.3.1 and 3.3.2 in hand, we can describe those filter continuous relations whose dual operations between lattices is a homomorphism.

Proposition 3.3.3. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be L-spaces. Let* $R \subseteq X \times Y$ *be a filter continuous relation. Then the function* $F_R : \mathbf{L_Y} \to \mathbf{L_X}$ *is a lattice homomorphism if and only if* R *satisfies the following conditions:*
(Reflection) If $R_f(x, y' \curlywedge z')$*, then there are* $y, z \in X_K$ *such that* $y \curlywedge z \leq x$ *and* $R_f(y, y')$ *and* $R_f(z, z')$,
(Idealization) If $R(x, y)$ *and* $R(x, z)$*, then there is a* $w \in X_L$ *such that* $y \leq w$ *and* $z \leq w$ *and* $R(x, w)$,
(Isolation) $xR1$ *iff* $x = 1$*, and*
(Totality) For all $x \in X$*, there is a* $y \in Y$ *such that* $R(x, y)$.

Proof. The property of Isolation holds iff $F_R[\{1_{\mathbf{Y}}\}] = \{1_{\mathbf{X}}\}$. Totality holds iff $F_R[Y] = X$. Join and meet preservation correspondences follow from Propositions 3.3.1 and 3.3.2, respectively. □

We could now compose L-space duality with the previous proposition to obtain that every L-space morphism corresponds to a filter continous relations with the properties reflection, idealization, isolation, and totality. Let us now show how to turn a binary filter continuous relation with these properties directly into an L-space morphism. Recall for the next lemma that all L-space $\mathbf{X}$ are also complete lattices. We will denote the join by $\curlyvee$.

Lemma 3.3.4. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be L-spaces. Let* $R \subseteq X \times Y$ *be a filter continuous relation satisfying idealization. For all* $x \in \mathbf{X}$, $\curlyvee R[x] \in R[x]$.

Proof. To prove this lemma, by Theorem 3.2.11 we may assume that there lattices $\mathbf{L}$ and $\mathbf{K}$ such that $\mathbf{X_L} = \mathbf{Y}$ and $\mathbf{X_K} = \mathbf{X}$. By proposition 3.3.2 and Theorem 3.2.11 we also obtain that there is some meet preserving $f : \mathbf{L} \to \mathbf{K}$ such that $R = R_f$. We recall that $\curlyvee R[x] = \bigcup \{[y_1 \cup .. \cup y_n) \mid y_1, ..., y_n \in R[x]\}$. So let $a \in \curlyvee R[x]$. Then there are $y_1, .., y_n \in R[x]$ such that $a \in [y_1 \cup .. \cup y_n)$. Since $R[x]$ is an ideal, we obtain that $[y_1 \cup .. \cup y_n) \in R[x]$. So by definition of R, we have that $fa \in x$. We therefore can conclude that $\curlyvee R[x] \in R[x]$. □

Proposition 3.3.5. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be L-spaces. Let* $R \subseteq X \times Y$ *be a filter continuous relation. Suppose that* R *satisfies reflection, idealization, isolation, and totality. Then* $f_R : \mathbf{X} \to \mathbf{Y}$ *defined* $f_R(x) = \bigcurlyvee R[x]$ *is an L-space morphism.*

Proof. For a reminder of the what an L-space morphism is, see Definition 2.2.4. For meet preservation, note that $\mathbf{Y}$ is not only a complete lattice but is also algebraic. It follows the from the fact that $R[x]$ is an ideal, and therefore directed, that $f_R(x \curlywedge y) = \bigcurlyvee R[x \curlywedge y] = \bigcurlyvee R[x] \curlywedge \bigcurlyvee R[y] = f_R(x) \curlywedge f_R(y)$ (see exercise 6 in section 4 of [9].) By isolation $f_R(x) = 1$ iff $x = 1$. For the back condition of an L-space morphism, we use reflection. Finally, for continuity, let U be a clopen filter of $\mathbf{Y}$. We claim that $f_R^{-1}[U] = F_R[U]$. So let $x \in f_R^{-1}[U]$. Then $f_R(x) = \bigcurlyvee R[x] \in U$. By Lemma 3.3.4, we have that $\bigcurlyvee R[x] \in U \in R[x]$ and so $x \in F_R[U]$. For the other direction, let $x \in F_R[U]$. then there is some $y \in U$ such that $R(x, y)$. But clearly $y \leq \bigcurlyvee R[x]$. So $\bigcurlyvee R[x] \in U$. We therefore have that $x \in f_R^{-1}[U]$. We conclude that f_R is an L-space morphism. $\square$

3.3.2 Recovering Modal L-spaces

Given the results above, and in particular proposition 3.3.2, it is clear that the natural way to represent the Modal Lattices of [6] in terms our representation of lattice expansions departs significantly from that of [6]. In particular, in [6] the $\Box$ and $\Diamond$ of a modal algebra are represented by a common relation while in our representation both operators will be associated with a distinct relations. Since the spaces of [6] are less complex in terms of the number relations and adhere to a more standard way of representing $\Box$ and $\Diamond$, it is worth considering how to recover their representation from ours.

Let us first recall the definition of modal lattice as presented in [6].

Definition 3.3.6. *A modal lattice* $\mathbf{L} = (L, \Box, \Diamond)$ *is a bounded lattice with two unary operations validating the following identities:*

$$\Box(a \wedge b) = \Box a \wedge \Box b \qquad \Box\top = \top \qquad \Diamond\bot = \bot$$

$$\Diamond a \leq \Diamond(a \vee b) \qquad \Diamond a \wedge \Box b \leq \Diamond(a \wedge b).$$

We now provide the necessary and sufficient conditions for the algebra of clopen filters of an L-space equipped with two filter continuous relations to be a modal lattice. To this end we define FC-modal L-spaces.

Definition 3.3.7. *An FC-modal L-space* $\mathbf{X} = (X, S, R)$ *is an L-space* $\mathbf{X}$ *with pair of filter continuous relations satisfying:*
(S-Idealization) If $S(x, y)$ *and* $S(x, z)$*, then there is a* $w \in X_L$ *such that* $y \leq w$ *and* $z \leq w$ *and* $S(x, w)$*,*
(Joint-idealization) If $R(x, y)$ *and* $S(x, z)$*, then there is a* $w \in X_L$ *such that* $y \leq w$ *and* $z \leq w$ *and* $R(x, w)$*,*
(S-totality) For all $x \in X$*, there is some* $y \in Y$ *such that* $S(x, y)$*, and*
(R-isolation) $R(x, 1)$ *iff* $x = 1$*.*

The following proposition shows that the algebra of clopen filters of an FC-modal L-space is a modal lattice.

Proposition 3.3.8. *Let* $\mathbf{X} = (X, S, R)$ *be FC-modal L-space, then* $\mathbf{L_X} = (\mathcal{F}i_{clp}(\mathbf{X}), F_S, F_R)$ *is a modal lattice.*

Proof. By proposition 3.3.2, and the assumption of (S-idealization) we know that $F_S[U \cap V] = F_R[U] \cap F_R[V]$. By Proposition 3.3.3 and the assumption (S-totality) and (R-isolation) we obtain that $F_S[Y] = X$ and $F_R[\{1_{\mathbf{Y}}\}] = \{1_{\mathbf{X}}\}$. Since F_R is monotone we have $F_R[U] \subseteq F_R[U \triangledown V]$. Finally, for $F_R[U] \cap F_S[V] \subseteq F_R[U \cap V]$. Let $x \in F_R[U] \cap F_S[V]$. Then there are $y \in U$ and $z \in V$ such that $R(x, y)$ and $S(x, z)$. By the assumption of condition (joint-idealization), we have that there is some $w \in X$ such that $y \leq w$ and $z \leq w$ and $R(x, w)$. Since $y \in U$ and $z \in V$, $w \in U \cap V$. So $x \in F_R[U \cap V]$, as desired. □

We now show that all modal lattices are represented with an L-space with two filter continuous relations and some extra conditions.

Proposition 3.3.9. *Every modal lattice is isomorphic to the modal lattice of clopen filters $\mathbf{L_X}$ of some FC-modal L-space $\mathbf{X} = (X, S, R)$.*

Proof. Let $\mathbf{L} = (L, \Box, \Diamond)$ be a modal lattice. By theorem 3.2.7, the representation theorem for monotone lattice expansions, we obtain that $\mathbf{L}$ is isomorphic to $\mathbf{L}_{\mathbf{X_L}}$.

By Propositions 3.3.2 and 3.3.3 we obtain that conditions (S-idealization), (S-totality), and (R-isolation) hold in $\mathbf{X_L}$. We just need to check (Joint-idealization). So let $R_\Diamond(x, y)$ and $R_\Box(x, z)$. Defined $w := [y \cup z)$. We claim that $R_\Diamond(x, w)$. So let $a \in w$. We will show $fa \in x$. If $a \in w = [y \cup z)$, then there are $b \in y$ and $c \in z$ such that $b \curlywedge c \leq a$. If $b \in y$ and $c \in z$, then from the assumption $R_\Diamond(x, y)$ and $R_\Box(x, z)$, we have that $\Diamond b \in x$ and $\Box c \in x$. So $\Diamond b \wedge \Box c \in x$. But $\Diamond b \wedge \Box c \leq \Diamond(b \wedge c)$ and $\Diamond(b \wedge c) \leq \Diamond a$, so $\Diamond a \in x$, as desired. We conclude that $R_\Diamond(x, w)$. □

The last thing we do in this section is show how to turn an FC-modal L-space into a modal L-space as defined in [6]. Recall the definition of modal L-space from [6].

Definition 3.3.10. *A modal L-space $\mathbf{X} = (X, R, \tau)$ is an L-space (X, τ) with a relation R such that:*
(1) $R(x, 1)$ iff $x = 1$
(2) if U is a clopen filter, then $\Diamond_R U = R^{-1}[U]$ and $\Box_R U = \{x \mid R[x] \subseteq U\}$ are clopen filters, and
(3) $R(x, y)$ iff (a) if $y \in U$, then $x \in \Diamond_R[U]$ and (b) if $x \in \Box_R U$, then $y \in U$.

Let us observe an important fact about relations with idealization. If $\mathbf{X}$ is an L-space and $S \subseteq X \times X$ is a filter continuous relation that satisfies idealization, then we can define a new relation $R^+ \subseteq X \times X$:

$$S^+(x, y) \text{ if and only if, for all clopen filters } U, \text{ if } x \in F_S[U], \text{ then } y \in U.$$

The operations $\Box_{S^+}$ and F_S coincide. In particular, for all clopen filters U we have $F_S[U] = \Box_{S^+} U = \{x \mid S^+[x] \subseteq U\}$. To see this, let $x \in \Box_{S^+} U$. Then $S^+[x] \subseteq U$. Now note that in virtue of S having the idealization property, $\curlyvee S[x] \in S^+[x]$. This is the case because whenever $x \in F_S[U]$, then there is some $y \in U$ such that $S(x, y)$. However, since $y \leq \curlyvee S[x]$ and, by idealization, $\curlyvee S[x] \in S[x]$ (See proof of proposition 3.3.5). Therefore, by definition of S^+, we have $\curlyvee S[x] \in S^+[x]$. We therefore get from the assumption that $x \in \Box_{S^+}$, that $\curlyvee S[x] \in U$ and $x \in F_S[U]$. Conversely, suppose that $x \in F_S[U]$. We must show that $S^+[x] \subseteq U$. So let $S^+(x, y)$. By definition, for all all clopen filters V, if $x \in F_S[V]$, then $y \in V$. So obtain that $y \in U$. We therefore, conclude that $S^+[x] \subseteq U$ and thus that $x \in \Box_{S^+} U$.

We just remarked on how to transform filter continuous S relations with idealization into another relation S^+ whose $\Box_+$ coincides with F_R. We can now propose that how this could provide a way to turn FC-modal L-spaces with pairs of filter continous relations into modal L-spaces with their single relation.

Conjecture 3.3.11. *Let* $\mathbf{X} = (X, R_1, S)$ *be an FC-modal L-space. The* $\mathbf{X} = (X, R_2)$ *is a modal L-space where* $R_2 = R_1 \cap S^+$*.*

Remark 3.3.12. *Reductions in the Number of Relations*

One consequence of the previous conjecture is that certain assumptions about how different filter continuous relations interact are sufficient to permit us to reduce two relations into one. Identifying a general method for reducing the number of relations along the lines of the one conjectured above would be a valuable tool for simplifying the representation of lattice expansions.

3.4 Conclusion

The theory supplied in this chapter is the general setting that grounds many of the results and developments to come. In particular, we will use the this theory to derive the OKHD-semantics and duality for $r\ell$-groupoids in chapters 5 and 6, respectively. In summary, this chapter we began in Section 3.1 with a characterization of products and coproducts of L-spaces (Theorem 3.1.10) and then used this characterization to obtain a representation of products of lattices (Theorem 3.1.11). Afterwards, in Section 3.2, we applied the latter developments to obtain a general representation of monotone operations between lattices and showed that each n-ary monotone operation between lattices could be represented by a unique $n+1$-ary filter continuous relation (Theorem 3.2.6). Then in Section 3.2.3 we showed that the category of lattice with monotone operations as morphisms is dually equivalent to the category of L-spaces with filter continuous relations as morphisms (Theorem 3.2.11). Finally, in Section 3.3, we described the filter continuous relations that correspond to meet and join preserving operations and suggested a strategy to recover the modal L-spaces of [6].

In the next chapter we develop the representation of completions of monotone lattice expansions in their dual L-spaces and extend the persistence results of [6] to arbitrary signatures that are interpreted to monotone operations. Our proof of this fact depends essentially on the duality between lattices with monotone operations and L-spaces with filter continuous relations.

Chapter 4

The Topological Representation of Completions and Π_1-Preservation

In this chapter we extend the topological representations of completions of modal lattices in [6] by exploiting the representation of monotone operations obtained in the previous chapter. We then use what is developed in this chapter and the last to show a general preservation result with respect to the Π_1-completion, again generalizing the specific case of modal lattices considered in [6]. In particular, we generalize Theorem 4.30 in [6] which shows every identity for modal lattice is preserved through the Π_1-completion. We obtain this generalization via an alternative proof that exploits the developments made in the previous chapter.

4.1 Completions of Monotone Lattice Expansions

We begin by reminding the reader of the definition of a completion of a poset $\mathbf{P}$, which is simply complete lattice that contains a copy of $\mathbf{P}$.

Definition 4.1.1. *A completion $(\mathbf{C}, e)$ of a partial order $\mathbf{P}$ is a complete lattice $\mathbf{C}$ together with an embedding $e : \mathbf{P} \to \mathbf{C}$ with the property that $a \leq b$ if and only if $e(a) \leq e(b)$.*

Of special interest to us will be the Π_1 completion of a lattice, introduce in by Gehrke and Priestly in [20]. The Π_1 completion of a lattice $\mathbf{L}$ is defined as the composition of the ideal completion with the filter completion of $\mathbf{L}$. Let us define the filter, ideal, and Π_1 completions.

Definition 4.1.2. *Let L be a lattice, then we define:*
(1) Filter Completion, $(\mathfrak{fe}(\mathbf{L}), \alpha : \mathbf{L} \to \mathfrak{fe}(\mathbf{L}))$, such that $fe(\mathbf{L}) := (\mathcal{F}i(\mathbf{L}), \leq^{\mathfrak{fe}})$ where $x \leq_{\mathfrak{fe}} y$ iff $y \subseteq x$ and $\alpha(a) =\uparrow_{\mathbf{L}} (a)$,
(2) Ideal Completion, $(\mathfrak{ie}(L), \beta : \mathbf{L} \to \mathfrak{ie}(\mathbf{L}))$, such that $\mathfrak{ie}(\mathbf{L}) := (\mathcal{I}d(\mathbf{L}), \leq_{\mathfrak{ie}})$ where $x \leq^{\mathfrak{ie}} y$ iff $x \subseteq y$ and $\beta(a) =\downarrow_{\mathbf{L}} (a)$, and
(3) Π_1-Completion, $(\Pi_1(\mathbf{L}), \pi : \mathbf{L} \to \Pi_1(\mathbf{L}))$, such that $\Pi_1(\mathbf{L}) := \mathfrak{ie}(\mathfrak{fe}(\mathbf{L}))$ and $\pi(a) = \{x \in \mathfrak{fe}(\mathbf{L}) \mid a \in x\} =\downarrow_{\mathfrak{fe}(\mathbf{L})} (\uparrow_{\mathbf{L}} (a))$.

Note that the joins are defined in terms of the closure operator that maps a set to the least filter or ideal containing it. Often we will treat $\mathbf{L}$ as a subset/sublattice of $\Pi_1(\mathbf{L})$, $\mathfrak{fe}(\mathbf{L})$, and $\mathfrak{ie}(\mathbf{L})$. This will be especially convenient when comparing elements that are in the image of the one of the embeddings with other elements of the completion. Similarly, we treat $\mathfrak{fe}(\mathbf{L})$ as a sublattice of $\Pi_1(\mathbf{L})$. We also define the closed elements $K(\Pi_1(\mathbf{L})$ of $\Pi_1(\mathbf{L})$ as the set $\{\bigwedge T \mid T \subseteq L\}$. It is not hard to see that $\mathfrak{fe}(\mathbf{L}) = K(\Pi_1(L))$.

4.1.1 Extending Monotone Operations

In this section we define the extension of a monotone operation to the filter, ideal, and Π_1-completions and show that the definition of these extensions guarantee that an monotone lattice expansion is a subalgebra of any one of the salient completions. As remarked on above, to lighten notation we will treat $\mathbf{L}$ as a subset of $\Pi_1(\mathbf{L})$, $\mathfrak{fe}(\mathbf{L})$, and $\mathfrak{ie}(\mathbf{L})$.

Definition 4.1.3. *(Extensions of functions between lattices) Let $f : \mathbf{L} \to \mathbf{K}$ be a function between lattices. We define the extension of f to the relevant completion as follows:*

$$f^{\mathfrak{fe}}(x) := \bigwedge\{f(a) \mid x \leq_{\mathfrak{fe}} a \ \&\ a \in L\} \tag{4.1}$$
$$f^{\mathfrak{ie}}(x) := \bigvee\{f(a) \mid a \leq_{\mathfrak{ie}} x \ \&\ a \in L\} \tag{4.2}$$
$$f^{\Pi_1}(x) := \bigvee\{\bigwedge\{f(a) \mid y \leq a \ \&\ a \in L\} \mid y \in K(\Pi_1(\mathbf{L}))\ \&\ y \leq x\} \tag{4.3}$$

Before going further, we will show that the filter, ideal, and Π_1-completion commutes with products. Note that in the trivial case when f is nullary, i.e. a constant, $f^{\mathfrak{fe}} = f^{\Pi_1} = \bigwedge\{a \mid f \leq a \ \&\ a \in L\} = f$.

Lemma 4.1.4. *The filter, ideal, and Π_1-completions commute with finite products:*

$$\mathfrak{fe}(\mathbf{L}_1) \times .. \times \mathfrak{fe}(\mathbf{L}_n) \cong \mathfrak{fe}(L_1 \times .. \times L_n) \tag{4.4}$$
$$\mathfrak{ie}(\mathbf{L}_1) \times .. \times \mathfrak{ie}(\mathbf{L}_n) \cong \mathfrak{ie}(L_1 \times .. \times L_n) \tag{4.5}$$
$$\Pi_1(\mathbf{L}_1) \times .. \times \Pi_1(\mathbf{L}_n) \cong \Pi_1(L_1 \times .. \times L_n). \tag{4.6}$$

Proof. The case of the filter completion follows directly from Lemma 3.1.5, which shows that filters commute with products for meet semilattices. An order dual argument shows that the products also commute with ideal completions. Composing these facts and recalling that $\Pi_1(L) \cong \mathfrak{ie}(\mathfrak{fe}(\mathbf{L})$, we obtain that Π_1-completion also commutes with products.

□

Remark 4.1.5. *Given that products commute with the Π_1-completion, it is possible to view the extension of maps to completions in an alternative and often more tangible way. For example, we can treat $f^{\Pi_1} : \Pi_1(L_1 \times ... \times L_n) \to \Pi_1(K)$ as essentially the same as a map $f^{\Pi_1} : \Pi_1(L_1) \times ... \times \Pi_1(L_n) \to \Pi_1(K)$ defined below.*

$$f^{fc}(x_1, .., x_n) := \bigwedge\{f(a_1, .., a_n) \mid \forall j \leq n(x_j \leq_{\mathfrak{fe}} a_j \ \&\ a_j \in L)\}$$
$$f^{ic}(x_1, .., x_n) := \bigvee\{f(a_1, .., a_n) \mid \forall j \leq n(a_j \leq_{\mathfrak{ie}} x_j \ \&\ a_j \in L)\}$$
$$f^{\Pi_1}(x_1, .., x_n) := \bigvee\{\bigwedge\{f(a_1, ..a_n) \mid (y_j \leq_{\Pi_1} a_j) \ \&\ a_i \in L\} \mid y_j \in K(\Pi_1(\mathbf{L}))\ \&\ y_j \leq x_j\}$$

From here on out we will use these maps interchangeably.

We now consider two examples. The first comes from [6] while the second is important for the chapters to come.

Example 4.1.6. *(Modal Lattices)*

Our first example is the case of modal lattices, which is presented from [6]. Given a modal lattice $\mathbf{L} = (L, \Box)$*, the schema from definition 4.1.3 prescribes us the following definition of the modal box operation thevarious completions we have considered.*

$$\begin{aligned}\Box^{fe}x &:= \bigwedge\{\alpha(\Box a) \mid x \leq \alpha(a)\ \}\\ \Box^{ie}x &:= \bigvee\{\beta(\Box a) \mid \alpha(a) \leq x\ \}\\ \Box^{\Pi_1}x &:= \bigvee\{\bigwedge\{\pi(\Box a) \mid a \in L \ \&\ y \leq \alpha(a)\} \mid y \in K(\Pi_1(L)) \ \&\ y \leq x\}\end{aligned}$$

Example 4.1.7. *(ℓ-groupoids)*

Our second example is the case of lattice ordered groupoids.

$$\begin{aligned}x \cdot^{fe} y &:= \bigwedge\{\alpha(a \cdot b) \mid x \leq \alpha(a) \ \&\ y \leq \alpha(b)\}e^{fe} && = \alpha(e),\\ x \cdot^{ie} y &:= \bigvee\{\beta(a \cdot b) \mid \beta(a) \leq x \ \&\ \beta(b) \leq y\}e^{ie} && = \beta(e),\\ x_1 \cdot^{\Pi_1} x_2 &:= \bigvee\{\bigwedge\{\pi(a_1 \cdot a_2) \mid a_i \in L \ \&\ y_i \leq \alpha(a_i)\} \mid y_i \in K(\Pi_1(L)) \ \&\ y_i \leq x_i\}e^{\Pi_1} && = \pi(e^{fe}).\end{aligned}$$

It is straightforward to check that for any monotone lattice expansion $\mathbf{L}$ the embeddings $\alpha : \mathbf{L} \to \mathfrak{fe}(\mathbf{L})$, and $\beta : \mathbf{L} \to \mathfrak{ie}(\mathbf{L})$, and $\pi : \mathbf{L} \to \Pi_1(\mathbf{L})$ are all homomorphism with respect to the operations of $\mathbf{L}$. And since $\Pi_1(L) \cong \mathfrak{ie}(\mathfrak{fe}(\mathbf{L})$, we also obtain that $fe(\mathbf{L})$ is a subalgebra of $\Pi_1(\mathbf{L})$.

Lemma 4.1.8. *Let* $\mathbf{L} = (L, \{f_i\}_{i \in I})$ *be a monotone lattice expansion. Then for each* f_i*, each* $a_1, .., a_n \in L$*, and each* $e \in \{\alpha, \beta, \pi\}$*, we have that* $e(f(a_1, .., a_n)) = f_i^C(e(a_1), .., e(a_n))$*.*

Proof. We show that case of $\alpha : \mathbf{L} \to \mathfrak{fe}(\mathbf{L})$ and note that the case of β has an order dual argument and the case of π follows in virtue of $\pi = \beta * \alpha$. $\Box$

4.2 The Representation of Completions

Similar to the situation in Priestly based dualities, various completions of a lattice L can be identified with lattices of subsets of the dual space $\mathbf{X_L}$ of L. In [6] the authors of demonstrated an analogous result by showing that the filter completion of L corresponds to the lattice of closed filters $\mathcal{F}i_k(\mathbf{X_L})$ of L's dual space, the ideal completion corresponds to the lattice of open filters $\mathcal{F}i_o(\mathbf{X_L})$ of L's dual space, and that the Π_1 completion of L corresponds to the lattice of all filters $\mathcal{F}i(\mathbf{X_L})$ of L's dual space. This is summarized precisely in the following lemma from [6].

Lemma 4.2.1. *Let* L *be a lattice and* $\mathbf{X_L}$ *be its dual* L*-space, then*

$$\mathfrak{fe}(L) \cong \mathcal{F}i_k(\mathbf{X_L}) \qquad \mathfrak{ie}(L) \cong \mathcal{F}i_o(\mathbf{X_L}) \qquad \Pi_1(L) \cong \mathcal{F}i(\mathbf{X_L}).$$

The operations that witness each of the isomorphisms as they will be useful in what is to come. We define $\overline{\phi} : \mathfrak{fe}(L) \to \mathcal{F}i_k(\mathbf{X_L})$ such that given an element c of $\mathfrak{fe}(L)$, the corresponding closed filter $\overline{\phi}(c)$ of $\mathbf{X_L}$ is defined

$$\overline{\phi}(a) = \bigcap\{\phi(a) \mid c \leq_{\mathfrak{fe}} a\}.$$

Similarly, we define $\underline{\phi} : \mathfrak{ie}(L) \to \mathcal{F}i_o(X_L)$ which given an element c of $\mathfrak{ie}(\mathbf{L})$, returns the corresponding open filter

$$\underline{\phi}(c) = \bigvee\{\phi(c) \mid a \leq_{\mathfrak{ie}} c \}$$

.

Finally, the correspondence between $\Pi_1(L)$ and $\mathcal{F}i(\mathbf{X_L})$ is witnessed by the map $\widehat{\phi} : \Pi_1(L) \to \mathcal{F}i(\mathbf{X_L})$. So given an element $c \in \Pi_1(\mathbf{L})$ we define $\widehat{\phi}(c)$ such that

$$\widehat{\phi}(c) = \bigvee\{\bigcap_{a\in T} \phi(a) \mid T \subseteq L \ \& \ \bigwedge T \leq c\}.$$

Beyond just showing the above correpsondence, the authors of [6] extended this result and showed the representation holds for all positive modal lattices as well. The following Lemma generalizes this result of[6] by extending it to all monotone lattice expansions. This is accomplished by showing that the operations defined in 4.1.3 agree with the operations defined in terms of filter continuous relations on the dual of that monotone lattice expansion.

Lemma 4.2.2. *Let $f : \mathbf{L}_1 \times ... \times \mathbf{L}_n \to \mathbf{K}$ be a monotone operation between lattices, then the following diagrams commute.*

$$\begin{tikzcd}
\mathcal{F}i_k(X_{L_1}) \times ... \times \mathcal{F}i_k(X_{L_n}) \arrow[r, "F_{R_f}"] \arrow[d, "{[\overline{\phi}_{L_1},..,\overline{\phi}_{L_n}]}"'] & \mathcal{F}i_k(X_K) \arrow[d, "\overline{\phi}_K"] \\
\mathfrak{fe}(L_1) \times ... \times \mathfrak{fe}(L_n) \arrow[r, "f^{\mathfrak{fe}}"'] & \mathfrak{fe}(K)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{F}i(X_{L_1}) \times ... \times \mathcal{F}i(X_{L_n}) \arrow[r, "F_{R_f}"] \arrow[d, leftrightarrow, "{[\widehat{\phi}_{L_1},..,\widehat{\phi}_{L_n}]}"'] & \mathcal{F}i(X_K) \arrow[d, leftrightarrow, "\widehat{\phi}_K"] \\
\Pi_1(L_1) \times ... \times \Pi_1(L_n) \arrow[r, "f^{\Pi_1}"'] & \Pi_1(K)
\end{tikzcd}$$

Proof. Let $f : \mathbf{L}_1 \times ... \times \mathbf{L}_n \to \mathbf{K}$ be a monotone operation between lattices. Suppose for notation's sake that $\overline{\phi}_L = [\overline{\phi}_{L_1}, .., \overline{\phi}_{L_n}]$ and $\widehat{\phi}_L = [\widehat{\phi}_{L_1}, .., \widehat{\phi}_{L_n}]$

Claim 1: $F_{R_f} * \overline{\phi}_L = \overline{\phi}_K * f^{\mathfrak{fe}}$.

Proof of claim 1: Let $c_1 \in \mathfrak{fe}(L_1), .., c_n \in \mathfrak{fe}(L_n)$, then:

$$\begin{aligned}
F_{R_f}[\overline{\phi}_{L_1}c_1, .., \overline{\phi}_{L_n}c_1] &= \bigcap\{F_{R_f}[\phi_{L_1}(a_1)...\phi_{L_n}(a_n)] \mid \overline{\phi}_{L_i}c_i \subseteq \phi_{L_i}(a_i) \ \& \ a_i \in L_i\} \\
&= \bigcap\{\phi_K(f(a_1, .., a_n) \mid \overline{\phi}_{L_i}c_i \subseteq \phi_{L_i}(a_i) \ \& \ a_i \in L\} \\
&= \overline{\phi}_K\big(\bigwedge\{f(a_1, .., a_n) \mid c_i \leq a_i \ \& \ a_i \in L_i\}\big) \\
&= \overline{\phi}_K(f^{\mathfrak{fe}}(c_1, .., c_n)).
\end{aligned}$$

The first identity holds by appeal to Lemma 3.2.3, which describes the extension of F_{R_f} to closed filters. The second identity holds in virtue of Theorem 3.2.6, which shows that $\phi_K(f(a_1, .., a_n)) = F_{R_f}[\phi_{L_1}(a_1)...\phi_{L_n}(a_n)]$. The third identity follows from the fact that $\overline{\phi}$ is a complete lattice isomorphism. Finally, the fourth identity holds in virtue of the definition of $f^{\mathfrak{fe}}$ in Remark 4.1.5.

Claim 2: $F_{R_f} * \widehat{\phi}_L = \widehat{\phi}_K * f^{\Pi}$.

Proof of claim 2: The proof is essentially the same as for claim 1. Let $d_1 \in \Pi_1(L_1), .., d_n \in \Pi_1(L_n)$, then:

$$\begin{aligned}
F_{R_f}[\widehat{\phi}_{L_1}(d_1),..,\widehat{\phi}_{L_n}(d_1)] &= \bigvee\{F_{R_f}[C_1,..,C_n] \mid C_i \subseteq \widehat{\phi}_{L_i}(d_i) \ \&\ C_i \in \mathcal{F}i_K(\mathbf{X_{L_1}})\} \\
&= \bigvee\{\overline{\phi}_K(f^{\mathfrak{fe}}(c_1,..,c_n) \mid \widehat{\phi}_{L_i}(c_i) \subseteq \phi_{L_i}(d_i) \ \&\ c_i \in \mathfrak{fe}(L_i)\} \\
&= \widehat{\phi}_K\big(\bigvee\{f(a_1,..,a_n) \mid c_i \leq d_i \ \&\ c_i \in K(\Pi_1(L_i))\}\big) \\
&= \widehat{\phi}_K(f^{\mathfrak{fe}}(d_1,..,d_n)).
\end{aligned}$$

The first, third, and fourth identities hold for more or less the same reasons that they did in the proof of Claim 1. The second identity holds in virtue of Claim 1 and the fact that $\overline{\phi}_K : \mathfrak{fe}(K) \to \mathcal{F}i_k(\mathbf{X_K})$ is an isomorphism. □

Theorem 4.2.3. *For all monotone lattice expansions* $\mathbf{L} = (L, \{f_i\}_{i\in I})$*:*

$$(\Pi_1(\mathbf{L}), \{f_i^{\Pi_1}\}_{i\in I}) \cong (\mathcal{F}i(X_L), \{F_{R_{f_i}}\}_{i\in I}).$$

Proof. The isomorphisms $\overline{\phi} : \mathfrak{fe}(L) \to \mathcal{F}i_k(\mathbf{X_L})$ and $\widehat{\phi} : \Pi_1(\mathbf{L}) \to \mathcal{F}i(\mathbf{X_L})$ are guaranteed to be homomorphisms of the relevant type by an application of Lemma 4.2.2 to the operation f_i for each $i \in I$. □

4.3 Π_1-Persistence

In the study of completions, one of the most common questions to ask is what sort of properties are preserved through a given completion. In particular given an (ordered) algebra, are the identities valid in that algebra also valid in its completion? A very notable answer to this question for canonical extensions was answered in logical form for Boolean Algebras with Operators by the Sahlqvist Completeness theorems [7, 10]. An algebraic proof of Sahlqvist Canonicity was later given by Jónsson in [26]. More generally, this question has been answered in similar fashion for arbitrary lattice expansions (See [19] chapter 6). In the context of Π_1-completions and lattice expansions, we show that for any identity between positive terms is preserved through the Π_1-completion where a positive term is a term built up from basic operation symbols that are evaluated as monotone operations.

Definition 4.3.1. *(Positive Terms and Identities) Let τ be a type of algebras with a definable order. A positive term of type τ is a term t where each of the basic operations f comprising t is such that for each algebra $\mathbf{A}$ of type τ, $f^{\mathbf{A}}$ is a monotone operation on $\mathbf{A}$. $T^+_\tau(V)$ is the set of positive terms of type τ over the set of variables V. A positive identity $t \approx s$ is an identity between terms s and t.*

At a high level, our method is analogous to that of [26] and can be explained with reference to clones. In particular, we basically show that the operation $(-)^{\Pi_1}$ mapping monotone operations of some lattice $\mathbf{L}$ to monotone operations on $\Pi_1(\mathbf{L})$ is a clone homomorphism between clones generated by the set of all monotone operations $M_{\mathbf{L}}$ on $\mathbf{L}$ and the set of monotone operations $M_{\Pi_1(\mathbf{L})}$ on $\Pi_1(\mathbf{L})$

and with the special property that the following diagram commutes.

$$\begin{tikzcd}
T^+_\tau(X) \arrow[d, "(-)^{\mathbf{L}}"'] \arrow[dr, "(-)^{\Pi_1(\mathbf{L})}"] & \\
Clo_{\mathbf{L}}(M_{\mathbf{L}}) \arrow[r, "(-)^{\Pi_1}"'] & Clo_{\Pi_1(\mathbf{L})}(M_{\Pi_1(\mathbf{L})})
\end{tikzcd}$$

That is just to say that for each positive term t, we have that $(t^{\mathbf{L}})^{\Pi_1} = t^{\Pi_1(\mathbf{L})}$. This is shown in Lemma 4.3.4 while the fact that $(-)^{\Pi_1}$ is essentially a clone homomorphism follows from Lemma 4.3.2 and Lemma 4.3.3, which jointly show that $(-)^{\Pi_1}$ commutes with generalized compositions of monotone operations. It then follows quickly that any positive identity $s \approx t$ will be persistent through the Π_1-completion since if $\mathbf{L} \vDash s \approx t$, then $s^{\Pi_1(\mathbf{L})} = (s^{\mathbf{L}})^{\Pi_1} = (t^{\mathbf{L}})^{\Pi_1} = t^{\Pi_1(\mathbf{L})}$ and so $\Pi_1(\mathbf{L}) \vDash s \approx t$.

While the previous paragraph explains our proof method at a very high level, and without reference to the representation of monotone operations developed in the previous chapter, at the level of particulars, the representation in Theorem 3.2.6 and the duality proved in Theorem 3.2.11 are key to showing that the operation $(-)^{\Pi_1}$ is behaves like a clone homomorphism. This essentially comes down to our demonstration that L-spaces and Filter continuous relations form a category dually isomorphic to the category of lattices with monotone operations in Theorem 3.2.11.

We now prove two of the main lemmas needed to show our persistence theorem. These lemmas follow in virtue of the representation of the Π_1-completion demonstrated in Theorem 4.2.3. Recall that given a collection of the n-ary operations $\{f_i : \mathbf{A}^n \to \mathbf{B}\}_{i \leq m}$ we define the operation $[f_1, .., f_m] : \mathbf{A}^n \to \mathbf{B}^m$ such that $[f_1, .., f_m](a_1, .., a_n) = (f_1((a_1, .., a_n)), .., f_m(a_1, .., a_n))$. This in the operation induced by the universal property of products.

Lemma 4.3.2. *Let $\mathbf{L}$ and $\mathbf{K}$ be lattices and $\{f_i : \mathbf{L}^n \to \mathbf{K}\}_{i \leq m}$ be a collection of n-ary monotone operations. Then $([f_1, .., f_m])^{\Pi_1} = [f_1^{\Pi_1}, .., f_m^{\Pi_1}]$.*

Proof. This fact follows from the universal property of products and the fact that we have identified $\Pi_1(\mathbf{K}^m)$ with $(\Pi_1(\mathbf{K}))^m$. □

We now show that the extension of an operation to the Π_1 completion commutes with function composition.

Lemma 4.3.3. *Let $\mathbf{L}$, $\mathbf{K}$, and $\mathbf{A}$ be lattices and $f : \mathbf{L} \to \mathbf{K}$ and $g : \mathbf{K} \to \mathbf{A}$ be a pair of monotone operations. Then $(g*f)^{\Pi_1} = g^{\Pi_1} * f^{\Pi_1}$.*

Proof. Using properties of the dual representation of f and g we have:

$$\begin{aligned}
(g * f)^{\Pi} &= \widehat{\phi}_L * F_{R_{g*f}} * \widehat{\phi}_A^{-1} && \text{(Lemma 4.2.2)} \\
&= \widehat{\phi}_L * F_{R_g} * F_{R_f} * \widehat{\phi}_A^{-1} && \text{(Lemma 3.2.8)} \\
&= \widehat{\phi}_L * F_{R_g} * \widehat{\phi}_K^{-1} * f^{\Pi} && \text{(Lemma 4.2.2)} \\
&= g^{\Pi} * f^{\Pi}. && \text{(Lemma 4.2.2)}
\end{aligned}$$

□

Lemma 4.3.4. *Let $\mathbf{L}$ be a lattice expansion of type τ and t be a positive term of the same type, then $(t^{\mathbf{L}})^{\Pi_1} = t^{\Pi_1(\mathbf{L})}$*

Proof. The proof proceeds by induction on the complexity of $t^{\mathbf{L}}$. The base case is covered by considering when either $t^{\mathbf{L}}$ is a projection or a constant. If $t^{\mathbf{L}} : \mathbf{L}^n \to \mathbf{L}$ is a projection, then by the identification of $(\Pi_1(\mathbf{L}))^n$ with $\Pi_n(\mathbf{L}^n)$, we are ensured that $(t^{\mathbf{L}})^{\Pi_1} = t^{\Pi_1(\mathbf{L})}$.

In the case that $t^{\mathbf{L}}$ is a constant c, we have by definition

$$t^{\Pi_1(\mathbf{L})} = t^{\mathbf{L}} = \bigwedge\{a \mid c \leq a \;\&\; a \in L\} = (t^{\mathbf{L}})^{\Pi_1}.$$

For the inductive step we must consider the case where $t^{\mathbf{L}} = g{*}[f_1, .., f_n]$. By Lemma 4.3.3, which shows that composition commutes with the extension of operations to the Π_1-completion, we have:

$$(g{*}[f_1, .., f_n])^{\Pi_1} = (g)^{\Pi_1}{*}([f_1, .., f_n])^{\Pi_1}$$

Howver we can then apply Lemma 4.3.2, which tells use that $([f_1, .., f_n])^{\Pi_1} = [f_1^{\Pi_1}, .., f_n^{\Pi_1}]$, to obtain:

$$(g)^{\Pi_1}{*}([f_1, .., f_n])^{\Pi_1} = (g)^{\Pi_1}{*}[f_1^{\Pi_1}, .., f_n^{\Pi_1}].$$

But then clearly we have that $(t^{\mathbf{L}})^{\Pi_1} = t^{\Pi_1(\mathbf{L})}$ in virtue of:

$$(t^{\mathbf{L}})^{\Pi_1} = (g{*}[f_1, .., f_n])^{\Pi_1} = (g)^{\Pi_1}{*}[f_1^{\Pi_1}, .., f_n^{\Pi_1}] = t^{\Pi_1(\mathbf{L})}.$$

We conclude for all positive terms that $(t^{\mathbf{L}})^{\Pi_1} = t^{\Pi_1(\mathbf{L})}$, as desired. □

Given an algebraic type τ, we say $\mathbf{L}$ *is a τ-subalgebra of* $\mathbf{B}$ to stress that $\mathbf{L}$ is a subalgebra of $\mathbf{B}$ with respect to the operation of τ. We now prove our preservation theorem for positive idenities.

Theorem 4.3.5. *Let $\mathbf{L} = (L, \{f_i\}_{i \in I})$ be a lattice expansion of type τ where $\mathbf{L}$ is a τ-subalgebra of $\Pi_1(\mathbf{L})$. Let $t \approx s$ be a positive identity of type τ. Then $\mathbf{L} \vDash t \approx s$ iff $\Pi_1(\mathbf{L}) \vDash t \approx s$.*

Proof. Suppose that $\mathbf{L} \vDash s \approx t$, then by Lemma 4.3.4 we have $s^{\Pi_1(\mathbf{L})} = (s^{\mathbf{L}})^{\Pi_1} = (t^{\mathbf{L}})^{\Pi_1} = t^{\Pi_1(\mathbf{L})}$ and so $\Pi_1(\mathbf{L}) \vDash s \approx t$ □

An immediate corollary of this fact is that any class of monotone lattice expansions defined by positive identities is closed under the Π_1-completion.

Corollary 4.3.6. *Let $\mathcal{K}$ be a class of lattice expansions of type τ that are defined by a set of positive identities. Suppose further that for each $\mathbf{A} \in \mathcal{K}$, $\mathbf{A}$ is a τ-subalgebra of $\Pi_1(\mathbf{A})$. Then if $\mathbf{L} \in K$, then $\Pi_1(\mathbf{L}) \in K$*

Our persistence result relies on the representability of the Π_1-completion and possibility of forming an algebra of relations to represent a clone. It would be interesting to attempt to generalize this method by defining a more general algebra of relations on an L-space.

4.4 Conclusion

In this short chapter we applied the duality and representation theory developed in Chapter 3 to provide representations of the filter, ideal, and Π_1-completions of a lattice expansion (Theorem 4.2.3). We then use this representation to provide a proof of the fact that all identities between positive terms are preserved through the Π_1-completion (Theorem 4.3.5). We will apply and adapt this result later to the case of $r\ell$-groupoids.

Chapter 5

From Topologies To Frames

The present chapter makes the move from L-spaces with filter continuous relations to what we call NRL-spaces, and then further to the OKHD-frames that undergird the semantics of Ono and Komori, Humberstone, and Došen. This chapter is therefore largely a conceptual contribution to the project of this thesis. It brings us from the very general picture established in Chapter 3 to the special case of of the topologies and the semantics that will entertain us in the chapters to come. Despite its conceptual nature, this chapter introduces NRL-spaces, which will be the focus of the next three chapters, and contains some key lemmas regarding these objects.

More specifically, in section 5.1 we provide the requirements for the clopen filters of an L-spaces equipped with a filter continuous to form and $r\ell$-groupoid. The specific class of objects we define here are called RML-spaces (See definition 5.1.1). Section 5.2 then introduces NRL-spaces, which can be seen as topological versions of the OKHD-frames, which were discussed briefly in the introduction and will be discussed in detail in Chapter 7. We show this fact in more precision in Proposition 5.2.11 by demonstrating that every NRL-space is also in an OKHD-frame. Next, in Section 5.3, we show that the category of NRL-spaces and RML-spaces are equivalent. In conjunction, the results of sections 5.2 and 5.3 guide us from the most general perspective of L-spaces and filter continuous relations to the semantics of substructural logics given in terms of OKHD-frames. We see these developments as showing how to derive the OKHD-semantics from the general theory of filter continuous relations. Finally, in Section 5.4, we define another class frames obtained by omitting the topological properties of an RML-space. We will call these frames *RML-frames*. We show that the functors that witness the equivalence between NRL-spaces and RML-space do not generalize to an equivalence between OKHD-frames and RML-frames. We note that a semantics in terms of these frame is possible and will be remarked on further in Chapter 7 on completeness via duality.

5.1 Residuation and Filter Continuous Relations

In this section we focus on the case of ternary filter continuous relations and add sufficient conditions to guarantee that that algebra of clopen filters form a pointed $r\ell$-groupoid (See Definition 2.1.5). These structures start us on the path to deriving the OKHD-semantics from the general theory of filter continuous relations.

We define *Residuated Merge L-Spaces* or simply *RML-Spaces*, as L-spaces with a ternary filter continuous relation and some additional conditions for that relation.

Definition 5.1.1. *A Residuated Merge L-Space or simply RML-Space is a tuple* $\mathbf{X} = (X, 1, \curlywedge, R, T, \tau)$ *where* $(X, 1, \curlywedge, \tau)$ *is an L-space,* T *is a clopen filter, and* $R \subseteq X^3$ *is a ternary filter continuous relation satisfying the following five constraints:*

If $U, V,$ *are clopen, then so are* $U\backslash_R V = \{z \mid \forall xy((y \in U \ \& \ Rxyz) \to x \in V)\}$ *and* $U\backslash_R V = \{y \mid \forall xz((z \in U \ \& \ Rxyz) \to x \in V)\}$ *are as well,*

2a) If $R(z, u \curlywedge u', y)$*, then there are* $t, t' \in X$ *such that* $t \curlywedge t' \leq z$ *and* $Rtuy$ *and* $Rt'u'y$*,*

2b) If $R(z, y, u \curlywedge u',)$*, then there are* $t, t' \in X$ *such that* $t \curlywedge t' \leq z$ *and* $Rtyu$ *and* $Rt'yu'$*,*

3a) For all $x, y \in X$*, if* $Ry1x$*, then* $y = 1$*, and finally*

3b) For all $x, y \in X$*, if* $Ryx1$*, then* $y = 1$*.*

Jointly, these condition are sufficient to guarantee that clopen filters of a salient RML-space are closed under the operation F_R, which we will henceforth denote by $\circ_R$ in the ternary case, and the operations $\backslash_R$ and $/_R$. We therefore obtain the following fact.

Corollary 5.1.2. *Let* $\mathbf{X} = (X, 1, \curlywedge, R, \tau)$ *be an RML-space. Then* $(\mathcal{F}i_{clp}(X), \cap, \nabla, X, \{1\}, \circ_R, \backslash_R, /_R, T)$ *is a pointed* $r\ell$*-groupoid.*

Proof. In virtue of the definition of filter continuous relations, $U \circ_R V = F_R[U, V]$ is a clopen filter. The fact that this algebra is pointed by T follows from the fact that T is a clopen filter. Finally, if we can show that $U\backslash_R V$ and $U/_R V$ are filters whenever U, V are, by condition (1) of the definition of RML-spaces we obtain that if U, V are clopen filters, then so are $U\backslash_R V$ and $U/_R V$.

So let us show that when U, V are filters, then so are $U\backslash_R V$ and $U/_R V$. Let us just consider the case of $U\backslash_R V$. For upward closure, let $z \leq z'$ and $z \in U\backslash_R V$. Now suppose that $y \in U$ and $Rxyz'$. By the order compatibility property of a filter continuous relation (see proposition 3.2.2), we then have that $Rxyz$. So since $z \in U\backslash_R V$ and $y \in U$, we have that $x \in V$. We therefore conclude that $z' \in U\backslash_R V$. For $\curlywedge$-closure, let $z, z' \in U\backslash_R V$. Suppose that $y \in U$ and $R(x, y, z \curlywedge z')$. By the condition (2b) from the definition of an RML-space, we obtain that there are t, t' such that $t \curlywedge t' \leq x$ and $Rtyz$ and $Rt'yz'$. However, by the assumption that $z, z' \in U\backslash_R V$, we obtain that $t, t' \in V$. Since V is a filter, $t \curlywedge t' \in V$ and thus $x \in V$, as desired. We conclude that $z \curlywedge z' \in U\backslash_R V$. Finally, if $Rxy1$, by condition (3b) from the definition of an RML-space $x = 1 \in V$, so $1 \in U\backslash_R V$.

The last thing we need to do is check that $\circ_R$, $\backslash_R$, and $/_R$ form a residuated family. The demonstration of this follows by a standard argument. For good faith, let us show that $U \circ_R \subseteq W$ iff $V \subseteq U\backslash_R W$. Suppose that $U \circ_R \subseteq W$ and $z \in V$. Now let $y \in U$ and suppose that $Rxyz$. Then $x \in U \circ_R \subseteq W$. Therefore, $z \in U\backslash_R W$ and thus $V \subseteq U\backslash_R W$. Conversely, suppose that $V \subseteq U\backslash_R W$. Now let $x \in U \circ_R V$. Then there are $y \in U$ and $z \in V$ such that $Rxyz$. However, if $z \in V$, then $z \in U\backslash_R W$. So from $Rxyz$, we obtain that $x \in W$, as desired. We conclude that $U \circ_R \subseteq W$. □

Since we would like to form a category of these objects, we have the following definition of morphisms between RML-spaces. The definition provided here guarantees that their inverses are $r\ell$-groupoid homomorphism between salient $r\ell$-groupoids of clopen filters.

Definition 5.1.3. *(M-space Morphism) An RML-space Morphism* $(X, \curlywedge, 1, R, \tau) \to (X', \curlywedge', 1', R', \tau')$ *is an L-space morpshism* $f : (X, \curlywedge, 1) \to (X', \curlywedge', 1')$ *satisfying the following additional constraints.*

1) if $Rzxy$*, then* $R'f(z)f(x)f(y)$*,*

2) If $Rf(z)x'y'$*, then there are* $x, y \in X$ *such that* $Rzxy$*,* $x' \leq f(x)$*, and* $y' \leq f(y)$*,* *3) if $R'fz'(x)y'$, then there are $y, z \in X$ such that $Rzxy$, $y' \leq f(y)$, and $f(z) \leq z'$,*

4) if $Rz''x'f(y)$, then there are $x, z \in X$ such that $Rxyz$, $x' \leq f(x)$, and $f(z) \leq z'$,

We will denote the category of Merge L-spaces and morphisms with **MLSp**.

In the next section we turn to what we refer to as NRL-spaces, which we will show to essentially be topological OKHD-frames. This will bring us one step closer to the explication of the the OKHD-semantics in terms of L-spaces and filter continuous relations.

5.2 NRL-Spaces and OKHD-Frames

We now arrive at the definitions of NRL-spaces their relation to OKHD-frames. The main result of the next few pages will be the demonstration that NRL-spaces are essentially OKHD-frames with an L-space topology. More precisely, by forgetting the topology of an NRL-space we obtain an OKHD-frame. Demonstrating this fact will be the focus of the following few pages. In the next section we will show how to derive NRL-spaces from the RML-spaces presented at the end of the last section. We therefore will establish the connection between the L-spaces and filter-continuous relations and the semantics of substructural logics developed by Ono and Komori, Humberstone, and Došen.

Definition 5.2.1. *An NRL-space $\mathbf{X} = (X, \curlywedge, 1, \otimes, \varepsilon, \tau)$ is such that $(X, \curlywedge, 1, \tau)$ is an L-spaces, $\otimes : X \times X \to X$ is a groupoid operation, ε is a designated element and:*

(1) For all clopen filters U, V, $U \circ_X V = {\uparrow}(\{x \otimes y \mid x \in U \ \&\ y \in V\})$, $U \backslash_X V = \{y \mid \forall x \in U(x \otimes y \in V)\}$, and $V/_X U = \{x \mid \forall y \in U(x \otimes y \in V)\}$ are clopen filters (see below),

(2) ${\uparrow}\varepsilon$ is clopen, and

(2) $x \otimes y \leq z$ iff for all $U, V \in \mathcal{F}_{clop}(X)$, if $x \in U$ and $y \in V$, then $z \in U \circ_X V$.

Let us consider two examples.

Example 5.2.2. *We define $\mathbf{X} = (\mathbb{N} \cup \{\omega\}, \curlywedge, 1_{\mathbf{X}}, \otimes, \varepsilon_{\mathbf{X}}, \tau)$ so that $n \curlywedge m := min(n, m)$, $1_{\mathbf{X}} := \omega$, and $n \otimes m := n + m$, and $\varepsilon_{\mathbf{X}} := 0$. Finally, we generate τ by the subbase $\{{\uparrow}n \mid n \in \mathbb{N} \cup \{\omega\}\} \cup \{{\downarrow}n \mid n \in \mathbb{N} \cup \{\omega\}\} \cup \{\emptyset\}$. To see that the clopen filters are closed under $\backslash$ and $/$, note that ${\uparrow}n \backslash {\uparrow}m = {\uparrow}m / {\uparrow}n = {\uparrow}(n - m)$.*

We now consider another example. We essentially stack the above NRL-space on top of the lattice $\mathbf{M}_3$ and equip the structure with an appropriate topology.

Example 5.2.3. *We define $\mathbf{Y} = (Y, \curlywedge, 1_{\mathbf{Y}}, \otimes, \varepsilon_{\mathbf{Y}}, \tau_{\mathbf{Y}})$ such that $Y = \{a, b, c, \bot\} \cup \mathbb{N} \cup \{\omega\}$. The semilattice structure is depicted below in the diagram on the following page. So $1_{\mathbf{Y}} = \omega$. $\otimes$ is defined by the following.*

$$x \otimes y = \begin{cases} x + y, & \text{if } x, y \in \mathbb{N} \cup \{\omega\}, \\ x \curlywedge y, & \text{otherwise.} \end{cases}$$

We define $\varepsilon_{\mathbf{Y}} := 0$, just as in the previous example. Note however that $\varepsilon_{\mathbf{Y}}$ is not an identity element for $\otimes$. Finally, the topology $\tau_{\mathbf{Y}}$ on Y is generated by the subbase: $\{{\uparrow}x \mid x \in Y\} \cup \{Y - {\uparrow}x \mid x \in Y\}$.

Note that $U \backslash V$ and U/V are clopen filters when U and V are. This shown by checking cases. If $U, V \subseteq \mathbb{N} \cup \{\omega\}$, then we reason as in the previous example. If $V \subseteq \mathbb{N} \cup \{\omega\}$ and $U = {\uparrow}x$ for $x \in \{a, b, c\}$, then we can show that ${\uparrow}x \backslash V \subseteq {\uparrow}x$. But since ${\uparrow}x$ is a linear order, we are ensured that ${\uparrow}x \backslash V$ is principal and thus a clopen filter. If instead $U \subseteq \mathbb{N} \cup \{\omega\}$ and $V = {\uparrow}x$ for $x \in \{a, b, c\}$, we *claim that* $\{a,b,c\}-\{x\} \nsubseteq U\backslash V$*. Therefore we have that* $U\backslash V \subseteq V$ *and so is principal. In both* $U = \uparrow x$ *and* $V = \uparrow y$ *some* $x, y \in \{a,b,c\}$*, it follows that* $U\backslash V = V/U = U$*. Finally, if* V *is any subset and* $U = \uparrow\perp$*, then* $\uparrow\perp\backslash V = V$*. Finally, if* U *is any subset and* $V = \uparrow\perp$*, then* $U\backslash\uparrow\perp = \uparrow\perp$*. So in all cases we have shown that if* U, V *are clopen filters, then so are* $U\backslash V$ *and* U/V*.*

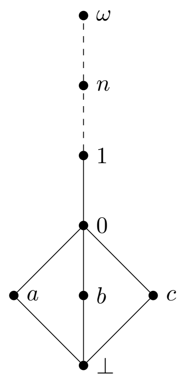


Figure 1. This is semi lattice structure of the NRL space in the present example.

A lemma useful for showing that NRL-spaces have an underlying OKHD-frame structure is that the clopen filters of an NRL-space forms a pointed $r\ell$-groupoid. This is also the key to our representation and duality results presented later in this chapter.

Proposition 5.2.4. *For any NRL-space* $\mathbf{X}$*, the algebra* $\mathbf{L_X} = (\mathcal{F}i_{clp}(X), \cap, \triangledown, X, \{1\}, \circ_X, \backslash_X, /_X, \uparrow\varepsilon)$ *is a pointed* $r\ell$*-groupoid.*

Proof. Since we are ensured by the definition of an NRL-space that $\circ_X$, $\backslash_X$, and $/_X$ are well defined operations on $\mathcal{F}i_{clp}(X)$, we just need to check that $\backslash_X$ and $/_X$ satisfy the residual law with respect to $\circ_X$. In particular, we need to show for all clopen filters U, V, and W:

$$U \circ_X V \subseteq W \Longleftrightarrow U \subseteq W/_X V \Longleftrightarrow V \subseteq U\backslash_X W.$$

We only show that $U \circ_X V \subseteq W$ iff $V \subseteq U\backslash_X W$ noting that the equivalence $U \circ_X V \subseteq W$ iff $U \subseteq W/_X V$ has a similar proof.

So suppose that $U \circ_X V \subseteq W$ and let $y \in V$ and $x \in U$. Clearly, $x \otimes y \in U \circ_X V$. So by the assumption $U \circ_X V \subseteq W$, we have that $x \otimes y \in W$. Then generalizing on x, we conclude that $y \in U\backslash_X W$ and then that $V \subseteq U\backslash_X W$. For the other direction, suppose that $V \subseteq U\backslash_X W$ and let $z \in U \circ_X V$. Then there are $x \in U$ and $y \in V$ such that $x \otimes y \leq z$. By the assumption $V \subseteq U\backslash_X W$, $y \in U\backslash_X W$. So by definition of $\backslash_X$, $x \otimes y \in W$. But since W is upward closed, we obtain that $z \in W$. We therefore conclude that $U \circ_X V \subseteq W$, as desired. □

Now, in order to show that NRL-spaces all posses OKHD-frame structure, we will need a few more lemmas. The first lemma, Lemma 5.2.5, is an analogue of Lemma 3.2.3 that showed how to extend the operations on filters associated with filter continuous relations to closed filters. It can be seen as an equivalent to condition (3) defining NRL-spaces.

Lemma 5.2.5. *Let* $\mathbf{X}$ *be an* $r\ell G$*-space. Let* U, V *be closed filters of* $\mathbf{X}$*, then*

$$U \circ_X V = \bigcap\{U' \circ_X V' \mid U', V' \in \mathcal{F}i_{clop}(X) \;\&\; U \subseteq U',\; V \subseteq V'\}.$$

Proof. Essentially follows immediately from condition (3) of the definition of an NRL-space (Definition 5.2.1) and Lemma 2.2.9. By Lemma 2.2.9, U and V are principal if they are closed filters. So there are $x, y \in X$ such that $\uparrow x = U$ and $\uparrow y = V$. The identity we are trying show then becomes:

$$\uparrow x \circ_X \uparrow y = \bigcap\{U \circ V \mid U, V \in \mathcal{F}i_{clp}(X) \ \&\ x \in U \ \&\ y \in V\}.$$

However, since $\uparrow x \circ_X \uparrow y = \uparrow(x \otimes y)$, this is essentially equivalent to condition (3) of the definition of an NRL-space (Definition 5.2.1).

□

The second lemma we need is a consequence of Lemma 5.2.5 and provides a useful condition for finding clopen filters of the form $U \circ V$. This lemma is analogous to Lemma 2.2.12

Lemma 5.2.6. *If If $\mathbf{X} = (X, \curlywedge, 1, \otimes, \varepsilon, \tau)$ is an NRL-space and U is a clopen filter of $\mathbf{X}$, then: if $x \otimes y \in U$, there are clopen filters V and W such that $x \in V$ and $y \in W$ and $V \circ W \subseteq U$.*

Proof. Let $x \otimes y \in U$. From Lemma 2.2.9 and HMS-separation, we know in general that for each $w \in X$, $\uparrow w = \bigcap\{U' \in \mathcal{F}i_{clp}(\mathbf{X}) \mid w \in U'\}$. Therefore by Lemma 5.2.5 we have that

$$\begin{aligned}
&\bigcap\{V' \circ W' \mid V', W' \in \mathcal{F}i_{clp}(\mathbf{X}) \ \&\ x \in V' \ \&\ y \in W'\} \\
&\qquad = \bigcap\{V' \in \mathcal{F}i_{clp}(\mathbf{X}) \mid x \in V'\} \circ \bigcap\{W' \in \mathcal{F}i_{clp}(\mathbf{X}) \mid y \in W'\} \\
&\qquad = \uparrow x \circ \uparrow y \\
&\qquad \subseteq U
\end{aligned}$$

It follows from compactness that there are $V_1', ..., V_n'$ with $x \in V_i'$ and $W_1', .., W_n'$ with $y \in W_i'$ such that $\bigcap\{V_i' \circ W_i' \mid i \leq n\} \subseteq U$. But then from monotonicity of $\circ$, we have

$$\bigcap\{V_i' \mid i \leq n\} \circ \{W_i' \mid i \leq n\} \subseteq \bigcap\{V_i' \circ W_i' \mid i \leq n\} \subseteq U.$$

However, since both $\bigcap\{V_i' \mid i \leq n\}$ and $\bigcap\{W_i' \mid i \leq n\}$ are clopen filters, we may generalize and establish that there are clopen filters V and W such that $x \in V$ and $y \in W$ and $V \circ W \subseteq U$. □

Our third lemma is quite simple and tells us that the $\otimes$ is monotone with respect to the order $\leq$. It is another consequence of condition (3) of the definition of an NRL-space.

Lemma 5.2.7. *If $\mathbf{X} = (X, \curlywedge, \otimes, 1, \tau)$ is an NRL-space, then: if $x \leq y$ and $x' \leq y'$, then $x \otimes y \leq x' \otimes y'$.*

Proof. Let $x \leq y$ and $x' \leq y'$. Let $x \in U$ and $x' \in V$. If $x \in U$ and $x' \in V$, then $y \in U$ and $y' \in U$. So $y \otimes y \in U \circ V$. Generalizing on U and V, condition (3) of Definition 5.2.1 implies that $x \otimes x' \leq y \otimes y'$. □

Let us now define OKHD-frames and provide a few examples.

Definition 5.2.8. *(OKHD-frames)*

An OKHD-frame $X = (X, \curlywedge, 1, \otimes, \varepsilon)$ is structure where $(X, \curlywedge, 1)$ is a semilattice, $(X, \otimes, \varepsilon)$ is a pointed groupoid, and (1) and (2) govern the relationship between $\curlywedge$, $\otimes$, and 1.

(1) $x \otimes (y \curlywedge z) = (x \otimes y) \curlywedge (x \otimes z)$ and $(y \curlywedge z) \otimes x = (y \otimes x) \curlywedge (z \otimes x)$, and

(2) $x \otimes 1 = 1 = 1 \otimes x$.

Before defining models on OKHD-frames, we provide the following examples of an OKHD-frame.

Example 5.2.9. *(The Tropical Semiring is an OKHD-frame) The Min-Tropical Semiring is the algebra* $(\mathbb{R} \cup \{\infty\}, \oplus, \infty, \otimes, 0)$ *where* $a \oplus b := min(a, b)$ *and* $a \otimes b = a + b$*. the Min-Tropical Semiring can be seen as an OKHD-frame since* $(\mathbb{R} \cup \{\infty\}, \oplus, \infty)$ *is a semilattice,* $(\mathbb{R} \cup \{\infty\}, \otimes, 0)$ *is a monoid, and:*

(1) $x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$ *and* $(y \oplus z) \otimes x = (y \otimes x) \oplus (z \otimes x)$*, and*

(2) $x \otimes \infty = \infty = \infty \otimes x$*.*

Note that this example cannot carry an NRL-space topology since the underlying semilattice of an L-space is always complete.

A less natural non-distributive example can formed by gluing copies of the positive extended reals $\mathbb{R}^+ \cup \{\infty\}$ at 0 and ∞. In the following we use $\mathbb{R}^+ = \{k \in \mathbb{R} \mid 0 < k\}$

Example 5.2.10. *Define* $X = \mathbb{R}^+_0 \uplus \mathbb{R}^+_1 \cup \{0, \infty\}$ *and define* $\curlywedge$ *and* $\otimes$ *such that:*

$$n \curlywedge m = \begin{cases} min(n, m), & \text{if } n, m \in \mathbb{R}^+_i \cup \{0, \infty\}, \\ 0, & \text{if } n \in \mathbb{R}_i \text{ and } m \in \mathbb{R}_j \text{ and } i \neq j. \end{cases}$$

$$n \otimes m = \begin{cases} n + m, & \text{if } n, m \in \mathbb{R}_i \cup \{\infty\}, \\ \infty, & \text{if } n \in \mathbb{R}_i \text{ and } m \in \mathbb{R}_j \text{ and } i \neq j, \end{cases}$$

Having seen a few examples, we now show that all NRL-spaces are additionally OKHD-frames, a fact that begins to formally establish the connection bwteen the semantics of Ono and Komori, Humberstone, and Došen's and the theory of L-spaces and filter continuous relations developed in this thesis. This result is not only conceptually significant but is also technically useful. We will often refer back to this property of NRL-spaces in subsequent chapters.

Proposition 5.2.11. *If* $\mathbf{X} = (X, \curlywedge, 1, \otimes, \varepsilon, \tau)$ *is an NRL-space, then* $(X, \curlywedge, 1, \otimes, \varepsilon)$ *is an OKHD-frame.*

Proof. For condition (1) of the definition of an OKHD-frame, let us just show $x \otimes (y \curlywedge z) = (x \otimes y) \curlywedge (x \otimes z)$, noting that the other identity follows from a very similar argument. The inequality, $x \otimes (y \curlywedge z) \leq (x \otimes y) \curlywedge (x \otimes z)$ is a straight forward consequence of Lemma 5.2.7. To show $(x \otimes y) \curlywedge (x \otimes z) \leq x \otimes (y \curlywedge z)$, Let U be a clopen filter and suppose that $(x \otimes y) \curlywedge (x \otimes z) \in U$. If we can show $x \otimes (y \curlywedge z) \in U$, then by HMS-separation we will obtain the desired inequality. By lemma 2.2.12, there are clopen filters V and W such that $V \triangledown W \subseteq U$ and $x \otimes y \in V$ and $x \otimes z \in W$. Now, by the similar lemma, Lemma 5.2.6, we find clopen filters V_1, V_2 and W_1, W_2 such that $x \in V_1$ and $x \in W_1$ and $y \in V_2$ and $z \in W_2$ and $V_1 \circ V_2 \subseteq V$ and $W_1 \circ W_2 \subseteq W$. It follows that $(Y \circ V_2) \triangledown (Y \circ W_2) \subseteq U$ where $Y = V_1 \cap W_1$ and that $(x \otimes y) \curlywedge (x \otimes z) \in (Y \circ V_2) \triangledown (Y \circ W_2)$. However, since $x \in Y$, $y \in V_2$, and $z \in W_2$, we also have that $x \otimes (y \curlywedge z) \in Y \circ (W_2 \triangledown V_2)$. Therefore, since the clopen filters of $\mathbf{X}$ form a pointed $r\ell$-groupoid (see Proposition 5.2.4), we have that $(Y \circ V_2) \triangledown (Y \circ W_2) = Y \circ (W_2 \triangledown V_2)$ and thus that $Y \circ (W_2 \triangledown V_2) \subseteq U$. So it follows that $x \otimes (y \curlywedge z) \in U$. Finally, by generalizing on U, we may conclude that $(x \otimes y) \curlywedge (x \otimes z) \leq x \otimes (y \curlywedge z)$ by contraposing HMS-separation.

Forcondition (2), let $x \in X$. From Lemma 2.2.9 we know that $\uparrow x = \bigcap \{U \in \mathcal{F}i_{clp}(\mathbf{X}) \mid x \in U\}$. By Lemma 5.2.5 we then have that

$$\{1\} \circ_X \bigcap \{U \in \mathcal{F}i_{clp}(\mathbf{X}) \mid x \in U\} = \bigcap \{\{1\} \circ_X U(\mathbf{X})\ U \in \mathcal{F}i_{clp} \ \& \mid x \in U\}.$$

However, since $\mathcal{F}i_{clp}(\mathbf{X})$ forms an $r\ell$-groupoid with $\{1\}$ as the bottom element, for all clopen filters U: $\{1\} \circ_X U = \{1\}$. So $\bigcap\{\{1\} \circ_X U(\mathbf{X})\ U \in \mathcal{F}i_{clp}\ \&\ |\ x \in U\} = \{1\}$ and thus we have the following.

$$\begin{aligned} x \otimes 1 &= \bigcurlywedge\big(\{1\} \circ_X \bigcap\{U \in \mathcal{F}i_{clp}(\mathbf{X}) \mid x \in U\}\big) \\ &= \bigcurlywedge\{1\} \\ &= 1. \end{aligned}$$

This concludes our proof. $\square$

We now demonstrate one more useful property relating the order to $\otimes$ and then show that the algebra of all filters of an NRL-space also form a pointed $r\ell$-groupoid.

Lemma 5.2.12. *Let $\mathbf{X} = (X, \curlywedge, \otimes, 1, \tau)$ be an OKHD-frame, then: if $x \otimes y \leq z$ and $x' \otimes y' \leq z'$, then $(x \curlywedge x') \otimes (y \curlywedge y') \leq z \otimes z'$.*

Proof. For (3), suppose that $x \otimes y \leq z$ and $x' \otimes y' \leq z'$. Then $(x \otimes y) \curlywedge (x' \otimes y') \leq z \otimes z'$ by the monotonicity of $\otimes$ (Lemma 5.2.7) . But we have that $x \curlywedge x' \leq x, x'$ and $y \curlywedge y' \leq y, y'$. So again by the monotonicity of $\otimes$, $x \curlywedge x' \otimes (y \curlywedge y') \leq (x \otimes y) \curlywedge (x' \otimes y')$. Therefore $x \curlywedge x' \otimes (y \curlywedge y') \leq z \otimes z'$. $\square$

We can now show that the operation $\circ_X$, $\backslash_X$, and $/_X$ are well defined on not only clopen filters, which is by definition, but is generally defined for the filters of an NRL-space.

Lemma 5.2.13. *If U and V are filters of an OKHD-frame, then so are:*
(i) $U \circ_X V = \uparrow(\{x \otimes y \mid x \in U\ \&\ y \in V\})$,
(ii) $U \backslash_X V = \{y \mid \forall x \in U (x \otimes y \in V)\}$, and
(iii) $U /_X V = \{x \mid \forall y \in U (x \otimes y \in V)\}$.

Proof. We will only show the cases of (i) and (ii) since (iii) is essentially the same as (ii).

We begin with (i). For upward closure, let $x \in U \circ_X V$ and suppose that $x \leq x'$. If $x \in U \circ_X V$, then there are $y \in U$ and $z \in V$ such that $y \otimes z \leq x$. Therefore $y \otimes z \leq x'$ and so $x' \in U \circ_X V$. For $\curlywedge$-closure, suppose that $x, x' \in U \circ_X V$. There are $y, y' \in U$ and $z, z' \in V$ such that $y \otimes z \leq x$ and $y' \otimes z' \leq x'$. Therefore, $(y \otimes z) \curlywedge (y' \otimes z') \leq x \curlywedge x'$. But then in virtue of the fact that $\otimes$ is monotone, we have that $(y \curlywedge' y) \otimes (z \curlywedge z') \leq (y \otimes z) \curlywedge (y' \otimes z') \leq x \curlywedge x'$ (see (2) in the preceding lemma). But since $y \curlywedge y' \in U$ and $z \curlywedge z' \in V$, we obtain that $x \curlywedge x' \in U \circ_X V$, as desired. Finally, to ensure $1 \in U \circ_X V$, note that $1 \in U$, $1 \in V$, and $1 \otimes 1 \leq 1$. It follows that $1 \in U \circ_X V$.

Let us now consider the case of $U \backslash_X V$. For upward closure, let $y \in U \backslash_X V$ and suppose that $y \leq y'$. Now suppose that $x \in U$. Since $y \in U \backslash_X V$, we have that $x \otimes y \in V$. Because $x \otimes y \leq x \otimes y'$ and V is a filter, we also have $x \otimes y' \in V$. Therefore, $y' \in U \backslash_X V$. For $\curlywedge$-closure, let $y, y' \in U \backslash_X V$. Suppose that $x \in U$. Then $x \otimes y$ and $x \otimes y'$ are both elements of V. Therefore $x \otimes (y \curlywedge y') = x \otimes y \curlywedge x \otimes y' \in V$ because V is filter. Finally, we are guaranteed that $1 \in U \backslash_X V$ since $1 \in U$ and $1 \otimes 1 = 1 \in V$.

The case of $V /_X U$ is nearly identical to that of $U \backslash_X V$. $\square$

Proposition 5.2.14. *Let $\mathbf{X}$ be an NRL-space, then $\mathcal{F}i(\mathbf{X})$ forms a pointed $r\ell$-groupoid with respect to the operations $\circ_X$, $\backslash_X$, and $/_X$ and the designated element $\uparrow\varepsilon_X$.*

Proof. In virtue of Lemma 5.2.13, the operations $\circ_X$, $\backslash_X$, and $/_X$ are well defined because they always return filters when applied to filters. The fact that $\circ_X$, $\backslash_X$, and $/_X$ jointly satisfy the residual law follows from the proof of Proposition 5.2.4. Since we did not rely on clopeness in our demonstration of the residual law in that context, we are ensured it also holds here. $\square$

In the next section we show how the Definition of NRL-spaces can be derived from the more general picture of filter continuous relations presented in the previous chapter. To this end we define morphisms between NRL-spaces as follows.

Definition 5.2.15. *An NRL-space morphism $f : \mathbf{X} \to \mathbf{Y}$ is an L-space morphism that satisfies the following conditions:*

($\otimes$-forth) $f(x) \otimes' f(y) \leq f(x \otimes y)$,

($\otimes$-back) If $x' \otimes' y' \leq f(z)$, then there are $x, y \in X$ such that $x' \leq fx$, $y' \leq fy$, and $x \otimes y \leq z$,

($/$-back) if $fx \otimes' y' \leq z'$, then there are $y, z \in X$ such that $y' \leq fy$, $fz \leq z'$, and $x \otimes y \leq z$,

($\backslash$-back) if $x' \otimes' fy \leq z'$, then there are $x, z \in X$ such that $x' \leq fx$, $fz \leq z'$, and $x \otimes y \leq z$,

(ε-forth) $\varepsilon' \leq f(\varepsilon)$, and

(ε-back) if $\varepsilon' \leq fx$, then $\varepsilon \leq x$.

5.3 The Equivalence of NRL-spaces with RML-spaces

In this section we show how to obtain NRL-spaces from RML-spaces and therefore how we can understand NRL-spaces in terms of our more general understanding of L-spaces with filter continuous relations.

Theorem 5.3.1. *The the category* **NRLSp** *is equivalent to the category* **RMLSp**.

Proof. In order to show the equivalence we will describe functors F : **NRLSp** $\to$ **RMLSp** and G : **RMLSp** $\to$ **NRLSp** and demonstrate that they are inverse to one another.

Let us first describe the functor F : **NRLSp** $\to$ **MLSp**, which is quite straight forward. We start at the level of objects. Given NRL-space $\mathbf{X} = (X, \curlywedge, 1, \otimes, \varepsilon, \tau)$, we can define an M-space $F(\mathbf{X}) = (X, \curlywedge, 1, R_\otimes, T, \tau)$ such that $T = {\uparrow}\varepsilon$ and

$$R_\otimes zxy \text{ iff } x \otimes y \leq z.$$

Let us show that $F(\mathbf{X})$ is a M-space. That $(X, \curlywedge, 1, \tau)$ is an L-space and that T is a clopen filter are both immediate from the definition of an NRL-space. We just need to check that $R_\otimes$ is a residuated filter continuous relation.

First, we show that $R_\otimes$ is filter continuous (Definition 3.2.1). For Clopen Continuity, we have that for all clopen filters of $\mathbf{X}$, $U \circ_X V = U \circ_{R_\otimes} V$ where we recall that $U \circ_X V = {\uparrow}\{x \otimes y \mid x \in U \;\&\; y \in V\}$ and $U \circ_{R_\otimes} V = \{z \mid \exists x \in U \exists y \in V (R_\otimes(z, x, y))\}$. So since $\mathcal{F}i_{clp}(\mathbf{X})$ is closed under $\circ_X$ by definition of an NRL-space, we have that $\mathcal{F}i_{clp}(\mathbf{X})$ is also closed under $\circ_R$. For Clopen Compatibility we again note that that $\circ_X = \circ_R$. In virtue this and the second condition in Definition 5.2.1 we are done.

Second, to show that $R_\otimes$ is residuated we first observe that $U \backslash_X V = U \backslash_{R_\otimes} V$ and $U /_X V = U /_{R_\otimes} V$ for all clopen filters U and V. This takes care of condition (1) from Definition **??**. For condition (2), suppose that $R_\otimes(z, x \curlywedge x', y)$. Then $(x \curlywedge x') \otimes y \leq z$ by definition of $R_\otimes$. However, from Proposition 5.2.11 we have that $x \otimes y \curlywedge x' \otimes y \leq (x \curlywedge x') \otimes y$. So then $x \otimes y \curlywedge x' \otimes y \leq z$. Therefore, by generalizing on $x \otimes y$ and $x' \otimes y$ respectively, we have found t and t' such that $t \curlywedge t' \leq z$ and $R_\otimes(t, x, y)$ and

$R_\otimes(t', x', t)$. The other case for condition (2) relies on a similar argument. Finally, to show condition (3) of Definition **??**, If $R_\otimes(z, 1, y)$, then $1 \otimes y \leq z$. But from Proposition 5.2.11, $1 \otimes y = 1$ and so $1 = z$.

Having shown that $R_\otimes$ is a residuated filter continuous relation, we can conclude that $F(\mathbf{X})$ is in fact a RML-space.

Now for morphisms. Given a morphism f in **NRLSp**, we define $F(f)$ to simply be f itself. The conditions required by definition 5.1.3 are met almost immediately in virtue of the properties associated with being an NRL-space morphism (See Def. 5.2.15).

Let us now describe the functor $G : \mathbf{MLSp} \to \mathbf{NRLSp}$. Beginning at the level of objects, let $\mathbf{X} = (X, \curlywedge, 1, R, T, \tau)$ be a RML-space. Then we define $G(\mathbf{X}) = (X, \curlywedge, 1, \otimes, \varepsilon\tau)$ such that $\varepsilon = \bigcurlywedge T$ and

$$x \otimes y = \bigcurlywedge\{z \mid Rzxy\}.$$

That the conditions (1) and (2) listed below hold, which are the defining conditions of an NRL-space, follows in virtue of the fact that R is filter-continuous (Definition 3.2.1).

(1) For all $U, V \in \mathcal{F}i_{clp}(X)$, $U \circ_X V \in \mathcal{F}i_{clp}(X)$, and

(2) $x \otimes_R y \leq z$ iff for all $U, V \in \mathcal{F}_{clop}(X)$, if $x \in U$ and $y \in V$, then $z \in U \circ_X V$,

We therefore conclude that $G(\mathbf{X})$ is a NRL-space.

Now for morphisms, let g be a morphism in the category **RMLSp**. $G(g) := g$. That g is a morphism in **NRLSp** follows quickly from the definitions of morphisms and of $\otimes_R$.

Having described our functors $F : \mathbf{NRLSp} \to \mathbf{RMLSp}$ and $G : \mathbf{RMLSp} \to \mathbf{NRLSp}$, to establish equivalence requires us to check that $G(F(\mathbf{X})) \cong \mathbf{X}$ and $F(G(\mathbf{Y})) \cong \mathbf{Y}$ for any $\mathbf{X}$ in **NRLSp** and $\mathbf{Y}$ in **RMLSp**. However, this is quickly confirmed by showing that:

(1) $x \otimes y = x \otimes_{R_\otimes} y$, and

(2) $Rzxy$ iff $R_{\otimes_R} zyx$

since at the level of the L-space component of these objects nothing has changed in the moves made between **NRLSp** and **RMLSp**. At the level of morphisms, by F and G are identities, so it is trivial that $FG(g) = g$ anf $GF(f) = f$. $\square$

We have just shown an equivalence between the category of NRL-spaces and a category of L-spaces with modified ternary filter continuous relations, which we called RML-spaces. In the previous section we showed that in Proposition 5.2.11 that by forgetting the topology of an NRL-space we obtain an OKHD-frame. Therefore, the composition of these two operations, first moving from L-spaces with filter continuous relations to NRL-spaces and then moving to OKHD-frames sketches the path from the our general theory of L-spaces and FC-relations to the semantics of substructural logics. In the final section of this chapter, we show that the category of frames that undergird RML-spaces is not equivalent to the category of the OKHD-frames.

5.4 The Non-Equivalence of OKHD-frames and RML-frames

In this final section of the chapter we show that functor $F : \mathbf{NRLSp} \to \mathbf{RMLSp}$ and $G : \mathbf{RMLSp} \to \mathbf{NRLSp}$ do not also general tp an equivalence between the category of OKHD-frames and and the category of RML-frames, which are the class of objects obtain from RML-spaces when the topological conditions that defined them are ignored. In particular, we show that there is an

RML-frame $\mathbf{X} = (X, \curlywedge, 1, R, T)$ whose ternary relation R is not definable by $Rxyz$ iff $y \otimes z \leq x$ for any groupoid operation, $\otimes : X \times X \to X$.

Let us define RML-frame in detail.

Definition 5.4.1. *A Residuated Merge L-frame or RML-frame* $\mathbf{X} = (X, \curlywedge, 1, R, T)$ *is semilattice* $(X, \curlywedge, 1)$ *equipped with a special subset* T *and a ternary relations* $R \subseteq X^3$ *that satisfies the following constraints:*

(1) If $Rxyz$ *and* $x \leq x'$*, then* $Rx'yz$*,*
(2) If $Rxyz$ *and* $Ruvw$*, then* $R(x \curlywedge u)(y \curlywedge v)(z \curlywedge w)$*,*
(3) If $Rz(x \curlywedge x')y$*, then there are* $t, t' \in X$ *such that* $Rtxy$ *and* $Rt'x'y$ *and* $t \curlywedge t' \leq z$*,*
(4) If $Rzx(y \curlywedge y')$*, then there are* $t, t' \in X$ *such that* $Rtxy$ *and* $Rt'xy'$ *and* $t \curlywedge t' \leq z$*,*
(5) For all $x, y \in X$*, if* $Rx1y$*, then* $x = 1$*,*
(6) For all $x, y \in X$*, if* $Rxy1$*, then* $x = 1$*.*

A consequence of (1) and (2) is the following fact.

Lemma 5.4.2. *For any RML-frame* $\mathbf{X} = (X, 1, \curlywedge, R, T)$*, If* $Rxyz$ *and* $x \leq x'$ *and* $y' \leq y$ *and* $z' \leq z$*, then* $Rx'y'z'$*.*

We have the following definition of morphisms between RML-frames.

Definition 5.4.3. *An RML-frame Morphsim* $(X, \curlywedge, 1, R) \to (X', \curlywedge', 1', R')$ *is a semi lattice homomorphism* $f : (X, \curlywedge, 1) \to (X', \curlywedge', 1')$ *satisfying the following additional constraints.*

(1) if $x \curlywedge y \leq f(z)$*, then there are* $x, y \in X$ *such that* $x \curlywedge y \leq z$ *and* $x' \leq f(x)$ *and* $y' \leq f(y)$*,*
(2) if $Rxyz$*, then* $R'f(x)f(y)f(z)$*,*
(3) If $Rx'y'f(z)$*, then there are* $x, y \in X$ *such that* $x' \leq f(x)$ *and* $y' \leq f(y)$ *and* $Rxyz$*,*
(4) if $R'f(x)y'z'$*, then there are* $y, z \in X$ *such that* $Rxyz$*,* $y' \leq f(y)$*, and* $f(z) \leq z'$*, and*
(5) if $R'x'f(y)z'$*, then there are* $x, z \in X$ *such that* $Rxyz$*,* $x' \leq f(x)$*, and* $f(z) \leq z'$*,*
(6) $x \in T_X$ *iff* $fx \in T_Y$*.*

We denote the category of RML-frames together with RML-frame morphisms with **RMLFrm.** We now show that every OKHD-frame is can be turned into an RML-frame.

Proposition 5.4.4. *For every OKHD-frame* $\mathbf{X} = (X, \curlywedge, 1, \otimes, \varepsilon)$*, the structure* $X^{\otimes} = (X, \curlywedge, 1, S^{\otimes}, {\uparrow}\varepsilon)$ *such that* $S^{\otimes}xyz$ *iff* $y \otimes z \leq x$ *is an RML-frame.*

Proof. We prove each condition in the definition of RML-frames (Definition 5.4.1).
Condition (1): suppose that $S^{\otimes}xyz$ and $x \leq x'$. Then $y \otimes z \leq x \leq x'$. Therefore $S^{\otimes}x'yz$. Condition (2): Suppose that If $S^{\otimes}xyz$ and $S^{\otimes}uvw$. Then $y \otimes z \leq x$ and $v \otimes w \leq u$. However, in virtue of the monotonicity of $\otimes$ we have $(y \curlywedge v) \otimes (z \curlywedge w) \leq (y \otimes z) \curlywedge (v \otimes w) \leq x \curlywedge u$. Therefore $S^{\otimes}(x \curlywedge u)(y \curlywedge v)(z \curlywedge w)$. For Conditions (3), (4), (5), and (6), see the proof of the equivalence of NRL-spaces and RML-spaces in Theorem 5.3.1. □

Despite the equivalence between NRL-spaces and RML-spaces demonstrated in the last section, there is no equivalence in the case of OKHD-frames and RML-frames. The following proposition shows by way of example that there is an RML-frame that is not an OKHD-frame.

Proposition 5.4.5. *There is an RML-frame* $X = (X, 1, \curlywedge, R, T)$ *such that there is no OKHD-frame* Y *such that* $Y^{\otimes} = X$ *(Where* $Y^{\otimes}$ *is defined as in Proposition 5.4.4).*

Proof. Consider the RML-frame $\mathbf{N} = (\mathbb{N} \cup \{\omega\}, \curlywedge, 0, R, T)$ where $x \curlywedge y$ is defined as $\max(x, y)$ as calculated in $\omega + 1$, 0 is the top element, $T = \mathbb{N} \cup \{\omega\}$ and $R = \{(x, y, z) \mid y = z = \omega \;\&\; x \in \mathbb{N}\} \cup \{(x, y, z) \mid y, z \in \mathbb{N} \cup \{\omega\} \;\&\; x = 0\}$. The semi-lattice structure is depicted in figure 1. Note why the meet is counter intuitively defined as max.

For showing that this structure is actually an RML frame, conditions (1) and (2) of the definition of an RML-frame (Definition 5.4.1) are straightforwardly verified. Similarly, the cases of conditions (5) and (6) are also almost immediate. The conditions (3) and (4) are also easy to check, but we will at least demonstrate (3) here and note that the argument for (4) is almost the same. So let $Rx(y \curlywedge y')z$. Then either (a) $y \curlywedge y' = z = \omega$ and $x \in \mathbb{N}$ or (b) $y \curlywedge y', z \in \mathbb{N} \cup \{\omega\}$ and $x = 0$. In the case of (a), If $y \curlywedge y' = \omega$, then either $y = \omega$ or $y' = \omega$. If $y = \omega$, then define $t := x$ and $t' := 0$. Then it is almost immediate by definition of R that $Rtyz$ and $Rt'y'z$ and that $t \curlywedge t' \leq x$. We similarly find t and t' if $y' = \omega$. We therefore conclude that there t, t' such that $Rtyz$ and $Rt'y'z$ and $t \curlywedge t' \leq x$. In the case of (b), then $y \curlywedge y', z \in \mathbb{N} \cup \{\omega\}$ and $x = 0$. So simply define $t = t' = 0$. It is immediate that $Rtyz$ and $Rt'y'z$ and $t \curlywedge t' \leq x$, so we are done.

Now suppose for contradiction that there is some operation $\otimes : N^2 \to N$ such that $S^{\otimes} = R$. Given that $S^{\otimes}xyz$ holds iff $x \otimes y \leq z$, we obtain for all $n \in \mathbb{N}$ that $\omega \otimes \omega \leq n$. But $\bigwedge \mathbb{N}$ exists in N and is ω. So we have that $\omega \otimes \omega \leq \bigwedge \mathbb{N} = \omega$. And thus we have that $S^{\otimes}\omega\omega\omega$. But then by the assumption that $S^{\otimes} = R$, we reach a contradiction since R be definition is not such that $R\omega\omega\omega$.

It is therefore the case that there is no OKHD-frame Y such that $Y^{\otimes} = X$. $\square$

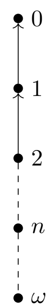


Figure 2. This is semi lattice structure of the merge frame defined in the proof of 5.4.5. In this semi lattice, $x \curlywedge y$ is defined as $\max(x, y)$ as calculated in $(\omega + 1, \leq)$.

We have just shown that not every RML-frame can be obtained from an OKHD-frame via the functor $(-)^{\otimes}$. We now consider the question of whether we can characterize the class of merge frames that *do* give rise to OKHD-frames via the operation we have been considering. The following proposition suffices to answer this question.

Proposition 5.4.6. *Suppose that* $\mathbf{X} = (X, \curlywedge, 1, R, T)$ *is RML-frame, then: for all* $x, y \in X$ *both* $\bigwedge\{w \mid Rxyw\}$ *exists in* X *and* $Rxy\bigwedge\{w \mid Rxyw\}$ *iff there is a sG-frame* $X_R = (X, \curlywedge, \otimes_R, 1)$ *where* $(X_R)^{\otimes_R} = X$ *(See Proposition 5.4.4 for definition of (-)*$^{\otimes}$*).*

Proof. We define $x \otimes_R y = \bigwedge\{z \mid Rxyz\}$. In virtue of proposition 5.4.4 we just need to check that when $X = (X, \curlywedge, 1, R, T)$ is residuated, $\otimes$-$\curlywedge$-distribution holds and that 1 is an absorbing element.

For $\otimes$-$\curlywedge$-distribution, note that in virtue of the monotonicity of $\otimes_R$, we have immediately that $x \otimes_R (y \curlywedge y') \leq (x \otimes_R y) \curlywedge (x \otimes_R y')$.

Now for the other inequality, note that $x \otimes_R (y \curlywedge y') = \bigwedge\{z \mid Rx(y \curlywedge y')z\}$ and that $R(x, (y \curlywedge y'), \bigwedge\{z \mid Rx(y \curlywedge y')z\})$. By condition 2 of the definition of a residuated L-frame, there are $t, t' \in X$

such that $Rxyt$ and $Rxy't'$ and $t \curlywedge t' \leq \bigcurlywedge\{z \mid Rx(y \curlywedge y')z\}$. But this implies that $x \otimes_R y \leq t$ and $x \otimes y' \leq t'$. So we have that

$$(x \otimes_R y) \curlywedge (x \otimes y') \leq t \curlywedge t' \leq \bigcurlywedge\{z \mid Rx(y \curlywedge y')z\} = x \otimes_R (y \curlywedge y').$$

We can therefore conclude that $\otimes$-$\curlywedge$-distribution holds.

Now, to see that 1 is absorbing, $x \otimes 1 = \bigcurlywedge\{y \mid Rx1y\} = 1$ by the condition (3) of an RML-frame. □

We have just shown that the functors that witness the equivalence between **NRLSp** and **RMLSp** do not generalize to an equivalence between the category of OKHD-frames and RML-frames.

5.5 Conclusion

In this chapter we discussed the move from the general theory of L-spaces with filter continuous relations to RML-spaces and NRL-spaces and then to OKHD-frames and RML-frames. We showed in Proposition 5.2.11 that every NRL-space is also an OKHD-frame. We then show that the category of RML-spaces and NRL–spaces is equivalent and thereby have demonstrated how to derive OKHD-frames from the general theory of L-spaces with filter continuous relations.

Chapter 6

Duality for Residuated Lattices

In this chapter we develop a point-set topological duality for not-necessarily-distributive residuated lattice ordered groupoids and similar algebras by extending a the recent duality for bounded lattices obtained by Bezhanishvili et.al [6] and introduced in detail in section 2.2 of Chapter 2. We show that the category of pointed $r\ell$-groupoids (see Definition 2.1.5) is dually isomorphic to the category of NRL-spaces (See definition 5.2.1). We then restrict to duality to obtain dualities for residuated lattices, FL-algebras, and involutive residuated lattices and consider some explicit correspondences between algebraic identities and topological properties. These results constitute some of the main contributions of this thesis. Coupled with the connections between NRL-spaces and OKHD-frames established in Chapter 5, these results will allow us to directly connect the algebraic semantics of substructural logics to topological semantics presented in the next chapter and the OKHD-semantics.

The chapter is structured as follows. In Section 6.1, we prove that that category on NRL-spaces is dually isomorphic to the category of $r\ell$-groupoids. We also derive duality with respect to RML-spaces. We end the section by considering a number of correspondences between algebraic identities and properties of NRL-spaces and RML-spaces. In Section 6.2, we explicitly prove dualitities for residuated lattices and FL-algebras in terms of special classes of NRL-spaces called RL-spaces and FL-spaces, respectively. Then in Section 6.3, we show duality for involutive residuated lattices and a class of spaces we call Involutive FL-spaces. Finally, is Section 6.4, we show that the representation of monotone lattice expansions can be extended to $r\ell$-groupoids and give a characterization of classes of $r\ell$-groupoids that are closed under the Π_1-completion.

The following table summarize some of the noteworthy dualities from this chapter.

Algebras	Spaces	Theorem 6.1.10
$r\ell$-Groupoids	NRL-Spaces	Theorem 6.1.10
Residuated Lattices	RL-Spaces	Theorem 6.2.3
FL-algebras	FL-Spaces	Theorem 6.2.7
Involutive Residuated Lattices	Involutive FL-spaces	Theorem 6.3.8

In addition we obtain duality for many other important classes of each of these algebras defined for example by weakening, idempotence, commutativity and so on.

6.1 Topological Duality for Pointed $r\ell$-Groupoids

In the last chapter we defined NRL-spaces. We showed that that by forgetting the topology of an NRL-space we obtain a OKHD-frame in Proposition 5.2.11 and that the category of NRL-spaces is equivalent to the categeory of RML-spaces, in Theorem 5.3.1. Together, these results allow us to understand the frames used in the semantics of Ono and Komori [32], Humberstone[23], and Došen [13] in terms of our general theory of L-spaces with filter continuous relations.

In this section we connect NRL-spaces to the algebraic semantics of substructural logics by demonstrating a duality between the category of NRL-spaces and the category of pointed $r\ell$-groupoids. We then use the equivalence between NRL-spaces and RML-spaces to derive another duality and representation theorem.

Let us recall the definition of NRL-space.

Definition 6.1.1. *An NRL-space $\mathbf{X} = (X, \curlywedge, 1, \otimes, \varepsilon, \tau)$ is such that $(X, \curlywedge, 1, \tau)$ is an L-spaces, $\otimes : X \times X \to X$ is a groupoid operation, ε is a designated element and:*

(1) For all clopen filters U, V, $U \circ_X V$, $U \backslash_X V$, and $V/_X U$ are clopen filters (see below),

(2) $\uparrow\varepsilon$ is clopen, and

(3) $x \otimes y \leq z$ iff for all $U, V \in \mathcal{F}_{clop}(X)$, if $x \in U$ and $y \in V$, then $z \in U \circ_X V$.

In the above definition we have that $U \circ_X V = \uparrow(\{x \otimes y \mid x \in U \ \& \ y \in V\})$ and $U \backslash_X V = \{y \mid \forall x \in U(x \otimes y \in V)\}$ and $V/_X U = \{x \mid \forall y \in U(x \otimes y \in V)\}$.

Of importance to showing that the category or NRL-spaces is dually equivalent to the categoty of $r\ell$-groupoids, we showed in Proposition 5.2.4 that the lattice of clopen filters of an NRL-space forms an $r\ell$-groupoid. In particular we demonstrated for any NRL-space $\mathbf{X} = (X, \curlywedge, 1, \otimes, \varepsilon, \tau)$, the algebra $\mathbf{L_X} = (\mathcal{F}i_{clp}((X), \cap, \triangledown, X, \{1\}, \circ, \backslash, /, \uparrow\varepsilon)$ was a pointed $r\ell$-groupoid. We restate the proposition here and reference the reader to the proof in the previous chapter (Proposition 5.2.4)

Proposition 6.1.2. *For any NRL-space $\mathbf{X}$, the algebra $\mathbf{G_X} = (\mathcal{F}i_{clp}(X), \cap, \triangledown, X, \{1\}, \circ_X, \backslash_X, /_X, \uparrow\varepsilon)$ is a pointed $r\ell$-groupoid.*

With an operation $\mathbf{X} \mapsto \mathbf{G_X}$ moving use from NRL-spaces to pointed $r\ell$-groupoids, proving duality will amount to characterizing the inverse of this operation and showing how to contravariantly transform morphisms from one category into morphisms of the other. With this goal in mind, the following proposition encodes the operation taking us from $r\ell$-groupoids to NRL-spaces. We will later show that this operation is inverse to the operation $\mathbf{X} \mapsto \mathbf{G_X}$.

Proposition 6.1.3. *(Pointed $r\ell$-groupoids to NRL-Spaces)*
For every pointed $r\ell$-groupoid $\mathbf{G} = (G, \vee, \wedge, \top, \bot, \cdot, \backslash, /, e)$, then $\mathbf{X_L} = (\mathcal{F}i(G), \cap, G, \otimes_G, \uparrow e, \tau)$ is an NRL-space where τ is generated by the subbase $\mathcal{S}$:

$$\mathcal{S} := \{\phi(a) \mid a \in L\} \cup \{X - \phi(a) \mid a \in L\}$$

Proof. We note that the operation $x \otimes_G y := \uparrow\{a \cdot b \mid a \in x \ \& \ b \in y\}$ is well defined on the collection of filters in virtue of a proof similar to the one given in Lemma **??**.

Given L-space duality (Theorem 2.2.8) we know that the topology τ on $\mathbf{X_G}$ is the L-space dual of the lattice $(G, \vee, \wedge, \top, \bot)$. We therefore only need to check that conditions (1)-(3) from the definition of NRL-spaces hold. However, again by Theorem 2.2.8, we know $\phi_G : G \to \mathbf{L_{X_G}}$ is a lattice isomorphism, so it is sufficient to check that ϕ is also a pointed $r\ell$-groupoid isomorphism since this will imply that the clopen filters are closed under the operation $\circ_X$, $\backslash_X$, and $\_X$ and contain $\uparrow_X(\uparrow_G e)$. In particular, we must check that $\phi(a \cdot b) = \phi(a) \circ_{X_G} \phi(b)$ and $\phi(a\backslash b) = \phi(a)\backslash\phi(b)$ and $\phi(a/b) = \phi(a)/\phi(b)$ and finally that $\phi(e) = \uparrow_X(\uparrow_G e)$.

Starting with the simplest case, we show $\phi(e) = \uparrow_X(\uparrow_G e)$. The subscript on $\uparrow$ indicates where the operation is calculated. Let $x \in \phi(e)$. Then $e \in x$ and so $\uparrow e \subseteq x$. This implies that $x \in \uparrow_X(\uparrow_G e)$, as desired. Let $x \in \uparrow_X(\uparrow_G e)$. Then $e \in \uparrow_G e \subseteq x$. So $x \in \phi(e)$.

For $\phi(a \cdot b) = \phi(a) \circ_{X_G} \phi(b)$, a simple argument can be given by noting that $\uparrow_G(a) \otimes_G \uparrow_G(b) = \uparrow_G(a \cdot b)$ and the fact that by L-space duality, $\phi(c) = \uparrow_{X_G}(\uparrow_G(c))$ for every $c \in G$. Again, the subscript on $\uparrow$ indicates where the operation is calculated.

For $\phi(a\backslash b) = \phi(a)\backslash\phi(b)$. Suppose that $y \in \phi(a\backslash b)$. Let $x \in \phi(a)$ and suppose that $x \circ y \subseteq z$. Then $a \cdot (a\backslash b) \in z$, so $b \in z$. For the other inclusion we reason by contraposition. Suppose that $y \notin \phi(a\backslash b)$. We show that there are filters x and z such that $a \in x$ and $x \circ y \subseteq z$ but $b \notin z$, which implies that $y \notin \phi(a)\backslash\phi(b)$. Therefore, we define $\mathbf{X} = \uparrow(a)$ and $z = \{c \mid \exists a' \in y\ (a \cdot a' \leq c)\}$.

To see that $z = \{c \mid \exists a' \in y\ (a \cdot a' \leq c)\}$ is indeed a filter, let $c_0, c_1 \in z$. Then there are b_0 and b_1 in y such that $a \cdot b_0 \leq c_0$ and $a \cdot b_1 \leq c_1$ hold. But then $b_0 \wedge b_1 \in y$ and so $a \cdot (b_0 \wedge b_1) \in y$. But given that the following two identities hold, we may infer that $c_0 \wedge c_1 \in z$.

$$(1)\ a \cdot (b_0 \wedge b_1) \leq a \cdot b_0 \leq c_0 \qquad (2)\ a \cdot (b_0 \wedge b_1) \leq a \cdot b_1 \leq c$$

Now, we must also check that $x \circ y \subseteq z$ holds. So take $a_0 \in x$ and $b_0 \in y$. Then because $a \cdot b \in z$ and $a \cdot b_0 \leq a_0 \cdot b_0$, we have that $a_0 \cdot b_0 \in z$, as desired. Finally, suppose that $b \in z$, then there is some $c \in y$ such that $a \cdot c \leq b$. But then $c \leq a\backslash b$ and so $a\backslash b \in y$, contradicting our assumption. Therefore, we may conclude that $b \notin z$. We have therefore shown that there are filters x and z such that $a \in x$ and $x \circ y \subseteq z$ but $b \notin z$, and thus that $y \notin \phi(a)\backslash\phi(b)$.

For $\phi(a/b) = \phi(a)/\phi(b)$, we note that the argument is nearly identical to the preceding one.

Finally, for condition (3), the direction from left to right is straight forwardly implied by the fact that clopen filters are upsets. From right to left, suppose that for all $U, V \in \mathcal{F}_{clop}(X)$, if $x \in U$ and $y \in V$, then $z \in U \circ_R V$. Now let $c \in x \otimes_G y$. Then there are $a \in x$ and $b \in y$ such that $a \cdot b \leq c$. But then $x \in \phi(a)$ and $y \in \phi(b)$ and $z \in \phi(a) \circ_X \phi(b) \subseteq \phi(c)$. But if $z \in \phi(c)$, then $c \in z$, as desired. □

Aside from taking us one step closer to our desired duality result, the above proposition also leads to a demonstration of the fact that every $r\ell$-groupoid can be represented as the alegbra of filters of some NRL-space.

Theorem 6.1.4. *(Representation of pointed $r\ell$-groupoids)*
Every pointed $r\ell$-groupoid $\mathbf{G}$, there is an NRL-space $\mathbf{X}$ such that $\mathbf{G}$ is isomorphic $\mathcal{F}i_{clp}(\mathbf{X})$.

Proof. In virtues of the proof of 6.1.3 every $r\ell$-groupoid G, it is the case that $\phi_G : G \to \mathcal{F}i_{clp}(\mathbf{X_G})$. □

The propositions 6.1.3 and 6.1.2 provide operations $\mathbf{G} \to \mathbf{G_X}$ and $\mathbf{X} \to \mathbf{G_X}$ We now turn to morphisms between NRL-spaces.

Definition 6.1.5. *An NRL-space morphism $f : \mathbf{X} \to \mathbf{Y}$ is an L-space morphism that satisfies the following conditions:*
($\otimes$-forth) $f(x) \otimes' f(y) \leq f(x \otimes y)$,

($\otimes$-back) If $x' \otimes' y' \le f(z)$, then there are $x, y \in X$ such that $x' \le fx$, $y' \le fy$, and $x \otimes y \le z$,
($/$-back) if $fx \otimes' y' \le z'$, then there are $y, z \in X$ such that $y' \le fy$, $fz \le z'$, and $x \otimes y \le z$,
($\backslash$-back) if $x' \otimes' fy \le z'$, then there are $x, z \in X$ such that $x' \le fx$, $fz \le z'$, and $x \otimes y \le z$,
(ε-forth) $\varepsilon' \le f(\varepsilon)$, and
(ε-back) if $\varepsilon' \le fx$, then $\varepsilon \le x$.

Lemma 6.1.6. *If $f : \mathbf{X} \to \mathbf{Y}$ is an NRL-space morphism, then $f^{-1} : \mathbf{G_Y} \to \mathbf{G_X}$ is a pointed $r\ell$-groupoid homomorphism.*

Proof. In virture of Lemma **??**, which proves that f^{-1} is a lattice homomorphism, we just need to check that f^{-1} preserves the groupoid operations and the designated element $\uparrow\varepsilon$.

If $x \in f^{-1}(U \circ_Y V)$ then, $f(x) \in U \circ V$, and then further there are $y \in U$ and $z \in V$ such that $y \otimes z \le f(x)$. By the back condition of f, there are x' and y' such that $x \le fx'$ and $y \le fy'$ and $x' \otimes_X y' \le x$. But then $x' \in f^{-1}(U)$ and $y' \in f^{-1}(V)$ since U and V are upward closed. It then follows that $x \in f^{-1}(U) \circ_X f^{-1}(V)$. Therefore, $f^{-1}(U \circ_Y V) \subseteq f^{-1}(U) \circ_X f^{-1}(V)$. For the other inclusion, use the forth condition of f.

We now need to show f^{-1} preserves the residual operations $\backslash$ and $/$. Let us show the case of $f^{-1}(U \backslash_Y V) = f^{-1}(U) \backslash_X f^{-1}(V)$noting that the case of $/$ is similar. Let $y \in f^{-1}(U \backslash_Y V)$. Let $x \in f^{-1}(U)$ and suppose that $x \otimes_X y \le z$. By the fact that f is a ℓG-space morphism, we have that $f(x) \otimes_Y f(y) \le f(z)$. We also have $f(x) \in U$ and $f(y) \in U \backslash_Y U$, so $f(z) \in V$. Therefore we have that $z \in f^{-1}(V)$, as desired. So $f^{-1}(U \backslash_Y V) \subseteq f^{-1}(U) \backslash_X f^{-1}(V)$. On the other hand, let $y \in f^{-1}(U) \backslash_X f^{-1}(V)$. Let $x \in U$ and suppose that $x \otimes_Y f(y) \le z$. Then by condition two in the definition of $r\ell G$-space morphism, we have that there are $x', z' \in X$ such that $x \le f(x')$ and $f(z') \le z$ and $x' \otimes y \le z'$. However, since $x \in U$, $f(x') \in U$, and thus $x' \in f^{-1}(U)$. This then implies in conjunction with the fact that $x' \otimes y \le z'$ and $y \in f^{-1}(U) \backslash_X f^{-1}(V)$, that $z' \in f^{-1}(V)$ and therefore that $f(z')$ and z are elements of V. We may then conclude that $f(y) \in U \backslash_Y V$ and thus that $y \in f^{-1}(U \backslash_Y V)$, as desired. We have therefore shown that $f^{-1}(U \backslash_Y V) = f^{-1}(U) \backslash_X f^{-1}(V)$.

Finally, we check that the f^{-1} preserves the designated element, $\uparrow\varepsilon$ i.e. $f^{-1}(\uparrow_Y(\varepsilon_Y)) = \uparrow_X(\varepsilon_X)$. For one inclusion, use the condition that if $\varepsilon_Y \le fx$, then $\varepsilon_X \le x$. For the other inclusion, use the fact that $\varepsilon_Y \le f(\varepsilon_X)$.

We therefore, conclude that f^{-1} is a pointed $r\ell$-groupoid homomorphism. $\square$

Lemma 6.1.7. *let $\mathbf{G} = (G, \vee, \wedge, \top, , \bot, \cdot, \backslash, /)$ be an rl-groupoid, then If $d \le b$, then $c/b \le c/d$.*

Proof. Suppose that $d \le b$. Then $c/b \cdot d \le c/b \cdot b \le c$. But then $c/b \le c/d$. $\square$

Lemma 6.1.8. *$f : \mathbf{G} \to \mathbf{H}$ is rl-groupoid homomorphism, then $f^{-1} : \mathbf{X_H} \to \mathbf{X_G}$ is a NRL-space morphism.*

Proof. let $f : \mathbf{G} \to \mathbf{H}$ be a ℓ-groupoid homomorphism. By L-space duality, f^{-1} is a L-space mprhism. We just show that $f^{-1} : \mathbf{X_H} \to \mathbf{X_G}$ satisfies the conditions from the defintion of an NRL-space morphism.

($\otimes$-forth) let $x \circ_H y \le z$ and suppose that $c \in f^{-1}(x) \otimes_G f^{-1}(y)$. Then there are $a \in f^{-1}(x)$ and $b \in f^{-1}(y)$ such that $a \cdot b \le c$. But then $f(a) \cdot f(b) = f(a \cdot b) \le f(c)$. So since $f(a) \in x$ and $f(b) \in y$, we have that $f(a) \cdot f(b) \in x \circ_H y$ and thus $f(c) \in x \circ_H y$. Therefore we obtain that $c \in f^{-1}(x \circ_H y) \subseteq f(z)$, as desired.

($\otimes$-back) Suppose that $x' \otimes_G y' \leq f^{-1}(z)$. $f[x']$ and $f[y']$ are both closed under taking meets, so $\uparrow f[x']$ and $\uparrow f[y']$ are both filters. It follows quickly that $x' \leq f^{-1}(\uparrow f[x'])$ and $y' \leq f^{-1}(\uparrow f[y'])$. Now let $c \in \uparrow f[x'] \circ_H \uparrow f[y']$. then there are $a \in \uparrow f[x']$ and $b \in \uparrow f[y']$ such that $a \cdot b \leq c$. But if $a \in \uparrow f[x']$ and $b \in \uparrow f[y']$, then there $f(a') \in f[x']$ and $f(b') \in f[y']$ such that $f(a') \leq a$. So $f(a' \cdot b') = f(a') \cdot f(b') \leq a \cdot b \leq c$. But $a' \in x'$ and $b' \in y'$, so $a' \cdot b' \in f^{-1}(z)$, which in turn implies that $f(a' \cot b') \in z$ and then that $c \in z$, as desired.

(/-Back) suppose that $f^{-1}(x) \otimes_G y' \subseteq z'$. We define $y = \uparrow f[y']$. We argued above that such a set is a filter and we also showed that $y' \subseteq f^{-1}(y)$. In contrast, we define $z = x \circ_H \uparrow f[y']$. It is then trivial that $x \circ_H y \leq z$. We just need to show then that $f^{-1}(z) \subseteq z'$. So let $c \in f^{-1}(x) = f^{-1}(x \circ_H \uparrow f[y']) = f^{-1}(x) \circ_H f^{-1}(\uparrow f[y'])$. Then there are $a \in f^{-1}(x)$ and $b \in f^{-1}(\uparrow f[y'])$ such that $a \cdot b \leq c$. Therefore $f(a) \in x$ and $f(b) \in \uparrow f[y']$ and $f(a) \cdot f(b) \leq f(c)$. And hence $f(a) \leq f(c)/f(b)$ and thus $f(c)/f(b) \in x$. But if $f(b) \in \uparrow f[y']$, then there is a $d \in y'$ such that $f(d) \leq f(b)$. So by Lemma 6.1.7, $f(c)/f(b) \leq f(c)/f(d)$. So we obtain that $f(c)/f(d) \in x$. But then $f(c/d) \in x$ and hence $c/d \in f^{-1}(x)$. Now recall that we had that $d \in y'$, so $c/d \cdot d \in f^{-1}(x) \otimes_G y'$, which in turn implies that $c \in f^{-1}(x) \otimes_G y'$ finally that $c \in z'$ because of our assumption that $f^{-1}(x) \otimes_G y' \subseteq z'$.

We have therefore shown from the assumption that $f^{-1}(x) \otimes_G y' \subseteq z'$ that there are $y, z \in X$ such that $x \otimes y \subseteq z$, $y' \subseteq f(y)$, and $f(z) \subseteq z'$, as was required.

For the condition (\-back), we note that the proof is sufficiently similar to the one given for (/-back) for us to omit.

Finally for the conditions associated with ε, the forth condition follows quickly. For the back condition, suppose that $\varepsilon_{X_G} \subseteq f^{-1}(x)$ for $x \in X_H$. Clearly then $e_G \in f^{-1}(x)$ and so $e_H = f(e_G) \in x$. But then $\varepsilon_K = \uparrow(e_K) \subseteq x$, as desired.

We therefore conclude that the f^{-1} is a NRL-space morphism.

$\square$

Before stating and proving the duality result of this section, we recall the following lemma which we proved in the previous chapter in Lemma 5.2.6.

Lemma 6.1.9. *If If $\mathbf{X} = (X, \curlywedge, 1, \otimes, \varepsilon, \tau)$ is an NRL-space and U is a clopen filter of $\mathbf{X}$, then: if $x \otimes y \in U$, there are clopen filters V and W such that $x \in V$ and $y \in W$ and $V \circ W \subseteq U$.*

With the lemma available, we prove duality between NRL-spaces and $r\ell$-groupoids.

Theorem 6.1.10. *The category **NRL** is dually isomorphic to the category **RLG**.*

Proof. By L-space duality, there is a L-space homeomorphism $\eta_X : \mathbf{X} \to \mathbf{X}_{\mathbf{G_X}}$ where $\mathbf{G_X}$ is the ℓ-groupoid of clopen filters of $\mathbf{X}$. To check check that η_X is also an NRL-space homeomorphism, it is sufficient to check that ε_X and $\varepsilon_{X_{L_X}}$ and the respective groupoid operations agree w.r.t to η_X i.e. that $\eta_X(\varepsilon_X) = \varepsilon_{X_{G_X}}$ and that $\eta_X(x \otimes y) = \eta_X(x) \circ_{X_{G_X}} \eta_X(y)$. Recall that $\eta_X(x) = \{U \in \mathcal{F}i_{clp}(X) \mid \uparrow x \subseteq U\}$.

To show $\eta_X(\varepsilon_X) = \varepsilon_{X_{G_X}}$ we have:

$$
\begin{aligned}
\eta_X(\varepsilon_X) &= \{U \in \mathcal{F}i_{clp}(X) \mid \uparrow_X(\varepsilon_X) \subseteq U\} && (6.1)\\
&= \uparrow_{G_X}(\uparrow_X(\varepsilon_X)) && (6.2)\\
&= \varepsilon_{X_{G_X}}. && (6.3)
\end{aligned}
$$

Note the identity in (1) holds by definition. The step to (2) is merely a rewriting. The last step to (3) holds again by definition since $\uparrow_X(\varepsilon_X)$ is a designated element of $\mathbf{G_X}$.

To show $\eta_X(x \otimes y) = \eta_X(x) \circ_{X_{G_X}} \eta_X(y)$ we reason as follows. The direction from right to left is more or less straight forward. Let $U \in \eta_X(x) \circ_{X_{G_X}} \eta_X(y)$. Then then there are $V \supseteq \uparrow x$ and $W \supseteq \uparrow y$ such that $W \circ_X V \subseteq U$, which are both clopen filters. But then since $x \in W$ and $y \in V$, $x \circ y \in U$ and hence $\uparrow(x \otimes y) \subseteq U$. Therefore $\eta_X(x) \circ_{X_{G_X}} \eta_X(y) \subseteq \eta_X(x \otimes y)$.

For the other direction, suppose that $U \in \eta_X(x \otimes y)$. Then $x \otimes y \in U$. By Lemma 6.1.9, we have that there are clopen filters V and W such that $x \in V$ and $y \in W$ and $V \circ W \subseteq U$. But then $V \in \eta_X(x)$ and $W \in \eta_X(y)$. So we conclude that $U \in \eta_X(x) \otimes \eta_X(y)$, as desired.

Now, by the proof of Theorem 6.1.4, we are also ensured that the map $\phi_G : G \to G_{X_G}$ is an isomorphism.

We have thus shown duality at the level of objects. It follows then by Theorem 2.2.8 and the fact that both of the salient categories are concrete that we also have duality at the level of morphisms. $\square$

We have concluded that a duality exists between the category of pointed $r\ell$-groupoids and the category of NRL-spaces. In virtue of the equivalence between the category of NRL-spaces and the category of RML-spaces established in the previous section (Theorem 5.3.1), we obtain another duality and representation theorem for pointed $r\ell$-groupoids.

Theorem 6.1.11. *(RML-Space Duality) The category of $r\ell$-groupoids is dually isomorphic to the category of RML-spaces.*

Proof. By Theorem 6.1.10 we have that $\mathbf{RLG}^{op} \cong \mathbf{NRLSp}$. By Theorem 5.3.1 we have $\mathbf{NRLSp} \cong \mathbf{RMLSp}$. Therefore $\mathbf{RLG}^{op} \cong \mathbf{RMLSp}$, as claimed. $\square$

Additionally, we obtain the following representation theorem. This result will enable another completeness theorem when we consider logical applications in Chapter 7.

Theorem 6.1.12. *(Representation Theorem) Every pointed $r\ell$-groupoid is isomorphic to the algebra of clopen filters of some RML-space.*

Proof. Let $\mathbf{G}$ be a pointed $r\ell$-groupoid. By theorem 6.1.4, $\mathbf{G}$ is isomorphic to the algebra of clopen filters $\mathbf{G}_{\mathbf{X_G}}$ of the NRL-space $\mathbf{X_G}$. In virtue of the proof of Theorem 5.3.1, we obtain that algebra of clopen filters of the RML-space $F(\mathbf{X_G})$ (See Theorem 5.3.1 for definition of F) and $\mathbf{G}_{\mathbf{X_G}}$ are isomorphic as pointed $r\ell$-groupoids. It follows that $\mathbf{G}$ is isomorphic as an $r\ell$-groupoid to the algebra of clopen filters of the RML-space $F(\mathbf{X_G})$. $\square$

We conclude with a remark on how to generalize the results to various reducts of pointed $r\ell$-groupoids.

Remark 6.1.13. *(Reducts of $r\ell$-groupoids)*

Given our duality proof it is possible to extract a number of other dualities for reducts of pointed $r\ell$-groupoids. This includes non-pointed $r\ell$-groupoids by removing the requirement of a designated element ε from the definition of an NRL-space. By only requiring the clopen filters to be closed under one of $\backslash_X$ and $/_X$ we can obtain dualities for algebras with only a left or right residual. These could be useful in the semantics of relevance logics like the system of entailment, E. Going further, we can also obtain duality just for ℓ-groupoids by non-longer requiring the clopen filters to be closed under the operations $\backslash_X$ and $/_X$. However, to guarantee that the algebra of clopen filters satisfies the identities $a \cdot (b \vee c) = (a \cdot b) \vee (a \cdot c)$ and $(b \vee c) \cdot a = (b \cdot a) \vee (c \cdot a)$, we require that the duals satisfy

$(x \otimes y) \curlywedge (x \otimes z) \leq x \otimes (y \curlywedge z)$ and $(y \otimes x) \curlywedge (z \otimes x) \leq (y \curlywedge z) \otimes x$. Similar remarks hold for obtaining dualities with respect to generalizations of RML-spaces.

6.1.1 Some Correspondences Between Identities and Properties of Dual Spaces

We list some noteworthy correspondences between identities that hold in an $r\ell$-groupoid and properties of $\otimes$ that hold in the dual space of that algebra. These correspondences lead to dualities for many other class of pointed $r\ell$-groupoids that have been studied in the literature. We then examine analagous correspondences in in terms of RML-space.

The correspondences for $\cdot$ and $\otimes$ are very similar to those for the logics presented by Došen in [13].

Proposition 6.1.14. *(Correspondences) For any pointed $r\ell$-groupoid $\mathbf{G}$ with dual NRL-space $\mathbf{X_G}$ we have the following correspondences between properties of $\cdot$ and $\otimes$ and properties of e and ε.*

Properties of $\cdot$	*Properties of* $\otimes$
$a \cdot (b \cdot c) \leq (a \cdot b) \cdot c$	$(x \otimes y) \otimes z \leq x \otimes (y \otimes z)$
$(a \cdot b) \cdot c \leq a \cdot (b \cdot c)$	$x \otimes (y \otimes z) \leq (x \otimes y) \otimes z$
$a \cdot b \leq b \cdot a$	$x \otimes y \leq y \otimes x$
$a \leq a \cdot a$	$x \otimes x \leq x$
$a \cdot a \leq a$	$x \leq x \otimes x$
$a \cdot b \leq a$	$x \leq x \otimes y$
$b \cdot a \leq a$	$x \leq y \otimes x$
$a \leq a \cdot b$	$x \otimes x \leq y$
$a \leq b \cdot a$	$y \otimes x \leq x$

Properties of e	*Properties of* ε
$a \leq e \circ a$	$\varepsilon \otimes x \leq x$
$a \leq a \circ e$	$x \otimes \varepsilon \leq x$
$e \circ a \leq a$	$x \leq \varepsilon \otimes x$
$a \circ e \leq a$	$x \leq x \otimes \varepsilon$

Proof. In the interest of space will only show a selection of these correspondences acknowledging that the other proofs are quite simple and or similar. First let us consider the correspondence between $a \cdot (b \cdot c) \leq (a \cdot b) \cdot c$ and $(x \otimes y) \otimes z \leq x \otimes (y \otimes z)$.

Let $\mathbf{G}$ be a pointed $r\ell$-groupoid. Suppose that for all $a, b, c \in G$, we have that $a \cdot (b \cdot c) \leq (a \cdot b) \cdot c$. Now let $x, y, z \in \mathbf{X_G}$ let $d \in (x \otimes y) \otimes z$. Then there $a, b, c \in G$ such that $(a \cdot b) \cdot c \leq d$ and $a \in x$, $b \in y$, and $c \in z$. By assumption, $a \cdot (b \cdot c) \leq (a \cdot b) \cdot c$ and so $a \cdot (b \cdot c) \leq d$. It follows that $d \in x \otimes (y \otimes z)$, as desired.

For the other direction of the correspondence, let $\mathbf{X}$ be an NRL space and suppose that for all $x, y, z \in \mathbf{X}$, $(x \otimes y) \otimes z \leq x \otimes (y \otimes z)$. Now let $U, V, W \in \mathcal{F}i_{clp}(\mathbf{X})$. By almost the exact same reasoning as in the previous paragraph, if $w \in U \circ (V \circ W)$, then there are $x \in U$, $y \in V$, and $z \in W$ such that $x \otimes (y \otimes z) \leq w$. But the $(x \otimes y) \otimes z \leq w$ and so $w \in (U \circ V) \circ W$.

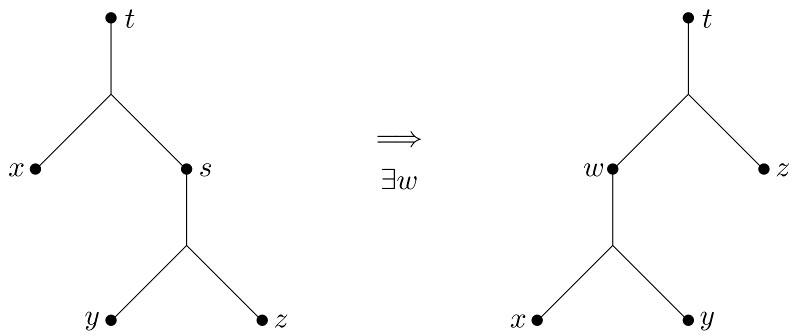
For our second sample we consider the correspondence between $a \cdot e \leq a$ and $\varepsilon \otimes x \leq x$. Let $\mathbf{G}$ be a pointed $r\ell$-groupoid and suppose for all $a \in G$ that $a \cdot e \leq a$. Now let $x \in \mathbf{X_G}$ and suppose that $a \in x$. We recall that $\varepsilon = \uparrow_G e$, so since $a \cdot e \leq a$ we obtain that $a \in x \circ \varepsilon$.

The other direction of the correspondence follows by an analogous argument. □

In addition to correspondences between properties of an $r\ell$-groupoid and it's dual NRL-space, we also obtain correspondences with respect to its dual RML-space. These are standard conditions from the literature on relevance logic and Routley-Meyer frames [33]. We record a few here.

Definition 6.1.15. *We say that a RML-space* $\boldsymbol{X} = (X, \curlywedge, 1, R, T, \tau)$ *satisfies:*
Permutation iff $\forall xyz(Rzxt \to Rzxy))$,
Right Rebracketing iff $\forall xyzts(Rtxs \;\&\; Rsyz \to \exists w(Rwxy \;\&\; Rtwz))$,
Left Rebracketing iff $\forall xyzts(Rtsz \;\&\; Rsxy \to \exists w(Rwyz \;\&\; Rtxw))$,
Right Omission iff $\forall xyz(Rzxy \to y \leq z)$,
Left Omission iff $\forall xyz(Rzxy \to x \leq z)$.

We may represent the Right-rebracketing condition graphically as follows.



We then have the following proposition that records properties of an algebra and its dual space.

Proposition 6.1.16. *Let* $\mathbf{G} = (G, \wedge, \vee, \top, \bot, \cdot, \backslash, /, e)$ *be an* $r\ell$*-groupoid and let* $\mathbf{X} = (X, \curlywedge, 1, R, T, \tau)$ *be the RML-space dual to* $\mathbf{G}$*, then we have that follows table of correspondences:*

Properties of $\cdot$	*Properties of* R
$a \cdot b \leq b \cdot a$	*R satisfies Permutation*
$a \cdot (b \cdot c) \leq (a \cdot b) \cdot c$	*R satisfies Right-Rebracketing*
$(a \cdot b) \cdot c \leq a \cdot (b \cdot c)$	*R satisfies Left-Rebracketing*
$a \cdot b \leq a$	*R satisfies Right Omission*
$b \cdot a \leq a$	*R satisfies Left Omission*

Proof. As with the correspondences between $\cdot$ and $\otimes$, we will only provide a sample here. In particular we will verify the correspondence between $(a \cdot b) \cdot c \leq a \cdot (b \cdot c)$ and R satisfies Right-Rebracketing.

So suppose that G satisfies n $a \cdot (b \cdot c) \leq (a \cdot b) \cdot c$ for all $a, b, c \in G$. Suppose for elements of $\mathbf{X}$, the dual of $\mathbf{G}$, that $Rtxs$ and $Rsyz$. Define w to be $x \otimes y$. It follows immediately that $Rwxy$. Now we must show that $Rtwz$. So let $d \in w$ and $c \in z$. Then there are $a \in x$ and $b \in y$ such that $a \cdot b \leq d$. So by monotonicity of $\cdot$, we have that $(a \cdot b) \cdot c \leq d \cdot c$. But since $a \cdot (b \cdot c) \in t$, in virtue of the assumptions that $Rtxs$ and $Rsyz$, and the fact that $a \cdot (b \cdot c) \leq (a \cdot b) \cdot c$, we arrive at the conclusion that $d \cdot c \in t$. Therefore we have found that $Rtwz$ as desired. We therefore conclude that Right-rebracketing holds.

For the other direction, suppose that Right-rebracketing holds. Let U, V, W be clopen filters of the RML-space $\mathbf{X}$. let $t \in U \circ (V \circ W)$. Then there is some $x \in U$ and $s \in V \circ W$ such that $Rtxs$. If $w \in V \circ W$, then there are $y \in V$ and $z \in W$ such that $Rsyz$. By Right-rebracketing, there is some w such that $Rwxy$ and $Rtwz$. There we obtain that $w \in U \circ V$ and that $t \in (U \circ V) \circ W$, as desired. $\square$

To ensure that designated clopen filter T of an RML-space behaves as an identity element for $\circ_R$ in the algebra of (clopen) filters, we can the following condition inspired by the notion of a T-set from Restall [33].

Definition 6.1.17. *We say that an RML-space* $\mathbf{X} = (X, 1, \curlywedge, R, T, \tau)$ *satisfies:*
Left-Interjection iff $\forall y, z(y \leq z \leftrightarrow \exists x \in T(Rzxy)$, *and*
Right-Interjection iff $\forall x, z(x \leq z \leftrightarrow \exists y \in T(Rzxy)$.

We then obtain the following correspondences.

Proposition 6.1.18. *Let* $\mathbf{G} = (G, \wedge, \vee, \top, \bot, \cdot, \backslash, /, e)$ *be an* $r\ell$*-groupoid and let* $\mathbf{X} = (X, \curlywedge, 1, R, T, \tau)$ *be the RML-space dual to* $\mathbf{G}$*, then we have that follows table of correspondences:*

Properties of $\mathbf{G}$	*Properties of* $\mathbf{X}$
$a \cdot e = a$	$\mathbf{X}$ *satisfies Right-Interjection*
$e \cdot a = a$	$\mathbf{X}$ *satisfies Left-Interjection*

In the next sections we will combine some of the facts we have just seen to obtain dualities for residuated lattices, FL-algebras, and Involutive algebras

6.2 Duality for Residuated Lattices and FL-Algebras

In the semantics of substructural logic, Residuated Lattices and FL-algebras are likely the most studied type of algebra. In this section we show how the various dualities presented throughout this chapter can be combined and extended to give dualities and representation theorems for Residuated Lattices and FL-algebras. We begin by defining RL-spaces, which are the NRL-space duals to residuated lattices. They are simply NRL-spaces where ε is an identity element for $\otimes$ and $\otimes$ is associative. We then define FL-spaces, which extend RL-spaces with a new constant μ that will generate a designated element in algebra of clopen filters and will make that algebra an FL-algebra.

Definition 6.2.1. *An RL-space* $\mathbf{X} = (X, \curlywedge, 1, \otimes, \varepsilon, \tau)$ *is an NRL-space where:*

$$x \otimes (y \otimes z) = (x \otimes y) \otimes z \text{ and } x \otimes \varepsilon = x = \varepsilon \otimes x$$

Theorem 6.2.2. *(Representation of Residuated Lattices) For every residuated lattice,* $\mathbf{L}$*, there is a RL-space* X *such that* $\mathbf{L}$ *is isomorphic to* $\mathcal{F}i_{clp}(\mathbf{X})$.

Proof. The proof of this fact follows from the representation theorem for $r\ell$-groupoids provided in Theorem 6.1.4 and the correspondence results in Proposition 6.1.14. □

Theorem 6.2.3. *The category of residuated lattices* **RLat** *is dually isomorphic to the category of RL-spaces,* **RLSp**

Proof. Follows immediately from Theorem 6.1.10, which established the duality between pointed $r\ell$-groupoids and NRL-spaces, and Lemma 6.1.14 which establishes the fact that associativity of a $r\ell$-groupoid corresponds to associativity of $\otimes$ in the dual space and similarly that when e is an identity element in an $r\ell$-groupoid, then ε is too. □

We note that analogous dualities are obtainable in virtue of RML-spaces by combining the duality result from Theorem 6.1.11 with the correspondence results for RML-spaces in 6.1.16 and 6.1.18.

From here, we extend even further to obtain the analogous results for FL-algebras.

Definition 6.2.4. *An FL-space* $\mathbf{X} = (X, \curlywedge, 1, \otimes, \varepsilon, \tau, \mu)$ *is an RL-space where* μ *is additional designated element and has the property that* $\uparrow\mu$ *is a clopen filter.*

The definition of morphism extends that of NRL-space morphisms by adding some additional conditions for μ.

Definition 6.2.5. *FL-space morphisms* $f : \mathbf{X} \to \mathbf{Y}$ *are NRL-space morphism satisfying the following two conditions:*

(μ-forth) $\mu_Y \leq f(\mu_X)$, *and*

(μ-back) if $\mu_Y \leq fx$, *then* $\mu_X \leq x$ *for all* $x \in X$.

Note that the above condition are the same as those for ε. Simple arguments show that FL-space morphism correspond to FL algebra homomorphism.

Lemma 6.2.6. *(Between Morphisms)*

(1) If $f : \mathbf{G} \to \mathbf{H}$ *is FL algebra homomorphism, then* $f^{-1} : \mathbf{X_H} \to \mathbf{X_G}$ *is a FL-space morphism.*

(2) If $f : \mathbf{X} \to \mathbf{Y}$ *is an FL-space morphism., then* $f^{-1} : \mathbf{G_Y} \to \mathbf{G_X}$ *is an FL algebra homomorphism.*

Proof. The proof of this lemma extends the argument showing the analagous correspondence for NRL-space morphisms and $r\ell$-groupoid homomorphisms in lemmas **??** and **??**. All that remains to be checked pertains to μ. However, the arguments for these properties follows from the same arguemnts for the case of ε. □

We can then prove duality.

Theorem 6.2.7. *(Duality for FL-algebras) The category of FL-algebras* **FLAlg** *is dually isomorphic to the category of FL-spaces,* **FLSp**.

Proof. In virtue of Theorem 6.2.3, which establishes duality for residuated lattices, we need to check that $\mathbf{X_L}$ is an FL-space when $\mathbf{L}$ is an FL-algebra and conversely that $\mathbf{L_X}$ is an FL-algebra when $\mathbf{X}$ is an FL-space. Then to prove duality, we only need to check that $\phi_L : \mathbf{L} \to \mathbf{L_{X_L}}$ is an FL-algebra homomorphism when $\mathbf{L}$ is an Fl-algebra and that $\eta_X : \mathbf{X} \to \mathbf{X_{L_X}}$ is a FL-space morphism when $\mathbf{X}$ is an FL-space.

So let $\mathbf{L}$ is an FL-algebra. $\mathbf{X_L}$ is an RL-space in virtue of Theorem 6.2.3. We then define $\mu := \uparrow_L(f)$ in $\mathbf{X_L}$. $\uparrow_{X_L}\mu = \uparrow_{X_L}(\uparrow(f)) = \phi_L(f)$, so $\uparrow_{X_L}$ is a clopen filter. We conclude then that $\mathbf{X_L}$ is an FL-space.

Conversely, if $\mathbf{X}$ is an FL-space, then $\mathbf{L_X}$ is an FL-algebra since $\uparrow\mu$ is a clopen filter.

Now, for showing $\phi_L : \mathbf{L} \to \mathbf{L_{X_L}}$ is an FL-algebra homomorphism, we have that $\phi_L(f) = \uparrow_{X_L}(\uparrow(f)) = \uparrow_{X_L}\mu$.

Last but not least, we must check that $\eta_X : \mathbf{X} \to \mathbf{X_{L_X}}$ is a FL-space morphism for all FL-spaces $\mathbf{X}$. □

Theorem 6.2.8. *For every FL-algebra,* $\mathbf{L}$, *there is a FL-space* $\mathbf{X}$ *such that* $\mathbf{L}$ *is isomorhic to* $\mathcal{F}i_{clp}(\mathbf{X})$.

Proof. Corollary of the proof of the duality theorem for FL-algebras in Theorem 6.2.7. □

6.3 Duality for Involutive Residuated Lattices

The final class of algebras we study duality for is the variety of Involutive Residuated Lattices. Having already proved duality for FL-algebras, then main task of this chapter is therefore to find a condition on FL-spaces that guarantees that an FL-algebra is involutive if and only if its dual space satisfies the relevant condition. In much the same way that preceding dualities for residuated lattices can be extended to provide duality fro Heyting Algebras, the duality presented here can be extended to give duality for Boolean algebras, even if it is not the most elegant duality.

Before defining Involutive FL-spaces, we prove a lemma about a relation definable in all FL-spaces.

Proposition 6.3.1. *Let $\mathbf{X} = (X, \curlywedge, 1, \otimes, \varepsilon, \mu)$ be an FL-space (definition 6.2.4). define a relation $C \subseteq X \times X$ such that:*

$$xCy \iff \mu \not\leq x \otimes y.$$

Then C satisfies the following properties:

Proof. It is easy to check that C satisfies all of the conditions to be a compatibility frame. For example, if $x \curlywedge yCz$, then $\mu \not\leq (x \curlywedge y) \otimes z = x \otimes z \curlywedge y \otimes z$. Therefore, either $\mu \not\leq x \otimes z$ or $\mu \not\leq y \otimes z$. And thus either xCz or yCz. The other conditions follow readily. □

Let us define $\sim_C U := \{x \in X \mid \forall y(xCy \to y \notin U)\}$ and similarly, $\neg_C U := \{x \in X \mid \forall y(yCx \to y \notin U)\}$.

Lemma 6.3.2. *Let $\mathbf{X} = (X, \mu)$ be an FL-space. Then we also obtain $\neg_C U = U \backslash {\uparrow}\mu$ and $\sim_C U = {\uparrow}\mu / U$.*

Proof. Let us show $\neg_C U = U \backslash {\uparrow}\mu$ noting that the other identity has an analogous proof. Let $x \in \neg_C U$. Now suppose that $y \otimes x \leq z$ and that $y \in U$. We need that $\mu \leq z$. If $y \in U$ and $x \in \sim_C U$, then $y \not{C} x$ and thus $\mu \leq y \otimes x$. But this then immediately implies that $\mu \leq z$, which is what we needed to show. It follows that $x \in U \backslash \uparrow \mu$. For the other inclusion, let $x \in U \backslash {\uparrow}\mu$. Suppose that yCx. We need that $y \notin U$. If we suppose for contradiction that $y \in U$, then we obtain from the assumption that $x \in U \backslash {\uparrow}\mu$ that $\mu \leq y \otimes x$. But this contradicts the supposition that Suppose that yCx. So $y \notin U$, as desired. It is then the case that $x \in \neg_C U$. □

Definition 6.3.3. *Let $\mathbf{X}$ be an FL-space and suppose that C is defined as above. We say an element $x \in X$ is C-separable if $x \neq 1$ and both of the following hold.*

$$\forall y(x \not\leq y \to \exists z(yCz \ \&\ {\uparrow}x \cap C^{-1}[z] = \emptyset)),\ \textit{and},$$
$$\forall y(x \not\leq y \to \exists z(zCy \ \&\ {\uparrow}x \cap C[z] = \emptyset)).$$

Definition 6.3.4. *We say an FL-space $\mathbf{X}$ is involutive if for all clopen filters U, $\bigcurlywedge U$ is C-separable.*

Lemma 6.3.5. *If $\mathbf{X}$ is an involutive FL-space, then $\mathbf{L_X}$ is an involutive residuated lattice.*

Proof. We define and C as in Proposition 6.3.1. We need to check that for all clopen filters U: $\sim\neg U \subseteq U$ and $\neg\sim U \subseteq U$.

Let us show $\sim\neg U \subseteq U$ noting that the other case of double negation elimination follows from an analagous proof. Suppose that $y \in \sim\neg U \subseteq U$ for some clopen filter U. Now let $x = \bigcurlywedge U$ (it follows that ${\uparrow}x = U$). Suppose for contradiction that $y \notin U$. Then $x \not\leq y$. Therefore, by the conditions that define an involutive split space, there is some z such that yCz and $C^{-1}[z] = \emptyset$. If $y \in \sim\neg U \subseteq U$, then

$z \notin \neg U$. Therefore, there is some $w \in U = \uparrow x$ such that such that wCz. But this is impossible given that $\uparrow x \cap C^{-1}[z] = \emptyset$. We conclude that $y \in U$, as desired. $\square$

Lemma 6.3.6. *If* $\mathbf{L}$ *is an involutive residuated lattice, then* $\mathbf{X_L}$ *is an involutive FL-space.*

Proof. Let $\mathbf{L}$ be an involutive residuated lattice and suppose that $\mathbf{X_L} = (\mathcal{F}i(L), \tau)$ is its dual space with C defined as above. Now suppose that $x \neq 1$ and that $\uparrow_X x$ is clopen. We will just show

$$\forall y(x \not\leq y \to \exists z(yCz \ \&\ \uparrow x \cap C^{-1}[z] = \emptyset))$$

noting that the other condition has an almost identical proof. Suppose that $x \not\leq y$. it follows that there is some $a \in L$ such that $\uparrow_L a = x$ (clopen sets of an L-space are always of the form $\uparrow_X(\uparrow_L a)$) and that there is some $b \in x$ such that $b \notin y$. Now define $z := \uparrow_L(\sim a)$. We first claim that yCz. So let $\sim c \in \uparrow_L(\sim a) = z$. We will show that $c \notin y$. If $\sim c \in \uparrow_L(\sim a)$, then $\sim a \leq \sim c$. Therefore we have that $c = \neg\sim c \leq \neg\sim a = a$ since L is involutive. It follows that $c \leq b$ since $a \leq b$ (recall $x = \uparrow_L a$ and $b \in x$). But since $b \notin y$ we obtain $c \notin y$, as desired.

We now claim that $C^{-1}[z] \cap \uparrow_X(x) = \emptyset$. Suppose otherwise. Then there is some $w \in C^{-1}[z] \cap \uparrow_X(x)$. Therefore wCz and $\uparrow a = x \subseteq w$. So $a \in w$ and thus $\neg\sim a \in w$. But since wCz, we then obtain that $\neg a \notin z = \uparrow(\sim a)$, which is a contradiction. Therefore, we conclude that $C^{-1}[z] \cap \uparrow_X(x) = \emptyset$. $\square$

With the various lemmas we have just proved, we arrive at the following representation theorem for involutive residuated lattices.

Theorem 6.3.7. *(Representation Theorem for Involutive Residuated Lattices)*

For every involutive residuated lattice $\mathbf{L}$*, there is an involutive FL-space* $\mathbf{X}$ *such that* $\mathbf{L}$ *is isomorphic to the involutive residuated lattice of clopen filters* $\mathbf{L_X}$*.*

Proof. Follows from Theorem 6.2.8, which establishes that every FL-algebra is representable by the alegrba of clopen filters of some FL-space and then Lemma 6.3.6 $\square$

Finally, we arrive at our final explicit duality theorem.

Theorem 6.3.8. *(Duality for Involutive Residuated Lattices) The category of involutive residuated lattices is dually equivalent to the category of involutive FL-spaces.*

Proof. The duality between FL-algebras and FL-space of Theorem 6.2.7 restricts to a duality between involutive residuated lattices and Involutive FL-spaces in virtue of Theorem 6.3.7. $\square$

Just as with $r\ell$-groupoids, residuated lattices, and FL-algebras, the correspondence results in Proposition 6.1.14 allow us to explicitly characterize a number of other varieties of involutive residuated lattices. Of particular note by adding contraction, weakening, and commutativity to the axioms of involutive residuated lattices we define Boolean algebras. Therefore, adding the corresponding properties to an involutive FL-space gives rise to spaces whose algebra of clopen filters are a Boolean algebra.

6.4 The Π_1-Completion of $r\ell$-Groupoids

In Chapter 4, we showed that the Π_1-completion of a lattice expansion $\mathbf{L}$ could be represented by the algebra of all filters in the dual L-space of $\mathbf{L}$. In this section we very briefly extend the topological representation of the Π_1-completion of monotone lattice expansions to the case of $r\ell$-groupoids. As a consequence we obtain a fairly general result on Π_1-persistent classes of $r\ell$-groupoids.

In Chapter 7 we will apply this persistence result to obtain a general completeness theorem with respect to the OKHD-semantics and a semantics based on RML-frames.

Definition 6.4.1. *Let $\mathbf{G} = (G, \cdot, \backslash, /e)$ be a pointed ℓ-groupoid, we define the following operations on $\Pi_1(\mathbf{G})$:*

$$x_1 \cdot^{\Pi_1} x_2 := \bigvee\{\bigwedge\{\pi(a_1 \cdot a_2) \mid a_i \in L \ \& \ y_i \leq \alpha(a_i)\} \mid y_i \in K(\Pi_1(L)) \ \& \ y_i \leq x_i\} \qquad e^{\Pi_1} = e.$$

$$x\backslash^{\Pi_1} y := \bigvee\{z \in \Pi_1(L) \mid x \cdot^{\Pi_1} z \leq y\} \qquad y/^{\Pi_1} x := \bigvee\{z \in \Pi_1(L) \mid z \cdot^{\Pi_1} x \leq y\}.$$

We must check that the definition of operations are defined in a way that yields an $r\ell$-groupoid.

Lemma 6.4.2. *If $\mathbf{G}$ is a unital $r\ell$-groupoid, then $\Pi_1(\mathbf{G}) = (\Pi_1(G), \cdot^{\Pi_1}, \backslash^{\Pi_1}, /^{\Pi_1}, e^{\Pi_1})$ is a pointed $r\ell$-groupoid.*

Proof. It is sufficient to check that $\Pi(\mathbf{G})$ has the property that for all $S \subseteq \Pi(\mathbf{G})$ and $x \in \Pi(\mathbf{G})$ that:

$$x \cdot^{\Pi_1} \bigvee S = \bigvee\{x \cdot^{\Pi_1} y \mid y \in S\} \text{ and } \bigvee S \cdot^{\Pi_1} x = \bigvee\{y \cdot^{\Pi_1} x \mid y \in S\}.$$

Given Theorem 4.2.3 in Chapter 4, we are ensured that the ℓ-groupoid reduct of $\mathbf{G}$ is isomorphic to ℓ-groupoid reduct of $\mathcal{F}i(\mathbf{X_G})$ where in this case $\mathbf{X_G}$ is the NRL-space dual to $\mathbf{G}$ (Note that the algebra of filters of the dual NRL-space coincides with the dual NRL-space so we can choose either in this argument). We can therefore check the identities in question hold in $\mathcal{F}i(\mathbf{X_G})$ and infer that they hold in $\Pi_1(\mathbf{G})$. Let us show $U \circ \bigtriangledown S = \bigtriangledown\{U \cdot^{\Pi_1} V \mid V \in S\}$ where $\{U\} \cup S \subseteq \mathcal{F}i(\mathbf{X_G})$. The inequality $\bigtriangledown\{U \cdot^{\Pi_1} V \mid V \in S\} \leq U \circ \bigtriangledown\{V \mid V \in S\}$ is an immediate consequence of monotonicity. For the other inequality, let $x \in U \cdot^{\Pi_1} \bigtriangledown S$. There are $y \in U$ and $z \in \bigtriangledown S$ such that $y \otimes z \leq x$. However, if $z \in \bigtriangledown S = \bigcup\{V_1 \triangledown .. \triangledown V_n \mid \forall i \leq n(V_i \in S)\}$, then there is a finite subset $S_0 \subseteq S$ such that $z \in \bigtriangledown S_0$. But then $x \in U \circ \bigtriangledown S_0 = \bigtriangledown\{U \cdot^{\Pi_1} V \mid V \in S_0\} \subseteq \bigtriangledown\{U \cdot^{\Pi_1} V \mid V \in S\}$, as desired.

It the follows by a standard argument that $\Pi_1(\mathbf{G})$ is a residuated lattice. $\square$

We can now show the following representation theorem for the Π_1-completion of

Theorem 6.4.3. *Let $(L, \cdot, e)$ be a pointed $r\ell$-groupoid and let $\mathbf{X_L}$ be its dual NRL-space, then:*

$$(\Pi_1(L), \cdot^{\Pi_1}, \backslash^{\Pi_1}, /^{\Pi_1}, e^{\Pi_1}) \cong (\mathcal{F}i(\mathbf{X_L}), \circ, \backslash, /, \uparrow\varepsilon).$$

Proof. We must show that the isomorphism $\widehat{\phi} : \Pi_1(\mathbf{G}) \to \mathcal{F}i(\mathbf{X_G})$ from Chapter 4 extends to an $r\ell$-groupoid isomorphism. In virtue of Theorem 4.2.1 Chapter 4, we know that $\widehat{\phi}(x \cdot^{\Pi_1} y) = \widehat{\phi}(x) \circ \widehat{\phi}(y)$. So it suffices to show $\widehat{\phi}(x\backslash^{\Pi_1} y) = \widehat{\phi}(x)\backslash\widehat{\phi}(y)$ and $\widehat{\phi}(x/^{\Pi_1} y) = \widehat{\phi}(x)/\widehat{\phi}(y)$. Let us show $\widehat{\phi}(x\backslash^{\Pi_1} y) = \widehat{\phi}(x)\backslash\widehat{\phi}(y)$ noting that the case of / is nearly identical.

In virtue of the definition of $\backslash^{\Pi_1}$, the fact that $\widehat{\phi}$ preserves arbitrary joins, and that $\widehat{\phi}$ is bijective, we have

$$\widehat{\phi}(x\backslash^{\Pi_1} y) = \widehat{\phi}(\bigvee\{z \in \Pi_1(L) \mid x \cdot^{\Pi_1} z \leq y\}) = \bigtriangledown\{U \mid \widehat{\phi}(x) \circ U \leq \widehat{\phi}(y)\}.$$

A standard argument shows that $\bigvee\{U \mid \widehat{\phi}(x) \circ U \leq \widehat{\phi}(y)\}$ satisfies the residual law with respect to $\circ$. However, since operations satisfying the residual law with respect to $\circ$ are unique, we arrive at the fact that $\widehat{\phi}\backslash\widehat{\phi}(y) = \bigvee\{U \mid \widehat{\phi}(x) \circ U \leq \widehat{\phi}(y)\}$. It follows then that $\widehat{\phi}(x\backslash^{\Pi_1} y) = \widehat{\phi}(x)\backslash\widehat{\phi}(y)$, as desired.

□

Given the previous two facts, we are ensured that $\Pi_1(\mathbf{G})$ is genuinely a completion of $\mathbf{G}$. We record this fact in the following corollary.

Corollary 6.4.4. *If* $\mathbf{G}$ *is a unital* $r\ell$*-groupoid, then* $\pi : \mathbf{G} \to \Pi_1(\mathbf{G})$ *is a unital* $r\ell$*-groupoid embedding.*

Proof. Note that $\pi = \widehat{\phi}^{-1} * \phi$ and that $\widehat{\phi}^{-1} * \phi$ is an embedding. □

By combining the topological representation for the Π_1-completion of $r\ell$-groupoids we just obtained in Theorem 6.4.3 with the Π_1-persistence results of Chapter 4, we obtain the following persistence result for pointed $r\ell$-groupoids.

Theorem 6.4.5. *Let* t *and* s *be terms in the signature* $\{e, \top, \bot, \cdot, \wedge, \vee\}$ *and let* $\mathbf{G}$ *be an* $r\ell$*-groupoid. If* $\mathbf{G} \models s \approx t$*, the* $\Pi_1(\mathbf{G}) \models s \approx t$*.*

Proof. A direct consequence of Theorem 4.3.5 and the topological representation of the Π_1-completion of $r\ell$-groupoids. □

As an immediate corollary we can provide a sufficient condition for class of $r\ell$-groupoids to be closed under the Π_1-completion.

Corollary 6.4.6. *Let* $\mathcal{K}$ *be a class of* $r\ell$*-groupoid defined by a set of identities in the signature* $\{e, \top, \bot, \cdot, \wedge, \vee\}$*, then if* $\mathbf{G} \in \mathcal{K}$*, then* $\Pi_1(\mathbf{G}) \in \mathcal{K}$*.*

Finally, we show that there is a class of residuated lattices that is not closed under the Π_1-completion.

Proposition 6.4.7. *There is a class of residuated lattices that is not closed under the* Π_1*-completion.*

Proof. It is well know that the filter completion $\mathfrak{fe}(()\mathbf{B})$ of a boolean algebra $\mathbf{B}$ need not be a boolean algebra. Since $\mathfrak{fe}(()\mathbf{B})$ embeds into $\Pi_1(\mathbf{B})$, we conclude that $\Pi_1(\mathbf{B})$ need not be a boolean algebra. Therefore, the class of boolean algebra is not closed under the Π_1-completion. □

In the next section we use our duality to obtain representations for products of pointed $r\ell$-groupoids and representations for congruences of residuated lattices.

6.5 Conclusion

In this chapter we have obtained various duality results for a range of variants of residuated lattices. In particular, we showed that the category of $r\ell$-groupoids is dually isomorphic to the category of NRL-spaces. Using the equivalence between NRL-spaces and RML-spaces we also derived a duality with respect to RML-spaces. We the restricted this duality to obtain dualities for residuated lattices, FL-algebras, and involutive residuated lattices. The following diagram summarizes these results. Horizontal arrows represent categorical inclusions while vertical arrows represent category isomorphisms.

$$\begin{tikzcd}
 & \mathbf{NRLSp} \arrow[dl, leftrightarrow] \arrow[dd, leftrightarrow] & \mathbf{RLSp} \arrow[l] \arrow[dd, leftrightarrow] & \mathbf{FLSp} \arrow[l] \arrow[dd, leftrightarrow] & \mathbf{InFLSp} \arrow[l] \arrow[dd, leftrightarrow] \\
\mathbf{RMLSp} \arrow[dr, leftrightarrow] & & & & \\
 & \mathbf{RLG}^{op} & \mathbf{RL}^{op} \arrow[l] & \mathbf{FL}^{op} \arrow[l] & \mathbf{InRL}^{op} \arrow[l]
\end{tikzcd}$$

In addition, from Proposition 6.1.14, each of these dualities was explicitly shown to restrict to classes defined by commutativity, weakening, and contraction. Finally, in Section 6.4 we showed that the Π_1-completion of $r\ell$-groupoids are representable in their dual NRL-spaces and use the Π_1-preservation results from Chapter 4 to obtain some sufficient conditions for an identity in the signature of $r\ell$-groupoids to be preserved through the Π_1-completion.

Chapter 7

The OKHD-Semantics, and Completeness via Duality

In this chapter we begin our exploration of how L-space duality and more particularly the duality and Π_1-persistence results from Chapter 6 can be applied to the semantics of substructural logics. We combine the duality for $r\ell$-groupoids, the notion of Π_1-persistence developed by the authors of [6], and the connection between between NRL-spaces and OKHD-frames to obtain a general completeness theorem with respect to the operational semantics for substructural logics developed by Ono and Komori [32], Humberstone [23], and Došen [13].

The chapter is structured as follows. We begin in Section 7.1 by reviewing the substructural logics and their algebraic semantics. In particular, we introduce the logic $\mathbf{NFL}^+$, which is the logic of $r\ell$-groupoids. In Section 7.2 we introduce the operational semantics of Ono and Komori, Humberstone, and Došen or, more succinctly, OKHD-semantics. We then introduce morphisms between OKHD-frames and show how they preserve and reflect satisfaction and validity. We will use these facts in the following chapter. We then briefly define the RML-frame based semantics in Section 7.3, which is more closely aligned with theory of filter continous relations developed in Chapter 3. In Section 7.4 we arrive at the main results of the chapter. First, in Theorem 7.4.3, we show that all extensions of $\mathbf{NFL}^+$ are complete with respect to a class of NRL-spaces. This is a simple consequence of alegbraic completenes and NRL-space duality. An analogous topological completeness theorem is also demonstrated with respect to RML-spaces. We then show in Theorem 7.4.7 that all sequents that do not contain the connectives $\backslash$ and $/$ are Π_1-persistent in the sense that if they are valid in an NRL-space, then they are also valid in the underlying OKHD-frame of that NRL-space. This leads to the general completeness result reported in Theorem 7.4.8. Finally, in Section 7.5 we discuss how the canonical model style proofs from the original papers from Ono and Korori, Humberstone, and Došen can be analyzed in terms of the completeness-via-duality methodology.

7.1 Substructural Logics

In this section we define the Positive Non-Associative Full Lambek Calculus, $\mathbf{NFL}^+$, and characterize some of its extensions. The reason we call this positive is because we do not include the "falsity constant" f, which is ususally included in the full Lambek Calculus. In [12], Dŏsen denotes $\mathbf{NFL}^+$ by $\mathbf{GL}$. $\mathbf{NFL}^+$ is one of the weakest logics with the additive connectives $\wedge$ and $\vee$, the residuated family of connectives \, and /, and the truth constant t. Below we give sequent style natural deduction rules that characterize the logic $\mathbf{NFL}^+$. These rules are inspired by the proof theory introduced in Restall's introductory book on substructural logic [33]. Dǒsen's original characterization of $\mathbf{NFL}^+$ was given as a Gentzen system with left and right rules for each connective. While Gentzen systems are often proof theoretically convenient, our considerations are primarily model theoretic and we therefore prefer the more intuitive set of rules presented below. For a thorough introduction to substructural logics see either [33] or [19].

The language $\mathcal{L}$ is built from atomic expressions $Prop = \{p_1, p_2, p_3, ..\}$ and the connectives I have already listed above.

Definition 7.1.1. *Language $\mathcal{L}$*

$$\varphi := ||\ p \mid \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \bullet \varphi \mid \varphi \backslash \varphi \mid \varphi / \varphi \mid t\ ||$$

We denote the set of all such expression by $\mathcal{L}$. The objects denoted by Γ, Δ and Σ below are called structures. They are built out of formulas and the punctuation mark $-;-$.

Definition 7.1.2. *$\mathcal{L}$-Structures $Str(\mathcal{L})$.*

$$\Gamma := ||\ \varphi \mid \Gamma; \Gamma\ ||$$

We denote the set of all structures by $Str(\mathcal{L})$. A context $\Gamma[\cdot]$ is structure with a special atom $\cdot$ that intuitively represents an empty position in Γ meant for substituting. $\Gamma[\Delta]$ is the structure obtained by substituting Δ for $\cdot$ in the context $\Gamma[\cdot]$.

Finally, a sequent is a pair $\Gamma \Rightarrow \varphi$ where $\Gamma \in Str(\mathcal{L})$ and $\varphi \in \mathcal{L}$.

We now define the basic logic $\mathbf{NFL}^+$. For us a logic is a set of sequents. Intuitively, a logic is simply a collection of argument pairs that are deemed acceptable.

Definition 7.1.3. *The Logic* $\mathbf{NFL}^+$ *is the least set of sequents all instances of axiom schemas*

$$(Ax)\ \varphi \Rightarrow \varphi \qquad (Ax\top)\ \Gamma \Rightarrow \top \qquad (Ax\bot)\ \Gamma[\bot] \Rightarrow \varphi$$

and closed under the rules:

$$(\wedge\text{-}out_r)\frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \varphi} \qquad (\wedge\text{-}out_l)\frac{\Gamma \Rightarrow \varphi \wedge \psi}{\Gamma \Rightarrow \psi}$$

$$(\vee\text{-}in_r)\frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \vee \psi} \qquad (\vee\text{-}out_l)\frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \vee \psi}$$

$$(\wedge\text{-}in)\ \frac{\Gamma \Rightarrow \varphi \qquad \Gamma \Rightarrow \psi}{\Gamma \Rightarrow \varphi \wedge \psi} \qquad (\vee\text{-}out)\ \frac{\Gamma[\varphi] \Rightarrow \chi \qquad \Gamma[\psi] \Rightarrow \chi}{\Gamma[\Delta] \Rightarrow \chi}$$

$$(/\text{-}in)\ \frac{\Gamma; \varphi \vdash \psi}{\Gamma \vdash \psi/\varphi} \qquad (/\text{-}out)\ \frac{\Gamma \vdash \varphi/\psi \qquad \Delta \vdash \psi}{\Gamma; \Delta \vdash \varphi}$$

$$(\backslash\text{-}in)\ \frac{\varphi; \Gamma \vdash \psi}{\Gamma \vdash \varphi\backslash\psi} \qquad (\backslash\text{-}out)\ \frac{\Gamma \vdash \varphi\backslash\psi \qquad \Delta \vdash \varphi}{\Delta; \Gamma \vdash \psi}$$

$$(\bullet\text{-}in)\ \frac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma ; \Delta \vdash \varphi \bullet \psi} \qquad (\bullet\text{-}out)\ \frac{\Gamma \vdash \varphi \bullet \psi \qquad \Delta[\varphi ; \psi] \vdash \xi}{\Delta[\Gamma] \vdash \xi}$$

$$(Cut)\ \frac{\Gamma \Rightarrow \varphi \qquad \Delta[\varphi] \Rightarrow \psi}{\Delta[\Gamma] \Rightarrow \psi}$$

To define extension of $\mathbf{NFL}^+$ we either add axioms or structural rules. Structural rules allow for control over how the structures on the left hand side of a sequent relate to the formulas on the right hand side. Intuitively, structural rules are a means of premise management and govern the relationship between premise and conclusion. We list a few common structural rules below.

$$(a)\ \frac{\Theta[\Gamma ; (\Delta ; \Sigma)] \Rightarrow \varphi}{\Theta[(\Gamma ; \Delta) ; \Sigma] \Rightarrow \varphi} \qquad (a^c)\ \frac{\Theta[(\Gamma ; \Delta) ; \Sigma] \Rightarrow \varphi}{\Theta[\Gamma ; (\Delta ; \Sigma)] \Rightarrow \varphi} \qquad (e)\ \frac{\Sigma[\Gamma ; \Delta] \Rightarrow \varphi}{\Sigma[\Delta ; \Gamma] \Rightarrow \varphi}$$

$$(w^r)\ \frac{\Sigma[\Gamma] \Rightarrow \varphi}{\Sigma[\Gamma ; \Delta] \Rightarrow \varphi} \qquad (w^l)\ \frac{\Sigma[\Delta] \Rightarrow \varphi}{\Sigma[\Gamma ; \Delta] \Rightarrow \varphi} \qquad (c)\ \frac{\Gamma[\Delta ; \Delta] \Rightarrow \varphi}{\Gamma[\Delta] \Rightarrow \varphi}$$

$$(t\text{-in}^l)\ \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow t \bullet \varphi} \qquad (t\text{-in}^r)\ \frac{\Gamma \Rightarrow \varphi}{\Gamma \Rightarrow \varphi \bullet t}$$

$$(t\text{-out}^l)\ \frac{\Gamma[t ; \Delta] \Rightarrow \varphi}{\Gamma[\Delta] \Rightarrow \varphi} \qquad (t\text{-out}^r)\ \frac{\Gamma[\Delta ; t] \Rightarrow \varphi}{\Gamma[\Delta] \Rightarrow \varphi}$$

While adding structural rules is the most common way of defining an extension of $\mathbf{NFL}^+$, for our purposes it will be more convenient to define extensions by adding axioms. Each system defined by adding some collection of the structural rules above can be equivalently characterized by adding a corresponding collection of the axioms listed below.

$(a)\ \varphi \bullet (\psi \bullet \xi) \Rightarrow (\varphi \bullet \psi) \bullet \xi$ $\quad (a^c)\ (\varphi \bullet \psi) \bullet \xi \Rightarrow \varphi \bullet (\psi \bullet \xi)$ $\quad (e)\ \varphi \bullet \psi \Rightarrow \psi \bullet \varphi$

$(w^r)\ \varphi \bullet \psi \Rightarrow \varphi$ $\quad (w^l)\ \varphi \bullet \psi \Rightarrow \psi$ $\quad (c)\ \varphi \Rightarrow \varphi \bullet \varphi$

$(t\text{-in}_1)\ \varphi \bullet t \Rightarrow \varphi$ $\quad (t\text{-in}_2)\ t \bullet \varphi \Rightarrow \varphi$ $\quad (t\text{-out}_1)\ \varphi \Rightarrow \varphi \bullet t$ $\quad (t\text{-out}_2)\ \varphi \Rightarrow t \bullet \varphi.$

We now define an extension of $\mathbf{NFL}^+$ as a set of sequents that contains the axioms of $\mathbf{NFL}^+$ is closed under the rules of $\mathbf{NFL}^+$.

Definition 7.1.4. *An extension of* $\mathbf{NFL}^+$ *is a collection of sequents* $\mathbf{L}$ *such that* $\mathbf{NFL}^+ \subseteq \mathbf{L}$ *and such that* $\mathbf{L}$ *is closed under the rules from Definition 7.1.3. We say that set of sequents* $\mathcal{S}$ *axiomatizes* $\mathbf{L}$ *is* $\mathbf{L}$ *is the least set of sequents obtain by closing* $\mathbf{NFL}^+ \cup \mathcal{S}$ *under the rules from Definition 7.1.3.*

Let us remark on some well know extensions of $\mathbf{NFL}^+$. We recover the Full Lambek calculus $\mathbf{FL}^+$ by extending $\mathbf{NFL}^+$ with the axioms a and a^c and all of the rules for t[1]. By adding the structural rule e to $\mathbf{FL}^+$ thus obtaining $\mathbf{FL}_e$, we obtain a fragment of linear logic. By adding back all of the axioms listed above we recover Intuitionistic Propositional Logic, **IPL**. In general, we denote extensions of **NFL** by some collection of axioms $r_1, ..r_n$ where each rule r_i is among $\{a, a^c, w^r, w^l, e, c\}$ with $\mathbf{NFL}_{r_1,..,r_n}$. We will shorten $\mathbf{NFL}_{a,a^c}$ to **FL**.

[1]**FL** has an additional constant f but we will ignore this for now.

7.1.1 Algebraic Semantics

In this section we briefly recall the algebraic semantics of the substructural substuctural logics in terms of $r\ell$-groupoids. We first define algebraic models and then define the Lindenbaum algebra of an extension of $\mathbf{NFL}^+$.

Definition 7.1.5. *An $r\ell$-groupoid model is a pair $(\mathbf{G}, \sigma)$ where $\mathbf{G}$ is an $r\ell$-groupoid and $\sigma : Prop \to G$. We extend σ to a homomorphism $\sigma^+ : \mathcal{L} \to \mathbf{G}$ as follows:*

$$\begin{array}{ll}
\sigma^+(\varphi \wedge \psi) = \sigma^+(\varphi) \wedge \sigma^+(\psi) & \sigma^+(\varphi \vee \psi) = \sigma^+(\varphi) \vee \sigma^+(\psi), \\
\sigma^+(\varphi \bullet \psi) = \sigma^+(\varphi) \cdot \sigma^+(\psi) & \sigma^+(t) = e \\
\sigma^+(\varphi \backslash \psi) = \sigma^+(\varphi) \backslash \sigma^+(\psi) & \sigma^+(\varphi / \psi) = \sigma^+(\varphi) / \sigma^+(\psi), \\
\sigma^+(\top) = \top & \sigma^+(\bot) = \bot, \\
\sigma^+(\Gamma; \Delta) = \sigma^+(\Gamma) \cdot \sigma^+(\Delta). &
\end{array}$$

We say that an $r\ell$-groupoid model $(\mathbf{G}, \sigma)$ *satisfies* a sequent $\Gamma \Rightarrow \varphi$ and write $\mathbf{G}, \sigma \vDash \Gamma \Rightarrow \varphi$ if $\sigma^+(\Gamma) \leq_{\mathbf{G}} \sigma^+(\varphi)$. We say a $r\ell$-groupoid *validates* a sequent $\Gamma \Rightarrow \varphi$ and write $\mathbf{G} \vDash \Gamma \Rightarrow \varphi$ if very every valuation $\sigma : Prop \to G$, $\mathbf{G}, \sigma \vDash \Gamma \Rightarrow \varphi$. Finally, we say that a class $\mathcal{K}$ of $r\ell$-groupoids validates a sequent $\Gamma \Rightarrow \varphi$ and write $\mathcal{K} \vDash \Gamma \Rightarrow \varphi$ if for all $\mathbf{G} \in \mathcal{K}$, $\mathbf{G} \vDash \Gamma \Rightarrow \varphi$.

We now define the Lindenbaum Algebra of an extension of $\mathbf{NFL}^+$. It is simply the congruence of the language with respect to interderivability in the salient logic.

Definition 7.1.6. *Let $\mathbf{L}$ be an extension of $\mathbf{NFL}^+$. The Lindenbaum algebra $\Lambda_{\mathbf{L}} = (/\equiv_{\mathbf{L}}, \wedge, \vee, \top, \bot, \cdot, \backslash, /, e)$ of $\mathbf{L}$ is defined such that $\mathcal{L}/\equiv_{\mathbf{L}} = \{[\varphi]_{\theta_{\mathbf{L}}} \mid \varphi \in \mathcal{L}\}$ and $\varphi \theta_{\mathbf{L}} \psi$ iff $\varphi \Rightarrow \psi \in \mathbf{L}$ and $\psi \Rightarrow \varphi \in \mathbf{L}$. The algebraic operations are then defined:*

$$\begin{array}{ll}
[\varphi]_{\theta_{\mathbf{L}}} \wedge [\psi]_{\theta_{\mathbf{L}}} := [\varphi \wedge \psi]_{\theta_{\mathbf{L}}} & [\varphi]_{\theta_{\mathbf{L}}} \vee [\psi]_{\theta_{\mathbf{L}}} := [\varphi \vee \psi]_{\theta_{\mathbf{L}}}, \\
[\varphi]_{\theta_{\mathbf{L}}} \cdot [\psi]_{\theta_{\mathbf{L}}} := [\varphi \cdot \psi]_{\theta_{\mathbf{L}}} & e = t_{\theta_{\mathbf{L}}} \\
[\varphi]_{\theta_{\mathbf{L}}} \backslash [\psi]_{\theta_{\mathbf{L}}} := [\varphi \backslash \psi]_{\theta_{\mathbf{L}}} & [\varphi]_{\theta_{\mathbf{L}}} / [\psi]_{\theta_{\mathbf{L}}} := [\varphi / \psi]_{\theta_{\mathbf{L}}}, \\
\top = \top_{\theta_{\mathbf{L}}} & \bot = \bot_{\theta_{\mathbf{L}}}
\end{array}$$

Using the construction of a Lindenbaum Algebra and the fact that each sequent corresponds to an algerbaic identity, it is possible to show that every extension $\mathbf{L}$ of $\mathbf{NFL}^+$ is complete with respect to a class of $r\ell$-groupoids. In particular, we can think of each formula of the logical language $\mathbf{L}$ as an algebraic term and each sequent $\psi \Rightarrow \varphi$ as the identity $\varphi \wedge \psi = \varphi$. In general, given a sequent α, we write $\alpha*$ for the corresponding identity. Likewise, given a set of sequents $\mathcal{S}$, the set of identities corresponding to elements of $\mathcal{S}$ is denoted $\mathcal{S}^*$.

Theorem 7.1.7. *(Algrbraic Completeness) Let $\mathbf{L}$ be an extension of $\mathbf{NFL}^+$. Let $\mathcal{K}_{\mathbf{L}}$ be that class of algebras validating all of the identities in $\mathbf{L}^*$.*

$$\text{If } \mathcal{K}_{\mathbf{L}} \vDash \Gamma \Rightarrow \varphi, \text{ then } \Gamma \Rightarrow \varphi \in \mathbf{L}.$$

Proof. We only sketch a proof since the result is standard. Suppose that $\Gamma \Rightarrow \varphi \notin \mathbf{L}$. Then $\Lambda_{\mathbf{L}} \nvDash \Gamma \Rightarrow \varphi$. However, $\Lambda_{\mathbf{L}} \vDash \mathbf{L}$ and it is then simple to show that $\Lambda_{\mathbf{L}} \vDash \mathbf{L}*$. We are therefore done. Note that in particular $\Lambda^+_{\mathbf{NFL}}$ is an $r\ell$-groupoid. □

In the next section we define the OKHD-semantics in detail and introduce morphisms between OKHD-frames and models.

7.2 The OKHD-semantics

In this section we define the operational OKHD-semantics. The OKHD-semantics are a form of frame based semantics developed for possibly non-distributive substructural logics that emerged independently in the work of Hiroakira Ono and Komori [32], Humberstone [23], and Kosta Dǒsen [13]. The key insight that made these semantics work is the treatment of disjunction as an intensional connective; an insight that opened the door to frame based semantics for non-distributive logics.

Definition 7.2.1. *(OKHD-frames)*

An OKHD-frame $X = (X, \curlywedge, 1, \otimes, \varepsilon)$ is structure where $(X, \curlywedge, 1)$ is a semilattice, $(X, \otimes, \varepsilon)$ is a pointed groupoid, and (1) and (2) govern the relationship between $\curlywedge$, $\otimes$, and 1.

(1) $x \otimes (y \curlywedge z) = (x \otimes y) \curlywedge (x \otimes z)$ and $(y \curlywedge z) \otimes x = (y \otimes x) \curlywedge (z \otimes x)$, and

(2) $x \otimes 1 = 1 = 1 \otimes x$.

We provided some example of OKHD-frames in Chapter 5 in Examples 5.2.9 and 5.2.10.

We define models based on OKHD-frames equipping frames with valuations. A valuation $V : At \to \mathcal{F}i(X)$ is a mapping from the atomic expressions At of the language to the filters of the structure $\mathcal{F}i(X)$. A OKHD-model is a OKHD-frame equipped with a valuation.

Satisfaction in a model is then given by the following rules.

Definition 7.2.2. *(Satisfaction)*

$X, V, x \Vdash p$ *iff* $x \in V(p)$

$X, V, x \Vdash \varphi \wedge \psi$ *iff* $X, V, x \Vdash \varphi$ *and* $X, V, x \Vdash \psi$

$X, V, x \Vdash \varphi \vee \psi$ *iff there are* $y, z \in X$ *such that* $y \curlywedge z \leq x$ *and* $X, V, y \Vdash \varphi$ *and* $X, V, z \Vdash \psi$

$X, V, x \Vdash \varphi \bullet \psi$ *iff there are* $y, z \in X$ *such that* $y \otimes z \leq x$ *and* $X, V, y \Vdash \varphi$ *and* $X, V, z \Vdash \psi$.

$X, V, x \Vdash \varphi \backslash \psi$ *iff for all* $y \in X$, *if* $X, V, y \Vdash \varphi$, *then* $X, V, y \otimes x \Vdash \psi$

$X, V, x \Vdash \psi / \varphi$ *iff for all* $y \in X$, *if* $X, V, y \Vdash \varphi$, *then* $X, V, x \otimes y \Vdash \psi$

$X, V, x \Vdash t$ *iff* $\varepsilon \leq x$.

$X, V, x \Vdash \top$ *iff* $x \in X$

$X, V, x \Vdash \bot$ *iff* $x = 1$.

Finally, we have satisfaction for structures and sequents.

$X, V, x \Vdash \Gamma; \Delta$ *iff there are* $y, z \in X$ *such that* $y \otimes z \leq x$ *and* $X, V, y \Vdash \Gamma$ *and* $X, V, z \Vdash \Delta$,

$X, V, x \Vdash \Gamma \Rightarrow \varphi$ *iff if* $X, V, x \Vdash X$, *then* $X, V, x \Vdash \varphi$.

We that a formula, structure, or sequent α is valid in an OKHD-frame $\mathbf{X}$, written $\mathbf{X} \vDash \alpha$ if for each valuation $V : Prop \to \mathcal{F}i(\mathbf{X})$ and each $x \in X$, $\mathbf{X}, V, x \Vdash \alpha$. Finally, we say that a class $\mathcal{K}$ of OKHD-frames validates a sequent $\Gamma \Rightarrow \varphi$ and write $\mathcal{K} \vDash \Gamma \Rightarrow \varphi$ if for all $\mathbf{X} \in \mathcal{K}$, $\mathbf{X} \vDash \Gamma \Rightarrow \varphi$.

For a formula or structure we α also define $[\![\alpha]\!]_{\mathbf{M}} = \{x \in X \mid \mathbf{M}, x \Vdash \alpha\}$ for the set of points which satisfy φ in the model $\mathbf{M} = (X, V)$. Often we omit the subscript $\mathbf{M}$ when confusion won't arise. We can then restate validity for a sequent: a sequent $\Gamma \Rightarrow \varphi$ is valid in an OKHD-frame $\mathbf{X}$ if for all valuations $V : Prop \to \mathcal{F}i(X\mathbf{X})$, $[\![\Gamma]\!] \subseteq [\![\varphi]\!]$. An important feature of the these semantics is that in each model $\mathbf{M} = (X, V)$, the function $[\![\cdot]\!] : \mathcal{L} \to \mathcal{P}(X)$ uniquely extends $V : Prop \to \mathcal{F}i(X)$ in such a way that guarantees that $[\![\varphi]\!]$ is always a filter.

Lemma 7.2.3. *(Heredity) For all formulas φ and all models $\mathbf{M} = (\mathbf{X}, V)$, $[\![\varphi]\!]_{\mathbf{M}}$ is a filter of X.*

A proof of this fact can be found in [30] or [13]. We note finally, that the logic $\mathbf{NFL}^+$ is sound with respect to the class of all OKHD-semantics.

Proposition 7.2.4. $\mathbf{NFL}^+$ *is sound with respect to the class of all OKHD-semantics: If $\Gamma \Rightarrow \varphi \in \mathbf{NFL}^+$, then for all OKHD-models $\mathbf{M}$, $[\![\Gamma]\!]_{\mathbf{M}} \subseteq [\![\varphi]\!]_{\mathbf{M}}$.*

A simple correspondence theory holds between sequents and frame conditions. The following table summarizes these results, which can be found in [13].

Sequents	Frame Conditions
$\varphi \bullet (\psi \bullet \chi) \Rightarrow (\varphi \bullet \psi) \bullet \chi$	$\forall xyz((x \otimes y) \otimes z \leq x \otimes (y \otimes z))$
$(\varphi \bullet \psi) \bullet \chi \Rightarrow \varphi \bullet (\psi \bullet \chi)$	$\forall xyz(x \otimes (y \otimes z) \leq (x \otimes y) \otimes z)$
$\varphi \bullet \psi \Rightarrow \psi \bullet \varphi$	$\forall xy(x \otimes y = y \otimes x)$
$\varphi \Rightarrow \varphi \bullet \varphi$	$\forall x(x \otimes x \leq x)$
$\varphi \bullet \psi \Rightarrow \varphi$	$\forall xy(x \leq x \otimes y)$
$\psi \bullet \varphi \Rightarrow \varphi$	$\forall xy(y \leq x \otimes y)$
$\varphi \Rightarrow t \bullet \varphi$	$\forall x(\varepsilon \otimes x \leq x)$
$\varphi \Rightarrow \varphi \bullet t$	$\forall x(x \otimes \varepsilon \leq x)$
$t \bullet \varphi \Rightarrow \varphi$	$\forall x(x \leq \varepsilon \otimes x)$
$\varphi \bullet t \Rightarrow \varphi$	$\forall x(x \leq x \otimes \varepsilon)$

We now move on to consider morphisms between models and frames and their effect on the preservation of satisfaction and validity.

7.2.1 Morphisms, Models, and Frames

In this section we introduce morphisms between OKHD-frames and models and prove some simple facts about the preservation of satisfaction and validity along these morphisms. As far as we know, the morphisms introduced here do not appear elsewhere in the literature. We find this surprising given that in the case of classical modal logic, much of the frame and model theory relies on the notion of p-morphism. Despite not being especially deep results, we hope that the consequences of these what is proved in this section and what is proved in the sections to come at least open the door to a wider array of applications.

Let us begin with morphisms between frames. Morphisms between OKHD-frames are essentially defined in virtue of NRL-space morphisms without the requirement of continuity.

Definition 7.2.5. *(OKHD-frame Morphism) Suppose that $\mathbf{X} = (X, \curlywedge, 1, \otimes, \varepsilon)$ and $\mathbf{Y} = (Y, \curlywedge', 1', \otimes', \varepsilon')$ are OKHD-frames. An OKHD-frame morphism is a semilattice homomorphism $f : X \to Y$ satisfying the following additional properties.*

(1-*backandforth*) $f(x) = 1'$ *iff* $x = 1$.

($\curlywedge$-back) If $x' \curlywedge' y' \leq f(z)$, then there are $x, y \in X$ such that $x' \leq fx$, $y' \leq fy$, and $x \curlywedge y \leq z$,

($\otimes$-forth) $f(x) \otimes' f(y) \leq f(x \otimes y)$,

($\otimes$-back) If $x' \otimes' y' \leq f(z)$, then there are $x, y \in X$ such that $x' \leq fx$, $y' \leq fy$, and $x \otimes y \leq z$,

($/$-back) if $fx \otimes' y' \leq z'$, then there are $y, z \in X$ such that $y' \leq fy$, $fz \leq z'$, and $x \otimes y \leq z$,

($\backslash$-back) if $x' \otimes' fy \leq z'$, then there are $x, z \in X$ such that $x' \leq fx$, $fz \leq z'$, and $x \otimes y \leq z$,

(ε-forth) $\varepsilon' \leq f(\varepsilon)$, *and*
(ε-back) *if* $\varepsilon' \leq fx$, *then* $\varepsilon \leq x$.

Given that a OKHD-morphism essentially is essentiall an NRL-space morphism without continuity, we obtain that follows simple fact in virtue of the results from the chapter on the duality. In particular we obtain that the inverse image of any filter is again a filter.

Lemma 7.2.6. *Let* $f : \mathbf{X} \to \mathbf{Y}$ *be an OKHD-frame morphism, then* $f^{-1} : \mathcal{F}i(Y) \to \mathcal{F}i(X)$ *is well defined.*

In order to extend this definition to OKHD-Models, we require in loose terms of that the models agree on the valuations of propositional letters with respect to a given morphism. More precisely, we require for each $p \in Prop$ that $V(p) = f^{-1}(V'(p))$. This leads to the following definition.

Definition 7.2.7. *(OKHD-Model Morphism) Let* $\mathbf{M} = (X, V)$ *and* $\mathbf{N} = (X', V')$ *be OKHD-Models. An OKHD-Model morphism* $f : \mathbf{M} \to \mathbf{N}$ *is a OKHD-frame morphism* $f : X \to X'$ *such that* $V = f^{-1} * V'$.

In the above definition $*$ denotes function composition. Having defined the notions of morphism for frame and models respectively, we will now show how our morphism can preserve semantic properties like satisfaction and validity.

Let us first restrict attention to OKHD-Model morphisms. By considering OKHD-Model morphisms and weakening our validity to satisfaction we obtain the following proposition which states that all given a morphism $f : \mathbf{M} \to \mathbf{N}$, x and fx satisfy all of the same formulas.

Proposition 7.2.8. *(Preservation of Satisfaction) Suppose that* $f : \mathbf{M} \to \mathbf{N}$ *and* $x \in \mathbf{M}$ *is an OKHD-Model morphism, then* $\mathbf{M}, x \Vdash \varphi$ *if and only if* $\mathbf{N}, fx \Vdash \varphi$.

Proof. The proof proceeds by an induction on the complexity of φ. In the interest of space and not repeating what has been done in [6], we will inly show the cases for $\bullet$ and $\backslash$.

Let $\mathbf{M} = (X, V)$ and $\mathbf{N} = (X', V')$ and suppose that $f : \mathbf{M} \to \mathbf{N}$ is OKHD-Model morphism.

Case $\bullet$: Suppose that $\mathbf{M}, x \Vdash \varphi \bullet \psi$. Then there are $y, z \in X$ such that $y \otimes z \leq x$ and $\mathbf{M}, y \Vdash \varphi$ and $\mathbf{M}, z \Vdash \psi$. By IH we obtain that $\mathbf{N}, fy \Vdash \varphi$ and $\mathbf{N}, fz \Vdash \psi$. But in virtue of $\otimes$-forth condition and monotonicity of f, we have $fy \otimes fz \leq f(y \otimes z) \leq fx$. Therefore $\mathbf{N}, fx \Vdash \varphi \bullet \psi$, as desired. Suppose for the converse that $\mathbf{N}, fx \Vdash \varphi \bullet \psi$. Then there are $y', z' \in X'$ such that $y' \otimes z' \leq fx$ and $\mathbf{N}, y' \Vdash \varphi$ and $\mathbf{N}, z' \Vdash \psi$.. By the $\otimes$-back condition there are $y, z \in X$ such that $y \otimes z \leq x$, $y' \leq fy$, and $z' \leq fz$. By persistence of φ and ψ, we obtain that $\mathbf{N}, fy \Vdash \varphi$ and $\mathbf{N}, fz \Vdash \psi$. So the IH guarantees that $M, y \Vdash \varphi$ and $\mathbf{M}, z \Vdash \psi$. So it follows from the fact that $y \otimes z \leq x$ that we can conclude $\mathbf{M}, x \Vdash \varphi \bullet \psi$.

Case $\backslash$: Suppose that $\mathbf{M}, y \Vdash \varphi \backslash \psi$. Suppose that $x' \otimes fy \leq z$ and that $\mathbf{N}, x' \Vdash \varphi$ for $y', z' \in X'$. We need to show that $\mathbf{N}, z' \Vdash \psi$. By the $\backslash$-back condition there are $x, z \in X$ such that $x \otimes y \leq z$ and $y' \leq fy$ and $fz \leq z'$. By persistence and the IH we have that $\mathbf{M}, x \Vdash \varphi$ and so given that $\mathbf{M}, y \Vdash \varphi \backslash \psi$ and $x \otimes y \leq z$, we obtain by the semantics of $\backslash$ that $\mathbf{M}, z \Vdash \psi$. Then again by the IH and persistence we obtain that $\mathbf{N}, z' \Vdash \psi$ and conclude that $\mathbf{N}, fy \Vdash \varphi \backslash \psi$.. For the converse, suppose that $\mathbf{N}, fy \Vdash \varphi \backslash \psi$. Suppose that $x \otimes y \leq z$ for $x, z \in X$. Then by the $\otimes$-forth condition and monotonicity of f we have that $fx \otimes fy \leq fz$. It then follows by IH that $\mathbf{N}, fx \Vdash \varphi$ and so by the semantics of $\backslash$ that $\mathbf{N}, fz \Vdash \psi$. So by IH we have that $\mathbf{M}, z \Vdash \psi$ and can conclude that $\mathbf{M}, y \Vdash \varphi \backslash \psi$. $\square$

We now consider the more general case of arbitrary OKHD-frame morphisms and demonstrate some of their properties. In particular, we will show that validity is reflected by injective OKHD-frame morphisms and is preserved by surjective frame morphisms. A lemma for accomplishing this goal follows. It provides a useful condition guaranteeing that a frame morphisms reflect satisfaction. It will be used in the proof showing that injective frame morphisms reflect validity.

Lemma 7.2.9. *Let (X, V) and (Y, V') be models, $f : X \to Y$ be a frame embedding, and φ be any sentence in $\mathcal{L}$. If for all $p \in Prop$, $V'(p) = \uparrow f[V(p)]$, then $(Y, V'), fx \Vdash \varphi$ iff $(X, V), x \Vdash \varphi$.*

Proof. We can show this by simply showing that $f^{-1}[V'(p)] = V(p)$, and thus that $f : (\mathbf{X}, V) \to (\mathbf{Y}, V')$ is an OKHD-model morphism. We then may apply Proposition 7.2.8. By definition of V', we have $f^{-1}[V'(p)] = f^{-1}[\uparrow f[V(p)]]$. So the inclusion $V(p) \subseteq f^{-1}[V'(p)]$ is straightforward. For the other inclusion, let $fx \in \uparrow f[V(p)]$. Then there is some $y \in V(p)$ such that $fy \leq fx$. Since f is injective and preserves $\curlywedge$, we have that $y \leq x$ and thus that $x \in V(p)$. $\square$

We now show that OKHD frame-morphisms have the properties mentioned above; namely reflection and preservation of validty for injections and surjections, respectively.

Proposition 7.2.10. *()*

(1) Suppose that $f : \mathbf{X} \to \mathbf{Y}$ is an injective OKHD-frame Morphism. If $\mathbf{Y} \vDash \varphi$, then $\mathbf{X} \vDash \varphi$.

(2) Suppose that $f : \mathbf{X} \to \mathbf{Y}$ is a surjective OKHD-frame Morphism . If $\mathbf{X} \vDash \varphi$, then $\mathbf{Y} \vDash \varphi$.

Proof. (1) Suppose that $f : \mathbf{X} \to \mathbf{Y}$ is an embedding and that $\mathbf{Y} \vDash \varphi$. Let $V : Prop \to \mathcal{F}i(X)$. Define $V' : Prop \to \mathcal{F}i(Y)$ as $V'(p) = \uparrow (f[V(p)])$. By the fact that $Y \Vdash \varphi$, $(Y, V'), fx \Vdash \varphi$. By Lemma 7.2.9 we have immediately that $(X, V), x \Vdash \varphi$. Generalizing on V and x, we have show that $X \vDash \varphi$.

(2) Suppose that $f : X \to Y$ is a surjective OKHD-frame morphism and that φ is valid on X: $X \vDash \varphi$. Now, let $V' : Prop \to \mathcal{F}i(Y)$ be a valuation and suppose that $x' \in Y$. We must show $(Y, V'), fx \vDash \varphi$. Define $V : Prop \to \mathcal{F}i(X)$ such that $V := f^{-1} * V'$, where $*$ is function composition. Given the definition of V, we are ensured that f is an OMH-*model* morphism from (X, V) to (Y, V'). However given the sujectivity of f we know there is some $x \in X$ such that $fx = x'$. So in virtue of Proposition 7.2.8 we obtain that $(X, V), x \vDash \varphi$ iff $(Y, V'), fx \vDash \varphi$. But of course we assumed that $X \vDash \varphi$, so then $(X, V), x \vDash \varphi$ and therefore $(Y, V'), fx \vDash \varphi$, as desired. Generalizing on x' and V', we may conclude that $Y \vDash \varphi$. $\square$

The facts we just proved are essential for the applications of the semantics we provide. We use them in both the characterization of the Disjunction Property and the new proof of the local deduction theorem. In the following section we will use duality to prove completeness give sufficient condition to guarantee that the algebraic completeness guarantees completeness on a class of OKHD-frames.

7.3 The RML-Frame Based Semantics

We briefly remark on the RML-frame based semantics. The semantics can be see as following through with a suggestion to generalize the OKHD-semantics by Dunn and Hardegree in [14]. We defined RML-frames in the last section of Chapter 5 in Definition RML-frame. In the interest of Space we do not repeat the definition here. We define a model $\mathbf{M} = (\mathbf{X}, V)$ to be an RML-frame $\mathbf{X} = (X, \curlywedge, 1, R, T)$ equipped with a valuation $V : Prop \to \mathcal{F}i(\mathbf{X})$. The satisfaction conditions for the language $\mathcal{L}$ differ from the OKHD-semantics only for the connectives $\bullet$, $\backslash$, $/$, the constant t, and the punctuation mark $-;-$. In particular we have that given a RML-model $\mathbf{M} = (\mathbf{X}, V)$ with $\mathbf{X} = (X, \curlywedge, 1, R, T)$:

$\mathbf{M}, x \Vdash \phi \bullet \psi$ iff there are $y, z \in X$ such that $Rxyz$ and $\mathbf{M}, y \Vdash \phi$ and $\mathbf{M}, z \Vdash \psi$.

$\mathbf{M}, x \Vdash \phi \backslash \psi$ iff for all $y \in X$, if $\mathbf{M}, y \Vdash \phi$ and $Rzyx$, then $\mathbf{M}, z \Vdash \psi$

$\mathbf{M}, x \Vdash \psi / \phi$ iff for all $y \in X$, if $\mathbf{M}, y \Vdash \phi$ and $Rzxy$, then $\mathbf{M}, z \Vdash \psi$,

$\mathbf{M}, x \Vdash t$ iff $x \in T$.

And for the punctuation we have:

$\mathbf{M}, x \Vdash \Gamma; \Delta$ iff there are $y, z \in X$ such that $Rxyx$ and $\mathbf{M}, y \Vdash \Gamma$ and $\mathbf{M}, z \Vdash \Delta$.

The semantics have the property of persistence. The proof can be adapted from the one for the OKHD-semantics.

Lemma 7.3.1. *(Persistence) For all formulas φ and all models $\mathbf{M} = (X, V)$, $[\![\varphi]\!]_{\mathbf{M}}$ is a filter of X.*

We can also adapt straight forwardly the results of preservations of validity and satisfaction for RML-frame morphisms.

Remark 7.3.2. *(Sahlqvist Correspondence)*

It is noteworthy that the Sahlqvist style correspondence results of [6] and the Thesis [] are readily adapted to this setting for positive formulas.

7.4 Completeness via Duality

In this section we use duality to derive various completeness theorems. In Theorem 7.4.3 we prove topological completeness for every extension of **NFL**. We then adapt this result to for a topoligical completeness theorem with respect to RML-spaces. Following our topological completeness theorem, we adapt an argument from [6] in order to give a generalization of existing completeness theorems with respect to OKHD-frames. In particular, we adapt the notion of a Π_1-persistent sequent from [6] (see Definition 7.4.6) and then show that any sequent in the signature only containing the propositional constants t, f, $\top$, and $\bot$, and the connectives $\vee$, $\wedge$, and $\bullet$ is Π_1-persistent. Finally, in Theorems 7.4.8 and 7.4.9 we show how Π_1-persistence guarantees completeness with respect to OKHD-frames and for any extension of **NFL** axiomatized by sequents in a Π_1-persistent signature. As far as we are aware, general completeness theorem with respect OKHD-frames have only gone as far as logics axiomatized by sequents in the signature consisting of $\bullet$ and t [30]. Our results therefore provide a significant generalization of existing completeness theorems.

7.4.1 Topological Completeness

We begin with topological completeness via our topological duality. Let us first define topological models.

Definition 7.4.1. *A Topological Model $\mathbf{M} = (\mathbf{X}, V)$ is a pair consisting of an NRL-space $\mathbf{X} = (X, \tau)$ and clopen-valuation $V : Prop \to \mathcal{F}i_{clp}(\mathbf{X})$.*

Satisfaction in a topological model is defined exactly the same as it is in ordinary OKHD-Models (see Definition 7.2.2). Validity in topological space is defined with respect to clopen valuations and validity in a class of spaces is defined with respect to validity in all frames of that class.

Definition 7.4.2. *We say that $\Gamma \Rightarrow \varphi$ is topologically-valid in an NRL-space and write $\mathbf{X} \vDash \Gamma \Rightarrow \varphi$ iff for every clopen valuation $V : Prop \to \mathcal{F}i_{clp}(X)$ and every point $x \in X$: $\mathbf{X}, V, x \Vdash \Gamma \Rightarrow \varphi$.*

We say that $\Gamma \Rightarrow \varphi$ is topologically-valid in a class $\mathcal{K}$ of NRL-spaces and write $\mathcal{K} \vDash \Gamma \Rightarrow \varphi$ iff for all $\mathbf{X} \in \mathcal{K}$, $\mathbf{X} \vDash \Gamma \Rightarrow \varphi$.

We define two operations $Log(-)$ and $Top_{\mathbf{NRL}}(-)$. For a class $\mathcal{K}$ of NRL-spaces $Log(\mathcal{K}) = \{\Gamma \Rightarrow \varphi \mid \mathcal{K} \vDash \Gamma \Rightarrow \varphi\}$. That is, $Log(\mathcal{K})$ is the set of sequents valid on every member of $\mathcal{K}$. Likewise, for a set of sequents $\mathcal{S}$, $Top_{\mathbf{NRL}}(\mathcal{S}) = \{\mathbf{X} \mid \mathbf{X}$ is an NRL-space and $\mathbf{X} \vDash \mathcal{S}\}$. In English, $Top_{\mathbf{NRL}}(\mathcal{S})$ is the set of NRL-spaces that validate every sequent in $\mathcal{S}$. It is not hard to see that these operations define a Galois connection between classes of NRL-spaces and sets of sequents. We will show that every extension of **NFL** is complete with respect to $Top_{\mathbf{NRL}}(\mathbf{L})$. Our completeness theorem essentially characterizes every extension of **NFL** as a fixed point of the composite operation $Log(Top_{\mathbf{NRL}}(-))$. In more standard terms, this notion of completeness means that if a sequent is valid on all topological spaces of a logic, then that logic proves that sequent.

Theorem 7.4.3. *Every extension $\mathbf{L}$ of NFL is complete with respect to a class of NRL-spaces. In particular we have:*

$$\mathbf{L} = Log(Top_{\mathbf{NRL}}(\mathbf{L})).$$

Proof. Suppose that $\Gamma \Rightarrow \varphi$ is not in $\mathbf{L}$. By algebraic completeness, there is an algebraic model $(\mathbf{K}, \sigma)$ of $\mathbf{L}$ such that $(\mathbf{K}, \sigma) \nvDash \Gamma \Rightarrow \varphi$. We can define a clopen valuation $V := \phi * \sigma$ on the dual space $\mathbf{X_L}$ of $\mathbf{K}$ by composing the isomorphism $\phi : K \to \mathcal{F}i_{clp}(\mathbf{X_K})$ with $\sigma : Prop \to K$. It is straight forward that $\phi \circ \sigma^+ = [\![-]\!]$. It then follows from $(\mathbf{K}, \sigma) \nvDash \Gamma \Rightarrow \varphi$ that $[\![X]\!] \nsubseteq [\![\varphi]\!]$ and therefore that there is some $x \in X$ such that $\mathbf{X}, V, x \Vdash \Gamma$ but $\mathbf{X}, V, x \nVdash \varphi$. It follows that $\Gamma \Rightarrow \varphi$ is not in $Log(Top_{\mathbf{NRL}}(\mathbf{L}))$ and therefore that $\mathbf{L}$ is complete with respect to $Top_{\mathbf{NRL}}(\mathbf{L})$. Moreover, it readily follows then that $\mathbf{L} = Log(Top_{\mathbf{NRL}}(\mathbf{L}))$, as desired. $\square$

The last thing we do in this section before moving to completeness with respect to frame based semantics is to a completeness theorem with respect to RML-spaces. RML-space model $\mathbf{M} = (\mathbf{X}, V)$ of the substructural language $\mathcal{L}$ is a RML-model (see Section 7.3) where $\mathbf{X} = (X, \tau)$ is an RML-space and $V : Prop \to \mathcal{F}i_{clp}(\mathbf{X})$ is a clopen valuation. We note that $Top_{RML}(-)$ is the operation mapping classes of sequents to RML-space models and $Log(-)$ is essentially as above.

Theorem 7.4.4. *Every extension $\mathbf{L}$ of NFL is complete with respect to a class of NRL-spaces. In particular we have:*

$$\mathbf{L} = Log(Top_{\mathbf{RML}}(\mathbf{L})).$$

Proof. Recall that there is an isomorphism $\mathbf{G} : \mathbf{RMLSp} \to \mathbf{NRLSp}$. We claim that $\mathbf{G}[Top_{\mathbf{RML}}(\mathbf{L})] = Top_{\mathbf{NRL}}(\mathbf{L})$. So let $\mathbf{X} \in \mathbf{G}[Top_{\mathbf{RML}}(\mathbf{L})]$. Then there is some $\mathbf{Y} \in Top_{\mathbf{RML}}(\mathbf{L})$ such that $\mathbf{G}(\mathbf{Y}) = \mathbf{X}$. A straightforward induction then shows that the for clopen valuations $\mathbf{V} : Prop \to \mathcal{F}i(\mathbf{Y})$, every $x \in X$, and every $\varphi \in \mathcal{L}$:

$$\mathbf{Y}, V, x \Vdash \varphi \text{ if and only if } \mathbf{X}, V, x \Vdash \varphi$$

Then $\mathbf{G}^{-1}(\mathbf{X}) \in Top_{\mathbf{RML}}$. Therefore we also obtain that

$$\mathbf{Y} \vDash \Gamma \Rightarrow \varphi \text{ if and only if } \mathbf{X} \vDash \Gamma \Rightarrow \varphi$$

for all $\Gamma \Rightarrow \varphi \in \mathbf{L}$. We then can conclude that $\mathbf{X} \in Top_{\mathbf{NRL}}(\mathbf{L})$.

The converse is shown by a similar argument. $\square$

This concludes the section on topological semantics and completeness. In the next section we apply these results to obtain completeness theorems with respect to the OKHD-semantics.

7.4.2 Frame Completeness and Π_1-Persistence

In this section we prove completeness with respect to OKHD-frames and RML-frames by combing the topological completeness theorems in the previous section with the Π_1-persistence results for $r\ell$-groupoids demonstrated at the end of Chapter 6. To begin we show completeness of $\mathbf{NFL}^+$ with respect to the class of all OKHD-frames. In definition 7.4.6 we define the notion of Π_1-persistence appropriate to the setting of our topological semantics and the OKHD-semantics. Using this definition, the Π_1-preservation Theorem 6.4.5, and topological completeness, we obtain the general completeness theorem in Theorem 7.4.8 for OKHD-semantics. Finally, we use Theorem 7.4.8 and the fact that we can always obtain an RML-frame from an OKHD-frame to show an analogous completeness theorem with respect to RML-frames.

In Chapter 5 we showed in Proposition 5.2.11 that by forgetting the topology of an NRL-space we are left with an OKHD-frame. We therefore recover the following fundamental completeness theorem proved by Dŏsen in [13] by way of our duality theory.

Theorem 7.4.5. $\mathbf{NFL}^+$ *is complete with respect of the class of all OKHD-frames.*

Proof. If $\Gamma \Rightarrow \varphi$ is not provable in $\mathbf{NFL}^+$, then by Theorem 7.4.3, there is a NRL-space $\mathbf{X}$, a clopen valuation $V : Prop \to \mathcal{F}i_{clp}(\mathbf{X})$, and an element $x \in X$ such that $\mathbf{X}, V, x \Vdash \Gamma$ and $\mathbf{X}, V, x \nVdash \varphi$. By Proposition 5.2.11 from chapter 5.2.11, we $\mathbf{X}$ is also an OKHD-frame and thus we are done. $\square$

Let us know define the notion of Π_1-persistence. This is the logical analogue of an identity being preserved through the Π_1-completion.

Definition 7.4.6. *We say a sequent* $\Gamma \Rightarrow \varphi$ *is* Π_1*-persistent if for any NRL-space* (X, τ)*, if* $\Gamma \Rightarrow \varphi$ *is topologically-valid in* $\mathbf{X} = (X, \curlywedge, 1, \otimes, \varepsilon, \tau)$*, then* $\Gamma \Rightarrow \varphi$ *is valid in the OKHD-frame* $(X, \curlywedge, 1, \otimes, \varepsilon)$*.*

We now arrive at one of the main results of this section.

Theorem 7.4.7. *Every sequent in the signature* $\{t, \top, \bot, \vee, \wedge, \bullet\}$ *is* Π_1*-persistent.*

Proof. Let $\mathbf{X} = (X, \tau)$ be an NRL-space. Let $\Gamma \Rightarrow \varphi$ be a sequent in the signature $\{t, \top, \bot, \vee, \wedge, \bullet\}$ that is topologically-valid in $\mathbf{X}$. This implies that $\mathbf{L_X} \vDash \Gamma \Rightarrow \varphi$ since any valuation on $\mathbf{L_X}$ can be transformed into a clopen valuation for $\mathbf{X}$ anf vise versa. With out loss of generality, we may assume that Γ is a single formula ψ[2] and we may treat $\psi \Rightarrow \varphi$ as an identity $\psi \wedge \varphi = \varphi$ in the algebraic language corresponding to $\mathcal{L}$. So the fact that $\psi \Rightarrow \varphi$ is in $\mathbf{L_X}$ means in algebraic language that $\mathbf{L_X} \vDash \psi \wedge \varphi = \varphi$. However, by Theorem 6.4.5, which assert that identities in the signature $\{t, \top, \bot, \vee, \wedge, \bullet\}$ are preserved through the Π_1-completion, we have that $\Pi_1(\mathbf{L}_X) \vDash \psi \wedge \varphi = \varphi$. So by the representation theorem (Theorem 6.4.3) for the Π_1-completion of $r\ell$-groupoids, we also have that $\mathcal{F}i(\mathbf{X}) \cong \Pi_1(\mathbf{L}_X)$. Now, let $V : Prop \to \mathcal{F}i(\mathbf{X})$ be valuation for the underlying OKHD-frame of $\mathbf{X}$ and let $x \in X$. From the fact that $\Pi_1(\mathbf{L}_X) \vDash \psi \wedge \varphi = \varphi$ and $\mathcal{F}i(\mathbf{X}) \cong \Pi_1(\mathbf{L}_X)$, we obtain that $[\![\psi]\!] \subseteq [\![\varphi]\!]$. We may therefore conclude that $\mathbf{X}, V, x \vDash \psi \Rightarrow \varphi$. Generalizing on V, we obtain that $\Gamma \Rightarrow \varphi$ is valid in $\mathbf{X}$ as an OKHD-frame. $\square$

[2] $\Gamma \Rightarrow \varphi$ is interderivable with $\psi \Rightarrow \varphi$ when ψ is obtained from Γ by substituting $\bullet$ for $-;-$.

We are finally in a position to provide our general completeness theorems. Just as in the case of logics and NRL-space, we define two operations $Log(-)$ and OKHD$(-)$ that map classes of frames to set of sequents and classes of sequents to classes of frames, respectively. More precisely, for a class $\mathcal{K}$ of OKHD-frames, we define $Log(\mathcal{K}) = \{\Gamma \Rightarrow \varphi \mid \mathcal{K} \vDash \Gamma \Rightarrow \varphi\}$. On the other hand given a set of sequents $\mathcal{S}$, we define $\mathrm{OKHD}(\mathcal{S}) = \{\mathbf{X} \mid \mathbf{X} \vDash \mathcal{S} \ \&\ \mathbf{X} \text{ is an OKHD-frame}\}$.

Theorem 7.4.8. *Every extension of* $\mathbf{L}$ *of* $\mathbf{NFL}^+$ *that is axiomatixed by a set of sequents* $\mathcal{S}$ *in the signature* $\{t, \top, \bot, \vee, \wedge, \bullet\}$ *is complete with respect to a class of OKHD-frames. In particular we have:*

$$\mathbf{L} = Log(OKHD(\mathbf{L}))$$

Proof. Clearly $\mathbf{L} \subseteq Log(\mathrm{OKHD}(\mathbf{L}))$. Fir the other inclusion let us reason contrapositively. Suppose that $\Gamma \Rightarrow \varphi \notin \mathbf{L}$. Then by Theorem 7.4.3, which demonstrates topological completeness, we obtain that there is some NRL-space $\mathbf{X}$ such that $\mathbf{X} \vDash \mathbf{L}$ but $\mathbf{X} \nvDash \Gamma \Rightarrow \varphi$. It follows that the OKHD-frame underlying $\mathbf{X}$ also fails to validate $\Gamma \Rightarrow \varphi$. However, since $\mathbf{L}$ is axiomatized by $\mathcal{S}$, which was assumed to consist only of sequents in the signature $\{t, \top, \bot, \vee, \wedge, \bullet\}$, we conclude from Theorem 7.4.7 that $\mathbf{X} \vDash \mathbf{L}$ as an OKHD-frame. The OKHD-frame $\mathbf{X}$ is therefore in $\mathrm{OKHD}(\mathbf{L})$ but $\Gamma \Rightarrow \varphi \notin Log(\mathrm{OKHD}(\mathbf{L}))$, as desird. We therefore conclude that $\mathbf{L} = Log(\mathrm{OKHD}(\mathbf{L}))$. □

In light of the previous theorem, we obtain an analogous theorem with respect to the RML-frame semantics. Let us define $\mathrm{RML}(\mathcal{S}) = \{\mathbf{X} \mid \mathbf{X} \vDash \mathcal{S} \ \&\ \mathbf{X} \text{ is an RML-frame}\}$.

Corollary 7.4.9. *Every extension of* $\mathbf{L}$ *of* $\mathbf{NFL}^+$ *that is axiomatixed by a set of sequents* $\mathcal{S}$ *in the signature* $\{t, \top, \bot, \vee, \wedge, \bullet\}$ *is complete with respect to a class of RML-frames. In particcluar, we have*

$$\mathbf{L} = Log(RML(\mathbf{L}))$$

Proof. Recall the operation $\mathbf{F} : \mathbf{OKHD} \to \mathbf{RMLFr}$ from Chapter 5 defined by sending an OKHD-frame $\mathbf{X} = (X, \curlywedge, 1, \otimes, \varepsilon)$ to the RML-frame $\mathbf{F}(\mathbf{X}) = (X, \curlywedge, 1, R^{\otimes}, T)$ where $T = \uparrow\varepsilon$ and $R^{\otimes}xyz$ holds iff $y \otimes z \leq x$. It is easy to show that for all sequents $\Gamma \Rightarrow \varphi$:

$$\mathbf{X} \Vdash \Gamma \Rightarrow \varphi \text{ if and only if } \mathbf{F}(\mathbf{X}) \Vdash \Gamma \Rightarrow \varphi.$$

Now let $\mathbf{L}$ be an extension of $\mathbf{NFL}^+$ axiomatized by sequents $\mathcal{S}$ in the signature $\{t, \top, \bot, \vee, \wedge, \bullet\}$. Suppose that $\Gamma \Rightarrow \psi \notin \mathbf{L}$. Then by Theorem 7.4.8, there is some OKHD-frame $\mathbf{X}$ such that $\mathbf{X} \vDash \mathcal{S}$ but $\mathbf{X} \nvDash \Gamma \Rightarrow \varphi$. It then follows that both $\mathbf{F}(\mathbf{X}) \vDash \mathcal{S}$ and $\mathbf{F}(\mathbf{X}) \nVdash \Gamma \Rightarrow \varphi$. It follows that $\mathbf{F}(\mathbf{X}) \in \mathrm{RML}(\mathbf{L})$ and thus that $\Gamma \Rightarrow \varphi \notin Log(\mathrm{RML}(\mathbf{L}))$ □

7.5 Analyzing Canonical Model Style Proofs

In this final section of the chapter we discuss how we can understand the success of the canonical model style completeness proofs of Ono and Komori [32], Humberstone [23], and Dǒsen [13] in terms of the completeness theorems we obtained in the previous section. In particular, we discuss how to explicate the notion canonicity implicit in the results of Ono and Komori, Humberstone, and Dǒsen in terms of algebraic completeness, topological duality, and Π_1-persistence. We will begin by defining canonical models of some extensions $\mathbf{L}$ of $\mathbf{NFL}^+$. We will then observe that there is a natural

NRL-space topology on these models and that these topologies are the duals of the Lindenbaum Algebras/free algebras associated with the logic $\mathbf{L}$. Using this insight, we can then explain then success of the canonical model style completeness proofs from [32, 23, 13] in virtue of the notions topological completeness via duality and Π_1-persistence.

7.5.1 Canonical Models and Topologies

Canonical models for **NFL** and its extensions can be constructed by definitionning appropriate operations on the set of all deductively closed sets of sentences of the language $\mathcal{L}$. In fact, more generally, given a set $Prop_\kappa$ of propositional letters of cardinality κ, we may define a language $\mathcal{L}_\kappa$. So the language $\mathcal{L}$ we defined in Section 7.1 is simply $\mathcal{L}_{\aleph_0}$.

First, given a set u of sentences in the language $\mathcal{L}_\kappa$, where κ is some cardinal, we define the *deductive closure* of u with respect to extension $\mathbf{L}$ of **NFL** by $Cl^{\mathbf{L}}_{\vdash}(u) = \{\phi \mid \exists V \subseteq_\omega u(\bigwedge V \vdash_{\mathbf{L}} \phi)\}$. Here $\subseteq_\omega$ is the finite subset relation. A set u of $\mathcal{L}$ expressions is called *deductively closed* if $Cl^{\mathbf{L}}_{\vdash}(u) = u$.

Definition 7.5.1. *(Canonical Model for an Extension* $\mathbf{L}$ *of* **NFL***)*

The canonical model $\mathfrak{M}^\kappa_{\mathbf{L}} = (X^{\mathbf{L}}, \curlywedge^{\mathbf{L}}, 1^{\mathbf{L}}, \otimes^{\mathbf{L}}, \varepsilon^{\mathbf{L}}, V^{\mathbf{L}})$ *is defined such that*

$*$ $X^{\mathbf{L}} = \{u \subseteq \mathcal{L}_\kappa \mid Cl^{\mathbf{L}}_{\vdash}(u) = u \;\&\; u \neq \emptyset\}$,

$*$ $u \curlywedge^{\mathbf{L}} v = u \cap v$

$*$ $1^{\mathbf{L}} = \mathcal{L}$

$*$ $u \otimes^{\mathbf{L}} v = \{\phi \in \mathcal{L} \mid \exists \psi \in u \exists \chi \in v(\psi \bullet \chi \vdash_{\mathbf{L}} \phi)\}$

$*$ $\varepsilon^{\mathbf{L}} = \{\phi \mid t \vdash_{\mathbf{L}} \phi\}$

$*$ $V^{\mathbf{L}} : Prop_\kappa \to \mathcal{P}(W^{\mathbf{L}})$ *such that* $V^{\mathbf{L}}(p) = \{u \in W^{\mathbf{L}} \mid p \in u\}$.

When $\kappa = \aleph_0$*, we just write* $\mathfrak{M}_{\mathbf{L}}$ *for* $\mathfrak{M}^{\aleph_0}_{\mathbf{L}}$*. We denote the underlying frame* $(X^{\mathbf{L}}, \curlywedge^{\mathbf{L}}, 1^{\mathbf{L}}, \otimes^{\mathbf{L}}, \varepsilon^{\mathbf{L}})$ *of a canonical model with* $\mathfrak{X}^\kappa_{\mathbf{L}}$.

A standard truth lemma is provable with respect to these models. For a proof of this fact see [13], for example.

Lemma 7.5.2. *(Truth Lemma) Let* $\mathbf{L}$ *be an extension of* $\mathbf{NFL}^+$*. Then for every* $\phi \in \mathcal{L}_\kappa$*:*

$$\phi \in u \text{ iff } \mathfrak{M}^\kappa_{\mathbf{L}}, u \Vdash \phi$$

As with standard canonical model constructions in modal logic, the canonical frame $\mathfrak{X}^\kappa_{\mathbf{L}}$ carries a topology. In particular, we may equip $\mathfrak{X}^\kappa_{\mathbf{L}}$ with a topology $\tau^{\mathbf{L}}$ generated by the subbase $\{[\![\phi]\!]_{V^{\mathbf{L}}} \mid \phi \in \mathcal{L}_\kappa\} \cup \{X^{\mathbf{L}} - [\![\phi]\!]_{V^{\mathbf{L}}} \mid \phi \in \mathcal{L}_\kappa\}$. We call $(\mathfrak{X}^\kappa_{\mathbf{L}}, \tau^{\mathbf{L}})$ the canonical topological frame and $(\mathfrak{M}^\kappa_{\mathbf{L}}, \tau^{\mathbf{L}})$ the topological canonical model. Note that that this topological model is well defined since $V^{\mathbf{L}}(p)$ is subbasic and thus clopen.

Again in parallel with ordinary modal logic, we can show that the canonical topology $(\mathfrak{X}^\kappa_{\mathbf{L}}, \tau^{\mathbf{L}})$ is the essentially the same as the dual NRL-space of the Lindenbaum Algebra $\Lambda_{\mathbf{L}} := \mathcal{L}/{\equiv_{\mathbf{L}}}$ of a extension $\mathbf{L}$ of $\mathbf{NFL}^+$.

Proposition 7.5.3. *For each extension* $\mathbf{L}$ *of* $\mathbf{NFL}^+$*:* $(\mathfrak{X}^\kappa_{\mathbf{L}}, \tau^{\mathbf{L}}) \cong \mathbf{X}_{\Lambda_{\mathbf{L}}}$.

Proof. We will only sketch the proof here. The relevant NRL-space isomorphism $f : (\mathfrak{X}^\kappa_{\mathbf{L}}, \tau^{\mathbf{L}}) \to \mathbf{X}_{\mathbf{A_L}}$ is defined such that $f(u) = u/\theta_{\mathbf{L}}$ where $u/\theta_{\mathbf{L}} := \{[\varphi]_{\theta_{\mathbf{L}}} \mid \varphi \in \mathcal{L}^\kappa\}$. The inverse is defined by taking the union of the a filter of $\Lambda_{\mathbf{L}}$. A precise proof that f is a bijective function satisfying the conditions of an NRL-space morphism can be adapted from Lemma 8.1.11 proved in the section duality for congruences.

To show that f is continuous, it is enough to show that $\varphi \in \mathcal{L}^\kappa$, $f^{-1}[\phi_{\Lambda_\mathbf{L}}(\varphi/\equiv_\mathbf{L})] = [\![\varphi]\!]_{V^\mathbf{L}}$. The following chain of equivalences witnesses fact.

$$\begin{aligned} x \in f^{-1}[\phi_{\Lambda_\mathbf{L}}(\varphi/\equiv_\mathbf{L})] &\iff fu \in \phi_{\Lambda_\mathbf{L}}(\varphi/\equiv_\mathbf{L}) \\ &\iff \varphi/\equiv_\mathbf{L} \in fu \\ &\iff \varphi \in u \\ &\iff u \in [\![\varphi]\!]_{V^\mathbf{L}}. \end{aligned}$$

The second equivalence holds in virtue of the definition of $\phi_{\Lambda_\mathbf{L}}$. The third equivalence holds by the definition of f. The last equivalence holds in virtue of the truth lemma. □

Having observed that the homeomorphism between the dual NRL-spaces of Lindenbaum alegbras and canonical topologies, we record a useful lemma we will make use of in the following chapter. We note that every OKHD-frame embeds into a topological canonical frame.

Lemma 7.5.4. *Suppose that* $\mathbf{L}$ *is a complete extension of* $\mathbf{NFL}$*. Every OKHD-frame for the logic* $\mathbf{L}$ *embeds into a (topological) canonical frame in* $Fr(\mathbf{L})$*.*

Proof. Every $r\ell$-groupoid is the homomorphic image image of a free $r\ell$-groupoid. The duals of free $r\ell$-groupoid are precisely the canonical topological frames in the category dual to the algebras of $\mathbf{L}$. If $\mathbf{X}$ is an OKHD-frame for the logic $\mathbf{L}$. $\mathcal{F}i(\mathbf{X})$ forms an $r\ell$-groupoid. $\mathcal{F}i(\mathbf{X})$ is the homomorphic image of a free $r\ell$-groupoid $\mathbf{F}$. It follows from NRL-space duality that $\mathbf{X}_{\mathcal{F}i(\mathbf{X})}$ embeds into the NRL-space $\mathbf{X}_\mathbf{F}$. What remains to be shown is that $\mathbf{X}$ embeds into $\mathbf{X}_{\mathcal{F}i(\mathbf{X})}$. However, this is witnessed by the map $e_\mathbf{X} : \mathbf{X} \to \mathbf{X}_{\mathcal{F}i(\mathbf{X})}$ defined by $e_\mathbf{X}(x) = \{U \in \mathcal{F}i(\mathbf{X}) \mid x \in U\}$. Showing it is an OKHD-frame morphism is somewhat straight forward and is similar to the non-topological parts of the proof showing $\eta_\mathbf{X}$ is a NRL-space morphism in Theorem 6.1.10. □

We end this section by observing that the above lemma guarantees the the caetgory of NRL-spaces fully determines the category of OKHD-frames.

Proposition 7.5.5. *Every OKHD frame embeds into an NRL-space.*

We now move on to discuss how Π_1-persistence together with the correspondence between Lindenbaum/free algebras and canonical models clarifies existing canonical model style completeness theorems with respect to the OKHD-semantics.

7.5.2 Completeness and Canonicity

In the terminology and notation of this chapter, canonical model style completeness proofs of Ono and Komori, Humberstone, and Dŏsen are demonstrated by showing that the canonical frame $\mathfrak{X}^\mathbf{L}$ is a member of OKHD($\mathbf{L}$), which is the class of all OKHD-frames validating the sequents of $\mathbf{L}$. Let us define the following notion of a canonical logic, which appears to generalize the notion of canonicity implicit in [32, 23, 13].

Definition 7.5.6. *An extension* $\mathbf{L}$ *of* $\mathbf{NFL}^+$ *is OKHD-canonical if the canonical frame* $\mathfrak{X}^\mathbf{L}$ *is an element of OKHD*($\mathbf{L}$)*.*

Using algebraic completeness, the dual correspondence between free/Lindenbaum algebras and topological canonical models, and the characterization of Π_1-persistent sequents, let us show how a OKHD-canonicity can be derived. Specifically, we observe that each of the logics considered in the articles [32, 23, 13] and more recently by [37] are axiomatized by sequents built up using connectives among $\bullet, \vee, \wedge, t, \top$ and $\bot$. In virtue of algebraic completeness, the Lindebaum algebras of these logics all validate the axioms of the logic. By the correspondence between Lindenbaum algebras and canonical topological frames, we also obtain the axioms of these logics are *topologically* valid in the canonical frame. Finally, since the axioms of all these logics are in the appropriate signature, we are ensured by Π_1-persistence that canonical frames validate the salient logical axioms as OKHD-frames. We have therefore explained the OKHD-canonicity of these logics as promised.

Let us consider a concrete example of the this process. Consider the logic $\mathbf{L_{BCC}}$ studied by Ono and Komori in [32]. $\mathbf{L_{BCC}}$ is essentially Intuitionistic Propositional Logic without distributivity or contraction. We can view this logic as $\mathbf{NFL}^+$ extended with the following four axioms.

$$\varphi \bullet (\psi \bullet \theta) \Rightarrow (\varphi \bullet \psi) \bullet \theta \qquad \varphi \bullet \psi \Rightarrow \psi \bullet \varphi \qquad \varphi \bullet \psi \Rightarrow \varphi \qquad \phi \Rightarrow \varphi \bullet t.$$

By algebraic completeness and the correspondence between the Lindenbaum algebra and the canonical frame we have that the axioms of $\mathbf{L_{BCC}}$ are topologically valid in $\mathfrak{X}_{\mathbf{L_{BCC}}}$. Now, since each of the axioms of $\mathbf{L_{BCC}}$ are Π_1-persistent, we also obtain that the axioms of $\mathbf{L_{BCC}}$ are valid in $\mathfrak{X}_{\mathbf{L_{BCC}}}$ as an OKHD frame and thus that $\mathfrak{X}_{\mathbf{L_{BCC}}}$ is an element of OKHD($\mathbf{L_{BCC}}$).

We summarize this example and the preceding discussion with the following proposition and the fact that all of the logics in [32, 23, 13] and more recently by [37] are axiomatized by Π_1-persistent axioms.

Proposition 7.5.7. *If an extension* $\mathbf{L}$ *of* $\mathbf{NFL}^+$ *is axiomatized by a* Π_1*-persistent set* $\mathcal{S}$ *of sequents, then* $\mathbf{L}$ *is OKHD-canonical.*

Proof. By algebraic completeness, $\Lambda_{\mathbf{L}} \vDash \mathcal{S}$. By proposition 7.5.3, $\mathcal{S}$ is topologically valid in $(\mathfrak{X}_{\mathbf{L}}, \tau_{\mathbf{L}})$. By Π_1-persistence, $\mathfrak{X}_{\mathbf{L}} \in$ OKHD($\mathbf{L}$). $\square$

7.6 Conclusion

In this chapter we used topological duality developed in the previous chapters to study the semantics of substructural logics. In particular we show how NRL-spaces and RML-spaces could be used to obtain topological semantics for substructural logics and how the topological semantics and the notion of Π_1-persistence could be used to generalize and understand existing OKHD-frame completeness theorems from articles like [32, 23, 13, 37].

Chapter 8

Expanding the Theory of NRL-spaces

This chapter extends the theory of NRL-space duality in two directions and then provides logical applications for both developments. The first development, in Section 8.1, regards congruences of residuated lattices from the perspective of NRL-space duality. We begin by demonstrating the lattice of congruences of any residuated lattice $\mathbf{L}$ is isomorphic to the lattice of positive, central, idempotents of the RL-space dual to $\mathbf{L}$ (Theorem 8.1.4). We use this representation to obtain a useful characterization of subspaces of RL-spaces (Proposition 8.1.8 and Theorem 8.1.12). We then apply our dual representation of congruences, and in particular the insights it provides into determining subspaces, to give a new proof of the parameterized local deduction theorem for the logic $\mathbf{FL}^+$. The second development to the theory of L-spaces provided in this chapter is the characterization of products and coproducts and then a representation theorem for products of $r\ell$-groupoids. After proving our representation we use our insight into the coproducts of NRL-spaces to demonstrate a simple characterization of when a substructural logic has the disjunction property.

We note that the logical results of this chapter are not particularly novel results. It is well known that the parameterized local deduction theorem holds for $\mathbf{FL}^+$. Further, in light of the duality theory we have developed in the present chapter and in Chapter 6, our characterization of logics with the disjunction property can be seen as a dualization of an existing algebraic characterization. However, this is not really the point of providing them here. Rather their inclusion is intended as proof of concept for the duality theory of NRL-spaces. We wish to stress this point since this is the primary contribution of this chapter.

8.1 The Dual Representation of Congruences and the Parameterized Local Deduction Theorem

Our first extension of the theory of NRL-space duality is a representation of congruences of $r\ell$-groupoids in their dual NRL-spaces. We then apply this representation to a new proof of the Parameterized Local Deduction Theorem for the logic $\mathbf{FL}^+$.

8.1.1 The Dual Representation of Congruences

Congruences are of central importance in universal algebra and algebraic logic. We will show that the congruences of a residuated lattices correspond to a special collection of points in their dual RL-spaces and that this correspondence leads to useful and tangible insights regarding subspaces of

RL-spaces. Specifically, Theorem 8.1.4 shows that lattice of congruences of some residuated lattice is isomorphic to the lattice of positive, central, idempotent elements (see Definition 8.1.2) of its dual RL-space. Then Theorem 8.1.12 shows that every subspace of an NRL-space can be defined in virtue of single positive central idempotent element. In virtue of these results, in the following section we will show that we can obtain a novel proof of the parameterized local deduction theorem for logic $\mathbf{FL}^+$.

A Representation of Congruences

We begin by recalling some basic facts about the correspondence between congruences and a special sort of filter, which we will call congruence filters. Although presented in a slightly different form, proofs of the following results can be found in [19] in section 3.6. The more concise summary below adapted from [1].

Recall the left and right conjugates $\lambda_u(a) = u\backslash au \wedge e$ and $\rho_u(a) = ua/u \wedge e$.

Definition 8.1.1. *Let L be a residuated lattice. A congruence filter F is a filter of L such that (1) $e \in F$, (2) if $a, b \in F$, then $ab \in F$, and (3) if $a \in F$ and $u \in L$ then $\lambda_u(a), \rho_u(a) \in F$.*

The value of this definition is that for any residuated lattice, the congruence filters of that algebra form a lattice isomorphic to its lattice of congruences. More formally stated, for any residuated lattice L, $\mathbf{Con}(L)$ is isomorphic to the lattice $\mathcal{F}i_{\mathbf{Con}}(L)$ of congruence filters of L. The operations that witness this isomorphism are

$$\theta \mapsto F_\theta = \bigcup\{a/\theta \mid e \leq a\} \text{ and } F \mapsto \theta_F = \{(a,b) \mid a\backslash b, b\backslash a \in F\}$$

Every $\theta \in \mathbf{Con}(L)$ and $F \in \mathcal{F}i_{\mathbf{Con}}(L)$. These operations are well defined in so far as $F_\theta = \bigcup\{a/\theta \mid e \leq a\}$ is a congruence filter whenever $\theta \in \mathbf{Con}(L)$ and $\theta_F = \{(a,b) \mid a\backslash b, b\backslash a \in F\}$ is a congruence of L whenever F is a congruence filter of $\mathbf{L}$.

Turning now to our representation of congruences in RL-spaces, we define the positive, central, idempotent elements of an RL-space.

Definition 8.1.2. *Let $\mathbf{X}$ be an RL-space. We say that that an element x is positive if $\varepsilon_X \leq x$. We say that x is idempotent if $x \otimes x = x$. Finally, we say x is central if for all $y \in X$ $x \otimes y = y \otimes x$.*

We denote the set of all positive, central, idemotent elements of X by $\mathfrak{C}^+(\mathbf{X})$.

The set of all positive, central, idempotent elements $\mathfrak{C}^+(\mathbf{X})$ form a lattice. This lattice will turn out to correspond exactly to the conruence filters of the algebra of clopen filters $\mathbf{L_X}$.

Proposition 8.1.3. *Let $\mathbf{X}$ be an RL-space. Then the set of all positive, central, idempotent elements $\mathfrak{C}^+(\mathbf{X})$ of $\mathbf{X}$ form a bounded distributive lattice $(\mathfrak{C}^+(\mathbf{X}), \curlywedge_X, \otimes_X, 1, \varepsilon_X)$ with $\curlywedge_X$ as meet, $\otimes_X$ as join, ε as bottom, and 1 as top.*

Proof. Distributivity follows straight from the defining condition on $\otimes$ and $\curlywedge$ in the definition of an RL-space. Clearly, ε and 1 are the bounds. We will show that $\mathfrak{C}^+(\mathbf{X})$ is closed under $\curlywedge$ and $\otimes$. The argument is simple but we produce it nevertheless. Let $x, y \in \mathfrak{C}^+(\mathbf{X})$. We begin with $\otimes$. To see that $x \otimes y$ is positive, we have that $\varepsilon = \varepsilon \otimes \varepsilon \leq x \otimes y$. For idempotence: $x \otimes y \otimes x \otimes y = x \otimes x \otimes y \otimes y = x \otimes y$. To show that $x \otimes y$ is central let $z \in X$, then: $z \otimes x \otimes y = x \otimes z \otimes y = x \otimes y \otimes z$.

Now, for $\curlywedge$ closure. Clearly if each x and y are positive, then $x \curlywedge y$ is positive. For idempotence, in one direction we have: $(x \curlywedge y) \otimes (x \curlywedge y) \leq x \otimes x \curlywedge y \otimes y = x \curlywedge y$ for the other direction we by the fact that $x \curlywedge y$ is positive we have $x \curlywedge y \leq (x \curlywedge y) \otimes (x \curlywedge y)$. For centrality, let $z \in X$, then we have $z \otimes (x \curlywedge y) = (z \otimes x) \curlywedge (z \otimes y) = (x \otimes z) \curlywedge (y \otimes z) = (x \curlywedge y) \otimes z$.

Finally, in order to ensure that $(\mathfrak{C}^+(\mathbf{X}), \curlywedge_X, \otimes_X, 1, \varepsilon_X)$ is a lattice we must check the absorption law holds. But this follows quickly from the fact that ε is bottom.

$\square$

We now prove that the lattice of congruences of a residuated lattice $\mathbf{L}$ is isomorphic to the lattice of positive, central, idempotent elements of the NRL-space $\mathbf{X_L}$. This is demonstrated by noticing that the positive, central, idempotent elements of the NRL-space $\mathbf{X_L}$ a simply the congruence filter of $\mathbf{L}$.

Theorem 8.1.4. *(Representation of Congruences) Let $\mathbf{X_L}$ be an RL-space that is dual to $\mathbf{L}$, then:*

$$Con(\mathbf{L}) \cong \mathfrak{C}^+(\mathbf{X}_L)$$

Proof. Let $\mathbf{X_L}$ be an RL-space dual to L. We will show that $\mathcal{F}i_{Con}(\mathbf{L})$ is equal to $\mathfrak{C}^+(\mathbf{X_L})$ and then infer from Proposition **??** that $Con(\mathbf{L}) \cong \mathfrak{C}^+(\mathbf{X_L})$.

Let is first show that $\mathcal{F}i_{Con}(\mathbf{L}) \subseteq \mathfrak{C}^+(\mathbf{X_L})$. Let $F \in \mathcal{F}i_{Con}(\mathbf{L})$. That F is positive follows directly from the fact that the identity of $\mathbf{L}$, e, is in F. That F is idempotent follows from closure under $\cdot$ and because F is positive. Since F is closed under $\cdot$ we have that $F \otimes F \leq F$. Since F is positive, we have that $F \leq F \otimes F$. To show that F is central, let $x \in \mathbf{X_L}$ and suppose that $a \in x \otimes F$. We will show that $a \in F \otimes x$. If $a \in x \otimes F$, then there are $b \in x$ and $c \in F$ such that $b \cdot c \leq a$. Since F is closed under conjugates, we know that $\rho_b(c) = b \cdot c/b \wedge e \in F$. So we have that $(b \cdot c/b) \cdot b \in F \otimes x$. But $(b \cdot c/b) \cdot b \leq b \cdot c \leq a$, so $a \in F \otimes x$. This shows that $x \otimes F \leq F \otimes x$. The argument for $F \otimes x \leq x \otimes F$ is exactly analogous except relies on the left conjugate λ.

Let us now show that $\mathfrak{C}^+(\mathbf{X}_L) \subseteq \mathcal{F}i_{Con}(\mathbf{L})$. So let $\mathfrak{c} \in \mathfrak{C}^+(\mathbf{X})$. We will show that $\mathfrak{c}$ is a congruence filter of $\mathbf{L}$. Given that $\mathfrak{c}$ is positive, we know $e \in F$. Since $\mathfrak{c}$ is idempotent we know that $\mathfrak{c}$ is closed under $\cdot$. Finally, we must show that $\mathfrak{c}$ is closed under conjugates. So $a \in \mathfrak{c}$ and let $b \in L$. We need to show that $\lambda_b(a), \rho_b(a) \in \mathfrak{c}$. We will show that $\rho_b(a) \in \mathfrak{c}$ by showing $\mathfrak{c} \in \phi(\rho_b(a))$. Recall that $\phi(\rho_b(a)) = \phi((b \cdot a)/b \wedge e) = [\phi(a \cdot b)/_X\phi(b)] \cap \phi(e)$. Since $\mathfrak{c}$ is positive we already have that $\mathfrak{c} \in \phi(e)$. Now suppose that $x \in \phi(b)$. Let us show that $\mathfrak{c} \otimes x \in \phi(a \cdot b)$ and thus showing that $\mathfrak{c} \in \phi(b \cdot a)/_X\phi(b)$. But clearly, if $a \in \mathfrak{c}$, then $b \cdot a \in x \otimes \mathfrak{c}$. And by centrality of $\mathfrak{c}$ then $b \cdot a \in \mathfrak{c} \otimes x$ and thus $\mathfrak{c} \otimes x \in \phi(a \cdot b)$, as desired. It follows that $\mathfrak{c} \in \phi(\rho_b(a))$ and therefore that $\rho_b(a) \in \mathfrak{c}$. An analogous argument shows that $\lambda_b(a) \in \mathfrak{c}$ as well. We may therefore conclude that $\mathfrak{c} \in \mathcal{F}i_{Con}(\mathbf{L})$.

Generalizing we conclude that $\mathcal{F}i_{Con}(L)$ is equal to $\mathfrak{C}^+(\mathbf{X}_L)$ and then by Proposition **??** that $Con(L) \cong \mathfrak{C}^+(\mathbf{X}_L)$. $\square$

We have just demonstrated that the lattice of congruence of any residuated lattices is isomorphic to the lattice of positive, central, idempotents of that residuated lattice's dual RL-space.

Corollary 8.1.5. *Let X be an RL-space: then L_X is s.i. iff $\uparrow(\mathfrak{C}^+(X) - \{\varepsilon_X\})$ is closed.*

Corollary 8.1.6. *Let X be an RL-space: then L_X is simple iff for all $x \in \mathfrak{C}^+(\mathbf{X})$, either $x = \varepsilon$ or $x = 1$.*

The Determination of Subspaces by Members of $\mathfrak{C}^+(\mathbf{X})$

In this section we show that every element $x \in \mathfrak{C}^+(\mathbf{X})$ determines a subspace $Sub(x)$ and conversely that every subspace arises this way. It is worth noting that that we only use topological properties to show that every subspace is of the form $Sub(x)$ for some $x \in \mathfrak{C}^+(\mathbf{X})$. The converse however holds also at the level of frames.

The general definition of subspace has the following definition.

Definition 8.1.7. *Let X be an RL-space. Then a subset Y of X is a subspace of X if Y is an RL-space and if the inclusion $i : Y \to X$ is an injective RL-space morphism.*

We can now show that the positive, central, idempotent elements of any RL-spaces determine the subspaces of that RL-space.

Proposition 8.1.8. *Let (X, τ) be a n RL-space and let $\mathfrak{c} \in \mathfrak{C}^+(\mathbf{X})$. Then $\mathbf{Sub}(\mathfrak{c}) = (Sub(\mathfrak{c}), \sigma)$ where $Sub(\mathfrak{c}) = \{x \in X \mid x \otimes \mathfrak{c} = x\}$ and the topology $\sigma = \{U \cap Sub(\mathfrak{c}) \mid U \in \tau\}$ is a subspace of X.*

Proof. Let $\mathfrak{c} \in \mathfrak{C}^+(\mathbf{X})$. Define $Sub(\mathfrak{c}) := \{x \in X \mid x \otimes \mathfrak{c} = x\}$ the topology $\sigma = \{U \cap Sub(\mathfrak{c}) \mid U \in \tau\}$ on $Sub(\mathfrak{c})$. We show that $Sub(\mathfrak{c})$ is a subspace of X.

First let us show that $\mathbf{Sub}(\mathfrak{c}) = (Sub(\mathfrak{c}), \sigma)$ is an RL-space. That $Sub(\mathfrak{c})$ is closed under $\curlywedge$ and $\otimes$ and contains 1 follows quickly from the definition. For $\curlywedge$ closure, suppose that $x, y \in Sub(\mathfrak{c})$. $(x \curlywedge y) \otimes \mathfrak{c} = (x \otimes \mathfrak{c}) \curlywedge (y \otimes \mathfrak{c}) = x \curlywedge y$. So $x \curlywedge y \in Sub(\mathfrak{c})$. $\otimes$ closure also follows.

For HMS-separation, suppose that $x \not\leq_Y y$. Then $x \not\leq_X y$. Therefore, there is some a clopen filter U of X such that $x \in U$ but $y \notin U$. Since U is clopen in X, $U \cap Y$ is clopen in Y and $x \in U \cap Y$ and $y \notin U \cap Y$. Therefore, HMS-separation holds.

For compactness note that the collection

$$\mathcal{S} := \{U \cap Y \mid U \in \mathcal{F}i_{clp}(\mathbf{X})\} \cup \{Y - (U \cap Y) \mid U \in \mathcal{F}i_{clp}(\mathbf{X})\}$$

forms a subbase for the topology σ. A more or less standard argument then guarantees compactness of $Sub(\mathfrak{c})$.

Clearly, the inclusion $i : Y \to X$ is continuous since for any open U, $i^-(U) = U \cap Sub(\mathfrak{c})$, which is open in σ. It remains to be shown that i has the properties of an RL-space morphism. To show this the map $\mathfrak{c} \otimes - : X \to Sub(\mathfrak{c})$ is left adjoint to the inclusion $i : Sub(\mathfrak{c}) \to X$:

$$\mathfrak{c} \otimes x \leq y \Leftrightarrow x \leq i(y)$$

for every $x \in X$ and $y \in Sub(\mathfrak{c})$. In virtue of this, it is not hard to see that i is an NRL-space morphism. Let us show the back condition for $\curlywedge$ and the back condition for $\backslash$.

($\curlywedge$-back) Suppose that $z \in Sub(\mathfrak{c})$. Let $x, y \in X$ and suppose that $x \curlywedge y \leq i(z)$. Then $\mathfrak{c} \otimes (x \curlywedge y) \leq z$ in virtue of the adjuction property. By idempotence (and centrality) of $\mathfrak{c}$, we have that $\mathfrak{c} \otimes x, \mathfrak{c} \otimes y \in Sub(\mathfrak{c})$. By $\otimes - \curlywedge$ distribution property we have that $\mathfrak{c} \otimes x \curlywedge \mathfrak{c} \otimes y \leq z$. And again in virtue of the adjunction property (or positivity of $\mathfrak{c}$) we have that $x \leq \mathfrak{c} \otimes x$ and $y \leq \mathfrak{c} \otimes y$. Therefore we ahve found $x' := \mathfrak{c} \otimes x$ and $y' := \mathfrak{c} \otimes y$ such that $x' \curlywedge y' \leq z$ and $x \leq i(x')$ and $y \leq i(y')$. Therefore i satisfies the $\curlywedge$-back condition.

($\backslash$-back). Let $z \in Sub(\mathfrak{c})$ and suppose that $x \otimes y \leq z$ for some $x, y \in X$. Define $z' := (\mathfrak{c} \otimes x) \otimes y$. We obtain $z' = (\mathfrak{c} \otimes x) \otimes y = x \otimes (\mathfrak{c} \otimes y) = x \otimes y \leq z$ by centrality and the fact that $y \in Sub(\mathfrak{c})$.

Further, we have that $x \leq \mathfrak{c} \otimes x$. Finally, it is by definition that $(\mathfrak{c} \otimes x) \otimes y \leq z'$. So we have found some $x' := \mathfrak{c} \otimes x$ and z' such that $x' \otimes y \leq z'$ and $x \leq i(x')$ and $i(z') \leq z$ and have obtained the back condition for $\backslash$. The other morphism conditions follow by similar reasoning.

We may therefore conclude that $\mathbf{Sub}(\mathfrak{c})$ is in fact a subspace of X. $\square$

The converse of the above can be shown. Specifically, we will show that all subspaces of an NRL-space can be defined in terms of a positive, central, idempotent element. First we observe the following few lemmas and the definition of a θ-invariant filter. This definition will provide more insight into how to think of $Sub(x)$ from the algebraic side of the duality.

Definition 8.1.9. *(θ-Invariant Filters) Let $\mathbf{L}$ be a residuated lattice and suppose that x is a filter of $\mathbf{L}$. We say that x is θ-invariant if $a \in x$ and $a\theta b$, then $b \in x$.*

We know that for any congruence θ of $\mathbf{L}$ that F_θ is a positive, central, idempotent element in virtue of Theorem 8.1.4. It turns out that $Sub(F_\theta)$ corresponds precisely to the θ-invariant filters of $\mathbf{L}$. While this fact is primarily useful for proving that subspaces are determined by elements of $\mathfrak{C}^+(\mathbf{X})$, we find it interesting in its own right since it characterize an element x of $\mathbf{X}_L$ with the property that $x \otimes F_\theta = x$ in virtue of the internal structure of x.

Lemma 8.1.10. *Let $\mathbf{L}$ be a residuated lattice and let $\theta \in Con(\mathbf{L})$, then:*

$$x \in Sub(F_\theta) \text{ if and only if } x \text{ is } \theta\text{-invariant.}$$

Proof. Let $x \in Sub(F_\theta)$. Suppose that $a \in x$ and $a\theta b$. If $a\theta b$, then $b\backslash a \in F_\theta$ in virtue of the correspondence between congruence-filters and congruences (See remarks below 8.1.1). So $a \cdot a\backslash b \in x \otimes F_\theta = x$. However since $a \cdot a\backslash b \leq b$, we then obtain $b \in x$. Therefore, x is θ-invariant.

Let x be θ-invariant. Let $a \in x \otimes F_\theta$. Then there are $b \in x$ and $c \in F_\theta$ such that $b \cdot c \leq a$. If $c \in F_\theta$, then there is some d such that $e \leq d$ and $c\theta d$. We then obtain $b \cdot c\theta b \cdot d$. However, $x \otimes {\uparrow_{\mathbf{L}}} e = x$ since $x \in \mathbf{X_L}$ and ${\uparrow_{\mathbf{L}}} e = \varepsilon_{\mathbf{X_L}}$. So $b \cdot d \in x \otimes {\uparrow_{\mathbf{L}}} e = x$. However, since we assumed that x was θ-invariant, this implies that $b \cdot c \in x$ and then that $a \in x$ since $b \cdot c \leq a$. We therefore have shown that $x \otimes F_\theta \leq x$. The other in equality follows since $\varepsilon_{\mathbf{X}} \leq f_\theta$. We concldue then that if x be θ-invariant, then $x \in Sub(F_\theta)$.

We have therefore shown that $x \in Sub(F_\theta)$ if and only if x is θ-invariant.

$\square$

The above lemma gives us insight into the nature of the element of $Sub(F_\theta)$. The next lemma uses these insights to bring us closer to the stated goal of showing that every subspace of an RL-spaces is of the form $Sub(\mathfrak{c})$ for some positive, central, idempotent element of $\mathbf{X_L}$.

Lemma 8.1.11. *Let $\mathbf{L}$ be a residuated lattice and suppose that $\theta \in Con(\mathbf{L})$. Then $\mathbf{X}_{\mathbf{L}/\theta} \cong \mathbf{Sub}(F_\theta)$.*

Proof. Define $f : X_{L/\theta} \to Sub(F_\theta)$ such that for each filter $x \in X_{L/\theta}$ of $\mathbf{L}/\theta$, $f(x) = \bigcup x$. To see that f is well defined we must check that fx is a filter of $\mathbf{L}$ and that is an element of $Sub(F_\theta)$. The first part, checking that fx is a filter, is straight forward. To show that $fx \in Sub(F_\theta)$ we note that fx is θ-invariant and then apply Lemma 8.1.10, which implies that if fx is θ-invariant, then $fx \in Sub(F_\theta)$. It follows that f is well defined.

For surjectivity of f, we claim for each $x \in Sub(F_\theta)$ that $x/\theta := {\uparrow}\{a/\theta \mid a \in x\}$ is filter and that $f(x/\theta) = x$. That x/θ is a filter follows from the fact that $\{a/\theta \mid a \in x\}$ is closed under meets. To see this, let $a/\theta, b/\theta \in fx$. Then there are $c, d \in x$ such that $c/\theta \leq a/\theta$ and $d/\theta \leq b/\theta$. It follows from the fact that $c \wedge d \in x$ and $(c \wedge d)/\theta = c/\theta \wedge d/\theta \leq a/\theta \wedge b/\theta$ that $a/\theta \wedge b/\theta \in fx$, as desired.

Now let us show that $f(x/\theta) = x$. Let us show the inclusion $f(x/\theta) \subseteq x$. So let $a \in f(x/\theta)$. Then $a/\theta \in x/\theta$. Therefore, there is some $b \in x$ such that $b/\theta \leq a/\theta$ and thus $a \wedge b\theta b$. Now, by the fact that $x \in Sub(F_\theta)$, we know by lemma 8.1.10 that x is θ-invariant. So $a \wedge b\theta b$ together with $b \in x$ implies that $a \curlywedge b \in x$ and therefore that $a \in x$, as desired. For the other inclusion, if $a \in x$, then $a/\theta \in x/\theta$ and so $a \in f(x/\theta)$. We therefore conclude that $f(x/\theta) = x$.

For injectivity of f suppose that $fx = fy$. We have $a/\theta \in x$ iff $a \in fx$ iff $a \in fy$ iff $a/\theta \in x$. So $x = y$, as desired. We conclude that f is a bijection.

Having concluded that f is a bijection, we must show that f is also an NRL-space morphism. It is sufficient to check that f preserves $\curlywedge$ and $\otimes$ and that $f(\varepsilon_{\mathbf{X}_{\mathbf{L}/\theta}}) = F_\theta$.

That f preserves $\curlywedge$ is almost immediate. Since taking unions is a monotone operation, f is monotone. So $f(x \wedge y) \leq fx \wedge fy$. If $a \in fx \wedge fy$, then $a\theta \in x$ and $a/\theta \in y$ and so $a/\theta \in x \cap y$. Therefore, $a \in f(x \curlywedge y)$.

We now show that f preserves $\otimes$. Let $a \in f(x \otimes y)$. Then there are $[b] \in x$ and $[c] \in y$ such that $[b \cdot c] \leq [a]$. Therefore, $(b \cdot c) \wedge a\theta b \cdot c$. However, if $[b] \in x$ and $[c] \in y$, then $b \cdot c \in fx \otimes fy$. And since $fx \otimes fy \in Sub(F_\theta)$, we obtain from Lemma 8.1.10 that $fx \otimes fy$ is θ-invariant. We then obtain from $(b \cdot c) \wedge a\theta b \cdot c$ that $(b \cdot c) \wedge a \in fx \otimes fy$ and then $a \in fx \otimes fy$, as desired. We have therefore shown that $f(x \otimes y) \leq fx \otimes fy$. The other inequality, $fx \otimes fy \leq f(x \otimes y)$ is straightforward.

We now observe that $f(\varepsilon_{\mathbf{X}_{\mathbf{L}/\theta}}) = F_\theta$. Let $a \in f(\varepsilon_{\mathbf{X}_{\mathbf{L}/\theta}})$. Then $e/\theta \leq a/\theta$. So $e\theta a \wedge e$ and hence $a \wedge e \in F_\theta$. But since F_θ is upward closed, $a \in F_\theta$. Now let $a \in F_\theta$. Then there is some $b \geq e$ such that $a\theta b$. But the $e/\theta \leq b/\theta = a/\theta$. So $a/\theta \in \varepsilon_{\mathbf{X}_{\mathbf{L}/\theta}}$ and hence $a \in f(\varepsilon_{\mathbf{X}_{\mathbf{L}/\theta}})$. We conclude that $f(\varepsilon_{\mathbf{X}_{\mathbf{L}/\theta}}) = F_\theta$.

Finally, we check that f is continuous. So let U be a clopen filter of $\mathbf{Sub}(F_\theta)$. Then there is a clopen filter $\phi_{\mathbf{L}}(a)$ of $\mathbf{X}_{\mathbf{L}}$ such that $Sub(F_\theta) \cap \phi_{\mathbf{L}}(a) = U$. We claim that $f^{-1}[U] = \phi_{\mathbf{L}/\theta}(a/\theta)$. To see this we provide the following chain of equivalences.

$$\begin{aligned} x \in f^{-}[U] &\Longleftrightarrow fx \in U \\ &\Longleftrightarrow fx \in \phi_{\mathbf{L}}(a) \cap Sub(F_\theta) \\ &\Longleftrightarrow a \in fx \ \&\ fx \in Sub(F_\theta) \\ &\Longleftrightarrow a/\theta \in x \ \&\ x \in \mathbf{X}_{\mathbf{L}/\theta} \\ &\Longleftrightarrow x \in \phi_{\mathbf{L}/\theta}(a/\theta). \end{aligned}$$

This concludes the proof that f is a RL-space homeomorphism and we can therefore conclude that $\mathbf{X}_{\mathbf{L}/\theta} \cong \mathbf{Sub}(F_\theta)$.

□

The claim that all subspaces are determined by a member of $C^+(\mathbf{X})$ is a direct consequence of the previous lemma.

Theorem 8.1.12. *Let* $\mathbf{Y}$ *be a sub RL-space of* $\mathbf{X}$*. Then there is some* $\mathfrak{c} \in \mathfrak{C}^+(\mathbf{X})$ *such that* $\mathbf{Sub}(\mathfrak{c}) \cong \mathbf{Y}$.

Proof. Use the first isomorphism theorem and the correspondence between congruences and elements of $\mathfrak{C}^+(\mathbf{X})$. More particularly, if $\mathbf{Y}$ is a sub RL-space of $\mathbf{X}$, then there is an injective NRL-space morphism $f : \mathbf{Y} \to \mathbf{X}$. By duality, $f^{-1} : \mathbf{L}_{\mathbf{X}} \to \mathbf{L}_{\mathbf{Y}}$ is a surject residuated lattice homomorphism.

Let F_f be the congruence filter of $\mathbf{L_X}$ corresponding to the kernel of f. By the preceding lemma, $\mathbf{X_{L_X/Ker(f)}} \cong \mathbf{Sub}(F_f)$. However, by the first isomorphism theorem we have that $\mathbf{L_Y} \cong \mathbf{L_X/Ker(f)}$. Therefore, we may conclude that $\mathbf{Y} \cong \mathbf{X}_{\mathbf{L_X}/ker(f)} \cong \mathbf{Sub}(F_f)$, which demonstrates the statement of this theorem. $\square$

A key insight of the previous theorems is that the elements of $\mathfrak{C}^+(\mathbf{X})$ correspond precisely to the identity elements of the subspaces of X. We now move from congruences to products.

A Remark on Negative Central Idempotents of a residuated lattice

It is known that that for finite residuated lattices the lattice of congruences is isomorphic to the lattice of negative central idempotents ([19], p198). Using our representation of the congruences together with the relation our duality holds to the Π_1-completion, we conjecture that we generalize this result by showing that the lattice of congruences for any residuated lattices is isomorphic to the lattice of closed negative central idempotents of the Π_1-completion. Given that the the Π_1 completion of a finite lattice is itself, this fact can be seen as a direct generalization of the existing result.

8.1.2 The Parameterized Local Deduction Theorem

We now apply some of the theory developed in the previous section on the representation of congruences to give a novel proof of the *Parameterized Local Deduction Theorem* (PLDT) for the logic $\mathbf{FL}^+$. A key step of the proof uses a positive, central, idempotent element to identify a sub-space. Algebraic proof can be found in [18]. A proof theoretic proof can be found in [19] starting on page 122.

It is important to note that in this section and the later section on the disjunction property we will work with a notion of consequence we have yet to discuss. In particular, given an extension $\mathbf{L}$ of $\mathbf{FL}^+$, the consequence relation $\vdash_\mathbf{L}$ relates sets of formulas to a single formula and intuitively characterizes truth preserving reasoning.

Definition 8.1.13. *Let $\Gamma \cup \{\varphi\}$ of formulas and $\mathbf{L}$ be an extension of $\mathbf{FL}^+$. We write $\Gamma \vdash_\mathbf{L} \varphi$ if whenever $t \Rightarrow \psi$ is provable in $\mathbf{L}$ for each $\psi \in \Gamma$, then $t \Rightarrow \varphi$ is provable in $\mathbf{L}$.*

In general the usual deduction theorem does not hold for $\vdash_{\mathbf{FL}^+}$. This reflects the fact that $\vdash_{\mathbf{FL}^+}$ is a more flexible notion of consequence that allows weakening and some degree of commutativity in the sense that $\{\varphi, \psi\}$ proves both $\varphi \bullet \psi$ and $\psi \bullet \varphi$ with respect to $\vdash_{\mathbf{FL}^+}$. However, the generalization of the deduction theorem PLDT does. The PLDT is stated in terms of conjungates, which are the logical analogue of the algebraic notion we used in earlier in the section when studying congruences from the perspective of our duality theory (see remarks preceding Definition 8.1.1). In particular the l*eft and right conjugates* are defined respectively as $\lambda(\varphi, \psi) := (\varphi \backslash (\psi \bullet \varphi)) \wedge t$ and $\rho(\varphi, \psi) := ((\varphi \bullet \psi)/\varphi) \wedge t$. An important property of these schemas is the potential of iterating them to form complex formulas. Given a set of formulas $\varphi_1, .., \varphi_n, \psi$, an *iterated conjugate* is a formula $\gamma(\psi)$ of the form:

$$\gamma_1(\varphi_1, \gamma_2(\varphi_2, ..(\gamma_n(\varphi_n, \psi))..))$$

where γ_i is λ or ρ for each $0 < i \leq n$. We call the formulas $\varphi_1, .., \varphi_n$ parameters. Intuitively, conjugates and iterated conjugates make the formulas they are applied to more flexible with regard to how they combine with certain other formulas. In particular, the permit some degree of commutativity and weakening.

We can then state the PLDT for $\mathbf{FL}^+$ as follows.

$$\Sigma, \Delta \vdash_{\mathbf{FL}^+} \phi \text{ if and only if } \Sigma \vdash_{\mathbf{FL}^+} \Pi_{i=1}^{n} \gamma_i(\psi_i)\backslash\varphi$$

For any collection of formulas $\Sigma \cup \Delta \cup \{\varphi\}$ where $\psi_1, .., \psi_n \in \Delta$ and each γ_i is an iterated conjugate containing parameters in the language $\mathcal{L}_{\mathbf{FL}^+}$. We have written $\Pi_{i=1}^{n}\psi_i$ for the fusion $\psi_1 \bullet ... \bullet \psi_n$. The intuition here is that the relatively inflexible $\backslash$ is only be able encode $\vdash_{\mathbf{FL}^+}$-consequences by exploiting the flexibility of the iterated conjugate formulas.

To prove the PDLT, we will require a few lemmas and definitions. The first lemma we require regards fusions of iterated conjugates. An algebraic version of this is proved in [19].

Lemma 8.1.14. *For any formulas φ and $\psi_1, .., \psi_n$, the following are provable in $\mathbf{FL}^+$:*

$$\Pi_{i\leq n}\lambda(\varphi, \gamma_i(\psi_i)) \Rightarrow \lambda(\varphi, \Pi_{i\leq n}\gamma_i(\psi_i)) \qquad \Pi_{i\leq n}\rho(\varphi, \gamma_i(\psi_i)) \Rightarrow \rho(\varphi, \Pi_{i\leq n}\gamma_i(\psi_i)).$$

Proof. Proof proceeds by an induction on n. When $n = 2$ we have the following proof in which we have assumed some $\mathbf{FL}^+$ provable sequents in order to make the proof more manageable and provide the essential steps. The assumed sequents are preceded by vertical dots to indicate that they are provable.

$$\dfrac{\dfrac{\dfrac{\vdots}{\lambda(\varphi,\psi_1)\bullet\lambda(\varphi,\psi_2) \Rightarrow (\varphi\backslash\psi_1\bullet\varphi)\bullet(\varphi\backslash\psi_1\bullet\varphi)} \quad \dfrac{\vdots}{(\varphi\backslash\psi_1\bullet\varphi)\bullet(\varphi\backslash\psi_1\bullet\varphi) \Rightarrow \varphi\backslash(\psi_1\bullet\psi_2\bullet\varphi)}}{\lambda(\varphi,\psi_1)\bullet\lambda(\varphi,\psi_2) \Rightarrow \varphi\backslash(\psi_1\bullet\psi_2\bullet\varphi)} \quad \dfrac{\vdots}{\lambda(\varphi,\psi_1)\bullet\lambda(\varphi,\psi_2) \Rightarrow t}}{\dfrac{\lambda(\varphi,\psi_1)\bullet\lambda(\varphi,\psi_2) \Rightarrow \varphi\backslash(\psi_1\bullet\psi_2\bullet\varphi)\wedge t}{\lambda(\varphi,\psi_1)\bullet\lambda(\varphi,\psi_2) \Rightarrow \lambda(\varphi,(\psi_1\bullet\psi_2))}}$$

The inductive step for this proof is carried out by a similar proof and the use of associativity of fusion. The case for ρ is symmetric.

$\square$

Now for some definitions. For a set of sentence Δ, we write $\gamma(\Delta)$ to denote the set of finite compositions of λ and ρ with parameters φ_i from the language.

$$\gamma(\Delta) = \{\gamma_1(\varphi_1, \gamma_2(\varphi_2, ..(\gamma_n(\varphi_n, \psi))..)) \mid n \in \omega \ \ \& \ \psi \in \Delta \ \& \ \gamma_i \in \{\lambda, \rho\} \ \& \ \varphi_i \in \mathcal{L}_{\mathbf{FL}}\}$$

Similarly, $\pi(\Delta)$ denotes the closure of Δ under the fusion connective.

$$\pi(\Delta) = \{\psi \mid \varphi_1 \bullet ... \bullet \varphi_n \Rightarrow \psi \text{ is provable in } \mathbf{FL}^+ \ \& \ \varphi_1, .., \varphi_n \in \Delta\}$$

Given a set of formulas Δ, the set $\pi(\gamma(\Delta))$ of formulas has some desirable features.

Lemma 8.1.15. *Let Δ be a set of formulas in the language of $\mathbf{FL}^+$:*
1) if $\varphi \in \pi(\gamma(\Delta))$ and $\varphi \Rightarrow \psi$ is provable in $\mathbf{F}L^+$, then $\psi \in \pi(\gamma(\Delta))$,
3) If φ and ψ are in $\pi(\gamma(\Delta))$, then $\varphi \wedge \psi \in \pi(\gamma(\Delta))$, and
2) $t \in \pi(\gamma(\Delta))$.

Proof. (1) one follows from the cut rule. (3) is a consequences of the fact that if $\varphi \in \Delta$, then $\lambda(t, \varphi) \in \pi(\gamma(\Delta))$. But $\lambda(t, \varphi) \Rightarrow \varphi \wedge t$ is provable in $\mathbf{FL}^+$, so by (1) we have $t \in \pi(\gamma(\Delta))$. Finally, (2) follows from Lemma 8.1.14.

To see this, if $\varphi, \psi \in \pi(\gamma(\Delta))$, then there are $\varphi_1, .., \varphi_n$ and $\psi_1, .., \psi_m$ in Δ such that both

$$\Pi_{i\leq n}\gamma_i(\varphi_i) \Rightarrow \varphi \qquad\qquad \Pi_{i\leq m}\gamma_i'(\psi_i) \Rightarrow \psi$$

are provable in $\mathbf{FL}^+$ when each γ_i and γ_j' have parameters in the $\mathcal{L}_{\mathbf{FL}^+}$. We also note that an easy proof shows that if $\theta_1 \Rightarrow \theta_2$ is provable, then $\lambda(\chi, \theta_1) \Rightarrow \lambda(\chi, \theta_2)$ is provable as well. We then have that

$$\lambda(t, \Pi_{i\leq n}\gamma_i(\varphi_i)) \Rightarrow \lambda(t, \varphi) \qquad\qquad \lambda(t, \Pi_{i\leq m}\gamma_i'(\psi_i)) \Rightarrow \lambda(t, \psi)$$

are both provable in $\mathbf{FL}^+$. However, by Lemma 8.1.14 and the transitivity of $\Rightarrow$ we then obtain that

$$\Pi_{i\leq n}\lambda(t, \gamma_i(\varphi_i)) \Rightarrow \lambda(t, \varphi) \qquad\qquad \Pi_{i\leq m}\lambda(t, \gamma_i'(\psi_i)) \Rightarrow \lambda(t, \psi)$$

are also both provable in $\mathbf{FL}^+$. So the following is also provable.

$$\Pi_{i\leq n}\lambda(t, \gamma_i(\varphi_i)) \bullet \Pi_{i\leq m}\lambda(t, \gamma_i'(\psi_i)) \Rightarrow \lambda(t, \varphi) \bullet \lambda(t, \psi)$$

Now, since $\lambda(t, \varphi) \bullet \lambda(t, \psi) \Rightarrow (\varphi \wedge t) \bullet (\psi \wedge t)$ and $(\varphi \wedge t) \bullet (\psi \wedge t) \Rightarrow \varphi \wedge \psi$ are both prvable, we obtain that

$$\Pi_{i\leq n}\lambda(t, \gamma_i(\varphi_i)) \bullet \Pi_{i\leq m}\lambda(t, \gamma_i'(\psi_i)) \Rightarrow \varphi \wedge \psi$$

is provable in $\mathbf{FL}^+$. So since $\Pi_{i\leq n}\lambda(t, \gamma_i(\varphi_i)) \bullet \Pi_{i\leq m}\lambda(t, \gamma_i'(\psi_i)) \in \pi(\gamma(\Delta))$, we conclude by (1) that $\varphi \wedge \psi \in \pi(\gamma(\Delta))$ as well. $\square$

We now show that for any set of formulas Δ, the set $\pi(\gamma(\Delta))$ is a positive central idempotent of the canonical frame. Positive central idempotents were defined in Definition 8.1.2 and are the idempotent elements above an NRL-spaces identity element that commute with respect to $\otimes$ with all other elements of the space.

Lemma 8.1.16. *Let Δ be a set of formulas in the language of $\mathbf{FL}^+$. Then $\pi(\gamma(\Delta))$ is a positive, central, idempotent element in the canonical (topological) frame $\mathfrak{X}_{\mathbf{FL}^+}$ i.e. $\pi(\gamma(\Delta)) \in \mathfrak{C}^+(\mathfrak{X}_{\mathbf{FL}^+})$.*

Proof. See proof for representation of congruences (Theorem 8.1.4), it is analogous. $\square$

The Importance of the previous lemma is that sets of formulas of the form $\pi(\gamma(\Delta))$ have the potential to determine submodels of the canonical model (See Theorem 8.1.12). We now demonstrate the Parameterized Local Deduction Theorem for $\mathbf{FL}^+$.

Theorem 8.1.17. *For an collection of formulas $\Sigma \cup \Delta \cup \{\varphi\}$:*

$$\Sigma, \Delta \vdash_{\mathbf{FL}^+} \phi \textit{ if and only if } \Sigma \vdash_{\mathbf{FL}^+} \Pi_{i=1}^n \gamma_i(\psi_i)\backslash\varphi$$

Where $\psi_1, .., \psi_n \in \Delta$ such that each γ_i has parameters in the language $\mathcal{L}_{\mathbf{FL}^+}$.

Proof. From left to right: Suppose $\Sigma, \Delta \vdash_{\mathbf{FL}^+} \varphi$. By soundness, we have that $\Sigma, \Delta \vDash_{\mathbf{FL}^+} \varphi$. Suppose also that $t \Rightarrow \theta$ is provable in $\mathbf{FL}^+$ for each element θ of Σ. The set $x_\Delta := \pi(\gamma(\Delta))$ is conjunction closed and is closed under $\Rightarrow$, so is a member of the (topological) canonical model $\mathfrak{M}_{\mathbf{FL}^+}$ of $\mathbf{FL}^+$. It is also the case that $\Delta \subseteq \pi(\gamma(\Delta))$ and, because $t \in \pi(\gamma(\Delta))$, we have $\Sigma \subseteq \pi(\gamma(\Delta))$ because we assumed $t \Rightarrow \theta$ was provable in $\mathbf{FL}^+$ for each element θ of Σ. Therefore, by the truth lemma, $\mathfrak{M}_{\mathbf{FL}^+}, x_\Delta \Vdash \Delta \cup \Sigma$

(Lemma 7.5.2). Now, by Lemma 8.1.16, we also have that $x_\Delta \in \mathfrak{C}^+(\mathfrak{M}_{\mathbf{FL}^+})$. So by Theorem 8.1.8, there is a sub-frame $Sub(x_\Delta)$ of $\mathfrak{M}_{\mathbf{FL}^+}$ and there is also a valuation $V : Prop \to \mathcal{F}i(Sub(x_\Delta))$ defined so that $V(p) = V^{\mathbf{FL}^+}(p) \cap Sub(x_\Delta)$. Clearly, $i^{-1}[V^{\mathbf{FL}^+}(p)] = V^{\mathbf{FL}^+}(p) \cap Sub(x_\Delta) = V(p)$, so the model $\mathfrak{N}_\Delta := (Sub(x_\Delta), V)$ is a well defined submodel of the canonical model $\mathfrak{M}_{\mathbf{FL}^+}$. Further, in virtue of Proposition 7.2.8 applied to the inclusion morphism $i : \mathfrak{N} \to \mathfrak{M}_{\mathbf{FL}^+}$, we obtain the that $\mathfrak{N}_\Delta, x_\Delta \Vdash \Delta \cup \Sigma$. But then since we assumed that $\Sigma, \Delta \vDash_{\mathbf{FL}^+} \varphi$, we obtain $\mathfrak{N}_\Delta x_\Delta \Vdash \varphi$. Therefore, an application of the truth lemma yields that $\varphi \in x_\Delta = \pi(\gamma(\Delta))$. But if $\varphi \in \pi(\gamma(\Delta))$, then there $\psi_1, .., \psi_n \in \Delta$ such that $\Pi_{i=1}^n \gamma_i(\psi_i) \Rightarrow \varphi$ is provable in $\mathbf{FL}^+$. We therefore obtain the conclusion that $t \Rightarrow \Pi_{i=1}^n \gamma_i(\psi_i) \backslash \varphi$ is also provable in $\mathbf{FL}^+$. So we conclude that $\Sigma \vdash_{\mathbf{FL}^+} \Pi_{i=1}^n \gamma_i(\psi_i) \backslash \varphi$ for some $\psi_1, .., \psi_n \in \Delta$ such that each γ_i has parameters in the language $\mathcal{L}_{\mathbf{FL}^+}$

For the other direction we assume $\Sigma \vdash_{\mathbf{FL}^+} \Pi_{i=1}^n \gamma_i(\psi_i) \backslash \varphi$ where $\psi_1, .., \psi_n \in \Delta$ and such that each γ_i has parameters in the language $\mathcal{L}_{\mathbf{FL}^+}$. More generally, we note that for any collections for formulas $\Theta \cup \{\xi\} \cup \{\zeta\}$, if $\Theta \vdash \xi$ and $\Theta \vdash \zeta$, then $\Theta \vdash \xi \bullet \zeta$. Further we note that for any ξ and ζ, we have $\xi \vdash_{\mathbf{FL}^+} \lambda(\xi, \zeta)$ and $\xi \vdash_{\mathbf{FL}^+} \rho(\xi, \zeta)$. Jointly we then obtain $\Delta \vdash_{\mathbf{FL}^+} \Pi_{i=1}^n \gamma_i(\psi_i)$ and thus that $\Sigma, \Delta \vdash_{\mathbf{FL}^+} \varphi$. □

The following section develops a characterization of coproducts in the category of NRL-spaces spaces and uses the NRL-space semantics to provide a new characterization of the disjunction property for extensions of the logic $\mathbf{NFL}^+$.

8.2 Products of NRL-spaces and The Disjunction Property

In this section we first study (co)products of NRL-spaces and then apply our insights to obtain a characterization of when an extension of $\mathbf{NFL}^+$ has the disjunction property.

8.2.1 Products of NRL-spaces

We now characterize the products and coproducts of NRL-spaces. To do so we build on the definition of products of L-spaces and extend the associated results to the case of of NRL-spaces. In particular, in Theorem 8.2.2 we show that finite products of NRL-spaces are again NRL-spaces. We then show that products of NRL-spaces also play the role of coproducts in the category of NRL-spaces (See Theorem 3.1.10). Specifically, we demonstrate that products of NRL-spaces have the universal property of coproducts with respect to the inclusion morphisms introduced in Definition 3.1.1. This is demonstrated in Theorem 8.2.4. Finally, in Theorem 8.2.4 we show that the NRL-space dual to a product of $r\ell$-groupoids is homeomorphic to a (co)product of the NRL-spaces dual to the factors of the salient product.

Now, let us begin by defining the products of NRL-spaces.

Definition 8.2.1. *(Products of NRL-spaces.) Let $\{(X_i, \tau_i)_i\}_{i \in I}$ be a family of NRL-spaces.*

The Product $\prod_{i \in I} \mathbf{X}_i = (X, \tau)$ is defined such that:

1) (X, τ) is the product of the underlying L-spaces i.e. $X = \prod X_i$ of the semilattices defining each $\mathbf{X}_i$ and τ is the product topology,

2) $\otimes_X : X \times X \to X$ is defined pointwise i.e. $\alpha \otimes \beta : I \to \bigcup X_i$ is the function defined such that $\alpha \otimes \beta(i) = \alpha(i) \otimes \beta(i)$, and

3) $\varepsilon_X : I \to \bigcup X_i$ is the function such that $\varepsilon(i) = \varepsilon_{X_i}$.

We recall the projections $p_i : X \to X_i$ and the inclusions $\gamma_i : X_i$. The projections are defined as usual and the inclusions were defined so that $\gamma_i(x)$ is the function $I \to \bigcup X_i$ such that $\gamma_i(x)(j) = 1_{\mathbf{X}_j}$ for all $j \neq i$ and $\gamma_i(x)(j) = x$ if $j = i$. We also remind the reader that in case of products of L-spaces, the subbase $\mathcal{S}_P$ was sufficient to generate the product topology of L-spaces. The same fact holds in the case of NRL-spaces.

$$\begin{aligned}\mathcal{S}_P := \{\textstyle\prod_{i\in I} U_i \mid U_i \in \mathcal{F}i_{clp}(X_i) \;\&\; |\{i \mid U_i \neq X_i\}| < \aleph_0\} \\ \cup \{X - (\textstyle\prod_{i\in I} U_i) \mid U_i \in \mathcal{F}i_{clp}(X_i) \;\&\; |\{i \mid U_i \neq \emptyset\}| < \aleph_0\}\end{aligned}$$

With these remarks in mind, we show that the category of NRL-spaces is closed under finite products. This result builds directly off the analogous fact for L-spaces demonstrated in Theorem 3.1.8.

Proposition 8.2.2. *A finite product of NRL-spaces is an NRL-space.*

Proof. Let $\{(X_i, \tau_i)_i\}_{i\leq n}$ be a finite family of NRL-spaces and let $\prod_{i\leq n} \mathbf{X}_i = (X, \tau)$ be the product topology.

In virtue of Theorem 3.1.8 from chapter 3, $\prod_{i\leq n} \mathbf{X}_i = (X, \tau)$ is an L-space. We then just need to check conditions (1) -(3) in Definition 5.2.1.

Beginning with the first condition, we must show that the set of clopen filters is closed under the operations $\circ_X$, $\backslash_X$ and $/_X$. So suppose that U and V are clopen filters of $\prod_{i\leq n} \mathbf{X}_i$. In again virtue of Lemma 3.1.7 and compactness we can show that any clopen filter is subbasic in $\mathcal{S}_P$. So U and V are subbasic and therefore we have that $U = \prod_{i\leq n} U_i$ and $V = \prod_{i\leq n} V_i$ for some families of clopen filters $\{U_i\}_{i\leq n}$ and $\{V_i\}_{i\leq n}$ where $U_i, V_i \subseteq X_i$ for all $i \in I$.

(Closure under $\circ$) It is the case that $\prod_{i\leq n} U_i \circ \prod_{i\leq n} V_i = \prod_{i\leq n}(U_i \circ V_i)$. To see this, let $\alpha \in \prod_{i\leq n} U_i \circ \prod_{i\leq n} V_i$. Then there are $\beta \in \prod_{i\leq n} U_i$ and $\beta' \in \prod_{i\leq n} V_i$ such that $\beta \otimes \beta' \leq \alpha$. Therefore, for all $i \in I$, $\beta(i) \otimes \beta'(i) \in U_i \circ V_i$ and $\beta(i) \otimes \beta'(i) \leq \alpha(i)$. So it follows that $\alpha(i) \in U_i \circ V_i$ for all $i \in I$ and thus that $\alpha \in \prod_{i\leq n}(U_i \circ V_i)$. For the converse, let $\alpha \in \prod_{i\leq n}(U_i \circ V_i)$. Then for all $i \in I$, $\alpha(i) \in U_i \circ V_i$. It then follows rather $\alpha \in \prod_{i\leq n} U_i \circ \prod_{i\leq n} V_i$.

Now, if t $\prod_{i\leq n} U_i \circ \prod_{i\leq n} V_i = \prod_{i\leq n}(U_i \circ V_i)$, then because $\prod_{i\leq n}(U_i \circ V_i)$ is clopen in the product topology, so is $\prod_{i\leq n} U_i \triangledown \prod_{i\leq n} V_i = U \triangledown V$.

(Closure under $\backslash$ and $/$) Another straight forward argument confirms that $\prod_{i\leq n} U_i \backslash \prod_{i\leq n} V_i = \prod_{i\leq n}(U_i \backslash V_i)$ and $\prod_{i\leq n} U_i / \prod_{i\leq n} V_i = \prod_{i\leq n}(U_i / V_i)$.

Condition (2) of an NRL-space requires that $\uparrow\varepsilon$ is clopen. However, $\uparrow\varepsilon = \prod_{i\leq n} \uparrow\varepsilon(i)$ (again by the definition of the subbase $\mathcal{S}_P$). So since $\uparrow\varepsilon(i)$ is clopen for each $i \in U$, $\uparrow\varepsilon$ is clopen too.

Finally, for condition (3) in Definition 5.2.1, we show the nontrivial direction. Suppose that for all clopen filters U and V of $\prod \mathbf{X}_i$ that if $\beta \in U$ and $\beta' \in V$, then $\alpha \in U \circ V$. We need to show that $\beta \otimes \beta' \leq \alpha$. First we will show that for each $i \leq n$ and each pair of clopen filters U' and V' in X_i that $\beta(i) \in U$ and $\beta'(i) \in V$, we have that $\alpha(i) \in U \circ V$. So let U' and V' be a pair of clopen filters in X_i for some $i \in I$. $\prod_{i\leq n} U_i$ is a clopen filter with $U_i = U'$ and $U_j = X_j$ for each each $j \neq i$. A similarly, $\prod_{i\leq n} V_i$ is a clopen filter when with $V_i = V'$ and $V_j = X_j$ for each each $j \neq i$. By construction, we get that $\beta \in \prod_{i\leq n} U_i$ and $\beta' \in \prod_{i\leq n} V_i$. Then by our assumption we obtain that $\alpha \in \prod_{i\leq n} U_i \circ \prod_{i\leq n} V_i$. Because we have $\prod_{i\leq n} U_i \circ \prod_{i\leq n} V_i = \prod_{i\leq n}(U_i \circ V_i)$, $\alpha(i) \in U' \circ V'$, as desired. It then follows from condition (3) of Definition 5.2.1 that $\beta(i) \otimes \beta'(i) \leq \alpha(i)$ for all $i \in I$. And therefore that $\beta \otimes \beta' \leq \alpha$, as desired. $\square$

Having concluded that finite products of NRL-spaces are NRL-spaces, let us prove a lemma that will support the proof of Theorem 8.2.4, which will show that products of NRLs-spaces have the universal property of co products.

Lemma 8.2.3. *$\{(X,\tau_i)_i\}_{i\in I}$ be a family of NRL-spaces. Let $X = \prod_{i\in I}\mathbf{X}_i$ be the product. Then for all $i \in I$, $\gamma_i : X_i \to X$ is an NRL-space morphism.*

Proof. In virtue of Lemma 3.1.9, we know that γ_i is an L-space morphism, so we just need to check that γ_i meets the additional requirements of being a NRL-space morphism.

The ($\otimes$-forth) condition is immediate since $\gamma_i(x \otimes y) = \gamma_i x \otimes \gamma_i y$. For ($\otimes$-back), let $\alpha \otimes \beta \leq \gamma_i(x)$. Then $\alpha(i) \otimes \beta(i) = \alpha \otimes \beta(i) \leq x$. However, in virtue of the adjunction property of γ_i, $\alpha \leq \gamma_i(\alpha(i))$ and $\beta \leq \gamma_i(\beta(i))$. So generalizing on $\alpha(i)$ and $\beta(i)$, we verify the $\otimes$-back condition.

For ($\backslash$-back), suppose that $\alpha \otimes \gamma_i(x) \leq \beta$. Then $\alpha(i) \otimes x \leq \beta(i)$. By the adjunction property of γ_i, $\alpha \leq \gamma_i(\alpha(i))$. Now, note that for all $j \neq i$, we have $\gamma_i(x)(j) = 1_{\mathbf{X}_j}$. It follows then that for all $j \neq i$, $\alpha(j) \otimes \gamma_i(x)(j) = \alpha(j) \otimes 1_{\mathbf{X}_j} = 1_{\mathbf{X}_j}$, since $1_{\mathbf{X}_j}$ is an absorbing element in any NRL-space. This in turn implies that $\beta(j) = 1_{\mathbf{X}_j}$ for all $j \neq i$ because of the assumption that $\alpha \otimes \gamma_i(x) \leq \beta$. We are then ensured that $\gamma_i(\beta(i)) = \beta$, which is enough for us to conclude that ($\backslash$-back) holds when we generalize on $\alpha(i)$ and $\beta(i)$. A similar argument guarantees that ($/$-back) is a also a property of γ_i.

Finally, we have the conditions (ε-forth) and (ε-back). For (ε-forth) it is by definition that $\varepsilon_{\mathbf{X}} \leq \gamma_i(\varepsilon_{\mathbf{X}_i})$. For ($\varepsilon$-back), if $\varepsilon_{\mathbf{X}} \leq \gamma_i(x)$, then $\varepsilon_{\mathbf{X}_i} = \varepsilon_{\mathbf{X}}(i) \leq \gamma_i(x)(i) = x$.

This concludes the proof that in addition to being an L-space morphism, γ_i is also an NRL-space morphism for each $i \in I$. $\square$

We now arrive the main result of this section. We show that the products of NRL-spaces play the role of co products in the category of NRL-spaces. This result extends Theorem 3.1.10, which showed that products of L-spaces also possess the universal property of coproducts when the inclusions γ_i are taken as the relevant inclusion morphism.

Proposition 8.2.4. *(Universal Property of Coproducts) Let $\{(X,\tau_i)_i\}_{i\leq n}$ and Y be a finite collection of NRL-spaces. Suppose that for each $i \leq n$, there is some NRL-space morphism $g_i : X_i \to Y$. Then there exists a map $g : \prod_{i\leq n} X_i \to Y$ that uniquely satisfies $g(\gamma_i(x)) = g_i(x)$ for all $x \in X_i$ and $i \leq n$.*

Proof. Let $\{\mathbf{X} = (X_i,\tau_i)_i\}_{i\leq n}$ and Y be NRL-spaces. Suppose we have $\mathbf{X} = \prod \mathbf{X}_i$ and that for each $i \leq n$, there is some NRL-space morphism $g_i : X_i \to Y$. Recall that NRL-spaces have all meets in virtue of being L-spaces (Lemma 2.2.13). In Theorem 3.1.10 we defined $g : \prod_{i\leq n} X_i \to Y$ such that:

$$g(\alpha) = \bigwedge_{i\leq n} g_i(\alpha(i)).$$

In virtue of this definition it was immediate that $g(\gamma_i(x)) = g_i(x)$. We also showed that g is an L-space morphism and that g uniquely satisfied the property $g(\gamma_i(x)) = g_i(x)$. We must now check the various conditions for g to be a morphism in the category of NRL-spaces. Specificlly, we just need to check that the conditions for $\otimes$ and ε are met.

For ($\otimes$-forth), we have:

$$g(\alpha) \otimes g(\beta) = \bigwedge_{i\leq n} g_i(\alpha(i)) \otimes \bigwedge_{i\leq n} g_i(\beta(i)) \leq \bigwedge_{i\leq n}(g_i(\alpha(i)) \otimes g_i(\beta(i) \leq \bigwedge_{i\leq n}(g_i(\alpha \otimes \beta(i))) = g(\alpha \otimes \beta).$$

For ($\otimes$-back), the argument for ($\curlywedge$-back) in the proof of Theorem 3.1.10 can be repurposed by substituting $\curlywedge$ for $\otimes$.

Now, for (/-back) we reason as follows. Let $x \otimes g(\alpha) \leq z$. For each $i \leq n$, by /-back for g_i, there are x_i and z_i such that $x \leq g_i(x_i)$ and $g(z_i) \leq x \otimes g_i(\alpha(i))$ and $x_i \otimes \alpha(i) \leq z_i$. Define $\beta, \beta' : n+1 \to \bigcup X_i$ such that for all $i \leq n$, $\beta(i) = x_i$ and $\beta'(i) = z_i$. It is immediate then that $x \leq g(\beta)$ and that $\beta \otimes \alpha \leq \beta'$ since $x_i \otimes \alpha(i) \leq z_i$. To show that $g(\beta') \leq z$. We note that $g(\beta') = \bigcurlywedge_{i \leq n} g_i(z_i) \leq x \otimes g_j(\alpha(j))$ for all $j \leq n$. So we have that $g(\beta') \leq \bigcurlywedge_{i \leq n} x \otimes g_j(\alpha(j))$. However, since n is finite, Lemma 5.2.11 tells us that $\bigcurlywedge_{i \leq n} x \otimes g_j(\alpha(j)) = x \otimes \bigcurlywedge_{i \leq n} g_i(\alpha(i)) = x \otimes g(\alpha)$. We obtain that $g(\beta') \leq x \otimes g(\alpha) \leq z$, as desired. By generalizing on β and β' we conclude that g satisfies the (/-back) condition.

A symmetric argument demonstrates (\-back) holds.

Finally, we consider the conditions (ε-forth) and (ε-back). For (ε-forth), we know by (ε-forth) for each g_i that $\varepsilon_{\mathbf{Y}} \leq g_i(\varepsilon_{\mathbf{X}_i})$. So we have that $\varepsilon_{\mathbf{Y}} \leq \bigcurlywedge_{i \leq n} g_i(\varepsilon_{\mathbf{X}_i}) = g(\varepsilon_{\mathbf{X}})$. For ($\varepsilon$-back), suppose that $\varepsilon_{\mathbf{Y}} \leq g(\alpha)$. Then for all $i \leq n$, $\varepsilon_{\mathbf{Y}} \leq g_i(\alpha(i))$. By ($\varepsilon$-back) for each g_i we obtain that $\varepsilon_{\mathbf{X}_i} \leq \alpha(i)$ for all $i \leq n$. We therefore arrive that the conclusion that $\varepsilon_{\mathbf{X}} \leq \alpha$ and thus that (ε-back) holsd for g.

Having checked these various conditions, we conclude that g is an NRL-space morphism and that the universal property for coproducts holds for the NRL-space $\mathbf{X} = \prod_{i \leq n} \mathbf{X}_i$.

□

We have just shown that products of NRL-spaces play double duty as coproducts in the category of NRL-spaces. This is essentially explained by the semilattice structure of NRL-spaces and thereby the inclusion maps γ_i.

We now show in Theorem 8.2.5 that the NRL-space $\mathbf{X_G}$ dual to a finite product of $r\ell$-groupoids $\mathbf{G} = \prod \mathbf{G}_i$ is essentially the same as the product $\mathbf{X} = \prod \mathbf{X_{G_i}}$ of NRL-spaces dual to each $\mathbf{G}_i$.

Unlike in the case of L-spaces, where we gave a concrete description of the homeomorphism, we will simply use the universal property of coproducts demonstrated in Proposition 8.2.4 to derive our representation. The uniqueness of the isomorphism we obtain together with the fact that there is a forgetful functor back to the category of L-spaces will ensure that the isomorphism has the same concrete description as in Theorem 3.1.11. We therefore will obtain the insight and usefulness of a concrete description, but without all the work.

Theorem 8.2.5. *Let $\{G_i\}_{i \leq n}$ be a finite family of $r\ell$-groupoids. Then:*

$$\mathbf{X}_{\prod_{i \leq n} G_i} \cong \prod_{i \leq n} X_{G_i}$$

Proof. Let $\{G_i\}_{i \leq n}$ be a finite family of $r\ell$-groupoids. Let $\mathbf{G} = \prod_{i \leq n} \mathbf{G}_i$ and let $\mathbf{X} = \prod_{i \leq n} \mathbf{X_{G_i}}$. In virtue of the fact that $\mathbf{RLG}^{op} \cong \mathbf{NRL}$, we know that$\mathbf{X_G}$ satisfies the universal property of coproducts in the category $\mathbf{NRL}$. Standard reasoning then implies that $\mathbf{X_G} \cong \mathbf{X}$.

□

8.2.2 The Disjunction Property

We apply our definition of products of NRL-spaces to provide a characterization of when an extension of the logic $\mathbf{NFL}^{+}$ has the disjunction property. We will then use our characterization to show some particular logics with weakening have the disjunction property. In the setting of substructural logics, the disjunction property can be stated as follows:

Definition 8.2.6. *Let* $\mathbf{L}$ *be an extension of* $\mathbf{NFL}^+$*. We say that* $\mathbf{L}$ *has the disjunction property if:*

$$\textit{(DP)} \qquad \textit{If } \vdash_{\mathbf{L}} \varphi \vee \psi, \textit{ then } \vdash_{\mathbf{L}} \varphi \textit{ or } \vdash_{\mathbf{L}} \psi.$$

Our characterization relies on what we call an ε-prime frame. An ε-prime frame $(X, \curlywedge, 1, \otimes, \varepsilon)$ is one where ε is $\curlywedge$-prime i.e. if $x \curlywedge y \leq \varepsilon$, then $x \leq \varepsilon$ or $y \leq \varepsilon$. With the concept of an ε-prime frame, our semantic characterization of when a logic has the disjunction property is that a Logic L extending $\mathbf{NFL}^+$ has the disjuction property iff the canonical model of L is ε-prime.

Definition 8.2.7. *(ε-Prime Frames and Spaces)*

An OKHD-frame $X = (X, \curlywedge, 1, \otimes, \varepsilon)$ *is ε-prime iff for all* $x, y \in X$*, if* $x \curlywedge y \leq \varepsilon$*, then* $x \leq \varepsilon$ *or* $y \leq \varepsilon$*. An NRL-space* $\mathbf{X}$ *is ε-prime if the underlying OKHD-frame of* $\mathbf{X}$ *is ε-prime.*

Being an ε-prime frame is simply the dual notion to being a well-connected algebra (see [34, 31]). From the logical perspective this fact amounts to the claim that the logic of a ε-prime frame has the disjunction property.

Lemma 8.2.8. *For any valuation* V *on an ε-prime OKHD-frame* $X = (X, \curlywedge, 1, \otimes, \varepsilon)$*, if* $X, V, \varepsilon \Vdash \phi \vee \psi$*, then* $X, V, \varepsilon \Vdash \phi$ *or* $X, V, \varepsilon \Vdash \psi$*.*

Proof. Suppose that $X, V, \varepsilon \Vdash \phi \vee \psi$. Then there are $x, y \in X$ such that $x \curlywedge y \leq \varepsilon$ and $X.V, x \Vdash \phi$ and $X, V, y \Vdash \psi$. By well connectedness, either $x \leq \varepsilon$ or $y \leq \varepsilon$. So since $[\![\phi]\!]$ and $[\![\psi]\!]$ are both upwards closed with respect to $\leq$, we have that either $X, V, \varepsilon \Vdash \phi$ or $X, V, \varepsilon \Vdash \psi$. □

Lemma 8.2.9. *If an extension* $\mathbf{L}$ *of* $\mathbf{NFL}^+$ *has the disjunction property, then any canonical model* $\mathfrak{M}_{\mathbf{L}}$ *of* $\mathbf{L}$ *is ε-prime.*

Proof. Suppose that $\mathbf{L}$ has the disjunction property. We reason indirectly to show that $\mathfrak{M}_{\mathbf{L}}$ is ε-prime. Suppose form $u, v \in X^{\mathbf{L}}$ that neither $u \leq \varepsilon^{\mathbf{L}}$ nor $v \leq \varepsilon^{\mathbf{L}}$. Then there are $\phi \in u$ and $\psi \in v$ such that $\phi\,\psi \notin \varepsilon^{\mathbf{L}}$. This in turn implies that neither $t \Rightarrow \phi$ nor $t \Rightarrow \psi$ are provable in $\mathbf{L}$ and thus by the disjunction property $t \Rightarrow \phi \vee \psi$ is not provable in $\mathbf{L}$. This gives us that $\phi \vee \psi \notin \varepsilon^{\mathbf{L}}$. But $\phi \vee \psi \in u \cap v$ since both $\phi \Rightarrow \phi \vee \psi$ and $\psi \Rightarrow \phi \vee \psi$ are derivable in any extension of $\mathbf{NFL}$. It thus follows that $u \cap v \nsubseteq \varepsilon^{\mathbf{L}}$.

□

With the lemmas we just proved at our disposal, we now move to the main result of this section. The result characterizes exactly when a logic has the disjunction property in terms of a property of the class of frames that corresponds to that logic.

Theorem 8.2.10. *An extensions* $\mathbf{L}$ *of* $\mathbf{NFL}^+$ *has the disjunction property iff for any two NRL-spaces* $\mathbf{X}$ *and* $\mathbf{Y}$ *in the class space for* $\mathbf{L}$*, there is a ε-prime NRL-space* $\mathbf{Z}$ *for* $\mathbf{L}$ *and an embedding* $f : X \times Y \to Z$*.*

Proof. For the direction from right to left, suppose that for any topological two NRLs-paces $\mathbf{X}$ and $\mathbf{Y}$ in the class of space for $\mathbf{L}$, there is a ε-prime NRL-space $\mathbf{Z}$ for $\mathbf{L}$) and an NRL-space embedding $f : X \times Y \to Z$. Now suppose that neither $t \Rightarrow \phi$ nor $t \Rightarrow \psi$ are provable in $\mathbf{L}$. By completeness, there are models $\mathbf{M} = (X, \tau, V)$ and $\mathbf{N} = (Y, \sigma, V')$ and elements x and y of those models such that $\mathbf{M}, x \Vdash t$ and $\mathbf{N}, y \Vdash t$ but $\mathbf{M}, x \nVdash \phi$ and $\mathbf{N}, y \nVdash \psi$. Since $\mathbf{M}, x \Vdash t$ and $\mathbf{N}, y \Vdash t$, we have that $\varepsilon_X \leq x$ and $\varepsilon_y \leq y$. By definition of the direct product of two NRL-spaces, $\varepsilon_{X \times Y} \leq (1_X, y)$ and $\varepsilon_{X \times Y} \leq (x, 1_Y)$. Now, by assumption there is an ε-prime NRL-space $\mathbf{Z}$ for $\mathbf{L}$) and an NRL-space embedding $f : X \times Y \to Z$. So by the definition of an embedding, $\varepsilon_Z \leq f((x, \top_Y))$ and $\varepsilon_Z \leq f(\top_X, y)$. Now, from a frame preservation fact (Lemma 7.2.10) we know know that embeddings reflect validity and therefore preserve refutation. So given that f and the maps $w \mapsto (w, \top_Y)$ and $w \mapsto (\top_X, w)$ are all embeddings, we have that $(Z, V_0), f((x, \top_Y)) \not\Vdash \phi$ and $(Z, V_1), f((\top_X, y)) \not\Vdash \psi$ for some valuations $V_0, V_1 : Prop \to \mathcal{F}i(Z)$. It follows that when we define the valuation V_2 such that $V_2(p) = V_0(p) \cap V_1(p)$ we also have $(Z, V_2), f((x, \top_Y)) \not\Vdash \phi$ and $(Z, V_2), f((\top_X, y)) \not\Vdash \psi$. So because $\varepsilon_Z \leq f((x, \top_Y))$ and $\varepsilon_Z \leq f(\top_X, y)$, we obtain $(Z, V_2), \varepsilon_Z \not\Vdash \phi$ and $(Z, V_2), \varepsilon_Z \not\Vdash \psi$. Finally, putting Lemma 8.2.8 to work, we arrive at the conclusion that $(Z, V_2), \varepsilon_Z \not\Vdash \phi \vee \psi$. By contraposition we have shown the disjunction property for the logic $\mathbf{L}$ in question.

Now for the direction from right to left. Suppose that the logic $\mathbf{L}$ has the disjunction property. Then the canonical frames of $\mathbf{L}$ are all ε-prime given Lemma 8.2.9. But by Lemma 7.5.4 any frame of $\mathbf{L}$ embeds into canonical frame. So in particular, the product of any two frames will embed into a ε-prime frame. Therefore we have shown what we wanted to show. $\square$

The above theorem's proof provides us with the following corollary, which provides a different perspective on when a substructural logic has the disjunction property. Such characterization may be preferred since the one given in the antecedent theorem is just a dualization of the algebraic characterization.

Corollary 8.2.11. *An extension* $\mathbf{L}$ *of* $\mathbf{NFL}$ *has the disjunction property iff the canonical models of the logic are* ε*-prime.*

Proof. If a logic $\mathbf{L}$ extending $\mathbf{NFL}^+$ has the disjunction property, then by Lemma 8.2.8, the canonical models of $\mathbf{L}$ are all ε-prime. Conversely, Assume that the canonical models of $\mathbf{L}$ are all ε-prime. by Lemma 7.5.4, the product of any frames will embed into a canonical model. Therefore, by Theorem 8.2.10, we obtain that $\mathbf{L}$ has the disjunction property. $\square$

The Disjunction Property in Logics with Weakening

Let us denote an extension of $\mathbf{NFL}^+$ with weakening by $\mathbf{NFL}^+_w$. Based on the results in the previous section and in particular Theorem 8.2.10 an easy method for showing an extension $\mathbf{L}$ of $\mathbf{NFL}^+_w$ presents itself. In particular, we may simply add a single point to the bottom of the product of two topological frames and then ensure that the new point behaves like an identity for $\otimes$. This is similar to the method used in modal logic where the disjoint union of two frames is given a new root.

Lemma 8.2.12. *Let* $\mathbf{W}$ *be the class of NRL-space satisfying the conditions*

$$\forall xy(x \leq x \otimes y) \qquad \forall xy(x \leq x \otimes y)$$

then if X *in* $\mathbf{K}$*, then the NRL-space* $\mathbf{X}^* = (\{*\} \oplus \mathbf{X}, \tau^*)$ *is in* $\mathbf{K}$ *where* $\{*\} \oplus \mathbf{X} = (\{*\} \uplus X, \curlywedge^*, 1^*, \otimes^*, \varepsilon^*)$ *is defined such that* $1^* = 1$, $\varepsilon^* = *$, *and*

$$y \curlywedge^* x = \begin{cases} x \curlywedge y \text{ if } x, y \in X \\ *, \text{ if either } x = * \text{ or } y = * \end{cases} \qquad y \otimes^* x = \begin{cases} x \otimes y \text{ if } x, y \in X \\ x, \text{ if } y = *, \\ y, \text{ if } x = *. \end{cases}$$

Finally, we have that τ^* is the topology generated by the subbase where $(-)^c$ is calculated in $\{*\} \oplus X$:

$$\mathcal{S}^* = \mathcal{F}i_{clp}(\mathbf{X}) \cup \{U^c \mid U \in \mathcal{F}i_{clp}(\mathbf{X})\} \cup \{\uparrow *\} \cup \{(\uparrow *)^c\}.$$

Proof. We must check that for all $x, y \in \{*\} \oplus X$ that $x \leq x \otimes^* y$ and $x \leq y \otimes^* x$ and $x \otimes^* \varepsilon^* = x = \varepsilon^* \otimes^* x$. But these are all almost immediate from the definition of $\otimes^*$ and ε^* given above.

Let us now observe that $\mathbf{X}^*$ is an NRL-space. $\mathbf{X}^*$ is 0-dimensional in virtue of the definition of the subbase $\mathcal{S}^*$.

For compactness, let $\{U_i\}_{i \in I} \subseteq \mathcal{S}^*$ be a sub basic cover of $\mathbf{X}^*$. We now have two cases to consider. In the case that there is some $i \in I$, $U_i = \uparrow *$, then trivially $\{\uparrow *\}$ is itself a cover of $\mathbf{X}$. If on the other hand there is no $i \in I$ such that $U_i = \uparrow *$, then observe that the collection $S^* \cap X = \{W \cap X \mid W \in \mathcal{F}i_{clp}(\mathbf{X})\}$ is a subbase for $\mathbf{X}$ since it contains all clopen filters of $\mathbf{X}$ and their complements. So $\{U_i \cap X\}_{i \in I}$ is a subbasic cover of $\mathbf{X}$. By compactness of $\mathbf{X}$, there is a finite $J \subseteq I$ such that $\{U_j \cap X\}_{j \in J}$ is a cover $\mathbf{X}$. Further, since there must be some i such that $* \in U_i$, this U_i must be equal to $(\{*\} \uplus X) - V$ for some $V \in \mathcal{F}i_{clp}(\mathbf{X})$. We then have that the following collection $\{U_j\}_{j \in J} \cup \{U_i\}$ is finite subcover of $\{U_i\}_{i \in I}$.

For HMS-separation, let $x \not\leq y$. If $x, y \in X$, then since HMS-separation holds in $\mathbf{X}$ and the fact all clopen filters of $\mathbf{X}$ are clopen filters in $\mathbf{X}^*$, we are done. If either $x = *$ or $y = *$, since X is a clopen filter we are done.

We now must check that the clopen filters are closed under the operations $\triangledown$, $\circ$, $\backslash$, and $/$. First note there is only one new clopen filter in $\mathbf{X}^*$, namely $\uparrow *$. So we just need to consider the cases where we are applying the operations in question to $\uparrow *$. With this in mind, closure under $\triangledown$ is straightforward since for any clopen filter U, $\uparrow * \triangledown U = \uparrow *$. In contrast, $\uparrow * \circ U = U = U \circ \uparrow *$ for all clopen filter U, so again we are ensured that the closure condition holds. A simple argument shows for any clopen filter U both $U \backslash \uparrow *$ and $\uparrow * /U$ are equal to $\uparrow *$ since $\uparrow *$ is the greatest clopen filter. In contrast, for any clopen filter U both $\uparrow * \backslash U$ and $U/\uparrow *$ are equal to U by applying the fact that $\uparrow *$ is the greatest clopen filter and the axioms $\forall xy(x \leq x \otimes y)$ and $\forall xy(x \leq x \otimes y)$. The clopen filters are therefore closed under the operations $\triangledown$, $\circ$, $\backslash$, and $/$.

Finally, we must check that if for all clopen filters U, V: if $x \in U$ and $y \in V$, then $z \in U \circ V$, then $x \otimes y \leq z$. Suppose that none of x, y, or z are $*$. Then $x, y, z \in X$ and so we can apply the relevant condition in $\mathbf{X}$ to obtain that $x \otimes y \leq z$. If either $x = *$, then $x \otimes y = y$. So suppose that $y \not\leq z$ for contradiction. Then by HMS-separation there is some clopen filter U such that $y \in U$ and $z \notin U$. But $x \in \uparrow *$, so by assumption that for all clopen filters U, V: if $x \in U$ and $y \in V$, then $z \in U \circ V$, we obtain that $z \in \uparrow * \circ U \subseteq U$, contradicting that we had $z \notin U$. It follows that $x \otimes y = y \leq y$, as desired. Roughly the same argument holds when $y = *$. Finally, if $z = *$, then it is not hard to see that $x = *$ and $y = *$ must also be the case and so we are again in one of the above cases.

We therefore conclude that $\mathbf{X}^*$ is an NRL-space. $\square$

Lemma 8.2.13. *Let $\mathbf{W}$ be as in the previous lemma. If $\mathbf{X} \in \mathbf{W}$, then $\{*\} \cup X$ is ε-prime and $\mathbf{X}$ embeds into $\mathbf{X}^*$.*

Proof. For ε-primeness, note that ε^* has exactly one element that covers it, namely ε_X[1]. So, if $x \curlywedge y \leq \varepsilon^*$, then either $x = \varepsilon^*$ or $y = \varepsilon^*$.

For the embedding, note that the identity function will do. The fact the the indentity satisfies all the morphism conditions isn't hard to see and follows quickly. In the case of ($\curlywedge$-back), slightly more

[1] x covers y iff $y < x$ and there is no z such that $y < z < x$.

care must be taken. the identity is continuous in virtue of our definition of the subbase, $\mathcal{S}^*$. $\square$

Proposition 8.2.14. *The logic* $\mathbf{NFL}_w^+$ *has the disjunction property.*

Proof. $\mathbf{W}$ is the class of NRL-space that corresponds to the logic $\mathbf{NFL}_w^+$. By Theorem 8.2.10, if the product of any NRL-spaces for $\mathbf{L}$ embeds into an ε-prime NRL-space for $\mathbf{L}$, then $\mathbf{L}$ has the disjunction property. So by Lemma 8.2.13, which tells us every NRL-space in $\mathbf{W}$ embeds into an ε-prime NRL-space, we conclude that $\mathbf{NFL}_w^+$ has the disjunction property. $\square$

We give one more particular instance of a logic with weakening and the disjunction property. In the section on canoncity at the end of Chapter 7, we remarked on the logic $\mathbf{L_{BCC}}$ considered by Ono and Komori [32], which was essentially characterized as intutitionistic logic without contraction or alternatively, as $\mathbf{FL}^+$ with commutativity and weakening.

Proposition 8.2.15. $\mathbf{L_{BCC}}$ *has the disjunction property.*

Proof. We need to check that operation $\otimes$ of $\{*\} \oplus X$ is associative and commutative whenever X is. For commutativity, let $x, y \in \{*\} \oplus X$. We just need to consider the case when $x = *$ since X is assumed to be commutative. But then clearly, $x \otimes y = y = y \otimes x$. Another very simple argument shows that $\otimes$ is associative. $\square$

This concludes the section on the disjunction property.

8.3 Conclusion

In this chapter we expanded the theory of NRL-spaces. First we proved a representation theorem for the congruences of residuated latticesin their dual RL-spaces (Theorem 8.1.4) and used this to obtain a characterization of subspaces of RL-spaces (Theorem 8.1.12). Second, we characterized products and coproducts of NRL-spaces (Theorem 8.2.4) and then gave a representation theorem for products of $r\ell$-groupoids (Theomem 8.2.5). We then applied these both of these developments to obtain logic results like a new proof of the parameterized local deduction theorem (Theorem **??**) and a characterization of when an extensions of $\mathbf{NFL}^+$ has the disjunction property (Theorem 8.2.10) and then gave a few specific examples of such logics (Propositions 8.2.14 and 8.2.15).

Conclusion

In this thesis we developed a novel topological duality for $r\ell$-groupoids and showed how this duality connects the theory of L-spaces developed by Bezhanishvili et al. [6] to the frame based semantics for substructural logics originating in the work of Ono and Komori, Humberstone [23], and Dosen [13]. We began in Chapter 3 and Chapter 4 by extending the representation theorems and Π_1 preservation results from Bezhanishvili et al. [6] to all monotone lattice expansions. In Chapter 5 we introduced NRL-spaces, RML-spaces, OKHD-frames, and RML-frames and established a number of connections between these types of objects. Most importantly we showed how OKHD-frames can be obtain from the general theory outlined in Chapter 3. In Chapter 6, we proved duality for the class of residuated lattices with respect to both NRL-spaces and RML-spaces. We then showed how these results restricted to obtain dualities for a number of other classes of residuated lattices. In Chapter 7 we reviewed substructural logic and in particular the OKHD-semantics. We then used our duality to define a topological semantics for substurctural logics and proved a general completeness theorem. Afterwards, we adapted the notion on Π_1-persistence to the setting of the OKHD-semantics and used duaity to show a general completeness theorem with respect to OKHD-frames that subsumes existing results. We ended the chapter with a discussion of how earlier canonical model style proofs could be understood in terms of duality and Π_1-persistence. Finally, in Chapter 8, we extend the theory of NRL-spaces. First we obtained a dual representation of congruences of residuated lattices and a characterization of subspaces. The we characterized products and coproducts of NRL-spaces and showed how this leads to a representation for products of $r\ell$-groupoids. We then applied the development to first obtain a new proof of the parameterized local deduction theorem and then a characterization of logics with the disjunction property. There are many places the theory presented in this thesis could be extended.

- Generalizations of the representation of monotone operations and monotone lattice expansions (Theorem 3.2.6 and Corollary 3.2.7) to include antitone operations as well.
- A characterization of identities on lattices expansions with antitione operations that are preserved by the Π_1-completion.
- A deeper study of the connection between the Π_1-completion, the Σ_1-completion, and the canonical extension from the point of view of L-spaces.
- A generalization of the characterization of classes of $r\ell$-groupoids closed under the Π_1-completion (Theorem 6.4.5) and there by a generization of our completeness theorem with respect to the OKHD-semantics (Theorem 7.4.8).
- The identification of interesting applications of the topological semantic to substructural logics.

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