Combinatorial Properties of the Raisonnier Filter

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

In 1970, Solovay [21] produced a model of $\mathsf{ZF} + \mathsf{DC}$ in which all sets of reals are Lebesgue measurable. In order to achieve this, he worked in ZFC with the assumption that an inaccessible cardinal exists (I). In 1984, Shelah [20] proved that this cannot be achieved without assuming the existence of an inaccessible cardinal, by showing that the theories $\mathsf{ZFC} + \mathsf{I}$ and $\mathsf{ZF} + \mathsf{DC} +$ *"every set of reals is Lebesgue measurable"* are equiconsistent. His main theorem states that if every $\mathbf{\Sigma}^1_3$ set is Lebesgue measurable, then $\aleph_1$ is an inaccessible cardinal in Gödel's constructible universe L. In the same year, Raisonnier [18] gave a simpler proof of this fact using a construction now known as the "Raisonnier filter". This filter remained underutilized outside of Raisonnier's proof.

This thesis is an investigation into the properties of the Raisonnier filter. First, we prove some basic facts about the Raisonnier filter constructed starting from various subsets of $\mathbf{2}^\omega$. Second, we generalize Raisonnier's proof method and obtain a converse to this generalized statement. This result is significant, as it suggests that the Raisonnier filter is strongly related to Lebesgue measurability and can thus not be used to obtain results about other regularity properties. Third, we obtain a new characterization of $\mathbf{\Sigma}^1_2$ Lebesgue measurability through the Raisonnier filter. This characterization is also related to the concept of "Laver measurability" introduced by Brendle and Löwe in [7]. Finally, we define a new ideal $\mathcal{R}$ consisting of the subsets of $\mathbf{2}^\omega$ for which the Raisonnier filter is rapid and we establish relationships between its cardinal characteristics and those of the ideal of null sets.

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Contents

Introduction

The study of well behaved sets is one of the most important aspects of the theory of topological spaces. The properties indicating that a set is well behaved are called *regularity properties*. Two notable examples are *Lebesgue measurability* and the *property of Baire*. In the three-dimensional space $\mathbb{R}^3$, the Lebesgue measure corresponds to our intuitive notion of the volume of a set. So, the Lebesgue measurable sets are those whose volume can be measured. The property of Baire expresses the fact that a set can be suitably approximated by simple sets.

These regularity properties are of great interest to the field of *descriptive set theory*, which is the study of definable sets of real numbers and their properties. A precursor to this area of research were Cantor's attempts to prove the Continuum Hypothesis by examining different sets of reals. Its origins, however, lie in the work of the French analysts Borel, Baire and Lebesgue, who were investigating the notions of definable functions and definable sets at the turn of the 20th century. Further foundational work was carried out in the following decades by Suslin, Lusin and Sierpiński.

Already in 1905, Vitali [23] constructed sets of real numbers that are not Lebesgue measurable and do not have the property of Baire. In 1924, Banach and Tarski [2] published their famous paradox, in which a ball in three-dimensional space can be decomposed into two balls identical to the original, through functions that intuitively ought to preserve volume. This decomposition made use of nonmeasurable sets, indicating that the possibility of such sets existing may lead to paradoxical results.

These results were formulated in Zermelo-Fraenkel set theory (ZF) with the Axiom of Choice (AC). In both cases, the construction of nonmeasurable sets relied on AC. As a result, Banach and Tarski's counter-intuitive result led to criticism towards the Axiom of Choice [24, p. 217]. This naturally led to the following question.

> *Is the Axiom of Choice necessary to construct counterexamples to Lebesgue measurability and the property of Baire?*

If the answer was negative, this could be proven by constructing such a set in ZF, without using the Axiom of Choice. To answer this question positively though, one would have to provide a model of ZF where every set of reals is Lebesgue measurable and has the property of Baire.

In 1970, Solovay [21] answered the above question positively, by defining a model of ZF along with Dependent Choice (DC), a weaker version of AC, in which every set of reals is Lebesgue Measurable and has the property of Baire. This proved that the Axiom of Choice is necessary in the construction of any counterexample to these properties.

In order to achieve this, though, he did not assume only the axioms of ZF or ZFC, but also *the existence of an inaccessible cardinal* (denoted as I). Inaccessible cardinals are a type of large cardinal and can be used to define models of ZFC. It follows by Gödel's Second Incompleteness Theorem that their existence cannot be proven in ZFC. As a result, Solovay's starting point of $\mathsf{ZFC} + \mathsf{I}$ is strictly stronger than ZFC. A new question thus arises.

> *Is it possible to construct a model of* $\mathsf{ZF} + \mathsf{DC}$ *in which every set of reals is Lebesgue measurable and has the property of Baire, without assuming the existence of an inaccessible cardinal?*

This was answered in 1984 by Shelah [20], with a surprising result: although mathematicians had considered the two properties to be similar and closely connected[1], this was not the case here. Using Cohen's technique of forcing, Shelah proved that from any model of ZFC we can obtain a model of $\mathsf{ZF} + \mathsf{DC}$ where every set of reals has the Baire property. On the other hand, under the assumption of $\mathsf{ZF} + \mathsf{DC}$ and that every set of real numbers is Lebesgue measurable, he proved that there exists a model of $\mathsf{ZFC}+\mathsf{I}$. Because of this, the theories $\mathsf{ZFC}+\mathsf{I}$ and $\mathsf{ZF}+\mathsf{DC}+$ "every set of real numbers is Lebesgue measurable" are *equiconsistent*. This meant that the existence of an inaccessible cardinal is necessary to obtain a model without nonmeasurable sets, but not necessary to obtain a model where every set has the property of Baire.

In addition to the above results, Solovay and Shelah obtain corresponding results in regard to ZFC instead of $\mathsf{ZF}+\mathsf{DC}$, by considering only definable sets of reals. These are the sets in the *projective hierarchy*, which are categorized into levels $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$, according to their complexity. In this context, the results by Solovay and Shelah give us the following statements in ZFC (see [11, Thm. 26.14] and [20, Thm. 5.1]):

Theorem (Solovay)**.** *Assuming the existence of an inaccessible cardinal, there exists a model of* ZFC *in which every projective set is Lebesgue measurable.*

Theorem (Shelah)**.** *If every* $\mathbf{\Sigma}^1_3$ *set of real numbers is Lebesgue measurable, then there exists a model of* $\mathsf{ZFC} + \mathsf{I}$*.*

It is natural to investigate whether analogous results can be obtained for other regularity properties. While this has been achieved for some properties, this remains a well-known open problem for the *property of Ramsey* (see [11, p. 524]). Whereas it can be proven that every projective set has the property of Ramsey in Solovay's model, it is yet unknown whether such a model can be constructed without assuming the existence of an inaccessible cardinal.

[1] Shelah in [20] writes that "the Baire property was considered the 'little sister' " of Lebesgue measurability.

An attempt to provide an answer could include generalizing or adapting Shelah's proof to the property of Ramsey. This proof, however, is technical and, as it relies on forcing, it uses metamathematical arguments. Raisonnier [18] provided a proof of Shelah's Theorem that is similarly technical, but does not make use of forcing. Bartoszyński and Judah then gave a more modern and intuitive version of Raisonnier's proof in their book [5]. The focal point of this proof is the construction a special set, now known as the *Raisonnier filter*.

Filters are collections of sets that can be considered large in some sense. Some filters can be viewed as sets of real numbers and we can therefore ask whether they satisfy various regularity properties. *Rapid* filters are notable for being nonmeasurable and not having the property of Baire. The main objective of Raisonnier's proof is showing that the Raisonnier filter is rapid and thus nonmeasurable. Despite its usefulness in this proof, the filter has remained underutilized in other contexts.

This thesis is an investigation of the Raisonnier filter and its applications. Moreover, it contains a generalization of Raisonnier's proof and connections between the Raisonnier filter and Lebesgue measurability. It is divided into five chapters.

In the first chapter, we provide an account of the necessary background, using the relevant material from various sources. This includes the aforementioned regularity properties, filters, and inaccessible cardinals, as well as Gödel's constructible universe and the cardinal characteristics of the continuum. For this, we assume familiarity with basic set theory and some elementary notions of topology.

In the second chapter, we give a detailed exposition of Bartoszyński and Judah's version of Raisonnier's proof, which can be found in their book [5]. We begin with the definition of the Raisonnier filter and its basic properties, while explaining the intuition behind them. We then describe the relation between Lebesgue measurability and the concept of *slaloms* in the Baire space ω^ω, which is established in [5]. Finally, we present Bartoszyński and Judah's method of deriving that Raisonnier's filter is rapid, under the assumption of $\mathbf{\Sigma}^1_2$ Lebesgue measurability.

In the third chapter, we investigate the properties of the Raisonnier filter $\mathcal{F}_X$ under various assumptions about the set $X \subseteq \mathbf{2}^\omega$ from which it is generated. We relate the Raisonnier filter generated by arbitrary sets to those generated by their subsets and unions. We also give explicit descriptions of the Raisonnier filter generated by some specific types of sets. Most notably, we prove that the Raisonnier filter of any open set is equal to Fréchet's filter.

In the fourth chapter, we first introduce the notion of *binary slaloms*, an analogue of slaloms for the Cantor space $\mathbf{2}^\omega$, and define suitable property-preserving translations nat_d and bin_d between $\mathbf{2}^\omega$ and ω^ω. We then use the binary slaloms to obtain a generalization of Raisonnier's method. Moreover, we prove the converse of this generalized statement, which leads to the following necessary and sufficient condition on a set X, for its Raisonnier filter $\mathcal{F}_X$ to be rapid.

Theorem 4.13. *The following are equivalent for every $X \subseteq \mathbf{2}^\omega$:*

1. *$\mathcal{F}_X$ is rapid,*

2. *for every $d \in \mathcal{P}$, X goes through a d-binary slalom, and*

3. *for every $d \in \mathcal{P}$, $\mathrm{nat}_d[X]$ goes through a slalom.*

In the fifth and final chapter, we use the maps defined in the fourth chapter, as well as Theorem 4.12 to establish some results about Lebesgue measurability. We introduce the ideal $\mathcal{R}$, defined as the collection of sets $X \subseteq \mathbf{2}^\omega$ for which $\mathcal{F}_X$ is rapid. We first establish a new characterization of the statement *"every $\mathbf{\Sigma}^1_2$ set of reals is Lebesgue measurable"*, and relate it to the notion of *Laver measurability* introduced by Brendle and Löwe in [7]. We then compare the cardinal characteristics of $\mathcal{R}$ to those of the ideal $\mathcal{N}$ of null sets. The main new results we obtain are the following cardinal relationships:

Theorem 5.9. $\mathrm{cof}(\mathcal{N}) \leq \max\{\mathrm{cov}(\mathcal{R}), \mathfrak{d}\}$.

Theorem 5.12. $\mathrm{add}(\mathcal{N}) = \min\{\mathrm{non}(\mathcal{R}), \mathfrak{b}\}$.

Chapter 1

Preliminaries

1.1 Notation

We use mostly standard notation for basic set-theoretic notions. We write Ord for the class of all ordinals. The set of natural numbers, identified as the first infinite ordinal, is denoted ω. Every $n \in \omega$ is also an ordinal and is identified with the set $\{k \in \omega \mid k < n\}$. So, we will often write $\mathbf{2}$ for the set $\{0, 1\}$. We will identify cardinal numbers with the corresponding initial ordinals in general.

A sequence of length α from a set X is defined as any function $s : \alpha \to X$. We denote the length of a sequence s by $|s|$. Given a set X and an ordinal α, the set of sequences from X of length α is denoted by X^α. The set of sequences from a set X of length less than α is defined by $X^{<\alpha}$. Given a set X, the set of its countably infinite subsets is denoted as $[X]^\omega$ and the set of its finite subsets is denoted as $[X]^{<\omega}$.

Given sequences $s : \alpha \to X$ and $t : \beta \to X$, where α and β are ordinals, we say that s *is an initial segment of* t and write $s \sqsubseteq t$ if and only if $\alpha \leq \beta$ and $s(\gamma) = t(\gamma)$ for all $\gamma \leq \alpha$. We say that s is a *strict* initial segment of t and write $s \sqsubset t$ if and only if $s \sqsubseteq t$ and $s \neq t$. Given two (finite) sequences s, t, we denote their concatenation as $s^\frown t$.

Given a sequence $s : \alpha \to X$, we will write $s{\restriction}I$ for the restriction of the sequence to the set $I \subseteq \alpha$. Usually, I will be an interval of α or some $n \in \omega$. Formally, $s{\restriction}I : I \to X$ is a function from $I \subseteq \alpha$ to X, but we will usually treat it as a sequence of length $|I|$.

We will also identify elements of $\mathbf{2}^\omega$ with subsets of ω, through the corresponding characteristic functions. Therefore, a real $x \in \mathbf{2}^\omega$ will be identified with the set $\{k \in \omega \mid x(k) = 1\}$ and a subset $s \subseteq \omega$ will be identified with its characteristic function in $\mathbf{2}^\omega$. In this sense, we will often write $x \cap n$ for the set $\{k \in x \mid k < n\}$. We will also sometimes identify finite sequences $s \in \mathbf{2}^{<\omega}$ with finite subsets of ω in the same way. For this reason, we will write $\varnothing$ for the empty set, as well as the empty sequence.

In addition to our usual first-order quantifiers, we will use $\exists^\infty$ to mean "there exist infinitely many" and $\forall^\infty$ to mean "for all but finitely many". We naturally have the equivalences

1. $\exists^\infty n \in \omega\ \varphi \iff \forall m \in \omega\ \exists n \geq m\ \varphi$,
2. $\forall^\infty n \in \omega\ \varphi \iff \exists m \in \omega\ \forall n \geq m\ \varphi$,
3. $\neg\exists^\infty x\ \varphi \iff \forall^\infty x\ \neg\varphi$ and $\neg\forall^\infty x\ \varphi \iff \exists^\infty x\ \neg\varphi$.

Given two elements f, g of ω^ω or $\mathbf{2}^\omega$, we have that $f = g$ if and only if $f(n) = g(n)$ for all $n \in \omega$. We also define the relation $\leq$ on ω^ω as $f \leq g$ if and only if $f(n) \leq g(n)$ for all $n \in \omega$. We say that f and g are *eventually equal* and write $f =^* g$, if and only if $f(n) = g(n)$ for all but finitely many $n \in \omega$. For any set $X \subseteq \mathbf{2}^\omega$ or $X \subseteq \omega^\omega$ and a relation R, we will write

$$[X]_R = \{y \in \mathbf{2}^\omega \mid \exists x \in X (yRx)\}.$$

We write $[x]_R$ for $[\{x\}]_R$. With this notation, $[X]_{=^*}$ is the set of reals that are eventually equal to some $x \in X$.

1.2 Models of Set Theory

We will now cover some basic notions regarding models of set theory. The language of set theory is that of first-order logic along with a binary predicate $\in$ expressing membership in a set. For this reason, any structure (M, E), where M is a domain and E is a binary relation, can be a model of set theory. We can define such structures while working in a theory such as ZFC. We will only work with models (M, E) where M is a class and E is the membership relation $\in$ between elements of M. Because of this, we will simply write M for the model (M, E). In this case, it is useful that the class M be *transitive*.

Definition 1.1. A class M is transitive if and only if for every $x \in M$ we have that $x \subseteq M$.

When a class M is a set, then it is possible to formally define the satisfaction relation $M \models \varphi$ within ZFC. This is not possible however for proper classes. In this case, we can only express that a proper class model satisfies a formula through *relativization*.

Definition 1.2. Let M be a class and φ be a formula of set theory. The *relativization of φ to M* is the formula φ^M defined inductively as

1. $(x \in y)^M \equiv x \in y$,
2. $(x = y)^M \equiv x = y$,
3. $(\neg\varphi)^M \equiv \neg\varphi^M$,
4. $(\varphi \wedge \chi)^M \equiv \varphi^M \wedge \chi^M$, and

5. $(\exists x\, \varphi)^M \equiv \exists x \in M\, \varphi^M$,

with the cases for $\vee$ and $\forall$ defined similarly.

The relativized formula φ^M expresses the fact that φ holds in the class M. Thus, determining whether a proper class model M satisfies a formula φ is examining whether the statement φ^M is true in our theory. We write this as $M \models \varphi$. If φ has free variables, we assume that they range over M. Similarly, if Φ is a collection of formulas in the language of set theory, the statement that M is a model of Φ, written as $M \models \Phi$, is the (metamathematical) statement that $M \models \varphi$ for every $\varphi \in \Phi$.

It is possible to define models $M \models \mathsf{ZFC}$ when working in ZFC. In this case, M cannot be a set, as this would mean that the notion $M \models \varphi$ would be definable for every formula of set theory within ZFC. This in turn would allow us to express the statement $\forall \varphi \in \mathsf{ZFC}\, M \models \varphi$ within ZFC, proving its consistency and contradicting Gödel's Second Incompleteness Theorem. Instead, such models are proper classes and $M \models \mathsf{ZFC}$ is a statement on the metamathematical level, which is not provable —or in fact expressible— in ZFC. In the following, we will say that a proper class M is a model of a theory T if and only if $M \models T$, that is if and only if φ^M is provable in our theory for every $\varphi \in T$.

A model $M \models \mathsf{ZFC}$, will satisfy every statement of ZFC in the sense mentioned above. This means that it will contain objects that the theory of ZFC necessitates. For example, in ZFC there exists the set $\aleph_1$, which is the smallest uncountable cardinal. The model M must also contain a set x for which "x is the smallest uncountable cardinal"^{M} holds, which we write as $\aleph_1^M$. In general, if x denotes a set or object with a particular definition, we write x^M for the object that satisfies the same definition in M.

It may be the case that $x^M \neq x$ for some definable object x. In our example above, $\aleph_1$ is defined as the smallest ordinal α such that no bijection between α and ω exists. If our model M contained an ordinal $\beta < \alpha$, but no bijection between it and ω, then it would be that $\aleph_1^M \neq \aleph_1$. Fortunately, this is not the case for many basic notions in set theory, which are *absolute* for transitive models.

Definition 1.3. Let φ be a formula in the language of set theory and let M be a model of set theory. We say that φ is *absolute for M* if and only if the statement $\varphi^M \leftrightarrow \varphi$ holds.

In other words, if an object in the universe V is defined by a formula φ which is absolute for M, then the same object occupies the same role in the model M. Many simple notions, such as the basic set-theoretic operations, natural numbers, ordinals and ω, are absolute for transitive models [11, pp. 163-165].

1.3 Descriptive Set Theory

Descriptive set theory is the study of definable sets of real numbers. Its origins lie in the work of Borel, Baire and Lebesgue in the turn of the 20th century. Soon after, it emerged as a distinct area of research and was advanced through the work of Luzin, Suslin and Sierpiński. In the 1930s, Gödel would provide some independence results in the field. His introduction of recursive functions would lead to the work of Kleene and Mostowski, who would provide an effective version of descriptive set theory in the following decades, connecting the field to logic.

In this section, we will provide a brief account of the basic notions relevant to this work, based on sections from the books [11] and [13]. An exposition of the basics of descriptive set theory can be found in the standard textbook [17]. Brief historical accounts of the development of the field can also be found in [17] and [13].

1.3.1 The Topological Perspective

Modern descriptive set theory concerns itself with complete separable metric spaces without isolated points. The spaces with the first three properties are called *Polish*, due to the origin of the mathematicians who first studied them, while the spaces with the fourth property are called *perfect*. The real line $\mathbb{R}$ is such a space and was the first to be studied extensively. However, formulating the theory in general terms leads to a wider applicability of its results. Other perfect Polish spaces are often used for convenience, as most of the properties that are of interest to descriptive set theory do not depend on the particular choice of the underlying space. Two such spaces are the *Baire* space ω^ω of infinite sequences of natural numbers and the *Cantor* space $\mathbf{2}^\omega$ of infinite sequences from the set $\mathbf{2} = \{0, 1\}$. Elements of these spaces are thus also called *real numbers* or *reals*.

The topology on ω^ω is formulated in terms of sets of reals with common initial segments. For a finite sequence $s \in \omega^{<\omega}$ we define the set

$$[s] = \{x \in \omega^\omega \mid s \sqsubset x\}$$

of all reals that have s as an initial segment. The longer s is, the more reals are excluded from $[s]$ and so the "smaller" we consider the set s to be. We define the basis of the topology on ω^ω to be the set $\{[s] \mid s \in \omega^{<\omega}\}$. The topology on $\mathbf{2}^\omega$ is defined analogously, with the basic open sets $[s]$ being determined by $s \in \mathbf{2}^{<\omega}$.

Open sets are defined in the standard way, as countable unions of basic open sets, while closed sets are defined as their complements. However, there is another characterization of closed sets, through *trees*.

Definition 1.4. A set $T \subseteq \omega^{<\omega}$ is called a tree if and only if it is closed under initial segments, that is, if $s \in T$ and $t \sqsubseteq s$, then $t \in T$. Given a tree T we define the set of *infinite paths* through T as

$$[T] = \{x \in \omega^\omega \mid s \in T \text{ for all } s \sqsubset x\}.$$

An element s of a tree T is called *maximal* if and only if there exists no $t \in T$ such that $s \sqsubset t$. The following proposition gives us a close connection between closed sets and trees.

Proposition 1.5 ([13, Prop. 12.10])**.** *For every tree T, $[T]$ is a closed subset of ω^ω, and every closed $X \subseteq \omega^\omega$ can be written as $X = [T]$ for some tree T without maximal elements.*

We will also sometimes refer to the following topological notions.

Definition 1.6. Let $X \subseteq \mathbf{2}^\omega$ (or $X \subseteq \omega^\omega$). The *interior of* X, denoted by $\mathrm{int}(X)$, is defined as the largest open subset of X. The *closure of* X, denoted by $\mathrm{cl}(X)$, is defined as the smallest closed superset of X.

The interior of a set X corresponds to those points $x \in X$ for which there exists a basic open set $O \subseteq X$ such that $x \in O$. Because X contains not only these points, but also a small "area" around them, represented by the basic open set, we think of them as being firmly "inside" of X. Dually, we think of the closure of a set as including it, along with the points that lie closest to it.

For any Polish space, the *Borel hierarchy* is defined through iterated applications of countable unions and intersections of sets in lower levels. In particular:

1. The class $\mathbf{\Sigma}^0_1$ is the collection of open sets.
2. The class $\mathbf{\Pi}^0_1$ is the collection of closed sets.
3. For any ordinal α, $\mathbf{\Sigma}^0_\alpha$ is the collection of countable unions $\bigcup_{n<\omega} X_n$, where each X_n is $\mathbf{\Pi}^0_\beta$ for some $\beta < \alpha$.
4. For any ordinal α, $\mathbf{\Pi}^0_\alpha$ is the collection of countable intersections $\bigcap_{n<\omega} X_n$, where each X_n is $\mathbf{\Sigma}^0_\beta$ for some $\beta < \alpha$. Equivalently, $\mathbf{\Pi}^0_\alpha$ is the collection of complements of $\mathbf{\Sigma}^0_\alpha$ sets.
5. For any ordinal α, $\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha$.

The *Borel sets* are the class $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha = \bigcup_{\alpha<\omega_1} \mathbf{\Pi}^0_\alpha$. Due to historical reasons, $\mathbf{\Sigma}^0_2$ sets are often called F_σ and $\mathbf{\Pi}^0_2$ sets are often called G_δ. The term "hierarchy" is warranted for the collection of these classes, as the following strict inclusions hold for uncountable Polish spaces.

Proposition 1.7 ([11, pp. 140-141])**.** *For every $\alpha, \beta \in \mathrm{Ord}$ with $\alpha < \beta$, $\mathbf{\Sigma}^0_\alpha \subset \mathbf{\Sigma}^0_\beta$, $\mathbf{\Sigma}^0_\alpha \subset \mathbf{\Pi}^0_\beta$, $\mathbf{\Pi}^0_\alpha \subset \mathbf{\Sigma}^0_\beta$, and $\mathbf{\Pi}^0_\alpha \subset \mathbf{\Pi}^0_\beta$.*

We can define another class of increasingly complex sets using the notion of a *projection* of a set.

Definition 1.8. Let $\mathcal{X}$ and $\mathcal{Y}$ be Polish spaces and $X \subseteq \mathcal{X} \times \mathcal{Y}$. We define the projection of X as the set

$$\operatorname{proj}(X) = \{x \in \mathcal{X} \mid \exists y \in \mathcal{Y}((x, y) \in X)\}$$

We call the projections of Borel sets *analytic sets.* Using this, we define the *projective hierarchy* as follows:

1. The class $\mathbf{\Sigma}^1_1$ is the collection of analytic sets.
2. For any $n < \omega$, $\mathbf{\Sigma}^1_{n+1}$ is the collection of projections of $\mathbf{\Pi}^1_n$ sets.
3. For any $n < \omega$, $\mathbf{\Pi}^1_n$ is the collection of complements of $\mathbf{\Sigma}^1_n$ sets.
4. For any $n < \omega$, $\mathbf{\Delta}^1_n = \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n$.

As with the Borel hierarchy, we have the following strict inclusions.

Proposition 1.9 ([11, pp. 144-145])**.** *For every $m, n \in \omega$ with $m < n$ we have that $\mathbf{\Sigma}^1_m \subset \mathbf{\Sigma}^1_n$, $\mathbf{\Sigma}^1_m \subset \mathbf{\Pi}^1_n$, $\mathbf{\Pi}^1_m \subset \mathbf{\Sigma}^1_n$, and $\mathbf{\Pi}^1_m \subset \mathbf{\Pi}^1_n$.*

$$\begin{tikzcd}
\vdots & & \vdots & & \vdots & & \vdots \\
\mathbf{\Sigma}^0_3 & & \mathbf{\Pi}^0_3 & & \mathbf{\Sigma}^1_3 & & \mathbf{\Pi}^1_3 \\
& \mathbf{\Delta}^0_3 \arrow[ul, phantom, sloped, "\supset"] \arrow[ur, phantom, sloped, "\subset"] & & & & \mathbf{\Delta}^1_3 \arrow[ul, phantom, sloped, "\supset"] \arrow[ur, phantom, sloped, "\subset"] & \\
\mathbf{\Sigma}^0_2 \arrow[ur, phantom, sloped, "\subset"] & & \mathbf{\Pi}^0_2 \arrow[ul, phantom, sloped, "\supset"] & & \mathbf{\Sigma}^1_2 \arrow[ur, phantom, sloped, "\subset"] & & \mathbf{\Pi}^1_2 \arrow[ul, phantom, sloped, "\supset"] \\
& \mathbf{\Delta}^0_2 \arrow[ul, phantom, sloped, "\supset"] \arrow[ur, phantom, sloped, "\subset"] & & & & \mathbf{\Delta}^1_2 \arrow[ul, phantom, sloped, "\supset"] \arrow[ur, phantom, sloped, "\subset"] & \\
\mathbf{\Sigma}^0_1 \arrow[ur, phantom, sloped, "\subset"] & & \mathbf{\Pi}^0_1 \arrow[ul, phantom, sloped, "\supset"] & & \mathbf{\Sigma}^1_1 \arrow[ur, phantom, sloped, "\subset"] & & \mathbf{\Pi}^1_1 \arrow[ul, phantom, sloped, "\supset"]
\end{tikzcd}$$

Figure 1.1: The Borel and projective hierarchies.

1.3.2 The Logical Perspective

Whereas the classical theory was defined in terms of topology, connections to logic were established through the development of *effective descriptive set theory.* The origins of this field of research lie in Kleene and Mostowski's investigations of definable relations on ω [13, pp. 151–152]. Addison would then establish the relation between the classical and effective theories and introduce the standard notation for the hierarchies in [1].

The effective theory in ω^ω concerns the relations that are definable in *second order arithmetic.* We can define this as the two-sorted structure:

$$\mathcal{A}^2 = (\omega, \omega^\omega, \mathrm{ap}, +, \times, \exp, <, 0, 1)$$

The two domains are ω and ω^ω, connected through the application function function ap : $\omega^\omega \times \omega \to \omega$, where $\mathrm{ap}(f, n) = f(n)$ for all $f \in \omega^\omega$ and $n \in \omega$. The constants $0, 1$ represent the corresponding natural numbers in ω and the functions $+$, $\times$ and exp are the usual arithmetical functions in ω.

Our language contains two types of variables. The first-order variables v_i^0 are evaluated as natural numbers and the second-order variables v_0^1 are valuated as reals from ω^ω. To each type of variable corresponds a type of quantifier. The first-order quantifiers $\exists^0$ and $\forall^0$ correspond to quantification over ω and the second-order quantifiers $\exists^1$ and $\forall^1$ correspond to quantification over ω^ω. The language also formally contains symbols for the functions and constants of $\mathcal{A}^2$. However, we will mostly write formulas of the language informally when their meaning is clear.

The above gives us a notion of definability for relations on $\omega^p \times (\omega^\omega)^k$ in general.

Definition 1.10. Let $p, k \in \omega$. We say that a relation $R \subseteq \omega^p \times (\omega^\omega)^k$ is *definable in second-order arithmetic by a formula* $\varphi(v_1^0, \ldots, v_p^0, v_1^1, \ldots, v_k^1)$ if and only if

$$R = \{(n_1, \ldots, n_p, f_1, \ldots, f_k) \mid \mathcal{A}^2 \models \varphi[n_1, \ldots, n_p, f_1, \ldots, f_k]\}$$

For the sake of simplicity, when writing formulas we will often use the symbol of a relation definable in second-order arithmetic in place of the formula defining it.

An example of a class of simple relations are those definable through *bounded quantifiers*. These are quantifiers of the form $\exists^0 v(v < \tau \wedge \varphi)$ and $\forall^0 v(v < \tau \to \varphi)$, which are written as $(\exists^0 v < \tau)\varphi$ and $(\forall^0 v < \tau)\varphi$ respectively. We say that a relation is Δ_0^0 if and only if it is definable in second-order arithmetic by a formula whose quantifiers are all bounded. Several useful encodings, such as encodings of finite sequences and real numbers, are Δ_0^0.

With this as a starting point, we can define the *arithmetical* and *analytical hierarchies*.

Definition 1.11 (Arithmetical Hierarchy)**.** For every $n < \omega$, we define the following classes of subsets of $\omega^p \times (\omega^\omega)^k$:

1. $R \in \Sigma_n^0 \iff R = \{\vec{w} \mid \mathcal{A}^2 \models \exists^0 v_1 \forall^0 v_2 \ldots Q_n v_n S(\vec{w}, v_1, v_2, \ldots v_n)\}$
2. $R \in \Pi_n^0 \iff R = \{\vec{w} \mid \mathcal{A}^2 \models \forall^0 v_1 \exists^0 v_2 \ldots Q'_n v_n S(\vec{w}, v_1, v_2, \ldots v_n)\}$
3. $\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$

where S is Δ_0^0, Q_n is $\exists^0$ if n is odd and $\forall^0$ if n is even and Q'_n is $\forall^0$ if n is odd and $\exists^0$ if n is even.

A relation is called *arithmetical* if and only if it is definable in second-order arithmetic by a function without real number quantifiers. Through the use of equivalences and Δ_0^0 encodings, it is possible to "pull out" quantifiers and replace consecutive quantifiers of the same kind with a single quantifier. As a result, every arithmetical relation can be defined by an Σ^0_n or Π^0_n formula and thus lies at some level of the arithmetical hierarchy. These encodings allow us to encode finite sequences of natural numbers into a natural number and countably many reals from ω^ω into one.

Definition 1.12 (Encodings)**.** Let $(p_i)_{i<\omega}$ denote the sequence of prime numbers. We define the encoding of a finite sequence $(n_0, \dots, n_k)$ of natural numbers as $\langle n_0, \dots, n_k \rangle = p_0^{n_0+1} p_1^{n_1+1} \dots p_k^{n_k+1}$. Given an $x \in \omega^\omega$, we define $(x)_i \in \omega^\omega$ as the real for which $(x)_i(n) = x(\langle i, n \rangle)$ for all $n \in \omega$ and $i < \omega$.

The class Σ^0_1 corresponds to the computably enumerable relations and the class Π^0_1 corresponds to their complements. Naturally, the class Δ^0_1 corresponds to the computable relations. By setting $\Sigma^1_0 = \Sigma^0_1$ and $\Pi^1_0 = \Pi^0_1$, we can define the analytical hierarchy.

Definition 1.13 (Analytical Hierarchy)**.** For every $n < \omega$ with $n > 0$, we define the following classes of subsets of $\omega^p \times (\omega^\omega)^k$:

1. $R \in \Sigma^1_n \iff R = \{\vec{w} \mid \mathcal{A}^2 \models \exists^1 v_1 \forall^1 v_2 \dots Q_n v_n S(\vec{w}, v_1, v_2, \dots v_n)\}$
2. $R \in \Pi^1_n \iff R = \{\vec{w} \mid \mathcal{A}^2 \models \forall^1 v_1 \exists^1 v_2 \dots Q'_n v_n S(\vec{w}, v_1, v_2, \dots v_n)\}$
3. $\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$

where S is arithmetical, Q_n is $\exists^1$ if n is odd and $\forall^1$ if n is even and Q'_n is $\forall^1$ if n is odd and $\exists^1$ if n is even.

Similarly to the arithmetical hierarchy, it can be shown that every relation definable in second-order arithmetic can be defined by a Σ^1_n or Π^1_n formula. The levels of the arithmetical and analytical hierarchies are related through strict inclusions as in Propositions 1.7 and 1.9.

The connection between these hierarchies and the Borel and projective hierarchies is established through *relativization*. Given an $a \in \omega^\omega$, we define *second-order arithmetic in a* to be the structure

$$\mathcal{A}^2(a) = (\omega, \omega^\omega, \mathrm{ap}, +, \times, \exp, <, 0, 1, a)$$

In this case, we have access to a real a as a constant when defining sets. So, by expanding the above notions and definitions to second-order arithmetic in a we obtain the relativized notions of *arithmetical in a*, $\Sigma^0_n(a)$, $\Pi^0_n(a)$, $\Delta^0_n(a)$, $\Sigma^1_n(a)$, $\Pi^1_n(a)$ and $\Delta^1_n(a)$. Using the encodings from Definition 1.12, we can encode countably many reals into one, and so we can write $\Sigma^0_n(a_0, a_1, \dots)$ to mean $\Sigma^0_n(a)$ for a real a that encodes every real a_i. The same applies to the rest of the complexity classes.

Through relativization, we get the following result which shows that the arithmetical hierarchy corresponds to the finite levels of the Borel hierarchy, while the analytical hierarchy corresponds to the projective hierarchy.

Theorem 1.14 ([13, pp.156-157])**.** *For every $n < \omega$ we have that*

1. $\boldsymbol{\Sigma}^0_n = \bigcup_{a \in \omega^\omega} \Sigma^0_n(a)$ *and* $\boldsymbol{\Pi}^0_n = \bigcup_{a \in \omega^\omega} \Pi^0_n(a)$,
2. $\boldsymbol{\Sigma}^1_n = \bigcup_{a \in \omega^\omega} \Sigma^1_n(a)$ *and* $\boldsymbol{\Pi}^1_n = \bigcup_{a \in \omega^\omega} \Pi^1_n(a)$.

1.3.3 Regularity Properties

One of the central topics of study in descriptive set theory is investigating certain properties that indicate that a set is regular in some sense. The notion of *Lebesgue measurability* is among the most important such regularity properties.

A *measure* is a function that assigns a real number in $\mathbb{R}$ (or ∞) to some subsets of a space and satisfies certain properties. A measure offers a notion of size for subsets of a space. The subsets of a space for which a measure is defined are called *measurable*.

We can think of the Lebesgue measure μ as generalizing the notions of area and volume. The definition of the Lebesgue measure for the spaces $\mathbf{2}^\omega$ and ω^ω is similar to that for the real line. We begin by defining the Lebesgue measure of basic open sets. In the space ω^ω, we do so recursively. First, we let $\mu([\varnothing]) = 1$. Having defined $\mu([s])$ for some $s \in \omega^{<\omega}$, we let $\mu([s^\frown n]) = \frac{1}{2^{n+1}}\,\mu([s])$. In $\mathbf{2}^\omega$ we define $\mu([s]) = 2^{-|s|}$ for all $s \in \mathbf{2}^{<\omega}$.

Any open set X can be written as $X = \bigcup_{s\in S}[s]$, where $[s] \cap [t] = \varnothing$ for all $s, t \in S$ such that $s \neq t$. We define

$$\mu(X) = \sum_{s\in S} \mu([s])\,.$$

Using this, for any subset X of $\mathbf{2}^\omega$ or ω^ω we define its *outer measure* μ^* to be $\mu^*(X) = \inf\{\mu(Y) \mid Y \text{ is open and } Y \supseteq X\}$. This leads to the following important notion that will occur frequently.

Definition 1.15. A set $X \subseteq \mathbf{2}^\omega$ (respectively $X \subseteq \omega^\omega$) is called *null* if and only if $\mu^*(X) = 0$.

Finally, we have the following standard definition of Lebesgue measurability.

Definition 1.16. A set $X \subseteq \mathbf{2}^\omega$ (respectively $X \subseteq \omega^\omega$) is called Lebesgue measurable if and only if for all $Y \subseteq \mathbf{2}^\omega$ (respectively $Y \subseteq \omega^\omega$),

$$\mu^*(X) = \mu^*(Y \cap X) + \mu^*(Y - X)$$

In this case, we simply say that the Lebesgue measure of X is $\mu(X) = \mu^*(X)$.

The *property of Baire* is another regularity property of interest to descriptive set theory. It essentially states that a set of real numbers can be approximated by an open set. Given two sets X, Y we define their *symmetric difference* $X \triangle Y$ as the set $(X - Y) \cup (Y - X)$. The symmetric difference of two sets contains the elements that are in X but not in Y and vice versa. For this reason, if it a small set, then we can view X and Y as being similar, or almost equal in a sense. We then define a particular notion of small set.

Definition 1.17. A set $X \subseteq \mathbf{2}^\omega$ (respectively $X \subseteq \omega^\omega$) is called *nowhere dense* if and only if $\mathrm{cl}(X)$ contains no open set. A set $X \subseteq \mathbf{2}^\omega$ (respectively $X \subseteq \omega^\omega$) is called *meagre* if and only if it is a countable union of nowhere dense sets.

A nowhere dense set is intuitively small. For its closure to contain no open set, it must not contain any basic open set $[s]$ either, no matter how "small" it is. Therefore, meagre sets, which are only countable unions of such sets are also thought of as being small. Using this notion, we can define what it means for a set to be approximated by an open set.

Definition 1.18. Let $X \subseteq \mathbf{2}^\omega$ (respectively $X \subseteq \omega^\omega$). We say that X *has the property of Baire* if and only if there exists an open $B \subseteq \mathbf{2}^\omega$ (respectively $B \subseteq \omega^\omega$) such that $X \triangle B$ is meagre.

Note that, because $X \triangle \varnothing = X$ for every set X, meagre sets have a meagre symmetric difference with $\varnothing$ and so always have the property of Baire.

The final propery we will mention is the *property of Ramsey*, which states that a subset of $\mathbf{2}^\omega$ either includes or avoids all infinite subsets of an infinite $a \in \mathbf{2}^\omega$. We have the following definition.

Definition 1.19. Let $X \subseteq [\omega]^\omega$. We say that an infinite $a \subseteq \omega$ is *homogeneous for* X if and only if either $[a]^\omega \subseteq X$ or $X \cap [a]^\omega = \varnothing$. A set $X \subseteq [\omega]^\omega$ *has the property of Ramsey* if and only if there exists an infinite $a \subseteq \omega$ that is homogeneous for X.

This property is of interest because it is still an open question whether we can obtain a model of ZFC where all projective sets satisfy it, without assuming the existence of an inaccessible cardinal. Further information on these results, as well as connections between the property of Ramsey and other concepts, can be found in Chapter 26 of [11].

Using the Axiom of Choice, it is easy to construct sets of reals that do not have the above properties. As we will see later, such counterexamples can also be obtained in the form of certain types of *filters on* ω, which are special subsets of $\mathbf{2}^\omega$.

1.4 Cardinal Characteristics of the Continuum

Another topic related to the spaces ω^ω and $\mathbf{2}^\omega$ is the study of their combinatorial properties. Central to this are the various *cardinal characteristics of the continuum.* These are cardinal numbers related to the structure of the space of real numbers and can consistently take values between $\aleph_0$ and $2^{\aleph_0}$. Some of them are related to *ideals* of sets with particular regularity properties. An ideal is a collection of sets that we consider to be small in some sense.

Definition 1.20. Let X be a non-empty set. A set $\mathcal{I} \subseteq \mathcal{P}(X)$ is called an *ideal on* X if and only if

1. $\varnothing \in \mathcal{I}$,
2. if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$, and
3. if $A, B \subseteq \mathcal{I}$, $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.

An ideal $\mathcal{I}$ is called *trivial* if and only if $X \in \mathcal{I}$.

These properties capture the notion of a collection of "small" sets: the empty set should always be considered small and the binary unions of small sets, as well as their subsets, should also be considered small. It is clear that, since ideals are closed under binary unions, they are closed under finite unions in general. Sometimes, we may ask that they are also closed under unions of more than finitely many sets.

Definition 1.21. An ideal $\mathcal{I}$ is called *κ-complete* if and only if for every collection $\mathcal{A} \subseteq \mathcal{I}$ of size less than κ we have that $\bigcup \mathcal{A} \in \mathcal{I}$. Ideals that are $\aleph_1$-complete are also called σ-complete.

Because countable unions of countable sets are countable, the collection $\mathcal{I}_{\text{ctbl}} = \{X \subseteq \omega^\omega \mid |X| \leq \aleph_0\}$ is an ideal. Let $\mathcal{N}$ be the collection of null sets and $\mathcal{M}$ be the collection of meagre sets. Similarly, it is easy to see that the countable union of null sets is always null and the countable union of meagre sets is always meagre. Therefore, $\mathcal{N}$ and $\mathcal{M}$ are also σ-complete ideals.

Definition 1.22. Let $\mathcal{I}$ be an ideal on ω^ω. We define the following cardinal coefficients.

1. The *additivity of* $\mathcal{I}$, $\operatorname{add}(\mathcal{I}) = \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I}, \bigcup \mathcal{A} \notin \mathcal{I}\}$, is the least amount of sets from $\mathcal{I}$ whose union is not in $\mathcal{I}$.

2. The *covering number of* $\mathcal{I}$, $\operatorname{cov}(\mathcal{I}) = \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I}, \bigcup \mathcal{A} = \omega^\omega\}$, is the least amount of sets from $\mathcal{I}$ needed to cover the entire space.

3. The *uniformity of* $\mathcal{I}$, $\operatorname{non}(\mathcal{I}) = \min\{|X| \mid X \notin \mathcal{I}\}$, is the least size of a set not in $\mathcal{I}$.

4. The *cofinality of* $\mathcal{I}$, $\operatorname{cof}(\mathcal{I}) = \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I}, \forall X \in \mathcal{I}\ \exists Y \in \mathcal{A}\ X \subseteq Y\}$, is the least size of a collection $\mathcal{A} \subseteq \mathcal{I}$ such that every set of $\mathcal{I}$ is contained in some set from $\mathcal{A}$.

It can easily be proven that $\operatorname{add}(\mathcal{I}) \leq \operatorname{cov}(\mathcal{I}) \leq \operatorname{cof}(\mathcal{I})$, and $\operatorname{add}(\mathcal{I}) \leq \operatorname{non}(\mathcal{I}) \leq \operatorname{cof}(\mathcal{I})$ for any σ-complete ideal $\mathcal{I}$ [5, p. 12]. Some of the cardinal characteristics of the continuum that are of interest include the above cardinal coefficients for the ideals $\mathcal{N}$ and $\mathcal{M}$. Although the above definition is for the space ω^ω, it is possible to define the above coefficients for the space $\mathbf{2}^\omega$ as well; in fact, doing so does not give different values [5, p. 16].

Other cardinal characteristics are defined as attributes of the space ω^ω. Two examples are the *bounding* and *dominating numbers*, defined through the *domination* relation.

Definition 1.23. Given $f, g \in \omega^\omega$, we say that g *dominates* f and write $f \leq^* g$, if and only if $f(n) \leq g(n)$ for all but finitely many $n \in \omega$. We say that a set $X \subseteq \omega^\omega$ *is dominated by a real* $d \in \omega^\omega$ if and only if $f \leq^* d$ for all $f \in X$.

In accordance with the notation introduced in Section 1.1, we will write $[d]_{\leq^*}$ for the set of reals dominated by d. Trivially, for every $d \in \omega^\omega$ we can find reals not dominated by it. As a result, the space ω^ω cannot be dominated by a single real. It is however worth investigating how big a subset $X \subseteq \omega^\omega$ needs to be so that it cannot be dominated by a real and how many reals we need to dominate the entire space. These questions lead to the following definitions.

Definition 1.24. A subset $F \subseteq \omega^\omega$ is called

1. an *unbounded family* if and only if for every $g \in \omega^\omega$ there exists an $f \in F$ such that $f \not\leq^* g$, and

2. a *dominating family* if and only if for all $g \in \omega^\omega$ there exists an $f \in F$ such that $g \leq^* f$.

We call $\mathfrak{b} = \min\{|F| \mid F \text{ is an unbounded family}\}$ the *bounding number* and $\mathfrak{d} = \min\{|F| \mid F \text{ is a dominating family}\}$ the *dominating number*.

Many results about the relations between the above cardinal characteristics have been discovered by various researchers in the past decades. They can be summarized in *Cichoń's diagramme*, which includes the inequalities that are provable in ZFC. For each inequality depicted, it is possible to find a model of ZFC in which it is strict. More information on this diagramme and the relations it depicts can be found in Bartoszyński and Judah's book [5], as well as Blass' chapter [6] in the *Handbook of Set Theory*.

$$
\begin{array}{ccccccccccc}
 & & \mathrm{cov}(\mathcal{N}) & \leq & \mathrm{non}(\mathcal{M}) & \leq & \mathrm{cof}(\mathcal{M}) & \leq & \mathrm{cof}(\mathcal{N}) & \leq & 2^{\aleph_0} \\
 & & & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & & & \\
 & & \rotatebox{90}{$\leq$} & & \mathfrak{b} & \leq & \mathfrak{d} & & \rotatebox{90}{$\leq$} & & \\
 & & & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & & & \\
\aleph_1 & \leq & \mathrm{add}(\mathcal{N}) & \leq & \mathrm{add}(\mathcal{M}) & \leq & \mathrm{cov}(\mathcal{M}) & \leq & \mathrm{non}(\mathcal{N}) & &
\end{array}
$$

Figure 1.2: Cichoń's diagramme.

Finally, the bounding and dominating numbers can be generalized to relations other than $\leq^*$, according to the following definition.

Definition 1.25 ([4])**.** Let X and Y be sets and $R \subseteq X \times Y$ be a relation between them. We define

1. $\mathfrak{b}(X, Y, R) = \min\{|F| \mid F \subseteq X,\ \forall y \in Y\ \exists x \in F\ \neg R(x, y)\}$

2. $\mathfrak{d}(X, Y, R) = \min\{|F| \mid F \subseteq Y,\ \forall x \in X\ \exists y \in F\ R(x, y)\}$

With this notation, we have that $\mathfrak{b} = \mathfrak{b}(\omega^\omega, \omega^\omega, \leq^*)$ and $\mathfrak{d} = \mathfrak{d}(\omega^\omega, \omega^\omega, \leq^*)$.

1.5 Filters on ω

In a general context, *filters* are subsets of partially ordered sets with certain properties. They have a wide range of applications to mathematics. In set theory and topology, we often work with filters that are subsets of the powerset of a set, ordered by the subset relation. These attempt to capture the notion of a collection of large subsets of a set.

Definition 1.26. Let X be a nonempty set. A set $\mathcal{F} \subseteq \mathcal{P}(X)$ is called a *filter on X* if and only if

1. $X \in \mathcal{F}$,
2. if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$, and
3. if $A, B \subseteq \mathcal{F}$, $A \in \mathcal{F}$ and $A \subseteq B$, then $B \in \mathcal{F}$.

A filter $\mathcal{F}$ is called *trivial* if and only if $\varnothing \in \mathcal{F}$.

These properties intuitively describe a collection of large subsets. It is clear from the above definition that filters are the dual notion to ideals and that $\mathcal{F}$ is a filter on X if and only if the set $\{X - A \subseteq X \mid A \in \mathcal{F}\}$ is an ideal. In order to define a filter we can use a collection of compatible subsets of a set.

Definition 1.27. Let X be a nonempty set. We say that a nonempty collection $\mathcal{B} \subseteq \mathcal{P}(X)$ is a *filter basis* if and only if for every $A, B \in \mathcal{B}$ there exists a $C \in \mathcal{B}$ such that $C \subseteq A \cap B$.

Proposition 1.28. *Let X be a nonempty set and $\mathcal{B}$ be a filter basis. The collection $\mathcal{F} = \{A \subseteq X \mid \exists B \in \mathcal{B}\,(B \subseteq A)\}$ is a filter on X.*

Proof. Since $\mathcal{B}$ is nonempty, there exists $B \in \mathcal{B}$ and $X \in \mathcal{F}$ as $B \subseteq X$ by definition. If $A, B \in \mathcal{F}$, then there exist $A', B' \in \mathcal{F}$ such that $A' \subseteq A$ and $B' \subseteq B$. By definition, there exists $C \in \mathcal{F}$ such that $C \subseteq A' \cap B' \subseteq A \cap B$ and thus $A \cap B \in \mathcal{F}$. Finally, if $A \in \mathcal{F}$ and $A \subseteq B$, then there exists a $C \in \mathcal{F}$ such that $C \subseteq A \subseteq B$ and $B \in \mathcal{F}$ as well. $\square$

Definition 1.29. The above $\mathcal{F}$ is called *the filter generated by the basis $\mathcal{B}$*.

In the following, we will be working with filters $\mathcal{F}$ on ω, which are formally subsets of $\mathcal{P}(\omega)$. The correspondence between $\mathcal{P}(\omega)$ and $\mathbf{2}^\omega$ allows us to view them as subsets of the Cantor space. Because of this, we can examine whether filters satisfy regularity properties, such as Lebesgue measurability. The importance of filters comes from the fact that certain types of them can provide counterexamples to these properties. Let us give some examples of filters.

Definition 1.30. Fréchet's filter $\mathscr{F}$ is the collection of all cofinite subsets of ω.

In the study of filters on ω, we often work with filters that contain $\mathscr{F}$. Note that such a filter $\mathcal{F} \supseteq \mathscr{F}$ either contains only infinite subsets of ω or is trivial. This is because, if there is a finite $s \in \mathcal{F}$, then $\omega - s \in \mathcal{F}$ as well and therefore $\varnothing = s \cap (\omega - s) \in \mathcal{F}$.

Remark 1.31. In the rest of the work, the term *filter* will refer to filters on ω that contain $\mathscr{F}$.

Fréchet's filter contains the cofinite subsets of ω and, although we cannot add any finite sets to it, we can add other infinite subsets of ω to obtain new filters. It is possible however that a filter contains so many elements that adding any further sets to it would cause it to become trivial.

Definition 1.32. A nontrivial filter $\mathcal{F} \subseteq \mathcal{P}(X)$ is called an *ultrafilter* if and only if it is *maximal*, that is, there exists no nontrivial filter $\mathcal{F}' \subseteq \mathcal{P}(X)$ with $\mathcal{F}' \supset \mathcal{F}$.

Ultrafilters can also be equivalently characterized by the following property.

Proposition 1.33 ([11, p. 74])**.** *A filter $\mathcal{F} \subseteq \mathcal{P}(X)$ is an ultrafilter if and only if for every $A \subseteq X$, either $A \in \mathcal{F}$ or $X - A \in \mathcal{F}$.*

Ultrafilters on ω are important to the study of regularity properties because they provide counterexamples to them. For example, when viewed as subsets of $\mathbf{2}^\omega$, they are neither Lebesgue measurable nor have the property of Baire, as proven by Sierpiński (see [5, p. 205]).

The method followed in Raisonnier's proof of Shelah's Theorem is the construction of a $\mathbf{\Sigma}^1_3$ nonmeasurable filter, which leads to a contradiction when combined with the assumption that every $\mathbf{\Sigma}^1_3$ set is measurable. Instead of an ultrafilter however, Raisonnier defines a *rapid* filter.

Definition 1.34. A filter $\mathcal{F}$ is *rapid* if and only if for every $f \in \omega^\omega$ there exists an $a \in \mathcal{F}$ with $|a \cap f(n)| \leq n$ for all $n \in \omega$.

The above property, introduced by Mokobodzki in [16], is of great importance because it provides us with another type of filter, other than ultrafilters, that do not have various regularity properties, including those defined in Section 1.3.3.

Theorem 1.35 ([5, p. 240] [1])**.** *Every nontrivial rapid filter is not Lebesgue measurable and does not have the property of Baire.*

Most of the results of Chapters 4 and 5 involve the property of rapidity. Note that the trivial filter $\mathcal{P}(\omega)$ on ω is always rapid. For any increasing $f \in \omega^\omega$ and $n \in \omega$, we have that $f[\omega] \subseteq \omega$ is an element of $\mathcal{P}(\omega)$ and

$$|f[\omega] \cap f(n)| = |\{f(0), \dots, f(n-1)\}| = n\,.$$

The intuition behind the term rapid comes from the fact that rapid filters contain subsets of ω that are arbitrarily sparse, in the sense that that their elements are far apart. As a result, the increasing enumerations of these sets increase rapidly. To see this, let $\mathcal{F}$ be a rapid filter and $f \in \omega^\omega$ be an increasing

[1] Both Bartoszyński and Judah in [5] and Schindler in [19] attribute this result to Mokobodzki, whereas Raisonnier in [18] attributes it to Talagrand [22].

function. By definition, $\mathcal{F}$ will contain a subset $a \subseteq \omega$ that contains at most n elements below $f(n)$. Therefore, its $(n+1)$-th element must be greater or equal to $f(n)$. If we pick a rapidly increasing f, then it follows that we can find an $a \in \mathcal{F}$ whose elements are spaced far apart. This also means that, if $\hat{a} \in \omega^\omega$ is the increasing enumeration of a, $\hat{a}(n) \geq f(n)$ for all $n \in \omega$.

This fact gives us an interesting property of rapid filters. If $\mathcal{F}$ is a rapid filter, then for any $f \in \omega^\omega$, by the above, we can find an $a \in \mathcal{F}$ such that $f(n) \leq \hat{a}(n)$ for all $n \in \omega$. In other words, we can always find an $a \in \mathcal{F}$ such that $\hat{a}$ dominates f. As a result, the set $\{\hat{a} \mid a \in \mathcal{F}\}$ is a dominating family.

As mentioned before, the trivial filter is a trivial example of a rapid filter. Unfortunately, it is not possible to find other examples in ZFC without additional assumptions, because of the following theorem.

Theorem 1.36 (Judah and Shelah, [12])**.** *Assuming the consistency of* ZF*, there exists a model of* ZFC *without nontrivial rapid filters.*

Instead of the original definition of rapid filers, we will often use similar characterizations that are more useful in practice. For example, when trying to prove that a filter is rapid, it is possible to replace the bound of $\varphi(n) = n$ in the definition with any convenient $\varphi \in \omega^\omega$.

Proposition 1.37 ([5, Lem. 4.6.2])**.** *The following conditions are equivalent for every filter $\mathcal{F}$:*

1. *$\mathcal{F}$ is rapid, and*
2. *there exists a $\varphi \in \omega^\omega$ such that for every $f \in \omega^\omega$ there exists an $a \in \mathcal{F}$ with $|a \cap f(n)| \leq \varphi(n)$ for all $n \in \omega$.*

Intuitively, this holds because the second property also guarantees that $\mathcal{F}$ will still have arbitrarily sparse elements. Therefore, even if we have a bound that is greater than the identity function, we can still find sets in the filter with at most n elements below $f(n)$, by applying (2) to a function appropriately defined, given $f \in \omega^\omega$.

In subsequent chapters we will be interested in the rapidity of the Raisonnier filter. For this we will consider partitions of ω into intervals. These partitions can be generated by the following set of reals.

Definition 1.38. Let $f \in \omega^\omega$ be a *partitioning real* if and only if it is strictly increasing and $f(0) = 0$. Let $\mathcal{P}$ denote the set of partitioning reals.

The requirement that $f(0) = 0$ is for convenience and we could simply consider the set of increasing functions instead. We arrive at the following equivalent characterization of rapidity.

Lemma 1.39. *The following are equivalent for every filter $\mathcal{F}$ and every increasing unbounded $\varphi \in \omega^\omega$ with $\varphi(0) = 0$:*

1. *$\mathcal{F}$ is rapid, and*

2. for every $f \in \mathcal{P}$ there exists an $a \in \mathcal{F}$ with $|a \cap f(n)| \leq \varphi(n)$ for all $n \in \omega$.

Proof. Assume that $\mathcal{F}$ is rapid, and let $\varphi \in \omega^\omega$ have the properties defined above. Given an $m \in \omega$, let $N_m = \{k \in \omega \mid \varphi(k) \leq m\}$. Let $f \in \mathcal{P}$ and define the function $g \in \omega^\omega$ such that $g(m) = f(\max N_m)$. For every $m \in \omega$, since $\varphi(0) = 0 \leq m$, the set N_m is nonempty. Moreover, because φ is increasing and unbounded, there exists $n \in \omega$ such that $\varphi(k) > m$ for all $k \geq n$. Therefore, N_m is finite and $\max N_m$ is well defined. As a result, the function g is also well defined. By assumption then, there exists an $a \in \mathcal{F}$ such that $|a \cap g(m)| \leq m$ for all $m \in \omega$.

Let $n \in \omega$. By definition, $n \in N_{\varphi(n)}$ and so $n \leq \max N_{\varphi(n)}$. Since f is strictly increasing by assumption,

$$f(n) \leq f(\max N_{\varphi(n)}) = g(\varphi(n)) \,.$$

As a result,

$$|a \cap f(n)| \leq |a \cap g(\varphi(n))| \leq \varphi(n) \,.$$

For the other direction, we will use Proposition 1.37. Assume that $\mathcal{F}$ satisfies (2) and let $f \in \omega^\omega$. We can define the function $g \in \omega^\omega$ as $g(0) = 0$ and $g(n+1) = \max\{f(n), g(n)+1\}$ for all $n \in \omega$. By definition, $g \in \mathcal{P}$, since $g(n+1) \geq g(n) + 1$ for all $n \in \omega$. By assumption then, there exists an $a \in \mathcal{F}$ such that $|a \cap g(n)| \leq \varphi(n)$ for all $n \in \omega$. For any $n \in \omega$, we have that

$$a \cap f(n) = \{k \in a \mid k < f(n)\} \subseteq \{k \in a \mid k < \max\{f(n), g(n)+1\}\} \,,$$

and so $|a \cap f(n)| \leq |a \cap g(n+1)| \leq \varphi(n+1)$. Therefore, $\mathcal{F}$ satisfies condition (2) of Proposition 1.37. □

Note that this theorem gives us a stronger characterization of rapid filters than Proposition 1.37. The previous proposition only allows us to prove that a filter is rapid using any convenient bound. It does not give us additional information when $\mathcal{F}$ is rapid, as the second property is a trivial consequence of the first. Lemma 1.39, on the other hand, gives us an equivalent definition of rapidity with any bound that satisfies the properties given. It is this characterization of rapid filters that we will use most often.

More information on rapid filters, as well as filters on ω in general, can be found in Chapter 4 and other relevant sections of [5].

1.6 Inaccessible Cardinals

Raisonnier's proof utilizes a result that involves the existence of models with *inaccessible cardinals*, which are a type of large cardinal. The existence of large cardinals in general cannot be proven in ZFC and, as their name suggests, they are larger than many cardinals that can be constructed in ZFC.

Inaccessible cardinals cannot be constructed by smaller cardinals through approximation or exponentiation. If α is a limit ordinal, we say that an increasing sequence $(\alpha_\iota)_{\iota<\beta}$ of length β is *cofinal* in α if and only if $\sup_{\iota<\beta} \alpha_\iota = \alpha$.

The *cofinality of* α, denoted by $\mathrm{cf}(\alpha)$, is defined as the least limit ordinal β for which there exists a sequence of length β that is cofinal in α. It can be shown that for limit ordinals α, $\mathrm{cf}(\alpha)$ is a cardinal. Therefore, we have the following definition.

Definition 1.40. A cardinal κ is called *regular* if and only if $\mathrm{cf}(\kappa) = \kappa$.

By the above definitions, a regular cardinal cannot be written as the supremum of an increasing sequence $(\lambda_\iota)_{\iota<\beta}$ of cardinals $\lambda_\iota < \kappa$ of length $\beta < \kappa$. This means that we cannot approximate it using only smaller cardinal numbers.

Definition 1.41. A cardinal κ is called a *strong limit* if and only if $\kappa > 2^\lambda$ for all cardinals $\lambda < \kappa$.

Strong limit cardinals cannot be obtained by smaller cardinals through exponentiation, since $\lambda^\lambda = 2^\lambda$ for all cardinals λ. Combining these two properties, we obtain the notion of an inaccessible cardinal.

Definition 1.42. A cardinal is called *weakly inaccessible* if and only if it is a regular limit cardinal. A cardinal is called *(strongly) inaccessible* if and only if it is regular and a strong limit.

Inaccessible cardinals can be used to obtain models of ZFC. In fact, it can be proven [13, p. 18] that if κ is an inaccessible cardinal, then $V_\kappa \models \mathsf{ZFC}$. As a consequence, because of Gödel's Second Incompleteness Theorem, their existence is not provable in ZFC. In the following, let I denote the formal sentence expressing that an inaccessible cardinal exists.

1.7 The Constructible Universe

The class of constructible sets was first defined by Gödel in 1938, in order to prove the consistency of ZF with the Axiom of Choice (AC) and the Generalized Continuum Hypothesis (GCH). Its first account was given in the publications [10] and [8], with a different presentation appearing in the monograph [9]. Our account of the basic definitions is based on Chapter 1, Section 3 of Kanamori's book [13], which also includes historical information on Gödel's notion of constructibility.

Constructibility in this context refers to definability on a structure through first-order formulas.

Definition 1.43. Let $\mathfrak{M} = (M, E)$ be a structure with domain M and a binary relation E. A set y is *definable over* $\mathfrak{M}$ if and only if there exists a first order formula $\varphi(u, v_0, \dots, v_n)$ in the language of set theory and parameters $a_0, \dots, a_n \in M$ such that $y = \{z \mid \mathfrak{M} \models \varphi[z, a_0, \dots, a_n]\}$. Given a set x, we write

$$\mathrm{def}(x) = \{y \subseteq x \mid y \text{ is definable over } (x, \in)\}$$

for the class of definable subsets of x.

Given a set x, the relation $\in$ on x can be formalized. This in turn means that $\mathrm{def}(x)$ is a set, as it is a definable subset of $\mathcal{P}(x)$. Using this notion, we can define the *constructible hierarchy* analogously to the von Neumann hierarchy, with the difference that in each step of the construction we add only definable subsets.

Definition 1.44. The levels of the *constructible hierarchy* are defined as

1. $L_0 = \varnothing$,
2. $L_{\alpha+1} = \mathrm{def}(L_\alpha)$ for all $\alpha \in \mathrm{Ord}$, and
3. $L_\alpha = \bigcup_{\iota<\alpha} L_\iota$ for all limit $\alpha \in \mathrm{Ord}$ with $\alpha > 0$.

The *constructible universe* is defined as $L = \bigcup_{\alpha\in\mathrm{Ord}} L_\alpha$.

The constructible hierarchy shares some similarities with von Neumann's hierarchy. For example, we have that each L_α is a transitive set, that $L_\alpha \subseteq L_\beta$ for all ordinals $\alpha \leq \beta$ and that $L_\alpha \cap \mathrm{Ord} = \alpha$ for every $\alpha \in \mathrm{Ord}$ [14, p. 135]. The latter fact also implies that L is a proper class, as it contains all ordinals. It can be shown that L is a model of $\mathsf{ZFC}+\mathsf{GCH}$. The result $L \models \mathsf{AC}$ is achieved by constructing a wellorder on L.

We can generalize the above definitions to obtain a notion of *relative constructibility*. In this generalization, we define sets using first-order formulas while also having access to a set as a parameter. This is achieved by adding a unary predicate to our language, which is interpreted as inclusion in the parameter. So, we can extend the notion of definability to structures (M, E, A) and formulas in the language of set theory with the additional unary predicate.

Definition 1.45. Given sets A and x, we write

$$\mathrm{def}^A(x) = \{y \subseteq x \mid y \text{ is definable over } (x, \in, A \cap x)\}$$

for the class of definable subsets of x relative to A.

Definition 1.46. Let A be a set. We define the hierarchy of sets *constructible relative to* A as

1. $L_0[A] = \varnothing$,
2. $L_{\alpha+1}[A] = \mathrm{def}^A(L_\alpha[A])$ for all $\alpha \in \mathrm{Ord}$, and
3. $L_\alpha[A] = \bigcup_{\iota<\alpha} L_\iota[A]$ for all limit $\alpha \in \mathrm{Ord}$ with $\alpha > 0$.

We define $L[A] = \bigcup_{\alpha\in\mathrm{Ord}} L[A]$.

The class $L[A]$ retains some properties of L. Most crucially, it is also a model of the Axiom of Choice and a wellorder $<_{L[A]}$ of all its elements can be constructed.

For the most part, we will be interested in the classes $L[a]$, where a is a real, usually from the Baire space. Using the encodings in Definition 1.12, we can encode countably many reals into one, in a way that is definable through first-order formulas. Therefore, if we want to obtain the sets constructible relative to the reals $(f_i)_{i<\omega}$, we can encode them into a single $f \in \omega^\omega$ and construct the class $L[f]$. Because of this, we will often write $L[f_0, f_1, \dots]$ for this class of sets. Naturally, for any reals $f, g \in \omega^\omega$, $L[f] \subseteq L[f, g]$.

Even though the notion of ordinals is absolute for models of ZFC, this is not the case for cardinals, since cardinality is defined in terms of the existence of bijections. Because of this, the cardinals of $L[a]$ may not coincide with those of V. However, because the notion of a bijection is also absolute, the inexistence of a bijection between a cardinal and a smaller ordinal in V also holds in $L[a]$, meaning that the cardinals of V are also cardinals in $L[a]$. However, what may happen is that they are larger according to $L[a]$, as $L[a]$ may contain new cardinals below them.

The cardinal $\aleph_1$ is the least ordinal that is not in bijection with ω. If $L[a]$ contains ordinals that are countable in V, but not bijections between them and ω, the least of these ordinals will be $\aleph_1^{L[a]}$. In this case, the object $\aleph_1^{L[a]}$ may be a countable ordinal in V and so $\aleph_1^{L[a]} < \aleph_1$. On the other hand, if $L[a]$ also contains a bijection between every countable ordinal and ω, $\aleph_1$ will also be the first uncountable cardinal according to $L[a]$ and $\aleph_1^{L[a]} = \aleph_1$. Therefore, in all cases, $\aleph_1^{L[a]} \leq \aleph_1$.

The following result gives us a sufficient condition for $\aleph_1$ to be an inaccessible cardinal in L.

Proposition 1.47 ([13, p. 135])**.** *If $\aleph_1^{L[a]} < \aleph_1$ for all $a \in \omega^\omega$, then $\aleph_1$ is inaccessible in $L[a]$ for all $a \in \omega^\omega$.*

As a consequence, if $\aleph_1^{L[a]} < \aleph_1$ for all $a \in \omega^\omega$, then $\aleph_1$ is inaccessible in $L = L[b]$, where $b \in \omega^\omega$ is any constructible real. This can thus be used to obtain a model of $\mathsf{ZFC} + \mathsf{I}$.

Using a generalization of the proof of $L \models \mathsf{GCH}$, it is possible to show that for any $a \in \omega^\omega$, $L[a]$ is a model of GCH as well [11, pp. 190-193]. This means that the set of *constructible reals* $\omega^\omega \cap L[a]$, or equivalently $\mathbf{2}^\omega \cap L[a]$, has cardinality $\aleph_1^{L[a]}$. So, as a consequence of $\aleph_1^{L[a]} \leq \aleph_1$, for any $a \in \omega^\omega$, we have two cases: $L[a]$ contains countably many or $\aleph_1$-many reals.

Another fact that we will make use of concerns the complexity of the constructible reals. First, we introduce the following relation.

Definition 1.48. Let $\prec$ be a wellorder. We define $\mathrm{IS}_\prec \subseteq (\omega^\omega)^2$ as

$$\mathrm{IS}_\prec(x, y) \iff \{(x)_i \mid i \in \omega\} = \{z \in \omega^\omega \mid z \prec y\}$$

for all $x, y \in \omega^\omega$

This relation expresses that $x \in \omega^\omega$ encodes the initial segment of $\prec$ which contains exactly the elements below $y \in \omega^\omega$. Since we can encode only countably many reals into one using the encoding from 1.12, this relation makes sense for wellorders with countable initial segments. Since the constructible reals of $L[a]$

are at most $\aleph_1$-many, the ordering $<_{L[a]}$ restricted on them is such a wellorder. We say that a relation $\prec$ is $\mathbf{\Sigma}^1_2$-*good* if and only if $\mathrm{IS}_\prec$ is $\mathbf{\Sigma}^1_2$. We then have the following result.

Theorem 1.49 ([11, pp. 494-495])**.** *The set $\omega^\omega \cap L[a]$ is $\Sigma^1_2(a)$ and the relation $<_{L[a]} \cap\, \omega^\omega$ is $\mathbf{\Sigma}^1_2$-good.*

Chapter 2

Raisonnier's Proof

Shelah, in his article [20], proved that the existence of a model of ZFC + "there exists an inaccessible cardinal", denoted as $\mathsf{ZFC}+\mathsf{I}$, can be obtained from the assumptions ZFC + "all projective sets are measurable". This is the result of the following theorem.

Theorem 2.1 (Shelah, [20, Thm. 5.1])**.** *If every $\mathbf{\Sigma}^1_3$ set of reals is Lebesgue measurable, then $\aleph_1$ is inaccessible in L.*

In his proof, the hypothesis of the theorem is assumed, along with $\aleph_1^{L[a]} = \aleph_1$ for some real a. Through some technical steps that involve forcing, a contradiction is reached and we obtain the conclusion through Proposition 1.47.

Raisonnier in [18] presented a simpler proof of Theorem 2.1. To achieve the same result, he constructed a particular filter and used it to reach a contradiction. His proof consists of the following steps.

1. *Assume that every $\mathbf{\Sigma}^1_3$ set is Lebesgue measurable and suppose towards contradiction that there exists an $a \in \omega^\omega$ with $\aleph_1^{L[a]} = \aleph_1$.*

2. *With $X = \mathbf{2}^\omega \cap L[a]$ as a starting point, define a specific filter $\mathcal{F}_X$.*

3. *Using the fact that $|X| = \aleph_1$, prove that $\mathcal{F}_X$ is nontrivial.*

4. *Using the assumption that every $\mathbf{\Sigma}^1_2$ set is Lebesgue measurable, prove that $\mathcal{F}_X$ is rapid. By Theorem 1.35, $\mathcal{F}_X$ is then nonmeasurable as a subset of $\mathbf{2}^\omega$.*

5. *Prove that $\mathcal{F}_X$ is a $\mathbf{\Sigma}^1_3$ subset of $\mathbf{2}^\omega$.*

6. *The filter $\mathcal{F}_X$ is a nonmeasurable $\mathbf{\Sigma}^1_3$ set, which contradicts our initial assumption. Therefore, for all $a \in \omega^\omega$, $\aleph_1^{L[a]} < \aleph_1$. Consequently, $\aleph_1$ is inaccessible in L by Theorem 1.47.*

This proof is simpler, in that it does not require the technique of forcing. Moreover, as will be shown, the third and fifth steps are straightforward. However, Raisonnier's method for the fourth step, which is the focal point of the proof, is quite technical and unintuitive.

In addition to Raisonnier's original paper, a very detailed exposition of this proof following Raisonnier's method, can be found in Schindler's book [19]. In this work, we will present a different version of the proof, given in the book [5] by Bartoszyński and Judah. In it, they relate the measurability of $\mathbf{\Sigma}^1_2$ sets to the notion of *slaloms*. Through this, they present a proof of Shelah's Theorem 2.1 that is clearer and easier to generalize.

2.1 The Raisonnier Filter

As mentioned above, Raisonnier's proof involves constructing a counterexample to Lebesgue measurability in the form of a rapid filter. This particular construction is now known as the "Raisonnier filter" and is denoted by $\mathcal{F}_X$, where $X \subseteq \mathbf{2}^\omega$. This filter is defined through countable covers of X and their *splitting points*. In Raisonnier's article, as well as the books by Schindler and Bartoszyński and Judah, X is assumed to have certain properties, or is a particular set. However, the definition of $\mathcal{F}_X$ is not dependent on any assumptions about X and so we will formulate it for any subset of $\mathbf{2}^\omega$.

Definition 2.2. Let $x, y \in \mathbf{2}^\omega$ such that $x \neq y$. We define their *splitting point* as the natural number $h(x, y) = \min\{n \in \omega \mid x(n) \neq y(n)\}$. Given a set $X \subseteq \mathbf{2}^\omega$, we define the set of *splitting points of* X as

$$H(X) = \{h(x, y) \mid x, y \in X,\, x \neq y\}\,.$$

We will also use the notion of splitting points for finite sequences $s, t \in \mathbf{2}^{<\omega}$ with the same notation. This is well defined when $s{\restriction}m \neq t{\restriction}m$, where $m = \min\{|s|, |t|\}$.

Definition 2.3 (The Raisonnier Filter)**.** We define the set $\mathcal{F}_X \subseteq \mathbf{2}^\omega$ as follows. Given an $a \in \mathbf{2}^\omega$, we let $a \in \mathcal{F}_X$ if and only if there exists a countable collection $(Y_n)_{n<\omega}$ of subsets of $\mathbf{2}^\omega$ such that $X \subseteq \bigcup_{n<\omega} Y_n$ and $a \supseteq \bigcup_{n<\omega} H(Y_n)$.

Proposition 2.4 (cf. [19, p. 166])**.** *For any set $X \subseteq \mathbf{2}^\omega$, $\mathcal{F}_X$ is a filter on ω .*

Proof. Let $X \subseteq \mathbf{2}^\omega$ and let $\mathcal{F}_X$ be defined as above. Notice that $H(\mathbf{2}^\omega) = \omega \in \mathcal{F}_X$, since for every $n \in \omega$ we can find two reals whose splitting point is n. Furthermore, if $a \in \mathcal{F}_X$ and $b \supseteq a$, by definition there exists a countable cover $(Y_n)_{n<\omega}$ of X for which $\bigcup_{n<\omega} H(Y_n) \subseteq a \subseteq b$, and therefore $b \in \mathcal{F}_X$.

Let now $a, b \in \mathcal{F}_X$. By definition, there exist countable covers $(X_n)_{n<\omega}$ and $(Y_n)_{n<\omega}$ of X, such that $a \supseteq \bigcup_{n<\omega} H(X_n)$ and $b \supseteq \bigcup_{n<\omega} H(Y_n)$. For every $m, n < \omega$ let $Z_{m,n} = X_m \cap Y_n$. We then have

$$X \subseteq \left(\bigcup_{m<\omega} X_m\right) \cap \left(\bigcup_{n<\omega} Y_n\right) = \bigcup_{m<\omega} \bigcup_{n<\omega} X_m \cap Y_n = \bigcup_{m<\omega} \bigcup_{n<\omega} Z_{m,n}\,,$$

which means that $(Z_{m,n})_{m,n<\omega}$ is a countable cover of X. Let $m, n < \omega$ and $k \in H(Z_{m,n})$. By definition, there exist $x, y \in Z_{m,n}$ such that $x \neq y$ and $h(x, y) = k$. Because $x, y \in X_m$ and $x, y \in Y_n$, $k \in H(X_m) \subseteq a$ and $k \in H(Y_n) \subseteq b$. As a result, $H(Z_{m,n}) \subseteq a \cap b$ for all $m, n < \omega$ and therefore

$$\bigcup_{m<\omega} \bigcup_{n<\omega} H(Z_{m,n}) \subseteq a \cap b \in \mathcal{F}_X .$$

The last thing we need to prove is that $\mathscr{F} \subseteq \mathcal{F}_X$. For this, let $a \subseteq \omega$ be a cofinite subset such that $a \supseteq \omega - m$. Note that for every $x \in X$ we have that $x \in [x \restriction m]$ by definition. As a result, $X \subseteq \bigcup_{s \in \mathbf{2}^m} [s]$. Moreover, for every $s \in \mathbf{2}^m$ and $k \in H([s])$, there exist $x, y \in [s]$ such that $x \neq y$ and $k = h(x, y)$. Because $x, y \in [s]$, they both have s as an initial segment and their splitting point k must be $k \geq m$. As a result, $k \in \omega - m$ and thus $H([s]) \subseteq a$. Therefore, $\bigcup_{s \in \mathbf{2}^m} H([s]) \subseteq a$ and $a \in \mathcal{F}_X$. $\square$

In Raisonnier's proof, the above construction is applied to the set $X = \mathbf{2}^\omega \cap L[a]$ for some $a \in \omega^\omega$ with $\aleph_1^{L[a]} = \aleph_1$. As mentioned, part of Raisonnier's proof includes showing that $\mathcal{F}_X$ is not trivial. In this case, this follows immediately from the fact that X is uncountable, because of the following property.

Proposition 2.5. *The filter $\mathcal{F}_X$ is trivial if and only if X is countable.*

Proof. Let X be countable. In this case, it can be written as $X = \{x_i \mid i < \omega\}$ and the collection $(\{x_i\})_{i<\omega}$ is a countable cover of X with $\varnothing = \bigcup_{i<\omega} H(\{x_i\}) \in \mathcal{F}_X$.

Let now $\mathcal{F}_X$ be trivial. By definition, we can find a countable cover $(Y_i)_{i<\omega}$ of X such that $\bigcup_{i<\omega} H(Y_i) \subseteq \varnothing$. This means that $H(Y_i) = \varnothing$ for all $i < \omega$. Suppose that for some $i < \omega$ there existed $x, y \in Y_i$ with $x \neq y$. In that case, it would be that $h(x, y) \in H(Y_i)$, which is a contradiction. Therefore, each Y_i can contain at most one element and X must be countable. $\square$

The elements of $\mathcal{F}_X$ are the sets of splitting points of countable covers of X. If X can be covered by a countable cover whose splitting points occur infrequently, this indicates that X is small, narrow, or simple in some sense. Moreover, since $\mathcal{F}_X$ extends Fréchet's filter, if $a \in \mathcal{F}_X$, then $\{k \in a \mid k > n\} \in \mathcal{F}_X$ as well, since filters are closed under finite intersections. This means that the elements of $\mathcal{F}_X$ can give us an indication, not of specific splitting points that must occur in every cover, as at least finitely many of them can be omitted, but of the splitting points that will always have to occur eventually.

Naturally, any countable cover of X must also eventually contain splitting points from X, unless X is countable itself. We can also disregard reals not in X when defining a countable cover of it. This leads to the following equivalent definition of $\mathcal{F}_X$ through countable partitions of X.

Proposition 2.6 ([5, p. 246])**.** *Let $X \subseteq \mathbf{2}^\omega$. For every $a \subseteq \omega$, $a \in \mathcal{F}_X$ if and only if there exists an equivalence relation R on $\mathbf{2}^\omega$ with countably many equivalence classes such that $a \supseteq \bigcup_{x \in X} H([x]_R \cap X)$.*

Here, since R is assumed to be an equivalence relation with countably many equivalence classes, the sets $[x]_R \cap X$ are countably many and form a partition of X. It must be noted that Proposition 2.6 was Raisonnier's original definition. However, Definition 2.3 is often easier to work with.

Another equivalent definition provides further intuition with respect to the information contained in the Raisonnier filter of X. If the elements of $\mathcal{F}_X$ contain splitting points that are eventually necessarily in X, if a large $k \in \omega$ is not one of them, there are no two reals in X that can diverge at the k-th digit and we can thus predict the k-th digit of a real $x \in X$, given its initial segment up to that point. Therefore, we can equivalently define $\mathcal{F}_X$ as follows.

Proposition 2.7 ([5, p. 246])**.** *Let $X \subseteq \mathbf{2}^\omega$. For every $a \subseteq \omega$, $a \in \mathcal{F}_X$ if and only if there exists an $f : \mathbf{2}^{<\omega} \to \{0,1\}$ such that for all $x \in X$ and all but finitely many $k \notin a$ we have that $x(k) = f(x{\restriction}k)$.*

In this definition, it is the function f that predicts the next digit of a real at a position k, provided that k is large enough and not in an element $a \in \mathcal{F}_X$. If $\mathcal{F}_X$ contains elements that are sparse enough, this means that for large enough k, the k-th digit of a real will eventually be predictable in most cases, that is $k \notin a$, given the initial segment of that real. In this sense, if $\mathcal{F}_X$ contains sparse elements, X can be seen as simple, as its elements are predictable to an extent.

If $\mathcal{F}_X$ is rapid for some $X \subseteq \mathbf{2}^\omega$, then it contains elements that are arbitrarily sparse in the sense mentioned in Section 1.5. Because of this, X can be partitioned into a countable collection of subsets with very sparse splitting points. This also means that, because of Proposition 2.7, every real $x \in X$ is eventually very predictable, in the sense that its digits that cannot be predicted are very far apart.

The fifth step of the proof involves proving that $\mathcal{F}_X$ is a $\mathbf{\Sigma}^1_3$ set. In fact, we can obtain a more general result about its complexity. For this, we can use the equivalent definition of the Raisonnier filter from Proposition 2.7.

Proposition 1.5 tells us that we are able to encode a closed set as a real by encoding the corresponding tree. As we can also encode countably many reals into one using the encoding in Definition 1.12, we can encode a countable cover of a set using a real. We can thus obtain the following result.

Lemma 2.8. *If a set $X \subseteq \mathbf{2}^\omega$ is $\Sigma^1_n(a)$ for some $a \in \omega^\omega$, then $\mathcal{F}_X$ is $\Sigma^1_{n+1}(a)$.*

Proof. Let $X \subseteq \mathbf{2}^\omega$ be a $\Sigma^1_n(a)$ set of reals. By Proposition 2.7, we can define

$$\mathcal{F}_X = \{b \in \mathbf{2}^\omega \mid \exists f : \mathbf{2}^{<\omega} \to \{0,1\}\, \forall x \in X\, \forall^\infty k \in \omega - b\, (x(k) = f(x{\restriction}k))\}$$

Given any function $f : \mathbf{2}^{<\omega} \to \{0,1\}$, we can define an equivalent $g \in \mathbf{2}^\omega$ that maps the encoding of a sequence $s \in \mathbf{2}^{<\omega}$ to the value $f(s)$, that is, $g(\langle s \rangle) = f(s)$ for all $s \in \mathbf{2}^{<\omega}$ (see Definition 1.12). Moreover, given a $g \in \mathbf{2}^\omega$ as above and an $x \in X$, the formula $\forall^\infty k \in \omega - b\, (x(k) = f(x{\restriction}k))$ can equivalently be written as

$$\exists m \in \omega\, \forall k \geq m\ (k \notin b \to x(k) = g(\langle x{\restriction}k \rangle))\,.$$

Therefore, we can define $\mathcal{F}_X$ in second-order arithmetic by the formula

$$\exists^1 g\,\forall^1 x(\ \ x \in X \to \exists^0 m\,\forall^0 k\,((k \geq m \wedge k \notin b) \to x(k) = g(\langle x{\restriction}k\rangle))\ \ ).$$

The condition $x \in X$ is in the antecedent of a conditional, meaning that it is in negated form. It can thus easily be seen that this conditional formula defines a $\Pi^1_n(a, g, x)$ set. As a result, $\mathcal{F}_X$ is a $\Sigma^1_{n+1}(a)$ set. □

Finally, we will give one more equivalent definition of the Raisonnier filter, which will become useful in Chapter 4. For this, we will need the following property, which tells us that a subset of $\mathbf{2}^\omega$ and its closure have the same splitting points.

Proposition 2.9 (cf. [19, pp. 166-167])**.** *For every $X \subseteq \mathbf{2}^\omega$, $H(X) = H(\mathrm{cl}(X))$.*

Proof. Let $X \subseteq \mathbf{2}^\omega$. Note that, since $X \subseteq \mathrm{cl}(X)$, we immediately have $H(X) \subseteq H(\mathrm{cl}(X))$. So, we only have to prove that $H(\mathrm{cl}(X)) \subseteq H(X)$.

Let $k \in H(\mathrm{cl}(X))$. By definition, there exist $x, y \in \mathrm{cl}(X)$ such that $x \neq y$ and $k = h(x, y)$. Suppose, towards contradiction, that $X \cap [x{\restriction}(k+1)] = \varnothing$. This would mean that $X \subseteq \mathbf{2}^\omega - [x{\restriction}(k+1)]$ and so $X \subseteq Y = \mathrm{cl}(X) \cap (\mathbf{2}^\omega - [x{\restriction}(k+1)])$, which is a contradiction, since Y is a closed superset of X with $Y \subset \mathrm{cl}(X)$. Therefore, there exists an $x' \in X$ such that $x' \in [x{\restriction}(k+1)]$, that is, $x'{\restriction}(k+1) = x{\restriction}(k+1)$. Similarly, there exists a $y' \in X$ such that $y'{\restriction}(k+1) = y{\restriction}(k+1)$. As a result, $k = h(x', y') \in H(X)$. □

This directly leads to the following alternative definition of $\mathcal{F}_X$ through covers of closed sets.

Corollary 2.10. *Let $X \subseteq \mathbf{2}^\omega$. For every $a \subseteq \omega$, $a \in \mathcal{F}_X$ if and only if there exists a countable collection of closed sets $(Y_n)_{n<\omega}$ such that $X \subseteq \bigcup_{n<\omega} Y_n$ and $a \supseteq \bigcup_{n<\omega} H(Y_n)$*

2.2 Lebesgue Measurability and Slaloms

The fourth step in Raisonnier's proof revolves around using the assumption that every $\mathbf{\Sigma}^1_2$ set is Lebesgue measurable to obtain that $\mathcal{F}_X$ is rapid for $X = \mathbf{2}^\omega \cap L[a]$, where $\aleph_1^{L[a]} = \aleph_1$. By the above discussion, this would indicate that, though uncountable, $\mathbf{2}^\omega \cap L[a]$ is narrow enough in the sense that it can be covered by countable collections of sets with sparse splitting points. Raisonnier's original proof of this was very technical and does not seem to offer much insight as to why this should be the case. In Chapter 2 of Bartoszyński and Judah's book [5], methods similar to those in Raisonnier's article are used to connect Lebesgue measurability and the concept of *slaloms*. This, in turn, offers a much more intuitive explanation as to *why* $\mathcal{F}_X$ is rapid. We begin with the basic definition.

Definition 2.11. Let $\varphi : \omega \to \omega$ be an unbounded function. A function $S : \omega \to [\omega]^{<\omega}$ is called a *φ-slalom* if and only if $|S(n)| \leq \varphi(n)$ for every $n \in \omega$. We will simply refer to slaloms instead of φ-slaloms when $\varphi(n) = n^2$. We say that a real $f \in \omega^\omega$ *goes through the (φ-)slalom S*, and write $f \in^* S$, if and only if $f(n) \in S(n)$ for all but finitely many $n \in \omega$. We will say that a set $X \subseteq \omega^\omega$ *goes through the (φ-)slalom S* if and only if $f \in^* S$ for every $f \in X$.

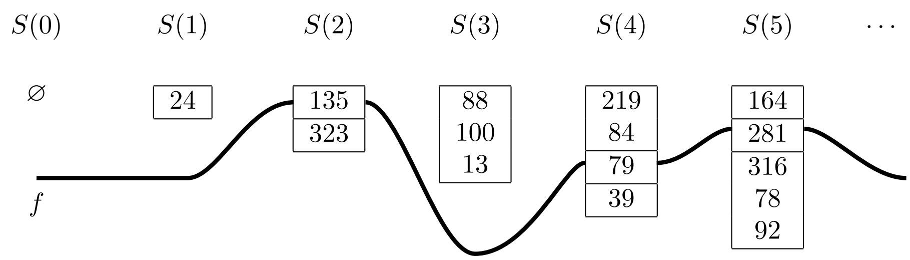

Figure 2.1: A φ-slalom S with $\varphi(n) = n$ and a real $f \in^* S$.

In other words, a slalom is a function assigning to each $n \in \omega$ a set of at most n^2 natural numbers. Based on an illustration in Bartoszyński's paper [3], their name appears to be inspired by slalom skiing, in which the skier must go through narrow pairs of poles. In this analogy, the real corresponds to the skier and must eventually go through each $S(n)$ of a slalom S.

While we define φ-slaloms for any unbounded $\varphi \in \omega^\omega$, for some purposes the particular choice of φ may not make a difference. This does not mean we may omit the bound from a definition of a slalom and instead only ask that each set $S(n)$ be finite [6, pp. 422-423].

Definition 2.12. We denote the set of φ-slaloms by $\mathcal{S}_\varphi$. When $\varphi(n) = n^2$, we will simply write $\mathcal{S}$ for $\mathcal{S}_\varphi$.

In the second chapter of [5], Bartoszyński and Judah show that by defining special maps between the set $\mathcal{S}$ and the ideal $\mathcal{N}$ of null sets, it is possible to obtain the following connection between slaloms and Lebesgue measurability.

Theorem 2.13. *The following are equivalent for every $a \in \omega^\omega$:*

1. *every $\Sigma^1_2(a)$ set of reals is Lebesgue measurable, and*
2. *there exists an $S \in \mathcal{S}$ such that every $f \in \omega^\omega \cap L[a]$ goes through S.*

This result is derived by applying their Theorem 2.3.11 to the constructible universe $L[a]$, along with a well-known characterization of $\boldsymbol{\Sigma}^1_2$ measurability by Solovay. Solovay's characterization relates the measurability of all $\Sigma^1_2(a)$ sets to a statement about the null sets of $L[a]$ and is established through the notions of *Borel codes* and *random reals*. More information can be found in Solovay's article [21] and in [13, pp. 178-179]. In addition to this, it is possible to obtain the following combinatorial characterizations of $\mathcal{N}$ in terms of slaloms.

Theorem 2.14 ([5, Thm. 2.3.9])**.** *We have that* $\mathrm{add}(\mathcal{N}) = \mathfrak{b}(\omega^\omega, \mathcal{S}, \in^*)$ *and* $\mathrm{cof}(\mathcal{N}) = \mathfrak{d}(\omega^\omega, \mathcal{S}, \in^*)$.

In other words, $\mathrm{add}(\mathcal{N})$ is the least cardinality of a set $X \subseteq \omega^\omega$ that does not go through any slalom and $\mathrm{cof}(\mathcal{N})$ is the least number of slaloms necessary for every real in ω^ω to go through at least one of them.

It is easy to observe that if a real $f \in \omega^\omega$ goes through a slalom $S \in \mathcal{S}$, then it must also be dominated by the real whose values consist of the maximum elements of each $S(n)$. Let us introduce the following notation.

Definition 2.15. Let $S \in \mathcal{S}$ be a slalom. Let $\max(S) \in \omega^\omega$ denote the real such that $\max(S)(n) = \max S(n)$.

Therefore, if $f \in^* S$, it will also be dominated by $\max(S)$. This fact, along with Theorem 2.14, gives us a simple proof of the following well-known inequalities.

Theorem 2.16. $\mathrm{add}(\mathcal{N}) \leq \mathfrak{b}$ *and* $\mathrm{cof}(\mathcal{N}) \geq \mathfrak{d}$.

Proof. If $F \subseteq \omega^\omega$ is an unbounded family of size $\mathfrak{b}$, then it cannot go through any slalom $S \in \mathcal{S}$, for it would be dominated by $\max(S)$. Therefore, $\mathrm{add}(\mathcal{N}) \leq |F| = \mathfrak{b}$.

Let now $\mathcal{C} \subseteq \omega^\omega$ be a collection of slaloms of size $\mathrm{cof}(\mathcal{N})$ such that for every $f \in \omega^\omega$ there exists an $S \in \mathcal{C}$ with $f \in^* S$. Then, for every $f \in \omega^\omega$, there exists an $S \in \mathcal{C}$ such that $f \leq^* \max(S)$. As a result $F = \{\max(S) \mid S \in \mathcal{C}\}$ is a dominating family and $\mathrm{cof}(\mathcal{N}) = |\mathcal{C}| \geq |F| \geq \mathfrak{d}$. $\square$

2.3 Proving Shelah's Theorem

We are now ready to give a proof of Shelah's Theorem 2.1, following the argument of Bartoszyński and Judah in [5, pp. 474-475]. For this, we will need the following result, which corresponds to the fourth step of Raisonnier's proof.

Theorem 2.17 (Raisonnier, [18])**.** *If every* $\mathbf{\Sigma}^1_2$ *set is Lebesgue measurable and* $a \in \omega^\omega$, *then* $\mathcal{F}_{\mathbf{2}^\omega \cap L[a]}$ *is a rapid filter.*

Proof. Let $a \in \omega^\omega$ and let $X = \mathbf{2}^\omega \cap L[a]$. We will show that $\mathcal{F}_X$ is rapid using Lemma 1.39. Let $f \in \mathcal{P}$ be a partitioning real. By our assumption of $\mathbf{\Sigma}^1_2$ Lebesgue measurability and Theorem 2.13, there exists a slalom S such that $\omega^\omega \cap L[a, f]$ goes through S. Let $s : \omega \to \mathbf{2}^{<\omega}$ be a computable bijective enumeration of all finite sequences from $\{0, 1\}$. Define $I_n = [f(n), f(n+1))$ for all $n \in \omega$. For any $x \in X$ and $n \in \omega$, $x \restriction I_n$ is a finite sequence in $\mathbf{2}^{<\omega}$ and thus there exists a unique $q_{x,n} \in \omega$ such that $s(q_{x,n}) = x \restriction I_n$. So, given an $x \in X$, we can define a unique real $g_x \in \omega^\omega$ such that $x \restriction I_n = s(g_x(n))$ for all n. Let $G = \{g_x \in \omega^\omega \mid x \in X\}$. The mapping $x \mapsto g_x$ is a bijection between X and G. Any $x \in X \subseteq \mathbf{2}^\omega \cap L[a, f]$ occurs at some level $L_\alpha[a, f]$ of the constructible hierarchy relative to a and f. For this reason, we can use the formula

$$(n, q) \in \omega^2 \wedge \forall i \in [f(n), f(n+1))\, (x(i) = s(q)(i))$$

to define g_x in $L_{\alpha+1}[a,f]$. Therefore, $G \subseteq \omega^\omega \cap L[a,f]$ and it goes through the slalom S.

Define $B : \omega \to \left[\mathbf{2}^{<\omega}\right]^{<\omega}$ such that for all $n \in \omega$, $B(n) = s[S(n)]$. The function B maps an $n \in \omega$ to the set of finite sequences corresponding to the natural numbers in $S(n)$. For every $x \in X$, $g_x \in^* S$, which means that for all but finitely many $n \in \omega$, $g_x(n) \in S(n)$. As a result, for every $x \in X$ we have that

$$x{\restriction}I_n = s(g_x(n)) \in s[S(n)] = B(n)$$

for all but finitely many $n \in \omega$.

This allows us to define a countable cover for X. For any $n \in \omega$ and $t \in \mathbf{2}^{f(n)}$, let

$$Y_{n,t} = \{x \in X \mid x \in [t],\ \forall m \geq n\,(x{\restriction}I_m \in B(m))\}$$

If $x \in X$, by the above, there exists an $n \in \omega$ such that $x{\restriction}I_m \in B(m)$ for all $m \geq n$ and so $x \in Y_{n,x\restriction f(n)}$. Therefore,

$$X \subseteq \bigcup_{n\in\omega} \bigcup_{t\in\mathbf{2}^{f(n)}} Y_{n,t}\ .$$

Define now the set

$$b = \{f(n) + h(t,u) \mid n \in \omega,\ t,u \in B(n)\}\ .$$

If $k \in \bigcup_{n\in\omega} \bigcup_{t\in\mathbf{2}^{f(n)}} H(Y_{n,t})$, then there exist $n \in \omega$, $t \in \mathbf{2}^{f(n)}$ and $x,y \in Y_{n,t}$ such that $h(x,y) = k$. As f is strictly increasing and $f(0) = 0$, there exists a unique $m \in \omega$ such that $k \in I_m$. By definition, it must be that $m \geq n$, as $x{\restriction}f(n) = y{\restriction}f(n) = t$. Since k is the splitting point of x and y, we can write that $k = f(m) + h(x{\restriction}I_m, y{\restriction}I_m)$. As $m \geq n$, $x{\restriction}I_m, y{\restriction}I_m \in B(m)$ and so $k \in b$. Therefore, $b \in \mathcal{F}_X$.

All that is left is to show that b is a witness according to Lemma 1.39. For any $n \in \omega$, we have that

$$\begin{aligned} b \cap f(n) &= \{f(m) + h(t,u) < f(n) \mid m \in \omega,\ t,u \in B(m)\} \\ &\subseteq \{f(m) + h(t,u) \mid m < n,\ t,u \in B(m)\} \\ &= \bigcup_{m=0}^{n-1} \{f(m) + h(t,u) \mid t,u \in B(m)\}\ , \end{aligned}$$

and therefore

$$|b \cap f(n)| \leq \sum_{m=0}^{n-1} |\{(t,u) \mid t,u \in B(m)\}| \leq \sum_{m=0}^{n-1} |B(m)|^2 \leq \sum_{m=0}^{n-1} m^4 \leq n^5\ .$$

$\square$

Proof of Theorem 2.1. Assume that every $\mathbf{\Sigma}^1_3$ set of reals is Lebesgue measurable and, towards contradiction, that there exists an $a \in \omega^\omega$ such that $\aleph_1^{L[a]} = \aleph_1$. The latter assumption implies that $\mathcal{F}_{\mathbf{2}^\omega \cap L[a]}$ is a nontrivial filter by Proposition 2.5. By our assumption, every $\mathbf{\Sigma}^1_2$ set of reals is Lebesgue measurable and, by Theorem 2.17, $\mathcal{F}_{\mathbf{2}^\omega \cap L[a]}$ is a rapid filter. Therefore, by Theorem 1.35, it is nonmeasurable. Finally, by Theorem 1.49, $\mathbf{2}^\omega \cap L[a]$ is $\mathbf{\Sigma}^1_2$ and, because of Lemma 2.8, $\mathcal{F}_{\mathbf{2}^\omega \cap L[a]}$ is a $\mathbf{\Sigma}^1_3$ subset of $\mathbf{2}^\omega$. This is a contradiction and we obtain that $\aleph_1^{L[a]} < \aleph_1$ for all $a \in \omega^\omega$, leading to the conclusion by Proposition 1.47. $\square$

Chapter 3

Properties of the Raisonnier Filter

Raisonnier introduced the filter $\mathcal{F}_X$ in his 1984 paper [18], giving the statement in Proposition 2.6 as the original definition. There, he defined it only for subsets of the Cantor space with a wellorder of type $\aleph_1$. Subsequent mentions of it in the literature have also mostly been in relation to the proof of Theorem 2.1 and the filter itself has remained underutilized outside of its original context. In this section, we will present some of the properties of $\mathcal{F}_X$ for various $X \subseteq \mathbf{2}^\omega$.

The first property relates the Raisonnier filter of a set to those of its subsets. As any countable cover of a set is also a cover of its subsets, we immediately obtain the following.

Proposition 3.1. *Let $X, Y \subseteq \mathbf{2}^\omega$ such that $X \subseteq Y$. Then, $\mathcal{F}_Y \subseteq \mathcal{F}_X$.*

Proof. Let $a \in \mathcal{F}_Y$. By definition, $a \supseteq \bigcup_{n<\omega} H(Y_n)$ for some countable cover $(Y_n)_{n<\omega}$ of Y. By assumption, it is a cover of $X \subseteq Y \subseteq \bigcup_{n<\omega} Y_n$ as well and $a \in \mathcal{F}_X$. $\square$

As mentioned in the previous chapter, the Raisonnier filter gives us an indication of the size or "width" of a set, as it contains those splitting points that must eventually occur in any countable cover or partition of the set. Therefore, the statement $\mathcal{F}_Y \subseteq \mathcal{F}_X$ tells us that X is possibly "narrower" that Y. This is because its Raisonnier filter extends that of Y, which in turn means that it contains sparser elements (as filters are upwards closed) and thus there are partitions of X with fewer splitting points. This can be the case when $X \subseteq Y$.

Another intuitive fact is that in order to obtain the Raisonnier filter of a countable union of sets, it is enough to take unions of elements of each Raisonnier filter. Countable covers of each set can be combined into a countable cover of their union and the corresponding set of splitting points will contain the splitting points of each cover.

Proposition 3.2. *Let* $\mathcal{X} = \{X_n \subseteq \mathbf{2}^\omega \mid n < \omega\}$ *be a countable collection of subsets of the Cantor space. Then,*

$$\mathcal{F}_{\bigcup \mathcal{X}} = \left\{ \bigcup_{n<\omega} a_n \,\middle|\, \forall n < \omega \, (a_n \in \mathcal{F}_{X_n}) \right\}.$$

Proof. Let $\mathcal{X}$ be as above and let $X = \bigcup \mathcal{X}$. Consider an arbitrary collection $\{a_n \in \mathcal{F}_{X_n} \mid n < \omega\}$. For every $n < \omega$ there exists a countable cover $(X_{n,m})_{m<\omega}$ of X_n such that $a_n \supseteq \bigcup_{m<\omega} H(X_{n,m})$. Let $a = \bigcup_{n<\omega} a_n$. We have that

$$X = \bigcup_{n<\omega} X_n \subseteq \bigcup_{n<\omega} \bigcup_{m<\omega} X_{n,m}$$

$$a = \bigcup_{n<\omega} a_n \supseteq \bigcup_{n<\omega} \bigcup_{m<\omega} H(X_{n,m})$$

meaning that $a \in \mathcal{F}_X$.

On the other hand, let $a \in \mathcal{F}_X$. For any $n < \omega$, $X_n \subseteq X$ and so $\mathcal{F}_X \subseteq \mathcal{F}_{X_n}$, meaning that $a \in \mathcal{F}_{X_n}$. So, $a = \bigcup_{n<\omega} a_n$ for $a_n = a \in \mathcal{F}_{X_n}$ for all $n < \omega$. $\square$

In regards to intersections, intersecting the countable covers of two sets will produce a countable cover of their intersection. Therefore, we can obtain the following weaker result.

Proposition 3.3. *If* $X, Y \subseteq \mathbf{2}^\omega$*, then* $\mathcal{F}_{X \cap Y} \supseteq \{b \cap c \mid b \in \mathcal{F}_X,\, c \in \mathcal{F}_Y\}$.

Proof. Let $X, Y \subseteq \mathbf{2}^\omega$. Let $b \in \mathcal{F}_X$ with $b \supseteq \bigcup_{n<\omega} H(X_n)$, where $X \subseteq \bigcup_{n<\omega} X_n$ and $c \in \mathcal{F}_Y$ with $c \supseteq \bigcup_{n<\omega} H(Y_n)$, where $Y \subseteq \bigcup_{n<\omega} Y_n$. Take the collection of sets $\{Z_{m,n} \mid m, n \in \omega\}$, where $Z_{m,n} = X_m \cap Y_n$.

If $x \in X \cap Y$, then there exists an $m \in \omega$ such that $x \in X_m$ and an $n \in \omega$ such that $n \in Y_n$, in which case $x \in X_m \cap Y_n = Z_{m,n}$. Therefore, $X \cap Y \subseteq \bigcup_{m<\omega} \bigcup_{n<\omega} Z_{m,n}$.

Moreover, let $m, n \in \omega$ and let $k \in H(Z_{m,n})$. By definition, there exist $x, y \in Z_{m,n}$ such that $h(x, y) = k$. Since $x, y \in X_m$, $k \in H(X_m)$ and, similarly, as $x, y \in Y_n$, $k \in H(Y_n)$. Thus, $k \in b$ and $k \in c$. As a result, $\bigcup_{m<\omega} \bigcup_{n<\omega} H(Z_{m,n}) \subseteq b \cap c$ and $b \cap c \in \mathcal{F}_{X \cap Y}$. $\square$

Our next result provides an example of a Raisonnier filter with a concrete characterization. First, let us introduce the following notation.

Definition 3.4. For any $a, b \subseteq \omega$, we say that *a is almost a subset of b*, and write $a \subseteq^* b$, if and only if $a - b$ is finite.

The term "almost" in the above definition is justified, since whenever $a, b \subseteq \omega$ and $a - b$ is finite, there are only finitely many $n \in a$ not in b. As a result, $n \in b$ for all but finitely many $n \in a$.

Assume that we have some infinite $a \subseteq \omega$. It is easy to see that, when viewed as reals in $\mathbf{2}^\omega$, its subsets can only have splitting points in the elements of a: if a natural number k is not in a, then any two subsets of a will have the value 0 at k and thus cannot have k as a splitting point. On the other hand, the reals in $\mathcal{P}(a)$ can take any value at every $k \in a$, as any subset of a may include or exclude k. This means that eventually every $k \in a$ must appear as a splitting point in a countable covering or partition. In other words, every $b \in \mathcal{F}_{\mathcal{P}(a)}$ must eventually contain every $k \in a$. Therefore, the filter $\mathcal{F}_{\mathcal{P}(a)}$ must consist of the almost supersets of a.

Proposition 3.5. *For every $a \in [\omega]^\omega$, $\mathcal{F}_{\mathcal{P}(a)} = \{b \in \mathbf{2}^\omega \mid a \subseteq^* b\}$.*

Proof. Let $a \in [\omega]^\omega$. First, we will show that $a \in \mathcal{F}_{\mathcal{P}(a)}$. If $k \in H(\mathcal{P}(a))$, then there exist $x, y \subseteq a$ such that $h(x, y) = k$. By definition it must be that $k \in a$, for otherwise it would be that $x(k) = y(k) = 0 = a(k)$, whereas, by assumption, $x(k) \neq y(k)$. On the other hand, for any $k \in a$, if x and y are the characteristic functions of $\varnothing$ and $\{k\}$ respectively, $h(x, y) = k$ and $k \in H(\mathcal{P}(a))$. Therefore, $a = H(\mathcal{P}(a))$ and so $a \in \mathcal{F}_{\mathcal{P}(a)}$.

Let now $b \in \mathbf{2}^\omega$ such that $a \subseteq^* b$. By assumption, there exists an $N \in \omega$ such that $k \in b$ for all $k \in a$ with $k \geq N$. Because $\mathcal{F}_{\mathcal{P}(a)}$ contains Fréchet's filter, it is closed under finite modifications of its elements and will also contain $c = a - N = \{k \in a \mid k \geq N\}$, as $c = a \cap (\omega - N)$. Therefore, $c \subseteq b \in \mathcal{F}_{\mathcal{P}(a)}$.

On the other hand, let $b \in \mathcal{F}_{\mathcal{P}(a)}$ and assume towards contradiction that $|a - b| = \aleph_0$. In this case, we have some $a' \subseteq a$ with $|a'| = \aleph_0$ and $a' \cap b = \varnothing$. Let $\hat{a}' \in \omega^\omega$ be its increasing enumeration. By definition, $b \supseteq \bigcup_{n<\omega} H(Z_n)$ for some countable covering $(Z_n)_{n<\omega}$ of $\mathcal{P}(a)$.

Through diagonalization, we can construct a real in $\mathcal{P}(a)$ that is not in the covering. We will define a sequence $\{x_n \in \mathbf{2}^{\hat{a}'(n)} \mid n < \omega\}$ as follows. First, let $x_0(k) = 0$ for all $k < \hat{a}'(0)$. Assuming we have defined $x_n \in \mathbf{2}^{\hat{a}'(n)}$, let $x_{n+1} \restriction \hat{a}'(n) = x_n$. We have the following three cases:

1. There exists no $z \in Z_n$ such that $z \restriction \hat{a}'(n) = x_n$. In this case, we can just let $x_{n+1}(\hat{a}'(n)) = 0$.

2. There is an $i \in \{0, 1\}$ such that $z(\hat{a}'(n)) = i$ for every $z \in Z_n$ with $z \restriction \hat{a}'(n) = x_n$. We can then set $x_{n+1}(\hat{a}'(n)) \neq i$.

3. There exist $z, y \in Z_n$ such that $z \restriction \hat{a}'(n) = y \restriction \hat{a}'(n)$ and $z(\hat{a}'(n)) \neq y(\hat{a}'(n))$. This case is impossible, for we would have that $\hat{a}'(n) = h(z, y) \in H(Z_n) \subseteq b$, whereas $a' \cap b = \varnothing$.

Finally, let $x_{n+1}(k) = a(k)$ for all $k \in \omega$ with $\hat{a}'(n) < k < \hat{a}'(n+1)$. Let $x = \bigcup_{n<\omega} x_n \in \mathbf{2}^\omega$, which is well defined, as $x_n \sqsubseteq x_{n+1}$ for all $n < \omega$ by construction. Assume that for some $k \in \omega$ we have $x(k) = 1$. Then, $x_{n+1}(k) = 1$ for some $n < \omega$ such that $\hat{a}'(n) \leq k < \hat{a}'(n+1)$. If $k = \hat{a}'(n)$, then, since $\hat{a}'$ is the increasing enumeration of a, $a(k) = 1$. Otherwise, by construction, $a(k) = x_{n+1}(k) = 1$ as well. Therefore, $x \subseteq a$ and $x \in \mathcal{P}(a)$.

However, for any $n < \omega$ we again the following two cases.

1. There exists no $z \in Z_n$ such that $z \restriction \hat{a}'(n) = x_n$. In this case, it cannot be that $x \in Z_n$, for x itself would be such a real.

2. There is an $i \in \{0,1\}$ such that $z(\hat{a}'(n)) = i$ for every $z \in Z_n$ with $z \restriction \hat{a}'(n) = x_n$. In this case, $x \notin Z_n$, for otherwise $x(\hat{a}'(n)) = x_{n+1}(\hat{a}'(n)) \neq z(\hat{a}'(n))$ for some $z \in Z_n$. This would mean that $h(x,z) = \hat{a}'(n) \in H(Z_n) \subseteq b$, which would be a contradiction.

Therefore, $x \notin Z_n$. We have thus found an $x \in \mathcal{P}(a) - \bigcup_{n<\omega} Z_n$, which is a contradiction, as we assumed that $\mathcal{P}(a) \subseteq \bigcup_{n<\omega} Z_n$. As a result, it must be that $|a-b| < \aleph_0$ and $a \subseteq^* b$. $\square$

Note that the above proposition can also be expressed in terms of the set $[a]_{=^*}$ of reals that are eventually equal to a. If $c =^* a$, then there exists an $m \in \omega$ such that $c(k) = a(k)$ for all $k \geq m$. So, there can only be finitely many $k \in a$ such that $k \notin c$ and $a \subseteq^* c$. As a result, c and every $b \supseteq c$ are elements of $\mathcal{F}_{\mathcal{P}(a)}$. If now $b \in \mathcal{F}_{\mathcal{P}(a)}$, then $b \subseteq^* a$, $a - b$ is finite, and there exists some $m \in \omega$ such that $k \in b$ for all $k \in a$ with $k \geq m$. Let now $c = a \cap b$ and $k \geq m$. If $a(k) = 0$, then $k \notin a \supseteq c$ and so $c(k) = 0$ as well. If $a(k) = 1$, then $b(k) = 1$ and $c(k) = 1$. Therefore, $b \supseteq c =^* a$. This directly leads to the following.

Corollary 3.6. *For every $a \in [\omega]^\omega$, $\mathcal{F}_{\mathcal{P}(a)}$ is the filter generated by $[a]_{=^*}$.*

Proposition 3.5 also gives us a direct characterization of the Raisonnier filter of the entire space $\mathbf{2}^\omega$.

Corollary 3.7. *$\mathcal{F}_{\mathbf{2}^\omega}$ is Fréchet's filter.*

Proof. By Proposition 3.5, we can write

$$\mathcal{F}_{\mathbf{2}^\omega} = \mathcal{F}_{\mathcal{P}(\omega)} = \{b \subseteq \omega \mid |\omega - b| < \aleph_0\}.$$

Therefore, every $b \in \mathcal{F}_{\mathbf{2}^\omega}$ is a cofinite subset of ω. Since $\mathcal{F}_{\mathbf{2}^\omega}$ extends Fréchet's filter, we obtain that $\mathcal{F}_{\mathbf{2}^\omega} = \mathscr{F}$. $\square$

This result is in agreement with our intuition about the Raisonnier filter: the entire space $\mathbf{2}^\omega$ is too "wide" to be partitioned into countably many subsets that do not eventually contain every splitting point. Because the elements of the Raisonnier filter depend on the splitting points that are eventually necessary, it is the eventual values of the reals of a set that determine its Raisonnier filter. This is confirmed by the following proposition, according to which the set $[X]_{=^*}$ of reals that are eventually equal to some $x \in X$ has the same Raisonnier filter as X.

Proposition 3.8. *For every $X \subseteq \mathbf{2}^\omega$, $\mathcal{F}_X = \mathcal{F}_{[X]_{=^*}}$.*

Proof. Let $X \subseteq \mathbf{2}^\omega$. As $X \subseteq [X]_{=^*}$ by the definition of $=^*$, we have that $\mathcal{F}_{[X]_{=^*}} \subseteq \mathcal{F}_X$. Therefore, it is enough to prove the converse.

Let $a \in \mathcal{F}_X$ such that $a \supseteq \bigcup_{n<\omega} H(Y_n)$, where $X \subseteq \bigcup_{n<\omega} Y_n$. Consider the following collection of sets: for $n < \omega$ and $s, t \in \mathbf{2}^{<\omega}$ we define

$$Z_{n,s,t} = \{x \in \mathbf{2}^\omega \mid x \in [s],\, \exists y \in Y_n \cap [t]\, \forall m \geq |t|\, (x(m) = y(m))\}.$$

Let $x \in [X]_{=^*}$. By definition, there exists a $y \in X$ such that $x =^* y$. Since $y \in X$, there exists an $n < \omega$ such that $y \in Y_n$. Moreover, because $x =^* y$, there exists an $N \in \omega$ such that $x(m) = y(m)$ for all $m \geq N$. Let $s = x \restriction N$ and $t = y \restriction N$. By definition, $x \in [s]$, $y \in Y_n \cap [t]$, $|s| = |t| = N$ and $x \in Z_{n,s,t}$. Therefore,

$$[X]_{=^*} \subseteq \bigcup \{Z_{n,s,t} \mid n < \omega,\, s,t \in \mathbf{2}^{<\omega}, |s| = |t|\}\,.$$

Let now $n < \omega$, $s,t \in \mathbf{2}^{<\omega}$ such that $|s| = |t| = N$ and $k \in H(Z_{n,s,t})$. By definition, there exist $x, y \in Z_{n,s,t}$ with $x \neq y$ and $h(x,y) = k$. Because $x, y \in [s]$, it must be that $k \geq |s|$. By definition, there exist $x', y' \in Y_n \cap [t]$ such that $x(m) = x'(m)$ and $y(m) = y'(m)$ for all $m \geq N$. Since x', y' share the same initial segment t up to N and are equal to x and y respectively from N onwards, whose splitting point comes after N, we have that $k = h(x,y) = h(x',y') \in H(Y_n)$. Therefore, $H(Z_{n,s,t}) \subseteq H(Y_n)$. As a result,

$$a \supseteq \bigcup_{n<\omega} H(Y_n) \supseteq \bigcup \{H(Z_{n,s,t}) \mid n < \omega,\, s,t \in \mathbf{2}^{<\omega},\, |s| = |t|\}$$

and thus $a \in \mathcal{F}_{[X]_{=^*}}$. We have therefore proven that $\mathcal{F}_X \subseteq \mathcal{F}_{[X]_{=^*}}$. □

Another intuitive observation is that shifting the elements of a set to the right by adding a finite sequence to the beginning leads to their splitting points being shifted as well. This in turn leads to a new Raisonnier filter, whose elements are also shifted to the right.

Definition 3.9. If $a \subseteq \omega$ and $n \in \omega$ we will write $a+n$ for the set $\{k+n \mid k \in a\}$. If $X \subseteq \mathbf{2}^\omega$ and $s \in \mathbf{2}^{<\omega}$, we will write $s^\frown X$ for the set $\{s^\frown x \mid x \in X\}$.

Proposition 3.10. *If $X \subseteq \mathbf{2}^\omega$ and $s \in \mathbf{2}^{<\omega}$, then $\mathcal{F}_{s^\frown X}$ is generated by the set $\{a + |s| \mid a \in \mathcal{F}_X\}$.*

Proof. Let $a \in \mathcal{F}_X$ such that $a \supseteq \bigcup_{n<\omega} H(Y_n)$, where $X \subseteq \bigcup_{n<\omega} Y_n$. For every $n < \omega$ let $Z_n = s^\frown Y_n$. If $y \in s^\frown X$, then $y = s^\frown x$ for some $x \in X$, which in turn means that $x \in Y_n$ for some $n < \omega$ and $y \in Z_n$. Therefore, $s^\frown X \subseteq \bigcup_{n<\omega} Z_n$. If now $k \in H(Z_n)$ for some $n < \omega$, there exist $x, y \in Y_n$ such that $x \neq y$, $k = h(s^\frown x, s^\frown y) = |s| + h(x,y)$. We have that $h(x,y) \in H(Y_n)$ and so $k \in a + |s|$. Therefore, $a + |s| \in \mathcal{F}_{s^\frown X}$.

On the other hand, let $b \in \mathcal{F}_{s^\frown X}$ such that $b \supseteq \bigcup_{n<\omega} H(Y_n)$, where $s^\frown X \subseteq \bigcup_{n<\omega} Y_n$. For any $n < \omega$, let $Z_n = \{x \in X \mid s^\frown x \in Y_n\}$. If $x \in X$, then $s^\frown x \in Y_n$ for some $n < \omega$ and $x \in Z_n$. Therefore, $X \subseteq \bigcup_{n<\omega} Z_n$ and $\bigcup_{n<\omega} H(Z_n) \in \mathcal{F}_X$. If now $k \in H(Z_n)$, then there exist $x, y \in X$ such that $s^\frown x, s^\frown y \in Y_n$, $x \neq y$ and $h(x,y) = k$. We then have that $k + |s| = h(s^\frown x, s^\frown y) \in H(Y_n)$. As a result, $\left(\bigcup_{n<\omega} H(Z_n)\right) + |s| \subseteq b$. □

Combining this result with Corollary 3.7, we obtain the following in regards to the Raisonnier filter of basic open sets.

Proposition 3.11. *For every $s \in \mathbf{2}^{<\omega}$, $\mathcal{F}_{[s]}$ is Fréchet's filter.*

Proof. For every $s \in \mathbf{2}^{<\omega}$ we have that $[s] = s^\frown \mathbf{2}^\omega$. If $a \in \mathcal{F}_{[s]}$, by Proposition 3.10, there exists a $b \in \mathcal{F}_{\mathbf{2}^\omega} = \mathscr{F}$ such that $b + |s| \subseteq a$. There exists an $n \in \omega$ such that $m \in b$ for every $m \geq n$ and therefore $m \in b + |s| \subseteq a$ for every $m \geq n + |s|$ as well, which means that a is cofinite. Thus, every element of $\mathcal{F}_{[s]}$ is cofinite and $\mathcal{F}_{[s]} = \mathscr{F}$. □

This agrees with our intuition that the elements of the Raisonnier filter depend on the eventual splitting points of a set. Since basic open sets are eventually as wide as the entire space, it is reasonable that their Raisonnier filter is equal to Fréchet's filter as well. In fact, this can be generalized to all open sets, as well as sets X whose interior $\mathrm{int}(X)$ is nonempty, using Proposition 3.1.

Corollary 3.12. *The following are true.*

1. *For every nonempty open set $X \subseteq \mathbf{2}^\omega$, $\mathcal{F}_X$ is Fréchet's filter.*
2. *For every $X \subseteq \mathbf{2}^\omega$, if $\mathrm{int}(X) \neq \varnothing$, then $\mathcal{F}_X$ is Fréchet's filter.*

Proof. In both cases there exists an $s \in \mathbf{2}^{<\omega}$ such that $[s] \subseteq X$ and therefore $\mathscr{F} \subseteq \mathcal{F}_X \subseteq \mathcal{F}_{[s]} = \mathscr{F}$. □

The converse of (2) does not hold, however, as we can find examples of sets with empty interior whose Raisonnier filter is nevertheless equal to $\mathscr{F}$. For example, if $a = \{2n \mid n \in \omega\}$ and $b = \{2n+1 \mid n \in \omega\}$, consider the set $X = \mathcal{P}(a) \cup \mathcal{P}(b)$. For every $s \in \mathbf{2}^{<\omega}$, the real $s^\frown 11\ldots$ is in $[s]$, but neither a subset of a nor b. As a result, X has empty interior. However, by Corollary 3.6 and Proposition 3.1, $\mathcal{F}_X$ is generated both by the sets $[a]_{=^*}$ and $[b]_{=^*}$. As a result, for any $c \in \mathcal{F}_X$ there exist $a' =^* a$ and $b' =^* b$ for which $a' \subseteq c$ and $b' \subseteq c$. Because of this, there exists some $n \in \omega$ such that both $c(2m) = a(2m) = 1$ and $c(2m+1) = b(2m+1) = 1$ for all $m \geq n$ and therefore c is cofinite. Since any $c \in \mathcal{F}_X$ is cofinite, $\mathcal{F}_X$ is Fréchet's filter.

We will end this section with the following interesting property regarding the Raisonnier filter of the constructible reals. It can easily be seen that every finite sequence in $\mathbf{2}^{<\omega}$ and the concatenation operation are definable by formulas. Therefore, whenever $s \in \mathbf{2}^{<\omega}$ and $x \in \mathbf{2}^\omega \cap L[a]$, we also have that $s^\frown x \in \mathbf{2}^\omega \cap L[a]$. This means that constructing the set $\bigcup_{s \in \mathbf{2}^n} s^\frown(\mathbf{2}^\omega \cap L[a])$ for some $n \in \omega$ gives us $\mathbf{2}^\omega \cap L[a]$ itself. Using Proposition 3.10 we can then see that its Raisonnier filter must also be closed under shifting to the right.

Proposition 3.13. *For every $a \in \omega^\omega$, if $b \in \mathcal{F}_{\mathbf{2}^\omega \cap L[a]}$ and $n \in \omega$, then $b + n \in \mathcal{F}_{\mathbf{2}^\omega \cap L[a]}$.*

Proof. Let $X = \mathbf{2}^\omega \cap L[a]$ for some $a \in \omega^\omega$. As mentioned before, it is easy to see that for any $n \in \omega$ we can write $X = \bigcup_{s \in \mathbf{2}^n} s^\frown X$. By Proposition 3.2, we then have that

$$\mathcal{F}_X = \left\{ \bigcup_{s \in \mathbf{2}^n} b_s \,\middle|\, \forall s \in \mathbf{2}^n \, (b_s \in \mathcal{F}_{s^\frown X}) \right\}.$$

By Proposition 3.10, for every $s \in \mathbf{2}^n$ is the set $\mathcal{G}$ generated by the set $\{b+n \mid b \in \mathcal{F}_X\}$. Therefore, we can write $\mathcal{F}_X$ as

$$\mathcal{F}_X = \left\{ \bigcup_{s \in \mathbf{2}^n} b_s \;\middle|\; \forall s \in \mathbf{2}^n \, (b_s \in \mathcal{G}) \right\}.$$

As a result, if $b \in \mathcal{F}_X$, then $b+n \in \mathcal{G}$ and thus $b+n \in \mathcal{F}_X$. $\square$

By the above, if $b \in \mathcal{F}_{\mathbf{2}^\omega \cap L[a]}$, then we can cover $\mathbf{2}^\omega \cap L[a]$ with countably many sets with splitting points only within b. However, we can also do so with a countable cover whose splitting points are all in $b+n$. Because filters are closed under intersections it is also the case that $b \cap (b+n) = \{k \in b \mid k+n\} \in \mathcal{F}_X$. While this fact is trivial if $\mathbf{2}^\omega \cap L[a]$ is countable, if $\aleph_1^{L[a]} = \aleph_1$, then $\mathbf{2}^\omega \cap L[a]$, although uncountable, can be covered by sets with few splitting points.

Chapter 4

A Generalization of Raisonnier's Theorem

The focal point of Raisonnier's proof, Theorem 2.17, gives us a sufficient condition for the filter $\mathcal{F}_{\mathbf{2}^\omega \cap L[a]}$ being rapid for some $a \in \omega^\omega$. The assumption of $\mathbf{\Sigma}^1_2$ Lebesgue measurability leads to the statement that every $\mathbf{2}^\omega \cap L[a,d]$ goes through a slalom. Recall that the set $\mathcal{P}$ of *partitioning reals* consists of the strictly increasing $d \in \omega^\omega$ such that $d(0) = 0$. Using such a $d \in \mathcal{P}$, we can limit the possible values of the constructible reals in the intervals defined by d and thus obtain the desired result.

Even though it was originally defined for the set $\mathbf{2}^\omega \cap L[a]$, this proof method does not depend on it. We can apply it for any subset of $\mathbf{2}^\omega$, given that we can similarly limit the values of its reals in arbitrarily defined intervals, as we did for $\mathbf{2}^\omega \cap L[a]$ through the use of slaloms. In fact, it is possible to obtain a converse of this result and thus a characterization of the rapidity of $\mathcal{F}_X$ through slaloms.

4.1 Binary Slaloms and Translations

In the proof of Raisonnier's Theorem 2.17, slaloms are used to obtain a restriction on the possible finite sequences $x \restriction I_n$, where $I_n = [f(n), f(n+1))$ for some $f \in \mathcal{P}$. Let us introduce the following generalized notation for the intervals defined by partitioning reals. Recall that we defined the partitioning reals $\mathcal{P}$ as the set of reals $d \in \omega^\omega$ that are strictly increasing and $d(0) = 0$.

Definition 4.1. Given a real $d \in \mathcal{P}$ and an $n < \omega$, let $I^d_n = [d(n), d(n+1))$.

The restriction of the reals in this proof consisted of a sequence of sets $\{B(n)\}_{n<\omega}$, where $|B(n)| \leq n^2$ for all $n < \omega$, to which every real $x \in L[a]$ had to eventually conform, in the sense that eventually $x \restriction I_n \in B(n)$ for large enough $n < \omega$. This is already similar to a slalom, with finite sequences in $\{0,1\}$ instead of natural numbers. However, these sequences need to be of length $|I_n|$, for the statement $x \upharpoonright I_n \in B(n)$ to have a meaning. We can generalize this in order to obtain a notion of slaloms for the Cantor space.

Definition 4.2 (Binary Slaloms)**.** Let $d \in \mathcal{P}$ and $\varphi \in \omega^\omega$. A function $B : \omega \to \left[\mathbf{2}^{<\omega}\right]^{<\omega}$ is called a *d-binary φ-slalom* if and only if

1. $|B(n)| \leq \varphi(n)$, and
2. $B(n) \subseteq \mathbf{2}^{|I_n^d|}$,

for every $n \in \omega$. We will simply refer to d-binary slaloms instead of d-binary φ-slaloms when $\varphi(n) = n^2$.

We say that a real $x \in \mathbf{2}^\omega$ *goes through the d-binary (φ-)slalom B*, and write $x \in^* B$, if and only if $x \upharpoonright I_n^d \in B(n)$ for all but finitely many $n \in \omega$. Similarly, we say that a set $X \subseteq \mathbf{2}^\omega$ goes through the d-binary (φ-)slalom B if and only if every $x \in B$ goes through it. Let $\mathcal{B}_\varphi^d$ denote the set of all d-binary φ-slaloms. When $\varphi(n) = n^2$, we will simply write $\mathcal{B}^d$.

Therefore, each set $B(n)$ of a d-binary slalom contains finite sequences from the set $\{0, 1\}$ of length $|I_n^d|$. We view these as possible segments of a real in the interval I_n^d.

In order to relate binary slaloms in $\mathbf{2}^\omega$ and slaloms in ω^ω we need functions that allow us to translate reals between the two spaces accordingly.

Definition 4.3. Let $\mathrm{nat} : \mathbf{2}^{<\omega} \to \omega$ be the function mapping a finite sequence in $\mathbf{2}^{<\omega}$ to the natural number represented by the sequence in the binary system. Given a partitioning real $d \in \mathcal{P}$, let $\mathrm{nat}_d : \mathbf{2}^\omega \to \omega^\omega$ be the function defined as $\mathrm{nat}_d(x)(n) = \mathrm{nat}(x \upharpoonright I_n^d)$ for all $x \in \mathbf{2}^\omega$ and $n \in \omega$.

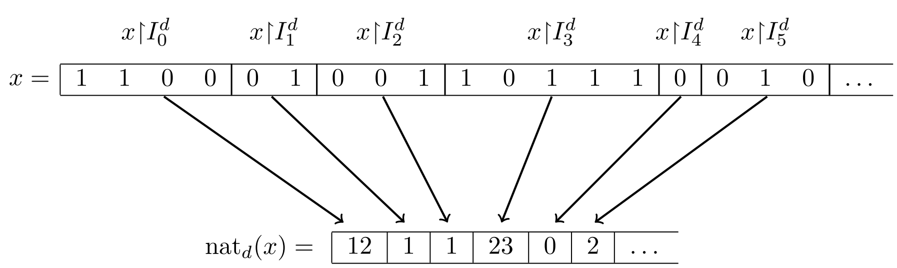

Figure 4.1: A real $x \in \mathbf{2}^\omega$ translated into $\mathrm{nat}_d(x) \in \omega^\omega$, using the partitioning real $d = (0, 4, 6, 9, 14, 15, 18, \dots)$.

For example, we have that $\mathrm{nat}(101) = 5$ and $\mathrm{nat}(1011) = \mathrm{nat}(01011) = 11$. The function nat_d maps a real $x \in \mathbf{2}^\omega$ to the real in ω^ω generated by partitioning x into the intervals I_n^d and interpreting the resulting sequences as natural numbers written in the binary system.

We can also define a converse translation from the Baire space to the Cantor space. In order to do this, given a real $d \in \mathcal{P}$ that partitions ω into intervals, we can take an $f \in \omega^\omega$, write each $f(n)$ in the binary system in the n-th interval I^d_n and concatenate these finite sequences.

Definition 4.4. Given a $k \in \omega$, let $\mathrm{bin}_k : \omega \to \mathbf{2}^k$ be the function mapping a natural number $m \in \omega$ to the last k digits of its binary representation (with leading zeroes, if necessary). Given a real $d \in \mathcal{P}$, let $\mathrm{bin}_d : \omega^\omega \to \mathbf{2}^\omega$ be the function mapping a real $f \in \omega^\omega$ to the real $\mathrm{bin}_d(f)$ such that for all $n \in \omega$, $\mathrm{bin}_d(f) \restriction I^d_n = \mathrm{bin}_{|I^d_n|}(f(n))$.

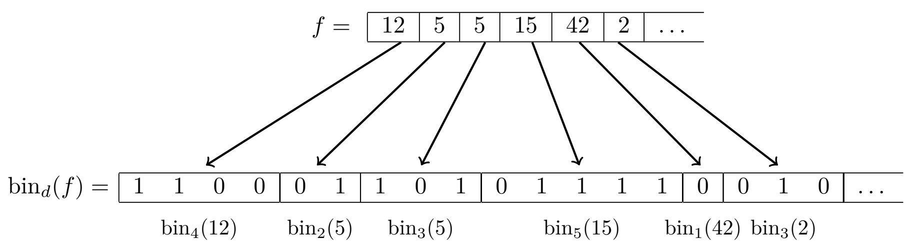

Figure 4.2: A real $f \in \omega^\omega$ translated into $\mathrm{bin}_d(x) \in \mathbf{2}^\omega$, using the partitioning real $d = (0, 4, 6, 9, 14, 15, 18, \dots)$.

Therefore, we have for example that $\mathrm{bin}_4(9) = 1001$ and $\mathrm{bin}_5(9) = 01001$. In order to compute $\mathrm{bin}_d(f)$ for a real $f \in \omega^\omega$, we need to write each $f(n)$ in the binary system using $|I^d_n| = (d(n+1) - d(n))$ digits and concatenate the finite sequences that are produced. The result can therefore also be written as

$$\mathrm{bin}_d(f) = \mathrm{bin}_{|I^d_0|}(f(0))^\frown \mathrm{bin}_{|I^d_1|}(f(1))^\frown \dots .$$

Note that, with this definition, the maps bin_k for $k \in \omega$ and bin_d for $d \in \mathcal{P}$ are not injective, since $\mathrm{bin}_k(n) = \mathrm{bin}_k(2^k + n)$ for all $n \in \omega$ with $n < 2^k$. This is because bin_k gives only the last k digits of the natural number written in binary. For example, we have that $\mathrm{bin}_4(7) = \mathrm{bin}_4(23) = 0111$. They are injective if we restrict bin_k to the set 2^k and bin_d to the set of reals $f \in \omega^\omega$ such that $f(n) < 2^{|I^d_n|}$ for all $n \in \omega$.

It is easy to see that these two maps give us a correspondence between the spaces $\mathbf{2}^\omega$ and ω^ω.

Lemma 4.5. *The following are true for the above maps* nat_d *and* bin_d *for every* $d \in \mathcal{P}$*:*

1. bin_d *is injective on the set* $\{f \in \omega^\omega \mid \forall n \in \omega\, (f(n) < 2^{|I^d_n|})\}$*;*

2. nat_d *is injective;*

3. $\mathrm{nat}(\mathrm{bin}_k(n)) = n$ *for every* $n \in \omega$ *such that* $n < 2^k$ *and* $\mathrm{nat}_d(\mathrm{bin}_d(f)) = f$ *for every* $f \in \omega^\omega$ *such that* $f(n) < 2^{|I^d_n|}$ *for all* $n \in \omega$*;*

4. $\mathrm{bin}_k(\mathrm{nat}(s)) = s$ *for every* $k \in \omega$ *and* $s \in \mathbf{2}^k$ *and* $\mathrm{bin}_d(\mathrm{nat}_d(x)) = x$ *for every* $x \in \mathbf{2}^\omega$*.*

Proof. For (1), let $f, g \in \omega^\omega$ such that $f(n) < 2^{|I_n^d|}$ and $g(n) < 2^{|I_n^d|}$ for all $n \in \omega$ and assume that $f \neq g$. This means that $f(n) \neq g(n)$ for some $n \in \omega$. Because $f(n), g(n) < 2^{|I_n^d|}$, the finite sequences $\mathrm{bin}_{|I_n^d|}(f(n))$ and $\mathrm{bin}_{|I_n^d|}(g(n))$ are exactly the binary representations of $f(n)$ and $g(n)$ respectively, and not just their last $|I_n^d|$ digits. It must then be that

$$\mathrm{bin}_d(f)(n) = \mathrm{bin}_{|I_n^d|}(f(n)) \neq \mathrm{bin}_{|I_n^d|}(g(n)) = \mathrm{bin}_d(g)(n)\,,$$

and thus $\mathrm{bin}_d(f) \neq \mathrm{bin}_d(g)$.

For (2), let $x, y \in \mathbf{2}^\omega$ such that $x \neq y$. By definition, there exists some $k \in \omega$ such that $x(k) \neq y(k)$. Because $d \in \mathcal{P}$, there exists an $n \in \omega$ such that $d(n) \leq k < d(n+1)$ and we have that $x{\restriction}I_n^d \neq y{\restriction}I_n^d$. By definition,

$$\mathrm{nat}_d(x)(n) = \mathrm{nat}(x{\restriction}I_n^d) \neq \mathrm{nat}(y{\restriction}I_n^d) = \mathrm{nat}_d(y)(n)$$

and thus $\mathrm{nat}_d(x) \neq \mathrm{nat}_d(y)$.

For (3), if $n \in \omega$ and $n < 2^k$, then it can be written in the binary system in k digits exactly as $\mathrm{bin}_k(n)$. By definition then, $\mathrm{nat}(\mathrm{bin}_k(n)) = n$. If $f \in \omega^\omega$ and $f(n) < 2^{|I_n^d|}$ for all $n \in \omega$, then we have that

$$\mathrm{nat}_d(\mathrm{bin}_d(f))(n) = \mathrm{nat}(\mathrm{bin}_d(f){\restriction}I_n^d) = \mathrm{nat}(\mathrm{bin}_{|I_n^d|}(f(n)) = f(n)$$

for all $n \in \omega$.

For (4), note that $\mathrm{bin}_k(\mathrm{nat}(s)) = s$ follows immediately from the definitions for every $s \in \mathbf{2}^k$, where $k \in \omega$, because $\mathrm{nat}(s) < 2^k$. If now $x \in \mathbf{2}^\omega$, then each $\mathrm{nat}_d(x)(n) = \mathrm{nat}(x{\restriction}I_n^d)$ is the interpretation of $x{\restriction}I_n^d$ as a natural number, which must be $\mathrm{nat}_d(x)(n) < 2^{|I_n^d|}$, since it is written in $|I_n^d|$ digits. Therefore,

$$\mathrm{bin}_d(\mathrm{nat}_d(x)){\restriction}I_n^d = \mathrm{bin}_{|I_n^d|}(\mathrm{nat}_d(x)) = \mathrm{bin}_{|I_n^d|}(\mathrm{nat}(x{\restriction}I_n^d)) = x{\restriction}I_n^d\,,$$

meaning that $\mathrm{bin}_d(\mathrm{nat}_d(x)) = x$. $\square$

Using these maps, we can translate between slaloms and binary slaloms in a way that maps corresponding sets of reals that go through them.

Definition 4.6. If $S \in \mathcal{S}$ is a slalom and $d \in \mathcal{P}$, we write $\mathrm{bin}_d[S] : \omega \to \left[\mathbf{2}^{<\omega}\right]^{<\omega}$ for the function such that $\mathrm{bin}_d[S](n) = \mathrm{bin}_{|I_n^d|}[S(n)]$ for all $n \in \omega$. If $d \in \mathcal{P}$ and $B \in \mathcal{B}^d$, we write $\mathrm{nat}[B] : \omega \to [\omega]^{<\omega}$ for the function such that $\mathrm{nat}[B](n) = \mathrm{nat}[B(n)]$ for all $n \in \omega$.

Proposition 4.7. *The following are true:*

1. *if $S \in \mathcal{S}$ is a slalom and $d \in \mathcal{P}$, then $\mathrm{bin}_d[S]$ is a d-binary slalom;*
2. *if $B \in \mathcal{B}^d$ is a d-binary slalom, then $\mathrm{nat}[B]$ is a slalom.*

Proof. For the first proposition, it suffices to observe that for every $k \in S(n)$, $\mathrm{bin}_{|I_n^d|}(k) \in \mathbf{2}^{|I_n^d|}$ by definition. The cardinality restriction follows from the fact that S is a slalom. Similarly, the second proposition follows immediately from the definition of nat and the fact that B is a d-binary slalom. $\square$

This correspondence preserves the property of going through a slalom. This is easy to see, since we can express the fact that a real goes through a slalom S using a corresponding binary slalom B and the maps nat_d and bin_d. Given an $f \in \omega^\omega$, $x = \mathrm{bin}_d(f)$ is the result of us writing the real f in binary, with each $f(n)$ being written in the section I_n^d. So, we can translate the possible naturals in $S(n)$ into their binary representations to define a binary slalom $B(n)$. Through this translation, asking if $f(n) \in S(n)$ is the same as asking whether $x \restriction I_n^d \in B(n)$.

Lemma 4.8. *The following hold.*

1. *For every $S \in \mathcal{S}$ and $d \in \mathcal{P}$, if $f \in^* S$, then* $\mathrm{bin}_d(f) \in^* \mathrm{bin}_d[S]$.
2. *For every $d \in \mathcal{P}$, $B \in \mathcal{B}^d$ and $x \in \mathbf{2}^\omega$, $x \in^* B$ if and only if* $\mathrm{nat}(x) \in^* \mathrm{nat}[B]$.

Proof. For (1), observe that, for every $n \in \omega$, by the above definitions, if $f(n) \in S(n)$, then

$$\mathrm{bin}_d(f) \restriction I_n^d = \mathrm{bin}_{|I_n^d|}(f(n)) \in \mathrm{bin}_{|I_n^d|}[S](n)\,.$$

Similarly, for (2) we obtain that

$$x \restriction I_n^d \in B(n) \iff \mathrm{nat}_d(x)(n) = \mathrm{nat}(x \restriction I_n^d) \in \mathrm{nat}[B](n)\,,$$

for all $n \in \omega$. The left-to-right direction is immediate. For the converse, if $\mathrm{nat}(x \restriction I_n^d) \in \mathrm{nat}[B](n)$, then there exists some $s \in B(n)$ such that $\mathrm{nat}(x \restriction I_n^d) = \mathrm{nat}(s)$. Because $s \in B(n)$, by definition it is of length $|I_n^d|$ and so is $x \restriction I_n^d$. Since they are of the same length and represent the same natural number in binary, it must be that $x \restriction I_n^d = s \in B(n)$. □

4.2 A Characterization of the Rapidity of $\mathcal{F}_X$

Using the notions defined in the previous section, we will now give a more general statement, using the method from the proof of Raisonnier's Theorem 2.17. We have the following statement.

Theorem 4.9. *Let $X \subseteq \mathbf{2}^\omega$. If for every partitioning real $d \in \mathcal{P}$, X goes through a d-binary slalom, then $\mathcal{F}_X$ is rapid.*

Proof. Let $d \in \mathcal{P}$ and $B \in \mathcal{B}^d$ such that X goes through B. Given an $n \in \omega$ and $s \in \mathbf{2}^{<\omega}$, let

$$X_{n,s} = \{x \in [s] \mid \forall m \geq n\, (x \restriction I_m^d \in B(m))\}\,.$$

If $x \in X$, then, by assumption, $x \in^* B$ and there exists some $n \in \omega$ such that for every $m \geq n$, $x \restriction I_m^d \in B(m)$. As a result, $x \in X_{n, x \restriction d(n)}$ and we have that

$$X \subseteq \bigcup \{X_{n,s} \mid n \in \omega,\ s \in \mathbf{2}^{d(n)}\}\,.$$

Let now $a = \{d(m) + h(t,u) \mid m \in \omega,\ t,u \in B(m)\}$. If $k \in H(X_{n,s})$ for some $n \in \omega$ and $s \in \mathbf{2}^{d(n)}$, then there exist $x, y \in X_{n,s}$ with $x \neq y$ and $k = h(x,y)$. Since $d \in \mathcal{P}$, there exists an $m \in \omega$ such that $d(m) \leq k < d(m+1)$. Since $x, y \in [s]$, it must be that $n \leq m$, for otherwise $h(x,y) = k < d(m+1) \leq d(n)$, whereas x and y share the same initial segment s up to $d(n)$. Therefore, $x{\restriction}I^d_m, y{\restriction}I^d_m \in B(m)$ and we can write $k = d(m) + h(x{\restriction}I^d_m, y{\restriction}I^d_m)$, which in turn means that $k \in a$. As a result,

$$a \supseteq \bigcup \{H(X_{n,s}) \mid n \in \omega,\ s \in \mathbf{2}^{d(n)}\}$$

and $a \in \mathcal{F}_X$. Finally, we have that

$$\begin{aligned} a \cap d(n) &= \{d(m) + h(t,u) < d(n) \mid m \in \omega,\ t,u \in B(m)\} \\ &\subseteq \{d(m) + h(t,u) \mid m < n,\ t,u \in B(m)\} \\ &= \bigcup_{m=0}^{n-1} \{d(m) + h(t,u) \mid t,u \in B(m)\}\ , \end{aligned}$$

and therefore

$$|a \cap d(n)| \leq \sum_{m=0}^{n-1} |\{(t,u) \mid t,u \in B(m)\}| \leq \sum_{m=0}^{n-1} |B(m)|^2 \leq \sum_{m=0}^{n-1} m^4 \leq n^5\,.$$

Because we can construct an $a \in \mathcal{F}_X$ such that $|a \cap d(n)| \leq n^5$ for all $n \in \omega$ for every $d \in \mathcal{P}$, $\mathcal{F}_X$ is rapid by Lemma 1.39. $\square$

Corollary 4.10. *Using the maps defined in the previous section, we obtain the following.*

1. *If $(X_d)_{d\in\mathcal{P}}$ is a collection of subsets of $\mathbf{2}^\omega$ such that every X_d goes through a d-binary slalom, then $\mathcal{F}_{\bigcap_{d\in\mathcal{P}} X_d}$ is rapid.*

2. *If $X \subseteq \omega^\omega$ goes through a slalom, then $\mathcal{F}_{\bigcap_{d\in\mathcal{P}} \mathrm{bin}_d[X]}$ is rapid.*

We can now also prove Raisonnier's Theorem 2.17 using Theorem 4.9. Assuming that every $\mathbf{\Sigma}^1_2$ set is Lebesgue measurable, we know that every $\omega^\omega \cap L[a]$ goes through a slalom. Let $a \in \omega^\omega$. Given a $d \in \mathcal{P}$, and any $x \in \mathbf{2}^\omega \cap L[a]$, $\mathrm{nat}_d(x)$ is definable by a formula and so we will have that $\mathrm{nat}_d(x) \in \omega^\omega \cap L[a,d]$. We then have that $x = \mathrm{bin}_d(\mathrm{nat}_d(x)) \in \mathrm{bin}_d[\omega^\omega \cap L[a,d]]$. This means that

$$\mathbf{2}^\omega \cap L[a] \subseteq \mathrm{bin}_d[\omega^\omega \cap L[a,d]]\,.$$

By our assumption, for every $d \in \mathcal{P}$, $\omega^\omega \cap L[a,d]$ goes through a slalom and, by Lemma 4.8, $\mathrm{bin}_d[\omega^\omega \cap L[a,d]]$, and thus $\mathbf{2}^\omega \cap L[a]$, go through a d-binary slalom. Because this holds for every $d \in \mathcal{P}$, $\mathcal{F}_{\mathbf{2}^\omega \cap L[a]}$ is rapid by Theorem 4.9.

In the proof of Theorem 4.9 we can observe that, when a set $X \subseteq \mathbf{2}^\omega$ goes through a d-binary slalom, there exists a restriction on the possible values of any $x \in X$, which can then give us a covering of X with few splitting points. However, if we know that $\mathcal{F}_X$ is rapid, we can define countable coverings of X with a bounded number of splitting points. Since any real in $x \in X$ must be in one of the sets in the cover, it follows that we can also obtain a restriction on the possible values of $x \in X$. As it turns out, if $\mathcal{F}_X$ is rapid, we can always define a d-binary slalom which X goes through, giving us the converse of Theorem 4.9.

Theorem 4.11. *Let $X \subseteq \mathbf{2}^\omega$. If $\mathcal{F}_X$ is rapid and $d \in \mathcal{P}$, then X goes through a d-binary slalom.*

Proof. Let $X \subseteq \mathbf{2}^\omega$ such that $\mathcal{F}_X$ is rapid and $d \in \mathcal{P}$. By the rapidity of $\mathcal{F}_X$, there must exist an $a \in \mathcal{F}_X$ such that $|a \cap d(n)| \leq \chi(n)$ for all $n \in \omega$, where

$$\chi(n) = \begin{cases} 0, \text{ if } n \leq 1 \\ \lfloor \log_2(n-1) \rfloor, \text{ otherwise} \end{cases}$$

By definition then, there exists a countable cover $(X_n)_{n<\omega}$ of X such that $a \supseteq \bigcup_{n<\omega} H(X_n)$. Because of Proposition 2.10 and the correspondence between closed sets and trees from Proposition 1.5, we can, without loss of generality, assume that each X_n is closed and that $X_n = [T_n]$ for some tree T_n without maximal elements.

Using this cover, we can define a collection $(B_n)_{n<\omega}$ of d-binary φ-slaloms, where $\varphi(m) = m$ for all $m \in \omega$. For any $n < \omega$ let $B_n(0) = \varnothing$ and, for $m \in \omega$ with $m > 0$, let

$$B_n(m) = \{x{\restriction}I^d_m \mid x \in X_n\}.$$

This is the set of all possible finite sequences of length $|I^d_m|$ that can occur as the I^d_m-sections of reals in X_n. Because $X_n = [T_n]$, for every $x \in X_n$ there exists an $s \in T_n$ with $s = x{\restriction}d(m+1)$. Conversely, because T_n contains no maximal elements, for every $s \in T_n$ such that $|s| = d(m+1)$ there exists an $x \in X$ such that $x{\restriction}d(m+1) = s$. Therefore, we can equivalently write that

$$B_n(m) = \{s{\restriction}I^d_m \mid s \in T_n,\ |s| = d(m+1)\}.$$

Let b be the increasing enumeration of $H(X_n)$ and observe that the set $W_k = \{s \in T_n \mid |s| = b(k)\}$ can have at most 2^k elements. For $k = 0$, since there are no splitting points before $b(0)$ in $X_n = [T_n]$, and thus T_n, it must be that W_0 has only one element. Every $s = W_{k+1}$ can be written as $s = t^\frown u$, where $t \in W_k$ and $u \in \mathbf{2}^{b(k+1)-b(k)}$. Therefore, if $|W_k| \leq 2^k$, there are at most 2^k options for the finite sequence t. Furthermore, because there is only one splitting point in the interval $[b(k), b(k+1))$, that is, $b(k)$, there are at most two options for u. Therefore, $|W_{k+1}| \leq 2^{k+1}$. In other words, this means that the tree T_n has at most 2^k branches before its $(k+1)$-th splitting point. As a result, we have that

$$|B_n(m)| \leq |\{s \in T_n \mid |s| = d(m+1)\}| \leq 2^{|H(X_n) \cap d(m+1)|} \leq 2^{|a \cap d(m+1)|},$$

and thus $|B_n(m)| \leq 2^{\lfloor \log_2(m) \rfloor} \leq m$ for all $n, m \in \omega$ with $m > 0$.

Let now $B : \omega \to \left[\mathbf{2}^{<\omega}\right]^{<\omega}$ such that for every $m \in \omega$, $B(m) = \bigcup_{n<m} B_n(m)$. We have that

$$|B(m)| \leq \sum_{n=0}^{m-1} |B_n(m)| \leq \sum_{n=0}^{m-1} m \leq m^2$$

Moreover, if $s \in B(m)$, then there exists some $n < m$ such that $s \in B_n(m)$, in which case, by construction, $s \in \mathbf{2}^{|I_m^d|}$. Therefore, B is a d-binary slalom. Finally, if $x \in X$, then there exists an $n < \omega$ such that $x \in X_n$. Then, for every $m \in \omega$ with $m > 0$, $x \restriction I_m^d \in B_n(m)$ by definition and so, for every $m > n$, $x \restriction I_m^d \in B_n(m) \subseteq B(m)$. This means that $x \in^* B$ and we have thus shown that X goes through the d-binary slalom B. $\square$

Finally, by combining Theorems 4.9 and 4.11, we arrive at the following characterization of the rapidity of $\mathcal{F}_X$.

Theorem 4.12. *The following are equivalent for every $X \subseteq \mathbf{2}^\omega$:*

1. *$\mathcal{F}_X$ is rapid,*
2. *for every $d \in \mathcal{P}$, X goes through a d-binary slalom, and*
3. *for every $d \in \mathcal{P}$, $\mathrm{nat}_d[X]$ goes through a slalom.*

Proof. Statements (1) and (2) are equivalent by Theorems 4.9 and 4.11. Therefore, we only need to show that (2) and (3) are equivalent. If $X \subseteq \mathbf{2}^\omega$ goes through a d-binary slalom B_d for every $d \in \mathcal{P}$, then by Lemma 4.8, for every $d \in \mathcal{P}$, $\mathrm{nat}_d[X]$ goes through $\mathrm{nat}[B_d] \in \mathcal{S}$. Assume now that for every $d \in \mathcal{P}$, $\mathrm{nat}_d[X]$ goes through a slalom $S_d \in \mathcal{S}$. Let $d \in \mathcal{P}$. By Lemma 4.5, $X = \mathrm{bin}_d[\mathrm{nat}_d[X]]$, which goes through the d-binary slalom $\mathrm{bin}_d[S_d]$ by Lemma 4.8. $\square$

Chapter 5

Connections to Lebesgue Measurability

So far, we have explored some properties of the Raisonnier filter and given a necessary and sufficient condition for it to be rapid. As mentioned before, the rapidity of $\mathcal{F}_X$ indicates that the set X is narrow, as it can be covered by countable collections of sets with arbitrarily sparse splitting points. Since sets with this property are in a sense "small", a natural question is whether they form an ideal. As it turns out, this is indeed the case. Moreover, because of the relation between the rapidity of $\mathcal{F}_X$ and slaloms established in the previous chapter, this ideal has connections to Lebesgue measurability and the ideal of null sets.

5.1 The Ideal $\mathcal{R}$

In this section, we will prove that the subsets $X \subseteq \mathbf{2}^\omega$ for which $\mathcal{F}_X$ is rapid form a σ-complete ideal which extends the ideal $\mathcal{I}_{\mathrm{ctbl}}$ of countable sets.

Definition 5.1. Let $\mathcal{R}$ denote the collection $\{X \subseteq \mathbf{2}^\omega \mid \mathcal{F}_X \text{ is rapid}\}$.

Remark 5.2. Recall that we also consider the trivial filter $\mathcal{P}(\omega)$ to be rapid, as it trivially satisfies Definition 1.34.

Proposition 5.3. *The collection $\mathcal{R}$ is an ideal.*

Proof. First, notice that $\mathcal{F}_\varnothing$ is the trivial filter by Proposition 2.5 and is thus also trivially rapid.

Let now $X, Y \in \mathcal{R}$. By Proposition 3.2,

$$\mathcal{F}_{X\cup Y} = \{a \cup b \mid a \in \mathcal{F}_X,\, b \in \mathcal{F}_Y\}$$

For any $f \in \omega^\omega$, by the assumption that $\mathcal{F}_X$ and $\mathcal{F}_Y$ are rapid, there exist $a \in \mathcal{F}_X$ and $b \in \mathcal{F}_Y$ such that $|a \cap f(n)| \leq n$ and $|b \cap f(n)| \leq n$ for every $n \in \omega$.

As a result, for every $n \in \omega$,

$$|(a \cup b) \cap f(n)| = |(a \cap f(n)) \cup (b \cap f(n))| \leq |a \cap f(n)| + |b \cap f(n)| \leq 2n\,.$$

Since $a \cup b \in \mathcal{F}_{X \cup Y}$, we have a witness for $f \in \omega^\omega$ according to Proposition 1.37 and thus $\mathcal{F}_{X \cup Y}$ is rapid, meaning that $X \cup Y \in \mathcal{R}$.

Finally, let $X \in \mathcal{R}$ and $Y \subseteq X$. For any $f \in \omega^\omega$ there exists an $a \in \mathcal{F}_X$ such that $|a \cap f(n)| \leq n$ for all $n \in \omega$. By Proposition 3.1, $\mathcal{F}_X \subseteq \mathcal{F}_Y$ and $a \in \mathcal{F}_Y$ as well. Therefore, $\mathcal{F}_Y$ is rapid and $Y \in \mathcal{R}$. □

We can now use Proposition 3.2, which gives us a concrete definition of the Raisonnier filter for countable unions of sets, to also prove that $\mathcal{R}$ must be σ-complete.

Proposition 5.4. *The ideal $\mathcal{R}$ is σ-complete.*

Proof. Let $\mathcal{X} = \{X_n \in \mathcal{R} \mid n < \omega\}$ and define $X = \bigcup \mathcal{X}$. Let $f \in \mathcal{P}$ be a partitioning real. As every $\mathcal{F}_{X_n}$ is rapid, for every $n < \omega$ there exists an $a_n \in \mathcal{F}_{X_n}$ such that $|a_n \cap f(m)| \leq m$ for all $m \in \omega$. Moreover, as every $\mathcal{F}_{X_n}$ extends Fréchet's filter, we have that for all $n < \omega$,

$$b_n = \{k \in a_n \mid k \geq f(n)\} \in \mathcal{F}_{X_n}$$

By Proposition 3.2, $b = \bigcup_{n<\omega} b_n \in \mathcal{F}_X$, and

$$|b \cap f(n)| = \left| \left(\bigcup_{m<\omega} b_m \right) \cap f(n) \right| = \left| \bigcup_{m<\omega} (b_m \cap f(n)) \right| \leq \sum_{m<\omega} |b_m \cap f(n)|\,.$$

However, for any $m \geq n$ we have that

$$b_m \cap f(n) = \{k \in a_m \mid k \geq f(m) \geq f(n)\} \cap f(n) = \varnothing\,.$$

Therefore for every $n \in \omega$,

$$|b \cap f(n)| \leq \sum_{m=0}^{n-1} |b_m \cap f(n)| \leq \sum_{m=0}^{n-1} |a_m \cap f(n)| \leq \sum_{m=0}^{n-1} n = n^2\,.$$

Consequently, $\mathcal{F}_X$ is rapid and $X \in \mathcal{R}$. □

In addition to the above, by Proposition 2.5, every countable $X \subseteq \mathbf{2}^\omega$ has a rapid Raisonnier filter. This means that $\mathcal{R}$ extends the ideal $\mathcal{I}_{\text{ctbl}}$ of countable sets. It is however not possible to determine in ZFC whether $\mathcal{I}_{\text{ctbl}} = \mathcal{R}$. Under the assumption that every $\mathbf{\Sigma}^1_2$ set is Lebesgue measurable and that there exists an $a \in \omega^\omega$ such that $\aleph_1^{L[a]} = \aleph_1$, Raisonnier's Theorem 2.17 tells us that $\mathcal{R}$ is strictly larger than $\mathcal{I}_{\text{ctbl}}$, as it contains the uncountable set $\mathbf{2}^\omega \cap L[a]$. On the other hand, Theorem 1.36, tells us that there exists a model of ZFC without any nontrivial rapid filters. In this model, every $X \in \mathcal{R}$ has a rapid Raisonnier filter $\mathcal{F}_X$, which must also be trivial. Therefore, every $X \in \mathcal{R}$ must be countable by Proposition 2.5. As a result, the statement $\mathcal{R} = \mathcal{I}_{\text{ctbl}}$ is independent of ZFC.

5.2 Lebesgue Measurability and $\mathcal{R}$

Raisonnier's Theorem establishes a connection between Lebesgue measurability and rapid filters. The assumption that every $\mathbf{\Sigma}^1_2$ set is Lebesgue measurable gives us that $\omega^\omega \cap L[a]$ goes through a slalom for every $a \in \omega^\omega$, which is used to prove that $\mathbf{2}^\omega \cap L[a] \in \mathcal{R}$. Theorem 4.9 indicates that it is possible to obtain a converse of this result. If $\mathbf{2}^\omega \cap L[a] \in \mathcal{R}$ for every $a \in \omega^\omega$, then every $\mathbf{2}^\omega \cap L[a]$ goes through a d-binary slalom for every $d \in \mathcal{P}$. If now the reals of $\omega^\omega \cap L[a]$ are dominated by a single $d \in \omega^\omega$, then d provides an eventual upper bound to their values. Because of this, we can always encode them using reals in $\mathbf{2}^\omega \cap L[a, d]$ and the maps from Section 4.1. Since the latter set goes through a d-binary slalom, $\omega^\omega \cap L[a]$ must go through a slalom. For this method, given a $d \in \omega^\omega$, we want to be able to partition ω into intervals such that in the n-th interval we can write all numbers up to $d(n)$. This can be achieved if the intervals have length $d(n)$.

Definition 5.5. Given $d \in \omega^\omega$ such that $d(n) > 0$ for all $n \in \omega$, let $\tilde{d} \in \mathcal{P}$ denote the real such that $\tilde{d}(n) = \sum_{m<n} d(m)$ for all $n \in \omega$.

Naturally, for each $n \in \omega$, we have that $|I^{\tilde{d}}_n| = d(n)$. Using the above method, we obtain a result that is close to the converse of Raisonnier's Theorem 2.17. For this, we need the additional assumption that for every set $\omega^\omega \cap L[a]$ we can find some real $d \in \omega^\omega$ dominating it, that is, $f \leq^* d$ for all $f \in \omega^\omega \cap L[a]$.

Lemma 5.6. *Assume that*

1. *for every $a \in \omega^\omega$, $\mathbf{2}^\omega \cap L[a] \in \mathcal{R}$, and*
2. *for every $a \in \omega^\omega$ there exists a $d \in \omega^\omega$ dominating $\omega^\omega \cap L[a]$.*

Then, for every $a \in \omega^\omega$ there exists a slalom $S \in \mathcal{S}$ such that $\omega^\omega \cap L[a]$ goes through S.

Proof. Let $a \in \omega^\omega$ and let $\omega^\omega \cap L[a]$ be dominated by $d \in \omega^\omega$. Without loss of generality, we can assume that $d(n) > 0$ for all $n \in \omega$. Because $\mathbf{2}^\omega \cap L[a, d] \in \mathcal{R}$, by Theorem 4.11, $\mathbf{2}^\omega \cap L[a, d]$ goes through a $\tilde{d}$-binary slalom B. By Lemma 4.8, $\mathrm{nat}_{\tilde{d}}[\mathbf{2}^\omega \cap L[a, d]]$ goes through the slalom $S = \mathrm{nat}[B] \in \mathcal{S}$.

Let $f \in \omega^\omega \cap L[a]$. By assumption, $f \leq^* d$ and it is easy to see that there exists a $g \in \omega^\omega$ such that $f =^* g$ and $g \leq d$. Since f and g differ in only finitely many values, $g \in L[a]$ as well and furthermore $g \in L[a, d]$. Because $g \leq d$, for every $n \in \omega$, we can write $g(n)$ in the binary system using at most $d(n)$ digits. In other words, we have that $g(n) \leq d(n) < 2^{d(n)} = 2^{|I^{\tilde{d}}_n|}$ for all $n \in \omega$. Since $\tilde{d}$ is definable given d, $\mathrm{bin}_{\tilde{d}}(g) \in \mathbf{2}^\omega \cap L[a, d]$. By Lemma 4.5, we have that

$$g = \mathrm{nat}_{\tilde{d}}(\mathrm{bin}_{\tilde{d}}(g)) \in \mathrm{nat}_{\tilde{d}}[\mathbf{2}^\omega \cap L[a, d]]$$

and thus $g \in^* S$. Because $f =^* g$, it must be that $f \in^* S$ as well. Therefore, $\omega^\omega \cap L[a]$ goes through the slalom S. $\square$

Unfortunately, it is not clear whether the assumption of every $\omega^\omega \cap L[a]$ being dominated by a real can be omitted. Nevertheless, this leads to the following equivalent characterization of the statement that every $\mathbf{\Sigma}^1_2$ set is Lebesgue measurable.

Theorem 5.7. *The following are equivalent.*

1. *Every $\mathbf{\Sigma}^1_2$ set is Lebesgue measurable.*

2. *For every $a \in \omega^\omega$, $\omega^\omega \cap L[a]$ is dominated by a real and $\mathbf{2}^\omega \cap L[a] \in \mathcal{R}$.*

Proof. Assuming (1), by Theorem 2.13, for every $a \in \omega^\omega$, $\omega^\omega \cap L[a]$ goes through a slalom. Let $a \in \omega^\omega$ and let $L[a]$ go through the slalom S. It is clear that the set $\omega^\omega \cap L[a]$ is dominated by the real $\max(S) \in \omega^\omega$, whose digits are the maximum elements of each $S(n)$. If now $\aleph_1^{L[a]} < \aleph_1$, then $\mathbf{2}^\omega \cap L[a]$ is countable and, by Proposition 2.5, $\mathbf{2}^\omega \cap L[a] \in \mathcal{R}$. Otherwise, Raisonnier's Theorem 2.17 gives us that $\mathbf{2}^\omega \cap L[a] \in \mathcal{R}$ as well. For the other direction, assuming (2), we obtain (1) by Lemma 5.6. □

Various regularity properties have been associated with forcing notions. This connection was already present in Solovay's article [21], where Cohen forcing and random forcing were associated with the property of Baire and Lebesgue measurability respectively (see also [13, pp.178-184]). Laver, in his 1976 article [15], introduced a forcing notion, now named after him, with which he proved the consistency of a conjecture by Borel with ZFC. In their article [7], Brendle and Löwe defined a concept of *Laver measurability* that is connected to Laver forcing and prove that the Laver measurability of every $\mathbf{\Sigma}^1_2$ set is equivalent to the statement *"for all $a \in \omega^\omega$, $\omega^\omega \cap L[a]$ is dominated by a real"*. Therefore, Theorem 5.7 leads to the following statement.

Corollary 5.8. *The following are equivalent:*

1. *Every $\mathbf{\Sigma}^1_2$ set is Lebesgue measurable.*

2. *Every $\mathbf{\Sigma}^1_2$ set is Laver measurable and $\mathbf{2}^\omega \cap L[a] \in \mathcal{R}$ for all $a \in \omega^\omega$.*

Theorem 5.7 gives us a connection between Lebesgue measurability, the ideal $\mathcal{R}$ and an additional statement about dominating reals. It is therefore natural that we can relate the cardinal characteristics of the ideal $\mathcal{N}$ to those of $\mathcal{R}$ and the bounding and dominating numbers. As in the proof of the theorem, this connection is established through slaloms, which are the link between Lebesgue measurability and the ideal $\mathcal{R}$. In particular, we will make use of Theorem 2.14, which tells us that $\mathrm{add}(\mathcal{N}) = \mathfrak{b}(\omega^\omega, \mathcal{S}, \in^*)$ and $\mathrm{cof}(\mathcal{N}) = \mathfrak{d}(\omega^\omega, \mathcal{S}, \in^*)$, or in other words that

1. $\mathrm{add}(\mathcal{N})$ is the smallest subset of ω^ω that does not go through any slalom, and

2. $\mathrm{cof}(\mathcal{N})$ is the smallest collection of slaloms $\mathcal{C}$ such that for every $f \in \omega^\omega$ there exists an $S \in \mathcal{C}$ for which $f \in^* S$.

We begin with the following result, which is proven using a method similar to Lemma 5.6.

Theorem 5.9. $\mathrm{cof}(\mathcal{N}) \leq \max\{\mathrm{cov}(\mathcal{R}), \mathfrak{d}\}$.

Proof. By Theorem 2.14, $\mathrm{cof}(\mathcal{N})$ is the size of the smallest collection $\mathcal{C}$ of slaloms needed so that every real goes through some $S \in \mathcal{C}$. Therefore, it is enough to construct such a collection of size $\max\{\mathfrak{d}, \mathrm{cov}(\mathcal{R})\}$.

Let $\mathcal{X} \subseteq \mathcal{R}$ be a collection of size $|\mathcal{X}| = \mathrm{cov}(\mathcal{R})$ that covers $\mathbf{2}^\omega$, that is, for every $x \in \mathbf{2}^\omega$ there exists an $X \in \mathcal{X}$ such that $x \in X$. Let $D \subseteq \omega^\omega$ be a dominating family of size $|D| = \mathfrak{d}$. Without loss of generality, we can assume that $d(n) > 0$ for every $d \in D$ and $n \in \omega$. By Theorem 4.12, for every $X \in \mathcal{X}$ and $d \in D$, $\mathrm{nat}_{\tilde{d}}[X]$ goes through a slalom $S_{X,d}$. Let

$$\mathcal{C} = \{S_{X,d} \mid X \in \mathcal{X},\, d \in D\}$$

denote the collection of these slaloms. Let $f \in \omega^\omega$. By assumption, there exists a $d \in D$ such that $f \leq^* d$, which in turn means that we can find a $g \in \omega^\omega$ such that $f =^* g \leq d$. By our assumption about $\mathcal{X}$, there exists an $X \in \mathcal{X}$ such that $\mathrm{bin}_{\tilde{d}}(g) \in X$. Because $g \leq d$, by Lemma 4.5, we have that

$$g = \mathrm{nat}_{\tilde{d}}(\mathrm{bin}_{\tilde{d}}(g)) \in \mathrm{nat}_{\tilde{d}}[X]$$

and, as a result, $g \in^* S_{X,d}$. Consequently, $f \in^* S_{X,d}$ as well. We have thus proven that for every $f \in \omega^\omega$ there exists a slalom $S_{X,d} \in \mathcal{C}$ such that $f \in^* S_{X,d}$ and, by our characterization of $\mathrm{cof}(\mathcal{N})$, we have that

$$\mathrm{cof}(\mathcal{N}) \leq |\mathcal{C}| \leq |X| \cdot |D| = \mathrm{cov}(\mathcal{R}) \cdot \mathfrak{d} = \max\{\mathrm{cov}(\mathcal{R}), \mathfrak{d}\}\,.$$

□

Because of the above, along with Theorem 2.16, we know that $\mathfrak{d} \leq \mathrm{cof}(\mathcal{N}) \leq \max\{\mathrm{cov}(\mathcal{R}), \mathfrak{d}\}$. Therefore, in models of ZFC in which $\mathrm{cof}(\mathcal{N}) > \mathfrak{d}$ holds, it must also be that $\mathrm{cov}(\mathcal{R}) > \mathfrak{d}$.

When working with cardinal characteristics of the continuum, a notion of duality is often —but not always— present. According to this, $\mathrm{add}(\mathcal{I})$ is dual to $\mathrm{cof}(\mathcal{I})$, $\mathrm{non}(\mathcal{I})$ is dual to $\mathrm{cov}(\mathcal{I})$, and $\mathfrak{b}$ is dual to $\mathfrak{d}$. More information on this duality can be found in [4], as well as relevant sections of [5]. In our case, we can obtain a dual inequality to Theorem 5.9.

Lemma 5.10. $\mathrm{add}(\mathcal{N}) \geq \min\{\mathrm{non}(\mathcal{R}), \mathfrak{b}\}$.

Proof. It is enough to show that, if $\mathrm{add}(\mathcal{N}) < \mathfrak{b}$, then $\mathrm{add}(\mathcal{N}) \geq \mathrm{non}(\mathcal{R})$. Therefore, under the assumption that $\mathrm{add}(\mathcal{N}) < \mathfrak{b}$, it is enough to construct a set $X \subseteq \mathbf{2}^\omega$ of size $\mathrm{add}(\mathcal{N})$ such that $X \notin \mathcal{R}$.

Let $F \subseteq \omega^\omega$ be a set of reals of size $|F| = \mathrm{add}(\mathcal{N})$ that does not go through any slalom in $\mathcal{S}$. As $|F| = \mathrm{add}(\mathcal{N}) < \mathfrak{b}$, F cannot be unbounded and is thus dominated by a $d \in \omega^\omega$. This means that $f \leq^* d$ for every $f \in F$. If $f \in F$, we can, using the Axiom of Choice, choose a $g_f \in \omega^\omega$ such that $g_f \leq d$ and $f =^* g$.

Let $X = \{g_f \mid f \in F\} \subseteq \omega^\omega$. For every $g \in X$ we have that $g(n) \leq d(n) < 2^{d(n)}$, meaning that $\text{bin}_{\tilde{d}}$ is injective on X by Lemma 4.5. Let $Y = \text{bin}_{\tilde{d}}[X]$, and we have that $|Y| = |X| \leq |F|$.

Assume towards contradiction that $Y \in \mathcal{R}$. Then, by Theorem 4.12, $\text{nat}_{\tilde{d}}[Y]$ goes through a slalom $S \in \mathcal{S}$. For every $f \in F$ we have that

$$f =^* g_f = \text{nat}_{\tilde{d}}(\text{bin}_{\tilde{d}}(g_f)) \in \text{nat}_{\tilde{d}}[Y] .$$

Since $g_f \in^* S$, it must then also be that $f \in^* S$. Consequently, F goes through the slalom S, which is a contradiction. As a result, $Y \notin \mathcal{R}$ and

$$\text{add}(\mathcal{N}) = |F| \geq |Y| \geq \text{non}(\mathcal{R}) .$$

□

We know by Theorem 2.16 that $\text{add}(\mathcal{N}) \leq \mathfrak{b}$. This means that, if we can prove that $\text{add}(\mathcal{N}) \leq \text{non}(\mathcal{R})$ as well, we can improve Lemma 5.10 and obtain the equality $\text{add}(\mathcal{N}) = \min\{\text{non}(\mathcal{R}), \mathfrak{b}\}$.

Lemma 5.11. $\text{add}(\mathcal{N}) \leq \text{non}(\mathcal{R})$.

Proof. In order to prove this inequality, it is enough to construct a subset of ω^ω of size $\text{non}(\mathcal{R})$ that does not go through any slalom. This can be achieved very easily by Theorem 4.12.

Let $X \subseteq \mathbf{2}^\omega$ of size $|X| = \text{non}(\mathcal{R})$ such that $X \notin \mathcal{R}$. By Theorem 4.12, there exists a $d \in \mathcal{P}$ such that $\text{nat}_d[X]$ does not go through any slalom. By Lemma 4.5, $|\,\text{nat}_d[X]| = |X|$ and so

$$\text{add}(\mathcal{N}) \leq |\,\text{nat}_d[X]| = |X| = \text{non}(\mathcal{R}) .$$

□

Theorem 5.12. $\text{add}(\mathcal{N}) = \min\{\text{non}(\mathcal{R}), \mathfrak{b}\}$.

This result tells us that in models of ZFC where $\text{add}(\mathcal{N}) < \mathfrak{b}$, we must also have that $\text{add}(\mathcal{N}) = \text{non}(\mathcal{R}) < \mathfrak{b}$. Moreover, as mentioned in Section 1.4, any inequality in Cichoń's diagramme can be made strict in some model of ZFC. Consequently, there exists a model in which $\text{add}(\mathcal{N}) > \aleph_1$, which in turn means that $\text{non}(\mathcal{R}) > \aleph_1$. Because of this, $\text{non}(\mathcal{R})$ does not always have the trivial value $\aleph_1$.

Note that for both Theorem 5.9 and Lemma 5.10 we used the direction from (1) to (3) of Theorem 4.12. Through this, we can obtain a subset of ω^ω that goes through a slalom from a set $X \in \mathcal{R}$. Note also that the dominating and bounding numbers were used meaningfully in the proofs of the two inequalities. This also intuitively corresponds to the direction from (2) to (1) in Theorem 5.7.

For the proof of Lemma 5.11 on the other hand, we used the contrapositive of the direction from (3) to (1) of Theorem 4.12. Due to the aforementioned duality, we would also expect that $\text{cof}(\mathcal{N}) \geq \text{cov}(\mathcal{R})$, and therefore $\text{cof}(\mathcal{N}) =$

$\max\{\mathrm{cov}(\mathcal{R}), \mathfrak{d}\}$, can be proven using the same methods. However, this is not as simple as the proof of Lemma 5.11. The most natural way to prove this statement would be to provide a family $\mathcal{C} \subseteq \mathcal{R}$ of size $|\mathcal{C}| = \mathrm{cov}(\mathcal{N})$ that covers $\mathbf{2}^\omega$. For this, we would have to construct a cover of sets in $\mathcal{R}$ given a family of slaloms that cover the entire space ω^ω, in the sense of every real going through at least one of them.

The difficulty in obtaining sets in $\mathcal{R}$ using Theorem 4.12 lies in defining the correct $X \subseteq \mathbf{2}^\omega$, as it requires X to go through a d-binary slalom, for every $d \in \mathcal{P}$. The existence of a family of slaloms as above gives us for each $d \in \mathcal{P}$ a collection $\mathcal{C}^d$ of d-binary slaloms such that for every $x \in \mathbf{2}^\omega$ there exists a $B \in \mathcal{C}^d$ for which $x \in^* B$. In other words, for every $d \in \mathcal{P}$ we can cover $\mathbf{2}^\omega$ by the collection $\mathcal{C}^d$ of d-binary slaloms. It is not obvious however how to use this to cover $\mathbf{2}^\omega$ with sets from $\mathcal{R}$. Corollary 4.10 indicates that it may be necessary to define suitable intersections of the slaloms in the collections $\mathcal{C}^d$. These intersections would not only need to be nonempty, but also cover the entire space $\mathbf{2}^\omega$.

It is easy to see that, since $\mathcal{R}$ extends $\mathcal{I}_{\mathrm{ctbl}}$, $\mathrm{cov}(\mathcal{R}) \leq 2^{\aleph_0}$, as every singleton has a trivial and thus rapid Raisonnier filter. Because we have not proven the inequality $\mathrm{cov}(\mathcal{R}) \leq \mathrm{cof}(\mathcal{N})$ however, it is not known whether there exist models in which $\mathrm{cof}(\mathcal{R}) < 2^{\aleph_0}$. If $\mathrm{cov}(\mathcal{R}) = 2^{\aleph_0}$, Theorem 5.9 is also trivial, as it is always the case that $\mathfrak{d} \leq 2^{\aleph_0}$.

Conclusion

In this work, we have explored various combinatorial properties of the Raisonnier filter. Through this investigation, it has become clear that its applications are not confined only to the proof of Shelah's Theorem.

Our intuition behind its definition and the results of Chapter 3 tell us that the Raisonnier filter can be used as a "measure" of the splitting points that must eventually occur in a subset of the Cantor space $\mathbf{2}^\omega$. In this sense, it is an indication of how "narrow" or "wide" a set of reals eventually becomes. The open subsets of $\mathbf{2}^\omega$, as well as those with an open interior, are the widest and their Raisonnier filter is as small as it can possibly be, that is, equal to Fréchet's filter, by Corollary 3.12. Then, Proposition 3.1 tells us that, as a set of reals becomes narrower, its corresponding Raisonnier filter expands. Finally, Proposition 2.5 confirms that the countable sets are the narrowest, having a trivial Raisonnier filter. Proposition 3.8 confirms that it is only the *eventual* splitting points of a set that determine its Raisonnier filter.

In the fourth chapter, we were able to generalize Raisonnier's Theorem, showing that the only necessary condition for the Raisonnier filter $\mathcal{F}_X$ to be rapid is that the set $X \subseteq \mathbf{2}^\omega$ goes through a d-binary slalom, for every partitioning real $d \in \mathcal{P}$. This tells us that the argument used by Bartoszyński and Judah in [5, pp. 474-475] to show that $\mathcal{F}_X$ is rapid does not depend on any intrinsic properties of $X = \mathbf{2}^\omega \cap L[a]$, but merely on that fact that every $\omega^\omega \cap L[a]$ goes through a slalom, assuming $\mathbf{\Sigma}^1_2$ Lebesgue measurability. The fact that this generalized Theorem 4.9 has a converse in Theorem 4.11, further reinforces the idea that the Raisonnier filter includes meaningful information about the structure of the set it is generated from.

In the final chapter, we obtained Lemma 5.6 as a form of a "converse" of Raisonnier's Theorem 2.17, showing that the assumptions that $\mathbf{2}^\omega \cap L[a]$ has a rapid filter and $\omega^\omega \cap L[a]$ is dominated for every $a \in \omega^\omega$ imply that every $\mathbf{\Sigma}^1_2$ set is Lebesgue measurable. This led to a new characterization of the Lebesgue measurability of $\mathbf{\Sigma}^1_2$ sets in the form of Theorem 5.7. This characterization appears to indicate that the Raisonnier filter cannot be used to yield an analogue of Shelah's Theorem for the property of Ramsey, as it is closely related to Lebesgue measurability.

This close connection is further confirmed by the cardinal equality $\text{add}(\mathcal{N}) = \min\{\text{non}(\mathcal{R}), \mathfrak{b}\}$ established in Theorem 5.12. One of its consequences is that in models where $\text{add}(\mathcal{N}) < \mathfrak{b}$, it must also be that $\text{add}(\mathcal{N}) = \text{non}(\mathcal{R})$. On the other hand, if $\mathrm{add}(\mathcal{N}) = \mathfrak{b}$, then we only know that $\mathrm{add}(\mathcal{N}) \leq \mathrm{non}(\mathcal{R})$. An interesting question is whether we can prove that $\mathrm{add}(\mathcal{N}) = \mathrm{non}(\mathcal{R})$ in ZFC, or if there exists a model in which $\mathrm{add}(\mathcal{N}) < \mathrm{non}(\mathcal{R})$.

The duality that is often present when working with cardinal characteristics of the continuum suggests that it may be also be possible to improve the result $\mathrm{cof}(\mathcal{N}) \leq \max\{\mathrm{cov}(\mathcal{R}), \mathfrak{d}\}$ of Theorem 5.9 into an equality. Further research is needed in order to determine whether this is provable in ZFC, or even with the assumption of some additional combinatorial principle.

In this work, we mostly encountered sets $X \subseteq \mathbf{2}^\omega$ whose Raisonnier filter was either Fréchet's filter $\mathscr{F}$, trivial, or nontrivial and rapid. Proposition 3.5 tells us that $\mathcal{F}_{\mathcal{P}(a)}$ is generated by $[a]_{=^*}$ for every infinite $a \in \mathbf{2}^\omega$, and so these filters are also similar to $\mathscr{F}$ in that they do not contain arbitrarily sparse elements. As we have seen, the existence of a set X such that $\mathcal{F}_X$ is nontrivial and rapid is consistent, but not provable in ZFC. Proposition 3.13 seems to indicate that, if $\aleph^{L[a]} = \aleph_1$ for some $a \in \omega^\omega$, then $\mathcal{F}_{\mathbf{2}^\omega \cap L[a]}$ may have increasingly sparse elements, without necessarily being rapid. This could perhaps give us a set $X \subseteq \mathbf{2}^\omega$ whose Raisonnier filter is neither Fréchet's filter nor trivial. A natural question is whether there exist sets $X \subseteq \mathbf{2}^\omega$ for which $\mathcal{F}_X$ is an ultrafilter. In general, it would be interesting to further investigate the possible forms that $\mathcal{F}_X$ can take under various assumptions.

One possible outcome of such an investigation would be the formulation of a new regularity property through the Raisonnier filter. The Lebesgue measure gives us a notion for the size of sets of real numbers and the null ideal consists of the smallest measurable sets. Similarly, the meagre ideal contains the sets that are smallest in regard to the property of Baire, in the sense that they have a meagre symmetric difference with the empty set. It may thus be possible to define a regularity property P using the Raisonnier filter, so that the sets that are "smallest" according to P are those in the ideal $\mathcal{R}$.

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