Modal Logics of Tame Topologies

MSc Thesis *(Afstudeerscriptie)*

written by

Simon Kreuzer

under the supervision of **Dr. Nick Bezhanishvili**, and submitted to the Examinations Board in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam*.

Date of the public defense:	**Members of the Thesis Committee:**
17 March 2026	Dr. Alexandru Baltag
	Dr. David Gabelaia
	Dr. Nick Bezhanishvili (supervisor)
	Dr. Balder ten Cate (chair)

Institute for Logic, Language and Computation

Abstract

This thesis investigates the modal logics of tame topologies on the Euclidean spaces. The central tool is the theory of stratifications, for which we show that closure algebras of tame topologies are determined by the decomposition spaces of their stratifications.

These methods are applied to several natural tame structures. First, we give an algebraic proof of the Euclidean hierarchy for chequered sets. We address tensor sums of modal algebras and show that varieties generated by such tensor sums of finite modal algebras are already generated by tensor sums of subdirectly irreducible algebras.

The main focus lies on algebraic and semialgebraic geometry. We prove that the modal logic of constructible sets in the n-dimensional affine spaces is axiomatized as $\mathbf{Grz}.\mathbf{2} \oplus \mathbf{bd}_{n+1}$, thus yielding a non-trivial Euclidean hierarchy. For compact semialgebraic sets, we use the semialgebraic Hauptvermutung (German for 'main conjecture') to prove that their semialgebraic logic coincides with the polyhedral logic of a polyhedron uniquely determined up to PL homeomorphism. Moreover, we show that the modal logic of semialgebraic subsets of the n-dimensional Euclidean space is $\mathbf{PL}_n$, the polyhedral logic of the n-simplex. These results show that polyhedral logic extends from polyhedra to a broad class of tame spaces.

Finally, we connect polyhedral logic with homotopy theory. We prove that the modal logic of all contractible polyhedra is $\mathbf{Grz}$, demonstrating that contractibility is not definable in polyhedral logic. To increase expressive power, we propose Homotopy Modal Logic, extending the modal language by a reachability operator to capture homotopies between maps.

Acknowledgements

Many thanks to ...

... Nick Bezhanishvili for his great supervision. He always took the time to listen to my ideas and to answer my questions. He challenged my results, motivating me to think beyond them, but made me feel at the same time that he had trust in my work. Especially in the last days before the deadline, I was very grateful for his availability and proofreading.

... David Gabelaia for his extremely helpful comments on my thesis. In particular, he pointed out that a simpler version of the proof of theorem 5.64 exists, which is implemented in the present version of this thesis.

... Mamuka Jiblaze and Matteo De Berardinis for their comments, suggestions and questions in our meetings.

... Sihan and Veronika for their thorough proofreading.

... every taxpayer who knowingly or unknowingly made the wonderful Master of Logic programme – and particularly my participation in it – possible.

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1 Introduction

1.1 Modal logic and topology

Modal logic

Classical propositional logic (CPC) has the property to be tabular, which means that the method of truth tables is adequate to determine the truth values of a propositional formula from the truth values of its propositional variables. While this implies good algorithmic properties, the expressivity of such a logic is limited by its staticity in the sense that it can only capture one state of the world. Modal logic (ML) enriches the language of CPC with modal operators $\Box$ and $\Diamond$, which are dual to each other and add the possibility to express statements about other states of the world.

Philosophical discussion of modal operators dates back to antiquity, mainly in the form of alethic modalities, where $\Box$ is interpreted as 'it is necessary' and $\Diamond$ as 'it is possible', but also in the form of temporal modalities. These modalities break the staticity of propositional logic by talking about possible states of the world in the case of alethic modalities, and about earlier or later states of the world in the case of temporal ones. Seemingly paradoxical phenomena occur when alethic and temporal modalities interact with each other and were especially object of research of ancient philosophers. A famous example is Aristotle's sea-battle problem (cf. [Fre85]). Other philosophical applications of modal logic include epistemic interpretations, where $\Box$ expresses 'it is known', and doxastic ones, where $\Box$ is interpreted as 'it is believed'.

In the modern understanding, a normal modal logic is a set of modal formulas that contains besides all theorems of CPC also the Kripke axiom $\Box(p \to q) \to (\Box p \to \Box q)$ and is closed under the rules modus ponens, necessitation $\varphi \vdash \Box\varphi$ and uniform substitution. Then, the dual modality $\Diamond\varphi$ can be defined as $\neg\Box\neg\varphi$.

Kripke semantics

Since the seminal work of Saul Kripke (e.g. [Kri59]), the most common semantics for modal logic is relational semantics, also known as Kripke semantics. This semantics enjoys great applicability in philosophy, because possible states of the world are explicitly modelled as elements of a set W. The truth values of the propositional variables can differ between those worlds, which explains the definition of a valuation as a map $v : \mathbf{Prop} \to \mathcal{P}(W)$, sending each variable to the set of worlds in which it holds true. The dynamics between the worlds are given by a binary accessibility relation $R \subseteq W^2$, defining a so-called Kripke frame $F = (W, R)$. A Kripke frame together with a valuation is called a Kripke model $M = (F, v)$. Given such a model, the modal formula $\Diamond\varphi$ is true at a point $x \in W$ if x has a successor xRy at which φ is true.

A modal formula is valid on a frame if it is true at every point under all valuations, or equivalently if it cannot be refuted by a valuation on some point. Given a class $\mathcal{C}$ of frames it can be shown that the set of modal formulas valid on all frames in $\mathcal{C}$ forms a normal modal logic. However, there are many normal modal logics that are not complete with respect to Kripke semantics, i.e. for that one fails to find such a class $\mathcal{C}$ of frames validating exactly the formulas

of the logic. Despite this shortcoming of relational semantics, it is an interesting question, when dealing with a normal modal logic, which relational structures it defines, i.e. which frames validate all of its formulas.

For example, the logic **K** is the least normal modal logic satisfying the definition from page 1 and is in fact complete with respect to the class of all frames. A less trivial example is **S4**, the least normal modal logic containing the axioms

$$\mathbf{T} : p \to \Diamond p, \qquad \mathbf{4} : \Diamond\Diamond p \to \Diamond p.$$

This logic is complete with respect to the class of reflexive and transitive frames and will play an important role in this thesis.

Both reflexivity and transitivity are definable by first-order sentences. However, one of the most fascinating features of modal logic is that its definable power is neither weaker nor stronger than first-order definability. For example, the class of irreflexive frames, i.e. frames that satisfy $\forall x \neg xRx$, is first-order but not modally definable. On the other hand, consider the Grzegorczyk formula

$$\mathbf{grz} : \Box(\Box(p \to \Box p) \to p) \to p.$$

A frame validates **grz** if and only if it is reflexive, transitive, antisymmetric (thus defining a partial order) and Noetherian, i.e. there exists no infinitely ascending chain. It can be shown by a compactness argument that this class of frames is not first-order definable.

Topological semantics

Topological semantics of modal logic predates Kripke semantics. Kazimierz Kuratowski's axiomatization of topological spaces by closure and interior operators from 1922 in [Kur22] is probably the first instance that studied modalities operating on a topological space, even though Kuratowski did not think about modal logic. He showed that a topological space X can be defined equivalently to the usual definition via open sets by a closure operator $\mathbf{c} : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying the following four axioms for all $A, B \subseteq X$:

K1 $\mathbf{c}\emptyset = \emptyset$,

K2 $A \subseteq \mathbf{c}A$,

K3 $\mathbf{cc}A \subseteq \mathbf{c}A$,

K4 $\mathbf{c}(A \cup B) = \mathbf{c}A \cup \mathbf{c}B$.

This closure operator **c** can be used to define topological semantics of modal logic. Like in the relational setting, a valuation is a map $v : \mathbf{Prop} \to \mathcal{P}(X)$, sending each propositional variable to the space region of a topological space (X, τ) where it is true. Then, the truth clause for the modal operator at a point $x \in X$ is given by:

$$X, x \models \Diamond\varphi \;\Leftrightarrow\; x \in \mathbf{c}[\![\varphi]\!],$$

where $[\![\varphi]\!] = \{x \in X \mid X, x \models \varphi\}$ denotes the space region satisfying φ. Using Kuratowski's axiomatization, one can now directly show that the modal logic of any topological space – the set of valid formulas – satisfies the axioms of **S4**. In particular, K1 and K4 correspond to the necessitation rule and the Kripke axiom, respectively, while K2 implies validity of the axiom **T** and K3 validity of **4**. Like for Kripke completeness, a normal modal logic is called topologically complete if one can find a class $\mathcal{C}$ of topological spaces such that the logic consists exactly of

the formulas valid in all spaces. We know from Kuratowski's axiom that every topologically complete logic must be an extension of **S4**.

But what is the modal logic of all topological spaces? This question was answered by McKinsey and Tarski in 1944 in [MT44] by proving what is now known as the McKinsey-Tarski theorem: The modal logic of any separable, dense-in-itself, metrizable space is **S4**. This result was later refined by Rasiowa and Sikorski in [RS63] who showed that the McKinsey-Tarski theorem still holds without the separability assumption. For example, **S4** is the modal logic of any Euclidean space $\mathbb{R}^n$ with the standard topology.

Kripke semantics for normal extensions of **S4** can in fact be recovered as a special case of topological semantics. Given a frame $F = (W, R)$, the collection of upwards closed sets (up-sets) – sets $U \subseteq W$ such that $x \in U$ and xRy imply $y \in U$ – constitutes the so-called Alexandrov topology on W, thus turning it into a topological space. The closure operator $\mathbf{c}$ on such Alexandrov spaces turns out to be the 'downset' operation $\mathbf{c}(A) = {\downarrow} A = \{x \in X \mid x \leq y \text{ for some } y \in A\}$. This shows that the relational interpretation of the modal operator $\Diamond$ on a frame is equivalent to its topological interpretation in the Alexandrov topology. Therefore, every Kripke complete normal extension of **S4** is also topologically complete, while there exist topologically complete logics that are not Kripke complete.

Algebraic semantics

A completeness result of an entirely different flavour is given by algebraic semantics. Here, logical connectives are modelled as algebraic operations on a set satisfying certain equations. In case of modal logic, such an algebra is called modal algebra. A formula φ is now valid in an algebra if it satisfies the equation $\varphi(\overline{a}) = 1$ – where 1 is a constant of the algebra – for all tuples $\overline{a}$ of elements of the algebra when substituted for the propositional variables of φ. It can now be shown that every normal modal logic is algebraically complete, i.e. that one can find a class of modal algebras $\mathcal{C}$ such that the formulas valid in all algebras in $\mathcal{C}$ are exactly the validities of the logic.

Kripke semantics and also topological semantics for normal extensions of **S4** can in fact be seen as special cases of algebraic semantics. In the latter case, given a topological space (X, τ), the power set $\mathcal{P}(X)$ turns out to be a modal algebra when $\Diamond$ is interpreted as the closure operator $\mathbf{c}$. Also, validity in the sense of topological semantics means exactly the same as validity in the modal algebra $\mathcal{P}(X)$. The reason is that a valuation map $\mathbf{Prop} \to \mathcal{P}(X)$ picks an element of the algebra $\mathcal{P}(X)$ for every propositional variable, so that the property that $X, v, x \models \varphi$ for all valuations v and points $x \in X$ is equivalent to saying that $\mathcal{P}(X)$ satisfies the equation $\varphi(\overline{A}) = X$ for all tuples $\overline{A}$ of elements of $\mathcal{P}(X)$.

The full power by algebraic completeness becomes unfolded by duality theory. The statement of the Jónsson-Tarski duality theorem is that every modal algebra can be realized as clopen – closed and open – subsets of a certain type of ordered topological spaces – called modal spaces – up to isomorphism. An important special case is that a modal space, or equivalently a modal algebra, is finite. Then, all finite modal spaces turn out to have trivial topology so that they are essentially Kripke frames.

Interestingly, this means that it is exactly a combination of relational semantics and topology with respect to which every normal modal logic is complete.

1.2 Tame topology

Wildness in the standard topology

The standard topologies on the real line $\mathbb{R}$ and the Euclidean spaces $\mathbb{R}^n$ are defined from seemingly innocent requirements. In case of $\mathbb{R}$, the standard topology is the least topology in which the generalized intervals $(x, \infty) = \{y \in \mathbb{R} \mid x < y\}$ and $(-\infty, x) = \{y \in \mathbb{R} \mid y < x\}$ are open for all $x \in \mathbb{R}$. In the higher-dimensional case, the open hypercubes – i.e. products of open intervals – form a basis for the topology. Despite these simple basic building blocks, it is the condition that *arbitrary* unions of open sets have to be open again that induces phenomena of 'wildness' in the standard topology, which defy our intuitive understanding of geometry. In the following, we are going to give two examples of pathological structures.

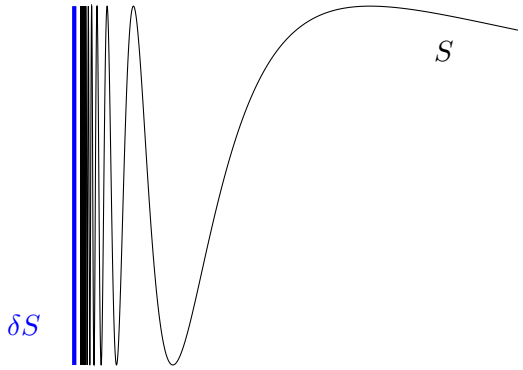

Figure 1.1: Plot of the set $S = \{(x, \sin(1/x)) \mid x > 0\} \subset \mathbb{R}^2$. In blue, its border $\delta S = \mathrm{Cl}(S) - S = \{(0, y) \mid -1 \leq y \leq 1\}$.

Example 1.1. Intuitively, the dimension of the border of a set is expected to be of lower dimension. For example, the border $\delta(0, 1) = \mathrm{Cl}(0, 1) - (0, 1) = \{0, 1\}$ of an open one-dimensional interval of $\mathbb{R}$ consists of its end points and is therefore zero-dimensional. For a counterexample, consider the graph of the function $x \mapsto \sin(1/x)$ for $x > 0$ depicted in figure 1.1. The graph S itself is one-dimensional but its border δS is also one-dimensional. Furthermore, the closure $\mathrm{Cl}(S)$ is connected but not path-connected: There is no path to reach e.g. the point $(0, 0) \in \mathbb{R}^2$ from some point on S.

Example 1.2. The reals $\mathbb{R}$ can be partitioned into n dense subsets for any $n \geq 1$. For $n = 2$, one possible partition is given by $\mathbb{R} = \mathbb{Q} \sqcup (\mathbb{R} - \mathbb{Q})$, decomposing it into the sets of rationals and irrationals, with both their closures being $\mathbb{R}$.

In the Euclidean plane $\mathbb{R}^2$, an even more surprising phenomenon occurs. There exist three connected sets A, B, C that partition $\mathbb{R}^2$ and share the same boundary $\partial A = \partial B = \partial C$. This phenomenon is known as 'Lakes of Wada'.

The quest for tame structures

In his famous research proposal 'Esquisse d'un Programme' from 1984, Alexander Grothendieck explained the necessity to look for a new notion of topology, which he calls 'modérée' or 'tame', which excludes pathological structures. Grothendieck writes:

> 'Very quickly, it appeared that it was out of the question to obtain such an ambitious statement in the context of topological spaces, because of the sempiternal

> phenomena of "wildness". [...] Some years later, I was informed of Hironaka's theory of sets which he calls, I believe, (real) "semi-analytic", which satisfy certain essential stability conditions (most probably even all of them) necessary for the development of a usable framework of "tame topology". As a result, this relaunches a reflection on the foundations of such a topology, whose necessity appears more and more clearly to me.' (Own translation from [Gro84, p. 27])

The theory of semialgebraic sets, which Grothendieck subsequently promotes as another candidate for a tame topology, is going to play the central role in chapter 5 of this thesis. Lou van den Dries in his book 'Tame Topology and O-minimal Topology' ([Dri98]) from 1998 generalized the theory of semialgebraic geometry to a bigger class of so-called o-minimal structures, which also include semianalytic geometry.

The probably biggest problem concerning tame topology is that it is left unclear by Grothendieck what kind of structure such a topology is supposed to be. Surely, it cannot be a topology in the classical sense because, as sketched on page 4, already openness of intervals and closure under arbitrary unions generate the standard topologies. A second problem is to find precise criteria to decide whether a candidate for a tame topology is in fact tame.

Van den Dries describes in the introduction to [Dri98] a general method to obtain tame topologies on Euclidean spaces:

> 'Consider 'nice' subsets of Euclidean spaces [...] Now perform elementary logical, geometric and topological operations on these sets: take unions, intersections, complements, closures and cartesian products, and project into lower-dimensional euclidean spaces. If we keep repeating these operations with the new sets that arise, then, roughly speaking, two kinds of things can happen:
>
> 1. After only a few such operations [...] a *stabilization* occurs, and performing further operations does not produce any new set.
> 2. Even more complicated sets arise: Cantor-like sets, Borel sets of arbitrary high complexity, and so on. [...]' [Dri98, p. 1]

The fact that the sets of the resulting structure will be closed under unions, intersections, complements and closures shows that it is an algebraic model of **S4**. However, it is a natural question for logicians to ask for more, namely for an exact characterization of the logic of such a tame structure. This leads to the main questions asked in this thesis:

Given a tame structure on a Euclidean space $\mathbb{R}^n$. What is its modal logic?

Euclidean hierarchies of modal logics

It is known from the McKinsey-Tarski theorem that the modal logic of any Euclidean space $\mathbb{R}^n$ is already **S4**. This result can be formulated in the style of [BBG03] as that there is no Euclidean hierarchy in modal logic. However, the proof of the McKinsey-Tarski theorem has to rely on structures of the standard topology on $\mathbb{R}^n$ that are 'wild' to some extent. For example, the partitions from example 1.2 have to be used to show that non-theorems of **S4** can be refuted.

Therefore, the question arises whether a hierarchy of modal logics appears if one considers tame structures on the Euclidean spaces. In [BBG03], just like in the definition of the standard topology, the generalized intervals $(x, \infty) = \{y \in \mathbb{R} \mid x < y\}$ and $(-\infty, x) = \{y \in \mathbb{R} \mid y < x\}$ were taken as the basic 'nice' sets. However, unlike when generating a topology, they were

not closed under arbitrary unions but under finite unions, finite intersections, complements and Cartesian products. The resulting subsets of $\mathbb{R}^n$ are called *chequered sets*. It is the main results of [BBG03] that the logics of these algebras of chequered sets are all different – but tabular. Therefore, a Euclidean hierarchy in fact may arise in tame topologies.

A more recent development was the discovery of polyhedral logic from 2015, which was published in [Bez+17b]. While the authors approach polyhedral logic from the perspective of intuitionistic logic, the transition to the modal perspective is standard. Here, the modal algebra of *polyhedral sets* in a compact polyhedron is considered, where a polyhedral set is a subset of the polyhedron that can be described as a Boolean combination of linear inequalities. Also here, a hierarchy arises: If two polyhedra have different dimensions, their polyhedral logics differ.

Although the modal logics of several tame structures have been explored in the last two decades, an analysis of the logic of Grothendieck's original candidate for tame topology – namely semialgebraic sets – is missing. In chapter 5, we are going to focus on this tame structure.

A common framework for tame logics

As mentioned, it is unclear what criteria a tame structure has to satisfy in order to actually qualify as tame. Interestingly, this questions becomes clearer when being asked from the logical perspective: If the modal logic of a tame structure is called *tame*, then what properties characterize tame logics? Both the logics of chequered sets and the polyhedral logic of a polyhedron share the property that they satisfy **grz** and that there exist certain classes of regular decompositions, which capture all logical information about the objects of consideration. In case of polyhedra, the main tool in examining the modal algebra of polyhedral sets is the triangulation lemma, stating that any finite collection of polyhedra has a common triangulation (e.g. in [Ada+22]). Similarly, [BBG03] considers the class of chequered valuations.

Triangulations can be seen as a special case of *stratifications* – a method to decompose spaces into building blocks that also engage each other nicely regarding the topology; specifically the closure of every building block (called stratum) has to be a finite union of strata. This observation leads to the approach towards logics of tame topologies in this thesis: We develop stratifications as a common framework of logics of tame topologies.

Later, in the chapters 4 and 5, we are going to address the main question 'what is the logic of a given tame topology' using the developed theory of stratifications from chapter 3. Specifically, we will work with these four tame topologies with the corresponding notions of stratification:

1. Polyhedral sets in a (compact) polyhedron. This is the ground setting for polyhedral logic and will work as a guiding example throughout the thesis. Also, in chapter 6, polyhedral logic will be the protagonist when studying relations to homotopy. The notion of stratification used here will be triangulations.

2. Chequered sets in the Euclidean space. The corresponding notion of stratifications will be called 'chequered' stratifications, which are decompositions of the Euclidean spaces into Cartesian products of points and open intervals.

3. Algebraic sets in the Euclidean space. Those do not form a closure algebra on their own but generate one whose elements are known as 'Zariski-constructibles'.

4. Semialgebraic sets in the Euclidean space. Here, the theory of stratifications is the most evolved. The used stratifications here have many good properties; for example their strata can be chosen to be algebraic manifolds. In particular, they have well-defined dimension. A special class of stratifications that we are going to use are Nash stratifications.

1.3 Overview of this thesis and its contributions

In chapter 2, we introduce the relevant preliminaries from order theory, model theory, algebraic modal logic, duality theory as well as the theory of simplicial complexes and polyhedral logic.

In chapter 3, we present and develop the theory of stratifications, which in the two subsequent chapters will be used to determine the modal logics of certain tame topologies. The main result from this chapter will be that if, for any finite collection of tame sets, there exists a stratification which simultaneously stratifies each of the tame sets (this will be called 'stratification property' for shortness) then the modal logic of the tame topology will be the logic of the decomposition frames of its stratifications.

The first example of a tame structure will be considered in chapter 4, namely the chequered sets from [BBG03]. Here, we will merely reproduce their result but give an algebraic proof. We present the preparatory work for this in section 4.1, where we consider the tensor sum of modal algebras – which in the finite case will just be the Cartesian product of frames – and give a simpler representation of the variety generated by a tensor sum of finite modal algebras in terms of subdirectly irreducible algebras. With this result and the stratification theory from chapter 3 at hand, the characterization of the modal logic of chequered sets will follow easily.

Chapter 5 will be the main part of this thesis. In the first half, section 5.1, we consider the modal algebra generated by algebraic subsets of the Euclidean spaces, i.e. sets that can be represented as zero sets of polynomials. We will use that these sets can be represented as the constructible sets of the Zariski topology, and use properties of this topology as well as stratification theory to prove the first main result of this thesis:

1. The modal logic of the Zariski-constructible sets in $\mathbb{R}^n$ is $\mathbf{Grz.2} \oplus \mathbf{bd}_{n+1}$.

In the second half of chapter 5, section 5.2, we will turn towards semialgebraic sets, which extend algebraic sets by also allowing polynomial inequalities in addition to equations. We will see that semialgebraic sets have many good tame properties. The main results of this section will show that there is a strong bond between the modal logic of semialgebraic sets and polyhedral logic. Specifically, we will establish the following main results:

2. For every compact semialgebraic set A, there exists a polyhedron P such that the modal logic of semialgebraic subsets of A is the polyhedral logic of P. Furthermore, P is unique up to PL isomorphism.

3. The modal logic of semialgebraic sets in $\mathbb{R}^n$ equals $\mathbf{PL}_n$, the polyhedral logic of the n-dimensional simplex.

These two results can be interpreted in the way that polyhedral logic is already quite universal – as an extension from polyhedra to a much bigger class of topological spaces does not yield a new logic – and thus underline the power of polyhedral logic.

Finally, the last chapter 6 will return to polyhedral logic. We will use the homotopy theory of finite spaces developed by Stong and McCord to show that the polyhedral logic of all contractible polyhedra is **Grz**. In the second part, section 6.2, we propose a more expressive variant of modal logic to express statements about homotopies.

2 Preliminaries

2.1 Relations, orders and lattices

In this section, we clarify some notation and terminology regarding relations and orders that we will use in this thesis. Also, we introduce Boolean algebras order-theoretically and state the result that certain topological spaces – Alexandrov spaces – are equivalent to preordered sets.

Regarding a lattice-theoretic treatment of Boolean and modal algebras, we refer to [DP02].

Given a binary relation $R \subseteq X^2$, the **image** of a set $U \subseteq X$ under R is denoted as $R(U) = \{x \in X \mid yRx \text{ for some } y \in U\}$, and the **preimage** is denoted as $R^{-1}(U) = \{x \in X \mid xRy \text{ for some } y \in U\}$. In case that $U = \{x\}$ consists of a single element, we write $R(x) = R(\{x\})$ and $R^{-1}(x) = R^{-1}(\{x\})$.

The relation R is called

preorder if it is reflexive and transitive,
partial order if it is an antisymmetric preorder,
linear order if it is connected and a partial order,
strict linear order if it is irreflexive, asymmetric, transitive and connected,
dense linear order if it is a dense and a strict linear order.

If $(X, \leq)$ is a preordered set and $U \subseteq X$, we write $\uparrow U$ for $\leq(U)$ and call it the **upset** of U, and we write $\downarrow U$ for $\leq^{-1}(U)$ and call it the **downset** of U. Generally, a set $U \subseteq X$ is called **upset** if $\uparrow U = U$ and **downset** if $\downarrow U = U$. Like for general relations, if $U = \{x\}$, we write $\uparrow x$ for $\uparrow\{x\}$ and call its members **successors** of x, and $\downarrow x$ for $\downarrow\{x\}$, the **predecessors** of x.

A **chain** in a partially ordered set (also called **poset**) is a subset that is linearly ordered in the induced order. The **length** of a chain C of n elements for $n \geq 0$ is defined as $n - 1$ and denoted as $\text{len}(C)$. The **height** of a poset $(X, \leq)$ is the maximum of the lengths of its chains. If no such maximum exists, the height is set to ∞.

We define the following notation for isomorphism classes of certain posets, with examples given in figure 2.1:

- C_n for the chain of length n,
- F_n for the **n-fork**, i.e. the poset with $n+1$ elements consisting of a root with n incomparable successors.

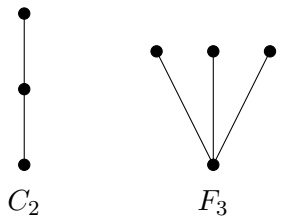

Figure 2.1: The 2-chain C_2 and the 3-fork F_3

A poset $(X, \leq)$ is **bounded** if there exist elements $0, 1 \in X$ such that $0 \leq a \leq 1$ for all

$a \in X$. It forms a **lattice** if every binary subset $\{a, b\} \subseteq X$ has a least upper bound $a \vee b$ – called **supremum** or **join** – and a greatest lower bound $a \wedge b$ – called **infimum** or **meet**. A lattice $(L, \leq)$ is called **complete** if every subset $A \subseteq L$ has a supremum $\bigvee A$ and infimum $\bigwedge A$. A bounded lattice is **complemented** if for every $a \in L$, there exists an element $\neg a$ – the **complement** of a – such that $a \vee \neg a = 1$ and $a \wedge \neg a = 0$.

A **Boolean algebra** is a bounded and complemented lattice that further satisfies either of the distributivity laws

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c), \qquad a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

An element $a \in A$ of a finite Boolean algebra is called **atom** if $a \neq 0$ and $b < a \Rightarrow b = 0$ for all $b \in A$. The set of all atoms of A is denoted as $\mathrm{At}(A)$.

Proposition 2.1. *Let A be a finite Boolean algebra. Then, every element of A equals the join of all atoms below a:*

$$a = \bigvee \{x \in \mathrm{At}(A) \mid x \leq a\}.$$

At this point, it might seem unintuitive that Boolean algebras are called algebras*. In section 2.3, we will give an alternative definition as an actual algebra in the traditional sense.*

An **Alexandrov space** X is a topological space in which any intersection of open sets is open. For an Alexandrov space X, define $P(X)$ to be the set X with the preorder $x \leq y \Leftrightarrow x \in \mathrm{Cl}(\{y\})$. For a preordered set $(X, \leq)$, define a topological space $T(X)$ by declaring $U \subseteq X$ open iff U is an upset, i.e. $x \in U$ and $x \leq y$ implies $y \in U$ for all $x, y \in X$. Let **Alex** be the category of Alexandrov spaces and continuous maps and **Prosets** the category of preordered sets with monotone maps.

Theorem 2.2. $T : \mathbf{Prosets} \longleftrightarrow \mathbf{Alex} : P$ *is an equivalence of categories.*

2.2 Model theory and universal algebra

The purpose of model theory in this thesis is twofold. First, the machinery used to study equationally defined classes of algebras is provided by universal algebra, which can be viewed as the subfield of model theory studying equational theories. Second, when dealing with o-minimal structures and theories in chapter 5, model theory offers a compact and natural perspective.

As a standard reference for the following content we refer to [Mar02].

A **language** or **signature** L is a set of constant symbols, function symbols and relation symbols, the latter two with an arity $n > 0$ for each symbol. A **structure** or **model** is a set M (the **domain**) together with **interpretations** of the language L, i.e. an element $c^M \in M$ for every constant symbol c, a function $f^M : M^n \to M$ for every function symbol f of arity n, and a relation $r^M \subseteq M^n$ for every relation symbol of arity n. If $L' \subseteq L$ and M is an L-structure, we obtain an L'-structure M' by interpreting L' like in M, in which case we say that M' is the L'-**reduct** of M or that M **expands** M'.

Let **Var** be a countably infinite set of variables. The set of **terms** is the smallest set that contains all variables and constants and satisfies that if f is an n-ary function symbol and $\bar{t}$ a list of n terms, the expression $f(\bar{t})$ is a term. An **atomic formula** has either the form $t_1 = t_2$ with terms t_1, t_2 or $r(\bar{t})$ with r an n-ary relation symbol and $\bar{t}$ a list of n terms. The set of

formulas is the smallest set that contains all atomic formulas and satisfies that $\neg\alpha$, $\alpha \vee \beta$, $\alpha \wedge \beta$, $\alpha \to \beta$, $\exists x\alpha$ and $\forall x\beta$ are formulas whenever α and β are formulas and x is a variable. A formula is called **sentence** if it does not contain free variables. The notation $\alpha(\overline{x})$ means that all free variables of α are in $\overline{x}$. A **theory** is a set of sentences.

Given a subset $A \subseteq M$, we can extend the language L to L_A by adding new constants a for every $a \in A$, which are interpreted as themselves in M, i.e. $a^M = a$. The **truth** or **satisfiability** relation between an L-structure M and L_M sentences is defined recursively as

$M \models t_1 = t_2$ iff $t_1^M = t_2^M$

$M \models r(\overline{t})$ iff $\overline{t}^M \in r^M$

$M \models \exists x\alpha$ iff there exists $m \in M$ such that $M \models \alpha(m)$

$M \models \forall x\alpha$ iff for all $m \in M$ it holds that $M \models \alpha(m)$

and for the Boolean connectives as expected. The theory of all L-sentences that are true in an L-structure M is denoted as $\mathrm{Th}(M)$. Two L-structures M and N are **elementarily equivalent**, denoted as $M \equiv N$, if $\mathrm{Th}(M) = \mathrm{Th}(N)$. An L-structure M is a **model of an L-theory** T if $M \models \alpha$ for all $\alpha \in T$. For theories T and S, we write $T \models S$ and say that T **entails** S if every model of T is also model of S. A theory T is **consistent** if it has a model and is **complete** if all models of T are elementarily equivalent.

Let M be an L-structure and $X \subseteq M^n$ a subset of the n-fold Cartesian product of its domain. We say that X is **definable** (with parameters from M) if there exists an L_M-formula $\alpha(x_1, \ldots, x_n)$ such that

$$X = \{(m_1, \ldots, m_n) \in M^n \mid M \models \alpha(m_1, \ldots, m_n)\}.$$

Definability is sometimes defined with and sometimes without parameters. For us, it makes sense to take parameters into account because the application will be in the theory of fields: Then the terms occuring in the formula α will be polynomials in n variables, with the parameters forming the coefficients. The set X will then be defined by a Boolean combination of zero sets of finitely many polynomials.

A map $h : M \to N$ between L-structures is called **homomorphism** if $h(c^M) = c^N$, $h(f^M(\overline{m})) = f^N(h(\overline{m}))$ and $\overline{m} \in r^M$ implies $h(\overline{m}) \in r^N$ for all constant symbols c, function symbols f, relation symbols r and lists of elements $\overline{m} \subseteq M$ of the right size. The map is called **embedding** if it is injective and $h(\overline{m}) \in r^N$ implies $\overline{m} \in r^M$, and **elementary embedding** if $M \models \alpha(\overline{m})$ iff $N \models \alpha(h(\overline{m}))$ for all L-formulas α and lists of elements $\overline{m} \subseteq M$. The map is called **isomorphism** if it is bijective and h^{-1} is a homomorphism. A subset $M' \subseteq M$ is a **substructure** if the inclusion $M' \hookrightarrow M$ is an embedding, and **elementary substructure** if the inclusion is also elementary.

A sentence is called **universal** if it has the form $\forall\overline{x}\alpha(\overline{x})$ for a quantifier-free formula α. A theory T is called **universal** if there is a set T' of universal sentences such that T and T' have the same models, i.e. $\mathrm{Mod}(T) = \mathrm{Mod}(T')$.

Theorem 2.3 (Łos-Tarski)**.** *The following are equivalent for a theory T:*

1. *T is universal.*
2. *If $M \in \mathrm{Mod}(T)$ and $M' \subseteq M$ is a substructure, then $M' \in \mathrm{Mod}(T)$.*

A theory T has **quantifier elimination** if for every L-formula $\alpha(\overline{x})$ there exists a quantifier-free L-formula $\beta(\overline{x})$ such that $T \models \forall\overline{x}(\alpha(\overline{x}) \leftrightarrow \beta(\overline{x}))$.

Lemma 2.4. *The following are equivalent for a theory T:*

1. *T has quantifier elimination.*
2. *T has quantifier elimination for all formulas of the form $\exists x \alpha(x, \overline{y})$ where α is a conjunction of atomic and negated atomic formulas.*

Proposition 2.5. *Let $e : M' \hookrightarrow M$ be an embedding between models M and M' of a theory T with quantifier elimination. Then, e is an elementary embedding.*

The elements from universal algebra occuring in this thesis are mostly taken from [BS84] unless otherwise indicated.

Universal algebra studies L-structures for signatures L that contain only constant symbols and function symbols (called **operations**). An **algebra** is an L-structure for such a signature. Two algebras A and B with the same signature are called **similar**. Given a class K of similar algebras, let $\mathbb{H}(K)$ be the class of homomorphic images, $\mathbb{S}(K)$ the class of subalgebras and $\mathbb{P}(K)$ the class of products of elements of K. A class of similar algebras is called **variety** if it is closed under $\mathbb{H}$, $\mathbb{S}$ and $\mathbb{P}$. A theory T over an algebraic signature is called **equational** if there is a set T' of equations, i.e. sentences of the form $\forall \overline{x}(t_1(\overline{x}) = t_2(\overline{x}))$ for terms t_1 and t_2, such that $\mathrm{Mod}(T) = \mathrm{Mod}(T')$. A class K of similar algebras is an **equational class** if there is an equational theory T such that $K = \mathrm{Mod}(T)$.

Theorem 2.6 (Tarski, Birkhoff)**.** *The following are equivalent for a class of algebras K:*

1. *K is a variety,*
2. *$\mathbb{HSP}(K) = K$,*
3. *K is an equational class.*

The variety $\mathrm{Var}(K) = \mathbb{HSP}(K)$ is called the variety **generated** by K and is in fact the smallest variety containing K.

Proposition 2.7. *The following are equivalent for a variety $\mathcal{V}$:*

1. *$\mathcal{V} = \mathrm{Var}(K)$,*
2. *$K \models \alpha \Rightarrow \mathcal{V} \models \alpha$ for all equations α,*
3. *$\mathcal{V} \not\models \alpha \Rightarrow K \not\models \alpha$ for all equations α.*

An algebra A is a **subdirect product** of algebras $\{B_i\}_{i \in I}$ if A is a subalgebra of the product $\prod_{i \in I} B_i$ such that all projections $\pi_i|_A : A \to B_i$ restricted to A are surjective. An algebra A is **subdirectly irreducible (s.i.)** if whenever $A \cong A' \leq \prod_{i \in I} B_i$ is represented as a subdirect product, then one of the projections π_i is an isomorphism.

Let A be an algebra, then a **congruence** is an equivalence relation $\sim \subseteq A^2$ such that whenever $a_i \sim b_i$ for all $1 \leq i \leq n$ then $f(a_1, \ldots, a_n) \sim f(b_1, \ldots, b_n)$ for all n-ary operations f. The set Cong(A) of congruences, ordered by inclusion, forms a lattice.

Theorem 2.8. *The class of subdirectly irreducible algebras coincides with the class of algebras A such that* Cong(A) *has a second-least element.*

Theorem 2.9 (Birkhoff)**.** *Every variety is generated by its subdirectly irreducible algebras.*

Theorem 2.9 gives a powerful tool to characterize varieties by determining their subdirectly irreducible algebras. However, if the variety itself is complicated, its subdirectly irreducible algebras may be complicated, too. The usual use case for the following theorem by Jónsson is to

determine the subdirectly irreducible algebras of finitely generated varieties, which are of simpler form.

Let $\mathbb{I}(K)$ be the class of algebras isomorphic to those in K, $\mathbb{P}_S$ the class of subdirect products and $\mathbb{P}_U$ the class of ultraproducts of algebras in K.

Theorem 2.10 (Jónsson's lemma)**.** *Let $\mathcal{V} = \mathrm{Var}(K)$ be a variety such that $\mathrm{Cong}(A)$ is distributive for all $A \in \mathcal{V}$. Then $\mathcal{V} = \mathbb{IP}_S\mathbb{HSP}_U(K)$.*

All varieties encountered in this thesis satisfy the distributivity condition in theorem 2.10, so we will presuppose it silently from now on.

Corollary 2.11. *Let K be a finite set of finite algebras, and let $\mathcal{V}_{si}$ denote the class of subdirectly irreducible algebras of $\mathcal{V} = \mathrm{Var}(K)$. Then, $\mathcal{V}_{si} \subseteq \mathbb{HS}(K)$.*

Let $S \subseteq A$ be a subset of an algebra A. We say that S **generates** A if no proper subalgebra of A contains S. The algebra A is **finitely generated** if there exist finitely many elements generating A. A variety $\mathcal{V}$ is **finitely generated** if $\mathcal{V} = \mathrm{Var}(A)$ for a finite algebra A, **locally finite** if every finitely generated algebra is finite, and has the **finite model property** (or is **finitely approximable**) if $\mathcal{V} = \mathrm{Var}(K)$ for K a class of finite algebras.

A category C is called **algebraic** if the objects are similar algebras and the morphisms are exactly the homomorphisms between them. It is called **equational** if its objects form a variety. Given a variety $\mathcal{V}$, we will identify it with its equational category. In the equational categories encountered in this thesis, monomorphisms will coincide with injective, epimorphisms with surjective and isomorphisms with bijective homomorphisms.

2.3 Boolean and modal algebras

In this section, we define Boolean and modal algebras – the protagonists of this thesis – as models of corresponding algebraic theories. Also, we formally define classical propositional logic (CPC), for which Boolean algebras are the algebraic models, and normal modal logics (NML), whose algebraic models are modal algebras.

For the algebraic approach to modal logic, see part III of [CZ97] and also chapter 5 of [BRV01].

A **Boolean algebra** is an algebra A over the signature $(\vee, \wedge, \neg, 0, 1)$ satisfying the following equations for all $a, b, c \in A$:

1. $a \vee b = b \vee a$,
2. $a \wedge b = b \wedge a$,
3. $a \vee (b \vee c) = (a \vee b) \vee c$,
4. $a \wedge (b \wedge c) = (a \wedge b) \wedge c$,
5. $a \vee a = a$,
6. $a \wedge a = a$,
7. $a \vee (a \wedge b) = a$,
8. $a \wedge (a \vee b) = a$,
9. $a \vee 0 = a$,
10. $a \wedge 1 = a$,
11. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,
12. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,
13. $a \vee \neg a = 1$,
14. $a \wedge \neg a = 0$.

The above axiomatization is far from minimal, however. For example, the absorption laws (7), (8), the associativity laws (3), (4) and one of the distributivity laws (11) and (12) can be derived from the other axioms.

The implication $a \to b$ in a Boolean algebra is defined for all $a, b \in A$ by

$$a \to b = \neg a \vee b.$$

The equations (1)–(10) axiomatize (bounded) lattices, and the axioms (1)–(12) bounded distributive lattices. By defining the order

$$a \leq b \quad \Leftrightarrow \quad a \vee b = b \quad \Leftrightarrow \quad a \wedge b = a,$$

Boolean algebras can be shown to be exactly those bounded and complemented distributive lattices that were defined in section 2.1.

A **modal algebra** is an algebra A over the signature $(\vee, \wedge, \neg, \Diamond, 0, 1)$ satisfying the axioms of Boolean algebras and additionally for all $a, b \in A$

1. $\Diamond 0 = 0$,
2. $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$.

The operator dual to $\Diamond$ is defined as $\Box a = \neg\Diamond\neg a$.

Let **Prop** be a countably infinite set of propositional variables; for convenience, identify this set with the set **Var** of variables in the first-order language. A **propositional formula** is a term in the first-order language $(\vee, \wedge, \neg, 0, 1)$ of Boolean algebras; and a **modal formula** is a term in the language $(\vee, \wedge, \neg, \Diamond, 0, 1)$ of modal algebras. For L one of the two, let A be an L-algebra and $\varphi(\overline{p})$ an L-term. Then, **validity** of φ in A is defined as:

$$A \models \varphi \quad \Leftrightarrow \quad A \models \forall \overline{p}\,(\varphi(\overline{p}) = 1).$$

The above definition of propositional and modal formulas has the advantage that validity in an algebra can be defined easily by viewing the algebra as a first-order model and resorting to its satisfiability relation. However, the separation of syntax and semantics is a bit blurred here: The connectives of the formulas use the same symbols as the operations of the algebra. The transition from syntax to semantics is hidden but still there: It happens during the interpretation of the terms in the algebra, when e.g. $\wedge$ is interpreted as the operation $\wedge^A$ of the algebra.

Still, it should be noted that the constants 0 and 1 are usually written as $\bot$ and $\top$ in propositional and modal formulas. Also, while first-order variables are usually chosen to be represented by the letters $x, y, z, \ldots$, propositional variables are preferably rather written with letters like $p, q, r, \ldots$.

For $\mathcal{C}$ a class of Boolean (or modal) algebras, we call

$$L(\mathcal{C}) = \{\varphi \mid A \models \varphi \text{ for all } A \in \mathcal{C}\}$$

– the set of all propositional (or modal) formulas valid on all algebras of $\mathcal{C}$ – the **propositional** (or **modal**) **logic** of $\mathcal{C}$. If $\mathcal{C} = \{A\}$ consists of a single algebra, we write $L(A)$ instead of $L(\{A\})$.

Let **BA** be the variety of Boolean algebras, then $L(\mathbf{BA})$ is called **classical propositional logic** and abbreviated as **CPC**. Let **2** be the Boolean algebra with two elements 0 and 1. Then, it can be shown that $\mathbf{BA} = \mathrm{Var}(\mathbf{2})$, and therefore that $\mathbf{CPC} = L(\mathbf{2})$. Furthermore, the only proper subvariety of **BA** is **Triv** – the variety of trivial algebras that satisfy $0 = 1$.

Similarly, let V be a variety of modal algebras. Then, the set of modal formulas $L(V)$ is a **normal modal logic**. For **MA** the variety of all modal algebras, the logic $L(\mathbf{MA})$ is called **K**. In contrast to CPC, there are many normal modal logics. Algebraic completeness establishes a one-to-one correspondence between normal modal logics and varieties of modal algebras.

Theorem 2.12. *The class of varieties of modal algebras and the class of normal modal logics both form lattices, which are dually isomorphic to each other via the map* $L(\cdot) : \Lambda(\mathbf{MA}) \to \mathrm{NExt}(\mathbf{K})$.

CPC and normal modal logics are usually defined via proof-theoretic systems. In case of normal modal logics, we have sketched such a definition on page 1. The result that those proof-theoretic systems yield exactly the same logics as the algebraic semantics presented here, is known as algebraic completeness. Our simplified definition of normal modal logics as the set of formulas valid on a variety silently presupposes this result.

In this thesis, we will mostly use the algebraic perspective in our argumentation. At exposed points as well as informally, however, we will switch to the logical perspective, i.e. in order to present our main results.

2.4 Stone and Jónsson-Tarski dualities

Reasoning about bigger concrete Boolean or modal algebras can be complicated even in the finite case. The duality theorems presented in this section state that every Boolean or modal algebra can be represented as algebras of certain subsets of topological spaces. The great advantage gained from this 'dual' perspective on algebras comes mainly from two facts: First, in the finite case, the dual spaces are exponentially smaller than the algebras and thus 'condense' their information. Second, arguments carried out in the dual spaces are easier to visualize – especially in the finite case, where modal spaces collapse to finite Kripke frames.

As a reference for Stone duality, we give chapter 11 of [DP02]. Jónsson-Tarski duality is covered in chapter 5 of [BRV01] and chapter 8 of [CZ97].

A **Stone space** X is a topological space that is compact and totally separated, i.e. for every disjoint pair $x, y \in X$, there exists a clopen (closed and open) set $U \subseteq X$ such that $x \in U$ and $y \notin U$. Given a Stone space X, the clopen sets form the Boolean algebra $X^* = (\mathrm{Clop}(X), \cup, \cap, (\cdot)^c, \emptyset, X)$.

A morphism between Stone spaces is a continuous map. Let **StoneS** denote the category of Stone spaces.

Let L be a bounded lattice. A **filter** is a non-empty upset $F \subseteq L$ that is closed under $\wedge$, i.e. whenever $a, b \in F$ then $a \wedge b \in F$. A filter is **proper** if $0 \notin F$, and **prime** if it is proper and satisfies that whenever $a \vee b \in F$ then $a \in F$ or $b \in F$. The **spectrum** of L, denoted as $\mathrm{Spec}(L)$ is the set of all prime filters of L equipped with the topology generated by the basis $\{\varphi(a) \mid a \in L\}$, where $\varphi(a) = \{x \in \mathrm{Spec}(L) \mid a \in x\}$.

The spectrum of a Boolean algebra A is also denoted as A_* and is a Stone space.

Theorem 2.13 (Stone duality)**.** $(\cdot)^* : \mathbf{StoneS} \longleftrightarrow \mathbf{BA} : (\cdot)_*$ *is a dual equivalence between the category of Boolean algebras with homomorphisms and the category of Stone spaces with continuous maps.*

A **modal space** (X, R) is a Stone space with a binary relation $R \subseteq X^2$ satisfying the following properties:

1. For every $x \in X$, the set $R(x)$ is closed,
2. For every clopen $U \subseteq X$, the set $R^{-1}(U)$ is clopen.

Given a modal space (X, R), the definition $\Diamond U = R^{-1}(U)$ for clopen sets $U \subseteq X$ yields the modal algebra $X^* = (\mathrm{Clop}(X), \cup, \cap, (\cdot)^c, \Diamond, \emptyset, X)$.

Morphisms between modal spaces are continuous **p-morphisms**, i.e. continuous maps $f : (X, R) \to (X, R')$ that further satisfy:

- $xRy \Rightarrow f(x)R'f(y)$,
- $f(x)R'y' \Rightarrow$ there exists xRy such that $f(y) = y'$.

Let **ModalS** denote the category of modal spaces.

Given a modal algebra A, the space $A_* = (\mathrm{Spec}(A), R)$ is a modal space with the relation R given by

$$xRy \;\Leftrightarrow\; (\forall a \in A : \Box a \in x \Rightarrow a \in y) \;\Leftrightarrow\; (\forall a \in A : a \in y \Rightarrow \Diamond a \in x).$$

Theorem 2.14 (Jónsson-Tarski duality)**.** $(\,\cdot\,)^* : \mathbf{ModalS} \longleftrightarrow \mathbf{MA} : (\,\cdot\,)_*$ *is a dual equivalence between the category of modal algebras with homomorphisms and the category of modal spaces with continuous p-morphisms.*

The morphism parts of the functors $(\,\cdot\,)^*$ and $(\,\cdot\,)_*$ in the duality theorems are given by inverse images, i.e. if $f : A \to B$ is a homomorphism of Boolean (or modal) algebras, then $f_* = f^{-1} : B_* \to A_*$ is a Stone (or modal) morphism. Conversely, if $f : X \to Y$ is a Stone (or modal) morphism, then $f^* = f^{-1} : Y^* \to X^*$ is a Boolean (or modal) homomorphism.

Modal algebras	Modal spaces
Homomorphism	Continuous p-morphism
Injective homomorphism	Surjective continuous p-morphism
Surjective homomorphism	Injective continuous p-morphism
Finite product	Disjoint union
Subalgebra	P-morphic image
Atom	Isolated point
S.i. algebra	Topo-rooted space

Table 2.1: Duality dictionary for Jónsson-Tarski duality

In the finite case, the property of Stone spaces to be totally separated forces the topology to be discrete. Therefore, finite Stone spaces collapse to finite sets, and finite modal spaces to a binary relation on a finite set. This structure will be called **frame** just like in relational semantics introduced on page 1. Also, morphisms between finite Stone spaces are simply functions, while morphisms between finite modal spaces are p-morphisms.

The duality theorems allow for 'translating' various notions back and forth between the dual perspectives of algebras and spaces. The relevant correspondences for Jónsson-Tarski duality are summarized in table 2.1.

A **generated subframe** of a modal space (X, R) is a subset $Y \subseteq X$ such that $R(Y) \subseteq Y$. A **root** of the modal space is a point $x \in X$ such that $R^*(x) = X$. The space is called **topo-rooted** if the set of roots is non-empty and open.

If X is a modal space, we are also going to write $\mathrm{Var}(X)$ for $\mathrm{Var}(X^*)$, the variety generated by the dual algebra, and $L(X)$ for $L(X^*)$ analogously.

2.5 Closure, skeletal and Grzegorczyk algebras

The contents of this section are taken from [Esa+19].

A modal algebra is called **closure algebra** or **S4-algebra** if it satisfies

1. $a \leq \Diamond a$, and
2. $\Diamond\Diamond a = \Diamond a$.

Note that (1) is equivalent to the equation $a \wedge \Diamond a = \Diamond a$, so that the class of closure algebras forms a variety by theorem 2.6.

Closure algebras can also be characterized dually:

Proposition 2.15. *A modal algebra A is a closure algebra if and only if its dual space $A_* = (X, R)$ is a preorder.*

For **CA** the variety of closure algebras, the logic $L(\mathbf{CA})$ is called **S4**.

Let (X, τ) be a topological space. Then, the powerset of X forms the S4-algebra $\mathcal{MP}(X) = (\mathcal{P}(X), \cup, \cap, (\,\cdot\,)^c, \mathrm{Cl}, \emptyset, X)$, where $\Diamond$ is interpreted as closure and $\Box$ consequently as interior.

Given a finite preordered set X, one has two ways at hand to obtain closure algebras from them: First, one can see X as a frame, i.e. as a finite S4-modal space. Second, one can apply the Alexandrov equivalence to obtain a finite topological space and take its modal algebra $\mathcal{MP}$. However, the resulting algebras will be the same. The reason for this is that closure in Alexandrov spaces corresponds to taking downsets, which is exactly how $\Diamond$ is interpreted in modal spaces.

Proposition 2.16. *Let $(X, \leq)$ be a finite preordered set. Then*

$$\mathcal{MP}(T(X)) = (X, \leq)^*.$$

Theorem 2.17 (McKinsey, Tarski; Rasiowa, Sikorski)**.** *Let X be any dense-in-itself and metrizable space. Then $L(\mathcal{MP}(X)) = \mathbf{S4}$.*

A variety V of S4-algebras is called **topological** if there exists a class $\mathcal{C}$ of topological spaces such that $V = \mathrm{Var}(\{\mathcal{MP}(X) \mid X \in \mathcal{C}\})$.

Let A be a closure algebra. An element $a \in A$ is called **open** if $\Box a = a$ and **closed** if $\Diamond a = a$. The **skeleton** $H(A)$ is the set of its open elements, and A is called **skeletal algebra** if it is generated by its skeleton. The class of skeletal algebras is denoted as **SkA**.

Theorem 2.18. *Let A be a modal algebra. The following are equivalent:*

1. *A is a skeletal algebra,*
2. *The relation of the dual space $A_* = (X, R)$ is a partial order.*

It is clear from the definition that every skeletal algebra is in particular a closure algebra. However, the definition of skeletal algebras is not equational so that one has no reason to assume that skeletal algebras form a variety – which they in fact do not. Still, they are in close relation to the variety of so-called Grzegorczyk algebras, which we are going to define next.

A **Grzegorczyk (Grz) algebra** A is a modal algebra that validates

$$\Box(\Box(a \to \Box a) \to a) \to a. \qquad \text{(the } \mathbf{grz} \text{ axiom)}$$

The variety of Grz algebras is denoted as **GA**.

Theorem 2.19.

1. $\mathbf{SkA} \subset \mathbf{GA} \subset \mathbf{CA}$,
2. *Every finite Grz algebra is a skeletal algebra,*
3. $\mathbf{GA}$ *has the finite model property.*

Corollary 2.20. $\mathbf{GA} = \mathrm{Var}(\mathbf{SkA})$.

2.6 Polyhedra and polyhedral logic

For simplicial complexes, we refer to the appendix of [Bar11] and also section IV.21 and IV.22 in [Bre93]. The information on polyhedral logic is taken from the papers [Bez+17b] and [Ada+22].

An **abstract simplex** (with the plural 'simplices') is a non-empty finite set σ. Non-empty subsets $F \subseteq \sigma$ are called **faces** of σ. The elements $v \in \Delta$ are the **vertices** (with the singular 'vertex') of σ. The **dimension** of a simplex σ is defined as the number $\dim \sigma = |\sigma| - 1$.

An **abstract simplicial complex** is a non-empty set K of simplices that is closed under taking faces, i.e. if $\sigma \in K$ and $\tau \subseteq \sigma$ then also $\tau \in K$. The **vertex set** of K is the union $V(K) = \bigcup K$ of all its simplices. If the dimension of simplices of K is bounded, one defines the **dimension** of K as $\dim K = \max\{\dim \sigma \mid \sigma \in K\}$ and as ∞ otherwise. A **facet** of K is a maximal simplex, i.e. a simplex $\sigma \in K$ that is not strictly contained in another simplex. A **subcomplex** of K is simply a subset $K' \subseteq K$ that is also an abstract simplicial complex.

Given an abstract simplicial complex K, one obtains a partially ordered set $\mathcal{X}(K) = (K, \subseteq)$ when ordering the simplices by set inclusion. Conversely, given a poset $(X, \leq)$, the set of its finite chains $\mathcal{K}(X) = \{C \subseteq X \mid C \text{ a finite chain}\}$ forms an abstract simplicial complex.

The maps $\mathcal{X}$ and $\mathcal{K}$ extend to functors

$$\mathcal{X} : \mathbf{Simp} \longleftrightarrow \mathbf{Posets} : \mathcal{K}$$

between the category of posets with monotone maps and the category of abstract simplicial complexes with simplicial maps. A **simplicial map** $f : K \to K'$ between abstract simplicial complexes is a function $V(K) \to V(K')$ preserving simplices, i.e. in $\sigma \in K$ then also $f[\sigma] \in K'$.

Note that the functors $\mathcal{X}$ and $\mathcal{K}$ are far from an equivalence: There are many posets that are far from being of the form $\mathcal{X}(K)$. However, we will see in chapter 6, that the functors are an equivalence up to homotopy equivalence.

Given a poset X, the poset obtained by the composition $\mathcal{X}\mathcal{K}(X)$ is called the **nerve** of X and denoted as $\mathcal{N}(X)$. There is always a canonical map $m_X : \mathcal{N}(X) \to X$ mapping a chain C of X to its maximal element.

Proposition 2.21. *For every poset X, the map m_X is a surjective p-morphism.*

Abstract simplicial complexes are purely combinatorial, they only describe the manner how simplices are glued together. For us, however, it will also be decisive whether such a combinatorial object can actually be realized in a Euclidean space. The resulting notion is that of a geometric simplicial complex.

A set $\{v_0, \ldots, v_k\}$ of $k+1$ points in $\mathbb{R}^n$ for $k \leq n$ is called **affinely independent** if the k vectors $v_1 - v_0, \ldots, v_k - v_0$ are linearly independent. A **(geometric) k-simplex** is the convex hull of $k+1$ affinely independent points

$$s = \{\alpha_0 v_0 + \ldots + \alpha_k v_k \mid \sum \alpha_i = 1 \text{ and } \alpha_i \geq 0 \text{ for all } i\}.$$

The points $\{v_i\}$ are called **vertices** of s. The **dimension** of a k-simplex is simply defined as the number k and coincides with its topological dimension in $\mathbb{R}^n$. A **face** of s is the convex hull of a non-empty subset of its vertices. If t is a face of s, we will write $t \leq s$. The **relative interior** of the simplex s is defined as the set of points

$$\widetilde{s} = \{\alpha_0 v_0 + \ldots + \alpha_k v_k \mid \sum \alpha_i = 1 \text{ and } \alpha_i > 0 \text{ for all } i\}.$$

A **(geometric) simplicial complex** is a set Δ of geometric simplices in some fixed Euclidean space $\mathbb{R}^n$ that satisfies

1. Δ is closed under taking faces, i.e. if $s \in \Delta$ and $t \leq s$ then $t \in \Delta$,
2. Δ is closed under taking intersections, i.e. if $s \in \Delta$, $t \in \Delta$ and $s \cap t \neq \emptyset$, then $s \cap t \in \Delta$.

The notions **subcomplex**, **dimension** and **facet** of a geometric simplicial complex are defined exactly like for abstract simplicial complexes. Like in the abstract case, the set of simplices of a geometric simplicial complex $\mathcal{X}(\Delta) = (\Delta, \leq)$ ordered by the 'face of'-relation yields a poset.

The **carrier** of a simplicial complex Δ is the union $|K| = \bigcup \Delta$ of its simplices as subsets of $\mathbb{R}^n$. A **polyhedron** P is a subset of $\mathbb{R}^n$ that is the carrier of some finite geometric simplicial complex. Such a finite simplicial complex is then called **triangulation** of the polyhedron P. The **dimension** of $P = |\Delta|$ is the dimension of the simplicial complex Δ, which is independent of the choice of Δ.

Theorem 2.22. *Let P be a polyhedron and Δ a triangulation of it.*

1. *The relative interiors of simplices*

$$P = \bigsqcup_{s \in \Delta} \widetilde{s}$$

   form a partition of P.

2. *For $s, t \in \Delta$, consider the order $\widetilde{s} \preccurlyeq \widetilde{t} \;\Leftrightarrow\; \widetilde{s} \subseteq \mathrm{Cl}(\widetilde{t})$. Then:*

$$\widetilde{s} \preccurlyeq \widetilde{t} \iff s \leq t.$$

Given a geometric simplicial complex Δ, consider the set

$$\mathcal{A}(\Delta) = \{V(s) \mid s \in \Delta\}$$

of the vertex sets of simplices of Δ, the **abstraction** of Δ. Then, $K = \mathcal{A}(\Delta)$ is an abstract simplicial complex, and Δ is called a **geometric realization** of K. For this, one writes $\Delta = |K|$.

Theorem 2.23. *Let K be an n-dimensional abstract simplicial complex. Then, K has a geometric realization in $\mathbb{R}^{2n+1}$.*

From now on, we will fix the notation that by 'simplex' and 'simplicial complex' we mean 'geometric simplex' and 'geometric simplicial complex', respectively. When talking about the abstract notions, we will explicitly add the adjective 'abstract'.

Let P and Q be two polyhedra. They are called **PL homeomorphic** ('PL' for 'piecewise linear') if there exist triangulations Δ of P and Σ of Q such that $\mathcal{A}(\Delta) \cong \mathcal{A}(\Sigma)$ are isomorphic (in the category **Simp**). For this relation, we will write $P \cong_{PL} Q$.

Let P be a polyhedron. A **polyhedral set** in P is a subset $X \subseteq P$ (possibly empty) that is **compatible** with some triangulation of P, i.e. there exists a triangulation Δ and a subset (not necessarily a subcomplex) $\Delta' \subseteq \Delta$ such that

$$X = \bigsqcup_{s \in \Delta'} \operatorname{relint}(s).$$

Theorem 2.24 (Triangulation lemma)**.** *Let P be a polyhedron, and $X_1, \ldots, X_n \subseteq P$ finitely many polyhedral sets in P. Then, there exists a triangulation Δ of P which is compatible with all X_i.*

Let $\mathcal{PS}(P)$ denote the set of polyhedral sets in a polyhedron P. For a polyhedral set $X \subseteq P$, let X^c denote the complement in P, i.e. the set $P - X$. Note that $\mathrm{Cl}(X)$ is unambiguous in that regard as the closure in P and in $\mathbb{R}^n$ coincide since P is closed.

The following theorems 2.25 and 2.26 are consequences of the triangulation lemma. The general pattern that the information about the logic of a polyhedron is contained in all its triangulations will be generalized to stratifications in chapter 3.

Theorem 2.25. *For a given polyhedron P, the collection $\mathcal{PS}(P)$ is closed under binary unions, binary intersections, closure and complement in P. The algebra*

$$\mathcal{MPS}(P) = (\mathcal{PS}(P), \cup, \cap, (\,\cdot\,)^c, \mathrm{Cl}, \emptyset, P)$$

is a locally finite Grz algebra.

Theorem 2.26. *Let P be a polyhedron. Then, $\mathrm{Var}(\mathcal{MPS}(P))$ is generated by its triangulations seen as posets:*

$$\mathrm{Var}(\mathcal{MPS}(P)) = \mathrm{Var}(\{\mathcal{X}(\Delta) \mid \Delta \text{ is a triangulation of } P\})$$

Theorem 2.27. *Let P be a polyhedron and Δ a triangulation of P. Then, $\mathrm{Var}(\mathcal{MPS}(P))$is generated by Δ and all of its iterated nerves:*

$$\mathrm{Var}(\mathcal{MPS}(P)) = \mathrm{Var}(\{\mathcal{N}^i\mathcal{X}(\Delta) \mid i \geq 0\})$$

A variety $\mathcal{V}$ of modal algebras is called **polyhedral** if there exists a class $\mathcal{C}$ of polyhedra such that $\mathcal{V} = \mathrm{Var}(\{\mathcal{MPS}(P) \mid P \in \mathcal{C}\})$.

Theorem 2.28. *A variety $\mathcal{V}$ of modal algebras is polyhedral if and only if it has the finite model property and its (finite) frames are closed under nerves, i.e. if for all finite frames F:*

$$F^* \in \mathcal{V} \;\Rightarrow\; \mathcal{N}(F)^* \in \mathcal{V}.$$

Theorem 2.29. *Let $P \cong_{PL} Q$ be two PL homeomorphic polyhedra. Then*

$$\mathrm{Var}(\mathcal{MPS}(P)) = \mathrm{Var}(\mathcal{MPS}(Q)).$$

The last theorem can be reformulated as: Polyhedral logic is an invariant for PL homeomorphisms. One of the main results of chapter 3 will be that this notion can even be enlarged: we will show that polyhedral logic is an invariant for semialgebraic homeomorphisms.

3 Stratification Theory

Abstract

We develop the theory of stratifications that we will apply in the chapters 4 and 5.
The main result of this chapter states that the closure algebras of tame topologies are determined by the decomposition spaces of their stratifications.

Escher, *Wand mosaïk in het Alhambra* (1922)

3.1 Stratifications and constructible sets

The theory of stratifications has its roots in 1957, when Hassler Whitney in [Whi57] searched for regular decompositions of algebraic sets in $\mathbb{R}^n$. An algebraic set need not be a submanifold but Whitney showed the remarkable fact that every algebraic set is a disjoint union of manifolds that satisfy certain regularity conditions along their boundaries. Such a structure is nowadays known as a 'stratified space'. In this sense, stratified spaces can be seen as finitely many manifolds glued together and therefore as a generalization of the concept of a manifold.

Since then, stratified spaces have been studied from both algebraic and topological perspectives. Depending on the application, Whitney's original definition has been weakened in different ways. In this chapter, we will consider an almost minimal notion of stratified spaces, where the strata – i.e. the building blocks – are just required to be locally closed and satisfy the so-called *frontier condition* (even though for some use cases in algebraic geometry even the frontier condition is commonly further weakened). However, we will restrict ourselves to *finite* stratifications.

A notion very reminescent of stratifications was developed in algebraic topology, where continuous surjections $X \twoheadrightarrow (I, \leq)$ onto a poset with the Alexandrov topology (cf. theorem 2.2) were studied. In the subsection 3.1.1, we will make this connection precise: There is a one-to-one correspondence between stratifications and such continuous surjections $X \twoheadrightarrow (I, \leq)$ that are open maps.

The word 'stratum' with the plural 'strata' is Latin and can be translated as 'paving'. This captures very well how stratifications are used: They partition a space as paving stones – notably without overlap, which are allowed in open covers for instance.

3.1.1 Stratifications and poset-stratifications

Before introducing stratifications, we will consider the first condition that one poses on strata, namely to be locally closed. There is a long list of properties that are equivalent to being locally closed, from which we will present three.

Proposition 3.1. *Let X be a topological space and $A \subseteq X$. The following are equivalent:*

1. *There exists an open set U and a closed set C such that $A = U \cap C$,*
2. *A is open in the subspace topology of $\mathrm{Cl}(A)$,*
3. *$\delta A = \mathrm{Cl}(A) - A$ is closed.*

Proof. $\boxed{1 \Rightarrow 2}$ Let $A = U \cap C$ for an open set U and a closed set C. Since $A \subseteq C$ and $\mathrm{Cl}(A)$ is the least closed set containing A, we have that $\mathrm{Cl}(A) \subseteq C$. Therefore, $U \cap \mathrm{Cl}(A) \subseteq U \cap C$. Also, it holds that $U \cap C = U \cap U \cap C = U \cap A \subseteq U \cap \mathrm{Cl}(A)$. In conclusion, $A = U \cap C = U \cap \mathrm{Cl}(A)$, which shows that A is open in $\mathrm{Cl}(A)$.

$\boxed{2 \Rightarrow 3}$ Suppose A is open in $\mathrm{Cl}(A)$. Then, there is an open set U such that $A = U \cap \mathrm{Cl}(A)$. Then, $\delta A = \mathrm{Cl}(A) - (U \cap \mathrm{Cl}(A)) = (\mathrm{Cl}(A) - U) \cup (\mathrm{Cl}(A) - \mathrm{Cl}(A)) = \mathrm{Cl}(A) - U$, which is closed.

$\boxed{3 \Rightarrow 1}$ Suppose δA is closed. Then, $A = \mathrm{Cl}(A) - \delta A = \mathrm{Cl}(A) \cap (\delta A)^c$ is an intersection of an open and a closed set. □

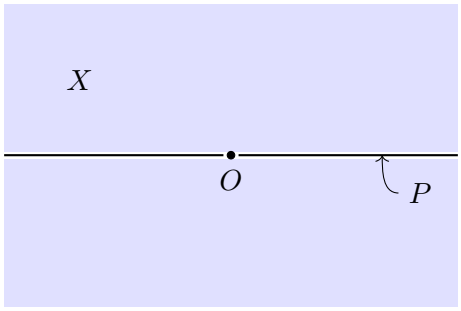

Figure 3.1: Partition of the Euclidean plane $\mathbb{R}^2$ into the locally closed sets
$O = \{(0,0)\}$,
$X = \{(x,y) \in \mathbb{R}^2 \mid y \neq 0\}$,
$P = \{(x,y) \in \mathbb{R}^2 \mid x \neq 0 \ \& \ y = 0\}$.

Definition 3.2 (local closedness)**.** A subset $A \subseteq X$ of a topologial space is *locally closed* if it satisfies any of the equivalent conditions from proposition 3.1.

Example 3.3. Consider the partition of $\mathbb{R}^2$ in figure 3.1. We claim that the set $X \cup O$ is not locally closed. If it were, by proposition 3.1, its border $\delta(X \cup O)$ would be closed, but this is the set P, which is not closed.

Now, we can define a stratification as a certain partition of a topological space by locally closed sets. A space together with a concrete stratification will be called *stratified space*.

Definition 3.4 (Stratification)**.** Let X be a topological space. A *stratification* of X is a finite set $\mathcal{S} \subseteq \mathcal{P}(X)$ satisfying the following properties:

1. $\mathcal{S}$ is a partition of X,
   i.e. $X = \bigcup \mathcal{S}$ and $A \cap B = \emptyset$ for all $A, B \in \mathcal{S}$ with $A \neq B$,

2. all $A \in \mathcal{S}$ are locally closed and non-empty,

3. if $A \cap \mathrm{Cl}(B) \neq \emptyset$ then $A \subseteq \mathrm{Cl}(B)$. (frontier condition)

The geometric intuition behind the frontier condition becomes clear in the following proposition. It is exactly the requirement that guarantees that the border (and thus also the closure) of each stratum is a finite union of strata.

Proposition 3.5. *Let $\mathcal{S}$ be a finite partition of a topological space X into locally closed and non-empty sets. The following are equivalent:*

1. *$\mathcal{S}$ is a stratification,*

2. *for every $A \in \mathcal{S}$, the border δA is a finite union of strata.*

Proof.

$\boxed{1 \Rightarrow 2}$ Suppose that $\mathcal{S}$ is a stratification, and let $A \in \mathcal{S}$. Given a point $x \in \delta A$, there exists a (unique) stratum $S_x \in \mathcal{S}$ such that $x \in S_x$. Since $x \in S_x \cap \mathrm{Cl}(A)$, the frontier condition implies that $S_x \subseteq \mathrm{Cl}(A)$. Since $x \in S_x$ but $x \notin A$, the strata S_x and A are disjoint, and we have that $S_x \subseteq \delta(A)$. Then, it follows that

$$\delta A = \bigcup \{S_x \mid x \in \delta A\},$$

which is finite because there are only finitely many strata.

$\boxed{2 \Rightarrow 1}$ Suppose that $\mathcal{S}$ satisfies the second property; then the frontier condition has to be shown. Let $A, B \in \mathcal{S}$ such that $A \cap \mathrm{Cl}(B) \neq \emptyset$. One can by assumption write the closure of B as

$$\mathrm{Cl}(B) = B \cup \delta B = B \sqcup S_1 \sqcup \ldots \sqcup S_n$$

with $S_i \in \mathcal{S}$. Because all sets $A, B, S_i \in \mathcal{S}$ and $\mathcal{S}$ is a partition, it already follows from $A \cap \mathrm{Cl}(B) \neq \emptyset$ that A has to be one of the sets B, A_i. In conclusion, $A \subseteq \mathrm{Cl}(B)$. □

Example 3.6. Consider figure 3.1 again. The sets X, P and O are stratification of $\mathbb{R}^2$. By proposition 3.5, their borders have to be finite unions of strata. Indeed, $\delta X = P \cup O$ and $\delta P = O$. By assuming the convention that the empty union is the empty set, we even have that $\delta O = \emptyset$ is a finite union of strata.

Proposition 3.7. *Given a stratification $\mathcal{S} = \{X_i\}_{i \in I}$ of a topological space X indexed by a finite set I. Define a relation on I by*

$$i \leq j \;\Leftrightarrow\; X_i \cap \mathrm{Cl}(X_j) \neq \emptyset.$$

Then, $(I, \leq)$ is a partially ordered set.

Proof. The relation $\leq$ is reflexive because $X_i \subseteq \mathrm{Cl}(X_i)$ and X_i is non-empty for all $i \in I$. To see that it is transitive, we need the frontier condition. Suppose that $X_i \cap \mathrm{Cl}(X_j) \neq \emptyset$ and $X_j \cap \mathrm{Cl}(X_k) \neq \emptyset$, then $X_i \subseteq \mathrm{Cl}(X_j)$ and $X_j \subseteq \mathrm{Cl}(X_k)$. Then, $X_i \subseteq \mathrm{Cl}(\mathrm{Cl}(X_k)) = \mathrm{Cl}(X_k)$ and therefore $i \leq k$.

It remains to show that $\leq$ is antisymmetric. Using the frontier condition, assume that $X_i \subseteq \mathrm{Cl}(X_j)$ and $X_j \subseteq \mathrm{Cl}(X_i)$ and $i \neq j$. Since $X_i \cap X_j = \emptyset$, we know that $X_i \subseteq \delta X_j$. By proposition 3.1, δX_j is closed, and therefore $\mathrm{Cl}(X_i) \subseteq \delta X_j$ by minimality of the closure. Then, $X_j \subseteq \mathrm{Cl}(X_j) \subseteq \delta X_j$. This is a contradiction to X_j being non-empty. In conclusion, $i = j$ must be the case; and $\leq$ is antisymmetric. □

In the following, we will introduce poset-stratifications and prove that they correspond exactly to stratifications. The study of this correspondence is relatively new and, for example, made precise in [WWY25] from 2025. The following proofs are loosely based on this publication.

Lemma 3.8. *Let $\mathcal{S} = \{X_i\}_{i\in I}$ be a stratification of a topological space X. Then, the map $\pi : X \to (I, \leq)$ mapping $x \in X$ to the unique $i \in I$ such that $x \in X_i$ is a surjective interior map.*

Proof. The map π is well-defined because $\mathcal{S}$ is a partition, and thus every $x \in X$ lies indeed in a unique stratum. Also, all strata X_i are non-empty, which implies that π is surjective.

π is continuous: Let $i \in I$, then the set $\downarrow i$ is closed in the Alexandrov topology of $(I, \leq)$. We know from the proof of proposition 3.5 that $\delta X_i = \bigcup\{S_x \mid x \in \delta X_i\}$. Note that the strata X_j for which there exists $x \in X_j$ such that $x \in \delta X_i$ are exactly those that satisfy $j < i$. Therefore, we have that $\delta X_j = \bigcup\{X_j \mid j < i\}$ and thus $\mathrm{Cl}(X_i) = \bigcup\{X_j \mid j \leq i\}$. This shows that $\pi^{-1}(\downarrow i) = \pi^{-1}\left(\bigcup\{\{j\} \mid j \leq i\}\right) = \bigcup\{\pi^{-1}(j) \mid j \leq i\} = \bigcup\{X_j \mid j \leq i\} = \mathrm{Cl}(X_i)$ is closed.

Now, let $D \subseteq I$ be closed set in the Alexandrov topology, i.e. a downset. Then, $\pi^{-1}(D) = \pi^{-1}\left(\bigcup\{\downarrow i \mid i \in D\}\right) = \bigcup\{\pi^{-1}(\downarrow i) \mid i \in D\}$. Therefore, $\pi^{-1}(D)$ is a finite union of closed sets and thus closed. This shows that π is continuous.

π is open: Let $U \subseteq X$ be open. We will show that $\pi(U)$ is an upset. Let $i \in \pi(U)$ and $i \leq j$. Then, $U \cap X_i \neq \emptyset$ and $X_i \subseteq \mathrm{Cl}(X_j)$. Therefore, also $U \cap \mathrm{Cl}(X_j) \neq \emptyset$. Suppose towards contradiction that $U \cap X_j = \emptyset$. Then $X_j \subseteq U^c$, and since U^c is closed, it follows that $\mathrm{Cl}(X_j) \subseteq U^c$. But then $U \cap \mathrm{Cl}(X_j) = \emptyset$, which contradicts the assumption. Therefore, $U \cap X_j \neq \emptyset$, which implies that $j \in \pi(U)$. In conclusion, $\pi(U)$ is an upset and π is an open map. □

Note that even if the index set I is not explicitely given, every stratification $\mathcal{S}$ of a space X is naturally indexed by its strata. Therefore, we can always choose the index set $(I, \leq)$ to be $(\mathcal{S}, \leq)$. In the rest of this thesis, we will adopt the convention to write $I_{\mathcal{S}}$ whenever we refer to $\mathcal{S}$ as a poset.

In the next lemma, we will see that, conversely, also every surjective interior map gives rise to a stratification.

Lemma 3.9. *Let X be a topological space, $(I, \leq)$ a finite partially ordered set and $\pi : X \to I$ a surjective interior map. Then, $\mathcal{S} = \{\pi^{-1}(i) \mid i \in I\}$ is a stratification.*

Proof. The set $\mathcal{S}$ is finite because I is finite; and the sets $\pi^{-1}(i)$ for $i \in I$ are non-empty because π is surjective. The property that $\mathcal{S}$ is a partition of X follows from π being a function as every $x \in X$ gets mapped to exactly one $i \in I$.

$\pi^{-1}(i)$ are locally closed: Let $i \in I$, then the set $\uparrow i$ is open in the Alexandrov topology on I, and the set $\downarrow i$ is closed. Because π is continuous, we have that $\pi^{-1}(i) = \pi^{-1}(\uparrow i \cap \downarrow i) = \pi^{-1}(\uparrow i) \cap \pi^{-1}(\downarrow i)$ is locally closed.

$\mathcal{S}$ satisfies the frontier condition: Let $i, j \in I$ such that $\pi^{-1}(i) \cap \mathrm{Cl}(\pi^{-1}(j)) \neq \emptyset$. Since π is open, the closure commutes with inverse images. The closure of $\{j\}$ in I is the downset $\downarrow j$. Therefore, we have that $\pi^{-1}(\{i\} \cap \downarrow j) \neq \emptyset$. But this implies that $\{i\} \subseteq \downarrow j$. Therefore, $\pi^{-1}(i) \subseteq \pi^{-1}(\downarrow j) = \mathrm{Cl}(\pi^{-1}(j))$, which instates the frontier condition. □

Definition 3.10 (Poset stratification)**.** Let X be a topological space. A *poset stratification* is a surjective interior (continuous and open) map $\pi : X \twoheadrightarrow (I, \leq)$ onto a finite poset with the Alexandrov topology.

To summarize the lemmata 3.8 and 3.9: Given a stratification $X = \bigsqcup_{i \in I} X_i$, the so-called *characteristic map* $\pi : X \to I$, mapping a point to its unique stratum, establishes a poset stratification. Conversely, given a poset stratification $\pi : X \to I$, the preimages of I under π form a stratification of X. The finite poset $(I, \leq)$ as defined in proposition 3.7 is called *decomposition space* of a stratification.

This establishes the following theorem:

Theorem 3.11. *There is a one-to-one correspondence between stratifications and poset-stratifications.*

By virtue of theorem 3.11, we will henceforth speak of 'stratifications' throughout the remainder of this thesis and use them as partitions or as interior surjections onto a finite poset however appropriate.

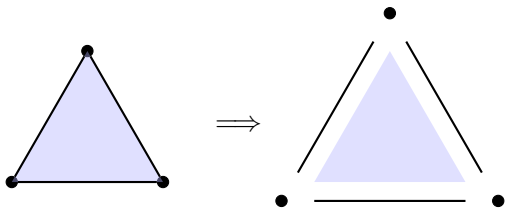

Figure 3.2: Stratification of a 2-simplex by the relative interiors of its faces

In the remainder of this subsection about stratifications, we would like to link back to polyhedra. Every triangulation of a polyhedron defines a partition by its relative interiors. Theorem 2.22 implies that this partition is in fact a stratification. Therefore, stratifications can be seen as a proper generalization of triangulations. To make this precise:

Proposition 3.12. *Let Δ be a triangulation of a polyhedron P. Then*

$$\widetilde{\Delta} = \{\widetilde{s} \mid s \in \Delta\}$$

is a stratification in the sense of definition 3.4.

Proof. First, note that the relative interiors of simplices are locally closed because their border is a finite union of (closed) faces.

Point 1 of theorem 2.22 then ensures that $\tilde{\Delta}$ is a finite partition of P into locally closed sets. So, it is left to show that the frontier condition holds.

Let $\widetilde{s}, \widetilde{t} \in \widetilde{\Delta}$ such that $\widetilde{s} \cap \mathrm{Cl}(\widetilde{t}) \neq \emptyset$. Note that $\mathrm{Cl}(\widetilde{t}) = t$. But t itself is a finite union of relative interiors by part 1 of theorem 2.22. Using point 2, it follows that $t = \bigsqcup\{\widetilde{r} \mid r \leq t\} = \bigsqcup\{\widetilde{r} \mid \widetilde{r} \preccurlyeq \widetilde{t}\}$. But the relative interiors are a partition, so that $\widetilde{s}$ must already be contained in the set $\{\widetilde{r} \mid \widetilde{r} \preccurlyeq \widetilde{t}\}$. By definition of the order $\preccurlyeq$, it follows that $\widetilde{s} \subseteq \mathrm{Cl}(\widetilde{t})$, which is the frontier condition. □

3.1.2 Products and sums

The original advantage of poset-stratifications in algebraic topology is that they can easily be enriched with a notion of morphisms to form a category. For example, it is an active area of research to formulate homotopy theory for stratified spaces. To our knowledge, however, the class of stratifications on a space has not yet been described as an algebraic structure. Even though the content of the present subsection is far from such an ambitious goal, we believe that this could be an interesting long-term object of research.

Every topological space X has the trivial stratification $\{X\}$ that just consists of one stratum. In the following we are going to examine how stratifications behave regarding *sums* and *products*. The use case in this thesis will become evident in chapter 4, where we will specify a class of stratifications on $\mathbb{R}$ and then use products and sums to generate a larger class of stratifications on the higher-dimensional spaces $\mathbb{R}^n$.

Let $(X, \leq_X)$ and $(Y, \leq_Y)$ be two partial orders. Their product order on $X \times Y$ is given by

$$(x, y) \leq (x', y') \iff x \leq_X x' \;\&\; y \leq_Y y'$$

and is again a partial order. If $(X, \leq_X)$ and $(Y, \leq_Y)$ are seen as Alexandrov spaces, the product order equals the product topology of the Alexandrov topologies.

Proposition 3.13. *Let $\pi_X : X \to I$ and $\pi_Y : Y \to J$ be poset-stratifications. Then, the map*

$$\pi_X \times \pi_Y : X \times Y \to I \times J,\ (x, y) \mapsto (\pi_X(x), \pi_Y(y))$$

defines a poset-stratification.

Proof. Obviously, $\pi_X \times \pi_Y$ is surjective, because a preimage of a point $(i, j) \in I \times J$ is given by a pair of componentwise preimages. It is well-known that products of continuous maps are again continuous. For openness, suppose that $U \subseteq X \times Y$ is open. By definition of the product topology, we can present U as a union $U = \bigcup_{i \in K} U_i \times V_i$, where the sets U_i are open in X and V_i in Y, respectively. Then, we have that

$$\begin{aligned} \pi_X \times \pi_Y(U) &= \bigcup_{i \in K} \pi_X \times \pi_Y(U_i \times V_i) \\ &= \bigcup_{i \in K} \pi_X(U_i) \times \pi_Y(V_i), \end{aligned}$$

which is a union of open rectangles and therefore open. □

The stratification considered in proposition 3.13 will be called *product stratification*. Using theorem 3.11, the equivalent description in terms of partitions is the following: Let $\mathcal{S}$ be a stratification of X and $\mathcal{T}$ a stratification of Y. Then, the strata of the stratification $\mathcal{S} \times \mathcal{T}$ will be exactly the rectangles $S \times T$ for $S \in \mathcal{S}$ and $T \in \mathcal{T}$.

Definition 3.14 (Sum partition)**.** Let $\mathcal{S} = \{X_i\}_{i \in I}$ and $\mathcal{T} = \{Y_i\}_{j \in J}$ be finite partitions of a topological space X. Their *sum* is the collection

$$\mathcal{S} + \mathcal{T} = \{X_i \cap Y_j \mid i \in I \;\&\; j \in J\} - \{\emptyset\}.$$

Proposition 3.15. *The sum of two partitions is a partition.*

Proof. Let $\mathcal{S}$ and $\mathcal{T}$ be as in definition 3.14. First, the parts of $\mathcal{S} + \mathcal{T}$ are locally closed as intersections of locally closed sets. Second, $\bigcup(\mathcal{S} + \mathcal{T}) = X$ because every $x \in X$ is contained in some $X_i \cap Y_i$. Third, this decomposition is disjoint. Let $X_1 \cap Y_1$ and $X_2 \cap Y_2$ be two different parts of $\mathcal{S} + \mathcal{T}$. Then, their intersection $(X_1 \cap Y_1) \cap (X_2 \cap Y_2) = (X_1 \cap X_2) \cap (Y_1 \cap Y_2) = \emptyset$ is empty because either $X_1 \neq X_2$ or $Y_1 \neq Y_2$ is the case. □

However, if $\mathcal{S}$ and $\mathcal{T}$ are stratifications, it is not guaranteed that their sum $\mathcal{S} + \mathcal{T}$ satisfies the frontier condition. We found the following sufficient condition to ensure this:

Definition 3.16 (Compatible stratifications)**.** Let $\mathcal{S} = \{X_i\}_{i\in I}$ and $\mathcal{T} = \{Y_i\}_{j\in J}$ be finite partitions of a topological space X. We call them *compatible* if the following holds for all $i \in I$ and $j \in J$:

$$X_i \cap Y_j \neq \emptyset \;\Rightarrow\; \mathrm{Cl}(X_i) \cap \mathrm{Cl}(Y_j) \subseteq \mathrm{Cl}(X_i \cap Y_j).$$

Proposition 3.17. *The sum of two compactible stratifications is a stratification.*

Proof. Let $\mathcal{S}$ and $\mathcal{T}$ be as in definition 3.16 and compatible. We know from proposition 3.15, that their sum $\mathcal{S}+\mathcal{T}$ is a partition, so it is left to show the frontier condition.

Let $X_1 \cap Y_1$ and $X_2 \cap Y_2$ be two strata of $\mathcal{S}+\mathcal{T}$ such that $X_1 \cap Y_1 \cap \mathrm{Cl}(X_2 \cap Y_2) \neq \emptyset$. It is to show that $X_1 \cap Y_1 \subseteq \mathrm{Cl}(X_2 \cap Y_2)$.

Since $X_1 \cap Y_1 \cap \mathrm{Cl}(X_2 \cap Y_2) \subseteq X_1 \cap Y_1 \cap \mathrm{Cl}(X_2) \cap \mathrm{Cl}(Y_2)$, we have that both $X_1 \cap \mathrm{Cl}(X_2) \neq \emptyset$ and $Y_1 \cap \mathrm{Cl}(Y_2) \neq \emptyset$. Because $\mathcal{S}$ and $\mathcal{T}$ are stratifications, it follows that $X_1 \subseteq \mathrm{Cl}(X_2)$ and $Y_1 \subseteq \mathrm{Cl}(Y_2)$. Then,

$$X_1 \cap Y_1 \subseteq \mathrm{Cl}(X_2) \cap \mathrm{Cl}(Y_2) \subseteq \mathrm{Cl}(X_2 \cap Y_2),$$

where the last inclusion follows from compatibility of $\mathcal{S}$ and $\mathcal{T}$ and the fact that $X_2 \cap Y_2 \neq \emptyset$. □

A property that will become useful in chapter 4 is that compatibility is preserved by taking product stratifications.

Proposition 3.18. *Let $\mathcal{S}$ and $\mathcal{T}$ be two compatible stratifications of a space X, and $\mathcal{R}$ and $\mathcal{Q}$ compatible stratifications of a space Y. Then, the product stratifications $\mathcal{S}\times\mathcal{R}$ and $\mathcal{T}\times\mathcal{Q}$ are compatible.*

Proof. Let $X_1 \times X_2$ be a stratum of $\mathcal{S}\times\mathcal{R}$, and $Y_1 \times Y_2$ a stratum of $\mathcal{T}\times\mathcal{Q}$ such that $(X_1 \times X_2) \cap (Y_1 \times Y_2) \neq \emptyset$. It holds that $(X_1 \times X_2) \cap (Y_1 \times Y_2) = (X_1 \cap Y_1) \times (X_2 \cap Y_2)$, from which we know that also $X_1 \cap Y_1 \neq \emptyset$ and $X_2 \cap Y_2 \neq \emptyset$.

Because $\mathcal{S}$ and $\mathcal{T}$ are compatible as well as $\mathcal{R}$ and $\mathcal{Q}$, compatibility tells us that $\mathrm{Cl}(X_1 \cap Y_1) = \mathrm{Cl}(X_1) \cap \mathrm{Cl}(Y_1)$ and $\mathrm{Cl}(X_2 \cap Y_2) = \mathrm{Cl}(X_2) \cap \mathrm{Cl}(Y_2)$. Then

$$\begin{aligned}
&\mathrm{Cl}(X_1 \cap Y_1) \times \mathrm{Cl}(X_2 \cap Y_2) = (\mathrm{Cl}(X_1) \cap \mathrm{Cl}(Y_1)) \times (\mathrm{Cl}(X_2) \cap \mathrm{Cl}(Y_2)) \\
\Rightarrow\quad &\mathrm{Cl}((X_1 \cap Y_1) \times (X_2 \cap Y_2)) = (\mathrm{Cl}(X_1) \times \mathrm{Cl}(X_2)) \cap (\mathrm{Cl}(Y_1) \times \mathrm{Cl}(Y_2)) \\
\Rightarrow\quad &\mathrm{Cl}((X_1 \times X_2) \cap (Y_1 \times Y_2)) = \mathrm{Cl}(X_1 \times X_2) \cap \mathrm{Cl}(Y_1 \times Y_2)
\end{aligned}$$

follows from the general facts that $\mathrm{Cl}(A \times B) = \mathrm{Cl}(A) \times \mathrm{Cl}(B)$ and $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$. Therefore, the compatibility condition for $\mathcal{S}\times\mathcal{T}$ and $\mathcal{R}\times\mathcal{Q}$ has been shown. □

Proposition 3.19. *Let $\mathcal{S}$ and $\mathcal{T}$ be two compatible stratifications of a space X, and $\mathcal{R}$ and $\mathcal{Q}$ compatible stratifications of a space Y. Then*

$$(\mathcal{S}\times\mathcal{R}) + (\mathcal{T}\times\mathcal{Q}) = (\mathcal{S}+\mathcal{T}) \times (\mathcal{R}+\mathcal{Q}).$$

Proof. First, note that all sums in the above equation exist due to the propositions 3.17 and 3.18.

Let $S \in \mathcal{S}$, $T \in \mathcal{T}$, $R \in \mathcal{R}$ and $Q \in \mathcal{Q}$ be any strata of the particular stratifications. Then, it holds that

$$(S \times R) \cap (T \times Q) = (S \cap T) \times (R \cap Q).$$

In particular, this holds for those strata for which either side is non-empty. □

Another property of stratifications that we will use in section 5.2 is that stratifications give rise to stratifications on closed subspaces:

Proposition 3.20. *Let X be a space, C a non-empty closed subspace and $\mathcal{S}$ a stratification of X. Then, the following is a stratification of C:*

$$\mathcal{S}_C = \{X \cap C \mid X \in \mathcal{S}\} - \{\emptyset\}.$$

Proof. It is immediate that $\mathcal{S}_C$ is a finite partition of C by non-empty sets. We have to show that the parts of $\mathcal{S}_C$ are locally closed and satisfy the frontier condition.

Because C is closed, the closures in C and in X coincide, i.e. $\mathrm{Cl}(A) = \mathrm{Cl}_C(A)$ for all $A \in \mathcal{S}_C$. For $A \in \mathcal{S}_C$, we have that $\mathrm{Cl}(A) \subseteq C$, and thus also that $\delta A \subseteq C$. But δA is closed in X and thus also closed in C, and we know that A is locally closed in C.

In order to show the frontier condition, suppose that $A \cap C, B \cap C \in \mathcal{S}_C$ for $A, B \in \mathcal{S}$ such that $A \cap C \cap \mathrm{Cl}(B \cap C) \neq \emptyset$. Because C is closed, we have that $\mathrm{Cl}(B \cap C) = \mathrm{Cl}(B) \cap C$. Therefore, $A \cap C \cap \mathrm{Cl}(B) \cap C \neq \emptyset$. In particular, $A \cap \mathrm{Cl}(B) \neq \emptyset$. But since $\mathcal{S}$ is a stratification, $A \subseteq \mathrm{Cl}(B)$ follows. Then, also $A \cap C \subseteq \mathrm{Cl}(B) \cap C = \mathrm{Cl}(B \cap C)$, which is the frontier condition for $\mathcal{S}_C$. □

3.1.3 The closure algebra of constructibles

The locally closed subsets of a topological space do not quite form a closure algebra. In the following, we will consider the closure algebra *generated by* the locally closed sets and show that it coincides with the closure algebra generated by the topology, that is, by all open sets. The following results follow section 5.15 of [Jon26] in parts.

Proposition 3.21. *Locally closed sets are closed under intersection and topological closure, but not under union and complement.*

Proof.

- Let A and B be locally closed sets. By proposition 3.1, there are opens U and U' and closed sets C and C' such that $A = U \cap C$ and $B = U' \cap C'$. Then, the intersection $A \cap B = (U \cap U') \cap (C \cap C')$ is the intersection of an open and a closed set and therefore locally closed.
- Every closed set is locally closed. Therefore, it is trivial that locally closed sets are closed under topological closure.
- For a counterexample for unions and complements, work figure 3.1 one last time. X and O are both locally closed but their union $X \cup O$ is not. Similarly, P is locally closed but its complement $P^c = X \cup O$ is not. □

It turns out that it suffices to close the locally closed sets under finite unions to obtain a closure algebra:

Definition 3.22 (Constructible set)**.** A *constructible set* (or short 'constructible' used as a noun) is a finite union of locally closed sets.

Proposition 3.23. *The collection $\mathcal{C}(X)$ of all constructible sets of a topological space is closed under binary intersection, binary unions, complements and topological closure.*

Proof. Let $A, B \in \mathcal{C}(X)$ be constructibles. Let $A = (U_1 \cap C_1) \cup \ldots \cup (U_n \cap C_n)$ and $B = (U'_1 \cap C'_1) \cup \ldots \cup (U'_m \cap C'_m)$ for opens U_i, U'_i and closed sets C_i, C'_i.

- $A \cap B = ((U_1 \cap C_1) \cup \ldots \cup (U_n \cap C_n)) \cap ((U'_1 \cap C'_1) \cup \ldots \cup (U'_m \cap C'_m))$. Using the distributivity laws, this can be rewritten as $\bigcup_{1 \leq i \leq n, 1 \leq j \leq m} U_i \cap U'_j \cap C_i \cap C'_j$. Each $U_i \cap U'_j \cap C_i \cap C'_j$ is the intersection of opens and closed sets and thus a locally closed; and therefore the whole union is constructible.
- Closedness under binary union is clear from the definition of constructibles.
- Closedness under closure is clear since closed sets are locally closed.
- Use de Morgan's law to write A^c as $A^c = (U_1^c \cup C_1^c) \cap \ldots \cap (U_n^c \cup C_n^c)$. The sets U_i^c are closed, and the sets C_i^c are open. Both are therefore constructible. We already know that constructibles are closed under binary unions and binary (and therefore finite) intersections. Therefore, A^c is constructible. □

Proposition 3.23 shows that $\mathcal{MC}(X) = (\mathcal{C}(X), \cup, \cap, (\,\cdot\,)^c, \mathrm{Cl}, \emptyset, X)$ is a subalgebra of $\mathcal{MP}(X)$ and thus a closure algebra. Even more, it is the subalgebra generated by the topology:

Proposition 3.24. *Let X be a topological space. $\mathcal{MC}(X)$ is the smallest subalgebra of $\mathcal{MP}(X)$ containing all open sets.*

Proof. Let $S \leq \mathcal{MP}(X)$ be any subalgebra containing all open sets, and let $A \in \mathcal{C}(A)$ be any constructible. Then by definition, A can be written as Boolean combination $A = (U_1 \cap (C_1^c)^c) \cup \ldots \cup (U_n \cap (C_n^c)^c)$ of open sets U_i and C_i^c. Therefore, $A \in S$. In conclusion, $\mathcal{C}(X) \subseteq S$. □

In [Esa+19], $\mathcal{MC}(X)$ is called the Boolean envelope of the skeleton $H(X)$. Note that some of the following properties of $\mathcal{MC}(X)$ also follow from the general theory of skeletal algebras.

The following alternative definition of constructible sets is an immediate consequence of the previous proposition:

Proposition 3.25. *Let $A \in \mathcal{C}(X)$ be a constructible set of some topological space X. Then, A is a finite union of pairwise disjoint locally closed sets.*

Proof. Consider the collection

$$\mathcal{D} = \{\text{finite disjoint unions of locally closed sets in } X\}.$$

We claim that $\mathcal{D}$ is closed under binary unions, binary intersections, complements and closure. The case of topological closure is obvious; and the case of binary unions follows from closure under binary intersections and complements via De Morgan's laws.

Let $A = X_1 \sqcup \ldots \sqcup X_n$ and $B = Y_1 \sqcup \ldots \sqcup Y_m$ be finite disjoint unions of locally closed sets X_i and Y_j. Using distributivity, their intersections has the form

$$A \cap B = \bigcup_{1 \leq i \leq n, 1 \leq j \leq m} X_i \cap X_j.$$

The intersections $X_i \cap Y_j$ are again locally closed; and furthermore this union is disjoint (because if $X_i \cap Y_i \cap X_{i'} \cap Y_{j'} \neq \emptyset$, then also $X_i \cap X_{i'} \neq \emptyset$ and $Y_j \cap Y_{j'} \neq \emptyset$ and thus $i = i'$ and $j = j'$).

Regarding closure under complements, we start by considering a single locally closed set $A = U \cap C$ with an open U and a closed set C. Then, $A^c = U^c \cup C^c = (U^c \cap C) \sqcup C^c$. Thus, A^c is a disjoint union of two locally closed sets.

Now, consider a general element $A = X_1 \sqcup \ldots \sqcup X_n$ of $\mathcal{D}$. We have that $A^c = X_1^c \cap \ldots \cap X_n^c$. Since we already know that the complements X_i^c of locally closed sets are elements of $\mathcal{D}$ and that $\mathcal{D}$ is closed under finite intersections, it follows that $A^c \in \mathcal{D}$.

We have shown that $\mathcal{D} \leq \mathcal{MP}(X)$ is a subalgebra. By proposition 3.24, it follows that $\mathcal{C}(X) \subseteq \mathcal{D}$. Therefore, every constructible set can be written as a disjoint union of locally closed sets. □

Recall from theorem 2.25 that the closure algebra $\mathcal{PS}(P)$ of polyhedral subsets of a polyhedron is a Grz algebra. If we consider P as a topological space, then $\mathcal{PS}(P) \leq \mathcal{MC}(P)$ is a subalgebra because every polyhedral set is constructible. The property of being a Grz algebra can still be recovered for the bigger algebra $\mathcal{MC}(P)$; and in fact for any topological space.

Theorem 3.26. *The closure algebra $\mathcal{MC}(X)$ is a Grz algebra.*

Proof. This was proven by Esakia in [Esa+19] and follows from theorem 2.19.

Consider the closure algebra $\mathcal{MP}(X)$, the condition $\mathrm{Int}(U) = U$ for $U \subseteq X$ identifies exactly the open sets in X. The algebra $\mathcal{MC}(X)$ is the subalgebra generated by the open sets and therefore skeletal.

By theorem 2.19, every skeletal algebra is a Grz algebra. □

But recall that theorem 2.25 said even more: The algebra $\mathcal{PS}(P)$ of polyhedral sets is not only a Grz algebra but even locally finite. For a general topological space X, we cannot expect such a result for $\mathcal{MC}(X)$.

The result on local finiteness in theorem 2.25 was an almost direct consequence of the triangulation lemma 2.24. Recall its statement:

> Let P be a polyhedron, and $X_1, \ldots, X_n \subseteq P$ finitely many polyhedral sets in P. Then, there exists a triangulation Δ of P which is compatible with all X_i.

Since the role of triangulations is now taken over by stratifications, the following definition almost writes itself:

Definition 3.27. Let X be a topological space. We say that X has the *stratification property* if for every finite collection of constructible sets $X_1, \ldots, X_n \subseteq X$, there exists a stratification $\mathcal{S}$ of X which is compatible with all X_i.

Here, just as in the context of triangulations, the statement that a constructible set A is *compatible* with a stratification $\mathcal{S}$ means that there exists a subset (possibly empty) $\mathcal{S}' \subseteq \mathcal{S}$ such that $A = \bigsqcup \mathcal{S}'$.

Proposition 3.28. *Let A be a finite Grz algebra. The atoms of A (or – strictly speaking – of the underlying Boolean algebra) are locally closed.*

Proof. Because A is a finite Grz algebra, its dual space $(X, R) = A_*$ is a partial order (with the discrete topology). Using the duality dictionary 2.1, the atoms of A correspond to the points of X. Note that an open element corresponds to an upset in (X, R), and a closed element to a downset. But because (X, R) is a partial order, every element $x \in X$ can be written as $\{x\} = {\uparrow}x \cap {\downarrow}x$. Therefore, every point is an intersection of an upset and a downset. The dual statement is that every atom is the meet of an open and a closed element – and therefore locally closed. □

Theorem 3.29. *The following are equivalent for a topological space X:*

1. *X has the stratification property.*

2. *$\mathcal{MC}(X)$ is locally finite,*

Proof. $\boxed{\Rightarrow}$ Suppose that X has the stratification property. Let $A_1, \ldots, A_n \in \mathcal{MC}(X)$ be finitely many constructible sets. Take a stratification $\mathcal{S}$ of X that is compatible with every set A_i. Consider the finite collection of sets

$$\mathcal{D} = \left\{ \bigsqcup \mathcal{S}' \mid \mathcal{S}' \subseteq \mathcal{S} \right\}.$$

We call elements of this collection *$\mathcal{S}$-objects*. The fact that all A_i are compatible with $\mathcal{S}$ means exactly that all $A_i \in \mathcal{D}$. It is obvious that $\mathcal{D}$ is closed under binary intersections and binary unions, and that $\emptyset \in \mathcal{D}$ and $X \in \mathcal{D}$. To see that $\mathcal{D}$ is also closed under closure, let $D = S_1 \sqcup \ldots \sqcup S_k \in \mathcal{D}$ be an $\mathcal{S}$-object. Then,

$$\mathrm{Cl}(D) = \mathrm{Cl}(S_1 \sqcup \ldots \sqcup S_k) = \mathrm{Cl}(S_1) \cup \ldots \cup \mathrm{Cl}(S_k)$$

is a (not necessarily disjoint) union of closures of strata. But we know from proposition 3.5 that every closure $\mathrm{Cl}(S_i)$ is a disjoint union of strata and therefore an $\mathcal{S}$-object. Since $\mathcal{D}$ is closed under finite unions, it follows that also $\mathrm{Cl}(D) \in \mathcal{D}$.

We have shown that $\mathcal{D}$ is a finite subalgebra of $\mathcal{MC}(X)$ containing the constructible sets $A_1, \ldots, A_n$. In conclusion, the generated subalgebra $\langle A_1, \ldots, A_n \rangle \subseteq \mathcal{D}$ is smaller than $\mathcal{D}$ and therefore finite. This shows that $\mathcal{MC}(X)$ is locally finite.

$\boxed{\Leftarrow}$ Suppose that $\mathcal{MC}(X)$ is locally finite. It is to show that X has the stratification property. Let $A_1, \ldots, A_n \in \mathcal{C}(X)$ be finitely many constructible sets. Because $\mathcal{MC}(X)$ is locally finite, the generated subalgebra $S = \langle A_1, \ldots, A_n \rangle$ is finite.

Define $\mathcal{S} = \mathrm{At}(S)$. We claim that this is a stratification as desired.

By definition of atoms, the strata in $\mathcal{S}$ are non-empty. We know from theorem 3.26 that $\mathcal{MC}(X)$ is a Grz algebra. Because they form a variety, they are closed under subalgebras such that S is a finite Grz algebra. By proposition 3.28, the atoms of S are locally closed.

By proposition 2.1, we know that $\bigcup \mathcal{S} = X$; and it follows from the definition of atoms that they are disjoint, i.e. that if $A \cap B \neq \emptyset$ for atoms $A, B \in \mathcal{S}$, then $A = B$. Also, $\mathcal{S}$ is compatible with all A_i because by proposition 2.1 every $A_i = \bigvee \{A \in \mathcal{S} \mid A \leq A_i\}$. Therefore, it just remains to show that $\mathcal{S}$ satisfies the frontier condition.

So, let $A, B \in \mathcal{S}$ such that $A \cap \mathrm{Cl}(B) \neq \emptyset$. Because S is a closure algebra, we have that $\mathrm{Cl}(B) \in S$. But A is an atom and $A \cap \mathrm{Cl}(B) \subseteq A$. Then, by definition of atoms, $A \cap \mathrm{Cl}(B) = A$ and thus $A \subseteq \mathrm{Cl}(B)$, which is the frontier condition. □

The concept $\mathrm{height}(\mathcal{MP}(X)_*)$ – the height of the dual space of the closure algebra of a topological space – is called *modal Krull dimension* in [Bez+17a]. It is known that the modal spaces $\mathcal{MP}(X)_*$ and $\mathcal{MC}(X)_*$ have the same height. Also, if $\mathcal{MC}(X)_*$ has finite height, the algebra $\mathcal{MC}(X)$ is locally finite. Therefore, it follows from theorem 3.29 that every space with finite modal Krull dimension has the stratification property.

However, note that is does not follow from $\mathcal{MC}(X)$ being locally finite that the generated variety $\mathrm{Var}(\mathcal{MC}(X))$ is locally finite.

Example 3.30. Take $X = (\mathbb{N}, \leq)$ with the Alexandrov topology and consider $\mathcal{MC}(X)$. Since the open sets of X are the cofinite upsets, it is easy to see that $\mathcal{MC}(X) = \mathrm{FinCofin}(X)$. The closure in X of a finite set corresponds to the downset operation and is thus a finite set, while the closure of a cofinite set is the whole space.

We claim that this algebra is locally finite. Take finitely many constructible sets $\mathcal{S}$. All sets in $\mathcal{S}$ are either finite or cofinite, so that we can write them as $\mathcal{S} = \{F_1, \ldots, F_n, F_{n+1}^c, \ldots, F_m^c\}$, with all F_i finite.

Then, the union $G = \bigcup_{1 \leq i \leq m} F_i$ is finite, and the following is a finite partition of X:

$$\mathcal{T} = \{\{x\} \mid x \in {\downarrow}G\} \cup \{G^c\}.$$

Consider the finite collection of sets

$$\mathcal{D} = \left\{ \bigsqcup \mathcal{T}' \mid \mathcal{T}' \subseteq \mathcal{T} \right\}.$$

$\mathcal{D}$ is closed under binary unions, binary intersections, complements, and obviously $\emptyset, X \in \mathcal{D}$. Also, $\mathcal{D}$ is closed under topological closure, because if $D \in \mathcal{D}$ is finite, its downset is the disjoint union of its points; and if D is cofinite, its downset is X.

This shows that $\mathcal{D}$ is a finite subalgebra of $\mathcal{MC}(X)$. By construction, $\mathcal{S} \subseteq \mathcal{D}$. Therefore, the generated subalgebra $\langle \mathcal{S} \rangle \subseteq \mathcal{D}$ has also to be finite. In conclusion, $\mathcal{MC}(X)$ is locally finite.

On the other hand, the dual space $\mathcal{MC}(X)_*$ contains an infinite chain, namely the principal primefilters (n) for each $n \in \mathbb{N}$. It is known due to Maksimova ([Mak75]) and Segerberg ([Seg71]) that a variety of closure algebras is locally finite if and only if it has bounded height. In conclusion, the variety $\mathrm{Var}(\mathcal{MC}(X))$ cannot be locally finite.

Still, it does follow from $\mathcal{MC}(X)$ being locally finite that the variety $\mathrm{Var}(\mathcal{MC}(X))$ has the finite model property, because every modal formula refuted on $\mathrm{Var}(\mathcal{MC}(X))$ will be refuted on a finite subalgebra of $\mathcal{MC}(X)$.

In the remainder of this section, we will show that the variety generated by the algebra $\mathcal{MC}(X)$ of constructible sets is generated by the decomposition spaces of its stratifications – given that X has the stratification property. This result reminds one of theorem 2.26 for polyhedra, which stated that the variety generated by the algebra of polyhedral subsets of a given polyhedron is generated by the face posets of its triangulations.

Proposition 3.31. *Let $f : X \to Y$ be a continuous and open map. Then, $f^{-1} : \mathcal{MC}(Y) \to \mathcal{MC}(X)$ is a homomorphism of modal algebras.*

Proof. The only critical question is whether f^{-1} is well-defined. Let $A \in \mathcal{C}(Y)$ be a constructible set in Y. Then, Y can be expressed from finitely many open sets using the operations $\cup$, $\cap$ and complement. The preimage map f^{-1} commutes with all of them; and furthermore preimages of opens are open because f is continuous. Therefore, $f^{-1}(Y)$ is again constructible. □

Proposition 3.32. *Let $(X, \leq)$ be a finite poset with the Alexandrov topology. Then $\mathcal{MC}(X) = \mathcal{MP}(X)$.*

Proof. Since $\leq$ is a partial order, every point $x \in X$ can be expressed as the intersection of its upset and downset, i.e. $\{x\} = {\uparrow}x \cap {\downarrow}x$. Therefore, every point is locally closed. Every subset of X is a finite union of its points because X is finite, and therefore constructible. This shows that $\mathcal{P}(X) = \mathcal{C}(X)$. □

Let $\mathcal{S}$ be a stratification with decomposition space $I_{\mathcal{S}}$. If we regard $I_{\mathcal{S}}$ as a finite partially ordered set, we will also call it *decomposition frame* of $\mathcal{S}$.

Lemma 3.33. *Let X be a topological space with the stratification property. Every modal formula that is refuted in $\mathcal{MC}(X)$ is refuted in (the dual algebra of) the decomposition frame of some stratification of X.*

Proof. Suppose $\mathcal{MC}(X) \not\models \varphi(p_1, \ldots, p_n)$. Then, there exist constructible sets $A_1, \ldots, A_n \in \mathcal{C}(X)$ such that $\varphi(A_1, \ldots, A_n) \neq X$. Because X has the stratification property, there exists a stratification $\mathcal{S}$ that is compatible with the sets $A_1, \ldots, A_n$. We know that this stratification gives rise to an open and continuous map $\pi : X \to I$, where I is a finite poset with the Alexandrov topology. Therefore, from proposition 3.31, we obtain a homomorphism of modal algebras $\pi^{-1} : \mathcal{MC}(I) \to \mathcal{MC}(X)$.

Let $\pi(A_i) = \{\pi(x) \mid x \in A_i\}$ be the image of the constructible sets A_i under π. Because π is in particular a decomposition of the set A_i, the preimage $\pi^{-1}(\pi(A_i))$ will be exactly A_i. Assume (towards contradiction) that $\varphi(\pi(A_1), \ldots, \pi(A_n)) = I$. Because π^{-1} is a homomorphism of modal algebras, we have that $\varphi(A_1, \ldots, A_n) = \varphi(\pi^{-1}(\pi(A_1)), \ldots, \pi^{-1}(\pi(A_n))) = X$. This is a contradiction to the assumption $\mathcal{MC}(X) \not\models \varphi$. In conclusion, $\varphi(\pi(A_1), \ldots, \pi(A_n)) \neq I$ and therefore $\mathcal{MC}(I) \not\models \varphi$.

Then, using the propositions 2.16 and 3.32, we know that φ is also refuted on $\mathcal{MP}(I)$ and $(I, \leq)^*$, the dual algebra of the decomposition frame of $\mathcal{S}$. □

Theorem 3.34. *Let X be a topological space with the stratification property. Then:*

$$\mathrm{Var}(\mathcal{MC}(X)) = \mathrm{Var}(\{I_{\mathcal{S}} \mid \mathcal{S} \text{ stratifies } X\}).$$

Proof. Let $\mathcal{S}$ be some stratification of X. Then, there exists a continuous and open surjection $\pi : X \to I_{\mathcal{S}}$. Proposition 3.31 yields a homomorphism $\pi^{-1} : \mathcal{MC}(I_{\mathcal{S}}) \to \mathcal{MC}(X)$. It suffices to show that this homomorphism is injective. Indeed, if $A \neq B$ for $A, B \in \mathcal{MC}(I_{\mathcal{S}})$, there exists $i \in I_{\mathcal{S}}$ such that w.l.o.g. $i \in A$ but $i \notin B$. Therefore, there exist a (non-empty) stratum $X_i = \pi^{-1}[i]$ and a point $x \in X_i$ such that $x \in \pi^{-1}[A]$ but $x \notin \pi^{-1}[B]$. Therefore, $\pi^{-1}[A] \neq \pi^{-1}[B]$ and π^{-1} is injective. Therefore, $\mathcal{MC}(I_{\mathcal{S}}) \leq \mathcal{MC}(X)$ is a subalgebra. Because varieties are closed under subalgebras, we have that $\mathcal{MC}(I_{\mathcal{S}}) \in \mathrm{Var}(\mathcal{MC}(X))$. By proposition 2.16 and 3.32, this implies that $I_{\mathcal{S}}^* \in \mathrm{Var}(\mathcal{MC}(X))$.

Because the stratification $\mathcal{S}$ was arbitrary, we showed that $\mathrm{Var}(\mathcal{MC}(X)) \supseteq \{I_{\mathcal{S}} \mid \mathcal{S} \text{ stratifies } X\}$. To show the desired equality of varieties, by proposition 2.7, it suffices to show that every modal formula φ that is refuted in $\mathcal{MC}(X)$ is refuted on one of the frames $I_{\mathcal{S}}$. But this was lemma 3.33. □

3.2 Closure algebras of tame topologies

In the previous section, we considered the closure algebra $\mathcal{MC}(X)$ generated by the topology of a space X and the stratification property. When we apply this to actual tame topologies in the chapters 4 and 5, however, two adjustments have to be made:

1. Usually, the building blocks of the tame structure, like e.g. polyhedra in case of polyhedral logic, do not generate the topology. Rather, they will generate a proper subalgebra A of $\mathcal{MC}(X)$.

2. Rather than just asking whether this subalgebra A has the stratification property or not, it will be useful to restrict oneself to a subclass of stratifications. For example, in section 5.2.1, we will meet 'Nash stratifications'. Therefore, we will in the following define the notion that A has the stratification property *with respect to* some class of stratifications.

Recall how Lou van den Dries described tame topology on page 5. One starts with 'nice' subsets of a topological space X and closes them under certain operations like binary unions, intersections, complements and topological closure (and possibly more). Assuming that

these elementary building blocks are locally closed, the resulting algebra will be a subalgebra of $\mathcal{MC}(X)$.

Definition 3.35. Let $A \leq \mathcal{MC}(X)$ be a subalgebra. An *A-stratification* is a stratification of X such that all strata are elements of A.

We say that A has the *stratification property* if for every finite collection of sets $X_1, \ldots, X_n \in A$, there exists an A-stratification $\mathcal{S}$ of X which is compatible with all X_i.

Theorem 3.36. *The following are equivalent for a subalgebra $A \leq \mathcal{MC}(X)$:*

1. *A has the stratification property.*
2. *A is locally finite.*

Proof. The proof of theorem 3.29 can be adjusted as follows:

$\boxed{\Rightarrow}$ Recall the workings of this direction. Starting with $A_1, \ldots, A_n \in A$, we took a stratification $\mathcal{S}$ compatible with all A_i and considered the closure algebra $\mathcal{D}$ of $\mathcal{S}$-objects.

Here, the stratification $\mathcal{S}$ is by assumption an A-stratification such that $\mathcal{D}$ is a finite subalgebra of A. Then, like before, we have that $\langle A_1, \ldots, A_n \rangle \subseteq \mathcal{D}$ is a finite subalgebra of A. In conclusion, A is locally finite.

$\boxed{\Leftarrow}$ Let $A, \ldots, A_n \in A$ be finitely many elements of A. In the proof of 3.29, we generated the subalgebra $S = \langle A_1, \ldots, A_n \rangle$. Now, this will be a finite subalgebra of A, and in particular a Grz algebra. By proposition 3.28, the atoms in $\mathrm{At}(S)$ are locally closed. Assuring that $\mathrm{At}(S)$ forms a stratification compatible with all A_i works exactly like in the proof of 3.29. Furthermore, it is an A-stratification because $\mathrm{At}(S) \subseteq S \subseteq A$. □

Next, we are going to address the second necessary adjustment, allowing us to restrict the class of stratifications in the stratification property.

Definition 3.37. Let $A \leq \mathcal{MC}(X)$ be a subalgebra and C be a family of A-stratifications. We say that A has the *stratification property with respect to C* if for every finite collection of sets $X_1, \ldots, X_n \in A$, there exists a stratification $\mathcal{S} \in C$ of X which is compatible with all X_i.

The following is the analogue of theorem 3.34 for the case that the class of stratifications is restricted:

Theorem 3.38. *Let X be a topological space and $A \leq \mathcal{MC}(X)$ a subalgebra with the stratification property with respect to a class C of stratifications. Then:*

$$\mathrm{Var}(A) = \mathrm{Var}(\{I_{\mathcal{S}} \mid \mathcal{S} \in C\}).$$

Proof. Let $\mathcal{S} \in C$ be some stratification. Then, there exists a continuous and open surjection $\pi : X \to I_{\mathcal{S}}$. Proposition 3.31 yields a homomorphism $\pi^{-1} : \mathcal{MC}(I_{\mathcal{S}}) \to \mathcal{MC}(X)$. We claim that this homomorphism maps into the subalgebra $A \leq \mathcal{MC}(X)$. To see this, let $K \in \mathcal{MC}(I_{\mathcal{S}})$. Then, K is a finite union of elements of $I_{\mathcal{S}}$. Then, the image

$$\pi^{-1}[K] = \bigcup_{i \in K} \pi^{-1}(i)$$

is a finite union of strata of $\mathcal{S}$. By the definition of the stratification property with respect to C, we have that $\mathcal{S}$ is an A-stratification, i.e. the strata are elements of A. Therefore, also $\pi^{-1}[K] \in A$.

The argument that π^{-1} is injective works like in the proof of 3.34. Therefore, $\mathcal{MC}(I_{\mathcal{S}}) \leq A$ is a subalgebra. Because varieties are closed under subalgebras, we have that $\mathcal{MC}(I_{\mathcal{S}}) \in \mathrm{Var}(A)$. By propositions 2.16 and 3.32, this implies that $I^*_{\mathcal{S}} \in \mathrm{Var}(A)$.

Like in the proof of theorem 3.34, it remains to show that every modal formula that is refuted on A is refuted on one of the frames $I_{\mathcal{S}}$ for some A-stratification $\mathcal{S} \in C$. This proof works exactly like the proof of lemma 3.33 with the only difference that $\mathcal{S}$ can be chosen from C. □

Until now, the perspective was that one starts with an algebra $A \leq \mathcal{MC}(X)$ and asks whether there exists a class of stratifications, which respect to which A has the stratification property. In the rest of this chapter, we present a reversed approach: We start with a class of stratifications and try to construct an algebra $A \leq \mathcal{MC}(X)$ that has the stratification property with respect to that class.

In the proof of theorem 3.29, we already used the term 'S-object'. In the following, we will explicitly define this notion and generalize it to classes of stratifications. The terminology is inspired by the notion of 'Whitney objects' which are exactly the C-objects for C the class of Whitney stratifications.

Definition 3.39. Let C be a class of stratifications of a topological space X. A C-object is a subset of X of the form

$$A = \bigsqcup \mathcal{S}'$$

for some $\mathcal{S}' \subseteq \mathcal{S} \in C$. The set of C-objects is denoted as $\mathrm{Ob}(C)$.

Lemma 3.40. *Let C be a non-empty class of stratifications of a space X. Suppose that all elements of C are pairwise compatible and that C is closed under sums. Then:*

1. $\mathrm{Ob}(C) \leq \mathcal{MC}(X)$ *is a subalgebra.*
2. $\mathrm{Ob}(C)$ *has the stratification property with respect to C.*

Proof.
Ad (1) We saw in the proof of 3.29 that the $\mathcal{S}$-objects for a given stratification $\mathcal{S}$ form a subalgebra of $\mathcal{MC}(X)$. Therefore, it is clear that $\mathrm{Ob}(C)$ is closed under the unary operations complement and topological closure. Also, it is clear that $\emptyset, X \in \mathrm{Ob}(C)$. For the binary operations $\cup$ and $\cap$, let $A, B \in \mathrm{Ob}(C)$. Then, there exist stratifications $\mathcal{S}$ and $\mathcal{T}$ such that $A = \bigsqcup \mathcal{S}'$ and $B = \bigsqcup \mathcal{T}'$ for subsets $\mathcal{S}' \subseteq \mathcal{S}$ and $\mathcal{T}' \subseteq \mathcal{T}$. By assumption, the sum $\mathcal{S} + \mathcal{T}$ exists and is an element of C.

Then, we have by distributivity that

$$A \cap B = \bigsqcup \mathcal{S}' \cap \bigsqcup \mathcal{T}' = \bigsqcup \{S \cap T \mid S \in \mathcal{S}' \ \&\ T \in \mathcal{T}'\}.$$

But the sets of the form $S \cap T$ for $S \in \mathcal{S}'$ and $T \in \mathcal{T}'$ are strata of the stratification $\mathcal{S} + \mathcal{T}$. Therefore, $A \cap B$ is an $\mathcal{S} + \mathcal{T}$-object.

Also, we have that

$$A \cup B = \bigsqcup \mathcal{S}' \cup \bigsqcup \mathcal{T}' = \bigsqcup \{S \cap T \mid S \in \mathcal{S}' \text{ or } T \in \mathcal{T}'\},$$

which shows that $A \cup B$ is an $\mathcal{S} + \mathcal{T}$-object.

Every C-object is a finite union of locally closed strata and therefore constructible. In conclusion, $\mathrm{Ob}(C)$ is a subalgebra of $\mathcal{MC}(X)$.

Ad (2) Let $A_1, \ldots, A_n \in \mathrm{Ob}(C)$ be finitely many C-objects. We need to find a stratification in C that is compatible with each A_i. By definition, there exist stratifications $\mathcal{S}_1, \ldots, \mathcal{S}_n$ such that all C-objects A_i are of the form $A_i = \bigsqcup \mathcal{S}_i'$ for $\mathcal{S}_i' \subseteq \mathcal{S}_i$. Then, consider

$$\mathcal{T} = \mathcal{S}_1 + \ldots + \mathcal{S}_n.$$

Because C is closed under sums, we have that $\mathcal{T} \in C$. We claim that $\mathcal{T}$ is further compatible with each A_i. To see this, we will consider the case $i = 1$ but the general case works the same. We have that:

$$\begin{aligned} A_1 = \bigsqcup \mathcal{S}_1' &= \bigsqcup \mathcal{S}_1' \cap \bigsqcup \mathcal{S}_1 \cap \ldots \cap \bigsqcup \mathcal{S}_n \\ &= \bigsqcup \{S_1 \cap \ldots \cap S_n \mid S_1 \in \mathcal{S}_1' \text{ and } S_k \in \mathcal{S}_k \text{ for } k \geq 2\}. \end{aligned}$$

The second equality holds because all $\mathcal{S}_i$ partition X; and the third equality follows from distributivity. This shows that all A_i are $\mathcal{T}$-objects, as desired. □

We close this chapter with the remark that nothing here is really new. The study of surjective interior maps from topological spaces to finite preordered sets in order to determine the modal logic of a topological space is a standard technique in modal logic. The significance of this chapter was rather to establish two equivalent perspectives on such interior maps: As stratifications, i.e. special partitions of spaces, and as poset-stratifications. We also saw that the geometric notion of the stratification property – that one can find a stratification refining a collection of finitely many sets – corresponds to the algebraic notion of local finiteness.

When reasoning about geometric objects, it is often more useful to understand interior surjective maps to finite frames as stratifications. For example, in section 5.2, we will meet a special class of stratifications whose strata are a special subclass of algebraically definable manifolds. This property is vital for reasoning about them in a similar way like reasoning about triangulations in polyhedral logic. In this sense, the endeavour of the following two chapters will be to demonstrate the usefulness of stratifications when investigating modal logics of tame topologies.

4 Euclidean Hierarchy Revisited

Abstract

We show that the varieties generated by a tensor sum of finite modal algebras is already generated by the tensor sums of their subdirectly irreducible algebras.
We use this result to reproduce the characterization of the logics of chequered sets from [BBG03].

Gris, *Still Life with Checked Tablecloth* (1915)

4.1 Tensor sum and varieties

The *tensor sum* of Heyting algebras was considered by Leo Esakia in the appendix to [Esa+19]. The definition, however, transfers directly to modal algebras. Tensor sums of finite modal algebras are still finite but may return algebras of much larger size: If $|A| = n$ and $|B| = m$, then the cardinality of their tensor sum $A \otimes B$ equals $n^{\log_2(m)}$. Before restricting to the finite case, we are going to prove that the general definition is well-formed.

Definition 4.1 (Tensor sum)**.** Let A and B be modal algebras. Their *tensor sum* is the modal algebra

$$A \otimes B := (A_* \times B_*)^*.$$

This is a definition using duality. Given modal spaces (X, R) and (Y, S), their Cartesian product is defined as the ordered topological space $X \times Y$ with the product topology and the product relation

$$(x, y)(R \times S)(x', y') \iff xRx' \;\&\; ySy'.$$

In the following, we might overload the relation symbols to simplify notation if it is clear from the compared elements to which ordered space they refer.

For definition 4.1 to be well-formed, one needs to assert that the Cartesian product of modal spaces is in fact a modal space.

Proposition 4.2. *Let X, Y be modal spaces. Then, their Cartesian product $X \times Y$ is again a modal space.*

Proof. We take it as given that the Cartesian product of Stone spaces is again a Stone space. Therefore, it is left to show that (1) $R[(x,y)]$ is closed for points $(x,y) \in X \times Y$ and (2) that $R^{-1}[U]$ is clopen whenever $U \subseteq X \times Y$ is clopen.

First of all, by definition of the product topology, products of open sets are open. The same applies to closed (and therefore also to clopen) sets. Indeed, if $C \subseteq X$ and $D \subseteq Y$ are closed, then $(C \times D)^c = (C^c \times Y) \cup (X \times D^c)$, which is open.

Ad (1)

$R[(x,y)]$ for some $(x,y) \in X \times Y$ consists by definition of the product order exactly of those points (x',y') such that xRx' in X and yRy' in Y. Therefore, $R[(x,y)] = R[x] \times R[y]$, which is a product of closed sets and therefore closed.

Ad (2)

Let $U \subseteq X \times Y$ be clopen. Since $X \times Y$ is a Stone space, it has a basis of clopen rectangles. Therefore, $U = \bigcup_{i \in I} U_i \times V_i$ with U_i and V_i clopen.

A set $R^{-1}[U_i \times V_i]$ consists by definition of the product order exactly of those points $(x,y) \in X \times Y$ such that $x \in R^{-1}[U_i]$ and $y \in R^{-1}[V_i]$. Therefore, we have that

$$R^{-1}[U] = \bigcup_{i \in I} R^{-1}[U_i \times V_i] = \bigcup_{i \in I} R^{-1}[U_i] \times R^{-1}[V_i].$$

Because X and Y are modal spaces, the sets $R^{-1}[U_i]$ and $R^{-1}[V_i]$ are clopens and therefore also their product. Note that U is a closed set in a compact space and therefore itself compact. Because the rectangles $U_i \times V_i$ are an open cover of U, by compactness I can be chosen to be finite. This shows that $R^{-1}[U]$ is a finite union of clopens and therefore clopen. □

In fact, it can be shown that for any collection $\{X_i\}_{i \in I}$ of modal spaces, the Cartesian product $\prod_{i \in I} X_i$ is again a modal space. The decisive step in this proof is that arbitrary products of compact spaces are compact, a theorem which is known as Tychonoff's theorem. For our application here, however, considering the binary case is sufficient.

Esakia notes that some varieties of Heyting algebras are closed under tensor sums, while others are not. Similarly, some varieties of modal algebras are closed under tensor sums, while others are not. It is easy to see that products of preorders are again preorders and that products of partial orders are again partial orders. Therefore, we know that the following classes are closed under tensor sums:

Proposition 4.3.

1. *If A, B are closure algebras, then so is $A \otimes B$,*
2. *If A, B are skeletal algebras, then so is $A \otimes B$.*

There is another important class of modal algebras that is closed under tensor sums, namely subdirectly irreducible algebras:

Proposition 4.4. *Let A, B be subdirectly irreducible modal algebras. Then, $A \otimes B$ is subdirectly irreducible.*

Proof. Using duality, it suffices to show the following statement: If X and Y are topo-rooted modal spaces, then $X \times Y$ is topo-rooted.

Let X_0 be the set of roots of X and Y_0 the sets of roots of Y. First, the set $X_0 \times Y_0$ is open in the product topology. It remains to show that $X_0 \times Y_0$ is exactly the set of roots of $X \times Y$.

To see this, let x_0 be a root of X, and y_0 a root of Y. This means that $X = R^*[x_0]$ and $Y = R^*[y_0]$. It follows once more from the definition of the product order that then $X \times Y = R^*[(x_0, y_0)]$. Conversely, let $(x_0, y_0) \in X \times Y$ be a root, then $X \times Y = R^*[(x_0, y_0)]$. Let $x \in X$ be arbitrary (the argument is the same for Y). Then, $(x, y_0) \in R^*[(x_0, y_0)]$. In particular, $x \in R^*[x_0]$. Therefore, x_0 is a root of X and analogously y_0 a root of Y. □

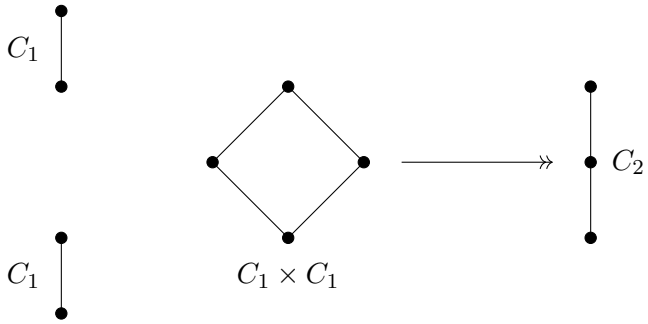

Figure 4.1: Consider the variety $\mathrm{Var}(C_1 \times C_1)$. Then, C_2 is a frame of this variety because it is a p-morphic image of $C_1 \times C_2$. Furthermore, C_2 is rooted (and therefore dual to a subdirectly irreducible algebra) but not a Cartesian product of two rooted frames of $\mathrm{Var}(C_1)$.

For classes $\mathcal{C}, \mathcal{D}$ of modal algebras, we define their tensor sum by

$$\mathcal{C} \otimes \mathcal{D} = \{A \otimes B \mid A \in \mathcal{C} \text{ and } B \in \mathcal{D}\}.$$

It is a natural question to ask how to characterize a variety $\mathrm{Var}(A \otimes B)$ generated by a tensor sum of modal algebras. It might be the case that $\mathrm{Var}(A \otimes B)$ contains subdirectly irreducible algebras that cannot be represented as tensor sums of subdirectly irreducible algebras of $\mathrm{Var}(A)$ and $\mathrm{Var}(B)$, respectively. For an example, see figure 4.1. However, we are going to prove in the remainder of this section that $\mathrm{Var}(A \otimes B)$ is generated by tensor sums of subdirectly irreducible algebras, given that A and B are *finite* modal algebras.

First, we observe that modal homomorphisms and properties of them are preserved by the tensor sum.

Proposition 4.5. *Let $f : (X, R) \to (Y, S)$ and $f' : (X', R') \to (Y', R')$ be continuous p-morphisms between modal spaces. Then*

1. *$f \times f' : (X \times X', R \times R') \to (Y \times Y', S \times S')$ is a continuous p-morphism,*
2. *if f and f' are surjective, then so is $f \times f'$,*
3. *if f and f' are injective, then so is $f \times f'$.*

Proof.

1. The Cartesian product of two continuous maps is again continuous. Also, monotonicity follows directly from the definition of the product relation.

   For boundedness, suppose $(f \times f')(x, x')(S \times S')(y, y')$ for $x \in X, y \in Y, x' \in X', y' \in Y'$. Then componentwise, $f(x)Sy$ and $f'(x')S'y'$. Because f and f' are p-morphisms, there exist $z \in X$ and $z' \in X'$ such that $f(z) = y$ and $f'(z') = y'$ as well as xRz and $x'R'z'$. This shows that there exists $(z, z') \in X \times X'$ such that $(f \times f')(z, z') = (y, y')$ and $(x, x')(R \times R')(z, z')$.

2. A preimage for any $(y, y') \in Y \times Y'$ under $f \times f'$ is given by componentwise preimages under f and f'.

3. Let $(x, x'), (y, y') \in X \times X'$ and $(x, x') \neq (y, y')$. W.l.o.g. suppose that $x \neq y$. Then, $f(x) \neq f(y)$ since f is injective. In conclusion, $f \times f'(x, x') = (f(x), f'(x')) \neq (f(y), f'(y')) = f \times f'(y, y')$. Therefore, $f \times f'$ is injective. □

The property that the product of p-morphisms yields a p-morphism, shows that the following construction of the tensor sum of modal homomorphisms is well-defined. Given modal homomorphisms $f : A \to B$ and $f' : A' \to B'$, we define modal homomorphism $f \otimes f' : A \otimes A' \to B \otimes B'$ dually by

$$f \otimes f' = (f_* \times f'_*)^*.$$

Lemma 4.6. *Let A, B, A' and B' be modal algebras.*

1. *if $A' \in \mathbb{H}(A)$ and $B' \in \mathbb{H}(B)$, then $A' \otimes B' \in \mathbb{H}(A \otimes B)$,*

2. *if $A' \in \mathbb{S}(A)$ and $B' \in \mathbb{S}(B)$, then $A' \otimes B' \in \mathbb{IS}(A \otimes B)$.*

Proof.

1. This follows via duality from point 2 of proposition 4.5.

2. If $A' \leq A$ and $B' \leq B$ are subalgebras, the inclusions $i : A' \hookrightarrow A$ and $j : B' \hookrightarrow B$ are injective homomorphisms. Dually, $i_* : A_* \twoheadrightarrow A'_*$ and $j_* : B_* \twoheadrightarrow B'_*$ are surjective continuous p-morphisms. By point 3 of proposition 4.5, $i_* \times j_* : A_* \times B_* \twoheadrightarrow A'_* \times B'_*$ is surjective. Changing back to the algebraic perspective, we get that $i \otimes j : A' \otimes B' \hookrightarrow A \otimes B$ is an injective homomorphism. Therefore, $A' \otimes B'$ is isomorphic to the image of $i \otimes j$, which is a subalgebra of $A \otimes B$. □

From now on, we will concentrate on finite algebras and spaces.

Theorem 4.7. *Let A and B be finite modal algebras and $\mathcal{V} = \mathrm{Var}(A)$ and $\mathcal{W} = \mathrm{Var}(B)$. Then, the following holds:*

$$\mathrm{Var}(A \otimes B) = \mathrm{Var}(\mathcal{V}_{si} \otimes \mathcal{W}_{si}).$$

Proof. $\boxed{\subseteq}$

It suffices to show that $A \otimes B \in \mathrm{Var}(\mathcal{V}_{si} \otimes \mathcal{W}_{si})$.

Because $A \in \mathcal{V}$, we know that A is a subdirect product of finitely many subdirectly irreducible algebras in $\mathcal{V}_{si}$ (this is corollary 8.7 in [BS84]). In particular, A is a subalgebra of a direct product of finitely many algebras $S_1, \ldots, S_n$ in $\mathcal{V}_{si}$. Similarly, B is a subalgebra of a direct product of finitely many algebras $T_1, \ldots, T_m$ in $\mathcal{W}_{si}$.

The dual statement is that A_* is a p-morphic image of a disjoint union of the dual spaces of the algebras S_i; and analogously for B:

$$\pi_A : \bigsqcup_{1 \leq i \leq n} S_{i*} \twoheadrightarrow A_* \qquad \pi_B : \bigsqcup_{1 \leq k \leq m} T_{k*} \twoheadrightarrow B_*$$

From proposition 4.5, one obtains that

$$\pi_A \times \pi_B : \left(\bigsqcup_{1 \leq i \leq n} S_{i*} \right) \times \left(\bigsqcup_{1 \leq k \leq m} T_{k*} \right) \twoheadrightarrow A_* \times B_*$$

is a surjective p-morphism. It also holds that

$$\left(\bigsqcup_{1\leq i\leq n} S_{i*}\right) \times \left(\bigsqcup_{1\leq k\leq m} T_{k*}\right) \cong \bigsqcup_{\substack{1\leq i\leq n\\ 1\leq k\leq m}} (S_{i*}\times T_{k*})\,.$$

This shows that $A_*\times B_*$ is a p-morphic image of a disjoint union of spaces of the form $S_{i*}\times T_{k*}$.

Changing back to the algebraic perspective, this corresponds to the dual statement that $A\otimes B$ is a subalgebra of a direct product of algebras of the form $S_i\otimes T_k$. But these tensor sums are elements of $\mathcal{V}_{si}\otimes\mathcal{W}_{si}$, which shows that $A\otimes B\in\mathbb{HSP}(\mathcal{V}_{si}\otimes\mathcal{W}_{si})$ – as desired.

$\supseteq$

Let $S\otimes T\in\mathcal{V}_{si}\otimes\mathcal{W}_{si}$. If suffices to show that $S\otimes T\in\mathrm{Var}(A\otimes B)$ because $\mathrm{Var}(\mathcal{V}_{si}\otimes\mathcal{W}_{si})$ is the least variety containing all algebras of the form $S\otimes T$ with $S\in\mathcal{V}_{si}$ and $T\in\mathcal{W}_{si}$.

The corollary 2.11 to Jónsson's lemma states that $S\in\mathbb{HS}(A)$ and $T\in\mathbb{HS}(B)$. Therefore, there exist subalgebras $A'\leq A$ and $B'\leq B$ such that $S\in\mathbb{H}(A')$ and $T\in\mathbb{H}(B')$.

By lemma 4.6, we have that $S\otimes T\in\mathbb{H}(A'\otimes B')$ and $A'\otimes B'\in\mathbb{IS}(A\otimes B)$. Therefore, $S\otimes T\in\mathbb{HISP}(A\otimes B)$, which concludes the proof. □

Corollary 4.8. *Let A, B, A' and B' be finite modal algebras such that* $\mathrm{Var}(A)=\mathrm{Var}(A')$ *and* $\mathrm{Var}(B)=\mathrm{Var}(B')$. *Then*

$$\mathrm{Var}(A\otimes B)=\mathrm{Var}(A'\otimes B').$$

Proof. By theorem 2.9, every variety is generated by its subdirectly irreducible algebras. Therefore, $\mathrm{Var}(A)_{si}=\mathrm{Var}(A')_{si}$ and $\mathrm{Var}(B)_{si}=\mathrm{Var}(B')_{si}$. By theorem 4.7, we have that

$$\begin{aligned}\mathrm{Var}(A\otimes B)&=\mathrm{Var}(\mathrm{Var}(A)_{si}\otimes\mathrm{Var}(B)_{si})\\&=\mathrm{Var}(\mathrm{Var}(A')_{si}\otimes\mathrm{Var}(B')_{si})\\&=\mathrm{Var}(A'\otimes B').\end{aligned}$$

□

4.2 Logic of chequered sets

We will define chequered sets as objects (in the sense of definition 3.39) of a certain class of stratifications. One-dimensional chequered sets are finite unions of points and open intervals. Intuitively, higher-dimensional chequered sets are then the finite unions of rectangles of one-dimensional ones. In figure 4.2, an example of a two-dimensional chequered set is visualized. Note how the blue area is composed of finitely many rectangles of points and open intervals.

In our opinion, the relevance of chequered sets among other tame topologies mainly comes from two properties. First, they can be regarded as a minimal notion of a tame topology. For example, also the semialgebraic and o-minimal geometries introduced in 5.2.1 satisfy that the one-dimensional sets are finite unions of points and open intervals, and that they are closed under Cartesian products. The higher-dimensional closure algebras of chequered sets are exactly the closure algebras generated by those Cartesian products.

Second, the definition of chequered sets is very similar to the definition of the closure algebra of $\mathbb{R}^n$ with the standard topology. There, the algebra $\mathcal{MP}(\mathbb{R}^n)$ is generated by arbitrary unions of open rectangles. From this perspective, the algebra of chequered sets is the resulting notion, if one restricts to finite unions.

Definition 4.9 (Serial stratification)**.** A *serial stratification* of $\mathbb{R}$ is a finite partition of $\mathbb{R}$ into non-empty convex open sets and points.

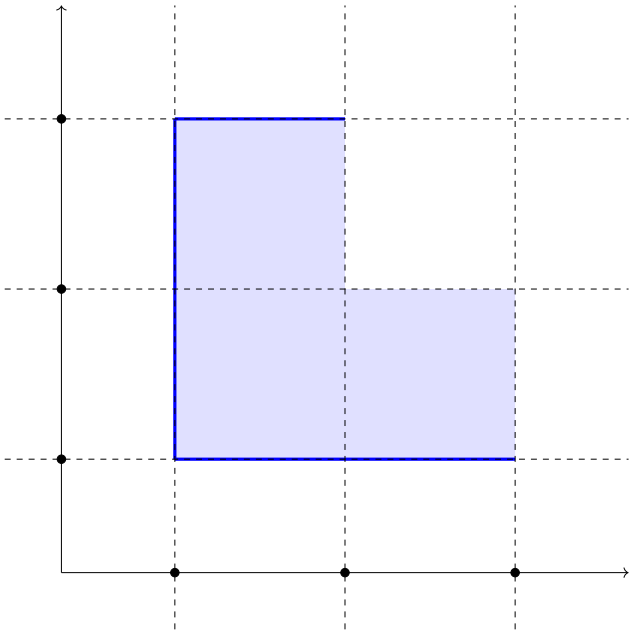

Figure 4.2: A chequered set and a product-stratification of the same

The convex open sets in $\mathbb{R}$ are exactly the generalized open intervals of the form (a, b) for $a, b \in \mathbb{R} \cup \{-\infty, \infty\}$ with $a < b$ (and the empty set).

It is easy to see that serial stratifications satisfy the frontier condition and are therefore actual stratifications. Using proposition 3.5, it suffices to see that the border of every stratum is a finite union of strata. For points, this is trivially true because they have empty border. For generalized open intervals, their borders consist of at most two points, which are strata.

Moreover, all serial stratifications are mutually compatible in the sense of definition 3.16:

Proposition 4.10. *Let $\mathcal{S}$ and $\mathcal{T}$ be two serial stratifications of $\mathbb{R}$. Then, they are compatible.*

Proof. Let X be a stratum of $\mathcal{S}$ and Y one of $\mathcal{T}$ such that $X \cap Y \neq \emptyset$. It is to show that $\mathrm{Cl}(X) \cap \mathrm{Cl}(Y) \subseteq \mathrm{Cl}(X \cap Y)$.

We start by excluding a few trivial cases. First, if one of the strata X and Y is a point, the statement is true. Second, if either $X \subseteq Y$ or $Y \subseteq X$ is the case, the statement is also trivially true.

The only remaining case is that X and Y are overlapping shifted open intervals. That means $X = (a, b)$ and $Y = (c, d)$. W.l.o.g. assume that X is the left-hand and Y the right-hand interval, i.e. $a < c < b < d$. In particular, $-\infty < c < b < \infty$, so that b and c are real numbers. Let $x \in \mathrm{Cl}(X) \cap \mathrm{Cl}(Y)$. Because b and c are real, this implies that $x \geq c$ and $x \leq b$. But then $x \in [c, b]$ and the closed interval $[c, b] = \mathrm{Cl}(c, b) = \mathrm{Cl}(X \cap Y)$. □

Definition 4.11 (Chequered stratifications, chequered sets)**.** Let $n \geq 1$. The classes of n-dimensional *chequered stratifications* are defined as

$$C_n = \left\{ \prod_{i=1}^{n} \mathcal{S}_i \mid \mathcal{S}_i \text{ are serial stratifications of } \mathbb{R} \right\}.$$

A *chequered set* in $\mathbb{R}^n$ is a C_n-object. The closure algebras of n-dimensional chequered sets are denoted as $\mathcal{CH}(\mathbb{R}^n)$.

It follows from results in chapter 3 that $\mathcal{CH}(\mathbb{R}^n)$ actually form closure algebras:

Proposition 4.12. *$\mathcal{CH}(\mathbb{R}^n)$ is a subalgebra of $\mathcal{MC}(\mathbb{R}^n)$ and has the stratification property with respect to C_n.*

Proof. Proposition 4.10 stated that all stratifications in C_1 are compatible. Furthermore, C_1 is closed under sums of stratifications because non-empty intersections of open intervals are again open intervals, and non-empty intersections with a point are always a point. Then, by proposition 3.18, all product stratifications in C_n are compatible. Furthermore, the classes C_n are closed under sums because proposition 3.19 showed that every sum of two product stratifications is a product of sums.

In conclusion, we have that all C_n are closed under sums. Then, it follows from lemma 3.40 that the C_n-objects $\mathcal{CH}(\mathbb{R}^n)$ form subalgebras of $\mathcal{MC}(\mathbb{R}^n)$ with the stratification property with respect to C_n. □

Next, we consider a special class of posets W_n for $n \geq 0$. The first three representants are shown in figure 4.3. Definition 4.13 determines them concretely but we are going to identify W_n with any isomorphic poset.

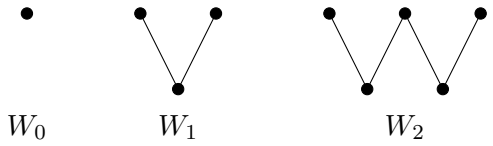

Figure 4.3: The frames W_0, W_1 and W_2

Definition 4.13. The partially ordered set $(W_n, \leq)$ is defined as the set $\{0, \ldots, n\} \cup \{\{m-1, m\} \mid 1 \leq m \leq n\}$ equipped with the order $a \leq b \;\Leftrightarrow\; b \in a$.

The frames W_n are the stratification frames of serial stratifications of $\mathbb{R}$. To see this, observe that a serial stratification of $\mathbb{R}$ is an alternating sequence of open intervals and points; also the least (left-most) and greatest (right-most) stratum have to be an open interval, which yields exactly the frames W_n. Therefore, the logic of one-dimensional chequered sets will be the logic of these frames.

The following proposition shows that the modal logic of any of the frames W_n is already the logic of the two-fork.

Proposition 4.14. *Let $n \geq 1$. Then,* $\mathrm{Var}(W_n) = \mathrm{Var}(W_1)$.

Proof.

$\boxed{\supseteq}$ Since $\mathrm{Var}(W_1)$ is the smallest variety containing the frame W_1 (or strictly speaking the algebra W_1^*), it suffices to show that $W_1 \in \mathrm{Var}(W_n)$. Because $n \geq 1$, the frame W_n has height one. Take any minimal element x and consider its upset $\uparrow x$. Since x has exactly two successors, the frame $\uparrow x$ is isomorphic to W_1. This shows that W_1 is (isomorphic to) a generated subframe of W_n. Using duality, this means that the algebra W_1^* is a homomorphic image of W_n^*. Therefore, $W_1^* \in \mathbb{H}(W_n^*) \subset \mathrm{Var}(W_n)$.

$\boxed{\subseteq}$ Again, it suffices to show that $W_n \in \mathrm{Var}(W_1)$. Varieties are closed under products, and finite products correspond dually to disjoint unions. Therefore the disjoint union $W_1 \sqcup \ldots \sqcup W_1$ of n copies of the frame W_1 is in $\mathrm{Var}(W_1)$.

We label the copies and their corresponding elements using superscript indices like $W_1^{(i)}$. Identifying maximal points of frames always constitutes a surjective p-morphism. Therefore, we obtain a surjective p-morphism from $W_1 \sqcup \ldots \sqcup W_1 \twoheadrightarrow W_n$ like follows:

$$W_1^{(1)} \sqcup \ldots \sqcup W_1^{(n)} \twoheadrightarrow W_n, \begin{cases} \{0^{(i)}, 1^{(i)}\} & \mapsto & \{i-1, i\} \\ 0^{(i)} & \mapsto & i-1 \\ 1^{(i)} & \mapsto & i \end{cases}$$

Intuitively, this map chains n copies of the frames W_1 together to form the frame W_n. An example for $n = 3$ is given in figure 4.4.

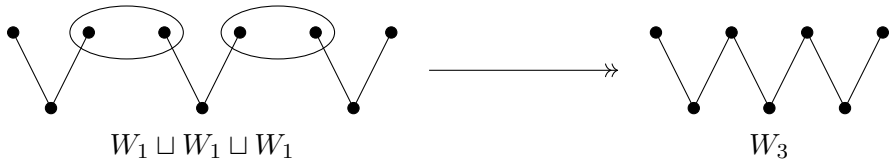

Figure 4.4: The surjective p-morphism $W_1 \sqcup W_1 \sqcup W_1 \twoheadrightarrow W_3$

This construction showed that W_n is a p-morphic image of disjoint unions of copies of W_1. Dually, W_n^* is (isomorphic to) a subalgebra of a power of W_1^*. Therefore, $W_n^* \in \mathbb{ISP}(W_1^*) \subset \mathrm{Var}(W_1)$. □

By now, we have all the tools at hand to determine the logic of the closure algebras $\mathcal{CH}(\mathbb{R}^n)$ of n-dimensional chequered sets.

Theorem 4.15. $\mathrm{Var}(\mathcal{CH}(\mathbb{R}^n)) = \mathrm{Var}(W_1^n)$.

Proof. By proposition 4.12, the algebra $\mathcal{CH}(\mathbb{R}^n)$ has the stratification property with respect to the class of chequered stratifications C_n. Then, by theorem 3.38, the variety of $\mathcal{CH}(\mathbb{R}^n)$ is generated by the decomposition frames of its stratifications, i.e.

$$\mathrm{Var}(\mathcal{CH}(\mathbb{R}^n)) = \mathrm{Var}\{I_{\mathcal{S}} \mid \mathcal{S} \in C_n\}.$$

Therefore, it suffices to show that $\mathrm{Var}\{I_{\mathcal{S}} \mid \mathcal{S} \in C_n\} = \mathrm{Var}(W_1^n)$.

First, we have that $W_1^n \in \mathrm{Var}\{I_{\mathcal{S}} \mid \mathcal{S} \in C_n\}$ because W_1^n is a decomposition frame of a stratification in C_n. To make this specific, the decomposition $\mathbb{R} = (-\infty, 0) \sqcup \{0\} \sqcup (0, \infty)$ is a serial stratification of $\mathbb{R}$ with decomposition frame W_1. Then, the n-fold product stratification is an element of C_n with decomposition frame W_1^n.

For the desired equality of varieties, we use proposition 2.7 and show that every modal formula that is refuted on $\mathrm{Var}\{I_{\mathcal{S}} \mid \mathcal{S} \in C_n\}$ is refuted on W_1^n. So, let φ be a modal formula such that $\mathrm{Var}\{I_{\mathcal{S}} \mid \mathcal{S} \in C_n\} \not\models \varphi$. Then, by proposition 2.7, there exists some decomposition frame I such that $I \not\models \varphi$. By definition of product stratification, I is an n-fold product of serial stratification frames, i.e. $I \cong W_{m_1} \times \ldots \times W_{m_n}$ with $m_i \geq 0$.

W.l.o.g. assume that all $m_i > 0$. We know from proposition 4.14 that $\mathrm{Var}(W_{m_i}) = \mathrm{Var}(W_1)$ if $m_i > 0$. Applying corollary 4.8 n times yields that $\mathrm{Var}(I) = \mathrm{Var}(W_1^n)$. Therefore, $\mathrm{Var}(W_1^n) \not\models \varphi$.

To justify the last assumption that all $m_i > 0$, observe that if $m_i = 0$, then the map

$$W_{m_1} \times \ldots \times W_1 \times \ldots \times W_{m_n} \twoheadrightarrow W_{m_1} \times \ldots \times W_0 \times \ldots \times W_{m_n},$$
$$(w_1, \ldots, w, \ldots, w_n) \mapsto (w_0, \ldots, \bullet, \ldots, w_n),$$

which maps in the ith component $w \in W_1$ to the unique element $\bullet \in W_0$ and which is the identity in the other components, is a surjective p-morphism. Therefore, also $W_{m_1} \times \ldots \times W_1 \times \ldots \times W_{m_n} \not\models \varphi$. □

Finally, we will reinterpret the previous results from chapter 4 from the logical perspective. The logical analogue of theorem 4.15 is that the modal logic L_n of $\mathcal{CH}(\mathbb{R}^n)$ is the logic of the frame W_1^n, i.e. the logic of the two-fork. Since this frame is a poset, $L_n \supset \mathbf{Grz}$ extends **Grz**. In particular, the logic is tabular and the truth-table (with entries in $(W_1^n)^*$) can be used to decide whether a formula is a theorem of L_n. As pointed out in [BBG03, corollary 4.6], L_n is thus finitely axiomatizable and decidable, since it is well-known that this is the case for tabular extensions of **Grz**.

5 Logic of Real Algebraic and Semialgebraic Geometry

Abstract

We show that the modal logic of Zariski-constructible sets in the n-dimensional affine space is $\mathbf{Grz.2} \oplus \mathbf{bd}_{n+1}$.
We prove that the modal logic of semialgebraic sets of a semialgebraic compactum is the polyhedral logic of a unique polyhedron and identify the modal logic of semialgebraic sets in the n-dimensional Euclidean space with the polyhedral logic of the n-simplex.

Arcimboldo, *The waiter* (1574)

5.1 Logic of affine spaces

We denote the ring of real polynomials in n variables $x_1, \ldots, x_n$ as $\mathbb{R}[x_1, \ldots, x_n]$ or also as $\mathbb{R}[n]$ in case the variables are clear. For a subset $A \subseteq \mathbb{R}[x_1, \ldots, x_n]$, we denote by $\mathcal{Z}(A)$ the common zero set of the polynomials, i.e.

$$\mathcal{Z}(A) = \{\vec{x} \in \mathbb{R}^n \mid P(\vec{x}) = 0 \text{ for all } P \in A\}.$$

Subsets of the Euclidean spaces $\mathbb{R}^n$ that have the form $\mathcal{Z}(A)$ for some $A \subseteq \mathbb{R}[n]$ are called *algebraic sets*. The set of algebraic sets in $\mathbb{R}^n$ is denoted as $\mathcal{A}(\mathbb{R}^n)$. It is a consequence of Hilbert's basis theorem (Lemma 10.31.2 in [Jon26]) that every algebraic set is of the form $\mathcal{Z}(A)$ for *finitely* many polynomials A.

Over the real numbers we have even more, namely that every algebraic set is the zero set of one polynomial. Note that e.g. for the complex numbers $\mathbb{C}$, the following argument would fail. Let $A = \{P_1, \ldots, P_k\} \subset \mathbb{R}[n]$ be a finite set of polynomials. Then, we have that

$$\big(P_1(\vec{x}) = 0 \;\&\; \ldots \;\&\; P_k(\vec{x}) = 0\big) \Leftrightarrow (P_1^2 + \ldots + P_k^2)(\vec{x}) = 0,$$

and therefore that $\mathcal{Z}(A) = \mathcal{Z}(P_1^2 + \ldots P_k^2)$. The crucial step in this argument is that over $\mathbb{R}$, a sum of non-negative numbers is zero if and only if all of them are zero, which cannot be reproduced e.g. over $\mathbb{C}$.

Algebraic sets exhibit good tame properties. For example, in the one-dimensional case, a real polynomial in one variable is either the zero polynomial or the number of its zeros is bounded

by its degree. Since the latter is always finite, the zero sets of a real polynomial in one variable is either $\mathbb{R}$ or finite. In contrast, the one-dimensional chequered sets from chapter 4 were finite unions of points and open intervals, so that it already seems like algebraic sets are even more restrictive.

However, algebraic sets alone do not yet form a closure algebra. For example, the set $\{0\} \subset \mathbb{R}$ is algebraic, because it is the zero set of the polynomial $x \in \mathbb{R}[x]$. Its complement $\mathbb{R} - \{0\}$, however, cannot be the zero set of a polynomial as it is neither finite nor the whole of $\mathbb{R}$.

In this thesis, we will see three ways to expand $\mathcal{A}(\mathbb{R}^n)$ to a closure algebra. The first – and slightly trivial – one is to consider the closure algebra $\mathcal{MP}(\mathbb{R}^n)$. It can be shown that every algebraic set is closed in the standard topology on $\mathbb{R}^n$, so that $\mathcal{A}(\mathbb{R}^n)$ will be contained in the closed elements of $\mathcal{MP}(\mathbb{R}^n)$. The second way is to also allow *inequalities* when defining subsets of Euclidean spaces, yielding so-called semialgebraic sets. This approach will be worked out in 5.2, the second half of the current chapter. The approach in the current section is the third way, namely to consider the subalgebra of $\mathcal{MP}(\mathbb{R}^n)$ generated by $\mathcal{A}(\mathbb{R}^n)$.

In the standard topology of $\mathbb{R}$, the closure algebra generated by $\mathcal{A}(\mathbb{R}^n)$ will be different from $\mathcal{MC}(\mathbb{R}^n)$, the algebra generated by all closed (or open) sets. It comes in handy, however, that there exists a coarser topology on $\mathbb{R}^n$ – the Zariski topology – in which the closure algebra generated by algebraic sets coincides with the closure algebra of constructibles. This allows us to work within the terminology and tools from section 3.1.

5.1.1 The Zariski topology

In this section, we collect some well-known facts about the Zariski topology. As a standard reference, we refer to [Mum95]. It is more common to define topologies by determining its open sets. In case of the Zariski topology, however, it is more practical to define the closed sets:

Definition 5.1 (Zariski topology, affine spaces, Zariski-constructibles)**.** The *Zariski topology* on $\mathbb{R}^n$ is the topology whose closed sets are exactly the algebraic sets. The resulting space will be called n-dimensional *affine space* and denoted by $\mathbb{A}^n$.

An element of $\mathcal{MC}(\mathbb{A}^n)$ – a finite Boolean combination of Zariski-closed sets – will be called *Zariski-constructible set*.

It is easy to see that definition 5.1 in fact constitutes a topology. The only non-trivial step for seeing that is to convince oneself that the closed sets of $\mathbb{A}^n$ are closed under binary unions. For that, let $A, B \subseteq \mathbb{R}[n]$ be two sets of polynomials. Then, it holds that

$$\mathcal{Z}(A) \cup \mathcal{Z}(B) = \mathcal{Z}\left(\{P \cdot Q \mid P \in A, Q \in B\}\right),$$

which shows that $\mathcal{Z}(A) \cup \mathcal{Z}(B)$ is again an algebraic set.

An important notion when studying the Zariski topology is irreducibility:

Definition 5.2 (Irreducible space, irreducible component)**.** A topological space X is *irreducible* if it is non-empty and cannot be presented as a union of two proper closed sets.

A subset $Z \subseteq X$ is *irreducible* if it is an irreducible space in the subspace topology, and an *irreducible component* if it a maximal closed irreducible set.

The following is an alternative characterization of irreducibility:

Proposition 5.3. *The following are equivalent for a topological space X:*

1. *X is irreducible.*

2. *X is non-empty and every non-empty open set of X is dense.*

Proof.

$\boxed{\Rightarrow}$ Assume (towards contradiction) that there was a non-empty open set $A \subseteq \mathbb{A}^n$ that is not dense. Then, $\mathrm{Cl}(A)$ is a proper closed subset. But then, $A^c \cup \mathrm{Cl}(A) = \mathbb{A}^n$, which contradicts the fact that $\mathbb{A}^n$ is irreducible.

$\boxed{\Leftarrow}$ By contraposition. Assume that $X = C \cup D$ for two proper closed sets. Then, C^c is a non-empty open set with $\mathrm{Cl}(C^c) \subseteq D \neq X$. □

Proposition 5.4. *In the Zariski topology, the following hold:*

1. *Every closed set in the Zariski topology is also closed in the standard topology, i.e. the Zariski topology is coarser than the standard topology.*
2. *$\mathbb{A}^n$ are* Noetherian *spaces, i.e. every descending chain $C_0 \supseteq C_1 \supseteq \ldots$ of closed subsets $C_i \subseteq \mathbb{A}^n$ becomes constant in the sense that there exists $N \in \mathbb{N}$ such that $C_i = C_{i+1}$ for all $i \geq N$.*
3. *Every non-empty open set $A \subseteq \mathbb{A}^n$ is dense, i.e. $\mathrm{Cl}(A) = \mathbb{A}^n$.*
4. *$\mathbb{A}^n$ is irreducible.*

Proof. All these facts are well-known so that we are only going to sketch proofs.

1. Let $P \in \mathbb{R}[n]$ be a polynomial in n variables. Evaluating P at points $\vec{x} \in \mathbb{R}^n$ yields a continuous map $P : \mathbb{R}^n \to \mathbb{R}$. Since this map is continuous and $\{0\}$ is closed, the zero set $\mathcal{Z}(P) = P^{-1}[0]$ is also closed.
2. Hilbert's basis theorem implies that the polynomial ring $\mathbb{R}[n]$ is Noetherian. The transition to the affine space $\mathbb{A}^n$ is given by Hilbert's Nullstellensatz, which relates the Zariski-closed sets with radical ideals of the ring $\mathbb{R}[n]$. For details, see [KN21] until Lemma 5.10.

3.&4. Corollary 5.7 in [KN21] states that $\mathbb{A}^n$ is irreducible. Proposition 5.3 states that this is equivalent to the third statement. □

Regarding points 2, 3 and 4 of proposition 5.4, it is a natural question to ask whether these properties are preserved in subspaces. For Noetherian spaces, it is easy to see that every subspace is also Noetherian. The property of being irreducible, however, is not preserved in general – not even by algebraic sets. For example, consider the polynomial $P = x_1 x_2 \in \mathbb{R}[x_1, x_2]$. Since P is the product of two polynomials, we have that $\mathcal{Z}(P) = \mathcal{Z}(x_1) \cup \mathcal{Z}(x_2)$, which is the union of the coordinate axes. This shows that the subspace $\mathcal{Z}(P)$ is not irreducible. Neither is the closure of every non-empty open dense, since e.g. $\mathcal{Z}(x_1) - \{(0,0)\}$ is open with closure $\mathcal{Z}(x_1)$.

The last example relied on a specific form of the polynomial P: It was a product of two smaller polynomials with different zero sets. This relationship can be made precise insofar as irreducible algebraic sets of $\mathbb{A}^n$ are in a one-to-one correspondence with the prime ideals of the polynomial ring $\mathbb{R}[n]$.

In the following presentation of the dimension theory of algebraic sets, we follow section 3.3.2 from [Cos02]; and for more details on the topological definition of Krull dimension, see section 5.10 in [Jon26].

The Lasker-Noether theorem states that every algebraic set $V \subseteq \mathbb{A}^n$ is a union of finitely many irreducible closed subsets $V_1, \ldots, V_k$ such that $V_i \not\subseteq V_j$ for $i \neq j$, which are furthermore unique.

The sets V_i are the irreducible components of V. For example, the irreducible components of the algebraic set $\mathcal{Z}(P)$ from above are the algebraic sets $\mathcal{Z}(x_1)$ and $\mathcal{Z}(x_2)$. On the other hand, since the affine spaces $\mathbb{A}^n$ themselves are irreducible, they only have one irreducible component.

Definition 5.5 (Krull dimension)**.** Let $V \subseteq \mathbb{A}^n$ be a Zariski-closed set. Then, its *(Krull) dimension* is defined as the supremum of the lengths of chains of irreducible closed subsets, i.e.

$$\dim V = \sup\{n \in \mathbb{N} \mid \text{there is a chain } C_0 \subsetneq \ldots \subsetneq C_n \subseteq V \text{ of irreducible closed sets } C_i\}.$$

For any set $A \subseteq \mathbb{A}^n$, define its (Krull) dimension as $\dim A = \dim \mathrm{Cl}(A)$, the dimension of its closure in the Zariski topology.

We present two easy properties of the Krull dimension:

Proposition 5.6.

1. *If $V \subseteq W \subseteq \mathbb{A}^n$ for closed sets V and W, then $\dim V \leq \dim W$.*
2. *$\dim(V \cup W) = \max\{\dim V, \dim W\}$ for closed sets V and W in $\mathbb{A}^n$.*

Proof.

1. This follows directly from definition 5.5 because every chain of irreducible closed sets in V exists also in W.
2. This is proposition 1 of chapter 8 in [Bou06]. □

For Zariski-constructible sets, it turns out that the Krull dimension defined as above coincides with our intuitive idea of dimension:

Proposition 5.7. *Let $A \subseteq \mathbb{A}^n$ be a Zariski-constructible set. Then, its Krull dimension $\dim A$ equals its topological dimension in $\mathbb{R}$ with the standard topology.*

Proof sketch. In the next section 5.2.1, we will introduce semialgebraic sets. For now, simply note that every Zariski-constructible set is semialgebraic. In theorem 3.20 of [Cos02], it is proven that the Krull dimension of a semialgebraic set equals its dimension as a semialgebraic set. As we will see in more detail, every semialgebraic set can be presented as a finite union of submanifolds of $\mathbb{R}^n$. Its semialgebraic dimension is defined as the maximal dimension of these submanifolds. But for submanifolds of $\mathbb{R}^n$, it is well-known that their dimension equals their topological dimension. □

In general, it is false that $\mathcal{MP}(\mathbb{A}^n) \leq \mathcal{MP}(\mathbb{R}^n)$ is a subalgebra. For example, the Euclidean closure of the positive reals $\{x \in \mathbb{R} \mid x > 0\} \subset \mathbb{R}$ is the set $\{x \in \mathbb{R} \mid x \geq 0\}$, while the closure in the Zariski topology must already be the whole of $\mathbb{R}$ because it is an infinite algebraic set. For Zariski-constructibles, however, it turns out that $\mathcal{MC}(\mathbb{A}^n) \leq \mathcal{MC}(\mathbb{R}^n)$ is in fact a subalgebra. This fact follows from the following properties of the Euclidean closure and the Zariski closure. In the following, we denote the Zariski closure by Cl_Z and the Euclidean closure by Cl_E.

Proposition 5.8. *Let $A \subseteq \mathbb{R}^n$. Then, the following hold:*

1. *$\mathrm{Cl}_\mathrm{E}(A) \subseteq \mathrm{Cl}_\mathrm{Z}(A)$.*
2. *If A is a Zariski-constructible, then $\mathrm{Cl}_\mathrm{Z}(A) = \mathrm{Cl}_\mathrm{E}(A)$.*

Proof.

1. Recall that the topological closure of a set A is the intersection of all closed sets containing A. Since the Euclidean topology is finer than the Zariski topology, and therefore has more closed sets, the Euclidean closure will be contained in the Zariski closure.

2. It is proven in theorem 2.33 in [Mum95] that every non-empty open set of an irreducible closed set is dense not only in the Zariski topology but also in the Euclidean topology. Mumford proves this theorem for the projective spaces, a generalization of the affine spaces.

   Let $A \subseteq \mathbb{R}^n$ be Zariski-constructible. Then, by definition, A is a finite union of locally closed sets. We know that any topological closure distributes over finite unions, so that if suffices to show the statement for a locally closed A. If $A = \emptyset$, the statement is trivially true, so assume that A is non-empty. Furthermore, the Zariski closure $\mathrm{Cl}_\mathrm{Z}(A)$ is a finite union of non-empty closed irreducible sets by the Lasker-Noether theorem. So let C_i be those irreducible components and write $\mathrm{Cl}_\mathrm{Z}(A) = C_1 \cup \ldots \cup C_k$. Then, we have that

   $$A = A \cap \mathrm{Cl}_\mathrm{Z}(A) = (A \cap C_1) \cup \ldots \cup (A \cap C_k).$$

   Observe that A is dense in $\mathrm{Cl}_\mathrm{Z}(A)$. Because of that, it cannot happen that $A \cap C_i = \emptyset$ for any i as then A would be contained in a proper closed subset of $\mathrm{Cl}_\mathrm{Z}(A)$. The sets $A \cap C_i$ are intersections of a locally closed and a closed set, and therefore locally closed. We claim that $\mathrm{Cl}_\mathrm{Z}(A \cap C_i) = C_i$. Since A is locally closed, by proposition 3.1, A is open in $\mathrm{Cl}_\mathrm{Z}(A)$. Therefore, $A \cap C_i$ is open in the subspace topology on C_i. But because C_i is irreducible, we have that every non-empty open is dense. Therefore, it follows that indeed $\mathrm{Cl}_\mathrm{Z}(A \cap C_i) = C_i$.

   The preceding argument showed that the locally closed A can be presented as a finite union of locally closed sets with irreducible Zariski closure. Again, since any topological closure distributes over finite unions, it suffices to prove the statement for locally closed A with irreducible Zariski closure.

   But since A is locally closed, proposition 3.1 tells us that A is open it its Zariski closure. Then, by proposition 5.3, A is even dense in its Zariski closure $\mathrm{Cl}_\mathrm{Z}(A)$ because the latter is irreducible by assumption. We know that then A is also Euclidean-dense in $\mathrm{Cl}_\mathrm{Z}(A)$, i.e. $\mathrm{Cl}_\mathrm{E}(A) = \mathrm{Cl}_\mathrm{Z}(A)$. □

Because the Zariski topology is coarser than the Euclidean topology, every Zariski-constructible set is also Euclidean-constructible. Proposition 5.8 showed that Euclidean closure and Zariski closure coincide for Zariski-constructible sets. Since the Boolean operations union, intersection and complement obviously coincide in $\mathbb{R}^n$ and $\mathbb{A}^n$, we obtain that the Zariski-constructible sets form a modal subalgebra of the Euclidean-constructibles:

Proposition 5.9. *$\mathcal{MC}(\mathbb{A}^n) \leq \mathcal{MC}(\mathbb{R}^n)$ is a subalgebra.*

In the remainder of this section, we will see that the affine spaces $\mathbb{A}^n$ have the stratification property, so that we can use the decomposition spaces of stratifications in order to determine the variety generated by $\mathcal{MC}(\mathbb{A}^n)$ and its corresponding logic in the next section.

Lemma 5.10. *The affine spaces $\mathbb{A}^n$ have the stratification property from definition 3.27.*

Proof. Let $A_1, \ldots, A_k \subseteq \mathbb{A}^n$ be finitely many constructible sets. Consider the finite Boolean algebra $B = \langle A_1, \ldots, A_k \rangle \leq \mathcal{P}(\mathbb{A}^n)$ generated by these sets. Then, the atoms $\mathrm{At}(B)$ are disjoint

constructible sets whose union is $\mathbb{A}^n$, i.e. $\mathrm{At}(B)$ is a finite partition of $\mathbb{A}^n$ by constructible sets. Furthermore, by proposition 2.1, each of the sets A_i is a union of atoms.

By proposition 3.25, every constructible is a finite disjoint union of locally closed sets. By applying this proposition to all atoms of B, we obtain a finite partition of $\mathbb{A}^n$ by locally closed sets. Still, each of the sets A_i is a union of parts.

Lemma 5.28.8 in [Jon26] states that in a Noetherian space any finite partition by locally closed sets can be refined by a stratification (note that the terminology there is slightly different). Applying this argument to our given partition of $\mathbb{A}^n$ – which is Noetherian by proposition 5.4 – yields a stratification compatible with the sets A_i. □

5.1.2 The closure algebra of Zariski-constructible sets

In this section, we will determine the variety generated by the modal algebras $\mathcal{MC}(\mathbb{A}^n)$, or from the logical perspective, the modal logic of the Zariski-constructible sets.

We saw in lemma 5.10 that the affine spaces $\mathbb{A}^n$ have the stratification property. Therefore, we can apply theorem 3.34 to determine the variety generated by $\mathcal{MC}(\mathbb{A}^n)$: It will be the variety generated by the decomposition spaces of all stratifications of $\mathbb{A}^n$.

Lemma 5.11. $\mathrm{Var}(\mathcal{MC}(\mathbb{A}^n)) = \mathrm{Var}\{I_{\mathcal{S}} \mid \mathcal{S} \textit{ stratifies } \mathbb{A}^n\}$.

It will turn out that $\mathrm{Var}(\mathcal{MC}(\mathbb{A}^n))$ can be axiomatized as $\mathbf{GA} \oplus \mathbf{g} \oplus \mathbf{bd}_{n+1}$, i.e. consists of those Grz algebras that validate the Geach formula (or Church-Rosser formula or axiom of confluence) $\mathbf{g}$ and the axiom of bounded depth $\mathbf{bd}_{n+1}$. In the following, we will start by collecting some properties of this variety, resulting in the insight that it is generated by finite ranked rooted trees of height $n-1$ with an extra top element.

We will refer to the logic $L(\mathbf{GA} \oplus \mathbf{g})$ of all Grz algebras that validate the Geach formula by $\mathbf{Grz.2}$. For the variety, however, we keep the notation $\mathbf{GA} \oplus \mathbf{g}$.

The proof will consist of two parts:

1. *Soundness*: It is to prove that $\mathcal{MC}(\mathbb{A}^n) \in \mathbf{GA} \oplus \mathbf{g} \oplus \mathbf{bd}_{n+1}$. Using lemma 5.11, it suffices to show that the decomposition spaces $I_{\mathcal{S}}$ validate $\mathbf{g}$ and $\mathbf{bd}_{n+1}$ for all stratifications $\mathcal{S}$ of $\mathbb{A}^n$.

2. *Completeness*: For equality of varieties, using proposition 2.7, it suffices to prove that every modal formula refuted by $\mathbf{GA} \oplus \mathbf{g} \oplus \mathbf{bd}_{n+1}$ is refuted on some stratification of $\mathbb{A}^n$. For that, we will show that for every finite ranked rooted tree of height $n-1$, there exists a stratification with an isomorphic decomposition space, which will be sufficient.

The Geach axiom $\mathbf{g}$ is the modal formula $\Diamond\Box a \to \Box\Diamond a$.

Proposition 5.12. *Let F be a finite rooted poset. Then, $F^* \models \mathbf{g}$ if and only if F has a maximum.*

Proof. Corollary 3.38 in [CZ97] implies that a finite frame validates $\mathbf{g}$ if and only if it is strongly connected, i.e. for all $x, y_1, y_2 \in F$ such that xRy_1 and xRy_2, there exists z such that y_1Rz and y_2Rz.

If F has a maximum, this condition is satisfied because the maximum can always be chosen as z. For the converse, assume (towards contradiction) that $F^* \models \mathbf{g}$ and there are two maximal elements m_1 and m_2. Because F is rooted with root r, we have that rRm_1 and rRm_2; and because F is strongly connected, there exists $p \in F$ such that m_1Rp and m_2Rp. Because m_1

and m_2 are maximal elements, we have that $m_1 = p = m_2$, which is a contradiction to m_1 and m_2 being distinct. Finally, since F is a finite poset, there exists at least one maximal element, which in conclusion must be a maximum. □

The axioms of bounded depth are defined recursively for $n \geq 0$:

$\mathbf{bd}_0 = \bot$,

$\mathbf{bd}_{n+1} = \Diamond(\Box a_{n+1} \wedge \neg \mathbf{bd}_n) \to a_{n+1}$ for new variables a_{n+1}.

Proposition 5.13. *Let A be a closure algebra. Then, $A \models \mathbf{bd}_{n+1}$ if and only if* $\mathrm{height}(A_*) \leq n$.

Proof. See proposition 3.44 in [CZ97] and lemma 2 in [Mak75]. □

Lemma 5.14. *Let $\mathcal{S}$ be a stratification of the affine space $\mathbb{A}^n$. Then, the decomposition frame $I_{\mathcal{S}}$ has a maximum.*

Proof. Let $\mathcal{S} = \{X_i\}_{i \in I}$ be a stratification of $\mathbb{A}^n$. Since I is a finite poset, it has at least one maximal element. It is to show that it has a unique one. Assume (towards contradiction) that I had distinct maximal elements $t_1, \ldots, t_p$ for $p > 1$.

Recall from proposition 3.5 that the border and thus the closure of a stratum is a union of strata. Therefore, we have that

$$\mathrm{Cl}(X_{t_i}) = \bigcup \{X_s \in \mathcal{S} \mid X_s \subseteq \mathrm{Cl}(X_{t_i})\} = \bigcup \{X_s \in \mathcal{S} \mid s \leq t_i\}$$

for all $1 \leq i \leq p$. Because t_i are maximal elements of I, we have that the strata X_{t_i} are only contained in $\mathrm{Cl}(X_{t_i})$ but in no other closures $\mathrm{Cl}(X_{t_j})$ for $j \neq i$. Therefore, all closed sets $\mathrm{Cl}(X_{t_i})$ are pairwise distinct and proper subsets of $\mathbb{A}^n$.

But since t_i are all maximal elements of I, it is the case that $\mathbb{A}^n = \mathrm{Cl}(X_{t_1}) \cup \ldots \cup \mathrm{Cl}(X_{t_p})$. This is a contradiction to $\mathbb{A}^n$ being irreducible by proposition 5.4. In conclusion, I has a maximum. □

Lemma 5.15. *Let $\mathcal{S}$ be a stratification of the affine space $\mathbb{A}^n$. Then, the decomposition frame $I_{\mathcal{S}}$ has height less or equal to n.*

Proof. We start by showing that $\dim(\delta A) < \dim(A)$ holds for every non-empty locally closed $A \subseteq \mathbb{A}^n$.

Because A is locally closed, A is open and also dense in $\mathrm{Cl}(A)$. It is known from basic topology that for every $Y \subseteq X$ it holds that $X - \mathrm{Int}(Y) = \mathrm{Cl}(X - Y)$. Therefore, we have that $\mathrm{Cl}(A) = \mathrm{Cl}(\mathrm{Cl}(A) - \delta A) = \mathrm{Cl}(A) - \mathrm{Int}(\delta A)$. This shows that $\mathrm{Int}(\delta A) = \emptyset$, i.e. the border δA has empty interior.

Let C be an irreducible component of $\mathrm{Cl}(A)$. We show that C has non-empty interior. First, if $\mathrm{Cl}(A)$ is irreducible, this is clear since A is open (and non-empty) in $\mathrm{Cl}(A)$. Second, suppose there are at least two irreducible components $C, C_1, \ldots, C_k$. Then, $\left(\bigcup_{i=1}^{k} C_i\right)^c$ is a non-empty open contained in C.

In summary, δA has empty interior but every irreducible component of $\mathrm{Cl}(A)$ has non-empty interior. This shows that no irreducible component of $\mathrm{Cl}(A)$ can be contained in δA.

Of course, every chain of irreducible closed sets in δA also exists in $\mathrm{Cl}(A)$, which shows that $\dim(\delta A) \leq \dim(\mathrm{Cl}(A))$. Assume (towards contradiction) that $\dim(\delta A) = \dim(\mathrm{Cl}(A)) = m$. Then, there exists a chain of irreducible closed sets in δA of length m. Take the maximal irreducible closed set C of this chain. We showed in the previous step that this C cannot be an

irreducible component of $\mathrm{Cl}(A)$ (since it is contained in δA). But it must be contained in some irreducible component of $\mathrm{Cl}(A)$, which shows that $\dim(\mathrm{Cl}(A)) > m$, which is a contradiction. In conclusion, $\dim(\delta A) < \dim(\mathrm{Cl}(A)) = \dim(A)$ has been established.

Assume (towards contradiction) that the decomposition frame $I_{\mathcal{S}}$ contained a chain D of length $n+1$. Recall that $i < j$ for $i, j \in D$ means that the strata $X_i \subseteq \delta X_j$. With the same argumentation as above (that any chain of irreducible closed sets in $\mathrm{Cl}(X_i)$ also exists in δX_j), it follows that $\dim(X_i) \leq \dim(\delta X_j)$, and therefore that $\dim(X_i) < \dim(X_j)$.

Therefore, $(\dim(X_i))_{i \in D}$ is a strictly increasing sequence of $n+2$ natural numbers (since D has length $n+1$ by assumption). But $\dim(\mathbb{A}^n) = n$, which e.g. follows from proposition 5.7. This is a contradiction because all $\dim(X_i) \leq n$. □

The last lemmata 5.14 and 5.15 combined show the soundness part:

Lemma 5.16. $\mathcal{MC}(\mathbb{A}^n) \in \mathbf{GA} \oplus \mathbf{g} \oplus \mathbf{bd}_{n+1}$.

Proof. We know from lemma 5.11 that $\mathrm{Var}(\mathcal{MC}(\mathbb{A}^n)) = \mathrm{Var}\{I_{\mathcal{S}} \mid \mathcal{S} \text{ stratifies } \mathbb{A}^n\}$. In particular, $\mathcal{MC}(\mathbb{A}^n) \in \mathbb{HSP}(\{I_{\mathcal{S}} \mid \mathcal{S} \text{ stratifies } \mathbb{A}^n\})$. Therefore, it suffices to show that $I_{\mathcal{S}}^* \in \mathbf{GA} \oplus \mathbf{g} \oplus \mathbf{bd}_{n+1}$ for all stratifications $\mathcal{S}$ of $\mathbb{A}^n$.

But that we already know: The lemmata 5.14 and 5.15 showed that $I_{\mathcal{S}}$ has a maximum and height at most n. Then, the propositions 5.12 and 5.13 yield that $I_{\mathcal{S}}^* \models \mathbf{g}$ and $I_{\mathcal{S}}^* \models \mathbf{bd}_{n+1}$. Also, all frames $I_{\mathcal{S}}$ are finite posets and thus dual to Grz algebras by theorem 2.19. In conclusion, all $I_{\mathcal{S}}^*$ are indeed elements of the equational class $\mathbf{GA} \oplus \mathbf{g} \oplus \mathbf{bd}_{n+1}$. □

In order to prove the completeness part, we will first narrow down the class of generators of the variety $\mathbf{GA} \oplus \mathbf{g} \oplus \mathbf{bd}_{n+1}$ further by showing that it is generated by finite graded topped trees of height n. This will simplify the following construction of stratifications in whose decomposition spaces non-theorems of the variety are refuted.

Proposition 5.17. *The variety* $\mathbf{GA} \oplus \mathbf{g} \oplus \mathbf{bd}_{n+1}$ *is generated by (the dual modal algebras of) finite rooted posets of height* n *that have a maximum.*

Proof. We know from proposition 5.13 that all algebras in the variety $\mathbf{GA} \oplus \mathbf{g} \oplus \mathbf{bd}_{n+1}$ have height at most n. It is known due to Maksimova ([Mak75]) and Segerberg ([Seg71]) that such a variety of closure algebras is locally finite and thus has the finite model property. Then, the variety is generated by its finite subdirectly irreducible algebras. We know from the propositions 5.12 and 5.13 that those are dual to finite rooted posets that have a maximum and have height at most n.

Let F be a finite rooted poset that has a maximum and has height $m < n$. Let r be the root of F. Then, take $n-m$ new points $x_1, \ldots, x_{n-m}$ and define the poset $T' = T \sqcup \{x_1, \ldots, x_{n-m}\}$ with the order $x_1 < \ldots < x_{n-m} < r$ on the new elements and the induced order on T. Intuitively, we put a chain of length $n-m$ underneath T to lift it to height n. Then, the map $T' \twoheadrightarrow T$ that is the identity on T and sends all x_i to r, is a p-morphism.

Furthermore, the poset T' is still rooted and has a maximum, but now also has height n. Because T is a p-morphic image of T', we have by duality that $T^* \in \mathbb{IS}(T'^*)$. Therefore, the variety $\mathbf{GA} \oplus \mathbf{g} \oplus \mathbf{bd}_{n+1}$ will also be generated by finite rooted posets with a maximum of height exactly n. □

Definition 5.18 (Topped tree, graded topped tree)**.** A *topped tree* T is a poset that is bounded – i.e. there exists a minimum (called 'root') r and a maximum t such that $r \leq x \leq t$ for all $x \in T$ – and satisfies that $\downarrow x$ is a chain for all $x \in T - \{t\}$.

We call a topped tree of height n *graded* if there exists a ranking function, i.e. a map $\rho : T \to \mathbb{N}_0$ such that

1. $\rho(r) = 0$ for the root $r \in T$, and
2. $\rho(y) = \rho(x) + 1$ for all $x, y \in T$ whenever y is an immediate successor of x.

Note that a tree is always graded, while for topped trees this is not necessarily the case.

Proposition 5.19. *The variety* $\mathbf{GA} \oplus \mathbf{g} \oplus \mathbf{bd}_{n+1}$ *is generated by (the dual modal algebras of) finite graded topped trees of height* n.

Proof. We already know from proposition 5.17 that $\mathbf{GA} \oplus \mathbf{g} \oplus \mathbf{bd}_{n+1}$ is generated by finite rooted posets with a maximum and height n.

It is well-known that every finite rooted frame is a p-morphic image of a finite tree of the same height. A proof is given in theorem 2.19 in [CZ97]. However, if we were to apply this construction directly, we would lose the property that our finite rooted posets have a maximum.

Instead, we want to show that every finite rooted poset of height n with a maximum is a p-morphic image of a finite topped tree of height n (not necessarily graded yet). We first intercept the case that $n = 0$. Then, a finite rooted poset of height 0 consists of a single point, which is in particular a finite topped tree of height 0. So assume w.l.o.g. that $n > 0$.

Let F be a finite rooted poset of height n with maximum $m \in F$. Since $n > 0$, the untopped frame $F - \{m\}$ is non-empty and has height $n - 1$. By theorem 2.19 ([CZ97]), there exists a finite tree T of height $n - 1$ and a p-morphism $f : T \to F - \{m\}$. Since finite trees are always graded, choose a ranking function ρ on T. Now, it is left to show that both the p-morphism f and the ranking function ρ lift to $T' = T \sqcup \{t\}$, when we add a new top element to T with $x < t$ for all $x \in T$.

Let x be a maximal element of T. Because T has height $n-1$, the rank of x will be $\rho(x) \leq n-1$. Suppose that $\rho(x) < n - 1$. Then, add an immediate successor x' of x to obtain the tree $T^+ = T \sqcup \{x'\}$. Because x was a maximal element before, it now has exactly one immediate successor in T^+, namely x'. In this situation, it is well-known that the map $\alpha : T^+ \to T$ mapping x' to x (and the identity otherwise) – called an *α-reduction* – is a p-morphism. Effectively, we have replaced a maximal element of rank $\rho(x)$ with a new one of rank $\rho(x) + 1$.

Repeat the last construction until all maximal elements in T^+ have rank $n - 1$. Then, the concatenations of all performed α-reductions with f yields a surjective p-morphism $T^+ \to F - \{m\}$. Therefore, we can w.l.o.g. assume that all maximal elements of T have rank $n - 1$.

In summary, we know now that $F - \{m\}$ is a p-morphic image of a finite graded tree of height $n - 1$, in which all maximal elements have rank $n - 1$. Then, extend ρ to T' by setting $\rho(t) = n$, which satisfies the definition of a ranking function.

Define the map $f' : T' \to F$ by $f'(t) = m$ and $f'(x) = f(x)$ for $x \in T$. The map f' is obviously surjective and monotone. For boundedness, suppose that $y' \geq f'(x)$ for some $y' \in F$. If $y' \neq m$, then $x \neq t$ must be the case. Then, because f' is the p-morphism f when restricted to T, there exists $y \in T$ such that $f'(y) = f(y) = y'$ and $x \leq y$. Therefore, the case $y' = m$ remains. But then, $f'(t) = m$ and certainly $x \leq t$ satisfies the definition of boundedness.

In summary, every finite rooted frame F of height n with a maximum is a p-morphic image of a finite graded topped tree T' of height n. By duality, this means that $F^* \in \mathbb{IS}(T'^*)$. Whenever $T'^* \in \mathcal{V}$ is a member of some variety, also $F^* \in \mathcal{V}$. Therefore, it follows from proposition 5.17 that $\mathbf{GA} \oplus \mathbf{g} \oplus \mathbf{bd}_{n+1}$ is generated by (the dual modal algebras of) finite graded topped trees of height n. □

Now, we are closer to proving completeness as we managed to isolate a smaller class of frames, for which we have to show that their non-validities are refuted on $\mathcal{MC}(\mathbb{A}^n)$. Using theorem 2.7, it suffices to show that every modal formula refuted in the variety $\mathbf{GA} \oplus \mathbf{g} \oplus \mathbf{bd}_{n+1}$ is refuted in $\mathcal{MC}(\mathbb{A}^n)$. For that, we need to construct a stratification of $\mathbb{A}^n$ from a given finite graded topped tree of height n. We will start by presenting this construction, and then use it to prove completeness.

Let T be a finite graded topped tree of height n with ranking function ρ. Let T be fixed for the following definitions.

We first observe that we can assign numbers to the elements of $T - \{t\}$, satisfying certain properties.

Definition 5.20 (Weighting function)**.** We call a function $w : T - \{t\} \to \mathbb{R}_{\geq 0}$ a *weighting function* if the following are satisfied:

1. If $x \leq y$, then $w(x) \leq w(y)$.
2. If $x \prec y$ and $x \prec y'$ are immediate successors of x, then either $w(z) < w(z')$ for all $z \geq y$ and $z' \geq y'$, or $w(z) > w(z')$ for all $z \geq y$ and $z' \geq y'$.

It is not difficult to find such weighting functions for concrete finite trees. For example, consider the topped trees in the subfigures (a) and (c) of figure 5.1 with (integer) weights assigned to their elements.

However, we want to ensure that generally such weighting functions always exist.

Lemma 5.21. *Let T be a finite topped graded tree of height n. Then, there exists a weighting function $w : T - \{t\} \to \mathbb{R}_{\geq 0}$.*

Proof. The existence of a weighting functions can be quite easily seen from the fact that any inhabited open interval in $\mathbb{R}_{\geq 0}$ contains any finite number of disjoint inhabited open intervals.

Let r be the root of the tree $T' = T - \{t\}$. Set $w(r) = 0$. Furthermore, assign to r the open interval $\iota(r) = (0, \infty)$.

Then, the weighting function can be defined recursively for the rest of the tree: Let $a \in T'$ with already set $w(a) = x \in \mathbb{R}_{\geq}$ and open interval $\iota(a)$.

Let $b_1, \ldots, b_k$ be the finitely many immediate successors of a. Assign to each b_i a different open interval $\iota(b_i) \subset \iota(a)$ and define the weight as its infimum $w(b_i) = \inf \iota(b_i)$.

Condition 1 in definition 5.20 is satisfied because: Let $x \prec y$ be an immediate predecessor. Then, $\iota(x) \supset \iota(y)$ and thus $\inf \iota(x) \leq \inf \iota(y)$, which implies by definition that $w(x) \leq w(y)$. If $x \leq y$, there exists a sequence of immediate successors $x \prec x_1 \prec \ldots \prec y$. Therefore, applying the last argument iteratively yields $w(x) \leq w(y)$ also in this case.

Condition 2 in definition 5.20 is satisfied because: Let y_1 and y_2 be two immediate successors of $x \in T'$. Then, $\iota(y_1)$ and $\iota(y_2)$ are by construction two disjoint open non-empty intervals. In particular, one of them has to be strictly smaller than the other in the sense that all reals in one are strictly smaller than all reals in the other. W.l.o.g. suppose that $s < t$ for all $s \in \iota(y_1)$ and $t \in \iota(y_2)$. Then, also $s < t$ for all $s \in \iota(y_1) \sqcup \{\inf \iota(y_1)\}$ and $t \in \iota(y_2) \sqcup \{\inf \iota(y_2)\}$ if we add the infima to the intervals. But by construction, $w(z_1) \in \iota(y_1) \sqcup \{\inf \iota(y_1)\}$ for all $z_1 \geq y_1$; and $w(z_2) \in \iota(y_2) \sqcup \{\inf \iota(y_2)\}$ for all $z_2 \geq y_2$. Therefore, $w(z_1) \leq w(z_2)$ for all $z_1 \geq y_1$ and $z_2 \geq y_2$ as been shown. □

Next, we will define a polynomial for each point $x \in T - \{t\}$ recursively:

Definition 5.22. Let T be a finite graded topped tree of height n with top element t. Then, $T - \{t\}$ is a tree of height $n-1$. Let w be a weighting function on T. For every $a \in T - \{t\}$, we define a polynomial $P_a \in \mathbb{R}[x_0, \ldots, x_{n-1}]$ as follows. Fix n variables $x_0, \ldots, x_{n-1}$.

- If $a = r$ is the root of the tree $T - \{t\}$, define

$$P_r = x_0.$$

- If $a > r$, there is an immediate predecessor $\hat{a}$. Then, $\rho(a) > 0$ has positive rank. Then, define P_a in terms of $P_{\hat{a}}$ as

$$P_a = P_{\hat{a}} - w(a) \cdot x_{\rho(a)}^2.$$

Using the polynomials from definition 5.22, we next assign a closed set of $\mathbb{A}^n$ to each element in $T - \{t\}$.

Definition 5.23. Let T be a finite graded topped tree of height n with top element t. Let w be a weighting function on T. Let $a \in T - \{t\}$ be an element of rank $\rho(a)$. Then, define the set of polynomials

$$A_a = \{P_a, x_{\rho(a)+1}, \ldots, x_{n-1}\},$$

and the closed set

$$Z_a = \mathcal{Z}(A_a),$$

the common zero set of all polynomials in A_a.

Let us recap what polynomials definition 5.22 yields. Since $T - \{t\}$ is a tree, the downsets $\downarrow a$ for any $a \in T - \{t\}$ is a chain starting at the root. Furthermore, this chain has length $\rho(a)$:

$$\downarrow a = \{a_0, \ldots, a_{\rho(a)}\} \text{ with } a_0 = r \text{ and } a_{\rho(a)} = a.$$

Then, by the recursive definition 5.22 of P_a, we obtain:

$$P_a = x_0 - w(a_1)x_1^2 - \ldots - w(a_{\rho(a)})x_{\rho(a)}^2.$$

The set $Z_a \subseteq \mathbb{A}^n$ is then defined as all $\vec{x} = (x_0, \ldots, x_{n-1}) \in \mathbb{A}^n$ such that $P_a(\vec{x}) = 0$ and $x_i = 0$ for all $i > \rho(a) + 1$. Because of the specific form of the polynomial P_a, we have a particular representation of Z_a. Note that

$$\begin{aligned} P_a = 0 &\Leftrightarrow x_0 - w(a_1)x_1^2 - \ldots - w(a_{\rho(a)})x_{\rho(a)}^2 = 0 \\ &\Leftrightarrow x_0 = w(a_1)x_1^2 + \ldots + w(a_{\rho(a)})x_{\rho(a)}^2. \end{aligned}$$

Consider the polynomial map

$$\psi_a : \mathbb{R}^{\rho(a)} \to \mathbb{R}, (x_1, \ldots, x_{\rho(a)}) \mapsto w(a_1)x_1^2 + \ldots + w(a_{\rho(a)})x_{\rho(a)}^2.$$

Then, the zero set $\mathcal{Z}(P_a)$ in $\mathbb{R}^{\rho(a)+1}$ is just the graph of the map ψ_a:

$$\Gamma(\psi_a) = \{(\vec{x}, x_0) \in \mathbb{R}^{\rho(a)} \times \mathbb{R} \mid x_0 = \psi_a(\vec{x})\} = \mathcal{Z}(P_a).$$

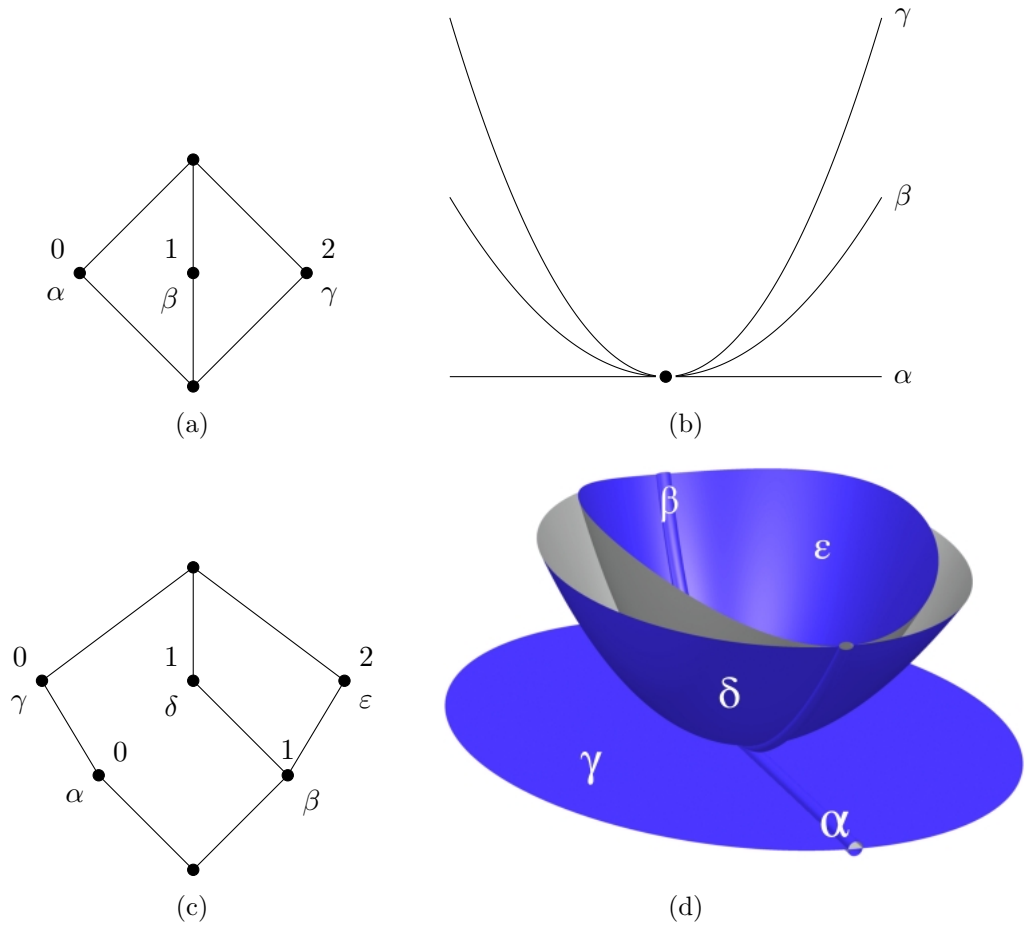

Figure 5.1: In (a) and (c), two graded topped trees are depicted together with possible weight functions. In (b) and (d), the corresponding zero sets Z_a of the polynomials assigned to the elements a of the untopped trees are visualized.

By definition, the set $Z_a \subseteq \mathbb{R}^n$ can then be obtained from $\Gamma(\psi_a)$ by setting all higher variables $x_{\rho(a)+1}, \ldots, x_{n-1}$ to zero, i.e. we have that

$$Z_a = \Gamma(\psi_a) \times \{0\}^{n-\rho(a)-1} \subseteq \mathbb{R}^n.$$

Two examples of finite graded topped trees of height 3 and 4 with the associated algebraic sets Z_a are visualized in figure 5.1. Note that all algebraic sets that are depicted there are indeed graphs.

The last insight, namely that the algebraic sets Z_a for $a \in T - \{t\}$ are essentially graphs of polynomials, simplifies the proof of the following proposition, which we will need later to build a stratification of $\mathbb{A}^n$ from the sets Z_a:

Proposition 5.24. *Let T a finite graded topped tree with maximum t. Let $a \in T - \{t\}$. Then, the following hold:*

1. $\dim Z_a = \rho(a)$.
2. *Z_a is irreducible.*

Proof.

1. We know that $Z_a = \Gamma(\psi_a) \times \{0\}^{n-\rho(a)-1} \subseteq \mathbb{R}^n$. Note that the polynomial map ψ_a gives rise to a bijective map $\phi_a : \mathbb{R}^{\rho(a)} \to \Gamma(\psi_a)$ given by $\vec{x} \mapsto (\vec{x}, \psi_a(\vec{a}))$. Because ϕ_a is continuous (in the Euclidean topology) as a polynomial map, and its inverse (the projection on the first $\rho(a)$ coordinates) is also continuous, we have that $\mathbb{R}^{\rho(a)} \cong \Gamma(\psi_a)$ in the Euclidean topology. Therefore, the topological dimension of $\Gamma(\psi_a)$ is $\rho(a)$. Then, by proposition 5.7, we obtain that also the Krull dimension $\dim \Gamma(\psi_a) = \rho(a)$.

2. This follows from the fact that polynomial maps are continuous in the Zariski topology. In particular, the map $\phi_a : \mathbb{A}^{\rho(a)} \to \Gamma(\psi_a) \subseteq \mathbb{A}^{\rho(a)+1}$ is continuous. We know from proposition 5.4 that $\mathbb{A}^n$ is an irreducible space. Assume (towards contracdiction) that $\Gamma(\psi_a)$ was reducible. Then, $\Gamma(\psi_a) = C_1 \cup C_2$ for two proper closed subsets. But since ϕ_a is bijective and continuous, then also $\mathbb{A}^n$ would be a proper union of the closed preimages of C_1 and C_2. In conclusion, $\Gamma(\psi_a)$ is irreducible, and therefore also Z_a because of $Z_a = \Gamma(\psi_a) \times \{0\}^{n-\rho(a)-1} \subseteq \mathbb{A}^n$. □

Next, we will show that the algebraic sets Z_a manage to capture all of the combinatorial information from the graded topped tree T.

Lemma 5.25. *Given a finite graded topped tree T with maximum t and of height n. Then, consider the poset*

$$Z(T) = \{Z_a \subseteq \mathbb{A}^n \mid a \in T - \{t\}\} \sqcup \{\mathbb{A}^n\}$$

ordered by set inclusion $\subseteq$. Then, $T \cong Z(T)$ as posets.

Proof. The isomorphism ι is given by the mapping $a \mapsto Z_a$ for $a \in T - \{t\}$ and $t \mapsto \mathbb{A}^n$, which is surjective. Thus, in order to show that it is an isomorphism, it suffices to show that it is an order embedding. As t is the maximum of T and all sets Z_a are contained in $\mathbb{A}^n$, it is clear that $a \leq t \Leftrightarrow \iota(a) \subseteq \mathbb{A}^n = \iota(t)$ for all $a \in T$. Therefore, we will in the following restrict ourselves to elements of $T - \{t\}$. Let $a, b \in T - \{t\}$. It is to show that $a \leq b \Leftrightarrow Z_a \subseteq Z_b$.

$\boxed{\Rightarrow}$ Suppose that $a \leq b$. Let $\downarrow a$ be the chain $\{a_0, \ldots, a_{\rho(a)}\}$ with $a_0 = r$ – the root of T – and $a_{\rho(a)} = a$. Because $a \leq b$, the chain $\downarrow b$ must be an extension of $\downarrow a$, i.e. it has the form

$$\downarrow b = \{a_0, \ldots, a_{\rho(a)}, \ldots, a_{\rho(b)}\}$$

with $a_{\rho(b)} = b$.

Let $\vec{r} = (r_0, \ldots, r_{n-1}) \in Z_a$. By definition of Z_a, this means that

- $P_a(\vec{r}) = r_0 - w(a_1)r_1^2 - \ldots - w(a_{\rho(a)})r_{\rho(a)}^2 = 0$ and
- $r_{\rho(a)+1} = \ldots = r_{n-1} = 0$.

But then also $\vec{r} \in Z_b$ because

$$P_b(\vec{r}) = r_0 - w(a_1)r_1^2 - \ldots - w(a_{\rho(a)})r_{\rho(a)}^2 \underbrace{-w(a_{\rho(a)+1})r_{\rho(a)+1}^2 - \ldots - w(a_{\rho(b)})r_{\rho(b)}^2}_{=0} = 0$$

and also $r_{\rho(b)+1} = \ldots = r_{n-1} = 0$ because $\rho(b) \geq \rho(a)$. This shows that indeed $Z_a \subseteq Z_b$.

$\boxed{\Leftarrow}$ By contraposition. Suppose that $a \not\leq b$. Because $T - \{t\}$ is a tree, there exists the meet $c = a \wedge b$, namely the largest element of $\downarrow a \cap \downarrow b$. Then, the chains $\downarrow a$ and $\downarrow b$ branch at c, i.e. they have the form

- $\downarrow a = \{a_0, \ldots, a_{\rho(a)}\}$ with $a_0 = r$ and $a_{\rho(a)} = a$ and
- $\downarrow b = \{b_0, \ldots, b_{\rho(b)}\}$ with $b_0 = r$ and $b_{\rho(b)} = b$,

such that for $0 \leq i \leq \rho(c)$ it holds that $a_i = b_i$, and $a_{\rho(c)} = c = b_{\rho(c)}$.

Next, we compute the intersection $Z_a \cap Z_b$. For any $\vec{r} = (r_0, \ldots, r_{n-1}) \in \mathbb{A}^n$, we have that $\vec{r} \in Z_a \cap Z_b$ if and only if the following conditions are satisfied:

1. $r_0 - w(a_1)r_1^2 - \ldots - w(a_{\rho(c)})r_{\rho(c)}^2 - w(a_{\rho(c)+1})r_{\rho(c)+1}^2 - \ldots - w(a_{\rho(a)})r_{\rho(a)}^2 = 0$
2. $r_0 - w(b_1)r_1^2 - \ldots - w(b_{\rho(c)})r_{\rho(c)}^2 - w(b_{\rho(c)+1})r_{\rho(c)+1}^2 - \ldots - w(b_{\rho(b)})r_{\rho(b)}^2 = 0$
3. $r_{\rho(a)+1} = \ldots = r_{n-1} = 0$
4. $r_{\rho(b)+1} = \ldots = r_{n-1} = 0$

Because we have that $a_i = b_i$ for all $0 \leq i \leq \rho(c)$, it follows from conditions (1) and (2) that:

$$w(a_{\rho(c)+1})r_{\rho(c)+1}^2 + \ldots + w(a_{\rho(a)})r_{\rho(a)}^2 = w(b_{\rho(c)+1})r_{\rho(c)+1}^2 + \ldots + w(b_{\rho(b)})r_{\rho(b)}^2.$$

Let $m = \min\{\rho(a), \rho(b)\}$. Then, we know from the conditions (3) and (4) that for all $m < i \leq n-1$, it is the case that $r_i = 0$. We obtain that

$$(w(a_{\rho(c)+1}) - w(b_{\rho(c)+1}))r_{\rho(c)+1}^2 + \ldots + (w(a_m) - w(b_m))r_m^2 = 0.$$

Observe that $a_{\rho(c)+1}$ and $b_{\rho(c)+1}$ are immediate successors of c. By the definition of weighting function, we have that either $w(a_i) < w(b_i)$ for all $i \geq \rho(c)+1$, or the other way around. W.l.o.g. assume that all coefficients $w(a_i) - w(b_i) > 0$ are strictly positive (else multiply the equation by -1). But a sum of positive real numbers can only equal zero if all of them equal zero. Since the coefficients are all positive, all $r_i = 0$. Also, the squares r_i^2 equal zero if and only if $r_i = 0$. In this way, we have derived:

$$r_{\rho(c)+1} = \ldots = r_{n-1} = 0.$$

Inserting this in the conditions (1) and (2) yields

$$r_0 - w(a_1)r_1^2 - \ldots - w(a_{\rho(c)})r_{\rho(c)}^2 = 0,$$

which shows that $\vec{r} \in Z_c$. In conclusion, we have that $Z_a \cap Z_b \subseteq Z_c$. Because $c \leq a$ and $c \leq b$, we obtain from the forwards direction '$\Rightarrow$' that $Z_c \subseteq Z_a$ and $Z_c \subseteq Z_b$. Together, this yields $Z_c \subseteq Z_a \cap Z_b$. Thus, we have established that $Z_a \cap Z_b = Z_c$.

Assume (towards contradiction) that $Z_a \subseteq Z_b$. Then, we know from the previous result that $Z_c = Z_a$. But by assumption, we have that $a \not\leq b$, which implies that $c < a$ has to be strictly below a. In particular, the rank $\rho(c) < \rho(a)$ is strictly smaller. But then we know from proposition 5.24 that $\dim Z_c < \dim Z_a$, which is a contradiction to $Z_c = Z_a$. In conclusion, we have that $Z_a \not\subseteq Z_b$, which concludes the proof. □

A consequence of the proof of lemma 5.25, which is worth being highlighted on its own, is the following:

Proposition 5.26. *The poset $Z(T)$ is closed under binary intersections.*

Proof. Let $Y, Z \in Z(T)$. If w.l.o.g. $Z = \mathbb{A}^n$, then the intersection $Y \cap Z = Y \in Z(T)$. So assume in the following that both Y and Z are distinct from $\mathbb{A}^n$.

Then, $Z = Z_a$ and $Y = Z_b$ for some $a, b \in T - \{t\}$. We saw in the proof of lemma 5.25 that $Z_a \cap Z_b = Z_{a \wedge b} \in Z(T)$. □

The remaining step is to construct a stratification of $\mathbb{A}^n$ from the poset $Z(T)$. Recall that for T a finite graded topped tree with maximum t and height n, the poset $Z(T)$ is defined as

$$Z(T) = \{Z_a \subseteq \mathbb{A}^n \mid a \in T - \{t\}\} \sqcup \{\mathbb{A}^n\}.$$

Then, define the collection of sets in $\mathbb{A}^n$:

$$S(T) = \left\{ Z - \bigcup \Downarrow Z \mid Z \in Z(T) \right\},$$

where $\Downarrow Z = \{Y \in Z(T) \mid Y \subsetneq Z\}$.

Lemma 5.27.

1. *$S(T)$ is a stratification of $\mathbb{A}^n$.*
2. *$Z(T) \cong I_{S(T)}$ as posets.*

Proof.

1. First, we show that $S(T)$ is a partition of $\mathbb{A}^n$. Because $\mathbb{A}^n \in Z(T)$, every point $x \in \mathbb{A}^n$ will be contained in at least one set of $Z(T)$. Because $Z(T)$ is finite, there will be a minimal such set $Z \ni x$ such that there is no $Y \in Z(T)$ with $Y \subsetneq Z$ and $x \in Y$. Then, $x \in Z - \bigcup \Downarrow Z$.

   Assume (towards contradiction) that some $x \in \mathbb{A}^n$ were contained in two distinct parts $A, B \in S(T)$. Then, A and B have the form $A = Z - \bigcup \Downarrow Z$ for some $Z \in Z(T)$ and $B = Y - \bigcup \Downarrow Y$ for some $Y \in Z(T)$. In particular, $x \in Z \cap Y$. Because A and B are distinct, also $Z \neq Y$, and thus $Z \cap Y \neq Z$ or $Z \cap Y \neq Y$. W.l.o.g. we assume that $Z \cap Y \neq Z$. Proposition 5.26 stated that $Z \cap Y \in Z(T)$. It follows that $Z \cap Y \in \Downarrow Z$; and thus that $Z \cap Y \cap A = \emptyset$. But x is an element of Z, Y and A, which is a contradiction. In conclusion, no element of $\mathbb{A}^n$ can be contained in more than one part of $S(T)$.

   To see that the elements of $S(T)$ are non-empty, assume (towards contradiction) that there were $Z \in Z(T)$ such that $Z = \bigcup \Downarrow Z$. If $Y < Z$ in $Z(T)$, then Y must be of the form Z_a for some $a \in T$. By proposition 5.24, we have that $\dim Y < \dim Z$, and thus using proposition 5.6 that $\dim \bigcup \Downarrow Z = \max\{\dim Y \mid Y < Z\}$. This is a contradiction to $Z = \bigcup \Downarrow Z$. In conclusion, $Z - \bigcup \Downarrow Z$ is non-empty for any $Z \in Z(T)$.

   The elements of $S(T)$ are locally closed. Note that all elements of $Z(T)$ are algebraic sets and thus Zariski-closed. Since finite unions of closed sets are closed, sets of the form $Z - \bigcup \Downarrow Z$ for $Z \in Z(T)$ are intersections of a closed and an open set and thus locally closed by proposition 3.1.

   It remains to show the frontier condition. We use proposition 3.5 and show instead that the border of any element $A \in S(T)$ is a finite union of strata. By definition, A has the form $A = Z - \bigcup \Downarrow Z$ for some $Z \in Z(T)$. We already know that A is non-empty. Since $\bigcup \Downarrow Z$ is closed, the complement $(\bigcup \Downarrow Z)^c$ is open; and thus A is a non-empty open in Z. But we know from proposition 5.24 that Z is irreducible (and in case that $Z = \mathbb{A}^n$, we

know it from proposition 5.4). Thus, it follows from proposition 5.3 that $\mathrm{Cl}(A) = Z$. Note that since Z is closed, the closures in Z and in $\mathbb{A}^n$ coincide. In conclusion, we have that

$$\delta A = \mathrm{Cl}(A) - A = Z - \left(Z - \bigcup \Downarrow Z\right) = \bigcup \Downarrow Z$$

and δ is a finite union of strata.

2. The isomorphism $\iota : Z(T) \to S(T)$ is given by $\iota(Z) = Z - \bigcup \Downarrow Z$ for $Z \in Z(T)$. Since this is exactly the definition of $S(T)$, ι is surjective. Therefore, it suffices to show that it is an order embedding.

   Let $Y, Z \in Z(T)$. We have to show that

   $$Y \subseteq Z \Leftrightarrow \iota(Y) \subseteq \mathrm{Cl}(\iota(Z)).$$

   We already know from the proof of the first point that $\mathrm{Cl}(\iota(Z)) = Z$ such that it suffices to show $Y \subseteq Z \Leftrightarrow \iota(Y) \subseteq Z$.

   $\boxed{\Rightarrow}$ This is clear because $\iota(Y) \subseteq Y$.

   $\boxed{\Leftarrow}$ If $\iota(Y) \subseteq Z$, then also $\mathrm{Cl}(\iota(Y)) \subseteq \mathrm{Cl}(Z)$ because closure is monotone. But we know from the proof of the first point that $\mathrm{Cl}(\iota(Y)) = Y$. Furthermore, Z is already closed, and we have derived $Y \subseteq Z$. □

Now, we are finally ready to bring all the previous results together and prove our characterization of the variety generated by $\mathcal{MC}(\mathbb{A}^n)$:

Theorem 5.28. $\mathrm{Var}(\mathcal{MC}(\mathbb{A}^n)) = \mathbf{GA} \oplus \mathbf{g} \oplus \mathbf{bd}_{n+1}$.

Proof. We know from lemma 5.16 that $\mathcal{MC}(\mathbb{A}^n) \in \mathbf{GA} \oplus \mathbf{g} \oplus \mathbf{bd}_{n+1}$. Using proposition 2.7, it suffices to prove that every modal formula refuted by $\mathbf{GA} \oplus \mathbf{g} \oplus \mathbf{bd}_{n+1}$ is refuted in $\mathcal{MC}(\mathbb{A}^n)$. Lemma 5.11 told us that is suffices to find a stratification of $\mathbb{A}^n$ whose decomposition frame refutes the formula.

So let φ be a modal formula such that $\mathbf{GA} \oplus \mathbf{g} \oplus \mathbf{bd}_{n+1} \not\models \varphi$. Proposition 5.19 implies that there exists a finite graded topped tree T of height n such that $T^* \not\models \varphi$. Then, the lemmata 5.25 and 5.27 together yield a stratification $S(T)$ of $\mathbb{A}^n$, whose decomposition frame $I_{S(T)}$ is isomorphic to T and thus refutes φ. □

Corollary 5.29. *$L(\mathcal{MC}(\mathbb{A}^n))$, the modal logic of constructibles of the n-dimensional affine space, is axiomatized as* $\mathbf{Grz.2} \oplus \mathbf{bd}_{n+1}$.

We close this section with a remark. Note that we saw in lemma 5.21 that we can always choose weighting functions as defined in definition 5.20. In particular, they satisfy condition (1), that whenever $x \leq y$ then $w(x) \leq w(y)$ for x and y elements of the untopped tree. However, it turned out that we did not use this condition in the proof. We still decided to keep this condition because it gives the zero sets Z_a a more regular and easier to visualize shape.

5.1.3 Towards a generalization to schemata

We have seen that the modal logic of the affine spaces $\mathbb{A}^n$ is $\mathbf{Grz.2} \oplus \mathbf{bd}_{n+1}$. The reason why $\mathcal{MC}(\mathbb{A}^n)$ validates the Geach axiom $\mathbf{g}$ was ultimately that $\mathbb{A}^n$ is irreducible – as this implied in lemma 5.14 that all decomposition frames of its stratifications have a maximum. The same argument would work for irreducible closed sets in $\mathbb{A}^n$. For arbitrary closed sets $C \subseteq \mathbb{A}^n$,

however, $\mathcal{MC}(C)$ does not have to validate $\mathbf{g}$. For example, consider the polynomial $P = x_1x_2 \in \mathbb{R}[x_1, x_2]$ from page 47. The reducible closed set defined by P was $\mathcal{Z}(P) = \mathcal{Z}(x_1) \cup \mathcal{Z}(x_2)$, which is just the coordinate cross. Then, the following is a stratification of $\mathcal{Z}(P)$:

$$\mathcal{Z}(P) = \big(\mathcal{Z}(x_1) - \{(0,0)\}\big) \sqcup \big(\mathcal{Z}(P) - \{(0,0)\}\big) \sqcup \{(0,0)\};$$

and furthermore its decomposition space has two maxima. This observation can be made more precise: Recall that by the Noether-Lasker theorem, every algebraic set can be decomposed into a unique number n of irreducible components. Then, a decomposition frame of a stratification can never have more then n maxima.

The preceding considerations give rise to a map

$$\Lambda_n : \mathbb{R}[x_1, \dots, x_n] \to \mathrm{NExt}(\mathbf{Grz} \oplus \mathbf{bd}_{n+1}),$$

which maps a polynomial $P \in \mathbb{R}[n]$ to the normal modal logic of the modal algebra $\mathcal{MC}(\mathcal{Z}(P))$, where $\mathcal{Z}(P)$ is equipped with the subspace topology from $\mathbb{A}^n$. It would be an interesting research question which algebraic properties of the polynomial P are captured by the logic $\Lambda_n(P)$. Irreducibility of P is certainly amongst them, for which we are going to sketch a proof. Recall that a polynomial is called *irreducible* if it cannot be written as a product of two non-constant polynomials.

Proposition 5.30. *A polynomial $P \in \mathbb{R}[x_1, \dots x_n]$ is irreducible if and only if $\Lambda_n(P) \vdash \mathbf{g}$.*

Proof. The polynomial ring $\mathbb{R}[n]$ is a so-called *factorial ring*, which implies that the ideal $(P) \subseteq \mathbb{R}[n]$ generated by P is a prime ideal if and only if P is a prime element if and only if P is irreducible. For more details, see section 10.120 in [Jon26], where factorial rings are called 'UFDs'.

But we already know that prime ideals of $\mathbb{R}[n]$ correspond to irreducible closed sets in $\mathbb{A}^n$. If $\mathcal{Z}(P)$ is irreducible, then $\mathcal{MC}(\mathcal{Z}(P))$ validates $\mathbf{g}$ like in lemma 5.14. Conversely, if $\mathcal{Z}(P)$ is not irreducible, there exist two proper closed subsets such that $\mathcal{Z}(P) = C_1 \cup C_2$. Because Noetherian spaces are closed under subspaces, the topological space $\mathcal{Z}(P)$ will still be Noetherian, and thus have the stratification property like in the proof of lemma 5.10. Then, there exists a stratification $\mathcal{S}$ of $\mathcal{Z}(P)$ such that C_1 and C_2 are unions of strata. Because C_1 and C_2 are closed, they must be unions of downsets in $\mathcal{S}$. Therefore, $\mathcal{S}$ is a union of two proper downsets; and can thus not have a maximum. □

One of the motivation of scheme theory (or even of modern algebraic geometry in general) is to generalize the affine (and projective) spaces to arbitrary commutative rings. The basic observation is that the points of the affine spaces $\mathbb{A}^n$ are in a one-to-one correspondence with the maximal ideals of the ring $\mathbb{R}[n]$. It was Grothendieck's seminal idea to instead consider the *spectrum* of a ring, i.e. the set of all prime ideals.

Recall that an *ideal* of a commutative ring R is a subset $\mathfrak{a} \subseteq R$ such that $0 \in \mathfrak{a}$ and for all $x, y \in \mathfrak{a}$ and $r \in R$, it holds that $x + y \in \mathfrak{a}$, $-x \in \mathfrak{a}$ and $rx \in \mathfrak{a}$. An ideal is called *prime* if it is proper and $xy \in \mathfrak{a}$ implies that either $x \in \mathfrak{a}$ or $y \in \mathfrak{a}$. The spectrum $\mathrm{spec}(R)$ is the set of all prime ideals of R.

The idea of the Zariski topology carries over naturally to $\mathrm{spec}(R)$. For any $T \subseteq R$, define $\varphi(T)$ as the set of all prime ideals containing T, i.e. $\varphi(T) = \{\mathfrak{p} \in \mathrm{spec}(R) \mid T \subseteq \mathfrak{p}\}$. Then, the sets of the form $\varphi(T)$ form the closed sets of the Zariski topology on $\mathrm{spec}(T)$. For more details, see section 10.17 of [Jon26].

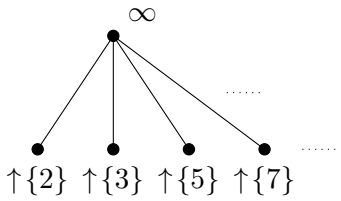

Figure 5.2: Dual space of the modal algebra $\mathcal{MC}(\mathrm{spec}(\mathbb{Z}))$

The objects $\mathrm{spec}(R)$ together with a certain sheaf of rings are also called *affine schemes*. For us, however, mainly the underlying topological space is of interest. Let **Rings** be the category of commutative rings. Then, we obtain a map on the objects

$$\Lambda : \mathbf{Rings} \to \mathbf{GA},$$

sending a ring R to the modal algebra $\mathcal{MC}(\mathrm{spec}(R))$, which is a Grz algebra by theorem 3.26.

For example, consider the ring $R = \mathbb{Z}$ of integers. Its spectrum consists of a closed point for each prime ideal (p) generated by prime numbers $p \in \mathbb{Z}$ and one point for the prime ideal $\{0\}$, whose closure is the whole of $\mathbb{Z}$. Then, the closed sets in $\mathrm{spec}(\mathbb{Z})$ are exactly the finite unions of closed points, as well as the whole space. Since the modal algebra $\mathcal{MC}(\mathrm{spec}(\mathbb{Z}))$ consists of Boolean combinations of these closed sets, the resulting algebra will be isomorphic to $\mathrm{FinCofin}(\mathbb{N})$ with the modal operator $\Diamond a = 1$ if $a \subseteq \mathbb{N}$ is cofinite, and $\Diamond a = a$ if a is finite.

Proposition 5.31. *The modal Krull dimension of* $\mathrm{spec}(\mathbb{Z})$ *is one, which equals the Krull dimension of* $\mathbb{Z}$.

Proof. It is well-known that the prime filters of $A = \mathrm{FinCofin}(\mathbb{N})$ are the principal prime filters $\{\uparrow\{n\} \mid n \in \mathbb{N}\}$ and one non-principal prime filter $\infty = \{C \in A \mid C \text{ cofinite}\}$. Recall that the order on the dual space is defined as $x \leq y$ if and only if $\forall U \in A : U \in y \Rightarrow \Diamond U \in x$.

We claim that the order is as displayed in figure 5.2. Let $x = \uparrow\{n\}$ be a principal prime filter for $n \in \mathbb{N}$ and $y = \uparrow\{m\}$ be a principal prime filter for $m \in \mathbb{N}$. If $x \neq y$, then choose $U = \{m\}$. Certainly, $U \in \uparrow\{m\}$ but $\Diamond U = U \notin \uparrow\{n\}$. This shows that no distinct principal prime filters are related. Now, let $y = \infty$ and x arbitrary. Then, for any $U \in \infty$, we know that $\Diamond U = \mathbb{N}$, which is contained in any primefilter. Therefore, $x \leq \infty$ for all x. We furthermore know that this will be the dual space of the algebra of constructibles of a topological space and thus a poset.

This frame has height one, so that indeed $\mathrm{mdim}(\mathrm{spec}(\mathbb{Z})) = 1$. □

A natural follow-up question is in which case $\Lambda : \mathbf{Rings} \to \mathbf{GA}$ can be extended to a functor. It holds that if $f : R \to S$ is a ring homomorphism, then $f^{-1} : \mathrm{spec}(S) \to \mathrm{spec}(R)$ is a continuous map. However, it may not be open, so that we cannot use proposition 3.31 to obtain a modal homomorphism $\mathcal{MC}(\mathrm{spec}(R)) \to \mathcal{MC}(\mathrm{spec}(S))$.

Without going into detail, proposition 7.18 in [KN21] states:

Proposition 5.32. *Let* $f : R \to S$ *be a ring homomorphism that satisfies the going-down property and has finite representation. Then,* $f^{-1} : \mathrm{spec}(S) \to \mathrm{spec}(R)$ *is an open map.*

For the relevant definitions and results, see sections 10.6 and 10.41 in [Jon26].

Although both the going-down property and the property to be of finite representation are preserved by compositions of ring homomorphisms and trivially satisfied by identities, the resulting category is not very natural in algebraic geometry. But there are 'good' notions of morphisms

that satisfy the prerequisites from proposition 5.32. Examples given in corollary 7.20 in [KN21] are *étale* and *smooth* maps.

It would be an interesting research direction to examine what properties of the categories of commutative rings with étale or smooth maps are captured 'logically' by the functor Λ. This, however, is far beyond the scope of this thesis.

5.2 Logic of semialgebraic sets

Algebraic sets exhibit some tame properties. For example, the only one-dimensional algebraic sets are finite unions of points as well as the whole space. There is one property that is widely considered as characteristic for tame structures, which algebraic sets violate, however, namely closure under projections. Not even are their projections Zariski-constructible in general. An example is given in figure 5.3: the parabola is an algebraic set, while its projection is not Zariski-constructible and thus not a Boolean combination of algebraic sets. In contrast, both the classes of polyhedral sets and chequered sets (from chapter 4) satisfy closure under projections and can therefore be considered tamer than Zariski-constructible sets.

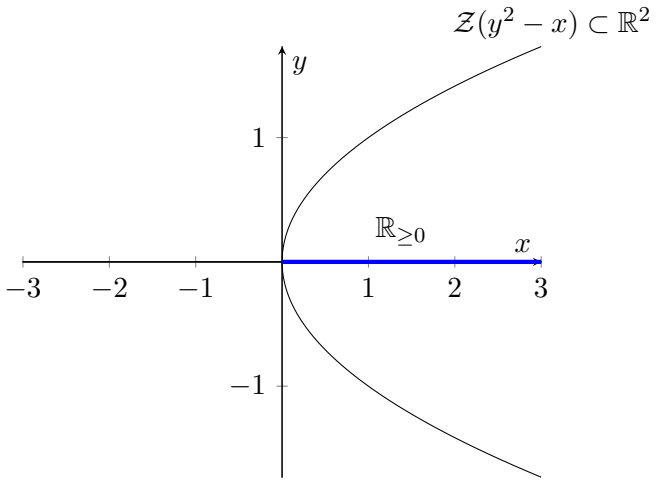

Figure 5.3: The zero set of the polynomial $y^2 - x \in \mathbb{R}[x, y]$ together with its projection $\mathbb{R}_{\geq 0}$ onto the x-axis (in blue). This shows that projections of algebraic sets may not be Zariski-constructible.

From a model-theoretic perspective, the failure of projections of Zariski-constructible sets to be Zariski-constructible shows that the theory $\mathrm{Th}(R)$ of the model $R = (\mathbb{R}, 0, 1, +, \cdot)$ in the language of rings $L_r = \{0, 1, +, \cdot\}$ does not admit quantifier elimination, which we will elaborate on in the following.

Proposition 5.33. $\mathrm{Th}(R)$ *lacks quantifier elimination.*

Proof. Consider the polynomial $P = y^2 - x$ from the counterexample in figure 5.3. P is a *term* in the language L_r of rings. The projection of its zero set onto the x-axis can then be defined by an L_r-formula in the following way:

$$\mathcal{Z}(P) = \{(x, y) \in \mathbb{R}^2 \mid \exists y\, (y^2 - x = 0)\}.$$

Observe that a set in $\mathbb{R}^n$ that is definable as solutions of a polynomial equation $P_1 = P_2$ can always be defined as the zero set of the polynomial $P_1 - P_2$. Suppose towards contradiction that the theory $\mathrm{Th}(R)$ had quantifier elimination. Then, there would exist a quantifier free formula $\phi(x, y)$ such that $\mathcal{Z}(P) = \{(x, y) \in \mathbb{R}^2 \mid \phi(x, y)\}$. But since ϕ would be quantifier free, it would be a Boolean combination of polynomial equations, i.e. $\mathcal{Z}(P)$ would have to be Zariski-constructible. But we know from proposition 5.8 that for Zariski-constructible sets the Euclidean and Zariski closures coincide. $\mathcal{Z}(P) = \mathbb{R}_{\geq 0}$ is Euclidean-closed, and thus has to be an algebraic set. But the only one-dimensional algebraic sets are finite unions of points and the whole space. In conclusion, $\mathrm{Th}(R)$ cannot have quantifier elimination. □

In the proof of the last proposition, we stated that the polynomial $y^2 - x$ is a term in the language of rings. But what about the polynomial $y^2 - \sqrt{2} \cdot x$? Surely, it only becomes a term if we allow parameters from $\mathbb{R}$. It turns out that, when a theory has quantifier elimination, then adding parameters from a model of the theory preserves quantifier elimination.

Proposition 5.34. *Let T be an L-theory with quantifier elimination and M a T-model. Then, $\mathrm{Th}(M)$ has quantifier elimination as an L_M-theory.*

Proof. Let $\phi(\overline{x})$ be an L_M-formula. Then, there exists an L-formula $\phi'(\overline{x}, \overline{y})$ such that $\phi(\overline{x}) = \phi'(\overline{x}, \overline{a})$ for $\overline{a} \in M^n$, obtained by substituting the parameters from M by new variables $\overline{y}$. Because T has quantifier elimination, there exists an quantifier-free L-formula $\psi(\overline{x}, \overline{y})$ such that

$$T \models \forall \overline{y} \forall \overline{x} \, (\phi'(\overline{x}, \overline{y}) \leftrightarrow \psi(\overline{x}, \overline{y})).$$

In particular, it holds that $M \models \forall \overline{y} \forall \overline{x} \, (\phi'(\overline{x}, \overline{y}) \leftrightarrow \psi(\overline{x}, \overline{y}))$. Semantics of the universal quantifier yields that also $M \models \forall \overline{x} \, (\phi'(\overline{x}, \overline{a}) \leftrightarrow \psi(\overline{x}, \overline{a}))$. This shows for the L_M-theory $\mathrm{Th}(M)$ that $\mathrm{Th}(M) \models \forall \overline{x} \, (\phi(\overline{x}) \leftrightarrow \psi(\overline{x}, \overline{a}))$. In conclusion, $\mathrm{Th}(M)$ has quantifier elimination. □

Then, the terms in the language L_R are exactly the polynomials with real coefficients. In fact, polynomials can be defined this way.

In the following section we will turn towards Grothendieck's original candidate for a tame topology: semialgebraic sets. These extend the Zariski-constructible sets by also allowing *inequalities*. We will see that they repair the deficit of Zariski-constructible sets presented above insofar as they are closed under projections.

5.2.1 Semialgebraic and o-minimal geometry

As a standard reference for the theory of semialgebraic sets, we refer to [Dri98] by van den Dries. Although we were working with the presentation in [Cos02], this is mostly based on van den Dries's book.

As already indicated, semialgebraic sets are Boolean combinations of polynomial equalities and inequalities. The following definition is taken from [Cos02]:

Definition 5.35 (Semialgebraic set)**.** The class $\mathcal{SA}_n \subset \mathcal{P}(\mathbb{R}^n)$ of *semialgebraic sets* is the smallest class such that

1. If $P \in \mathbb{R}[n]$ is a polynomial, then the sets $\{\vec{x} \in \mathbb{R}^n \mid P(\vec{x}) = 0\} \in \mathcal{SA}_n$ and $\{\vec{x} \in \mathbb{R}^n \mid P(\vec{x}) > 0\} \in \mathcal{SA}_n$.
2. If $A, B \in \mathcal{SA}_n$, then $A \cap B, A \cup B, A^c \in \mathbb{R}^n$.

We already observed in section 5.1 that algebraic sets in $\mathbb{R}^n$ can be defined using a single polynomial. Similarly, semialgebraic sets have a certain normal form:

Proposition 5.36. *Let $A \in \mathcal{SA}_n$ be a semialgebraic set. Then, there exist finitely many polynomials $P_i, Q_{i1}, \ldots, Q_{k_i} \in \mathbb{R}[n]$ for $0 \leq i \leq p$ such that*

$$A = \bigcup_{i=0}^{p} \{\vec{x} \in \mathbb{R}^n \mid P_i(\vec{x}) = 0 \ \& \ Q_{i1}(\vec{x}) > 0 \ \& \ \ldots \ \& \ Q_{ik_i}(\vec{x}) > 0\}.$$

Proof. This is proposition 2.1 in [Cos02]. □

In the following, we will relate semialgebraic sets to definable sets.

From now on, we will work in the language $L_{or} = \{0, 1, +, \cdot, <\}$ of ordered rings, extending the language of rings by a binary order relation. The theory $\mathrm{Th}(R)$ of the model $R = (\mathbb{R}, 0, 1, +, \cdot, <)$ in the language L_{or} is also called **RCF**, the theory of *real closed fields*. It was shown by Tarski in [Tar51] that **RCF** can be finitely axiomatized. For our use, however, this is not very important, so that we will just refer to this theory as $\mathrm{Th}(R)$ in the following.

Theorem 5.37. *The theory* $\mathrm{Th}(R)$ *has quantifier elimination.*

Proof. This was proven by Tarski in the 1930s but published later ([Tar51]). For a proof see also [Mar02, theorem 3.3.15]. □

Note that by proposition 5.34, also the L_R-theory $\mathrm{Th}(R)$ with parameters from $\mathbb{R}$ has quantifier elimination. In the following, we will present a few easy corollaries of theorem 5.37:

Corollary 5.38.

1. *The definable sets of* $\mathrm{Th}(R)$ *are exactly the semialgebraic sets.*
2. *(Tarski-Seidenberg theorem) Let* $A \in \mathcal{SA}_{n+1}$ *be a semialgebraic set, and let* $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ *the projection onto the first* n *components. Then,* $\pi(A)$ *is semialgebraic.*
3. *Let* $A \in \mathcal{SA}_n$ *be a semialgebraic set. Then, the Euclidean closure* $\mathrm{Cl}(A)$ *is semialgebraic.*
4. *Let* $s \subset \mathbb{R}^n$ *be a* k*-simplex. Then, both* s *and the relative interior* $\widetilde{s}$ *are semialgebraic.*

Proof.

1. All semialgebraic sets are definable using their normal forms from proposition 5.36. Conversely, let $A = \{\vec{x} \in \mathbb{R}^n \mid \phi(\vec{x})\}$ be some definable set for some L_R-formula ϕ. Because $\mathrm{Th}(R)$ has quantifier elimination, we can assume ϕ to be quantifier-free. Then, ϕ is a Boolean combination of finitely many polynomial equations $P_i = P_i'$ and inequations $Q_i < Q_i'$. But, as already pointed out, these are equivalent to $P_i' - P_i = 0$ and $Q_i' - Q_i > 0$. The definition 5.35 of semialgebraic sets ensures that such Boolean combinations are indeed semialgebraic sets.

2. This is similar to proposition 5.33. Let $A \in \mathcal{SA}_{n+1}$ be a semialgebraic set. Then, there exists an L_R-formula $\phi(x, \vec{y})$ such that $A = \{(x, \vec{y}) \in \mathbb{R} \times \mathbb{R}^n \mid \varphi(x, \vec{y})\}$. The projection $\pi(A)$ can be defined as $\{\vec{y} \in \mathbb{R}^n \mid \exists x\, \phi(x, \vec{y})\}$. Therefore, $\pi(A)$ is definable and by point (1) semialgebraic.

3. Let $A \in \mathcal{SA}_n$ be semialgebraic; then, there exists an L_R-formula ϕ such that $A = \{\vec{x} \in \mathbb{R}^n \mid \phi(\vec{x})\}$. The Euclidean closure is defined in first-order logic:

$$\mathrm{Cl}(A) = \left\{ \vec{x} \in \mathbb{R}^n \mid \forall \epsilon (\epsilon > 0 \to \exists \vec{y} \, (\phi(\vec{y}) \wedge \sum_i (x_i - y_i)^2 < \epsilon^2)) \right\},$$

   where the sum and squares are abbreviations. Thus, this set is semialgebraic by point (1).

4. Let s be the given k-simplex. Then, by definition, there exist vertices $v_0, \ldots, v_k \in \mathbb{R}^n$ such that

$$\begin{aligned} s &= \left\{ \alpha_0 v_0 + \ldots + \alpha_k v_k \mid \sum \alpha_i = 1 \text{ and } \alpha_i \geq 0 \text{ for all } i \right\} \\ &= \Bigg\{ \vec{x} \in \mathbb{R}^n \mid \exists \alpha_0 \ldots \exists \alpha_k \\ &\left(\vec{x} = \alpha_0 v_0 + \ldots + \alpha_k v_k \wedge \sum_i \alpha_i = 1 \wedge \neg \alpha_0 < 0 \wedge \ldots \wedge \neg \alpha_k < 0 \right) \Bigg\}, \end{aligned}$$

where $\vec{x} = \alpha_0 v_0 + \ldots + \alpha_k v_k$ is an abbreviation for the conjunction over the componentwise equations. This shows that s is definable and thus semialgebraic by point (1). The argument for the relative interior $\widetilde{s}$ is analogous. □

The third point of corollary 5.38 showed that $\mathcal{SA}_n$ are closed under Euclidean closure. By definition, they are closed under binary intersections, unions and complements. In summary, we obtain that they form a closure algebra. Also, all Zariski-constructible sets are semialgebraic, so that we observe that the closure algebras of Zariski-constructibles, semialgebraic sets and Euclidean-constructibles form a chain of subalgebras:

Corollary 5.39. *$\mathcal{MC}(\mathbb{A}^n) \leq \mathcal{SA}_n \leq \mathcal{MC}(\mathbb{R}^n)$ are subalgebras.*

To see that semialgebraic sets are in fact constructible in the standard topology on $\mathbb{R}^n$, observe that algebraic sets of the form $\{\vec{x} \in \mathbb{R}^n \mid P(\vec{x}) = 0\}$ are closed in the standard topology (by proposition 5.4), while sets of the form $\{\vec{x} \in \mathbb{R}^n \mid P(\vec{x}) > 0\}$ are open. In conclusion, semialgebraic sets are Boolean combinations of open and closed sets and thus constructible.

It is a natural question, what the logic of the closure algebra $\mathcal{SA}_n$ is. We will give an answer in section 5.2.4. But for now, let us look at a few examples of semialgebraic sets.

Example 5.40. The following are semialgebraic sets in $\mathcal{SA}_n$:

- All Zariski-constructible sets and in particular all algebraic sets are semialgebraic due to definition 5.35.
- We saw in corollary 5.38 that geometric simplices and their relative interiors are semialgebraic. Recall that polyhedra are finite unions of simplices, and that polyhedral sets are finite unions of relative interiors of simplices. Therefore, polyhedra and polyhedral sets are semialgebraic.
- The one-dimensional semialgebraic sets $\mathcal{SA}_1$ are finite unions of points and intervals in $\mathbb{R} \cup \{\pm\infty\}$. We saw in chapter 4 that these are exactly the one-dimensional chequered sets. Therefore, we obtain that $\mathcal{SA}_1 = \mathcal{CH}(\mathbb{R})$. In higher dimensions, however, there are much more semialgebraic than chequered sets.

We saw in corollary 5.38 that projections of semialgebraic sets are semialgenraic, which is also known as the Tarski-Seidenberg theorem. In fact, proving the Tarski-Seidenberg theorem algebraically is the crucial step in proving Tarski's theorem 5.37. We have in fact more, namely that every semialgebraic set is a projection of even an *algebraic* set, which is a well-known exercise in semialgebraic geometry:

Proposition 5.41. *Let $A \in \mathcal{SA}_n$ be a semialgebraic set. Then, there exists an algebraic set $V \in \mathcal{A}_{n+k}$ for some $k \geq 0$ such that $A = \pi(V)$, where $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$ denotes the projection onto the first n coordinates.*

Proof. Using the normal form from proposition 5.36, we can express $A = A_1 \cup \ldots \cup A_p$ as a finite union of sets of the form

$$A_i = \{\vec{x} \in \mathbb{R}^n \mid P_i(\vec{x}) = 0 \ \& \ Q_{i1}(\vec{x}) > 0 \ \& \ \ldots \ \& \ Q_{ik_i}(\vec{x}) > 0\}.$$

We first show that each set A_i is a projection of an algebraic set in $\mathbb{R}^{n+k_i}$. Fix the variables $x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k_i}$. We define the following algebraic set:

$$V_i = \{\vec{x} \in \mathbb{R}^{n+k_i} \mid P_i(\vec{x}) = 0 \ \& \ x_{n+1}^2 Q_{i1}(\vec{x}) - 1 = 0 \ \& \ \ldots \ \& \ x_{n+k_i}^2 Q_{ik_i}(\vec{x}) - 1 = 0\}.$$

Then, the projection onto the first n coordinates is given by:

$$\pi(V_i) = \{\vec{x} \in \mathbb{R}^n \mid \exists x_{n+1} \ldots \exists x_{n+k_i} \big(P_i(\vec{x}) = 0 \ \& \ x_{n+1}^2 Q_{i1}(\vec{x}) - 1 = 0 \ \& \ \ldots \ \& \ x_{n+k_i}^2 Q_{ik_i}(\vec{x}) - 1 = 0\big)\}.$$

Note that all polynomials $P_i, Q_{i1}, \ldots, Q_{ik_i} \in \mathbb{R}[n]$. Therefore, the 'new' variables $x_{n+1}, \ldots, x_{n+k_i}$ occur in exactly one of the conjuncts. It is well-known from first-order logic that (1) $\exists x\,(\phi \wedge \psi) \equiv \phi \wedge \exists x\,\psi$, whenever x does not occur freely in ϕ. Applying the rule (1) k_i times yields the following form:

$$\pi(V_i) = \{\vec{x} \in \mathbb{R}^n \mid P_i(\vec{x}) = 0 \ \& \ \exists x_{n+1}\, x_{n+1}^2 Q_{i1}(\vec{x}) - 1 = 0 \ \& \ \ldots \ \& \ \exists x_{n+k_i}\, x_{n+k_i}^2 Q_{ik_i}(\vec{x}) - 1 = 0\}.$$

But now, the formulas of the form $\exists x\, x^2 Q(\vec{x}) - 1 = 0$ express that there exists a non-negative real number $r \geq 0$ such that $Q(\vec{x}) \cdot r = 1$. This is equivalent to $Q(\vec{x}) > 0$. In conclusion, we showed that $\pi(V_i) = A_i$.

Finally, define $k = \max_{1 \leq i \leq p} k_i$. Identify $\mathbb{R}^{k_i} \subseteq \mathbb{R}^k$ as the projection onto the first k_i coordinates. Then, by the above construction, we have that $V = V_1 \cup \ldots \cup V_p \subseteq \mathbb{R}^{n+k}$ and because images commute with unions, we obtain that $\pi(V) = A_1 \cup \ldots \cup A_p = A$. In conclusion, A is a projection of an algebraic set. □

Recall that Zariski-constructible sets were obtained from closing the collection $\mathcal{A}(\mathbb{R}^n)$ of algebraic sets under Boolean operations. In light of proposition 5.41, we have a similar characterization of semialgebraic sets: They are the closure of $\mathcal{A}(\mathbb{R}^n)$ under projections.

Semialgebraic sets also come with a natural notion of morphisms:

Definition 5.42 (Semialgebraic map, semialgebraic homeomorphism)**.** Let $A \subseteq \mathbb{R}^m$ and $B \subseteq \mathbb{R}^n$ be semialgebraic sets. A *semialgebraic map* is a function $f : A \to B$ such that the graph $\Gamma(f) \subseteq A \times B$ is a semialgebraic set in $\mathbb{R}^{m+n}$.

A *semialgebraic homeomorphism* $f : A \to B$ is a homeomorphism that is also a semialgebraic map.

Note from definition 5.42 that it is not necessary to require the inverse map f^{-1} of a semialgebraic homeomorphism to be semialgebraic – this can be proven. The following corollary follows from the Tarski-Seidenberg theorem and is corollary 2.9 in [Cos02]:

Corollary 5.43. *Let $f : A \to B$ and $g : B \to C$ be semialgebraic maps between semialgebraic sets. Then:*

1. *The image $f(A)$ and the preimage $f^{-1}(B)$ are semialgebraic sets.*
2. *The composition $g \circ f : A \to C$ is semialgebraic.*

The following examples of semialgebraic maps are given on page 29 of [Cos02]:

Example 5.44.

- If $f : A \to B$ is a polynomial map from $A \subseteq \mathbb{R}^m$ to $B \subseteq \mathbb{R}^n$ – i.e. all coordinates are given by polynomials – then f is a semialgebraic map.
- If $f : A \to B$ is a function such that all coordinates are given by fractions of semialgebraic maps that do not vanish on A, then f is a semialgebraic map.
- If $f : A \to \mathbb{R}_{\geq 0}$ is a semialgebraic map, then the square root $\sqrt{f} : A \to \mathbb{R}_{\geq 0}$ is a semialgebraic map.

In the remainder of this section, we will introduce a vast generalization of semialgebraic geometry in the shape of o-minimal structures, where the 'o' is short for 'order'. While the focus in this thesis lies on semialgebraic geometry, many of the results can be generalized to o-minimal geometry. The following definition is taken from [Mar02]:

Definition 5.45 (O-minimal structure)**.** Let $L = \{<, \ldots\}$ be a language extending the language L_{dlo} of dense linear orders (DLO). Let $(M, <, \ldots)$ be an L-structure such that the L_{dlo}-reduct models DLO. Then, M is called *o-minimal* if the only definable sets in M are finite unions of intervals in $M \cup \{\pm\infty\}$ and points.

The $\{<\}$-reduct of the structure $R = (\mathbb{R}, 0, 1, +, \cdot, <)$ is of course a dense linear order. Also, we saw in example 5.40 that the only one-dimensional definable sets are indeed finite unions of points and intervals. This shows that R is an o-minimal structure.

It is an active area of research in o-minimal geometry to study such o-minimal structures expanding a real closed field R, i.e. a model of the theory **RCF**. Many of the results that we will use in this thesis can be recovered in this more general setting, like for example the cell decomposition and triangulation theorems from the next two sections. For an overview of this research field, we refer to [Cos99].

In some sense, o-minimal geometry is the search for answers to a problem formulated by Tarski: If we add some analytic functions to the theory **RCF**, does the resulting theory have quantifier elimination? Alex Wilkie gave an answer in 1996 in a celebrated paper [Wil96] concerning the real exponential function $\exp : \mathbb{R} \to \mathbb{R}$. If we consider the structure $R = (\mathbb{R}, 0, 1, +, \cdot, <, \exp)$, then the theory $\mathrm{Th}(R)$ does not have full quantifier elimination but the structure R is o-minimal.

From now on, we will solely focus on semialgebraic geometry and leave the generalization to o-minimal geometry for future work.

5.2.2 Nash stratifications and semialgebraic dimension

In this section, we are going to establish that the closure algebras $\mathcal{SA}_n$ have the stratification property. Furthermore, the stratifications can be chosen to have a specific shape – namely to be so-called Nash stratifications. Therefore, we will work within the setting of definition 3.37, where the stratification property was defined with respect to some class of stratifications.

The term 'Nash stratification' is non-standard; for example, in [Cos02] the following definition is simply referred to as 'stratification'. For us, however, this term is already used in definition 3.4. Because the cells of the stratifications occurring in definition 5.46 are Nash manifolds, we will call them *Nash stratifications* to avoid ambiguity.

Definition 5.46 (Nash stratification)**.** A stratification $\mathcal{S} = \{X_i\}_{i \in I}$ of $\mathbb{R}^n$ is called *Nash stratification*, if the following conditions are satisfied:

1. The strata $X_i \in \mathcal{S}$ (also called 'cells') are *Nash manifolds*, i.e. they are semialgebraically homeomorphic to an open hypercube $(0,1)^{d_i}$ for some $d_i \leq n$. The natural number d is called *dimension* of X and denoted as $\dim X$.

2. Whenever $i < j$ – i.e. $X_i \subseteq \delta X_j$, then $\dim X_i < \dim X_j$.

Note that in [Cos02, corollary 3.8] it is not explicitly stated that the strata of a Nash stratification are locally closed, which we required in 3.4. However, it is well-known that submanifolds of $\mathbb{R}^n$ are locally closed.

At the end of the present section, we will give more detail on the structure of Nash stratifications that are obtained when stratifying a finite number of semialgebraic sets. For now, we just state the following result:

Theorem 5.47. *Let $A_1, \ldots, A_p$ be finitely many semialgebraic sets in $\mathbb{R}^n$. Then, there exists a Nash stratification of $\mathbb{R}^n$ compatible with all A_i.*

Proof. This is corollary 3.8 in [Cos02]. Note that there the semialgebraic sets A_i are assumed to be subsets of a semialgebraic set S. But we can just set $S = \mathbb{R}^n$. □

Theorem 5.47 establishes the stratification property of the algebra $\mathcal{SA}_n$ of semialgebraic sets with respect to the class of Nash stratifications. Then, the premise of theorem 3.38 is satisfied and we obtain the following description of the variety generated by $\mathcal{SA}_n$:

Corollary 5.48. $\mathrm{Var}(\mathcal{SA}_n) = \mathrm{Var}\{I_{\mathcal{S}} \mid \mathcal{S} \text{ a Nash stratification of } \mathbb{R}^n\}$.

In the following, we will use the existence of Nash stratifications to define the dimension of a semialgebraic set and state a few properties. This part follows section 3.3 of [Cos02].

Definition 5.49 (Dimension)**.** Let $A \subseteq \mathbb{R}^n$ be a semialgebraic set. By theorem 5.47, A can be presented as a finite disjoint unions of cells $A = \bigsqcup_{i=1}^{p} C_p$ such that each C_i is semialgebraically homeomorphic to $(0,1)^{d_i}$. Then, the *dimension* of A is defined as $\dim A = \max_{i=1}^{p} d_i$.

In summary, the dimension of a semialgebraic set is defined as the maximal dimension of its cells. Note the similarity to the definition of the dimension of a polyhedron, which was the maximal dimension of a simplex of some triangulation.

It turns out that semialgebraic dimension is well-defined, i.e. it does not depend on the chosen Nash stratification:

Proposition 5.50. *Let $A \subseteq \mathbb{R}^n$ be a semialgebraic set. Its dimension $\dim A$ as defined in definition 5.49 does not depend on the chosen decomposition.*

Proof. This is proposition 3.15 in [Cos02]. □

The dimension of semialgebraic sets behaves naturally insofar as closures do not increase the dimension, while borders have strictly smaller dimension:

Proposition 5.51. *Let $A \subseteq B \subseteq \mathbb{R}^n$ and $D \subseteq \mathbb{R}^n$ be semialgebraic sets. Then*

1. $\dim A \leq \dim B$,

2. $\dim A \cup \dim D = \max\{\dim A, \dim D\}$,

3. $\dim \delta A < \dim A$,

4. $\dim \mathrm{Cl}(A) = \dim A$.

Proof. For (1), choose a Nash stratification $\mathcal{S}$ of $\mathbb{R}^n$ that is compatible with A and B. Therefore, every cell of A will also be a cell of B. In conclusion, $\dim A = \max\{\dim C \mid C \in \mathcal{S} \;\&\; C \subseteq A\} \leq \max\{\dim C \mid C \in \mathcal{S} \;\&\; C \subseteq B\} = \dim B$.

For (2), choose a Nash stratification $\mathcal{S}$ compatible with A, D and $A \cup D$. Let C be a cell of $A \cup D$. Then, C is either a cell of A or of D. To see this, let $x \in C$. Then, either $x \in A$ or $x \in D$. W.l.o.g. let $x \in A$. Because $\mathcal{S}$ is a partition of A, there is a cell of A containing X. But this must already be C because all cells are disjoint. This shows that $\dim A \cup D \geq \dim A$ and $\dim(A \cup D) \geq \dim D$, and therefore $\dim(A \cup D) \geq \max\{\dim A, \dim D\}$. The converse follows from point (1) since $A, D \subseteq A \cup D$.

The points (3) and (4) are proposition 3.16 in [Cos02] and follow from the definition of Nash stratifications: By theorem 5.47, there exists a Nash stratification $\mathbb{R}^n = \bigsqcup_{i \in I} C_i$ compatible with A, i.e. there is $J \subseteq I$ such that $A = \bigsqcup_{i \in J} C_i$. Then,

$$\delta A = \mathrm{Cl}A - A = \bigcup_{i \in J} (\mathrm{Cl}(C_i) - A).$$

Observe that $\mathrm{Cl}(C_i) = \delta C_i \sqcup C_i$. But $C_i \subseteq A$, so that we obtain that $\mathrm{Cl}(C_i) - A \subseteq \delta C_i$. This shows that $\delta A \subseteq \bigcup_{i \in J} \delta C_i$. By definition of Nash stratifications, we have that $\dim \delta C_i < \dim C_i$. Therefore, we have the following using the points (1) and (2):

$$\dim \delta A \leq \dim \bigcup_{i \in J} \delta C_i = \max_{i \in J} \dim \delta C_i < \max_{i \in J} \dim C_i = \dim A.$$

Finally, (4) follows from $\mathrm{Cl}(A) = A \cup \delta A$. Since $A \subseteq \mathrm{Cl}(A)$, we have that $\dim A \leq \dim \mathrm{Cl}(A)$ by (1). Conversely, $\dim \mathrm{Cl}(A) = \max\{\dim A, \dim \delta A\} = \dim A$ due to (2) and (3). □

We know from example 5.40 that polyhedra and polyhedral sets are semialgebraic. Recall how the dimension of a polyhedron P was defined on page 18: Choose a triangulation $|\Delta| = P$ and define $\dim P$ as the maximal dimension of simplices of Δ. Our definition of semialgebraic dimension is conceptually very similar.

We already saw in proposition 3.12 that the decomposition of a polyhedron in the relative interiors of a triangulation is a stratification. It can also be shown that it is in fact a Nash stratification because every k-dimensional relative interior is semialgebraically homeomorphic to $(0,1)^k$. Therefore, the Nash stratification on a polyhedron or polyhedral set that is used to define its semialgebraic dimension can be chosen to be the decomposition given by relative interiors of a triangulation. This shows that the semialgebraic dimension of a polyhedron coincides with the dimension as defined on page 18.

In particular, the statements about semialgebraic dimension from proposition 5.51 transfer to polyhedra and polyhedral sets. An elementary proof of these facts which uses the triangulation lemma instead of theorem 5.47 can be found in appendix 7.2. The proof there uses the same key steps as the proof of 5.51 and is based on unpublished notes by David Gabelaia.

In the remainder of this section, we will see that $\mathcal{SA}_n$ has the stratification property even with respect to a smaller class of stratifications, i.e. the Nash stratifications in theorem 5.47 can always be chosen to have a more specific form. Although we will not use properties of this smaller class in this thesis, we will briefly describe it in the following.

The main step in finding a Nash stratification as demanded in theorem 5.47 is to find a cylindrical algebraic decomposition (CAD) of the Euclidean spaces. The first algorithm to compute such a CAD was described by Collins in [Col75]. The following definition is taken from [Cos02]:

Definition 5.52 (CADs, cells, graphs, bands)**.** A *CAD* of $\mathbb{R}^n$ is a sequence $\mathcal{C}_1, \ldots, \mathcal{C}_n$ such that each $\mathcal{C}_k \subseteq \mathcal{P}(\mathbb{R}^k)$ is a finite partition of $\mathbb{R}^k$ into semialgebraic sets – called *cells* – and the following properties are satisfies:

1. $\mathcal{C}_1$ consists of non-empty convex open sets of $\mathbb{R}$ and points.
2. For every $C \in \mathcal{C}_k$ for $1 \leq k < n$, there exist finitely many semialgebraic maps $\xi_{C,1} < \ldots < \xi_{C,l_C} : C \to \mathbb{R}$ (where the inequalities are pointwise) such that the cylinder $C \times \mathbb{R} \subset \mathbb{R}^{k+1}$ is a disjoint union of cells from $\mathcal{C}_{k+1}$ with are all of one of the following two forms:
   - A *graph* $\Gamma(\xi_{C,i})$ of one of the maps $\xi_{C,i}$ for $1 \leq i \leq l_C$, or
   - a *band* of two maps $\xi_{C,i}$ and $\xi_{C,i+1}$ for $0 \leq i \leq l_C$ (where we set $\xi_{C,0} = -\infty$ and $\xi_{C,l_C+1} = \infty$ to cover the unbounded bands below the least and above the greatest map), i.e. a set of the form

$$\{(\vec{x}, x_{k+1}) \in C \times \mathbb{R} \mid \xi_{C,i}(\vec{x}) < x_{k+1} < \xi_{C,i+1}(\vec{x})\}.$$

Intuitively, the cylinder over an k-dimensional cell C is a sequence of $k+1$-dimensional volumes (the bands) separated by k-dimensional surfaces (the graphs). An example is visualized in figure 5.4.

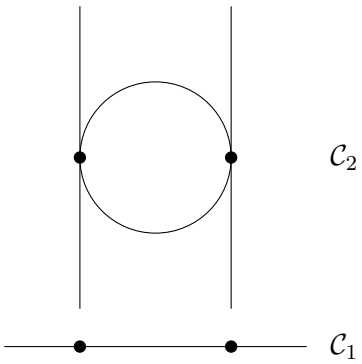

Figure 5.4: A CAD of $\mathbb{R}^2$. The cells of $\mathcal{C}_1$ are points and open intervals of $\mathbb{R}$. For every $C \in \mathcal{C}_1$, the cylinders $C \times \mathbb{R}$ is a disjoint union of graphs and points.

CADs are reminiscent of what we called chequered stratifications in chapter 4.11. Recall that the chequered stratifications of $\mathbb{R}$ also consisted of points and non-empty open convex sets. Therefore, for the one-dimensional space $\mathbb{R}$, the notions of chequered stratifications and CADs coincide. In higher dimensions, however, we only have that CADs look locally combinatorially like chequered stratifications. One can in fact recover chequered stratifications from CADs by requiring that the maps $\xi_{C,i}$ in definition 5.52 are first constant and second independent of C.

The property that cells are Nash manifolds is already present in CADs (this is proposition 2.15 in [Cos02]). Therefore, also CADs can be used perfectly well to define the semialgebraic dimension of a semialgebraic set.

However, not all CADs are stratifications. We can only ensure that the frontier condition is satisfied within a cylinder (if we consider the cylinder with the subspace topology): Then, the closure of a band will be the union of the graphs with which it is in contact. But just from the definition of CADs, we have no control over what happens topologically if we move from one cylinder to another.

In order to instantiate the stratification property for semialgebraic sets, we want that for any finite number $A_1, \ldots, A_p$ of semialgebraic sets, there exists a stratification compatible with all of them, in the sense that each A_i is a union of strata. For CADs, however, this is *false* in general: There exist semialgebraic sets which do not allow CADs satisfying the frontier condition. But luckily, all such pathological examples can be overcome by applying a linear automorphism to $\mathbb{R}^n$.

Theorem 5.53. *Let $A_1, \ldots, A_p$ be finitely many semialgebraic sets in $\mathbb{R}^n$. Then, there exists a linear automorphism $u : \mathbb{R}^n \to \mathbb{R}^n$ and a CAD of $\mathbb{R}^n$ which is compatible with the images $u(A_1), \ldots, u(A_p)$ and satisfies the frontier condition.*

Proof. This follows from the theorems 3.6 and 3.7 and corollary 3.8 in [Cos02]. □

In particular, a linear automorphism is a semialgebraic homeomorphism. Suppose that $A_1, \ldots, A_p$ are finitely many semialgebraic subsets and $\mathcal{C}_n$ a CAD of $\mathbb{R}^n$ satisfying the frontier condition and compatible with all $u(A_1), \ldots, u(A_p)$ for a linear automorphism u, whose existence is guaranteed by theorem 5.53. Then, the preimages $u^{-1}(C)$ of the cells $C \in \mathcal{C}_n$ are a partition of $\mathbb{R}^n$ compatible with all A_i. Because u is a semialgebraic map, the preimages $u^{-1}(A_i)$ are still semialgebraic by corollary 5.43; and because u is a homeomorphism, they satisfy the frontier condition. This shows that the collection $\mathcal{S} = \{u^{-1}(C) \mid C \in \mathcal{C}_n\}$ is a Nash stratification of $\mathbb{R}$ compatible with all sets A_i.

5.2.3 Logic of compact semialgebraic sets

Before we determine the logic of the closure algebras $\mathcal{SA}_n$, we will examine the closure algebra of semialgebraic subsets of a *compact* semialgebraic set. This question is easier to approach because the theory of semialgebraic geometry provides us with a triangulation theorem for such semialgebraic compacta. In the next section 5.2.4, we will use the results from the present section to finally approach the algebras $\mathcal{SA}_n$. Let throughout this section C denote a fixed compact semialgebraic set in $\mathbb{R}^n$.

Definition 5.54 (Semialgebraic logic of a compactum)**.** Let $C \subseteq \mathbb{R}^n$ be a compact semialgebraic set. Denote the class of all semialgebraic subsets of C by $\mathcal{SA}(C)$. Then, the *semialgebraic logic* of C is the modal logic of the Grz algebra $(\mathcal{SA}(C), \cup, \cap, (\,\cdot\,)^c, \mathrm{Cl}, \emptyset, C)$, where $(\,\cdot\,)^c$ denotes the complement in C.

We show that this is well-defined:

Proposition 5.55. *$\mathcal{SA}(C)$ is a subalgebra of $\mathcal{MC}(C)$ and a Grz algebra.*

Proof. Since C is compact in the Euclidean topology on $\mathbb{R}^n$, it is closed. Therefore, the closure of any subset $A \in \mathcal{SA}(C)$ will be contained in C.

Also, by corollary 5.39, every $A \in \mathcal{SA}(C)$ is constructible in $\mathbb{R}^n$. By definition, A is then a finite union of locally closed sets $A = X_1 \cup \ldots \cup X_k$. Every X_i is contained in A and thus in C. Then, there exist a closed set D_i and an open set U_i such that each $X_i = U_i \cap D_i$. By definition of the subspace topology, $D_i \cap C$ is closed in C and $U_i \cap C$ is open in C. Therefore, all X_i are intersections of open and closed sets in C and thus locally closed also in C. In conclusion, A is constructible in C.

Obviously, $\mathcal{SA}(C)$ is closed under binary intersections, unions and complements (with respect to C). In conclusion, $\mathcal{SA}(C)$ is a subalgebra of $\mathcal{MC}(C)$ and thus itself a Grz algebra. □

We will in the following show that the variety $\mathrm{Var}(\mathcal{SA}(C))$ generated by $\mathcal{SA}(C)$ is polyhedral, and in fact the polyhedral logic of one polyhedron. The decisive ingredient is the following triangulation theorem:

Theorem 5.56. *Let $C \subset \mathbb{R}^n$ be a compact semialgebraic set and let $A_1, \dots, A_n \in \mathcal{SA}(C)$. Then, there exists a polyhedron $P = |\Delta|$ in $\mathbb{R}^n$ with a triangulation Δ and a semialgebraic homeomorphism $h : P \to C$ such that each set A_i is the image of a polyhedral set compatible with Δ.*

Proof. This is theorem 3.12 in [Cos02]. □

The idea to find a simplicial complex whose geometric realization is homeomorphic to a given topological space is ubiquitous in algebraic topology. For example, triangulations of a topological space can be used to compute its homology groups. The requirement for such techniques to be applicable is that a space is triangulable, i.e. it is homeomorphic to a simplicial complex. The *Hauptvermutung* (German for 'main conjecture') of geometric topology stated that whenever a topological space is triangulable, then the homeomorphic simplicial complex is unique up to PL homeomorphism. Then, topological invariants of the simplicial complexes would automatically be invariants for the topological spaces as well.

The Hauptvermutung was eventually rejected by Milnor in 1961 in [Mil61], who constructed two simplicial complexes that are homeomorphic but fail to be PL homeomorphic. But as Shiota pointed out in [Shi14], the existence of such pathological examples is due to wildness of the constructed homeomorphism: Milnor constructed the simplicial complexes by repeating a finite procedure infinitely often. On the other hand, semialgebraic sets are inherently finite by definition and exhibit many tame properties. Therefore, the question asked by the Hauptvermutung was revived in the context of semialgebraic geometry. And indeed, Shiota and Yokoi proved in 1984 that the semialgebraic Hauptvermutung is true. Later, Shiota generalized this to o-minimal geometry and proved in 2014 in [Shi14] that also the o-minimal variant of the Hauptvermutung is true.

Theorem 5.57 (Semialgebraic Hauptvermutung)**.** *Let P and Q be two polyhedra that are semialgebraically homeomorphic. Then, they are PL homeomorphic.*

Proof. This is corollary 4.3 in [SY84]. □

Recall from theorem 2.29 that two PL homeomorphic polyhedra have the same polyhedral logic. Therefore, the following corollary is a formulation of the semialgebraic Hauptvermutung in terms of polyhedral logic:

Corollary 5.58. *Let P and Q be semialgebraically homeomorphic polyhedra. Then, the polyhedral logics $L(\mathcal{MPS}(P)) = L(\mathcal{MPS}(P))$ coincide.*

Therefore, polyhedral logic can be understood as a topological invariant for semialgebraic homeomorphisms.

We will use the existence of triangulations together with the semialgebraic Hauptvermutung to determine the modal logic of $\mathcal{SA}(C)$:

Theorem 5.59. *Let $C \subset \mathbb{R}^n$ a semialgebraic compactum. Then, there exists a polyhedron P in $\mathbb{R}^n$ such that $\mathrm{Var}(\mathcal{SA}(C)) = \mathrm{Var}(\mathcal{MPS}(P))$. Furthermore, P is unique up to PL homeomorphism.*

Proof. By theorem 5.56, there exists a polyhedron P in $\mathbb{R}^n$ which is semialgebraically homeomorphic to C via $h : P \to C$. Suppose there is a second polyhedron Q semialgebraically homeomorphic to C via $g : Q \to C$. Then, $g^{-1} \circ h : P \to Q$ is a semialgebraic homeomorphism. Using the Hauptvermutung 5.57, we have that $P \cong_{\mathrm{PL}} Q$ are PL homeomorphic.

It remains to show that $\mathrm{Var}(\mathcal{SA}(C)) = \mathrm{Var}(\mathcal{MPS}(P))$. We use the fact from theorem 2.6 that varieties are equationally definable classes. Therefore, using algebraic completeness, it suffices to show that $\mathrm{Var}(\mathcal{SA}(C))$ and $\mathrm{Var}(\mathcal{MPS}(P))$ satisfy the same modal formulas.

Let φ be a modal formula such that $\mathrm{Var}(\mathcal{MPS}(P)) \not\models \varphi$. Then, by proposition 2.7, we have that $\mathcal{MPS}(P) \not\models \varphi$. Denote the propositional variables in φ as $\varphi(x_1, \dots, x_k)$. Then, there exist polyhedral sets $A_1, \dots, A_k$ in P such that $\varphi(A_1, \dots, A_k) \neq P$. By corollary 5.43, we know that the images $h(A_i)$ are semialgebraic sets in C. Because h is a homeomorphism, both h and h^{-1} commute with unions, intersections, complements and topological closure. Since h is bijective, we can write $\varphi(h^{-1} \circ h(A_1), \dots, h^{-1} \circ h(A_k)) \neq h^{-1} \circ h(P)$. Using that h^{-1} commutes with all Boolean operations and closure, we obtain $h^{-1}(\varphi(h(A_1), \dots, h(A_k))) \neq h^{-1}(h(P))$. But if the preimages are distinct, also $\varphi(h(A_1), \dots, h(A_k)) \neq h(P)$ must be distinct. Since $h(P) = C$, we found semialgebraic subsets $h(A_1), \dots, h(A_k) \in \mathcal{SA}(C)$ refuting the formula φ. In conclusion, $\mathcal{SA}(C) \not\models \varphi$ and therefore also $\mathrm{Var}(\mathcal{SA}(C)) \not\models \varphi$.

Conversely, let φ now be a (new) formula such that $\mathrm{Var}(\mathcal{SA}(C)) \not\models \varphi$. Again, by proposition 2.7, we have that $\mathcal{SA}(C) \not\models \varphi$. Denote the propositional variables in φ as $\varphi(x_1, \dots, x_k)$. There exist semialgebraic sets $A_1, \dots A_k \in \mathcal{SA}(C)$ such that $\varphi(A_1, \dots, A_k) \neq C$. By theorem 5.56, there exists a polyhedron P' in $\mathbb{R}^n$ and a semialgebraic homeomorphism $g : P' \to C$ together with a triangulation Δ of P' such that every A_i is the image of a polyhedral set compatible with Δ. Like before, we use the homeomorphism to write $\varphi(g \circ g^{-1}(A_1), \dots, g \circ g^{-1}(A_k)) \neq g \circ g^{-1}(C)$. Using that g commutes with Boolean operations and closure, we obtain that $g(\varphi(g^{-1}(A_1), \dots, g^{-1}(A_k))) \neq g \circ g^{-1}(C)$ and thus that the preimages $\varphi(g^{-1}(A_1), \dots, g^{-1}(A_k)) \neq g^{-1}(C) = P$ differ. But by assumption the preimages $g^{-1}(A_i)$ are polyhedral sets. Therefore, we found polyhedral sets $g^{-1}(A_1), \dots, g^{-1}(A_k) \in \mathcal{MPS}(P')$ refuting φ. In conclusion, $\mathcal{MPS}(P') \not\models \varphi$ and thus also $\mathrm{Var}(\mathcal{MPS}(P')) \not\models \varphi$.

In the final step, we observe that P and P' are PL homeomorphic by the semialgebraic Hauptvermutung. Using theorem 2.29, we obtain that also $\mathrm{Var}(\mathcal{MPS}(P)) \not\models \varphi$, which concludes the proof. □

In logical terms, we obtain the following corollary:

Corollary 5.60. *Let C be a semialgebraic compactum. Then, the semialgebraic logic of C equals the polyhedral logic of some polyhedron, which is semialgebraically homeomorphic to C and unique up to PL homeomorphism.*

The last result can be understood in the way that polyhedral logic does not just apply to polyhedra but also to a larger class of topological spaces. On the one hand, this reminds one of the history of algebraic topology, in which many methods like homology and cohomology were originally only defined for simplicial complexes and later generalized to larger classes of spaces. On the other hand, this confirms the intuition that even finite simplicial complexes have a great power in approximation.

5.2.4 Logic of semialgebraic sets

In this section, we are going to show that the logic of $\mathcal{SA}_n$ is $\mathbf{PL}_n$, the polyhedral logic of the standard n-simplex Δ_n. In [Ada+23], where this logic appeared for the first time, it is

shown that $\mathbf{PL}_n$ is also the polyhedral logic of the class of all convex polyhedra, which follows readily from the fact that every convex polyhedron is PL homeomorphic to Δ_n. Furthermore, it is proven in [Ada+23] that $\mathbf{PL}_n$ can be finitely axiomatized and that it is in fact the largest polyhedrally complete logic of height n.

The proof of $L(\mathcal{SA}_n) = L(\mathcal{MPS}(\Delta_n))$ will consist of two steps: First, we will convince ourselves that the standard n-simplex is semialgebraically homeomorphic to the closed n-ball $\mathbf{D}^n$. Second, the main part will consist of showing that $\mathcal{SA}_n$ and $\mathcal{SA}(\mathbf{D}^n)$ validate the same modal formulas.

Recall that the closed n-ball is defined as $\mathbf{D}^n = \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| \leq 1\}$. It is easy to see from the definition that any two n-simplexes are PL homeomorphic, so that they have the same polyhedral logic by theorem 2.29. In the following, we will just use that Δ_n is a compact and convex body in $\mathbb{R}^n$.

Furthermore, we will for the rest of this section assume that $n \geq 1$. In case that $n = 0$, the statement $L(\mathcal{SA}_n) = L(\mathcal{MPS}(\Delta_n))$ is trivially true since both $\mathbb{R}^0$ and Δ_0 are singletons.

Proposition 5.61. *Let $\vec{c} \in \mathrm{Int}(\Delta_n) \subset \mathbb{R}^n$ be a point in the interior of the standard n-simplex. Then, there exists a semialgebraic homeomorphism $h : \mathbf{D}^n \to \Delta_n$ such that $h(\vec{0}) = \vec{c}$.*

Proof. We follow the construction from [Bre93, pp. 56, 57]. Surely, the affine translation $t : \mathbb{R}^n \to \mathbb{R}^n, \vec{x} \mapsto \vec{x} - \vec{c}$ is a semialgebraic homeomorphism. Therefore, we can w.l.o.g. assume $\vec{c} = \vec{0}$ to be the origin.

So let $\vec{0} \in \mathrm{Int}(\Delta_n) \subset \mathbb{R}^n$. Then, we are in the conditions for proposition 16.3 in [Bre93], which states that then the map

$$f : \partial\Delta_n \to \mathbf{S}^{n-1} = \partial\mathbf{D}^n,\ \vec{x} \mapsto \vec{x}/\|\vec{x}\|$$

is a homeomorphism. Recall from example 5.44 that the map $\vec{x} \mapsto \|\vec{x}\| = \sqrt{x_1^2 + \ldots + x_n^2}$ is semialgebraic as the square root of a polynomial map. Then, f is componentwise given as a fraction of semialgebraic maps and thus itself semialgebric. In conclusion, f is a semialgebraic homeomorphism. It is well-known that then also the inverse map $f^{-1} : \partial\mathbf{D}^n \to \partial\Delta_n$ is a semialgebraic homeomorphism.

Bredon then continues to show in proposition 16.4 of [Bre93] that the map

$$k : \mathbf{D}^n \to \Delta_n,\ \vec{x} \mapsto \begin{cases} \|\vec{x}\| f^{-1}(\vec{x}/\|\vec{x}\|), & \text{if } \vec{x} \neq \vec{0} \\ \vec{0}, & \text{if } \vec{x} = \vec{0} \end{cases}$$

is a homeomorphism. We already know that f^{-1} is semialgebraic and also that that $\vec{x} \mapsto \|\vec{x}\|$ is semialgebraic. It is corollary 2.9.3 in [Cos02] that semialgebraic maps to $\mathbb{R}$ form a ring, i.e. in particular they are closed under products. This shows that the restricted map $k|_{\mathbf{D}^n - \{\vec{0}\}} : \mathbf{D}^n - \{\vec{0}\} \to \Delta_n - \{\vec{0}\}$ is semialgebraic. Also, the restricted map $k|_{\{\vec{0}\}}$ is obviously semialgebraic.

But if $A, B \subseteq \mathbb{R}^n$ are semialgebraic and disjoint, and $\xi : A \to \mathbb{R}^m$ and $\zeta : B \to \mathbb{R}^m$ are semialgebraic maps, then the graph of the map $\xi \cup \zeta : A \sqcup B \to \mathbb{R}^m$ is the union of $\Gamma(\xi)$ and $\Gamma(\zeta)$. Since semialgebraic sets are closed under unions, also $\xi \cup \zeta$ is semialgebraic. Applied to the situation with $k|_{\mathbf{D}^n - \{\vec{0}\}}$ and $k|_{\{\vec{0}\}}$, we obtain that k is a semialgebraic homeomorphism. □

Therefore, we obtain from theorem 5.59 that $\mathrm{Var}(\mathcal{SA}(\mathbf{D}^n)) = \mathrm{Var}(\mathcal{MPS}(\Delta_n))$. In the following, we will show that $\mathrm{Var}(\mathcal{SA}(\mathbf{D}^n)) = \mathrm{Var}(\mathcal{SA}_n)$. But before that, we prove an easy consequence of the definition of Nash stratifications, which we will use to prove the lemma thereafter:

Proposition 5.62. *Let $\mathcal{S}$ be a Nash stratification of $\mathbb{R}^n$ and $A \in \mathcal{S}$. Then, $\mathrm{Cl}(A)$ contains a zero-dimensional stratum, i.e. a point.*

Proof. By induction on $k = \dim A$.

If $\dim A = 0$, the stratum is a (closed) point; and as such its closure contains a zero-dimensional stratum.

Suppose that $\dim A = k+1$ and that the statement is known for all strata A' of dimension $\dim A' \leq k$. Because A is homeomorphic to $(0,1)^d$ with $d > 0$, we know that $\mathrm{Cl}(A) = A \sqcup \delta A$ with non-empty border A. Furthermore, $\delta A = A_1 \sqcup \ldots \sqcup A_p$ with $\dim A_i < \dim A$ for all $1 \leq i \leq p$ by definition of Nash stratifications. Choose A_* as one of the A_i. By induction hypothesis, we have that $\mathrm{Cl}(A_*)$ contains a zero-dimensional stratum. But $\mathrm{Cl}(A_*) \subseteq \mathrm{Cl}(A)$, so that also $\mathrm{Cl}(A)$ contains one. □

Lemma 5.63. *If φ is a modal formula such that $\mathcal{SA}_n \not\models \varphi$ then $\mathcal{SA}(\mathbf{D}^n) \not\models \varphi$.*

Proof. By corollary 5.48, we have that

$$\mathrm{Var}(\mathcal{SA}_n) = \mathrm{Var}\{I_{\mathcal{S}} \mid \mathcal{S} \text{ a Nash stratification of } \mathbb{R}^n\}.$$

Therefore, there exists a Nash stratification $\mathcal{S}$ of $\mathbb{R}^n$ such that $I_{\mathcal{S}} \not\models \varphi$.

Let $P \subseteq \mathcal{S}$ be the subset of zero-dimensional cells, i.e. the points of the stratification. Note that for $x \in P$ the Euclidean norm $\|x\|$ is well-defined as x is a singleton set. Then define the real number

$$R = \max\{\|x\| \mid x \in P\}.$$

The number R determines the maximal distance of the points of $\mathcal{S}$ from the origin. Since every cell of $\mathcal{S}$ has at least one point in its border, the number R intuitively captures how far from the origin the 'interesting' structure of $\mathcal{S}$ extends.

We take the n-dimensional ball with radius $R+1$, which is a closed set, and consider the subspace stratification, which is indeed a stratification by proposition 3.20. Let $\mathbf{B} = \mathbf{D}^n_{R+1} = \{\vec{x} \in \mathbb{R}^n \mid \|\vec{x}\| \leq R+1\}$. Then define

$$\mathcal{S}_{\mathbf{B}} = \{A \cap \mathbf{B} \mid A \in \mathcal{S}\} - \{\emptyset\},$$

a stratification of $\mathbf{B}$. We claim that $I_{\mathcal{S}} \cong I_{\mathcal{S}_{\mathbf{B}}}$ via the map $\iota : \mathcal{S} \to \mathcal{S}_{\mathbf{B}}, A \mapsto A \cap \mathbf{B}$.

First, we have to show that this map is well-defined, i.e. that $A \cap \mathbf{B} \neq \emptyset$ for any $A \in \mathcal{S}$. Assume (towards contradiction) that $A \cap \mathbf{B} = \emptyset$ for some $A \in \mathcal{S}$. Then, $A \subseteq \mathbf{B}^c$ must lie completely outside of $\mathbf{B}$. We know from proposition 5.62 that $\mathrm{Cl}(A)$ contains a zero-dimensional stratum $x \in P$. Then, for every $\epsilon > 0$, the ϵ-ball $B^n_\epsilon(x)$ around x must intersect non-trivially with A. Choose $\epsilon = 1/2$. But by definition of the ball $\mathbf{B}$, we have that $B^n_{1/2}(x) \subseteq \mathbf{B}$. This is a contradiction to $A \subseteq \mathbf{B}^c$. In conclusion, $A \cap \mathbf{B} \neq \emptyset$ and the map ι is well-defined.

By definition of $\mathcal{S}_B$, this map is surjective, so that it suffices to show that ι is an order-embedding, i.e. that for all strata $A, B \in \mathcal{S}$, it is the case that $A \leq B \Leftrightarrow A \cap \mathbf{B} \leq B \cap \mathbf{B}$, were the order is given by $A \leq B \Leftrightarrow A \subseteq \mathrm{Cl}(B)$.

$[\Rightarrow]$ Suppose that $A \subseteq \mathrm{Cl}(B)$ for $A, B \in \mathcal{S}$. Then, also $A \cap \mathbf{B} \subseteq \mathrm{Cl}(B) \cap \mathbf{B} = \mathrm{Cl}(B \cap \mathbf{B})$, where the last equality is due to $\mathbf{B}$ being closed. In conclusion, $\iota(A) \leq \iota(B)$.

$[\Leftarrow]$ Suppose that $A \cap \mathbf{B} \subseteq \mathrm{Cl}(B \cap \mathbf{B}) = \mathrm{Cl}(B) \cap \mathbf{B}$. Because $A \cap \mathbf{B} \neq \emptyset$, there exists $x \in A \cap \mathrm{Cl}(B)$, witnessing that $A \cap \mathrm{Cl}(B) \neq \emptyset$. Since $\mathcal{S}$ is a stratification, we obtain that $A \subseteq \mathrm{Cl}(B)$, showing that indeed $A \leq B$.

Therefore, since $I_{\mathcal{S}} \not\models \varphi$, we have that also $I_{\mathcal{S}_{\mathbf{B}}} \not\models \varphi$. We further know from proposition 5.55 that $\mathcal{SA}(\mathbf{B}) \leq \mathcal{MC}(\mathbf{B})$ is a subalgebra. Note that $\mathcal{SA}(\mathbf{B})$ has the stratification property (because $\mathbb{R}^n$ has the stratification property and we can turn any stratification of $\mathbb{R}^n$ into one of $\mathbf{B}$ by means of proposition 3.20). Then, it follows by theorem 3.38 that indeed $\mathcal{SA}(\mathbf{B}) \not\models \varphi$.

The final step to conclude that $\mathcal{SA}(\mathbf{D}^n) \not\models \varphi$ is easy since $\mathbf{D}^n$ and $\mathbf{B} = \mathbf{D}^n_{R+1}$ are semialgebraically homeomorphic via the map

$$h : \mathbf{D}^n \to \mathbf{D}^n_{R+1},\ \vec{x} \mapsto (R+1)\vec{x}. \qquad \square$$

Now, we are ready to prove the main result of this section:

Theorem 5.64. $\mathrm{Var}(\mathcal{SA}_n) = \mathrm{Var}(\mathcal{MPS}(\Delta_n))$.

Proof. We use again that varieties are equational classes and prove that $\mathcal{SA}_n$ and $\mathcal{MPS}(\Delta_n)$ validate the same modal formulas.

Let φ be a modal formula such that $\mathcal{SA}_n \not\models \varphi$. By lemma 5.63, we know that $\mathcal{SA}(\mathbf{D}^n) \not\models \varphi$. We further know from proposition 5.61 that Δ_n is semialgebraically homeomorphic to Δ_n. But Δ_n is a polyhedron, so that we obtain using theorem 5.59 that $\mathrm{Var}(\mathcal{SA}(\mathbf{D}^n)) = \mathrm{Var}(\mathcal{MPS}(\Delta_n))$. Therefore, φ must also be refuted on $\mathcal{MPS}(\Delta_n)$, which concludes this direction.

For the converse direction, let φ be a modal formula such that $\mathcal{MPS}(\Delta_n) \not\models \varphi$. By theorem 2.26 there exists a triangulation Σ of Δ_n such that $\mathcal{X}(\Sigma) \not\models \varphi$. Let $\mathcal{S} = \{\mathrm{relint}(s) \mid s \in \Sigma\}$ be the stratification of Δ_n by the relative interiors of the simplices of Σ. We know from theorem 2.22 that $(\mathcal{X}(\Sigma), \subseteq) \cong (\mathcal{S}, \leq)$ are isomorphic as posets with the order $s \leq t \Leftrightarrow s \subseteq \mathrm{Cl}(t)$. In conclusion, $I_{\mathcal{S}} \not\models \varphi$.

We know from proposition 5.61 that there exists a semialgebraic homeomorphism $h : \Delta_n \to \mathbf{D}^n$ (the inverse of the homeomorphism given there). Consider the family

$$\mathcal{T} = \{h(A) \mid A \in \mathcal{S}\}.$$

Relative interiors of simplices are semialgebraic sets by corollary 5.38. Due to corollary 5.43, their images under a semialgebraic map are again semialgebraic. Furthermore, because h is a homeomorphism, $\mathcal{T}$ will be a stratification of $\mathbf{D}^n$ such that $I_{\mathcal{S}} \cong I_{\mathcal{T}}$ as posets. In conclusion, we have that also $I_{\mathcal{T}} \not\models \varphi$.

In the next step, we will turn $\mathcal{T}$ into a stratification of $\mathbb{R}^n$. First, consider the map

$$m : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\},\ \vec{x} \mapsto \vec{x}/\|\vec{x}\|^2.$$

Intuitively, m mirrors the space $\mathbb{R}^n - \{0\}$ without the origin on $\partial\mathbf{D}^n$. It is easy to see that m is continuous as concatenation of continuous maps. Furthermore, observe that $m \circ m = 1_{\mathbb{R}^n - \{0\}}$ is self-inverse and therefore a homeomorphism. Also, with the same argumentation as in the proof of proposition 5.61, m is a semialgebraic map and thus a semialgebraic homeomorphism.

Second, consider the surjective map

$$p : \mathbb{R}^n \to \mathbf{D}^n, \vec{x} \mapsto \begin{cases} \vec{x}, & \text{if } \|\vec{x}\| \leq 1, \\ m(\vec{x}), & \text{if } \|\vec{x}\| \geq 1. \end{cases}$$

That p is continuous follows from the well-known 'pasting lemma': If $A, B \subseteq X$ are closed sets such that $X = A \cup B$, and $f : X \to Y$ is a function such that $f|_A$ and $f|_B$ are continuous, then f is continuous. Here, set $A = \mathbf{D}^n \subseteq \mathbb{R}^n$ and $B = \mathbb{R}^n - \mathrm{Int}(\mathbf{D}^n)$. Then, p is continuous because $p|_A = 1_A$ and $p|_B = m|_B$ are continuous.

We claim that p is an open map. To see this, let $U \subseteq \mathbb{R}^n$ be open. We have that $p(U) = p((U \cap \mathbf{D}^n) \cup (U \cap (\mathbf{D}^n)^c)) = p(U \cap \mathbf{D}^n) \cup p(U \cap (\mathbf{D}^n)^c) = (U \cap \mathbf{D}^n) \cup m(U \cap (\mathbf{D}^n)^c)$. Because U is open, we have that $U \cap \mathbf{D}^n$ is open in $\mathbf{D}^n$. Also, $U \cap (\mathbf{D}^n)^c$ is open in $(\mathbf{D}^n)^c$; and because $(\mathbf{D}^n)^c$ is open in $\mathbb{R}^n - \{0\}$, we have that $U \cap (\mathbf{D}^n)^c$ is also open in $\mathbb{R}^n - \{0\}$. Because $m : \mathbb{R}^n - \{0\} \to \mathbb{R}^n - \{0\}$ is a homeomorphism, $m(U \cap (\mathbf{D}^n)^c)$ is open in $\mathbb{R}^n - \{0\}$. If follows from the definition of m that $m(U \cap (\mathbf{D}^n)^c)$ is contained in the interior of $\mathbf{D}^n - \{0\}$, so that we know that $m(U \cap (\mathbf{D}^n)^c)$ is open in $\mathbf{D} - \{0\}$. Finally, because $\mathbf{D}^n - \{0\}$ is open in $\mathbf{D}^n$, we have that $m(U \cap (\mathbf{D}^n)^c)$ in open in $\mathbf{D}^n$. In summary, both $U \cap \mathbf{D}^n$ and $m(U \cap (\mathbf{D}^n)^c)$ are open in $\mathbf{D}^n$, and consequently also their union, which is $p(U)$.

Let $\pi : \mathbf{D}^n \to I_{\mathcal{T}}$ be the characteristic map of $\mathcal{T}$. Because $\mathcal{T}$ is a stratification, we know that π is an interior surjection. It follows from the previous observations that the composition $\pi \circ p : \mathbb{R}^n \to I_{\mathcal{T}}$ is also a surjective interior map.

Therefore, $\pi \circ p$ defines a stratification of $\mathbb{R}^n$ with decomposition space $I_{\mathcal{T}}$, about which we know that $I_{\mathcal{T}} \not\models \varphi$. Using theorem 3.38, we obtain that $\mathcal{SA}_n \not\models \varphi$.

In summary, we have shown that $\mathcal{SA}_n$ and $\mathcal{MPS}(\Delta_n)$ refute exactly the same modal formulas, and we conclude that $\mathrm{Var}(\mathcal{SA}_n) = \mathrm{Var}(\mathcal{MPS}(\Delta_n))$. □

The logical presentation of theorem 5.64 is the following corollary:

Corollary 5.65. *The semialgebraic logic of $\mathbb{R}^n$ equals the polyhedral logic of the n-simplex.*

In summary, we have seen in the last two sections that, first, the semialgebraic logic of any compact semialgebraic set is polyhedral, and second, that the semialgebraic logic of the Euclidean spaces $\mathbb{R}^n$ is polyhedral. This gives rise to the following question, which we are leaving for future work: Is there a closed semialgebraic set whose semialgebraic logic is not polyhedral?

6 Polyhedral Logic and Homotopy

Abstract

We show that the modal logic of all n-dimensional contractible polyhedra is $\mathbf{Grz} \oplus \mathbf{bd}_{n+1}$ and that the modal logic of all contractible polyhedra is $\mathbf{Grz}$.
In order to enhance the expressivity of logic with regard to homotopy, we propose 'Homotopy modal logic', an application of reachability logic to function and loop spaces.

Balla, *Dynamism of a Dog on a Leash* (1912)

6.1 Logic of contractible polyhedra

In [Ada+23], the polyhedral logic of convex polyhedra was studied. Every convex polyhedron is, in particular, a contractible topological space, i.e. is homotopically trivial. Therefore, it is a natural continuation to study the polyhedral logic of all contractible polyhedra. In this section, we will prove that the polyhedral logic of n-dimensional contractible polyhedra is $\mathbf{Grz} \oplus \mathbf{bd}_{n+1}$, and that the polyhedral logic of all contractible polyhedra is $\mathbf{Grz}$.

The methods we will use for this were developed by Robert Evert Stong in [Sto66] and Michael Campbell McCord in [McC66] in the 1960s. The bundled presentation we have worked with is that of the book 'Algebraic topology of finite topological spaces and applications' ([Bar11]), which is based on Jonathan Barmak's PhD thesis.

Stong studied homotopy theory of finite spaces via the Alexandrov equivalence from theorem 2.2 and showed that every homotopy equivalence between finite spaces can be expressed as a sequence of adding and removing so-called 'linear points', which gives homotopy theory of finite spaces a combinatorial character. This raises the question whether this combinatorial approach to homotopy could also be applied to study more familiar infinite spaces. However, it can be shown that no infinite Hausdorff space is homotopy equivalent to a finite space, so that the naïve application to consider finite homotopy equivalent spaces is out of the question.

A more elaborate approach was developed by McCord in [McC66]. He considered the functors $\mathcal{X} : \mathbf{FinSimp} \longleftrightarrow \mathbf{FinPosets} : \mathcal{K}$ between the categories of finite abstract simplicial complexes and finite posets, and studied how they behave with regard to homotopy types of finite spaces and finite simplicial complexes. He showed that homotopy of finite spaces and of simplicial complexes do not directly coincide, but instead that homotopy equivalences of finite spaces correspond to the notion of *strong homotopy equivalences* between simplicial complexes, and conversely that homotopy equivalences between simplicial complexes correspond to *weak homotopy equivalences* between finite spaces.

Weak homotopy equivalences are much harder to capture combinatorially then homotopy equivalences between finite spaces. Intuitively, this makes sense: It is possible to study homotopy types between simplicial complexes via weak homotopy equivalences between their face posets. But it is a difficult problem to determine whether two simplicial complexes are homotopy equivalent, so the corresponding notion for finite spaces should have the same complexity.

Barmak further investigated how the functors $\mathcal{K}$ and $\mathcal{X}$ behave with other notions from algebraic topology like e.g. Cartesian products and topological joins. Here, he showed that joins between simplicial complexes correspond to ordinal sums between their face posets, while Cartesian products are preserved and reflected as such.

In this section, we want to show that **Grz** is the modal logic of all contractible polyhedra; or equivalently, that the variety **GA** of Grz algebras is generated by algebras of the form $\mathcal{MPS}(P)$ for contractible polyhedra P. In principle, this would mean that we have to occupy ourselves with finite posets that are weakly contractible, i.e. weakly homotopy equivalent to a point. However, we will in fact show that **GA** is generated by *strongly* contractible polyhedra – which implies the result for all contractible polyhedra – so that we can resort to the simpler notion of homotopy equivalence between finite posets.

Our occupation with the modal logics of semialgebraic sets as presented in chapter 5 in the course of this master's thesis started from the insight that polyhedral logic is not very expressive with regard to homotopy types of polyhedra. However, we saw in section 5.2.3 that – at least for compacta – the logics of semialgebraic sets and polyhedral logic coincide. Therefore, the quest for a modal logic that is more expressive with respect to homotopy types continues. In the second half of this chapter, we will propose a new variant of reachability logic that could be a candidate for such a logic.

In figure 6.1, the conceptual setting of this chapter is visualized: Both finite posets and polyhedra give semantics for modal logics, with the relations specified in theorems 2.26 and 2.27.

In the following, we will compactly introduce the relavant notions from homotopy theory and present the central results from [Bar11]. For details on the standard definitions from homotopy theory, we refer to [Bre93].

Let $I = [0, 1]$ denote the standard interval with the inherited subspace topology from $\mathbb{R}$. A useful notion in homotopy theory are pairs of spaces (X, A) with $A \subseteq X$. Then, a map $f : (X, A) \to (Y, B)$ between such pairs is a continuous map $f : X \to Y$ such that $f(A) \subseteq B$.

Definition 6.1 (Homotopy)**.** Let X and Y be topological spaces and $A \subseteq X$. Two continuous maps $f, g : X \to Y$ are called *homotopic rel* A (relative to) if their exists a *homotopy* between them, which is a continuous map $h : X \times I \to Y$ such that

- $h(x, 0) = f(x)$ for all $x \in X$,
- $h(x, 1) = g(x)$ for all $x \in X$, and
- $h(a, t) = h(a, t')$ for all $a \in A$ and $t, t' \in I$

Then, one writes $f \simeq g$ rel A.

The maps f and g are called *homotopic* if they are homotopic rel $\emptyset$, for which one just writes $f \simeq g$.

It can be shown that $\simeq$ defines an equivalence relation in $[X, Y]$, the set of all continuous maps from X to Y. Next, homotopy equivalences are defined as a weaker form of homeomorphism, where a pair of maps has to be inverse to each other only up to homotopy:

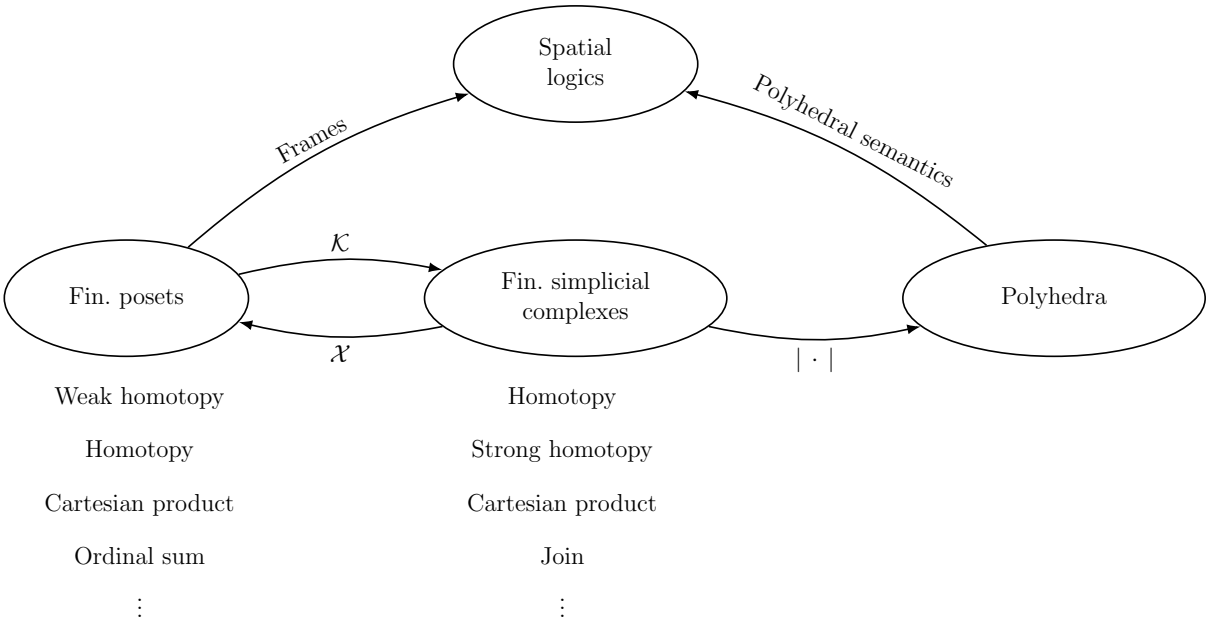

Figure 6.1: The conceptual landscape of the interplay between polyhedral logic and homotopy theory. The functor $\mathcal{K}$ and $\mathcal{X}$ establish correspondences between various notions from algebraic topology between finite posets and finite simplicial complexes. The geometric realizations of the latter are polyhedra, which interpret modal formulas in their algebra of polyhedral sets.

Definition 6.2 (Homotopy equivalence, homotopy type, contractible space)**.** A continuous map $f : X \to Y$ is called *homotopy equivalence* if there exists a map $g : Y \to X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$.

Two spaces X and Y have the same *homotopy type* if there exists a homotopy equivalence between them. A space X is *contractible* if it is homotopy equivalent to the space with one element $\{*\}$.

In order to define the notion of weak homotopy equivalences, we first need the notion of homotopy groups, whose definition we will sketch in the following. Let $\mathbf{S}^n = \{\vec{x} \in \mathbb{R}^{n+1} \mid \|\vec{x}\| = 1\}$ denote the n-dimensional spheres for $n \geq 0$. Fix for every sphere a canonical base point $* \in \mathbf{S}^n$.

Let X be a topological space and let $x \in X$ be a base point. Then, consider the set of all continuous maps $\mathbf{S}^n \to X$ such that $*$ is mapped to x_0:

$$[(\mathbf{S}^n, \{*\}), (X, \{x_0\})].$$

Then, the relation $f \sim g \Leftrightarrow f \simeq g \text{ rel } *$ defines an equivalence relation on this set. The underlying sets of the homotopy groups are defined as the quotient under this equivalence relation:

$$\pi_n(X, x_0) = [(\mathbf{S}^n, \{*\}), (X, \{x_0\})]/_\sim$$

If $n \geq 1$, the objects $\pi_n(X, x_0)$ can be equipped with a group operation, which yields the *homotopy groups*. The intuition behind this group operation is the clearest in the case $n = 1$.

Here, elements of $\pi_1(X, x_0)$ are equivalence classes of closed paths – loops – starting and ending at x_0. Two such loops can be concatenated to yield a composite loop. This construction can be generalized to higher dimensions $n > 1$.

If $n = 0$, however, we do not obtain a group structure on $\pi_n(X, x_0)$ in a similar way. Instead, the set $\pi_0(X)$ turns out to be isomorphic to the set of path components of X. Since this does not depend on the choice of the base point x_0, it is usually dropped from the notation in this case.

Most importantly, assigning homotopy groups to topological spaces is functorial. If $f : (X, x_0) \to (Y, y_0)$ is a continuous map, we obtain an induced group homomorphism $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ if $n \geq 1$ as well as a set-theoretic function $f_* : \pi_0(X) \to \pi_0(Y)$.

For details on the definition of the homotopy groups, see pages 127 to 132 in [Bre93]. For us, however, the use of homotopy groups lies solely in the following definition of weak homotopy equivalences:

Definition 6.3 (Weak homotopy equivalence, weak homotopy type, weakly contractible space)**.** A continuous map $f : X \to Y$ is called *weak homotopy equivalence* if for all $x \in X$ the induced maps $f_* : \pi_n(X, x) \to \pi_n(Y, f(x))$ are group isomorphisms for all $n \geq 1$ and bijective for $n = 0$.

Two spaces X and Y have the same *weak homotopy type* if there exists a weak homotopy equivalence between them. A space X is *weakly contractible* if it is weakly homotopy equivalent to the space with one element $\{*\}$.

The reason why it suffices in most applications in topology to only consider homotopy equivalences lies in the following theorem, with the second part due to Henry Whitehead in [Whi49]:

Theorem 6.4.

1. *Every homotopy equivalence is a weak homotopy equivalence.*
2. *Every weak homotopy equivalence between CW-complexes is a homotopy equivalence. (the Whitehead Theorem)*

The Whitehead Theorem is the reason why the category of CW-complexes constitutes a natural setting for a categorical approach to homotopy theory. For finite spaces, however, the Whitehead Theorem can fail dramatically. An example is given in figure 6.2.

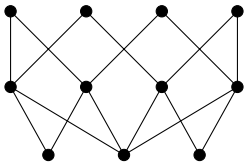

Figure 6.2: The 'Wallet' as seen on page 54 of [Bar11]. This finite topological space is weakly contractible but fails to be contractible.

A topological space X satisfies the T_0 separation axiom – and is then called *T_0-space* – if for every pair of distinct points $x \neq y$, there exists an open set $U \subseteq T$ containing exactly one of them. Using Alexandrov equivalence from theorem 2.2, it is easy to see that a finite space is a T_0-space if and only if it is equivalent to a partially ordered set.

Proposition 6.5. *Let X be a finite space. Then, X is a T_0-space if and only if $P(X)$ is a poset.*

Proof. First, note that every finite topological space is an Alexandrov space because finite intersections of open sets are always open.
$\boxed{\Rightarrow}$ Suppose that X satisfies T_0. Let $x, y \in X$ be points such that $x \leq y$ and $y \leq x$. Assume (towards contradiction) that $x \neq y$. Then, because X is T_0, there exists an open set U such that either $x \in U$ and $y \notin U$ holds, or $y \in U$ and $x \notin U$. But open sets correspond to upsets via Alexandrov duality so that both cases contradict $x \leq y$ and $y \leq x$. In conclusion, $x = y$ and $P(X)$ is a poset.
$\boxed{\Leftarrow}$ Let $x \neq y$ be two distinct points of X. Because $P(X)$ is a partial order by assumption, either $x \not\leq y$ or $y \not\leq x$ must be the case. Assume w.l.o.g. that $x \not\leq y$. Then, $\uparrow x$ is an open set in the Alexandrov topology containing x but not y. □

When studying homotopy types of finite spaces, it usually suffices to consider T_0-spaces because of the following result, which implies that every finite topological space has the homotopy type of a finite T_0-space.

Proposition 6.6. *Let X be a finite space and $X/_\sim$ the quotient space under the equivalence relation $x \sim y \Leftrightarrow (x \leq y \;\&\; y \leq x)$. Then, the quotient map $X \to X/_\sim$ is a homotopy equivalence.*

Proof. Proposition 1.3.1 in [Bar11]. □

In the following, we will compactly present Stong's beautiful characterization of the homotopy types of finite spaces. The most important notion here is that of 'linear' and 'colinear' points of a space. Barmak calls these 'up beat points' and 'down beat points', respectively.

Definition 6.7 (linear points, colinear points, minimal space)**.** Let X be a finite T_0-space. A point $x \in X$ is called ...
... *linear* if it has exactly one immediate successor (then $\Uparrow x$ has a minimum) and
... *colinear* if it has exactly one immediate predecessor (then $\Downarrow x$ has a maximum).
The space X is called *minimal* if it has neither linear nor colinear points.

For example, consider the wallet from figure 6.2. It lacks any linear or colinear points and is therefore a minimal space.

The importance of linear and colinear points comes from the fact that deleting (and therfore also adding) them does not change the homotopy type of any finite T_0-space:

Proposition 6.8. *Let X be a finite T_0-space and $x \in X$ a linear (or colinear) point. By definition, x has a unique successor (or predecessor) y. Then, the map $r : X \to X - \{x\}$ defined by $r(x) = y$ (and the identity else) is a homotopy equivalence.*

Proof. This follows from proposition 1.3.4 in [Bar11]. □

Given a finite T_0-space X, we can always find a minimal space that is homotopy equivalent to X by applying proposition 6.8 iteratively until the space does not have linear or colinear points anymore. Since X is finite, this procedure has to terminate. The resulting minimal space is called a *core* of X and denoted as $C(X)$.

Theorem 6.9. *Every homotopy equivalence between minimal finite spaces is a homeomorphism.*

Proof. This is the first part of corollary 1.3.7 in [Bar11]. □

A direct consequence of theorem 6.9 is that the core of a finite T_0-space is unique up to homeomorphism. Indeed, if X has two cores C_1 and C_2, then both are constructed from a sequence of homotopy equivalences, whose concatenation yields a homotopy equivalence between C_1 and C_2. Therefore, it is justified to speak of *the* core of X.

Another consequence of theorem 6.9 is the following corollary, which provides a complete classification of homotopy types of finite spaces:

Corollary 6.10. *Two finite T_0 spaces are homotopy equivalent if and only if they have homeomorphic cores.*

In particular, we can see now that the wallet in figure 6.2 is not contractible. Since it is a minimal space, it is its own core. If it had the same homotopy type as a single point (which is also a minimal space), they would have to be homeomorphic by corollary 6.10, which is clearly not the case.

Applying the notion of linear and colinear points to modal spaces, we obtain the following result:

Proposition 6.11. *Let A be a non-trivial subdirectly irreducible finite Grz algebra. Then, the dual space A_* seen as a poset is contractible.*

Proof. We know from the duality dictionary 2.1 that A_* has a root.

Now, consider two cases: (1) The root has no successor. Then the space is already a single point. (2) The root has an immediate successor y. Then, y is colinear because it has a unique predecessor. Therefore, deleting y is a homotopy equivalence by 6.8. Continue until all successors of the root are deleted.

Therefore, the core of A_* is a single point, which implies via corollary 6.10 that A_* is contractible (in the Alexandrov topology). □

Next, we present McCord's results on the relationship between finite T_0-spaces and simplicial complexes. Given a finite T_0-space X, there exists the so-called *McCord map* $\mu_X : |\mathcal{K}(X)| \to X$, which turns out to be a weak homotopy equivalence. Recall that the elements of $\mathcal{K}(X)$ are chains in X. For $C \in \mathcal{K}(X)$, the image under the McCord map is then defined as the minimum of C.

The fact that the McCord map is a weak homotopy equivalence is proven in theorem 1.4.6 in [Bar11]. The following theorem is a consequence of that result:

Theorem 6.12.

1. *Let X and Y be finite T_0-spaces. Then, $X \simeq_w Y$ are weakly homotopy equivalent if and only if $|\mathcal{K}(X)| \simeq |\mathcal{K}(Y)|$ are homotopy equivalent.*

2. *Let K and L be finite simplicial complexes. Then, $|K| \simeq |L|$ are homotopy equivalent if and only if $\mathcal{X}(K) \simeq_w \mathcal{X}(L)$ are weakly homotopy equivalent.*

Proof. This is corollary 1.4.18 in [Bar11]. □

Using the first part of theorem 6.12, the way to show that the wallet from 6.2 is weakly contractible would be to compute the polyhedron $|\mathcal{K}(X)|$ and show that it is contractible.

Our main result of this section is the following:

Theorem 6.13. *Let $\mathcal{C}_n$ denote the class of all contractible n-dimensional polyhedra. Their generated polyhedral variety is axiomatized by:*

$$\mathrm{Var}\{\mathcal{MPS}(P) \mid P \in \mathcal{C}_n\} = \mathbf{GA} \oplus \mathbf{bd}_{n+1}.$$

Let $\mathcal{C}$ denote the class of all polyhedra. Then:

$$\mathrm{Var}\{\mathcal{MPS}(P) \mid P \in \mathcal{C}\} = \mathbf{GA}.$$

Proof. We know from theorem 2.26 that for any polyhedron P:

$$\mathrm{Var}(\mathcal{MPS}(P)) = \mathrm{Var}(\{\mathcal{X}(\Delta) \mid \Delta \text{ is a triangulation of } P\}).$$

We start by showing soundness, i.e. that $\mathcal{MPS}(P) \in \mathbf{GA} \oplus \mathbf{bd}_{n+1}$ for every $P \in \mathcal{C}_n$. Surely, $\mathcal{MPS}(P) \in \mathbf{GA}$ by theorem 2.25. Therefore, using theorem 2.26, it suffices to show that $\mathcal{X}(\Delta)^* \models \mathbf{bd}_{n+1}$ for every triangulation Δ of P.

Recall that the dimension of P is defined as the dimension of its triangulations. Let Δ be a triangulation of P. Assume (towards contradiction) that $\mathcal{X}(\Delta)$ contained a chain C of length $n+1$. For geometric simplices $s, t \in C$ such that $s < t$, we have that s in a proper face of t, and therefore $\dim s < \dim t$ (recall that the dimension is defined by the number of vertices). Let m be the maximal element of the chain C. Then, we have that the dimension of m has to be at least $n+1$, which is a contradiction to the dimension of P being n. In conclusion, it follows that $\mathrm{height}(\mathcal{X}(\Delta)) \leq n$. Then, by proposition 5.13, we have that $\mathcal{X}(\Delta)^* \models \mathbf{bd}_{n+1}$, which proves soundness.

For completeness, we have to show that whenever $\mathbf{GA} \oplus \mathbf{bd}_{n+1} \not\models \varphi$ for a modal formula φ, there exists a polyhedron $P \in \mathcal{C}_n$ such that $\mathcal{MPS}(P) \not\models \varphi$. Then, equality of varieties follows from proposition 2.7. So, let φ be a modal formula such that $\mathbf{GA} \oplus \mathbf{bd}_{n+1} \not\models \varphi$. It follows from Maksimova ([Mak75]) and Segerberg ([Seg71]) that such a variety of closure algebras of bounded height is locally finite and thus has the finite model property. Then, the variety is generated by its finite subdirectly irreducible algebras, which by proposition 5.13 are those dual to finite rooted posets of height at most n. Therefore, there exists a finite rooted poset F of height at most n such that $F^* \not\models \varphi$. With the same argument as in the proof of proposition 5.17, we can attach a chain of $n - \mathrm{height}(F)$ new points under F such that this new frame has height exactly n, still refutes φ and is still rooted, i.e. the dual algebra is still subdirectly irreducible. Therefore, we can w.l.o.g. assume that F has height exactly n.

We showed in proposition 6.11 that F is contractible because its dual algebra is subdirectly irreducible. By the first part of theorem 6.4, it is in particular weakly contractible. Then, it follows from the first part of theorem 6.12 that $|\mathcal{K}(F)|$ is contractible. Note that by theorem 2.23, the simplicial complex $\mathcal{K}(F)$ can be realized in some Euclidean space, so that $|\mathcal{K}(F)|$ is a well-defined polyhedron. Also, it is well-known that $\mathrm{height}(\mathcal{K}(F)) = \mathrm{height}(F)$ so that we obtain that $\dim |\mathcal{K}(F)| = n$.

In summary, we have shown that $\mathcal{K}(F)$ is a triangulation of a polyhedron $|\mathcal{K}(F)| \in \mathcal{C}_n$. Recall that there is a surjective p-morphims

$$m_F : \mathcal{N}(F) = \mathcal{XK}(F) \twoheadrightarrow F,$$

which shows by duality that $F^* \in \mathbb{IS}(\mathcal{N}(F)^*) \subset \mathbb{HSP}(\mathcal{N}(F)^*)$. Therefore, we also have that $\mathcal{N}(F)^* \not\models \varphi$ by proposition 2.7. Since with $\mathcal{K}(F)$ a triangulation of the polyhedron $|\mathcal{K}(F)| \in \mathcal{C}_n$ refutes φ, it follows from theorem 2.26 that also $\mathcal{MPS}(|\mathcal{K}(F)|) \not\models \varphi$. This completes the main part of proof.

In the second part, it remains to show that $\mathrm{Var}\{\mathcal{MPS}(P) \mid P \in \mathcal{C}\} = \mathbf{GA}$. It again follows from theorem 2.25 that $\mathcal{MPS}(P) \in \mathbf{GA}$ for any $P \in \mathcal{C}$. So it is left to show that every modal formula φ refuted in $\mathbf{GA}$ is refuted in $\mathcal{MPS}(P)$ for some $P \in \mathcal{C}$. Suppose that $\mathbf{GA} \not\models \varphi$. By theorem 2.19, the variety of Grz algebras has the finite model property. Therefore, $A \not\models \varphi$ for a finite Grz algebra A. In particular, A_* has finite height, so that $A \in \mathbf{GA} \oplus \mathbf{bd}_{k+1}$ for some k. Then, we know from the first part that $\mathcal{MPS}(P) \not\models \varphi$ for some $P \in \mathcal{C}_n \subset \mathcal{C}$. Therefore, equality of the varieties $\mathrm{Var}\{\mathcal{MPS}(P) \mid P \in \mathcal{C}\} = \mathbf{GA}$ follows by proposition 2.7. □

Changing to the logical perspective, we obtain the following corollary.

Corollary 6.14. *The polyhedral logic of all contractible polyhedra is* **Grz**. *The polyhedral logic of all n-dimensional contractible polyhedra is* $\mathbf{Grz} \oplus \mathbf{bd}_{n+1}$.

Therefore, the polyhedral logics of the classes of all polyhedra and of all contractible polyhedra are the same – this means that the property to be contractible cannot be expressed by a modal formula. Note that all convex polyhedra are contractible. The polyhedral logic of all convex polyhedra was proven to be a proper extension of **Grz** in [Ada+23]. Therefore, there exist modal formulas that are valid on the class of convex polyhedra but refuted on some contractible polyhedron.

6.2 Towards a modal logic of homotopy

The last result in corollary 6.14 can be understood in the way that polyhedral logic is not very expressive with regard to homotopy equivalences. It is known ([Ada+22]) that **Grz** is the polyhedral logic of all polyhedra. Therefore, polyhedral logic is not capable of distinguishing the class of contractible polyhedra from the class of all polyhedra.

In this section, we propose to use the approach of extending the modal language by a 'reachability' operator to increase the expressivity of modal logic with regard to homotopies. In this way, we resort to the original concrete definition of homotopy, namely that there exists a homotopy between certain maps. The existence of such a homotopy can be understood as a higher level of reachability, which we will make precise in the following.

Reachability logic has its roots in spatial model checking (cf. [Bez+22]) and has been examined with frame semantics, topological semantics and polyhedral semantics. Furthermore, in [Bez+24], a complete proof system for polyhedral reachability logic was found.

In this section, we will focus on topological semantics and generalize reachability logic to function spaces. For an informal recap of topological semantics of modal logic, see page 2. In the following, we present the definition of the reachability operator γ from [Bez+24] and connect it to homotopy:

Spatial reachability and homotopy

Let (X, V) be a topological model consisting of a topological space and a valuation $V : \mathbf{Prop} \to \mathcal{P}(X)$. Extending the standard syntax of modal logic with the standard topological semantics, the modal logic of spatial reachability has an extra binary 'Until'-like operator γ with the semantics

$$X, V, x \models \gamma(\varphi, \psi) \;\Leftrightarrow\; \begin{array}{l}\text{there exists a continuous map } \pi : I \to X \text{ s.t.} \\ \pi(0) = x,\ \pi(1) \in [\![\psi]\!], \text{ and } \pi[\mathrm{int}(I)] \subseteq [\![\varphi]\!],\end{array}$$

where $I = [0,1] \subset \mathbb{R}$ denotes the standard interval. Intuitively, the operator γ expresses whether the space region $[\![\psi]\!]$ can be reached via a path through $[\![\varphi]\!]$ starting from x.

The connection to homotopy arises when we choose the topological space X to be of a specific form, namely a mapping space. Let S, X be topological spaces. The set $[S, X]$ can be equipped with the *compact-open topology*, which is generated by the subbasis $\{\mathcal{O}_{K,A} \mid K \subseteq S \text{ compact}, A \subseteq X \text{ open}\}$ with $\mathcal{O}_{K,A} = \{f : S \to X \mid f(K) \subseteq A\}$. This topology has the property that if S is locally compact, the exponential law holds:

$$\mathrm{Hom}(S \times I, X) \cong \mathrm{Hom}(I, [S, X]).$$

For more details on mapping spaces see e.g. the pages 437 – 440 of [Bre93]. When defining the homotopy groups, we considered the spaces $[\mathbf{S}^n, X]$, i.e. continuous maps from the n-dimensional spheres to X. Since the spheres are locally compact, we obtain from the exponential law that a homotopy between $f, g : \mathbf{S}^n \to X$ is nothing else than a path in the space $[\mathbf{S}^n, X]$ from f to g.

Note that in the definitions in section 6.1 we in fact considered a subspace of $[\mathbf{S}^n, X]$, namely the space of all maps that map the base point $* \in \mathbf{S}^n$ to the base point $x_0 \in X$. For the sake of simplicity, we drop this constraint in the present section and instead consider arbitrary mapping spaces $[S, X]$ for locally compact S. However, the case of $[(\mathbf{S}^n, *), (X, x_0)]$ is still nicely covered due a subtlety in the definition of homotopy groups: We assumed homotopies between $f, g \in [(\mathbf{S}^n, *), (X, x_0)]$ to be *rel* $*$. This is in fact equivalent to the assumption that there is a path between f and g in $[(\mathbf{S}^n, *), (X, x_0)]$ because the base points are constant throughout the homotopy.

So suppose that S is locally compact. Let $V : \mathbf{Prop} \to \mathcal{P}([S, X])$ be a valuation on the mapping space. Then, the definition of the reachability operator becomes:

$$[S, X], V, f \models \gamma(\varphi, \psi) \;\Leftrightarrow\; \begin{array}{l}\text{there exists a continuous map } \pi : I \to [S, X] \\ \text{s.t. } \pi(0) = f,\ \pi(1) \in [\![\psi]\!], \text{ and } \pi[\mathrm{int}(I)] \subseteq [\![\varphi]\!],\end{array}$$

which means, using the exponential law, that there exists a homotopy $h : S \times I \to X$ such that $h(\,\cdot\,, 0) = f$, $h(\,\cdot\,, 1) \in [\![\psi]\!]$, and $h(\,\cdot\,, t) \in [\![\varphi]\!]$ for all $0 < t < 1$.

Homotopy modal logic (HTML)

The main step from the above observation to what we call 'homotopy modal logic' is to actively include the space S in the truth clauses as an object *at which* a formula is evaluated – as opposed to just interpreting at points of the space $[S, X]$.

Let X and S be topological spaces, $V : \mathbf{Prop} \to \mathcal{P}([S, X])$ a valuation and $f \in [S, X]$ a continuous map. Then define the non-Boolean truth clauses as:

- $X, S, V, f \models p \quad \Leftrightarrow \quad f \in V(p)$,
- $X, S, V, f \models \gamma(\varphi, \psi) \quad \Leftrightarrow \quad$ there exists a homotopy $h : S \times I \to X$ such that
  - $h(\,\cdot\,, 0) = f$,
  - $X, S, V, h(\,\cdot\,, 1) \models \psi$, and
  - $X, S, V, h(\,\cdot\,, t) \models \varphi$ for all $0 < t < 1$.

When it comes to the definition of validity, we now have two notions of validity. The first one is the familiar notion that a formula is true at all points (note that in our case these are in fact maps) of the space. But now there is a second one, namely that a formula is valid in the

mapping spaces $[S, X]$ for all S – possibly restricted to a certain subclass, such as all spheres $\mathbf{S}^n$. If we want to be able to express e.g. that a space is homotopically trivial for all n, it is important to have a notion of validity of the second kind.

$$\begin{aligned} X, S \models \varphi \quad &\Leftrightarrow \quad X, S, V, f \models \varphi \text{ for all } f \in [S, X] \text{ and } V : \mathbf{Prop} \to \mathcal{P}([S, X]). \\ X \models_{\mathcal{C}} \varphi \quad &\Leftrightarrow \quad X, S \models \varphi \text{ for all } S \in \mathcal{C}, \end{aligned}$$

for some class of spaces $\mathcal{C}$.

There is a natural subclass of valuations, namely those that come from valuations on X. Let $V : \mathbf{Prop} \to \mathcal{P}(X)$ be a valuation, then we can define $V^S : \mathbf{Prop} \to \mathcal{P}([S, X])$ by setting $V^S(p) = \{f : S \to X \mid f(S) \subseteq V(p)\}$. This would mean that valuations actually specify a spatial region of X (as opposed to abstractly a region of $[S, X]$). Then, maps $S \to X$ satisfy propositional letters when they map into that spatial region.

Example 6.15. As a trivial example, let the space S be a point $\{*\}$. Then $[S, X] \cong X$ in the compact-open topology, and homotopies become just paths. Therefore, homotopy modal logic collapses back to reachability logic in this case.

Example 6.16. Choose the space S to be the one-dimensional sphere $\mathbf{S}^1$. We want to express the property that any pair of maps $\mathbf{S}^1 \to X$ are homotopic, i.e. that the space X is simply connected. With a global modality $\exists$, suppose this formula is valid:

$$X, \mathbf{S}^1 \models \exists p \to \gamma(\top, p).$$

This formula expresses that whenever there is some loop in X satisfying p, then every other loop in X can reach it via a homotopy. Given loops $f, g : \mathbf{S}^1 \to X$, choose a valuation with $V(p) = \{g\}$. Then, we have that $X, \mathbf{S}^1, V, f \models \gamma(\top, p)$, implying that g is reachable from f via a homotopy. Conversely, if the formula is not valid, there exist maps $f, g : \mathbf{S}^1 \to X$ such that g is not reachable from f via a homotopy.

Truth-preserving maps

In this last section, we will meet a class of truth-preserving maps for homotopy modal logic. We always have the property that the composition of a homotopy with a continuous map yields a homotopy. In the other direction, this is not guaranteed, i.e. we cannot always pull homotopies back via continuous maps. This leads to the following class of maps.

A map $f : X \to Y$ has the *homotopy lifting property (HLP)* w.r.t. the space S if the following commutative diagram can always be completed as indicated:

$$\begin{array}{ccc} S \times \{0\} & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ S \times I & \longrightarrow & Y \end{array}$$

The map f is further called *Hurewicz fibration* if it has the HLP for all spaces S.

Proposition 6.17. *Let S, X and Y be topological spaces, and $V : \mathbf{Prop} \to \mathcal{P}([S, X])$ and $W : \mathbf{Prop} \to \mathcal{P}([S, Y])$ valuations. Let $f : X \to Y$ be a continuous map with the HLP w.r.t. S such that for all $p \in \mathbf{Prop}$ it holds that $\pi \in V(p)$ if and only if $f \circ \pi \in W(p)$. Let $\pi : S \to X$. Then:*

$$X, S, V, \pi \models \varphi \quad \Leftrightarrow \quad Y, S, W, f \circ \pi \models \varphi.$$

Proof. By induction on the complexity of φ.

- The cases $\varphi = p \in \mathbf{Prop}$ as well as the Boolean cases are immediate.
- Suppose $\varphi = \gamma(\psi, \chi)$ and $X, S, V, \pi \models \gamma(\psi, \chi)$. By definition, there exists a homotopy $h : S \times I \to X$ such that
  - $h(\,\cdot\,, 0) = \pi$,
  - $X, S, V, h(\,\cdot\,, 1) \models \chi$, and
  - $X, S, V, h(\,\cdot\,, t) \models \psi$ for all $0 < t < 1$.

  Then, $f \circ h : S \times I \to Y$ is a homotopy such that $f \circ h(\,\cdot\,, 0) = f \circ \pi$. Furthermore, we know from the induction hypothesis that $Y, S, W, f \circ h(\,\cdot\,, 1) \models \chi$, and $Y, S, W, f \circ h(\,\cdot\,, t) \models \psi$ for all $0 < t < 1$. This shows that $Y, S, W, f \circ \pi \models \gamma(\psi, \chi)$.
- Suppose $Y, S, W, f \circ \pi \models \gamma(\psi, \chi)$. By definition, there exists a homotopy $h : S \times I \to Y$ such that
  - $h(\,\cdot\,, 0) = f \circ \pi$,
  - $Y, S, W, h(\,\cdot\,, 1) \models \chi$, and
  - $Y, S, h(\,\cdot\,, t) \models \psi$ for all $0 < t < 1$.

  Because f has the HLP, there exists a homotopy $h' : S \times I \to X$ such that $h = f \circ h'$ and $h'(\,\cdot\,, 0) = \pi$. Therefore, we have that $Y, S, W, f \circ h'(\,\cdot\,, 1) \models \chi$ and $Y, S, W, f \circ h'(\,\cdot\,, t) \models \psi$ for all $0 < t < 1$. Using the induction hypothesis, we get that $X, S, V, h'(\,\cdot\,, 1) \models \chi$ and $X, S, V, h'(\,\cdot\,, t) \models \psi$ for all $0 < t < 1$. Since also $h'(\,\cdot\,, 0) = \pi$, this shows that $X, S, V, \pi \models \gamma(\varphi, \psi)$. □

However, it is not clear how this notion of truth-preserving map lifts to validity. For example, if $f : X \to Y$ is a surjective Hurewicz fibration, we cannot conclude $Y \models \varphi$ from $X \models \varphi$ because the induced map $f \circ _ : [S, X] \to [S, Y]$ will almost never be surjective.

We leave further working out this approach for future work.

7 Conclusion

7.1 Review of this thesis

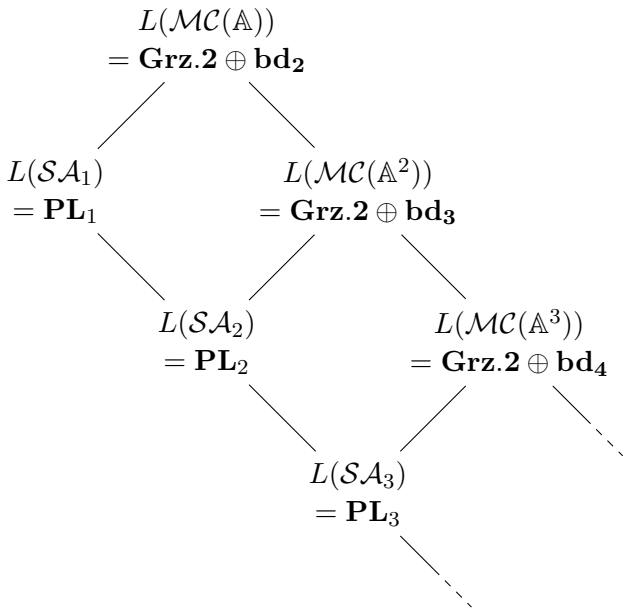

Figure 7.1: The lattice of modal logics of Zariski-constructible sets $L(\mathcal{MC}(\mathbb{A}^n))$ and semialgebraic sets $L(\mathcal{SA}_n)$ establish two new Euclidean hierarchies.

The contributions of this thesis can be grouped into different lines. The first line was to find non-trivial Euclidean hierarchies on tame topologies on the Euclidean spaces $\mathbb{R}^n$. Concretely, we looked in chapter 5 at two such structures: The modal logic of constructible sets of the Zariski topology were studied in section 5.1, while the modal logic of semialgebraic sets was subject of section 5.2. In both cases, we showed that there exist Euclidean hierarchies, i.e. the modal logics in different dimensions differ. This is in contrast to the classic McKinsey-Tarski theorem, which showed that the modal logics of $\mathbb{R}^n$ with the standard topology do not exhibit hierarchies.

The lattice of the logics encountered in this line is visualized in figure 7.1. The inclusions $L(\mathcal{SA}_{n+1}) \subset L(\mathcal{SA}_n)$ follow either directly from the axiomatization of the logics $\mathbf{PL}_n$ given in [Ada+23] or since the projections $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ are surjective semialgebraic interior maps and thus can be used to show that $\mathcal{SA}_n \leq \mathcal{SA}_{n+1}$ is (isomorphic to) a subalgebra. Analogously, the inclusions $L(\mathcal{MC}(\mathbb{A}^{n+1})) \subset L(\mathcal{MC}(\mathbb{A}^n))$ follow directly from the axiomatization as $\mathbf{Grz.2} \oplus \mathbf{bd}_n$ or since the projections $\pi : \mathbb{A}^{n+1} \to \mathbb{A}^n$ are surjective interior maps in the Zariski topology (which is non-trivial to show). We know from corollary 5.39 that $\mathcal{MC}(\mathbb{A}^n) \leq \mathcal{SA}_n$ are subalgebras such that we also get the inclusion $L(\mathcal{SA}_n) \subset L(\mathcal{MC}(\mathbb{A}^n))$ of logics.

The most important tool we used to determine those modal logics was the theory of stratifications developed in chapter 3 and specifically the theorems 3.34 and 3.38, which expressed the varieties generated by closure algebras of tame structures in terms of the finite decomposition

spaces of their stratifications. This was a generalization of theorem 2.26, which stated that the polyhedral logic of a polyhedron was the logic of the face posets of its triangulations.

Recall from page 1.2 that two problems concerning tame topologies were, first, what exact structures they consisted of and, second, to find criteria to decide whether they are in fact tame. The approach taken in this thesis was clear in that regard: We understood tame structures as a locally finite subalgebra $A \leq \mathcal{MC}(X)$, which then by the theorems 3.36 has the stratification property.

The second thematic line in this thesis was to examine the closure algebra of semialgebraic subsets of a compact semialgebraic set. We showed in theorem 5.59 that there exists a polyhedron P, which is unique up to PL homeomorphism, whose polyhedral logic is the same as the semialgebraic logic of the compactum. Given that Grothendieck considered semialgebraic geometry as the most natural notion of a tame topology, this result can be understood in the way that polyhedral logic does not restrict to polyhedra but is more universal: It is the logic of tame topologies.

A minor third line in this thesis was present in chapter 4, where the result from [BBG03] that the closure algebras $\mathcal{CH}(\mathbb{R}^n)$ of n-dimensional chequered sets exhibit a Euclidean hierarchy was approached in an algebraic way. As a central part of obtaining this result algebraically, we examined the tensor sum of modal algebras in section 4.1. Specifically, we showed that a variety generated by a tensor sum of finite modal algebras is generated by all tensor sums of subdirectly irreducible algebras of the summands.

The fourth and last thematic line diverged from the others insofar as it did not address Euclidean hierarchies but connected polyhedral logic and homotopy theory. Results by Stong and McCord established a deep correspondence between the homotopy theory of polyhedra and finite posets. Since polyhedral logic also can be seen as such a correspondence – by theorem 2.26, the logic of a polyhedron is the logic of its finite triangulations – the two approaches connect naturally to each other. However, it turned out that polyhedral logic is not very expressive regarding homotopy types: We used the existing theory to prove in section 6.1 that the polyhedral logic of the class of all n-dimensional contractible (i.e. homotopically trivial) polyhedra equals already $\mathbf{Grz} \oplus \mathbf{bd}_{n+1}$, which is the logic of *all* n-dimensional polyhedra. In particular, the class of contractible polyhedra is not definable in polyhedral logic.

To enhance the expressivity of modal logic regarding homotopies, we introduced 'Homotopy Modal Logic' in the last section 6.2, which extends the modal language by a binary operator 'γ' from reachability logic, allowing to express that some spatial region is reachable via paths in another spatial region. In contrast to standard reachability logic, however, we applied this logic to mapping spaces from some base space S into a topological space X to capture the notion of 'reachability via a homotopy' between maps $S \to X$.

7.2 Future work

This thesis explored new areas like the modal logics of algebraic and semialgebraic sets as well as connections between homotopy theory and polyhedral logic. Therefore, many follow-up questions arise. We end this thesis by presenting a list of some open questions that can be addressed in future work.

1. In section 5.1.3, we described a map $\Lambda_n : \mathbb{R}[n] \to \mathrm{NExt}(\mathbf{Grz} \oplus \mathbf{bd}_{n+1})$, which takes a polynomial P and maps it to the logic of the modal algebra $\mathcal{MC}(\mathcal{Z}(P))$. We saw specifically in proposition 5.30 that irreducibility of P is modally definable using this map.

Here, the question arises which other algebraic properties of polynomials the map Λ_n can capture. Also, we know that $\mathbb{R}[n]$ forms a ring. How do the ring operations translate via Λ_n, i.e. if we know $\Lambda_n(P)$ and $\Lambda_n(Q)$, what can we say about $\Lambda_n(P \cdot Q)$ and $\Lambda_n(P + Q)$?

2. A related question is: What can we say about the semialgebraic logic of $\mathcal{Z}(P)$? Certainly, $\mathcal{Z}(P)$ is closed for any polynomial P but might not be compact. We can define a semialgebraic analogue of the map Λ_n, which maps a polynomial P to the normal modal logic of the algebra $\mathcal{SA}(\mathcal{Z}(P))$. Is this logic always polyhedral? In case of e.g. the constant polynomial 0, whose zero set is $\mathbb{R}^n$, we saw in section 5.2.4 that the answer is 'yes': We obtained the polyhedral logic $\mathbf{PL}_n$.

3. In the second half on section 5.1.3, we addressed the generalization of the affine spaces to schemata. Specifically, we defined a map $\Lambda : \mathrm{Ob}(\mathbf{Rings}) \to \mathrm{Ob}(\mathbf{GA})$, which maps a commutative ring R to the modal algebra $\mathcal{MC}(\mathrm{spec}(R))$ of constructibles of its Zariski spectrum. Of course this map extends to a map from commutative rings to normal modal logics.

   - We saw specifically for $R = \mathbb{Z}$ that the dual space of the modal algebra $\Lambda(\mathbb{Z})$ has height 1. In the terminology of [Bez+17a], this means that the *modal Krull dimension* mdim of spec($\mathbb{Z}$) equals one. The Krull dimension of a commutative ring is defined as the Krull dimension kdim of its spectrum. Proposition 7.1 in [Bez+17a] stated that $\mathrm{kdim}X \leq \mathrm{mdim}X$ for any spectral space X. In case of spec($\mathbb{Z}$), we therefore have that the Krull dimension and the modal Krull dimension coincide. This gives rise to the question: What is an *algebraic* condition on a ring R to ensure that its Krull dimension equals its modal Krull dimension?
   - We pointed out in section 5.1.3 that when $f : R \to S$ is either an étale map or a smooth map between commutative rings, then the induced map $\Lambda(R) \to \Lambda(S)$ is a homomorphism of modal algebras. Further working out this approach might lead to insights what these types of maps capture logically and establish a bond between modal logic and algebraic geometry.

4. In section 6.2, we introduced homotopy modal logic. Since the application of modal reachability logic to homotopy theory is new, many questions arise, of which we are going to highlight two:

   - What properties of the homotopy types of topological spaces are expressible in homotopy modal logic? We saw in example 6.16 that simple connectedness is expressible if we add an existential quantifier.
   - We saw in proposition 6.17 that continuous maps with the homotopy lifting property are natural truth-preserving maps for homotopy modal logic. However, it is not clear how they lift to validity. Our hope is to connect the validity notion $X \models_{\mathbf{Top}} \varphi$ – i.e. that $X, S \models \varphi$ for all topological spaces S to Hurewicz fibrations. Working out this approach could be a promising next step.

   The ultimate goal would be to be able to capture the formulation of homotopy groups logically (see page 82 for the construction). Determining the higher homotopy groups of topological spaces is computationally very difficult – even for simplicial complexes. If we were able to capture homotopy groups logically, we could possibly use decision procedures of homotopy modal logic to obtain new algorithms.

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Appendix: Dimension for Polyhedra

Claim A.1. *The closure of a polyhedral set is a polyhedron.*

Proof. Since a cell is defined as the relative interior of a (geometric) simplex, the closure of a cell is a simplex. Note that closure distributes over finite unions. Therefore, the closure of a polyhedral set is a finite union of simplices and thus a polyhedron. □

The dimension of a simplicial complex Σ is defined as $\dim \Sigma = \max\{\dim \sigma \mid \sigma \in \Sigma\}$. The dimension of a polyhedron is defined as the dimension of any of its triangulations. The dimension of a polyhedral set is defined as the dimension of its closure.

Claim A.2. *For an n-cell S, we have that $\dim \delta S < \dim S$.*

Proof. By definition, the closure of S is a (geometric) n-simplex. Its boundary is the union of its non-maximal faces; and they all have dimension $\leq n-1$. □

Claim A.3. *Let A and B be polyhedral sets and $A \subseteq B$. Then $\dim A \leq \dim B$.*

Proof. We know from claim A.1 that $\overline{A}$ and $\overline{B}$ are polyhedra and that $\overline{A} \subseteq \overline{B}$. Choose a common triangulation Σ of $\overline{A}$ and $\overline{B}$. Then every face of $\Sigma|\overline{A}$ is also face of Σ, showing that $\dim A \leq \dim B$. □

Claim A.4. *Let A and B be polyhedral sets, then $\dim(A \cup B) = \max\{\dim A, \dim B\}$.*

Proof. We use the property that if Δ is a subcomplex of Σ and $\sigma \in \Sigma$ such that $\text{relint}(\sigma) \subseteq |\Delta|$, then $\sigma \in \Delta$. We know from claim A.1 that $\overline{A}$, $\overline{B}$ and $\overline{A \cup B}$ are polyhedra and that $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Choose a triangulation Σ of $\overline{A \cup B}$ that also triangulates $\overline{A}$ and $\overline{B}$. Let σ be a simplex such that $\sigma \notin \overline{A}$. Since $\Sigma|\overline{A}$ is a subcomplex, $\text{relint}(\sigma) \notin \overline{A}$. But the relative interiors partition both $\overline{A}$ and $\overline{B} - \overline{A}$, and therefore $\text{relint}(\sigma) \subseteq \overline{B}$. Because $\Sigma|\overline{B}$ is a subcomplex, we have that $\sigma \subseteq \overline{B}$. Therefore, every simplex of Σ is a simplex of $\Sigma|\overline{A}$ or $\Sigma|\overline{B}$, showing that $\dim(A \cup B) \leq \max\{\dim A, \dim B\}$. Equality follows from claim A.3 as $A, B \subseteq A \cup B$. □

Lemma A.5. *Let A be a non-empty polyhedral set. Then $\dim \delta A < \dim A$.*

Proof. $A = S_1 \cup \ldots \cup S_n$ is a finite union of cells S_i. Observe that

$$\begin{aligned}
\delta A &= \delta(S_1 \cup \ldots \cup S_n) \\
&= \overline{S_1 \cup \ldots \cup S_n} - (S_1 \cup \ldots \cup S_n) \\
&= (\overline{S_1} - (S_1 \cup \ldots \cup S_n)) \cup \ldots \cup (\overline{S_n} - (S_1 \cup \ldots \cup S_n)) \\
&\subseteq (\overline{S_1} - S_1) \cup \ldots \cup (\overline{S_n} - S_n) \\
&= \delta S_1 \cup \ldots \cup \delta S_n.
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
\dim \delta A &\leq \dim(\delta S_1 \cup \ldots \cup \delta S_n) && \text{(claim A.3)} \\
&= \max_{i=1}^{n} \dim \delta S_i && \text{(claim A.4)} \\
&< \max_{i=1}^{n} \dim S_i && \text{(claim A.2)} \\
&= \dim(S_1 \cup \ldots \cup S_n) = \dim A. && \text{(claim A.4)}
\end{aligned}$$

□

A polyhedral map is a map $f : P \to (X, \leq)$ from a polyhedron to a poset with the property that the preimage of any open set is an open sub-polyhedron or, equivalently, that the preimage of any closed set is a closed sub-polyhedron.

Lemma A.6. *Let $f : P \to (X, \leq)$ be a surjective open polyhedral map from a polyhedron P to a poset. If $x < y$, then $f^{-1}(x) \subseteq \delta f^{-1}(y)$ and $\dim f^{-1}(x) < \dim f^{-1}(y)$.*

Proof. Because the preimages of opens are in particular open, f is continuous and open. This implies that $f^{-1}(\overline{A}) = \overline{f^{-1}(A)}$ for any $A \subseteq X$. We have that $\delta f^{-1}(y) = \overline{f^{-1}(y)} - f^{-1}(y) = f^{-1}(\overline{y}) - f^{-1}(y) = f^{-1}(\delta\{y\})$. Because $\delta\{y\} = {\downarrow} y - \{y\}$ and $x < y$, it follows that $\{x\} \subseteq \delta\{y\}$ and thus that $f^{-1}(x) \subseteq f^{-1}(\delta\{y\}) = \delta f^{-1}(y)$. Note that we did not use that f is surjective.

For the second part, $f^{-1}(x)$ is a polyhedral set because $f^{-1}(x) = f^{-1}({\uparrow} x \cap {\downarrow} x) = f^{-1}({\uparrow} x) \cap f^{-1}({\downarrow} x)$ is an intersection of an open and a closed sub-polyhedron. In the same way, also $f^{-1}(y)$ and $\delta f^{-1}(y) = \overline{f^{-1}(y)} - f^{-1}(y)$ are polyhedral sets – the latter because we know from claim A.1 that $\overline{f^{-1}(y)}$ is a polyhedron. Then, we have that

$$\dim f^{-1}(x) \leq \dim \delta f^{-1}(y) < \dim f^{-1}(y),$$

where the first inequality follows from the first part of this lemma and claim A.3, and the second inequality from lemma A.5. Also $f^{-1}(y)$ is non-empty as f is surjective. □