Transfinite Games and Regularity Properties in Higher Baire Spaces

MSc Thesis *(Afstudeerscriptie)*

written by

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Contents

Abstract

Since their formal introduction in 1953 by Gale and Stewart [GS53], infinite games of length ω have been a central part of the study of the continuum. However, games of uncountable length have received limited attention partly because of the inconsistency of the axiom of determinacy (AD) with ZF, and the fact that many determinacy results fail in higher cardinalities, even for very simple sets. However, with the rise of generalized descriptive set theory, specific types of transfinite games corresponding to regularity properties have been studied in the context of higher Baire spaces, such as the Banach-Mazur game and the perfect set game.

This thesis is a rigorous analysis of transfinite games in higher Baire spaces κ^κ for an uncountable cardinal κ satisfying $\kappa^{<\kappa} = \kappa$. The underlying goal is to provide determinacy results for *simple* subsets of the spaces κ^κ and 2^κ in ZFC, while also trying to isolate specific properties of games that are connected to their determinacy.

First, we review some essential facts about standard games in the transfinite setting. While the Gale-Stewart theorem (i.e., closed and open determinacy of standard games) fails in higher Baire spaces, we show that an analogue of this does hold if the set of non-losing positions is sufficiently closed either for Player I or Player II.

Second, we discuss the class of asymmetric games first introduced by Kechris in [Kec77] and then generalized to the transfinite setting in [SSz], in which Player I plays elements of $\kappa^{<\kappa}$, while II imposes requirements that Player I must satisfy in the next round. The determinacy of such games for sets definable from a κ-sequence of ordinals is consistent, relative to an inaccessible cardinal, by results of [SSz]. We show that Kechris's games can be coded as instances of the standard game via a function which is continuous in a strong sense.

The majority of our new results concern variants of Kechris's games, where Player I can now only play sequences whose length is bounded by some $\mu < \kappa$. We prove the existence of closed non-determined subsets of κ^κ for games of this form. Our results confirm the intuition that determinacy of such games is closely related to whether or not Player I can play sequences of any length less than κ. Furthermore, they provide deeper insight on how games behave in the transfinite setting.

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Introduction

It is widely accepted that the systematic study of *two-player perfect-information games* in the context of set theory has its roots in Zermelo's 1913 paper [Zer13]. By "perfect-information games" we mean games where players alternate moves that are known to both of them at any time of the game, and the outcome only depends on their choices. In the decades that followed, as descriptive set theory and the study of Polish spaces such as the Baire space ω^ω and the Cantor space 2^ω developed, infinite games attracted the attention of many mathematicians, largely because of their connection to regularity properties of sets of reals such as the Baire property, the perfect set property and Lebesgue measurability.

In the early 1960s, the Polish mathematicians Jan Mycielski and Hugo Steinhaus [MS62] introduced the *Axiom of Determinacy* (AD), which stated that every game is determined, in the sense that one of the two players has a winning strategy. This axiom has major implications for subsets of the continuum: it implies that every set of reals has the Baire property, the perfect set property, and is Lebesgue measurable. However, AD's drawback is its incompatibility with the axiom of choice (AC). This was known to Mycielski and Steinhaus at the time they introduced the axiom, since already Gale and Stewart had proved, in [GS53], the existence of a non-determined set using AC. In [MS62], Mycielski and Steinhaus wrote:

> It is not the purpose of this paper to depreciate the classical mathematics with its fundamental 'absolute' intuitions on the universum of sets (to which belongs the axiom of choice), but only to propose another theory which seems very interesting although its consistency is problematic. Our axiom can be considered as a restriction of the classical notion of a set leading to a smaller universum, say of determined sets, which reflect some physical intuitions which are not fulfilled by the classical sets (e.g. paradoxical decompositions of the sphere are eliminated by AD). Our axiom could be considered as an axiom added to the classical set theory claiming the existence of a class of sets satisfying AD and the classical axioms without the axiom of choice.

In a follow-up paper, Mycielski [Myc64] proves several consequences of

AD. He proves that, even though AD and AC are incompatible, it can still provide some form of choice, namely, *countable* choice. This, along with the regularity properties it implies, shows that ZF + AD, even though it was never considered a serious alternative to ZFC, provides a natural framework for the study of the continuum. In the same paper, Mycielski uses AD to show that there is no uncountable well-order of reals, which implies the existence of a non-determined game of length ω_1 *without* AC.

During the late 1980s and early 1990s, motivated mainly by model theory, the study of *higher* Baire spaces and *generalized* descriptive set theory has started to emerge. The goal was clear from the beginning: systematically replace ω with an uncountable κ such that $\kappa^{<\kappa} = \kappa$ [1] and see how much structure is preserved from the classical case. This resulted in an extremely rich and interesting theory which quickly became a field of interest in itself. It was soon evident that many of the classical non-trivial results were simply not true in this setting or, in many cases, independent from the axioms of ZFC. However, by adopting specific large cardinal hypotheses, one can preserve some of the structural properties from the classical setting.

Games of length longer than ω have received limited attention, partly because the relative determinacy axiom is inconsistent with ZF. Another reason may be the fact that many basic results fail in the generalized setting. However, there has been some work done on games that are used to characterize regularity properties. Väänänen in [Vää91], defines a game of length ω_1 to describe perfect subsets of the space $\omega_1^{\omega_1}$. Schlicht in [Sch17] proves that the perfect set game characterizes a generalized version of the perfect set property and that the Banach-Mazur game corresponds to a variant of the (generalized) Baire property which is now known as the asymmetric Baire property. In the same paper, he shows that, after a Lévy-collapse of an inaccessible cardinal to κ^+, the Banach-Mazur and the perfect set game are determined for subsets definable by a κ-sequence of ordinals.

This thesis provides a rigorous analysis of games in higher Baire spaces. The underlying goal is to provide determinacy results for *simple* subsets of the spaces κ^κ and 2^κ (where κ an uncountable cardinal such that $\kappa = \kappa^{<\kappa}$) in ZFC, while also trying to isolate specific properties of games that are connected to their determinacy. We study two specific types of games: the standard games, where each player plays ordinals smaller than κ, and asymmetric games where Player I plays sequences of bounded length while Player II provides *requirements* that Player I has to satisfy in each round. In addition, this thesis also serves as a documentation of essential results about games in higher Baire spaces that are scattered across the literature.

The first chapter is introductory. We present the essential facts about the

[1] This assumption is equivalent to κ being regular and such that $2^{<\kappa} = \kappa$. Although this is considered standard, there is some work done in cardinals that do not satisfy this, see e.g. [DM25; HMV25]

topology of the Baire space ω^ω and prove some of its combinatorial properties. In addition, we introduce the reader to the notion of *higher Baire spaces* κ^κ (also called the *generalized Baire space*) and explain some of its differences with the classical case. Alongside the well-established results presented in this chapter, we introduce the notion of a *supercontinuous* function in κ^κ and in Proposition 1.3.7 we prove a property one would expect such a function to have: inverse images of superclosed sets with respect supercontinuous functions are supercontinuous.

The second chapter serves as an introduction to infinite games in the classical setting and is divided in two parts. The first part is a presentation of essential definitions and facts on the determinacy of *standard* games (i.e., games where both players play natural numbers alternately in each round). In addition, we discuss the Banach-Mazur and perfect set game and study their connection to regularity properties for sets of reals. In the second part of the chapter, we briefly discuss AD and its implications while also presenting some of its limitations when we make a jump to higher cardinalities.

The third chapter is concerned with generalizations of both standard and asymmetric games to the transfinite setting. In the first part of the chapter, we discuss how replacing ω with an uncountable cardinal κ affects classical determinacy results related to the standard games. Results like Theorem 3.1.1 and Proposition 3.1.4 are known to experts in the field but have not been sufficiently documented in the literature. In addition, in Corollary 3.1.3, we give a sufficient condition for an analogue of the Gale-Stewart theorem (see Theorem 2.1.8) for κ^κ. In the second part of the chapter, we study games related to regularity properties such as the Banach-Mazur game and the perfect set game and see what their determinacy implies in the uncountable setting. The majority of this part is based on the works of Bogomil Kovachev [Kov09] and Philipp Schlicht [Sch17].

The fourth chapter focuses on a specific class of asymmetric games which Kechris introduced in [Kec77]. Motivated by his paper, we present a generalized version of such games and discuss the connection between their determinacy and regularity properties for subsets of κ^κ. The roles of the players in these game are asymmetric: Player I's moves are elements of $\kappa^{<\kappa}$ while Player II's moves are elements of an arbitrary set R called the set of *requirements*. Player II's moves play the role of conditions that Player I has to satisfy in the next round. The relevance of this class of games stems from the fact that several well-known games characterizing regularity properties can be obtained as special cases, including the Banach-Mazur game and the perfect set game. Similarly to these two special cases, generalized Kechris's games are determined for all $A \subseteq \kappa^\kappa$ that are definable by a κ-sequence of ordinals, after a Lévy-collapse of an inaccessible cardinal to κ^+, by results in [SSz]. All results in this part are due to unpublished work of Dorottya Sziráki and Philipp Schlicht in the same preprint. In the second part of the chapter, we provide a coding procedure from Kechris's games to standard games. To do so, we follow the idea behind the coding of such games in the classical case and, in Proposition 4.2.1, we prove that the coding can be achieved using a supercontinuous function and a superclosed set. Hence the codes of superclosed winning sets are also superclosed.

In the fifth and final chapter, we present most of the new results of this thesis. The motivation behind them comes from our intuition that games like the ones described in Chapter 4 can be consistently determined for "simple" winning sets only when Player I is allowed to play sequences of *any* length less than κ and become non-determined if the set of moves Player I is allowed to play is bounded below κ. The chapter is divided in three parts. In the first part, we modify Kechris's games from Chapter 4 and force Player I to play elements of $\kappa^{<\omega}$ in successor rounds. In Corollaries 5.1.7 and 5.1.10 we provide closed non-determined subsets of 2^κ and κ^κ respectively for such games. In the second part, we are concerned with games where Player I can play elements of $\kappa^{<\mu}$, for an uncountable μ such that $\mu = \mu^{<\mu}$. We expand the general class of games we have been working thus far and, in Corollary 5.2.6, we provide a closed non-determined subset of κ^κ for such games. However, the types of closed counterexamples may become determined if the game is changed in such a way that Player I is allowed to play sequences of any length in limit rounds, by Lemma 5.2.2. In the third part of the chapter, we study four applications of these results in variations of the standard game. In particular, the application is concerned with a variant of the standard game where Player I is now allowed to play elements of $\kappa^{<\kappa}$ instead of ordinals. This is a special case of games described in Chapter 4 and therefore its determinacy for closed sets (and even for sets definable from a κ-sequence of ordinals) is consistent relative to an inaccessible by results in [SSz]. The fourth application describes the same game but instead of elements of $\kappa^{<\kappa}$, Player I is only allowed to play elements of $\kappa^{<\mu}$ where $\omega \leq \mu < \kappa$ such that $\mu^{<\mu} = \mu$, which by Corollary 5.2.6 is non-determined for a closed subset of κ^κ. This aligns with our initial intuition that the determinacy of games of this form is strongly connected to whether or not Player I can play sequences of any length less than κ.

Chapter 1

Preliminaries

1.1 Notation

We mostly use standard set theoretic notation throughout this thesis. We write ω to denote the set of natural numbers and Lim for the set of all limit ordinals.

For a set X and an ordinal α we write X^α for the set of functions $s : \alpha \to X$. The ordinal α is also going to be called the *length* of s and we denote it with $|s|$. We also write $X^{<\alpha}$ for the set of functions $s : \beta \to X$ where $\beta < \alpha$. In most cases we are going to refer to elements of X^α as *sequences of elements of X of length α*. We will write $\varnothing$ for the empty sequence, i.e., the sequence of length 0.

Given $s : \alpha \to X$ and $t : \beta \to X$ we say that t is an *initial segment* of s and write $t \subseteq s$ if $\beta \leq \alpha$ and $t(\gamma) = s(\gamma)$ for all $\gamma \leq \beta$. If $t \nsubseteq s$ and $s \not\subset t$, we say that t and s are *incompatible* and write $t \perp s$. We write $t \subset s$ if $t \subseteq s$ and $t \neq s$ and say that s is a *successor* of t. If $t \subset s$ and for all $s' \supset t$ either $s \subseteq s'$ or $s \perp s'$, then we will call s an *immediate successor* of t. We also write $s{\restriction}\gamma$ for $\gamma \leq \alpha$ for the restriction of s to γ and treat it like a sequence of elements of X of length γ. For a set $A \subseteq X^{<\alpha}$, we denote with $\mathrm{dw}(A)$ its downwards closure, i.e., the set $\mathrm{dw}(A) := \{t \in X^{<\alpha} : t \subseteq s \text{ for } s \in A\}$.

We write $s^\frown t$ to denote the concatenation of s and t. We will also use $\oplus_{\alpha<\beta} s_\alpha$ to denote the concatenation of s_α for all $\alpha < \beta$.

1.2 The Baire Space

1.2.1 Topology on the Baire space

Classical descriptive set theory is concerned with the study of Polish spaces and properties of their subsets. A Polish space is a topological space which is completely separable and metrizable with a complete metric. Examples of such spaces include the real line $\mathbb{R}$, the Baire space ω^ω, and the Cantor space

2^ω. In this part of the thesis, we are going to provide some basic definitions concerning ω^ω (all of which can easily be translated to 2^ω) and state some well-known results about its properties.

The Baire space ω^ω is the space of infinite sequences of natural numbers, equipped with the topology generated by basic open sets of the following form, for $s \in \omega^{<\omega}$:

$$U_s = \{x \in \omega^\omega : s \subset x\}$$

We can also see the Baire space as the product of infinitely many copies of the set of natural numbers, equipped with the product topology (where each copy of ω is equipped with the discrete topology). Notice that the basic open sets are also closed: take $x \notin U_s$ for some $s \in \omega^{<\omega}$ and see that $s(n) \neq x(n)$ for some $n < |s|$. Now take $s' = x \restriction n$ and see that $x \in U_{s'}$ and $U_{s'} \cap U_s = \varnothing$. Therefore $\omega^\omega \setminus U_s$ is open and U_s is clopen. It can also easily be checked that the topology generated by such sets coincides with the topology induces by the metric:

$$d(x,y) := \begin{cases} 0 & \text{if } x = y \\ 1/2^n & \text{where } n \text{ is the least s.t. } x(n) \neq y(n) \end{cases}$$

We also have the following properties:

Lemma 1.2.1 Let $s, t \in \omega^{<\omega}$, then the following hold:

i. $s \subseteq t$ if and only if $U_t \subseteq U_s$

ii. $U_s \cap U_t$ is either $\varnothing$, U_s or U_t.

The idea behind the use of initial segments is that they can serve as approximations of elements of ω^ω. This motivates the following definition of convergence in the Baire space:

Definition 1.2.2 Let $\langle x_n \mid n < \omega\rangle$ be an infinite sequence of elements of ω^ω. We say that it converges to $x \in \omega^\omega$ if the following holds:

$$(\forall s \subset x)(\exists m < \omega)(\forall n \geq m)\ (s \subset x_n)$$

We call x the *limit* of $\langle x_n \mid n < \omega\rangle$.

There are also two equivalent definitions for convergence which we are going to state for the sake of completeness:

1. We say that $\langle x_n \mid n < \omega\rangle$ converges to x if for all basic open neighborhoods U_s of x, there exists m such that for all $n \geq m$ it holds that $x_n \in U_s$.

2. We say that $\langle x_n \mid n < \omega \rangle$ converges to x if

$$(\forall i < \omega)(\exists m)(\forall n \geq m) \ (x_n(i) \text{ is constant and equal to } x(i))$$

Since ω^ω is a topological space we can use the standard topological definition of continuity, that is, $f : \omega^\omega \to \omega^\omega$ is *continuous* if for $U \subseteq \omega^\omega$ open, $f^{-1}[U]$ is also open. But, in the case of ω^ω, there is also the following equivalent definition:

Definition 1.2.3 A function $f : \omega^\omega \to \omega^\omega$ is *continuous* if for each $x \in \omega^\omega$ the following holds:

$$(\forall s \subset f(x))(\exists t \subset x)(\forall y \in \omega^\omega) \ (t \subset y \to s \subset f(y))$$

1.2.2 Trees

In descriptive set theory, trees are one of the most useful combinatorial tools. Throughout this thesis our use of tree notions will be extensive and we will rely a lot on the intuition behind them. The notion of a tree itself pops up in a lot of different areas of mathematics like combinatorics, group theory, probability theory, etc. The definition we are going to use in this context is the following:

Definition 1.2.4 A set $T \subseteq \omega^{<\omega}$ is called a *tree* if T is closed under initial segments. A *branch through* T is any $x \in \omega^\omega$ such that $x \restriction n \in T$ for all $n < \omega$. We write $[T]$ for the set of branches through T.

It is very helpful to also think of the whole space ω^ω as a tree where each branch through it is an infinite sequence:

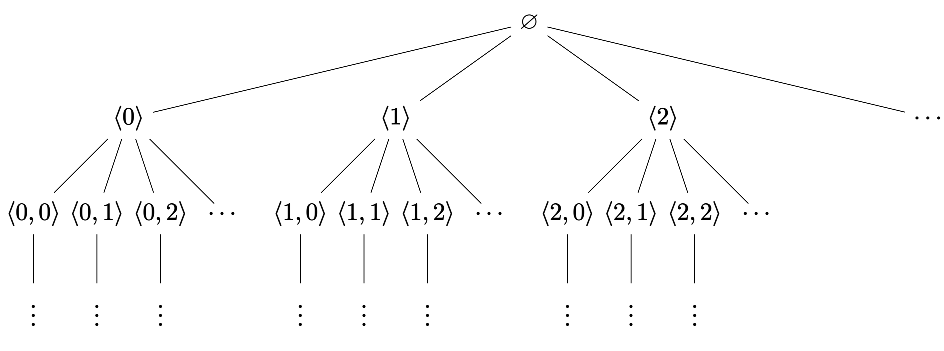

Figure 1.1: The space ω^ω as tree

Not all subsets of the Baire space can be seen as trees. To see this, note that there are $2^{2^{\aleph_0}}$ subsets of ω^ω while there are only $2^{\aleph_0}$ subsets of $\omega^{<\omega}$.

Definition 1.2.5 For $A \subseteq \omega^\omega$, the *tree of* A is the set

$$T_A := \{t \in \omega^{<\omega} : t \subset x \text{ for some } x \in A\}$$

If $x \in A$ then every $s \subset x$ is in T_A therefore $x \in [T_A]$ and $A \subseteq [T_A]$. On the other hand, the reverse implication cannot hold for every $A \subseteq \omega^\omega$. The following theorem highlights a nice correspondence between trees and closed subsets of the Baire space.

Theorem 1.2.6 Let $T \subseteq \omega^{<\omega}$ be a tree and $C \subseteq \omega^\omega$ be a closed set. The following hold:

(i) $[T]$ is closed.

(ii) $[T_C] = C$.

Proof. For (i), let $x \notin [T]$, which means that there is $n < \omega$ such that $x \restriction n \notin T$. Hence we have a neighborhood of $x \in \omega^\omega$, namely $U_{x\restriction n}$, that is disjoint from $[T]$ therefore $\omega^\omega \setminus [T]$ is open and $[T]$ closed.

For (ii), we've already proven that $C \subseteq [T_C]$ so, suppose that $x \in [T_C]$ but $x \notin C$. Since C is closed, its complement will be open and therefore there exists an $s \subset x$ such that $U_s \cap C = \varnothing$. But since $x \in [T_C]$ and $s \subset x$, we have that $s \in T_C$ which means that s is an initial segment of an element of C. But this contradicts the fact that $U_s \cap C = \varnothing$. Therefore, $x \in C$ and $[T_C] \subseteq C$. □

1.3 Higher Baire spaces

1.3.1 Topology on higher Baire spaces

This section is concerned with the generalization of notions we have mentioned thus far to higher cardinalities. A natural way to do that is to replace any instance of ω in our definitions with an uncountable cardinal κ. This gives rise to *higher Baire spaces* κ^κ (and, in the same manner, *higher Cantor spaces* 2^κ) which are spaces of functions from κ to κ equipped with the topology generated by basic open sets of the following form for $s \in \kappa^{<\kappa}$:

$$U_s = \{x \in \kappa^\kappa : s \subset x\}$$

Since our goal is to preserve as much structure from the classical case, one essential assumption is that κ is a regular cardinal such that $2^{<\kappa} = \kappa$ (which can be summarized as $\kappa^{<\kappa} = \kappa$) [1]. Among other things, this guarantees the existence of a coding of basic open sets with ordinals below κ which will play a crucial role in Chapters 4 and 5. Moving forward, we will always assume this unless stated otherwise.

[1] Although this is considered a standard assumption in the field of generalized descriptive set theory, there has been some work done *without* assuming $\kappa^{<\kappa} = \kappa$. See, for example, [DM25; HMV25]

As before, the space κ^κ can be seen as the product of κ copies of κ, equipped with the $<\kappa$-box topology[2], where all copies of κ are equipped with the discrete topology. When making this jump to higher cardinalities, many structural properties of the classical case disappear. The following proposition is an example of such a property.

Proposition 1.3.1 Higher Baire spaces are not metrizable.

Proof. We know from topology that if a space X is metrizable then for every $x \in X$ there exists a countable collection of open sets $\mathscr{V}_x$, such that for every open neighborhood U of x there exists $V \in \mathscr{V}_x$ with $V \subseteq U$. So, suppose that κ^κ is metrizable. Let $x \in \kappa^\kappa$ and $\mathscr{V}_x$ be the collection with the above property. Let $V \in \mathscr{V}_x$ and let a basic open set $U_s \subseteq V$ such that $x \in U_s$. Since $\mathscr{V}_x$ is countable (and κ is regular) we can find $s' \in \kappa^{<\kappa}$ such that $s \subset s'$ for every $V \in \mathscr{V}_x$. We therefore have $x \in U_{s'}$ and $V \not\subseteq U_{s'}$ for all $V \in \mathscr{V}_x$. $\square$

On the other hand, the next lemma highlights some properties that translate well in the generalized setting:

Lemma 1.3.2 The following hold in κ^κ:

(i) The intersection of $<\kappa$-many basic open sets is either empty or basic open.

(ii) The intersection of $<\kappa$-many open sets is open.

(iii) Every basic open set is also closed.

The Definitions 1.2.2 and 1.2.3 for convergence and continuity translate to uncountable κ. In fact, due to the structure of κ^κ, we can further strengthen the definition of a continuous function. To do this, we will need the following:

Definition 1.3.3 Let $j : \kappa^{<\kappa} \to \kappa^{<\kappa}$:

(i) We call j *strict order preserving* if for $s \subset t$ we have $j(s) \subset j(t)$.

(ii) We call j *continuous* if for all $\lambda < \kappa$ and every strictly increasing sequence $\langle s_\alpha : \alpha < \lambda \rangle$ we have $\bigcup_{\alpha<\lambda} j(s_a) = j(\bigcup_{\alpha<\lambda} s_\alpha)$.

(iii) For strict order preserving $j : \kappa^{<\kappa} \to \kappa^{<\kappa}$ we define:

$$f_j(x) = \bigcup_{\alpha<\kappa} j(x \restriction \alpha)$$

Lemma 1.3.4 If $j : \kappa^{<\kappa} \to \kappa^{<\kappa}$ is a strict order preserving map then $f_j : \kappa^\kappa \to \kappa^\kappa$ is continuous.

[2]This is the topology whose basic open sets are of the form $\prod_{\alpha<\kappa} V_\alpha$ where each V_α is an open subset of κ and $V_\alpha \neq \kappa$ holds only $<\kappa$ many times. Note that, the $<\omega$-box topology is the same as the product topology

Proof. Let $j : \kappa^{<\kappa} \to \kappa^{<\kappa}$ be a strict order preserving map and $s \subset f_j(x)$ for some $x \in \kappa^\kappa$. This means that $\bigcup_{\alpha<\kappa} j(x\restriction\alpha) \supset s$, so let $\beta < \kappa$ be the smallest ordinal such that $j(x\restriction\beta) \supset s$. Now, for arbitrary $y \in \kappa^\kappa$ we have that if $x\restriction\beta \subset y$ then $j(x\restriction\beta) \supset f_j(y)$ and hence $s \subset f_j(y)$. This shows that f_j is continuous. Also, for every $x \in \kappa^\kappa$, we have $x\restriction\alpha \in \mathrm{dom}(j)$ for all $\alpha < \kappa$. The sequence $\langle x\restriction\alpha : \alpha < \kappa\rangle$ is strictly increasing and of length κ, therefore $\langle j(x\restriction\alpha) : \alpha < \kappa\rangle$ is also a strictly increasing sequence of length κ. For every $x \in \kappa^\kappa$, therefore, $f_j(x)$ is defined which means $\mathrm{dom}(f) = \kappa^\kappa$. □

Definition 1.3.5 A function $f : \kappa^\kappa \to \kappa^\kappa$ is called *supercontinuous* if there exists a strict order preserving and continuous $j : \kappa^{<\kappa} \to \kappa^{<\kappa}$ such that for all $x \in \kappa^\kappa$:

$$f(x) = f_j(x) = \bigcup_{\alpha<\kappa} j(x\restriction\alpha)$$

While several properties of ω^ω break down when making the jump to the uncountable, the structure of higher Baire spaces allow us to further strengthen topological notions such as continuity. Notice that, supercontinuous function in ω^ω are just continuous functions due to the lack of limits $\lambda < \omega$. This will be a common theme in our study of higher Baire spaces and, moving on, we will also see other instances of such strengthenings.

1.3.2 Trees

Just as in the case of ω^ω, trees play a big role in the study of κ^κ. One of the important properties that the space κ^κ preserves is the correspondence between closed sets and trees. In addition, just as we did with continuous and supercontinuous functions, we can strengthen the definition of a closed set in κ^κ as follows:

Definition 1.3.6 Let $T \subseteq \kappa^{<\kappa}$. We call T *$<\kappa$-closed* if for every $\lambda < \kappa$ and every strictly increasing sequence $\langle s_\alpha : \alpha < \lambda\rangle$ of elements of T, we have $\bigcup_{\alpha<\lambda} s_\alpha \in T$. A set $C \subseteq \kappa^\kappa$ is called *superclosed* if there exists a $<\kappa$-closed tree T such that $[T] = C$.

Notice that every closed subset $C \subseteq \omega^\omega$ is trivially superclosed. Indeed, let $T_C \subseteq \omega^{<\omega}$ be the tree of C and $\langle x_n : n < k\rangle$ a strictly increasing sequence of elements of T_C, for $k < \omega$. Since k is a natural number we have $\bigcup_{n<k} x_n = x_{k-1} \in T_C$ which shows that T_C is $<\omega$-closed and C superclosed.

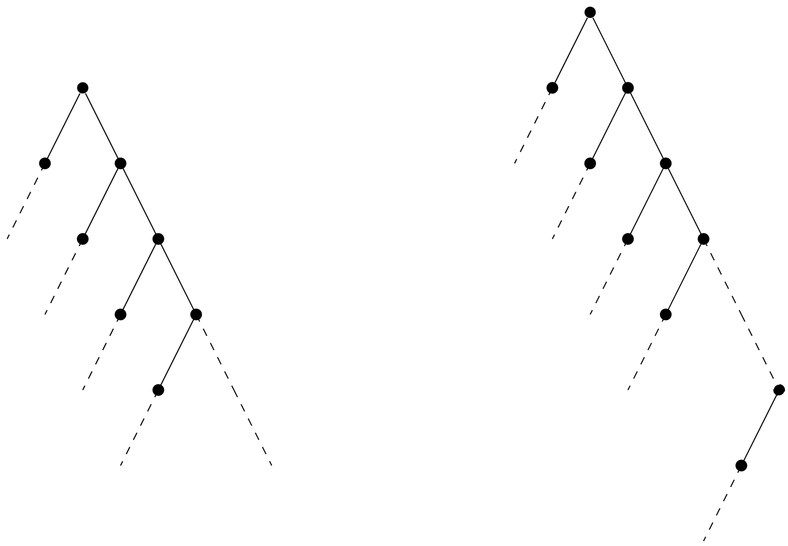

Figure 1.2: A tree that is not $<\kappa$-closed compared to a $<\kappa$-closed tree

In topology, the inverse image of a closed set through a continuous function is closed. It is natural, therefore, to expect superclosed sets and supercontinuous functions to work in the same way. The following proposition shows that this is indeed the case:

Proposition 1.3.7 Let $C \subseteq \kappa^\kappa$ be a superclosed set and $f : \kappa^\kappa \to \kappa^\kappa$ a supercontinuous function. Then, $f^{-1}[C]$ is superclosed.

Proof. Let T_C be the $<\kappa$-closed tree such that $[T_C] = C$ and also let $j : \kappa^{<\kappa} \to \kappa^{<\kappa}$ be the function that induces f. We claim that for $S = \{s \in \kappa^{<\kappa} : j(s) \in T_C\}$ we have $[S] = f^{-1}[C]$. To prove this, first suppose that $x \in [S]$. This means that for all $\alpha < \kappa$ we have $x{\restriction}\alpha \in S$ or equivalently $j(x{\restriction}\alpha) \in T_C$. The sequence $\langle j(x{\restriction}\alpha) : \alpha < \kappa\rangle$ is a strictly increasing sequence of elements of T_C and therefore $\bigcup_{\alpha<\kappa} j(x{\restriction}\alpha) = f(x) \in C$ and $x \in f^{-1}[C]$. For the other direction, let $x \in f^{-1}[C]$ which means $f(x) \in C$ and therefore $\bigcup_{\alpha<\kappa} j(x{\restriction}\alpha) \in C$. For each $\alpha < \kappa$ we have $f(x) \supset j(x{\restriction}\alpha)$ and therefore $j(x{\restriction}\alpha) \in T_C$. This gives us $x{\restriction}\alpha \in S$ for all $\alpha < \kappa$ and hence $x \in S$.

Now, we prove that $[S]$ is indeed superclosed. Let $\langle s_\alpha : \alpha < \lambda\rangle$ a strictly increasing sequence on S where $\lambda < \kappa$ is an arbitrary limit ordinal. Since j is strictly order preserving we have that $\langle j(s_\alpha) : \alpha < \lambda\rangle$ is also a strictly increasing sequence. Now, since T_C is superclosed we have $\bigcup_{\alpha<\lambda} j(s_\alpha) \in T_C$ and by continuity of j we get $j(\bigcup_{\alpha<\lambda} s_\alpha) \in T_C$ and $\bigcup_{\alpha<\lambda} s_\alpha \in S$. □

We now give some standard definitions regarding trees that we are going to use extensively throughout the thesis. The closed sets induced by these tree properties are going to play a vital role. Note that these definitions can easily be adjusted to the classical case.

Definition 1.3.8 Let $T \subseteq \kappa^{<\kappa}$ be a tree.

(i) A node $s \in T$ is called *μ-splitting* if it has at least μ-many immediate successors in T. A node $s \in T$ is called fully-splitting if for every $\alpha < \kappa$, the node $t^\frown\langle\alpha\rangle$ is a successor of t.

(ii) T is called *μ-perfect* if it is $<\kappa$-closed and the set of μ-splitting nodes is cofinal in T, i.e., there is a μ-splitting node above every node of T. We will call a tree *perfect* if it is 2-perfect.

Chapter 2

Games of length ω

In this part of the thesis we are going to discuss infinite two-player games of perfect information of countable length and study their applications in set theory. By games of perfect information we mean games where players have access to every move they and their opponent have made at any time during the game. After being formally introduced by Gale and Stewart in [GS53], such games have been extensively studied, especially because of their connection to regularity properties of the real line.

This chapter is divided in three sections. The first section introduces *standard* games of length ω and presents essential definitions and terminology that is going to be used throughout this thesis. The second section is concerned with the correspondence between specific games and regularity properties while, in the third section, we discuss the *Axiom of Determinacy* and see how it can provide a natural framework for the study of the continuum as well as its limitations.

2.1 Games on ω^ω

2.1.1 Basic notions

We define *standard* games G_ω of length ω in ω^ω as follows: Fix a set $A \subseteq \omega^\omega$ which we are going to call the *payoff set* of the game. In each round, Player I plays a natural number n_i and Player II answers with a natural n_{i+1}:

I	n_0		n_2		$\dots$
II		n_1		n_3	$\dots$

At the end of the game, the players have constructed an element of ω^ω, namely $x = \langle n_0, n_1, n_2, n_3, \dots \rangle$. Player I wins if $x \in A$, otherwise Player II wins. The sequence x is called a *play* of $G_\omega(A)$ and each initial segment p of x is called a *partial play* or *position* of the game $G_\omega(A)$.

Naturally, since we are talking about games, we will also have to talk about strategies for $G_\omega(A)$. Formally, a strategy σ for Player I is a function assigning a natural number to any finite sequence of natural numbers of even length and, analogously, a strategy τ for Player II assigns a natural in every sequence of odd length. The formal definition is given below:

Definition 2.1.1 Let $A \subseteq \omega^\omega$ and $G_\omega(A)$ be the standard game of length ω on A.

(i) A strategy σ for Player I is a function:

$$\sigma : \bigcup_{n<\omega} \omega^{2n} \to \omega$$

(ii) A strategy τ for Player II is a function:

$$\tau : \bigcup_{n<\omega} \omega^{2n+1} \to \omega$$

Strategies are rules that dictate how each player should play. If a strategy guarantees the player to win it is called a *winning strategy.* For $z \in \omega^\omega$ we denote with $\sigma * z$ the play where Player II plays z and Player I plays according to σ. Symmetrically, we write $z * \tau$ for the play where Player I plays z and Player II plays according to τ. Therefore, a strategy σ for Player I (τ for Player II) in the game $G_\omega(A)$ is called *winning* if for all $z \in \omega^\omega$ we have $\sigma * z \in A$ ($z * \tau \notin A$).

We can also use the same notation to describe partial plays of the game where one of the players plays according to some strategy. In particular, if σ is a strategy for Player I, we write $\sigma * t$ for the partial play where Player I plays according to σ and Player II plays $t \in \omega^{<\omega}$. Note that if $|t| = n$, then $|\sigma * \tau| = 2n$ which means that Player I uses their strategy only in their first n moves. In the same manner we can define $t * \tau$, where τ is a strategy for Player II. The following lemma shows that winning strategies work as one would expect:

Lemma 2.1.2 For $A \subseteq \omega^\omega$ and game $G_\omega(A)$, Player I and Player II cannot both have winning strategies.

Definition 2.1.3 A game $G_\omega(A)$ is called *determined* if either Player I or Player II has a winning strategy.

Since games only depend on their payoff set, if the game $G_\omega(A)$ is determined, the set A will also be called *determined.* It turns out that not all games are determined, but to prove this, the use of the Axiom of Choice is necessary.

Theorem 2.1.4 (AC) [GS53] There exists a non-determined set $D \subseteq \omega^\omega$.

Proof. We start by enumerating the strategies for Player I and Player II. Since a strategy is a function from $\omega^{<\omega}$ to ω and $\omega^{<\omega}$ is a countable set, the number of possible strategies is $2^{\aleph_0}$. Using the axiom of choice, we can enumerate strategies for Player I and II as $\{\sigma_\alpha : \alpha < 2^{\aleph_0}\}$ and $\{\tau_\alpha : \alpha < 2^{\aleph_0}\}$ respectively. We will inductively define sets $X = \{x_\alpha : \alpha < 2^{\aleph_0}\}$ and $Y = \{y_\alpha : \alpha < 2^{\aleph_0}\}$. For the base case, arbitrarily pick x_0 such that $x_0 = a * \tau$ for some $a \in \omega^\omega$ and y_0 such that $y_0 = \sigma * b$ for some $b \in \omega^\omega$ and, in addition, $x_0 \neq y_0$.

For the inductive step, suppose that we have already constructed the sets $\{x_\beta : \beta < \alpha\}$ and $\{y_\beta : \beta < \alpha\}$. Pick x_α such that $x_\alpha = a * \tau$ for some $a \in \omega^\omega$. Since $|\{y_\beta : \beta < \alpha\}| < 2^{\aleph_0}$ we can choose x_α which differs from all y_β in the set. We then use a similar argument to pick y_α different from each $\{x_\beta : \beta < \alpha\} \cup \{x_\alpha\}$.

By construction, the sets X, Y are disjoint. We now prove that X is non-determined. Suppose, for contradiction, that Player I has a winning strategy σ in $G_\omega(X)$. For this strategy σ, there exists an $\alpha < 2^{\aleph_0}$ such that $\sigma = \sigma_\alpha$ but, in stage α of the above induction, we picked $y_\alpha = \sigma * b$ for some $b \in \omega^\omega$ such that $y_\alpha \notin X$. Therefore if Player II plays b we can't have $\sigma * b \in X$ which is a contradiction. A similar argument can be used to prove that Player II does not have a winning strategy. □

Winning strategies are not the only type of strategies that one can define in such games. There are cases where neither of the players has a winning strategy at the beginning of the game but, after n moves, one of them might be in a winning position. This can happen if, for example, their opponent makes a mistake at some point. This is the reason we are going to introduce the notion of a *defensive strategy* which essentially guarantees that the player makes no mistake when their opponent does not have a winning strategy. Before that, let us define the following:

Definition 2.1.5 We call a position p of even length of the game $G_\omega(A)$, *non-losing for Player I* if Player II does not have a winning strategy in the game $G_\omega(A;p)$ where $A;p := \{x \in \omega^\omega : p^\frown x \in A\}$. We can define a *non-losing position for Player II* in the same manner.

Definition 2.1.6 Let $G_\omega(A)$ be a standard game of length ω:

(i) A strategy ∂_I for Player I is called *defensive* if for every $t \in \omega^{<\omega}$, the position $\partial_I * t$ is non-losing for Player I.

(ii) A strategy ∂_{II} for Player II is called *defensive* if for every $t \in \omega^{<\omega}$, the position $t * \partial_{II}$ is non-losing for Player II.

Clearly, when one of the players has a winning strategy, it is impossible for the other to have a defensive one. The next lemma shows that defensive strategies on standard games actually exist.

Lemma 2.1.7 Let $A \subseteq \omega^\omega$ and $G_\omega(A)$ be the standard game of length ω.

(i) If Player II does not have a winning strategy in $G_\omega(A)$, then Player I has a defensive strategy.

(ii) If Player I does not have a winning strategy in $G_\omega(A)$, then Player II has a defensive strategy.

Proof. For (i), we are going to define strategy ∂_I for Player I such that for all $t \in \omega^{<\omega}$, the position $\partial_I * t$ is non-losing. We do this by induction on the length of t. First, for the base case, let $t = \varnothing$. Then $\partial_I * t$ will be the initial position of the game which, by our hypothesis, is non-losing for Player I. Hence, there exists $x_0 < \omega$ such that for all $y < \omega$, the position $\langle x_0, y \rangle$ is non-losing for I so, let $\partial_I(\varnothing) = x_0$. Next, let t of arbitrary length be such that $\partial_I * t$ is non-losing for Player I. From now on, let $p := \partial_I * t$. Then, there exists some $x_n < \omega$ such that for all $y < \omega$ the position $p^\frown\langle x_n, y \rangle$ is still non-losing for Player I because, if that was not the case, it would mean that for all $x < \omega$, there exists $y < \omega$ such that $p^\frown\langle x, y \rangle$ is a winning position for Player II. But this implies that p was already a winning position for II which is a contradiction so, let $\partial_I(p) = x_n$.

The proof of (ii) is analogous, with the roles of Player I and II interchanged. □

The existence of defensive strategies in standard games of length ω implies a fundamental result for the study of infinite games and it is due to [GS53].

Theorem 2.1.8 (Gale-Stewart) For open or closed $A \subseteq \omega^\omega$, the game $G_\omega(A)$ is determined.

Proof. Let $A \subseteq \omega^\omega$ be closed and suppose that Player II does not have a winning strategy for $G_\omega(A)$, then, by Lemma 2.1.7, Player I has a defensive strategy ∂_I. We will show that ∂_I is in fact a winning strategy. Let $x \in \omega^\omega$ and suppose, for contradiction, that $\partial_I * x \in \omega^\omega \setminus A$. Since $\omega^\omega \setminus A$ is open, there exists $s \subset \partial_I * x$ such that for all $y \supset s$ we have $y \in \omega^\omega \setminus A$. Therefore, s is a winning position for Player II in the game $G(A)$ which is a contradiction because Player I plays defensively according to ∂_I. Therefore, ∂_I is a winning strategy.

The proof for an open set A is analogous, with the roles of Player I and II interchanged. □

2.1.2 Games and Regularity Properties in ω^ω

We have already mentioned that infinite games can give us a new perspective on various set-theoretic properties. In this section we are going to see this in action while we investigate the implications of determinacy of different games in the Baire space. Specifically, we are going to talk about games that imply regularity properties for ω^ω, namely: the Banach-Mazur game for the Baire property and the perfect set game for the perfect set property. All of the results we are going to present here are well established and can be found in various set theory books (e.g. see [Jec07], [Kan03]).

Both the Banach-Mazur game and the perfect set game look a bit different from the standard games we have discussed thus far. In the Banach-Mazur game both players play finite sequences, each extending the one previously played, instead of natural numbers. At the end of the game, we take the union of those sequences and check whether it belongs to the payoff set. On the other hand, the perfect set game looks like a hybrid between the standard game and the Banach-Mazur game. Player I plays finite sequences while Player II answers with natural numbers. In addition, Player I's moves must be such that their first digit is different from Player II's last move. At the end of the game, we concatenate Player I's moves and check whether they belong to the payoff set.

Games like the ones described above may look different from what we have discussed thus far but, one can easily view them as standard games using a suitable coding procedure. We briefly present this procedure after introducing each game. We return to it in Chapter 4 where we discuss a similar coding procedure for a very general class of games, in the setting of higher Baire spaces.

The Baire Property Let us then begin with the Banach-Mazur game G^{**}_ω, introduced by Mazur in [Mau81], which describes the Baire property (see Definition 2.1.11). There are several equivalent ways to formulate the game, so we are going to choose the most intuitive for now and see other formulations in later sections, when they become relevant. In the game G^{**}_ω the players play elements of $\omega^{<\omega}$, each one properly extending the previous one:

I	s_0		s_2		$\dots$
II		s_1		s_3	$\dots$

where $s_0 \subset s_1 \subset s_2 \subset s_3 \subset \dots$. Player I wins the game if and only if $\bigcup_{n<\omega} s_n \in A$. We can easily code the game $G^{**}_\omega(A)$ to a standard game. Let $\langle t_n : n < \omega \rangle$ be an enumeration of the elements of $\omega^{<\omega}$. For

$$A^{**} := \{x \in \omega^\omega : \bigcup_{n<\omega} t_{x(n)} \in A\}$$

we have that Player I has a winning strategy for $G^{**}_\omega(A)$ if and only if they have one for $G_\omega(A^{**})$, and the same holds for Player II. Hence the games $G^{**}_\omega(A)$ and $G_\omega(A^{**})$ are equivalent.

In order to establish the connection between G^{**}_ω and the Baire property we will need the following terminology:

Definition 2.1.9 Let $A \subseteq \omega^\omega$. A *good position* $p = \langle s_0, s_1, \dots, s_n \rangle$ for the game $G^{**}_\omega(A)$ where Player II has a winning strategy τ, is a sequence of even length where its odd values are moves of Player II according to τ. We write p^* for the union of p, i.e., the sequence constructed by the players thus far.

Proposition 2.1.10 Let τ be a winning strategy for Player II in the game $G^{**}_\omega(A)$. For $x \in \omega^\omega$, if for every good position $p = \langle s_0, s_1, \dots, s_n \rangle$ with $s_n \subset x$, there exists $s \in \omega^{<\omega}$ with $s_n \subset s$ such that $\tau(p^\frown s) \subset x$, then $x \in \omega^\omega \setminus A$.

The proof of the proposition is almost immediate from the fact that in each step of the sequence, Player II plays according to τ and the sequence converges to x. Similar propositions to 2.1.10 are also going to be useful later, when we describe the general class of Kechris's games in Chapter 4.

Definition 2.1.11 The set A satisfies the *Baire property* if it is either meager or differs from an open set by a meager set, i.e., $A \triangle U$ is meager for some open set $U \subseteq \omega^\omega$.

Lemma 2.1.12 ([Mau81], [Oxt58]) For $A \subseteq \omega^\omega$ we have the following:

(i) Player II has a winning strategy in the game $G^{**}_\omega(A)$ if and only if A is meager.

(ii) Player I has a winning strategy in the game $G^{**}_\omega(A)$ if and only if for some $s \in \omega^{<\omega}$, $U_s \setminus A$ is meager.

Proof. For the left-to-right direction of (i), suppose that Player II has a winning strategy τ. For every good position $p = \langle s_0, s_1, \dots, s_n \rangle$ define the set:

$$K_p = \{x \in \omega^\omega : s_n \subset x \text{ and } (\forall s \supset s_n)(\tau(p^\frown s) \not\subset x)\}$$

By the contrapositive of Proposition 2.1.10 see that for every $x \in A$ there exists a good position p such that $x \in K_p$ therefore we have

$$A \subseteq \bigcup \{K_p : p \text{ good position}\}$$

Now, for each $p = \langle s_0, s_1, \dots, s_n, \rangle$ the set $U_{s_n} \setminus K_p$ is an open dense set in U_{s_n} and hence K_p is a closed nowhere dense set. The set of good positions is countable, therefore A is meager. For the other direction, suppose that A is meager. This means that $A = \bigcup_{n<\omega} C_n$ where C_n is closed and nowhere dense for all $n < \omega$. Hence, $\omega^\omega \setminus C_n$ is open and dense for all $n < \omega$, therefore for each of Player I's moves s_i, Player II can respond with $s_{i+1} \supset s_i$ such that $U_{s_{i+1}} \subseteq \omega^\omega \setminus C_i \subseteq \omega^\omega \setminus A$. Therefore, the sequence $s_0 \subset s_1 \subset s_2 \subset \dots$ converges to some $x \in \omega^\omega \setminus A$.

For the left-to-right direction of (ii), suppose that Player I has a winning strategy σ in $G^{**}_\omega(A)$. Let $\sigma(\varnothing) = s$, then Player II can use this strategy in the game $G^{**}_\omega(U_s \setminus A)$. Then, by (i), $U_s \setminus A$ is meager. For the other direction, let $s \in \omega^{<\omega}$ such that $U_s \setminus A$ is meager. This means that Player II has a winning strategy for the game $G^{**}(U_s \setminus A)$. Player I of $G^{**}_\omega(A)$ now plays s for their first move and then follows the strategy of Player II in $G^{**}(U_s \setminus A)$. This guarantees that the resulting sequence is an element of $A \cap U_s \subseteq A$, therefore it is a winning strategy. □

The following shows the connection between the game G^{**}_ω and the Baire Property:

Lemma 2.1.13 Let $\mathbf{\Gamma}$ be a boldface pointclass[1]. If for every $A \in \mathbf{\Gamma}$, either A is meager or there exists $s \in \omega^{<\omega}$ such that $U_s \setminus A$ is meager, then every A has the Baire property.

Proof. Let $A \in \mathbf{\Gamma}$. If A is meager, then it is immediate that A has the Baire property. Otherwise, define U_A as in the previous Theorem:

$$U_A = \bigcup \{U_s : U_s \setminus A \text{ is meager}\}$$

It is clear that $U_A \setminus A$ is meager and see that since $\mathbf{\Gamma}$ is closed under intersections with closed sets, $A \setminus U_A \in \mathbf{\Gamma}$. If $A \setminus U_A$ is not meager, by assumption, there exists $s \in \omega^{<\omega}$ such that $U_s \setminus (A \setminus U_A)$ is meager which implies that $U_s \setminus A$ is meager therefore $U_s \subseteq U$. It also implies that $U_s \setminus (A \setminus U_A) = U_s$ is meager which is a contradiction. Therefore, $A \setminus U_A$ is meager and A has the Baire property. □

Corollary 2.1.14 If for every $A \in \mathbf{\Gamma}$, the game $G^{**}_\omega(A)$ is determined, then every set in $\mathbf{\Gamma}$ has the Baire property.

The Perfect Set Property We now turn to another property that we can describe using infinite games, namely the perfect set property. To do so, we are going to define a new game G^*_ω of length ω, due to [Dav64]. The game G^*_ω is a hybrid version of the standard game G_ω and G^{**}_ω. Player I plays elements of $\omega^{<\omega} \setminus \varnothing$ while Player II answers with elements of ω as follows:

I	s_0		s_1		$\dots$
II		n_0		n_1	$\dots$

Player I wins the game if and only if $\oplus_{n<\omega} s_n \in A$ and, in addition, $s_{i+1}(0) \neq n_i$ for all $i < \omega$. The roles of the players in this game are asymmetrical. At each round, one can view Player II's moves as challenges for Player I to overcome. In other words, Player II can influence the sequence produced in the game only through Player I's moves. Their role is not to construct the sequence but to provide Player I with instructions on how they should do

[1] A collection $\mathbf{\Gamma} \subseteq \mathscr{P}(\omega^\omega)$ is called a *boldface pointclass* if it is closed under continuous pre-images and intersections with closed sets. Examples of pointclasses are: the collection of all Borel sets, F_σ sets, G_δ sets, $\mathbf{\Sigma}^1_1$ sets etc.

it. This is an important intuition that will be helpful later, when we discuss Kechris's games, in Chapter 4.

Just as for the Banach-Mazur game, this game can be coded as an instance of the standard game. In particular, $G^*_\omega(A)$ is equivalent to the game $G_\omega(A^*)$ where

$$A^* := \{x \in \omega^\omega : \oplus_{n<\omega} t_{x(2n)} \text{ and } t_{x(2n+2)} \neq x(2n+1)\}$$

Notation 2.1.15 In a standard game $G_\omega(A)$, where σ is a strategy for Player I, we use the notation $\sigma * x$ for the sequence constructed by the players if Player I plays according to σ and Player II plays $x \in \omega^\omega$. For the case of $G^*_\omega(A)$ games, we write $\sigma * x$ for the concatenation of Player I's moves according to σ when Player II plays $x \in \omega^\omega$. On the other hand, if τ is a strategy for Player II, a sequence $\overline{x} \in (\omega^{<\omega})^\omega$ is called *legal according to* τ if in the game:

I	$\overline{x}(0)$		$\overline{x}(1)$		$\ldots$	$\overline{x}(k)$		$\ldots$
II		n_0		n_1	$\ldots$		n_k	$\ldots$

where Player II plays according to τ and Player I plays $\overline{x}$, we have $\overline{x}(i+1)(0) \neq n_i$ for all $i < \omega$.

Theorem 2.1.16 [[Dav64]] Let $A \subseteq \omega^\omega$.

(i) Player I has a winning strategy in $G^*_\omega(A)$ if and only if A contains a perfect subset.

(ii) Player II has a winning strategy in $G^*_\omega(A)$ if and only if A is countable.

Proof. For (i), first suppose that Player I has a winning strategy σ in $G^*_\omega(A)$. It is easy to see that $\{\sigma * x : x \in \omega^\omega\}$ is a perfect subset of A. For the converse, suppose that A contains a perfect subset. This means that A contains a tree T such that for every node in T, there exists a 2-splitting node above it. We define strategy σ for Player I inductively as follows: start by playing $\sigma(\varnothing) = s_0 \in T$ such that s_0 is 2-splitting. This guarantees that Player I can stay on the perfect tree regardless of Player II's move. Suppose that $p = \langle s_0, n_0, \ldots, s_k, n_k\rangle$ is a partial play of the game G^*_ω where Player I plays according to σ. Define $p^* := \oplus_{i\leq k} s_i$ and let $\sigma(p) = s_{k+1}$ where s_{k+1} is such that $s_{k+1}(0) \neq n_k$ and $p^*{}^\frown s_{k+1}$ is a 2-splitting node. Such s_{k+1} exists because we assumed that, in p, Player I was playing according to σ and that T is a perfect tree, i.e. infinitely 2-splitting.

For (ii) first suppose that Player II has a winning strategy τ in $G^*_\omega(A)$. Let $p = \langle s_0, n_0, \ldots, s_k, n_k\rangle$ be a position of the game where Player II plays according to τ. For $x \in A$, we say that p is *compatible* with x if there exists $s_{k+1} \in \omega^{<\omega}$ such that $s_{k+1}(0) \neq n_k$ and $p^*{}^\frown s_{k+1} \subset x$. We say that p *rejects* x if it is compatible with x and for all $s_{k+1} \in \omega^{<\omega}$ such that $s_{k+1}(0) \neq n_k$ the position $p^*{}^\frown\langle s_{k+1}, \tau(p^\frown\langle s_{k+1}\rangle)\rangle$ is not compatible with x. In other words p is maximally compatible with x.

Now, suppose that for some $x \in A$ there is no p that rejects it. Then, at any point of the game, Player I can play $s \in \omega^{<\omega}$ such that $s_0{}^\frown \ldots {}^\frown s \subset x$ and after that they are in a winning position. But, Player II is playing according to τ which is a winning strategy, so this is a contradiction. Next, suppose that $p = \langle s_0, n_0, \ldots, s_k, n_k \rangle$ rejects both x and y and $x \neq y$. This means that p is compatible with both x and y. Therefore they can play $s \in \omega^{<\omega}$ such that $s(0) \neq n_k$ and any further extension of $p^{*\frown} s$ cannot be an initial segment of both x and y. Consider $n_{k+1} = \tau(p^{*\frown}\langle s \rangle)$ and see that $p^\frown \langle s, n_{k+1} \rangle$ is compatible either with x or with y which is a contradiction. Hence, p does not reject both x and y.

Finally, define $K_p := \{x \in \omega^\omega : p \text{ rejects } x\}$ and see that $A \subseteq \bigcup_p K_p$ and each K_p is a singleton. Since there are only countable p's we get that A is countable.

For the other direction of (ii), suppose that A is countable and let $\langle a_i \in A : i < \omega \rangle$ be an enumeration of its elements. Then, at each round $k < \omega$, Player II plays $n_k = a_k(| \oplus_{i \leq k} s_i|)$. This guarantees that the final sequence constructed by Player I differs from all elements of A and therefore is a winning strategy for Player II. □

2.2 The Axiom of Determinacy

In the previous section we discussed infinite games and the connection between regularity properties of sets of ω^ω and their determinacy. To see the full strength of infinite games one has to consider a new axiom, the Axiom of Determinacy (abbreviated as AD). The Axiom of Determinacy was introduced by Mycielski and Steinhaus in [MS62] and is the following statement:

$$\textit{Every standard game of length } \omega \textit{ on } \omega \textit{ is determined.} \qquad \text{(AD)}$$

At first glance, AD may not look as useful as it actually is, especially because of its incompatibility with AC (see Theorem 2.1.4). However, by adopting AD in our theory, we get several desired properties for the subsets of the reals such as the Baire property and the perfect set property. In addition, Mycielski and Świerczkowski in [MŚ64], provided a game for Lebesgue measurability which we are not going to discuss since it falls outside the scope of this thesis.

Even though AD is inconsistent with AC, we can still use it to get the principle of countable choice:

Lemma 2.2.1 (ZF + AD) Every countable family of non-empty sets of elements of ω^ω has a choice function.

Proof. Let $\mathscr{A} = \{A_n : n \in \omega\}$ be a countable family of sets of reals. Consider the following game: If player I plays $a = \langle a_0, a_1, a_2, \ldots \rangle$ and player II plays

$b = \langle b_0, b_1, b_2, \ldots \rangle$ then II wins if $b \in A_{a_0}$. We can immediately see that I doesn't have a winning strategy since if they play a_0, player II finds $b \in A_{a_0}$, plays b and wins. Therefore II must have a winning strategy τ and we define the choice function:

$$f(A_n) = \langle n, 0, 0, \ldots \rangle * \tau$$

□

It was already noticed by Mycielski [Myc64] that attempting to extend either the length of the game or the possible set of choices of the players beyond countable ordinals, would result in an axiom of determinacy that would be inconsistent even without AC. To see this, we will need the following proposition which was also proven in the same paper by Mycielski:

Proposition 2.2.2 (ZF + AD) [2] There exists no well-ordered uncountable set of reals.

This implies the following two limitations of AD:

Proposition 2.2.3 (ZF + AD) There exists $A \subseteq \omega_1^\omega$ such that $G_\omega(A)$ is non-determined.

Proof sketch. For $x \in \omega_1^\omega$, let $x_{II} := \langle x(1), x(3), \ldots, x(2k+1), \ldots \rangle$ and consider the set $A = \{x \in \omega_1^\omega : x(0) \geq \omega$ and x_{II} is not a code for $x(0)\}$. In the game $G_\omega(A)$, Player I plays an ordinal $\alpha \geq \omega$ and Player II tries to play a code for α. Player I cannot have winning strategy in $G_\omega(A)$, therefore II has a winning strategy τ. But, this strategy gives rise to a function that assigns to each countable ordinal one of its codes. Hence, there is an injection $\omega_1 \to \omega^\omega$ definable from τ which contradicts Proposition 2.2.2. □

Proposition 2.2.4 (ZF + AD) There exists $A \subseteq \omega^{\omega_1}$ such that $G_{\omega_1}(A)$ is non-determined.

Proof sketch. The argument behind this proof is very similar to the previous one. Consider the following game of length ω_1: Player I must, at some point, play 1, otherwise they lose. Suppose that Player I plays 1 for the first time in round $\alpha < \omega$. After this, Player II, in their next ω moves, must respond with an element of ω^ω that codes $\alpha < \omega_1$. It is clear that Player I does not have a winning strategy. If the game is determined, Player II must have a winning strategy τ. But, τ would give rise to an injection $\omega_1 \to \omega^\omega$ which contradicts Proposition 2.2.2. □

[2] For a proof of this proposition, see e.g. [Kan03, Proposition 27.11]

Chapter 3

Games of uncountable length

After discussing games of length ω, it is very natural to consider games of length $\kappa > \omega$ and see how the previous results generalize in this case. As we have already seen, when one makes the jump to higher cardinalities, many of the classical results fail to translate. The axiom of determinacy is no exception since the relevant formulation for longer games is inconsistent with ZF. To be more precise, let $\mathsf{AD}(\kappa, \lambda)$ be the following statement:

$$\textit{Every game of length } \lambda \text{ on } \kappa \text{ is determined.} \qquad (\mathsf{AD}(\kappa,\lambda))$$

Notice that, by Propositions 2.2.3 and 2.2.4, $\mathsf{AD}(\kappa, \lambda)$ is inconsistent if either $\kappa > \omega$ or $\lambda > \omega$.

Nevertheless, games like the Banach-Mazur game G^{**}_κ and the perfect set game G^*_κ, can still describe regularity properties for subsets of κ^κ. In section 3.2, we will study the correspondence between G^{**}_κ and a variant of the Baire property called the *asymmetric Baire property* and the connection between G^*_κ and the (generalized) perfect set property. Even without a relevant axiom of determinacy, some sets can be proven to be consistently determined if we adopt specific large cardinal hypotheses. Schlicht in [Sch17] shows that, after a Lévy-collapse of an inaccessible cardinal to κ^+, the games $G^{**}_\kappa(A)$ and $G^*_\kappa(A)$ are determined for every $A \subseteq \kappa^\kappa$ defined by a κ-sequence of ordinals. We will also discuss similar results in Chapter 4.

3.1 Standard games

Consider a higher Baire space κ^κ where κ is an uncountable cardinal such that $\kappa^{<\kappa} = \kappa$. For $A \subseteq \kappa^\kappa$, the *standard* game $G_\kappa(A)$ of length κ is played the same way as the game of length ω with the additional rule that Player I resumes the play at limit stages.

I	x_0		x_2		$\dots$	x_λ		$\dots$
II		x_1		x_3	$\dots$		$x_{\lambda+1}$	$\dots$

The winning condition for Player I is again $\langle x_0, x_1, \ldots, x_\alpha, x_{\alpha+1}, \ldots \rangle \in A$ otherwise Player II wins. It is sometimes very useful to think of such a game as a concatenation of κ-many games of length ω. We will see how this is relevant in Chapter 5.

In the generalized case, many basic results from the classical setting collapse. The most significant of them is the Gale-Steward theorem (Theorem 2.1.8):

Theorem 3.1.1 (AC) (Folklore) There exists a non-determined clopen set $A \subseteq \kappa^\kappa$.

Proof. We know that there exists a non-determined set $D \subseteq \kappa^\omega$, even without using the axiom of choice. Consider the following clopen set:

$$A := \bigcup_{s \in D} U_s$$

If any of the players had a winning strategy for A this would imply that they would also have a winning strategy for D, which is a contradiction. $\square$

If one tries to replicate the proof of Theorem 2.1.8 in the generalized case, one would need the notion of a defensive strategy. It immediately becomes apparent that we cannot prove the existence of such strategies, due to the existence of limit stages in the generalized game. To tackle this problem, let $A \subseteq \kappa^\kappa$ and $\mathbb{P}_A$ be the set of non-losing positions of Player I in the game $G_\kappa(A)$ defined in the analogous way to Definition 2.1.5. Let also $\mathbb{Q}_A$ be the set of non-losing positions of Player II in $G_\kappa(A)$. We prove the following:

Lemma 3.1.2 Let $A \subseteq \kappa^\kappa$ and $G_\kappa(A)$ be the standard game of length κ.

(i) If $\mathbb{P}_A$ is $<\kappa$-closed and Player II does not have a winning strategy in $G_\kappa(A)$, then Player I has a defensive strategy.

(ii) If $\mathbb{Q}_A$ is $<\kappa$-closed and Player I does not have a winning strategy in $G_\kappa(A)$, then Player II has a defensive strategy.

Proof. For (i), we are going to define strategy ∂_I for Player I such that for all $t \in \kappa^{<\kappa}$, the position $\partial_I * t$ is non-losing. We do this by induction on the length of t. First, for the base case, let $t = \varnothing$. Then $\partial_I * t$ will be the initial stage of the game which, by our hypothesis, is a non-losing position for Player I. Hence, there exists $x_0 < \kappa$ such that for all $y < \kappa$, the position $\langle x_0, y \rangle$ is non-losing for I so, let $\partial_I(\varnothing) = x_0$.

For the successor step, let t of successor length be such that $\partial_I * t$ is non-losing for Player I. From now on, let $p := \partial_I * t$. Then, there exists some $x_\alpha < \kappa$ such that for all $y < \kappa$ the position $p^\frown\langle x_\alpha, y \rangle$ is still non-losing for Player I because, if that was not the case, it would mean that for all $x < \kappa$, there exists $y < \kappa$ such that $p^\frown\langle x, y \rangle$ is a winning position for Player

II. But this implies that p was already a winning position for II which is a contradiction so, let $\partial_I(p) = x_\alpha$. Finally, let t be such that $|t| \in \mathrm{Lim}$ and, for all $\alpha < |t|$, the position $\partial_I * t \restriction \alpha$ is non-losing for Player I. The sequence $\langle \partial_I * t \restriction \alpha \mid \alpha < |t| \rangle$ is a strictly increasing sequence of elements of $\mathbb{P}_A$ such that

$$\bigcup_{\alpha < |t|} \partial_I * t \restriction \alpha = \partial_I * t$$

therefore, since $\mathbb{P}_A$ is $<\kappa$-closed and $\partial_I * t \restriction \alpha \in \mathbb{P}_A$ for all $\alpha < |t|$, we have $\partial_I * t \in \mathbb{P}_A$.

The proof of (ii) is analogous, with the roles of Player I and II interchanged. □

Corollary 3.1.3 The following holds:

(i) If $A \subseteq \kappa^\kappa$ is closed and the set $\mathbb{P}_A$ of non-losing positions of Player I is $<\kappa$-closed, then A is determined.

(ii) If $A \subseteq \kappa^\kappa$ is open and the set $\mathbb{Q}_A$ of non-losing positions of Player II is $<\kappa$-closed, then A is determined.

Proof. For (i) let $A \subseteq \kappa^\kappa$ be closed and suppose that the set $\mathbb{P}_A$ of non-losing positions for Player I is $<\kappa$-closed. Suppose that Player II does not have a winning strategy for $G_\kappa(A)$, then, by the previous lemma, Player I has a defensive strategy ∂_I. We will show that ∂_I is in fact a winning strategy. Let $x \in \kappa^\kappa$ and suppose, for contradiction, that $\partial_I * x \in \kappa^\kappa \setminus A$. Since $\kappa^\kappa \setminus A$ is open, there exists $s \subset \partial_I * x$ such that for all $y \supset s$ we have $y \in \kappa^\kappa \setminus A$. Therefore, s is a winning position for Player II in the game $G_\kappa(A)$ which is a contradiction because Player I plays defensively according to ∂_I. Therefore, ∂_I is a winning strategy.

The proof of (ii) is analogous, with the roles of Player I and II interchanged. □

In Section 1.3 we discussed how the notion of being (topologically) closed in κ^κ might not always be an accurate enough analogue to the classical case. One would naturally conjecture that being superclosed (or superopen, meaning the complement of a superclosed set) would be a sufficient condition for sets to be determined. Unfortunately, this also does not hold:

Proposition 3.1.4 (Folklore) There exists a non-determined superclosed set $A \subseteq \kappa^\kappa$.

Proof. We already know that there is a non-determined set $D \subseteq \kappa^\omega$ so consider the following set:

$$A := \left(\bigcup_{s \in D} U_s \right) \cup \left(\bigcup_{s \in \kappa^\omega \setminus D} U_{s ^\frown \langle 0,0 \rangle} \right)$$

which is superclosed. For contradiction, if Player I has a winning strategy for A then either I has a winning strategy for $\bigcup_{s \in D} U_s$ and hence for D, which is a contradiction, or has a winning strategy for $\bigcup_{s \in \kappa^\omega \setminus D} U_{s^\frown \langle 0,0 \rangle}$ which is also a contradiction because Player II can play $\alpha \neq 0$ in the $(\omega+1)$-st move. On the other hand, if Player II has a winning strategy then they would also have one for D, which is a contradiction. Hence A is non-determined. □

Remark 3.1.5 Theorem 3.1.1 and Proposition 3.1.4, can also be stated for the generalized Cantor space 2^κ. However, in order to get the desired non-determined set $D \subseteq 2^\omega$ needed for the proof, we need to also assume AC.

3.2 Games and Regularity properties in κ^κ

Even though many important results for standard games do not translate nicely to κ^κ, asymmetric games, like we previously discussed, still give us some connection between their determinacy and properties of subsets of κ^κ. In this part of the thesis we are going to present analogues of the $G^*_\kappa, G^{**}_\kappa$ games and see what their determinacy implies in this generalized context. This discussion will set the stage for the rest of the thesis where games of this asymmetric form will be at the core of our study.

The Banach-Mazur game G:** Let $A \subseteq \kappa^\kappa$. In the game G^{**}_κ each of the players play elements of $\kappa^{<\kappa}$, each of them extending the previous one. By the end of the game, the players have played a strictly increasing sequence $\langle s_\alpha : \alpha < \kappa \rangle$. Player I wins the game if and only if $\bigcup_{\alpha<\kappa} s_\alpha \in A$.

I	s_0		s_2		...
II		s_1		s_3	...

In the classical case, the game G^{**}_ω describes the Baire property for subsets of ω^ω. Unfortunately, this is not the case for κ^κ. In this setting, the game G^{**}_κ characterizes a variant of the Baire property called the *asymmetric-Baire property.* The results discussed in this part are all due to [Sch17] unless stated otherwise. To kick thing off, we first need to adjust our definitions to the κ-case:

Definition 3.2.1 A set A is called *κ-meager* if it is the union of κ-many nowhere dense sets.

First, let us see which parts of the proof of Lemma 2.1.12 work the same in the general case:

Definition 3.2.2 Let $A \subseteq \kappa^\kappa$. A *good position* $p = \langle s_0, s_1, s_2, \ldots \rangle$ for the game $G^{**}_\kappa(A)$, where Player II has a winning strategy τ, is a bounded sequence of length $\alpha < \kappa$ where α is even and the s_i's, for odd i, are the bounded sequences played by Player II according to the strategy τ. We denote with p^* the union of p i.e. the sequence constructed by the Players in the game thus far.

Proposition 3.2.3 Let $x \in \kappa^\kappa$. If for all good positions p such that $p^* \subseteq x$ there exists $t \supseteq p^*$ such that $\tau(p^\frown\langle t\rangle) \subseteq x$ then $x \notin A$.

Proof. Let p_0 a good position for the game $G^{**}_\kappa(A)$ such that $p_0^* \subseteq x$. There exists $t \supseteq p_0^*$ such that $\tau(p_0^\frown\langle t\rangle) \subseteq x$. Now we have that $p_0^\frown\langle t, \tau(p_0^\frown t)\rangle$ is another good position and this way we can create a sequence of good positions such that $p^* = \bigcup_{\alpha<\kappa} p_\alpha^* = x$. Therefore, since Player II plays according to τ, we have $x \notin A$. □

Lemma 3.2.4 ([Kov09]) Let $A \subseteq \kappa^\kappa$, then A is κ-meager if and only if Player II has a winning strategy in the game $G^{**}_\kappa(A)$.

Proof. For the left-to-right direction, let A be a κ-meager set which means that A is of the form $\bigcup_{\alpha<\kappa} C_\alpha$, where C_α are closed and nowhere dense. Therefore $\kappa^\kappa \setminus C_\alpha$ are open dense subsets of κ^κ. This means that in each move s_α of Player I, Player II can answer with $t_\alpha \supset s_\alpha$ such that $U_{t_\alpha} \subseteq \kappa^\kappa \setminus C_\alpha$. The final sequence $s_0 \subset t_0 \subset \cdots \subset s_\alpha \subset t_\alpha \subset \dots$ will converge to some $x \in \bigcap_{\alpha<\kappa} \kappa^\kappa \setminus C_\alpha \subseteq A^c$.

For the other direction, by the previous proposition, for

$$K_p = \{x \in \kappa^\kappa : p^* \subseteq x \text{ and } (\forall t \supseteq p^*)(\tau(p^\frown\langle t\rangle))\}$$

we have $A \subseteq \bigcup\{K_p \mid p \text{ a good position}\}$. For every good position we also have that $U_{p^*} \setminus K_p$ is open dense in U_{p^*}, so K_p is nowhere dense, and since there are at most κ good positions (because of our assumption $\kappa = \kappa^{<\kappa}$) we have that $\bigcup K_p$ is κ-meager and therefore A is κ-meager. □

Definition 3.2.5 A function $j : \kappa^{<\kappa} \to \kappa^{<\kappa}$ is called *dense* if for all $s \in \kappa^{<\kappa}$ and all $t \supseteq j(s)$ there is some $\alpha < \kappa$ such that $j(s^\frown\langle\alpha\rangle) \supseteq t$.

We will now use this and Definition 1.3.3 to characterize co-meager sets in κ^κ.

Lemma 3.2.6 ([Sch17]) Let $t \in \kappa^{<\kappa}$ and $A \subseteq \kappa^\kappa$. The set A is comeager in U_t (in other words $U_t \setminus A$ is meager) if and only if there exists a strict order preserving, dense, and continuous $j : \kappa^{<\kappa} \to \kappa^{<\kappa}$ such that $j(\varnothing) = t$ and $\operatorname{ran}(f_j) \subseteq A$.

Proof. For the left-to-right implication suppose that $A \subseteq \kappa^\kappa$ is comeager in U_t. We will construct $j(s)$ by induction on the length of s for all $s \in \kappa^{<\kappa}$. First see that since A is comeager in U_t, there exist open dense sets $D_\alpha \subseteq U_t$ for $\alpha < \kappa$ such that $\bigcap_{\alpha<\kappa} D_\alpha \subseteq A$. Throughout the proof we will also assume that for $\alpha \leq \beta < \kappa$ we have $D_\beta \subseteq D_\alpha$.

For the base case of our induction, let $j(\varnothing) = t$. For the successor steps, suppose that $|s| = \gamma$ a successor ordinal and we need to define $j(s^\frown\langle\alpha\rangle)$ for all $\alpha < \kappa$. Take the set

$$K = \{u \supset j(s) \mid U_u \subseteq D_\gamma\}$$

and see that, since D_γ is open and dense in U_t, for every $v \supseteq j(s)$ there exists $u \in K$ such that $v \subseteq u$. Since $\kappa^{<\kappa} = \kappa$ holds, we can choose an enumeration $\langle t_\alpha \mid \alpha < \kappa\rangle$ for the elements of K. Now we define $j(s^\frown\langle\alpha\rangle) = t_\alpha$ for every $\alpha < \kappa$.

For the limit steps, suppose that $|s| = \lambda$ is a limit ordinal and that $j(s{\restriction}\alpha)$ is defined for all $\alpha < \lambda$. We define $j(s) = \bigcup_{\alpha<\lambda} j(s{\restriction}\alpha)$ and this completes the induction. Now, by construction we have a dense continuous homomorphism such that $\operatorname{ran}(f_j) \subseteq \bigcap_{\alpha<\kappa} D_\alpha \subseteq A$.

For the right-to-left direction suppose that j satisfies all the desired properties. For every $x \in \kappa^\kappa$ we define the set:

$$K_x = \{s \in \kappa^{<\kappa} \mid t \subseteq s \text{ and } j(s) \subseteq x\}$$

which is obviously non-empty since $t \in A$.

Claim For every $x \in U_t$, if K_x doesn't have maximal elements then $x \in A$.

Proof. Suppose that K_x doesn't have maximal elements. Therefore we can construct a strictly increasing sequence $\langle s_\alpha \mid \alpha < \kappa\rangle$ such that $s_0 = t$ and $j(s_\alpha) \subseteq x$ for all $\alpha < \kappa$. Hence we will have that $x \in \operatorname{ran}(f_j) \subseteq A$. ■

Now, for every $s \in \kappa^{<\kappa}$ with $t \subseteq s$ let

$$C_s = \{x \in U_t \mid s \text{ maximal element of } K_x\}$$

which is a closed, nowhere dense subset of $U_{j(s)}$. It follows from the claim that $U_t \setminus A \subseteq \bigcup_{s\supseteq t} C_s$ and therefore A is comeager in U_t.

□

Definition 3.2.7 A set $A \subseteq \kappa^\kappa$ has the *asymmetric Baire Property* if there exists a strict order preserving and dense $j : \kappa^{<\kappa} \to \kappa^{<\kappa}$ with one of the following properties:

(i) $\operatorname{ran}(f_j) \subseteq A$

(ii) $\operatorname{ran}(f_j) \subseteq \kappa^\kappa \setminus A$ and j is continuous with $j(\varnothing) = \varnothing$.

Let us now see the connection of this to the Baire property in the space ω^ω. Suppose that $A \subseteq \omega^\omega$ has the asymmetric Baire property and let $j : \omega^{<\omega} \to \omega^{<\omega}$ be a strict order preserving and dense map. Since every strict order preserving map is also continuous in the classical case we have, by

Lemma 3.2.6, that either A is comeager in $U_{j(\varnothing)}$ or A is meager. Therefore, by Lemma 2.1.13, if $\mathbf{\Gamma}$ is a boldface pointclass and every set in $\mathbf{\Gamma}$ satisfies the asymmetric Baire property, then every set also satisfies the Baire property.

In the following lemma we will see how the asymmetric Baire property is related to the Banach-Mazur game G^{**}_κ:

Lemma 3.2.8 ([Sch17]) For $A \subseteq \kappa^\kappa$ we have the following:

(i) Player I has a winning strategy in the game G^{**} if and only if there exists a strict order preserving and dense $j : \kappa^{<\kappa} \to \kappa^{<\kappa}$ with $\operatorname{ran}(f_j) \subseteq A$.

(ii) Player II has a winning strategy in the game G^{**} if and only if there exists a strict order preserving, dense and continuous with $\operatorname{ran}(f_j) \subseteq \kappa^\kappa \setminus A$ and $j(\varnothing) = \varnothing$.

Proof. For the left-to-right direction of (i) let σ be a winning strategy for Player I in the game G^{**}. For every $t \in \kappa^{<\kappa}$ we will define by induction on $|t|$ the function $j(t)$ and partial runs:

$$\overline{s}_t = \langle s_t(\alpha) \mid \alpha \leq 2 \cdot |t| + 1 \rangle$$

where Player I plays according to σ such that for $t \subseteq u$ we have $\overline{s}_t \subseteq \overline{s}_u$ and $j(t) = s_t(2 \cdot |t|)$. For the base case, suppose that $\sigma(\varnothing) = v$ is the first move of Player I and let $j(\varnothing) = v$ and $\overline{s}_\varnothing = \langle v \rangle$. For the successor cases let $t \in \kappa^{<\kappa}$ and suppose that $\overline{s}_t$ and $j(t)$ are already defined. We choose an enumeration $\langle u_\alpha \mid \alpha < \kappa \rangle$ for the possible answers of Player II in $\overline{s}_t$. Also, let v_α be the answer of Player I to $\overline{s}_t{}^\frown \langle u_\alpha \rangle$ according to σ. We now define $\overline{s}_{t^\frown \langle \alpha \rangle} = \overline{s}_t{}^\frown \langle u_\alpha, v_\alpha \rangle$ and $j(t^\frown \langle \alpha \rangle) = v_\alpha$.

For the limit step let $|t|$ be a limit ordinal and suppose that $\overline{s}_{t \restriction \alpha}$ and $j(t \restriction \alpha)$ are already defined for $\alpha < |t|$. Now, if v is the answer of Player I to $\bigcup_{\alpha < |t|} \overline{s}_{t \restriction \alpha}$ according to σ, let

$$\overline{s}_t = (\bigcup_{\alpha < |t|} \overline{s}_{t \restriction \alpha})^\frown \langle v \rangle$$

and $j(t) = v$. This completes the induction and we have a dense homomorphism with $\operatorname{ran}(f_j) \subseteq A$ by construction.

For the right-to-left direction, let $j : \kappa^{<\kappa} \to \kappa^{<\kappa}$ be a dense homomorphism with $\operatorname{ran}(f_j) \subseteq A$. We will define a strategy σ by induction on partial runs $\overline{s}$ of even length. To do that, we will also define $t_{\overline{s}} \in \kappa^{<\kappa}$ such that $l(t_{\overline{s} \restriction 2 \cdot \alpha}) = \alpha$, $\sigma(\overline{s} \restriction 2 \cdot \alpha) = j(t_{\overline{s} \restriction 2 \cdot \alpha})$ and $t_{\overline{s} \restriction 2 \cdot \alpha} \subseteq t_{\overline{s} \restriction 2 \cdot \beta}$ with $2 \cdot \alpha \leq 2 \cdot \beta \leq l(\overline{s})$. This $t_{\overline{s}}$ arises from the density of j and is used to define the moves of Player I.

For the base case let $\sigma(\varnothing) = j(\varnothing)$. In the successor step suppose that $l(\overline{s})$ is even and that $t_{\overline{s} \restriction \alpha}$, $\sigma(\overline{s} \restriction \alpha)$ are already defined for all even $\alpha \leq l(\overline{s})$. Also, let $u \in \kappa^{<\kappa}$ be a possible answer of Player II to the partial run $\overline{s}^\frown \langle \sigma(\overline{s}) \rangle$

with $\sigma(\overline{s}) \subset u$. Since $l(\overline{s})$ has even length, we have that $j(t_{\overline{s}}) = \sigma(\overline{s}) \subset u$ and since j is dense there exists suitable $\alpha < \kappa$ such that $j(t_{\overline{s}}{}^\frown\langle\alpha\rangle) \supset u$. We now define:

$$t_{\overline{s}^\frown\langle\sigma(\overline{s}),u\rangle} = t_{\overline{s}}{}^\frown\langle\alpha\rangle \text{ and } \sigma(\overline{s}^\frown\langle\sigma(\overline{s},u\rangle)) = j(t_{\overline{s}^\frown\langle\sigma(\overline{s},u\rangle)}) = j(t_{\overline{s}}{}^\frown\langle\alpha\rangle)$$

For the limit stages, suppose that $|\overline{s}|$ is a limit ordinal and let $t_{\overline{s}\restriction\alpha}$, $\sigma(\overline{s}\restriction\alpha)$ are already defined for all $\alpha < |\overline{s}|$. We now define

$$t_{\overline{s}} = \bigcup_{\alpha<|\overline{s}|} t_{\overline{s}\restriction\alpha} \text{ and } \sigma(\overline{s}) = j(t_{\overline{s}})$$

This completes the induction and σ is a winning strategy for Player I by construction.

To prove (ii) first suppose that Player II has a winning strategy for the game $G^{**}(A)$. By Lemma 3.2.4 we have that this holds if and only if A is meager which means that $\kappa^\kappa \setminus A$ is comeager and by Lemma 3.2.6 this holds if and only if there exists a dense continuous homomorphism $j : \kappa^{<\kappa} \to \kappa^{<\kappa}$ such that $j(\varnothing) = \varnothing$ and $\operatorname{ran}(f_j) \subseteq A$. □

Let us now conclude our discussion about the Banach-Mazur game by comparing the κ-Baire property and the asymmetric Baire property for subsets of κ^κ. A set $A \subseteq \kappa^\kappa$ satisfies the κ-Baire property if there exists an open set U such that $A \triangle U$ is κ-meager. By a result of Halko and Shelah [HS01], the Baire property fails for κ-analytic subsets of κ^κ, namely

$$\text{Club}_\kappa := \{x \in \kappa^\kappa : \{\alpha < \kappa : x(\alpha) < 0\} \text{ contains a club}\}$$

is a κ-analytic set that does not satisfy the κ-Baire property. To see this, suppose that Club_κ is κ-comeager in some basic open set U_t. This means that, there are open dense sets $\langle U_\alpha : \alpha < \kappa\rangle$ such that $\bigcap_{\alpha<\kappa} U_\alpha \subseteq \text{Club}_\kappa$. Let us inductively construct a strictly increasing sequence $\langle t_\alpha : \alpha < \kappa\rangle$ of elements of $\kappa^{<\kappa}$ in the following way: First, let $t_0 = t$. For $\alpha = \beta+1$, pick $t_\alpha \supset t_\beta$ such that $U_{t_\alpha} \subseteq U_\beta$. For limit α, let $t_\alpha := \bigcup_{\beta<\alpha} t_\beta \cup \{\langle\alpha, 0\rangle\}$. Take $x := \bigcup_{\alpha<\kappa} t_\alpha$ and see that $x \notin \text{Club}_\kappa$ but $x \in \bigcap_{\alpha<\kappa} U_\alpha$ which is a contradiction. Using a similar argument, one can show that Club_κ is not κ-meager and therefore does not have property of Baire.

This goes to show that, for uncountable κ, the Baire property is stronger than the asymmetric Baire property since the former fails in ZFC for κ-analytic sets while the latter holds for all subsets of κ^κ that are definable by a κ-sequence of ordinals after a Lévy-collapse of an inaccessible cardinal [Sch17].

The perfect set game $\mathbf{G}^*_\kappa$: We now turn to the game G^*_κ which corresponds to the perfect set property for subsets of κ^κ. For $A \subseteq \kappa^\kappa$, the game $G^*_\kappa(A)$ is played as follows: In each round α of the game, Player I plays $s_\alpha \in \kappa^{<\kappa}$ and Player II responds with $\gamma_\alpha < \kappa$.

I	s_0		s_1		$\dots$	s_α		$\dots$
II		γ_0		γ_1	$\dots$		γ_α	$\dots$

Player I wins the game $G^*_\kappa(A)$ if and only if $\oplus_{\alpha<\kappa} s_\alpha \in A$ and $s_{\alpha+1}(0) \neq \gamma_\alpha$ for all $\alpha < \kappa$. We will now see how G^*_κ characterizes the perfect set property for subsets of κ^κ. Let us first give the following definition before stating the main result which is due to [Kov09].

Definition 3.2.9 A set $A \subseteq \kappa^\kappa$ is called *perfect* if it is superclosed and there exists a perfect tree [1] T such that $A = [T]$.

Lemma 3.2.10 [Kov09] Let $A \subseteq \kappa^\kappa$.

(i) Player II has a winning strategy in the game $G^*_\kappa(A)$ if and only if $|A| \leq \kappa$

(ii) Player I has a winning strategy in the game $G^*_\kappa(A)$ if and only if A contains a perfect subset.

Proof. For (i), let p be a position of the game where Player II plays according to τ. We write p^* for the concatenation of Player I's moves in the game thus far. If $p = \langle s_0, \gamma_0, \dots, s_\alpha, \gamma_\alpha \rangle$ is of successor length we say that p is a *good* position for x if there exists $s \in \kappa^{<\kappa}$ with $s(0) \neq \gamma_\alpha$ such that $p^{*\frown} s \subset x$. If $p = \langle s_0, \gamma_0, \dots, s_\alpha, \gamma_\alpha, \dots \rangle$ is of limit length, we say that p is *good* if $p^* \subseteq x$. Let

$$K_p = \{x \in \kappa^\kappa : p \text{ is a maximal good position for } x\}$$

and suppose that there exists $x \in A$ such that there is no maximal good position for x. This means that we can recursively construct a strictly increasing sequence of good positions such that $\bigcup_{i<\kappa} p_i^* = x$ which is a contradiction because Player II plays according to τ. We will now show that each K_p contains at most one element. Let $x, y \in K_p$ such that $x \neq y$. If p is of successor length, we use the same argument as in the proof of Theorem 2.1.16. If p is of limit length, let $s \in \kappa^{<\kappa}$ such that $p^{*\frown} s \subset x$ and $p^{*\frown} s \subset y$ but no other extension is an initial segment of both x and y. Let $\tau(p^\frown\langle s\rangle) = \gamma$ and see that $p^\frown\langle s, \gamma\rangle$ is a good position for exactly one of x and y which is a contradiction. It follows that

$$A \subseteq \bigcup \{K_p : p \text{ is position of the game where Player II plays } \tau\}$$

and therefore $|A| \leq |\bigcup_p K_p| \leq \kappa$ since there are at most κ-many positions of the game.

For the other direction, suppose that $|A| \leq \kappa$ and let $\langle a_i \in A : i < \kappa \rangle$ be an enumeration of its elements. Then, at each round $i < \kappa$, Player II plays

[1] see (ii) of Definition 1.3.8

$\gamma_i = a_i(| \oplus_{j \leq i} s_j|)$. This guarantees that the final sequence constructed by Player I differs from all elements of A and therefore is a winning strategy for Player II.

For (ii), first suppose that σ is a winning strategy for Player I. Let $P = \{\sigma * y \mid y \in \kappa^\kappa\}$ and since σ is a winning strategy we have $P \subseteq A$. This is a cofinally 2-splitting tree and, in addition, since Player I also plays at limit stages, the tree is also $<\kappa$-closed and therefore perfect.

For the other direction, let $P \subseteq A$ be a perfect subset of A. This means that P is of the form $[T]$ for some perfect tree T. Now, Player I can always jump to a splitting node in the tree T which will guarantee that they always stay inside the tree after Player II's move. Furthermore, Player I can always find suitable node to play at limit stages because of $<\kappa$-closeness. $\square$

Chapter 4

Kechris's Games

In his 1977 paper [Kec77], Kechris defines a general class of games that differ from the standard ones in the following sense: instead of alternately playing digits, players have asymmetric roles. In each round, Player I plays bounded sequences, while Player II plays a requirement that Player I's next move must satisfy. The asymmetry arises from the fact that Player II does not contribute to the sequence directly since, at the end of the game, the winner is determined by whether the concatenation of Player I's moves belongs to the payoff set. We have already seen examples of such games in previous chapters, with the most obvious one being the perfect set game.

In the same paper, Kechris associates this class of games with specific notions of smallness in the classical Baire space. These often coincide with notions such as meagerness or countability, making this general class an excellent blueprint for games that characterize regularity properties.

This chapter is divided in two sections. The first section is concerned with the generalization of the class of games defined in [Kec77] in the context of κ^κ (or, analogously, 2^κ). In addition, we study their connection to regularity properties and discuss the implications of their determinacy for subsets of κ^κ. The results of this section are all due to [SSz] unless stated otherwise. In the second section, we show how asymmetric games can be viewed as standard games [1] and prove several desired properties of the coding procedure. Games like the ones studied here will also work as building blocks for the next chapter, where we consider some modified versions and see how this affects their determinacy.

4.1 The class of G^F_κ games

Let R be an arbitrary set which we are going to call the set of *requirements.* Let also $F : R \to \mathscr{P}(\kappa^{<\kappa} \setminus \{\varnothing\})$ be a function called the *satisfaction function*

[1]We have already discussed how to code games like G^{**}_ω and G^*_ω to standard games in Chapter 2

which maps $r \in R$ (moves of Player II), to a set of suitable answers for Player I. For $A \subseteq \kappa^\kappa$, the game $G^F_\kappa(A)$ is played as follows:

$$\begin{array}{c||ccccccccc} \text{I} & s_0 & & s_1 & & \dots & s_\alpha & & \dots \\ \hline \text{II} & & r_0 & & r_1 & \dots & & r_\alpha & \dots \end{array}$$

In each round i, Player I plays $s_i \in \kappa^{<\kappa}$ and Player II answers with $r_i \in R$. Player I's moves must also satisfy $s_{i+1} \in F(r_i)$ for all $i < \kappa$. As before, Player I resumes the play in limit stages and wins if and only if $\oplus_{i<\kappa} s_i \in A$. Notice that, in contrast with standard games, only Player I's moves matter for the final sequence. Player II's role in this game is to give Player I challenges in order to make them miss the payoff set. Before moving forward, let us mention that, for the game $G^F_\kappa(A)$, we adapt the (generalized) analogue of Notation 2.1.15.

Notation 4.1.1 Let $A \subseteq \kappa^\kappa$. We have the following notation for the game $G^F_\kappa(A)$:

(i) If σ is a strategy for Player I and $x \in R^{\leq\kappa}$, we write $\sigma * x$ for the concatenation of Player I's moves if they play according to σ and Player II plays x.

(ii) If τ is a strategy for Player II and $\overline{x} \in (\kappa^{<\kappa})^\kappa$, we call $\overline{x}$ *legal according to* τ if in the game:

$$\begin{array}{c||ccccccccc} \text{I} & \overline{x}(0) & & \overline{x}(1) & & \dots & \overline{x}(\alpha) & & \dots \\ \hline \text{II} & & r_0 & & r_1 & \dots & & r_\alpha & \dots \end{array}$$

where Player II plays according to τ and Player I plays $\overline{x}$, we have $\overline{x}(\alpha+1) \in F(r_\alpha)$ for all $\alpha < \kappa$. In the same manner, we define *legal* $\overline{x} \in (\kappa^{<\kappa})^\beta$ for $\beta < \kappa$.

Let us now give some examples that highlight the range of different possible games that belong to this class.

Example 4.1.2 Let $A \subseteq \kappa^\kappa$.

(i) For $R = \kappa$ and $F(r) = \{s \in \kappa^{<\kappa} : s(0) \neq r\}$ the game $G^F_\kappa(A)$ is the perfect set game $G^*_\kappa(A)$.

(ii) For $R = \kappa^{<\kappa}$ and $F(r) = \{s \in \kappa^{<\kappa} : r \subseteq s\}$ the game $G^F_\kappa(A)$ is the Banach-Mazur game $G^{**}_\kappa(A)$.

(iii) For $R = \kappa$ and $F(r) = \{s \in \kappa^{<\kappa} : s(0) > r\}$ the game $G^F_\kappa(A)$ is the game for a variant of the Hurewicz Dichotomy in κ^κ. [2]

[2]The Hurewicz Dichotomy is the following statement for a set $A \subseteq \kappa^\kappa$: either there exists a sequence $\langle T_\alpha : \alpha < \kappa \rangle$ of $<\kappa$-splitting trees such that $A \subseteq \bigcup_{\alpha<\kappa}[T_\alpha]$ or A contains a κ-perfect tree. This specific variant of the Hurewicz dichotomy was studied in [LMS18] and also [SSz26, Section 6.1].

The examples show that one can use such games to characterize various regularity properties for subsets of κ^κ. In fact, since G^F_κ games only depend on the requirement set R and the function F, Kechris associated the pair $\mathscr{E} = \langle R, F\rangle$ with a specific notion of smallness. To do so, he also made the following extra assumptions for $\mathscr{E}$ [Kec77, p. 203]:

(K1) If $t \in F(r)$ and $t \subseteq s$, then $s \in F(r)$ (each $F(r)$ is upright-closed).

(K2) For all $\alpha < \kappa$ there exists $r \in R$ such that $t(0) \neq \alpha$ for all $t \in F(r)$ (there exists a requirement r that can force the first member of all sequences $t \in F(r)$ to be different that α).

In Kechris's original paper there is also a third requirement for $\mathscr{E}$ which, in the generalized case, already fails for the Banach-Mazur game. Let us now give the following definition which connects $\mathscr{E}$ to a notion of smallness for the generalized Baire space:

Definition 4.1.3 Let $T \subseteq \kappa^{<\kappa}$ and $j : R^{<\kappa} \to \kappa^{<\kappa}$ a strict order preserving map.

(i) T is called *$\mathscr{E}$-dense above* $t \in \kappa^{<\kappa}$ if for all $r \in R$ there exists $s \in F(r)$ such that $t^\frown s \in T$.

(ii) The map j is called *$\mathscr{E}$-dense* if for every $t \in R^{<\kappa}$, the set $\{j(t)^\frown r : r \in R\}$ is dense above $j(t)$. In other words, if for every $t \in R^{<\kappa}$ and every $r \in R$ there exists $s \in F(r)$ such that $j(t)^\frown s = j(t^\frown\langle r'\rangle)$ for some $r' \in R$.

(iii) T is called *$\mathscr{E}$-nowhere dense* if for all $t \in \kappa^{<\kappa}$, there exists $r \in R$ such that for all $s \in F(r)$ we have $t^\frown s \notin T$.

(iv) A set $A \subseteq \kappa^\kappa$ is called *$\mathscr{E}$-meager* if $A = \bigcup_{\alpha<\kappa} D_\alpha$ where each D_α is closed and T_{D_α} is $\mathscr{E}$-nowhere dense.

For the rest of this chapter we will always assume that, for a requirement set R and a satisfaction function $F : R \to \mathscr{P}(\kappa^{<\kappa} \setminus \{\varnothing\})$, the conditions (K1) and (K2) hold and, in addition, $|R| \leq \kappa$. The next two lemmas establish the connection between these notions and the determinacy of G^F_κ games and are both due to [SSz].

Lemma 4.1.4 ([SSz]) Player II has a winning strategy in $G^F_\kappa(A)$ if and only if A is $\mathscr{E}$-meager.

Proof. First, assume that τ is a winning strategy for Player II in the game $G^F_\kappa(A)$ and let $p = \langle t_\alpha, r_\alpha | \alpha < \beta\rangle$ be a position of the game where Player II plays according to τ. Let us write p^* for the concatenation of Player I's moves in the game thus far. If there exists $\beta' < \kappa$ such that $\beta = \beta' + 1$ then p

is called a *good* position for $x \in \kappa^\kappa$ if for some $s \in F(r_{\beta'})$ we have $p^{*\frown}s \subset x$. If β is a limit ordinal, p is a *good* position for x if $p^* \subseteq x$. The set

$$K_p = \{x \in A : p \text{ is a maximal good position for } x\}$$

is a closed subset of A and, moreover, $\mathscr{E}$-nowhere dense. To see this, let $t \in \kappa^{<\kappa}$ and notice that if $p^* \perp t$ then there is nothing to prove. Now, suppose that $t \subset p^*$ which means that there exists $u \in \kappa^{<\kappa}$ such that $t^\frown u = p^*$. By (K2) there exists $r \in R$ such that for all $s \in F(r)$ we have $s(0) \neq u(0)$ and therefore $t^\frown s \perp p^*$ and $t^\frown s \notin T_{K_p}$. So, the only interesting case is $p^* \subset t$. Let $u \in \kappa^{<\kappa}$ such that $p^{*\frown}u = t$ and suppose for contradiction that for all $r \in R$ there exists $s \in F(r)$ such that $t^\frown s \in T_{K_p}$. Let $\tau(p^\frown u) = r_{\beta+1}$ and notice that for $p^\frown\langle u, r_{\beta+1}\rangle$ there exists $s \in F(r_{\beta+1})$ such that $p^{*\frown}u^\frown s \in T_{K_p}$ therefore $p^{*\frown}u^\frown s \subset x$ for some $x \in A$. Hence, $p^\frown\langle u, r_{\beta+1}\rangle$ is a good position for this x which is a contradiction. Finally, let us prove the following:

$$A = \bigcup\{K_p : p \text{ is a position of the game } G^F_\kappa(A) \text{ where Player II plays } \tau\}$$

For simplicity's sake let us write $\bigcup_p K_p$ for the right-hand side of the equality. It is clear that $\bigcup_p K_p \subseteq A$ so, for the other direction, let $x \notin \bigcup_p K_p$. We will recursively construct a strictly increasing sequence of good positions $\langle p_\alpha : \alpha < \kappa\rangle$ for x. For the base case, see that $\varnothing = p_0$ is, trivially, a good position for x. For the successor step, suppose that p_α has already been constructed. By our assumption, p_α is not a maximal good position for x therefore there exists some good position $p_{\alpha+1}$ such that $p_\alpha \subset p_{\alpha+1}$. Finally, for the limit case, suppose that p_α has already been constructed for all $\alpha < \beta$ and β is a limit ordinal. Let $p_\beta = \bigcup_{\alpha<\beta} p_\alpha$ and see that $p^*_\alpha \subset x$ for all $\alpha < \beta$ therefore $p^*_\beta = \bigcup_{\alpha<\beta} p^*_\alpha \subset x$ which exactly means that p_β is a good position for x. Since, in each p_α, Player II plays according to τ and the sequence $\langle p^*_\alpha : \alpha < \kappa\rangle$ converges to x, we have $x \notin A$.

For the other direction, let A be an $\mathscr{E}$-nowhere dense set, which means that $A = \bigcup_{\alpha<\kappa} D_\alpha$ where each D_α is $\mathscr{E}$-nowhere dense. The strategy for Player II is the following: in each position $p = \langle t_0, r_0, \ldots, t_\alpha\rangle$ of the game, Player II plays r_α such that for every $t \in F(r_\alpha)$ we have $p^{*\frown}t \notin T_{D_\alpha}$. □

Lemma 4.1.5 ([SSz]) Player I has a winning strategy in $G^F_\kappa(A)$ if and only if there exists a strict order preserving and $\mathscr{E}$-dense map $j : R^{<\kappa} \to \kappa^{<\kappa}$ with $\mathrm{ran}(f_j) \subseteq A$.

Proof. The proof of this lemma is very similar to (i) of Lemma 3.2.8. First, suppose that σ is a winning strategy for Player I. We construct $j : R^{<\kappa} \to \kappa^{<\kappa}$ by induction on the length of $t \in R^{<\kappa}$. To do so, we will also define positions p_t of odd length such that $t \subseteq u$ implies $p_t \subseteq p_u$. For a position p, we write p^* for the concatenation of Player I's moves thus far.

For the base case, let $j(\varnothing) = \sigma(\varnothing) = v$ and $p_\varnothing = \langle v\rangle$. For successor cases, let $t \in R^{<\kappa}$ and $\langle \overline{r}_\alpha : \alpha < \kappa \rangle$ an enumeration of the elements of $R^{<\kappa}$. Also, let $\langle r_\alpha : \alpha < \kappa \rangle$ be an enumeration of R and $u_\alpha = \sigma(p_t{}^\frown\langle r_\alpha \rangle)$. Now, define $p_{t^\frown\langle r_\alpha \rangle} = p_t{}^\frown\langle r_\alpha, u_\alpha \rangle$ and $j(t^\frown\langle r_\alpha \rangle) = p_t^*{}^\frown u_\alpha$. For limit cases, suppose that $|t|$ is a limit ordinal and $p_{t\restriction\alpha}$, $j(t\restriction\alpha)$ are defined for all $\alpha < |t|$. Now, define $p_t = (\bigcup_{\alpha<|t|} p_{t\restriction\alpha})^\frown v$ where $v = \sigma(\bigcup_{\alpha<|t|} p_{t\restriction\alpha})$. Also, let $j(t) = (\bigcup_{\alpha<|t|} p_{t\restriction\alpha})^{*\frown} v$. This concludes the induction and the map j has the desired properties by construction.

For the other direction, let $j : R^{<\kappa} \to \kappa^{<\kappa}$ be an $\mathscr{E}$-dense strict order preserving map with $\operatorname{ran}(f_j) \subseteq A$. We will define winning strategy σ for Player I by induction on the positions p of the game of even length. To do so, we will also define $t_p \in R^{<\kappa}$ with $|t_{p\restriction 2\alpha}| = \alpha$ and $(p\restriction 2\alpha)^* = j(t_{p\restriction 2\alpha})$.

For the base case, let $\sigma(\varnothing) = j(\varnothing)$ and $t_\varnothing = \varnothing$. Now, let p be a position of even length and suppose that $t_{p\restriction\alpha}$, $\sigma(p\restriction\alpha)$ are already defined for all even $\alpha \leq |p|$. Let $r \in R$ be a possible answer of Player II to $p^\frown\sigma(p)$. Since j is $\mathscr{E}$-dense, let $s \in F(r)$ and suitable $r' \in R$ such that $j(t_p)^\frown s = j(t_p{}^\frown\langle r' \rangle)$. Now, define $\sigma(p^\frown\langle \sigma(p), r \rangle) = s$ and $t_{p^\frown\langle \sigma(p), r \rangle} = t_p{}^\frown\langle r' \rangle$. For limit $|p|$, suppose that $t_{p\restriction\alpha}$ and $\sigma(p\restriction\alpha)$ have already been defined for all $\alpha < |p|$. First, let $\bigcup_{\alpha<|t|} t_{p\restriction\alpha} = t_p$ and now, by j's $\mathscr{E}$-density there exist suitable $r' \in R$ and $s \in \kappa^{<\kappa}$ such that $j(t_p)^\frown s = j(t_p{}^\frown\langle r' \rangle)$. Finally, define $\sigma(p) = s$, which concludes the induction. □

Those results show that such games can be used to characterize various dichotomies in the generalized Baire space by choosing a suitable set of requirements R and a function F. Notice that, Lemmas 3.2.8 and 3.2.10 are special cases of 4.1.4 and 4.1.5.

Games of this general form are especially relevant mainly because of their connection to regularity properties which are consistently true for subsets of κ^κ definable from κ-sequences of ordinals. In particular, [SSz] shows that, after Lévy-collapsing an inaccessible cardinal $\lambda > \kappa$ to κ^+, the games $G_\kappa^F(A)$ with $|\operatorname{dom}(F)| = |R| = \kappa$ are determined for every $A \subseteq \kappa^\kappa$ definable from a κ-sequence of ordinals [3].

4.2 Coding G_κ^F games

In this section, we assume $\omega \leq \kappa = \kappa^{<\kappa}$ to discuss how games like the ones we have talked about so far can be seen as standard games and be treated as such. An essential assumption for the coding to work is that the requirement set R is of cardinality at most κ. For simplicity's sake, we will

[3] In more detail, [SSz] shows that the determinacy of $G_\kappa^F(A)$ follows from the restriction of the open dihypergraph dichotomy for A to the class of dihypergraphs definable from a κ-sequence of ordinals (see [SSz26, Definition 1.3]). The latter statement holds for all subsets of κ^κ definable from a κ-sequence of ordinals after a Lévy-collapse of an inaccessible cardinal to κ^+ by [SSz26, Theorem 1.4(a)]

discuss the case of $|R| = \kappa$. Fix bijections $\varphi : \kappa \to \kappa^{<\kappa} \setminus \{\varnothing\}$ and $\psi : \kappa \to R$. Our coding function $f : \kappa^\kappa \to \kappa^\kappa$ is the following:

$$f(x) = \varphi(x(0))^\frown \varphi(x(2))^\frown \dots ^\frown \varphi(x(2\alpha))^\frown \dots$$

One can easily notice that f is used for coding the moves of Player I in the game $G^F_\kappa(A)$ for some payoff set $A \subseteq \kappa^\kappa$. Next, let us see how to code the requirements that these moves must satisfy. Consider the following set:

$$C := \{x \in \kappa^\kappa : \forall\alpha < \kappa\ (\varphi(x(2\alpha+2)) \in F(\psi(x(2\alpha+1)))\}$$

See that Player I wins the game $G^F_\kappa(A)$ if and only if Player I wins the standard game $G_\kappa(A^F)$ where $A^F := f^{-1}[A] \cap C$. Moreover, the same holds for Player II, hence the game $G^F_\kappa(A)$ is equivalent to the game $G_\kappa(A^F)$ In the classical case, the coding function f is continuous and the set C is closed. In the generalized case we have something stronger:

Proposition 4.2.1 For $|R| = \kappa$ and $F : R \to \mathscr{P}(\kappa^{<\kappa} \setminus \{\varnothing\})$:

(i) The function $f : \kappa^\kappa \to \kappa^\kappa$ with

$$f(x) := \varphi(x(0))^\frown \varphi(x(2))^\frown \dots ^\frown \varphi(x(2\alpha))^\frown \dots$$

is supercontinuous.

(ii) The set

$$C = \{x \in \kappa^\kappa : \forall\alpha < \kappa\ (\varphi(x(2\alpha+2)) \in F(\psi(x(2\alpha+1))))\}$$

is superclosed.

Proof. For (i), consider $j : \kappa^{<\kappa} \to \kappa^{<\kappa}$ such that for any $s = \langle s(0), s(1), \dots\rangle$ we have

$$j(s) := \varphi(s(0))^\frown \varphi(s(2))^\frown \dots$$

it is immediate that j is strict order preserving and continuous. Also, for every x, we have $f(x) = \bigcup_{\alpha<\kappa} j(x\restriction\alpha)$.

For (ii), first see that it is immediate by definition that C is a closed set so C is of the form $[T_C]$ for a tree T_C. We can write C as

$$C = \bigcap_{\alpha<\kappa} C_\alpha = \bigcap_{\alpha<\kappa} \{z \in \kappa^\kappa : \varphi(z(2\alpha+2)) \in F(\psi(x(2\alpha+1)))\}$$

so it suffices to show that C_α is superclosed. For each C_α there exists a tree T_α, such that $C_\alpha = [T_\alpha]$, of the following form:

$$T_\alpha = \{t \in \kappa^{<\kappa} : |t| \geq 2\alpha+3 \to \varphi(t(2\alpha+2)) \in F(\psi(t(2\alpha+1))\}$$

Let $\langle s_\gamma : \gamma < \lambda\rangle$ be a strictly increasing sequence of elements of T_α for arbitrary $\lambda < \kappa$. Let $\beta < \lambda$ be the least ordinal such that $|s_\beta| \geq 2\alpha+3$

.If such β doesn't exist then it is immediate that the limit of the sequence belongs to T_α. For every $\beta \leq \xi < \lambda$ we have $\varphi(s_\xi(2\alpha+2) \in F(\psi(t(2\alpha+1)))$ and, since we have a strictly increasing sequence, we have $s = \bigcup_{\gamma<\lambda} s_\gamma$ is such that $\varphi(s(2\alpha+2)) \in F(\psi(t(2\alpha+1))$ therefore $s \in T_\alpha$ and T_α is superclosed. □

By Proposition 1.3.7, it is clear that for a superclosed set A, the set $A^F = f^{-1}(A) \cap C$ is also superclosed.

This procedure of coding asymmetric to standard games, highlights the expressivity of infinite and transfinite games in both the classical and the generalized context. In the classical setting, we use the coding to translate determinacy results from standard games to asymmetric games. This happens either with the help of AD or by other determinacy results for specific subsets of ω^ω. In higher Baire spaces, however, this stops being the case. Many essential classical determinacy results do not generalize to higher cardinalities, and AD also fails in this context. On the other hand, we have already discussed that asymmetric games can be consistently determined for specific subsets of κ^κ (see [Sch17; SSz26; SSz]), which is what leads us to view them as individual mathematical objects, separated from standard games. With this in mind, the coding presented in this chapter works more as a different way to to represent asymmetric games and less as a reduction principle, as opposed to the classical case.

Chapter 5

Non-determinacy of bounded games in higher Baire spaces

In this chapter, we modify Kechris's games presented in Chapter 4 by no longer requiring upwards closure for the sets of Player I's possible moves in each round [1]. In particular, we study variants where the possible moves of Player I are sequence of length bounded by an infinite $\mu < \kappa$. We present a way to reduce the non-determinacy of games of length κ to that of games of shorter length in order to make use of already known results, such as the non-determinacy of Bernstein subsets of ω^ω and 2^ω. The bound on the length of Player I's moves will allow us to produce closed non-determined subsets of κ^κ and 2^κ, which highlights the importance of such an assumption for the results of Chapter 4.

The chapter is divided in three parts: in the first part we discuss Kechris's games where Player I's moves are elements of $\kappa^{<\omega}$ in successor rounds. In the second part, we extend this class of games by introducing a second satisfaction function E that takes care of Player I's moves in rounds 0 and limit. We then fix an infinite $\mu < \kappa$ such that $\mu^{<\mu} = \mu$, and study games where Player I's moves are elements of $\kappa^{<\mu}$ in *all* rounds of the game. Finally, the third part is concerned with applications of what has been discussed thus far on variations of the standard game.

Our results confirm the intuition mentioned in the Introduction, that for the variants of the standard game where Player I plays sequences instead of ordinals, determinacy for closed sets depends on whether or not the length of sequences Player I's plays is bounded below κ.

Before moving forward we should note that throughout this chapter we work in ZFC as most of our results heavily rely on the existance of Bernstein sets and AC is necessary to prove their existence both in the classical case as well as in the generalized setting (see Proposition 5.2.4).

[1] This corresponds to condition (K1) for $\mathscr{E} = \langle R, F \rangle$ presented in section 4.1

5.1 Kechris's Games with moves of finite length for Player I

We start off by presenting a version of Kechris's games with a slight change in the satisfaction function F, namely, we force Player I to play finite sequences in successor rounds. To be more precise, fix a requirement set R such that $|R| \leq \kappa$ and a satisfaction function $F : R \to \mathscr{P}(\kappa^{<\omega} \setminus \{\varnothing\})$. For a limit ordinal $\lambda \leq \kappa$ and $A \subseteq \kappa^{\lambda}$, the game $G^F_{\kappa,\lambda}(A)$ of length λ is played as follows:

I	t_0		t_1		$\dots$	t_α		$\dots$
II		r_0		r_1	$\dots$		r_α	$\dots$

Player I plays $t_\gamma \in \kappa^{<\lambda}$ in rounds $\gamma \in \mathrm{Lim} \cup \{0\}$ and $t_{\alpha+1} \in \kappa^{<\omega}$ such that $t_{\alpha+1} \in F(r_\alpha)$ for every $\alpha < \lambda$. Player I wins the game $G^F_{\kappa,\lambda}(A)$ if and only if $\oplus_{\alpha<\kappa} t_\alpha \in A$.

Notice that for $\lambda = \omega$ all the moves of Player I are finite sequences. For $\lambda = \kappa$ we write G^F_κ instead of $G^F_{\kappa,\kappa}$. The following lemma highlights an important property of such games:

Lemma 5.1.1 Let $B \subseteq \kappa^\omega$ and $A_B = \{\oplus_{i<\kappa} z_i : z_i \in B\}$. The following hold:

(i) If Player I wins $G^F_\kappa(A_B)$ then they also win $G^F_{\kappa,\omega}(B)$.

(ii) If Player II wins $G^F_\kappa(A_B)$ then they also win $G^F_{\kappa,\omega}(B)$.

Proof. First, to prove (i), let σ be a winning strategy for Player I in $G^F_\kappa(A_B)$. We will define strategy σ' for Player I in $G^F_{\kappa,\omega}(B)$ by induction on positions of even length. For the base case, suppose that the first move of Player I in $G^F_\kappa(A_B)$ is $\sigma(\varnothing) = t_0 = u_0{}^\frown v_0$ with $|u_0| \in \mathrm{Lim}$ and $|v_0| < \omega$. Define $\sigma'(\varnothing) = v_0$. Now, let $p = \langle v_0, r_0, \dots, t_n, r_n\rangle$ be a position of the game and define $\sigma'(p) = \sigma(\langle t_0, r_0, \dots, t_n, r_n\rangle)$ for all $0 \leq n < \omega$.

To prove that σ' is indeed a winning strategy for Player I, let $q' = \langle v_0, r_0, t_1, r_1, \dots\rangle$ be a run of the game $G^F_{\kappa,\omega}(B)$ where Player I plays according to σ' and $q = \langle t_0, r_0, t_1, r_1, \dots\rangle$ be a run of the game $G^F_\kappa(A_B)$ where Player I plays according to σ and Player II plays the same r_n's in the first ω rounds. We have, by the definition of σ', that $t_0 = u_0{}^\frown v_0$ for $u_0 \in \kappa^{<\kappa}$ of limit length. Let $z = v_0{}^\frown \oplus_{1\leq n<\omega} t_n$ and see that $x := u_0{}^\frown z^\frown \oplus_{\omega\leq\alpha<\kappa} t_\alpha \in A_B$ since σ is a winning strategy. Hence, $z \in B$ and σ' is a winning strategy for Player I in $G^F_{\kappa,\omega}(B)$.

For (ii), suppose that Player II has a winning strategy τ for $G^F_\kappa(A_B)$ but not for $G^F_{\kappa,\omega}(B)$. We are going to inductively construct a run $\langle t_\alpha, r_\alpha \mid \alpha < \kappa\rangle$ of the game $G^F_\kappa(A_B)$ where Player II uses τ. We are going to do this by constructing each $[\omega{\cdot}\beta, \omega{\cdot}\beta + \omega)$ segment of the run. First, let τ_0 be the strategy τ of Player II restricted to the first ω moves of the game $G^F_\kappa(A_B)$.

Since, τ_0 cannot be winning for II in the game $G^F_{\kappa,\omega}(B)$, let $\langle t_n, r_n \mid n < \omega\rangle$ be a run of the game where Player II plays according to τ_0 but does not win, i.e. $z_0 := \oplus_{i<\omega} t_i \in B$. Suppose now that $\langle t_\alpha, r_\alpha \mid \alpha < \omega{\cdot}\beta\rangle$ is already defined for $\beta < \kappa$. First, define strategy τ_β for Player II in $G^F_{\kappa,\omega}(B)$ such that

$$\tau_\beta(\langle s_0, r_0, \dots, s_n\rangle) = \tau(\langle t_\alpha, r_\alpha \mid \alpha < \omega{\cdot}\beta\rangle^\frown\langle s_0, r_0, \dots, s_n\rangle)$$

for all $n < \omega$. Since τ_β is not a winning strategy in the game $G^F_{\kappa,\omega}(B)$, let $\langle t_{\omega\cdot\beta+i}, r_{\omega\cdot\beta+i} \mid i < \omega\rangle$ be a run where II plays according to τ_β but doesn't win the game, i.e. $z_\beta = \oplus_{i<\omega} t_{\omega\cdot\beta+i} \in B$. In this way, we constructed a run of the game where Player II uses τ but for all $i < \kappa$ we have $z_i \in B$, therefore $z := \oplus_{i<\kappa} z_i \in A_B$ which is a contradiction. □

Corollary 5.1.2 If $G^F_{\kappa,\omega}(B)$ is non-determined, so is $G^F_\kappa(A_B)$.

Notice that such sets $A_B \subseteq \kappa^\kappa$ are closed regardless of the complexity of B. We will use this construction to provide closed non-determined subsets of κ^κ.

Before moving forward we will need to introduce some additional terminology which will be necessary for the results that follow.

Definition 5.1.3 Suppose $2 \leq d \leq \kappa$ and D a set. A map $\varphi : d^{<\kappa} \to D^{<\kappa}$ is called:

(i) *$\perp$-preserving* if for $s, t \in d^{<\kappa}$ such that $s \perp t$ we have $\varphi(s) \perp \varphi(t)$.

(ii) *length preserving* if $|s| = |\varphi(s)|$ for every $s \in d^{<\kappa}$.

(iii) *strong* if it is strict order, length and $\perp$-preserving.

Lemma 5.1.4 If $\varphi : 2^{<\kappa} \to R^{<\kappa}$ is strong, where R is an arbitrary set, then $T_\varphi := \mathrm{dw}(\mathrm{ran}(\varphi))$ is a perfect tree and $\mathrm{ran}(f_\varphi) = [T_\varphi]$.

Proof. See [SSz26, Corollaries 5.1.2, 5.1.6] □

By adjusting the definitions of the game $G^F_{\kappa,\lambda}$ to the generalized Cantor space 2^κ we can easily produce closed non-determined subsets of 2^κ. To do so, we also require the set R and the function F to satisfy the following two conditions:

(G1) $\forall r \in R\ \exists t, u \in F(r)\ (t \perp u)$

(G2) $\forall i < 2\ \exists r \in R\ \forall t \in F(r)\ (t(0) \neq i)$

The most important difference between these games and games from Chapter 4 is the absence of upwards closure of the sets $F(r)$. We've replaced therefore the condition (K1) with something weaker and left condition (K2) untouched. This might look like a slight change in our framework but it has major implication on the determinacy of these games. Before moving forward, we will need the following definition:

Definition 5.1.5 A set $B \subseteq 2^\omega$ is called a *Bernstein set* if it intersects every perfect subset of 2^ω but contains none of them. In the same manner we can define Berstein sets $B \subseteq \omega^\omega$. [2]

Notice that, if $B \subseteq 2^\omega$ is a Bernstein set, then $2^\omega \setminus B$ is also Bernstein. Such sets are commonly used as counterexamples to regularity properties since most games on B are non-determined. Their existance both in ω^ω and 2^ω, as well as in κ^κ and 2^κ (see Proposition 5.2.4) requires the axiom of choice. In the next lemma we use such a subset of 2^ω to produce a non-determined subset of 2^κ.

Lemma 5.1.6 Let $B \subseteq 2^\omega$ be a Bernstein set. For arbitrary set R such that $|R| \leq \kappa$ and $F : R \to \mathscr{P}(2^{<\omega} \setminus \{\varnothing\})$ which satisfy (G1) and (G2), the game $G^F_{2,\omega}(B)$ is non-determined.

Proof. For contradiction, suppose that $G^F_{2,\omega}(B_e)$ is determined. First, let σ be a winning strategy for Player I. We will inductively construct a strong map $\varphi : 2^{<\omega} \to R^{<\omega}$ and a map $\psi : 2^{<\omega} \to 2^{<\omega}$ such that:

(i) $\psi(t) = \sigma * \varphi(t)$

(ii) $\psi(t^\frown\langle 0\rangle) \perp \psi(t^\frown\langle 1\rangle)$

Then, T_ψ will be a perfect subtree of $T_\sigma := \mathrm{dw}(\{\sigma * s : s \in R^<\omega\})$ since it is strict order preserving by (i) and $\perp$-preserving by (ii). The map φ defines the moves of Player II that we will use to construct the tree. For the base case of the construction, let $\varphi(\varnothing) = \varnothing$ and $\psi(\varnothing) = \sigma(\varnothing)$. Then, suppose that, for $s \in 2^{<\omega}$ of successor length, $\varphi(s)$ is defined and $\psi(s) = \sigma * \varphi(s)$. To define $\varphi(s^\frown\langle i\rangle)$ and $\psi(s^\frown\langle i\rangle)$ for $i = 0, 1$, let p be the position of the game where Player I plays according to σ and Player II answers with $\varphi(s)$. By (G2) there exists $r_{s,0} \in R$ such that for all $t \in F(r_{s,0})$ we have $t(0) \neq 0$. Similarly, there exists $r_{s,1} \in R$ such that for all $t' \in F(r_{s,1})$ we have $t'(0) \neq 1$. Define

$$t_{s,0} := \sigma(p^\frown\langle r_{s,0}\rangle) \text{ and } t_{s,1} := \sigma(p^\frown\langle r_{s,1}\rangle)$$

Notice that, $t_{s,0} \perp t_{s,1}$, hence let $\varphi(s^\frown\langle i\rangle) := \varphi(s)^\frown\langle r_{s,i}\rangle$ and $\psi(s^\frown\langle i\rangle) := \psi(s)^\frown t_{s,i}$ for $i = 0, 1$.

The map ψ is by construction strict order preserving and $\perp$-preserving therefore T_ψ is a perfect tree. For $x \in 2^\omega$ we have $f_\psi(x) = \sigma * f_\varphi(x)$ therefore $[T_\psi] \subseteq P_\sigma \subseteq B$ which is a contradiction since B is Bernstein.

Next, suppose that τ is a winning strategy for Player II. We construct strict order preserving and $\perp$-preserving map $\psi : 2^{<\omega} \to 2^{<\omega}$ such that $\mathrm{ran}(f_\psi) \subseteq Q_\tau$ where $Q_\tau = \{\overline{x} * \tau : \overline{x} \in (2^{<\omega})^\omega \text{ legal according to } \tau\}$. First, we will define the map $\varphi : 2^{<\omega} \to 2^{<\omega}$ such that

[2] The construction of a Bernstein set can be found e.g. in [Kec95, Example 8.24, pp 48] or see Lemma 5.2.4 for a proof of the existence of Bernstein subsets of κ^κ for $\kappa = \kappa^{<\kappa}$. We should also point that, to construct such sets, one need to make use of the axiom of choice.

(a) $\langle \varphi(s\restriction n) : n \leq |s|\rangle$ is legal according to τ.

(b) $\varphi(s^\frown\langle 0\rangle)\perp\varphi(s^\frown\langle 1\rangle)$

We do this by induction on $|s|$ for each $s \in 2^{<\omega}$. If $s = \varnothing$, we arbitrarily define $\varphi(s) := t \in 2^{<\omega}$. Suppose that $\langle \varphi(s\restriction n) : n \leq |s|\rangle$ is already defined. We will construct $\varphi(s^\frown\langle i\rangle)$ for $i = 0, 1$. Let p be the position of the game where Player I plays $\langle \varphi(s\restriction n) : n \leq |s|\rangle$ and Player II's answers are determined by τ. Let r be II's last move in p and, by (G1), there exist $t_0, t_1 \in F(r)$ such that $t_0 \perp t_1$. Define

$$\varphi(s^\frown\langle 0\rangle) := t_0 \text{ and } \varphi(s^\frown\langle 1\rangle) := t_1$$

Now, using φ, let us define

$$\psi(s) := \bigoplus_{n\leq|s|} \varphi(s\restriction n)$$

which, by construction, is strict order preserving and $\perp$-preserving therefore $[T_\psi]$ is perfect. Also, for every $x \in 2^\omega$ we have $f_\psi(x) = \bigcup_{n<\omega} \psi(x\restriction n) = \bigcup_{n<\omega}\left(\oplus_{k\leq|x\restriction n|} \varphi((x\restriction n)\restriction k)\right) = \oplus_{k<\omega}\varphi(x\restriction k)$. Therefore $[T_\psi] \subseteq Q_\tau \subseteq B$ which is a contradiction.

□

Corollary 5.1.7 For the set $A_B = \{\oplus_{i<\kappa} z_i : z_i \in B\} \subseteq 2^\kappa$, where $B \subseteq 2^\omega$ is Bernstein, the game $G^F_{2,\kappa}(A)$ is non-determined.

Proof. Immediate from Lemma 5.1.1. □

By tweaking the conditions (G1) and (G2) we can produce a similar closed non-determined subset of κ^κ. To do that, we will also need the following definition.

Definition 5.1.8 Let $e : \kappa \to \omega$ be a surjection. For $z \in \kappa^{\leq\kappa}$, we define $z_e \in \omega^{\leq\kappa}$ where $z_e(\alpha) = e(z(\alpha))$ for all $\alpha < \kappa$.

Now, for $e : \kappa \to \omega$ a surjection and for R such that $|R| \leq \kappa$ and $F : R \to \mathscr{P}(\kappa^{<\omega} \setminus \{\varnothing\})$ suppose that the following hold:

(G1e) $(\forall r \in R)(\exists t, s \in F(r))(t_e \perp s_e)$

(G2e) $(\forall i < \omega)(\exists r \in R)(\forall t \in F(r))(e(t(0)) \neq i)$

Lemma 5.1.9 Let $B \subseteq \omega^\omega$ be a Bernstein set and $e : \kappa \to \omega$ be a surjection. Then the game $G^F_{\kappa,\omega}(B_e)$ is non-determined for $B_e := \{z \in \kappa^\omega : z_e \in B\}$

Proof. For contradiction, suppose that $G^F_{\kappa,\omega}(B_e)$ is determined. First, let σ be a winning strategy for Player I. We will inductively construct a strong map $\varphi : 2^{<\omega} \to R^{<\omega}$ and a map $\psi : 2^{<\omega} \to \omega^{<\omega}$ such that:

(i) $\psi(t) = (\sigma * \varphi(t))_e$

(ii) $\psi(t^\frown\langle 0\rangle)\perp\psi(t^\frown\langle 1\rangle)$

Then, T_ψ will be a perfect subtree of $T_{\sigma,e} := \{(\sigma * s)_e : s \in R^{<\omega}\}$ since it is strict order preserving by (i) and $\perp$-preserving by (ii). The map φ defines the moves of Player II that we will use in the construction of T_ψ. For the base case of the construction, let $\varphi(\varnothing) = \varnothing$ and $\psi(\varnothing) = (\sigma(\varnothing))_e$. Now, suppose that $\varphi(s)$ is defined for $s \in 2^{<\omega}$ and $\psi(s) = (\sigma * \varphi(s))_e$. To define $\varphi(s^\frown\langle i\rangle)$ and $\psi(s^\frown\langle i\rangle)$ for $i = 0, 1$, let p be the position of the game where Player I plays according to σ and Player II answers with $\varphi(s)$. Fix arbitrary $r_{s,0} \in R$ and, by (G2e), there exists $r_{s,1} \in R$ such that for all $t \in F(r_{s,1})$ we have $e(t(0)) \neq e(\sigma(p^\frown\langle r_{s,0}\rangle)(0))$. Define

$$t_{s,0} := \sigma(p^\frown\langle r_{s,0}\rangle) \text{ and } t_{s,1} := \sigma(p^\frown\langle r_{s,1}\rangle)$$

Notice that, $(t_{s,0})_e\perp(t_{s,1})_e$, hence let $\varphi(s^\frown\langle i\rangle) = \varphi(s)^\frown\langle r_{s,i}\rangle$ and $\psi(s^\frown\langle i\rangle) = \psi(s)^\frown(t_{s,i})_e$ for $i = 0, 1$.

The map ψ is by construction strict order preserving and $\perp$-preserving therefore T_ψ is a perfect tree. For $x \in 2^{<\omega}$ we have $f_\psi(x) = (\sigma * f_\varphi(x))_e$ therefore $[T_\psi] \subseteq P_{\sigma,e} := \{(\sigma * x)_e : x \in R^\omega\} \subseteq B$ which is a contradiction since B is Bernstein.

Next, suppose that τ is a winning strategy for Player II. We construct strict order preserving and $\perp$-preserving map $\psi : 2^{<\omega} \to \kappa^{<\omega}$ such that $\mathrm{ran}(f_\psi) \subseteq Q_{\tau,e}$ where $Q_\tau = \{\overline{x} * \tau : \overline{x} \in (\kappa^{<\omega})^\omega \text{ legal according to } \tau\}$ and $Q_{\tau,e} = \{(\overline{x} * \tau)_e : \overline{x} \in (\kappa^{<\omega})^\omega \text{ legal according to } \tau\}$. First, we will define the map $\varphi : 2^{<\omega} \to \kappa^{<\omega}$ such that

(a) $\langle \varphi(s{\restriction}\alpha) : \alpha \leq |s|\rangle$ is legal according to τ.

(b) $(\varphi(s^\frown\langle 0\rangle))_e\perp(\varphi(s^\frown\langle 0\rangle))_e$

We do this by induction on $|s|$ for each $s \in 2^{<\omega}$. If $|s| = 0$, we arbitrarily choose $\varphi(s) \in \kappa^{<\omega}$. Then, suppose that $\langle \varphi(s{\restriction}n) : n \leq |s|\rangle$ has already been defined. We will construct $\varphi(s^\frown\langle i\rangle)$ for $i = 0, 1$. Let p be the position of the game where Player I plays $\langle \varphi(s{\restriction}n) : n \leq |s|\rangle$ and Player II's answers are determined by τ. Let r be II's last move in p and, by (G1e), there exist $t_0, t_1 \in F(r)$ such that $(t_0)_e\perp(t_1)_e$. Define

$$\varphi(s^\frown\langle 0\rangle) := t_0 \text{ and } \varphi(s^\frown\langle 1\rangle) := t_1$$

Now, using φ, let us define

$$\psi(s) := \bigoplus_{n\leq|s|} (\varphi(s{\restriction}n))_e$$

which, by construction, is strict order preserving and $\perp$-preserving therefore $[T_\psi]$ is perfect. Also, for every $x \in 2^\omega$ we have $f_\psi(x) = \bigcup_{n<\omega} \psi(x{\restriction}n) =$

$\bigcup_{n<\omega}\left(\oplus_{k\leq|x\restriction n|}\left(\varphi((x\restriction n)\restriction k)\right)_e\right) = \oplus_{k<\omega}(\varphi(x\restriction k))_e$. Therefore $[T_\psi] \subseteq Q_{\tau,e} \subseteq B$ which is a contradiction.

□

Corollary 5.1.10 For the set

$$A_{B,e} = \{\oplus_{i<\kappa} z_i : z_i \in \kappa^\omega \text{ and } (z_i)_e \in B \text{ for all } i<\kappa\} \subseteq \kappa^\kappa$$

where $B \subseteq \omega^\omega$ is Bernstein, the game $G^F_{2,\kappa}(A)$ is non-determined.

Proof. Immediate from Lemma 5.1.1. □

5.2 Kechris's Games with moves of bounded length for Player I

5.2.1 Games with moves of bounded length in successor rounds only

We will now see what happens when we try to generalize the results of the previous section for sequences bounded by an uncountable $\omega < \mu < \kappa$ such that $\mu^{<\mu} = \mu$. Player I is now allowed to play sequences of length $<\mu$ in successor stages.

Let R be an arbitrary set of requirements such that $|R| \leq \kappa$ and let $F : R \to \mathscr{P}(\kappa^{<\mu} \setminus \{\varnothing\})$ and consider the game $G^F_{\kappa,\mu}$.

Definition 5.2.1 A set $B \subseteq \kappa^\mu$ is called a *dense subset of* κ^μ if $B \cap V_t \neq 0$, for all $V_t = \{x \in \kappa^\mu : t \subseteq x\}$ where $t \in \kappa^{<\mu}$.

We will use the following lemma to construct a counterexample to the analogue of Lemma 5.1.1 for $\mu < \kappa$.

Lemma 5.2.2 If $B \subseteq \kappa^\mu$ is a dense subset of $\kappa^{<\mu}$, then Player I has a winning strategy in $G^F_\kappa(A_B)$ for

$$A_B := \{\oplus_{i<\kappa} z_i : z_i \in B\}$$

Proof. We will define strategy in rounds $\beta \in [\omega\cdot\alpha,\ \omega\cdot\alpha+\omega)$ for all $\alpha < \kappa$, by induction on α. For the base case, define $t_0 = \varnothing$ and for each of Player II's moves $r_n \in R$, Player I arbitrarily plays $t_{n+1} \in F(r_n)$. For the successor step, let $1 \leq \alpha < \kappa$ and suppose that, for all $\beta \leq \alpha$, the moves in $[\omega\cdot\beta,\ \omega\cdot\beta+\omega)$ have already been defined. We define the moves in $[\omega\cdot(\alpha+1),\ \omega\cdot(\alpha+1)+\omega)$. First, notice that $T_B = \{t \in \kappa^{<\mu} : (\exists x \in B)(t \subseteq x)\} = \kappa^{<\mu}$ by our assumption about B. Let

$$u_\alpha := \bigoplus_{n<\omega} t_{\omega\cdot\alpha+(n+1)} \in \kappa^{<\mu}$$

Since B is dense in κ^μ, and in particular since $B \cap V_{u_\alpha} \neq \varnothing$, there exists suitable $t_{\omega\cdot(\alpha+1)} \in \kappa^\mu$ such that $u_\alpha{}^\frown t_{\omega\cdot(\alpha+1)} \in B$. For the rest of the moves of

Player I in $[\omega\cdot(\alpha+1), \omega\cdot(\alpha+1)+\omega)$, Player I arbitrarily plays $t_{\omega\cdot(\alpha+1)+(n+1)} \in F(r_{\omega\cdot(\alpha+1)+n})$ for all $n < \omega$. Finally, let $\alpha \in \mathrm{Lim}$. Define $t_{\omega\cdot\alpha} = \varnothing$ and the let Player I arbitrarily play $t_{\omega\cdot\alpha+(n+1)} \in F(r_{\omega\cdot\alpha+n})$. Now, for each

$$z_\alpha := \Big(\bigoplus_{n<\omega} t_{\omega\cdot\alpha+(n+1)}\Big)^\frown t_{\omega\cdot\alpha+\omega}$$

we have that $z_\alpha \in B$ therefore $\oplus_{\alpha<\kappa} z_\alpha \in A_B$. □

Under any non-trivial assumptions for R and F, using this lemma can give us counterexamples of the analogue of Lemma 5.1.1. Take for example $B = \{t^\frown\langle 0\rangle^\mu : t \in \kappa^{<\mu}\}$ where $\langle 0\rangle^\mu$ denotes the constant sequence of length μ with value 0 and see that if F is non-trivial (i.e. for every $\alpha < \kappa$ there exists a requirement $r \in R$ such that for every $t \in F(r)$, we have $t(0) \neq \alpha$), Player I has a winning strategy in $G^F_\kappa(A_B)$ but Player II has a winning strategy in $G^F_{\mu,\kappa}(B)$ namely, in each round Player II forces Player I to play $t \in \kappa^{<\mu}$ such that $t(0) \neq 0$.

5.2.2 Games with moves of bounded length in every round

In order to explore similar situations, we expand the class of Kechris's games by introducing a second satisfaction function that puts requirements on the moves of Player I in round 0 and limit rounds. To make this concrete, fix $\mu < \kappa$ be an infinite cardinal such that $\mu^{<\mu} = \mu$. Let also R be an arbitrary set of requirements $|R| \leq \kappa$ and fix $F : R \to \mathscr{P}(\kappa^{<\mu} \setminus \{\varnothing\})$ and $E : \kappa \cap (\mathrm{Lim} \cup \{0\}) \to \mathscr{P}(\kappa^{<\mu})$. For a payoff set $A \subseteq \kappa^\lambda$, where $\mu \leq \lambda < \kappa$ and $\lambda^{<\lambda} = \lambda$, define the following game $G^{F,E}_{\kappa,\lambda}$ of length λ:

I	t_0		t_1		$\dots$	t_α		$\dots$
II		r_0		r_1	$\dots$		r_α	$\dots$

where Player I's moves are such that $t_\alpha \in E(\alpha)$ if $\alpha \in \kappa \cap (\mathrm{Lim} \cup \{0\})$ and $t_{\alpha+1} \in F(r_\alpha)$ for every $\alpha < \lambda$. As always, Player I's wins the game if and only if $\oplus_{\alpha<\mu} t_\alpha \in A$. The game $G^{F,E}_\kappa(B)$ of length κ is defined similarly. The only change compared to the previous definition is the length of the game. The following lemma can be proved by adapting the proof of Lemma 5.1.1 for $\mu > \omega$ in a straightforward way:

Lemma 5.2.3 Let $B \subseteq \kappa^\mu$ and $A_B = \{\oplus_{i<\kappa} z_i : z_i \in B\}$. The following hold:

(i) If Player I wins $G^{F,E}_\kappa(A_B)$ then they also win $G^{F,E}_{\kappa,\mu}(B)$.

(ii) If Player II wins $G^{F,E}_\kappa(A_B)$ then they also win $G^{F,E}_{\kappa,\mu}(B)$.

Proof. This proof is similar to the one of Lemma 5.1.1. For (i), let σ be a winning strategy for Player I in $G^{F,E}_\kappa(A_B)$. Let σ' be the strategy σ of Player I in $G^{F,E}_\kappa(A_B)$ restricted to the first μ moves.

To prove that σ' is indeed a winning strategy for Player I, let $q' = \langle v_0, r_0, t_1, r_1, \ldots \rangle$ a run of the game $G^{F,E}_{\kappa,\omega}(B)$ where Player I plays according to σ' and $q = \langle t_0, r_0, t_1, r_1, \ldots \rangle$ be a run of the game $G^{F,E}_{\kappa}(A_B)$ where Player I plays according to σ and Player II plays the same r_i's in the first μ rounds. We have, $z := \oplus_{\alpha<\mu} t_\alpha$ and see that $z^\frown \oplus_{\mu \leq \alpha < \kappa} t_\alpha \in A_B$ since σ is a winning strategy. Hence, $z \in B$ and σ' is a winning strategy for Player I in $G^{F,E}_{\kappa,\mu}(B)$.

For (ii), suppose that Player II has a winning strategy τ for $G^{F,E}_{\kappa}(A_B)$ but not for $G^{F,E}_{\kappa,\mu}(B)$. We are going to inductively construct a run $\langle t_\alpha, r_\alpha | \alpha < \kappa \rangle$ of the game $G^{F,E}_{\kappa}(A_B)$ where Player II uses τ but does not win. We do this by constructing each $[\mu{\cdot}\beta, \mu{\cdot}\beta + \mu)$ segment of the run. First, let τ_0 be the strategy τ of Player II restricted to the first μ moves of the game $G^{F,E}_{\kappa}(A_B)$. Since, τ_0 cannot be winning for II in the game $G^{F,E}_{\kappa,\mu}(B)$, let $\langle t_\alpha, r_\alpha | \alpha < \mu \rangle$ be a run of the game where Player II plays according to τ_0 but does not win, i.e. $z_0 := \oplus_{\alpha<\mu} t_\alpha \in B$. Suppose now that $\langle t_\alpha, r_\alpha | \alpha < \mu{\cdot}\beta \rangle$ is already defined for $\beta < \kappa$. First, define strategy τ_β for Player II such that

$$\tau_\beta(\langle s_i, r_i | i < \gamma \rangle) = \tau(\langle t_\alpha, r_\alpha | \alpha < \mu{\cdot}\beta \rangle^\frown \langle s_i, r_i | i < \gamma \rangle)$$

for all $\gamma < \mu$. Since τ_β is not a winning strategy in the game $G^{F,E}_{\kappa,\mu}(B)$, let $\langle t_{\mu\cdot\beta+i}, r_{\mu\cdot\beta+i} | i < \mu \rangle$ be a run where II plays according to τ but doesn't win the game, i.e., $z_\beta = \oplus_{i<\mu} t_{\mu\cdot\beta+i} \in B$. In this way, we have constructed a run of the game where Player II uses τ but for all $i < \kappa$ we have $z_i \in B$, therefore $z := \oplus_{i<\kappa} z_i \in A_B$ which is a contradiction. □

Our next goal is to prove an analogue of Lemma 5.1.9. To do so we will need the following:

Lemma 5.2.4 (AC) There exists a Bernstein set $B \subseteq \mu^\mu$.

Proof. The construction of Bernstein subsets of μ^μ is analogous to the one in ω^ω. Each perfect subset $P \subseteq \mu^\mu$ is induced by a perfect tree $T_P \subseteq \mu^{<\mu}$ therefore we can enumerate perfect subsets of μ^μ as $\langle P_\alpha : \alpha < 2^\mu \rangle$. Each perfect subset of $\mu^{<\mu}$ has cardinality at least 2^μ [3]. We inductively construct disjoint sets X, Y as follows: for the base case let $x_0 \in P_0$ and $y_0 \in P_0$ such that $x_0 \neq y_0$. Define $X_0 = \{x_0\}$ and $Y_0 = \{y_0\}$. Next, suppose that $X_\alpha = \{x_\beta : \beta \leq \alpha\}$ and $Y_\alpha = \{y_\beta : \beta \leq \alpha\}$ have already been constructed and choose $x_{\alpha+1} \in P_{\alpha+1}$ and $y_{\alpha+1} \in P_{\alpha+1}$ such that $y_{\alpha+1} \notin X_\alpha \cup \{x_{\alpha+1}\}$. Now, define $Y_{\alpha+1} = Y_\alpha \cup \{y_{\alpha+1}\}$ and $X_{\alpha+1} = X_\alpha \cup \{x_{\alpha+1}\}$. For the limit case, define $X_\lambda = \bigcup_{\beta<\lambda} X_\beta$ and $Y_\lambda = \bigcup_{\beta<\lambda} Y_\beta$ which concludes the induction. Finally, define:

$$X := \bigcup_{\beta<2^\kappa} X_\beta \text{ and } Y := \bigcup_{\beta<2^\kappa} Y_\beta$$

and see that X is a Bernstein subset of μ^μ. □

[3] If P is a perfect set then P contains a homeomorphic image of 2^μ. See e.g. [LMS16, Lemma 2.9]

Similarly as before, let $e : \kappa \to \mu$ be a surjection. We make the following additional assumptions for F and R:

(G1e) $(\forall r \in R)(\exists t, s \in F(r))(t_e \perp s_e)$

(G2e) $(\forall i < \mu)(\exists r \in R)(\forall t \in F(r))(e(t(0)) \neq i)$

Lemma 5.2.5 Let $B \subseteq \mu^\mu$ be a Bernstein set and $e : \kappa \to \mu$ be a surjection. Then the game $G^{F,E}_{\kappa,\mu}(B_e)$ is non-determined for $B_e := \{z \in \kappa^\mu : z_e \in B\}$

Proof. For contradiction, suppose that $G^{F,E}_{\kappa,\mu}(B_e)$ is determined. First, let σ be a winning strategy for Player I. We will inductively construct a strong map $\varphi : 2^{<\mu} \to R^{<\mu}$ and a map $\psi : 2^{<\mu} \to \mu^{<\mu}$ such that:

(i) $\psi(t) = (\sigma * \varphi(t))_e$

(ii) $\psi(t^\frown\langle 0\rangle) \perp \psi(t^\frown\langle 1\rangle)$

Then, T_ψ will be a perfect subtree of $T_{\sigma,e} := \mathrm{dw}(\{(\sigma * s)_e : s \in R^{<\mu}\})$ since it is strict order preserving by (i) and $\perp$-preserving by (ii). The map φ defines the moves of Player II that we will use to construct the tree. For the base case of the construction, let $\varphi(\emptyset) = \emptyset$ and $\psi(\emptyset) = (\sigma(\emptyset))_e$. Then, for successor stages, suppose that, for $s \in 2^{<\mu}$ of successor length, $\varphi(s)$ is defined and $\psi(s) = (\sigma * \varphi(s))_e$. To define $\varphi(s^\frown\langle i\rangle)$ and $\psi(s^\frown\langle i\rangle)$ for $i = 0, 1$, let p be the position of the game where Player I plays according to σ and Player II answers with $\varphi(s)$. Fix arbitrary $r_{s,0} \in R$ and, by (G2e), there exists $r_{s,1} \in R$ such that for all $t \in F(r_{s,1})$ we have $e(t(0)) \neq e(\sigma(p^\frown\langle r_{s,0}\rangle)(0))$. Define

$$t_{s,0} := \sigma(p^\frown\langle r_s, 0\rangle) \text{ and } t_{s,1} := \sigma(p^\frown\langle r_{s,1}\rangle)$$

Notice that, $(t_{s,0})_e \perp (t_{s,1})_e$, hence let $\varphi(s^\frown\langle i\rangle) := \varphi(s)^\frown\langle r_{s,i}\rangle$ and $\psi(s^\frown\langle i\rangle) := \psi(s)^\frown(t_{s,i})_e$ for $i = 0, 1$. Finally, let $s \in 2^{<\mu}$ with $|s| \in \mathrm{Lim}$ such that $\varphi(t)$ and $\psi(t)$ have already been defined for every $t \subset s$. Define $\varphi(s) = \bigcup_{t \subset s} \varphi(t)$ and $\psi(s) = (\sigma * \varphi(t))_e$.

The map ψ is by construction strict order preserving and $\perp$-preserving therefore T_ψ is a perfect tree. For $x \in 2^{<\mu}$ we have $f_\psi(x) = (\sigma * f_\varphi(x))_e$ therefore $[T_\psi] \subseteq P_{\sigma,e} := \{(\sigma * x)_e : x \in R^\mu\} \subseteq B$ which is a contradiction since B is Bernstein.

Next, suppose that τ is a winning strategy for Player II. We construct strict order preserving and $\perp$-preserving map $\psi : 2^{<\mu} \to \mu^{<\mu}$ such that $\mathrm{ran}(f_\psi) \subseteq Q_{\tau,e}$ where $Q_\tau = \{\overline{x} * \tau : \overline{x} \in (\kappa^{<\mu})^\mu \text{ legal according to } \tau\}$ and $Q_{\tau,e} = \{(\overline{x} * \tau)_e : \overline{x} \in (\kappa^{<\mu})^\mu \text{ legal according to } \tau\}$. First, we will define the map $\varphi : 2^{<\mu} \to \kappa^{<\mu}$ such that

(a) $\langle \varphi(s \restriction \alpha) : \alpha \leq |s|\rangle$ is legal according to τ.

(b) $(\varphi(s^\frown\langle 0\rangle))_e \perp (\varphi(s^\frown\langle 0\rangle))_e$

We do this by induction on $|s|$ for each $s \in 2^{<\mu}$. If $|s| \in \mathrm{Lim} \cup \{0\}$, we arbitrarily choose $\varphi(s) \in E(|s|)$. For the successor step, let $\langle \varphi(s \restriction \alpha) : \alpha \leq |s| \rangle$ be defined. We will construct $\varphi(s^\frown\langle i \rangle)$ for $i = 0, 1$. Let p be the position of the game where Player I plays $\langle \varphi(s \restriction \alpha) : \alpha \leq |s| \rangle$ and Player II's answers are determined by τ. Let r be II's last move in p and, by (G1e), there exist $t_0, t_1 \in F(r)$ such that $(t_0)_e \perp (t_1)_e$. Define

$$\varphi(s^\frown\langle 0 \rangle) := t_0 \text{ and } \varphi(s^\frown\langle 1 \rangle) := t_1$$

Now, using φ, let us define

$$\psi(s) := \bigoplus_{\alpha \leq |s|} (\varphi(s \restriction \alpha))_e$$

which, by construction, is strict order preserving and $\perp$-preserving therefore $[T_\psi]$ is perfect. Also, for every $x \in 2^\mu$ we have

$$f_\psi(x) = \bigcup_{\alpha<\mu} \psi(x \restriction \alpha) = \bigcup_{\alpha<\mu} \left(\bigoplus_{\beta \leq |x \restriction \alpha|} (\varphi((x \restriction \alpha) \restriction \beta))_e \right) =$$
$$= \bigoplus_{\beta<\mu} (\varphi(x \restriction \beta))_e$$

Therefore $[T_\psi] \subseteq Q_{\tau,e} \subseteq B$ which is a contradiction.

□

Corollary 5.2.6 For the set

$$A_{B,e} := \{\oplus_{\alpha<\kappa} z_\alpha : z_\alpha \in \kappa^\mu \text{ and } (z_\alpha)_e \in B \text{ for all } \alpha < \kappa\} \subseteq \kappa^\kappa$$

where $B \subseteq \mu^\mu$ is Bernstein, the game $G_\kappa^F(A_{B,e})$ is non-determined.

Proof. Immediate from Lemma 5.2.3. □

5.3 Applications to the standard game

In this section, we use ideas and results from Section 5.1 and Chapter 4 to study four variations of the standard game. In the first variation, Player I plays an element of $\kappa^{<\kappa}$ as their first move and then continues playing ordinals as usual for the rest of the game. In the second, Player I plays elements of $\kappa^{<\kappa}$ in rounds 0 and limit rounds and plays ordinals as usual in successor rounds. In the third, Player I plays elements of $\kappa^{<\kappa}$ in all rounds, while in the fourth, they play elements of $\kappa^{<\mu}$ for some $\omega \leq \mu < \kappa$. Player II on the other hand, plays ordinals in all four variations as usual. As we will see, the latter three variations can be obtained as specific cases of Kechris's games and its bounded variants.

Let us start with the first variation $G_\kappa^{1\times}$, which looks as follows:

I	s_0		y_1		$\dots$
II		x_0		x_1	$\dots$

Where $s_0 \in \kappa^{<\kappa}$, $x_0 < \kappa$ and $x_i, y_i < \kappa$ for $1 \leq i < \kappa$. Player I wins if and only if $s_0^\frown\langle x_0, y_1, x_1, \dots\rangle \in A$.

Lemma 5.3.1 There exists a set A such that $G^{1\times}_\kappa(A)$ is non-determined.

Proof. Let $D \subseteq \kappa^\omega$ be a non-determined subset of κ^ω. Recall that such a set exists by Theorem 2.1.4. Consider the set $A := \{\oplus_{i<\kappa} z_i : z_i \in D\}$ and suppose that the game $G^{1\times}_\kappa(A)$ is determined. The idea behind this proof is similar to that of Lemma 5.1.1. First suppose that σ is a winning strategy for Player I in the game $G^{1\times}_\kappa(A)$ and let $p = \langle s_0, x_0, y_0, \dots x_i, y_i, \dots\rangle$ for $s_0 \in \kappa^{<\kappa}$, $x_0 < \kappa$ and $x_i, y_i < \kappa$ for all $i < \omega$, be the position of the game $G^{1\times}_\kappa(A)$ after the first ω rounds where Player I uses τ and II plays arbitrary ordinals. For position p' of $G_\omega(D)$ of even length, define

$$\sigma'(p') := \sigma(p^\frown p')$$

This is a winning strategy for Player I in $G_\omega(D)$ which is a contradiction.

Now suppose that Player II has a winning strategy for $G^{1\times}_\kappa(A)$. This means that at some segment $[\omega{\cdot}\beta, \omega{\cdot}\beta + \omega)$ of the game, Player II's strategy gives a sequence $z_{\omega\cdot\beta} \notin D$. If Player I's first move is a sequence $s \in \kappa^{<\kappa}$ such that $|s| = 1$ then, at each segment of the game, we have a copy of the standard game $G_\omega(A)$ of length ω. This means that, at some point, Player II's strategy for $G^{1\times}_\kappa(A)$ gives us a winning strategy for Player II in $G_\omega(D)$ which is a contradiction. $\square$

The next variation we are going to look at is similar to the above with the additional rule that Player I plays elements of $\kappa^{<\kappa}$ also in limit rounds. For payoff set $A \subseteq \kappa^\kappa$ the game looks as follows:

I	s_0		y_1		$\dots$	s_λ		$y_{\lambda+1}$	$\dots$
II		x_0		x_1	$\dots$		x_λ		$\dots$

Player I wins the game if and only if

$$s_0{}^\frown\langle x_0, y_1, \dots\rangle^\frown s_\omega{}^\frown\langle x_\omega, y_{\omega+1}\dots\rangle^\frown \dots \in A$$

Games of this form are equivalent to the spacial case of bounded Kechris games from Section 5.1 for $R = \kappa$ and for $F(\beta) = \{\langle\beta, \alpha\rangle : \alpha < \kappa\}$, for a payoff set A, the game $G^F_\kappa(A)$ looks as follows:

I	s_0		$\langle\beta_0, \alpha_1\rangle$		$\langle\beta_1, \alpha_2\rangle$	$\dots$	s_λ		$\langle\beta_\lambda, \alpha_{\lambda+1}\rangle$	$\dots$
II		β_0		β_1		$\dots$		β_λ		$\dots$

Player I wins the game $G^F_\kappa(A)$ if and only if

$$\bigoplus_{\gamma<\kappa}(s_\gamma{}^\frown(\oplus_{n<\omega}\langle\beta_{\omega\cdot\gamma+n},\alpha_{\omega\cdot\gamma+n+1}\rangle))\in A$$

It is also immediate that for all surjections $e:\kappa\to\omega$, the function F and the set R satisfy the conditions (G1e) and (G2e). Therefore, by Lemma 5.1.9 the game $G^F_{\kappa,\omega}(B_e)$, where $B_e=\{z\in\kappa^\omega : z_e\in B\}$, is non-determined and by Lemma 5.1.1 we have that for the set

$$A_B=\{\oplus_{i<\kappa}z_i : z_i\in B_e\}\subseteq\kappa^\kappa$$

the game $G^F_\kappa(A)$ is non-determined.

Thirdly, let us discuss the game where Player I plays elements of $\kappa^{<\kappa}$ in each round while Player II plays ordinals. The winning condition for Player I is that the concatenation of the players moves belongs to the payoff set. More concretely, for payoff set $A\subseteq\kappa^\kappa$, the game looks as follows:

I	t_0		t_1		$\dots$
II		γ_0		γ_1	$\dots$

Player I wins if and only if $\oplus_{\alpha<\kappa}(t_\alpha{}^\frown\langle\gamma_\alpha\rangle)\in A$. One immediately sees that such games can be seen as instances of Kechris's games like the one defined in Chapter 4. For $R=\kappa$ and $F:R\to\mathscr{P}(\kappa^{<\kappa}\setminus\langle\rangle)$ such that $F(\gamma)=\{t\in\kappa^{<\kappa} : t(0)=\gamma\}$, the games $G^{<\kappa}_\kappa(A)$ and $G^F_\kappa(A)$ are equivalent.

Each $F(\gamma)$ is an upwards closed subset of $\kappa^{<\kappa}$. Therefore, by result of [SSz] mentioned in Section 5.1, the game $G^{<\kappa}_\kappa(A)$ is determined for all sets $A\subseteq\kappa^\kappa$ which are definable from a κ-sequence of ordinals, after a Lévy-collapse of an inaccessible cardinal to κ^+. The next lemma characterizes when Player I has a winning strategy in the game $G^{<\kappa}_\kappa(A)$.

Lemma 5.3.2 For $A\subseteq\kappa^\kappa$, Player I has a winning strategy in the game $G^F_\kappa(A)$ if and only if A contains a cofinally fully-splitting $<\kappa$-closed tree.

Proof. First suppose that σ is a winning strategy for Player I. Let $P=\{\sigma * y \mid y\in\kappa^\kappa\}$ and since σ is a winning strategy we have $P\subseteq A$. The tree T_P is cofinally fully-splitting. T_P is also $<\kappa$-closed and $P=[T_P]$, by [SSz26, Corollaries 5.1.2 and 5.1.6], since Player I also plays at limit stages and $\langle\gamma\rangle\subseteq F(\gamma)$ for all $\gamma<\kappa$.

For the other direction, suppose $[T]\subseteq A$ for a cofinally fully-splitting tree T. Now, Player I can always jump to a fully-splitting node in the tree T which will guarantee that they always stay inside the tree after Player II's move. Furthermore, Player I can always find suitable node to play at limit stages because T is $<\kappa$-closed. □

Finally, let us consider the following variant $G^{<\mu}_\kappa$ of the previous game, where the length of suitable moves for Player I is bounded by an infinite cardinal μ such that $\omega \leq \mu = \mu^{<\mu} < \kappa$. For $A \subseteq \kappa^\kappa$, the game $G_\kappa^{<\mu}(A)$ is played just like $G_\kappa^{<\kappa}(A)$, except Player I has to play $t_\alpha \in \kappa^{<\mu}$ in every round $\alpha < \kappa$, i.e., the length of Player I's moves cannot exceed μ.

I	t_0		t_1		$\dots$
II		γ_0		γ_1	$\dots$

Notice that, the game $G_\kappa^{<\mu}(A)$ is equivalent to the game $G_\kappa^{F,E}(A)$ for $R = \kappa$, $F(\gamma) = \{t \in \kappa^{<\mu} : t(\gamma) := 0\}$ and $E(\alpha) = \kappa^{<\mu}$ for all $\alpha \in \kappa \cap (\mathrm{Lim} \cup \{0\})$. This is a special case of the games described in Section 5.2.2. Hence, for a Bernstein set $B \subseteq \mu^{<\mu}$, the game $G_\kappa^{<\mu}(A_{B,e})$ is non-determined for

$$A_{B,e} := \{\oplus_{\alpha<\kappa} z_\alpha : z_\alpha \in \kappa^\mu \text{ and } (z_\alpha)_e \in B \text{ for all } \alpha < \kappa\}$$

where $e : \kappa \to \mu$ is a surjection, by Corollary 5.2.6.

Conclusions and Further Questions

In this thesis, we have explored both standard and asymmetric transfinite games in the context of higher Baire spaces. The central theme throughout this work has been to find specific properties or sufficient conditions that make such games determined.

It is true that there are very few results that concern the determinacy of standard games in higher Baire spaces. This absence indicates that determinacy is a much harder property to satisfy in this generalized context than in the classical case. Our first significant result is that of Proposition 3.1.3, which provides a sufficient condition for an analogue of the Gale-Stewart theorem (see Theorem 3.1.3) for κ^κ. While it is not immediately clear whether there are specific classes of sets that satisfy this condition, further work may lead to more concrete examples. Another question one could ask about this result is whether $<\kappa$-closure of $\mathbb{P}_A$ and $\mathbb{Q}_A$ can be weakened.

The largest part of the thesis is concerned with asymmetric games. Such games have been studied extensively in the classical case and they generalize naturally to κ^κ. In Chapter 4, we studied the transfinite version of a general class of games due to Kechris [Kec77]. Specific instances of such games characterize regularity properties in κ^κ such as the perfect set property or the asymmetric Baire property. We have also seen that the determinacy of these games for sets $A \subseteq \kappa^\kappa$ definable from a κ-sequence of ordinals, is consistent relative to an inaccessible cardinal by results of [SSz].

Our initial intuition was that the determinacy of such games is connected to whether or not the sets of possible moves for Player I are upwards closed. In Chapter 5 we investigate this idea. By modifying Kechris's games in such a way that Player I is forced to play finite sequences in every successor round, we construct non-determined subsets of both 2^κ and κ^κ. We then expand the class of Kechris's games presented in Chapter 4 by introducing a second satisfaction function that dictates how Player I should play in limit rounds and round 0 of the game. Using this class, we describe similar games, with the main difference being that the length of Player I's moves is bounded by an infinite $\mu < \kappa$ such that $\mu^{<\mu} = \mu$. By a similar argument as before, we are able to provide closed non-determined sets which aligns with our initial intuition that determinacy is strongly connected with upwards closure of the sets of possible moves for Player I. The applications of these results to variants of the standard game in Section 5.3 further confirms this.

We conjecture that asymmetric games like the ones described in this thesis, can be placed in a spectrum from "good" to "bad" depending on the properties of their requirement set R and the satisfaction functions F and E. Towards the "good" end of the spectrum are games like Kechris's original games. These are consistently determined for subsets of κ^κ definable from a κ-sequence of ordinals. On the other side of this spectrum, we have games like the ones studied in Section 5.2.2, where the length of Player I's moves is bounded by the same infinite cardinal $\mu < \kappa$ in all rounds. For such games we can easily construct simple (e.g. closed) non-determined payoff sets in ZFC. Games like the ones studied in 5.2.1 for example, might be closer to the "good" end than the ones from 5.2.2 since Player I can play sequences of arbitrary length $< \kappa$ at limit stages. The fact that an argument similar to that applied in Corollaries 5.1.1 and 5.2.3 does not work for that case may also hint towards this direction.

In our view, one could approach this idea of a spectrum of games in two different ways. On the one hand, one could ask where do specific types of games sit in the spectrum, e.g. games where the bound $\mu_\alpha \leq \kappa$ is different is each round $\alpha < \kappa$.[4] On the other hand, one could ask whether there are games that sit in specific points of this spectrum, e.g. games that are determined for all superclosed sets, or even for all closed sets. Such games would sit very far towards the "good" side. The study of this spectrum highlights the richness of these notions in the generalized case and can give us a deeper insight on how determinacy behaves in this setting.

[4] We have already discussed a special case of this where $\mu_\alpha = \omega$ for successor $\alpha < \kappa$ and $\mu_\alpha = \kappa$ for limit $\alpha < \kappa$ in Section 5.1. Games of this form are non-determined for closed subsets of κ^κ and therefore sit towards the "bad" side of the spectrum.

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