Lattices of Tense Logics

Tabularity, Completeness and Decidability

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Science Park 107
1098 XG Amsterdam
phone: +31-20-525 6051
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The research for/publication of this doctoral thesis received financial assistance from the China Scholarship Council (CSC).

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Tabularity, Completeness and Decidability

Dissertation submitted to

Tsinghua University

in partial fulfillment of the requirement

for the degree of

Doctor of Philosophy

in

Philosophy

by

CHEN Qian

Thesis Committee

Supervisor:	Prof. Dr. Fenrong Liu	Tsinghua University
Co-supervisor:	Dr. Nick Bezhanishvili	University of Amsterdam
External referees:		
	Prof. Dr. Xuefeng Wen	Sun Yat-sen University
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	Dr. Junhua Yu	Tsinghua University
	Dr. Nick Bezhanishvili	University of Amsterdam

The Tsinghua - UvA Joint Research Centre for Logic
Department of Philosophy, School of Humanities
Tsinghua University

Tsinghua version: May, 2026
ILLC version: June, 2026

Contents

Preface to the ILLC Version

I would like to thank Nick Bezhanishvili for inviting me to include this thesis in the ILLC Dissertation Series.

The content of this document is essentially the same as that of the thesis submitted to Tsinghua University in May 2026 and deposited in the Tsinghua University Library. For this edition, I have reformatted the document to better match the style of the ILLC Dissertation Series, expanded the acknowledgements, and made a few minor corrections throughout the text.

Qian Chen
Beijing, June 24, 2026

Acknowledgments

First, I would like to express my deepest gratitude to my supervisor, Fenrong Liu, for her steadfast support during my Ph.D. studies. It was through her guidance that I gradually developed a wider academic perspective and came to see a much broader and more fascinating world of logic. I am grateful for her encouragement to pursue the questions that genuinely interested me, and for giving me the opportunity to study at Tsinghua and the ILLC. Over the years, Fenrong has profoundly shaped the way I approach academia.

I am equally grateful to my supervisor, Nick Bezhanishvili, for his generous guidance and support, not only during my years at the ILLC but also afterwards. I am especially thankful for the extraordinary time and care he devoted to reading my work, discussing ideas with me, and encouraging me to think more deeply. Nick's door was always open to me, and I will always cherish the many conversations we shared over the years. Beyond logic, I learned a great deal from Nick as a scholar, a teacher, and a person. Through Nick, I came to see the kind of logician and teacher I aspire to become.

My sincere thanks go to Minghui Ma, my undergraduate thesis supervisor, who led me into the world of logic. His support made it possible for me to join the Tsinghua Logic Center. Over the years, he has remained a generous mentor, collaborator, and friend, and I have benefited enormously from our collaborations and discussions. Looking back, it is hard for me to imagine my journey in logic without his guidance and support.

I would like to thank all my coauthors and collaborators during my Ph.D. studies, including Rodrigo N. Almeida, Nick Bezhanishvili, Penghao Du, Sujata Ghosh, Qingyu He, Dazhu Li, Fenrong Liu, Minghui Ma, Chenwei Shi, Tenyo Takahashi, Yaxin Tu, and Yiyan Wang. I am grateful for the many ideas, discussions, and collaborations we have shared over the years.

Thanks also go to the members of the committee, Nick Bezhanishvili, Yanjing Wang, Yi Wang, Xuefeng Wen, Junhua Yu, and Donghua Zhu, for their careful reading of this thesis and for their valuable comments and suggestions.

I am grateful to all the teachers, colleagues, and friends whose paths crossed mine during my years in Beijing. I would especially like to thank Johan van Benthem, Qi Feng, Jeremy Seligman, Martin Stokhof, Dag Westerståhl, and Junhua Yu for their teaching, guidance, and inspiration. I am also grateful to Mingjun Chen, Yumin Ji, Lei Li, Xin Li, Zhizhen Ma, Wenfei Ouyang, Wei Wang, Kaibo Xie, Xi Yang, Weijun Yu, Wenlong Zheng, and Mingjia Zuo for the many conversations and moments we shared in Room 329.

Special thanks go to Dazhu Li, Yiyan Wang, and Jialiang Yan, who generously shared their experiences, advice, and encouragement with me. I am grateful to Fengxiang Cheng, Penghao Du, and Qingyu He, who entered the Tsinghua Logic Center together with me and shared many of the challenges and joys of graduate life. I am also grateful to Han Xiao, who offered me generous support and friendship after my return from Amsterdam. Finally, I am especially grateful to Bin Liu, my undergraduate classmate, who has accompanied me on this journey from our student days to our Ph.D. studies.

I was fortunate to spend two years at the ILLC as a visiting researcher, where I met many wonderful colleagues and friends. I would like to thank Benno van den Berg, Balder ten Cate, Tianyi Chu, Marianna Girlando, Ruiting Jiang, Yurii Khomskii, Johannes Kloibhofer, Søren Knudstorp, Daniël Otten, Yde Venema, Haoyu Wang, and Woxuan Zhou, for making my visit especially memorable.

I am especially grateful to Tenyo Takahashi for his friendship and for the many conversations we shared about logic and life. I will always cherish the time and memories we shared in many different countries. I am also grateful to Rodrigo N. Almeida. I have always admired both his breadth of knowledge and his kindness toward others, and I am glad that our friendship has continued to this day. I owe special thanks to Jian Jin and Enming Zhang, whose companionship became one of the brightest parts of my years in Europe and with whom I shared many of my happiest memories during that time.

My heartfelt gratitude goes to my parents, Li Yang and Qingneng Chen, for their unwavering trust, respect, and support throughout every stage of my life. They have always given me the freedom to follow my own path and the confidence to pursue what I truly care about. I am grateful that, in their quiet way, they have always been proud of me. I am deeply grateful to my girlfriend, Xiaotong Sun, who has walked beside me through these years, for her gentleness and love, and for everything beyond what my words can properly hold. We have grown, learned, and celebrated together, and I am thankful for every step of the journey we have shared. Finally, I should maybe also thank myself: for being strict with my aspirations, yet gentle with my limitations.

Qian Chen
Beijing, June 24, 2026

Chapter 1

Introduction

In this thesis, we study classes of tense logics, understood as normal bimodal logics equipped with a pair of adjoint modalities $\Box$ and $\blacklozenge$. This thesis aims to provide a systematic analysis of the lattices of tense logics and to obtain a deeper understanding of the interactions between modalities by clarifying the relation between the lattices of tense logics and those of modal logics. In particular, we study logical properties such as tabularity, Post-completeness, finite model property and Kripke completeness of tense logics, which will be introduced later in this chapter. We also investigate the undecidability of logical properties in lattices of tense logics.

Since the theory of modal logics provides the conceptual background for our study, we first review basic facts and several key developments in modal logic.

1.1 Modal Logic

Modal logic is a well-investigated branch of mathematical logic concerned with necessity-like and possibility-like operators, whose algebraic and relational semantics are firmly grounded [96, 65, 75, 74]. The formal language $\mathscr{L}_\Box$ of modal logic is obtained by adding to the propositional language a unary modality $\Box$. A set $L \subseteq \mathsf{Form}_m$ of modal formulas is called a *normal modal logic* if (i) the *classical propositional logic* CPC is contained in L; (ii) $\Box(p \to q) \to (\Box p \to \Box q) \in L$; and (iii) L is closed under the rules (MP), (Sub) and (Nec). Let K denote the least normal modal logic.

Modal logic has developed into a central area of mathematical logic since the introduction of the systems $\mathsf{S1}$ to $\mathsf{S5}$ by Lewis [77]. An important development in modal logic was the establishment of relational semantics based on possible worlds and accessibility relations, which was introduced by Kripke [75, 74] and now known as Kripke semantics. A *Kripke frame* is a pair $\mathfrak{F} = (X, R)$ where X is a non-empty set and $R \subseteq X \times X$ is a binary relation on X. We write $\mathfrak{F} \models \varphi$

if φ is *valid* in $\mathfrak{F}$. Let Fr and Fin denote the class of all Kripke frames and finite Kripke frames, respectively. For all sets $\Sigma \subseteq \mathsf{Form}_m$ of modal formulas and classes $\mathcal{K} \subseteq \mathsf{Fr}$ of frames, we define

$$\mathcal{K}(\Sigma) \coloneqq \{\mathfrak{F} \in \mathcal{K} : \forall \varphi \in \Sigma(\mathfrak{F} \vDash \varphi)\} \text{ and } \mathsf{Log}(\mathcal{K}) \coloneqq \{\varphi \in \mathsf{Form}_m : \forall \mathfrak{F} \in \mathcal{K}(\mathfrak{F} \vDash \varphi)\}.$$

For example, given a modal logic L, we see that $\mathsf{Fin}(L)$ and $\mathsf{Fr}(L)$ are the classes of all finite frames and frames for L, respectively. We call $\mathsf{Log}(\mathcal{K})$ the *modal logic* of $\mathcal{K}$. It is well known that $\mathsf{K4}$ is the modal logic of *transitive* frames, that is, frames (X, R) such that for all $x, y, z \in X$, if Rxy and Ryz, then Rxz.

1.1.1 Logical properties of modal logics

Logical properties such as *tabularity*, the *finite model property* and *Kripke completeness* have been extensively investigated for individual modal systems.

Kripke completeness

A central logical property in modal logic is Kripke completeness, which has been extensively studied since the 1960s. A modal logic L is *Kripke complete* if L is the modal logic of its Kripke frames, that is, $L = \mathsf{Log}(\mathsf{Fr}(L))$. The modal logics K, $\mathsf{K4}$ and $\mathsf{S4}$ are proved to be Kripke complete (see, e.g., [12, 29]). Moreover, Fine [52] proved that an extension L of $\mathsf{K4}$ is Kripke complete if $\mathsf{bw}_n \in L$ for some natural number n, where the formula bw_n is defined as follows:

$$\mathsf{bw}_n \coloneqq \bigwedge_{i \leq n} \Diamond p_i \to \bigvee_{i \neq j \leq n} \Diamond(p_i \wedge (p_j \vee \Diamond p_j)).$$

There was a conjecture that every normal modal logic is Kripke complete, which was disproved by Fine [51] and Thomason [121]: Fine [51] found a Kripke incomplete extension of $\mathsf{S4}$ and Thomason [121] presented a finitely axiomatizable Kripke incomplete sublogic of $\mathsf{S4}$. Simpler examples of Kripke incomplete modal logics were provided later by van Benthem [125]. Cresswell [39] even established the existence of a decidable Kripke incomplete normal modal logic which is finitely axiomatizable.

Finite model property

A modal logic L enjoys the *finite model property* (FMP) if L is the modal logic of its finite frames, that is, $L = \mathsf{Log}(\mathsf{Fin}(L))$. Equivalently, a modal logic L enjoys the FMP if every modal formula $\varphi \notin L$ is refuted by some finite frame for L. The finite model property is one of the most important logical properties in the study of modal logics. Note that a modal logic is Kripke complete whenever it has the FMP. It is worth noting that if a finitely axiomatizable modal logic L enjoys the

FMP, then L is *decidable*, that is, there exists an effective algorithm for deciding whether any given formula φ is in L.

On the positive side, many modal logics enjoy the FMP and there are several well-understood techniques for proving this property, most notably filtration and selective filtration (see, e.g., [12, 29]). These constructions provide systematic ways of transforming models into finite ones. For example, by showing that each of K, $\mathsf{K4}$ and $\mathsf{S4}$ admits filtration, we obtain that all of them have the FMP and thus are decidable. Moreover, Segerberg [115] proved that an extension L of $\mathsf{K4}$ has the FMP if $\mathsf{bd}_n \in L$ for some natural number n, where the formulas bd_n are defined inductively as follows:

$$\begin{aligned}\mathsf{bd}_1 &:= \Diamond\Box p_0 \to p_0,\\ \mathsf{bd}_{n+1} &:= \Diamond(\Box p_n \wedge \neg\mathsf{bd}_n) \to p_n.\end{aligned}$$

These modal logics are now known as modal logics with finite depth.

However, on the negative side, not every modal logic enjoys the FMP: Makinson [86] presented a logic $L \subseteq \mathsf{S4}$ that lacks the FMP, while Fine [49] exhibited a logic $L \supseteq \mathsf{S4}$ without the FMP. These examples demonstrate that the FMP is a nontrivial property and cannot be taken for granted even for logics closely related to $\mathsf{S4}$.

Tabularity

Among the logical properties studied in lattices of modal logics, tabularity is of particular interest due to the strong restrictions it imposes on logics. Recall that a modal logic L is *consistent* if $L \neq \mathsf{Form}_m$. A consistent modal logic L is *tabular* if and only if $L = \mathsf{Log}(\mathfrak{F})$ for some $\mathfrak{F} \in \mathsf{Fin}$. The notion of tabularity was introduced by Kuznetsov [76], motivated by the observation that tabular logics can be characterized by 'truth tables' and thus enjoy desirable properties. For example, the well-understood logic CPC is tabular, which explains the effectiveness of truth tables as a semantic method.

By general results on universal algebra by Funayama and Nakayama [56] and Baker [2] that every tabular normal modal logic L is *finitely axiomatizable*, that is, L is axiomatized by a finite set of formulas. Tabularity in normal modal logics can be characterized by a family of modal formulas (see, e.g. [29, Chapter 12]). Recall that for each natural number n, the modal formulas alt_n and tra_n are defined as follows:

$$\begin{aligned}\mathsf{alt}_n &:= \Box p_0 \vee \Box(p_0 \to p_1) \vee \cdots \vee \Box(p_0 \wedge \cdots \wedge p_{n-1} \to p_n),\\ \mathsf{tra}_n &:= \bigwedge_{i \leq n} \Box^i p \to \Box^{n+1} p.\end{aligned}$$

Then a modal logic L is tabular if and only if $\mathsf{alt}_n \wedge \mathsf{tra}_n \in L$ for some $n \in \omega$ (see, e.g., [29, p. 417]). By this characterization, none of K, $\mathsf{K4}$ and $\mathsf{S4}$ is tabular.

1.1.2 Lattices of modal logics

The study of modal logic initially focused on particular modal logics such as K, $\mathsf{K4}$ and $\mathsf{S4}$. As these individual logics became better understood, attention gradually shifted to classes of modal logics, for example, all extensions of a given particular logic and extensions with particular axioms. This shift allows logical properties to be analyzed at a higher level, thereby deepening our understanding of logics and logical properties.

Later in this section, we will unavoidably encounter notions from lattice theory and universal algebra, as we review research on lattices formed by modal logics viewed as algebraic structures. To keep the exposition clear, we do not introduce all of these notions here, but instead postpone their definitions to Chapter 2. For more details, we refer the reader to [6], [21], and [60]. Nevertheless, we make an effort to keep the presentation accessible to readers who are not familiar with lattice theory or universal algebra.

For each normal modal logic L, let $\mathsf{NExt}(L)$ denote the set of all normal extensions of L, i.e.,

$$\mathsf{NExt}(L) \coloneqq \{L' \supseteq L : L' \text{ is a normal modal logic}\}.$$

It turns out that $(\mathsf{NExt}(L), \subseteq)$ is a *complete lattice* for every modal logic L. We simply write $\mathsf{NExt}(L)$ for the lattice $(\mathsf{NExt}(L), \subseteq)$. For several modal logics, logical properties of their extensions have been fully investigated and even the lattices of their extensions have been fully characterized. For example, consider the logic

$$\mathsf{S4.3} \coloneqq \mathsf{S4} \oplus \Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond q) \vee \Diamond(p \wedge q) \vee \Diamond(q \wedge \Diamond p),$$

which is known to be the modal logic of *linear frames*. Bull [19] showed that every extension of the modal logic $\mathsf{S4.3}$ has the FMP by algebraic methods and Fine [48] proved the same result via Kripke semantics. Moreover, Fine provided a complete characterization of all extensions of $\mathsf{S4.3}$ and showed that all of these logics are finitely axiomatizable and so decidable. Further examples of fully characterized lattices include $\mathsf{NExt}(\mathsf{KAlt}_1)$, described by Bellissima [4], and $\mathsf{NExt}(\mathsf{K5})$, characterized by Nagle and Thomason [99].

In contrast, many modal logics have lattices of extensions that are too complex to admit a full characterization. Fine [50] showed that the lattice $\mathsf{NExt}(\mathsf{S4})$ has cardinality continuum, i.e., $|\mathsf{NExt}(\mathsf{S4})| = 2^{\aleph_0}$, illustrating the combinatorial richness of such lattices. Nevertheless, even for these lattices of modal logics, some characterizations of the structure of these lattices can still be obtained. Makinson [87] proved that the lattice $\mathsf{NExt}(\mathsf{K})$ contains exactly two *co-atoms*. Recall that a consistent logic is called *Post-complete* if it has no proper consistent extension. Then it follows that there exist precisely two Post-complete normal modal logics. Post-complete extensions of modal logics were further investigated by Makinson and Segerberg [88], Sambin and Valentini [113] and Bellissima [5]. Blok [13, 16] investigated the cover relation in the lattices of varieties of modal algebras. Rautenberg [108] studied the splitting varieties of Boolean algebras with operators and characterized the splitting logics in the lattices of *pre-transitive* modal logics.

Studying the structure of lattices of logics yields deep and general results concerning logical properties. Rather than simply asking whether a single logic has a given property, one can now study how logical properties are distributed among lattices of logics.

From Kripke completeness to degree of Kripke incompleteness. As the existence of Kripke incomplete modal logics has been established in [51, 121, 125, 39], one may ask the following questions: Do Kripke incomplete modal logics occur only occasionally? How are Kripke (in)complete modal logics distributed in the lattice $\mathsf{NExt}(\mathsf{K})$? To answer these questions, we need systematic investigation of Kripke (in)completeness.

Fine [51] introduced the notion of the *degree of Kripke incompleteness*. For any lattice $\mathcal{L}$ of modal logics and $L, L' \in \mathcal{L}$, we say that L and L' are Fr-*equivalent* (notation: $L \equiv_{\mathsf{Fr}} L'$) if they have the same class of frames, i.e., $\mathsf{Fr}(L) = \mathsf{Fr}(L')$. Then the degree $\mathsf{df}_{\mathcal{L}}(L)$ of Kripke incompleteness of L in $\mathcal{L}$ is defined to be the cardinality $|[L]_{\mathsf{Fr}}|$ of $[L]_{\mathsf{Fr}}$, where $[L]_{\mathsf{Fr}} := \{L' \in \mathcal{L} : L' \equiv_{\mathsf{Fr}} L\}$ is the set of all modal logics Fr-equivalent to L. A modal logic L is called *strictly Kripke complete* in $\mathcal{L}$ if $\mathsf{df}_{\mathcal{L}}(L) = 1$.

A celebrated result in this field was obtained by Blok [14], now known as *Blok's dichotomy theorem* for the degree of Kripke incompleteness in $\mathsf{NExt}(\mathsf{K})$: every modal logic $L \in \mathsf{NExt}(\mathsf{K})$ is either strictly Kripke complete or has the degree of Kripke incompleteness $2^{\aleph_0}$. This theorem was first proved in [14] algebraically by showing that union-splittings in $\mathsf{NExt}(\mathsf{K})$ are exactly the consistent strictly Kripke complete logics and all other consistent logics have the degree $2^{\aleph_0}$. Blok's dichotomy theorem shows that, somewhat surprisingly, Kripke complete modal logics occur only occasionally in $\mathsf{NExt}(\mathsf{K})$. However, since Blok's proof relies heavily on non-transitive frames, it is natural to ask whether the dichotomy theorem holds for sublattices of $\mathsf{NExt}(\mathsf{K})$, especially for the lattices of transitive modal logics such as $\mathsf{NExt}(\mathsf{K4})$ and $\mathsf{NExt}(\mathsf{S4})$. These problems remain open, see [29, Problem 10.5].

From FMP to degree of FMP. General results on the FMP can also be obtained by studying lattices of modal logics. For example, Kracht [72] studied the relation between splittings and the FMP of modal logics. Fine [53] proved a general result on the FMP that every subframe modal logic in $\mathsf{NExt}(\mathsf{K4})$ has the FMP. Zakharyaschev [138, 139] generalized this result to cofinal subframe logics above $\mathsf{K4}$. Later, Bezhanishvili et al. [10] further generalized these results to cofinal subframe logics in $\mathsf{NExt}(\mathsf{wK4})$, where the modal logic $\mathsf{wK4}$ is defined by

$$\mathsf{wK4} := \mathsf{K} \oplus \Diamond\Diamond p \to p \vee \Diamond p.$$

In a similar vein to the study of Kripke completeness, one may naturally ask how the FMP is distributed in lattices of modal logics. Recently, Bezhanishvili et al. [9] introduced the notion of the *degree of FMP*, an analogue of the degree of Kripke incompleteness, obtained by replacing Fr with Fin in the definition of the latter. Intuitively, a modal logic L has the degree of FMP α if there exist α many modal logics share the same class of finite frames with L. This notion provides a systematic framework for studying the FMP.

Note that every strictly Kripke complete modal logic has been proved to enjoy the FMP (see [14]), the dichotomy theorem for the degree of FMP holds for K: every modal logic $L \in \mathsf{NExt}(\mathsf{K})$ has the degree of FMP either 1 or $2^{\aleph_0}$. However, Bezhanishvili et al. [9] proved the following anti-dichotomy theorem for $\mathsf{K4}$ and $\mathsf{S4}$: For every cardinal κ, if $1 \leq \kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, then there exists $L \in \mathsf{NExt}(\mathsf{K4})$ ($L \in \mathsf{NExt}(\mathsf{S4})$) such that L has the degree of FMP κ in $\mathsf{NExt}(\mathsf{K4})$ ($\mathsf{NExt}(\mathsf{S4})$).

From tabularity to pretabularity. One important problem in modal logic is to decide whether a given modal logic is tabular. In order to answer this problem and study tabularity systematically, Kuznetsov [76] introduced the concept of *pretabularity*. A modal logic L is called *pretabular* if L is not tabular while all of its proper extensions are tabular. Esakia and Meskhi [44] and Maksimova [90] provided full characterizations of pretabular modal logics in $\mathsf{NExt}(\mathsf{S4})$. Bellissima [4] studied the finitely alternative modal logics KAlt_n and proved that there exists exactly one pretabular logic in $\mathsf{NExt}(\mathsf{KAlt}_1)$. Note that every pretabular logic mentioned above is decidable. Therefore, tabularity is decidable in both $\mathsf{NExt}(\mathsf{S4})$ and $\mathsf{NExt}(\mathsf{KAlt}_1)$. On the other hand, Blok [17] showed that there exist continuum many pretabular extensions of $\mathsf{K4}$, indicating that the lattice $\mathsf{NExt}(\mathsf{K4})$ is considerably more complex and that decidability of tabularity cannot be established in the same way. In fact, Blok [16] further showed that tabularity in $\mathsf{NExt}(\mathsf{K})$ is undecidable, while the decidability of tabularity in $\mathsf{NExt}(\mathsf{K4})$ remains unknown [105, Section 5].

(Un)decidability of logical properties. To study the distribution of logical properties such as Kripke completeness and tabularity in lattices of modal logics and obtain a deeper understanding of these properties, the decidability of such properties has also been investigated. More precisely, we call a logical property P *decidable* in a class $\mathcal{C}$ of logics if there exists an algorithm such that, for any finitely axiomatizable logic $L \in \mathcal{C}$, given by its finite axiomatization, the algorithm decides whether L has the property P.[1] We refer to [29, Section 17.6] and [136]

[1] We have to restrict to finitely axiomatizable logics because an input of an algorithm must be a finite object; see Section 6.1.1 and [29, Section 17.1] for more discussions.

for historical accounts.

On the positive side, it is well-known that consistency is decidable for normal monomodal logics, in the sense that the set $\{\varphi \in \mathsf{Form}_m : \mathsf{K} \oplus \varphi = \mathsf{Form}_m\}$ is decidable [87] (see also [29, Section 17.2]). Tabularity was proved to be decidable in both $\mathsf{Ext}(\mathsf{IPC})$ and $\mathsf{NExt}(\mathsf{S4})$, where $\mathsf{Ext}(\mathsf{IPC})$ is the lattice of all *superintuitionistic propositional logics* [76, 89, 90, 44]. Recently, Takahashi [119] proved that being a union-splitting is decidable in $\mathsf{NExt}(\mathsf{K})$, which entails that being strictly Kripke complete is also decidable.

However, Thomason [122] proved that Kripke completeness is undecidable in the lattice $\mathsf{NExt}(\mathsf{K})$, that is, the set $\{\varphi \in \mathsf{Form}_m : \mathsf{K} \oplus \varphi \text{ is Kripke complete}\}$ is undecidable. Further undecidability results were obtained by Chagrov, who showed that several logical properties, including the finite model property, decidability and canonicity, are undecidable in lattices of modal logics [25, 24, 26, 27]. These results illustrate the high complexity of logical properties in lattices of modal logics.

1.2 Tense Logic

Having briefly reviewed research in modal logic and the study of lattices of modal logics, we now turn to *tense logic*, which was introduced for reasoning about temporal notions such as past and future. Temporal logic has been studied for many decades. The modern modal approach to temporal logic was initiated by Prior [102, 103, 104] and has been extensively developed since then. Linear-time temporal logic with 'since' and 'until' operators was introduced by Kamp [67] and further studied by Burgess [20] and Xu [137]. Branching-time temporal logic, originating in the work of Prior [103] and now widely applied in computer science, has also been extensively studied (see [43, 110, 61, 69, 118]).

In this thesis, we focus on tense logic, the basic temporal logic introduced by Prior [103]. Formally, the *formal language $\mathscr{L}_t$ of tense logic* is obtained by adding to $\mathscr{L}_\Box$ the diamond operator $\blacklozenge$. The set Form_t of all *tense formulas* is defined by

$$\mathsf{Form}_t \ni \varphi ::= p \mid \bot \mid (\varphi \to \varphi) \mid \Box\varphi \mid \blacklozenge\varphi,$$

where $p \in \mathsf{Prop}$. The dual modality $\blacksquare$ of $\blacklozenge$ is defined by $\blacksquare\varphi := \neg\blacklozenge\neg\varphi$. Formulas of the form $\Box\varphi$ are read as *'It is always going to be the case that φ'*, and formulas of the form $\blacklozenge\varphi$ are read as *'It was at some point in the past that φ'*.

For the relational semantics, Kripke frames and models are defined as in the modal case. Intuitively, a frame $\mathfrak{F} = (X, R)$ can be viewed as a set X of *time instants* equipped with a *temporal precedence* relation R. Then for all time instants $t_1, t_2 \in X$ such that $t_1 R t_2$, we say that t_1 is a *history* of t_2 and t_2 is a *future* of t_1. The truth condition for formulas $\blacklozenge\varphi$ is given as follows:

$$\mathfrak{M}, x \models \blacklozenge\varphi \quad \text{if and only if} \quad \mathfrak{M}, y \models \varphi \text{ for some } y \text{ such that } (y, x) \in R.$$

We choose $\Box$ and $\blacklozenge$ to be primitive modalities in $\mathscr{L}_t$, since the core of tense logic is the following adjointness of the modalities $\Box$ and $\blacklozenge$: A set $L \subseteq \mathsf{Form}_t$ is called a *tense logic* if $\mathsf{CPC} \subseteq L$ and L is closed under (MP), (Sub) and the adjointness rule (Adj), where

$$\blacklozenge\varphi \to \psi \in L \text{ if and only if } \varphi \to \Box\psi \in L. \tag{Adj}$$

From an algebraic perspective, tense logic may therefore be regarded as the "logic of adjointness". On the other hand, equivalently, tense logics can be defined as normal bimodal logics containing the following axioms:

$$p \to \Box\blacklozenge p \text{ and } p \to \blacksquare\Diamond p.$$

Intuitively, the former means that the current moment is always a possible history of its possible future. The latter has a similar interpretation. Properties of time and temporal notions can be studied using tense logic. However, since the aim of this thesis is to obtain a deeper understanding of the interactions between modalities, we do not engage with the philosophical analysis of time. Instead, we study tense logics in the framework of polymodal logic, viewing them as bimodal logics whose modalities interact naturally.

1.2.1 Transformations between modal and tense logics

Tense logics are closely related to modal logics. Let K_t denote the least tense logic and $\mathsf{NExt}(L)$ the lattice of extensions of L for each tense logic L. For a normal modal logic $L \in \mathsf{NExt}(\mathsf{K})$, let L^+ denote the *minimal tense extension* of L. For a tense logic $L \in \mathsf{NExt}(\mathsf{K}_t)$, let L_+ and L_- denote the set of all $\mathscr{L}_\Box$-formulas and $\mathscr{L}_\blacklozenge$-formulas in L, respectively. Then the transformations yield maps

$$(\cdot)^+ : \mathsf{NExt}(\mathsf{K}) \to \mathsf{NExt}(\mathsf{K}_t) \quad \text{and} \quad (\cdot)_+, (\cdot)_- : \mathsf{NExt}(\mathsf{K}_t) \to \mathsf{NExt}(\mathsf{K}),$$

relating modal logics and tense logics. For example, consider the tense logic $\mathsf{K4}_t := \mathsf{K}_t \oplus \Box p \to \Box\Box p$, which is known to be the tense logic of transitive frames. Then one can easily check that $(\mathsf{K4}_t)_+ = \mathsf{K4}$ and $\mathsf{K4}^+ = \mathsf{K4}_t$.

The preservation of logical properties under the maps $(\cdot)^+$ and $(\cdot)_+$ has been thoroughly investigated: Wolter [128] proved that a subframe logic $L \in \mathsf{NExt}(\mathsf{K4})$ is *elementary* if and only if its minimal tense extension L^+ is elementary, where a logic is called elementary if the class of its Kripke frames can be defined by a set of first-order formulas. Zakharyaschev [139] further proved that for each cofinal subframe modal logic $L \in \mathsf{NExt}(\mathsf{K4})$, we have that L is elementary if and only if L is *compact*, that is, every L-consistent set of modal formulas is satisfied in some $\mathfrak{F} \in \mathsf{Fr}(L)$. These results have been extended to their minimal tense extensions by Wolter [129, Theorem 25]. Kikot et al. [68] showed that for each modal logic

L, its minimal tense extension L^+ admits filtration whenever L admits filtration and L^+ is Kripke complete.

The maps $(\cdot)^+$, $(\cdot)_+$ and $(\cdot)_-$ establish systematic connections between modal and tense logics, allowing results obtained in one setting to be transferred to the other. For instance, Bellissima [4] showed that every *finitely alternative modal logic* is Kripke complete, where a modal logic L is called *finitely alternative* if $L \supseteq \mathsf{KAlt}_n \coloneqq \mathsf{K} \oplus \mathsf{alt}_n$ for some natural number n. Kracht [71] proved later that every *finitely alternative tense logic*, i.e., a tense logic L such that both L_+ and L_- are finitely alternative, is also Kripke complete. Further results illustrating the close relationship between modal and tense logics can be found in the literature (see, e.g., [133, 134]). However, a number of negative results are known concerning the minimal tense extension map $(\cdot)^+$ and the modal fragment map $(\cdot)_+$. For example, the composition $((\cdot)^+)_+$ of these maps is, in general, not the identity map: there exists $L \in \mathsf{NExt}(\mathsf{K})$ such that $(L^+)_+ \neq L$ (see [128, p.84]). Wolter [130] constructed a transitive modal logic $L \supseteq \mathsf{K4}$ with the FMP whose minimal tense extension L^+ is Kripke incomplete. It follows that the FMP and Kripke completeness are not preserved under $(\cdot)^+$. It is still unknown whether the map $(\cdot)^+ : \mathsf{NExt}(\mathsf{K4}) \to \mathsf{NExt}(\mathsf{K4}_t)$ is injective (cf. [128]). These results illustrate the difference between modal and tense logics, as well as their lattices.

1.2.2 Lattices of tense logics

Note that the map $(\cdot)^+ : \mathsf{NExt}(\mathsf{K}) \to \mathsf{NExt}(\mathsf{K}_t)$ is not surjective: there exist tense logics that are not of the form L^+ for any modal logic L (see, e.g. [128]). Consequently, to study the structure of the lattice $\mathsf{NExt}(\mathsf{K}_t)$ of tense logics, it is not enough to focus only on tense logics of the form L^+. Accordingly, our approach is not limited to the study of tense logics via $(\cdot)^+$. In this thesis, we proceed with a direct analysis of the lattices of extensions of tense logics. In fact, lattices of tense logics turn out to be essentially different from lattices of modal logics. Recall that the modal logic $\mathsf{S4.3}$ of linear frames has been extensively studied by Bull [19] and Fine [48]. Later, the linear tense logic $\mathsf{Lin}_t \coloneqq \mathsf{K4}_t \oplus (\Diamond\blacklozenge p \vee \blacklozenge\Diamond p \to p \vee \Diamond p \vee \blacklozenge p)$ has also been thoroughly investigated in a series of papers by Wolter [131, 132, 129] (see also [136] for an overview). It has been proved in [130] that there are countably many tense logics in $\mathsf{NExt}(\mathsf{S4.3}_t)$ that do not have the FMP (cf. [19]). Chagrov and Shehtman [30] provided a full characterization of tabular tense logics and showed that tabularity is not decidable in $\mathsf{NExt}(\mathsf{K4}_t)$.

However, the lattices of tense logics are still not completely understood, certainly not to the same extent as those of modal logics. In particular, several basic questions concerning their structure and the distribution of logical properties remain open.

Degree of Kripke incompleteness (FMP). To the best of our knowledge, neither the degree of Kripke incompleteness nor the degree of FMP in lattices of tense logics has been studied. Rautenberg [107] studied lattices of tense logics and obtained basic results on splitting tense logics, and Kracht [71] characterized all splitting logics in the lattices $\mathsf{NExt}(\mathsf{K}_t)$, $\mathsf{NExt}(\mathsf{K4}_t)$ and $\mathsf{NExt}(\mathsf{S4}_t)$. It is also worth noting that tense logics are closely related to *bi-superintuitionistic logics*, whose degrees of FMP have recently been studied by Chernev [38]. However, the connection between splitting tense logics and strictly Kripke complete tense logics remains unclear. It is also not known whether results on bi-superintuitionistic logics can be fully generalized to the tense setting. To clarify these issues, two natural questions arise:

(Q1) Do analogues of Blok's dichotomy theorem hold for lattices of tense logics?

(Q2) Are union splittings in these lattices exactly the strictly Kripke complete ones?

(Pre)tabularity and Post-completeness. As mentioned above, tabular modal logics are fully characterized (see, e.g. [29]) and the characterization of tabular modal logics can be easily generalized to tense logic. However, pretabularity in tense logic has not yet been well understood. In particular, while pretabular modal logics over $\mathsf{S4}$ have been characterized [44, 90], a characterization of pretabular tense logics over $\mathsf{S4}_t$ cannot be obtained directly from the corresponding modal results. Indeed, pretabularity is a logical property concerning not only a single logic, but all of its extensions. Thus, to understand pretabularity in tense logic, we need to study the structure of lattices of tense logics. In fact, the following problem has been open for several decades [107, p. 23]:

(Q3) How many pretabular tense logics are there in the lattice $\mathsf{NExt}(\mathsf{S4}_t)$?

Similarly, Post-completeness of a logic depends strongly on the lattice to which it belongs. Recall that there are exactly two Post-complete normal modal logics and both of them are tabular, while there exists a bimodal logic that has $2^{\aleph_0}$ many Post-complete extensions (see, e.g., [87, 73]). It is also known that Post-completeness is related to tabularity (see, e.g., [29]). However, Post-completeness in lattices of tense logics has not yet been systematically studied. For example, one may ask:

(Q4) How many Post-complete tense logics are there, and how can they be characterized?

(Q5) What is the relation between Post-completeness and tabularity?

(Un)decidability of logical properties. Results on (un)decidability of logical properties indicate that interactions of modalities significantly affect the complexity of lattices of logics. Wolter [131, 133] studied the decidability of logical properties in $\mathsf{NExt}(\mathsf{Lin}_t)$: the decision problems for Kripke completeness, the finite model property, and decidability are all decidable. Chagrov and Shehtman [30] proved that tabularity is undecidable in $\mathsf{NExt}(\mathsf{K4}_t)$, and even that consistency is undecidable in $\mathsf{NExt}(\mathsf{K4}_t)$. However, decidability of logical properties in tense logic is not fully investigated yet. For example, little is known about the decidability of logical properties in $\mathsf{NExt}(\mathsf{S4}_t)$ and the decidability of other logical properties such as Kripke completeness, the FMP and decidability remain open for $\mathsf{NExt}(\mathsf{K4}_t)$.

The questions mentioned above illustrate that the study of lattices of tense logics is of considerable theoretical interest, as these lattices differ substantially from those of modal logics. In this thesis, we contribute to this line of research by establishing several new results concerning these questions. Our aim is to obtain a clearer picture of the structure of the lattices of tense logics and of the distribution of logical properties within them. This will also lead to a deeper understanding of the interactions between modalities.

In the following section, we present the organization of the thesis, explain how these questions are addressed, and summarize the main contributions.

1.3 Organization and Main Contributions

In this section, we give a brief overview of the organization of the thesis, the main problems studied in each chapter, and the main techniques used in their proofs.

1.3.1 Organization

The thesis is divided into two parts. The first part consists of Chapters 3 and 4, which study Post-complete, tabular and pretabular tense logics. As indicated in Figure 1.1, tabular logics form an upset in the lattice of tense logics, while Post-complete logics and pretabular logics form anti-chains. All of these logics lie near the top of the lattices of tense logics. In this part, we focus on the following logical properties:

Chapter 3

(1) Post-completeness in the lattice $\mathsf{NExt}(\mathsf{K}_t)$;

(2) tabularity in the lattice $\mathsf{NExt}(\mathsf{K}_t)$.

Chapter 4

(3) Pretabularity in the lattice $\mathsf{NExt}(\mathsf{S4}_t)$.

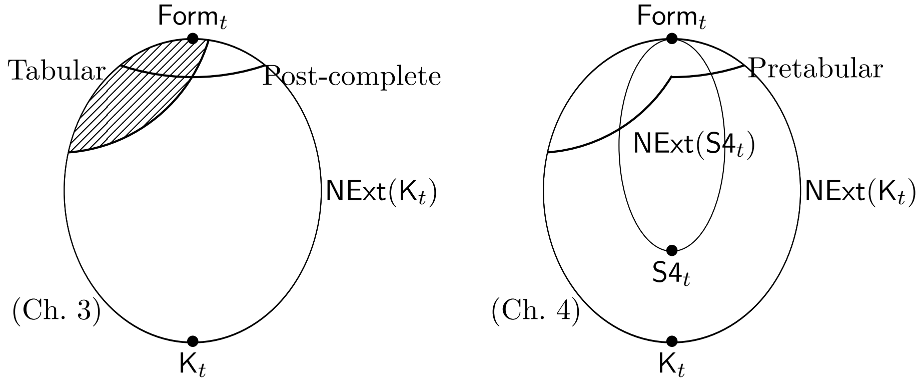

Figure 1.1: The location of Post-complete, tabular and pretabular tense logics

The second part of the thesis consists of Chapters 5 and 6. In this part, we turn from the study of logics near the top of the lattice to the study of the structure of lattices of tense logics as a whole. In particular, we investigate the following properties:

Chapter 5

(4) Degrees of Kripke incompleteness in $\mathsf{NExt}(\mathsf{K}_t)$, $\mathsf{NExt}(\mathsf{K4}_t)$ and $\mathsf{NExt}(\mathsf{S4}_t)$.

Chapter 6

(5) (Un)decidability of logical properties in $\mathsf{NExt}(\mathsf{K}_t)$, $\mathsf{NExt}(\mathsf{K4}_t)$ and $\mathsf{NExt}(\mathsf{S4}_t)$.

1.3.2 Main contributions

In this section, we summarize the main contributions of this thesis. We outline, chapter by chapter, the principal results established and briefly indicate the main ideas and techniques used in their proofs.

In Chapter 3, we study tabularity and Post-completeness in lattices of tense logics. We give a new criterion for tabularity in $\mathsf{NExt}(\mathsf{K}_t)$ by introducing formulas tab_n^T for $n \geq 1$: a tense logic L is tabular if and only if $\mathsf{tab}_n^T \in L$ for some $n \in \omega$. The relation between our characterization and the one given by Chagrov and Shehtman [30] is clarified. For Post-completeness in $\mathsf{NExt}(\mathsf{K}_t)$, three characterization theorems for Post-complete tense logics are obtained using closed formulas: (i) the first theorem gives three equivalent conditions for Post-completeness of the tense logic of finite rooted frames; (ii) a tabular tense logic L is Post-complete if and only if L has only one rooted frame up to isomorphism; and (iii) a consistent tense logic L is Post-complete if and only if it satisfies two conditions on closed formulas. The second theorem connects Post-completeness and tabularity, which partially answers (Q5). Moreover, these results allow us to determine the Post-numbers, that is, the numbers of Post-complete extensions, of several tense logics, including $\mathsf{K4}_t$, $\mathsf{D4}_t$ and B_t. We prove that the Post-number of $\mathsf{D4}_t$ is $2^{\aleph_0}$, which implies that there exist continuum many Post-complete tense logics thereby answering (Q4).

In Chapter 4, we study pretabular tense logics in $\mathsf{NExt}(\mathsf{S4}_t)$ with the aim of determining the cardinality of pretabular extensions of $\mathsf{S4}_t$. We begin by studying sublattices of $\mathsf{NExt}(\mathsf{S4}_t)$, namely, lattices of tense logics with bounded parameters, and obtain a full characterization of pretabular fully bounded tense logics. We then investigate some concrete tense logics where some of the parameters are infinite: full characterizations for pretabular logics extending $\mathsf{S4.3}_t$ and $\mathsf{S4BP}^{2,\omega}_{2,2}$ are provided. The main result in this chapter is that there exist continuum many pretabular extensions of $\mathsf{S4BP}^{2,\omega}_{2,3}$. The proof relies on several new techniques, most notably generalized Thue-Morse sequences, generalized Jankov formulas and local t-morphisms. Consequently, we obtain the anti-dichotomy theorem for the cardinality of pretabular extensions in $\mathsf{NExt}(\mathsf{S4}_t)$. In particular, we determine that the cardinality of $\mathsf{PTAB}(\mathsf{S4}_t)$ is $2^{\aleph_0}$, thereby giving a full solution to the open problem (Q3) presented in [107].

In Chapter 5, we study Kripke completeness in lattices of tense logics. We start with the lattice $\mathsf{NExt}(\mathsf{K}_t)$. Inspired by the proof of Blok's dichotomy theorem in [29], we establish a dichotomy theorem for tense logics: every tense logic $L \in \mathsf{NExt}(\mathsf{K}_t)$ has the degree of Kripke incompleteness either 1 or $2^{\aleph_0}$. The proof proceeds by showing that union-splittings in $\mathsf{NExt}(\mathsf{K}_t)$ are exactly the strictly Kripke complete logics, while all other logics have the degree $2^{\aleph_0}$. A key technique in the proof is the method of reflective unfolding. By similar arguments, we obtain the dichotomy theorem of the degree of Kripke incompleteness for $\mathsf{NExt}(\mathsf{K4}_t)$. Finally, we turn to the lattice $\mathsf{NExt}(\mathsf{S4}_t)$. We provide the following characterization of the degree of Kripke incompleteness in $\mathsf{NExt}(\mathsf{S4}_t)$: iterated splittings are strictly Kripke complete and all other logics are of degree $2^{\aleph_0}$. The dichotomy theorem of the degree of Kripke incompleteness for $\mathsf{NExt}(\mathsf{S4}_t)$ follows immediately from the characterization. This resolves the open problem (Q1) in the previous section. Consequently, strictly Kripke complete logics are no longer the union-splittings in the lattice $\mathsf{NExt}(\mathsf{S4}_t)$, thereby answering the problem (Q2).

In Chapter 6, we study decidability problems for logical properties in lattices of tense logics. We begin with the lattice $\mathsf{NExt}(\mathsf{K4}_t)$. First, we show that strict Kripke completeness is decidable. We then turn to undecidable logical properties and establish a general criterion (Theorem 6.2.5) for a logical property to be undecidable in $\mathsf{NExt}(\mathsf{K4}_t)$. This criterion yields the undecidability of several logical properties, including Kripke completeness, the FMP and decidability. These results generalize those obtained by Chagrov and Shehtman [30]. We then turn to the lattice $\mathsf{NExt}(\mathsf{S4}_t)$. In this setting, strict Kripke completeness remains decidable. Moreover, consistency is decidable and there exist countably many tabular logics for which the coincidence problem is decidable. On the other hand, we show directly that logical properties, including Kripke completeness, tabularity and decidability, are undecidable. Furthermore, there exist countably many tabular logics for which the coincidence problem is undecidable. The proofs of these results rely on a reduction technique inspired by Chagrov's method. We adapt this method to the tense setting by reducing the decision problem for a logical property in $\mathsf{NExt}(\mathsf{K4}_t)$ and $\mathsf{NExt}(\mathsf{S4}_t)$ to an undecidable problem concerning Minsky machines. This approach provides a uniform framework for establishing undecidability results for a wide range of properties in lattices of tense logics.

Finally, we conclude the introduction with the following list of main contributions:

- *Tabularity and Post-completeness*: We provide a new criterion for tabularity in $\mathsf{NExt}(\mathsf{K}_t)$ and clarify its relation to the characterization of Chagrov and Shehtman [30]. We obtain several characterizations of Post-complete tense logics and determine the cardinality of Post-complete extensions of logics such as $\mathsf{K4}_t$, $\mathsf{D4}_t$ and B_t. Consequently, we show that there are $2^{\aleph_0}$ many Post-complete tense logics.

- *Pretabular tense logics over* $\mathsf{S4}_t$: We determine the cardinality of pretabular extensions of several sublattices of $\mathsf{NExt}(\mathsf{S4}_t)$. In particular, we prove that there are $2^{\aleph_0}$ many pretabular logics over $\mathsf{S4}_t$, thereby resolving an open problem posed in [107].

- *Degrees of Kripke incompleteness of tense logics*: We prove dichotomy theorems for the degree of Kripke incompleteness in $\mathsf{NExt}(\mathsf{K}_t)$, $\mathsf{NExt}(\mathsf{K4}_t)$ and $\mathsf{NExt}(\mathsf{S4}_t)$, generalizing Blok's theorem from the modal logic K to the tense setting.

- *Undecidability of logical properties*: We show that strict Kripke completeness is decidable in $\mathsf{NExt}(\mathsf{K4}_t)$ and $\mathsf{NExt}(\mathsf{S4}_t)$. For $\mathsf{NExt}(\mathsf{K4}_t)$, we provide a general criterion for a logical property to be undecidable, which yields the undecidability of properties such as Kripke completeness, the FMP and decidability. For $\mathsf{NExt}(\mathsf{S4}_t)$, we prove directly that logical properties including Kripke completeness, tabularity and decidability are undecidable. Moreover, there exist countably many tabular logics L for which the coincidence problem is undecidable.

Sources of the chapters

- Chapter 3 is based on:

Q. Chen and M. Ma (2024). Tabularity and Post-completeness in Tense Logic. *The Review of Symbolic Logic* 17(2):475-492.

- Chapter 4 is based on:

  Q. Chen (2026). Pretabular Tense Logics over $\mathsf{S4}_t$. *Annals of Pure and Applied Logic* 177(10): 103807.

- Chapter 5 is based on:

  Q. Chen (2025). Degree of Kripke incompleteness of Tense Logics [Preprint]. arXiv:2507.04533.

- Chapter 6 is based on:

  Q. Chen and T. Takahashi (2026). Most Properties are Undecidable for Transitive Tense Logic. *Accepted by Advances in Modal Logic 2026.* To appear.

  Q. Chen and T. Takahashi (2026). Undecidable Properties in $\mathsf{NExt}(\mathsf{S4}_t)$. *Manuscript.*

Contribution statement. The research problems studied in the joint work underlying Chapter 3 were originally proposed by Minghui Ma. The author carried out most of the technical development and proofs. The writing and revision of the paper were completed collaboratively by both authors.

The work underlying Chapter 6 was carried out in close collaboration with Tenyo Takahashi. The constructions of frames and formulas used in the proofs were developed primarily by the author, while Tenyo Takahashi contributed key techniques and insights related to undecidability. The writing and revision of the papers were completed collaboratively by both authors.

Chapter 2

Preliminaries

In this chapter, we review the basic notions and results concerning propositional modal logics and tense logics that will be used throughout the thesis. Although some minimal background was already introduced in the previous chapter, we present here a more systematic account of the required preliminaries for the sake of readability.

We presume familiarity with naive set theory. Let ω denote the least non-zero limit ordinal, i.e., the set of all natural numbers. We write $\mathbb{O}$ and $\mathbb{E}$ for the sets of odd and even numbers in ω respectively. Let $\mathbb{Z}$ and $\mathbb{Z}^+$ be the sets of all integers and positive integers respectively. The cardinality of a set X is denoted by $|X|$. The power set of X is denoted by $\mathcal{P}(X)$. We use Boolean operations $\cap$, $\cup$ and $-(\cdot)$ on $\mathcal{P}(X)$. Let X, X' be sets. We write $f : X \to X'$ if f is a function from X to X'. Moreover, $f \subseteq X \times X'$ is called a partial function from X to X' if $f : Y \to X'$ for some $Y \subseteq X$. Given any (partial) function f, we write $\mathsf{dom}(f)$ and $\mathsf{ran}(f)$ for its domain and range, respectively.

2.1 Modal Logic

Modal logics are sets of formulas closed under certain rules, and their formal languages are obtained by extending a given base language with modalities (or modal operators). In this thesis, we focus on modal logics based on the classical propositional logic CPC which contains only finitely many unary modalities. We therefore start with the basic definitions and results concerning CPC. We then review the syntax, relational semantics, and algebraic semantics for modal logics.

2.1.1 Classical propositional logic

In this section, we review basic definitions and results on the classical propositional logic CPC. For more details, we refer the reader to [12, 29, 42, 82].

2.1.1. Definition. Let $\mathscr{L}_p$ denote a *propositional language* consisting of a denumerable set $\mathsf{Prop} = \{p_i : i \in \omega\}$ of propositional variables, a propositional constant $\bot$ (falsum) and three binary propositional connectives $\wedge$ (conjunction), $\vee$ (disjunction) and $\to$ (implication). The set Form_p of all well-formed $\mathscr{L}_p$-formulas is defined as follows:

$$\mathsf{Form}_p \ni \varphi ::= p \mid \bot \mid (\varphi \wedge \varphi) \mid (\varphi \vee \varphi) \mid (\varphi \to \varphi),$$

where $p \in \mathsf{Prop}$. Let $\top$ abbreviate $\bot \to \bot$. For all formulas $\varphi, \psi \in \mathsf{Form}_p$, we write $\varphi \leftrightarrow \psi$ for $(\varphi \to \psi) \wedge (\psi \to \varphi)$. For every finite set Γ of formulas, let $\bigvee \Gamma$ and $\bigwedge \Gamma$ be the disjunction and conjunction of all formulas in Γ, respectively. In particular, let $\bigvee \varnothing = \bot$ and $\bigwedge \varnothing = \top$. A formula φ is called *atomic* if $\varphi \in \mathsf{Prop} \cup \{\bot\}$. A *substitution* is a function $(\cdot)^s : \mathsf{Form}_p \to \mathsf{Form}_p$ such that

- $\bot^s = \bot$, and
- $(\varphi \circ \psi)^s = \varphi^s \circ \psi^s$ for all $\circ \in \{\wedge, \vee, \to\}$.

Every substitution $(\cdot)^s$ is induced by the map $s : \mathsf{Prop} \to \mathsf{Form}_p$ such that $s(p) = p^s$. We identify $(\cdot)^s$ and s if there is no danger of confusion. For each $\varphi \in \mathsf{Form}_p$, we write $\varphi(q_1, \cdots, q_n)$ if the propositional variables occurring in φ are among $q_1, \cdots, q_n$. We write $[\gamma_1/q_1, \cdots, \gamma_n/q_n]$ for the substitution $(\cdot)^s$ induced by:

$$s(p) = \begin{cases} \gamma_i, & \text{if } p = q_i; \\ p, & \text{otherwise.} \end{cases}$$

Moreover, we write $\varphi[\gamma_1/q_1, \cdots, \gamma_n/q_n]$ (sometimes $\varphi(\gamma_1, \cdots, \gamma_n)$) for the formula obtained from φ by applying the substitution $[\gamma_1/q_1, \cdots, \gamma_n/q_n]$.

2.1.2. Definition. The classical propositional logic CPC is the smallest set of formulas containing the following axioms:

(A1) $p \to (q \to p)$,

(A2) $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$,

(A3) $p \wedge q \to p$,

(A4) $p \wedge q \to q$,

(A5) $p \to (q \to p \wedge q)$,

(A6) $p \to p \vee q$,

(A7) $q \to p \vee q$,

(A8) $(p \to r) \to ((q \to r) \to ((p \vee q) \to r))$,

(A9) $\bot \to p$,

(A10) $p \vee (p \to \bot)$,

and is closed under the rules *Modus Ponens* (MP) and *uniform substitution* (Sub):

(MP) from α and $\alpha \to \beta$ infer β,

(Sub) from $\alpha(p_0, \cdots, p_{n-1})$ infer $\alpha(\psi_0, \cdots, \psi_{n-1})$.

In this thesis, the notion of a logic is defined as follows:

2.1.3. Definition. Let $\mathscr{L}$ be a language and Form the set of all formulas in $\mathscr{L}$. Then a *logic* over $\mathscr{L}$ is a set $L \subseteq \mathsf{Form}$ that is closed under (MP) and (Sub).

Clearly, the classical propositional logic CPC satisfies Definition 2.1.3. Recall that the *intuitionistic propositional logic* IPC is defined to be the smallest set of formulas containing the axioms (A1) - (A9) and closed under the rules (MP) and (Sub). Hence, IPC also satisfies Definition 2.1.3.

Next, we review the notions of *derivation* and *proof*.

2.1.4. Definition. Given a set Γ of formulas, a finite sequence of formulas $(\varphi_0, ..., \varphi_n)$ is called a *derivation* of φ from Γ in CPC if:

(1) $\varphi_n = \varphi$;

(2) For each $i \leq n$, φ_i satisfies one of the following:

  (a) φ_i is an axiom, i.e., one of (A1) - (A10);

  (b) $\varphi_i \in \Gamma$;

  (c) There exist $j, k < i$ such that $\varphi_k = \varphi_j \to \varphi_i$;

  (d) There exist $j < i$ and a substitution s such that φ_j is an axiom and $\varphi_i = s(\varphi_j)$.

The set Γ is called the *assumption* of the derivation. A derivation is called a *proof* if its assumption is the empty set. We write $\Gamma \vdash_{\mathsf{CPC}} \varphi$ if there is a derivation of φ from Γ. We write $\vdash_{\mathsf{CPC}} \varphi$ if $\varnothing \vdash_{\mathsf{CPC}} \varphi$. Moreover, Γ is CPC-*consistent* if $\Gamma \nvdash_{\mathsf{CPC}} \bot$.

2.1.5. Proposition. *For all* $\varphi \in \mathsf{Form}_p$,

$$\varphi \in \mathsf{CPC} \text{ if and only if } \vdash_{\mathsf{CPC}} \varphi.$$

It is well known that the deduction theorem holds for CPC:

2.1.6. Theorem (Deduction Theorem). *If* $\Gamma \subseteq \mathsf{Form}_p$ *and* $\varphi, \psi \in \mathsf{Form}_p$*, then*

$$\Gamma \cup \{\varphi\} \vdash_{\mathsf{CPC}} \psi \textit{ if and only if } \Gamma \vdash_{\mathsf{CPC}} \varphi \to \psi.$$

Now we move to the semantics for the classical propositional logic CPC.

2.1.7. Definition. A classical model is a subset $\mathfrak{M} \subseteq \mathsf{Prop}$. Truth of formulas φ in the model $\mathfrak{M}$ (notation: $\mathfrak{M} \vDash \varphi$) is defined inductively as follows:

$\mathfrak{M} \vDash \bot$	never	
$\mathfrak{M} \vDash p$	if and only if	$p \in \mathfrak{M}$
$\mathfrak{M} \vDash \varphi \wedge \psi$	if and only if	$\mathfrak{M} \vDash \varphi$ and $\mathfrak{M} \vDash \psi$
$\mathfrak{M} \vDash \varphi \vee \psi$	if and only if	$\mathfrak{M} \vDash \varphi$ or $\mathfrak{M} \vDash \psi$
$\mathfrak{M} \vDash \varphi \to \psi$	if and only if	$\mathfrak{M} \vDash \varphi$ implies $\mathfrak{M} \vDash \psi$

We say that φ is *true* in $\mathfrak{M}$ or $\mathfrak{M}$ satisfies φ if $\mathfrak{M} \vDash \varphi$. We write $\mathfrak{M} \nvDash \varphi$ if $\mathfrak{M} \vDash \varphi$ does not hold. In this case, we say that φ is *false* in $\mathfrak{M}$ or $\mathfrak{M}$ falsifies φ. A set Σ of formulas is *true* in $\mathfrak{M}$ if $\mathfrak{M} \vDash \sigma$ for all $\sigma \in \Sigma$. A formula φ is called *valid* (notation: $\vDash \varphi$) if $\mathfrak{M} \vDash \varphi$ for all models $\mathfrak{M}$. Moreover, we say that φ is a *semantic consequence* of Σ (notation: $\Sigma \vDash \varphi$) if $\mathfrak{M} \vDash \varphi$ for all $\mathfrak{M} \vDash \Sigma$.

It is well known that CPC is sound and complete with respect to the semantics given above, that is, the following theorem holds (see, e.g., [29, 42]).

2.1.8. Theorem. *For all* $\varphi \in \mathsf{Form}_p$ *and* $\Sigma \subseteq \mathsf{Form}_p$*,*

$$\Sigma \vdash_{\mathsf{CPC}} \varphi \textit{ if and only if } \Sigma \vDash \varphi.$$

Moreover, by the following proposition, the connectives $\wedge$ and $\vee$ can be defined by $\to$ and $\bot$ and so the $\{\to, \bot\}$-fragment $\mathscr{L}'_p$ of $\mathscr{L}_p$ is as expressive as $\mathscr{L}_p$.

2.1.9. Proposition. *For all* $\varphi, \psi \in \mathsf{Form}_p$*, the following holds*

(1) $\vDash (\varphi \wedge \psi) \leftrightarrow ((\varphi \to (\psi \to \bot)) \to \bot)$,

(2) $\vDash (\varphi \vee \psi) \leftrightarrow ((\varphi \to \bot) \to \psi)$.

The logic CPC is well studied and enjoys many desirable logical properties. We now review some basic logic properties such as *consistency* and *interpolation property* (see, e.g., [29]). These properties apply not only to logics over Form_p, but also to modal and tense logics that will be introduced later.

Consistency

A logic L over a formal language $\mathscr{L}$ is *consistent* if $L \neq \mathsf{Form}$, where Form is the set of all $\mathscr{L}$-formulas. Note that $\nvDash \bot$. By Theorem 2.1.8, we see that $\bot \notin \mathsf{CPC}$ and so the following theorem holds:

2.1.10. Theorem. CPC *is consistent.*

Craig interpolation property

A logic L enjoys the *Craig interpolation property* (CIP) if for all pairwise disjoint sets of propositional variables $\overline{p}, \overline{q}, \overline{r}$ and formulas $\varphi(\overline{p}, \overline{q})$ and $\psi(\overline{p}, \overline{r})$ such that $\vdash_L \varphi \to \psi$, there exists $\gamma(\overline{p})$ satisfying $\vdash_L \varphi \to \gamma$ and $\vdash_L \gamma \to \psi$.

2.1.11. THEOREM. CPC *has the Craig interpolation property.*

The CIP is one of the most important syntactic properties of logic and has been extensively studied in proof theory. In this thesis, however, it will not be a central focus. For further discussion of interpolation properties, we refer the reader to [124, 57, 22].

Local tabularity

Let L be a logic and φ, ψ are formulas. Then we say that φ and ψ are L*-equivalent* if $\varphi \leftrightarrow \psi \in L$. Otherwise, we say that they are L-non-equivalent.

Recall that we write $\varphi(p_1, \cdots, p_n)$ if the propositional variables occurring in φ are among $p_1, \cdots, p_n$. Then a logic L is *locally tabular* if for each $n \in \omega$, the logic L contains only finitely many L-non-equivalent formulas of the form $\varphi(p_1, \cdots, p_n)$. Formally, L is locally tabular if for all $n \in \omega$, there exists a finite set Ψ_n of formulas such that:

$$\text{for all } \varphi(p_1, \cdots, p_n), \text{ there exists } \psi \in \Psi_n \text{ such that } \varphi \leftrightarrow \psi \in L.$$

For the logic CPC, it is well known that a propositional formula $\varphi(p_1, \cdots, p_n)$ can be characterized by a unique function $f : 2^n \to 2$. Thus, the following theorem holds:

2.1.12. THEOREM. CPC *is locally tabular.*

Decidability

A logic L is *decidable* if there exists an effective method to decide whether a formula φ is in L. For each formula $\varphi \in \mathsf{Form}_p$, one can decide whether $\varphi \in \mathsf{CPC}$ by examining its truth table. Thus, the following theorem holds:

2.1.13. THEOREM. CPC *is decidable.*

Post-completeness

A logic L is *Post-complete* if L is consistent and there exists no consistent logic L' such that $L \subsetneq L'$. For example, since $\mathsf{IPC} \subsetneq \mathsf{CPC} \subsetneq \mathsf{Form}_p$, we see that IPC is not Post-complete. It is also well known that the following theorem holds:

2.1.14. THEOREM. CPC *is Post-complete.*

2.1.2 Syntax and relational semantics for modal logic

Now we move from the classical propositional logic to modal logics. For more details, the reader can consult [12] and [29]. We first review basic definitions and facts of modal logics. The formal language $\mathscr{L}_\Box$ of modal logic is obtained from $\mathscr{L}_p'$ by adding a unary modality $\Box$, where $\mathscr{L}_p'$ is the $\{\to, \bot\}$-fragment of $\mathscr{L}_p$. The set Form_m of all modal formulas is defined by

$$\mathsf{Form}_m \ni \varphi ::= p \mid \bot \mid (\varphi \to \varphi) \mid \Box\varphi,$$

where $p \in \mathsf{Prop}$. For all formulas $\varphi, \psi \in \mathsf{Form}_m$, we let $\neg\varphi$, $\varphi \wedge \psi$ and $\varphi \vee \psi$ abbreviate $\varphi \to \bot$, $\neg(\varphi \to \neg\psi)$ and $\neg\varphi \to \psi$, respectively. The dual modal operator $\Diamond$ of $\Box$ is defined by $\Diamond\varphi := \neg\Box\neg\varphi$. Substitutions are defined as usual.

Recall that a set $L \subseteq \mathsf{Form}_m$ of formulas is called a *normal modal logic* if

(1) $\mathsf{CPC} \subseteq L$ and L is closed under rules (MP) and (Sub);

(2) $\Box(p \to q) \to (\Box p \to \Box q) \in L$;

(3) L is closed under Necessity (Nec): $\alpha \in L$ implies $\Box\alpha \in L$, for all $\alpha \in \mathsf{Form}_m$.

Let K denote the least normal modal logic. In this thesis, we simply use the term *modal logic* to mean *normal modal logic* whenever there is no danger of confusion. For every modal logic L and set Σ of formulas, let $L \oplus \Sigma$ denote the smallest modal logic containing $L \cup \Sigma$. A modal logic L is *consistent* if $\bot \notin L$; and L is *finitely axiomatizable* if there is a finite set Σ of formulas such that $L = \mathsf{K} \oplus \Sigma$.

A modal logic L_1 is a *sublogic* of L_2 (or L_2 is an *extension* of L_1) if $L_1 \subseteq L_2$. Moreover, L_2 is called a *proper extension* of L_1 if $L_2 \supsetneq L_1$. For each modal logic L, let $\mathsf{NExt}(L)$ denote the set of all normal extensions of L, that is

$$\mathsf{NExt}(L) := \{L' \supseteq L : L' \text{ is a normal moal logic}\}.$$

A formula φ is *deducible* in L from a set Γ of formulas (notation: $\Gamma \vdash_L \varphi$), if $\varphi \in L$ or there exist $\psi_1, \cdots, \psi_n \in \Gamma$ with $(\psi_1 \wedge \cdots \wedge \psi_n) \to \varphi \in L$. A set Γ of formulas is *L-consistent* if $\Gamma \nvdash_L \bot$. A set Γ of formulas is *maximal L-consistent* if Γ is L-consistent and it has no L-consistent proper extension.

The Kripke semantics for modal logic is given as follows:

2.1.15. DEFINITION. A *Kripke frame* is a pair $\mathfrak{F} = (X, R)$ where $X \neq \varnothing$ and $R \subseteq X \times X$. We call X the *domain* of $\mathfrak{F}$ and R the *accessibility relation* in $\mathfrak{F}$. We write Rxy (or xRy) if $(x, y) \in R$. For every $Y \subseteq X$, we define

$$R[Y] := \{z \in X : \exists y \in Y(Ryz)\}.$$

For all $x \in X$, we write $R[x]$ for $R[\{x\}]$.

The cardinality of a Kripke frame $\mathfrak{F} = (X, R)$ is defined to be the cardinality $|X|$ of its domain X. If $|X| < \aleph_0$, then we say that $\mathfrak{F}$ is a finite Kripke frame. Let Fr and Fin denote the class of all Kripke frames and finite Kripke frames, respectively.

In this thesis, we write frame for Kripke frame if there is no danger of confusion.

2.1.16. Definition. Let $\mathfrak{F} = (X, R)$ be a frame. Then a *valuation* in $\mathfrak{F}$ is a function $V : \mathsf{Prop} \to \mathcal{P}(X)$. A *Kripke model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F}$ is a frame and V a valuation in $\mathfrak{F}$. For every point $x \in X$ and formula φ, we define the relation $\mathfrak{M}, x \vDash \varphi$ which is read as *'φ is true at x in $\mathfrak{M}$'* as follows:

$\mathfrak{M}, x \vDash \bot$	never	
$\mathfrak{M}, x \vDash p$	if and only if	$x \in V(p)$, for all $p \in \mathsf{Prop}$
$\mathfrak{M}, x \vDash \varphi \to \psi$	if and only if	$\mathfrak{M}, x \nvDash \varphi$ or $\mathfrak{M}, x \vDash \psi$
$\mathfrak{M}, x \vDash \Box\varphi$	if and only if	$\mathfrak{M}, y \vDash \varphi$ for all $y \in R[x]$

Moreover, we say that (i) φ is *valid* at x in $\mathfrak{F}$ (notation: $\mathfrak{F}, x \vDash \varphi$) if $\mathfrak{F}, V, x \vDash \varphi$ for every valuation V in $\mathfrak{F}$; and (ii) φ is *valid* in $\mathfrak{F}$ (notation: $\mathfrak{F} \vDash \varphi$) if $\mathfrak{F}, x \vDash \varphi$ for every $x \in X$.

As we shall see, Kripke frames provide an adequate semantic framework for many modal logics. For example, the minimal normal modal logic K consists of all formulas that are valid in every frame, and many modal logics can likewise be characterized by suitable classes of frames. However, not every modal logic is characterized by Kripke frames. This motivates the introduction of general frames, which generalize Kripke frames and offer increased expressive power.

2.1.17. Definition. A *general frame* is a triple $\mathbb{F} = (X, R, A)$ where (X, R) is a frame and $A \subseteq \mathcal{P}(X)$ is a set such that

(1) $\varnothing \in A$,

(2) $U \cap V \in A$ for all $U, V \in A$,

(3) $X \setminus U \in A$ and $R^{-1}[U] \in A$, for all $U \in A$.

We call $\kappa\mathbb{F} = (X, R)$ the *underlying frame* of $\mathbb{F}$ and $A \subseteq \mathcal{P}(X)$ the set of *internal sets* in $\mathbb{F}$. Let GFr denote the class of all general frames.

2.1.18. Definition. Let $\mathbb{F} = (X, R, A)$ be a general frame. Then a map $V : \mathsf{Prop} \to A$ is called an *admissible valuation* in $\mathbb{F}$. An admissible valuation V is extended to the map $V : \mathsf{Form}_m \to A$ as follows:

$$V(\bot) = \varnothing, \quad V(\varphi \to \psi) = -V(\varphi) \cup V(\psi), \quad V(\Box\varphi) = -R^{-1}[-V(\varphi)].$$

A *general model* is a pair $\mathfrak{M} = (\mathbb{F}, V)$ where $\mathbb{F} \in \mathsf{GFr}$ and V an admissible valuation in $\mathbb{F}$. We shall often refer to general models simply as models, and write (X, R, V) instead of $(\mathbb{F}, V)$. Let φ be a formula and $x \in X$. Then (i) φ is *true* at x in $\mathfrak{M}$ (notation: $\mathfrak{M}, x \vDash \varphi$) if $x \in V(\varphi)$; (ii) φ is *valid* at x in $\mathbb{F}$ (notation: $\mathbb{F}, x \vDash \varphi$) if $x \in V(\varphi)$ for every admissible valuation V in $\mathbb{F}$; (iii) φ is *valid* in $\mathbb{F}$ (notation: $\mathbb{F} \vDash \varphi$) if $\mathbb{F}, x \vDash \varphi$ for every $x \in X$; and (iv) φ is *valid* in a class of general frame $\mathcal{K}$ (notation: $\mathcal{K} \vDash \varphi$) if $\mathbb{F} \vDash \varphi$ for every $\mathbb{F} \in \mathcal{K}$. For all sets $\Sigma \subseteq \mathsf{Form}_m$ of formulas and classes $\mathcal{K} \subseteq \mathsf{GFr}$ of general frames, let

$$\mathcal{K}(\Sigma) \coloneqq \{\mathbb{F} \in \mathcal{K} : \mathbb{F} \vDash \Sigma\} \text{ and } \mathsf{Log}(\mathcal{K}) \coloneqq \{\varphi : \mathcal{K} \vDash \varphi\}.$$

We call $\mathsf{Log}(\mathcal{K})$ the *modal logic* of $\mathcal{K}$.

Let $\mathbb{F} = (X, R, A)$ be a general frame. We sometimes identify $\mathbb{F}$ with its underlying frame $\kappa\mathbb{F}$ if $A = \mathcal{P}(X)$. In this sense, every Kripke frame can be viewed as a general frame of the form $(X, R, \mathcal{P}(X))$ and so $\mathsf{Fr} \subseteq \mathsf{GFr}$. In fact, most of the notions for general frames that will be introduced later in this thesis can be naturally generalized to Kripke frames. So we say that the general frame semantics generalizes the Kripke semantics. Some classical modal logics and their frames are reviewed in the following example.

2.1.19. Example. The modal logics $\mathsf{K4}$, $\mathsf{S4}$ and $\mathsf{S5}$ are defined as follows:

- $\mathsf{K4} \coloneqq \mathsf{K} \oplus \Box\Box p \to \Box p$;
- $\mathsf{S4} \coloneqq \mathsf{K4} \oplus \Box p \to p$;
- $\mathsf{S5} \coloneqq \mathsf{S4} \oplus p \to \Box\Diamond p$.

Let R be a binary relation on a non-empty set X. Then we say that R is

- *transitive* if for all $x, y, z \in X$, if Rxy and Ryz, then Rxz;
- *reflexive* if Rxx holds for all $x \in X$;
- *symmetric* if Rxy implies Ryx for all $x, y \in X$.

A frame $\mathfrak{F} = (X, R)$ is said to be *transitive* if R is transitive. Moreover, we say that $\mathfrak{F}$ is a *pre-order* if R is both transitive and reflexive; and $\mathfrak{F}$ is an *equivalence frame* if $\mathfrak{F}$ is a pre-order and R is symmetric.

It is well known that $\mathsf{K4}$, $\mathsf{S4}$ and $\mathsf{S5}$ are the modal logics of transitive frames, pre-orders and equivalence frames, respectively.

2.1.20. Definition. Let $\mathbb{F} = (X, R, A)$ be a general frame. Then $\mathbb{F}$ is called

(1) *differentiated*, if for all distinct $x, y \in X$, $x \in U$ and $y \notin U$ for some $U \in A$;

(2) *tight*, if for all $x, y \in X$ such that $y \notin R[x]$, there exists an internal set $U \in A$ such that $y \in U$ and $R[x] \cap U = \varnothing$;

(3) *compact*, if $\bigcap B \neq \varnothing$ for any $B \subseteq A$ with finite intersection property;[1]

Moreover, we say that $\mathbb{F}$ is a

(4) *refined frame*, if $\mathbb{F}$ is both differentiated and tight;

(5) *descriptive frame*, if $\mathbb{F}$ is both refined and compact.

Let RFr and DFr denote the class of all refined and descriptive frames, respectively.

Next, we recall the main operations on general frames for modal logics: generated subframes, disjoint unions and p-morphisms.

Generated subframes

2.1.21. Definition. Let $\mathbb{F} = (X, R, A)$ be a general frame and $Y \subseteq X$. The *subframe* $\mathbb{F}{\restriction}Y$ *of* $\mathbb{F}$ *induced by* Y is defined by $\mathbb{F}{\restriction}Y = (Y, R{\restriction}Y, A{\restriction}Y)$, where

$$R{\restriction}Y \coloneqq R \cap (Y \times Y) \text{ and } A{\restriction}Y \coloneqq \{U \cap Y : U \in A\}.$$

The *subframe of* $\mathbb{F}$ *generated by* Y is defined to be $\mathbb{F}{\restriction}R^\omega[Y]$, where

$$R^\omega[Y] \coloneqq \bigcup_{n\in\omega} R^n[Y]$$

and the sets $R^n[Y]$ are defined inductively by $R^0[Y] \coloneqq Y$ and $R^{n+1}[Y] \coloneqq R[R^n[Y]]$. A general frame $\mathbb{G}$ is called a *generated subframe* of $\mathbb{F}$ if $\mathbb{G} = \mathbb{F}{\restriction}R^\omega[Y]$ for some $Y \subseteq X$. For all $x \in X$, we write $R^\omega[x]$ for $R^\omega[\{x\}]$. We call $\mathbb{F}_x \coloneqq \mathbb{F}{\restriction}R^\omega[x]$ the *subframe of* $\mathbb{F}$ *generated by* x. We call $\mathbb{F}$ a *rooted frame* if $\mathbb{F} = \mathbb{F}_x$ for some $x \in X$.

Generated general subframes preserve validity, that is,

2.1.22. Proposition. *Let* $\mathbb{F} = (X, R, A)$ *be a general frame and* $\mathbb{G}$ *a generated subframe of* $\mathbb{F}$. *Then* $\mathsf{Log}(\mathbb{F}) \subseteq \mathsf{Log}(\mathbb{G})$.

2.1.23. Remark. In much of the literature, for a descriptive frame (sometimes called a modal space) $\mathbb{F} = (X, R, A)$ and a subset $Y \subseteq X$, the general frame $\mathbb{F}{\restriction}Y$ is considered a subframe of $\mathbb{F}$ only under the additional requirement that $Y \in A$ (see, e.g., [8, 10]). In this thesis, we work primarily with general frames and therefore do not impose this requirement.

[1]For all sets X and $A \subseteq \mathcal{P}(X)$, we say that A has the *finite intersection property*, if $\bigcap B \neq \varnothing$ for any non-empty finite subset B of A.

Disjoint unions

2.1.24. Definition. Let $\mathcal{F} = (\mathbb{F}_i = (X_i, R_i, A_i))_{i\in I}$ be a family of general frames. Then the *disjoint union* of $\mathcal{F}$ is defined as $\biguplus_{i\in I} \mathbb{F}_i = (X, R, A)$, where

- $X \coloneqq \bigcup_{i\in I} X_i \times \{i\}$,
- $R \coloneqq \{((x,i),(y,i)) : R_i xy \text{ and } i \in I\}$, and
- $A \coloneqq \{U \subseteq X : \forall i \in I(\{y \in X_i : (y,i) \in U\} \in A_i)\}$.

Validity of modal formulas is preserved under disjoint unions:

2.1.25. Proposition. *Let $\mathcal{F} = (\mathbb{F}_i = (X_i, R_i, A_i))_{i\in I}$ be a family of general frames. Then for all $i \in I$, $x \in X_i$ and $\varphi \in \mathsf{Form}_m$, the following holds:*

$$\biguplus_{i\in I} \mathbb{F}_i, (x,i) \nvDash \varphi \text{ if and only if } \mathbb{F}_i, x \nvDash \varphi.$$

As a corollary, we have $\mathsf{Log}(\biguplus_{i\in I} \mathbb{F}_i) = \bigcap_{i\in I} \mathsf{Log}(\mathbb{F}_i)$.

P-morphisms

2.1.26. Definition. Let $\mathbb{F} = (X, R, A)$ and $\mathbb{F}' = (X', R', A')$ be general frames. A map $f : X \to X'$ is called a *p-morphism* from $\mathbb{F}$ to $\mathbb{F}'$ (notation: $f : \mathbb{F} \to \mathbb{F}'$), if

- for all $Y' \in A'$, we have $f^{-1}[Y'] \in A$;
- for all $x, y \in X$, if Rxy, then $R'f(x)f(y)$; and
- for all $x \in X$ and $y' \in X'$, if $R'f(x)y'$, then there exists $y \in R[x]$ with $f(y) = y'$.

We write $f : \mathbb{F} \twoheadrightarrow \mathbb{F}'$ if f is a surjective p-morphism from $\mathbb{F}$ to $\mathbb{F}'$. Moreover, $\mathbb{F}'$ is called a *p-morphic image* of $\mathbb{F}$ (notation: $\mathbb{F} \twoheadrightarrow \mathbb{F}'$) if there exists $f : \mathbb{F} \twoheadrightarrow \mathbb{F}'$. For a class $\mathcal{K}$ of general frames, let $\mathsf{M}_p(\mathcal{K})$ denote the class of all p-morphic images of frames in $\mathcal{K}$.

It is well known that surjective p-morphisms preserve validity:

2.1.27. Proposition. *Let $\mathbb{F} = (X, R, A)$ and $\mathbb{F}' = (X', R', A')$ be general frames and $f : \mathbb{F} \twoheadrightarrow \mathbb{F}'$. Then for all $x \in X$ and $\varphi \in \mathsf{Form}_m$,*

$$\mathbb{F}, x \vDash \varphi \text{ implies } \mathbb{F}', f(x) \vDash \varphi.$$

As a corollary, $\mathsf{Log}(\mathbb{F}) \subseteq \mathsf{Log}(\mathbb{F}')$.

2.1.28. Remark. Note that Kripke frames can be viewed as special general frames. The operations of taking subframes, disjoint unions, and p-morphic images defined above therefore apply to Kripke frames as well.

2.1.3 Algebraic semantics for modal logic

In this section, we review the algebraic semantics for modal logics, focusing on basic definitions, key properties, and their connection to relational semantics. We make an effort to keep this section self-contained. However, we assume familiarity with basic notions from lattice theory and universal algebra. For concepts that are not defined here, we refer the reader to standard textbooks on universal algebra, such as [6], [21], and [60].

Universal algebra

Let us begin with some basic notions of universal algebra. In this thesis, we consider only algebras of *finite similarity type*. Thus, a *similarity type* is a finite tuple $\sigma = \langle k_i : i \in n \rangle$ of natural numbers. For each similarity type, we may fix a set $F = \{f_i : i \in n\}$ of function symbols. Then, an *algebra* $\mathfrak{A}$ of similarity type σ is a tuple $\langle A; f_0^{\mathfrak{A}}, \cdots, f_{n-1}^{\mathfrak{A}} \rangle$, where A is a nonempty set and $f_i^{\mathfrak{A}}$ is a k_i-ary function on A for each $i \in n$. We write $\langle A; F \rangle$ for $\mathfrak{A}$ if there is no danger of confusion. Throughout this thesis, all algebras under consideration are assumed to have the same similarity type.

Lattices serve as a good example of algebras. In fact, lattices play a particularly important role in universal algebra: on the one hand, lattice theory provides a general framework for analyzing algebraic structures, and on the other hand, lattices themselves form a well-behaved class of algebras. We therefore recall the definition of lattices.

2.1.29. Definition. A *lattice* is an algebra $\mathfrak{L} = \langle L; \wedge, \vee \rangle$ of similarity type $\langle 2, 2 \rangle$ such that the following equations hold for all $a, b, c \in L$:

$$\begin{aligned} a \wedge b &= b \wedge a, & a \vee b &= b \vee a, \\ (a \wedge b) \wedge c &= a \wedge (b \wedge c), & (a \vee b) \vee c &= a \vee (b \vee c), \\ (a \wedge b) \vee a &= a, & (a \vee b) \wedge a &= a. \end{aligned}$$

A lattice $\mathfrak{L}$ is said to be *distributive* if for all $a, b, c \in L$,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

2.1.30. Definition. A *bounded lattice* is an algebra $\mathfrak{L} = \langle L; \wedge, \vee, 0, 1 \rangle$ of similarity type $\langle 2, 2, 0, 0 \rangle$ where $\langle L; \wedge, \vee \rangle$ is a lattice and for all $a \in L$:

$$a \wedge 0 = 0, \qquad a \vee 1 = 1.$$

It is well known that lattices can also be defined order-theoretically.

2.1.31. Definition. A pre-order (X, R) is called a *partially ordered set*, *poset* for short, if R is *antisymmetric*:

$$\forall xy \in X(Rxy \wedge Ryx \to x = y).$$

Let (X, R) be a poset and $Y \subseteq X$. Then an element $a \in X$ is called an *upper bound* of Y if $a \geq y$ for all $y \in Y$. We call $a \in L$ the *supremum* or the *least upper bound* of Y (notation: $\sup(Y)$), if $a \leq b$ for all upper bounds b of Y. Analogously, an element $a \in L$ is called a *lower bound* of Y if $a \leq y$ for all $y \in Y$. We call $a \in L$ the *infimum* or the *greatest lower bound* of Y (notation: $\inf(Y)$), if $a \geq b$ for all lower bounds b of Y.

An equivalent definition of lattice is given as follows:

2.1.32. Definition. A poset $\mathfrak{L} = (L, \leq)$ is called a *lattice* if every pair $\{a, b\}$ of elements has a supremum and an infimum. It is called *bounded* if L has a least and greatest element. Moreover, $\mathfrak{L}$ is *complete* if every subset of L has a supremum and infimum.

2.1.33. Definition. Let L be a bounded lattice. An element $a \in L$ is called a *co-atom* if $a \neq 1$ and there is no $c \in L$ such that $a < c < 1$.

The algebraic definition and the order-theoretic definition of lattice are equivalent: Let $\langle L; \wedge, \vee\rangle$ be a lattice. Then the relation $\leq$ defined by

$$a \leq b \text{ if and only if } a \wedge b = a$$

is a partial order on L, and $(L, \leq)$ is a lattice whose infima and suprema are given by $\wedge$ and $\vee$, respectively. Conversely, if $(L, \leq)$ is a lattice, then defining

$$a \wedge b = \inf(\{a, b\}) \text{ and } a \vee b = \sup(\{a, b\})$$

yields a lattice $\langle L; \wedge, \vee\rangle$.

2.1.34. Example. Since modal logics are closed under intersections, we obtain that $(\mathsf{NExt}(L), \subseteq)$, or $\langle \mathsf{NExt}(L); \cap, \oplus, L, \mathsf{Form}_m\rangle$, is a complete lattice for every normal modal logic L. We simply write $\mathsf{NExt}(L)$ for the lattice $(\mathsf{NExt}(L), \subseteq)$.

Next, we review the notion of *terms* and *equations*. Let $\sigma = \langle k_i \in \omega : i \in n\rangle$ be a similarity type and $F = \{f_i : i \in n\}$ the set of function symbols for σ. Let $\mathsf{var} = \{x_i : i \in \omega\}$ be a set of variables. Then the set T_σ of σ-terms is defined to be the minimal set such that

(1) $\mathsf{var} \subseteq T_\sigma$;

(2) $f_i(t_1, \cdots, t_{k_i}) \in T_\sigma$ for all $t_1, \cdots, t_{k_i} \in T_\sigma$ and $i \in n$.

An *equation* is a string of the form $t_1 = t_2$, where $t_1, t_2 \in T_\sigma$.

Let $\mathfrak{A}$ be an algebra of the similarity type σ. Then every term $t \in T_\sigma$ induces a function $t^{\mathfrak{A}}$ on A naturally. If $t_1^{\mathfrak{A}} = t_2^{\mathfrak{A}}$, then we say that an equation $t_1 = t_2$ is satisfied in $\mathfrak{A}$ (notation: $\mathfrak{A} \vDash t_1 = t_2$). Let Σ be a set of equations. Then we say that Σ is satisfied in $\mathfrak{A}$ (notation: $\mathfrak{A} \vDash \Sigma$), if $\mathfrak{A} \vDash \alpha$ for all $\alpha \in \Sigma$. For each equation α, we write $\mathsf{Alg}(\alpha)$ for the class of all algebras satisfying α, that is, $\mathsf{Alg}(\alpha) := \{\mathfrak{A} : \mathfrak{A} \vDash \alpha\}$. Similarly, for every set Σ of equations, we write $\mathsf{Alg}(\Sigma)$ for the class $\{\mathfrak{A} : \mathfrak{A} \vDash \Sigma\}$. On the other hand, for each algebra $\mathfrak{A}$, we write $\mathsf{Equ}(\mathfrak{A})$ for the *theory* of $\mathfrak{A}$, that is, the set of all equations satisfied in $\mathfrak{A}$. Thus, $\mathsf{Equ}(\mathfrak{A}) := \{\alpha : \mathfrak{A} \vDash \alpha\}$. Again, for each class $\mathcal{K}$ of algebras, we write $\mathsf{Equ}(\mathcal{K})$ for the *theory* of $\mathcal{K}$, namely, $\bigcap_{\mathfrak{A} \in \mathcal{K}} \mathsf{Equ}(\mathfrak{A})$. A class $\mathcal{K}$ is called an *equational class* if $\mathcal{K} = \mathsf{Alg}(\mathsf{Equ}(\mathcal{K}))$.

As the reader might have noticed from Definition 2.1.30, the class of all (distributive) lattices is an equational class. In universal algebra, equational classes are of central interest (see, e.g., [6]). By the celebrated Birkhoff's Theorem, a class $\mathcal{K}$ of algebras is an equational class if and only if $\mathcal{K}$ is a *variety*, that is, $\mathcal{K}$ is closed under *subalgebras*, *homomorphic images* and *direct products*. Thus, we recall these three algebraic operators.

2.1.35. Definition. Let $\mathfrak{A} = \langle A; F \rangle$ and $\mathfrak{B} = \langle B; F \rangle$ be algebras of the same similarity type $\sigma = \langle k_i \in \omega : i \in n \rangle$. A map $h : A \to B$ is called a *homomorphism* from $\mathfrak{A}$ to $\mathfrak{B}$ (notation: $h : \mathfrak{A} \to \mathfrak{B}$), if for each $f_i \in F$ and $a_0, \cdots, a_{k_i - 1} \in A$,

$$h(f_i^{\mathfrak{A}}(a_0, \cdots, a_{k_i-1})) = f_i^{\mathfrak{B}}(h(a_0), \cdots, h(a_{k_i-1})).$$

If h is surjective, we say that $\mathfrak{B}$ is a *homomorphic image* of $\mathfrak{A}$ and write $h : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ or simply $\mathfrak{A} \twoheadrightarrow \mathfrak{B}$. The map $h : \mathfrak{A} \to \mathfrak{B}$ is called an *embedding* (notation: $h : \mathfrak{A} \rightarrowtail \mathfrak{B}$) if h is injective. $\mathfrak{A}$ is called a *subalgebra* of $\mathfrak{B}$ if $A \subseteq B$ and the *inclusion map* $i : A \to B$ is an embedding from $\mathfrak{A}$ to $\mathfrak{B}$. Moreover, we say that $h : \mathfrak{A} \to \mathfrak{B}$ is an *isomorphism* (notation: $h : \mathfrak{A} \cong \mathfrak{B}$) if h is bijective. The algebras $\mathfrak{A}$ and $\mathfrak{B}$ are called *isomorphic* if there is an isomorphism $h : \mathfrak{A} \cong \mathfrak{B}$.

For each algebra $\mathfrak{A}$, we write $\mathsf{H}(\mathfrak{A})$, $\mathsf{S}(\mathfrak{A})$ and $\mathsf{I}(\mathfrak{A})$ for the class of all homomorphic images, subalgebras and isomorphic images of $\mathfrak{A}$, respectively. For each class $\mathcal{K}$ of algebras, we write $\mathsf{H}(\mathcal{K})$, $\mathsf{S}(\mathcal{K})$ and $\mathsf{I}(\mathcal{K})$ for $\{\mathfrak{B} : \exists \mathfrak{A} \in \mathcal{K}(\mathfrak{B} \in \mathsf{H}(\mathfrak{A}))\}$, $\{\mathfrak{B} : \exists \mathfrak{A} \in \mathcal{K}(\mathfrak{B} \in \mathsf{S}(\mathfrak{A}))\}$ and $\{\mathfrak{B} : \exists \mathfrak{A} \in \mathcal{K}(\mathfrak{B} \in \mathsf{I}(\mathfrak{A}))\}$, respectively.

Homomorphisms from an algebra are closely related to the *congruence relations* on it. For every algebra $\mathfrak{A} = \langle A; F \rangle$, a binary relation $\equiv$ on A is called a *congruence relation* if

- $\equiv$ is an equivalence relation,

- for all k-ary function symbol $f \in F$ and $a_1, \cdots, a_k, b_1, \cdots, b_k \in A$, if $a_i \equiv b_i$ for all $i \in \{1, \cdots, k\}$, then
$$f^{\mathfrak{A}}(a_1, \cdots, a_k) \equiv f^{\mathfrak{A}}(b_1, \cdots, b_k).$$

Let $\mathsf{Con}(\mathfrak{A})$ denote the set of all congruence relations on $\mathfrak{A}$. Note that $1_{\mathfrak{A}} = A \times A$ and $0_{\mathfrak{A}} = \{(x, x) : x \in A\}$ are always congruence relations on $\mathfrak{A}$. Moreover, $\mathsf{Con}(\mathfrak{A})$ is a bounded lattice whose bottom element and top element are $0_{\mathfrak{A}}$ and $1_{\mathfrak{A}}$, respectively.

Let $\mathfrak{A}$ be an algebra and θ be a congruence relation on $\mathfrak{A}$. Then we define the algebra $\mathfrak{A}/\theta = \langle A/\theta; F\rangle$ as follows:

- $A/\theta \coloneqq \{[a]_\theta : a \in A\}$, where $[a]_\theta \coloneqq \{b \in A : (a, b) \in \theta\}$ for all $a \in A$;
- $f^{\mathfrak{A}/\theta}([a_1]_\theta, \cdots, [a_k]_\theta) \coloneqq [f^{\mathfrak{A}}(a_1, \cdots, a_k)]_\theta$, for all k-ary function symbol $f \in F$.

We call $\mathfrak{A}/\theta$ the *quotient algebra* of $\mathfrak{A}$ induced by θ. Moreover, we see that the function $[\cdot]_\theta : a \mapsto [a]_\theta$ is a surjective homomorphism from $\mathfrak{A}$ to $\mathfrak{A}/\theta$. Thus, $\mathfrak{A}/\theta \in \mathsf{H}(\mathfrak{A})$. On the other hand, for every homomorphism $f : \mathfrak{A} \to \mathfrak{B}$, we define the *kernel* $\ker(f)$ of f by
$$\ker(f) \coloneqq \{(x, y) \in A \times A : f(x) = f(y)\}.$$
Then the reader can readily check that $\ker(f)$ is a congruence relation on $\mathfrak{A}$.

Now we are ready to state the Fundamental Homomorphism Theorem, which shows the deep connection between congruence relations and homomorphisms:

2.1.36. THEOREM. *Let $\mathfrak{A}, \mathfrak{B}$ be algebras and $h : \mathfrak{A} \to \mathfrak{B}$. Then there exists a unique injective homomorphism $\overline{h} : \mathfrak{A}/\ker(h) \rightarrowtail \mathfrak{B}$ such that $\overline{h} \circ [\cdot]_{\ker(h)} = h$. Moreover, if h is surjective, then $\overline{h}$ is an isomorphism.*

In order to state Birkhoff's Theorem, it remains to review the notion of *direct product of algebras*. Recall that for a family $(A_i)_{i\in I}$ of sets, the *direct product* $\prod_{i\in I} A_i$ of $(A_i)_{i\in I}$ is defined as follows:
$$\prod_{i\in I} A_i \coloneqq \{(a : I \to \bigcup_{i\in I} A_i) : \forall i \in I(a(i) \in A_i)\}.$$

2.1.37. DEFINITION. Let $(\mathfrak{A}_l = \langle A_l; F\rangle)_{l\in L}$ be a family of algebras of the same similarity type $\sigma = \langle k_i \in \omega : i \in n\rangle$. Then the *direct product* $\prod_{l\in L} \mathfrak{A}_l$ of $(\mathfrak{A}_l)_{l\in L}$ is defined to be the algebra $\langle \prod_{l\in L} A_l; F\rangle$ such that for all $i \in n$, $l \in L$ and $a_0, \cdots, a_{k_i-1} \in \prod_{l\in L} A_l$,
$$f_i^{\prod_{l\in L} \mathfrak{A}_l}(a_0, \cdots, a_{k_i-1})(l) = f_i^{\mathfrak{A}_l}(a_0(l), \cdots, a_{k_i-1}(l)).$$
For each $l \in L$, the function π_l defined by $\pi_l : a \mapsto a(l)$ is called the *l-projection map*. For each class $\mathcal{K}$ of algebras, we write $\mathsf{P}(\mathcal{K})$ for the class of all direct products of $\mathcal{K}$.

Let $\mathcal{K}$ be a class of algebras. Then the *variety* $\mathsf{V}(\mathcal{K})$ generated by $\mathcal{K}$ is defined to be the minimal class of algebras containing $\mathcal{K}$ and is closed under H, S and P. It is well-known that $\mathsf{V}(\mathcal{K}) = \mathsf{HSP}(\mathcal{K})$ (see, e.g., [6]).

2.1.38. THEOREM (Birkhoff's Theorem). *Let $\mathcal{K}$ be a class of algebras. Then $\mathcal{K}$ is an equational class if and only if $\mathcal{K}$ is a variety.*

Since our focus is on equational classes of algebras, i.e., varieties of algebras, it is unavoidable to review the notion of *subdirectly irreducible* algebras. Subdirectly irreducible algebras constitute the basic components in *subdirect representations* of algebras within a variety [11] (see also [6, 21]). As a result, many structural and representation-theoretic properties of equational classes can be reduced to the study of their subdirectly irreducible members.

2.1.39. DEFINITION. Let $\langle \mathfrak{A}_i : i \in I \rangle$ be a family of algebras. Then an algebra $\mathfrak{B}$ is called a *subdirect product* of $\langle \mathfrak{A}_i : i \in I \rangle$ if $\mathfrak{B}$ is a subalgebra of $\prod_{i \in I} \mathfrak{A}_i$ and for all $i \in I$, the restriction of the i-projection $\pi_i : \prod_{i \in I} \mathfrak{A}_i \to \mathfrak{A}_i$ to B is surjective.

Let $\mathfrak{C}$ be an algebra and $f : \mathfrak{C} \rightarrowtail \prod_{i \in I} \mathfrak{A}_i$. Then f is a *subdirect embedding*, or f is a *subdirect representation of* $\mathfrak{C}$, if $f[\mathfrak{C}]$ is a subdirect product of $\langle \mathfrak{A}_i : i \in I \rangle$.

2.1.40. DEFINITION. An algebra $\mathfrak{A}$ is called *subdirectly irreducible*, if for every family $\langle \mathfrak{A}_i : i \in I \rangle$ of algebras and subdirect embedding $f : \mathfrak{A} \to \prod_{i \in I} \mathfrak{A}_i$, there exists $j \in I$ such that $\pi_j \circ f : \mathfrak{A} \cong \mathfrak{A}_j$. For each class $\mathcal{K}$ of algebras, let $\mathcal{K}_{\mathrm{si}}$ denote the class of all subdirectly irreducible algebras in $\mathcal{K}$.

As the following theorem shows, subdirect irreducibility of an algebra $\mathfrak{A}$ turns out to be a property of the lattice $\mathsf{Con}(\mathfrak{A})$.

2.1.41. THEOREM. *Let $\mathfrak{A}$ be an algebra. Then $\mathfrak{A}$ is subdirectly irreducible if and only if there exists $\theta \in \mathsf{Con}(\mathfrak{A})$ such that*

- $0_{\mathfrak{A}} \neq \theta$;
- $\theta \subseteq \theta'$ *for all* $\theta' \in \mathsf{Con}(\mathfrak{A})$.

Recall that an algebra $\mathfrak{A}$ is called *trivial* if $|A| = 1$. Then the celebrated Subdirect Representation Theorem by Birkhoff [11] can be stated as follows:

2.1.42. THEOREM (Birkhoff). *Every non-trivial algebra is isomorphic to a subdirect product of subdirectly irreducible algebras.*

Modal algebra

We now turn from general universal algebras to algebras for modal logics. We begin by a brief historical review of the establishment of the algebraic semantics for modal logic.

McKinsey and Tarski [96] studied the algebra of topologies and proved that modal logic S4 is sound and complete with respect to the class of *closure algebras*. The notion of closure algebras was later generalized by *Boolean algebras with operators* (BAOs), which became a fundamental tool for investigating modal logics. The close connection between algebraic and relational semantics was subsequently clarified through representation and duality results. In particular, inspired by the representation theorem for BAOs proved by Jónsson and Tarski [65, 66], Goldblatt [58, 59] discovered the duality between BAOs and descriptive frames (see also [29, Chapter 7]). A closely related development was obtained by Esakia [47] (see also [46]), who established the duality between closure algebras and topological Kripke frames for the modal logic S4. Together, these duality results further reinforced the conceptual link between algebraic and relational semantics.

There are many results obtained via the connection between algebras and logics. For example, Maksimova [92] proved that there are at most 37 extensions of S4 having the CIP by showing that for every extension L of S4, L has the CIP if and only if the variety of BAOs for L has the *super-amalgamation property*.[2] This characterization of the CIP allows us to study the CIP, a syntactic property of modal logic, in purely algebraic terms. Maksimova [93] studied further the relation between the CIP of modal logics and the super-amalgamation property of their varieties of BAOs.

We now recall the definition of Boolean algebras.

2.1.43. Definition. A *Boolean algebra* is an algebra $\mathfrak{B} = \langle B; \wedge, \vee, \neg, 0, 1\rangle$ of similarity type $\langle 2, 2, 1, 0, 0\rangle$ such that

- $\langle B; \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice, and
- for all $a \in B$, $a \wedge \neg a = 0$ and $a \vee \neg a = 1$.

We call $\neg a$ the *complement* of a. As usual, we write B for $\langle B; \wedge, \vee, \neg, 0, 1\rangle$ if there is no danger of confusion. Let BA denote the class of all Boolean algebras.

2.1.44. Remark. Since this thesis adopts $\to$ and $\bot$ as the primitive Boolean connectives, we introduce an alternative definition of Boolean algebras. Let $\mathfrak{B} =$

[2] Similar results for superintuitionistic logics have been obtained by Maksimova [91]. Recently, Santschi and Voojis [114] completed Maksimova's classification of modal logics in NExt(S4) with interpolation.

$\langle B; \to, 0\rangle$ be an algebra of the similarity type $\langle 2, 0\rangle$. Then we call $\mathfrak{B}$ a Boolean algebra if $\langle B; \wedge', \vee', \neg', 0', 1'\rangle$ is a Boolean algebra defined in Definition 2.1.43, where the functions $\wedge'$, $\vee'$, $\neg'$, $0'$ and $1'$ are defined as follows:

- $0' := 0$, $1' := (0 \to 0)$;
- $\neg'(x) := (x \to 0)$, for all $x \in B$;
- $\wedge'(x, y) := \neg'(x \to \neg' y)$ and $\vee'(x, y) := (\neg' x \to y)$, for all $x, y \in B$.

Conversely, given a Boolean algebra $\langle B; \wedge, \vee, \neg, 0, 1\rangle$, one can verify that $\langle B; \to', 0\rangle$ is again a Boolean algebra, where $a \to' b := \neg a \vee b$.

Next, we recall the definition of modal algebras.

2.1.45. DEFINITION. A *modal algebra* is an algebra $\mathfrak{B} = \langle B; \Box\rangle$ where B is a Boolean algebra, and $\Box : B \to B$ is a function such that for all $a, b \in B$, the following holds

- $\Box(a \wedge b) = \Box a \wedge \Box b$;
- $\Box 1 = 1$.

Let MA denote the class of all modal algebras.

Note that every modal formula $\varphi \in \mathsf{Form}_m$ can be viewed as a term in the corresponding algebraic language. For example, the formula $(p_1 \wedge p_2) \to p_3$ can be viewed as the term $(x_1 \wedge x_2) \to x_3$. Then for all modal algebras $\mathfrak{A}$, we say that φ is satisfied in $\mathfrak{A}$ (notation: $\mathfrak{A} \models \varphi$) if $\mathfrak{A} \models (\varphi = 1)$. For each class $\mathcal{K}$ of modal algebras, let

$$\mathsf{Log}(\mathcal{K}) := \{\varphi \in \mathsf{Form}_m : \forall \mathfrak{A} \in \mathcal{K}(\mathfrak{A} \models \varphi)\}.$$

We call $\mathsf{Log}(\mathcal{K})$ the *modal logic* of $\mathcal{K}$. Conversely, for each set Σ of modal formulas, let $\mathsf{MA}(\Sigma)$ denote the class of all Σ-algebras, i.e., modal algebras validating Σ. Formally,

$$\mathsf{MA}(\Sigma) := \{\mathfrak{A} \in \mathsf{MA} : \forall \varphi \in \Sigma(\mathfrak{A} \models \varphi)\}.$$

Now we review the Lindenbaum-Tarski construction of modal algebras, from which the completeness of algebraic semantics follows. Recall that Form_m is the set of all modal formulas. Then the *modal term algebra* $\mathfrak{Form}_m$ is defined to be the algebra $\langle \mathsf{Form}_m; \to, \Box, \bot\rangle$ such that

- for all $\varphi \in \mathsf{Form}_m$, $\Box(\varphi) := \Box\varphi$;
- for all $\varphi, \psi \in \mathsf{Form}_m$, $\to(\varphi, \psi) := (\varphi \to \psi)$.

The modal term algebra is also called the *modal formula algebra*. Let $L \in \mathsf{NExt}(\mathsf{K})$ be a modal logic. Then the binary relation $\equiv_L$ on Form_m is defined as follows:

$$\varphi \equiv_L \psi \text{ if and only if } \varphi \leftrightarrow \psi \in L.$$

The reader can readily check that $\equiv_L$ is a congruence relation on Form_m. Let $\mathfrak{A}_L$ denote the quotient algebra $\mathfrak{Form}_m/{\equiv_L}$, which is called the *Lindenbaum-Tarski* L-algebra. By [29, Theorem 7.2], we have $\mathsf{Log}(\mathfrak{A}_L) = L$. Thus, we obtain

2.1.46. THEOREM. *Let $L \in \mathsf{NExt}(\mathsf{K})$. Then $L = \mathsf{Log}(\mathfrak{A}_L) = \mathsf{Log}(\mathsf{MA}(L))$.*

By Theorem 2.1.46, a modal logic is always complete with respect to its modal algebras. Moreover, the following theorem holds

2.1.47. THEOREM. *Let $\Lambda(\mathsf{MA})$ be the lattice of all subvarieties of MA. Then the map $\mathsf{MA} : L \mapsto \{\mathfrak{A} \in \mathsf{MA} : \mathfrak{A} \models L\}$ is an isomorphism from $\mathsf{NExt}(\mathsf{K})$ to $\Lambda(\mathsf{MA})$.*

For a proof of Theorem 2.1.47, we refer the reader to [29, Chapter 7]. For a more general version, see [126]. As a corollary, we conclude that there exist at most continuum many subvarieties of MA.

Finally, we briefly mention the relation between the categories of modal algebras and descriptive frames. Here we assume familiarity with the basic notions of category theory and refer the reader to [85, 101] for the relevant definitions and background. Let $\mathbb{MA}$ denote the category whose objects are modal algebras and whose morphisms are homomorphisms of modal algebras. Moreover, let $\mathbb{DFR}$ denote the category whose domain is the class DFr of all descriptive frames and morphisms are p-morphisms. By [126, Theorem 67], the following theorem holds:

2.1.48. THEOREM. *$\mathbb{MA}$ and $\mathbb{DFR}$ are dually equivalent, i.e., $\mathbb{DFR} \simeq^{op} \mathbb{MA}$.*

We do not go into details here. Instead, in Section 2.2, we show more details for the duality between the categories of tense algebras and descriptive frames for tense logics. It follows from Theorem 2.1.48 that every modal logic L is equal to the modal logic of its descriptive frames, that is, $L = \mathsf{Log}(\mathsf{DFr}(L))$.

2.1.4 Properties of modal logics

In this subsection, we review some important logical properties of modal logics, some of which have been introduced in Chapter 1. Recall that the logical properties of consistency, local tabularity, decidability and Post-completeness have been discussed in Section 2.1.1.

Canonicity

A modal logic L is *canonical* if it coincides with the logic of its canonical frame $\mathfrak{F}^L$, which is defined as follows:

2.1.49. Definition. The *canonical model* $\mathfrak{M}^L$ for a modal logic L is defined as $\mathfrak{M}^L = (X^L, R^L, V^L)$ where

(1) X^L is the set of all maximal L-consistent sets of formulas;

(2) $R^L := \{(x, y) \in X^L \times X^L : \varphi \in y \text{ for all } \Box\varphi \in x\}$; and

(3) $V^L(p) := \{x \in X^L : p \in x\}$ for each $p \in \mathsf{Prop}$.

The *canonical general frame* $\mathbb{F}^L$ for L is defined as $\mathbb{F}^L = (X^L, R^L, A^L)$, where

$$A^L := \{V^L(\varphi) : \varphi \in \mathsf{Form}_m\}.$$

The *canonical frame* $\mathfrak{F}^L$ for L is defined to be the underlying frame $\kappa\mathbb{F}^L$ of $\mathbb{F}^L$.

A general result on canonical modal logics was obtained by Sahlqvist [112], who introduced a family of formulas, now known as *Sahlqvist formulas*. It was proved in [112] that every *Sahlqvist logic*, i.e., modal logic axiomatized by Sahlqvist formulas, is canonical. For example, modal logics K, K4, S4 and S5 are all *Sahlqvist logics* and so canonical. For the definition of Sahlqvist formulas and more details on Sahlqvist's theorem, we refer the reader to [12, Chapter 3.6] and [29, Chapter 10.3].

2.1.50. Theorem (Sahlqvist's Theorem). *Let L be a modal logic axiomatized by Sahlqvist formulas. Then L is canonical, that is, $L = \mathsf{Log}(\mathfrak{F}^L)$.*

Kripke completeness

In this thesis, we will be mostly concerned with the notion of Kripke completeness. A modal logic L is *Kripke complete* if $L = \mathsf{Log}(\mathsf{Fr}(L))$. Note that every canonical modal logic is Kripke complete. Consequently, many results on Kripke completeness have been obtained using the canonical frame method (see, e.g., [29, 12]). For example, by Theorem 2.1.50, we have

2.1.51. Theorem. *Every Sahlqvist modal logic is Kripke complete.*

Consequently, each of K, K4, S4 and S5 is Kripke complete.

Finite model property

A modal logic L enjoys the *finite model property* (FMP) if $L = \mathsf{Log}(\mathsf{Fin}(L))$. In this case, we also say that L is *finitely approximable*. Clearly, since $\mathsf{Fin} \subseteq \mathsf{Fr}$, the FMP implies Kripke completeness. In fact, the FMP is strictly stronger than Kripke completeness, even in the lattice $\mathsf{NExt}(\mathsf{S4})$ (see, e.g., [49]).

General results have been obtained for subframe logics and cofinal subframe logics (see [53, 138, 139, 10]), which are defined as follows:

2.1.52. Definition. Let $\mathbb{F} = (X, R, A)$ be a descriptive frame. Then we call $\mathbb{F}'$ a *subframe* of $\mathbb{F}$ if $\mathbb{F}' = \mathbb{F}{\restriction}Y$ for some $Y \in A$. Moreover, a subframe $\mathbb{F}{\restriction}Y$ of $\mathbb{F}$ is called *cofinal* if $R[Y] \subseteq Y \cup R^{-1}[Y]$.

A modal logic L is called a *(cofinal) subframe logic*, if the class $\mathsf{DFr}(L)$ of its descriptive frames is closed under (cofinal) subframes. That is, for all $\mathbb{F} \in \mathsf{DFr}(L)$ and $\mathbb{F}' \in \mathsf{DFr}$, if $\mathbb{F}'$ is a (cofinal) subframe of $\mathbb{F}$, then $\mathbb{F}' \in \mathsf{DFr}(L)$.

Let $\mathsf{wK4} \coloneqq \mathsf{K} \oplus \Diamond\Diamond p \to p \vee \Diamond p$. The following theorem was proved in [10].

2.1.53. Theorem. *Every cofinal subframe logic* $L \in \mathsf{NExt}(\mathsf{wK4})$ *has the FMP.*

2.1.54. Example. Consider the modal logic $\mathsf{wK4.1} \coloneqq \mathsf{wK4} \oplus \Box\Diamond p \to \Diamond\Box p$. By Chen and Ma [35], $\mathsf{wK4.1}$ is a cofinal subframe logic. By Theorem 2.1.53, $\mathsf{wK4.1}$ has the FMP. A direct proof can be found in [35, Section 4].

It is worth noting that the FMP is closely related to decidability. A well known theorem is as follows:

2.1.55. Theorem. *Let L be a modal logic. Suppose L is finitely axiomatizable and has the FMP. Then L is decidable.*

Since each of K, $\mathsf{wK4}$, $\mathsf{K4}$, $\mathsf{wK4.1}$, $\mathsf{S4}$ and $\mathsf{S5}$ has the FMP, they are decidable.

Tabularity and pretabularity

The last logical properties we review in this section are *tabularity* and *pretabularity*. A modal logic L is said to be *tabular* if it is the modal logic of some finite modal algebra $\mathfrak{A}$. By the duality between finite frames and finite modal algebras (see, e.g., [126, 29]), a consistent modal logic L is tabular if and only if $L = \mathsf{Log}(\mathfrak{F})$ for some $\mathfrak{F} \in \mathsf{Fin}$. A characterization of tabular modal logics is provided as follows (see, e.g., [29, p. 417]).

2.1.56. Theorem. *A modal logic $L \in \mathsf{NExt}(\mathsf{K})$ is tabular if and only if $\mathsf{alt}_n \wedge \mathsf{tra}_n \in L$ for some $n \in \omega$, where alt_n and tra_n are the formulas defined as follows:*

$$\mathsf{alt}_n \coloneqq \Box p_0 \vee \Box(p_0 \to p_1) \vee \cdots \vee \Box(p_0 \wedge \cdots \wedge p_{n-1} \to p_n),$$
$$\mathsf{tra}_n \coloneqq \bigwedge_{i \le n} \Box^i p \to \Box^{n+1} p.$$

As a corollary, we see that none of K, $\mathsf{wK4}$, $\mathsf{K4}$, $\mathsf{wK4.1}$, $\mathsf{S4}$ and $\mathsf{S5}$ is tabular. Tabular tense logics will be studied in Chapter 3.

In order to study tabularity systematically, Kuznetsov [76] introduced the concept of *pretabularity.* A modal logic L is called *pretabular* if L is not tabular but all of its proper extensions are tabular. A famous result about pretabular modal logics is the following theorem obtained by Maksimova [90] and Esakia and Meskhi [44] independently:

2.1.57. THEOREM. *There are exactly* 5 *pretabular extensions of* $\mathsf{S4}$ *and all of them are finitely axiomatizable and finitely approximable.*

We will study pretabular tense logics in Chapter 4.

2.1.5 Polymodal logic

So far, we have reviewed the preliminaries on normal modal logics with a single modal operator $\Box$. Most of the definitions and properties can be generalized to normal polymodal logics, that is, modal logics with multiple modalities. Formally, for each $n \in \mathbb{Z}^+$, the language $\mathscr{L}_n$ of n-modal logic is obtained by adding modal operators $\Box_0, \cdots, \Box_{n-1}$ to $\mathscr{L}_p$. The set Form_n of n-modal formulas is defined by

$$\mathsf{Form}_n \ni \varphi ::= p \mid \bot \mid (\varphi \to \varphi) \mid \Box_i \varphi,$$

where $p \in \mathsf{Prop}$ and $i \in n$. A normal n-modal logic is a set $L \subseteq \mathsf{Form}_n$ such that (i) $\mathsf{CPC} \subseteq L$; (ii) $\Box_i(p \to q) \to (\Box_i p \to \Box_i q) \in L$ for all $i \in n$; and (iii) L is closed under (MP), (Sub) and (Nec$_i$) for all $i \in n$. Let K_n denote the least normal n-modal logic. Relational semantics and algebraic semantics can be naturally generalized to n-modal logics. As in Sections 2.1.1 and 2.1.4, the logical properties that have been discussed so far can be defined for polymodal logics as well (see, e.g., [29]). The following completeness theorem holds for all $n \in \mathbb{Z}^+$:

2.1.58. THEOREM. *Let* $L \in \mathsf{NExt}(\mathsf{K}_n)$. *Then* $L = \mathsf{Log}(\mathsf{DFr}_n(L)) = \mathsf{Log}(\mathsf{MA}_n(L))$.

In the next section, we review tense logics, which are a particular class of 2-modal logics. For a more comprehensive discussion of polymodal logics, we refer the reader to [126].

2.2 Tense Logic

In this section, we review preliminaries on *tense logics.* Tense logics are modal logics with two modalities $\Box$ and $\blacklozenge$. The *formal language* $\mathscr{L}_t$ *of tense logic* is obtained from $\mathscr{L}_\Box$ by adding the diamond operator $\blacklozenge$. The set Form_t of all tense formulas is defined by

$$\mathsf{Form}_t \ni \varphi ::= p \mid \bot \mid (\varphi \to \varphi) \mid \Box\varphi \mid \blacklozenge\varphi,$$

where $p \in \mathsf{Prop}$. Intuitively, formulas of the form $\Box\varphi$ are read as *'It is always going to be the case that φ'*, and formulas of the form $\blacklozenge\varphi$ are read as *'It was at some point in the past that φ'*. We choose $\Box$ and $\blacklozenge$ to be primitive modalities in $\mathscr{L}_t$, since the adjointness of $\Box$ and $\blacklozenge$ (see Definition 2.2.1) is one of the most important properties of tense logics.

The connectives $\top$, $\wedge$, $\vee$ and the modality $\Diamond$ are defined as usual. The dual $\blacksquare$ of $\blacklozenge$ is defined by $\blacksquare\varphi := \neg\blacklozenge\neg\varphi$. Let $var(\varphi)$ denote the set of all propositional variables in φ. We call φ a *closed formula* if $var(\varphi) = \varnothing$. For each $n \in \omega$, let

$$\mathsf{Prop}(n) := \{p_i : i < n\} \text{ and } \mathsf{Form}_t(n) := \{\varphi \in \mathsf{Form}_t : var(\varphi) \subseteq \mathsf{Prop}(n)\}.$$

A *substitution* is an endomorphism $(\cdot)^s : \mathsf{Form}_t \to \mathsf{Form}_t$ of the formula algebra $\mathfrak{Form}_t$. The *modal depth* $md(\varphi)$ of φ is defined as follows:

$$\begin{aligned} md(p) &:= 0, \\ md(\bot) &:= 0, \\ md(\varphi \to \psi) &:= \max\{md(\varphi), md(\psi)\}, \\ md(\Box\varphi) &:= md(\varphi) + 1, \\ md(\blacklozenge\varphi) &:= md(\varphi) + 1. \end{aligned}$$

2.2.1. DEFINITION. A *tense logic* is a set of formulas $L \subseteq \mathsf{Form}_t$ such that

(Tau) $\mathsf{CPC} \subseteq L$;

(Adj) $\blacklozenge\varphi \to \psi \in L$ if and only if $\varphi \to \Box\psi \in L$;

(MP) if $\varphi, \varphi \to \psi \in L$, then $\psi \in L$;

(Sub) if $\varphi \in L$, then $\varphi^s \in L$ for every substitution s.

The least tense logic is denoted by K_t. The proof-theoretic notions such as *consistency*, *L-consistency* and *deduction* are defined analogously to the modal case.

2.2.2. REMARK. Let L be a tense logic. Then the dual (Adj$^\partial$) of (Adj) holds:

$$(\text{Adj}^\partial)\ \Diamond\varphi \to \psi \in L \text{ if and only if } \varphi \to \blacksquare\psi \in L.$$

Indeed, assume $\Diamond\varphi \to \psi \in L$. Then $\neg\psi \to \Box\neg\varphi \in L$. By (Adj), $\blacklozenge\neg\psi \to \neg\varphi \in L$ and so $\varphi \to \blacksquare\psi \in L$. The other direction is similar.

Note that we could also choose $\blacksquare$ as a primitive modality instead of $\blacklozenge$. The following propositions show that tense logics can in fact be defined as normal bimodal logics equipped with additional axioms.

2.2.3. PROPOSITION. *Let L be a tense logic. Then for all $\boxtimes \in \{\Box, \blacksquare\}$:*

(1) $\boxtimes(p \wedge q) \leftrightarrow (\boxtimes p \wedge \boxtimes q) \in L$; and

(2) $\boxtimes\top \in L$.

Proof:
Suppose L is a tense logic. For (1), we prove only $\Box(p \wedge q) \leftrightarrow (\Box p \wedge \Box q) \in L$. Since $\Box p \wedge \Box q \to \Box p \in L$, by (Adj), we have $\blacklozenge(\Box p \wedge \Box q) \to p \in L$. Analogously, $\blacklozenge(\Box p \wedge \Box q) \to q \in L$. Thus, we obtain that $\blacklozenge(\Box p \wedge \Box q) \to p \wedge q \in L$ and so $(\Box p \wedge \Box q) \to \Box(p \wedge q) \in L$. Since $\Box(p \wedge q) \to \Box(p \wedge q) \in L$, by (Adj), we have $\blacklozenge\Box(p \wedge q) \to (p \wedge q) \in L$. Then $\blacklozenge\Box(p \wedge q) \to p \in L$ and $\blacklozenge\Box(p \wedge q) \to q \in L$. By (Adj) again, $\Box(p \wedge q) \to \Box p \in L$ and $\Box(p \wedge q) \to \Box q \in L$. Hence, $\Box(p \wedge q) \to \Box p \wedge \Box q \in L$. Now we see that the case $\boxtimes = \blacksquare$ follows from (Adj$^\partial$).

For (2), since $\Diamond\top \to \top \in L$ and $\blacklozenge\top \to \top \in L$, by (Adj) and (Adj$^\partial$), we have $\top \to \blacksquare\top \in L$ and $\top \to \Box\top \in L$. Thus, $\blacksquare\top \in L$ and $\Box\top \in L$. □

2.2.4. PROPOSITION. *Let $L \subseteq \mathsf{Form}_t$. Then L is a tense logic if and only if*

(1) L is a normal bimodal logic, and

(2) $p \to \Box\blacklozenge p \in L$ and $p \to \blacksquare\Diamond p \in L$.

Proof:
Suppose L is a tense logic. Then (1) follows from Proposition 2.2.3. For (2), since $\blacklozenge p \to \blacklozenge p \in L$ and $\Diamond p \to \Diamond p \in L$, by (Adj) and (Adj$^\partial$), we have $p \to \Box\blacklozenge p \in L$ and $p \to \blacksquare\Diamond p \in L$.

Suppose $L \subseteq \mathsf{Form}_t$ satisfies (1) and (2). Let $\varphi, \psi \in \mathsf{Form}_t$. Suppose $\varphi \to \Box\psi \in L$. By (1), $\blacklozenge\varphi \to \blacklozenge\Box\psi \in L$. By (2), $\blacklozenge\Box\psi \to \psi \in L$. Thus, $\blacklozenge\varphi \to \psi \in L$. Suppose $\blacklozenge\varphi \to \psi \in L$. Then by (1) and (2), $\Box\blacklozenge\varphi \to \Box\psi \in L$ and $\varphi \to \Box\blacklozenge\varphi \in L$, which entails that $\varphi \to \Box\psi \in L$. Thus, (Adj) holds. Note that (Tau), (MP) and (Sub) follow from (1). Hence, L is a tense logic. □

Similar to the modal case, for every tense logic L and set of formulas Σ, let $L \oplus \Sigma$ denote the smallest tense logic containing $L \cup \Sigma$. A tense logic L_1 is a *sublogic* of L_2 (or L_2 is an *extension* of L_1) if $L_1 \subseteq L_2$. L_2 is called a *proper extension* of L_1 if $L_2 \supsetneq L_1$. For every tense logic L, let $\mathsf{NExt}(L)$ be the set of all extensions of L, that is,

$$\mathsf{NExt}(L) \coloneqq \{L' \supseteq L : L' \text{ is a tense logic}\}.$$

Clearly, $(\mathsf{NExt}(L), \subseteq)$ is a complete lattice with top Form_t and bottom L. For tense logics $L_1 \subseteq L_2$, we define the *interval* between L_1 and L_2 as follows:

$$[L_1, L_2] \coloneqq \{L \in \mathsf{NExt}(\mathsf{K}_t) : L_1 \subseteq L \subseteq L_2\}.$$

2.2.5. REMARK. In this thesis, we use the notation $\mathsf{NExt}(L)$ for both modal and tense logics. The intended interpretation will always be clear from the context.

2.2.1 Relational semantics

Now we move to the Kripke semantics for tense logics.

2.2.6. DEFINITION. A *Kripke frame* for tense logic is a pair $\mathfrak{F} = (X, R)$ where $X \neq \varnothing$ and $R \subseteq X \times X$. The *inverse of* R is defined as $R^{-1} = \{(y, x) : Rxy\}$. The *inverse of* $\mathfrak{F}$ is defined as the frame $\mathfrak{F}^{-1} = (X, R^{-1})$. For every $Y \subseteq X$, let

$$R^{-1}[Y] \coloneqq \{z \in X : \exists y \in Y(Rzy)\}.$$

For all $x \in X$, we write $R^{-1}[x]$ for $R^{-1}[\{x\}]$.

Let Fr and Fin denote the class of all frames and finite frames, respectively.

The Kripke semantics for tense logic is given as follows:

2.2.7. DEFINITION. A *Kripke model* is a pair $\mathfrak{M} = (\mathfrak{F}, V)$ where $\mathfrak{F} = (X, R)$ is a frame and $V : \mathsf{Prop} \to \mathcal{P}(X)$ is a valuation in $\mathfrak{F}$. For every point $x \in X$ and formula φ, we define the relation *'φ is true at x in $\mathfrak{M}$'* (notation: $\mathfrak{M}, x \vDash \varphi$) by

$\mathfrak{M}, x \vDash \bot$	never	
$\mathfrak{M}, x \vDash p$	if and only if	$x \in V(p)$, for all $p \in \mathsf{Prop}$
$\mathfrak{M}, x \vDash \varphi \to \psi$	if and only if	$\mathfrak{M}, x \nvDash \varphi$ or $\mathfrak{M}, x \vDash \psi$
$\mathfrak{M}, x \vDash \Box\varphi$	if and only if	$\mathfrak{M}, y \vDash \varphi$ for all $y \in R[x]$
$\mathfrak{M}, x \vDash \blacklozenge\varphi$	if and only if	$\mathfrak{M}, y \vDash \varphi$ for some $y \in R^{-1}[x]$

As the reader might have noticed, Kripke frames for tense logics and those for modal logics are exactly the same as mathematical objects (see Definitions 2.2.6 and 2.1.15). However, general frames for tense logics are different from those for modal logics.

2.2.8. DEFINITION. A *tense general frame* is a triple $\mathbb{F} = (X, R, A)$ where (X, R) is a frame and $A \subseteq \mathcal{P}(X)$ is a set such that

(1) $\varnothing \in A$;

(2) $U \cap V \in A$ for all $U, V \in A$;

(3) $X \setminus U, R[U], R^{-1}[U] \in A$ for all $U \in A$.

Again, we call $\kappa\mathbb{F} = (X, R)$ the *underlying frame of* $\mathbb{F}$ and $A \subseteq \mathcal{P}(X)$ the set of *internal sets* in $\mathbb{F}$. Let GFr denote the class of all tense general frames. We simply write general frame for tense general frame if there is no danger of confusion.

Differentiated and compact tense general frames are defined as in the modal case. A tense general frame $\mathbb{F} = (X, R, A)$ is *tight* if for all $x, y \in X$ such that $y \notin R[x]$, there are internal sets $U, V \in A$ such that $x \in U \setminus R^{-1}[V]$ and $y \in V \setminus R[U]$. Again, $\mathbb{F}$ is *refined* if $\mathbb{F}$ is both differentiated and tight; and $\mathbb{F}$ is *descriptive* if $\mathbb{F}$ is both refined and compact. Let RFr and DFr denote the classes of all refined and descriptive tense general frames, respectively.

We use the notations GFr, RFr and DFr for classes of tense general frames, whereas in the previous section the same notation was used for modal general frames. Throughout this thesis, whenever the meaning is clear from the context, we allow this kind of ambiguity in order to simplify notation.

The general frame semantics for tense logic is given as follows:

2.2.9. DEFINITION. Let $\mathbb{F} = (X, R, A)$ be a general frame. Then a function $V : \mathsf{Prop} \to A$ is called an *admissible valuation* in $\mathbb{F}$. An admissible valuation $V : \mathsf{Prop} \to A$ is extended to $V : \mathsf{Form}_t \to A$ as follows:

$$V(\bot) := \varnothing, \qquad V(\varphi \to \psi) := -V(\varphi) \cup V(\psi),$$
$$V(\blacklozenge\varphi) := R[V(\varphi)], \qquad V(\Box\varphi) := -R^{-1}[-V(\varphi)].$$

A *general model* is a pair $\mathfrak{M} = (\mathbb{F}, V)$ where $\mathbb{F} \in \mathsf{GFr}$ and V is an admissible valuation in $\mathbb{F}$. Truth and validity are defined analogously to the modal case, and we use the same notation. For all sets $\Sigma \subseteq \mathsf{Form}_t$ and classes $\mathcal{K} \subseteq \mathsf{GFr}$ of general frames, let

$$\mathcal{K}(\Sigma) := \{\mathbb{F} \in \mathcal{K} : \mathbb{F} \vDash \Sigma\} \text{ and } \mathsf{Log}(\mathcal{K}) := \{\varphi \in \mathsf{Form}_t : \mathcal{K} \vDash \varphi\}.$$

We call $\mathsf{Log}(\mathcal{K})$ the *tense logic* of $\mathcal{K}$.

The reader can readily check that the general frame semantics generalizes the Kripke semantics for tense logics. Next, we recall the main operations on general frames: generated subframes, disjoint unions, and tense-morphisms (t-morphisms). These operations are defined differently for tense general frames than for modal ones. In addition, we also recall ultraproducts of general frames.

Generated subframes

Let $\mathbb{F} = (X, R, A) \in \mathsf{GFr}$. Since we have modalities for both R and R^{-1}, the definition of generated subframes is closely connected to the relation $R_\sharp$ defined as follows.

2.2.10. Definition. Let $\mathfrak{F} = (X, R)$ be a Kripke frame and $x \in X$. For all $n \in \omega$, we define $R^n_\sharp[x]$ inductively as follows:

$$R^0_\sharp[x] \coloneqq \{x\} \text{ and } R^{n+1}_\sharp[x] \coloneqq R^n_\sharp[x] \cup R[R^n_\sharp[x]] \cup R^{-1}[R^n_\sharp[x]].$$

Moreover, we define $R_\sharp[x] \coloneqq R^1_\sharp[x]$ and $R^\omega_\sharp[x] \coloneqq \bigcup_{k\in\omega} R^k_\sharp[x]$.

The reader can readily check that for each Kripke frame $\mathfrak{F} = (X, R)$, the binary relation $R_\sharp$ is exactly $(= \cup R \cup R^{-1})$ on X. In other words,

$$R_\sharp = \{(x, y) \in X \times X : x = y \text{ or } Rxy \text{ or } Ryx\}.$$

It follows by induction on n that for all $x, y \in X$, if $y \in R^n_\sharp[x]$, then there exists $k \leq n$ and a sequence $\langle x_i : i \leq k\rangle$ such that

(1) $x = x_0$,

(2) $y = x_k$, and

(3) $x_{i+1} \in R_\sharp[x_i]$ for all $i < k$.

In this case, we call $\langle x_i : i \leq k\rangle$ a *path* from x to y of length k.

Before introducing the definition of generated subframes, we introduce the diamond modalities $\Delta^{\leq n}$ associated with the binary relations $R^n_\sharp$.

2.2.11. Definition. For each $n \in \omega$ and $\varphi \in \mathsf{Form}_t$, we define $\Delta^{\leq n}\varphi$ by:

$$\Delta^{\leq 0}\varphi \coloneqq \varphi \text{ and } \Delta^{\leq n+1}\varphi \coloneqq \Delta^{\leq n}\varphi \vee \Diamond\Delta^{\leq n}\varphi \vee \blacklozenge\Delta^{\leq n}\varphi.$$

The dual $\nabla^{\leq n}$ of $\Delta^{\leq n}$ is defined by $\nabla^{\leq n}\varphi \coloneqq \neg\Delta^{\leq n}\neg\varphi$.

For example, we have

$$\begin{aligned}
\Delta^{\leq 1}p &= p \vee \Diamond p \vee \blacklozenge p,\\
\Delta^{\leq 2}p &= \Delta^{\leq 1}p \vee \Diamond\Delta^{\leq 1}p \vee \blacklozenge\Delta^{\leq 1}p\\
&= p \vee \Diamond p \vee \blacklozenge p \vee \Diamond(p \vee \Diamond p \vee \blacklozenge p) \vee \blacklozenge(p \vee \Diamond p \vee \blacklozenge p)\\
&\leftrightarrow p \vee \Diamond p \vee \blacklozenge p \vee \Diamond^2 p \vee \Diamond\blacklozenge p \vee \blacklozenge\Diamond p \vee \blacklozenge^2 p.
\end{aligned}$$

The following lemma is straightforward.

2.2.12. Lemma. *Let $\mathbb{F} = (X, R, A)$ be a general frame and $\mathfrak{M} = (\mathbb{F}, V)$ a model. Then for all $x \in X$, $\varphi, \psi \in \mathsf{Form}_t$ and $n \in \omega$,*

(1) $\mathfrak{M}, x \vDash \Delta^{\leq n}\varphi$ if and only if $\mathfrak{M}, y \vDash \varphi$ for some $y \in R^n_\sharp[x]$.

(2) $\mathbb{F}, x \vDash \Delta^{\leq n}\varphi$ whenever $\mathbb{F}, y \vDash \varphi$ for some $y \in R^n_\sharp[x]$.

Proof:
(1) follows from an easy induction on n and (2) follows from (1) immediately. □

Now we introduce the definition of generated subframes.

2.2.13. Definition. Let $\mathbb{F} = (X, R, A)$ be a general frame and $Y \subseteq X$. The *subframe* $\mathbb{F}{\restriction}Y$ *of* $\mathbb{F}$ *induced by* Y is defined by $\mathbb{F}{\restriction}Y = (Y, R{\restriction}Y, A{\restriction}Y)$, where

$$R{\restriction}Y := R \cap (Y \times Y) \text{ and } A{\restriction}Y := \{U \cap Y : U \in A\}.$$

The *subframe of* $\mathbb{F}$ *generated by* Y is defined to be $\mathbb{F}{\restriction}R_\sharp^\omega[Y]$. A general frame $\mathbb{G}$ is called a *generated subframe* of $\mathbb{F}$ if $\mathbb{G} = \mathbb{F}{\restriction}R_\sharp^\omega[Y]$ for some $Y \subseteq X$.

For all $x \in X$, let $\mathbb{F}_x$ denote the frame $\mathbb{F}{\restriction}R_\sharp^\omega[x]$. Then we call $\mathbb{F}_x$ the *subframe of* $\mathbb{F}$ *generated by* x and x *a root of* $\mathbb{F}_x$. We call $\mathbb{F}$ a *rooted frame* if $\mathbb{F} = \mathbb{F}{\restriction}R_\sharp^\omega[x]$ for some $x \in X$. For each class $\mathcal{K}$ of general frames, we write $\mathcal{K}_\mathrm{r}$ for the subclass of all rooted elements in $\mathcal{K}$, that is, $\mathcal{K}_\mathrm{r} := \{\mathbb{F} \in \mathcal{K} : \mathbb{F} \text{ is rooted}\}$.

In general, given a rooted frame $\mathfrak{F}$ for bimodal logic, it is not true that every point in $\mathfrak{F}$ is a root of $\mathfrak{F}$. However, for rooted frames in tense logic, this is true. The key observation is that given a path $\langle x_i : i \leq k\rangle$ from x to y, the sequence $\langle x_{k-i} : i \leq k\rangle$ is always a path from y to x. As the following proposition shows, every point in a rooted frame can be regarded as a root.

2.2.14. Proposition. *Let* $\mathbb{F} = (X, R, A)$ *be a rooted general frame. Then* $R_\sharp^\omega[x] = X$ *for all* $x \in X$. *As a corollary,* $\mathbb{F} \cong \mathbb{F}_x$ *for all* $x \in X$.

The following lemma captures a basic stabilization property of $R_\sharp^n$.

2.2.15. Lemma. *Let* $\mathbb{F} = (X, R, A) \in \mathsf{GFr}$ *and* $x \in X$. *Then for every* $n \in \omega$,

$$R_\sharp^n[x] = R_\sharp^{n+1}[x] \text{ if and only if } R_\sharp^n[x] = R_\sharp^\omega[x].$$

Proof:
The right-to-left direction is trivial. Suppose $R_\sharp^n[x] = R_\sharp^{n+1}[x]$. Then by induction on $k > n$, we have $R_\sharp^k[x] = R_\sharp^n[x]$. Thus, $R_\sharp^\omega[x] = \bigcup_{k\in\omega} R_\sharp^k[x] = R_\sharp^n[x]$. □

It should be clear that if a general frame is differentiated or tight, then so are its generated subframes. Moreover, the following proposition holds:

2.2.16. Proposition. *Let* $\mathbb{F} = (X, R, A) \in \mathsf{DFr}$ *and* $x \in X$. *Then* $\mathbb{F}_x \in \mathsf{RFr}$.

Next, we establish several propositions concerning the truth of formulas in GFr.

2.2.17. PROPOSITION. *Let* $\mathbb{F} = (X, R, A)$ *be a general frame,* $x \in Y \subseteq X$ *and* $n \in \omega$*. Suppose* $R^n_\sharp[x] \subseteq Y$*. Then for all* $\varphi \in \mathsf{Form}_t$ *such that* $md(\varphi) \leq n$*,*

$$\mathbb{F}, x \vDash \varphi \textit{ if and only if } \mathbb{F}{\restriction}Y, x \vDash \varphi.$$

Proof:
By induction on $md(\varphi)$. □

Consequently, generated subframes preserve validity, that is,

2.2.18. PROPOSITION. *Let* $\mathbb{F} = (X, R, A)$ *be a general frame and* $\mathbb{G}$ *a generated subframe of* $\mathbb{F}$*. Then* $\mathsf{Log}(\mathbb{F}) \subseteq \mathsf{Log}(\mathbb{G})$*.*

The following lemma shows that, over rooted general frames, taking intersections of tense logics corresponds exactly to taking unions of their frames.

2.2.19. LEMMA. *Let* L_1, L_2 *be tense logics and* $\mathcal{K} \subseteq \mathsf{GFr}_r$*. Then* $\mathcal{K}(L_1 \cap L_2) = \mathcal{K}(L_1) \cup \mathcal{K}(L_2)$*.*

Proof:
Clearly $\mathcal{K}(L_1 \cap L_2) \supseteq \mathcal{K}(L_1) \cup \mathcal{K}(L_2)$. Take any $\mathbb{F} = (X, R, A) \in \mathcal{K}(L_1 \cap L_2)$. Suppose $\mathbb{F} \notin \mathcal{K}(L_1) \cup \mathcal{K}(L_2)$. Then there are $\psi_1 \in L_1$, $\psi_2 \in L_2$ and $x, y \in X$ such that $\mathbb{F}, x \nvDash \psi_1$ and $\mathbb{F}, y \nvDash \psi_2$. Moreover, we may assume there is no common variable in ψ_1 and ψ_2. Thus, there exists a valuation V in $\mathbb{F}$ such that $\mathbb{F}, V, x \vDash \neg\psi_1$ and $\mathbb{F}, V, y \vDash \neg\psi_2$. Since $\mathbb{F}$ is rooted, $y \in R^n_\sharp[x]$ for some $n \in \omega$. Thus, $\mathbb{F}, V, x \vDash \neg\psi_1 \wedge \neg\nabla^{\leq n}\psi_2$. Note that $\psi_1 \vee \nabla^{\leq n}\psi_2 \in L_1 \cap L_2$. Then we have $\mathbb{F} \notin \mathcal{K}(L_1 \cap L_2)$, which leads to a contradiction. □

Disjoint unions

The definition of disjoint unions is essentially the same as in the modal case.

2.2.20. DEFINITION. Let $\mathcal{F} = (\mathbb{F}_i = (X_i, R_i, A_i))_{i \in I}$ be a family of general frames. Then the *disjoint union* $\biguplus_{i \in I} \mathbb{F}_i = (X, R, A)$ of $\mathcal{F}$ is defined as follows:

- $X \coloneqq \bigcup_{i \in I} X_i \times \{i\}$,
- $R \coloneqq \{((x, i), (y, i)) : R_i xy \text{ and } i \in I\}$, and
- $A \coloneqq \{U \subseteq X : \forall i \in I(\{y \in X_i : (y, i) \in U\} \in A_i)\}$.

Validity of tense formulas are preserved under disjoint unions:

2.2.21. PROPOSITION. *Let $\mathcal{F} = (\mathbb{F}_i = (X_i, R_i, A_i))_{i\in I}$ be a family of general frames. Then for all $i \in I$, $x \in X_i$ and $\varphi \in \mathsf{Form}_t$, the following holds:*

$$\biguplus_{i\in I} \mathbb{F}_i, (x, i) \nvDash \varphi \text{ if and only if } \mathbb{F}_i, x \nvDash \varphi.$$

As a corollary, we have $\mathsf{Log}(\biguplus_{i\in I} \mathbb{F}_i) = \bigcap_{i\in I} \mathsf{Log}(\mathbb{F}_i)$.

Next, we introduce *t-morphisms*, i.e., p-morphisms for tense general frames.

T-morphisms

2.2.22. DEFINITION. Let $\mathbb{F} = (X, R, A)$ and $\mathbb{F}' = (X', R', A')$ be general frames. A map $f : X \to X'$ is called a *t-morphism* from $\mathbb{F}$ to $\mathbb{F}'$ (notation: $f : \mathbb{F} \to \mathbb{F}'$), if

- for all $Y' \in A'$, $f^{-1}[Y'] \in A$; and
- for all $x \in X$, $f[R[x]] = R'[f(x)]$ and $f[R^{-1}[x]] = R'^{-1}[f(x)]$.

We write $f : \mathbb{F} \twoheadrightarrow \mathbb{F}'$ and $f : \mathbb{F} \cong \mathbb{F}'$ if f is a surjective and bijective t-morphism from $\mathbb{F}$ to $\mathbb{F}'$, respectively. Moreover, $\mathbb{F}'$ is said to be (i) a *t-morphic image* of $\mathbb{F}$ (notation: $\mathbb{F} \twoheadrightarrow \mathbb{F}'$) if there exists $f : \mathbb{F} \twoheadrightarrow \mathbb{F}'$; and (ii) *isomorphic* to $\mathbb{F}$ (notation: $\mathbb{F} \cong \mathbb{F}'$) if there exists $f : \mathbb{F} \cong \mathbb{F}'$. For each class $\mathcal{K} \subseteq \mathsf{GFr}$, let $\mathsf{M}_t(\mathcal{K})$ and $\mathsf{I}(\mathcal{K})$ denote the classes of all t-morphic images and isomorphic images of general frames in $\mathcal{K}$, respectively.

The following proposition can serve as an alternative definition of t-morphisms:

2.2.23. PROPOSITION. *Let $\mathbb{F} = (X, R, A)$ and $\mathbb{F}' = (X', R', A')$ be general frames. Then $f : \mathbb{F} \to \mathbb{F}'$ if and only if the following conditions hold:*

(1) for all $Y' \in A'$, we have that $f^{-1}[Y'] \in A$;

(2) for all $x, y \in X$, if xRy, then $f(x)R'f(y)$;

(3) for all $x \in X$ and $y' \in X'$, if $f(x)R'y'$, there exists $y \in R[x]$ with $f(y) = y'$;

(4) for all $x \in X$ and $y' \in X'$, if $y'R'^{-1}f(x)$, there exists $y \in R^{-1}[x]$ with $f(y) = y'$.

2.2.24. FACT. Let $\mathfrak{F} = (X, R)$ be a rooted frame, $\mathfrak{F}' = (X', R')$ a frame and the function $f : \mathfrak{F} \to \mathfrak{F}'$ an injective t-morphism. Then for all $x, y \in X$, Rxy if and only if $R'f(x)f(y)$. As a corollary, if $\mathfrak{F}'$ is rooted, then $f : \mathfrak{F} \cong \mathfrak{F}'$.

2.2.25. LEMMA. *Let $\mathbb{F} = (X, R, A)$, $\mathbb{G} = (Y, S, B)$ be general frames, $f : \mathbb{F} \twoheadrightarrow \mathbb{G}$ and $x, y \in X$. Suppose $f(x) = f(y)$. Then*

$$f[R[x]] = f[R[y]] \text{ and } f[R^{-1}[x]] = f[R^{-1}[y]].$$

Proof:
It follows from $f : \mathbb{F} \twoheadrightarrow \mathbb{G}$ that $f[R[x]] = S[f(x)] = S[f(y)] = f[R[y]]$. Analogously, we obtain that $f[R^{-1}[x]] = f[R^{-1}[y]]$. □

By the following proposition, onto t-morphisms preserve validity of formulas:

2.2.26. Proposition. *Let* $\mathbb{F} = (X, R, A)$ *and* $\mathbb{F}' = (X', R', A')$ *be general frames and* $f : \mathbb{F} \twoheadrightarrow \mathbb{F}'$. *Then for all* $x \in X$ *and* $\varphi \in \mathsf{Form}_t$,

$$\mathbb{F}, x \vDash \varphi \textit{ implies } \mathbb{F}', f(x) \vDash \varphi.$$

As a corollary, $\mathsf{Log}(\mathbb{F}) \subseteq \mathsf{Log}(\mathbb{F}')$.

Ultraproducts

We review the definition and basic results about ultraproducts of general frames (c.f. [73]). Recall that for a Boolean algebra B, a subset $F \subseteq B$ is called a *filter* in B if all of the following hold:

(1) $1 \in F$;

(2) for all $a, b \in F$, $a \wedge b \in F$; and

(3) for all $a \in F$ and $b \in B$, $a \leq b$ implies $b \in F$.

A filter F in B is called (i) a *proper filter* if $0 \notin F$; (ii) an *ultrafilter* if F is proper and for all $b \in B$, exactly one of $b \in F$ and $\neg b \in F$ holds. Moreover, a filter F is called a *principal filter* if $F = \{b \in B : b \geq a\}$ for some $a \in B$. For a non-empty set I, we say F is a (proper, ultra-) *filter* over I if F is a (proper, ultra-, principal) filter in $\mathcal{P}(I)$.

Now we are ready to define ultraproducts of frames and general frames. For more details, we refer the reader to [73, Chapter 5.7].

2.2.27. Definition. Let $(\mathfrak{F}_i = (X_i, R_i))_{i \in I}$ be a family of frames and U an ultrafilter over I. Then the *ultraproduct* $\prod_U \mathfrak{F}_i = (X, R)$ of $(\mathfrak{F}_i)_{i \in I}$ is defined by:

- X is the *ultraproduct* of $(X_i)_{i \in I}$, i.e., $X = \prod_U X_i \coloneqq \{[x] : x \in \prod_{i \in I} X_i\}$, where
$$[x] \coloneqq \{y \in \prod_{i \in I} X_i : \{i \in I : x(i) = y(i)\} \in U\};$$
- for all $x, y \in \prod_{i \in I} X_i$, $([x], [y]) \in R$ if and only if $\{i \in I : R_i x(i) y(i)\} \in U$.

2.2.28. Definition. Let $(\mathbb{F}_i = (\mathfrak{F}_i, A_i))_{i\in I}$ be a family of general frames and U an ultrafilter over I. Then the *ultraproduct* $\prod_U \mathbb{F}_i$ of $(\mathbb{F}_i)_{i\in I}$ is defined as $(\prod_U \mathfrak{F}_i, A)$, where the set A of internal sets is

$$\{[P] : P \in \prod_{i\in I} A_i\},$$

where

$$[P] := \{[x] \in \prod_U X : \{i \in I : x(i) \in P(i)\} \in U\}.$$

2.2.29. Proposition. *Let $(\mathbb{F}_i = (X_i, R_i, A_i))_{i\in I}$ be a family of general frames, U an ultrafilter over I and $\prod_U \mathbb{F}_i$ the ultraproduct of $(\mathbb{F}_i)_{i\in I}$. Then for all $P, Q \in \prod_{i\in I} A_i$, the following holds:*

$$-[P] = [\langle -P(i) : i \in I\rangle], \qquad [P] \cap [Q] = [\langle P(i) \cap Q(i) : i \in I\rangle],$$
$$R[[P]] = [\langle R_i[P(i)] : i \in I\rangle], \qquad R^{-1}[[P]] = [\langle R_i{}^{-1}[P(i)] : i \in I\rangle].$$

As a corollary, $\prod_U \mathbb{F}_i$ is a general frame. Moreover, if $\mathbb{F}_i$ is differentiated (respectively, tight) for each $i \in I$, then $\prod_U \mathbb{F}_i$ is differentiated (respectively, tight).

Proof:
The fact that $\prod_U \mathbb{F}_i$ is a general frame follows from [73, Proposition 5.7.1]. Suppose $\mathbb{F}_i$ is differentiated for each $i \in I$. Take any distinct $[x], [y] \in X$. Then $J = \{j \in I : x(j) \neq y(j)\} \in U$. For each $j \in J$, there exists $P_j \in A_j$ such that $x(j) \in P_j$ and $y(j) \notin P_j$. Let $[P] \in A$ be such that $P(j) = P_j$ for all $j \in J$. Then clearly, $[x] \in [P]$ and $[y] \notin [P]$, which entails $\prod_U \mathbb{F}_i$ is differentiated. The case for tightness is similar. □

2.2.30. Theorem. *Let $(\mathfrak{M}_i = (\mathbb{F}_i, V_i))_{i\in I}$ be a family of models, $V = \langle V_i : i \in I\rangle$ and U an ultrafilter over I. Let $[V]$ be the valuation in $\prod_U \mathbb{F}_i$ such that $[V] : p \mapsto [\langle V_i(p) : i \in I\rangle]$. Then for all $x = \langle x_i : i \in I\rangle$ and $\varphi \in \mathsf{Form}_t$,*

(1) $\prod_U \mathbb{F}_i, [V], [x] \models \varphi$ if and only if $\{i \in I : \mathfrak{M}_i, x_i \models \varphi\} \in U$.

(2) $\biguplus_{i\in I} \mathbb{F}_i \models \varphi$ implies $\prod_U \mathbb{F}_i \models \varphi$.

Proof:
(1) follows from [73, Theorem 5.7.2]. For (2), suppose $\biguplus_{i\in I} \mathbb{F}_i \models \varphi$. Take any $y = \langle y_i : i \in I\rangle$ and valuation V' in $\prod_U \mathbb{F}_i$. Note that for all $j \in \omega$, $V'(p_j) = [P_j]$ for some $P_j \in \prod_{i\in I} A_i$. For all $i \in I$, let V_i' be the valuation in $\mathbb{F}_i$ such that $V_i'(p_j) = P_j(i)$ for all $j \in \omega$. Then we see $V' = [\langle V_i' : i \in I\rangle]$. Since $\biguplus_{i\in I} \mathbb{F}_i \models \varphi$, we have $\{i \in I : \mathbb{F}_i, V_i', y_i \models \varphi\} = I \in U$. Then by (1), $\prod_U \mathbb{F}_i, V', [y] \models \varphi$. □

2.2.31. Remark. Let L be a tense logic. By Theorem 2.2.30, $\mathsf{GFr}(L)$ is always closed under ultraproducts. However, $\mathsf{Fr}(L)$ is not closed under ultraproducts in general. Consider the logic $L = \mathsf{K}_t \oplus \mathsf{grz}^+$, where grz^+ is the *Grzegorczyk formula*:

$$\mathsf{grz}^+ := \Box(\Box(p \to \Box p) \to p) \to p.$$

As we will see later in Section 2.2.4, a frame $\mathfrak{F}$ validates grz^+ if and only if $\mathfrak{F}$ is a *Noetherian* pre-order, i.e., a pre-order containing no infinite ascending chain (see also [29, Proposition 3.48]). For each $n \in \mathbb{Z}^+$, let $\mathfrak{Ch}_n$ be the reflexive-transitive chain of n elements, that is, $\mathfrak{Ch}_n = (n, \leq)$. Let U be a non-principal ultrafilter over ω. Then clearly, $\mathfrak{Ch}_n \in \mathsf{Fr}(L)$ for all $n \in \mathbb{Z}^+$. However, it is not hard to verify that $\prod_U \mathfrak{Ch}_n$ contains an infinite ascending chain, which entails that $\prod_U \mathfrak{Ch}_n \notin \mathsf{Fr}(L)$.

2.2.2 Tense algebras

In this section, we review the algebraic semantics for tense logics. Let us begin with the definition of tense algebras.

2.2.32. Definition. A *tense algebra* is an algebra $\mathfrak{A} = \langle A; \wedge, \vee, \neg, \Box, \blacklozenge, 0, 1\rangle$ of the similarity type $\langle 2, 2, 1, 1, 1, 0, 0\rangle$ where

- $\langle A; \wedge, \vee, \neg, 0, 1\rangle$ is a Boolean algebra;
- for all $x, y \in A$, $\blacklozenge x \leq y$ if and only if $x \leq \Box y$.

We often write $\langle A, \Box, \blacklozenge\rangle$ for $\mathfrak{A}$ if there is no danger of confusion. Let TA denote the class of all tense algebras. Moreover, let $\mathbb{TA}$ denote the category whose objects are tense algebras and whose morphisms are homomorphisms of tense algebra.

As in the modal case, tense logics are closely related to varieties of tense algebras.

2.2.33. Theorem. *Let $\Lambda(\mathsf{TA})$ be the lattice of all subvarieties of TA. Then the function $\mathsf{TA} : L \mapsto \{\mathfrak{A} \in \mathsf{TA} : \mathfrak{A} \vDash L\}$ is an isomorphism from $\mathsf{NExt}(\mathsf{K}_t)$ to $\Lambda(\mathsf{TA})$.*

As a consequence, every tense logic is algebraically complete.

2.2.34. Corollary. *For every tense logic $L \in \mathsf{NExt}(\mathsf{K}_t)$, we have that*

$$L = \mathsf{Log}(\mathsf{TA}(L)).$$

Let $\mathbb{DFR}$ denote the category of tense descriptive frames, that is, the category whose objects are elements of DFr and whose morphisms are t-morphisms. Then we have the following duality between $\mathbb{DFR}$ and $\mathbb{TA}$:

2.2.35. DEFINITION. The functor $(\cdot)_* : \mathbb{TA} \to \mathbb{DFR}$ is defined as follows:

- For each $\mathfrak{A} = \langle A; \Box, \blacklozenge \rangle \in \mathsf{TA}$, the descriptive frame $\mathfrak{A}_* = (X_\mathfrak{A}, R_\mathfrak{A}, A_\mathfrak{A})$ is defined by
  - $X_\mathfrak{A}$ is the set of all ultrafilters in $\mathfrak{A}$;
  - for all $x, y \in X_\mathfrak{A}$, $R_\mathfrak{A}xy$ if and only if $\{a \in A : \Box a \in x\} \subseteq y$; and
  - $A_\mathfrak{A} := \{\varphi(a) : a \in A\}$, where $\varphi(a) := \{x \in X_\mathfrak{A} : a \in x\}$ for each $a \in A$.
- For each homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ of tense algebras, the map $f_* : \mathfrak{B}_* \to \mathfrak{A}_*$ is defined by $f_*(y) = f^{-1}[y]$ for all $y \in X_\mathfrak{B}$.

The functor $(\cdot)^* : \mathbb{DFR} \to \mathbb{TA}$ is defined as follows:

- For each $\mathbb{F} = (X, R, A) \in \mathsf{DFr}$, we define the tense algebra
$$\mathbb{F}^* = \langle A_\mathbb{F}; \cap, \cup, X \setminus (\cdot), \Box_\mathbb{F}, \blacklozenge_\mathbb{F}, \varnothing, X \rangle$$
by taking $A_\mathbb{F} = A$, $\Box_\mathbb{F} : a \mapsto X \setminus (R^{-1}[X \setminus a])$ and $\blacklozenge_\mathbb{F} : a \mapsto R[a]$.
- For each t-morphism $f : \mathbb{F} \to \mathbb{G}$, we define $f^* : \mathbb{G}^* \to \mathbb{F}^*$ by $f^*(y) = f^{-1}[y]$ for all $y \in A_\mathbb{G}$.

For all $\mathfrak{A} \in \mathsf{TA}$ and $\mathbb{F} \in \mathsf{DFr}$, the maps $\varphi_\mathfrak{A} : \mathfrak{A} \to {\mathfrak{A}_*}^*$ and $\varepsilon_\mathbb{F} : \mathbb{F} \to {\mathbb{F}^*}_*$ are defined by

$$\varphi_\mathfrak{A}(a) := \{x \in X_\mathfrak{A} : a \in x\} \text{ and } \varepsilon_\mathbb{F}(x) := \{U \in A_\mathbb{F} : x \in U\}.$$

The reader can check that the functors $(\cdot)^*$ and $(\cdot)_*$ are well-defined and φ, ε are natural isomorphisms. Moreover, the following theorem holds.

2.2.36. THEOREM (Duality). *$((\cdot)_*, (\cdot)^*, \varphi, \varepsilon)$ is a duality between $\mathbb{TA}$ and $\mathbb{DFR}$.*

As a corollary, $\mathbb{DFR} \simeq^{op} \mathbb{TA}$ and the following proposition holds:

2.2.37. PROPOSITION. *For all tense logic L, we have that*

$$L = \mathsf{Log}(\mathsf{DFr}(L)) = \mathsf{Log}(\mathsf{RFr_r}(L)).$$

Proof:
The equation $L = \mathsf{Log}(\mathsf{DFr}(L))$ follows from Theorems 2.2.36 and 2.2.33. Let $\mathbb{F} = (X, R, A) \in \mathsf{DFr}(L)$ and $x \in X$. By Proposition 2.2.16, the subframe $\mathbb{F}_x$ of $\mathbb{F}$ generated by x is refined. Note that $\mathsf{Log}(\mathbb{F}) = \mathsf{Log}(\{\mathbb{F}_x : x \in X\})$. Thus, we have

$$L = \mathsf{Log}(\mathsf{DFr}(L)) \supseteq \mathsf{Log}(\{\mathbb{F}_x : \mathbb{F} \in \mathsf{DFr}(L) \text{ and } x \in \mathbb{F}\}) \supseteq \mathsf{Log}(\mathsf{RFr_r}(L)) \supseteq L,$$

which entails that $L = \mathsf{Log}(\mathsf{RFr_r}(L))$. □

2.2.3 Finitely alternative tense logics

Throughout this thesis, we will encounter many tense logics with bounded parameters, for example, tense logics of bounded depth, bounded width and bounded degree of reachability (see Chapter 4). These logics will play important roles in our study. In this section, we recall finitely alternative tense logics from [84]. This gives an initial picture of how we approach the study of lattices and properties of tense logics.

Let $n, m \in \omega$. A tense logic L is called (n, m)-*alternative* if $L \supseteq \mathsf{T}_{m,n} = \mathsf{K}_t \oplus \{\mathsf{alt}_n^+, \mathsf{alt}_m^-\}$ where the formulas alt_n^+ and alt_m^- are defined by:

$$\mathsf{alt}_n^+ := \Box p_0 \vee \Box(p_0 \to p_1) \vee \cdots \vee \Box(p_0 \wedge \cdots \wedge p_{n-1} \to p_n)$$
$$\mathsf{alt}_m^- := \blacksquare p_0 \vee \blacksquare(p_0 \to p_1) \vee \cdots \vee \blacksquare(p_0 \wedge \cdots \wedge p_{m-1} \to p_m).$$

By definition, $\mathsf{T}_{m,n}$ is the minimal (n, m)-alternative tense logic. We say that L is *finitely alternative* if $L \supseteq \mathsf{T}_{n,m}$ for some $n, m \in \omega$.

2.2.38. LEMMA. *For every differentiated* $\mathbb{F} = (X, R, A) \in \mathsf{GFr}$*,* $x \in X$ *and* $n, m \in \omega$*,*

(1) $\mathbb{F}, x \models \mathsf{alt}_n^+$ *if and only if* $|R[x]| \leq n$*;*

(2) $\mathbb{F}, x \models \mathsf{alt}_m^-$ *if and only if* $|R^{-1}[x]| \leq m$*.*

Proof:
For (1), suppose $|R[x]| > n$. Let $W = \{w_0, \cdots, w_n\} \subseteq R[x]$ such that $|W| > n$. Since $\mathbb{F}$ is differentiated, there exists a valuation V in $\mathbb{F}$ such that $V(p_i) = W \setminus \{w_0, \ldots, w_i\}$ for all $i \leq n$. Clearly $\mathbb{F}, V, x \not\models \mathsf{alt}_n^+$, which entails $\mathbb{F}, x \not\models \mathsf{alt}_n^+$. The other direction is straightforward. The proof of (2) is similar. □

Now we show that every finitely alternative tense logic is Kripke complete.

2.2.39. DEFINITION. Let $\mathbb{F} = (X, R, A) \in \mathsf{GFr}$. Then $\mathbb{F}$ is said to be *image-finite* if $|R_\sharp[x]| < \aleph_0$ for all $x \in X$.

2.2.40. LEMMA. *Let* $\mathbb{F} = (X, R, A) \in \mathsf{GFr}$ *be image-finite. Then*

(1) $|R_\sharp^n[x]| < \aleph_0$ *for all* $x \in X$ *and* $n \in \omega$*.*

(2) If $\mathbb{F}$ *is differentiated, then* $\mathsf{Log}(\mathbb{F}) = \mathsf{Log}(\kappa\mathbb{F})$*.*

Proof:
For (1), we prove by induction on n. The case $n = 0$ is trivial. Suppose $n > 0$. Then by induction hypothesis, we have

$$|R_\sharp^n[x]| \leq \sum_{y \in R_\sharp^{n-1}[x]} |R_\sharp[y]| < \aleph_0.$$

For (2), it suffices to show $\mathsf{Log}(\mathbb{F}) \subseteq \mathsf{Log}(\kappa\mathbb{F})$. Take any $\varphi \notin \mathsf{Log}(\kappa\mathbb{F})$. Then there exists $x \in X$ and a valuation V in $\kappa\mathbb{F}$ such that $\kappa\mathbb{F}, V, x \not\models \varphi$. Let $d(\varphi) = m$. By (1), $R_\sharp^m[x]$ is finite. Since $\mathbb{F}$ is differentiated, there exists a valuation V' in $\mathbb{F}$ such that $V(p) \cap R_\sharp^m[x] = V'(p) \cap R_\sharp^m[x]$ for all $p \in \mathsf{Prop}$. Thus, $(\kappa\mathbb{F}, V){\restriction}R_\sharp^m[x] \cong (\mathbb{F}, V'){\restriction}R_\sharp^m[x]$. Note that $d(\varphi) = m$. Hence, we have $\mathbb{F}, V', x \not\models \varphi$, which entails $\varphi \notin \mathsf{Log}(\mathbb{F})$. ☐

As a corollary, we obtain canonicity of finitely alternative tense logics.

2.2.41. Lemma. *Let $\mathbb{F} \in \mathsf{GFr}(\mathsf{T}_{n,m})$ be differentiated. Then $\mathsf{Log}(\mathbb{F}) = \mathsf{Log}(\kappa\mathbb{F})$.*

2.2.42. Theorem. *Every $L \in \mathsf{NExt}(\mathsf{T}_{n,m})$ is canonical and so Kripke complete.*

For proof details of Lemma 2.2.41 and Theorem 2.2.42, see [84].

2.2.4 Transitive tense logics

In this section, we focus on transitive general frames and transitive tense logics. A tense logic L is called *transitive* if $L \in \mathsf{NExt}(\mathsf{K4}_t)$, where $\mathsf{K4}_t := \mathsf{K}_t \oplus \Box p \to \Box\Box p$.

Let us start by reviewing some famous transitive tense logics $\mathsf{S4}_t$, Lin_t, $\mathsf{S4.3}_t$ and $\mathsf{S5}_t$, which are defined as follows:

- $\mathsf{S4}_t := \mathsf{K4}_t \oplus \Box p \to p$;
- $\mathsf{Lin}_t := \mathsf{K4}_t \oplus (\Diamond\blacklozenge p \vee \blacklozenge\Diamond p \to p \vee \Diamond p \vee \blacklozenge p)$;
- $\mathsf{S4.3}_t := \mathsf{S4}_t \oplus (\Diamond\blacklozenge p \vee \blacklozenge\Diamond p \to p \vee \Diamond p \vee \blacklozenge p)$;
- $\mathsf{S5}_t := \mathsf{S4}_t \oplus p \to \Box\Diamond p$.

A frame $\mathfrak{F} = (X, R)$ is called *linear* if R is a *linear relation* on X, that is,

$$\forall xy \in X(Rxy \vee Ryx \vee x = y).$$

Then the following proposition holds:

2.2.43. Proposition. *Let $\mathfrak{F} = (X, R)$ be a frame. Then*

(1) $\mathfrak{F} \models \mathsf{S4}_t$ if and only if $\mathfrak{F}$ is a pre-order;

(2) $\mathfrak{F} \models \mathsf{Lin}_t$ if and only if $\mathfrak{F}$ is transitive and linear;

(3) $\mathfrak{F} \models \mathsf{S4.3}_t$ if and only if $\mathfrak{F}$ is a linear pre-order;

(4) $\mathfrak{F} \vDash \mathsf{S5}_t$ *if and only if* $\mathfrak{F}$ *is an equivalence frame.*

We then review some basic notions and facts about transitive general frames.

2.2.44. DEFINITION. Let $\mathbb{F} = (X, R, A) \in \mathsf{GFr}(\mathsf{K4}_t)$ and $x \in X$. Then the *cluster* generated by x, denoted by $C(x)$, is defined as follows:

$$C(x) = (R[x] \cap R^{-1}[x]) \cup \{x\}$$

A subset $C \subseteq X$ is called a *cluster in* $\mathbb{F}$ if $C = C(x)$ for some $x \in X$. We call C a (i) *proper cluster* if $|C| \geq 2$; (ii) *non-degenerated cluster* if $C \cap R \neq \varnothing$; and (iii) *degenerated cluster* if $|C| = 1$ and $R{\upharpoonright}C = \varnothing$. Moreover, we call $\mathfrak{F}$ a *cluster* if X is a cluster in $\mathfrak{F}$.

2.2.45. DEFINITION. Let $\mathbb{F} = (X, R, A)$ be a transitive general frame and $n \in \mathbb{Z}^+$. We say $\mathbb{F}$ is of *girth* n (notation: $\mathsf{gir}(\mathbb{F}) = n$) if there exists a cluster C in $\mathbb{F}$ such that $|C| = n$ and there is no cluster in $\mathbb{F}$ of larger size. We say that the girth of $\mathbb{F}$ is infinite and write $\mathsf{gir}(\mathbb{F}) = \aleph_0$, if for all $k \in \mathbb{Z}^+$, there exists a cluster C in $\mathbb{F}$ such that $|C| > k$.

As the reader can see, girth measures the maximal size of clusters. Unlike the parameters that will be introduced in Chapter 4, there exists no formula that bounds girth. Nevertheless, there exist formulas whose validity guarantees that the girth is bounded. Consider the formulas grz^+ and grz^- defined as follows:

$$\mathsf{grz}^+ := \Box(\Box(p \to \Box p) \to p) \to p$$
$$\mathsf{grz}^- := \blacksquare(\blacksquare(p \to \blacksquare p) \to p) \to p.$$

The formula (grz^+) is known as the Grzegorczyk formula. Recall that a frame $\mathfrak{F} = (X, R)$ is called *Noetherian* if it contains no infinite ascending chain. By [29, Proposition 3.48], a frame validates grz^+ if and only if $\mathfrak{F}$ is reflexive, transitive, antisymmetric and Noetherian. Recall that for a frame $\mathfrak{F} = (X, R)$, we write $\mathfrak{F}^{-1}$ for the frame (X, R^{-1}). Then the following proposition holds:

2.2.46. PROPOSITION. *Let* $\mathfrak{F} = (X, R) \in \mathsf{Fr}(\mathsf{S4}_t)$ *and* $x \in X$. *Then*

(1) $\mathfrak{F}, x \vDash \mathsf{grz}^+$ *if and only if* $\mathfrak{F}$ *is antisymmetric and Noetherian;*

(2) $\mathfrak{F}, x \vDash \mathsf{grz}^-$ *if and only if* $\mathfrak{F}^{-1}$ *is antisymmetric and Noetherian;*

(3) $\mathfrak{F}, x \vDash \mathsf{grz}^-$ *if and only if* $\mathfrak{F}^{-1}, x \vDash \mathsf{grz}^+$.

The final task in this section is to show that the girth of *finitely generated* refined frames are bounded. We begin with the definition of finitely generated frames. Recall that a tense algebra $\mathfrak{A} = \langle A; \Box, \blacklozenge \rangle$ is *finitely generated* if there exists a finite subset B of A such that for all $a \in A$, there exist $b_1, \cdots, b_n \in B$ and $\varphi(p_1, \cdots, p_n) \in \mathsf{Form}_t$ such that $a = \varphi(b_1, \cdots, b_n)$. Then a general frame $\mathbb{F} = (X, R, A)$ is called *finitely generated*, if the tense algebra $\mathbb{F}^* = \langle A; \Box_{\mathbb{F}}, \blacklozenge_{\mathbb{F}} \rangle$ is finitely generated.

Let $\mathfrak{M} = (X, R, V)$ be a model and Σ a set of formulas. Then for all $x, y \in X$, we say x and y are *equivalent with respect to* Σ (notation: $x \equiv_\Sigma y$) if

$$\{\varphi \in \mathsf{Form}_t : \mathfrak{M}, x \models \varphi\} = \{\varphi \in \mathsf{Form}_t : \mathfrak{M}, y \models \varphi\}.$$

2.2.47. Lemma. *Let $\mathfrak{M} = (X, R, V)$ be a model and $C \subseteq X$ be a cluster in the frame (X, R). Suppose $n \in \omega$, $x, y \in C$ and $x \equiv_{\mathsf{Prop}(n)} y$. Then $x \equiv_{\mathscr{L}_t(n)} y$.*

Proof:
The proof proceeds by induction on φ. The case $\varphi \in \mathsf{Prop}(n)$ follows from $x \equiv_{\mathsf{Prop}(n)} y$ immediately and the Boolean cases are standard. Consider the case $\varphi = \Box\psi$. Suppose $\mathfrak{M}, x \models \varphi$. Then $\mathfrak{M}, z \models \psi$ for all $z \in R[y]$. Since C is a cluster and $x, y \in C$, we see $R[x] = R[y]$. By induction hypothesis, $\mathfrak{M}, z \models \psi$ for all $z \in R[y]$. Thus, $\mathfrak{M}, y \models \varphi$. Symmetrically, $\mathfrak{M}, y \models \varphi$ implies $\mathfrak{M}, x \models \varphi$. Note that $R^{-1}[x] = R^{-1}[y]$, the proof for the case $\varphi = \blacklozenge\psi$ is similar. □

Now we are ready to prove the following theorem:

2.2.48. Theorem. *Let $\mathbb{F} \in \mathsf{RFr}$ be finitely generated. Then $\mathsf{gir}(\mathbb{F}) < \aleph_0$.*

Proof:
Let $\mathbb{F} = (X, R, A)$ be generated by $U_0, \cdots, U_{n-1} \in A$ for some $n \in \omega$. Let V be a valuation in $\mathbb{F}$ such that $V(p_i) = U_i$ for all $i < n$. Then $A = \{V(\varphi) : \varphi \in \mathsf{Form}_t(n)\}$. Let $\mathcal{M} = (\mathbb{F}, V)$. Since $\mathbb{F}$ is differentiated, $x \not\equiv_{\mathsf{Form}_t(n)} y$ for any distinct $x, y \in X$. Take any cluster C in $\mathbb{F}$. By Lemma 2.2.47, $x \not\equiv_{\mathsf{Prop}(n)} y$ for any distinct $x, y \in X$. Thus, $|C| \leq 2^n < \aleph_0$. Since C is chosen arbitrarily, $\mathsf{gir}(\mathbb{F}) \leq 2^n < \aleph_0$. □

So far, we have provided the necessary preliminaries on modal and tense logics, including their semantic and algebraic foundations. With this background in place, we now turn to the main subject of this thesis. Beginning with the next chapter, we undertake a systematic study of the lattices of tense logics.

Chapter 3

Tabularity and Post-Completeness in Tense Logic

In this chapter, which is based on [34], we study tabularity and Post-completeness in lattices of tense logics. Let us recall some notions and results from [29, 73]. Recall that a polymodal logic $L \subseteq \mathsf{Form}_n$ is *tabular* if $L = \mathsf{Log}(\mathfrak{A})$ for some finite modal algebra $\mathfrak{A}$. Blok and Köhler [18] provided a finite axiomatization for every tabular quasi-normal logic.[1] Finite axiomatizability of tabular normal modal logics follows from general results on universal algebra by Funayama and Nakayama [56] and Baker [2]. It is well-known that tabularity in normal modal logics can be characterized by a family of modal formulas alt_n and tra_n (see, e.g. [29, Chapter 12]). Similar result holds for polymodal logics [29, p.430]. Every tabular modal logic has finitely many extensions and all of them are tabular, and all tabular modal logics are finitely axiomatizable (see [29, 136]).

Recall that a (normal) polymodal logic $L \subseteq \mathsf{Form}_n$ is *Post-complete* if $L \neq \mathsf{Form}_n$ and there exists no consistent proper (normal) extension of L. Research on the Post-complete modal logics dates back to McKinsey [95]. Post-complete extensions of modal logics were studied by Makinson and Segerberg [88], Sambin and Valentini [113] and Bellissima [5]. Recall that a is a *co-atom* in a bounded lattice $\mathcal{L}$ if $a \neq 1$ and there exists no $b \in \mathcal{L}$ such that $a < b < 1$. Algebraically, a normal polymodal logic $L \subseteq \mathsf{Form}_n$ is Post-complete if L is a co-atom in the lattice $\mathsf{NExt}(\mathsf{K}_n)$. In general, Post-completeness is not merely a property of a logic, but also a property of the lattice to which it belongs. In lattices of quasi-normal monomodal logics and polymodal logics, Post-completeness is quite complicated (see [29, 88, 116, 55]). For example, Chagrov [23] proved that there are $2^{\aleph_0}$ Post-complete quasi-normal modal logics extending $\mathsf{K4}$ (see also [29, Theorem 13.15]) and there exists a bimodal logic that has $2^{\aleph_0}$ many Post-complete extensions (see, e.g., [73]). However, Makinson [87] showed that there are exactly two Post-

[1] A quasi-normal modal logic is a set $L \supseteq \mathsf{K}$ closed under (MP) and (Sub).

complete logics in $\mathsf{NExt}(\mathsf{K})$ (see also [29, 73]). Post-completeness is also related to tabularity. A consistent quasi-normal modal logic is called *anti-tabular* if it has no finite models. A quasi-normal modal logic L is anti-tabular if and only if all Post-complete extensions of L are not tabular. If a quasi-normal modal logic $L \supseteq \mathsf{K4}$ has infinitely many Post-complete extensions, it has an anti-tabular extension (see, e.g., [29, Theorem 13.22]).

In this chapter, we study tabularity and Post-completeness in lattices of tense logics. A characterization of tabularity in $\mathsf{NExt}(\mathsf{K}_t)$ has been given by Chagrov and Shehtman [30]. They proved that a tense logic L is tabular if and only if $\alpha_n \wedge \beta_n \in L$ for some formulas α_n and β_n. We give a new criterion of tabularity in $\mathsf{NExt}(\mathsf{K}_t)$ by defining formulas tab_n^T with $n \geq 1$ (Theorem 3.1.8). We discuss the relation between our characterization and the one given by Chagrov and Shehtman [30] in Remark 3.1.11. As far as Post-completeness in $\mathsf{NExt}(\mathsf{K}_t)$ is concerned, it is known that there exist infinitely many Post-complete tense logics [132, 84]. In this chapter, we give three characterization theorems for Post-completeness in $\mathsf{NExt}(\mathsf{K}_t)$: (i) the first theorem gives three equivalent conditions for Post-completeness of a tense logic $\mathsf{Log}(\mathfrak{F})$ where $\mathfrak{F}$ is a finite rooted frame (Theorem 3.2.13) — Rautenberg's characterization of tabularity of tense logics in [107] follows from this result; (ii) a tabular tense logic L is Post-complete if and only if L has only one rooted frame up to isomorphism (Theorem 3.2.19); and (iii) a consistent tense logic L is Post-complete if and only if it satisfies two conditions on *closed formulas*, i.e., formulas without propositional variables (Theorem 3.3.4). Using these results, we determine the Post-numbers, the number of Post-complete extensions, of tense logics such as $\mathsf{K4}_t$, $\mathsf{D4}_t$ and B_t. As a corollary, we obtain that there exist continuum many Post-complete tense logics.

This chapter is structured as follows. Section 3.1 gives some preliminaries on tabularity of tense logics and a new characterization of tabularity in $\mathsf{NExt}(\mathsf{K}_t)$. Section 3.2 proves two characterization theorems on Post-completeness in the set of all tabular tense logics. In Section 3.3, we give a characterization theorem for Post-completeness in $\mathsf{NExt}(\mathsf{K}_t)$. Section 3.4 provides some concluding remarks.

3.1 A New Characterization of Tabularity in $\mathsf{NExt}(\mathsf{K}_t)$

In this section, we first review the definition and basic properties of tabular tense logics. Then we present a new characterization of tabularity in tense logics and compare our characterization with the one given by Chagrov and Shehtman [30].

3.1.1. Definition. A tense logic L is called *tabular* if $L = \mathsf{Log}(\mathfrak{A})$ for some finite algebra $\mathfrak{A}$; and *non-tabular* if L is not tabular. Let TAB be the set of all tabular tense logics.

By the duality between finite tense algebras and finite frames, we have

3.1.2. PROPOSITION. *Let L be a tense logic. Then L is tabular if and only if (i) $L = \mathsf{Log}(\mathfrak{F})$ for some finite frame $\mathfrak{F}$; or (ii) L is inconsistent, i.e., $L = \mathsf{Form}_t$.*

In what follows, we adopt Proposition 3.1.2 as an alternative definition of tabular tense logics and use it without further mention.

For a class $\mathcal{K}$ of general frames, recall that the quotient $\mathcal{K}/\cong$ of $\mathcal{K}$ by isomorphism is defined by $\mathcal{K}/\cong := \{[\mathbb{F}]_{\cong} : \mathbb{F} \in \mathcal{K}\}$, where $[\mathbb{F}]_{\cong} := \{\mathbb{F}' \in \mathcal{K} : \mathbb{F} \cong \mathbb{F}'\}$. We say that $\mathcal{K}$ is *finite up to isomorphism* (notation: $|\mathcal{K}| < \aleph_0$) if $|\mathcal{K}/\cong| < \aleph_0$. Moreover, we write $|\mathcal{K}| = n$ if $|\mathcal{K}/\cong| = n$ for some $n \in \omega$.

It is not hard to verify that $\mathcal{K}$ is finite up to isomorphism if and only if there exists a finite set $\mathcal{K}' = \{\mathbb{F}_i : i < n\}$ such that $\mathcal{K}$ is included in the class of all isomorphic images of $\mathcal{K}'$, i.e., $\mathcal{K} \subseteq \mathsf{I}(\mathcal{K}')$.

3.1.3. PROPOSITION. *Let L be a tense logic. If $L = \mathsf{Log}(\mathcal{K})$ for some class $\mathcal{K}$ of finite frames such that $|\mathcal{K}| < \aleph_0$, then L is tabular.*

Proof:
Suppose $L = \mathsf{Log}(\mathcal{K})$ for some class $\mathcal{K}$ of finite frames such that $|\mathcal{K}| < \aleph_0$. If $\mathcal{K} = \varnothing$, then L is inconsistent and so tabular. Suppose $\mathcal{K} \neq \varnothing$. Then there exists a finite set $\mathcal{K}' = \{\mathbb{F}_i : i < n\}$ of general frames such that $\mathcal{K} \subseteq \mathsf{I}(\mathcal{K}')$. By Proposition 2.2.21,

$$L = \mathsf{Log}(\mathcal{K}) = \mathsf{Log}(\mathcal{K}') = \mathsf{Log}(\biguplus_{i<n} \mathbb{F}_i).$$

Note that since $\biguplus_{i<n} \mathbb{F}_i$ is a finite frame, we conclude that L is tabular. □

As the reader might have already noticed, tabularity is a rather strong logical property. By definition, if a tense logic L is tabular, then L has the FMP and is therefore Kripke complete. Moreover, the following theorem holds:

3.1.4. THEOREM. *Let L be a tabular tense logic. Then*

(1) L is canonical;

(2) L is finitely axiomatizable;

(3) L is decidable.

Proof:
Take any $L \in \mathsf{TAB}$. For (1), if L is inconsistent, then L is canonical trivially. Suppose L is consistent. Then $L = \mathsf{Log}(\mathfrak{F})$ for some frame $\mathfrak{F}$ with $|\mathfrak{F}| = k \in \omega$. It follows that $\mathfrak{F} \models \mathsf{alt}_k^+ \wedge \mathsf{alt}_k^-$. Thus, $\mathsf{T}_{k,k} \subseteq L$. By Theorem 2.2.42, L is canonical. For (2), note that $L = \mathsf{Log}(\mathfrak{A}) = \mathsf{Log}(\mathsf{V}(\mathfrak{A}))$ for some finite tense algebra $\mathfrak{A}$. Since

$\mathsf{V}(\mathfrak{A})$ is a finitely generated congruence-distributive variety of tense algebras, L is finitely axiomatizable (see, e.g., [6, Corollary 5.40]). (3) follows from (2) since L has the FMP. ☐

Let us now introduce our new characterization for tabular tense logics. Our characterization is based on the formulas tab_n^T, which are defined as follows:

3.1.5. Definition. For each $n \in \omega$, the formula tab_n^T is defined as

$$\mathsf{tab}_n^T := \neg(\Delta^{\leq n}\psi_0 \wedge \cdots \wedge \Delta^{\leq n}\psi_n),$$

where $\psi_i := \neg p_0 \wedge \cdots \wedge \neg p_{i-1} \wedge p_i$ for each $i \leq n$. Note that $\psi_0 = p_0$.

For example, we have

$$\begin{aligned}\mathsf{tab}_1^T &= \neg(\Delta^{\leq 1}\psi_0 \wedge \Delta^{\leq 1}\psi_1)\\ &= \neg(\Delta^{\leq 1}p_0 \wedge \Delta^{\leq 1}(\neg p_0 \wedge p_1))\\ &= \neg((p_0 \vee \Diamond p_0 \vee \blacklozenge p_0) \wedge ((\neg p_0 \wedge p_1) \vee \Diamond(\neg p_0 \wedge p_1) \vee \blacklozenge(\neg p_0 \wedge p_1))).\end{aligned}$$

Now let us take a closer look at the formulas tab_n^T. We first show that for any general frame, if the formula tab_n^T is valid at a point x, then x can access at most n points in n steps. Recall from Definition 2.2.10 that $R_\sharp^n[x]$ is the set of all points accessible from x in n steps. Then we have the following lemma holds:

3.1.6. Lemma. *Let $\mathbb{F} = (X, R, A) \in \mathsf{RFr}$ and $x \in X$. Then for all $n \in \omega$,*

$$\mathbb{F}, x \vDash \mathsf{tab}_n^T \text{ if and only if } |R_\sharp^n[x]| \leq n.$$

Proof:
Let $\mathbb{F} = (X, R, A)$ be a refined frame. Take any $x \in X$. Suppose $|R_\sharp^n[x]| > n$. Then there exists $W = \{w_0, \cdots, w_n\} \subseteq R_\sharp^n[x]$ such that $|W| = n + 1$. Since $\mathbb{F}$ is differentiated and W is finite, there exists admissible sets $U_0, \cdots, U_n \in A$ such that $U_i \cap W = \{w_i\}$ for all $i \leq n$. Let V be a valuation in $\mathbb{F}$ such that $V(p_i) = U_i$ for all $i \leq n$. By Lemma 2.2.12(1), we have $\mathbb{F}, V, x \vDash \Delta^{\leq n}\psi_i$ for all $i \leq n$. Hence, $\mathbb{F}, x \nvDash \mathsf{tab}_n^T$.

Suppose $\mathbb{F}, x \nvDash \mathsf{tab}_n^T$. Then $\mathbb{F}, V, x \vDash \neg\mathsf{tab}_n^T$ for some valuation V in $\mathbb{F}$. Let $\mathfrak{M} = (\mathbb{F}, V)$. Take any $i \leq n$. Then $\mathfrak{M}, x \vDash \Delta^{\leq n}\psi_i$. By Lemma 2.2.12(1), $\mathfrak{M}, w_i \vDash \psi_i$ for some $w_i \in R_\sharp^n[x]$. Note that $\{w_0, \cdots, w_n\} \subseteq R_\sharp^n[x]$ and $w_i \neq w_j$ for $i \neq j \leq n$, we see that $|R_\sharp^n[x]| > n$. ☐

Moreover, by the construction of tab_n^T, we can even show that if a point x validates tab_n^T, then x can access at most n points in arbitrarily many steps. Thus, as we will explain in Chapter 4, the formulas tab_n^T imply a form of 'tense pre-transitivity'.

3.1.7. THEOREM. *Let* $\mathbb{F} = (X, R, A) \in \mathsf{RFr}$ *and* $x \in X$. *Then for all* $n \in \omega$,

$$\mathbb{F}, x \vDash \mathsf{tab}_n^T \text{ if and only if } |R_\sharp^\omega[x]| \leq n.$$

Proof:
Note that $R_\sharp^n[x] \subseteq R_\sharp^\omega[x]$, the right-to-left direction follows from Lemma 3.1.6 immediately. For the other direction, suppose $\mathbb{F}, x \vDash \mathsf{tab}_n^T$. By Lemma 3.1.6, $|R_\sharp^n[x]| \leq n$. It suffices to show that $R_\sharp^n[x] = R_\sharp^\omega[x]$. Towards a contradiction, suppose $R_\sharp^n[x] \neq R_\sharp^\omega[x]$. By Lemma 2.2.15, $R_\sharp^i[x] \neq R_\sharp^{i+1}[x]$ for all $i \leq n$. Thus, $1 = |R_\sharp^0[x]| < |R_\sharp^1[x]| < \cdots < |R_\sharp^n[x]|$, which entails $|R_\sharp^n[x]| > n$ and contradicts the assumption $|R_\sharp^n[x]| \leq n$. Hence, $R_\sharp^n[x] = R_\sharp^\omega[x]$ and so $|R_\sharp^\omega[x]| \leq n$. □

Now we are ready to prove our new characterization theorem.

3.1.8. THEOREM. *For every tense logic* $L \in \mathsf{NExt}(\mathsf{K}_t)$, *we have that*

$$L \in \mathsf{TAB} \text{ if and only if } \mathsf{tab}_n^T \in L \text{ for some } n \in \mathbb{Z}^+.$$

Proof:
The case when L is inconsistent is trivial. Suppose L is consistent. Suppose $L \in \mathsf{TAB}$. Then $L = \mathsf{Log}(\mathfrak{F})$ for some frame $\mathfrak{F}$ with $|\mathfrak{F}| = n \in \omega$. By Lemma 3.1.6, $\mathfrak{F} \vDash \mathsf{tab}_n^T$, which entails $\mathsf{tab}_n^T \in L$. Suppose $\mathsf{tab}_n^T \in L$ for some $n \in \mathbb{Z}^+$. Take any rooted refined frame $\mathbb{F} = (X, R, A) \in \mathsf{RFr}_{\mathrm{r}}(L)$ and $x \in X$. Since $\mathsf{tab}_n^T \in L$, we have $\mathbb{F}, x \vDash \mathsf{tab}_n^T$. By Theorem 3.1.7, $|R_\sharp^\omega[x]| \leq n$, which entails $|\mathbb{F}| \leq n$. Thus, $|\mathbb{F}| \leq n$ for every $\mathbb{F} \in \mathsf{RFr}_{\mathrm{r}}(L)$, which entails $|\mathsf{RFr}_{\mathrm{r}}(L)| < \aleph_0$. It follows from Proposition 2.2.37 that $L = \mathsf{Log}(\mathsf{RFr}_{\mathrm{r}}(L))$. Hence, by Proposition 3.1.3, we have that $L \in \mathsf{TAB}$. □

Given the characterization, we obtain the following theorem:

3.1.9. THEOREM. *Let* $L \in \mathsf{TAB}$. *Then the following hold:*

(1) $\mathsf{NExt}(L) \subseteq \mathsf{TAB}$*; and*

(2) $\mathsf{NExt}(L)$ *is finite.*

Proof:
For (1), take any $L' \in \mathsf{NExt}(L)$. By Theorem 3.1.8, $\mathsf{tab}_n^T \in L \subseteq L'$ for some $n \in \omega$. By Theorem 3.1.8 again, L' is tabular. For (2), note that $\mathsf{V}(L)$ has only finitely many subvarieties (see [6, Corollary 5.12]), by duality, $\mathsf{NExt}(L)$ is finite. □

A tense logic L has *codimension* n with $n \in \omega$, if there exists a chain $L = L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n = \mathsf{Form}_t$ which cannot be refined (see, e.g., [73, 109]). By Theorem 3.1.9, we have the following corollary

3.1.10. Corollary. *Every tabular tense logic is of finite codimension.*

In the lattice $\mathsf{NExt}(\mathsf{S4})$ of modal logics, every tabular logic has only finitely many immediate predecessors, and all of those are tabular. Thus, in $\mathsf{NExt}(\mathsf{S4})$, modal logics of finite codimension are exactly tabular modal logics. For more details, see [29, Chapter 12]. However, as we will see in Section 3.3, in the lattice $\mathsf{NExt}(\mathsf{K}_t)$, a tense logic of finite codimension is not necessarily tabular. In Chapter 5, we will see that even in $\mathsf{NExt}(\mathsf{S4}_t)$, there exists non-tabular logics of codimension 3.

3.1.11. Remark. The characterization of tabularity given by Chagrov and Shehtman [30] uses formulas α_n and β_n with $n \geq 1$ which are defined as follows:

(1) α_n is the conjunction of all formulas of the form

$$\neg(\gamma_1 \wedge M_1(\gamma_2 \wedge M_2(\gamma_3 \wedge \cdots \wedge M_{n-1}\gamma_n))\cdots),$$

(2) β_n is the conjunction of all formulas of the form

$$\neg M_1 \cdots M_s(M_{s+1}\gamma_1 \wedge \cdots \wedge M_{s+n}\gamma_n),$$

where $s < n$, each $M_i \in \{\Diamond, \blacklozenge\}$ with $1 \leq i \leq s+n$, and for each $1 \leq i \leq n$, $\gamma_i = p_1 \wedge \cdots \wedge p_{i-1} \wedge \neg p_i \wedge p_{i+1} \wedge \cdots \wedge p_n$. For every frame $\mathfrak{F} = (X, R)$ and $x \in X$, (i) $\mathfrak{F}, x \not\models \alpha_n$ if and only if there exists a route $\langle x_0, \cdots, x_{n-1}\rangle$ with $x = x_0$ and $x_i \neq x_j$ for all $i \neq j < n$; (ii) $\mathfrak{F}, x \not\models \beta_n$ if and only if there exist $s < n$ and $y \in R^s_\sharp[x]$ with $|R[y] \cup R^{-1}[y]| \geq n$. It follows that

$$(\dagger) \text{ if } \mathfrak{F}_x, x \models \alpha_n \wedge \beta_n, \text{ then } |\mathfrak{F}_x| < f(n) = \textstyle\sum_{k=0}^{n-1}(n-1)^k.$$

By these results, the tabularity in tense logic is characterized as follows ([30]):

(‡) A consistent logic $L \in \mathsf{TAB}$ if and only if $\alpha_n \wedge \beta_n \in L$ for some $n \geq 1$.

Theorem 3.1.9 also follows from (‡). For every $n \geq 1$, we can show the difference between tab^T_n and $\alpha_n \wedge \beta_n$. Consider frames $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{G}_1$ and $\mathfrak{G}_2$ in Figure 3.1. Clearly, $\mathfrak{F}_1, w_1 \models \alpha_n \wedge \beta_m$ if and only if $\mathfrak{F}_2, w_2 \models \alpha_n \wedge \beta_m$ for each $n, m \geq 1$. It follows that for any $n, m \in \mathbb{Z}^+$, the formulas $\alpha_n \wedge \beta_n$ cannot distinguish the pointed-frames $(\mathfrak{F}_1, w_1)$ from $(\mathfrak{F}_2, w_2)$. However, $\mathfrak{F}_1, w_1 \models \mathsf{tab}^T_7$ and $\mathfrak{F}_2, w_2 \not\models \mathsf{tab}^T_7$. On the other hand, for all $n \geq 1$, $\mathfrak{G}_1 \models \mathsf{tab}^T_n$ if and only if $\mathfrak{G}_2 \models \mathsf{tab}^T_n$. However, $\mathfrak{G}_1, u_1 \models \alpha_3 \wedge \beta_3$ and $\mathfrak{G}_2, u_2 \not\models \alpha_3 \wedge \beta_3$.

Intuitively, the formulas tab^T_n give a better characterization of the cardinality of frames, while the formulas $\alpha_n \wedge \beta_m$ provide more information about the structure of frames. By Lemma 3.1.6, we can replace the function $f(n)$ in (†) with n when the cardinality of a frame is concerned. For every rooted frame $\mathfrak{F}$, the formula $\alpha_n \wedge \beta_n$ gives only the upper bound $f(n)$ for the cardinality $|\mathfrak{F}|$, while the formulas tab^T_n tell the exact cardinality of $\mathfrak{F}$, in the sense that $|\mathfrak{F}| = n$ if and only if $\mathfrak{F} \models \mathsf{tab}^T_n$ and $\mathfrak{F} \not\models \mathsf{tab}^T_{n-1}$.

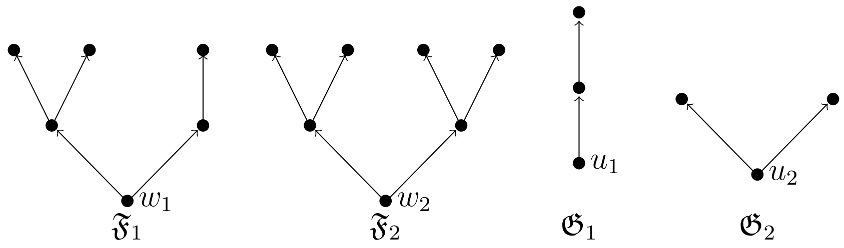

Figure 3.1: Frames $\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{G}_1$ and $\mathfrak{G}_2$

3.2 Post-completeness in TAB

A characterization of Post-completeness in TAB has been given in [107, Proposition 2] without giving a proof. In this section, we provide two characterizations. Let us first recall the definition of Post-completeness and Post-numbers:

3.2.1. DEFINITION. A tense logic L is *Post-complete* if L is consistent and there is no consistent proper extension of L. Let PCOM be the set of all Post-complete tense logics. The *Post-number of* a tense logic L is the cardinality $\mathsf{PN}(L) = |\mathsf{NExt}(L) \cap \mathsf{PCOM}|$.

In this section, we present a new characterization that is called *the first Post-completeness theorem* (Theorem 3.2.13). Our characterization is closely connected to *closed formulas*. Recall that a formula $\varphi \in \mathsf{Form}_t$ is called a *closed formula* if it contains no propositional variable, i.e., $var(\varphi) = \varnothing$. Let Form_t^0 denote the set of all closed formulas. Note that Form_t forms the formula algebra, and Form_t^0 forms the closed formula algebra. A *closed substitution* is a homomorphism $(\cdot)^s : \mathsf{Form}_t \to \mathsf{Form}_t^0$.

Let us begin with Rautenberg's characterization of Post-complete tabular tense logics [107], in which the notion of *contraction*, a special kind of *partition*, plays a central role. A *partition* of a nonempty set X is a subset $\delta \subseteq \mathcal{P}(X)$ such that $\varnothing \notin \delta$, $\bigcup \delta = X$ and $A_1 \cap A_2 = \varnothing$ for all $A_1, A_2 \in \delta$. We write $\mathrm{Part}(X)$ for the set of all partitions of X. For $\delta \in \mathrm{Part}(X)$ and $x \in X$, we write $\delta(x)$ for the element $A \in \delta$ such that $x \in A$, and call $\delta(x)$ the *block* of x. The *trivial partition* of X is $Id_X = \{\{x\} : x \in X\}$. A partition δ_1 is a *refinement* of δ_2, if for every $A \in \delta_1$ there exists $B \in \delta_2$ with $A \subseteq B$.

Partitions and equivalence relations are two sides of the same coin. Recall that a binary relation $\equiv$ on a set X is called an *equivalence relation* if it is reflexive, transitive and symmetric. We write $\mathrm{Eq}(X)$ for the set of all equivalence relations on X. Let X be a non-empty set. Given any partition δ of X, we define

$$\equiv_\delta := \{(x, y) \in X \times X : \delta(x) = \delta(y)\}$$

and call it the equivalence relation induced by δ. On the other hand, given an equivalence relation $\equiv$ on X, we define

$$\delta_{\equiv} := \{[x]_{\equiv} : x \in X\}$$

and call it the partition induced by $\equiv$. It is not hard to show that the functions $(\cdot)_\delta$ and $(\cdot)_\equiv$ are the inverse functions of each other.

3.2.2. DEFINITION. Let $\mathfrak{F} = (X, R)$ be a frame. A partition δ of X is called a *contraction*, if for all $x, y \in X$, the following conditions hold:

(C1) if Rxy and $x' \in \delta(x)$, there exists $y' \in \delta(y)$ with $Rx'y'$.

(C2) if Ryx and $x' \in \delta(x)$, there exists $y' \in \delta(y)$ with $Ry'x'$.

Let $Ctr(\mathfrak{F})$ be the set of all contractions of $\mathfrak{F}$.

Obviously, for every frame $\mathfrak{F} = (X, R)$, the trivial partition Id_X belongs to $Ctr(\mathfrak{F})$. Then a frame $\mathfrak{F}$ has no nontrivial contraction if and only if $|Ctr(\mathfrak{F})| = 1$. Given the relation between partitions and equivalence relations, we have

3.2.3. DEFINITION. Let $\mathfrak{F} = (X, R)$ be a frame and $\equiv$ an equivalence relation on X. Then we say that $\equiv$ induces a contraction if $\delta_\equiv$ is a contraction, that is, for all $x, y \in X$,

(C1′) if Rxy and $x \equiv x'$, there exists $y' \equiv y$ such that $Rx'y'$.

(C2′) if Ryx and $x \equiv x'$, there exists $y' \equiv y$ such that $Ry'x'$.

It was claimed in [107] that, a tabular tense logic L is Post-complete if and only if $L = \mathsf{Log}(\mathfrak{F})$ for some finite rooted frame $\mathfrak{F}$ with $|Ctr(\mathfrak{F})| = 1$. Let us take a closer look at the contractions to get a better understanding of the claim above given by Rautenberg. We prove the following lemmas to show the relation between contractions and t-morphisms.

Recall that for all functions $f : X \to Y$, the kernel $\ker(f)$ is defined as follows:

$$\ker(f) := \{(x, x') \in X \times X : f(x) = f(x')\}.$$

Then the following lemma holds:

3.2.4. LEMMA. *Let* $\mathfrak{F} = (X, R)$ *and* $\mathfrak{F}' = (X', R')$ *be frames and* $f : \mathfrak{F} \twoheadrightarrow \mathfrak{F}'$. *Then* $\ker(f)$ *induces a contraction of* $\mathfrak{F}$.

Proof:
It suffices to show that (C1′) and (C2′) hold for $\ker f$. Take any $x, x', y \in X$ such that $f(x) = f(x')$ and Rxy. By $f : \mathfrak{F} \twoheadrightarrow \mathfrak{F}'$, we obtain that $R'f(x)f(y)$. Thus, $R'f(x')f(y)$ and there exists $y' \in R[x']$ such that $f(y') = f(y)$. Hence, (C1′) holds. Analogously, we see that (C2′) also holds. □

3.2.5. LEMMA. *Let $\mathfrak{F} = (X, R)$ be a frame and $\delta \in Ctr(\mathfrak{F})$. Then the function $\delta : x \mapsto \delta(x)$ is a t-morphism from $\mathfrak{F}$ to $\mathfrak{F}/\delta = (X/\delta, R/\delta)$, where*

$$X/\delta = \delta \text{ and } R/\delta = \{(\delta(x), \delta(y)) \in \delta \times \delta : \exists x' \in \delta(x) \exists y' \in \delta(y)(Rx'y')\}.$$

Proof:
It is clear that Rxy implies $(\delta(x), \delta(y)) \in R/\delta$ for all $x, y \in X$. Take any $x \in X$ and $A = \delta(z) \in \delta$. Suppose $(\delta(x), A) \in R/\delta$. Then there exists $x' \in \delta(x)$ and $z' \in \delta(z)$ such that $Rx'z'$. Note that $x \in \delta(x')$, by (C1), there exists $y \in R[x]$ such that $y \in \delta(z')$. Then we have $\delta(y) = A$. Symmetrically, by (C2), for all $x \in X$ and $B \in \delta$, if $(\delta(x), B) \in (R/\delta)^{-1}$, then there exists $y \in R^{-1}[x]$ such that $\delta(y) = B$. Hence, $\delta : \mathfrak{F} \twoheadrightarrow \mathfrak{F}/\delta$. □

By Lemma 3.2.5, we see that every contraction of $\mathfrak{F}$ could induce a t-morphism. On the other hand, by Lemma 3.2.4, every onto t-morphism $f : \mathfrak{F} \twoheadrightarrow \mathfrak{F}'$ induces a contraction of $\mathfrak{F}$. Thus, by Rautenberg's claim, a tabular tense logic L is Post-complete if and only if $L = \mathsf{Log}(\mathfrak{F})$ for some finite rooted frame $\mathfrak{F}$ such that $\mathsf{M}_t(\mathfrak{F}) = \mathsf{I}(\mathfrak{F})$, i.e., every t-morphic image of $\mathfrak{F}$ is isomorphic to $\mathfrak{F}$. By the duality between finite frames and tense algebras, we see that a tabular tense logic L is Post-complete if and only if $L = \mathsf{Log}(\mathfrak{A})$ for some finite subdirectly irreducible tense algebra $\mathfrak{A}$ admits no proper subalgebras.

Now we give a new characterization of Post-completeness in TAB utilizing closed formulas. In what follows, for each frame $\mathfrak{F} = (X, R)$ and $x \in X$, the *closed theory* $CT(x)$ of x in $\mathfrak{F}$ is defined as $CT(x) = \{\varphi \in \mathsf{Form}_t^0 : \mathfrak{F}, x \models \varphi\}$.

3.2.6. DEFINITION. Let $\mathfrak{F} = (X, R)$ be a frame. The 0-*filtration* $\mathfrak{F}^c$ of $\mathfrak{F}$ is defined by $\mathfrak{F}^c = (X^c, R^c)$ where

(1) $X^c = \{[x]_c : x \in X\}$, where $[x]_c = \{y \in X : CT(x) = CT(y)\}$.

(2) $R^c[x]_c[y]_c$ if and only if $\Diamond\varphi \in CT(x)$ for every $\varphi \in CT(y)$.

We write $\delta^c_{\mathfrak{F}}$ for X^c and call it the 0-*partition* of X by $\mathfrak{F}$.

It is not hard to verify that $\delta^c_{\mathfrak{F}}$ is a partition of $\mathfrak{F}$. Moreover, we have

3.2.7. LEMMA. *Let $\mathfrak{F} = (X, R)$ be a frame and $\delta \in Ctr(\mathfrak{F})$. Then*

(1) for all $x, y \in X$, $y \in \delta(x)$ implies $CT(x) = CT(y)$;

(2) δ is a refinement of $\delta^c_{\mathfrak{F}}$.

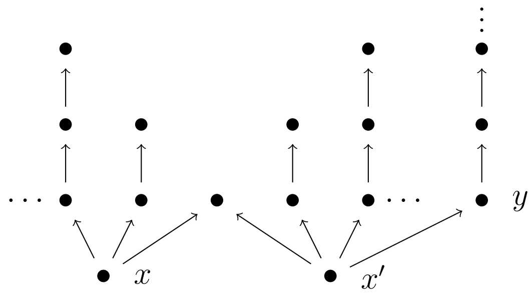

Figure 3.2: A frame $\mathfrak{F}$ where $\delta^{\mathsf{c}}_{\mathfrak{F}}$ is not a contraction

Proof:
For (1), take any $x, y \in X$ such that $y \in \delta(x)$. It suffices to show that for every formula $\varphi \in \mathsf{Form}^0_t$, $\mathfrak{F}, x \vDash \varphi$ if and only if $\mathfrak{F}, y \vDash \varphi$. We prove by induction on φ. The atomic and Boolean cases are trivial. Let $\varphi = \Box\psi$. Suppose $\mathfrak{F}, x \nvDash \varphi$. Then there exists $y \in R[x]$ such that $\mathfrak{F}, y \nvDash \psi$. By (C1), there exists $y' \in \delta(y) \cap R[x']$. By induction hypothesis, $\mathfrak{F}, y' \nvDash \psi$, which entails $\mathfrak{F}, x' \nvDash \varphi$. Suppose $\mathfrak{F}, x' \nvDash \varphi$. By (C2), we see that $\mathfrak{F}, x \nvDash \varphi$. The case $\varphi = \blacklozenge\psi$ can be proved analogously.

(2) follows from (1) immediately. $\Box$

Lemma 3.2.7 shows that every contraction is a refinement of $\delta^{\mathsf{c}}_{\mathfrak{F}}$. Thus, if $\delta^{\mathsf{c}}_{\mathfrak{F}}$ itself is a contraction, then it is the maximal one. However, in general, $\delta^{\mathsf{c}}_{\mathfrak{F}}$ is not a contraction of $\mathfrak{F}$. Consider the non-transitive frame $\mathfrak{F}$ depicted in Figure 3.2. The reader can readily check that $CT(x) = CT(x')$ and $[y]_c = \{y\}$, thereby $\delta^{\mathsf{c}}_{\mathfrak{F}}$ is not a contraction of $\mathfrak{F}$.

In what follows, we introduce the notion of 0-*saturated* frames and show that for every 0-saturated frame $\mathfrak{F}$, the partition $\delta^{\mathsf{c}}_{\mathfrak{F}}$ is the maximal contraction of $\mathfrak{F}$. It follows that for all finite rooted frame $\mathfrak{F}$, $\delta^{\mathsf{c}}_{\mathfrak{F}}$ is the maximal contraction of $\mathfrak{F}$.

3.2.8. Definition. Let $\mathfrak{F} = (X, R)$ be a frame and $Y \subseteq X$. Then we say that a set of closed formulas Σ is *satisfiable* in Y, if there exists $y \in Y$ such that $\mathfrak{F}, y \vDash \varphi$ for all $\varphi \in \Sigma$; and Σ is *finitely satisfiable* in Y, if every finite subset of Σ is satisfiable in Y.

A frame $\mathfrak{F}$ is called 0-*saturated*, if for all $x \in X$ and $\Sigma \subseteq \mathsf{Form}^0_t$:

(1) If Σ is finitely satisfiable in $R[x]$, then Σ is satisfiable in $R[x]$;

(2) If Σ is finitely satisfiable in $R^{-1}[x]$, then Σ is satisfiable in $R^{-1}[x]$.

3.2.9. Lemma. *Let $\mathfrak{F} = (X, R)$ be 0-saturated and $f : X \to X^c$ be the function such that $f(x) = [x]_c$ for all $x \in X$. Then $f : \mathfrak{F} \twoheadrightarrow \mathfrak{F}^c$ is a surjective t-morphism.*

Proof:
Clearly, f is surjective. Take any $x, y \in X$. Suppose Rxy. Then clearly, for all $\varphi \in \mathsf{Form}_t^0$, we have that $\mathfrak{F}, y \models \varphi$ implies $\mathfrak{F}, x \models \Diamond\varphi$. Thus, $R^c[x]_c[y]_c$. Assume $R^c[x]_c[y]_c$. It suffices to show that there exists $z \in R[x]$ with $[z]_c = [y]_c$. We show that $CT(y)$ is finitely satisfiable in $R[x]$. Take any finite subset $\Theta = \{\varphi_1, \cdots, \varphi_n\}$ of $CT(y)$. Then $\Diamond(\varphi_1 \wedge \cdots \wedge \varphi_n) \in CT(x)$, i.e., $\mathfrak{F}, x \models \Diamond(\varphi_1 \wedge \cdots \wedge \varphi_n)$. Then Θ is satisfiable in $R[x]$. Hence, $CT(y)$ is finitely satisfiable in $R[x]$. Since $\mathfrak{F}$ is 0-saturated, there exists $z \in R[x]$ with $\mathfrak{F}, z \models \psi$ for all $\psi \in CT(y)$. Note that $CT(y) \subseteq CT(z)$. Suppose $\chi \notin CT(y)$. Then $\neg\chi \in CT(y) \subseteq CT(z)$ and so $\chi \notin CT(z)$. Hence, $CT(y) = CT(z)$ and so $[y]_c = [z]_c$.

Similarly, $R^c[y]_c[x]_c$ implies $[z]_c = [y]_c$ for some $z \in R^{-1}[x]$. Hence, $f : \mathfrak{F} \twoheadrightarrow \mathfrak{F}^c$ is a surjective t-morphism. □

It follows from Lemmas 3.2.9 and 3.2.4 that the following lemma holds:

3.2.10. LEMMA. *If $\mathfrak{F} = (X, R)$ is a 0-saturated frame, then $\delta^c_{\mathfrak{F}} \in Ctr(\mathfrak{F})$.*

By Lemmas 3.2.10 and 3.2.7(2), we see that for a 0-saturated frame $\mathfrak{F}$, the 0-partition $\delta^c_{\mathfrak{F}}$ is the maximal contraction of $\mathfrak{F}$. Then the following theorem holds:

3.2.11. THEOREM. *If $\mathfrak{F} = (X, R)$ is a 0-saturated frame, then*

$$|Ctr(\mathfrak{F})| = 1 \text{ if and only if } \delta^c_{\mathfrak{F}} = Id_X$$

Proof:
By Lemma 3.2.10, $\delta^c_{\mathfrak{F}} \in Ctr(\mathfrak{F})$. Since $Id_X \in Ctr(\mathfrak{F})$, we see that $|Ctr(\mathfrak{F})| = 1$ implies $\delta^c_{\mathfrak{F}} = Id_X$. Suppose $\delta^c_{\mathfrak{F}} = Id_X$. By Lemma 3.2.7(2), we have $\delta \subseteq \delta^c_{\mathfrak{F}} = Id_X$ for all $\delta \in Ctr(\mathfrak{F})$. Thus, $|Ctr(\mathfrak{F})| = 1$. □

Now, let us focus again on tabular tense logics. We show that every finite frame is 0-saturated, and then apply the results obtained above to study Post-completeness of tabular tense logics.

3.2.12. LEMMA. *If $\mathfrak{F}$ is a finite frame, then $\mathfrak{F}$ is 0-saturated and $\mathsf{Log}(\mathfrak{F}) \subseteq \mathsf{Log}(\mathfrak{F}^c)$.*

Proof:
Let $\mathfrak{F} = (X, R)$ be finite, $x \in X$ and $\Sigma \subseteq \mathsf{Form}_t^0$. Let $R[x] = \{x_0, \cdots, x_n\}$ and $|R[x]| = n + 1$. Suppose that Σ is not satisfiable in $R[x]$. Then for every $i \leq n$, there exists $\varphi_i \in \Sigma$ with $\mathfrak{F}, x_i \models \neg\varphi_i$. Hence, $\mathfrak{F}, x \models \Box\neg(\varphi_0 \wedge \cdots \wedge \varphi_n)$. Let $\Theta = \{\varphi_0, \cdots, \varphi_n\}$. Then Θ is not satisfiable in $R[x]$, which entails that Σ is not finitely satisfiable in $R[x]$. Similarly, if Σ is finitely satisfiable in $R^{-1}[x]$, then Σ is satisfiable in $R^{-1}[x]$. Thus, $\mathfrak{F}$ is 0-saturated. By Lemma 3.2.9, $\mathfrak{F} \twoheadrightarrow \mathfrak{F}^c$. By

Proposition 2.2.26, $\mathsf{Log}(\mathfrak{F}) \subseteq \mathsf{Log}(\mathfrak{F}^c)$. $\square$

Next, we present and prove the first Post-completeness theorem.

3.2.13. THEOREM (The first Post-completeness theorem). *Let $\mathfrak{F} = (X, R)$ be a finite rooted frame. The following are equivalent:*

(1) $\mathsf{Log}(\mathfrak{F})$ *is Post-complete;*

(2) $\mathfrak{F} \cong \mathfrak{F}^c$*;*

(3) for every $x, y \in X$, $CT(x) = CT(y)$ if and only if $x = y$;

(4) $|Ctr(\mathfrak{F})| = 1$.

Proof:
By Theorem 3.2.11, (2), (3) and (4) are equivalent. Now we show that (1) implies (3). Assume $\mathsf{Log}(\mathfrak{F}) \in \mathsf{PCOM}$. For a contradiction, suppose there exists $x \neq y \in X$ such that $CT(x) = CT(y)$. Recall that $[x]_c = \{z \in X : CT(z) = CT(x)\}$. Since $\mathfrak{F}$ is finite, there exists $\varphi_x \in CT(x)$ such that $\mathfrak{F}, z \models \neg\varphi_x$ for all $z \notin [x]_c$. Since $\mathfrak{F}$ is rooted, $y \in R_\sharp^n[x]$ for some $n \in \omega$. Let $\mathfrak{M} = (\mathfrak{F}, V)$ be a model such that $V(p) = \{x\}$. Then it is not hard to verify that $\mathfrak{M}, x \not\models (\varphi_x \wedge p) \to \nabla^{\leq n}(\varphi_x \to p)$. Thus, $\mathfrak{F} \not\models (\varphi_x \wedge p) \to \nabla^{\leq n}(\varphi_x \to p)$. Let $\mathfrak{F}^c$ be the 0-filtration of $\mathfrak{F}$. Since $\mathfrak{F}$ is finite, by Lemma 3.2.12, $\mathsf{Log}(\mathfrak{F}) \subseteq \mathsf{Log}(\mathfrak{F}^c)$. Note that $[x]_c$ is the only point in X^c validating φ_x. Then $\mathfrak{F}^c \models (\varphi_x \wedge p) \to \nabla^{\leq n}(\varphi_x \to p)$. Hence, $\mathsf{Log}(\mathfrak{F}) \subsetneq \mathsf{Log}(\mathfrak{F}^c)$ which contradicts the assumption $\mathsf{Log}(\mathfrak{F}) \in \mathsf{PCOM}$.

Now we show that (2) implies (1). Suppose $\mathfrak{F} \cong \mathfrak{F}^c$ and $\mathsf{Log}(\mathfrak{F}) \subseteq L$ for some consistent tense logic L. Let $X = \{x_0, \cdots, x_n\}$ and $|X| = n+1$. Since $\mathfrak{F} \cong \mathfrak{F}^c$, for every $i \neq j \leq n$, $CT(x_i) \neq CT(x_j)$. Thus, for each $i \leq n$, there exists $\varphi_i \in \mathsf{Form}_t^0$ such that for all $x \in X$, $\mathfrak{F}, x \models \varphi_i$ if and only if $x = x_i$. Let $\mathfrak{G} = (Y, S)$ be a rooted frame for L. Now we show that $\mathfrak{F}^c \cong \mathfrak{G}^c$. It suffices to show $Y^c = X^c$. Take any $y \in Y$. Clearly, $\bigvee_{i \leq n} \varphi_i \in \mathsf{Log}(\mathfrak{F}) \subseteq L$. Then $\mathfrak{G}, y \models \varphi_{i_y}$ for some $i_y \leq n$. Note that $\{\varphi_{i_y} \to \psi \in \mathsf{Form}_t^0 : \mathfrak{F}, x_{i_u} \models \psi\} \subseteq \mathsf{Log}(\mathfrak{F})$, we have $CT(y) = CT(x_{i_y})$. Thus, $Y^c \subseteq X^c$. Take any $x_k \in X$. Note that since $\Delta^{\leq n}\varphi_k \in \mathsf{Log}(\mathfrak{F})$, there exists $y \in Y$ such that $CT(y) = CT(x_k)$. Hence, $\mathfrak{F}^c \cong \mathfrak{G}^c$. Since $|\mathfrak{F}| = n+1$, by Lemma 3.1.6, we obtain that $\mathfrak{F} \models \mathsf{tab}_{n+1}^T$. Then $\mathfrak{G} \models \mathsf{tab}_{n+1}^T$. Since $\mathfrak{G}$ is rooted, by Lemma 3.1.6, $\mathfrak{G}$ is finite. By Lemma 3.2.12, $L \subseteq \mathsf{Log}(\mathfrak{G}) \subseteq \mathsf{Log}(\mathfrak{G}^c) = \mathsf{Log}(\mathfrak{F}) \subseteq L$. Then $\mathsf{Log}(\mathfrak{F}) = L$. Hence, $\mathsf{Log}(\mathfrak{F}) \in \mathsf{PCOM}$. $\square$

3.2.14. REMARK. We could also prove that $\mathfrak{F} \cong \mathfrak{F}^c$ implies Post-completeness of $\mathsf{Log}(\mathfrak{F})$ by duality. Suppose $\mathfrak{F} \cong \mathfrak{F}^c$. Let $L = \mathsf{Log}(\mathfrak{F})$. Then $V = \mathsf{V}(L) = \mathsf{V}(\mathfrak{F}^*)$. Since $\mathfrak{F}^*$ is finite and $\mathsf{V}(L)$ is congruence-distributive, by Jónsson's Lemma, $V_{\mathrm{si}} =$

$\mathsf{HS}(\mathfrak{F}^*)$. Since $\mathfrak{F} \cong \mathfrak{F}^c$, by Lemmas 3.2.4 and 3.2.7, every t-morphic image of $\mathfrak{F}$ is isomorphic to $\mathfrak{F}$. Then $\mathsf{S}(\mathfrak{F}^*) = \mathsf{I}(\mathfrak{F}^*)$. Since $\mathfrak{F}$ is rooted, we have that $\mathsf{H}(\mathfrak{F}^*) = \mathsf{I}(\{\mathtt{1}, \mathfrak{F}^*\})$, where $\mathtt{1}$ is the trivial tense algebra. Thus, $V_{\text{si}} = \mathsf{I}(\{\mathtt{1}, \mathfrak{F}^*\})$, which implies by Birkhoff's theorem that there exists no nontrivial subvariety of V. Thus, L is Post-complete.

By Theorem 3.2.13, we can check Post-completeness of $\mathsf{Log}(\mathfrak{F})$ for certain finite frames $\mathfrak{F}$. For example, consider frames $\mathfrak{H}_1$, $\mathfrak{H}_2$, $\mathfrak{H}_3$, $\mathfrak{H}_4$ in Figure 3.3 and their logics L_1, L_2, L_3 and L_4 respectively. For $\mathfrak{H}_1, \mathfrak{H}_2$ and $\mathfrak{H}_3$, we can distinguish distinct points by closed formulas as in Figure 3.3. However, for the frame $\mathfrak{H}_4$, we have that $CT(x) = CT(y)$ while $x \neq y$. By Theorem 3.2.13, L_1, L_2 and L_3 are Post-complete but L_4 is not. In fact, $\mathfrak{H}_4^c \cong \mathfrak{H}_2$ and $L_4 \subsetneq L_2$. Note that L_1 was incorrectly claimed not to be Post-complete in [107].

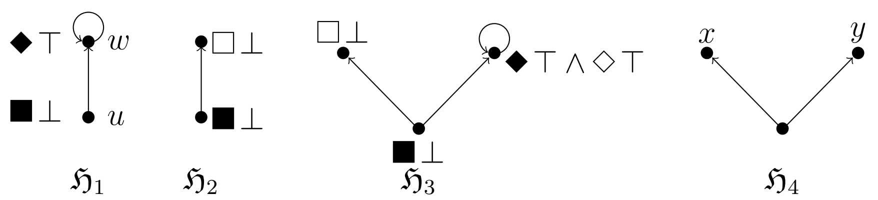

Figure 3.3: Frames $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3$ and $\mathfrak{H}_4$

3.2.15. Example. For each $n \in \mathbb{Z}^+$, let $\mathfrak{Ch}_n^<$ denote the strict chain of n elements, that is, $\mathfrak{Ch}_n^< = (n, <)$. The frame $\mathfrak{Ch}_n^<$ is depicted in Figure 3.4. It is clear that for each $i < n$, the formula $\Box^{n-i} \bot \wedge \Diamond^{n-(i+1)} \top$ is true only at i. By Theorem 3.2.13, $\mathsf{Log}(\mathfrak{Ch}_n^<)$ is Post-complete for all $n \in \mathbb{Z}^+$.

Consequently, the following theorem holds:

3.2.16. Theorem. *There exists $\aleph_0$ Post-complete tabular tense logics.*

By Theorem 3.2.13, we also obtain the following corollary on decidability of Post-completeness for tabular tense logics:

3.2.17. Corollary. *Post-completeness of the tense logic $\mathsf{Log}(\mathfrak{F})$ for a given finite frame $\mathfrak{F}$ is decidable. That is, given any finite frame $\mathfrak{F}$, there exists an effective way to decide whether the tense logic $\mathsf{Log}(\mathfrak{F})$ is Post-complete.*

Proof:
Let $\mathfrak{F} = (X, R)$ be a finite frame. Then $Ctr(\mathfrak{F})$ is finite. By Theorem 3.2.13, it suffices to check if there exists a non-trivial contraction in $Ctr(\mathfrak{F})$. It should be clear that this procedure terminates. □

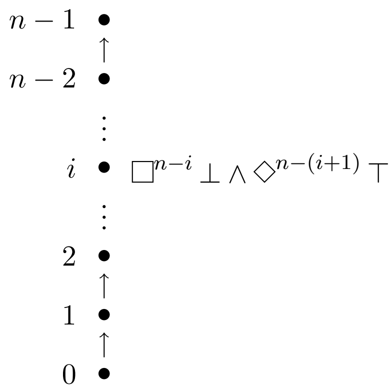

Figure 3.4: The frame $\mathfrak{Ch}_n^<$

The reader should note that in Corollary 3.2.17, we take a finite frame as the input. It does not follow that given any formula φ, we could decide whether $\mathsf{K}_t \oplus \varphi$ is Post-complete. In fact, as we are going to show in Chapter 6, the set

$$\{\varphi \in \mathsf{Form}_t : \mathsf{K}_t \oplus \varphi \text{ is Post-complete}\}$$

is undecidable. In what follows, we are going to give another theorem on Post-completeness of tabular tense logics, which characterizes Post-completeness of tabular tense logics via the number of rooted frames for L up to isomorphism. As a corollary, we show that the set

$$\{\varphi \wedge \mathsf{tab}_n^T \in \mathsf{Form}_t : \mathsf{K}_t \oplus \varphi \text{ is Post-complete}\}$$

is decidable for all $n \in \omega$.

Recall that for a class $\mathcal{K}$ of general frames, we write $|\mathcal{K}| = n$ if $|\mathcal{K}/\cong| = n$, i.e., the cardinality of the quotient $\mathcal{K}/\cong$ of $\mathcal{K}$ by isomorphism is n.

3.2.18. Lemma. *Let $\mathfrak{F} = (X, R)$ be a finite rooted frame. If $L = \mathsf{Log}(\mathfrak{F}) \in$ PCOM, then* $|\mathsf{Fr}_{\mathrm{r}}(L)| = 1$.

Proof:
Let $\mathfrak{F} = (X, R)$ be a finite rooted frame and $X = \{x_0, \cdots, x_{n-1}\}$. Assume $L = \mathsf{Log}(\mathfrak{F}) \in \mathsf{PCOM}$. By Theorem 3.2.13, for all $x, y \in X$, $CT(x) = CT(y)$ if and only if $x = y$. Then there exist $\varphi_0, \cdots, \varphi_{n-1} \in \mathsf{Form}_t^0$ such that

(†) for all $y \in X$ and $i < n$, we have that $\mathfrak{F}, y \models \varphi_i$ if and only if $y = x_i$.

Indeed, we have the following claims:

(i) $\mathsf{tab}_n^T \in L$. Since $\mathfrak{F}$ is finite and rooted, by Lemma 3.1.6, $\mathfrak{F} \models \mathsf{tab}_n^T$.

(ii) $\Delta^{\leq n}\varphi_i \in L$ for every $i < n$. Since $\mathfrak{F}$ is rooted, for every $y \in X$, we obtain $R^n_\sharp[y] = X$. Thus, $\mathfrak{F} \vDash \Delta^{\leq n}\varphi_i$ for every $i < n$.

(iii) $\varphi_i \wedge \varphi_j \to \bot \in L$ whenever $i \neq j < n$. This follows from (†).

(iv) if Rx_ix_j, then $\varphi_i \to \Diamond\varphi_j \in L$. This follows from (†).

Suppose $\mathfrak{G} = (Y, S) \in \mathsf{Fr}_\mathrm{r}(L)$. By (i) and Lemma 3.1.6, $|Y| \leq n$. By (ii) and (iii), for every $i < n$, there exists $y \in Y$ with $\mathfrak{G}, y \vDash \varphi_i \wedge \bigwedge_{j\neq i} \neg\varphi_j$. Hence, $|Y| = n$. Let $Y = \{y_0, \cdots, y_{n-1}\}$. Without loss of generality, let $\mathfrak{G}, y_i \vDash \varphi_i$ for each $i < n$. Let $f : X \to Y$ be the function such that $f(x_i) = y_i$ for each $i < n$. Clearly f is bijective. Assume Rx_ix_j. By (iv), $\varphi_i \to \Diamond\varphi_j \in L$. Since in $\mathfrak{G}$, φ_i holds only in y_i and φ_j holds only in y_j, we have Sy_iy_j. Similarly, Sy_iy_j implies Rx_ix_j. Then $\mathfrak{F} \cong \mathfrak{G}$. Hence, $\mathsf{Fr}_\mathrm{r}(L)$ is a singleton up to isomorphism and so $|\mathsf{Fr}_\mathrm{r}(L)| = 1$. □

3.2.19. Theorem (The second Post-completeness theorem). *Let $L \in \mathsf{TAB}$. Then L is Post-complete if and only if $|\mathsf{Fr}_\mathrm{r}(L)| = 1$.*

Proof:
Let $L \in \mathsf{TAB}$. Assume $|\mathsf{Fr}_\mathrm{r}(L)| = 1$. Suppose $L \subseteq L'$ where L' is a consistent tense logic. By Lemma 3.1.9 (1), L' is tabular. By Theorem 3.1.4, L' is Kripke complete and so $L' = \mathsf{Log}(\mathsf{Fr}_\mathrm{r}(L'))$. Clearly $\mathsf{Fr}_\mathrm{r}(L') \subseteq \mathsf{Fr}_\mathrm{r}(L)$. Then $L = \mathsf{Log}(\mathsf{Fr}_\mathrm{r}(L)) \subseteq \mathsf{Log}(\mathsf{Fr}_\mathrm{r}(L')) = L'$. Hence, $L = L'$. It follows that $L \in \mathsf{PCOM}$.

For the other direction, suppose $L \in \mathsf{PCOM}$. Since $L \in \mathsf{TAB}$, there exists a finite frame $\mathfrak{F} = (X, R)$ with $L = \mathsf{Log}(\mathfrak{F})$. Let $x \in X$. By Proposition 2.2.18, $\mathsf{Log}(\mathfrak{F}) \subseteq \mathsf{Log}(\mathfrak{F}_x)$. Since $L \in \mathsf{PCOM}$, $L = \mathsf{Log}(\mathfrak{F}) = \mathsf{Log}(\mathfrak{F}_x)$. By Lemma 3.2.18, $|\mathsf{Fr}_\mathrm{r}(L)| = 1$. □

3.2.20. Remark. From an algebraic point of view, we see that a tense logic $L \in \mathsf{TAB}$ is Post-complete if and only if the variety $\mathsf{V}(L)$ contains only one subdirectly irreducible tense algebra up to isomorphism.

The second Post-completeness theorem gives a new characterization of Post-completeness in TAB. Consider frames in Figure 3.3 and their tense logics which are tabular. Note that $\mathsf{Fr}_\mathrm{r}(L_i) = \{\mathfrak{H}_i\}$ for $i = 1, 2, 3$, and $\mathsf{Fr}_\mathrm{r}(L_4) = \{\mathfrak{H}_2, \mathfrak{H}_4\}$. Then $L_1, L_2, L_3 \in \mathsf{PCOM}$ and $L_4 \notin \mathsf{PCOM}$. Moreover, we have

3.2.21. Corollary. *Given any $n \in \omega$ and formula $\varphi = \psi \wedge \mathsf{tab}^T_n$, it is decidable whether the tense logic $\mathsf{K}_t \oplus \varphi$ is Post-complete.*

Proof:
Consider the set X of all rooted frames whose domains are subsets of n. It is straightforward to verify that X is finite and every rooted frame for $\mathsf{K}_t \oplus \varphi$ is isomorphic to some frame in X. By Theorem 3.2.19, it suffices to check one by one whether a rooted frame $\mathfrak{F} \in X$ validates φ. Clearly, this procedure terminates. Hence, Post-completeness of $\mathsf{K}_t \oplus \varphi$ is decidable. □

3.3 Post-completeness in $\mathsf{NExt}(\mathsf{K}_t)$

In this section, we explore Post-completeness in the lattice $\mathsf{NExt}(\mathsf{K}_t)$ and prove *the third Post-completeness theorem* (Theorem 3.3.4). Note that Theorem 3.2.13 gives a characterization of Post-completeness of $\mathsf{Log}(\mathfrak{F})$ where $\mathfrak{F}$ is a finite rooted frame. We first show that there exists a Post-complete tense logic which has no finite frames. In what follows, the 0-*general frame* based on a frame $\mathfrak{F} = (X, R)$ is defined as $\mathfrak{F}^{\clubsuit} := (W, R, A^{\clubsuit})$ where $A^{\clubsuit} = \{V(\varphi) : \varphi \in \mathsf{Form}_t^0\}$ for arbitrary valuation V in $\mathfrak{F}$. The definition of $A^{\clubsuit}$ does not depend on the choice of valuation V. From the view of duality, $A^{\clubsuit}$ is exactly the subalgebra of the dual $\mathfrak{F}^*$ of $\mathfrak{F}$ generated by the bottom element $\varnothing$.

3.3.1. Proposition. *Let $\mathfrak{N} = (\omega, <)$ where $<$ is the strict natural order on ω. Let $L = \mathsf{Log}(\mathfrak{N}^{\clubsuit})$. Then $\mathsf{Fin}(L) = \varnothing$ and L is Post-complete.*

Proof:
Towards a contradiction, suppose $\mathfrak{F} \models L$ for some finite frame $\mathfrak{F} = (X, R)$. Note that (i) $\blacklozenge\blacksquare\bot \vee \blacksquare\bot \in L$, (ii) $\mathfrak{N}^{\clubsuit}, 0 \models \Diamond(\blacklozenge^n \top \wedge \blacksquare^{n+1} \bot)$ for all $n \in \omega$, and (iii) $\mathfrak{N}^{\clubsuit}, m \not\models \blacksquare\bot$ for all $m > 0$. Thus, $\{\blacksquare\bot \to \Diamond(\blacklozenge^n \top \wedge \blacksquare^{n+1} \bot) : n \in \omega\} \subseteq L$. Since $\mathfrak{F} \models L$, by (i), we have $\mathfrak{F} \models \blacklozenge\blacksquare\bot \vee \blacksquare\bot$. Then for every $n \in \omega$, $\{x \in X : \mathfrak{F}, x \models \blacklozenge^n \top \wedge \blacksquare^{n+1} \bot\} \neq \varnothing$. It follows immediately that $\mathfrak{F}$ is infinite. Thus, $\mathsf{Fin}(L) = \varnothing$.

To show that L is Post-complete, it suffices to show that $L \oplus \varphi$ is inconsistent for any $\varphi \notin L$. Take any $\varphi \notin L$. Then $\mathfrak{N}^{\clubsuit}, V, x \models \neg\varphi$ for some admissible valuation V in $\mathfrak{N}^{\clubsuit}$ and $x \in X$. Let $var(\varphi) = \{p_1, \cdots, p_n\}$. For every $p \in var(\varphi)$, we choose a closed formula $\psi_p \in \mathsf{Form}_t^0$ with $V(p) = V(\psi_p)$. Let s be a substitution with $s(p_k) = \psi_{p_k}$ for each $1 \leq k \leq n$. Then $\mathfrak{N}^{\clubsuit}, x \models \neg\varphi^s$ and so $\blacksquare\bot \to (\Diamond\neg\varphi^s \vee \neg\varphi^s) \in L$. Suppose $L \oplus \varphi$ is consistent. Then $\mathbb{F} \models L \oplus \varphi$ for some descriptive frame $\mathbb{F} = (X, R, A)$. By $\mathbb{F} \models L$, we have $\mathbb{F}, x_0 \models \blacksquare\bot$ for some $x_0 \in X$. By $\blacksquare\bot \to (\Diamond\neg\varphi^s \vee \neg\varphi^s) \in L$, we have $\mathbb{F}, y \models \neg\varphi^s$ for some $y \in X$. Then $\mathbb{F} \not\models \varphi$ which contradicts $\mathbb{F} \models L \oplus \varphi$. Then $L \oplus \varphi$ is not consistent. Hence, every proper extension of L is inconsistent and so $L \in \mathsf{PCOM}$. □

3.3.2. Remark. Note that the notion of a 0-general frame generalizes that of 0-filtration, in the sense that 0-filtrations apply only to 0-saturated frames, whereas a 0-general frame can be constructed for an arbitrary frame. Both of them aim to preserve only the information that can be characterized by closed formulas. Moreover, the reader can readily verify that for any 0-saturated frame $\mathfrak{F}$, we have $\mathsf{Log}(\mathfrak{F}^c) = \mathsf{Log}(\mathfrak{F}^\clubsuit)$.

3.3.3. Lemma. *Let* $\mathfrak{F} = (X, R)$ *be a frame and* $L = \mathsf{Log}(\mathfrak{F}^\clubsuit)$. *For every formula* $\varphi \in \mathsf{Form}_t$, *we have that* $\varphi \notin L$ *implies* $\varphi^s \notin L$ *for some closed substitution* s.

Proof:
Assume $\varphi \notin L$. Then $\mathfrak{F}^\clubsuit, V, x \vDash \neg\varphi$ for some $x \in X$ and admissible valuation V in $\mathfrak{F}^\clubsuit$. For each $p \in \mathsf{Prop}$, we choose $\psi_p \in \mathsf{Form}_t^0$ with $V(p) = V(\psi_p)$. Let s be the closed substitution with $s(p) = \psi_p$ for each $p \in \mathsf{Prop}$. Then $\mathfrak{F}^\clubsuit, x \vDash \neg\varphi^s$. By $\mathfrak{F}^\clubsuit \vDash L$, we have $\varphi^s \notin L$. □

Now we are ready to state and prove the third Post-completeness theorem.

3.3.4. Theorem (The third Post-completeness theorem). *A consistent tense logic* L *is Post-complete if and only if the following conditions hold:*

(1) for every $\psi \in \mathsf{Form}_t^0$, *if* $\neg\psi \notin L$, *then* $\Delta^{\leq n}\psi \in L$ *for some* $n \in \omega$; *and*

(2) for every $\varphi \in \mathsf{Form}_t$, *if* $\varphi \notin L$, *then* $\varphi^s \notin L$ *for some closed substitution* s.

Proof:
Assume $L \in \mathsf{PCOM}$. For (1), suppose that there exists $\psi \in \mathsf{Form}_t^0$ such that $\neg\psi \notin L$ and $\Delta^{\leq n}\psi \notin L$ for all $n \in \omega$. Now we show that $\Sigma = \{\nabla^{\leq n}\neg\psi : n \in \omega\}$ is L-consistent. Suppose not. There exist $n_1, \cdots, n_k \in \omega$ with $\neg(\nabla^{\leq n_1}\neg\psi \wedge \cdots \wedge \nabla^{\leq n_k}\neg\psi) \in L$, i.e, $\Delta^{\leq n_1}\psi \vee \cdots \vee \Delta^{\leq n_k}\psi \in L$. Clearly, if $1 \leq i \leq j \leq k$, then $\Delta^{\leq n_i}\psi \to \Delta^{\leq n_j}\psi \in L$. Let $h = max\{n_1, \cdots, n_k\}$. Then $\Delta^{\leq h}\psi \in L$ which contradicts $\Delta^{\leq h}\psi \notin L$. Hence, Σ is L-consistent. Let x be a maximal L-consistent set with $\Sigma \subseteq x$. Then $\mathbb{F}^L_x \vDash L$. By $\mathbb{F}^L_x \vDash \Sigma$, we have $\mathbb{F}^L_x \vDash \neg\psi$. Hence, $\mathbb{F}^L_x \vDash L \oplus \neg\psi$. Then $L \subsetneq \mathsf{Log}(\mathbb{F}^L_x)$ which contradicts $L \in \mathsf{PCOM}$. For (2), $\mathsf{Log}(\mathbb{F}^L) \subseteq \mathsf{Log}((\mathfrak{F}^L)^\clubsuit)$. By $L \in \mathsf{PCOM}$, $L = \mathsf{Log}(\mathbb{F}^L) \subseteq \mathsf{Log}((\mathfrak{F}^L)^\clubsuit) = L$. By Lemma 3.3.3, (2) holds.

Assume $L \notin \mathsf{PCOM}$. For a contradiction, suppose that both (1) and (2) hold. By the assumption, there exists a formula $\varphi \notin L$ such that $L' = L \oplus \varphi$ is consistent. By $\varphi \notin L$ and (2), $\varphi^s \notin L$ for some closed substitution s. Then $\neg\neg\varphi^s \notin L$. By (1), $\Delta^{\leq n}\neg\varphi^s \in L$ for some $n \in \omega$. Then $\Delta^{\leq n}\neg\varphi^s \in L'$. By $\varphi^s \in L'$, we have $\nabla^{\leq n}\varphi^s \in L'$ which contradicts $\Delta^{\leq n}\neg\varphi^s \in L'$. □

The third Post-completeness theorem (Theorem 3.3.4) provides a syntactic characterization of Post-completeness. We now turn to tense logics of general frames. Our first observation is that for any finite rooted frame, the tense logic of the 0-general frame based on it is always Post-complete. More precisely, we obtain the following corollary:

3.3.5. Corollary. *If* $\mathfrak{F} = (X, R) \in \mathsf{Fin}$ *is rooted, then* $\mathsf{Log}(\mathfrak{F}^\clubsuit)$ *is Post-complete.*

Proof:
Let $\mathfrak{F} = (X, R)$ be a finite rooted frame. It suffices to show that the conditions (1) and (2) in Theorem 3.3.4 hold. For (1), assume $\psi \in \mathsf{Form}^0_t$ and $\neg\psi \notin \mathsf{Log}(\mathfrak{F}^\clubsuit)$. Then there exists $x \in X$ with $\mathfrak{F}^\clubsuit, x \not\models \neg\psi$. Then $\mathfrak{F}^\clubsuit, x \models \psi$. Let $n = |X|$. Then $\mathfrak{F}^\clubsuit \models \Delta^{\leq n}\psi$. Moreover, (2) follows from Lemma 3.3.3. □

We could easily generalize Corollary 3.3.5 to infinite general frame:

3.3.6. Corollary. *Let* $\mathfrak{F} = (X, R)$ *be a rooted frame. If* $X = R^n_\sharp[x]$ *for some* $x \in X$ *and* $n \in \omega$*, then* $\mathsf{Log}(\mathfrak{F}^\clubsuit)$ *is Post-complete.*

Proof:
Suppose $X = R^n_\sharp[x]$ for some $x \in X$ and $n \in \omega$. It suffices to show that conditions (1) and (2) in Theorem 3.3.4 hold. For (1), assume $\psi \in \mathsf{Form}^0_t$ and $\neg\psi \notin \mathsf{Log}(\mathfrak{F}^\clubsuit)$. Then $\mathfrak{F}^\clubsuit, y \models \psi$ for some $y \in X$. By the assumption, $x \in R^n_\sharp[y]$ and so $R^{2n}_\sharp[y] = X$, which yields $\Delta^{\leq 2n}\psi \in \mathsf{Log}(\mathfrak{F}^\clubsuit)$. Note that (2) follows from Lemma 3.3.3. □

It was pointed out by an anonymous referee of [34] that our results contradict those of Kracht [71]. In the following remark we clarify this issue. For readers familiar with Kracht's work [71], our results may appear to contradict his. In the following remark, we clarify this issue.

3.3.7. Remark. Given a normal modal logic S and tense logic L, let S^+ be the smallest tense logic containing S, and L_+ be the normal modal logic $L \cap \mathsf{Form}_m$. This gives maps $(\cdot)^+ : \mathsf{NExt}(\mathsf{K}) \to \mathsf{NExt}(\mathsf{K}_t)$ and $(\cdot)_+ : \mathsf{NExt}(\mathsf{K}_t) \to \mathsf{NExt}(\mathsf{K})$. If S is Kripke complete, then $(S^+)_+ = S$. However, the map $((\cdot)^+)_+$ is in general not injective; there exists $S \in \mathsf{NExt}(\mathsf{K})$ such that $(S^+)_+ \neq S$ (see, e.g., [128, p.84]).

Recall that a tense logic L has *codimension* n for some $n \in \omega$ if there exists a chain $L = L_0 \subsetneq L_1 \subsetneq \cdots \subsetneq L_n = \mathsf{Form}_t$ which cannot be refined. Kracht [71, Corollary 16] claims that every extension of $\mathsf{K4}_t$ of finite codimension is Kripke complete and finitely alternative. However, this claim is incorrect. Here we give a counterexample. Consider the general frame $\mathfrak{N}^\clubsuit$ in Proposition 3.3.1. In fact, $\mathsf{Log}(\mathfrak{N}^\clubsuit) \in \mathsf{NExt}(\mathsf{K4}_t)$ is of codimension 1. Since $\mathfrak{N}$ is transitive and has no infinite descending chain, we have $\mathfrak{N}^\clubsuit \models \blacksquare(\blacksquare p \to p) \to \blacksquare p$. For every $\varphi \in \mathsf{Form}^0_t$, $V(\varphi) \in A^\clubsuit$ is either finite or cofinite. Then $\mathfrak{N}^\clubsuit \models \neg(\Diamond p \wedge \Box(p \to \Diamond(\neg p \wedge \Diamond p)))$. Every transitive frame validating $\neg(\Diamond p \wedge \Box(p \to \Diamond(\neg p \wedge \Diamond p)))$ does not contain infinite ascending chains. Every frame validating $\blacksquare(\blacksquare p \to p) \to \blacksquare p$ is irreflexive, i.e., contains no reflexive point. Then for every $\mathfrak{F} \in \mathsf{Fr}(\mathsf{Log}(\mathfrak{N}^\clubsuit))$, $\mathfrak{F} \models \Diamond\top \to \Diamond\Box\bot$. Clearly $\mathfrak{N}^\clubsuit \not\models \Diamond\top \to \Diamond\Box\bot$. Hence, $\mathsf{Log}(\mathfrak{N}^\clubsuit)$ is Kripke incomplete and so it is a counterexample for Kracht's claim. Furthermore, by checking Kracht's proof, we find that it is based on the following claims:

(K1) If $L \in \mathsf{NExt}(\mathsf{K4}_t)$ is of finite codimension, then L_+ is tabular.

(K2) For all $S \in \mathsf{NExt}(\mathsf{K})$ and $L \in \mathsf{NExt}(\mathsf{K}_t)$, $S \subseteq L_+$ if and only if $S^+ \subseteq L$.

Here the operations $(\cdot)_+$ and $(\cdot)^+$ are explained above. Note that (K2) holds. This is shown as follows. If $S \subseteq L_+ \subseteq L$, then $S \subseteq L$ and so $S^+ \subseteq L$. Assume $S^+ \subseteq L$. Let $\varphi \in S$. Then $\varphi \in S^+ \subseteq L$ and so $\varphi \in L \cap \mathsf{Form}_m = L_+$. Hence, $S \subseteq L_+$. However, the statement (K1) does not hold. For a contradiction, suppose (K1) holds. Consider again the logic $L = \mathsf{Log}(\mathfrak{N}^\clubsuit) \in \mathsf{PCOM}$. Obviously $\mathsf{alt}_n^+ \notin L$ for every $n \in \omega$. Then $L \oplus \mathsf{alt}_n^+ = \mathsf{Form}_t$ for all $n \in \omega$. By (K1), L_+ is tabular. Then $\mathsf{K} \oplus \mathsf{alt}_m \subseteq L_+$ for some $m \in \omega$. By (K2), $\mathsf{K}_t \oplus \mathsf{alt}_m^+ = (\mathsf{K} \oplus \mathsf{alt}_m)^+ \subseteq L$ which is impossible. Hence, (K1) does not hold.

Using the third Post-completeness theorem and its consequences, we can show some results on the Post-number of some tense logics. Recall that the Post-number $\mathsf{PN}(L)$ of a tense logic L is the number of Post-complete extensions of L. For tense logics L_1 and L_2, if $L_1 \subseteq L_2$, then $\mathsf{PN}(L_2) \leq \mathsf{PN}(L_1)$. In particular, since $|\mathsf{NExt}(\mathsf{K}_t)| = 2^{\aleph_0}$, we have $\mathsf{PN}(L_1) = 2^{\aleph_0}$ implies $\mathsf{PN}(L_2) = 2^{\aleph_0}$. Let us consider tense logics in Table 3.1.

$\mathsf{D}_t^+ = \mathsf{K}_t \oplus \Diamond\top$	$\mathsf{D}_t^- = \mathsf{K}_t \oplus \blacklozenge\top$
$\mathsf{D}_t = \mathsf{K}_t \oplus \Diamond\top \wedge \blacklozenge\top$	$\mathsf{B}_t = \mathsf{K}_t \oplus p \to \Box\Diamond p$
$\mathsf{T}_t = \mathsf{K}_t \oplus p \to \Diamond p$	$\mathsf{K4}_t = \mathsf{K}_t \oplus \Diamond\Diamond p \to \Diamond p$
$\mathsf{D4}_t^+ = \mathsf{D}_t^+ \oplus \Diamond\Diamond p \to \Diamond p$	$\mathsf{D4}_t^- = \mathsf{D}_t^- \oplus \Diamond\Diamond p \to \Diamond p$

Table 3.1: Tense logics axiomatized by $\mathsf{4}$, D, B and T

As we will see in Chapter 4, for every tense logic L listed in Table 3.1, we have $|\mathsf{NExt}(L)| = 2^{\aleph_0}$. However, as the following propositions show, their Post-numbers differ significantly.

3.3.8. Proposition. $\mathsf{PN}(\mathsf{D}_t) = \mathsf{PN}(\mathsf{T}_t) = 1$ *and* $\mathsf{PN}(\mathsf{B}_t) = 2$.

Proof:
(1) Let $L \in \mathsf{PCOM} \cap \mathsf{NExt}(\mathsf{D}_t)$. Let $\mathbb{F} = (X, R, A)$ be a general frame with $\mathbb{F} \models L$. Then $\mathbb{F} \models \Diamond\top \wedge \blacklozenge\top$. Hence, $R[x] \neq \varnothing \neq R^{-1}[x]$ for every $x \in X$. Let $\mathbb{F}_\circ = (\{\circ\}, \{(\circ, \circ)\}, \{\varnothing, \{\circ\}\})$. Let $f : X \to \{\circ\}$ be the function with $f(x) = \circ$

for all $x \in X$. The reader can readily check that f is a t-morphism from $\mathbb{F}$ to $\mathbb{F}_\circ$. Then $\mathbb{F}_\circ \vDash L$. Hence, $L = \mathsf{Log}(\mathbb{F}_\circ) = \mathsf{K}_t \oplus p \leftrightarrow \Box p$. By Theorem 3.2.13, $\mathsf{Log}(\mathbb{F}_\circ)$ is Post-complete. Then $\mathsf{PCOM} \cap \mathsf{NExt}(\mathsf{D}_t) = \{\mathsf{Log}(\mathbb{F}_\circ)\}$ and so $\mathsf{PN}(\mathsf{D}_t) = 1$. By $\mathsf{D}_t \subseteq \mathsf{T}_t$, we see $\mathsf{PN}(\mathsf{T}_t) \leq \mathsf{PN}(\mathsf{D}_t) = 1$.

(2) Let $L \in \mathsf{PCOM} \cap \mathsf{NExt}(\mathsf{B}_t)$. Let $\mathbb{F} = (X, R, A)$ be a rooted general frame with $\mathbb{F} \vDash L$. Then $L = \mathsf{Log}(\mathbb{F})$. Now we have two cases:

(a) $R = \varnothing$. Since $\mathbb{F}$ is rooted, $X = \{\bullet\}$ and $A = \{\varnothing, \{\bullet\}\}$. Hence, $L = \mathsf{K}_t \oplus \Box\bot$.

(b) $R \neq \varnothing$. Then $R[x] \cup R^{-1}[x] \neq \varnothing$ for every $x \in X$. Note that $\Diamond\top \leftrightarrow \blacklozenge\top \in \mathsf{B}_t \subseteq L$. Thus, $R[x] \neq \varnothing$ and $R^{-1}[x] \neq \varnothing$ for every $x \in X$. Hence, $\Diamond\top, \blacklozenge\top \in L$, i.e., $\mathsf{D}_t \subseteq L$. By the proof of (1), $L = \mathsf{K}_t \oplus p \leftrightarrow \Box p$.

Hence, $\mathsf{PCOM} \cap \mathsf{NExt}(\mathsf{B}_t) = \{\mathsf{K}_t \oplus p \leftrightarrow \Box p, \mathsf{K}_t \oplus \Box\bot\}$ and so $\mathsf{PN}(\mathsf{B}_t) = 2$. □

3.3.9. Proposition. *$\mathsf{PN}(\mathsf{D4}_t^+) = \mathsf{PN}(\mathsf{D4}_t^-) = 2^{\aleph_0}$ and hence $\mathsf{PN}(\mathsf{D}_t^+) = \mathsf{PN}(\mathsf{D}_t^-) = \mathsf{PN}(\mathsf{K4}_t) = 2^{\aleph_0}$.*

Proof:
(1) $\mathsf{PN}(\mathsf{D4}_t^+) = 2^{\aleph_0}$. For every subset $I \subseteq \omega\backslash\{0,1\}$, let $I^* = \{i^* : i \in I\}$. Let $\mathfrak{F}_I = (X_I, R_I)$ be the frame where $X_I = \omega \cup I^*$ and $R_I = \{(n,m) \in \omega \times \omega : n < m\} \cup \{(n^*, m) \in I^* \times \omega : n \leq m\}$. (Examples of frames $\mathfrak{F}_\varnothing, \mathfrak{F}_{\{2,5\}}$ and $\mathfrak{F}_\mathbb{P}$ where $\mathbb{P}$ is the set of prime numbers are presented in Figure 3.5.) Clearly, $X_I = (R_I)^2_\sharp[0]$ for every $I \subseteq \omega$.

For every $I \subseteq \omega\backslash\{0,1\}$, let $L_I = \mathsf{Log}(\mathfrak{F}_I^\clubsuit)$. Recall that for every general frame $\mathbb{F} = (X, R, A)$, $\mathbb{F} \vDash \mathsf{D4}_t^+$ if and only if $R[x] \neq \varnothing$ and $R[R[x]] \subseteq R[x]$ for all $x \in X$. Then it is clear that for every $I \subseteq \omega$, $\mathfrak{F}_I^\clubsuit \vDash \mathsf{D4}_t^+$ and so $L_I \in \mathsf{NExt}(\mathsf{D4}_t^+)$. Moreover, by Corollary 3.3.6, L_I is Post-complete. Thus, to show that $\mathsf{PN}(\mathsf{D4}_t^+) = 2^{\aleph_0}$, it suffices to show that $L_I \neq L_J$ when $I \neq J$.

Suppose $I \neq J$. Let $i \in I \setminus J$ without loss of generality. By $\mathfrak{F}_I^\clubsuit, i \vDash \blacksquare^{i+1}\bot$ and $i^* R_I i$, we have $\mathfrak{F}_I^\clubsuit, i^* \vDash \Diamond\blacksquare^{i+1}\bot$. Note that $R_I[i^*] = \{k \in \omega : k \geq i\}$. Then $\mathfrak{F}_I^\clubsuit, k \vDash \blacklozenge^i\top$ for each $k \in R_I[i^*]$. Note that ${R_I}^{-1}[i^*] = \varnothing$. Then $\mathfrak{F}_I^\clubsuit, i^* \vDash \blacksquare\bot \wedge \Diamond\blacksquare^{i+1}\bot \wedge \Box\blacklozenge^i\top$. By $X_I = (R_I)^2_\sharp[i^*]$, we have $\Delta^{\leq 2}(\blacksquare\bot \wedge \Diamond\blacksquare^{i+1}\bot \wedge \Box\blacklozenge^i\top) \in L_I$.

Now we show that $\neg(\blacksquare\bot \wedge \Diamond\blacksquare^{i+1}\bot \wedge \Box\blacklozenge^i\top) \in L_J$. Towards a contradiction, suppose $\neg(\blacksquare\bot \wedge \Diamond\blacksquare^{i+1}\bot \wedge \Box\blacklozenge^i\top) \notin L_J$. Then there exists $y \in X_J$ with $\mathfrak{F}_J^\clubsuit, y \vDash \blacksquare\bot \wedge \Diamond\blacksquare^{i+1}\bot \wedge \Box\blacklozenge^i\top$. Then ${R_J}^{-1}(y) = \varnothing$ which yields $y \in J^* \cup \{0\}$. There are two cases:

(a) $y = j^*$ for some $j > i$. Then $R_J[y] \subseteq \{l \in \omega : l > i\}$ and so $\mathfrak{F}_J^\clubsuit, y \nvDash \Diamond\blacksquare^{i+1}\bot$, which contradicts the assumption.

(b) $y \in \{l^* : l < i\} \cup \{0\}$. Note that $\mathfrak{F}_J^\clubsuit, i-1 \vDash \blacksquare^i\bot$ and $yR_J(i-1)$, we have $\mathfrak{F}_J^\clubsuit, y \vDash \Diamond\blacksquare^i\bot$, which contradicts $\mathfrak{F}_J^\clubsuit, y \vDash \Box\blacklozenge^i\top$.

Hence, $\neg(\blacksquare\bot \wedge \Diamond\blacksquare^{i+1}\bot \wedge \Box\blacklozenge^i\top) \in L_J$, which entails $\nabla^{\leq 2}\neg(\blacksquare\bot \wedge \Diamond\blacksquare^{i+1}\bot \wedge \Box\blacklozenge^i\top) \in L_J$ and so $\Delta^{\leq 2}(\blacksquare\bot \wedge \Diamond\blacksquare^{i+1}\bot \wedge \Box\blacklozenge^i\top) \in L_I \setminus L_J$. Hence, $L_I \neq L_J$.

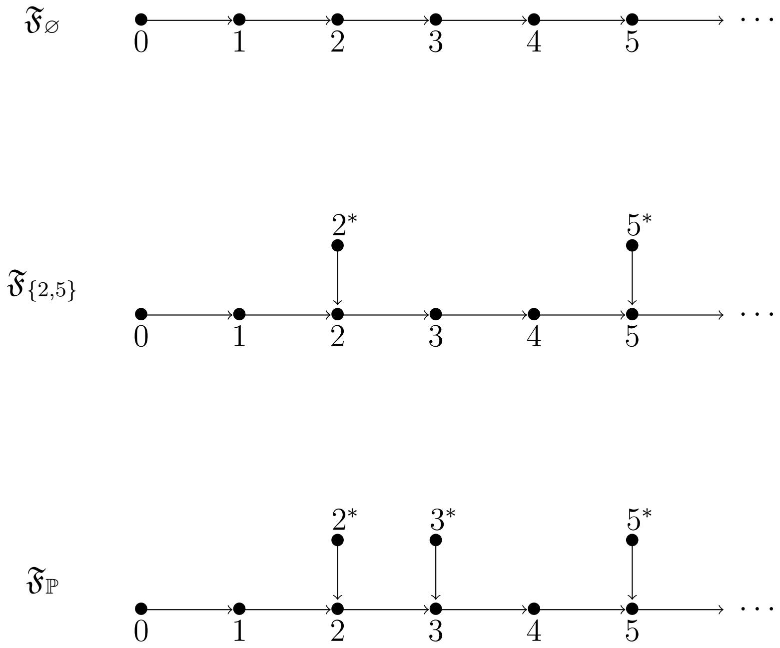

Figure 3.5: Frames $\mathfrak{F}_{\varnothing}, \mathfrak{F}_{\{2,5\}}$ and $\mathfrak{F}_{\mathbb{P}}$

(2) $\mathsf{PN}(\mathsf{D4}_t^-) = 2^{\aleph_0}$. The proof is similar to (1). It suffices to observe that for each $I \subseteq \omega \setminus \{0, 1\}$, $\mathfrak{F}_I = (X_I, (R_I)^{-1})$ is a frame for $\mathsf{D4}_t^-$. ☐

Let us conclude this section with the following anti-dichotomy theorem for Post-numbers of tense logic:

3.3.10. Theorem. *For every κ such that $0 < \kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, there exists a consistent tense logic $L \in \mathsf{NExt}(\mathsf{K}_t)$ such that $\mathsf{PN}(L) = \kappa$.*

Proof:
Recall that $\mathsf{Log}(\mathfrak{Ch}_j^<) \in \mathsf{PCOM}$ for all $j \in \mathbb{Z}^+$. For every $n \in \mathbb{Z}^+$, let $L_n = \mathsf{Log}(\{\mathfrak{Ch}_i^< : 1 \leq i \leq n\})$. Then we see that $\mathsf{PN}(L_n) = n$. Moreover, we have that $\mathsf{PN}(\bigcap_{i \in \omega} \mathsf{Log}(\mathfrak{Ch}_i^<)) = \aleph_0$. By Theorem 3.3.9, $\mathsf{PN}(\mathsf{D4}_t^+) = 2^{\aleph_0}$. ☐

3.4 Summary

In this chapter, we obtained a series of results on tabularity and Post-completeness in tense logic. In particular, we established a new characterization of tabularity, two characterization theorems for Post-completeness in tabular tense logic, and a characterization of Post-completeness in the lattice of all tense logics. We also determined the Post-numbers of several tense logics and proved an anti-dichotomy theorem for Post-numbers.

In the next chapter, we are going to study the boundary of tabular tense logics, namely, the pretabular tense logics.

Chapter 4

Pretabularity in Tense Logics

In this chapter, which is based on [33], we further investigate tabularity by examining the boundary of tabular tense logics, namely, *pretabular* tense logics. Recall that a logic is *pretabular* if it is not tabular but all of its proper consistent extensions are tabular. For any normal modal and tense logic L, let $\mathsf{PTAB}(L)$ denote the set of all pretabular tense logics in $\mathsf{NExt}(L)$. In general, given a lattice $\mathcal{L}$ of logics, tabularity in $\mathcal{L}$ is decidable if the following holds:

(i) every non-tabular logic has a pretabular extension;

(ii) there are finitely many pretabular logics in $\mathcal{L}$;

(iii) every pretabular extension of $\mathcal{L}$ is decidable.

Indeed, by (i), a finitely axiomatizable logic $L \in \mathcal{L}$ is non-tabular if and only if $L \subseteq L'$ for some pretabular logic $L' \in \mathcal{L}$, and the latter condition is decidable by (ii) and (iii). Consider the intuitionistic proposition logic IPC the lattice $\mathsf{Ext}(\mathsf{IPC})$ of all intermediate logics. Kuznetsov [76] showed that (i) holds for $\mathsf{Ext}(\mathsf{IPC})$ and every pretabular extension of IPC has the FMP. Maksimova [89] provided a full characterization of pretabular intermediate logics, which entails that there exist exactly 3 pretabular superintuitionistic logics and all of them are finitely axiomatizable. Combining the results above, we see that tabularity in $\mathsf{Ext}(\mathsf{IPC})$ is decidable in the following sense:

Given any $\varphi \in \mathsf{Form}$, one can effectively decide whether $\mathsf{IPC} \oplus \varphi$ is tabular.

Similar results have been obtained for the modal logic $\mathsf{S4}$: Esakia and Meskhi [44] and Maksimova [90] provided independently full characterizations of $\mathsf{PTAB}(\mathsf{S4})$ and showed that $|\mathsf{PTAB}(\mathsf{S4})| = 5$. On the other hand, Blok [17] showed that there exist $2^{\aleph_0}$ pretabular logics extending $\mathsf{K4}$, which means that the lattice $\mathsf{NExt}(\mathsf{K4})$ is highly complex, and the decidability of tabularity cannot be established via a complete characterization of pretabular logics.

In this chapter, we study pretabular tense logics in $\mathsf{NExt}(\mathsf{S4}_t)$. Our aim is to determine the cardinality of $\mathsf{PTAB}(\mathsf{S4}_t)$. As indicated by Kracht [71], the lattice $\mathsf{NExt}(\mathsf{S4}_t)$ is far more complex than its modal counterpart $\mathsf{NExt}(\mathsf{S4})$. We therefore begin by studying sublattices of $\mathsf{NExt}(\mathsf{S4}_t)$. For modal logics, transitive modal logics with bounded forth-width and depth have been thoroughly studied, see [50, 53, 29]. In the tense setting, however, the presence of a past-looking modality motivates to study logics not only with bounded depth and forth-width, but also with bounded back-width and *degree of reachability* (*r-degree* for short). Intuitively, a tense logic L is of *r-degree* l if every rooted frames for L can be generated within l back-and-forth steps from any point. As we shall see later, the r-degree is a crucial parameter for tense logics.

For all $n, m, k, l \in \mathbb{Z}^+ \cup \{\omega\}$, we define $\mathsf{S4BP}^{k,l}_{n,m}$ to be the tense logic of all frames with forth-width, back-width, depth and r-degree no more than n, m, k and l, respectively. We give first full characterizations of $\mathsf{PTAB}(\mathsf{S4BP}^{k,l}_{n,m})$ where n, m, k, l are finite. Then we move to lattices $\mathsf{NExt}(\mathsf{S4BP}^{k,l}_{n,m})$ where some of the parameters are infinite. The first logic we consider is $\mathsf{S4BP}^{\omega,1}_{1,1}$, where the depth is unbounded and all other parameters are bounded by 1. Note that $\mathsf{S4BP}^{\omega,1}_{1,1}$ is exactly the tense logic $\mathsf{S4.3}_t$ of all linear frames. We give a full characterization of $\mathsf{PTAB}(\mathsf{S4.3}_t)$ and prove that $|\mathsf{PTAB}(\mathsf{S4.3}_t)| = 5$. Moreover, we show that every logic in $\mathsf{PTAB}(\mathsf{S4.3}_t)$ is finitely axiomatizable and has the FMP. As a corollary, tabularity in $\mathsf{NExt}(\mathsf{S4.3}_t)$ is decidable. Then we turn our attention to the lattice $\mathsf{NExt}(\mathsf{S4BP}^{2,\omega}_{2,2})$, where the logic Ga is proved to be the unique pretabular logic of infinite r-degree. We provide a characterization of the rooted frames of Ga and identify an error in the characterization of $\mathsf{NExt}(\mathsf{Ga})$ given in [71]. Moreover, we provide a full characterization of $\mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,2})$ and show that every logic in $\mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,2})$ has the FMP. It follows from the characterization that $|\mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,2})| = \aleph_0$.

Based on the above, we can already observe a key difference between $\mathsf{PTAB}(\mathsf{S4})$ and $\mathsf{PTAB}(\mathsf{S4}_t)$: the former is finite, whereas the latter is infinite. Moreover, there exist pretabular tense logics, such as Ga, whose modal fragment are not pretabular. Next, in order to address the problem of determining the cardinality of $\mathsf{PTAB}(\mathsf{S4}_t)$, we move from the lattice $\mathsf{NExt}(\mathsf{S4BP}^{2,\omega}_{2,2})$ to $\mathsf{NExt}(\mathsf{S4BP}^{2,\omega}_{2,3})$, where the bound of back-width is increased from 2 to 3. The main result we obtain is that $|\mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,3})| = 2^{\aleph_0}$. To prove this result, we introduce the generalized Thue-Morse sequences, generalized Jankov formulas and local t-morphisms. It follows that the following anti-dichotomy theorem for the cardinality of pretabular extensions in $\mathsf{NExt}(\mathsf{S4}_t)$ holds:

For all $\kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, there exists $L \in \mathsf{NExt}(\mathsf{S4}_t)$ such that $|\mathsf{PTAB}(L)| = \kappa$.

Consequently, we determine that the cardinality of $\mathsf{PTAB}(\mathsf{S4}_t)$ is $2^{\aleph_0}$. This gives a full solution to the problem presented in [107]. At the same time, this result also indicates that decidability of tabularity in $\mathsf{NExt}(\mathsf{S4}_t)$ cannot be established via pretabular logics.

This chapter is structured as follows. Section 4.1 introduces generalized Jankov formulas and local t-morphisms. In Section 4.2, we present tense logics $\mathsf{S4BP}^{k,l}_{n,m}$ with bounded parameters including depth, width and r-degree; and studies their basic properties. Section 4.3 gives a characterization of $\mathsf{PTAB}(\mathsf{S4BP}^{k,l}_{n,m})$ for each $n, m, k, l \in \mathbb{Z}^+$. In Section 4.4 we provide a characterization of $\mathsf{PTAB}(\mathsf{S4.3}_t)$. Section 4.5 gives a characterization of $\mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,2})$ and proves the anti-dichotomy theorem for the cardinality of pretabular logics extending $\mathsf{S4BP}^{2,\omega}_{2,2}$. It also highlights the connection between Kracht's work on the lattice $\mathsf{NExt}(\mathsf{Ga})$ in [71, Section 4] and our results. Section 4.6 introduces generalized Thue-Morse sequences and defines a continual family of pretabular logics in $\mathsf{NExt}(\mathsf{S4BP}^{2,\omega}_{2,3})$, which shows that $|\mathsf{PTAB}(\mathsf{S4}_t)| = 2^{\aleph_0}$. Section 4.7 summarize this chapter.

4.1 Generalized Jankov Formulas and Local t-morphisms

Jankov formulas are widely used in the study of lattices of intermediate logics and modal logics (see [63, 40, 72, 108, 133]). Fine [50] developed frame formulas for finite rooted $\mathsf{S4}$-frames, which are similar to Jankov formulas for finite subdirectly irreducible Heyting algebras. For a survey of Jankov formulas and Fine formulas, we refer to [7]. The Fine formula for $\mathfrak{F}$ is refuted by an $\mathsf{S4}$-frame $\mathfrak{G}$ if and only if $\mathfrak{F}$ is a p-morphic image of a generated subframe of $\mathfrak{G}$. In this section, we introduce the generalized Jankov formulas for image-finite Kripke frames and local t-morphisms, which will be one of the main tools we use in this chapter. Given any rooted image-finite frame $\mathfrak{F} = (X, R)$, $x \in X$ and $k \in \mathbb{Z}^+$, we associate the generalized Jankov formula $\mathcal{J}^k(\mathfrak{F}, x)$ to it. We show that for any frame $\mathfrak{G} = (Y, S)$ and $y \in Y$, $(\mathfrak{G}, y)$ validates $\neg\mathcal{J}^k(\mathfrak{F}, x)$ if and only if there is no k-t-morphism $f : (\mathfrak{G}, y) \to^k (\mathfrak{F}, x)$. This generalizes Jankov formulas for finite algebras or finite rooted frames defined in [40, 50].

Recall that a general frame $\mathbb{F} = (X, R, A) \in \mathsf{GFr}$ is said to be *image-finite* if $|R_\sharp[x]| < \aleph_0$ for all $x \in X$. By Lemma 2.2.40(1), $R^k_\sharp[x]$ is finite for all $k \in \omega$. Now we introduce generalized Jankov formulas for image-finite frames.

4.1.1. Definition. Let $\mathfrak{F} = (X, R)$ be an image-finite frame and $x \in X$. Let $k \in \mathbb{Z}^+$ and $\langle x_i : i \in n\rangle$ be an enumeration of $R^k_\sharp[x]$ where $x = x_0$. Then the formula $\mathcal{J}^k(\mathfrak{F}, x)$ is defined to be the conjunction of the following formulas:

(1) $p_0 \wedge \nabla^{\leq k}(p_0 \vee \cdots \vee p_{n-1})$

(2) $\nabla^{\leq k}(p_i \to \neg p_j)$, for all $i \neq j$

(3) $\nabla^{\leq k-1}((p_i \to \Diamond p_j) \wedge (p_j \to \blacklozenge p_i))$, for all Rx_ix_j

(4) $\nabla^{\leq k-1}((p_i \to \neg\Diamond p_j) \wedge (p_j \to \neg\blacklozenge p_i))$, for all $x_j \notin R[x_i]$

$\mathcal{J}^k(\mathfrak{F}, x)$ is called the *Jankov formula of* $(\mathfrak{F}, x)$ *of degree* k.

Intuitively, $\mathcal{J}^k(\mathfrak{F}, x)$ is a formula that describes the subframe $\mathfrak{F}{\upharpoonright}R_\sharp^k[x]$ of an image-finite frame $\mathfrak{F}$, while the classical Jankov formula describes its corresponding finite frames. Let V be a valuation in $\mathfrak{F}$ such that $V(p_i) = \{x_i\}$ for all $i \in n$. Then it is not hard to check that $\mathfrak{F}, V, x \vDash \mathcal{J}^k(\mathfrak{F}, x)$. Thus, $\mathfrak{F}, x \nvDash \neg\mathcal{J}^k(\mathfrak{F}, x)$ for any $k \in \mathbb{Z}^+$. If $X = R_\sharp^k[x]$, then $\mathfrak{F}$ is finite and the formula $\mathcal{J}^k(\mathfrak{F}, x)$ captures all the information about $\mathfrak{F}$. In this sense, our formulas generalize Jankov formulas.

Next, we introduce *local t-morphisms* of frames:

4.1.2. Definition. Let $\mathfrak{F} = (X, R)$ and $\mathfrak{F}' = (X', R')$ be frames, $x \in X$ and $x' \in X'$. For each $k \in \mathbb{Z}^+$, a partial function $f : X \to X'$ is called a *k-t-morphism* from $(\mathfrak{F}, x)$ to $(\mathfrak{F}', x')$ (notation: $f : (\mathfrak{F}, x) \to^k (\mathfrak{F}', x')$), if $\mathsf{dom}(f) \supseteq R_\sharp^k[x]$, $f(x) = x'$ and

for all $y \in R_\sharp^{k-1}[x]$, we have $f[R[y]] = R'[f(y)]$ and $f[R^{-1}[y]] = R'^{-1}[f(y)]$.

We call $f : X \to X'$ a *local t-morphism* if it is a k-t-morphism for some $k \in \mathbb{Z}^+$. Moreover, $f : (\mathfrak{F}, x) \to^k (\mathfrak{F}', x')$ is said to be *full* if $\mathsf{ran}(f) \supseteq (R'_\sharp)^\omega[x']$.

As the reader might have noticed, a local t-morphism $f : (\mathfrak{F}, x) \to^k (\mathfrak{F}', x')$ is in some sense a 't-morphism for points in $R_\sharp^{k-1}[x]$'. By definition, f is a t-morphism from $\mathfrak{F}$ to $\mathfrak{F}'$ if $R_\sharp^{k-1}[x] = X$ and f is full. It is also not hard to verify that if $f : \mathfrak{F} \to \mathfrak{F}'$ is a t-morphism and $f(x) = x'$, then $f : (\mathfrak{F}, x) \to^k (\mathfrak{F}', x')$ for all $k \in \mathbb{Z}^+$.

Proposition 2.2.26 shows that t-morphisms preserve validity. We now show that k-t-morphisms also preserve validity, but only for formulas of modal depth at most k.

4.1.3. Lemma. *Let* $\mathfrak{F} = (X, R)$ *and* $\mathfrak{F}' = (X', R')$ *be frames,* $x \in X$*,* $x' \in X'$ *and* $k \in \mathbb{Z}^+$*. Suppose* $f : (\mathfrak{F}, x) \to^k (\mathfrak{F}', x')$*. Then for all* $\varphi \in \mathsf{Form}_t$ *with* $md(\varphi) \leq k$*,*

$$\mathfrak{F}, x \vDash \varphi \text{ implies } \mathfrak{F}', x' \vDash \varphi.$$

Proof:
Suppose $\mathfrak{F}', V', x' \nvDash \varphi$. Let V be a valuation in $\mathfrak{F}$ such that $V(p) = f^{-1}[V'(p)]$ for all $p \in \mathsf{Prop}$. It suffices to prove the following claim:

4.1.4. Claim. *For all* $l \leq k$*,* $y \in R_\sharp^{k-l}[x]$ *and* $\psi \in \mathsf{Form}_t$ *with* $\mathsf{md}(\psi) \leq l$*,*

$$\mathfrak{F}, V, y \vDash \psi \text{ if and only if } \mathfrak{F}', V', f(y) \vDash \psi.$$

Proof:
By induction on l. The case $l = 0$ follows from the definition of V immediately. Let $l > 0$. By induction hypothesis, y and $f(y)$ agree on all formulas with modal depth at most $l-1$. Take any $\gamma \in \mathsf{Form}_t$ with $\mathsf{md}(\gamma) \leq l-1$. Suppose $\mathfrak{F}, V, y \vDash \blacklozenge\gamma$. Then $\mathfrak{F}, V, z \vDash \gamma$ for some $z \in R^{-1}[y]$. Since $z \in R_\sharp^{k-(l-1)}[x]$, by induction hypothesis, $\mathfrak{F}', V', f(z) \vDash \gamma$. Since $f(z) \in f[R^{-1}[y]] = R'^{-1}[f(y)]$, we obtain that $\mathfrak{F}', V', f(y) \vDash \blacklozenge\gamma$. Suppose $\mathfrak{F}', V', f(y) \vDash \blacklozenge\gamma$. Then $\mathfrak{F}', V', z' \vDash \gamma$ for some $z' \in R'^{-1}[f(y)]$. Since $R'^{-1}[f(y)] = f[R^{-1}[y]]$, there exists $z \in R^{-1}[y]$ such that $f(z) = z'$. Because $z \in R_\sharp^{k-(l-1)}[x]$, by induction hypothesis, $\mathfrak{F}, V, z \vDash \gamma$, which entails $\mathfrak{F}, V, y \vDash \blacklozenge\gamma$. Since $f[R[y]] = R'[f(y)]$, by a similar proof, it follows that $\mathfrak{F}, V, y \vDash \Box\gamma$ if and only if $\mathfrak{F}', V', f(y) \vDash \Box\gamma$. $\Box$

By Claim 4.1.4, since $x \in R_\sharp^0[x]$, $f(x) = x'$ and $\mathsf{md}(\varphi) \leq k$, we have $\mathfrak{F}, x \nvDash \varphi$. $\Box$

By Jankov's lemma [63], refutation of a Jankov formula means existence of particular p-morphisms. We now generalize Jankov's lemma by showing that if a generalized Jankov formula $\mathcal{J}^k(\mathfrak{G}, y)$ is refuted by some pointed frame $(\mathfrak{F}, x)$, then there exists a k-t-morphism from $(\mathfrak{F}, x)$ to $(\mathfrak{G}, y)$.

4.1.5. Lemma. *Let $\mathfrak{F} = (X, R)$ be a frame, $\mathfrak{G} = (Y, S)$ an image-finite frame, $x \in X$ and $y \in Y$. Then for all $k \in \mathbb{Z}^+$,*

$$\mathfrak{F}, x \nvDash \neg\mathcal{J}^k(\mathfrak{G}, y) \text{ if and only if there exists } f : (\mathfrak{F}, x) \to^k (\mathfrak{G}, y).$$

Proof:
Suppose $f : (\mathfrak{F}, x) \to^k (\mathfrak{G}, y)$. Since $\mathfrak{G}, y \nvDash \neg\mathcal{J}^k(\mathfrak{G}, y)$ and $\mathsf{md}(\mathcal{J}^k(\mathfrak{G}, y)) = k$, by Lemma 4.1.3, we have $\mathfrak{F}, x \nvDash \neg\mathcal{J}^k(\mathfrak{G}, y)$. Then there exists a valuation V in $\mathfrak{F}$ such that $\mathfrak{F}, V, x \nvDash \neg\mathcal{J}^k(\mathfrak{G}, y)$. Let $\langle y_i : i \in n\rangle$ be an enumeration of $S_\sharp^k[y]$ used in the definition of $\mathcal{J}^k(\mathfrak{G}, y)$. We define the function $f : R_\sharp^k[x] \to Y$ as follows:

$$\text{for all } z \in R_\sharp^k[x], \text{ we let } f(z) = y_i \text{ if and only if } z \vDash p_i.$$

Since $x \vDash \nabla^{\leq k}(p_0 \vee \cdots \vee p_{n-1})$ and $x \vDash \nabla^{\leq k}(p_i \to \neg p_j)$ for all $i \neq j$, we obtain that $R_\sharp^k[x] \subseteq \bigcup_{i \in n} V(p_i)$ and $R_\sharp^k[x] \cap V(p_i) \cap V(p_j) = \varnothing$ for all $i \neq j$. Thus, f is well-defined. It suffices to show that $f : (\mathfrak{F}, x) \to^k (\mathfrak{G}, y)$. Since $y = y_0$ and $x \vDash p_0$, $f(x) = y$. Take any $z \in R_\sharp^{k-1}[x]$. Then $f(z) = y_i$ and $z \vDash p_i$ for some $i \in n$. For all $y_j \in Y$, we have

$$y_j \in S[f(z)] \Longrightarrow x \vDash \nabla^{\leq k-1}(p_i \to \Diamond p_j) \Longrightarrow z \vDash \Diamond p_j \Longrightarrow y_j \in f[R[z]];$$

and we also have

$$y_j \notin S[f(z)] \Longrightarrow x \vDash \nabla^{\leq k-1}(p_i \to \neg\Diamond p_j) \Longrightarrow z \vDash \neg\Diamond p_j \Longrightarrow y_j \notin f[R[z]].$$

Thus, we obtain that $f[R[z]] = S[f(z)]$. Similarly, the reader can check that $f[R^{-1}[z]] = S^{-1}[f(z)]$. Hence, $(\mathfrak{F}, x) \to^k (\mathfrak{G}, y)$. □

For the purposes of this thesis, we are primarily interested in full local t-morphisms. By definition, to verify that a local t-morphism $f : (\mathfrak{F}, x) \to^k (\mathfrak{G}, y)$ is full, one needs to show that $\mathsf{ran}(f) \supseteq S_\sharp^\omega[y]$, where $\mathfrak{G} = (Y, S)$. However, we are not always able to obtain direct information about $S_\sharp^\omega[y]$. Thus, in what follows, we provide a sufficient condition (Lemma 4.1.7) for showing that f is full, one that requires information only about f itself. To this end, we first introduce the notion of *sufficient local t-morphisms*.

4.1.6. Definition. Let $f : (\mathfrak{F}, x) \to^k (\mathfrak{F}', x')$ be a local t-morphism. Then a subset $Y \subseteq R_\sharp^{k-1}[x]$ is called *sufficient* if $f[R_\sharp[y]] \subseteq f[Y]$ for all $y \in Y$. We say that f is *sufficient* if there is a nonempty sufficient set $Y \subseteq \mathsf{dom}(f)$.

4.1.7. Lemma. *Let $\mathfrak{F} = (X, R)$ and $\mathfrak{F}' = (X', R')$ be frames, $x \in X$ and $x' \in X'$. Suppose $k \in \mathbb{Z}^+$ and $f : (\mathfrak{F}, x) \to^k (\mathfrak{F}', x')$ is sufficient. Then f is full.*

Proof:
Suppose $Y \subseteq \mathsf{dom}(f)$ is sufficient and $y \in Y$. We show by induction on n that $(R')_\sharp^n[f(y)] \subseteq f[Y]$ for all $n \in \omega$. The case $n = 0$ is trivial. Let $n > 0$. Take any $z' \in (R')_\sharp^{n-1}[f(y)]$. By induction hypothesis, $f(z) = z'$ for some $z \in Y$. Since Y is sufficient, we have that $(R'_\sharp)[z'] = f[R_\sharp[z]] \subseteq f[Y]$. Thus,

$$\bigcup_{z' \in (R')_\sharp^{n-1}[f(y)]} (R')_\sharp[z'] \subseteq f[Y],$$

which entails $(R')_\sharp^n[f(y)] \subseteq f[Y]$. Hence, $(R')_\sharp^\omega[f(y)] \subseteq f[Y] \subseteq \mathsf{ran}(f)$. □

As a corollary, we obtain

4.1.8. Theorem. *Let $\mathfrak{F} = (X, R)$ and $\mathfrak{F}' = (X', R')$ be rooted frames, $x \in X$ and $x' \in X'$. Let $k \in \mathbb{Z}^+$ and $f : (\mathfrak{F}, x) \to^k (\mathfrak{F}', x')$. If $R_\sharp^{k-1}[x] = X$, then $f : \mathfrak{F} \twoheadrightarrow \mathfrak{F}'$ is a surjective t-morphism.*

Proof:
Since $f : (\mathfrak{F}, x) \to^k (\mathfrak{F}', x')$ and $R_\sharp^{k-1}[x] = X$, we see that $f : X \to X'$ is a t-morphism. Since $X \subseteq R_\sharp^{k-1}[x]$ is sufficient, by Lemma 4.1.7, f is full. Note that $\mathfrak{F}'$ is rooted. Thus, $f : \mathfrak{F} \twoheadrightarrow \mathfrak{F}'$ is a surjective t-morphism. □

By Lemma 4.1.5 and Theorem 4.1.8, the following theorem holds:

4.1.9. Theorem. *Let $k \in \mathbb{Z}^+$, $\mathfrak{F} = (X, R)$ be a rooted frame, $\mathfrak{G} = (Y, S)$ a rooted image-finite frame and $y \in Y$. Suppose $X = R_\sharp^{k-1}[x]$ for all $x \in X$. Then*

$$\mathfrak{F} \twoheadrightarrow \mathfrak{G} \text{ if and only if } \mathfrak{F} \nvDash \neg \mathcal{J}^k(\mathfrak{G}, y).$$

4.2 Tense Logics over $\mathsf{S4}_t$ with Bounded Parameters

Recall that $\mathsf{S4}_t$ is the tense logic of pre-orders. As was stated in the introduction of this chapter, our aim is to determine the cardinality of $\mathsf{PTAB}(\mathsf{S4}_t)$. Thus, from this section onward, we focus on tense logics in $\mathsf{NExt}(\mathsf{S4}_t)$. Unless otherwise specified, general frames are always assumed to be reflexive and transitive.

To study a lattice with a complex structure, it is often useful to investigate its simpler sublattices. For example, a better understanding of the lattice $\mathsf{NExt}(\mathsf{K4})$ of transitive modal logics can be obtained by studying transitive modal logics with bounded depth and width. Segerberg [115] proved that every modal logic of finite depth enjoys the FMP and Fine [52] showed that every modal logic of finite width is Kripke complete.

Inspired by these results, in this section we introduce *tense logics with bounded parameters*, that is, extensions of $\mathsf{S4}_t$ with bounded *depth*, *forth-width*, *back-width* and *r-degree*. We investigate their basic properties and study their pretabular extensions.

Let us start by introducing *chains* in general frames and the notions of depth, forth-width and back-width of general frames.

4.2.1. Definition. Let $\mathbb{F} = (X, R, A)$ be a general frame for $\mathsf{S4}_t$, α an ordinal and $\mathcal{Y} = \langle y_i \in X : i < \alpha \rangle$ a sequence of elements in X. Then

- $\mathcal{Y}$ is called a *chain* in $\mathbb{F}$ if $Ry_\lambda y_\gamma$ for all $\lambda < \gamma < \alpha$;
- $\mathcal{Y}$ is called a *strict chain* in $\mathbb{F}$ if it is a chain and $y_\lambda \notin R[y_\gamma]$ for all $\lambda < \gamma < \alpha$;
- $\mathcal{Y}$ is called a (strict) *co-chain* in $\mathbb{F}$ if it is a (strict) chain in $\mathbb{F}^{-1}$;
- $\mathcal{Y}$ is called an *anti-chain* in $\mathbb{F}$ if $y_\lambda \notin R[y_\gamma]$ for all $\lambda \neq \gamma < \alpha$.

The length $l(\mathcal{Y})$ of a strict chain $\mathcal{Y} = \langle y_i \in X : i < \alpha \rangle$ is defined to be α.

Let $n \in \mathbb{Z}^+$. We say that $x \in X$ is of *depth* n (notation: $\mathsf{dep}(x) = n$), if there exists a strict chain $\mathcal{Y}$ in $\mathbb{F}{\upharpoonright}R[x]$ with $l(\mathcal{Y}) = n$ and there is no strict chain of greater length. Otherwise, x is said to be of infinite depth and we write $\mathsf{dep}(x) = \aleph_0$. We define the *depth* $\mathsf{dep}(\mathbb{F})$ *of* $\mathbb{F}$ by $\mathsf{dep}(\mathbb{F}) = \sup\{\mathsf{dep}(x) : x \in X\}$. Let L be a tense logic. Then the *depth* $\mathsf{dep}(L)$ of L is defined to be $\sup\{\mathsf{dep}(\mathbb{F}) : \mathbb{F} \in \mathsf{RFr}(L)\}$. We say that a tense logic L (a general frame $\mathbb{F}$) is of depth n if $\mathsf{dep}(L) = n$ ($\mathsf{dep}(\mathbb{F}) = n$).

We say that $x \in X$ is of *forth-width* n (notation: $\mathsf{wid}^+(x) = n$), if there exists an anti-chain $Y \subseteq R[x]$ with $|Y| = n$ and there is no anti-chain in $R[x]$ with greater size. Otherwise, we write $\mathsf{wid}^+(x) = \aleph_0$. Similarly, we define $\mathsf{wid}^+(\mathbb{F}) = \sup\{\mathsf{wid}^+(x) : x \in X\}$ and $\mathsf{wid}^+(L) = \sup\{\mathsf{wid}^+(\mathbb{F}) : \mathbb{F} \in \mathsf{RFr}(L)\}$.

We define the *back-width* of x in $\mathbb{F}$ to be the forth-width of x in $\mathbb{F}^{-1}$. The notations $\mathsf{wid}^-(x)$, $\mathsf{wid}^-(\mathbb{F})$ and $\mathsf{wid}^-(L)$ are defined analogously.

Intuitively, depth characterizes the maximal length of chains, while width captures the maximal size of anti-chains under some restriction.

Next, we introduce the notion of *reachability degree*, an important parameter for tense logics and, more generally, for polymodal logics. Intuitively, the reachability degree measures how many steps are needed to generate the whole frame from a single point. More precisely, we have the following definition:

4.2.2. Definition. Let $\mathbb{F} = (X, R, A) \in \mathsf{GFr}$ be a general frame and $x \in X$. Let $k \in \mathbb{Z}^+$. Then we say x is of *reachability-degree* (*r-degree*) k (notation: $\mathsf{rdg}(x) = k$), if $R_\sharp^{k-1}[x] \neq R_\sharp^k[x] = R_\sharp^\omega[x]$. Specially, $\mathsf{rdg}(x) = 0$ if $R_\sharp^\omega[x] = \{x\}$ and $\mathsf{rdg}(x) = \aleph_0$ if $R_\sharp^k[x] \neq R_\sharp^{k+1}[x]$ for any $k \in \omega$.

We define the *r-degree* $\mathsf{rdg}(\mathbb{F})$ of $\mathbb{F}$ by $\mathsf{rdg}(\mathbb{F}) = \sup\{\mathsf{rdg}(x) : x \in X\}$. For a tense logic L, we define the *r-degree* $\mathsf{rdg}(L)$ of L by $\mathsf{rdg}(L) = \sup\{\mathsf{rdg}(\mathbb{F}) : \mathbb{F} \in \mathsf{RFr}(L)\}$.

4.2.3. Remark. Recall that a normal modal logic L is said to be *pre-transitive* if there exists $n \in \omega$ such that $\Box^n p \to \Box^{n+1} p \in L$. In the tense setting, a tense logic of finite r-degree is similar to a pre-transitive modal logic in the sense that the existence of a master modality, say $\Delta^{\leq n}$, is guaranteed. As we will see later, a tense logic L is of finite r-degree if and only if $\Delta^{\leq n+1}p \to \Delta^{\leq n}p \in L$ for some $n \in \omega$. It is worth noting that the tense logic $\mathsf{S4}_t$ is not of finite r-degree.

Let us introduce formulas that bound the parameters mentioned above:

4.2.4. Definition. For each $n \in \mathbb{Z}^+$, we define formulas br_n, bw_n^+ and bw_n^- by

$$\begin{aligned}
\mathsf{br}_n &:= \Delta^{\leq n+1}p \to \Delta^{\leq n}p \\
\mathsf{bw}_n^+ &:= \bigwedge_{i \leq n} \Diamond p_i \to \bigvee_{i \neq j \leq n} \Diamond(p_i \wedge (p_j \vee \Diamond p_j)) \\
\mathsf{bw}_n^- &:= \bigwedge_{i \leq n} \blacklozenge p_i \to \bigvee_{i \neq j \leq n} \blacklozenge(p_i \wedge (p_j \vee \blacklozenge p_j))
\end{aligned}$$

Moreover, we recall the definition of bd_n for $n \in \mathbb{Z}^+$ (see, e.g., [29, Chapter 3]).

$$\begin{aligned}
\mathsf{bd}_1 &:= \Diamond\Box p_0 \to p_0 \\
\mathsf{bd}_{k+1} &:= \Diamond(\Box p_k \wedge \neg \mathsf{bd}_k) \to p_k
\end{aligned}$$

Specially, we define $\mathsf{bd}_\omega = \mathsf{br}_\omega = \mathsf{bw}_\omega^+ = \mathsf{bw}_\omega^- = \top$.

The following proposition shows that the formulas bd_n, bw_n^+, bw_n^-, and br_n bound the depth, forth-width, back-width, and r-degree of refined frames, respectively.

4.2.5. PROPOSITION. *Let $\mathbb{F} = (X, R, A) \in \mathsf{RFr}(\mathsf{S4}_t)$, $x \in X$ and $n \in \mathbb{Z}^+$. Then*

(1) $\mathbb{F}, x \models \mathsf{bd}_n$ *if and only if* $\mathsf{dep}(x) \leq n$.

(2) $\mathbb{F}, x \models \mathsf{bw}_n^+$ *if and only if* $\mathsf{wid}^+(x) \leq n$.

(3) $\mathbb{F}, x \models \mathsf{bw}_n^-$ *if and only if* $\mathsf{wid}^-(x) \leq n$.

(4) $\mathbb{F}, x \models \mathsf{br}_n$ *if and only if* $\mathsf{rdg}(x) \leq n$.

Proof:
The proofs of (1) - (3) are analogous to those of [29, Propositions 3.42 and 3.44], while the proof of (4) is similar to that of [29, Corollary 3.35]. □

4.2.6. REMARK. Proposition 4.2.5 holds for also refined frames for $\mathsf{K4}_t$, we leave the proofs to the reader.

The tense logic of rooted frame with uniformly bounded r-degree enjoys desirable property. In particular, finite rooted frames validating such logics can be obtained from the corresponding generating frames by taking generated subframes. More precisely, we have the following theorem.

4.2.7. THEOREM. *Let $\{\mathfrak{F}_i \in \mathsf{Fr}_\mathrm{r} : i \in I\}$ be a family of rooted frames and $k \in \omega$. Suppose $L = \mathsf{Log}(\{\mathfrak{F}_i : i \in I\})$ and $\mathsf{br}_k \in L$. Then $\mathsf{Fin}_\mathrm{r}(L) \subseteq \bigcup_{i\in I} \mathsf{M}_t(\mathfrak{F}_i)$.*

Proof:
Take any $\mathfrak{G} = (Y, S) \in \mathsf{Fin}_r(L)$ and $y \in Y$. Since $\mathfrak{G} \not\models \neg\mathcal{J}^{k+1}(\mathfrak{G}, y)$, it follows that $\neg\mathcal{J}^{k+1}(\mathfrak{G}, y) \notin L$. By $L = \mathsf{Log}(\{\mathfrak{F}_i : i \in I\})$, we have $\mathfrak{F}_i \not\models \neg\mathcal{J}^{k+1}(\mathfrak{G}, y)$ for some $i \in I$. Note that $\mathsf{br}_k \in L$ and $\mathfrak{F}_i \models L$. Thus, $\mathsf{rdg}(\mathfrak{F}_i) \leq k$. Then by Theorem 4.1.9, we obtain $\mathfrak{F}_i \twoheadrightarrow \mathfrak{G}$. Hence, $\mathfrak{G}$ is a t-morphic image of $\mathfrak{F}_i$. By arbitrariness of $\mathfrak{G}$, we obtain that $\mathsf{Fin}_\mathrm{r}(L) \subseteq \bigcup_{i\in I} \mathsf{M}_t(\mathfrak{F}_i)$. □

By the duality between finite tense algebras and finite frames, we obtain

4.2.8. THEOREM. *Let $\mathcal{K}$ be a family of subdirectly irreducible tense algebras and $\mathcal{K}'$ be the class of all finite members in $\mathsf{V}_\mathrm{si}(\mathcal{K})$. Suppose $\mathcal{K} \models \mathsf{br}_k$. Then $\mathcal{K}' \subseteq \mathsf{S}(\mathcal{K})$.*

Now we are ready to define the notion of tense logics with bounded parameters.

4.2.9. DEFINITION. Let $k, l, n, m \in \mathbb{Z}^+ \cup \{\omega\}$. We define the logic $\mathsf{S4BP}^{k,l}_{n,m}$ by

$$\mathsf{S4BP}^{k,l}_{n,m} := \mathsf{S4}_t \oplus \{\mathsf{bd}_k, \mathsf{br}_l, \mathsf{bw}_n^+, \mathsf{bw}_m^-\}.$$

We say that a tense logic L is *with bounded parameters* if $L \supseteq \mathsf{S4BP}^{k,l}_{n,m}$ for some $k, l, n, m \in \mathbb{Z}^+ \cup \{\omega\}$ such that $\min\{k, l, n, m\} < \omega$. Moreover, we say that a tense logic is *fully bounded* if $L \supseteq \mathsf{S4BP}^{k,l}_{n,m}$ for some $k, l, n, m \in \mathbb{Z}^+$. Let $\mathcal{S}\mathsf{4BP}^{<\aleph_0}$ denote the set of all fully bounded tense logics.

Lots of well-studied logics are closely related to tense logics with bounded parameters. For example, the tense logic $\mathsf{S4.3}_t$ ($\mathsf{Lin}_t \oplus \mathsf{Dens}_1$ in [132]) of linear frames turns out to be equal to $\mathsf{S4BP}^{\omega,1}_{1,1}$. The tense logic Ga of garlands introduced by Kracht [71] is exactly $\mathsf{S4BP}^{2,\omega}_{2,2} \oplus \mathsf{grz}^+$.

In this section, our final task is to show that tense logics of finite depth and width are Kripke complete. Recall that a proper cluster C in a general frame $\mathbb{F} = (X, R, A)$ is a cluster C of X such that $|C| \geq 2$. One of our key observation is the following lemma:

4.2.10. Lemma. *Let $k, n, m \in \mathbb{Z}^+$ and $\mathbb{F} = (X, R, A)$ be a refined frame for $\mathsf{S4BP}^{k,\omega}_{n,m}$. Suppose $\mathbb{F}$ contains no proper cluster. Then $\mathbb{F}$ is image-finite.*

Proof:
We first prove that $R[x]$ is finite for all $x \in X$ by induction on $\mathsf{dep}(x)$. Suppose $\mathsf{dep}(x) = 1$. Then $R[x] = \{x\}$ and so $|R[x]| = 1$. Suppose $\mathsf{dep}(x) > 1$. Consider the set $Y = \{y \in R[x] : \mathsf{dep}(y) = \mathsf{dep}(x) - 1\}$. Since $\mathbb{F}$ contains no proper cluster, for all $y, y' \in Y$, neither Ryy' nor $Ry'y$. Thus, Y forms an anti-chain in $R[x]$. By Proposition 4.2.5, Y is finite. Note that $R[x] = \{x\} \cup \bigcup_{y \in Y} R[y]$. By induction hypothesis, $R[x]$ is finite. Similarly, by induction on $n - \mathsf{dep}(x)$, we show that $R^{-1}[x]$ is finite for all $x \in X$. Thus, $R_\sharp[x] = \{x\} \cup R[x] \cup R^{-1}[x]$ is finite for all $x \in X$ and so $\mathbb{F}$ is image-finite. $\square$

Recall that a general frame $\mathbb{F} = (X, R, A)$ is called finitely generated if $\langle A; \Box_R, \blacklozenge_R \rangle$ is a finitely generated tense algebra. Recall also that a general frame $\mathbb{F}$ is of finite girth if there exists an $n \in \omega$ such that every cluster in $\mathbb{F}$ is of size at most n. By Theorem 2.2.48, every finitely generated refined frame $\mathbb{F} \in \mathsf{RFr}(\mathsf{K4}_t)$ is of finite girth. Thus, by Lemma 4.2.10, we obtain:

4.2.11. Lemma. *Let $n, m, k \in \mathbb{Z}^+$ and $\mathbb{F} = (X, R, A)$ be a finitely generated refined frame for $\mathsf{S4BP}^{k,\omega}_{n,m}$. Then $\mathbb{F}$ is image-finite.*

As a consequence, the following theorem holds:

4.2.12. Theorem. *Let $n, m, k, l \in \mathbb{Z}^+$ and $L \in \mathsf{NExt}(\mathsf{S4BP}^{k,\omega}_{n,m})$. Then L is Kripke complete. Moreover, if $L \in \mathsf{NExt}(\mathsf{S4BP}^{k,l}_{n,m})$, then L has the FMP.*

Proof:
Let $\mathcal{K}$ be the class of all rooted finitely generated refined frames of L. By Lemma 4.2.11, every frame in $\mathcal{K}$ is image-finite. By Lemma 2.2.40(2), we obtain that $L = \mathsf{Log}(\mathcal{K}) = \mathsf{Log}(\{\kappa\mathbb{F} : \mathbb{F} \in \mathcal{K}\})$, which entails that L is Kripke complete.

Suppose $L \in \mathsf{NExt}(\mathsf{S4BP}^{k,l}_{n,m})$. Take any $\mathbb{F} = (X, R, A) \in \mathcal{K}$ and $x \in X$. Since $\mathbb{F}$ is rooted, by Lemma 2.2.40(1) and Proposition 4.2.5, we see that $X = R^l_\sharp[x]$ is finite. Thus, every frame in $\mathcal{K}$ is finite, which entails that L has the FMP. □

4.2.13. Corollary. *If $L \in \mathcal{S}4\mathsf{BP}^{<\aleph_0}$ and $\mathsf{grz}^+ \in L$, then L is tabular.*

Proof:
By Theorem 4.2.12, L has the FMP. Take any $\mathfrak{F} \in \mathsf{Fin}_{\mathrm{r}}(L)$. Because $\mathsf{grz}^+ \in L$, by Proposition 2.2.46, there is no proper cluster in $\mathfrak{F}$. By Lemma 2.2.40(1) and Proposition 4.2.5, the cardinality of the rooted frames of L without proper cluster has an upper bound, i.e., there exists $d \in \mathbb{Z}^+$ such that $|\mathfrak{G}| < d$ for any rooted frame $\mathfrak{G} \in \mathsf{Fr}(L)$ without proper cluster. Thus, $\mathsf{tab}^T_d \in \mathsf{Log}(\mathsf{Fin}_{\mathrm{r}}(L)) = L$. By Theorem 3.1.8, L is tabular. □

4.2.14. Remark. The reader can notice that reflexivity of frames plays almost no role in the above proofs. In fact, it is also natural to drop axiom T and define tense logics $\mathsf{K4BP}^{k,l}_{n,m}$ with bounded parameters. Basic properties above can be easily generalized to that case. In particular, Theorem 4.2.12 holds for $\mathsf{K4BP}^{k,l}_{n,m}$.

4.3 Pretabularity in Fully Bounded Tense Logics

In this section, we study the fully bounded pretabular logics, that is, pretabular logics in $\mathcal{S}4\mathsf{BP}^{<\aleph_0}$. We obtain a full characterization of pretabular logics which are fully bounded. By Theorem 4.2.12, every logic in $\mathcal{S}4\mathsf{BP}^{<\aleph_0}$ is Kripke complete. As we shall see later, it turns out that a tense logic $L \in \mathcal{S}4\mathsf{BP}^{<\aleph_0}$ is pretabular if and only if $L = \mathsf{Log}(\mathfrak{F})$ for some rooted frame $\mathfrak{F}$ with certain conditions.

Recall that a frame $\mathfrak{F} = (X, R)$ is called a *skeleton* if it contains no proper cluster. Let us now generalize the notation of skeleton to *pre-skeleton*, which will play a central role in our characterization of pretabular logics in $\mathcal{S}4\mathsf{BP}^{<\aleph_0}$.

4.3.1. Definition. Let $\mathfrak{F} = (X, R)$ be a frame. Then for each cardinal λ and $x \in X$, we define the frame $\mathfrak{F}^x_\lambda = (X^x_\lambda, R^x_\lambda)$ as follows:

- $X^x_\lambda = X \uplus N^x_\lambda$, where $N^x_\lambda = \{x_i : 0 < i \leq \lambda \text{ and } i \in \omega\}$;

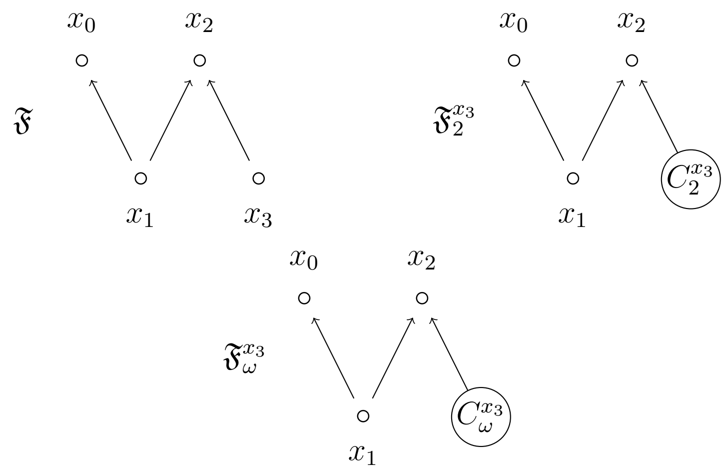

Figure 4.1: Examples of skeleton and pre-skeleton

- $R_\lambda^x = R \cup (C_\lambda^x \times R[x]) \cup (R^{-1}[x] \times C_\lambda^x) \cup (C_\lambda^x \times C_\lambda^x)$, where $C_\lambda^x = N_\lambda^x \cup \{x\}$.

A frame $\mathfrak{G}$ is called a λ-*pre-skeleton* if $\mathfrak{G} \cong \mathfrak{F}_\lambda^x$ for some skeleton $\mathfrak{F}$. Moreover, we call $\mathfrak{G}$ a *pre-skeleton* if $\mathfrak{G}$ is a λ-pre-skeleton for some $\lambda > 0$.

Intuitively, $\mathfrak{F}_\lambda^x$ is the frame obtained from $\mathfrak{F}$ by replacing one reflexive point x in $\mathfrak{F}$ by a cluster with $1 + \lambda$ points. The reader can readily verify that pre-skeletons are those frames containing exactly 1 proper cluster. It should be clear that $\mathfrak{F}$ and $\mathfrak{F}_\lambda^x$ share the same skeleton and have the same r-degree, width and depth. In what follows, without loss of generality, we always assume that $X \cap N_\lambda^x = \varnothing$.

4.3.2. Example. Concrete examples of skeletons and pre-skeletons are provided in Figure 4.1: the frame $\mathfrak{F}$ is a skeleton and the frames $\mathfrak{F}_2^{x_3}$ and $\mathfrak{F}_\omega^{x_3}$ are pre-skeletons, where $C_2^{x_3} = \{x_3, x_{31}, x_{32}\}$ is a 3-cluster and $C_\omega^{x_3} = \{x_3\} \cup \{x_{3i} : 0 < i \leq \omega\}$ is an ω-cluster.

Let us take a closer look at the skeletons for fully bounded tense logics. Recall that for a frame $\mathfrak{F} = (X, R)$, we write $\mathbb{F}^S = (X^S, R^S)$ for the skeleton of $\mathbb{F}$ defined by $X^S = \{C(x) : x \in X\}$ and $R^S = \{\langle C(x), C(y)\rangle : Rxy\}$.

4.3.3. Lemma. *Let* $\mathbb{F} = (X, R, A) \in \mathsf{RFr_r}(\mathsf{S4BP}_{n,m}^{k,l})$ *for some* $k, l, n, m \in \mathbb{Z}^+$. *Then*

(1) the skeleton $(\kappa\mathbb{F})^S$ *of* $\kappa\mathbb{F} = (X, R)$ *is finite.*

(2) $C \in A$ *for every cluster* $C \subseteq W$.

Proof:
For (1), by Proposition 4.2.5, $\kappa\mathbb{F} \in \mathsf{RFr}_{\mathrm{r}}(\mathsf{S4BP}^{k,l}_{n,m})$. Note that $\kappa\mathbb{F}$ contains no proper cluster. By Lemma 4.2.10, $\mathbb{F}$ is image-finite and so (1) follows from Lemma 2.2.40(1). For (2), let $\langle C_i \rangle_{i \leq s}$ be an enumeration of clusters in $\mathbb{F}$ such that $C = C_s$. Let $c \in C$. For each $i < s$, take a point $c_i \in C_i$. Suppose $c_i \notin R[c]$. Since $\mathbb{F}$ is tight, $c_i \in U_i$ and $c \notin \Diamond U_i$ for some $U_i \in A$. Since $\mathbb{F}$ is differentiated, $c_i \in U_i'$ and $c \notin U_i'$ for some $U_i' \in A$. Let $V_i = (U_i \cap U_i') \cup R^{-1}[U_i \cap U_i']$. Then $c_i \in V_i$ and $c \notin V_i$. Since $V_i \subseteq R^{-1}[V_i]$, we have $C_i \subseteq V_i$ and $C \cap V_i = \varnothing$. Suppose $c_i \in R[c]$. Since $c_i \notin C$, we see $c_i \notin R^{-1}[c]$. By a similar argument, there exists $V_i \in A$ with $C_i \subseteq V_i$ and $C \cap V_i = \varnothing$. Then it is not hard to see that $C = X \setminus \bigcup_{i<s} V_i \in A$. □

By Lemma 4.3.3, we see that for each rooted refined frame $\mathbb{F}$ for fully bounded tense logics, the skeleton of its underlying frame is always finite and every cluster in $\mathbb{F}$ is an admissible set. Thus, we can safely obtain t-morphic images of $\mathbb{F}$ that has girth 1.

4.3.4. LEMMA. *Let $\mathfrak{F} = (X, R)$ be a frame, $x \in X$ and α a cardinal. Then there exists an onto t-morphism $f : \mathfrak{F}^x_\alpha \twoheadrightarrow \mathfrak{F}$. Moreover, suppose $\mathbb{F} = (\mathfrak{F}^x_\alpha, A) \in \mathsf{RFr}_{\mathrm{r}}(\mathsf{S4BP}^{k,l}_{n,m})$ for some $k, l, n, m \in \mathbb{Z}^+$. Then $\mathbb{F}' = (\mathfrak{F}, \{f[a] : a \in A\})$ is a t-morphic image of $\mathbb{F}$。*

Proof:
Recall that $\mathfrak{F}^x_\alpha = (X^x_\alpha, R^x_\alpha)$, where $X^x_\alpha = X \cup N^x_\alpha$. Let $f : X^x_\alpha \twoheadrightarrow X$ be the function defined as follows:

$$f(y) = \begin{cases} y & \text{if } y \in X, \\ x & \text{otherwise.} \end{cases}$$

By the definition of $\mathfrak{F}^x_\alpha$, it is clear that f is a surjective t-morphism from $\mathfrak{F}^x_\alpha$ to $\mathfrak{F}$. Let $A' = \{f[a] : a \in A\}$. By Lemma 4.3.3(2), we obtain that $f^{-1}[a'] \in A$ for all $a' \in A'$. Thus, $\mathbb{F}' = (\mathfrak{F}, A')$ is a t-morphic image of $\mathbb{F}$. □

Lemma 4.3.4 indicates that we can always obtain t-morphic images of a fully bounded refined frame by 'squeezing' clusters in it. As a consequence, we have

4.3.5. THEOREM. *Let $\mathfrak{F} = (X, R)$ be a frame of girth α. Suppose $\mathbb{F} = (\mathfrak{F}, A)$ is a rooted refined frame for $\mathsf{S4BP}^{k,l}_{n,m}$ for some $k, l, n, m \in \mathbb{Z}^+$. Then there exists a t-morphic image $\mathbb{F}'$ of $\mathbb{F}$ whose underlying frame $\kappa\mathbb{F}'$ is an α-pre-skeleton.*

Next, we show that every fully bounded refined frame based on an infinite pre-skeleton can be simulated by a set of finite frames.

4.3.6. Lemma. *Let $k, l, n, m \in \mathbb{Z}^+$, $\mathfrak{F} = (X, R) \in \mathsf{Fin}_r$ and $x \in X$. Suppose $\lambda \geq \aleph_0$ and $\mathbb{F} = (\mathfrak{F}^x_\lambda, A) \in \mathsf{RFr}_r(\mathsf{S4BP}^{k,l}_{n,m})$. Then $\mathfrak{F}^x_s \in \mathsf{M}_t(\mathbb{F}^x_\lambda)$ for all $s \in \omega$.*

Proof:
Let $s \in \omega$ and C^x_λ denote the cluster in $\mathbb{F}$ generated by x. By Lemma 4.3.3, $C^x_\lambda \in A$. Note that C^x_λ is infinite and $\mathbb{F}$ is differentiated, there are pairwise disjoint $U_0, \cdots, U_s \in A$ such that $\bigcup_{i \leq s} U_i = C^x_\lambda$. Let $D = \{d_0, \cdots, d_s\} = C^x_s$ denote the cluster in $\mathfrak{F}^x_s$ generated by x. We define the map $f : X^x_\lambda \to X^x_s$ by

$$f(y) = \begin{cases} d_i & \text{if } y \in U_i, \\ y & \text{otherwise.} \end{cases}$$

Note that since $\mathfrak{F}$ is a finite frame, the reader can readily check that $f^{-1}[y] \in A$ for all $y \in X^x_s$. Hence, f is a t-morphism from $\mathbb{F}$ onto $\mathfrak{F}^x_s$. □

Recall that we write $\mathsf{Prop}(n)$ for the set $\{p_i : i < n\}$ of propositional variables and $\mathsf{Form}_t(n)$ for the set of all tense formulas built up from $\mathsf{Prop}(n)$.

4.3.7. Definition. Let $\mathfrak{M} = (X, R, V)$ be a model and $C \subseteq X$ a cluster. We call $D \subseteq C$ an *n-approximation* of C if for all $c \in C$, $c \equiv_{\mathsf{Prop}(n)} d$ for some $d \in D$.

4.3.8. Lemma. *Let $\mathfrak{M} = (X, R, V)$ be a model and $C \subseteq X$ be a cluster in $\mathfrak{F}$. Let Y be a subset of X such that $X \setminus C \subseteq Y$ and $Y \cap C$ is an n-approximation of C. Then for all $y \in Y$ and $\varphi \in \mathsf{Form}_t(n)$, we have*

$$\mathfrak{M}, y \models \varphi \text{ if and only if } \mathfrak{M}{\upharpoonright}Y, y \models \varphi.$$

Proof:
Let $\mathfrak{M}{\upharpoonright}Y = (Y, S)$. The proof proceeds by induction on φ. The case $\varphi \in \mathsf{Prop}(n)$ is trivial and the Boolean cases are standard. Consider the case $\varphi = \Box\psi$. Suppose $\mathfrak{M}, y \models \varphi$. Then $\mathfrak{M}, z \models \psi$ for all $z \in R[y]$. Since $S[y] \subseteq R[y]$, by induction hypothesis, $\mathfrak{M}{\upharpoonright}Y, z \models \psi$ for all $z \in S[y]$. Thus, $\mathfrak{M}{\upharpoonright}Y, y \models \varphi$. Suppose $\mathfrak{M}, y \not\models \varphi$. Then $\mathfrak{M}, z \not\models \psi$ for some $z \in R[y]$. Assume $z \notin C$. Since $X \setminus C \subseteq Y$, we see $z \in Y$. By induction hypothesis, $\mathfrak{M}{\upharpoonright}Y, z \not\models \psi$ and so $\mathfrak{M}{\upharpoonright}Y, y \not\models \varphi$. Assume $z \in C$. Since $Y \cap C$ is an n-approximation of C, there is $z' \in C \cap Y$ such that $z \equiv_{\mathsf{Prop}(n)} z'$. By Lemma 2.2.47, $\mathfrak{M}, z' \not\models \psi$. By induction hypothesis, $\mathfrak{M}{\upharpoonright}Y, z' \not\models \psi$ and so $\mathfrak{M}{\upharpoonright}Y, y \not\models \varphi$. The case $\varphi = \blacklozenge\psi$ can be proved analogously. □

Lemma 4.3.8 shows that to refute a formula, it is not necessary to have an infinite cluster. Then we can prove the following lemma:

4.3.9. Lemma. *Let $\mathfrak{F} = (X, R)$ be a skeleton and $x \in X$. Then for all $\lambda \geq \aleph_0$,*

$$\mathsf{Log}(\mathfrak{F}^x_\lambda) = \bigcap_{n \in \omega} \mathsf{Log}(\mathfrak{F}^x_n).$$

Proof:
Note that $\mathfrak{F}^x_\lambda \twoheadrightarrow \mathfrak{F}^x_n$ for each $n \in \omega$. Thus, $\mathsf{Log}(\mathfrak{F}^x_\lambda) \subseteq \bigcap_{n\in\omega} \mathsf{Log}(\mathfrak{F}^x_n)$. Suppose $\varphi(p_0, \cdots, p_{m-1}) \notin \mathsf{Log}(\mathfrak{F}^x_\lambda)$. Then $\mathfrak{F}^x_\lambda, V \not\models \varphi$ for some valuation V in $\mathfrak{F}^x_\lambda$. Let $\mathfrak{M} = (\mathfrak{F}^x_\lambda, V)$. Since $\mathsf{Prop}(m)$ is finite, there exists a finite m-approximation C of C^x_λ such that $x \in C$. Let $Y = X \cup C$. Clearly, $\mathfrak{F}^x_{|C|-1} \cong \mathfrak{F}^x_\lambda{\upharpoonright}Y$. Note that $X^x_\lambda \setminus C^x_\lambda \subseteq Y$. By Lemma 4.3.8, $\mathfrak{M}{\upharpoonright}Y \not\models \varphi$, which entails $\varphi \notin \mathsf{Log}(\mathfrak{F}^x_{|C|-1})$ and so $\varphi \notin \bigcap_{n\in\omega} \mathsf{Log}(\mathfrak{F}^x_n)$. □

As a consequence, by Lemma 4.3.6, we obtain the following theorem

4.3.10. Theorem. *Let $\mathfrak{F} = (X, R)$ be a finite skeleton, $x \in X$ and $\mathbb{F} = (\mathfrak{F}^x_\lambda, A) \in \mathsf{RFr_r}(\mathsf{S4BP}^{k,l}_{n,m})$ for some $k, l, n, m \in \mathbb{Z}^+$ and $\lambda \geq \aleph_0$. Then*

$$\mathsf{Log}(\mathfrak{F}^x_\lambda) = \mathsf{Log}(\mathfrak{F}^x_\omega) = \mathsf{Log}(\mathbb{F}) = \bigcap_{n\in\omega} \mathsf{Log}(\mathfrak{F}^x_n).$$

Next, we introduce the notion of *c-irreducible pre-skeleton*, which will play a central role in our characterization of fully bounded pretabular tense logics.

4.3.11. Definition. Let $\lambda > 0$. Then a λ-pre-skeleton $\mathfrak{F}^x_\lambda$ is called *c-irreducible* if

$$\mathsf{M}_t(\mathfrak{F}^x_\lambda) = \mathsf{I}(\{\mathfrak{F}^x_m : 0 < m \leq \lambda\}) \cup \mathsf{M}_t(\mathfrak{F}).$$

Otherwise, $\mathfrak{F}^x_\lambda$ is *c-reducible.*

Intuitively, a pre-skeleton $\mathfrak{F}^x_\lambda$ is c-irreducible if and only if there exists no t-morphic image of $\mathfrak{F}^x_\lambda$ that is not isomorphic to $\mathfrak{F}^x_\lambda$ but has the same girth as $\mathfrak{F}^x_\lambda$. In other words, every t-morphism from $\mathfrak{F}^x_\lambda$ 'squeezes' the cluster C^x_λ.

4.3.12. Example. Consider the frames $\mathfrak{F}_1$ and $\mathfrak{F}_2$ in Figure 4.2. We see that $\mathfrak{F}_1$ is c-reducible, since $\mathfrak{F}_2$ is a t-morphic image of $\mathfrak{F}_1$. However, $\mathfrak{F}_2$ is c-irreducible, since for every non-injective t-morphism f from $\mathfrak{F}_2$, the f-image is always a skeleton.

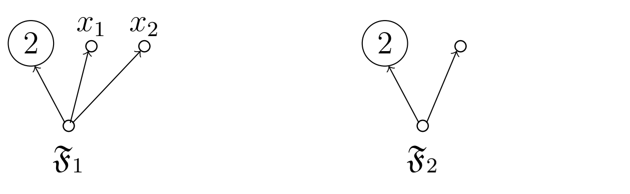

Figure 4.2: A c-reducible frame and a c-irreducible frame

In what follows, our aim is to show that for all finite rooted skeleton $\mathfrak{F}$, the tense logic $\mathsf{Log}(\mathfrak{F}^x_\omega)$ is pretabular if and only if $\mathfrak{F}^x_\omega$ is c-irreducible. To show this, we first study t-morphisms from pre-skeletons.

4.3.13. LEMMA. *Let $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, S)$ be frames of finite depth and $x \in X$. Suppose $f : \mathfrak{F} \twoheadrightarrow \mathfrak{G}$ and $|C(f(x))| \geq 2$. Then*

(1) $|C(y)| \geq 2$ for some $y \in R[x]$.

(2) $|C(y')| \geq 2$ for some $y' \in R^{-1}[x]$.

Moreover, if $\mathfrak{F} = (\mathfrak{F}')^x_n$ is an n-pre-skeleton for some $n \geq 1$, then

(3) for all $z \in X$, $C(z)$ is proper if and only if $C(f(z))$ is proper.

(4) $f[C(x)] = C(f(x))$.

Proof:
For (1), let $f(x) = y_0$ and $\mathsf{dep}(\mathfrak{F}) = k$. Since $|C(y_0)| \geq 2$, there exists $y_1 \in C(y_0)$ with $y_0 \neq y_1$. Since f is a t-morphism and $Sf(x)y_1$, there exists $x_1 \in X$ such that $f(x_1) = y_1$ and Rxx_1. Again, since $Sf(x_1)y_0$, there exists $x_2 \in X$ such that $f(x_2) = y_0$ and Rx_1x_2. By repeating this construction, we get an R-chain $\langle x_i \in X : i \leq 2k+1 \rangle$ such that $x = x_0$, $f(x_{2i}) = y_0$ and $f(x_{2i+1}) = y_1$ for all $i \leq k$. Since $\mathsf{dep}(\mathfrak{F}) = k < 2k+1$, we see that $x_m = x_l$ for some $m < l \leq 2k+1$. By transitivity of $\mathfrak{F}$, we obtain $x_{m+1}Rx_l = x_m$ and so $C(x_m) = C(x_{m+1})$. Note that since $f(x_m) \neq f(x_{m+1})$, we have $x_m \neq x_{m+1}$ and so $|C(x_m)| \geq 2$. Item (2) can be proved analogously.

For (3), suppose $C(f(z))$ is proper. Then $|C(f(z))| \geq 2$. By (1) and (2), there are $z_1 \in R[z]$ and $z_2 \in R^{-1}[z]$ such that $|C(z_1)| \geq 2$ and $|C(z_2)| \geq 2$. Since $\mathfrak{F}$ is a pre-skeleton, there is exactly 1 proper cluster C in $\mathfrak{F}$. Thus, $C(z_1) = C(z_2) = C$. Since z_2RzRz_2, we have $z \in C$ and so $C(z) = C$, which implies $C(z)$ is proper. Suppose $C(z)$ is proper. Then $z \in C(x)$ and so $C(f(z)) = C(f(x))$ is proper.

For (4), it is clear that $f(y) \in C(f(x))$ for all $y \in C(x)$, which entails $f[C(x)] \subseteq C(f(x))$. Let $u \in C(f(x))$. Since f is onto, $f(z) = u$ for some $z \in X$. Since $C(u) = C(f(x))$ is proper, by (3), $C(z)$ is proper. Note that $C(x)$ is the only proper cluster in $\mathfrak{F}$, we obtain that $z \in C(x)$ and so $u \in f[C(x)]$. Thus, $f[C(x)] = C(f(x))$. □

Lemma 4.3.13 shows that a t-morphism between proper pre-skeletons always sends proper clusters to proper clusters. It is worth noting that the assumption that $\mathfrak{F}$ is a pre-skeleton in Lemma 4.3.13(3) is necessary. Consider the t-morphism f depicted in Figure 4.3. The 1-cluster $\{x\}$ is mapped to a proper cluster.

The following lemma shows another central property of t-morphisms between pre-skeletons.

4.3.14. LEMMA. *Let $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, S)$ be finite skeletons. Then the following are equivalent:*

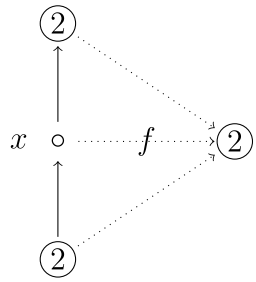

Figure 4.3: A t-morphism maps a 1-cluster to a proper cluster

(1) $\mathfrak{F}^x_n \twoheadrightarrow \mathfrak{G}^y_m$ *for some* $1 \leq m \leq n \leq \omega$*;*

(2) $\mathfrak{F}^x_k \twoheadrightarrow \mathfrak{G}^y_l$ *for all* $l \leq k \leq \omega$*.*

Proof:
Clearly (1) follows from (2). Suppose $f : \mathfrak{F}^x_n \twoheadrightarrow \mathfrak{G}^y_m$. Let $X_0 = X \setminus C(x)$ and $Y_0 = Y \setminus C(y)$. By Lemma 4.3.13(3-4), $f[C^x_n] = C^y_m$ and $f[X_0] = Y_0$. For any $k \geq l$, there exists $g : C^x_k \twoheadrightarrow C^y_l$. Let $f' = f \upharpoonright X_0$ and $h = g \cup f'$. It is straightforward to verify that $h : \mathfrak{F}^x_k \twoheadrightarrow \mathfrak{G}^y_l$. □

Now we focus on c-irreducible ω-pre-skeletons and their tense logics. By Lemma 4.3.14, we have

4.3.15. Lemma. *Let* $\mathfrak{F} = (X, R)$ *and* $\mathfrak{G} = (Y, S)$ *be finite rooted skeletons,* $x \in X$ *and* $y \in Y$*. Suppose* $\mathfrak{F}^x_\omega$ *and* $\mathfrak{G}^y_\omega$ *are c-irreducible. Then*

$$\mathsf{Log}(\mathfrak{F}^x_\omega) = \mathsf{Log}(\mathfrak{G}^y_\omega) \text{ implies } \mathfrak{F}^x_1 \cong \mathfrak{G}^y_1.$$

Proof:
Let $k = \mathsf{rdg}(\mathfrak{F}) + \mathsf{rdg}(\mathfrak{G})$. Since $\mathfrak{F}^x_1 \models \mathsf{Log}(\mathfrak{F}^x_\omega)$ and $\mathfrak{F}^x_1, x \not\models \neg\mathcal{J}^k(\mathfrak{F}^x_1, x)$, we obtain $\neg\mathcal{J}^k(\mathfrak{F}^x_1, x) \notin \mathsf{Log}(\mathfrak{G}^y_\omega)$ and so $\mathfrak{G}^y_\omega \not\models \neg\mathcal{J}^k(\mathfrak{F}^x_1, x)$. By Theorem 4.1.5 and Theorem 4.1.8, we have $\mathfrak{G}^y_\omega \twoheadrightarrow \mathfrak{F}^x_1$. It follows from Lemma 4.3.14 that $\mathfrak{G}^y_1 \twoheadrightarrow \mathfrak{F}^x_1$. Analogously, we can show that $\mathfrak{F}^x_1 \twoheadrightarrow \mathfrak{G}^y_1$. Then $|\mathfrak{G}^y_1| = |\mathfrak{F}^x_1|$ and by Fact 2.2.24, we obtain that $\mathfrak{F}^x_1 \cong \mathfrak{G}^y_1$. □

In the following lemma, we show that an ω-pre-skeleton is c-irreducible whenever the corresponding 1-pre-skeleton is c-irreducible.

4.3.16. Lemma. *Let* $\mathfrak{F} = (X, R) \in \mathsf{Fin}$ *be a skeleton and* $x \in X$ *reflexive. Then*

$$\mathfrak{F}^x_1 \text{ is c-irreducible if and only if } \mathfrak{F}^x_\omega \text{ is c-irreducible.}$$

Proof:
The right-to-left direction follows immediately from Lemma 4.3.13(4). For the left-to-right direction, take any $\mathfrak{H}' \in \mathsf{M}_t(\mathfrak{F}^x_\omega)$ such that $\mathfrak{H}'$ is not isomorphic to $\mathfrak{F}^x_\omega$. Let $h : \mathfrak{F}^x_\omega \to \mathfrak{H}'$. By Lemma 4.3.13, $\mathfrak{H}'$ is of the form $\mathfrak{H}^z_m$ for some $m \leq \omega$. If $m = 0$, then $\mathfrak{H}' \in \mathsf{M}_t(\mathfrak{F})$ by Lemma 4.3.14. Suppose $m > 0$. By Lemma 4.3.14, we have that $\mathfrak{H}^z_1 \in \mathsf{M}_t(\mathfrak{F}^x_1)$. Note that $\mathsf{M}_t(\mathfrak{F}^x_1) = \mathsf{I}(\{\mathfrak{F}^x_1\}) \cup \mathsf{M}_t(\mathfrak{F})$ and there is no frame in $\mathsf{M}_t(\mathfrak{F})$ of girth greater than 1. Thus, $\mathfrak{H}^z_1 \cong \mathfrak{F}^x_1$ and so $\mathfrak{H}' \cong \mathfrak{H}^z_m \cong \mathfrak{F}^x_m$. □

Lemma 4.3.16 allows us to decide the c-irreducibility of an ω-pre-skeleton by examining its finite counterpart, which enables the use of generalized Jankov formulas. Now we are ready to provide a characterization of pretabularity for tense logics of pre-skeletons.

4.3.17. Lemma. *Let $\mathfrak{F} = (X, R)$ be a rooted finite skeleton and $x \in X$. Then*

$$\mathsf{Log}(\mathfrak{F}^x_\omega) \text{ is pretabular if and only if } \mathfrak{F}^x_1 \text{ is c-irreducible.}$$

Proof:
Suppose $\mathfrak{F}^x_1$ is c-reducible. Then there exists a frame $\mathfrak{G}_0$ such that $\mathfrak{G}_0 \ncong \mathfrak{F}^x_1$ and $\mathfrak{G}_0 \in \mathsf{M}_t(\mathfrak{F}^x_1) \setminus \mathsf{M}_t(\mathfrak{F})$. Let $f : \mathfrak{F}^x_1 \twoheadrightarrow \mathfrak{G}_0$. Note that C^x_1 is the unique proper cluster in $\mathfrak{F}^x_1$. We claim $|f[C^x_1]| > 1$. Suppose $|f[C^x_1]| = 1$. Then $f[C^x_1] = \{y_0\}$ for some $y_0 \in \mathfrak{G}_0$. Then we define $f' : \mathfrak{F} \to \mathfrak{G}_0$ by $f'(x) = y_0$ and $f'(z) = f(z)$ for all $z \in X \setminus \{x\}$. It is clear that $f' : \mathfrak{F} \twoheadrightarrow \mathfrak{G}_0$. Thus, $\mathfrak{G}_0 \in \mathsf{M}_t(\mathfrak{F})$, which is a contradiction. Thus, $f[C^x_1] = C(f(x))$ is a proper cluster in $\mathfrak{G}_0$. By Lemma 4.3.13(3), $\mathfrak{G}_0$ is a 2-pre-skeleton and so $\mathfrak{G}_0 \cong \mathfrak{G}^y_1$ for some $\mathfrak{G} = (Y, S)$. It suffices to show $\mathsf{Log}(\mathfrak{G}^y_\omega) \supsetneq \mathsf{Log}(\mathfrak{F}^x_\omega)$. Since $\mathfrak{F}^x_1 \twoheadrightarrow \mathfrak{G}^y_1$, by Lemma 4.3.14, $\mathfrak{F}^x_\omega \twoheadrightarrow \mathfrak{G}^y_\omega$ and so $\mathsf{Log}(\mathfrak{G}^y_\omega) \supseteq \mathsf{Log}(\mathfrak{F}^x_\omega)$. Consider the formula $\varphi = \neg\mathcal{J}^k(\mathfrak{F}^x_1, x)$ where $k = \mathsf{rdg}(\mathfrak{F}) + 1$. We show that $\varphi \in \mathsf{Log}(\mathfrak{G}^y_\omega) \setminus \mathsf{Log}(\mathfrak{F}^x_\omega)$. Clearly, $\varphi \notin \mathsf{Log}(\mathfrak{F}^x_\omega)$. Suppose $\varphi \notin \mathsf{Log}(\mathfrak{G}^y_\omega)$. By Lemma 4.3.9, $\mathfrak{G}^y_m \not\models \varphi$ for some $m \in \omega$. By Theorem 4.1.9, we have that $\mathfrak{G}^y_m \twoheadrightarrow \mathfrak{F}^x_1$. It follows from Lemma 4.3.14 that $\mathfrak{G}_0 = \mathfrak{G}^y_1 \twoheadrightarrow \mathfrak{F}^x_1$. Thus, $|\mathfrak{F}^x_1| = |\mathfrak{G}_0|$ and so $f : \mathfrak{F}^x_1 \twoheadrightarrow \mathfrak{G}_0$ is injective. By Fact 2.2.24, we obtain that $\mathfrak{G}_0 \cong \mathfrak{F}^x_1$, which contradicts our assumption.

Suppose $\mathfrak{F}^x_1$ is c-irreducible. By Lemma 4.3.16, we have that $\mathfrak{F}^x_\omega$ is c-irreducible. Take any $L \supsetneq \mathsf{Log}(\mathfrak{F}^x_\omega)$. By Theorem 4.2.12, $L = \mathsf{Log}(\mathsf{Fin}_r(L))$. Then it follows from Theorem 4.2.7 that

$$\mathsf{Fin}_r(L) \subsetneq \mathsf{Fin}_r(\mathsf{Log}(\mathfrak{F}^x_\omega)) \subseteq \mathsf{M}_t(\mathfrak{F}^x_\omega) = \mathsf{I}(\{\mathfrak{F}^x_m : 0 < m \leq \omega\}) \cup \mathsf{M}_t(\mathfrak{F}).$$

Thus, there exists $n \in \mathbb{Z}^+$ such that $\mathsf{Fin}_r(L) \subseteq \mathsf{I}(\{\mathfrak{F}^x_m : 0 < m \leq n\}) \cup \mathsf{M}_t(\mathfrak{F})$, which entails that $L = \mathsf{Log}(\mathsf{Fin}_r(L))$ is tabular. Clearly, $\mathsf{Log}(\mathfrak{F}^x_\omega)$ is non-tabular. Hence, $\mathsf{Log}(\mathfrak{F}^x_\omega)$ is pretabular. □

Our final task in this section is to characterize pretabularity for fully bounded tense logics. The key observation is that every pretabular tense logic can be represented as the logic of an infinite rooted refined frame, which we establish using ultraproducts of general frames.

4.3.18. THEOREM. *Let L be a non-tabular tense logic. Then $\mathsf{Fin}_r(L) \neq \mathsf{RFr}_\mathrm{r}(L)$, i.e., there exists an infinite rooted refined frame $\mathbb{F}$ such that $\mathbb{F} \vDash L$.*

Proof:
Since L is non-tabular, by Theorem 3.1.8, $\mathsf{tab}_n^T \notin L$ for any $n \in \omega$. By Proposition 2.2.37, $L = \mathsf{Log}(\mathsf{RFr}_\mathrm{r}(L))$. Thus, for all $n \in \omega$, there exists $\mathbb{F}_n = (X_n, R_n, A_n) \in \mathsf{RFr}_\mathrm{r}(L)$ and $x_n \in X_n$ such that $\mathbb{F}_n, x_n \nvDash \mathsf{tab}_n^T$. By Lemma 3.1.6, we have $\mathbb{F}_n, x_n \nvDash \mathsf{tab}_m^T$ for all $m \geq n$. By Theorem 2.2.30(1), $\prod_U \mathbb{F}_n, [x] \nvDash \mathsf{tab}_n^T$ for any $n \in \omega$, where U is a non-principal ultrafilter over ω and $x : n \mapsto x_n$ for all $n \in \omega$. Let $\mathbb{G} = (\prod_U \mathbb{F}_n)_{[x]}$. By Theorem 2.2.30, $\prod_U \mathbb{F}_n \vDash L$, which implies $\mathbb{G} \vDash L$. Note that $\mathbb{G} \in \mathsf{RFr}_\mathrm{r}$ and $\mathbb{G}, [x] \nvDash \mathsf{tab}_n^T$ for any $n \in \omega$, by Lemma 3.1.6, we see that $\mathbb{G} \in \mathsf{RFr}_\mathrm{r}(L) \setminus \mathsf{Fin}_r(L)$. □

4.3.19. COROLLARY. *If L is a pretabular tense logic, then $L = \mathsf{Log}(\mathbb{F})$ for some rooted refined frame $\mathbb{F}$.*

Proof:
Since L is non-tabular, by Theorem 4.3.18, there exists an infinite rooted refined frame $\mathbb{F}$ such that $\mathbb{F} \vDash L$. Note that $\mathsf{Log}(\mathbb{F}) \supseteq L$ is non-tabular and L is pretabular. Thus, $L = \mathsf{Log}(\mathbb{F})$, which concludes the proof. □

Now we provide the characterization of pretabular fully bounded tense logics.

4.3.20. THEOREM. *Let $L \in \mathcal{S}4\mathsf{BP}^{<\aleph_0}$. Then L is pretabular if and only if $L = \mathsf{Log}(\mathfrak{F}^x_\omega)$ for some c-irreducible rooted pre-skeleton $\mathfrak{F}^x_\omega$.*

Proof:
The right-to-left direction follows from Lemma 4.3.17 immediately. For the other direction, suppose L is pretabular. By Theorem 4.3.18, $L = \mathsf{Log}(\mathbb{F}'')$ for some infinite rooted refined frame $\mathbb{F}''$. By Theorem 4.3.5, there exists $\lambda \geq \aleph_0$ and a t-morphic image $\mathbb{F}'$ of $\mathbb{F}''$ such that $\kappa\mathbb{F}'$ is a λ-pre-skeleton. Then $\mathbb{F}'$ is of the form $(\mathfrak{F}^x_\lambda, A)$, where $\mathfrak{F}$ is a finite skeleton. Since $\mathsf{S4BP}^{k,l}_{n,m} \subseteq L$, by Theorem 4.3.10, we see that $L \subseteq \mathsf{Log}(\mathbb{F}') = \mathsf{Log}(\mathfrak{F}^x_\omega)$. Because L is pretabular, $L = \mathsf{Log}(\mathfrak{F}^x_\omega)$. By Lemma 4.3.17, $\mathfrak{F}^x_\omega$ is c-irreducible. □

By Lemma 4.3.9 and Theorem 4.3.20, we have

4.3.21. THEOREM. *Every pretabular logic in* $\mathcal{S}4\mathsf{BP}^{<\aleph_0}$ *has the FMP.*

Let us conclude this section with the following theorem about the cardinality of pretabular extensions of fully bounded tense logics:

4.3.22. THEOREM. *For all* $k, l, n, m \in \mathbb{Z}^+$, *we have that* $|\mathsf{PTAB}(\mathsf{S4BP}^{k,l}_{n,m})| < \aleph_0$.

Proof:
Take any skeleton $\mathfrak{F} \in \mathsf{Fr}_r(\mathsf{S4BP}^{k,l}_{n,m})$ and $x \in \mathfrak{F}$. It is not hard to see that $|\mathfrak{F}| \leq |R^l_\sharp[x]| < (k(m+n))^l$. Thus, there are only finitely many skeletons validating $\mathsf{S4BP}^{k,l}_{n,m}$. By Theorem 4.3.20, $|\mathsf{PTAB}(\mathsf{S4BP}^{k,l}_{n,m})| < \aleph_0$. □

4.3.23. REMARK. We will see in Section 4.5 that there exist countably many pretabular logics in $\mathcal{S}4\mathsf{BP}^{<\aleph_0}$. In fact, as we have mentioned, it always makes sense to drop reflexivity and consider fully bounded tense logics above $\mathsf{K4}_t$. Then we could generalize our results and show now that there exists countably many such pretabular logics. Recall that $\mathfrak{Ch}^<_n = (n, <)$ is the strict chain of n-elements. For each $n \in \mathbb{Z}^+$, consider the tense logic L_n of the ω-pre-skeleton $(\mathfrak{Ch}^<_n)^0_\omega$. Clearly, for all $i, j \in \mathbb{Z}^+$, we have $L_i \neq L_j$ whenever $i \neq j$. It is also easy to see that $(\mathfrak{Ch}^<_n)^0_1$ is c-irreducible for all $n \in \omega$. By generalizing Lemma 4.3.17, we see that L_n is pretabular for all $n \in \omega$.

4.4 Pretabular Tense Logics in $\mathsf{NExt}(\mathsf{S4.3}_t)$

In Section 4.3, we studied pretabular fully bounded tense logics and gave their full characterizations. In the present section, we turn our attention to tense logics with weaker restrictions. As a representative case, we consider the tense logic $\mathsf{S4.3}_t$ defined by

$$\mathsf{S4.3}_t = \mathsf{S4}_t \oplus (\Diamond\blacklozenge p \vee \blacklozenge\Diamond p \to p \vee \Diamond p \vee \blacklozenge p).$$

It is known that $\mathsf{S4.3}_t$ is the tense logic of linear pre-orders. Note that $\mathsf{S4.3}_t = \mathsf{S4BP}^{\omega,1}_{1,1}$, which indicates that frames of this logic has strong restrictions on forth-width, back-width and r-degree, while imposing no restriction on depth.

Linearity is important in the research of tense logics. The tense logic $\mathsf{S4.3}_t$ and its modal fragment $\mathsf{S4.3}$ have been well investigated. Bull [19] and Fine [48] showed that every extension of $\mathsf{S4.3}$ has the FMP. By characterizations of pretabular logics over $\mathsf{S4}$ in [44, 90], we have that $|\mathsf{PTAB}(\mathsf{S4.3})| = 3$. Wolter [132] proved that every extension of $\mathsf{S4.3}_t$ is finitely axiomatizable. The aim of this section is to prove the following:

There are exactly 5 pretabular tense logics in $\mathsf{NExt}(\mathsf{S4.3}_t)$.

To prove the theorem above, we first show that the tense logic $\mathsf{Grz.3}_t$ is the unique pretabular logic in $\mathsf{NExt}(\mathsf{S4.3}_t)$ with infinite depth, where

$$\mathsf{Grz.3}_t = \mathsf{S4.3}_t \oplus \{\mathsf{grz}^+, \mathsf{grz}^-\}.$$

Recall that for each $n \in \mathbb{Z}^+$, we define the n-chain $\mathfrak{Ch}_n$ by $\mathfrak{Ch}_n = (n, \leq)$. It is well known that $\mathsf{Grz.3}_t$ is the tense logic of all finite chains, that is,

$$\mathsf{Grz.3}_t = \bigcap_{n=1}^{\omega} \mathsf{Log}(\mathfrak{Ch}_n).$$

4.4.1. LEMMA. *Let* $\mathbb{F} \in \mathsf{RFr}_{\mathrm{r}}(\mathsf{S4.3}_t)$ *and* $n \in \mathbb{Z}^+$. *Then*

$$\mathsf{dep}(\mathbb{F}) > n \textit{ implies } \mathfrak{Ch}_{n+1} \in \mathsf{M}_t(\mathbb{F}).$$

Proof:
Let $\mathbb{F} = (X, R, A)$. Since $\mathsf{dep}(\mathbb{F}) > n$, by Proposition 4.2.5(1), $\mathbb{F} \not\models \mathsf{bd}_n$. Then there exists a valuation V and a co-chain $\langle x_i : i \leq n \rangle$ in $\mathbb{F}$ such that $\mathbb{F}, V, x_i \models \Box p_i$ and $\mathbb{F}, V, x_{i+1} \models \neg p_i$ for all $i < n$. We define the function $f : X \to n + 1$ by:

$$f(x) = \begin{cases} n - i & \text{if } x \in V(\Box p_i) \setminus V(\Box p_{i+1}) \text{ and } i < n \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $f^{-1}[D] \in P$ for all $D \subseteq n + 1$. Take any $j < n$ and $x \in V(\Box p_j)$. Note that $\mathbb{F}$ is linear, by $x_{j+1} \models \Box p_{j+1} \wedge \neg p_j$, we see $x \in R[x_{j+1}]$ and so $x \in V(\Box p_{j+1})$. Thus, $V(\Box p_i) \subsetneq V(\Box p_{i+1})$ for all $i < n$. Hence, $f : \mathbb{F} \twoheadrightarrow \mathfrak{Ch}_{n+1}$. □

It follows from Lemma 4.4.1 that $\mathsf{Grz.3}_t$ is the maximal logic with infinite depth. More precisely, the following theorem holds:

4.4.2. THEOREM. *Let* $L \in \mathsf{NExt}(\mathsf{S4.3}_t)$. *Then* $L \subseteq \mathsf{Grz.3}_t$ *iff* $\mathsf{dep}(L) = \aleph_0$.

Proof:
The left-to-right direction is trivial. Suppose $\mathsf{dep}(L) = \aleph_0$. Then for each $n \in \mathbb{Z}^+$, there exists $\mathbb{F}_n \in \mathsf{RFr}_{\mathrm{r}}(L)$ such that $\mathsf{dep}(\mathbb{F}_n) > n$. By Lemma 4.4.1, we have $\mathbb{F}_n \twoheadrightarrow \mathfrak{Ch}_{n+1}$, which entails $\mathfrak{Ch}_{n+1} \models L$. Thus, $L \subseteq \bigcap_{n=1}^{\omega} \mathsf{Log}(\mathfrak{Ch}_n) = \mathsf{Grz.3}_t$. □

Now we show that $\mathsf{Grz.3}_t$ is the only pretabular logic with infinite depth.

4.4.3. LEMMA. $\mathsf{Grz.3}_t$ *is pretabular.*

Proof:
Let $L \supsetneq \mathsf{Grz.3}_t$. Note that $\mathsf{br}_1 \in \mathsf{Grz.3}_t$, by Theorem 4.2.7, $\mathsf{Fin}_r(\mathsf{Grz.3}_t) = \bigcup_{n \in \mathbb{Z}^+} \mathsf{M}_t(\mathfrak{Ch}_n)$. By Theorem 4.4.2 and Proposition 4.2.5(1), $\mathsf{dep}(L) < \aleph_0$ and so $\mathsf{bd}_n \in L$ for some $n \in \mathbb{Z}^+$. Thus, $\mathsf{Fin}_r(L) = \mathsf{M}_t(\{\mathfrak{Ch}_n\})$ for some $n \in \mathbb{Z}^+$.

By Theorem 4.2.12, L has the FMP. Thus, $L = \mathsf{Log}(\mathsf{Fin}_r(L)) = \mathsf{Log}(\mathfrak{Ch}_n)$, which entails that L is tabular. Hence, $\mathsf{Grz.3}_t \in \mathsf{PTAB}(\mathsf{S4.3}_t)$. □

To characterize pretabular logics with finite depth, we introduce some auxiliary definitions. Recall that for a skeleton $\mathfrak{F} = (X, R)$ and $x \in X$, we write $\mathfrak{F}^x_\lambda$ for the λ-pre-skeleton generated by $(\mathfrak{F}, x)$.

4.4.4. Definition. Let $\mathsf{tp} = \{\pm, +, -, \circ\}$. For all λ such that $\lambda \leq \omega$, we define

$$\mathfrak{Ch}^\circ_\lambda = (\mathfrak{Ch}_1)^0_\lambda,\ \mathfrak{Ch}^+_\lambda = (\mathfrak{Ch}_2)^0_\lambda,\ \mathfrak{Ch}^-_\lambda = (\mathfrak{Ch}_2)^1_\lambda \text{ and } \mathfrak{Ch}^\pm_\lambda = (\mathfrak{Ch}_3)^1_\lambda.$$

For all $t \in \mathsf{tp}$, let $L^t = \mathsf{Log}(\mathfrak{Ch}^t_\omega)$.

The frames $\mathfrak{Ch}^\circ_\lambda$, $\mathfrak{Ch}^+_\lambda$, $\mathfrak{Ch}^-_\lambda$ and $\mathfrak{Ch}^\pm_\lambda$ are depicted in Figure 4.4.

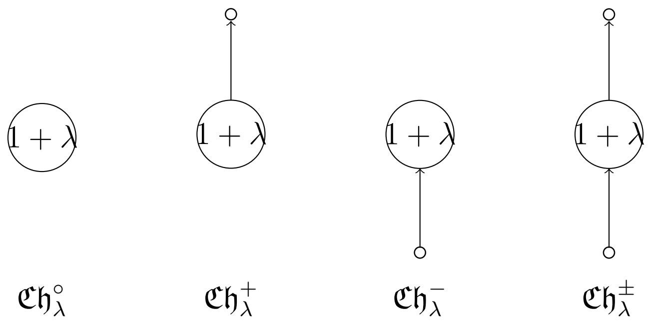

Figure 4.4: Frames $\mathfrak{Ch}^\circ_\lambda$, $\mathfrak{Ch}^+_\lambda$, $\mathfrak{Ch}^-_\lambda$ and $\mathfrak{Ch}^\pm_\lambda$

4.4.5. Proposition. *For each $t \in \mathsf{tp}$, the tense logic L^t is finitely axiomatizable. More precisely, the following holds:*

(1) $L^\circ = \mathsf{S5}_t = \mathsf{S4.3}_t \oplus (\Diamond p \to \Box\Diamond p)$,

(2) $L^+ = \mathsf{S4.3}_t \oplus \{\mathsf{bd}_2, \Box\Diamond p \to \Diamond\Box p\}$,

(3) $L^- = \mathsf{S4.3}_t \oplus \{\mathsf{bd}_2, \blacksquare\blacklozenge p \to \blacklozenge\blacksquare p\}$, *and*

(4) $L^\pm = \mathsf{S4.3}_t \oplus \{\mathsf{bd}_3, \Box\Diamond p \to \Diamond\Box p, \blacksquare\blacklozenge p \to \blacklozenge\blacksquare p\}$.

Proof:
The proof is standard. □

In what follows, we show that $L^\pm, L^+, L^-$ and L° are exactly the pretabular logics in $\mathsf{NExt}(\mathsf{S4.3}_t)$ of finite depth. As a corollary, we obtain that

$$\mathsf{PTAB}(\mathsf{S4.3}_t) = \{L^\pm, L^+, L^-, L^\circ, \mathsf{Grz.3}_t\}$$

4.4.6. Lemma. *For all* $t \in \mathsf{tp}$*, the logic* L^t *is pretabular.*

Proof:
Since $\mathfrak{Ch}_1^t$ is c-irreducible, by Lemma 4.3.17, $L^t = \mathsf{Log}(\mathfrak{Ch}_\omega^t)$ is pretabular. □

4.4.7. Lemma. *Let* $t, s \in \mathsf{tp}$*. Then* $t = s$ *if and only if* $L^t = L^s$*.*

Proof:
Follows immediately from Lemma 4.3.15. □

By Lemmas 4.4.6 and 4.4.7, we see that $L^{\pm}, L^+, L^-$ and L° are pairwise different pretabular logics of finite depth. It remains to show that there is no other linear pretabular logics of finite depth. By Theorem 4.3.20, we could focus on the logics of c-irreducible ω-pre-skeletons.

4.4.8. Lemma. *Let* $\mathfrak{F} = (X, R) \in \mathsf{Fr}_r(\mathsf{S4.3}_t)$ *be a skeleton of finite depth and* $x \in X$*. Then the following holds:*

$\mathfrak{F}_\omega^x$ *is c-irreducible if and only if* $\mathfrak{F}_\omega^x$ *is isomorphic to* $\mathfrak{Ch}_\omega^t$ *for some* $t \in \mathsf{tp}$*.*

Proof:
The right-to-left direction is trivial. For the other direction, suppose $\mathfrak{F}_\omega^x$ is c-irreducible. Let $Y^+ = R[x] \setminus C(x)$ and $Y^- = R^{-1}[x] \setminus C(x)$. We claim that $|Y^+| < 2$ and $|Y^-| < 2$. Suppose $|Y^+| \geq 2$. Then let $\mathfrak{G} = (X_0, R_0)$ where $X_0 = Y^- \cup C(x) \cup \{y\}$ and R_0 is the reflexive-transitive closure of $(Y^- \times C(x)) \cup (C(x) \times \{y\})$. It is obvious that $\mathfrak{G} \in \mathsf{M}_t(\mathfrak{F}_\omega^x) \setminus \mathsf{I}(\{\mathfrak{F}_n^x : n \leq \omega\})$, which contradicts the fact that $\mathfrak{F}_\omega^x$ is c-irreducible. Similarly, $|Y^-| \geq 2$ implies $\mathfrak{F}_\omega^x$ is c-reducible, which is impossible. Thus, $|Y^+| < 2$ and $|Y^-| < 2$. Then we have the following 4 cases:

(1) $\langle |Y^+|, |Y^-| \rangle = \langle 0, 0 \rangle$. In this case, $\mathfrak{F}_\omega^x \cong \mathfrak{Ch}_\omega^\circ$.

(2) $\langle |Y^+|, |Y^-| \rangle = \langle 0, 1 \rangle$. Then $\mathfrak{F}_\omega^x \cong \mathfrak{Ch}_\omega^-$.

(3) $\langle |Y^+|, |Y^-| \rangle = \langle 1, 0 \rangle$. Then $\mathfrak{F}_\omega^x \cong \mathfrak{Ch}_\omega^+$.

(4) $\langle |Y^+|, |Y^-| \rangle = \langle 1, 1 \rangle$. Then $\mathfrak{F}_\omega^x \cong \mathfrak{Ch}_\omega^\pm$.

Hence, $\mathfrak{F}_\omega^x$ is isomorphic to $\mathfrak{Ch}_\omega^t$ for some $t \in \mathsf{tp}$. □

4.4.9. Theorem. *There exists exactly 5 pretabular logics in* $\mathsf{NExt}(\mathsf{S4.3}_t)$*. More precisely,* $\mathsf{PTAB}(\mathsf{S4.3}_t) = \{L^{\pm}, L^+, L^-, L^\circ, \mathsf{Grz.3}_t\}$*.*

Proof:
By Lemmas 4.4.3, 4.4.6 and 4.4.7, we see that $\{L^{\pm}, L^{+}, L^{-}, L^{\circ}, \mathsf{Grz.3}_t\} \subseteq \mathsf{PTAB}(\mathsf{S4.3}_t)$ and $|\{L^{\pm}, L^{+}, L^{-}, L^{\circ}, \mathsf{Grz.3}_t\}| = 5$. Take any $L \in \mathsf{PTAB}(\mathsf{S4.3}_t)$. Suppose $\mathsf{dep}(L) = \aleph_0$. By Theorem 4.4.2 and Lemma 4.4.3, we obtain that $L = \mathsf{Grz.3}_t$. Suppose $\mathsf{dep}(L) = n < \aleph_0$. By Proposition 4.2.5, $L \in \mathsf{NExt}(\mathsf{S4BP}^{n,1}_{1,1})$. It follows from Theorem 4.3.20 that $L = \mathsf{Log}(\mathfrak{F}^x_\omega)$ for some c-irreducible rooted $\mathfrak{F}^x_\omega$. By Lemma 4.4.8, $\mathfrak{F}^x_\omega$ is isomorphic to $\mathfrak{Ch}^t_\omega$ for some $t \in \mathsf{tp}$. Hence, $L = \mathsf{Log}(\mathfrak{Ch}^t_\omega) \in \{L^t : t \in \mathsf{tp}\}$. □

By Proposition 4.4.5 and Theorems 4.2.12 and 4.4.9, we have

4.4.10. THEOREM. *Let $L \in \mathsf{PTAB}(\mathsf{S4.3}_t)$. Then L is finitely axiomatizable and has the FMP. Hence, L is decidable.*

Moreover, by Theorems 4.4.9 and 4.4.10, we have

4.4.11. THEOREM. *Tabularity in $\mathsf{NExt}(\mathsf{S4.3}_t)$ is decidable. That is, given any formula φ, it is decidable whether $\mathsf{S4.3}_t \oplus \varphi$ is tabular.*

Generally speaking, Theorem 4.4.11 concerns a higher level of decidability: it is not merely about the decidability of logics, but about the decidability of logical properties within lattices of logics. For further discussion of the decidability of logical properties, see Chapter 6.

4.5 Pretabular Tense Logics in $\mathsf{NExt}(\mathsf{S4BP}^{2,\omega}_{2,2})$

In this section, we study pretabular logics over $\mathsf{S4BP}^{2,\omega}_{2,2}$. Comparing to $\mathsf{S4.3}_t$, the tense logic $\mathsf{S4BP}^{2,\omega}_{2,2}$ has weaker constraints on the width of logics (allows forth and back width no more than 2 instead of 1), but a stronger constraint on the depth (allows only depth 2). As we will see in this section, rooted frames of $\mathsf{S4BP}^{2,\omega}_{2,2}$ are 'garlands' and 'hoops'. We provide a full characterization of $\mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,2})$, and it turns out that the cardinality of $\mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,2})$ is $\aleph_0$. Thus, we have a constructive proof for the claim in [107] that $\mathsf{PTAB}(\mathsf{S4}_t)$ is infinite.

Before characterizing the pretabular logics in $\mathsf{NExt}(\mathsf{S4BP}^{2,\omega}_{2,2})$, we need to define some finite skeletons of $\mathsf{S4BP}^{2,\omega}_{2,2}$, namely, the (co)-garlands and hoops. These finite skeletons will play an important role in our proof.

4.5.1. DEFINITION. Let $\mathfrak{G}_{\mathbb{Z}}$ denote the frame $(\mathbb{Z}, R_z)$ where

$$R_z = \{\langle i, i\rangle : i \in \mathbb{Z}\} \cup \{\langle 2i+1, 2i\rangle : i \in \mathbb{Z}\} \cup \{\langle 2i+1, 2i+2\rangle : i \in \mathbb{Z}\}.$$

For all $\lambda \leq \omega$, we define $\mathfrak{G}_\lambda$ as the frame $\mathfrak{G}_{\mathbb{Z}}{\restriction}(1+\lambda)$. For each $n \in \mathbb{O}$, we define $\mathfrak{H}_n$ as the frame $(1+n, (R_z{\restriction}(1+n)) \cup \{\langle n, 0\rangle\})$. We call $\mathfrak{G}_n$ an n-garland and $\mathfrak{H}_n$ an n-hoop. Let $\mathcal{G} = \{\mathfrak{G}_n : n \in \omega\}$, $\mathcal{G}^{-1} = \{(\mathfrak{G}_n)^{-1} : n \in \omega\}$ and $\mathcal{H} = \{\mathfrak{H}_n : n \in \mathbb{O}\}$.

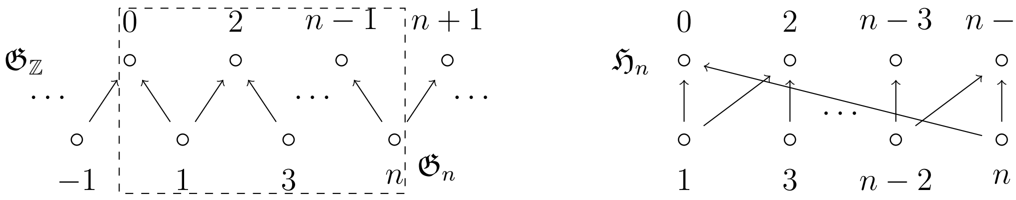

Figure 4.5: The garlands $\mathfrak{G}_n$ and hoops $\mathfrak{H}_n$ for some $n \in \mathbb{O}$

The frames $\mathfrak{G}_\mathbb{Z}$, $\mathfrak{G}_n$ and $\mathfrak{H}_n$ are depicted in Figure 4.4.

We first show that every (co)-garland and hoop is a t-morphic image of $\mathfrak{G}_\mathbb{Z}$.

4.5.2. Lemma. $\mathcal{G} \cup \mathcal{G}^{-1} \cup \mathcal{H} \cup \{\mathfrak{G}_\omega, (\mathfrak{G}_\omega)^{-1}\} \subseteq \mathsf{M}_t(\mathfrak{G}_\mathbb{Z})$.

Proof:
It is easy to see $\text{abs}(\cdot) : \mathfrak{G}_\mathbb{Z} \twoheadrightarrow \mathfrak{G}_\omega$, where $\text{abs}(\cdot)$ is the absolute value function defined as follows:

$$\text{abs}(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{otherwise.} \end{cases}$$

Take any $n \in \omega$. Then the function $h_n : x \mapsto x \mod (1+n)$ is a t-morphism from $\mathfrak{G}_\mathbb{Z}$ to $\mathfrak{H}_n$. Moreover, $g_n : \mathfrak{G}_\omega \twoheadrightarrow \mathfrak{G}_n$, where

$$g(x) = \begin{cases} x \mod 2n & \text{if } (x \mod 2n) \leq n; \\ 2n - (x \mod 2n) & \text{otherwise.} \end{cases}$$

Thus, $\mathcal{G} \cup \mathcal{H} \cup \{\mathfrak{G}_\omega\} \subseteq \mathsf{M}_t(\mathfrak{G}_\mathbb{Z})$ and so $\mathcal{G}^{-1} \cup \{(\mathfrak{G}_\omega)^{-1}\} \subseteq \mathsf{M}_t((\mathfrak{G}_\mathbb{Z})^{-1})$. Note that $s : x \mapsto x+1$ is an isomorphism between $\mathfrak{G}_\mathbb{Z}$ and $(\mathfrak{G}_\mathbb{Z})^{-1}$. Thus, $\mathcal{G} \cup \mathcal{G}^{-1} \cup \mathcal{H} \cup \{\mathfrak{G}_\omega, (\mathfrak{G}_\omega)^{-1}\} \subseteq \mathsf{M}_t(\mathfrak{G}_\mathbb{Z})$. □

Next, we show that rooted skeletons for $\mathsf{S4BP}^{2,\omega}_{2,2}$ are exactly the garlands and hoops. To do this, we introduce the following auxiliary fact:

4.5.3. Fact. Let $\mathfrak{F} = (X, R) \in \mathsf{Fr}_r(\mathsf{S4}_t)$ be a skeleton. Then the following holds:

(1) If $\mathfrak{F} \models \mathsf{bd}_2$, then for all $x \in X$, $R[x] = R_\sharp[x]$ or $R^{-1}[x] = R_\sharp[x]$.

(2) If $\mathfrak{F} \models \mathsf{bd}_2 \wedge \mathsf{bw}_2^+$, then for all $x \in X$, $|R[x]| \leq 3$.

(3) If $\mathfrak{F} \models \mathsf{bd}_2 \wedge \mathsf{bw}_2^-$, then for all $x \in X$, $|R^{-1}[x]| \leq 3$.

Proof:
The proof is standard, we omit it here. □

4.5.4. LEMMA. *Let* $\mathfrak{F} = (X, R) \in \mathsf{Fr}_r(\mathsf{S4BP}^{2,\omega}_{2,2})$ *be a skeleton and* $x \in X$*. Then*

(1) $\mathfrak{F}{\restriction}R^n_\sharp[x] \in \mathsf{I}(\mathcal{G} \cup \mathcal{G}^{-1} \cup \mathcal{H})$ *for all* $n \in \omega$*.*

(2) $\mathfrak{F}{\restriction}R^\omega_\sharp[x] \in \mathsf{I}(\mathcal{G} \cup \mathcal{G}^{-1} \cup \mathcal{H} \cup \{\mathfrak{G}_\omega, (\mathfrak{G}_\omega)^{-1}, \mathfrak{G}_\mathbb{Z}\})$*.*

Proof:
For (1), the proof proceeds by induction on n. When $n = 0$, we see $\mathfrak{F}{\restriction}R^0_\sharp[x] \cong \mathfrak{G}_0$. Let $n > 0$. By induction hypothesis, $\mathfrak{F}{\restriction}R^{n-1}_\sharp[x] \in \mathsf{I}(\mathcal{G} \cup \mathcal{G}^{-1} \cup \mathcal{H})$. Suppose $\mathfrak{F}{\restriction}R^{n-1}_\sharp[x] \in \mathsf{I}(\mathcal{H})$. Then $|R_\sharp[y] \cap R^{n-1}_\sharp[x]| = 3$ for all $y \in X$. By Fact 4.5.3, we have $R^n_\sharp[x] = R^{n-1}_\sharp[x]$ and so $\mathfrak{F}{\restriction}R^n_\sharp[x] \in \mathsf{I}(\mathcal{H})$. Suppose $\mathfrak{F}{\restriction}R^{n-1}_\sharp[x] \in \mathsf{I}(\mathcal{G})$. Then there exists $f : \mathfrak{G}_m \cong \mathfrak{F}{\restriction}R^{n-1}_\sharp[x]$ for some $m \in \omega$. Since $R^{n-1}_\sharp[x] \subseteq R^n_\sharp[x] \neq X$, by Lemma 2.2.15, $R^{n-1}_\sharp[x] \neq R^n_\sharp[x]$ and so $R_\sharp[f(0)] \nsubseteq R^{n-1}_\sharp[x]$ or $R_\sharp[f(m)] \nsubseteq R^{n-1}_\sharp[x]$. Then we have the following three cases:

- $R_\sharp[f(0)] \subseteq R^{n-1}_\sharp[x]$ and $R_\sharp[f(m)] \nsubseteq R^{n-1}_\sharp[x]$. Note that $\{f(m-1), f(m)\} \subseteq R_\sharp[f(m)]$, by Fact 4.5.3, there exists a unique point $y \in R_\sharp[f(m)] \setminus R^{n-1}_\sharp[x]$. Clearly, $R_\sharp[y] \cap R^{n-1}_\sharp[x] = \{f(m)\}$. Thus, $f \cup \{\langle m+1\rangle, y\} : \mathfrak{G}_{m+1} \cong \mathfrak{F}{\restriction}R^n_\sharp[x]$.

- $R_\sharp[f(0)] \nsubseteq R^{n-1}_\sharp[x]$ and $R_\sharp[f(m)] \subseteq R^{n-1}_\sharp[x]$. By a similar argument and Fact 4.5.3, there is a unique point $y \in R^{-1}[f(0)] \setminus R^{n-1}_\sharp[x]$, and we see that $g : (\mathfrak{G}_{m+1})^{-1} \cong \mathfrak{F}{\restriction}R^n_\sharp[x]$, where $g(0) = y$ and $g(1+k) = f(k)$ for all $0 < k \leq m$.

- $R_\sharp[f(0)] \nsubseteq R^{n-1}_\sharp[x]$ and $R_\sharp[f(m)] \nsubseteq R^{n-1}_\sharp[x]$. Again, by Fact 4.5.3, there exists a unique point $y \in R_\sharp[f(m)] \setminus R^{n-1}_\sharp[x]$ and a unique point $z \in R^{-1}[f(0)] \setminus R^{n-1}_\sharp[x]$. Suppose $y \neq z$. Then $g : (\mathfrak{G}_{m+2})^{-1} \cong \mathfrak{F}{\restriction}R^n_\sharp[x]$, where $g(0) = z$, $g(m+1) = y$ and $g(1+k) = f(k)$ for all $0 < k \leq m$. Otherwise, $y = z$ and we see that $f \cup \{\langle m+1, y\rangle\} : \mathfrak{H}_{m+1} \cong \mathfrak{F}{\restriction}R^n_\sharp[x]$.

Suppose $\mathfrak{F}{\restriction}R^{n-1}_\sharp[x] \in \mathsf{I}(\mathcal{G}^{-1})$. Then $\mathfrak{F}^{-1}{\restriction}R^{n-1}_\sharp[x] \in \mathsf{I}(\mathcal{G})$, which entails $\mathfrak{F}^{-1}{\restriction}R^n_\sharp[x] \in \mathsf{I}(\mathcal{G})$. Thus, $\mathfrak{F}{\restriction}R^n_\sharp[x] \in \mathsf{I}(\mathcal{G}^{-1})$. Since $\mathfrak{F}$ is rooted, (2) follows from (1). □

Let $\mathsf{Ga} = \mathsf{Log}(\mathfrak{G}_\mathbb{Z})$. The following theorem shows that Ga is the maximal logic in $\mathsf{NExt}(\mathsf{S4BP}^{2,\omega}_{2,2})$ with infinite r-degree.

4.5.5. THEOREM. *Let* $L \in \mathsf{NExt}(\mathsf{S4BP}^{2,\omega}_{2,2})$*. Then*

$$L \subseteq \mathsf{Ga} \text{ if and only if } \mathsf{br}_n \notin L \text{ for any } n \in \mathbb{Z}^+.$$

Proof:
The left-to-right direction is trivial. For the other direction, suppose $\mathsf{br}_n \notin L$ for any $n \in \mathbb{Z}^+$. Take any $\varphi \notin \mathsf{Ga}$ with $md(\varphi) = k$. Then $\mathfrak{G}_\mathbb{Z}, m \nvDash \varphi$ for some $m \in \mathbb{Z}$. Since $\mathsf{br}_{4k+2} \notin L$ and L is Kripke complete, there exists a frame $\mathfrak{F} = (X, R) \in \mathsf{Fr}_r(L)$ and $x \in X$ such that $\mathfrak{F}, x \nvDash \mathsf{br}_{4k+2}$. Then $\mathfrak{F}^S, C(x) \nvDash \mathsf{br}_{4k+2}$ and so $\mathsf{rdg}(C(x)) \geq 4k + 2$. Let $\mathfrak{G} = \mathfrak{F}^S{\upharpoonright}(R^S)^{4k+1}_\sharp[C(x)]$. By Lemma 4.5.4, $\mathfrak{G} \in \mathsf{I}(\mathcal{G} \cup \mathcal{G}^{-1} \cup \mathcal{H})$. Since $\mathsf{rdg}(C(x)) \geq 4k + 2$, $\mathfrak{G} \notin \mathsf{I}(\mathcal{H})$. Suppose $g : \mathfrak{G} \cong \mathfrak{G}_l$ for some $l \in \omega$. Since $\mathsf{rdg}(C(x)) \geq 4k + 2$, we see $l \geq 4k + 1$. Let $f : 1 + l \to \mathbb{Z}$ be the function such that $f(x) = x + m + (m \mod 2) - 2k$. Then we see $f \circ g : (\mathfrak{F}, y) \to^{l-1} (\mathfrak{G}_\mathbb{Z}, m)$, where y is such that $g(y) = 2k - (m \mod 2)$. By Lemma 4.1.3, $\mathfrak{F}, x \nvDash \varphi$ and so $\varphi \notin L$. □

4.5.6. Corollary. *Let $\mathcal{F} \subseteq \mathcal{G} \cup \mathcal{G}^{-1} \cup \mathcal{H}$ be infinite. Then* $\mathsf{Ga} = \mathsf{Log}(\mathcal{F})$.

It follows that Ga is pretabular:

4.5.7. Theorem. $\mathsf{Ga} \in \mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,2})$.

Proof:
Clearly, Ga is non-tabular. Take any $L \supsetneq \mathsf{Ga}$. By Theorem 4.5.5, $\mathsf{br}_n \in L$ for some $n \in \mathbb{Z}^+$. By Lemma 4.5.4, $\mathsf{Fr}_r(L) \subseteq \mathsf{I}(\mathcal{G} \cup \mathcal{G}^{-1} \cup \mathcal{H})$. It follows from Corollary 4.5.6 that $\mathsf{Fr}_r(L) \subseteq \mathsf{I}(\mathcal{F})$ for some finite $\mathcal{F} = \{\mathfrak{F}_i : i < n\} \subseteq \mathcal{G} \cup \mathcal{G}^{-1} \cup \mathcal{H}$. By Theorem 4.2.12, we see that $L = \mathsf{Log}(\bigoplus_{i<n} \mathfrak{F}_i)$ is tabular. □

In what follows, we give a characterization for pretabular logics over $\mathsf{S4BP}^{2,\omega}_{2,2}$ of finite r-degree. Note that every tense logic in $\mathsf{NExt}(\mathsf{S4BP}^{2,\omega}_{2,2})$ of finite r-degree is fully bounded. Thus, to study pretabularity, we now focus on c-irreducible pre-skeletons. The following lemmas (Lemma 4.5.8 and 4.5.9) characterize c-irreducible pre-skeletons based on finite garlands.

4.5.8. Lemma. *Let $m \leq n \in \omega$, $\mathfrak{F}^x_1$ be a pre-skeleton and $f : (\mathfrak{G}_n)^m_1 \twoheadrightarrow \mathfrak{F}^x_1$. Then*

(1) $f^{-1}[f(m)] = \{m\}$ and $f^{-1}[f(m_1)] = \{m_1\}$;

(2) for all different $k, l \in 1 + n$, $f(k) = f(l)$ implies $k + l = 2m$;

(3) if $2m < n$, then $f : (\mathfrak{G}_n)^m_1 \cong \mathfrak{F}^x_1$.

Proof:
Recall that the domain of $(\mathfrak{G}_n)^m_1$ is $(1 + n) \cup \{m_1\}$. Since $f : (\mathfrak{G}_n)^m_1 \twoheadrightarrow \mathfrak{F}^x_1$, by Lemma 4.3.13, $f[\{m, m_1\}] = \{x, x_1\}$ is the unique proper cluster in $\mathfrak{F}^x_1$. For (1), take any $k \in (\mathfrak{G}_n)^m_1$. Suppose $f(k) = f(m)$. Then $C(f(k))$ is a proper cluster. By Lemma 4.3.13, $C(k)$ is proper. Since $f(m) \neq f(m_1)$ and $\{m, m_1\}$ is the unique

proper cluster in $(\mathfrak{G}_n)_1^m$, we see $k = m$. Hence, $f^{-1}[f(m)] = \{m\}$. Similarly, we obtain that $f^{-1}[f(m_1)] = \{m_1\}$.

For (2), take any different $k, l \in 1+n$ such that $f(k) = f(l)$. By (1), $k \neq m \neq l$. Let $\mathfrak{F}_1^x = (X, R)$. Then there exists $a \in \mathbb{Z}^+$ such that $f(m) \in R_\sharp^a[f(k)] \setminus R_\sharp^{a-1}[f(k)]$. Thus, $f(m) \in f[R_\sharp^a[k]] \setminus f[R_\sharp^{a-1}[k]]$ and $f(m) \in f[R_\sharp^a[l]] \setminus f[R_\sharp^{a-1}[l]]$. By (1), $m \in R_\sharp^a[k] \setminus R_\sharp^{a-1}[k]$ and $m \in R_\sharp^a[l] \setminus R_\sharp^{a-1}[l]$. Since $k \neq l$, we see $\{k, l\} = \{m-a, m+a\}$. Thus, $k + l = 2m$.

For (3), suppose $2m < n$. By (1) and (2), $f^{-1}[f[y]] = \{y\}$ for all $y \in \{m, m_1\} \cup \{k \in 1+n : k > 2m\}$. Since $2m+1 \in R_\sharp^m[m+a] \setminus R_\sharp^m[m-a]$ for all $a \leq m$, by (2), we see $f^{-1}[f[k]] = \{k\}$ for all $k \leq 2m$. Thus, f is injective and so $f : (\mathfrak{G}_n)_1^m \cong \mathfrak{F}_1^x$. □

4.5.9. Lemma. *Let $m, n \in \omega$. Suppose $n > 0$ and $2m \leq n$. Then*

(1) $(\mathfrak{G}_n)_1^m$ is c-irreducible if and only if $2m \neq n$.

(2) $((\mathfrak{G}_n)^{-1})_1^m$ is c-irreducible if and only if $2m \neq n$.

Proof:
Note that $\mathsf{M}_t(((\mathfrak{G}_n)^{-1})_1^m) = \{\mathfrak{F}^{-1} : \mathfrak{F} \in \mathsf{M}_t((\mathfrak{G}_n)_1^m)\}$, (2) follows from (1) immediately. For (1), suppose $2m \neq n$. By Lemma 4.5.8(3), $(\mathfrak{G}_n)_1^m$ is c-irreducible. For the other direction, suppose $2m = n$. Let $f : X \to (1+m) \cup \{m_1\}$ be defined by:

$$f(x) = \begin{cases} x & \text{if } x \leq m \text{ or } x = m_1; \\ n - x & \text{otherwise.} \end{cases}$$

Thus, $f : (\mathfrak{G}_n)_1^m \twoheadrightarrow (\mathfrak{G}_m)_1^m$. Since $n > 0$, we see $n > m$ and so $(\mathfrak{G}_m)_1^m \ncong (\mathfrak{G}_n)_1^m$. By Lemma 4.3.13, $(\mathfrak{G}_m)_1^m \notin \mathsf{M}_t(\mathfrak{G}_n)$. Thus, $(\mathfrak{G}_n)_1^m$ is c-reducible. □

It is worth noting that there exists no c-irreducible pre-skeleton based on hoops. More precisely, the following lemma holds:

4.5.10. Lemma. *For all $n \in \mathbb{O}$ and $m \leq n$, the frame $(\mathfrak{H}_n)_1^m$ is c-reducible.*

Proof:
Suppose $n = 4k+1$ for some $k \in \omega$ and $m \in \mathbb{E}$. Then we let the function $f : (4k+2) \cup \{m_1\} \to (4k+2) \cup \{(2k)_1\}$ be defined as follows:

$$f(x) = \begin{cases} (x + 2k - m) \mod (4k+1) & \text{if } x \in (4k+2); \\ (2k)_1 & \text{otherwise.} \end{cases}$$

It is clear that $f : (\mathfrak{H}_n)^m_1 \cong (\mathfrak{H}_n)^{2k}_1$. We define the function $g : (4k+2) \cup \{(2k)_1\} \to (2k+1) \cup \{0_1\}$ by

$$g(x) = \begin{cases} l & \text{if } \mathrm{abs}(x - 2m) = l \text{ and } x \in (n+1); \\ 0_1 & \text{otherwise.} \end{cases}$$

Then we see that $g : (\mathfrak{H}_n)^{2k}_1 \twoheadrightarrow (\mathfrak{G}_{2k+2})^0_1$, which entails $g \circ f : (\mathfrak{H}_n)^m_1 \twoheadrightarrow (\mathfrak{G}_{2k+2})^0_1$. Note that $(\mathfrak{G}_{2k+2})^0_1 \notin \mathsf{M}_t(\mathfrak{H}_n)$ and $(\mathfrak{G}_{2k+2})^0_1 \not\cong (\mathfrak{H}_n)^m_1$, $(\mathfrak{H}_n)^m_1$ is c-reducible. Assume $m \in \mathbb{O}$. Then we can construct maps $f' : (\mathfrak{H}_n)^m_1 \cong (\mathfrak{H}_n)^{2k+1}_1$ and $g' : (\mathfrak{H}_n)^{2k+1}_1 \twoheadrightarrow ((\mathfrak{G}_{2k+1})^{-1})^0_1$ similarly, which also implies that $(\mathfrak{H}_n)^m_1$ is c-reducible. Analogous arguments work for the case $n = 4k+3$ for some $k \in \omega$. □

Now we are ready to prove the main theorem of this section:

4.5.11. THEOREM. *The set* $\mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,2})$ *is characterized as follows:*

$$\mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,2}) = \{\mathsf{Ga}, \mathsf{S5}_t\} \cup \{\mathsf{Log}((\mathfrak{G}_n)^m_\omega), \mathsf{Log}(((\mathfrak{G}_n)^{-1})^m_\omega) : 2m < n \in \mathbb{Z}^+\}.$$

Proof:
Note that $\mathsf{S5}_t = L^\circ$ and $\mathsf{S5}_t \supseteq \mathsf{S4BP}^{2,\omega}_{2,2}$, by Theorem 4.4.9, $\mathsf{S5}_t$ is pretabular. By Theorem 4.5.7, $\mathsf{Ga} \in \mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,2})$. Take any $2m < n \in \omega$. By Lemma 4.5.9 and Lemma 4.3.16, $(\mathfrak{G}_n)^m_\omega$ is c-irreducible. By Theorem 4.3.20, $\mathsf{Log}((\mathfrak{G}_n)^m_\omega)$ is pretabular.

Take any $L \in \mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,2})$. Suppose $\mathsf{br}_n \notin L$ for any $n \in \mathbb{Z}^+$. Then by Theorem 4.5.5, $L \subseteq \mathsf{Ga}$. Since L is pretabular, $L = \mathsf{Ga}$. Suppose $\mathsf{br}_n \in L$ for some $n \in \mathbb{Z}^+$. Then $L \in \mathsf{PTAB}(\mathsf{S4BP}^{2;n}_{2,2})$. By Theorem 4.3.20, $L = \mathsf{Log}(\mathfrak{F}^x_\omega)$ for some c-irreducible rooted finite pre-skeleton. By Lemma 4.5.4, $\mathfrak{F} \in \mathsf{I}(\mathcal{G} \cup \mathcal{G}^{-1} \cup \mathcal{H})$. If $|\mathfrak{F}| = 1$, then $\mathfrak{F}^x_\omega \cong (\mathfrak{G}_0)^0_\omega$ and so $L = \mathsf{S5}_t$. Suppose $|\mathfrak{F}| \neq 1$, then $|\mathfrak{F}| = 1 + n$ for some $n \in \mathbb{Z}^+$. By Lemma 4.5.9 and Lemma 4.5.10, $\mathfrak{F}^x_\omega \cong (\mathfrak{G}_n)^m_\omega$ or $\mathfrak{F}^x_\omega \cong ((\mathfrak{G}_n)^{-1})^m_\omega$ for some $m \in \omega$ such that $2m \neq n$. If $2m < n$, then we are done. If $2m > n$, then we see that $\mathfrak{F}^x_\omega \cong (\mathfrak{G}_n)^{n-m}_\omega$ or $\mathfrak{F}^x_\omega \cong ((\mathfrak{G}_n)^{-1})^{n-m}_\omega$. Thus, $L \in \{\mathsf{Log}((\mathfrak{G}_n)^m_\omega) : 2m < n \in \omega\}$. □

4.5.12. EXAMPLE. Note that the frame $\mathfrak{F}^{x_3}_\omega$ given in Figure 4.1 is isomorphic to $((\mathfrak{G}_3)^{-1})^0_\omega$, by Theorem 4.5.11, we see that $\mathsf{Log}(\mathfrak{F}^{x_3}_\omega)$ is pretabular.

4.5.13. THEOREM. $|\mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,2})| = \aleph_0$.

Proof:
By Theorem 4.5.11, $|\mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,2})| \leq \aleph_0$. Take any $n, m, k, l \in \omega$ such that $2m < n$ and $2l < k$. It is clear that $(\mathfrak{G}_n)^m_\omega$, $((\mathfrak{G}_n)^{-1})^m_\omega$, $(\mathfrak{G}_k)^l_\omega$ and $((\mathfrak{G}_k{}^{-1}))^l_\omega$ are pairwise non-isomorphic. By Lemma 4.3.15, their logics are pairwise different.

Thus, by Theorem 4.5.11, $|\mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,2})| = \aleph_0$. □

Moreover, we show the following anti-dichotomy theorem for the cardinality of pretabular extensions in $\mathsf{NExt}(\mathsf{S4BP}^{2,\omega}_{2,2})$:

4.5.14. Theorem. *For all $\kappa \leq \aleph_0$, there exists $L \in \mathsf{NExt}(\mathsf{S4}_t)$ such that*

$$|\mathsf{PTAB}(L)| = \kappa.$$

Proof:
By Theorem 4.5.13, $|\mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,2})| = \aleph_0$. Obviously, $\mathsf{Log}(\mathfrak{Ch}_1) \supseteq \mathsf{S4BP}^{2,\omega}_{2,2}$ and $|\mathsf{PTAB}(\mathsf{Log}(\mathfrak{Ch}_1))| = 0$. Take any $\kappa \in \mathbb{Z}^+$. Let $\mathcal{F} = \{(\mathfrak{G}_{n+1})^0_\omega : n < \kappa\}$ and $L = \bigcap_{\mathfrak{F}\in\mathcal{F}} \mathsf{Log}(\mathfrak{F})$. Then $|\mathcal{F}| = \kappa$. Note that $\mathsf{bd}_1 \notin L$ and $\mathsf{br}_{\kappa+1} \in L$, $\{\mathsf{Ga}, \mathsf{S5}_t\} \cap \mathsf{PTAB}(L) = \varnothing$. Take any $\mathfrak{F}^x_\omega \in \{(\mathfrak{G}_n)^m_\omega, ((\mathfrak{G}_n)^{-1})^m_\omega : 2m < n \in \mathbb{Z}^+\}$. Suppose $\mathfrak{F}^x_\omega \notin \mathcal{F}$. Then similar to the proof of Lemma 4.3.15, we see $\mathcal{J}^{\mathsf{rdg}(\mathfrak{F})+\kappa}(\mathfrak{F}^x_1, x) \notin L$. Thus, $\mathfrak{F} \in \mathcal{F}$ if and only if $\mathsf{Log}(\mathfrak{F}) \supseteq L$. By Theorem 4.5.11, $\mathsf{PTAB}(L) = \{\mathsf{Log}(\mathfrak{F}) : \mathfrak{F} \in \mathcal{F}\}$. By Lemma 4.3.15, $|\mathsf{PTAB}(L)| = \kappa$. □

As a corollary, we have

4.5.15. Theorem. *Every pretabular logic in $\mathsf{NExt}(\mathsf{S4BP}^{2,\omega}_{2,2})$ has the FMP.*

Proof:
Follows from Lemma 4.3.9, Corollary 4.5.6 and Theorem 4.5.11 immediately. □

4.5.16. Remark. The results obtained in this section are closely related to the ones in [71, Section 4]. The logic Ga was defined to be $\mathsf{S4}_t \oplus \{\mathsf{grz}, \mathsf{alt}^+_3, \mathsf{alt}^-_3, \mathsf{bd}_2\}$. Garlands $\mathfrak{G}_n$ were also defined there. It was proved that $\mathsf{Ga} = \mathsf{Log}(\mathfrak{G}_\omega) = \bigcap_{n\in\omega} \mathsf{Log}(\mathfrak{G}_n)$ is pretabular.

However, there are some problematic claims in [71, Section 4], which makes the characterization of $\mathsf{NExt}(\mathsf{Ga})$ given there incomplete. It was claimed that a rooted frame $\mathfrak{F}$ validates Ga if and only if $\mathfrak{F} \cong \mathfrak{G}_n$ for some $n \in \omega$. It follows that $\{\mathsf{Log}(\mathfrak{G}_n) : n \text{ is prime}\}$ is the set of all logics in $\mathsf{NExt}(\mathsf{Ga})$ which are of codimension 3. However, as Lemma 4.5.4 shows, the class of rooted frame for Ga is $\mathsf{I}(\mathcal{G} \cup \mathcal{G}^{-1} \cup \mathcal{H} \cup \{\mathfrak{G}_\omega, (\mathfrak{G}_\omega)^{-1}, \mathfrak{G}_\mathbb{Z}\})$, which is not the same as $\mathsf{I}(\mathcal{G})$. Consider the frames $\mathfrak{H}_3$ and $(\mathfrak{G}_2)^{-1}$, as shown in Figure 4.6. Then we see that $\mathfrak{H}_3 \models \mathsf{Ga}$ but $\mathfrak{H}_3 \ncong \mathfrak{G}_n$ for any $n \in \omega$. Moreover, note that $\mathsf{tab}^T_4 \wedge \neg\mathcal{J}^4(\mathfrak{G}_3, 0) \in \mathsf{Log}(\mathfrak{H}_3)$ and $\mathsf{tab}^T_3 \notin \mathsf{Log}(\mathfrak{H}_3)$, we see that $\mathsf{Log}(\mathfrak{H}_3) \notin \{\bigcap_{i\in I} \mathsf{Log}(\mathfrak{G}_i) : I \subseteq \omega\}$. Thus, $\mathsf{Log}(\mathfrak{H}_3)$ is missing in the characterization given by Kracht [71]. It is also straightforward to show that $\mathsf{Log}((\mathfrak{G}_2)^{-1}) \notin \{\bigcap_{i\in I} \mathsf{Log}(\mathfrak{G}_i) : I \subseteq \omega\}$.

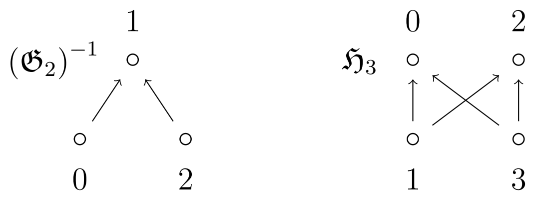

Figure 4.6: The garland $(\mathfrak{G}_2)^{-1}$ and hoop $\mathfrak{H}_3$

4.6 Pretabular Tense Logics in $\mathsf{NExt}(\mathsf{S4BP}^{2,\omega}_{2,3})$

So far, we have studied several families of tense logics with bounded parameters and constructed countably many pretabular logics in $\mathsf{NExt}(\mathsf{S4}_t)$. In this section, we investigate the pretabular tense logics extending $\mathsf{S4BP}^{2,\omega}_{2,3}$, which imposes a weaker constraint on back-width than $\mathsf{S4BP}^{2,\omega}_{2,2}$: namely, back-width at most 3 is allowed instead of 2. As we will see shortly, the class of rooted frames for $\mathsf{S4BP}^{2,\omega}_{2,3}$ is much more complicated than that for $\mathsf{S4BP}^{2,\omega}_{2,2}$.

The aim of this section is to show that $|\mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,3})| = 2^{\aleph_0}$, that is, there exists continuum many pretabular logics extending $\mathsf{S4BP}^{2,\omega}_{2,3}$. The strategy is as follows: We first construct a continual family of sequences of integers by generalizing the Thue-Morse sequences. Based on these sequences, we construct corresponding 'umbrella-like' frames. Then we show that their logics are pairwise different and pretabular. As a corollary, $|\mathsf{PTAB}(\mathsf{S4}_t)| = 2^{\aleph_0}$, which answers the open problem about the cardinality of $\mathsf{PTAB}(\mathsf{S4}_t)$ given in [107].

In the following subsection, we review the preliminaries on sequences and introduce the notion of generalized Thue-Morse sequences.

4.6.1 Generalized Thue-Morse sequences

For all $i, j \in \mathbb{Z}$, we write $[i, j]$ for $\{k \in \mathbb{Z} : i \leq k \leq j\}$. A subset I of $\mathbb{Z}$ is said to be an *interval in* $\mathbb{Z}$ if for all $i, j \in I$, $[i, j] \subseteq I$. A map $t : \mathbb{Z} \to \mathbb{Z}$ is called a *translation* if there exists $k \in \mathbb{Z}$ such that $t(i) = i + k$ for all $i \in \mathbb{Z}$. We write $t : a \mapsto b$ for the translation t such that $t : i \mapsto (i + b - a)$ for all $i \in \mathbb{Z}$. Let X be a non-empty set. An X*-sequence* is a partial function $f : \mathbb{Z} \to X$ where $\mathsf{dom}(f)$ is an interval in $\mathbb{Z}$. Let $\mathsf{Seq}(X)$ and $\mathsf{Seq}^{<\aleph_0}(X)$ denote the sets of all X-sequences and finite X-sequences, respectively.

Let $\alpha : [a, b] \to X$ be a nonempty finite sequence such that $\alpha(i) = x_{i-a}$ for all $i \in [a, b]$. Then we write $\langle a, \langle x_0, \cdots, x_b \rangle\rangle$ for α. We write $\langle x_0, \cdots, x_b\rangle$ for α if $a = 0$.

Let $\alpha, \beta \in \mathsf{Seq}(X)$. Then we say (i) α *is embedded into* β *(notation:* $\alpha \trianglelefteq \beta$*)*, if $\alpha \circ s \subseteq \beta$ for some translation s. (ii) α *is finitely covered by* β *(notation:* $\alpha \preceq \beta$*)*, if $\gamma \trianglelefteq \beta$ for all finite sequence $\gamma \subseteq \alpha$. (iii) α *and* β *are similar (notation:* $\alpha \approx \beta$*)*, if $\alpha \preceq \beta$ and $\beta \preceq \alpha$. α and β are *dissimilar* if $\alpha \not\approx \beta$.

4.6.1. Remark. As the reader might have already noticed, our notion of a sequence is slightly more general than the usual one: X-sequences are partial functions from $\mathbb{Z}$ to X, but not simply functions from ordinals to X. The main reason for this choice is that we need to consider infinite sequences that extend in both directions. This feature turns out to be essential for the construction in Definition 4.6.12 as well as for the proofs such as the one of Lemma 4.6.16.

Next, we introduce some operations on sequences. The first is the notion of *concatenation* of sequences. Since the domains of finite sequences are intervals of integers instead of natural numbers, we need a slightly general definition of concatenation of sequences.

4.6.2. Definition (Concatenation). Let $\alpha : [a, b] \to X$ and $\beta : [c, d] \to X$ be finite X-sequences for some nonempty set X. Then we define $\alpha * \beta$ by

$$\alpha * \beta = \langle \alpha(a), \cdots, \alpha(b), \beta(c), \cdots, \beta(d) \rangle.$$

The sequences $\alpha^\dagger * \beta$ and $\alpha * \beta^\dagger$ are defined as follows:

$$\alpha^\dagger * \beta = \langle a, \alpha * \beta \rangle \text{ and } \alpha * \beta^\dagger = \langle c + a - (b + 1), \alpha * \beta \rangle.$$

Let $\langle \alpha_i : i \in n \rangle$ be a finite tuple of finite X-sequences. Then we write $\alpha_0 * \alpha_1 * \cdots * \alpha_{n-1}$ or $\alpha_0\alpha_1 \cdots \alpha_{n-1}$ for $(\cdots(\alpha_0 * \alpha_1) * \cdots * \alpha_{n-2}) * \alpha_{n-1}$. Moreover, let

$$\alpha_0 * \cdots * {\alpha_m}^\dagger * \cdots * \alpha_{n-1} = ((\alpha_0 * \cdots * \alpha_{m-1}) * {\alpha_m}^\dagger)^\dagger * (\alpha_{m+1} * \cdots * \alpha_{n-1}).$$

The notation in Definition 4.6.2 looks complicated, but the idea behind is simple. Given any finite tuple $A = \langle \alpha_i : i \in n \rangle$ of finite X-sequences, the sequence $\alpha_0 * \cdots * {\alpha_m}^\dagger * \cdots * \alpha_{n-1}$ is designed to be the concatenation of A which always preserves the index of α_m. An example is given in Table 4.1.

4.6.3. Definition. Let X be a non-empty set, $A \subseteq \mathsf{Seq}(X)$ and $n \in \omega$. Then the set $\mathsf{Conc}(A, n)$ of all n-concatenations of A is defined as follows:

$$\mathsf{Conc}(A, n) = \{\alpha_0 \cdots \alpha_{n-1} : \forall i < n(\alpha_i \in A)\}.$$

Let $\mathsf{Conc}(A) = \bigcup_{i \in \omega} \mathsf{Conc}(A, n)$.

In what follows, we will mainly focus on 2-sequences, that is, $\{0, 1\}$-sequences. For each 2-sequence $\alpha : \mathbb{Z} \to 2$, the *inverse* $\overline{\alpha}$ of α is defined by

$$\overline{\alpha}(x) = 1 - \alpha(x), \text{ for all } x \in \mathsf{dom}(\alpha).$$

$\mathbb{Z}$	$\cdots$	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	$\cdots$
α						a	b	c	d					
β					x	y	z							
γ					u	v	w							
$\alpha * \beta$							a	b	c	d	x	y	z	
$\alpha^\dagger * \beta$						a	b	c	d	x	y	z		
$\alpha * \beta * \gamma^\dagger * \alpha$	$\cdots$	x	y	z	u	v	w	a	b	c	d			

Table 4.1: Example of concatenations of sequences

Now we are ready to introduce the Thue-Morse sequence, which was originally introduced by Thue [123] and rediscovered by Morse [98]. The Thue-Morse sequence α^t is the 2-sequence defined by $\alpha^t = \bigcup_{i\in\omega} \alpha^t_i$, where

- $\alpha^t_0 = \langle 0 \rangle$; and
- $\alpha^t_i = \alpha_i * \overline{\alpha_{i-1}}$ for all $i > 0$.

The sequence α^t has many nice properties. For example, α^t is shown to be overlap-free, i.e., $x\beta x\beta x \not\trianglelefteq \alpha^t$ for any 2-sequence β and $x \in \{0, 1\}$ (see, e.g., [54, Proposition 5.1.6]). Another notable property of the Thue-Morse sequence is that it is *uniformly recurrent* (see, e.g., [83, p.30–31]), a notion defined as follows:

4.6.4. Definition. Let $\alpha : \mathbb{Z} \to 2$ be a map. We say that α is *uniformly recurrent* if for all finite subsequence β of α, there is $n \in \omega$ such that $\beta \trianglelefteq \zeta$ for all $\zeta \trianglelefteq \alpha$ with $|\zeta| > n$.

As we shall see later, uniformly recurrent 2-sequences play a central role in our proofs. To construct continuum many pretabular logics, we need to obtain a continual family of uniformly recurrent 2-sequences. We generalize the Thue-Morse sequence as follows:

4.6.5. Definition (Generalized Thue-Morse sequence). Let $f : \omega \to 2$. For each $i \in \omega$, the finite binary sequence χ_i is defined as follows:

- $\chi^f_0 = \langle 0, 0, 1 \rangle$;
- $\chi^f_{2i+1} = {\chi^f_{2i}}^\dagger * \langle f(2i) \rangle * \overline{\chi^f_{2i}}$;
- $\chi^f_{2i+2} = \overline{\chi^f_{2i+1}} * \langle f(2i+1) \rangle * {\chi^f_{2i+1}}^\dagger$.

Let $\chi^f = \bigcup_{i\in\omega} \chi^f_i$. Then we see χ^f is a function from $\mathbb{Z}$ to 2. The sequence χ^f is called the generalized Thue-Morse sequence generated by f.

4.6.6. Example. Consider the maps $f:\omega\to\{0\}$ and $g:\omega\to\{1\}$. Then the sequences χ^f and χ^g are constructed as follows:

$\mathbb{Z}$	$\cdots$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	$\cdots$
χ^f_0										0	0	1						
χ^g_1										0	0	1	1	1	1	0		
χ^f_2		1	1	0	1	0	0	1	0	0	0	1	0	1	1	0		
χ^g_2		1	1	0	0	0	0	1	1	0	0	1	1	1	1	0		
									$\vdots$									
χ^f	$\cdots$	1	1	0	1	0	0	1	0	0	0	1	0	1	1	0	0	$\cdots$
χ^g	$\cdots$	1	1	0	0	0	0	1	1	0	0	1	1	1	1	0	1	$\cdots$

We now study the properties of the generalized Thue-Morse sequences. The first task is to show that every generalized Thue-Morse sequence is uniformly recurrent (Lemma 4.6.9).

4.6.7. Lemma. *Let $f\in 2^\omega$, $j\le i\in\omega$ and $\nu(j)=2^{i-j}$. Then there are $\alpha_0,\cdots,\alpha_{\nu(j)}\in\{\chi^f_j,\overline{\chi^f_j}\}$ and $n_0,\cdots,n_{\nu(j)-1}\in 2$ such that*

$$\chi^f_i\approx\alpha_0 n_0\cdots\alpha_{\nu(j)-1}n_{\nu(j)-1}\alpha_{\nu(j)}.$$

Proof:
The proof proceeds by induction on $i-j$. The case $i-j=0$ is trivial. Suppose $i-j>0$. By induction hypothesis, $\chi^f_i\approx\alpha_0 n_0\cdots\alpha_{\nu(j+1)}n_{\nu(j)}\alpha_{\nu(j+1)}$ for some $\alpha_0,\cdots,\alpha_{\nu(j+1)}\in\{\chi^f_{j+1},\overline{\chi^f_{j+1}}\}$ and $n_0,\cdots,n_{\nu(j+1)-1}\in 2$. Note that for all $l\le\nu(j+1)$, we have $\alpha_l\approx\beta^1_l m_l\beta^2_l$ for some $\beta^1_l,\beta^2_l\in\{\chi^f_j,\overline{\chi^f_j}\}$ and $m_l\in 2$. Thus, $\chi^f_i\approx\beta^1_0 m_0\beta^2_0 n_0\cdots n_{\nu(j+1)-1}\beta^1_{\nu(j+1)}m_{\nu(j+1)}\beta^2_{\nu(j+1)}$, which concludes the proof. □

As a corollary, we have

4.6.8. Lemma. *Let $f\in 2^\omega$. Then for all $j\in\omega$,*

$$\chi^f\preceq\mathsf{Conc}(\{\chi^f_j 0,\chi^f_j 1,\overline{\chi^f_j}0,\overline{\chi^f_j}1\}).$$

Intuitively, Lemma 4.6.8 says that the generalized Thue-Morse sequence χ^f is builded up by iterating elements in $\{\chi^f_j 0,\chi^f_j 1,\overline{\chi^f_j}0,\overline{\chi^f_j}1\}$. It follows that the generalized Thue-Morse sequences are uniformly recurrent:

4.6.9. Lemma. *Let $f\in 2^\omega$. Then χ^f is uniformly recurrent.*

Proof:
Take any finite subsequence $\alpha \trianglelefteq \chi^f$. Then $\alpha \trianglelefteq \chi^f_{i-1}$ for some $i \in \omega$. Let $n = 2(|\chi^f_i| + 1)$. Take any $\gamma \trianglelefteq \chi^f$ with $|\gamma| > n$. By Lemma 4.6.8, we see that $\beta \trianglelefteq \gamma$ for some $\beta \in \{\chi^f_i 0, \chi^f_i 1, \overline{\chi^f_i} 0, \overline{\chi^f_i} 1\}$. Note that since $\chi^f_{i-1} \trianglelefteq \chi^f_i$ and $\chi^f_{i-1} \trianglelefteq \overline{\chi^f_i}$, we obtain that $\alpha \trianglelefteq \beta \trianglelefteq \gamma$. □

The final task in this subsection is to show that given distinct functions $f, g : \omega \to 2$, the sequences χ^f and χ^g are dissimilar (Lemma 4.6.11).

4.6.10. Lemma. *Let $f \in 2^\omega$ and $i \in \mathbb{Z}^+$. Let $\alpha : [a, b] \to 2$ and $\beta : [c, d] \to 2$ be 2-sequences such that $\alpha \approx \chi^f_i$ and $\beta \approx \alpha x \overline{\alpha}$. Then for every translation $t : \mathbb{Z} \to \mathbb{Z}$,*

(1) $\alpha \circ t \subseteq \beta$ implies $t : a \mapsto c$;

(2) $\alpha \circ t \subseteq \overline{\beta}$ implies $t : b \mapsto d$.

Proof:
We prove (1) and (2) together by induction on i. Suppose $i = 1$. Then we see that $\alpha \approx 001f(0)110$, $\beta \approx 110\overline{f(0)}001x001f(0)110$ and $\overline{\beta} \approx 001f(0)110\overline{x}110\overline{f(0)}001$. It is not hard to verify that both (1) and (2) hold. Let $i > 1$. Assume $i \in \mathbb{O}$. Then $\alpha \approx \chi^f_{i-1} f(i-1) \overline{\chi^f_{i-1}}$ and $\beta \approx \chi^f_{i-1} f(i-1) \overline{\chi^f_{i-1}} x \overline{\chi^f_{i-1}}\, \overline{f(i-1)} \chi^f_{i-1}$. Suppose $\alpha \circ t \subseteq \beta$. Let $\gamma = \alpha \restriction [a, a + |\chi^f_i| - 1]$, $\gamma' = \alpha \restriction [b + 1 - |\chi^f_i|, b]$, $\lambda = \beta \restriction [c, c + |\alpha| - 1]$ and $\lambda' = \beta \restriction [d + 1 - |\alpha|, d]$. Then either $\gamma \circ t \subseteq \lambda$ or $\gamma' \circ t \subseteq \lambda'$. By induction hypothesis, we see $t : a \mapsto c$. The case for $\overline{\beta}$ is similar. □

4.6.11. Lemma. *For all $f, g \in 2^\omega$, we have that $f = g$ if and only if $\chi^f \preceq \chi^g$.*

Proof:
The left-to-right direction is trivial. For the other direction, suppose that $f \neq g$ and $\chi^f \preceq \chi^g$. Then there exists $i \in \omega$ such that $f(i) \neq g(i)$ and $f(j) = g(j)$ for all $j < i$. Without loss of generality, assume $f(i) = 0$ and $i \in \mathbb{E}$. Then $\chi^f_2 \approx \overline{\chi^f_i} 1 \chi^f_i f(i+1) \chi^f_i 0 \overline{\chi^f_i}$. By Lemma 4.6.8, $\chi^g \preceq \mathsf{Conc}(\{\chi^g_{i+1} 0, \chi^g_{i+1} 1, \overline{\chi^g_{i+1}} 0, \overline{\chi^g_{i+1}} 1\})$. Since $g(i) = 1$ and $f(j) = g(j)$ for all $j < i$, $\chi^g_{i+1} = \chi^f_i 1 \overline{\chi^f_i}$. Then we see that $\chi^g \preceq \mathsf{Conc}(\{\chi^f_i 1 \overline{\chi^f_i} 0, \chi^f_i 1 \overline{\chi^f_i} 1, \overline{\chi^f_i} 0 \chi^f_i 0, \overline{\chi^f_i} 0 \chi^f_i 1\})$. However, by Lemma 4.6.10, neither $\chi^f_i 1 \overline{\chi^f_i} \trianglelefteq \chi^f_2 \approx \overline{\chi^f_i} 1 \chi^f_i f(i+1) \chi^f_i 0 \overline{\chi^f_i}$ nor $\overline{\chi^f_i} 0 \chi^f_i \trianglelefteq \chi^f_2$ holds. Thus, $\chi^f_{i+2} \not\trianglelefteq \chi^g$, which contradicts $\chi^f \preceq \chi^g$. □

4.6.2 Umbrellas and their properties

We are now ready to construct the frames for $\mathsf{S4BP}^{2,\omega}_{2,3}$, which we call *umbrellas*. For each sequence $\alpha : \mathbb{Z} \to 2$, we define a frame $\mathfrak{Z}_\alpha$ and call it the umbrella induced by α. Using a key property of $\mathfrak{Z}_\alpha$ (Lemma 4.6.23), we show that $\mathsf{Log}(\mathfrak{Z}_\alpha)$ is pretabular for every uniformly recurrent $\alpha : \mathbb{Z} \to 2$.

We start by introducing the definition of umbrellas.

4.6.12. Definition. Let $\mathfrak{Z}_0 = (Z_0, R_0)$ and $\mathfrak{Z}_1 = (Z_1, R_1)$ be frames as depicted in Figure 4.7. To be precise, we define

- $Z_0 = \{a_i : i < 6\} \cup \{b_0, b_1\} \cup \{c_i : i < 2\}$,
- $Z_1 = Z_0 \cup \{a_6, a_7\}$, and
- $R_0 = R_1 \restriction Z_0$,

where R_1 is the reflexive-transitive closure of the union of the following sets

- $\{\langle a_{2i}, a_{2i+1}\rangle : i < 4\}$;
- $\{\langle a_{2i+2}, a_{2i+1}\rangle : i < 3\}$;
- $\{\langle b_0, b_1\rangle, \langle b_0, a_1\rangle, \langle c_0, c_1\rangle, \langle c_2, c_1\rangle, \langle c_0, a_3\rangle\}$.

For each $\alpha : \mathbb{Z} \to 2$, we define the frame $\mathfrak{Z}_\alpha = (Z_\alpha, R_\alpha)$ as follows:

- $Z_\alpha = \biguplus_{i\in\mathbb{Z}} Z_{\alpha(i)} = \{\langle x, i\rangle : x \in Z_{\alpha(i)} \text{ and } i \in \mathbb{Z}\}$
- $\langle x, i\rangle R_\alpha \langle y, j\rangle$ if and only if one of the following holds:
  - $i = j$ and $R_{\alpha(i)}xy$;
  - $j = i + 1$, $f(i) = 0$, $x = a_5$ and $y = a_0$;
  - $j = i + 1$, $f(i) = 1$, $x = a_7$ and $y = a_0$.

The frame $\mathfrak{Z}_\alpha$ is called the *umbrella* induced by α.

In what follows, let $\alpha : \mathbb{Z} \to 2$ be an arbitrarily fixed sequence on $\mathbb{Z}$ and $\mathfrak{Z}_\alpha = (Z, R)$. Our main task is to show that $\mathsf{Log}(\mathfrak{Z}_\alpha)$ is pretabular. To make the proofs below easier to read, we re-indexed the elements in Z by an onto map $l : Z \to \mathbb{Z}$ where:

- $l(\langle a_0, 0\rangle) = 0$;
- for all $\langle a_i, j\rangle, \langle a_{i'}, j'\rangle \in Z$, $l(\langle a_i, j\rangle) < l(\langle a_{i'}, j'\rangle)$ if and only if (i) $j < j'$ or, (ii) $j = j'$ and $i < i'$;

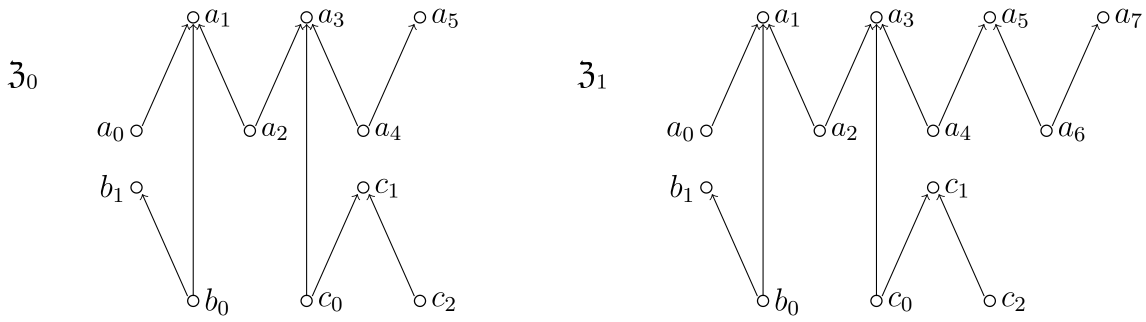

Figure 4.7: The frames $\mathfrak{Z}_0$ and $\mathfrak{Z}_1$

- for all $i < 2$ and $j \in \mathbb{Z}$, $l(\langle b_i, j\rangle) = l(\langle a_1, j\rangle)$;
- for all $i < 3$ and $j \in \mathbb{Z}$, $l(\langle c_i, j\rangle) = l(\langle a_3, j\rangle)$.

It is easy to see such map f exists and is unique. To simplify notations, we write $a^i_{l(\langle x_i,j\rangle)}$, $b^i_{l(\langle y_i,j\rangle)}$ and $c^i_{l(\langle z_i,j\rangle)}$ for $\langle a_i, j\rangle$, $\langle b_i, j\rangle$ and $\langle c_i, j\rangle$, respectively. We also write a_j for a^i_j. A fragment of the re-indexed frame $\mathfrak{Z}_\alpha$ is shown in Figure 4.8. For all $i, j \in \mathbb{Z}$ such that $i \leq j$, we define $Z[i, j] = \{z \in Z : i \leq l(z) \leq j\}$.

Note that for all $a^{i_1}_{j_1}, b^{i_2}_{j_2}, c^{i_3}_{j_3} \in Z$, we have $i_1 + j_1 \in \mathbb{E}$, and $j_2, j_3 \in \mathbb{O}$.

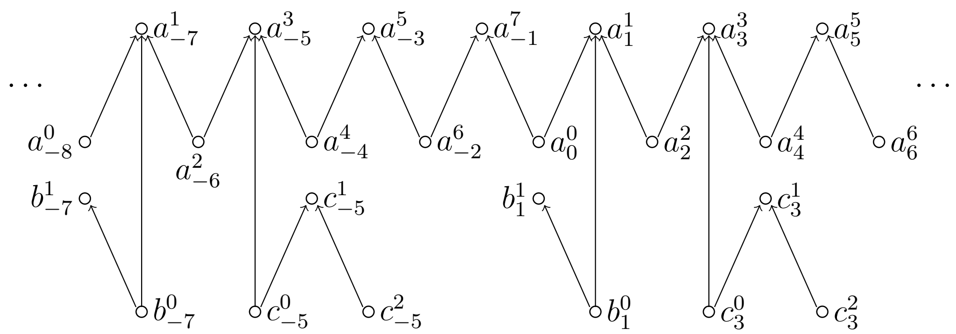

Figure 4.8: Fragment of relabelled $\mathfrak{Z}_\alpha$

Before proving the main lemmas and theorems of this section, we present the following auxiliary propositions and lemmas:

4.6.13. PROPOSITION. *Let $f : (\mathfrak{F}, x) \to^k (\mathfrak{F}', x')$ and $Y \subseteq R^{k-1}_\sharp[x]$. Then Y is sufficient if one of the following holds:*

(1) for all $y \in Y$, there are $u, v \in Y$ with $f(y) = f(u) = f(v)$ and $R[u] \cup R^{-1}[v] \subseteq Y$;

(2) there exists $Z \subseteq Y$ *such that* $R_\sharp[Z] \subseteq Y$ *and* $f[Y] = f[Z]$.

Proof:
Suppose that (1) holds. Take any $y \in Y$. Then we see that $f[R[y]] = R'[f(y)] = R'[f(u)] = f[R[u]] \subseteq f[Y]$. Similarly, $f[R^{-1}[y]] = f[R^{-1}[v]] \subseteq f[Y]$. Thus, $f[R_\sharp[y]] \subseteq f[Y]$. Because y is arbitrarily chosen, we obtain that f is sufficient. Note that since (2) implies (1), we see that f is sufficient if (2) holds. □

4.6.14. Lemma. *Let* $\mathfrak{G} = (Y, S)$ *be a rooted frame and* $f : (\mathfrak{Z}_\alpha, z_0) \to^n (\mathfrak{G}, y_0)$. *Then for all* $i \leq j$ *such that* $Z[i,j] \subseteq R^n_\sharp[z_0]$, $Z[i,j]$ *is sufficient if* $f[\{x_i, x_j\}] \subseteq f[Z[i,j] \setminus \{x_i, x_j\}]$.

Proof:
Note that $R_\sharp[Z[i,j] \setminus \{x_i, x_j\}] \subseteq Z[i,j]$. By Proposition 4.6.13, $f[\{x_i, x_j\}] \subseteq f[Z[i,j] \setminus \{x_i, x_j\}]$ implies that $Z[i,j]$ is sufficient. □

4.6.15. Proposition. *Let* $\mathfrak{F} = (X, R)$, $\mathfrak{F}' = (X', R')$ *be frames,* $x \in X$, $x' \in X'$ *and* $f : (\mathfrak{F}, x) \to^k (\mathfrak{F}', x')$. *Then for all* $m < k$, $z, z' \in R^m_\sharp[x]$ *and* $f(z) = f(z')$,

$$f[R^n_\sharp[z]] = f[R'^n_\sharp[z']] \text{ for all } n \in \omega \text{ such that } n + m < k.$$

Proof:
The proof proceeds by induction on n. The case $n = 0$ is trivial. Let $n > 0$. Take any $w \in R^n_\sharp[z]$. Then there exists $u \in R^{n-1}_\sharp[z]$ such that $w \in R_\sharp[u]$. By induction hypothesis, $f(u) \in f[R^{n-1}_\sharp[z]] = f[R^{n-1}_\sharp[z']]$ and so $f(u) = f(u')$ for some $u' \in R^{n-1}_\sharp[z']$. Note that since $u, u' \in R^{k-1}_\sharp[x]$, we have

$$f[R_\sharp[u]] = R'_\sharp[f(u)] = R'_\sharp[f(u')] = f[R_\sharp[u']].$$

Thus, $f(w) \in f[R_\sharp[u]] = f[R_\sharp[u']] \subseteq f[R^n_\sharp[z']]$. By arbitrariness of w, $f[R^n_\sharp[z]] \subseteq f[R^n_\sharp[z']]$. Analogously, $f[R^n_\sharp[z']] \subseteq f[R^n_\sharp[z]]$. Hence, $f[R^n_\sharp[z]] = f[R^n_\sharp[z']]$. □

Intuitively, Lemma 4.6.14 provides a simple criterion for sufficient local t-morphisms whose domain is $\mathfrak{Z}_\alpha$. Moreover, Proposition 4.6.15 shows that if two points have the same image under a local t-morphism f, then their neighborhoods are ‘equivalent’ w.r.t f.

We now turn to the most combinatorial part of this section. Our aim is to prove Lemma 4.6.24, which states that if $\mathfrak{G}$ is an infinite rooted frame for $\mathsf{Log}(\mathfrak{Z}_\alpha)$, then every finite fragment of $\mathfrak{G}$ is also a finite fragment of $\mathfrak{Z}_\alpha$. To establish this result, we need the following lemmas (Lemmas 4.6.16, 4.6.21 and 4.6.23):

4.6.16. LEMMA. *Let* $\mathfrak{G} = (Y, S) \in \mathsf{Fr}_r(\mathsf{S4}_t)$, $z_0 \in Z$, $y_0 \in Y$ *and* $n \in \omega$. *Suppose* $f : (\mathfrak{Z}_\alpha, z_0) \to^{n+19} (\mathfrak{G}, y_0)$ *and* f *is not sufficient. Then for all* $x, y \in \{a, b, c\}$ *and* $i, j, k, l \in \mathbb{Z}$ *such that* $f(x^i_j) = f(y^k_l)$, *the following holds:*

(1) if $x^i_j, y^k_l \in R^{n+18}_\sharp[z_0]$, *then* $i + k \in \mathbb{E}$;

(2) if $x^i_j, y^k_l \in R^{n+16}_\sharp[z_0]$, *then* $x = b$ *implies* $y \neq c$;

(3) if $x^i_j, y^k_l \in R^{n+6}_\sharp[z_0]$, *then* $x = b$ *implies* $y \neq a$;

(4) if $x^i_j, y^k_l \in R^{n}_\sharp[z_0]$, *then* $x = c$ *implies* $y \neq a$.

Proof:
For (1), suppose $x^i_j, y^k_l \in R^{n+18}_\sharp[z_0]$ and $i + k \notin \mathbb{E}$. Assume $i \in \mathbb{E}$. Then for each $z \in \{x^i_j, y^k_l\}$, we have $f(x) = f(x^i_j) = f(y^k_l)$ and $R[y^k_l] \cup R^{-1}[x^i_j] \subseteq \{x^i_j, y^k_l\}$. By Proposition 4.6.13(1), $\{x^i_j, y^k_l\}$ is sufficient. Assume $i \in \mathbb{O}$. Then $R^{-1}[y^k_l] \cup R[x^i_j] \subseteq \{x^i_j, y^k_l\}$, which also implies that $\{x^i_j, y^k_l\}$ is sufficient.
For (2), suppose $x^i_j, y^k_l \in R^{n+16}_\sharp[z_0]$, $x = b$ and $y = c$. Then we have two cases:

(2.1) $i = 1$. By (1), $k = 1$. By Proposition 4.6.15, we see $\{f(b^0_j), f(b^1_j)\} = f[R_\sharp[b^1_j]] = f[R_\sharp[c^1_l]] = \{f(c^0_l), f(c^1_l), f(c^2_l)\}$, which entails $f(c^0_l) = f(c^2_l) = f(b^0_j)$. Let $X = \{b^0_j, b^1_j, c^2_l, c^1_l\}$. Then $R_\sharp[b^1_j] \cup R_\sharp[c^2_l] \subseteq X$. Note that $X \subseteq R^{n+18}_\sharp[z_0]$, $f(c^2_l) = f(b^0_j)$ and $f(c^1_l) = f(b^1_j)$, by Proposition 4.6.13, we see X is sufficient.

(2.2) $i \neq 1$. Then $i = 0$ and so $k \in \{0, 2\}$. By (2.1), $f(b^1_j) \neq f(c^1_j)$. If $k = 2$, then by Proposition 4.6.15 and (1), $f(b^1_j) = f(c^1_j)$, which is impossible. Thus, $k = 0$. By Proposition 4.6.15 and (1), we see $f(a^1_j) = f(c^1_l)$ and $f(a^1_l) = f(b^1_j)$. Consider the set $X' = Z[j, j] \cup Z[l, l]$. Clearly, $X' \subseteq R^2_\sharp[x^i_j] \cup R^2_\sharp[y^k_l] \subseteq R^{n+18}_\sharp[z_0]$. Note that $R_\sharp[X' \setminus \{a^1_j, a^1_l\}] \subseteq X'$ and $f[\{a^1_j, a^1_l\}] \subseteq f[X' \setminus \{a^1_j, a^1_l\}]$, we see X' is sufficient. Thus, this case is impossible.

For (3), let $x = b$ and $y = a$. We first prove the following two claims:

4.6.17. CLAIM. *Suppose* $i = 1$. *If* $x^i_j, y^k_l \in R^{n+15}_\sharp[z_0]$, *then* $j = l + 2$.

Proof:
By $x^i_j, y^k_l \in R^{n+15}_\sharp[z_0]$, we see $Z[\min(j, l), \max(j, l)] \subseteq R^{n+18}_\sharp[z_0]$. Suppose $l = j$. Then $f(b^1_j) = f(a^1_j)$. By Lemma 4.6.14, $Z[j, j]$ is sufficient. Suppose $l = j + 2$. Then by Proposition 4.6.15 and (1), we have $f(b^0_j) = f(c^0_l)$, which contradicts (2.1). Suppose $l > j + 2$. By (1) and Proposition 4.6.15, we see $f(b^0_j) = f(a_{l-1})$ and $f(a_j) = f(a_{l-2})$. Note that $a_{l-2} \neq a_j$, we see $f[\{a_j, a_l\}] \subseteq f[Z[j, l] \setminus \{a_j, a_l\}]$. By Lemma 4.6.14, $Z[j, l]$ is sufficient. Suppose $l + 2 < j$. By (1) and Proposition 4.6.15, we see $f(b^0_j) = f(a_{l+1})$ and $f(a_j) = f(a_{l+2})$. Note that $a_{l+2} \neq a_j$, we see $f[\{a_j, a_l\}] \subseteq f[Z[l, j] \setminus \{a_j, a_l\}]$. By Lemma 4.6.14, $Z[l, j]$ is sufficient. □

4.6.18. CLAIM. *Suppose* $i = 1$. *If* $x^i_j, y^k_l \in R^{n+7}_\sharp[z_0]$, *then* $j \neq l + 2$.

Proof:
Suppose $j = l + 2$. Then $k \in \{5, 7\}$.

(a) $k = 5$. Consider the frame $\mathfrak{Z}_\alpha$ with labels in Figure 4.9. By Proposition 4.6.15 and (1), points with same label have the same f-image. Then $f(c^0_{l-2}) \in f[\{a_{j+1}, b^0_j\}]$. By (2), $f(c^0_{l-2}) \neq f(b^0_j)$ and so $f(a_{j+1}) = f(c^0_{l-2})$. Note that $f(a_{l-2}) = f(a_j)$ and $Z[l-2, j+1] \subseteq R^{n+18}_\sharp[z_0]$, by Lemma 4.6.14, $Z[l-2, j+1]$ is sufficient, which is impossible.

(b) $k = 7$. Consider the frame $\mathfrak{Z}_\alpha$ with labels given in Figure 4.10(a). By Proposition 4.6.15 and (1), we see that points with same label have the same f-image. Moreover, we see that $f(c^0_{l-4}) \in f[\{a_{j+3}, c^0_{j+2}\}]$. Suppose $f(c^0_{l-4}) = f(a_{j+3})$. Note that $Z[l-4, j+3] \subseteq R^{n+18}_\sharp[z_0]$, by Lemma 4.6.14, $Z[l-4, j+3]$ is sufficient. Suppose $f(c^0_{l-4}) = f(c^0_{j+2})$. By (1) and Proposition 4.6.15, we can verify that in Figure 4.10(b), points with same label have the same f-image. Then $f(b^1_{l-6}) \in f[\{a_{j+2}, a_{j+6}\}]$. Note that $\{a_{j+2}, a_{j+6}, b^1_{l-6}\} \subseteq R^{n+15}_\sharp[z_0]$, by Claim 1, $f(b^1_{l-6}) \notin f[\{a_{j+2}, a_{j+6}\}]$. □

Now we are ready to prove (3). Suppose $x^i_j, y^k_l \in R^{n+6}_\sharp[z_0]$. By Claims 4.6.17 and 4.6.18, we have $i \neq 1$. Then $i = 0$. By (1) and Proposition 4.6.15, we have that $f(b^1_j) \in \{f(a_{l-1}), f(a_{l+1})\}$. Note that $\{b^1_j, a_{l-1}, a_{l+1}\} \subseteq R^{n+7}_\sharp[z_0]$, by Claims 4.6.17 and 4.6.18, $f(b^1_j) \notin \{f(a_{l-1}), f(a_{l+1})\}$, which is impossible.
For (4), let $x = c$ and $y = a$. Again, we first prove the following two claims:

4.6.19. Claim. *Suppose $x^i_j, y^k_l \in R^{n+2}_\sharp[z_0]$. Then $i \neq 2$.*

Proof:
Suppose $x^i_j, y^k_l \in R^{n+2}_\sharp[z_0]$ and $i = 2$. By (1), $j + l \in \mathbb{O}$. Suppose $l = j + 1$. By Proposition 4.6.15 and (1), $f(a_l) = f(c^1_l)$. By Lemma 4.6.14, $Z[j, l]$ is sufficient. Similarly, $l = j - 1$ implies $Z[l, j]$ is sufficient. Suppose $l > j + 3$. By Proposition 4.6.15 and (1), $f(a_l) = f(a_{j-3})$. By Lemma 4.6.14, $Z[j, l]$ is sufficient. Similarly, $l < j - 3$ implies $Z[l, j]$ is sufficient. Suppose $l = j - 3$. By Proposition 4.6.15 and (1), we obtain that $f(b^0_{l-1}) \in f[R^2_\sharp[a_l]] = f[R^2_\sharp[c^2_j]] = f[\{c^0_j, c^1_j, c^2_j\}]$. Since $\{b^0_{l-1}, c^0_j, c^1_j, c^2_j\} \subseteq R^{n+16}_\sharp[z_0]$, by (2), $f(b^0_{l-1}) \notin f[\{c^0_j, c^1_j, c^2_j\}]$. Suppose $l = j + 3$. Then $k = 0$ or $k = 6$. If $k = 0$, then by (1) and Proposition 4.6.15, we see $f(b^0_{l+1}) \in f[R^2_\sharp[a_l]] = f[R^2_\sharp[c^2_j]] = f[\{c^0_j, c^1_j, c^2_j\}]$, which contradicts (2). Suppose $k = 6$. Consider the relabelled frame in Figure 4.11. By Proposition 4.6.15 and (1), points with same label have the same f-image. Thus, $f(b^0_{l+3}) \in \{f(c^0_j), f(a_{j-1})\}$. Note that $\{b^0_{l+3}, c^0_j, a_{j-1}\} \subseteq R^{n+6}_\sharp[z_0]$, by (2) and (3), $f(b^0_{l+3}) \notin \{f(c^0_j), f(a_{j-1})\}$. □

4.6.20. Claim. *Suppose $x^i_j, y^k_l \in R^{n+1}_\sharp[z_0]$. Then $i \neq 1$.*

Proof:
Suppose $x^i_j, y^k_l \in R^{n+1}_\sharp[z_0]$ and $i = 1$. Note that $R_\sharp[a^k_l] \subseteq R^{n+2}_\sharp[z_0]$, by (1) and Proposition 4.6.15, $f(c^2_j) \in f[R_\sharp[a^k_l]]$. By (2) and (3), $k = 3$ and $f(c^0_l) = f(c^2_j)$. By Proposition 4.6.15, $f(c^1_j) = f(c^1_l)$, and thus $\{c^0_l, c^1_l, c^2_l, c^1_j, c^2_j\}$ is sufficient. □

Finally, we are ready to prove (4). Suppose $x^i_j, y^k_l \in R^n_\sharp[z_0]$. By Claims 4.6.19 and 4.6.20, we see that $i \notin \{1, 2\}$. Then $i = 0$. By (1) and Proposition 4.6.15, we have $f(c^1_j) \in \{f(a_{l-1}), f(a_{l+1})\}$. Note that $\{c^1_j, a_{l-1}, a_{l+1}\} \subseteq R^{n+1}_\sharp[z_0]$, by Claims 4.6.19 and 4.6.20, $f(c^1_j) \notin \{f(a_{l-1}), f(a_{l+1})\}$, which is impossible. □

4.6.21. Lemma. *Let $\mathfrak{G} = (Y, S) \in \mathsf{Fr}_r(\mathsf{S4}_t)$, $z_0 \in Z$, $y_0 \in Y$ and $n \in \omega$. Suppose $f : (\mathfrak{Z}_\alpha, z_0) \to^{n+26} (\mathfrak{G}, y_0)$ and f is not sufficient. Then for all $x, y \in \{a, b, c\}$ and $i, j, k, l \in \mathbb{Z}$ such that $f(x^i_j) = f(y^k_l)$, the following holds:*

(1) if $x^i_j, y^k_l \in R^{n+4}_\sharp[z_0]$, then $i = k$;

(2) if $x^i_j, y^k_l \in R^n_\sharp[z_0]$, then $j = l$.

Proof:
Take any $x^i_j, y^k_l \in R^n_\sharp[z_0]$ such that $f(x^i_j) = f(y^k_l)$. By Lemma 4.6.16, $x = y$ and $i + k \in \mathbb{E}$. For (1), consider the following three cases:

(1.1) $x = b$. Then $i, k < 2$. Since $i + k \in \mathbb{E}$, we see $i = k$.

(1.2) $x = c$. Then $i, k \in \{0, 1, 2\}$. Suppose $i \neq k$. Then $\{i, k\} = \{0, 2\}$. Suppose $i = 0$. By Proposition 4.6.15, $f(a_i) \in f[R_\sharp[c^0_j]] = f[R_\sharp[c^2_l]] = \{f(c^2_l), f(c^1_l)\}$. Since $f : (\mathfrak{Z}_\alpha, z_0) \to^{n+26} (\mathfrak{G}, y_0)$ and $\{a_i, c^2_l, c^1_l\} \subseteq R^{n+7}_\sharp[z_0]$, by Lemma 4.6.16(4), we have $f(a_i) \notin \{f(c^0_l), f(c^1_l)\}$, which is a contradiction. Analogously, $i = 0$ is also impossible.

(1.3) $x = a$. Then $i, k \leq 7$. We first show the following claim:

4.6.22. Claim. *If $x^i_j, y^k_l \in R^{n+5}_\sharp[z_0]$, then $i \in \mathbb{E}$ implies $i = k$.*

Proof:
Consider the following cases:

(a) $2 \in \{i, k\}$. Suppose $i = 2$. By Proposition 4.6.15, $f(b^0_{j-1}), f(c^0_{j+1}) \in f[R^2_\sharp[a^k_l]]$. Since $R^2_\sharp[a^k_l] \subseteq R^{n+7}_\sharp[z_0]$, by Lemma 4.6.16, $k \in \mathbb{E}$ and there are $b^{k_1}_{l_1}, c^{k_2}_{l_2} \in R^2_\sharp[a^k_l]$, which entails $k = 2$. Analogously, $k = 2$ implies $i = 2$.

(b) $0 \in \{i, k\}$. Suppose $i = 0$. By Proposition 4.6.15, we see $f(b^0_{j+1}) \in f[R^2_\sharp[a^k_l]]$, which entails $k \in \{0, 1, 2\}$. By Lemma 4.6.16(1) and (a), we have that $k = 0$. Analogously, $k = 0$ implies $i = k$.

(c) $4 \in \{i, k\}$. Similar to the argument for (b).

Since $i + k \in \mathbb{E}$ and $i, k \leq 7$, by (a)-(c), we see $i = 6$ if and only if $k = 6$. □

Now we are ready to complete the proof for case (1.3). If $i \in \mathbb{E}$, then $i = k$ follows from Claim 4.6.22 immediately. Suppose $i \in \mathbb{O}$. By Proposition 4.6.15 and the definition of Z, we have $f[\{a^{i-1}_{j-1}, a^{i'}_{j+1}\}] \subseteq f[\{a^{k-1}_{l-1}, a^{k'}_{l+1}\}]$. Note that $\{a_{j-1}, a_{j+1}\} \subseteq R^{n+5}_\sharp[z_0]$, by Claim 4.6.22, $\{i-1, i'\} = \{k-1, k'\}$. Suppose $i \neq k$. Then $i - 1 = k' \in \{k+1, 0\}$ and $k - 1 = i' \in \{i+1, 0\}$. If $i - 1 = 0$, then $k \in \{5, 7\}$, which contradicts the fact that $k - 1 \in \{i+1, 0\} = \{2, 0\}$. Thus, $i - 1 = k + 1$ and so $k - 1 = 0$. Then $i \in \{5, 7\}$, which contradicts the fact that $i - 1 \in \{k+1, 0\} = \{2, 0\}$. Hence, $i = k$.

For (2), suppose $j < l$. By Lemma 4.6.16 and (1), $x = y$ and $i = k$. By Proposition 4.6.15, $f(a_j) = f(a_l)$. Note that $a_j, a_l \in R^3_\sharp[x^i_j] \cup R^3_\sharp[y^k_l]$, we see $R_\sharp[a_j] \cup R_\sharp[a_l] \subseteq R^{n+4}_\sharp[z_0]$. Then by (1), $f(a_{j-1}) = f(a_{l-1})$. Thus, $Z[j-1, l]$ is sufficient. Analogously, $j > l$ implies that $Z[l-1, j]$ is sufficient. By assumption, f is not sufficient, which leads to a contradiction. Hence, $j = l$. □

4.6.23. Lemma. *Let $\mathfrak{G} = (Y, S) \in \mathsf{Fr}_r$ be an infinite frame. Let $z_0 \in Z$, $y_0 \in Y$ and $n \in \omega$. Suppose $f : (\mathfrak{Z}_\alpha, z_0) \to^{n+26} (\mathfrak{G}, y_0)$. Then $g = f{\restriction}R^n_\sharp[z_0]$ is injective.*

Proof:
Take any $z_1, z_2 \in \mathsf{dom}(g)$. Then there exists $x, y \in \{a, b, c\}$ and $i, j, k, l \in \mathbb{Z}$ such that $z_1 = x^i_j$ and $z_2 = y^k_l$. Suppose $f(z_1) = f(z_2)$. Then by Lemmas 4.6.16 and 4.6.21, $x = y$, $i = k$ and $j = l$, which entails $z_1 = z_2$ immediately. □

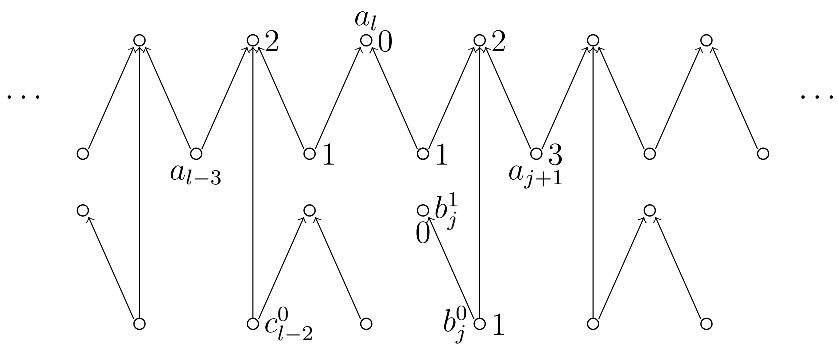

Figure 4.9: For the proof of Claim 4.6.18(a)

4.6.24. Lemma. *Let $\mathfrak{G} = (Y, S) \in \mathsf{Fr}_r(\mathsf{Log}(\mathfrak{Z}_\alpha))$ be an infinite frame. Then for all $k \in \omega$ and $y \in Y$, there exists $x \in Z$ such that $\mathfrak{G}{\restriction}S^k_\sharp[y] \cong \mathfrak{Z}_\alpha{\restriction}R^k_\sharp[x]$.*

Proof:
Since $\mathfrak{G} \models \mathsf{Log}(\mathfrak{Z}_\alpha)$ and $\mathfrak{G}, y \not\models \neg\mathcal{J}^{k+26}(\mathfrak{G}, y)$, there exists $x \in Z$ such that $\mathfrak{Z}_\alpha, x \not\models$

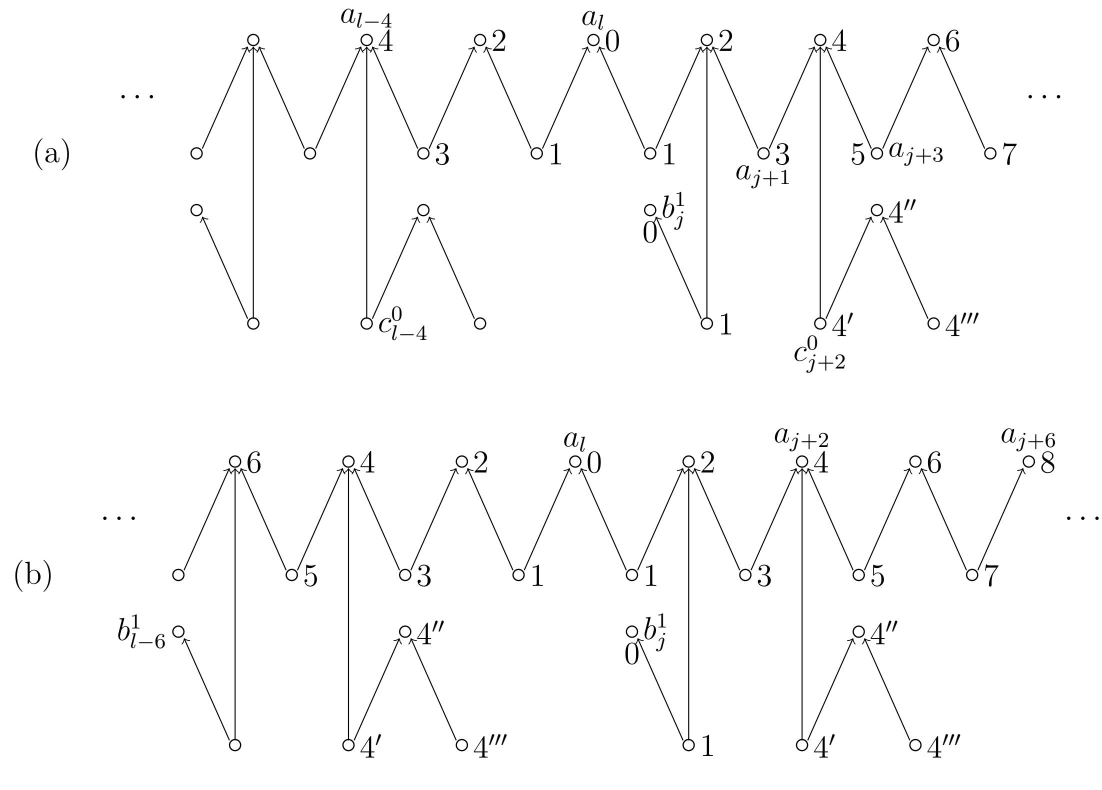

Figure 4.10: For the proof of Claim 4.6.18(b)

$\neg\mathcal{J}^{k+26}(\mathfrak{G}, y)$. By Proposition 4.1.5, there exists a map $f : (\mathfrak{Z}_\alpha, x) \to^{k+26} (\mathfrak{F}, y)$. It follows from Lemma 4.6.23 that $g = f{\restriction}R^k_\sharp[x]$ is injective. By Fact 2.2.24, $g : R^k_\sharp[x] \cong R^k_\sharp[y]$. □

Now we are ready to prove the following lemma:

4.6.25. Lemma. *If $\alpha : \mathbb{Z} \to 2$ is uniformly recurrent, then $\mathsf{Log}(\mathfrak{Z}_\alpha)$ is pretabular.*

Proof:
Let $L \supseteq \mathsf{Log}(\mathfrak{Z}_\alpha)$ be non-tabular. By Theorem 4.3.18, $L \subseteq \mathsf{Log}(\mathbb{F})$ for some rooted refined frame $\mathbb{F}$. Note that $\mathbb{F} \models \mathsf{alt}^+_3 \wedge \mathsf{alt}^-_4$, we see that $\mathbb{F}$ is image-finite. By Lemma 2.2.40, $\mathsf{Log}(\mathbb{F}) = \mathsf{Log}(\kappa\mathbb{F})$. Let $\kappa\mathbb{F} = \mathfrak{G} = (Y, S)$. It suffices to show that $\mathsf{Log}(\mathfrak{Z}_\alpha) \supseteq \mathsf{Log}(\mathfrak{G})$. Take any $\varphi \notin \mathsf{Log}(\mathfrak{Z}_\alpha)$. Then $\mathfrak{Z}_\alpha, z \not\models \varphi$ for some $z \in Z$ and there exists a finite subsequence $\beta \subseteq \alpha$ such that $\mathfrak{Z}_\alpha{\restriction}R^{\mathsf{md}(\varphi)}_\sharp[z] \subseteq \mathfrak{Z}_\beta$. Recall that α is uniformly recurrent, there exists $n_\beta \in \omega$ such that $\beta \trianglelefteq \gamma$ for all $\gamma \trianglelefteq \alpha$ with $|\gamma| > n_\beta$.

Let $m = 8(n_\beta + 3)$. Take any $y \in Y$. By Lemma 4.6.24, $\mathfrak{G}{\restriction}S^m_\sharp[y] \cong \mathfrak{Z}_\alpha{\restriction}R^m_\sharp[x]$ for some $x \in Z$. By the construction of $\mathfrak{Z}_\alpha$, m is a large enough number such that

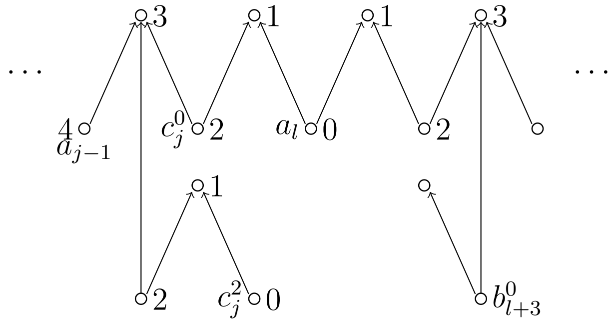

Figure 4.11: For the proof of Claim 4.6.19

$\mathfrak{Z}_\gamma \subseteq \mathfrak{Z}_\alpha \restriction R_\sharp^m[x]$ for some $\gamma \trianglelefteq \alpha$ with $|\gamma| \geq n_\beta$. Thus, $\mathfrak{Z}_\alpha \restriction R_\sharp^{\mathsf{md}(\varphi)}[z] \subseteq \mathfrak{Z}_\beta \subseteq \mathfrak{Z}_\gamma \subseteq \mathfrak{Z}_\alpha \restriction R_\sharp^m[x] \cong \mathfrak{G} \restriction S_\sharp^m[y]$, which implies $\varphi \notin \mathsf{Log}(\mathfrak{G})$. Hence, $\mathsf{Log}(\mathfrak{Z}_\alpha) \supseteq \mathsf{Log}(\mathfrak{G}) \supseteq L \supseteq \mathsf{Log}(\mathfrak{Z}_\alpha)$ and so $\mathsf{Log}(\mathfrak{Z}_\alpha)$ is pretabular. □

By Lemmas 4.6.9 and 4.6.25, we have the following theorem holds:

4.6.26. Theorem. *For all $f : \omega \to 2$, the logic $\mathsf{Log}(\mathfrak{Z}_{\chi^f})$ is pretabular.*

To obtain continuum many pretabular logics, it remains to show that for all functions $f, g : \omega \to 2$, if $f \neq g$, then $\mathsf{Log}(\mathfrak{Z}_\alpha) \neq \mathsf{Log}(\mathfrak{Z}_\beta)$. By Lemma 4.6.11, it suffices to show the following lemma:

4.6.27. Lemma. *For all sequences $\alpha, \beta : \mathbb{Z} \to 2$,*

$$\mathsf{Log}(\mathfrak{Z}_\alpha) \subseteq \mathsf{Log}(\mathfrak{Z}_\beta) \textit{ implies } \beta \preceq \alpha.$$

Proof:
Suppose $\mathsf{Log}(\mathfrak{Z}_\alpha) \subseteq \mathsf{Log}(\mathfrak{Z}_\beta)$. Then $\mathfrak{Z}_\beta \in \mathsf{Fr}_r(\mathsf{Log}(\mathfrak{Z}_\alpha))$. Take any finite $\gamma \trianglelefteq \beta$. By Lemma 4.6.24, we see that $\mathfrak{Z}_\gamma$ can be embedded into $\mathfrak{Z}_\alpha$. By the construction of $\mathfrak{Z}_\gamma$ and $\mathfrak{Z}_\alpha$, we have $\gamma \trianglelefteq \alpha$. Since γ is chosen arbitrarily, we obtain $\beta \preceq \alpha$. □

As consequences, the following theorems hold:

4.6.28. Theorem. $|\mathsf{PTAB}(\mathsf{S4}_t)| \geq |\mathsf{PTAB}(\mathsf{S4BP}_{2,3}^{2,\omega})| = 2^{\aleph_0}$.

Proof:
By Lemmas 4.6.11 and 4.6.27, $|\{\mathsf{Log}(\mathfrak{Z}_{\chi^f}) : f \in 2^\omega\}| = 2^{\aleph_0}$. By Theorem 4.6.26, we see that $\{\mathsf{Log}(\mathfrak{Z}_{\chi^f}) : f \in 2^\omega\} \subseteq \mathsf{PTAB}(\mathsf{S4BP}_{2,3}^{2,\omega}) \subseteq \mathsf{PTAB}(\mathsf{S4}_t)$. □

4.6.29. Theorem. *For all cardinal κ such that $\kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, we have that $|\mathsf{PTAB}(L)| = \kappa$ for some $L \in \mathsf{NExt}(\mathsf{S4}_t)$.*

4.6.30. THEOREM. *For all cardinal κ such that $\kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, there exists $L \in \mathsf{NExt}(\mathsf{S4}_t)$ such that L has κ Kripke complete pretabular extensions.*

This gives a full solution to the open problem on the cardinality of pretabular extensions of $\mathsf{S4}_t$ raised by Rautenberg [107].

4.6.31. REMARK. Assume the Axiom of Choice (AC), the condition "$\kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$" is the same as "$\kappa \leq 2^{\aleph_0}$". It is natural to ask, for example, whether Theorems 4.6.29 and 4.6.30 still holds without using the axiom of choice. To solve this kind of questions, it might be useful to apply techniques from descriptive set theory (see, e.g., [1]). We leave it as an open problem.

4.6.32. REMARK. As it is shown in this section, the logics $\mathsf{Log}(\mathfrak{Z}_{\chi^f})$ are pretabular, Kripke complete and of finite depth. There exist modal logics in $\mathsf{NExt}(\mathsf{S4})$, say $\mathsf{S5}$, which also satisfy these properties.

However, there are substantial differences between the lattices $\mathsf{NExt}(\mathsf{S4})$ and $\mathsf{NExt}(\mathsf{S4}_t)$. For example, by [29, Theorems 12.7 and 12.11], the following claims are true in $\mathsf{NExt}(\mathsf{S4})$, even in $\mathsf{NExt}(\mathsf{K4})$:

(i) every tabular logic has finitely many immediate predecessors;

(ii) every pretabular logic enjoys the finite model property.

On the other hand, we have the following conjecture: $\mathsf{Fin}_r(\mathsf{Log}(\mathfrak{Z}_{\chi^f})) = \mathsf{I}(\{\mathfrak{Ch}_1, \mathfrak{Ch}_2\})$ for any $f \in 2^\omega$. This will give that $\mathsf{Log}(\mathfrak{Ch}_2)$ has a continuum of immediate predecessors and $\mathsf{Log}(\mathfrak{Z}_{\chi^f})$ is pretabular but lacks the FMP. Thus, neither (i) nor (ii) may hold in $\mathsf{NExt}(\mathsf{S4}_t)$. In order to prove this conjecture, we would need to show that the critical exponent of χ^f is always finite. We leave this to future research.

4.7 Summary

In this chapter, we introduced tense logics with bounded parameters and studied their pretabular extensions. We gave a full characterization of pretabular fully bounded tense logics. Full characterizations for pretabular logics extending $\mathsf{S4.3}_t$ and $\mathsf{S4BP}^{2,\omega}_{2,2}$ were provided. Moreover, we studied pretabular tense logics in $\mathsf{NExt}(\mathsf{S4BP}^{2,\omega}_{2,3})$. It follows from Theorem 4.6.28 that the cardinality of $\mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,3})$ is $2^{\aleph_0}$.

In fact, Theorem 4.6.28 suggests that a full characterization of $\mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,3})$ or $\mathsf{PTAB}(\mathsf{S4}_t)$ is unattainable. Likewise, the decidability of tabularity in $\mathsf{NExt}(\mathsf{S4}_t)$ cannot be obtained via pretabular logics. However, this does not mean that research on pretabular tense logics in $\mathsf{NExt}(\mathsf{S4}_t)$ is exhausted; much remains to be explored. Some open problems have already been mentioned in the remarks (see Remark 4.2.14 and Remark 4.5.16) and we outline a few further topics below.

One possible direction for future work is to investigate pretabular logics in other sublattices of $\mathsf{NExt}(\mathsf{S4}_t)$. For example, consider the tense logic $\mathsf{S4BP}^{3,\omega}_{2,2}$, which has the forth-width and back-width 2 and the depth 3. The cardinality of $\mathsf{PTAB}(\mathsf{S4BP}^{3,\omega}_{2,2})$ remains unknown. Similar to pretabular pre-transitive modal logics, the pretabular logics of finite r-degrees are not well-understood.

Another direction for future work is to investigate pretabular logics in $\mathsf{NExt}(\mathsf{S4}_t)$ with the FMP. Pretabular logics can be viewed as boundaries of tabular logics. It is natural to consider that pretabular logics with the FMP act as the limit of certain set of tabular logics. It is known that every pretabular modal logic in $\mathsf{NExt}(\mathsf{K4})$ has the FMP (see [29, Theorem 12.11]). By Theorem 4.5.15, we obtain that every pretabular tense logic in $\mathsf{NExt}(\mathsf{S4BP}^{2,\omega}_{2,2})$ has the FMP. However, suppose that our conjecture in Remark 4.6.32 is correct. Then there exists a family of Kripke complete pretabular tense logics lacking the FMP in $\mathsf{NExt}(\mathsf{S4BP}^{2,\omega}_{2,3})$. This raises the following questions: (i) When does $\mathsf{NExt}(L)$ contain pretabular logics lacking the FMP? (ii) How many pretabular logics with the FMP exist in $\mathsf{NExt}(\mathsf{S4}_t)$? Exploring these questions will deepen our understanding of the lattices of tense logics.

So far, we have explored the upper part of the lattices of tense logics. We first investigated the Post-complete tense logics, which are exactly the co-atoms of these lattices. We then studied tabular and pretabular tense logics in $\mathsf{NExt}(\mathsf{K}_t)$ and in sublattices of $\mathsf{NExt}(\mathsf{S4}_t)$, which are likewise located near the top of the corresponding lattices. In the following chapter, we turn to the study of the degree of Kripke incompleteness, which measures how far a logic is from being Kripke complete and thus shifts our focus from the top to the entire lattice.

Chapter 5

Degree of Kripke incompleteness of Tense Logics

In this chapter, which is based on [32], we study Kripke completeness at a higher level by investigating the *degree of Kripke incompleteness* of tense logics. Recall that a logic L is Kripke complete if L is the logic of some class of Kripke frames. Existence of Kripke incomplete modal and tense logics has been established by Fine [50], Thomason [120, 121] and van Benthem [125]. The notion of the degree of Kripke incompleteness was introduced by Fine [51]. For a lattice $\mathcal{L}$ of logics and $L, L' \in \mathcal{L}$, we say that L and L' are Fr-*equivalent* (notation: $L \equiv_{\mathsf{Fr}} L'$) if they have the same class of frames, i.e., $\mathsf{Fr}(L) = \mathsf{Fr}(L')$. Then the degree of Kripke incompleteness of L in $\mathcal{L}$ is defined to be the cardinality of $[L]_{\mathsf{Fr}}$, where $[L]_{\mathsf{Fr}}$ is the set of all logics Fr-equivalent to L.

A celebrated result in this field is Blok's dichotomy theorem for the degree of Kripke incompleteness in $\mathsf{NExt}(\mathsf{K})$: every modal logic $L \in \mathsf{NExt}(\mathsf{K})$ is either strictly Kripke complete or is of the degree of Kripke incompleteness $2^{\aleph_0}$, where strictly Kripke complete means that the degree of Kripke incompleteness is 1, i.e., there exists no logic L' other than L such that $\mathsf{Fr}(L) = \mathsf{Fr}(L')$. This theorem was first proved in [14] algebraically by showing that union-splittings in $\mathsf{NExt}(\mathsf{K})$ are exactly the consistent strictly Kripke complete logics and all other consistent logics have the degree $2^{\aleph_0}$. Blok's characterization shows the connection between strictly Kripke complete normal modal logics and splittings of lattices of logics in $\mathsf{NExt}(\mathsf{K})$. For further research on splittings of lattices of modal, tense and subframe logics, we refer the reader to [135, 72, 70, 108]. A proof based on relational semantics was given later in [29, Section 10.5]. The characterization of the degree of Kripke incompleteness given by Blok indicates locations of Kripke complete logics in the lattice $\mathsf{NExt}(\mathsf{K})$. Since Blok's proof relies heavily on non-transitive frames, it is natural to ask whether the dichotomy theorem holds for sublattices of $\mathsf{NExt}(\mathsf{K})$, especially for the lattices of transitive modal logics such as $\mathsf{K4}$ and $\mathsf{S4}$. These problems remain open, see [29, Problem 10.5].

Generally speaking, except for Kripke incompleteness, one can always consider $\mathcal{C}$*-incompleteness* for classes $\mathcal{C}$ of mathematical structures, for example, the class MA of all modal algebras, the class NF of all neighborhood frames and the class Fin of all finite frames. Since every normal modal logic is complete with respect to modal algebras (see, e.g., [29, Theorem 7.73]) we obtain that every logic in $\mathsf{NExt}(\mathsf{K})$ is strictly MA-complete. The degree of modal incompleteness with respect to neighborhood semantics has been extensively investigated, e.g., by Chagrova [31], Dziobiak [41] and Litak [78]. Litak [78] studied modal incompleteness with respect to Boolean algebras with operators (BAOs) and showed the existence of a continuum of neighborhood-incomplete modal logics extending Grz. For more on modal incompleteness from an algebraic view, we refer the reader to [79].

Bezhanishvili et al. [9] introduced the notion of the degree of FMP. The anti-dichotomy theorem for the degree of FMP for extensions of the intuitionistic propositional logic IPC was proved in [9]: for each cardinal κ with $0 < \kappa \leq \aleph_0$ or $\kappa = 2^{\aleph_0}$, there exists $L \in \mathsf{Ext}(\mathsf{IPC})$ such that the degree of FMP of L in $\mathsf{Ext}(\mathsf{IPC})$ is κ. It was also shown in [9] that the anti-dichotomy theorem for the degree of FMP holds for $\mathsf{NExt}(\mathsf{K4})$ and $\mathsf{NExt}(\mathsf{S4})$. Degrees of FMP in lattices of bi-intuitionistic logics were studied in [38]. However, as far as we are aware, the degree of Kripke incompleteness in lattices of tense logics has not been investigated systematically.

In this chapter, we study Kripke incompleteness in lattices of tense logics. We start with the lattice $\mathsf{NExt}(\mathsf{K}_t)$ of all tense logics. Inspired by the proof for Blok's dichotomy theorem in [29], we prove the dichotomy theorem for tense logics, that is, every tense logic $L \in \mathsf{NExt}(\mathsf{K}_t)$ is of degree of Kripke incompleteness either 1 or $2^{\aleph_0}$. This is proved by showing that union-splittings in $\mathsf{NExt}(\mathsf{K}_t)$ are exactly the strictly Kripke complete logics and all other logics have the degree $2^{\aleph_0}$. By a similar argument, we prove the dichotomy theorem of the degree of Kripke incompleteness for $\mathsf{NExt}(\mathsf{K4}_t)$. Finally, we turn to the lattice $\mathsf{NExt}(\mathsf{S4}_t)$. We provide the following characterization of the degree of Kripke incompleteness in $\mathsf{NExt}(\mathsf{S4}_t)$: iterated splittings are strictly Kripke complete and all other logics are of degree $2^{\aleph_0}$, where iterated splittings are intuitively splittings in the lattice of splitting logics (For more details, we refer the reader to [135]). The dichotomy theorem of the degree of Kripke incompleteness for $\mathsf{NExt}(\mathsf{S4}_t)$ follows from the characterization immediately. We show that proper iterated splittings in $\mathsf{NExt}(\mathsf{S4}_t)$ are exactly extensions of $\mathsf{S5}_t$, which entails that there are countably many strictly Kripke complete logics in $\mathsf{NExt}(\mathsf{S4}_t)$. It also follows that in the lattice $\mathsf{NExt}(\mathsf{S4}_t)$, strictly Kripke complete logics are no longer the union-splittings.

This chapter is structured as follows: Section 5.1 introduces reflective unfolding of Kripke frames, which is one of the most important methods used in this chapter. Section 5.2 gives preliminaries on splittings. In Sections 5.3 and 5.4, we give characterizations of the degree of Kripke incompleteness and generalize

Blok's dichotomy theorem to the lattices $\mathsf{NExt}(\mathsf{K}_t)$, $\mathsf{NExt}(\mathsf{K4}_t)$ and $\mathsf{NExt}(\mathsf{S4}_t)$. In Section 5.5, we give some concluding remarks.

5.1 Reflective Unfolding of Kripke Frames

In this section, we review the *reflective unfolding* of Kripke frames, which has been introduced and studied by Kracht [71]. Recall the notion of reachability degree from Chapter 4. Intuitively, for any $n \in \omega$, by taking reflective unfolding of a (transitive) non-cluster $\mathfrak{F}$ that refutes φ, we obtain a (transitive) Kripke frame $\mathfrak{F}\langle n \rangle$ of reachability degree greater than n in which φ is still refuted. The reflective unfolding turns out to be one of the crucial constructions in our proofs of the characterization theorems: to construct a set of pairwise different tense logics which shares the same class of frames, as will be explained in the following sections, we need frames of large enough r-degree. In general, we could obtain such frames by the unrevealing method (see [12]). However, if preservation of transitivity is required, then unrevealing is insufficient and the technique of reflective unfolding is required.

Let us start by introducing the formal definition of *combinations of frames.* Recall that for a binary relation R on X, we write R^t for the *transitive closure* of R, that is,

$$R^t := \bigcap \{R' \supseteq R : R' \text{ is a transitive binary relation on} X\}.$$

Moreover, we call (X, R^t) the *transitive closure* of (X, R).

5.1.1. Definition. Let $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, S)$ be frames such that $X \cap Y = \varnothing$, $w \in X$ and $u \in Y$. Then we define the *combination* $\langle \mathfrak{F}w{+}u\mathfrak{G} \rangle = (Z, T)$ *of* $\mathfrak{F}$ *and* $\mathfrak{G}$ *at* $\langle w, u \rangle$ by $Z = (X \cup Y) \setminus \{u\}$ and

$$T = (R \cup S \cup (\{w\} \times S[u]) \cup (S^{-1}[u] \times \{w\})) \cap (Z \times Z).$$

Moreover, we write $\langle \mathfrak{F}w{+}^t u\mathfrak{G} \rangle$ for the transitive closure (Z, T^t) of $\langle \mathfrak{F}w{+}u\mathfrak{G} \rangle$ and call it the *transitive combination* of $\mathfrak{F}$ and $\mathfrak{G}$ at $\langle w, u \rangle$.

In Remark 5.1.2 and Example 5.1.4, we explain the notion of combination of frames defined above.

5.1.2. Remark. Let $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, S)$ be frames such that $X \cap Y = \varnothing$, $w \in X$ and $u \in Y$. Consider the combination $\langle \mathfrak{F}w{+}u\mathfrak{G} \rangle = (Z, T)$. Note that we could give another equivalent definition of T as follows

- for all $x, y \in X$, Txy if and only if Rxy;
- for all $x, y \in Y \setminus \{u\}$, Txy if and only if Sxy;

- for all $y \in Y \setminus \{u\}$, Twy if and only if Suy;
- for all $y \in Y \setminus \{u\}$, Tyw if and only if Syu.

Intuitively, the frame $\langle \mathfrak{F}w{+}u\mathfrak{G}\rangle = (Z, T)$ is obtained by taking the disjoint union of $\mathfrak{F}$ and $\mathfrak{G}$ and then identifying the points w and u. Note that for all frames $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, S)$, we may always assume that $X \cap Y = \varnothing$, since we are free to rename the points of $\mathfrak{F}$ and $\mathfrak{G}$. More precisely, this motivates the following generalized definition:

5.1.3. Definition. Let $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, S)$ be frames, $w \in X$ and $u \in Y$. Then we define the *combination* $\langle \mathfrak{F}w{+}u\mathfrak{G}\rangle = (Z, T)$ *of* $\mathfrak{F}$ *and* $\mathfrak{G}$ *at* $\langle w, u\rangle$ by $Z = (X \times \{0\}) \cup (Y \times \{1\}) \setminus \{\langle u, 1\rangle\}$ and

$$\begin{aligned} T =& \{\langle\langle x, a\rangle, \langle y, a\rangle\rangle \in Z \times Z : \langle x, y\rangle \in R \cup S \text{ and } a \in \{0, 1\}\} \cup \\ & (\{\langle w, 0\rangle\} \times \{\langle x, 1\rangle : x \in S[u]\}) \cup (\{\langle x, 1\rangle : x \in S^{-1}[u]\} \times \{\langle w, 0\rangle\}). \end{aligned}$$

In what follows, when considering the combinations of frames, we always presume that the domains of frames are disjoint and follow Definition 5.1.1.

5.1.4. Example. A simple example of combination of frames is presented in Figure 5.1.

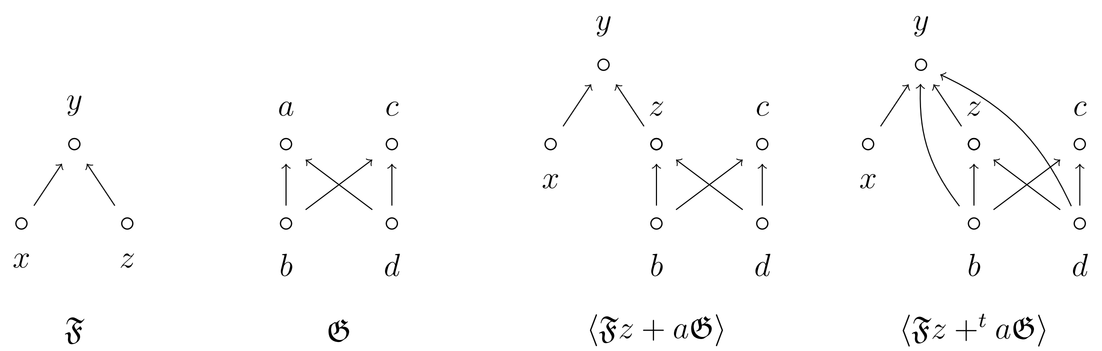

Figure 5.1: The combinations $\langle \mathfrak{F}z{+}a\mathfrak{G}\rangle$ and $\langle \mathfrak{F}z{+}^t a\mathfrak{G}\rangle$ of $\mathfrak{F}$ and $\mathfrak{G}$ at $\langle z, a\rangle$

As illustrated in Example 5.1.4, it is worth noting that b is not directly related to y in $\langle \mathfrak{F}z + a\mathfrak{G}\rangle$, while they are directly linked in $\langle \mathfrak{F}z{+}^t a\mathfrak{G}\rangle$. Next, we show how points become linked in the transitive combinations of frames.

5.1.5. Proposition. *Let* $\mathfrak{F} = (X, R)$ *and* $\mathfrak{G} = (Y, S)$ *be transitive frames,* $w \in X$ *and* $u \in Y$. *Consider the transitive combination* $\langle \mathfrak{F}w{+}^t u\mathfrak{G}\rangle = (Z, T^t)$. *Then for all* $x, x' \in X$ *and* $y, y' \in Y \setminus \{u\}$, *the following holds:*

(1) Rxx' if and only if $T^t xx'$;

(2) Syy' if and only if $T^t yy'$;

(3) $T^t xy$, if and only if, (i) Rxw or $x = w$ and (ii) Suy;

(4) $T^t yx$, if and only if, (i) Syu or $y = u$ and (ii) Rwx.

Proof:
Follows immediately from the construction of $\langle \mathfrak{F} w {+^t} u \mathfrak{G} \rangle$. □

Now we are in a position to introduce the notion of reflective unfolding:

5.1.6. Definition. Let $\mathfrak{F} = (X, R)$ be a frame. For each $n \in \omega$, we write $\mathfrak{F}\langle n \rangle$ for the frame $(X\langle n \rangle, R\langle n \rangle)$, where $X\langle n \rangle = \{x_n : x \in X\}$ and $R\langle n \rangle = \{\langle x_n, y_n \rangle : xRy\}$. For all $w, u \in X$ and $n \in \mathbb{Z}^+$, we define the *n-r-unfolding* $\mathfrak{F}^n_{w,u} = (X^n_{w,u}, R^n_{w,u})$ of $\mathfrak{F}$ by (w, u) inductively as follows:

- $\mathfrak{F}^1_{w,u} = \mathfrak{F}\langle 0 \rangle$;
- $\mathfrak{F}^{2n}_{w,u} = \langle \mathfrak{F}^{2n-1}_{w,u} u_{2n-2} + u_{2n-1} \mathfrak{F}\langle 2n-1 \rangle \rangle$;
- $\mathfrak{F}^{2n+1}_{w,u} = \langle \mathfrak{F}^{2n}_{w,u} w_{2n-1} + w_{2n} \mathfrak{F}\langle 2n \rangle \rangle$.

Moreover, if $\mathfrak{F}$ is transitive, then we call the transitive closure $(\mathfrak{F}^n_{w,u})^t$ of $\mathfrak{F}^n_{w,u}$ the *transitive n-r-unfolding* of $\mathfrak{F}$ by (w, u). A frame $\mathfrak{G}$ is called a (transitive) reflective unfolding of $\mathfrak{F}$ if $\mathfrak{G} \cong \mathfrak{F}^n_{w,u}$ ($\mathfrak{G} \cong (\mathfrak{F}^n_{w,u})^t$) for some $n \in \mathbb{Z}^+$ and $w, u \in X$.

Intuitively, $\mathfrak{F}^n_{w,u}$ is obtained by taking n copies of $\mathfrak{F}$ and combining them together step by step at the points w_i and u_i. It should be clear that the reflective unfolding preserves reflexivity and rootedness, i.e., $\mathfrak{F}^n_{w,u}$ is rooted and reflexive if $\mathfrak{F}$ is rooted and reflexive, respectively. An example of reflective unfolding is given in Figure 5.2.

The following lemmas (Lemmas 5.1.7 and 5.1.8) show that, for every frame $\mathfrak{F} = (X, R)$ and all $w, u \in X$, there always exist t-morphisms from the n-r-unfolding $\mathfrak{F}^n_{w,u}$ onto $\mathfrak{F}$. Moreover, under certain additional conditions, there always exists a (transitive) reflective unfolding of $\mathfrak{F}$ with arbitrarily large r-degree.

5.1.7. Lemma. *Let $\mathfrak{F} = (X, R)$ be a frame and $w, u \in X$. For each $n \in \mathbb{Z}^+$, let $n\varphi : X^n_{w,u} \to X$ be the map defined by $n\varphi : x_k \mapsto x$ for all $x \in X$ and $k < n$. Then $n\varphi$ is a surjective t-morphism from $\mathfrak{F}^n_{w,u}$ to $\mathfrak{F}$. Moreover, if $\mathfrak{F}$ is transitive, then $n\varphi$ is a surjective t-morphism from $(\mathfrak{F}^n_{w,u})^t$ to $\mathfrak{F}$.*

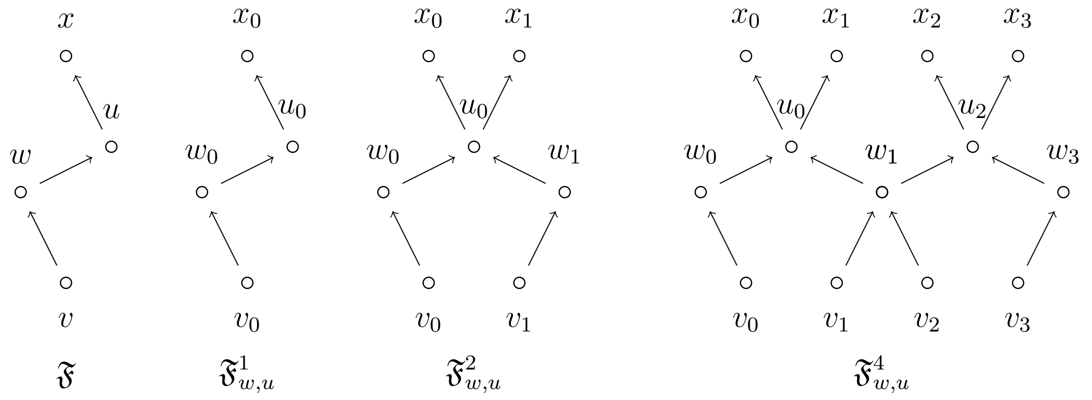

Figure 5.2: Examples for the reflective unfolding

Proof:
For all $x_i \in X^n_{w,u}$, we have $n\varphi[R^n_{w,u}[x_i]] = n\varphi[R\langle i\rangle[x_i]] = R[x] = R[n\varphi(x_i)]$ and similarly $n\varphi[R^{-1}{}^n_{w,u}[x_i]] = R^{-1}[x] = R^{-1}[n\varphi(x_i)]$. For the transitive case, the key observation is that for all $i, j < n$ and $x, y \in X$, if $(R^n_{w,u})^t x_i y_j$, then $R^t xy$ and so Rxy. The rest is straightforward. □

The following lemma implies that for each rooted frame $\mathfrak{F} = (X, R)$ and $n \in \omega$, if $|X| > 1$, then there is a reflective unfolding of $\mathfrak{F}$ with r-degree greater than n.

5.1.8. Lemma. *Let $\mathfrak{F} = (X, R)$ be a rooted frame, $w, u \in X$ and $n \in \omega$. Suppose $w \neq u$. Let $\mathfrak{F}^{n+2}_{w,u} = (Y, S)$. Then for all $x \in X$ and $k < n+1$ the following holds:*

(1) If $k \in \mathbb{E}$ and $x \neq u$, then $S_\sharp[x_k] \subseteq \bigcup_{i\leq k} X\langle i\rangle$;

(2) If $k \in \mathbb{E}$ and $x = u$, then $S_\sharp[x_k] \subseteq X\langle k\rangle \cup X\langle k+1\rangle$;

(3) If $k \in \mathbb{O}$ and $x \neq w$, then $S_\sharp[x_k] \subseteq \bigcup_{i\leq k} X\langle i\rangle$;

(4) If $k \in \mathbb{O}$ and $x = w$, then $S_\sharp[x_k] \subseteq X\langle k\rangle \cup X\langle k+1\rangle$;

(5) $S_\sharp[x_k] \subseteq \bigcup_{i\leq k+1} X\langle i\rangle$.

(6) $S^n_\sharp[x_0] \neq Y$.

Proof:
Items (1) - (4) hold by the definition of $\mathfrak{F}^{n+2}_{w,u}$. Note that (5) follows from (1) - (4) immediately. For (6), we obtain from (5) and an easy induction on n that $S^n_\sharp[x_0] \subseteq \bigcup_{i\leq n} X\langle i\rangle$. Since $w \neq u$, we see that $X\langle n+1\rangle \nsubseteq \bigcup_{i\leq n} X\langle i\rangle$. Thus,

$S_\sharp^n[x_0] \neq Y$. □

As a consequence, we have

5.1.9. LEMMA. *Let $\varphi \in \mathsf{Form}_t$. Suppose φ is satisfied in some finite rooted frame $\mathfrak{G}$ which is not isomorphic to $\mathfrak{Ch}_1^<$. Then for each $n \in \omega$, there exists a finite rooted frame $\mathfrak{F} = (X, R)$ such that φ is satisfied in $\mathfrak{F}$ and $\mathsf{rdg}(\mathfrak{F}) \geq n$.*

Proof:
Suppose φ is satisfied in some rooted frame $\mathfrak{G} = (Y, S) \not\cong \mathfrak{Ch}_1^<$. Suppose $|Y| = 1$. Then $S = Y \times Y$ and so $\mathfrak{G} \cong \mathfrak{Ch}_1$. Let $\mathfrak{F} = \mathfrak{G}_{n+1}$, where $\mathfrak{G}_{n+1}$ is the n-garland defined in Definition 4.5.1. Note that $\mathfrak{Ch}_1$ is a t-morphic image of $\mathfrak{F}$. Since $\mathfrak{G} \cong \mathfrak{Ch}_1$ and φ is satisfied in $\mathfrak{G}$, we obtain that φ is also satisfied in $\mathfrak{F}$.

Suppose $|Y| \geq 2$. Then there exists $w, u \in Y$ such that $\mathfrak{G}, w \not\models \neg\varphi$ and $w \neq u$. It now follows from Lemmas 5.1.7 and 5.1.8(2) that $\mathfrak{G}_{w,u}^{n+3}, w_0 \not\models \neg\varphi$ and $\mathsf{rdg}(\mathfrak{G}_{w,u}^{n+3}) \geq n$. □

Next, we prove that a similar lemma holds for transitive rooted frames. In general, transitivity is not preserved under r-unfolding. For example, for every r-unfolding of the cluster $\mathfrak{Cl}_2$, its transitive closure is always of r-degree 1. Thus, we need a lemma to ensure that the transitive closure of r-unfolding has large enough r-degree.

5.1.10. LEMMA. *Let $\mathfrak{F} = (X, R)$ be a transitive rooted frame and $w, u \in X$ such that $w \neq u$ and $\langle w, u\rangle \notin R$. Let $n \in \omega$ and $(\mathfrak{F}_{w,u}^{2n+2})^t = (Y, S)$. Then for all $x \in X$ and $k < 2n+1$, the following holds:*

(1) $S_\sharp[x_k] \subseteq \bigcup_{i \leq k+2} X\langle i\rangle$;

(2) $S_\sharp^n[x_0] \neq Y$.

Proof:
For (1), suppose $k \in \mathbb{O}$. Take any $y_j \in S_\sharp[x_k]$. It suffices to show that $j \leq k+2$. Suppose $j > k+2$. Since $y_j \in S_\sharp[x_k]$, either Sx_ky_j or Sy_jx_k. Suppose Sx_ky_j. Recall that R^r is the reflexive closure of R. By Proposition 5.1.5, we obtain that R^rxu and Su_ky_j. Again, by applying Proposition 5.1.5 twice, we have (i) $Sw_{k+1}y_j$ and R^ruw and (ii) $Su_{k+2}y_j$ and R^rwu. Since $w \neq u$, we have Rwu, which contradicts $\langle w, u\rangle \notin R$. For the case Sy_jx_k, by a similar argument, we see that R^rwu and so Rwu, which also gives a contradiction. Thus, $j \leq k+2$. The case $k \in \mathbb{E}$ is similar. Hence, (1) holds.

For (2), by an easy induction on n, we have $S_\sharp^n[x_0] \subseteq \bigcup_{i \leq 2n} X\langle i\rangle$. Since $w \neq u$, we see that $X\langle 2n+1\rangle \nsubseteq \bigcup_{i \leq 2n} X\langle i\rangle$. Thus, $S_\sharp^n[x_0] \neq Y$. □

As a consequence of Lemma 5.1.10, we obtain

5.1.11. Lemma. *Let $\varphi \in \mathsf{Form}_t$ and $L \in \{\mathsf{K4}_t, \mathsf{S4}_t\}$. Suppose φ is satisfied in some rooted non-symmetric frame $\mathfrak{G} \in \mathsf{Fin}_{\mathrm{r}}(L)$. Then for each $n \in \omega$, there exists a rooted frame $\mathfrak{F} = (X, R) \in \mathsf{Fin}_r(L)$ such that φ is satisfied in $\mathfrak{F}$ and $\mathsf{rdg}(\mathfrak{F}) \geq n$.*

Proof:
Let $\mathfrak{G} = (Y, S) \in \mathsf{Fin}_{\mathrm{r}}(L)$ be non-symmetric. Then there exists $w \neq u$ with $\langle w, u\rangle \notin S$. Since $\mathfrak{G}$ satisfies φ, there exists $y \in Y$ such that $\mathfrak{G}, y \nvDash \neg\varphi$. It follows from Lemmas 5.1.7 and 5.1.10 that $(\mathfrak{G}^{2n+2}_{w,u})^t, y_0 \nvDash \neg\varphi$ and $\mathsf{rdg}((\mathfrak{G}^{2n+2}_{w,u})^t) \geq n$. Note that $(\mathfrak{G}^{2n+2}_{w,u})^t$ is reflexive whenever $\mathfrak{G}$ is reflexive. Hence, taking $\mathfrak{F} = (\mathfrak{G}^{2n+2}_{w,u})^t$ completes the proof. ☐

5.2 Splittings of Lattices of Tense Logics

Splittings were introduced as a concept of lattice theory by Whitman [127] and systematically studied in the context of lattices of logics by McKenzie [94]. Later, the notion of splitting played a core role in Blok's characterization of the degree of Kripke incompleteness [14]. In this section, we review the notion of splitting and some results on the splittings of lattices of modal and tense logics obtained by Blok [14] and Kracht [71]. For more on splittings of logics, we refer the reader to [70]. Let us first recall the definition of splittings of lattices:

5.2.1. Definition. Let $\mathcal{L} = \langle L; \wedge, \vee, 0, 1\rangle$ be a complete lattice and $a, b \in L$. Then $\langle a, b\rangle$ is called a *splitting pair* in $\mathcal{L}$ if, for all $x \in \mathcal{L}$, exactly one of $x \leq a$ and $x \geq b$ holds. In this case, we call b a *splitting* of $\mathcal{L}$ and a a *co-splitting* of $\mathcal{L}$. The element b is uniquely determined by a and denoted by $\mathcal{L}/a$ or $0/a$.

An element $x \in L$ is called a *join-splitting* in $\mathcal{L}$ if there exists a family $\{b_i : i \in I\}$ of splittings in $\mathcal{L}$ such that $x = \bigvee_{i \in I} b_i$.

Recall that for every lattice $\mathcal{L} = \langle L; \wedge, \vee\rangle$ and $a \in L$, we write $\uparrow a$ and $\downarrow a$ for the sets $\{b \in L : b \geq a\}$ and $\{b \in L : b \leq a\}$, respectively. Then we see that $\langle a, b\rangle$ is a splitting pair in $\mathcal{L}$ if and only if $b \nleq a$, $\uparrow b \cup \downarrow a = L$ and $\uparrow b \cap \downarrow a = \varnothing$. We sometimes view $\uparrow a$ as a sublattice of $\mathcal{L}$.

The notion of *iterated splitting* is defined as follows:

5.2.2. Definition. Let $\mathcal{L} = \langle L; \wedge, \vee, 0, 1\rangle$ be a complete lattice. Then *iterated splittings* in $\mathcal{L}$ are defined inductively as follows:

- 0 is an iterated splitting in $\mathcal{L}$;
- if $x \in L$ is an iterated splitting in $\mathcal{L}$ and y is a splitting in $\uparrow x$, then y is an iterated splitting in $\mathcal{L}$.

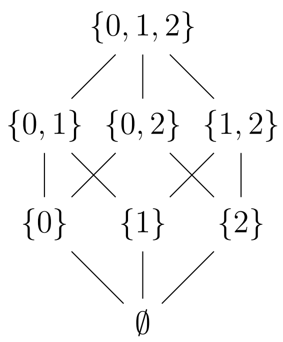

Figure 5.3: The Boolean algebra $\mathcal{P}(3)$

An iterated splitting x is called *proper* if $x \neq 0$.

The reader can readily check that for each iterated splitting $x \in L$, there exist $a_1, \cdots, a_n \in L$ such that $x = (0/a_1)/\cdots/a_n$.

5.2.3. Example. Consider the Boolean algebra $\mathfrak{B} = \langle \mathcal{P}(3); \cap, \cup \rangle$ (see Figure 5.3). Then $\langle \{0,1\}, \{2\} \rangle$ is a splitting pair in $\mathfrak{B}$, which implies that $\{0,1\}$ splits $\mathfrak{B}$ and $\{2\}$ is a splitting in $\mathfrak{B}$. Note that $\langle \{0,2\}, \{1,2\} \rangle$ is again a splitting pair in $\uparrow\{2\}$. Thus, $\{1,2\}$ is an iterated splitting in $\mathfrak{B}$.

Splittings of lattices are closely related to prime elements in lattices.

5.2.4. Definition. Let $\mathcal{L} = \langle L; \wedge, \vee, 0, 1 \rangle$ be a complete lattice and $x \in L$. Then x is called a *meet-prime element* if for all $\{y_i : i \in I\} \subseteq L$, we have that $\bigwedge_{i \in I} y_i \leq x$ implies $y_i \leq x$ for some $i \in I$. Dually, x is called a *join-prime element* if for all $\{y_i : i \in I\} \subseteq L$, we have that $x \leq \bigvee_{i \in I} y_i$ implies $x \leq y_i$ for some $i \in I$.

The following proposition shows the relation between (co-)splittings and join-prime (meet-prime) elements.

5.2.5. Proposition ([94]). *Let $\mathcal{L} = \langle L; \wedge, \vee, 0, 1 \rangle$ be a complete lattice and $x \in L$. Then the following holds:*

(1) x is a splitting in $\mathcal{L}$ if and only if x is join-prime in $\mathcal{L}$;

(2) x is a co-splitting in $\mathcal{L}$ if and only if x is meet-prime in $\mathcal{L}$.

For splittings in lattices of polymodal logics, we have

5.2.6. Definition. Let L_0 be a polymodal logic and $L_1, L_2 \in \mathsf{NExt}(L_0)$. Then $\langle L_1, L_2 \rangle$ is called a *splitting pair* in $\mathsf{NExt}(L_0)$ if, for all $L \in \mathsf{NExt}(L_0)$, exactly one of $L \subseteq L_1$ and $L \supseteq L_2$ holds. In this case, we say that

- L_1 *splits* the lattice $\mathsf{NExt}(L_0)$; and
- L_2 is the splitting of $\mathsf{NExt}(L_0)$ by L_1.

We write L_0/L_1 for L_2. A polymodal logic L is called a *union-splitting* in $\mathsf{NExt}(L_0)$ if L is a join-splitting, i.e., there exists a family $\{L_i : i \in I\}$ of splittings in $\mathsf{NExt}(L_0)$ such that $L = \bigoplus_{i\in I} L_i$. *Iterated splittings* in $\mathsf{NExt}(L_0)$ are defined inductively as follows:

- L_0 is an iterated splitting in $\mathsf{NExt}(L_0)$;
- if L is an iterated splitting in $\mathsf{NExt}(L_0)$ and L' is a splitting in $\mathsf{NExt}(L)$, then L' is also an iterated splitting in $\mathsf{NExt}(L_0)$.

In other words, L is an iterated splitting in $\mathsf{NExt}(L_0)$ if there are $L_1, \cdots, L_n \in \mathsf{NExt}(L_0)$ such that $L = (L_0/L_1)/\cdots/L_n$.

Let us review the splittings in $\mathsf{NExt}(\mathsf{K})$. A frame $\mathfrak{F} = (X, R)$ is said to be *cycle-free* if there exists no cycle $x_1, x_2, \cdots, x_n$ such that $x_1Rx_2R\cdots Rx_nRx_1$. Clearly, a finite frame $\mathfrak{F}$ is cycle-free if $\mathfrak{F} \vDash \Box^n\bot$ for some $n \in \omega$. It was proved by Blok [14] that a normal modal logic L splits $\mathsf{NExt}(\mathsf{K})$ if and only if $L = \mathsf{Log}(\mathfrak{A})$ for some finite subdirectly irreducible modal algebra $\mathfrak{A}$ such that $\mathfrak{A} \vDash \Box^n\bot$ for some $n \in \omega$. By the duality between finite frames and finite modal algebras, L splits $\mathsf{NExt}(\mathsf{K})$ if and only if $L = \mathsf{Log}(\mathfrak{F})$ for some finite rooted cycle-free frame $\mathfrak{F}$ (see [29, Theorems 10.49 and 10.53]).

5.2.7. Example. A toy example of splittings in $\mathsf{NExt}(\mathsf{K})$ is provided in Figure 5.4. We see that $\langle \mathsf{Log}(\mathfrak{Ch}_1^<), \mathsf{D}\rangle$ is a splitting pair in $\mathsf{NExt}(\mathsf{K})$, where $\mathsf{D} = \mathsf{K} \oplus \Diamond\top$. In fact, the modal logic D is the greatest splitting in $\mathsf{NExt}(\mathsf{K})$ and $\langle \mathsf{Log}(\mathfrak{Ch}_1), \mathsf{Form}_m\rangle$ is the unique splitting pair in $\mathsf{NExt}(\mathsf{D})$ [14]. This also entails that $\mathsf{Form}_m = (\mathsf{K}/\mathsf{Log}(\mathfrak{Ch}_1^<))/\mathsf{Log}(\mathfrak{Ch}_1)$ is an iterated splitting in $\mathsf{NExt}(\mathsf{K})$.

It was also proved by Blok [14] that a normal modal logic L is a consistent iterated splitting in $\mathsf{NExt}(\mathsf{K})$ if and only if L is the union of finitely many splittings. Moreover, the following propositions hold:

5.2.8. Proposition. *There are $2^{\aleph_0}$ union-splittings in* $\mathsf{NExt}(\mathsf{K})$.

Proof:
For each $n \in \mathbb{Z}^+$, consider the frame $\mathfrak{F}_n$ depicted in Figure 5.5. Let $n \in \mathbb{Z}^+$. Since $\mathfrak{F}_n$ is cycle-free, $\mathsf{Log}(\mathfrak{F}_n)$ splits $\mathsf{NExt}(\mathsf{K})$. For each non-empty $I \subseteq \mathbb{Z}^+$, let $L_I = \bigoplus_{i\in I} \mathsf{K}/\mathsf{Log}(\mathfrak{F}_i)$. Then we see that L_I is a union-splitting.

It suffices to show that for all non-empty sets $I, J \subseteq \mathbb{Z}^+$ such that $I \neq J$, we have $L_I \neq L_J$. Without loss of generality, suppose there exists $i \in I \setminus J$. Consider the formula

$$\varphi = (\Diamond p \wedge \Diamond\neg p) \to \Diamond^{i+2}\top \wedge \Box^{i+3}\bot.$$

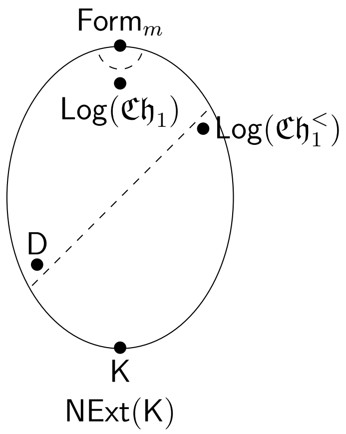

Figure 5.4: Examples of splittings and iterated splittings

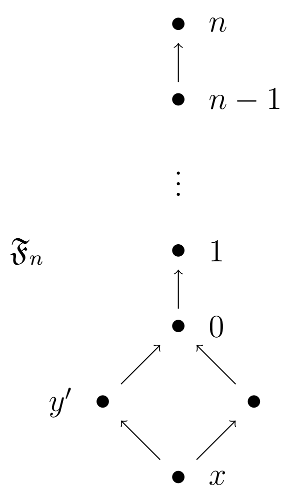

Figure 5.5: The frame $\mathfrak{F}_n$

It is not hard to see that $\mathfrak{F}_i \models \varphi$ and $\mathfrak{F}_k \not\models \varphi$ for all $k \neq i$. Thus, $\mathsf{K} \oplus \varphi \subseteq \mathsf{Log}(\mathfrak{F}_i)$, which entails that $\mathsf{K} \oplus \varphi \not\supseteq \mathsf{K}/\mathsf{Log}(\mathfrak{F}_i)$ and so $\mathsf{K} \oplus \varphi \not\supseteq L_I$. On the other hand, take any $j \in J$, we see that $\mathsf{K} \oplus \varphi \nsubseteq \mathsf{Log}(\mathfrak{F}_j)$ and so $\mathsf{K} \oplus \varphi \supseteq \mathsf{K}/\mathsf{Log}(\mathfrak{F}_j)$. Thus, $\mathsf{K} \oplus \varphi \supseteq L_J$. Hence, $L_I \neq L_J$. By arbitrariness of I and J, there are $2^{\aleph_0}$ union-splittings in $\mathsf{NExt}(\mathsf{K})$. □

It follows that there exist countably many iterated splittings in $\mathsf{NExt}(\mathsf{K})$.

Lattices of tense logics are substantially different from those of modal logics. Splittings of lattices of tense logics have been investigated in [71]. The following theorem gives a necessary condition for a logic to be a co-splitting in a lattice of tense logics:

5.2.9. THEOREM. *Let L_0 be a tense logic that enjoys the FMP. Suppose L_1 splits the lattice $\mathsf{NExt}(L_0)$. Then $L_1 = \mathsf{Log}(\mathfrak{F})$ for some finite rooted frame.*

Proof:
Since L_0 enjoys the finite model property, $L_0 = \bigcap\{\mathsf{Log}(\mathfrak{F}) : \mathfrak{F} \in \mathsf{Fin}_{\mathrm{r}}(L_0)\}$. Since L_1 splits $\mathsf{NExt}(L_0)$, by Proposition 5.2.5, L_1 is meet-prime. Thus, $\mathsf{Log}(\mathfrak{G}) \subseteq L_1$ for some $\mathfrak{G} \in \mathsf{Fin}_{\mathrm{r}}$. By Theorem 3.1.8, L_1 is tabular. Thus,

$$L_1 = \bigcap\{\mathsf{Log}(\mathfrak{F}) : \mathfrak{F} \in \mathsf{Fin}_{\mathrm{r}}(L_1)\}.$$

Again, since L_1 is meet-prime, we obtain that $\mathsf{Log}(\mathfrak{F}) \subseteq L_1$ for some $\mathfrak{F} \in \mathsf{Fin}_{\mathrm{r}}(L_1)$. Thus, we have $L_1 = \mathsf{Log}(\mathfrak{F})$. □

By Theorem 5.2.9, if a logic L splits the lattice $\mathsf{NExt}(L_0)$ where L_0 has the FMP, then $L = L_0/\mathsf{Log}(\mathfrak{F})$ for some finite rooted frame $\mathfrak{F}$. In this case, we write $L_0/\mathfrak{F}$ for L. In fact, Theorem 5.2.9 follows from general results from universal algebra (see, e.g., [6, Chapter 4]) and the duality between polymodal spaces and polymodal algebras (see [126]). A generalization of Theorem 5.2.9 is as follows:

5.2.10. THEOREM. *Let L_0 be a polymodal logic that enjoys the FMP. Suppose L_1 splits the lattice $\mathsf{NExt}(L_0)$. Then $L_1 = \mathsf{Log}(\mathfrak{F})$ for some finite rooted frame.*

Kracht [71] proved, using reflective unfolding, that there exists exactly one splitting pair in $\mathsf{NExt}(\mathsf{K}_t)$ and in $\mathsf{NExt}(\mathsf{K4}_t)$, and exactly two splitting pairs in $\mathsf{NExt}(\mathsf{S4}_t)$. More precisely, the following theorem holds:

5.2.11. THEOREM ([71, Theorems 21 and 22]). *Let $L \in \mathsf{NExt}(\mathsf{K}_t)$. Then*

(1) L splits $\mathsf{NExt}(\mathsf{K}_t)$ if and only if $L = \mathsf{Log}(\mathfrak{Ch}_1^<)$.

(2) L splits $\mathsf{NExt}(\mathsf{K4}_t)$ if and only if $L = \mathsf{Log}(\mathfrak{Ch}_1^<)$.

(3) L splits $\mathsf{NExt}(\mathsf{S4}_t)$ if and only if $L \in \{\mathsf{Log}(\mathfrak{Ch}_1), \mathsf{Log}(\mathfrak{Ch}_2)\}$.

Note that there are countably many splittings and continuum many union-splittings in $\mathsf{NExt}(\mathsf{K})$. Although there exists only 1 splitting in $\mathsf{NExt}(\mathsf{K}_t)$ and $\mathsf{NExt}(\mathsf{K4}_t)$, as we are going to prove in Section 5.3, union-splittings in $\mathsf{NExt}(\mathsf{K}_t)$ and $\mathsf{NExt}(\mathsf{K4}_t)$ are still exactly the strictly Kripke complete logics. Moreover, we will show that consistent iterated splittings in $\mathsf{NExt}(\mathsf{K}_t)$ and $\mathsf{NExt}(\mathsf{K4}_t)$ are exactly the union-splittings. However, in Section 5.4, we show that strictly Kripke complete logics in $\mathsf{NExt}(\mathsf{S4}_t)$ are exactly iterated splittings and there exist iterated splittings which are not union-splittings in $\mathsf{NExt}(\mathsf{S4}_t)$.

5.3 Kripke Incompleteness in $\mathsf{NExt}(\mathsf{K}_t)$ and $\mathsf{NExt}(\mathsf{K4}_t)$

In this section, we generalize Blok's dichotomy theorem for $\mathsf{NExt}(\mathsf{K})$ to the lattices $\mathsf{NExt}(\mathsf{K}_t)$ and $\mathsf{NExt}(\mathsf{K4}_t)$ of tense logics. Blok [14] proved that a normal modal logic L has the degree of Kripke incompleteness 1 in $\mathsf{NExt}(\mathsf{K})$ if and only if L is a union-splitting of $\mathsf{NExt}(\mathsf{K})$. Bezhanishvili et al. [9] introduced the notion of *degree of FMP* and proved the anti-dichotomy theorem for the degree of FMP for the lattices $\mathsf{NExt}(\mathsf{K4})$, $\mathsf{NExt}(\mathsf{S4})$ and $\mathsf{Ext}(\mathsf{IPC})$.

For the lattices $\mathsf{NExt}(\mathsf{K}_t)$ and $\mathsf{NExt}(\mathsf{K4}_t)$ of tense logics, similarly, we prove the dichotomy theorems by showing that the union-splittings of $\mathsf{NExt}(\mathsf{K}_t)$ and $\mathsf{NExt}(\mathsf{K4}_t)$ are exactly those having the degree of Kripke incompleteness 1, and all other tense logics have the degree of Kripke incompleteness $2^{\aleph_0}$, respectively. Moreover, we show that the degree of Kripke incompleteness coincides with the degree of FMP in both $\mathsf{NExt}(\mathsf{K}_t)$ and $\mathsf{NExt}(\mathsf{K4}_t)$.

Let us start by introducing the precise definition of the degree of Kripke incompleteness in lattices of polymodal logics.

5.3.1. Definition. Let L_0 be a polymodal logic and $L \in \mathsf{NExt}(L_0)$. Then we define the *degree of Kripke incompleteness* $\mathsf{deg}_{\mathsf{NExt}(L_0)}(L)$ of L in $\mathsf{NExt}(L_0)$ by

$$\mathsf{deg}_{\mathsf{NExt}(L_0)}(L) = |\{L' \in \mathsf{NExt}(L_0) : \mathsf{Fr}(L') = \mathsf{Fr}(L)\}|.$$

To simplify notation, we write deg_{L_0} for $\mathsf{deg}_{\mathsf{NExt}(L_0)}$. Moreover, we define the *degree of finite model property* (degree of FMP) $\mathsf{df}_{\mathsf{NExt}(L_0)}(L)$ of L in $\mathsf{NExt}(L_0)$ by

$$\mathsf{df}_{\mathsf{NExt}(L_0)}(L) = |\{L' \in \mathsf{NExt}(L_0) : \mathsf{Fin}(L') = \mathsf{Fin}(L)\}|.$$

Analogously, we simply write df_{L_0} for $\mathsf{df}_{\mathsf{NExt}(L_0)}$.

The following proposition follows from the fact that $\mathsf{Fin}(L) \subseteq \mathsf{Fr}(L)$:

5.3.2. Proposition. *Let $L \supseteq L_0$ be polymodal logics. Then $\mathsf{deg}_{L_0}(L) \leq \mathsf{df}_{L_0}(L)$.*

5.3.1 Degree of Kripke incompleteness in $\mathsf{NExt}(\mathsf{K}_t)$

In this subsection, we focus on the degree of Kripke incompleteness and the degree of FMP in $\mathsf{NExt}(\mathsf{K}_t)$. We write deg and df for $\mathsf{deg}_{\mathsf{K}_t}$ and $\mathsf{df}_{\mathsf{K}_t}$, respectively.

Our main task is to prove the dichotomy theorem of the degree of Kripke incompleteness for $\mathsf{NExt}(\mathsf{K}_t)$ (Theorem 5.3.18). As indicated by Blok [14], the degree of Kripke incompleteness is closely connected to splittings. Thus, we first investigate splittings in $\mathsf{NExt}(\mathsf{K}_t)$. By Theorem 5.2.11, $\langle \mathsf{Log}(\mathfrak{Ch}_1^<), \mathsf{K}_t/\mathfrak{Ch}_1^< \rangle$ is the unique splitting pair in $\mathsf{NExt}(\mathsf{K}_t)$. Clearly, $\mathsf{Log}(\mathfrak{Ch}_1^<) = \mathsf{K}_t \oplus (\Box\bot \wedge \blacksquare\bot)$. The reader can readily check that $\{\mathsf{K}_t, \mathsf{K}_t/\mathfrak{Ch}_1^<\}$ is exactly the set of union-splittings of $\mathsf{NExt}(\mathsf{K}_t)$. Moreover, since K_t enjoys the FMP, we see immediately that $\mathsf{deg}(\mathsf{K}_t) = \mathsf{df}(\mathsf{K}_t) = 1$. Now, we take a closer look at the logic $\mathsf{K}_t/\mathfrak{Ch}_1^<$ by showing the following propositions:

5.3.3. PROPOSITION. *Let* $\mathbb{F} \in \mathsf{GFr}_{\mathrm{r}}$ *be a rooted general frame. Then* $\mathbb{F} \vDash \Diamond\top \vee \blacklozenge\top$ *if and only if* $\mathbb{F} \not\cong \mathfrak{Ch}_1^<$.

Proof:
The left-to-right direction is trivial. Suppose $\mathbb{F} \not\cong \mathfrak{Ch}_1^<$. Since $\mathbb{F}$ is rooted, we see that $R[x] \cup R^{-1}[x] \neq \varnothing$ for any $x \in X$. Thus, $\mathbb{F} \vDash \Diamond\top \vee \blacklozenge\top$. □

5.3.4. PROPOSITION. $\mathsf{K}_t/\mathfrak{Ch}_1^< = \mathsf{K}_t \oplus (\Diamond\top \vee \blacklozenge\top)$.

Proof:
Since $\mathfrak{Ch}_1^< \nvDash \Diamond\top \vee \blacklozenge\top$, we have $\mathsf{K}_t/\mathfrak{Ch}_1^< \subseteq \mathsf{K}_t \oplus (\Diamond\top \vee \blacklozenge\top)$. Take any $\mathbb{F} \in \mathsf{GFr}_{\mathrm{r}}(\mathsf{K}_t/\mathfrak{Ch}_1^<)$. Then $\mathbb{F} \not\cong \mathfrak{Ch}_1^<$. By Proposition 5.3.3, $\mathbb{F} \vDash \Diamond\top \vee \blacklozenge\top$. Hence, $\Diamond\top \vee \blacklozenge\top \in \mathsf{Log}(\mathsf{GFr}_{\mathrm{r}}(\mathsf{K}_t/\mathfrak{Ch}_1^<)) = \mathsf{K}_t/\mathfrak{Ch}_1^<$. □

5.3.5. PROPOSITION. *Let* $L \in \mathsf{NExt}(\mathsf{K}_t)$. *Then* $L \subsetneq \mathsf{K}_t/\mathfrak{Ch}_1^<$ *if and only if* $L = \mathsf{K}_t$.

Proof:
Suppose $L \subsetneq \mathsf{K}_t/\mathfrak{Ch}_1^<$. Then $L \not\supseteq \mathsf{K}_t/\mathfrak{Ch}_1^<$ and so $L \subseteq \mathsf{Log}(\mathfrak{Ch}_1^<)$. Thus, $\mathfrak{Ch}_1^< \in \mathsf{Fr}(L)$. By Proposition 5.3.3, $\mathsf{GFr}_{\mathrm{r}}(L) = \mathsf{GFr}_{\mathrm{r}}(\mathsf{K}_t)$, which entails $L \subseteq \mathsf{Log}(\mathsf{GFr}_{\mathrm{r}}(L)) \subseteq \mathsf{Log}(\mathsf{GFr}_{\mathrm{r}}(\mathsf{K}_t)) = \mathsf{K}_t$. □

So far, we have obtained a finite axiomatization of $\mathsf{K}_t/\mathfrak{Ch}_1^<$ and showed that the interval $[\mathsf{K}_t, \mathsf{K}_t/\mathfrak{Ch}_1^<]$ contains only two logics. Next, we show that union-splittings and iterated splittings coincide in $\mathsf{NExt}(\mathsf{K}_t)$ (Theorem 5.3.7).

5.3.6. LEMMA. *No logic splits* $\mathsf{NExt}(\mathsf{K}_t/\mathfrak{Ch}_1^<)$.

Proof:
Towards a contradiction, suppose there exists a logic L that splits $\mathsf{NExt}(\mathsf{K}_t/\mathfrak{Ch}_1^<)$. Since $L \supseteq \mathsf{K}_t/\mathfrak{Ch}_1^<$, by Theorem 5.2.11, L does not split $\mathsf{NExt}(\mathsf{K}_t)$. By Proposition 5.2.5, $\bigcap_{i\in I} L_i \subseteq L$ for some family of logics $\mathcal{K} = \{L_i \in \mathsf{NExt}(\mathsf{K}_t) : L_i \nsubseteq L, i \in I\}$. For each $i \in I$, let $L_i' = L_i \oplus (\Diamond\top \vee \blacklozenge\top)$. Let $\mathcal{K}' = \{L_i' : i \in I\}$. Take any $\varphi \notin L$. Since $\mathsf{K}_t/\mathfrak{Ch}_1^< \subseteq L$, we have $(\Diamond\top \vee \blacklozenge\top) \to \varphi \notin L$. Since $\bigcap_{i\in I} L_i \subseteq L$, we have that $(\Diamond\top \vee \blacklozenge\top) \to \varphi \notin L_i$ for some $i \in I$. Then there exists $\mathbb{F} \in \mathsf{GFr}(L_i)$ such that $\mathbb{F} \nvDash (\Diamond\top \vee \blacklozenge\top) \to \varphi$. By Proposition 5.3.3, it is not hard to see that $\mathbb{F} \vDash (\Diamond\top \vee \blacklozenge\top)$ and $\mathbb{F} \nvDash \varphi$. Thus, $\varphi \notin L_i \oplus (\Diamond\top \vee \blacklozenge\top) = L_i'$. Since φ is arbitrarily chosen, $\bigcap \mathcal{K}' \subseteq L$. Note that $L_i' \in \mathsf{NExt}(\mathsf{K}_t/\mathfrak{Ch}_1^<)$ and $L_i' \nsubseteq L$ for any $i \in I$. Since L splits $\mathsf{NExt}(\mathsf{K}_t/\mathfrak{Ch}_1^<)$, by Proposition 5.2.5, there exists $i \in I$ such that $L_i' \subseteq L$. Thus, $L_i \subseteq L_i' \subseteq L$, which contradicts the definition of $\mathcal{K}$. □

As a corollary, the following theorem holds:

5.3.7. THEOREM. *Let* $L \in \mathsf{NExt}(\mathsf{K}_t)$. *Then the following are equivalent:*

(1) L *is an iterated splitting in* $\mathsf{NExt}(\mathsf{K}_t)$*;*

(2) L *is a union-splitting in* $\mathsf{NExt}(\mathsf{K}_t)$*;*

(3) $L \in \{\mathsf{K}_t, \mathsf{K}_t/\mathfrak{Ch}_1^<\}$.

Proof:
By Lemma 5.3.6, K_t and $\mathsf{K}_t/\mathfrak{Ch}_1^<$ are the only two iterated splittings in $\mathsf{NExt}(\mathsf{K}_t)$. Thus, (1) and (3) are equivalent. It follows from Theorem 5.2.11 that (2) and (3) are equivalent. □

We are now ready to show a part of the characterization theorem (Theorem 5.3.18) of the degree of Kripke incompleteness for $\mathsf{NExt}(\mathsf{K}_t)$:

5.3.8. THEOREM. *Let* L *be a union-splitting in* $\mathsf{NExt}(\mathsf{K}_t)$. *Then* $\mathsf{df}(L) = 1$.

Proof:
By Theorem 5.3.7, $L \in \{\mathsf{K}_t, \mathsf{K}_t/\mathfrak{Ch}_1^<\}$. Clearly, $\mathsf{df}(\mathsf{K}_t) = 1$. Let $L = \mathsf{K}_t/\mathfrak{Ch}_1^<$. Suppose $\mathsf{Fin}(L') = \mathsf{Fin}(\mathsf{K}_t/\mathfrak{Ch}_1^<)$ for some $L' \neq \mathsf{K}_t/\mathfrak{Ch}_1^<$. Note that $\mathsf{K}_t/\mathfrak{Ch}_1^< = \mathsf{K}_t \oplus (\Diamond\top \vee \blacklozenge\top)$ has the FMP, we see $L' \subsetneq \mathsf{K}_t/\mathfrak{Ch}_1^<$. By Proposition 5.3.5, $L' = \mathsf{K}_t$. Thus, $\mathfrak{Ch}_1^< \in \mathsf{Fin}(L')$, which contradicts the fact that $\mathsf{Fin}(L') = \mathsf{Fin}(\mathsf{K}_t/\mathfrak{Ch}_1^<)$. □

To complete the proof of the main theorem (Theorem 5.3.18), it is now sufficient to fix a tense logic $L \in \mathsf{NExt}(\mathsf{K}_t) \setminus \{\mathsf{K}_t, \mathsf{K}_t/\mathfrak{Ch}_1^<\}$ and show that $\mathsf{deg}(L) = 2^{\aleph_0}$. The proof idea is as follows: we construct for each $I \subseteq \mathbb{Z}^+$ a general frame $\mathbb{F}_I$ such that $\mathsf{Fr}(L \cap \mathsf{Log}(\mathbb{F}_I)) = \mathsf{Fr}(L)$. Let $L_I = L \cap \mathsf{Log}(\mathbb{F}_I)$. Then $\mathsf{deg}(L) \geq |\{L_I : I \subseteq \mathbb{Z}^+\}|$ and we are done once we show the following:

$$L_I \neq L_J \text{ for any distinct } I, J \subseteq \mathbb{Z}^+.$$

In what follows, let $I \subseteq \mathbb{Z}^+$ be arbitrarily fixed and we start with the construction of $\mathbb{F}_I$. Intuitively, the general frame $\mathbb{F}_I$ will be a combination of a finite rooted frame $\mathfrak{F}_L$ and a general frame $\mathbb{F}'_I$. On the one hand, the finite frame $\mathfrak{F}_L$ is designed to refute some formula in L, since we have to ensure that $L_I \neq L_J$ for any different $I, J \subseteq \mathbb{Z}^+$, which requires $L \neq L \cap \mathsf{Log}(\mathbb{F}_I)$. On the other hand, we have to construct $\mathbb{F}'_I$ properly to make the logics L_I pairwise different and share the same frames as L. The trick here is to choose $\mathfrak{F}_L$ to be a finite frame of sufficiently large r-degree, which ensures that $\mathbb{F}'_I$ and $\mathfrak{F}_L$ both work well after being combined. More precisely, the following lemma holds:

5.3.9. LEMMA. *For all* $\varphi \notin \mathsf{K}_t/\mathfrak{Ch}_1^<$ *and* $n \in \omega$, *there exists* $\mathfrak{F} \in \mathsf{Fin}_r$ *such that* $\mathfrak{F} \nvDash \varphi$ *and* $\mathsf{rdg}(\mathfrak{F}) \geq n$.

Proof:
Take any $\varphi \notin \mathsf{K}_t/\mathfrak{Ch}_1^<$. By Proposition 5.3.4, $\mathsf{K}_t/\mathfrak{Ch}_1^< = \mathsf{K}_t \oplus (\Diamond\top \vee \blacklozenge\top)$ has the FMP and so $\mathfrak{G} \not\models \varphi$ for some $\mathfrak{G} \in \mathsf{Fin}_r(\mathsf{K}_t/\mathfrak{Ch}_1^<)$. By Propositions 5.3.3, $\mathbb{F} \not\cong \mathfrak{Ch}_1^<$. Then the existence of required $\mathfrak{F}$ follows from Lemma 5.1.9. □

By Proposition 5.3.5, there exists a formula $\varphi_L \in L$ such that $\varphi_L \notin \mathsf{K}_t/\mathfrak{Ch}_1^<$. By Lemma 5.3.9, there is a finite rooted frame $\mathfrak{F}_L = (X_L, R_L)$ and $w_L, u_L \in X_L$ such that $\mathfrak{F}_L, w_L \not\models \varphi_L$ and $u_L \notin R_\sharp^{\mathsf{md}(\varphi_L)}[w_L]$.

5.3.10. Example. Consider the tense logic $\mathsf{K4}_t$. Let $\varphi_{\mathsf{K4}_t} = \Diamond\Diamond p \to \Diamond p$. Then we see that $\varphi_{\mathsf{K4}_t} \in \mathsf{K4}_t \setminus (\mathsf{K}_t/\mathfrak{Ch}_1^<)$. Moreover, we could take $\mathfrak{F}_{\mathsf{K4}_t} = (X_{\mathsf{K4}_t}, R_{\mathsf{K4}_t})$ to be the non-transitive frame depicted in Figure 5.6. Clearly, $(R_{\mathsf{K4}_t})_\sharp^2[w_0] \neq X_{\mathsf{K4}_t}$ and $w_0 \not\models \varphi_{\mathsf{K4}_t}$.

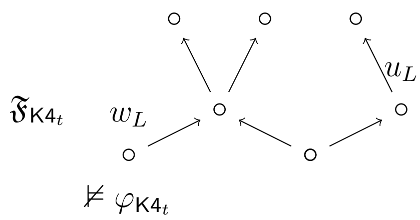

Figure 5.6: A possible choice of $\mathfrak{F}_{\mathsf{K4}_t}$

5.3.11. Definition. For each $I \subseteq \mathbb{Z}^+$, let $\mathbb{F}'_I = (Y_I, S_I, B_I)$ be the general frame defined as follows:

- $Y_I = \omega \cup \{i^* : i \in I \setminus \{1\}\} \cup \{0^*\}$.
- $S_I = \{\langle n, m\rangle \in \omega \times \omega : n < m\} \cup \{\langle i^*, j\rangle : i \in I \setminus \{1\} \text{ and } i \leq j\} \cup \{\langle 0, 0^*\rangle\}$.
- B_I is the internal set generated by $\varnothing$.

The general frames $\mathbb{F}'_I$ were introduced in [34]. The tense logic of $\mathbb{F}'_I$ has no consistent proper extension and no Kripke frame [34, Proposition 5.7]. Now we define the general frame $\mathbb{F}_I$ to be $(\langle \mathfrak{F}_L u_L + 0^*\kappa\mathbb{F}'_I\rangle, A_I)$, where A_I is the internal set generated by $\mathcal{P}(X_L)$.

5.3.12. Example. Consider again the frame $\mathfrak{F}_{\mathsf{K4}_t}$. Let $\mathbb{P}$ be the set of all prime numbers. Then $\mathbb{F}_\mathbb{P}$ is depicted in Figure 5.7. We see that the formula φ_{K4_t} is refuted at w_0.

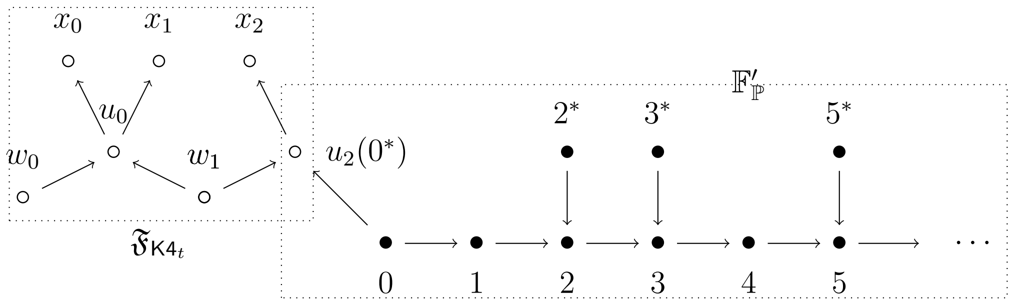

Figure 5.7: The general frame $\mathbb{F}_{\mathbb{P}}$ (for $\mathsf{K4}_t$)

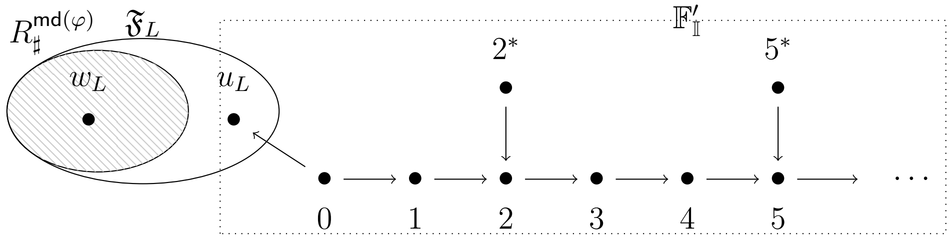

Figure 5.8: The general frame $\mathbb{F}_{\mathbb{I}}$ where $2, 5 \in I$

In general, the general frames $\mathbb{F}_I$ are of the shape depicted in Figure 5.8.

Now we are in a position to show that for any $I \subseteq \mathbb{Z}^+$, the tense logic $\mathsf{Log}(\mathbb{F}_I)$ has no Kripke frame (Lemma 5.3.14). Let $k \in \omega$ be such that $|\mathfrak{F}_L| < k$ and $X_I = (R_I)_\sharp^k[v]$ for all $v \in X_I$. For each $n \in \omega$ and $m \in \mathbb{Z}^+$, we define the formulas γ_n and γ_m^* as follows:

- $\gamma_0 = \blacksquare \bot \wedge \Diamond \blacksquare^2 \bot \wedge \Diamond^k \blacksquare^{k+1} \bot$ and $\gamma_{l+1} = \blacklozenge \gamma_l \wedge \blacksquare^2 \neg \gamma_l$.
- $\gamma_m^* = \Diamond \gamma_m \wedge \Box \neg \gamma_{m-1} \wedge \blacksquare \bot \wedge \Box \Diamond^k \top$.

5.3.13. Lemma. *For all $n \in \omega$, $m \in \mathbb{Z}^+$ and $x \in X_I$,*

(1) $\mathbb{F}_I, x \vDash \gamma_n$ if and only if $x = n$;

(2) $\mathbb{F}_I, x \vDash \gamma_m^*$ if and only if $m \in I$ and $x = m^*$.

Proof:
For (1), we prove by induction on n. Let $n = 0$. Note that ${R_I}^{-1}[0] = \varnothing$, $k \in R_I^k[0]$ and $\mathbb{F}_I, k \vDash \blacksquare^{k+1} \bot$, we have $\mathbb{F}_I, 0 \vDash \gamma_0$. Suppose $\mathbb{F}_I, x \vDash \gamma_0$. Then $\mathbb{F}_I, x \vDash \Diamond^k \blacksquare^{k+1} \bot$, which entails that there exists a strict chain $\langle x_i : i \leq k \rangle$ with $x = x_0$. Since $|\mathfrak{F}_L| < k$, we see $x \notin X_L$. Since $\mathbb{F}_I, x \vDash \blacksquare \bot \wedge \Diamond \blacksquare^2 \bot$, we have $x \notin \mathbb{Z}^+ \cup \{i^* : i \in I \setminus \{1\}\}$. Thus, $x = 0$. Let $n > 0$. By induction

hypothesis, for all $y \in X_I$, we have $\mathbb{F}_I, y \models \gamma_{n-1}$ if and only if $y = n-1$. Since $R_I[n-1] \setminus R_I^2[n-1] = \{n\}$, we have $\mathbb{F}_I, n \models \gamma_n$ if and only if $x = n$.

(2) The right-to-left direction is trivial. Suppose $\mathbb{F}_I, x \models \gamma_m^*$. By $\mathbb{F}_I, x \models \Diamond\gamma_m \wedge \Box\neg\gamma_{m-1}$, we have $x \notin X_L \cup \{l, l^* \in X_I : l < m-1 \text{ or } l > m\} \cup \{m, (m-1)^*\}$. Since $\mathbb{F}_I, x \models \blacksquare\bot \wedge \Box\Diamond^k\top$, $x \neq m-1$. By $x \in X_I = X_L \uplus (\omega \cup \{i^* : i \in I\})$, we see $m \in I$ and $x = m^*$. □

5.3.14. Lemma. $\mathsf{Fr}(\mathsf{Log}(\mathbb{F}_I)) = \varnothing$.

Proof:
Suppose there exists $\mathfrak{G} = (Y, S) \in \mathsf{Fr}(\mathsf{Log}(\mathbb{F}_I))$. By an easy induction, we see that $V \cap \omega$ is either finite or cofinite for all $V \in A_I$. Thus,

$$\mathbb{F}_I, 1 \models \Box(\Box(p \to \Box p) \to p) \to \Box p.$$

By Lemma 5.3.13(1), $\mathbb{F}_I \models \gamma_i \to \Diamond\gamma_{i+1}$ for all $i \in \omega$. Thus, we have

$$\mathfrak{G} \models \gamma_1 \to (\Box(\Box(p \to \Box p) \to p) \to \Box p) \text{ and } \mathfrak{G} \models \{\gamma_i \to \Diamond\gamma_{i+1} : i \in \omega\}.$$

Note that $\mathbb{F}_I \models \Delta^{\leq k}\gamma_1$ and γ_1 is variable-free, there exists $y \in Y$ such that $\mathfrak{G}, y \models \gamma_1$. Since $\mathfrak{G} \models \{\gamma_i \to \Diamond\gamma_{i+1} : i \in \omega\}$, we see that there exists an infinite strictly ascending S-chain $\langle u_i : i \in \mathbb{Z}^+\rangle$ such that $y = u_1$ and $\mathfrak{G}, u_i \models \gamma_i$ for all $i \in \mathbb{Z}^+$. Let U be a valuation in $\mathfrak{G}$ such that $U(p) = \{u_{2i} : i \in \mathbb{Z}^+\}$. Then we see that $\mathfrak{G}, U, y \not\models \Box(\Box(p \to \Box p) \to p) \to \Box p$. Hence, $\mathfrak{G} \not\models \gamma_1 \to (\Box(\Box(p \to \Box p) \to p) \to \Box p)$, which contradicts $\mathfrak{G} \in \mathsf{Fr}(\mathsf{Log}(\mathbb{F}_I))$. □

As a corollary, we have

5.3.15. Lemma. $\mathsf{Fr}(L) = \mathsf{Fr}(L_I)$ *for all* $I \in \mathcal{P}(\mathbb{Z}^+)$.

Proof:
By Lemmas 2.2.19 and 5.3.14, $\mathsf{Fr}_r(L_I) = \mathsf{Fr}_r(L) \cup \mathsf{Fr}_r(\mathsf{Log}(\mathbb{F}_I)) = \mathsf{Fr}_r(L)$. □

By Lemma 5.3.15, we see that L_I has the same frame as L for all $I \subseteq \mathbb{Z}^+$. It remains to prove that distinct subsets I, J of $\mathbb{Z}^+$ induce different tense logics:

5.3.16. Lemma. *For all* $I, J \in \mathcal{P}(\mathbb{Z}^+)$, *if* $I \neq J$ *then* $L_I \neq L_J$.

Proof:
Take any distinct $I, J \in \mathcal{P}(\mathbb{Z}^+)$. Let $i \in I \setminus J$. It suffices to show that

$$\neg\varphi_L \to \Delta^{\leq k}\gamma_i^* \in L_I \setminus L_J,$$

where the modality $\Delta^{\leq k}$ is defined in Definition 2.2.11. By Lemma 5.3.13(2), $\mathbb{F}_I, i^* \models \gamma_i^*$ and so $\mathbb{F}_I \models \Delta^{\leq k}\gamma_i^*$. Since $\varphi_L \in L$, we see that $\neg\varphi_L \to \Delta^{\leq k}\gamma_i^* \in L \cap \mathsf{Log}(\mathbb{F}_I) = L_I$. Since $i \notin J$, by Lemma 5.3.13(2), we obtain $\mathbb{F}_J \models \neg\gamma_i^*$. Note that $\mathbb{F}_J\restriction R_{J^\sharp}^{\mathsf{md}(\varphi_L)}[w_L] \cong \mathfrak{F}_L\restriction R_{L^\sharp}^{\mathsf{md}(\varphi_L)}[w_L]$. Thus, we have $\mathbb{F}_J, w_L \not\models \varphi_L$. Hence, $\mathbb{F}_J, w_L \not\models \varphi_L \vee \Delta^{\leq k}\gamma_i^*$ and so $\neg\varphi_L \to \Delta^{\leq k}\gamma_i^* \notin L_J$. □

Since L is chosen to be an arbitrary tense logic which is not a union-splitting in $\mathsf{NExt}(\mathsf{K}_t)$, by Lemmas 5.3.15 and 5.3.16, we have

5.3.17. Theorem. *Let* $L \in \mathsf{NExt}(\mathsf{K}_t) \setminus \{\mathsf{K}_t, \mathsf{K}_t/\mathfrak{Ch}_1^<\}$. *Then* $\mathsf{deg}(L) = 2^{\aleph_0}$.

We conclude this section by the following characterization theorems for $\mathsf{NExt}(\mathsf{K}_t)$:

5.3.18. Theorem. *Let* $L \in \mathsf{NExt}(\mathsf{K}_t)$. *Then the following are equivalent:*

(1) L *is a union-splitting in* $\mathsf{NExt}(\mathsf{K}_t)$.

(2) L *is an iterated splitting in* $\mathsf{NExt}(\mathsf{K}_t)$.

(3) $\mathsf{df}(L) = 1$.

(4) $\mathsf{deg}(L) = 1$.

(5) $\mathsf{df}(L) \neq 2^{\aleph_0}$.

(6) $\mathsf{deg}(L) \neq 2^{\aleph_0}$.

Proof:
The equivalence of (1) and (2) follows from Theorem 5.3.7. By Theorem 5.3.8, (1) implies (3). Since $|\mathsf{NExt}(\mathsf{K}_t)| \leq 2^{\aleph_0}$, by Proposition 5.3.2, (3) implies (4), and (5) implies (6). Clearly (3) implies (5), and (4) implies (6). Finally, by Theorem 5.3.17, (6) implies (1), which concludes the proof. □

The following dichotomy theorem follows immediately:

5.3.19. Theorem. *For all* $L \in \mathsf{NExt}(\mathsf{K}_t)$, $\mathsf{deg}(L) = \mathsf{df}(L) \in \{1, 2^{\aleph_0}\}$.

By Theorem 5.3.19, we see the similarity and difference between $\mathsf{NExt}(\mathsf{K}_t)$ and $\mathsf{NExt}(\mathsf{K})$. Blok's dichotomy theorem of the degree of Kripke incompleteness hold for both. However, the dichotomy theorem of the degree of FMP holds for exactly one of them, namely, $\mathsf{NExt}(\mathsf{K}_t)$ (cf. [9]). As we will see in the next subsection, dichotomy theorems also hold for $\mathsf{NExt}(\mathsf{K4}_t)$, while it is still unknown whether the dichotomy theorem of the degree of Kripke incompleteness for $\mathsf{NExt}(\mathsf{K4})$ holds.

5.3.2 Degree of Kripke incompleteness in $\mathsf{NExt}(\mathsf{K4}_t)$

Let us now move from the lattice $\mathsf{NExt}(\mathsf{K}_t)$ to $\mathsf{NExt}(\mathsf{K4}_t)$. The proof idea of the dichotomy theorem for $\mathsf{NExt}(\mathsf{K4}_t)$ is similar to the one for $\mathsf{NExt}(\mathsf{K}_t)$. In this subsection, we always presume that frames are transitive. By [71, Theorem 10], $\langle \mathsf{Log}(\mathfrak{Ch}_1^<), \mathsf{K4}_t/\mathfrak{Ch}_1^< \rangle$ is the unique splitting pair in $\mathsf{NExt}(\mathsf{K4}_t)$. Moreover, in this subsection, we write deg and df for $\mathsf{deg}_{\mathsf{K4}_t}$ and $\mathsf{df}_{\mathsf{K4}_t}$, respectively.

Similar to the case for $\mathsf{NExt}(\mathsf{K}_t)$, the logics $\mathsf{K4}_t$ and $\mathsf{K4}_t/\mathfrak{Ch}_1^<$ are exactly the union-splittings in $\mathsf{NExt}(\mathsf{K4}_t)$. Since $\mathsf{K4}_t$ has the FMP, $\mathsf{deg}(\mathsf{K4}_t) = \mathsf{df}(\mathsf{K4}_t) = 1$. Moreover, by arguments similar to those in Section 5.3.1, we have:

5.3.20. PROPOSITION. *The following statements hold:*

(1) for all $\mathbb{F} \in \mathsf{GFr}_\mathrm{r}(\mathsf{K4}_t)$, $\mathbb{F} \models \Diamond\top \vee \blacklozenge\top$ *if and only if* $\mathbb{F} \ncong \mathfrak{Ch}_1^<$;

(2) $\mathsf{K4}_t/\mathfrak{Ch}_1^< = \mathsf{K4}_t \oplus (\Diamond\top \vee \blacklozenge\top)$; *and*

(3) for all $L \in \mathsf{NExt}(\mathsf{K4}_t)$, $L \subsetneq \mathsf{K4}_t/\mathfrak{Ch}_1^<$ *if and only if* $L = \mathsf{K4}_t$.

By Proposition 5.3.20, the following theorem holds:

5.3.21. THEOREM. *Let* $L \in \mathsf{NExt}(\mathsf{K4}_t)$. *Then the following are equivalent:*

(1) L *is an iterated splitting in* $\mathsf{NExt}(\mathsf{K4}_t)$;

(2) L *is a union-splitting in* $\mathsf{NExt}(\mathsf{K4}_t)$;

(3) $L \in \{\mathsf{K4}_t, \mathsf{K4}_t/\mathfrak{Ch}_1^<\}$.

Proof:
By replacing every occurrence of K_t in the proof of Theorem 5.3.7 by $\mathsf{K4}_t$, we obtain the desired result. □

As a corollary, we have

5.3.22. THEOREM. *For all union-splittings* L *in* $\mathsf{NExt}(\mathsf{K4}_t)$, $\mathsf{df}(L) = \mathsf{deg}(L) = 1$.

Similar to the case for K_t, we have already proved half of the main theorem (Theorem 5.3.26). It is sufficient now to prove that $\mathsf{deg}(L) = 2^{\aleph_0}$ for all non-union-splittings L, i.e., $L \in \mathsf{NExt}(\mathsf{K4}_t) \setminus \{\mathsf{K4}_t, \mathsf{K4}_t/\mathfrak{Ch}_1^<\}$. The method we use here is similar to the one in Section 5.3.1 and the key lemma is the following:

5.3.23. LEMMA. *Let* $\varphi \notin \mathsf{K4}_t/\mathfrak{Ch}_1^<$. *Then*

(1) φ *is refuted by some non-symmetric* $\mathfrak{G} \in \mathsf{Fin}_r(\mathsf{K4}_t)$;

(2) for all $n \in \omega$*, there exists* $\mathfrak{F} \in \mathsf{Fin}_r(\mathsf{K4}_t)$ *such that* $\mathfrak{F} \not\models \varphi$ *and* $\mathsf{rdg}(\mathfrak{F}) \geq n$*.*

Proof:
For (1), take any $\varphi \notin \mathsf{K4}_t/\mathfrak{Ch}_1^<$. By Proposition 5.3.20, $\mathsf{K4}_t/\mathfrak{Ch}_1^< = \mathsf{K4}_t \oplus (\Diamond\top \vee \blacklozenge\top)$ has the FMP. Then there exists $\mathfrak{F}' = (X', R') \in \mathsf{Fin}_r(\mathsf{K4}_t/\mathfrak{Ch}_1^<)$ such that $\mathfrak{F}' \not\models \varphi$. If $\mathfrak{F}'$ is already non-symmetric, then take $\mathfrak{G} = \mathfrak{F}'$ and we are done. Suppose $\mathfrak{F}'$ is symmetric. Since $\mathfrak{F}'$ is rooted and transitive, $\mathfrak{F}$ is a cluster. Note that $\mathfrak{F}' \models \Diamond\top \vee \blacklozenge\top$, we see that $R'_\sharp[x'] \neq \varnothing$ for each $x' \in X'$ and so $R' = X' \times X'$. Let $\mathfrak{G} = (X, R)$ where $X = X' \times \{0, 1\}$ and $R = \{\langle\langle x, a\rangle, \langle y, b\rangle\rangle \in X \times X : a \leq b\}$. Consider the map $f : X \to X'$ defined by $f(\langle x, a\rangle) = x$ for all $\langle x, a\rangle \in X$. Obviously, f is a t-morphism from $\mathfrak{G}$ to $\mathfrak{F}'$. Thus, $\mathfrak{G} \not\models \varphi$ and (1) holds.

By Lemma 5.1.11, (2) follows immediately from (1). □

We can now prove the following theorem:

5.3.24. Theorem. *Let* $L \in \mathsf{NExt}(\mathsf{K4}_t) \setminus \{\mathsf{K4}_t, \mathsf{K4}_t/\mathfrak{Ch}_1^<\}$*. Then* $\mathsf{deg}(L) = 2^{\aleph_0}$*.*

Proof:
Let $L \in \mathsf{NExt}(\mathsf{K4}_t) \setminus \{\mathsf{K4}_t, \mathsf{K4}_t/\mathfrak{Ch}_1^<\}$. By Proposition 5.3.20, $L \not\subseteq \mathsf{K4}_t/\mathfrak{Ch}_1^<$ and so there exists $\varphi_L \in L \setminus \mathsf{K4}_t/\mathfrak{Ch}_1^<$. By Lemma 5.3.23, there is a finite rooted frame $\mathfrak{F}_L = (X_L, R_L)$ and $w_L, u_L \in X$ such that $\mathfrak{F}_L, w_L \not\models \varphi_L$ and $u_L \notin R_\sharp^{\mathsf{md}(\varphi)}[w_L]$. For each $I \in \mathbb{Z}^+$, we define $\mathbb{F}_I$ to be the general frame $(\langle \mathfrak{F}_L u_L +^t 0^* \kappa \mathbb{F}'_I\rangle, A_I)$, where A_I is the internal set generated by $\mathcal{P}(X_L)$. Then $\mathbb{F}_I$ is transitive. By essentially the same proofs as those of Lemmas 5.3.13, 5.3.14, 5.3.15 and 5.3.16, we have

- $\mathsf{Fr}(L) = \mathsf{Fr}(L_I)$ for all $I \subseteq \mathbb{Z}^+$;
- $I \neq J$ implies $L_I \neq L_J$ for all $I, J \subseteq \mathbb{Z}^+$.

It follows that $2^{\aleph_0} = |\{L_I : I \subseteq \mathbb{Z}^+\}| \leq \mathsf{deg}(L) \leq 2^{\aleph_0}$. Hence, $\mathsf{deg}(L) = 2^{\aleph_0}$. □

5.3.25. Remark. In the proof of Theorem 5.3.24, in order to preserve transitivity throughout the construction, we defined the general frames $\mathbb{F}_I$ to be $(\langle \mathfrak{F}_L u_L +^t 0^* \kappa \mathbb{F}'_I\rangle, A_I)$ rather than $(\langle \mathfrak{F}_L u_L + 0^* \kappa \mathbb{F}'_I\rangle, A_I)$. In the basic modal case, this construction no longer works. This is precisely the reason why we employ the technique of transitive reflective unfolding.

By Theorems 5.3.21, 5.3.22 and 5.3.24, the following theorem holds:

5.3.26. Theorem. *Let* $L \in \mathsf{NExt}(\mathsf{K4}_t)$*. Then the following are equivalent:*

(1) L *is a union-splitting in* $\mathsf{NExt}(\mathsf{K4}_t)$*.*

(2) L *is an iterated splitting in* $\mathsf{NExt}(\mathsf{K4}_t)$*.*

(3) $\mathsf{df}(L) = 1$.

(4) $\mathsf{deg}(L) = 1$.

(5) $\mathsf{df}(L) \neq 2^{\aleph_0}$.

(6) $\mathsf{deg}(L) \neq 2^{\aleph_0}$.

Again, we conclude the subsection with the dichotomy theorem:

5.3.27. THEOREM. *For all* $L \in \mathsf{NExt}(\mathsf{K4}_t)$*,* $\mathsf{deg}(L) = \mathsf{df}(L) \in \{1, 2^{\aleph_0}\}$.

5.4 Kripke Incompleteness in $\mathsf{NExt}(\mathsf{S4}_t)$

In this section, we move from the lattices $\mathsf{NExt}(\mathsf{K}_t)$ and $\mathsf{NExt}(\mathsf{K4}_t)$ to the lattice $\mathsf{NExt}(\mathsf{S4}_t)$. In what follows, we write $\mathsf{deg}(L)$ for the degree of Kripke incompleteness of L in $\mathsf{NExt}(\mathsf{S4}_t)$. Our aim is to prove the dichotomy theorem for $\mathsf{NExt}(\mathsf{S4}_t)$. We first characterize the iterated splittings in $\mathsf{NExt}(\mathsf{S4}_t)$ and show that every iterated splitting has the degree of Kripke incompleteness 1.

To show that all other tense logics L in $\mathsf{NExt}(\mathsf{S4}_t)$ are of the degree of Kripke incompleteness $2^{\aleph_0}$, we construct general frames closely connected to the *Rieger-Nishimura ladder*, which is the underlying frame of the dual space of the 1-generated free Heyting algebra (for more details, see [100, 111, 3, 117]). With the help of these general frames, we obtain continuum many tense logics in $[L]_{\mathsf{Fr}} = \{L' \in \mathsf{NExt}(\mathsf{S4}_t) : \mathsf{Fr}(L) = \mathsf{Fr}(L')\}$. This gives a characterization theorem for the degree of Kripke incompleteness in $\mathsf{NExt}(\mathsf{S4}_t)$ and completes the proof of the dichotomy theorem.

Let $n \in \mathbb{Z}^+$. Recall that we write $\mathfrak{Ch}_n$ for the reflexive transitive chain of length n and $\mathfrak{Cl}$ for the n-cluster, that is, $\mathfrak{Ch}_n = (n, \leq)$ and $\mathfrak{Cl}_n = (n, n \times n)$. We first give a characterization of the iterated splittings in $\mathsf{NExt}(\mathsf{S4}_t)$. The key observation is the following characterization of the lattice $\mathsf{NExt}(\mathsf{S5}_t)$:

5.4.1. THEOREM. $(\mathsf{NExt}(\mathsf{S5}_t), \subseteq) \cong (\omega + 1, \geq)$.

Proof:
By Proposition 4.4.5 and Theorem 4.3.10, we have

$$\mathsf{NExt}(\mathsf{S5}_t) = \{\mathsf{S5}_t, \mathsf{Form}_t\} \cup \{\mathsf{Log}(\mathfrak{Cl}_i) : i \in \mathbb{Z}^+\}.$$

Thus, the lattice $\mathsf{NExt}(\mathsf{S5}_t)$ can be characterized as follows:

$$\mathsf{S5}_t \subseteq \cdots \subseteq \mathsf{Log}(\mathfrak{Cl}_i) \subseteq \mathsf{Log}(\mathfrak{Cl}_{i-1}) \subseteq \cdots \subseteq \mathsf{Log}(\mathfrak{Cl}_1) \subseteq \mathsf{Form}_t.$$

Hence, we conclude that $(\mathsf{NExt}(\mathsf{S5}_t), \subseteq) \cong (\omega + 1, \geq)$. □

As a consequence, the following theorem holds:

5.4.2. THEOREM. *Let* $L \in \mathsf{NExt}(\mathsf{S4}_t)$*. Then* L *is an iterated splitting if and only if* $L \in \mathsf{NExt}(\mathsf{S5}_t) \cup \{\mathsf{S4}_t\}$*.*

Proof:
For the right-to-left direction, note first that $\mathsf{S4}_t$ and $\mathsf{S5}_t = \mathsf{NExt}(\mathsf{S4}_t)/\mathfrak{Ch}_2$ are iterated splittings. By Theorem 5.4.1, $\mathsf{Form}_t = \mathsf{S5}_t/\mathfrak{Cl}_1$ and $\mathsf{Log}(\mathfrak{Cl}_k) = \mathsf{S5}_t/\mathfrak{Cl}_{k+1}$ for all $k \in \mathbb{Z}^+$.

For the other direction, suppose $L = (\mathsf{S4}_t/L_1)/\cdots/L_n$ is an iterated splitting. If $n = 0$, then $L = \mathsf{S4}_t$. If $n = 1$, by Theorem 5.2.11(3), $L \in \{\mathsf{S5}_t, \mathsf{Form}_t\} \subseteq \mathsf{NExt}(\mathsf{S5}_t)$. Otherwise, $n \geq 2$. Then $L \supseteq \mathsf{S4}_t/L_1 \supseteq \mathsf{S5}_t$, which entails that $L \in \mathsf{NExt}(\mathsf{S5}_t) \cup \{\mathsf{S4}_t\}$. □

Now we can show that every iterated splitting L is of the degree of FMP 1, which entails that the degree of Kripke incompleteness of L is also 1.

5.4.3. THEOREM. *Let* L *be an iterated splitting in* $\mathsf{NExt}(\mathsf{S4}_t)$*. Then* $\mathsf{df}(L) = 1$*.*

Proof:
By Theorem 5.4.2, $L \in \mathsf{NExt}(\mathsf{S5}_t) \cup \{\mathsf{S4}_t\}$. Since $\mathsf{S4}_t$ has the FMP, $\mathsf{df}(\mathsf{S4}_t) = 1$. Let $L \in \mathsf{NExt}(\mathsf{S5}_t)$. Take any $L' \in \mathsf{NExt}(\mathsf{S4}_t)$ such that $\mathsf{Fin}(L') = \mathsf{Fin}(L)$. Then $\mathfrak{Ch}_2 \not\models L'$. By Theorem 5.2.11, $\langle \mathsf{Log}(\mathfrak{Ch}_2), \mathsf{S5}_t \rangle$ is a splitting pair in $\mathsf{NExt}(\mathsf{S4}_t)$, which entails $L' \supseteq \mathsf{S5}_t$, i.e., $L' \in \mathsf{NExt}(\mathsf{S5}_t)$. Note that every extension of $\mathsf{S5}_t$ has the FMP. Thus, $L = \mathsf{Log}(\mathsf{Fin}(L)) = \mathsf{Log}(\mathsf{Fin}(L')) = L'$. Since L is arbitrarily chosen in $\mathsf{NExt}(\mathsf{S5}_t)$, we obtain that $\mathsf{df}(L) = 1$ for all $L \in \mathsf{NExt}(\mathsf{S5}_t)$. □

5.4.4. REMARK. By Theorem 5.2.11, $\langle \mathsf{Log}(\mathfrak{Ch}_2), \mathsf{S5}_t \rangle$ and $\langle \mathsf{Log}(\mathfrak{Ch}_1), \mathsf{Form}_t \rangle$ are the only two splitting pairs in $\mathsf{NExt}(\mathsf{S4}_t)$. Then $\{\mathsf{Form}_t, \mathsf{S5}_t, \mathsf{S4}_t\}$ is exactly the set of union-splittings in $\mathsf{NExt}(\mathsf{S4}_t)$. By Theorem 5.4.3, every union-splitting in $\mathsf{NExt}(\mathsf{S4}_t)$ has the degree of Kripke incompleteness 1. However, by Theorem 5.4.3, we see that $\mathsf{Log}(\mathfrak{Cl}_2)$ is of the degree of Kripke incompleteness 1, while it is not a union-splitting. Thus, not every strictly Kripke complete logic is a union-splitting. In this sense, Blok's characterization of the degree of Kripke incompleteness for $\mathsf{NExt}(\mathsf{K})$ can not be generalized to $\mathsf{NExt}(\mathsf{S4}_t)$.

In what follows, we show that a tense logic in $\mathsf{NExt}(\mathsf{S4}_t)$ has the degree $2^{\aleph_0}$ if it is not an iterated splitting. Let $L \in \mathsf{NExt}(\mathsf{S4}_t)$ be an arbitrarily chosen logic which is not an iterated splitting in $\mathsf{NExt}(\mathsf{S4}_t)$. Then it suffices to show that $\mathsf{deg}(L) = 2^{\aleph_0}$. The first key observation is the following lemma, which is an analogue of Lemma 5.3.23.

5.4.5. LEMMA. *Let* $\varphi \notin \mathsf{S4}_t$*. Then*

(1) φ *is refuted by some non-symmetric* $\mathfrak{G} \in \mathsf{Fin}_r(\mathsf{S4}_t)$*;*

(2) for all $n \in \omega$*, there exists* $\mathfrak{F} \in \mathsf{Fin}_r(\mathsf{S4}_t)$ *such that* $\mathfrak{F} \not\models \varphi$ *and* $\mathsf{rdg}(\mathfrak{F}) \geq n$*.*

Proof:
For (1), take any $\varphi \notin \mathsf{S4}_t$. Since $\mathsf{S4}_t$ has the FMP, there exists $\mathfrak{F}' = (X', R') \in \mathsf{Fin}_r(\mathsf{S4}_t)$ such that $\mathfrak{F}' \not\models \varphi$. If $\mathfrak{F}'$ is non-symmetric, then take $\mathfrak{G} = \mathfrak{F}'$ and we are done. Suppose $\mathfrak{F}'$ is symmetric. Then $R' = X' \times X'$. Let $\mathfrak{G} = (X, R)$ where $X = X' \times \{0, 1\}$ and $R = \{\langle\langle x, a\rangle, \langle y, b\rangle\rangle \in X \times X : a \leq b\}$. Then the map $f : X \to X'$ defined by $f(\langle x, a\rangle) = x$ for all $\langle x, a\rangle \in X$ is a t-morphism from $\mathfrak{G}$ to $\mathfrak{F}'$. Thus, $\mathfrak{G} \not\models \varphi$ and (1) holds. By Lemma 5.1.11, (2) follows from (1). □

Since $L \supsetneq \mathsf{S4}_t$, there exists $\varphi_L \in L \setminus \mathsf{S4}_t$. By Lemma 5.4.5, there exists $\mathfrak{F}_L \in \mathsf{Fin}_r(\mathsf{S4}_t)$ and $w_L, u_L \in X_L$ with $\mathfrak{F}_L, w_L \not\models \varphi_L$ and $u_L \notin R_\sharp^{\mathsf{md}(\varphi)}[w_L]$. Next, we construct a continuum-sized family of general frames $\langle \mathbb{F}_I : I \subseteq \mathbb{Z}^+\rangle$ such that $\mathsf{Fr}(L) = \mathsf{Fr}(L \cap \mathsf{Log}(\mathbb{F}_I))$ and $L \cap \mathsf{Log}(\mathbb{F}_I) \neq L \cap \mathsf{Log}(\mathbb{F}_J)$ for any $I \neq J \subseteq \mathbb{Z}^+$.

5.4.6. Definition. Let $I \in \mathcal{P}(\mathbb{Z}^+)$. The frame $\mathfrak{F}'_I = (Y_I, S_I)$ is defined by:

- $Y_I = A \cup B \cup C_I \cup \{x_0, x_1, x_2, y_0, y_1, r_0, r_1, r_2, r'\}$, where $A = \{a_i : i \in \omega\}$, $B = \{b_i : i \in \omega\}$ and $C_I = \{c_i : i \in I \cup \{0\}\}$;
- S_I is the reflexive-transitive closure of the union of the following binary relations:
  - $\{\langle x_0, x_1\rangle, \langle x_2, x_1\rangle, \langle x_2, a_0\rangle, \langle y_1, y_0\rangle, \langle y_1, b_0\rangle, \langle r_0, r'\rangle, \langle r_0, r_1\rangle, \langle r_2, r_1\rangle\}$;
  - $\{\langle c_i, c_j\rangle : i > j \text{ and } i, j \in I \cup \{0\}\}$;
  - $\{\langle a_i, a_j\rangle : i > j \in \omega\} \cup \{\langle a_i, b_j\rangle : i > j \in \omega\}$;
  - $\{\langle b_i, b_j\rangle : i > j \in \omega\} \cup \{\langle b_i, a_j\rangle : i > j + 1 \in \omega\}$.

Then the general frame $\mathbb{F}_I$ is defined to be $(\mathfrak{F}_I, A_I)$, where $\mathfrak{F}_I = (X_I, R_I) = \langle \mathfrak{F}_L u_L +^t r_3 \mathfrak{F}'_I\rangle$ and A_I is generated by $\mathcal{P}(X_L)$.

An example of the underlying frame $\mathfrak{F}_I$ of $\mathbb{F}_I$ is as depicted in Figure 5.9. Clearly, $\mathbb{F}_I \in \mathsf{GFr}(\mathsf{S4}_t)$. The reader might notice that $\mathfrak{F}_I{\restriction}(\{a_i, b_i : i \in \omega\})$ is exactly the Rieger-Nishimura ladder.

Let us prove some basic properties of the general frames $\mathbb{F}_I$.

5.4.7. Lemma. *Let* $k \in \omega$ *be such that* $|\mathfrak{F}_L| + 2 < k$ *and* $\mathsf{rdg}(\mathbb{F}_I) < k$*. Then for all* $x \in X_I$*,*

(1) $\mathbb{F}_I, x \models \mathsf{bw}_k^+ \wedge \mathsf{bw}_k^- \wedge \mathsf{br}_k$*;*

(2) either $\mathbb{F}_I, x \models \mathsf{grz}^+ \wedge \mathsf{grz}^-$ *or* $\mathbb{F}_I, x \models \mathsf{alt}_k^+ \wedge \mathsf{alt}_k^-$*.*

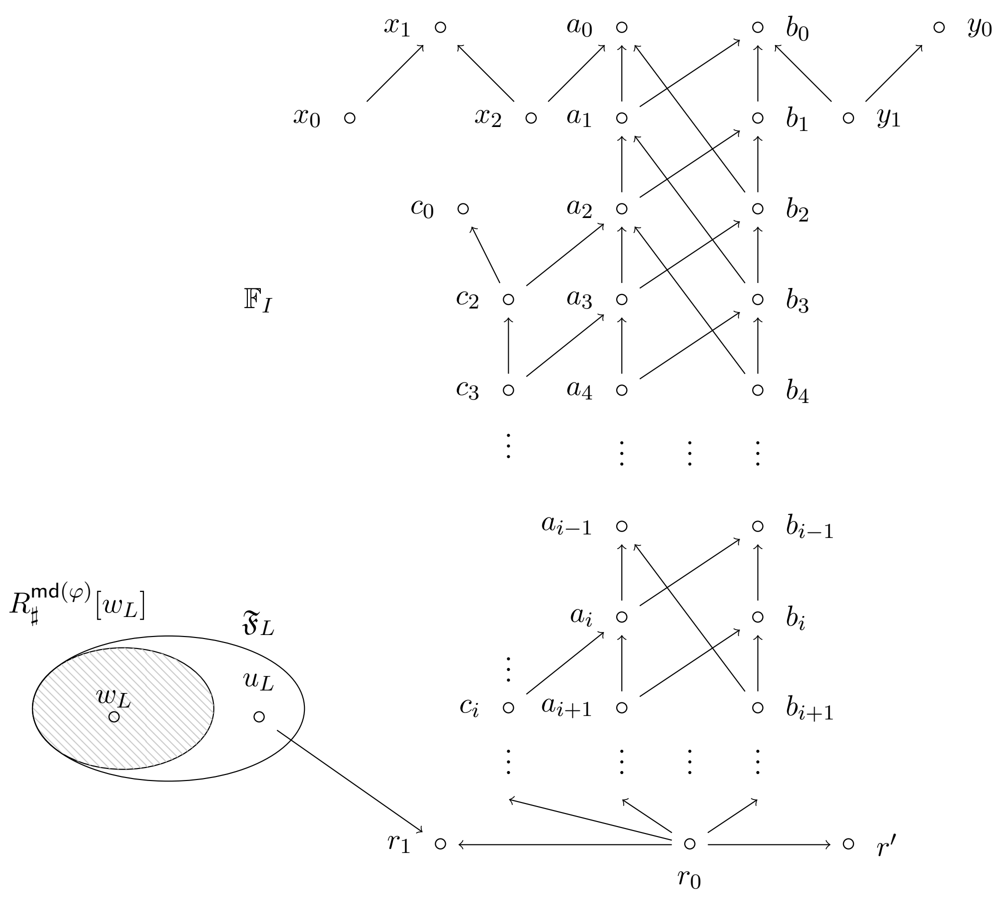

Figure 5.9: The frame $\mathfrak{F}_I$ where $1 \notin I$ and $2, 3, i \in I$

Proof:
For (1), since $X_I = (R_I)^k_\sharp[x]$, we have $\mathbb{F}_I, x \vDash \mathsf{br}_k$. Note that there exists no anti-chain or zigzag of size greater than k. By Fact 4.2.5, $\mathbb{F}_I, x \vDash \mathsf{bw}_k^+ \wedge \mathsf{bw}_k^-$. For (2), by Fact 4.2.5, if $x \in X_L \cup \{r_1\}$, then $\mathbb{F}_I, x \vDash \mathsf{alt}_k^+ \wedge \mathsf{alt}_k^-$ and we are done. Suppose $x \notin X_L \cup \{r_1\}$. Then by Proposition 2.2.46, $\mathbb{F}_I, x \vDash \mathsf{grz}^+$. To show that $\mathbb{F}_I, x \vDash \mathsf{grz}^-$, we first prove the following:

(†) for all $U \in A_I$ and co-chain $D = \langle d_i : i \in \omega \rangle$, either $U \cap D$ or $D \setminus U$ is finite.

We prove (†) by induction on the construction of U. Since D is an infinite co-chain, the reader can readily check that every element of D is in the set $A \cup B \cup C_I$ and so $U \cap D = \varnothing$ for all $U \in \mathcal{P}(X_L)$. The Boolean cases are straightforward. Let $U = (R_I)^{-1}[V]$. If $D \cap U = \varnothing$ then we are done. Suppose $D \cap U \neq \varnothing$. Then $d_i \in U$ for some $i \in \omega$. Note that D is closed under $(R_I)^{-1}$, this yields that $d_j \in (R_I)^{-1}[V]$ for all $j \geq i$. Then $D \setminus U \subseteq \{d_j \in D : j < i\}$ is finite. Let $U = R_I[V]$. If $D \setminus U = \varnothing$, then we are done. Suppose $D \setminus U \neq \varnothing$. Then there exists $d_i \in D \setminus U$. Since U is closed under R_I, we obtain that $d_j \notin U$ for any $j \geq i$. Thus, $D \setminus U \subseteq \{d_j : j < i\}$ is finite, and so (†) holds.

Towards a contradiction, suppose $\mathbb{F}_I, x \nvDash \mathsf{grz}^-$. Then $\mathbb{F}_I, V, x \nvDash \mathsf{grz}^-$ for some valuation V in $\mathbb{F}_I$. Let $d_0 = x$. Then $d_0 \notin V(p)$ and there exists $d_1 \in (R_I)^{-1}[d_0]$ such that $d_1 \in V(\neg p \wedge \blacklozenge p)$. Thus, there exists $d_2 \in (R_I)^{-1}[d_1]$ such that $d_2 \notin V(p)$. Note that there exists no proper cluster in $\mathbb{F}_I$. Thus, $d_2 \notin R_I[d_0]$. By repeating the argument above, for every $l \in \mathbb{Z}^+$, if d_{2l} is defined, then there exist points d_{2l+1} and d_{2l+2} such that (i) $d_{2l+2} R_I d_{2l+1} R_I d_{2l}$; (ii) $d_{2l+2} \notin R_I[d_{2l}]$; and $d_{2l+2} \notin V(p)$. Thus, there exists a co-chain $D = \{d_i : i \in \omega\} \subseteq R^{-1}[x]$ such that $D \cap V(\neg p) = \{d_{2l} : l \in \omega\}$. Thus, $|D \cap V(p)| = |D \setminus V(p)| = \aleph_0$, which contradicts (†). Hence, $\mathbb{F}_I, x \vDash \mathsf{grz}^-$ and so $\mathbb{F}_I, x \vDash \mathsf{grz}^+ \wedge \mathsf{grz}^-$ for any $x \notin X_L \cup \{r_1\}$. □

Next, we show that the tense logic $\mathsf{Log}(\mathbb{F}_I)$ has no infinite rooted Kripke frame. For this, we use a celebrated theorem by Ramsey from set theory (for more details, see, e.g., [64, Theorem 9.1]). For each set X and $n \in \mathbb{Z}^+$, we write $[X]^n$ for the set of all n-element subsets of X, that is,

$$[X]^n = \{Y \subseteq X : |Y| = n\}.$$

Ramsey's theorem can then be stated as follows:

5.4.8. Theorem. *Let $n, k \in \mathbb{Z}^+$ and $f : [\omega]^k \to \{1, \cdots, n\}$. Then there exists an infinite subset $X \subseteq \omega$ such that f is constant on $[X]^k$, i.e., $|f[[X]^k]| = 1$.*

Now we are ready to show the following lemma:

5.4.9. Lemma. $\mathsf{Fin}_r(\mathsf{Log}(\mathbb{F}_I)) = \mathsf{Fr}_r(\mathsf{Log}(\mathbb{F}_I))$.

Proof:
Take any $\mathfrak{G} = (Y, S) \in \mathsf{Fr}_r(\mathsf{Log}(\mathbb{F}_I))$. Towards a contradiction, suppose $\mathfrak{G}$ is infinite. By Lemma 5.4.7(1), $\mathfrak{G} \models \mathsf{br}_k$. Thus, we see that $\mathfrak{G}$ is not image-finite and so there exists $y \in Y$ such that $|S[y]| \geq \aleph_0$ or $|S^{-1}[y]| \geq \aleph_0$. Assume $|S[y]| \geq \aleph_0$. By Lemma 5.4.7(1), $\mathfrak{G}, y \models \mathsf{bw}_k^+$ and so there exists no infinite anti-chain in $S[y]$. Since $S[y]$ is infinite, by Lemmas 2.2.38 and 5.4.7(2), it follows that $\mathfrak{G}, y \models \mathsf{grz}^+ \wedge \mathsf{grz}^-$. By Fact 2.2.46, there is no proper cluster in $S[y]$. By Lemma 5.4.7(2) again, $\mathfrak{G}, z \models \mathsf{grz}^+ \wedge \mathsf{grz}^-$ for all $z \in S[y]$ and so there exists no infinite ascending chain or descending chain in $S[y]$. Now, take a countable set $Z = \{z_i : i \in \omega\} \subseteq S[y]$ and consider the function $f : [\omega]^2 \to \{1, 2, 3\}$ such that for all $i < j \in \omega$:

$$f(\{i, j\}) = \begin{cases} 1, & \text{if } z_i S z_j; \\ 2, & \text{if } z_j S z_i; \\ 3, & \text{otherwise.} \end{cases}$$

By Theorem 5.4.8, there exists an infinite subset $I \subseteq \omega$ such that f is constant on $[I]^2$. Then we have three cases:

(1) $f[[I]^2] = \{1\}$. Then $Z' = \{z_i : i \in I\}$ forms an infinite ascending chain in $S[y]$, which is impossible.

(2) $f[[I]^2] = \{2\}$. Then $Z' = \{z_i : i \in I\}$ forms an infinite descending chain in $S[y]$, which is also impossible.

(3) $f[[I]^2] = \{3\}$. Then $Z' = \{z_i : i \in I\}$ forms an infinite anti-chain in $S[y]$, which is again impossible.

Thus, $S[y]$ is finite. By a similar argument, we see that $S^{-1}[y]$ is also finite. Hence, $\mathfrak{G}$ is finite. As $\mathfrak{G}$ was arbitrary, we obtain that $\mathsf{Fin}_r(\mathsf{Log}(\mathbb{F}_I)) = \mathsf{Fr}_r(\mathsf{Log}(\mathbb{F}_I))$. □

Let $L_I = L \cap \mathsf{Log}(\mathbb{F}_I)$. Now, our task is to show $\mathsf{Fr}(L) = \mathsf{Fr}(L_I)$. Note that $\mathfrak{Ch}_1$ and $\mathfrak{Ch}_2$ validate $\mathsf{Log}(\mathbb{F}_I)$ for all $I \subseteq \mathbb{Z}^+$. Unlike the situation in Section 5.3, we can no longer show that $\mathsf{Log}(\mathbb{F}_I)$ has no Kripke frames. However, we can still show that adding the general frame $\mathbb{F}_I$ does not have any Kripke frame that is not isomorphic to either $\mathfrak{Ch}_1$ or $\mathfrak{Ch}_2$. Thus, $\mathbb{F}_I$ does not introduce any new Kripke frames into $\mathsf{Fr}(L)$.

In the proof of the following lemma, we use local t-morphisms and generalized Jankov formulas introduced in Section 4.1.

5.4.10. LEMMA. $\mathsf{Fr}_r(\mathsf{Log}(\mathbb{F}_I)) = \mathsf{M}_t(\mathfrak{Ch}_2)$.

Proof:
To simplify the notation, we drop the subscripts I if there is no danger of confusion. Note that since $\mathsf{Log}(\mathbb{F}_I) \notin \mathsf{NExt}(\mathsf{S5}_t)$ and $\langle \mathsf{Log}(\mathfrak{Ch}_2), \mathsf{S5}_t \rangle$ is a splitting pair in $\mathsf{NExt}(\mathsf{S4}_t)$, we have $\mathfrak{Ch}_2 \models \mathsf{Log}(\mathbb{F}_I)$ and so $\mathsf{Fr}_r(\mathsf{Log}(\mathbb{F}_I)) \supseteq \mathsf{M}_t(\mathfrak{Ch}_2)$. Take any $\mathfrak{G} \in \mathsf{Fr}_r(\mathsf{Log}(\mathbb{F}_I))$. By Lemma 5.4.9, $\mathfrak{G}$ is finite. Then there exists $k \in \omega$ such that $\mathsf{rdg}(\mathfrak{G}) < k$, $|\mathfrak{F}_L| + 2 < k$ and $\mathsf{rdg}(\mathbb{F}_I) < k$. Let $\mathcal{J}^k(\mathfrak{G})$ be the Jankov-formula of $\mathfrak{G}$ of degree k. Since $\mathfrak{G} \not\models \neg\mathcal{J}^k(\mathfrak{G})$, we see that $\mathbb{F}_I \not\models \neg\mathcal{J}^k(\mathfrak{G})$. Since $\mathbb{F}_I$ is rooted and $\mathsf{rdg}(\mathbb{F}_I) < k$, by Theorem 4.1.9, $\mathfrak{G}$ is a t-morphic image of $\mathbb{F}_I$. Let $f : \mathbb{F}_I \twoheadrightarrow \mathfrak{G}$ be a surjective t-morphism. Now it suffices to show that $\mathfrak{G} \in \mathsf{M}_t(\mathfrak{Ch}_2)$. Suppose $\mathfrak{G} \notin \mathsf{M}_t(\mathfrak{Ch}_2)$. Then we have

5.4.11. Claim. *For all $u \in X \setminus X_L$, $f(u) = f(x_0)$ implies $u = x_0$.*

Proof:
Take any $u \in X_I \setminus X_L$ such that $f(u) = f(x_0)$. Towards a contradiction, suppose also $u \neq x_0$. Then one of the following cases holds:

(1) $u \in \{x_1, y_0, r'\}$. By Lemma 2.2.25, $S[f(x_0)] = S[f(u)] = f[R[u]] = \{f(u)\}$ and $S^{-1}[f(x_0)] = f[R^{-1}[x_0]] = \{f(x_0)\}$. Thus, $\mathfrak{G} \cong \mathfrak{Ch}_1$, which contradicts the assumption.

(2) $u = x_2$. Then clearly $\{x_0, x_1, x_2\}$ is sufficient, which entails $Y = \{f(x_0), f(x_1)\}$. Note that $f(x_0) \notin S[f(x_1)]$. Thus, $\mathfrak{G} \cong \mathfrak{Ch}_2$, which contradicts the assumption.

(3) $u = y_1$. Similar to (2), we see $\{x_0, x_1, y_0, y_1\}$ is sufficient and so $\mathfrak{G} \cong \mathfrak{Ch}_2$, which contradicts the assumption.

(4) None of (1)-(3) holds. Then $u \in R[r_0] \setminus \{r'\}$. By Lemma 2.2.25, $f(r_0) \in f[R^{-1}[u]] = f[R^{-1}[x_0]] = \{f(x_0)\}$. Again, by Lemma 2.2.25, we see that $f(r') \in \{f(x_0), f(x_1)\}$. By (1), $f(r') = f(x_1)$, which entails that $\{x_0, x_1, r_0, r'\}$ is sufficient. Thus, $\mathfrak{G} \cong \mathfrak{Ch}_2$, which again contradicts the assumption.

Hence, we conclude that $f(u) = f(x_0)$ implies $u = x_0$ for all $u \in X \setminus X_L$. □

5.4.12. Claim. *$f(a_0) \neq f(b_0) \neq f(b_1) \neq f(a_0)$.*

Proof:
Suppose $f(a_0) = f(b_0)$. By Lemma 2.2.25, $f(x_0) \in f[R^{-1}[R[R^{-1}[b_0]]]]$. By Claim 5.4.11, the only possible case is that $f(r_0) = f(x_2)$, $f(r_1) = f(x_1)$ and $f(x_0) = f(u)$ for some $u \in X_L \cap R^{-1}[r_1]$. Then we see that $f(r') \in f[R[x_2]] = \{f(a_0), f(x_1), f(x_2)\}$. It is not hard to see that $f(r') = f(x_2)$ implies $\mathfrak{G} \cong \mathfrak{Ch}_1$, and $f(r') = f(x_1)$ contradicts Claim 5.4.11. Thus, $f(r') = f(a_0) = f(b_0)$,

which entails $f(y_1) \in f[R^{-1}[r']] = \{f(a_0), f(x_2)\}$. Since $\mathfrak{G} \notin \mathsf{M}_t(\mathfrak{Ch}_2)$, we have $f(y_1) = f(x_2)$ and $f(y_0) = f(x_1)$, which again contradicts Claim 5.4.11. Thus, $f(a_0) \neq f(b_0)$. Note that since $f(a_0) = f(b_1)$ implies $f(a_0) = f(b_0)$, we obtain that $f(a_0) \neq f(b_1)$.

Suppose $f(b_0) = f(b_1)$. Since $f(y_1) \neq f(b_1)$, $f(y_1) = f(v)$ for some $v \in R^{-1}[b_1] \setminus \{b_1\}$. Note that $v \in R^{-1}[a_0]$ and $f(a_0) \neq f(b_0)$, it is not hard to show that $f(y_0) = f(a_0)$, which entails that $f(x_0) \in f[R^3_\sharp[y_0]]$ and contradicts Claim 5.4.11. □

5.4.13. Claim. *For all* $n \in \omega$, $|f[\{a_i : i \leq n\} \cup \{b_i : i \leq n+1\}]| = 2n + 3$.

Proof:
For each $n \in \omega$, we write Z_n for the set $f[\{a_i : i \leq n\} \cup \{b_i : i \leq n+1\}]$. The proof proceeds by induction on $n \in \omega$. The case $n = 0$ follows from Claim 5.4.12 immediately. Let $n > 0$. By induction hypothesis, it suffices to show that $f(a_n), f(b_{n+1}) \notin f[Z_{n-1}]$ and $f(a_n) \neq f(b_{n+1})$. Since $\{f(a_{n-1}), f(b_{n-1})\} \subseteq f[R[a_n]] \cap f[R[b_{n+1}]]$ and $\{f(a_{n-1}), f(b_{n-1})\} \nsubseteq f[R[v]]$ for any $v \in Z_{n-1}$, we have $f(a_n), f(b_{n+1}) \notin f[Z_{n-1}]$. Note that $f(b_n) \in f[R[b_{n+1}]] \setminus f[R[a_n]]$, we obtain that $f(a_n) \neq f(b_{n+1})$. □

By Claim 5.4.13, $f[A \cup B]$ is infinite and so $\mathfrak{G}$ is infinite, which contradicts Lemma 5.4.9. Thus, $\mathfrak{G} \in \mathsf{M}_t(\mathfrak{Ch}_2)$ and hence $\mathsf{Fr}_r(\mathsf{Log}(\mathbb{F}_I)) = \mathsf{M}_t(\mathfrak{Ch}_2)$. □

5.4.14. Lemma. *For all* $I \subseteq \mathbb{Z}^+$, $\mathsf{Fr}(L) = \mathsf{Fr}(L_I)$.

Proof:
Since $L \notin \mathsf{NExt}(\mathsf{S5}_t)$ and $\langle \mathsf{Log}(\mathfrak{Ch}_2), \mathsf{S5}_t \rangle$ is a splitting pair in $\mathsf{NExt}(\mathsf{S4}_t)$, we have $\mathfrak{Ch}_2 \models L$ and so $\mathsf{M}_t(\mathfrak{Ch}_2) \subseteq \mathsf{Fr}_r(L)$. By Lemmas 2.2.19 and 5.4.10, $\mathsf{Fr}_r(L_I) = \mathsf{Fr}_r(L) \cup \mathsf{Fr}_r(\mathsf{Log}(\mathbb{F}_I)) = \mathsf{Fr}_r(L)$. Hence, $\mathsf{Fr}(L) = \mathsf{Fr}(L_I)$ for all $I \in \mathbb{Z}^+$. □

So far, we have obtained a family of logics which share the same class of frames as L. It remains to show that the set $\{L_I : I \subseteq \mathbb{Z}^+\}$ of tense logics is of the cardinality $2^{\aleph_0}$. To do this, we generalize the modalities $\Delta^{\leq n}$ and $\nabla^{\leq n}$ as follows:

5.4.15. Definition. For all $n \in \omega$ and $\varphi, \psi \in \mathsf{Form}_t$, we define $\Delta^{\leq n}_\psi \varphi$ by:

$$\Delta^{\leq 0}_\psi \varphi = \psi \wedge \varphi \text{ and } \Delta^{\leq k+1}_\psi \varphi = \Delta^{\leq k}_\psi \varphi \vee \Diamond(\psi \wedge \Delta^{\leq k}_\psi \varphi) \vee \blacklozenge(\psi \wedge \Delta^{\leq k}_\psi \varphi)$$

As usual, we define the dual operator $\nabla^{\leq n}_\psi$ of $\Delta^{\leq n}_\psi$ by $\nabla^{\leq n}_\psi \varphi := \neg \Delta^{\leq n}_\psi \neg \varphi$.

5.4.16. PROPOSITION. *Let $\mathfrak{M} = (X, R, V)$ be a model, $x \in X$ and $\varphi, \psi \in \mathsf{Form}_t$. Then for all $k \in \omega$, the following holds:*

(1) $\mathfrak{M}, x \models \Delta_{\psi}^{\leq k}\varphi$ *if and only if there exists an $R_\sharp$-path $\langle x_i : i < k' \rangle$ such that*

- $k' \leq k$;
- $x = x_0$;
- $\mathfrak{M}, x_{k'-1} \models \varphi$ *and* $\mathfrak{M}, x_i \models \psi$ *for all* $i < k'$.

(2) $\mathfrak{M}, x \models \Delta_{\top}^{\leq k}\varphi$ *if and only if* $\mathfrak{M}, x \models \Delta^{\leq k}\varphi$.

Proof:
By induction on k. □

Recall that for all formula $\varphi(\overline{p})$, we write $\varphi[\overline{q}/\overline{p}]$ for the formula obtained from φ by substituting $\overline{p}$ with $\overline{q}$.

5.4.17. DEFINITION. Let $\varphi_0 := \neg\mathsf{bd}_k[q_0, \cdots, q_k/p_0, \cdots, p_k] \wedge \blacksquare\neg p$, where $p, q_0, \cdots, q_k \in \mathsf{Prop}$ are propositional variables which do not occur in φ_L. Then we define

- $\varphi_{x_0} := \Delta^{\leq k}\neg\varphi_L \wedge \Delta_p^{\leq 4}\varphi_0 \wedge \nabla^{\leq 3}\neg\varphi_0$,
- $\varphi_{x_1} := \blacklozenge\varphi_{x_0} \wedge \neg\varphi_{x_0}$ and $\varphi_{x_2} := \Diamond\varphi_{x_1} \wedge \neg\varphi_{x_1}$;
- $\varphi_{y_0} := \Delta_p^{\leq 7}\varphi_{x_0} \wedge \Delta^{\leq 6}\neg\varphi_{x_0}$ and $\varphi_{y_1} := \Diamond\varphi_{y_0} \wedge \neg\varphi_{y_0}$;
- $\varphi_{a_0} := \blacklozenge\varphi_{x_2} \wedge \neg\varphi_{x_2}$, $\varphi_{b_0} := \blacklozenge\varphi_{y_1} \wedge \neg\varphi_{y_1}$ and $\varphi_{b_1} := \Diamond\varphi_{b_0} \wedge \Box\neg\varphi_{a_0} \wedge \neg\varphi_{b_0}$;
- $\varphi_{AB} = \Box(\varphi_{b_0} \vee \varphi_{b_1} \vee \Diamond\blacklozenge\Diamond\blacklozenge\varphi_{x_0})$.

Moreover, for all $l \in \mathbb{Z}^+$, we define

- $\varphi_{a_l} := \varphi_{AB} \wedge \Diamond\varphi_{a_{l-1}} \wedge \Diamond\varphi_{b_{l-1}} \wedge \Box\neg\varphi_{b_l}$;
- $\varphi_{b_{l+1}} := \varphi_{AB} \wedge \Diamond\varphi_{a_{l-1}} \wedge \Diamond\varphi_{b_l} \wedge \Box\neg\varphi_{a_l}$;
- $\varphi_{c_l} := \neg\varphi_{AB} \wedge \Diamond\varphi_{a_l} \wedge \Box\neg\varphi_{a_{l+1}}$.

The formulas defined in Definition 5.4.17 characterize points in $\mathbb{F}_I$.

5.4.18. LEMMA. *Let $U = A \cup B \cup \{x_0, x_1, x_2, y_0, y_1\}$. For all $u \in U$ and $v \in X_I$,*

(1) $\mathbb{F}_I, u \not\models \varphi_u \to \nabla^{\leq k}\varphi_L$,

(2) for all valuation V in $\mathbb{F}_I$, $V(\varphi_{x_0}) \neq \varnothing$ implies $V(\varphi_{AB}) = A \cup B$ and $V(\varphi_u) = \{u\}$.

(3) $\mathbb{F}_I, v \not\models \neg\varphi_u$ *implies* $u = v$,

(4) $\mathbb{F}_I \models \neg\varphi_{c_j}$ *for any* $j \notin I$.

Proof:
For (1), recall first that $\mathfrak{F}_L, w_L \not\models \varphi_L$. Then there exists a valuation $V' : \mathsf{Prop} \to \mathcal{P}(X_L)$ such that $\mathfrak{F}_L, V', w_L \models \neg\varphi_L$. Since $\mathbb{F}_I$ is differentiated and X_L is finite, there exists a valuation V in $\mathbb{F}_I$ such that the following conditions hold: (i) $V{\upharpoonright}X_L = V'$, (ii) $V(p) = R[b_k] \cup \{x_0, x_1, x_2, y_0, y_1\}$ and (iii) $V(q_i) = R[b_i]$ for all $i \leq k$. Let $\mathfrak{M} = (\mathbb{F}_I, V)$. Then $\mathfrak{M}, b_{k+1} \models \varphi_0$ and $\mathfrak{M}, w_L \models \neg\varphi_L$. Since $\mathsf{rdg}(\mathbb{F}_I) < k$, $\mathfrak{M}, x_0 \models \nabla^{\leq 3}\mathsf{bd}_k$ and there exists a p-path from x_0 to b_{k+1} of length 4, we see that $\mathfrak{M}, x_0 \models \varphi_{x_0}$. Note that $\mathfrak{M}, r_0 \models \neg p$ and $V(\varphi_0) \subseteq Y_I$. Then $\mathfrak{M}, u \models \Delta^{\leq 3}\varphi_0 \vee \neg\nabla_p^{\leq 4}\varphi_0$ for each point $u \in X_I \setminus \{x_0\}$. Thus, x_0 is the unique point satisfying φ_{x_0}. By the construction of the formulas φ_u, the reader can now easily check that $V(\varphi_{AB}) = A \cup B$ and $\mathfrak{M}, u \models \varphi_u$ for all $u \in U$. Since $w_L \in R_\sharp^k[u]$, we have $\mathfrak{M}, u \models \varphi_u \wedge \Delta^{\leq k}\neg\varphi_L$.

For (2), let V be a valuation in $\mathbb{F}_I$ and $\mathfrak{M} = (\mathbb{F}_I, V)$. Suppose $\mathfrak{M}, v_0 \models \varphi_{x_0}$ for some $v_0 \in X_I$. Then $\mathfrak{M}, v_0 \models \Delta_p^{\leq 4}\varphi_0$ and $\mathfrak{M}, v_1 \models \varphi_0$ for some $v_1 \in X_I$. By Proposition 4.2.5, $R[v_1]$ contains a chain of length greater than k. Thus, $v_1 \in R[r_0] \setminus \{a_0, b_0, b_1, c_0\}$ and so $\mathfrak{M}, r_0 \models \neg p$. Then $\mathfrak{M}, w \models \Delta^{\leq 3}\varphi_0 \vee \neg\nabla_p^{\leq 4}\varphi_0$ for each point $w \in X_I \setminus \{x_0\}$, which entails that $V(\varphi_{x_0}) = \{x_0\}$. Then clearly, $V(\varphi_{AB}) = A \cup B$ and $V(\varphi_w) = \{w\}$ for all $w \in U$.

For (3), take any $u \in U$ and $v \in X_I$. Suppose $\mathbb{F}_I, v \not\models \neg\varphi_u$. Then there exists a valuation V in $\mathbb{F}_I$ such that $\mathbb{F}_I, V, v \models \varphi_u$. Let $\mathfrak{M} = (\mathbb{F}_I, V)$. By the construction of φ_u, we always have $\models \varphi_u \to \Delta^{\leq m}\varphi_{x_0}$ for some $m \in \omega$. Thus, $V(\varphi_{x_0}) \neq \varnothing$. By (2), $V(\varphi_w) = \{w\}$ for all $w \in U$, which entails $u = v$.

For (4), take any $j \notin I$. Suppose $\mathbb{F}_I \not\models \neg\varphi_{c_j}$. Then there exists $v \in X_I$ and a valuation V in $\mathbb{F}_I$ such that $\mathbb{F}_I, V, v \models \varphi_{c_j}$. Thus, $V(\varphi_{x_0}) \neq \varnothing$. By (2),$V(\varphi_{AB}) = A \cup B$ and $V(\varphi_w) = \{w\}$ for $w \in \{a_j, a_{j+1}\}$. Since $j \notin I$, we have $R^{-1}[a_j] \setminus (A \cup B \cup R^{-1}[a_{j+1}]) = \varnothing$ and so $\mathfrak{M} \models \neg\varphi_{c_j}$, which is a contradiction. □

Now we are ready to prove the following lemma:

5.4.19. Lemma. *For all* $I, J \in \mathcal{P}(\mathbb{Z}^+)$, $I \neq J$ *implies* $L_I \neq L_J$.

Proof:
Take any distinct $I, J \in \mathcal{P}(\mathbb{Z}^+)$. Let $i \in I \setminus J$. It suffices to show that

$$\varphi_{c_i} \to \nabla^{\leq k}\varphi_L \in L_J \setminus L_I.$$

By Lemma 5.4.18(1), $\varphi_{c_i} \to \nabla^{\leq k}\varphi_L \notin \mathsf{Log}(\mathbb{F}_I) \supseteq L_I$. By Lemma 5.4.18(4), $\neg\varphi_{c_i} \in \mathsf{Log}(\mathbb{F}_J)$. Since $\varphi_L \in L$, we see $\nabla^{\leq k}\varphi_L \in L$. By substitutions, we may always assume that φ_L and φ_{c_i} contains no common variable, $\neg\varphi_{c_i} \vee \nabla^{\leq k}\varphi_L \in$

$L \cap \mathsf{Log}(\mathbb{F}_J)$. Hence, $\varphi_{c_i} \to \nabla^{\leq k}\varphi_L \in L_J$. □

Note that $L \in \mathsf{NExt}(\mathsf{S4}_t)$ is chosen to be an arbitrary logic which is not an iterated splitting. By Lemmas 5.4.14 and 5.4.19, the following theorem holds:

5.4.20. THEOREM. *Let $L \in \mathsf{NExt}(\mathsf{S4}_t)$. If L is not an iterated splitting, then $\mathsf{deg}(L) = 2^{\aleph_0}$.*

Finally, we obtain our main results of this section:

5.4.21. THEOREM. *Let $L \in \mathsf{NExt}(\mathsf{S4}_t)$. Then the following are equivalent:*

(1) L is an iterated splitting in $\mathsf{NExt}(\mathsf{S4}_t)$.

(2) $\mathsf{df}(L) = 1$.

(3) $\mathsf{deg}(L) = 1$.

(4) $\mathsf{df}(L) \neq 2^{\aleph_0}$.

(5) $\mathsf{deg}(L) \neq 2^{\aleph_0}$.

Proof:
By Theorem 5.4.3, (1) implies (2). By Proposition 5.3.2, (2) implies (3). Since $|\mathsf{NExt}(\mathsf{S4}_t)| \leq 2^{\aleph_0}$, (4) implies (5). It is obvious that (3) implies (5), and (2) implies (4). It remains to notice that (5) implies (1) follows from Theorem 5.4.20 and Theorem 5.4.2. □

As a corollary, we obtain the following dichotomy theorem:

5.4.22. THEOREM. *For all $L \in \mathsf{NExt}(\mathsf{S4}_t)$, $\mathsf{deg}(L) = \mathsf{df}(L) \in \{1, 2^{\aleph_0}\}$.*

5.4.23. REMARK. As we can see now, union-splittings in $\mathsf{NExt}(\mathsf{S4}_t)$ are still strictly Kripke complete, while there exist strictly Kripke complete logics in $\mathsf{NExt}(\mathsf{S4}_t)$ which are not union-splittings (see Remark 5.4.4). In fact, in the lattice $\mathsf{NExt}(\mathsf{S5}_t)$, it follows from Theorem 5.4.1 that every logic is strictly Kripke complete and every logic is an iterated splitting. Hence, so far, the notion of iterated splitting fits better with strictly Kripke completeness in the lattices of tense logics, in the sense that for $L \in \{\mathsf{K}_t, \mathsf{K4}_t, \mathsf{S4}_t, \mathsf{S5}_t\}$, the iterated splittings in $\mathsf{NExt}(L)$ are exactly the strictly Kripke complete logics.

5.5 Summary

In this chapter, we obtained a series of results on the degree of Kripke incompleteness in lattices of tense logics. We started by introducing the notion of reflective unfolding and reviewing splittings of lattices. We gave characterizations of the degree of Kripke incompleteness in $\mathsf{NExt}(\mathsf{K}_t)$, $\mathsf{NExt}(\mathsf{K4}_t)$ and $\mathsf{NExt}(\mathsf{S4}_t)$ and generalized Blok's dichotomy theorem from $\mathsf{NExt}(\mathsf{K})$ to these lattices of tense logics. By showing that the degree of Kripke incompleteness coincides with the degree of FMP in all the lattices mentioned above, we also obtain a dichotomy theorem for the degree of FMP in these lattices. These results show the similarities and differences of the degree of Kripke incompleteness and the degree of FMP between the lattices of modal and tense logics (cf. [9, 14]).

We claim that we could obtain more results on the degree of Kripke incompleteness in lattices of tense logics by the method given in this chapter. For example, consider the tense logic $\mathsf{K4D}_t^+ = \mathsf{K4}_t \oplus \Diamond\top$, which is the tense logic of serial frames. It is not hard to see that $\mathfrak{F}_L \models \Diamond\top$ implies $\mathbb{F}_I \models \Diamond\top$ for all $\mathbb{F}_I$ defined in Section 5.3.2. Note that serial frames are closed under reflective unfolding, we claim that $\mathsf{deg}_{\mathsf{K4D}_t^+}(L) = 2^{\aleph_0}$ for all proper extension L of $\mathsf{K4D}_t^+$. Similarly, since $\mathfrak{F}_L \models \mathsf{grz}^+$ implies $\mathbb{F}_I \models \mathsf{grz}^+$ for all $\mathbb{F}_I$ defined in Section 5.4, we claim that the dichotomy theorem holds for $\mathsf{Grz}^+ = \mathsf{K}_t \oplus \mathsf{grz}^+$. Given that the frames $\mathfrak{F}_I$ are frames for the bi-intuitionistic logic biInt and every finite bi-p-morphic image of $\mathfrak{F}_I$ is a bi-p-morphic image of $\mathfrak{Ch}_2$, we claim that the dichotomy theorem for the degree of FMP holds for $\mathsf{Ext}(\mathsf{biInt})$. This is left for future work.

Beyond identifying possible applications of the reflective unfolding method, there remain many worthwhile directions for future research. We outline a few such topics below:

By Blok's characterization theorem, the union-splittings in $\mathsf{NExt}(\mathsf{K})$ are exactly the strictly Kripke complete logics. We obtained in this work that Blok's characterization theorem can be generalized to the lattices $\mathsf{NExt}(\mathsf{K}_t)$ and $\mathsf{NExt}(\mathsf{K4}_t)$. However, as we mentioned in Remark 5.4.23, every union-splitting in $\mathsf{NExt}(\mathsf{S4}_t)$ is strictly Kripke complete while the inverse does not hold. Instead, iterated splittings fit perfectly with strictly Kripke complete logics. So it is natural to ask: what is the relation between union-splittings, iterated splittings and strictly Kripke complete logics in the lattices of tense logics? For example, is an iterated splitting always a union-splitting? Is it true that for all tense logic L_0 and L, L is a union-splitting in $\mathsf{NExt}(L_0)$ implies $\mathsf{deg}_{L_0}(L) = 1$? Is a strictly Kripke complete tense logic always an iterated splitting?

As the reader might already notice, our method relies heavily on the reflective unfolding of Kripke frames. In fact, the method used in this chapter applies to only those lattices $\mathsf{NExt}(L_0)$ where L_0 is preserved under reflective unfolding. For example, if L is of finite r-degree, then our method fails. An important future work is to study the degree of Kripke incompleteness in lattices of tense logics of finite r-degree, for example, $\mathsf{NExt}(\mathsf{S4.2}_t)$ and $\mathsf{NExt}(\mathsf{S4.3}_t)$. By [71, Proposition 23], there are infinite splittings in both of these lattices, which indicates that the case is quite different from the one for $\mathsf{NExt}(\mathsf{S4}_t)$. We believe these topics will require new methods and techniques.

As we have shown in Sections 5.3 and 5.4, the degree of FMP and the degree of Kripke incompleteness coincide in the lattices $\mathsf{NExt}(\mathsf{K}_t)$, $\mathsf{NExt}(\mathsf{K4}_t)$ and $\mathsf{NExt}(\mathsf{S4}_t)$. The dichotomy theorem for the degree of FMP holds for all these three lattices. Bezhanishvili et al. [9] showed the anti-dichotomy theorem for the degree of FMP for $\mathsf{NExt}(\mathsf{K4})$ and $\mathsf{NExt}(\mathsf{S4})$. Here is a natural follow-up question: Is there any tense logic L such that the anti-dichotomy theorem holds for $\mathsf{NExt}(L)$?

In the next chapter, which will be the final chapter on tense logics in the thesis, we study the decidability of logical properties in the lattices of tense logics.

Chapter 6

(Un)decidable Logical Properties of Tense Logic

In this chapter, which is based on [36] and [37], we study the decidability of logical properties of tense logics. A central part of the study of modal logic is to determine whether a logic has a certain logical property, such as Kripke completeness, the finite model property, tabularity, Post-completeness and decidability. As mentioned in Chapter 1, from a global viewpoint, this gives rise to algorithmic problems for classes of logics: in a class of logics, is it decidable whether a logic in this class has a certain logical property? Recall that a property P is *decidable* in a class $\mathcal{C}$ of logics if there exists an algorithm such that, for any finitely axiomatizable logic $L \in \mathcal{C}$, given by its finite axiomatization, the algorithm decides whether L has the property P.

The decidability of logical properties in lattices of modal logics has been studied. Thomason [122] showed the undecidability of Kripke completeness, and a series of works by Chagrov and his co-authors [25, 24, 28, 27, 26] introduced a general method to show the undecidability of various logical properties, including the finite model property, first-order definability, decidability, tabularity, and the coincidence with a fixed tabular logic. On the positive side, it was recently proved in [119] that the property of being a union-splitting is decidable in $\mathsf{NExt}(\mathsf{K})$, which entails that being strictly Kripke complete is also decidable. As we have mentioned in Chapter 4, the notion of pretabularity was introduced in order to study tabularity, and it was proved that tabularity is decidable in both $\mathsf{Ext}(\mathsf{IPC})$ and $\mathsf{NExt}(\mathsf{S4})$ (see [76, 89, 90, 44]). In fact, by Theorem 4.4.11, tabularity is also decidable in $\mathsf{NExt}(\mathsf{S4.3}_t)$.

The (un)decidability of logical properties in a lattice of logics can also be seen as a measure of the complexity of that lattice. Let us restrict to $\mathsf{NExt}(\mathsf{K4})$, which is a sublattice of $\mathsf{NExt}(\mathsf{K})$. Then Kripke completeness, the finite model property, and the interpolation property are still undecidable [25, 24]. However, since every tabular logic has only finitely many immediate predecessors and all of those are tabular [15], the coincidence problem for any fixed tabular logic turns out to be decidable [62, 106] (see also [29, Theorem 17.3]). As far as we know, the decision problem of tabularity in $\mathsf{NExt}(\mathsf{K4})$ is still open [105]. The decision problem of the FMP in $\mathsf{NExt}(\mathsf{S4})$ is proved to be undecidable by Chagrov and Zakharyaschev [28], while the decision problem of Kripke completeness is still open for $\mathsf{NExt}(\mathsf{S4})$. These results reflect our intuition that $\mathsf{NExt}(\mathsf{K4})$ is less complex than $\mathsf{NExt}(\mathsf{K})$ and is more complex than $\mathsf{NExt}(\mathsf{S4})$.

Now we move to tense logics. Wolter [131, 133] studied decidability of logical properties in $\mathsf{NExt}(\mathsf{Lin}_t)$: Kripke completeness, the finite model property, and canonicity are all decidable. Note that Lin_t is an extension of K_2 and the logics $\mathsf{K}_2 \subseteq \mathsf{K}_t \subseteq \mathsf{K4}_t \subseteq \mathsf{Lin}_t$ form a chain in $\mathsf{NExt}(\mathsf{K}_2)$. Thus, results on decidability of logical properties indicate that interactions of modalities significantly affect the complexity of lattices of logics. This naturally raises the question of where the boundary of decidability lies. Given that we know little about decidability of logical properties in $\mathsf{NExt}(\mathsf{S4})$, it is also natural to study the decision problems in $\mathsf{NExt}(\mathsf{S4}_t)$. Chagrov and Shehtman [30] proved that tabularity is undecidable in $\mathsf{NExt}(\mathsf{K4}_t)$, and even that the coincidence problem for any fixed tabular tense logic is undecidable in $\mathsf{NExt}(\mathsf{K4}_t)$. However, the decidability of other logical properties such as Kripke completeness, strict Kripke completeness, the FMP and decidability remained open for the lattices $\mathsf{NExt}(\mathsf{K}_t)$, $\mathsf{NExt}(\mathsf{K4}_t)$ and $\mathsf{NExt}(\mathsf{S4}_t)$. We will resolve these problems in this chapter.

For the lattice $\mathsf{NExt}(\mathsf{K4}_t)$, we first show that strict Kripke completeness is decidable. Then we turn to study undecidable properties. We provide a general criterion (Theorem 6.2.5) for a logical property to be undecidable in $\mathsf{NExt}(\mathsf{K4}_t)$, which yields the undecidability of, for example, Kripke completeness, the FMP and decidability; see Corollary 6.2.14 for a more complete list of undecidable properties in $\mathsf{NExt}(\mathsf{K4}_t)$ following from the theorem. This illustrates the complexity of $\mathsf{NExt}(\mathsf{K4}_t)$ is much higher than that of $\mathsf{NExt}(\mathsf{K4})$. Note that the results imply that these properties are also undecidable in $\mathsf{NExt}(\mathsf{K}_t)$. Next, we move to the lattice $\mathsf{NExt}(\mathsf{S4}_t)$. We prove that strict Kripke completeness is again decidable. Moreover, we show directly that properties including Kripke completeness, tabularity and decidability are undecidable. We also show that there exist countably tabular logics L such that the coincidence problem for L is undecidable. On the other hand, we prove that consistency is now decidable and there exist countably many tabular logics L in $\mathsf{NExt}(\mathsf{S4}_t)$ for which the coincidence problem is decidable.

Our proofs of Theorems 6.2.5 and 6.3.26 adapt Chagrov's method to the tense setting, reducing the decision problem of a logical property to an undecidable problem regarding Minsky machines. While the overall strategy is similar, we exploit the interaction between the two temporal modalities, which leads to a more efficient encoding and a simpler construction. *Minsky machines*, also called counter machines or register machines, are a type of mathematical model of computation as strong as Turing machines [97]. There exist a Minsky machine $\mathcal{M}$ and a configuration c_0 of $\mathcal{M}$ such that it is undecidable whether a given configuration of $\mathcal{M}$ is reachable from c_0 by a computation of $\mathcal{M}$ (see, e.g., [29, Theorem 16.3]). Let us call this undecidable problem Q. For the $\mathsf{K4}_t$ case, given a logical property P, we will find a logic $L \in \mathsf{NExt}(\mathsf{K4}_t)$ that has P and construct a computable reduction from Q to the decision problem of P as follows. For a configuration c, the reduction produces a logic $L(c) \in \mathsf{NExt}(\mathsf{K4}_t)$ satisfying the following:

(1) if c is reachable from c_0, then $L(c) = L$, which implies that $L(c)$ has P;

(2) if c is not reachable from c_0, then $L(c)$ does not have P.

Thus, if we could decide whether a logic in $\mathsf{NExt}(\mathsf{K4}_t)$ has P, we would be able to decide the problem Q: Given a configuration c, we compute the logic $L(c)$ and ask if it has the property P, the answer of which is also the answer of Q. As Q is undecidable, it follows that P is undecidable.

Table 6.1 summarizes the results on the decidability of some major properties discussed so far. The entries marked with † correspond to results established in this thesis. It is worth noting that our new results do not follow directly from the minimal tense extension map $(\cdot)_+ : \mathsf{NExt}(\mathsf{K}) \to \mathsf{NExt}(\mathsf{K}_t)$ and undecidability results for the modal cases: It is known that the map $(\cdot)_+$ is not injective [128], while it remains unknown whether $(\cdot)_+ \upharpoonright \mathsf{NExt}(\mathsf{K4})$ is injective (see [134, p. 158]). Wolter [130] presented a modal logic $L \supseteq \mathsf{K4}$ with the FMP such that L_+ is Kripke incomplete. This result implies that the map $(\cdot)_+$ preserves neither Kripke completeness nor the FMP. It remains open whether decidability is preserved (see [134, p. 132]).

This chapter is organized as follows. Section 6.1 introduces preliminaries on decision problems and Minsky machines. In Section 6.2, we prove the main theorem for $\mathsf{NExt}(\mathsf{K4}_t)$ and apply it to show the undecidability of logical properties. In Section 6.3, we study decidability of logical properties in $\mathsf{NExt}(\mathsf{S4}_t)$. Finally, Section 6.4 concludes this chapter with an overview of future work.

6.1 Decision Problems and Minsky Machines

6.1.1 Decision problem of logical properties

Recall that a tense logic L is *decidable* if it is decidable as a set, i.e., there exists an effective method to decide whether a formula φ is in L. In this section, instead of decidability of logics, we focus on decidability of logical properties. We refer to [29, Chapter 17] and [136] for a detailed introduction and survey of the decision problems of logical properties for modal logics. We identify a property P in the lattice $\mathsf{NExt}(L_0)$ with the set of logics in $\mathsf{NExt}(L_0)$ that satisfy P, that is, $P = \{L \in \mathsf{NExt}(L_0) : L \text{ satisfies } P\}$.

	Cons.	Tab.	Fixed Tab.	KC	FMP	sKC
$\mathsf{NExt}(\mathsf{K})$	✓	×	×	×	×	✓
$\mathsf{NExt}(\mathsf{K4})$	✓	?	✓	×	×	?
$\mathsf{NExt}(\mathsf{S4})$	✓	✓	✓	?	×	?
$\mathsf{NExt}(\mathsf{K}_2)$	×	×	×	×	×	?
$\mathsf{NExt}(\mathsf{K}_t)$	×	×	×	×†	×†	✓†
$\mathsf{NExt}(\mathsf{K4}_t)$	×	×	×	×†	×†	✓†
$\mathsf{NExt}(\mathsf{S4}_t)$	✓	×†	depends†	×†	×†	✓†
$\mathsf{NExt}(\mathsf{Lin}_t)$	✓	?	✓	✓	✓	?

Abbreviations. Cons.: consistency; Tab.: tabularity; Fixed Tab.: coincidence with a fixed tabular logic;

KC: Kripke completeness; FMP: finite model property; sKC: strict Kripke completeness.

Table 6.1: (Un)decidability of logical properties in monomodal and bimodal logics

6.1.1. Definition. Let L_0 be a tense logic. A logical property P is *decidable* in $\mathsf{NExt}(L_0)$ iff the set $\{\varphi : L_0 \oplus \varphi \in P\}$ is decidable.

Following the convention, we restrict ourselves to finitely axiomatizable logics because an input for an algorithm must be a finite object. We do not consider all recursively axiomatizable logics since, as Kuznetsov showed, otherwise the only decidable properties would be the trivial ones (see [29, Section 17.1]). Since most logics we encounter in practice are finitely axiomatizable, this is not a serious drawback. A finitely axiomatizable logic is encoded by a finite set of formulas axiomatizing the logic, or equivalently, a single formula axiomatizing the logic.

Note that determining a logical property in a larger lattice of logics is at least as hard as in a smaller one, in the following sense.

6.1.2. Proposition. *Let $L \in \mathsf{K}_n$ and L' be an extension of L with finitely many axioms. If a property P is decidable in $\mathsf{NExt}(L)$, then it is decidable in $\mathsf{NExt}(L')$.*

Proof:
We may assume $L' = L \oplus \varphi$ for a formula φ. Let P be a decidable property in $\mathsf{NExt}(L)$. Then, given a formula ψ, we can determine whether $L' \oplus \psi$ has P by asking whether $L \oplus (\varphi \wedge \psi)$ has P since the two logics are the same. □

Decidable logical properties in a lattice $\mathcal{L}$ of logics are always closely related to the splittings of $\mathcal{L}$. A general result is that for any fixed splitting L, if L is decidable, then the coincidence problem for L is decidable. To be precise, the following theorem holds:

6.1.3. THEOREM. *Let $L \in \mathsf{NExt}(\mathsf{K}_n)$ be a polymodal logic that enjoys the FMP and L_2 a splitting in $\mathsf{NExt}(L)$. Then*

(1) $\{\varphi \in \mathsf{Form}_n : L \oplus \varphi \supseteq L_2\}$ is decidable;

(2) if L_2 is decidable, then $\{\varphi \in \mathsf{Form}_n : L \oplus \varphi = L_2\}$ is decidable.

Proof:
For (1), since L_2 is a splitting in $\mathsf{NExt}(L)$, there exists $L_1 \in \mathsf{NExt}(L)$ such that $L_2 = L/L_1$. By Theorem 5.2.10, $L_1 = \mathsf{Log}(\mathfrak{F})$ for some finite rooted frame $\mathfrak{F}$. It suffices to show that

$$(\dagger)\ \{\varphi \in \mathsf{Form}_n : L \oplus \varphi \supseteq L_2\} = \{\varphi \in \mathsf{Form}_n : \mathfrak{F} \nvDash \varphi\}.$$

Take any $\varphi \in \mathsf{Form}_n$. Since $L_2 = L/L_1$ and $L_1 = \mathsf{Log}(\mathfrak{F})$, we have

$$\mathfrak{F} \nvDash \varphi \text{ iff } L \oplus \varphi \nsubseteq L_1 \text{ iff } L \oplus \varphi \supseteq L_2.$$

Thus, $(\dagger)$ holds. Because $\mathfrak{F}$ is finite, $\{\varphi \in \mathsf{Form}_n : \mathfrak{F} \nvDash \varphi\}$ is decidable and so (1) holds. For (2), suppose L_2 is decidable. Then $\{\varphi \in \mathsf{Form}_n : L \oplus \varphi \subseteq L_2\} = L_2$ is also decidable. Note that $\{\varphi \in \mathsf{Form}_n : L \oplus \varphi = L_2\} = \{\varphi \in \mathsf{Form}_n : L \oplus \varphi \supseteq L_2\}$ $\cap\{\varphi \in \mathsf{Form}_n : L \oplus \varphi \subseteq L_2\}$. Thus, we conclude that (2) holds. □

6.1.2 Minsky machines

The most commonly used method for proving the undecidability of a decision problem is to construct a computable reduction to the problem from another problem that is already known to be undecidable. In this chapter, we will use an undecidable problem about Minsky machines. We recall the basics of Minsky machines in the rest of this section and refer to [29, Section 16.1] and [97] for more details; see also [29, Sections 16 and 17] for various applications of Minsky machines to obtain undecidability results.

A *Minsky machine* with two registers (also called a register machine with two registers) is a finite set of instructions acting on two registers. We will only use Minsky machines with two registers, so we simply call them Minsky machines. A Minsky machine has finitely many *states*. A *register* can store a natural number and is assumed to be unbounded. So, a situation of a Minsky machine is represented by a tuple $\langle s, n, m\rangle$, called a *configuration*, where s is the current state and n and m are the natural numbers on each register. An instruction operates on the state and one of the two registers: it increments the number in the register, or tests if the number in the register is zero and decrements it if not. More specifically, an instruction I has one of the following four forms:

- $I = t \to \langle t', 1, 0\rangle$ means that I turns the state t into t' and increments the first register,
- $I = t \to \langle t', 0, 1\rangle$ means that I turns the state t into t' and increments the second register,
- $I = t \to \langle t', -1, 0\rangle(\langle t'', 0, 0\rangle)$ means that I turns the state t into t' and decrements the first register if the number in the first register is non-zero, and turns the state t into t'' otherwise,
- $I = t \to \langle t', 0, -1\rangle(\langle t'', 0, 0\rangle)$ means that I turns the state t into t' and decrements the second register if the number in the second register is non-zero, and turns the state t into t'' otherwise.

For example, after applying the instruction $I = s \to \langle s', -1, 0\rangle(\langle s'', 0, 0\rangle)$ to the configuration $\langle s, n, m\rangle$, we obtain the configuration $\langle s', n-1, m\rangle$ if $n \geq 1$ and the configuration $\langle s'', n, m\rangle$ if $n = 0$.

In this chapter, Minsky machines are assumed to be *deterministic*, that is, for each state t there is at most one instruction that acts on the state t. For a Minsky machine $\mathcal{M}$, we write $\mathcal{M} : \langle s, n, m\rangle \rightsquigarrow \langle t, k, l\rangle$ if, starting from the configuration $\langle s, n, m\rangle$, by applying the instructions in $\mathcal{M}$, we can reach the configuration $\langle t, k, l\rangle$ in finitely many (possibly 0) steps. For each configuration $\langle s, n, m\rangle$, let $\mathcal{R}each_{\mathcal{M}}(\langle s, n, m\rangle)$ denote the set

$$\{\langle t, k, l\rangle : (\mathcal{M} : \langle s, n, m\rangle \rightsquigarrow \langle t, k, l\rangle)\}.$$

We call $\mathcal{R}each_{\mathcal{M}}(\langle s, n, m\rangle)$ the *reachability set* of $\langle s, n, m\rangle$. We drop $\mathcal{M}$ if it is clear from the context.

6.1.4. Example. Let $\mathcal{M} = \{\langle s, 1, 0\rangle\}$. Then, for any $n, m \in \omega$,

$$\{\langle t, k, l\rangle : \langle s, n, m\rangle \rightsquigarrow \langle t, k, l\rangle\} = \{\langle s, n+i, m\rangle : i \in \omega\}.$$

We will use the following undecidable problem, which is called the *second configuration problem* in [29, Theorem 16.3].

6.1.5. Theorem. *There exist a Minsky machine $\mathcal{M}$ and a configuration $\langle s, n, m\rangle$ such that the reachability from $\langle s, n, m\rangle$ in $\mathcal{M}$ is undecidable, that is, the set $\{\langle t, k, l\rangle : \langle s, n, m\rangle \rightsquigarrow \langle t, k, l\rangle\}$ is undecidable.*

In what follows, let $\mathcal{M}$ be a Minsky machine and $\langle s, n, m\rangle$ a configuration of $\mathcal{M}$ such that the set $\mathcal{R}each_{\mathcal{M}}(\langle s, n, m\rangle) = \{\langle t, k, l\rangle : (\mathcal{M} : \langle s, n, m\rangle \rightsquigarrow \langle t, k, l\rangle)\}$ is undecidable. The existence of such $\mathcal{M}$ and $\langle s, n, m\rangle$ is given by Theorem 6.1.5. To simplify notation, we write $\mathcal{R}each$ for the set $\mathcal{R}each_{\mathcal{M}}(\langle s, n, m\rangle)$.

6.2 (Un)decidable Logical Properties in $\mathsf{NExt}(\mathsf{K4}_t)$

In this section, we focus on the decidable and undecidable logical properties in the lattice $\mathsf{NExt}(\mathsf{K4}_t)$. We first show that strict Kripke completeness is decidable. More precisely, the following theorem holds:

6.2.1. THEOREM. *The following sets are decidable:*

(1) $\{\varphi \in \mathsf{Form}_t : \mathsf{K4}_t \oplus \varphi = \mathsf{K4}_t/\mathfrak{Ch}_1^{<}\}$*;*

(2) $\{\varphi \in \mathsf{Form}_t : \mathsf{K4}_t \oplus \varphi$ *is a union-splitting*$\}$*;*

(3) $\{\varphi \in \mathsf{Form}_t : \mathsf{K4}_t \oplus \varphi$ *is strictly Kripke complete*$\}$*.*

Proof:
(1) follows immediately from Theorem 6.1.3. By Theorem 5.3.21, we obtain that $\{\mathsf{K4}_t, \mathsf{K4}_t/\mathfrak{Ch}_1^{<}\}$ is the set of union-splittings. Note that $\mathsf{K4}_t$ is decidable and

$$\{\varphi : \mathsf{K4}_t \oplus \varphi \text{ is a union-splitting}\} = \mathsf{K4}_t \cup \{\varphi : \mathsf{K4}_t \oplus \varphi = \mathsf{K4}_t/\mathfrak{Ch}_1^{<}\}.$$

Thus, we conclude that (2) holds. By Theorem 5.3.26, (3) follows from (2). □

6.2.2. REMARK. Note that $\mathsf{K}_t/\mathfrak{Ch}_1^{<}$ is the unique splitting in $\mathsf{NExt}(\mathsf{K}_t)$. By a similar argument, we obtain that being strictly Kripke complete is also decidable in $\mathsf{NExt}(\mathsf{K}_t)$.

Next, we turn to prove the general undecidability result (Theorem 6.2.5), which implies the undecidability of various logical properties summarized in Corollary 6.2.14. The proof idea follows Chagrov's method of using Minsky machines [25, 24] (see also [136]). Since the Minsky machine $\mathcal{M}$ is finite, we may assume that $\mathcal{M}$ contains t_0 many states, labeled as $0, \dots, t_0 - 1$. To state our main theorem, we introduce the following general frame that encodes the information of the problem $\mathcal{R}each$.

6.2.3. DEFINITION. Let $\mathbb{F} = (X, R, A)$ be the general frame defined as follows:

- $X = \mathcal{R}each \cup \{a_n, b_n, c_n : n < \omega\} \cup \{a', b', b''\}$;
- R is the transitive closure of the union of the following binary relations:
  - $\{(c_i, c_j) : j < i < \omega\}$;
  - $\{(a_i, a_j) : j < i < \omega\} \cup \{(a', a_0)\}$;
  - $\{(b_i, b_j) : j < i < \omega\} \cup \{(b', b_0), (b', b'')\}$;
  - $\{(\langle t, k, l\rangle, c_t), (\langle t, k, l\rangle, a_k), (\langle t, k, l\rangle, b_l) : \langle t, k, l\rangle \in \mathcal{R}each\}$.

– $\mathcal{R}each \times \mathcal{R}each$.

- $A = \{U \subseteq X : U \text{ is finite or cofinite on } \{c_i : i \in \omega\}\}$.

It is clear that A is closed under $\cap$, $-(\cdot)$, $R[\cdot]$ and $R^{-1}[\cdot]$, so $\mathbb{F}$ is well-defined as a general frame. The underlying frame (X, R) of $\mathbb{F}$ is depicted in Figure 6.1.

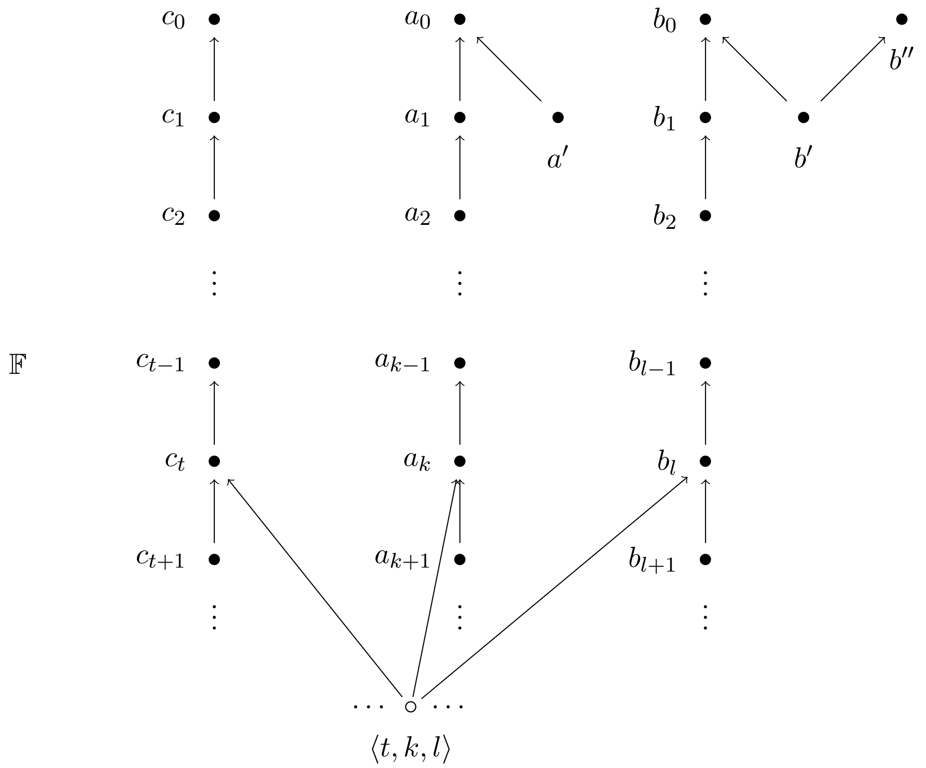

Figure 6.1: The underlying frame of $\mathbb{F}$

By Proposition 4.2.5(4), it follows that

6.2.4. LEMMA. *$\mathbb{F} \models \Delta^{\leq 6}p \to \Delta^{\leq 5}p$.*

Now we state the main theorem of this section. Let

$$\mathsf{KC} = \{L \in \mathsf{NExt}(\mathsf{K4}_t) : L \text{ is Kripke complete}\}$$

and

$$\mathsf{DEC} = \{L \in \mathsf{NExt}(\mathsf{K4}_t) : L \text{ is decidable}\}.$$

6.2.5. THEOREM. *Let P be a property and $\alpha \in \mathsf{Form}_t$ be a formula such that*

(1) $\mathbb{F} \not\models \alpha$,

(2) $\mathsf{K4}_t \oplus \alpha \in P$, *and*

(3) $P \subseteq \mathsf{KC} \cup \mathsf{DEC}$.

Then P *is undecidable, i.e., the set*

$$\{\varphi \in \mathsf{Form}_t : \mathsf{K4}_t \oplus \varphi \in P\}$$

is undecidable.

The rest of this section is dedicated to proving this theorem. We start by introducing formulas that define a point or a set of points in $\mathbb{F}$. Their meaning is summarized in Lemma 6.2.6. For each $n \in \omega$, we define the formulas φ_{c_n}, φ_{a_n} and φ_{b_n} as follows:

- $\varphi_{c_0} \coloneqq \Box\bot \wedge \blacksquare\blacklozenge\top$;
- $\varphi_{c_n} \coloneqq \Diamond^n \varphi_{c_0} \wedge \neg\Diamond^{n+1} \varphi_{c_0}$ for all $n \in \mathbb{Z}^+$;
- $\varphi_{a_0} \coloneqq \Box\bot \wedge \blacklozenge(\blacksquare\bot \wedge \Box\blacklozenge^2 \top)$;
- $\varphi_{a_n} \coloneqq (\varphi_{a_0} \vee \Diamond\varphi_{a_0}) \wedge \Box\neg\varphi_{c_0} \wedge \blacklozenge\top \wedge \Diamond^n \varphi_{a_0} \wedge \neg\Diamond^{n+1} \varphi_{a_0}$ for all $n \in \mathbb{Z}^+$;
- $\varphi_{b_0} \coloneqq \Box\bot \wedge \blacklozenge\blacklozenge\top \wedge \blacklozenge\Diamond\blacksquare^2 \bot$;
- $\varphi_{b_n} \coloneqq (\varphi_{b_0} \vee \Diamond\varphi_{b_0}) \wedge \Box\neg\varphi_{c_0} \wedge \blacklozenge\top \wedge \Diamond^n \varphi_{b_0} \wedge \neg\Diamond^{n+1} \varphi_{b_0}$ for all $n \in \mathbb{Z}^+$.

Note that these formulas are all variable-free. Intuitively, for each $x \in \{a, b, c\}$ and $n \in \omega$, the formula φ_{x_n} is designed to be true at exactly x_n in $\mathbb{F}$, regardless of valuations. Moreover, we define

- $\varphi_A \coloneqq (\varphi_{a_0} \vee \Diamond\varphi_{a_0}) \wedge \Box\neg\varphi_{c_0} \wedge \blacklozenge\top$;
- $\varphi_B \coloneqq (\varphi_{b_0} \vee \Diamond\varphi_{b_0}) \wedge \Box\neg\varphi_{c_0} \wedge \blacklozenge\top$.

Similarly, φ_A and φ_B are true exactly at $\{a_n : n \in \omega\}$ and $\{b_n : n \in \omega\}$ in $\mathbb{F}$, respectively. To simulate the action of $+1$ and -1 on the two tapes of $\mathcal{M}$, we define the following formulas:

- $\psi_A \coloneqq \varphi_A \wedge p_A \wedge \neg\Diamond p_A$;
- $\psi_A^+ \coloneqq \varphi_A \wedge \Diamond p_A \wedge \neg\Diamond\Diamond p_A$;
- $\psi_B \coloneqq \varphi_B \wedge p_B \wedge \neg\Diamond p_B$;
- $\psi_B^+ \coloneqq \varphi_B \wedge \Diamond p_B \wedge \neg\Diamond\Diamond p_B$,

where p_A, p_B are fresh variables. Intuitively, if ψ_A is true at some point, then the point must be a_i, where $i = \min\{j : a_j \vDash p_A\}$; then ψ_A^+ is true at the next point, namely, a_{i+1}. A similar intuition applies to ψ_B and ψ_B^+ as well. Moreover, a key syntactic observation is: if s is the substitution $[\Diamond^k \varphi_{a_0}/p_A, \Diamond^l \varphi_{b_0}/p_B]$ for some $k, l \in \omega$, then $s(\psi_A) = \varphi_{a_k}$, $s(\psi_A^+) = \varphi_{a_{k+1}}$, $s(\psi_B) = \varphi_{b_l}$, and $s(\psi_B^+) = \varphi_{b_{l+1}}$. This will be used in Lemma 6.2.7.

Finally, for each state t of $\mathcal{M}$ and formulas $\pi, \kappa \in \mathsf{Form}_t$, we define:

$$\sigma(t,\pi,\kappa) := \Diamond \varphi_{c_t} \wedge \Box\neg\varphi_{c_{t+1}} \wedge \Diamond \pi \wedge \Box\neg(\Diamond \pi \wedge \Box\neg\varphi_{c_0}) \wedge \Diamond \kappa \wedge \Box\neg(\Diamond \kappa \wedge \Box\neg\varphi_{c_0}).$$

As we will see in Lemma 6.2.6, the formula $\sigma(t,\pi,\kappa)$ is true exactly at the point $\langle t, k, l\rangle$ if the formulas π and κ are true exactly at a_k and b_l, respectively.

6.2.6. LEMMA. *For any $x \in X$, valuation V on $\mathbb{F}$, and $n \in \omega$, the following holds:*

(1) $\mathbb{F}, x \vDash \varphi_{c_n}$ *if and only if* $x = c_n$.

(2) $\mathbb{F}, x \vDash \varphi_{a_n}$ *if and only if* $x = a_n$.

(3) $\mathbb{F}, x \vDash \varphi_{b_n}$ *if and only if* $x = b_n$.

(4) $\mathbb{F}, x \vDash \varphi_A$ *if and only if* $x \in \{a_i : i \in \omega\}$.

(5) $\mathbb{F}, x \vDash \varphi_B$ *if and only if* $x \in \{b_i : i \in \omega\}$.

(6) $\mathbb{F}, V, x \vDash \psi_A$ *if and only if* $V(\psi_A) = \{a_i\} = \{x\}$ *and* $V(\psi_A^+) = \{a_{i+1}\}$ *for some* $i \in \omega$.

(7) $\mathbb{F}, V, x \vDash \psi_A^+$ *if and only if* $V(\psi_A) = \{a_i\}$ *and* $V(\psi_A^+) = \{a_{i+1}\} = \{x\}$ *for some* $i \in \omega$.

(8) $\mathbb{F}, V, x \vDash \psi_B$ *if and only if* $V(\psi_B) = \{b_i\} = \{x\}$ *and* $V(\psi_B^+) = \{b_{i+1}\}$ *for some* $i \in \omega$.

(9) $\mathbb{F}, V, x \vDash \psi_B^+$ *if and only if* $V(\psi_B) = \{b_i\}$ *and* $V(\psi_B^+) = \{b_{i+1}\} = \{x\}$ *for some* $i \in \omega$.

(10) If $V(\pi) = \{a_k\}$ *and* $V(\kappa) = \{b_l\}$*, then* $\mathbb{F}, V, x \vDash \sigma(t,\pi,\kappa)$ *if and only if* $x = \langle t, k, l\rangle$.

Proof:
We only prove (1), (4), (6), and (10) and leave the rest to the reader. The proof of (1) proceeds by induction on n. Let $n = 0$. The right-to-left direction is clear. Suppose $\mathbb{F}, x \vDash \varphi_{c_0}$. Then $R[x] = \varnothing$ and $R^{-1}[y] \neq \varnothing$ for all $y \in R^{-1}[x]$, which

entails $x = c_0$. Let $n > 0$. Suppose $\mathbb{F}, x \vDash \varphi_{c_n}$. By the induction hypothesis, $c_{n-1} \in R[x] \setminus R[R[x]]$, thus $x = c_n$, and (1) follows.

For (4), the right-to-left direction is straightforward. For the other direction, suppose $\mathbb{F}, x \vDash \varphi_A$. Then $\mathbb{F}, x \vDash \varphi_{a_0} \vee \Diamond \varphi_{a_0}$, which entails $x \in R^{-1}[a_0] \cup \{a_0\}$. Since $\mathbb{F}, x \vDash \Box \neg \varphi_{c_0} \wedge \blacklozenge \top$, we see that $x \notin \mathcal{R}each$ and $x \neq a'$. Thus, $x \in \{a_i : i \in \omega\}$.

For (6), the right-to-left direction is again straightforward. For the other direction, suppose $\mathbb{F}, x \vDash \psi_A$. Take any $y \in X$ such that $\mathbb{F}, V, y \vDash \psi_A$. By (4) $y = a_i$ for some $i \in \omega$. By $\mathbb{F}, V, y \vDash p_A \wedge \neg \Diamond p_A$, we see that y is an R-maximal point in $V(p_A) \cap \{a_i : i \in \omega\}$. Since R is a linear order on $V(p_A) \cap \{a_i : i \in \omega\}$, we have $V(\psi_A) = \{a_i\} = \{x\}$. As $V(\Diamond p_A \wedge \neg \Diamond \Diamond p_A) \cap \{a_i : i \in \omega\} = \{a_{i+1}\}$, we have $V(\psi_A^+) = \{a_{i+1}\}$.

For (10), suppose $V(\pi) = \{a_k\}$ and $V(\kappa) = \{b_l\}$. Then clearly, $\mathbb{F}, V, \langle t, k, l\rangle \vDash \sigma(t, \pi, \kappa)$. For the other direction, suppose $\mathbb{F}, V, x \vDash \sigma(t, \pi, \kappa)$. Then, $\mathbb{F}, V, x \vDash \Diamond \varphi_{c_t} \wedge \Diamond \pi$, which entails $x = \langle t', k', l'\rangle$ for some $\langle t', k', l'\rangle$. By $\mathbb{F}, V, x \vDash \Diamond \varphi_{c_t} \wedge \Box \neg \varphi_{c_{t+1}}$, we have $x \in R^{-1}[c_t] \setminus R^{-1}[c_{t+1}]$ and so $t' = t$. Note that $\mathbb{F}, V, a_{k+1} \vDash \Diamond \pi \wedge \Box \neg \varphi_{c_0}$. By $\mathbb{F}, V, x \vDash \Diamond \pi \wedge \Box \neg(\Diamond \pi \wedge \Box \neg \varphi_{c_0})$, we see that $x \in R^{-1}[a_k] \setminus R^{-1}[a_{k+1}]$ and so $k' = k$. Similarly, we obtain $l = l'$. Thus, $x = \langle t, k, l\rangle$. $\Box$

Next, we encode the behavior of the Minsky machine $\mathcal{M}$ by formulas. With each instruction I in $\mathcal{M}$, we associate a formula AxI as follows.

- $AxI := \neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \psi_A, \psi_B) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t', \psi_A^+, \psi_B)$, if $I = t \to \langle t', 1, 0\rangle$.
- $AxI := \neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \psi_A, \psi_B) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t', \psi_A, \psi_B^+)$, if $I = t \to \langle t', 0, 1\rangle$.
- $AxI := [\neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \psi_A^+, \psi_B) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t', \psi_A, \psi_B)] \wedge [\neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \varphi_{a_0}, \psi_B) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t'', \varphi_{a_0}, \psi_B)]$, if $I = t \to \langle t', -1, 0\rangle(\langle t'', 0, 0\rangle)$.
- $AxI := [\neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \psi_A, \psi_B^+) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t', \psi_A, \psi_B)] \wedge [\neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \psi_A, \varphi_{b_0}) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t'', \psi_A, \varphi_{b_0})]$, if $I = t \to \langle t', 0, -1\rangle(\langle t'', 0, 0\rangle)$.

Each formula encodes the behavior of the corresponding instruction.

Let $AxM := \bigwedge_{I \in \mathcal{M}} AxI$. This is a well-defined formula since there are only finitely many instructions in $\mathcal{M}$.

6.2.7. Lemma. *For each configuration $\langle t, k, l\rangle$, if $\langle t, k, l\rangle \in \mathcal{R}each$, then*

$$\neg\alpha \wedge \Delta^{\leq 5}\sigma(s, \varphi_{a_n}, \varphi_{b_m}) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \varphi_{a_k}, \varphi_{b_l}) \in \mathsf{K4}_t \oplus AxM.$$

Proof:

We prove by induction on the length of the computation of $\mathcal{M}$. The base case $\langle t, k, l\rangle = \langle s, n, m\rangle$ is clear since the corresponding formula is a tautology. Consider the computation of the form $\langle s, n, m\rangle \rightsquigarrow \langle t, k, l\rangle \to \langle \tilde{t}, \tilde{k}, \tilde{l}\rangle$, where I is the last instruction applied. By induction hypothesis, we have

$$\neg\alpha \wedge \Delta^{\leq 5}\sigma(s, \varphi_{a_n}, \varphi_{b_m}) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \varphi_{a_k}, \varphi_{b_l}) \in \mathsf{K4}_t \oplus AxM.$$

So, it suffices to show

$$\neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \varphi_{a_k}, \varphi_{b_l}) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(\tilde{t}, \varphi_{a_{\tilde{k}}}, \varphi_{b_{\tilde{l}}}) \in \mathsf{K4}_t \oplus AxM.$$

We distinguish cases according to the form of I.

- $I = t \to \langle t', 1, 0\rangle$. Then $\langle \tilde{t}, \tilde{k}, \tilde{l}\rangle = \langle t', k+1, l\rangle$. From AxI we have

  $$\neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \psi_A, \psi_B) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t', \psi_A^+, \psi_B) \in \mathsf{K4}_t \oplus AxM.$$

  By applying the substitution $[\Diamond^k \varphi_{a_0}/p_A, \Diamond^l \varphi_{b_0}/p_B]$ to AxI, we obtain

  $$\neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \varphi_{a_k}, \varphi_{b_l}) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t', \varphi_{a_{k+1}}, \varphi_{b_l}) \in \mathsf{K4}_t \oplus AxM.$$

- $I = t \to \langle t', 0, 1\rangle$. This case is similar to the previous one.

- $I = t \to \langle t', -1, 0\rangle(\langle t'', 0, 0\rangle)$. We further distinguish cases depending on whether $k = 0$.

  * $k \geq 1$. Then $\langle \tilde{t}, \tilde{k}, \tilde{l}\rangle = \langle t', k-1, l\rangle$. From AxI we have

    $$\neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \psi_A^+, \psi_B) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t', \psi_A, \psi_B) \in \mathsf{K4}_t \oplus AxM.$$

    By applying the substitution $[\Diamond^{k-1} \varphi_{a_0}/p_A, \Diamond^l \varphi_{b_0}/p_B]$ to AxI, we have

    $$\neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \varphi_{a_k}, \varphi_{b_l}) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t', \varphi_{a_{k-1}}, \varphi_{b_l}) \in \mathsf{K4}_t \oplus AxM.$$

  * $k = 0$. Then $\langle \tilde{t}, \tilde{k}, \tilde{l}\rangle = \langle t'', 0, l\rangle$. From AxI we have

    $$\neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \varphi_{a_0}, \psi_B) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t'', \varphi_{a_0}, \psi_B) \in \mathsf{K4}_t \oplus AxM.$$

    Applying the substitution $[\Diamond^l \varphi_{b_0}/p_B]$, we have

    $$\neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \varphi_{a_0}, \varphi_{b_l}) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t'', \varphi_{a_0}, \varphi_{b_l}) \in \mathsf{K4}_t \oplus AxM.$$

- $I = t \to \langle t', 0, -1\rangle(\langle t'', 0, 0\rangle)$. The proof is analogous.

Thus, we conclude our induction. □

6.2.8. LEMMA. $\mathbb{F} \vDash AxM$.

Proof:
Take any instruction $I \in \mathcal{M}$. Suppose that I has the form $t \to \langle t', 1, 0\rangle$. Take any point x and any valuation V on $\mathbb{F}$ such that $\mathbb{F}, V, x \vDash \neg\alpha \wedge \Delta^{\leq 5}(\sigma(t, \psi_A, \psi_B))$. Then $\mathbb{F}, V, y \vDash \sigma(t, \psi_A, \psi_B)$ for some $y \in X$. Since $\mathbb{F}, V, y \vDash \Diamond\psi_A \wedge \Diamond\psi_B$, by Lemma 6.2.6(6) and (8), there exists $k, l \in \omega$ such that $V(\psi_A) = \{a_k\}$ and $V(\psi_B) = \{b_l\}$. By Lemma 6.2.6(10), $y = \langle t, k, l\rangle$ and so $\langle s, n, m\rangle \rightsquigarrow \langle t, k, l\rangle$ by the definition of X. Then, since $I \in \mathcal{M}$, we have $\langle s, n, m\rangle \rightsquigarrow \langle t', k+1, l\rangle$. By Lemma 6.2.6 (7), (8) and (10), $\mathbb{F}, V, \langle t', k+1, l\rangle \vDash \sigma(t', \psi_A^+, \psi_B)$. So, we have $\mathbb{F}, V, x \vDash \neg\alpha \wedge \Delta^{\leq 5}(\sigma(t', \psi_A^+, \psi_B))$. Thus, $\mathbb{F} \vDash AxI$. Similarly, $\mathbb{F} \vDash AxI$ for instructions $I \in \mathcal{M}$ of other forms. Hence, $\mathbb{F} \vDash AxM$. □

Now we construct the reduction. For each configuration $\langle t, k, l\rangle$, we define the logic

$$\begin{aligned} L(t,k,l) := \mathsf{K4}_t &\oplus AxM \\ &\oplus (\neg\alpha \wedge \Delta^{\leq 5}\sigma(s, \varphi_{a_n}, \varphi_{b_m}) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \varphi_{a_k}, \varphi_{b_l})) \to \alpha \\ &\oplus (\neg\alpha \to \nabla^{\leq 5}(\varphi_{c_{t_0}} \to \varphi_0) \wedge \Delta^{\leq 5}\varphi_{c_{t_0}}) \end{aligned}$$

where

$$\varphi_0 = (\blacksquare(\blacksquare(p \to \blacksquare p) \to p) \to \blacksquare p) \wedge \blacksquare((\Box q \wedge \neg q) \to \blacklozenge(\Box^2 q \wedge \Diamond\neg q)) \wedge \blacklozenge\Box^{t_0+2}\bot$$

and $p, q \in \mathsf{Prop}$ are fresh variables with respect to AxM and α. Note that the axioms of $L(t,k,l)$ are computable from a configuration $\langle t, k, l\rangle$.

6.2.9. Remark. It is known that the inverse $\Box(\Box(p \to \Box p) \to p) \to \Box p$ of the formula $\blacksquare(\blacksquare(p \to \blacksquare p) \to p) \to \blacksquare p$ can be used to axiomatize the modal logic K4.Grz, which is a generalization of Grz. Frames for K4.Grz are Noetherian, but need not be reflexive. For more details, we refer the reader to [45, 80, 81].

The subsequent lemmas show some properties of $L(t,k,l)$. Intuitively, the formula φ_0 is designed so that any extension of $\mathsf{K4}_t \oplus \varphi$ is Kripke incomplete. More precisely, we have

6.2.10. Lemma. *Let $\mathfrak{F} = (Y, S) \in \mathsf{Fr}(\mathsf{K4}_t)$. Then $\mathfrak{F}, y \nvDash \varphi_0$ for any $y \in Y$.*

Proof:
Suppose that $\mathfrak{F}, y \vDash \varphi_0$ for a contradiction. Since $\mathfrak{F}, y \vDash \blacklozenge\Box^{t_0+2}\bot$, $\mathfrak{F}, b' \vDash \Box^{t_0+2}\bot$ for some irreflexive $b' \in S^{-1}[y]$. Now take any irreflexive point $z \in S^{-1}[y]$. Let V_z be a valuation in $\mathfrak{F}$ such that $V_z(q) = S[z]$. Then $\mathfrak{F}, z \vDash \Box q \wedge \neg q$. By $\mathfrak{F}, y \vDash \blacksquare((\Box q \wedge \neg q) \to \blacklozenge(\Box^2 q \wedge \Diamond\neg q))$, we have $\mathfrak{F}, z \vDash \blacklozenge(\Box^2 q \wedge \Diamond\neg q)$ and so $\mathfrak{F}, V_z, z' \vDash \Box^2 q \wedge \Diamond\neg q$ for some $z' \in S^{-1}[z]$. Then, z' is again irreflexive because otherwise $\mathfrak{F}, V_z, z \vDash q$, and $z' \in S^{-1}[y]$ by the transitivity. Thus, there

exists an infinite S^{-1}-chain $\{z_i : i \in \omega\}$ of irreflexive points in $S^{-1}[y]$. Since $\mathfrak{F}$ is transitive, we see that $z_i \neq z_j$ for any $i \neq j$. Let V be a valuation in $\mathfrak{F}$ such that $V(b) = \{z_{2j} : j \in \omega\}$. Then it is straightforward to check that $\mathfrak{F}, V, y \not\models \blacksquare(\blacksquare(p \to \blacksquare p) \to p) \to \blacksquare p$, which contradicts $\mathfrak{F}, y \models \varphi_0$. □

On the other hand, the following lemma holds:

6.2.11. Lemma. $\mathbb{F}, c_{t_0} \models \varphi_0$.

Proof:
First, we show that $\mathbb{F}, c_{t_0} \models \blacksquare(\blacksquare(p \to \blacksquare p) \to p) \to \blacksquare p$. Take any valuation V on $\mathbb{F}$. Then $V(p) \in A$, so either $V(p) \cap \{c_i : i \in \omega\}$ is finite or $\{c_i : i < \omega\} \setminus V(p)$ is finite. Let $\mathfrak{M} = (\mathbb{F}, V)$. Suppose $\mathfrak{M}, c_{t_0} \not\models \blacksquare(\blacksquare(p \to \blacksquare p) \to p) \to \blacksquare p$ for a contradiction. Then $\mathfrak{M}, c_{t_0} \models \blacksquare(\blacksquare(p \to \blacksquare p) \to p)$ and $\mathfrak{M}, c_{t_0} \models \blacklozenge\neg p$. Thus, $R^{-1}[c_{t_0}] \setminus V(p) \neq \emptyset$, and for every $x \in R^{-1}[c_{t_0}] \setminus V(p)$, we have $\mathfrak{M}, x \models \blacklozenge(p \wedge \blacklozenge\neg p)$. Since $R^{-1}[c_{t_0}] = \{c_i : i > t_0\}$, it follows that both $\{c_i : i < \omega\} \cap V(p)$ and $\{c_i : i < \omega\} \setminus V(p)$ are infinite, which is a contradiction. Thus, $\mathbb{F}, c_{t_0} \models \blacksquare(\blacksquare(p \to \blacksquare p) \to p) \to \blacksquare p$.

To show that $\mathbb{F}, c_{t_0} \models \blacksquare((\Box q \wedge \neg q) \to \blacklozenge(\Box^2 q \wedge \Diamond\neg q))$. Take any point $x \in R^{-1}[c_{t_0}]$ and any valuation V in $\mathbb{F}$. Note that $R^{-1}[c_{t_0}] = \{c_i : i > t_0\}$ since $\mathcal{M}$ contains t_0 many states labeled as $0, \ldots, t_0 - 1$. Then $x = c_i$ for some $i > t_0$. Let $\mathfrak{M} = (\mathbb{F}, V)$. Suppose $\mathfrak{M}, c_i \models \Box q \wedge \neg q$. Then $\mathfrak{M}, c_{i+1} \models \Box^2 q \wedge \Diamond\neg q$, which entails $\mathfrak{M}, c_i \models \blacklozenge(\Box^2 q \wedge \Diamond\neg q)$. Thus, $\mathbb{F}, c_i \models (\Box q \wedge \neg q) \to \blacklozenge(\Box^2 q \wedge \Diamond\neg q)$, and so $\mathbb{F}, c_{t_0} \models \blacksquare((\Box q \wedge \neg q) \to \blacklozenge(\Box^2 q \wedge \Diamond\neg q))$.

Finally, $\mathbb{F}, c_{t_0} \models \blacklozenge\Box^{t_0+2}\bot$ follows from $\mathbb{F}, c_{t_0+1} \models \Box^{t_0+2}\bot$. Thus, we conclude that $\mathbb{F}, c_{t_0} \models \varphi_0$. □

Now we prove the following lemmas that summarize how the reduction works.

6.2.12. Lemma. *Let $\langle t, k, l\rangle$ be a configuration such that $\langle t, k, l\rangle \notin \mathcal{R}each$. Then*

(1) $\mathbb{F} \models L(t, k, l)$.

(2) $L(t, k, l)$ is Kripke incomplete.

(3) $L(t, k, l)$ is undecidable.

Proof:
For (1), we know from Lemma 6.2.8 that $\mathbb{F} \models AxM$. Since $\langle t, k, l\rangle \notin X$, by Lemma 6.2.6 (2), (3), and (10), $\mathbb{F} \models \neg\sigma(t, \varphi_{a_k}, \varphi_{b_l})$ and so $\mathbb{F} \models \neg\Delta^{\leq 5}\sigma(t, \varphi_{a_k}, \varphi_{b_l})$. Similarly, since $\langle s, n, m\rangle \rightsquigarrow \langle s, n, m\rangle$, we have $\mathbb{F}, \langle s, n, m\rangle \models \sigma(s, \varphi_{a_n}, \varphi_{b_m})$. By Lemma 6.2.4, $\mathbb{F} \models \Delta^{\leq 5}\sigma(s, \varphi_{a_n}, \varphi_{b_m})$. Take any $x \in X$ and valuation V in $\mathbb{F}$. If $\mathbb{F}, V, x \not\models \alpha$, then $\mathbb{F}, V, x \models \neg\alpha \wedge \Delta^{\leq 5}\sigma(s, \varphi_{a_n}, \varphi_{b_m}) \wedge \neg\Delta^{\leq 5}\sigma(t, \varphi_{a_k}, \varphi_{b_l})$, i.e.

$\mathbb{F}, V, x \nvDash \neg\alpha \wedge \Delta^{\leq 5}\sigma(s, \varphi_{a_n}, \varphi_{b_m}) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \varphi_{a_k}, \varphi_{b_l})$. Thus, $\mathbb{F} \vDash (\neg\alpha \wedge \Delta^{\leq 5}\sigma(s, \varphi_{a_n}, \varphi_{b_m}) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \varphi_{a_k}, \varphi_{b_l})) \to \alpha$. By Lemmas 6.2.11 and 6.2.6 (1), $\mathbb{F} \vDash \varphi_{c_{t_0}} \to \varphi_0$ and so $\mathbb{F} \vDash \nabla^{\leq 5}(\varphi_{c_{t_0}} \to \varphi_0)$. By Lemmas 6.2.4 and 6.2.6 (1), $\mathbb{F} \vDash \Delta^{\leq 5}\varphi_{c_{t_0}}$. Thus, $\mathbb{F} \vDash \nabla^{\leq 5}(\varphi_{c_{t_0}} \to \varphi_0) \wedge \Delta^{\leq 5}\varphi_{c_{t_0}}$. Hence, $\mathbb{F} \vDash L(t, k, l)$.

For (2), it suffices to show that $\alpha \in \mathsf{Log}(\mathsf{Fr}(L(t, k, l)))$, since $\alpha \notin L(t, k, l)$ follows from $\mathbb{F} \vDash L(t, k, l)$ and the assumption $\mathbb{F} \nvDash \alpha$. Take any Kripke frame $\mathfrak{F} = (Y, S) \in \mathsf{Fr}(L(t, k, l))$. Suppose $\mathfrak{F} \nvDash \alpha$. Then $\mathfrak{F}, V, x \vDash \neg\alpha$ for some $x \in Y$ and valuation V on $\mathfrak{F}$. Since $\mathfrak{F} \vDash L(t, k, l)$, we have $\mathfrak{F}, x \vDash \neg\alpha \to \nabla^{\leq 5}(\varphi_{c_{t_0}} \to \varphi_0) \wedge \Delta^{\leq 5}\varphi_{c_{t_0}}$. Thus, for any valuation V' that agrees with V on the variables occurring in α, we have $\mathfrak{F}, V', x \vDash \neg\alpha$ and so $\mathfrak{F}, V', x \vDash \nabla^{\leq 5}(\varphi_{c_{t_0}} \to \varphi_0) \wedge \Delta^{\leq 5}\varphi_{c_{t_0}}$. Note that φ_0 and α have no common variable and $\varphi_{c_{t_0}}$ is variable-free. So, we have $\mathfrak{F}, x \vDash \nabla^{\leq 5}(\varphi_{c_{t_0}} \to \varphi_0) \wedge \Delta^{\leq 5}\varphi_{c_{t_0}}$. Since $\varphi_{c_{t_0}}$ is variable-free, there exists $y \in S^5_\sharp[x]$ such that $\mathfrak{F}, y \vDash \varphi_{c_{t_0}}$, and so $\mathfrak{F}, y \vDash \varphi_0$, which contradicts Lemma 6.2.10. Thus, $\mathfrak{F} \vDash \alpha$, and we conclude $\alpha \in \mathsf{Log}(\mathsf{Fr}(L(t, k, l)))$.

For (3), it suffices to show that for any configuration $\langle t', k', l' \rangle$, the following are equivalent:

(i) $\langle s, n, m \rangle \rightsquigarrow \langle t', k', l' \rangle$;

(ii) $\neg\alpha \wedge \Delta^{\leq 5}\sigma(s, \varphi_{a_n}, \varphi_{b_m}) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t', \varphi_{a_{k'}}, \varphi_{b_{l'}}) \in L(t, k, l)$.

This yields a reduction from the problem $\{\langle t', k', l' \rangle : \langle s, n, m \rangle \rightsquigarrow \langle t', k', l' \rangle\}$ to the decision problem of $L(t, k, l)$, which implies the undecidability of $L(t, k, l)$. By Lemma 6.2.7, (i) implies (ii). Suppose $\langle s, n, m \rangle \not\rightsquigarrow \langle t', k', l' \rangle$. Since $\mathbb{F} \nvDash \alpha$, similar to the proof of (1), we see that $\mathbb{F} \nvDash \neg\alpha \wedge \Delta^{\leq 5}\sigma(s, \varphi_{a_n}, \varphi_{b_m}) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t', \varphi_{a_{k'}}, \varphi_{b_{l'}})$. By (1) $\mathbb{F} \vDash L(t, k, l)$, we have $\neg\alpha \wedge \Delta^{\leq 5}\sigma(s, \varphi_{a_n}, \varphi_{b_m}) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t', \varphi_{a_{k'}}, \varphi_{b_{l'}}) \notin L(t, k, l)$, which proves the converse direction. □

6.2.13. Lemma. *For each configuration* $\langle t, k, l \rangle \in \mathcal{R}each$,

$$L(t, k, l) = \mathsf{K4}_t \oplus \alpha.$$

Proof:
Let $L = \mathsf{K4}_t \oplus \alpha$. It is clear that $L(t, k, l) \subseteq L$. Suppose that $\langle s, n, m \rangle \rightsquigarrow \langle t, k, l \rangle$. By Lemma 6.2.7, $\neg\alpha \wedge \Delta^{\leq 5}\sigma(s, \varphi_{a_n}, \varphi_{b_m}) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \varphi_{a_k}, \varphi_{b_l}) \in \mathsf{K4}_t \oplus AxM$. So, $\alpha \in L(t, k, l)$ since $(\neg\alpha \wedge \Delta^{\leq 5}\sigma(s, \varphi_{a_n}, \varphi_{b_m}) \to \neg\alpha \wedge \Delta^{\leq 5}\sigma(t, \varphi_{a_k}, \varphi_{b_l})) \to \alpha \in L(t, k, l)$, and thus $L(t, k, l) = L$. □

Now we are ready to prove the main theorem.
Proof of Theorem 6.2.5. Let P be a property and $\alpha \in \mathsf{Form}_t$ be a formula such that $\mathbb{F} \nvDash \alpha$ and $\mathsf{K4}_t \oplus \alpha \in P$ and $P \subseteq \mathsf{KC} \cup \mathsf{DEC}$. The reduction $\langle t, k, l \rangle \mapsto L(t, k, l)$ satisfies the following:

- If $\langle t, k, l\rangle \in \mathcal{R}each$, then we obtain $L(t, k, l) = \mathsf{K4}_t \oplus \alpha$ by Lemma 6.2.13, and so $L(t, k, l) \in P$.
- If $\langle t, k, l\rangle \notin \mathcal{R}each$, then $L(t, k, l) \notin \mathsf{KC} \cup \mathsf{DEC}$ by Lemma 6.2.12. Thus, we have $L(t, k, l) \notin P$.

Hence, we obtain a reduction from the set $\mathcal{R}each$, which is undecidable, to the set $\{\varphi \in \mathsf{Form}_t : \mathsf{K4}_t \oplus \varphi \in P\}$, which is therefore also undecidable. ■

As a corollary of Theorem 6.2.5, we obtain the following undecidability results:

6.2.14. Corollary. *The following sets are undecidable:*

(1) $\{\varphi \in \mathsf{Form}_t : \mathsf{K4}_t \oplus \varphi$ *is tabular*$\}$,

(2) $\{\varphi \in \mathsf{Form}_t : \mathsf{K4}_t \oplus \varphi$ *is locally tabular*$\}$,

(3) $\{\varphi \in \mathsf{Form}_t : \mathsf{K4}_t \oplus \varphi$ *has the FMP*$\}$,

(4) $\{\varphi \in \mathsf{Form}_t : \mathsf{K4}_t \oplus \varphi$ *is decidable*$\}$,

(5) $\{\varphi \in \mathsf{Form}_t : \mathsf{K4}_t \oplus \varphi$ *is canonical*$\}$,

(6) $\{\varphi \in \mathsf{Form}_t : \mathsf{K4}_t \oplus \varphi$ *is Kripke complete*$\}$,

(7) $\{\varphi \in \mathsf{Form}_t : \mathsf{K4}_t \oplus \varphi = L\}$, *where L is an arbitrarily fixed tabular logic,*

(8) $\{\varphi \in \mathsf{Form}_t : \mathsf{K4}_t \oplus \varphi$ *is consistent*$\}$.

Proof:
All properties in (1) - (6) are contained in $\mathsf{KC} \cup \mathsf{DEC}$. Since the logic $\mathsf{Log}(\bullet) = \mathsf{K4}_t \oplus \Box\bot \wedge \blacksquare\bot$ is tabular, it has all properties in (1) - (6). Also, it is clear that $\mathbb{F} \not\models \Box\bot \wedge \blacksquare\bot$. So, applying Theorem 6.2.5 with $\alpha = \Box\bot \wedge \blacksquare\bot$, we obtain the undecidability for (1) - (6). For (7), take any tabular tense logic L. It follows from Theorem 3.1.4 that there exists a formula α such that $L = \mathsf{K4}_t \oplus \alpha$, and every extension of L is again tabular. Since $\mathsf{Log}(\mathbb{F})$ is non-tabular, we have $\mathbb{F} \not\models \alpha$. By Theorem 6.2.5, (7) is undecidable. For (8), it suffices to show the complement, namely, the inconsistency is undecidable. This follows from (7) and the fact that the inconsistent logic is tabular. □

6.2.15. Remark. Since $\mathsf{K4}_t = \mathsf{K}_t \oplus \Diamond\Diamond p \to \Diamond p$, it follows from Proposition 6.1.2 that the properties mentioned in Corollary 6.2.14 are also undecidable in the lattice $\mathsf{NExt}(\mathsf{K}_t)$.

6.2.16. Remark. Recall that consistency is decidable in $\mathsf{NExt}(\mathsf{K})$ because the lattice $\mathsf{NExt}(\mathsf{K})$ has exactly two co-atoms, namely, maximal consistent logics. The result that consistency is undecidable in $\mathsf{NExt}(\mathsf{K4}_t)$ aligns with the fact that there are $2^{\aleph_0}$ co-atoms in the lattice $\mathsf{NExt}(\mathsf{K4}_t)$ (Theorem 3.3.9).

6.3 (Un)decidable Logical Properties in $\mathsf{NExt}(\mathsf{S4}_t)$

In this section, we move from the lattice $\mathsf{NExt}(\mathsf{K4}_t)$ to its sublattice $\mathsf{NExt}(\mathsf{S4}_t)$. As we are going to show, there exist some logical properties such as consistency that become decidable, while most of the properties considered in Corollary 6.2.14 remain undecidable.

6.3.1 Decidable properties in $\mathsf{NExt}(\mathsf{S4}_t)$

Note that Form_t is a splitting in $\mathsf{NExt}(\mathsf{S4}_t)$. By Theorem 6.1.3, consistency is decidable in $\mathsf{NExt}(\mathsf{S4}_t)$. More precisely, the following theorem holds:

6.3.1. Theorem. *The set* $\{\varphi \in \mathsf{Form}_t : \mathsf{S4}_t \oplus \varphi = \mathsf{Form}_t\}$ *is decidable.*

Moreover, the following logical properties are decidable:

6.3.2. Theorem. *The following sets are decidable:*

(1) $\{\varphi \in \mathsf{Form}_t : \mathsf{S4}_t \oplus \varphi$ *is an iterated splitting*$\}$,

(2) $\{\varphi \in \mathsf{Form}_t : \mathsf{S4}_t \oplus \varphi$ *is strictly Kripke complete*$\}$,

(3) $\{\varphi \in \mathsf{Form}_t : \mathsf{S4}_t \oplus \varphi = L\}$, *where L is an arbitrarily fixed iterated splitting.*

Proof:
For (1), recall from Theorem 5.4.2 that $\mathsf{NExt}(\mathsf{S5}_t) \cup \{\mathsf{S4}_t\}$ is exactly the set of iterated splittings in $\mathsf{NExt}(\mathsf{S4}_t)$. Clearly, the property of coinciding with $\mathsf{S4}_t$ is decidable. Since $\mathsf{S5}_t$ is a splitting in $\mathsf{NExt}(\mathsf{S4}_t)$, by Theorem 6.1.3, the property of extending $\mathsf{S5}_t$ is also decidable. Thus, $\{\varphi \in \mathsf{Form}_t : \mathsf{S4}_t \oplus \varphi$ is an iterated splitting$\}$ is decidable. By Theorem 5.4.21, (2) follows from (1) immediately. For (3), let L be an iterated splitting in $\mathsf{NExt}(\mathsf{S4}_t)$. The case $L = \mathsf{S4}_t$ is trivial. The case $L \in \{\mathsf{S5}_t, \mathsf{Form}_t\}$ follows from Theorem 6.1.3. It suffices to consider the case $L = \mathsf{Log}(\mathfrak{Cl}_n)$ for some $n \in \mathbb{Z}^+$. Note that L is a splitting in $\mathsf{S5}_t$ and $\{\varphi : \mathsf{S4}_t \oplus \varphi = L\} = \{\varphi : \mathsf{S4}_t \oplus \varphi \supseteq \mathsf{S5}_t\} \cap \{\varphi : \mathsf{S5}_t \oplus \varphi = L\}$. By Theorem 6.1.3, $\{\varphi \in \mathsf{Form}_t : \mathsf{S4}_t \oplus \varphi = L\}$ is decidable and so (3) holds. ☐

Recall that in $\mathsf{NExt}(\mathsf{K4}_t)$, there exists no tabular logic $L \in \mathsf{NExt}(\mathsf{K4}_t)$ such that $\{\varphi \in \mathsf{Form}_t : \mathsf{K4}_t \oplus \varphi = L\}$ is decidable. However, by Theorem 6.3.2, there exists a tabular logic $L' \in \mathsf{NExt}(\mathsf{S4}_t)$ such that $\{\varphi \in \mathsf{Form}_t : \mathsf{S4}_t \oplus \varphi = L'\}$ is decidable. This shows the difference between $\mathsf{NExt}(\mathsf{K4}_t)$ and $\mathsf{NExt}(\mathsf{S4}_t)$. As we are going to show in Section 6.3.2, there exists a tabular logic $L'' \in \mathsf{NExt}(\mathsf{S4}_t)$ such that $\{\varphi \in \mathsf{Form}_t : \mathsf{S4}_t \oplus \varphi = L''\}$ is undecidable.

6.3.2 Undecidable properties in $\mathsf{NExt}(\mathsf{S4}_t)$

In this section, we focus on undecidable logical properties in $\mathsf{NExt}(\mathsf{S4}_t)$. Instead of providing a general undecidability result in $\mathsf{NExt}(\mathsf{S4}_t)$, we show directly that logical properties listed in Theorem 6.3.26 are undecidable. Recall from Section 6.1 that $\mathcal{M}$ is a Minsky machine and $\langle s, n, m\rangle$ a configuration of $\mathcal{M}$ such that the set $\mathcal{R}each = \{\langle t, k, l\rangle : (\mathcal{M} : \langle s, n, m\rangle \rightsquigarrow \langle t, k, l\rangle)\}$ is undecidable. We may assume that $\mathcal{M}$ contains t_0 many states, labeled as $0, \dots, t_0 - 1$. Again, to state the main theorem, we introduce the following general frame.

6.3.3. Definition. Let $\mathbb{F} = (W, R, A)$ be the general frame defined as follows:

- $W = AB \cup C \cup \{x_i : i \leq 5\} \cup \{e\} \cup \{\langle t, k, l\rangle_i : i \leq 3 \text{ and } \langle t, k, l\rangle \in \mathcal{R}each\}$, where $AB = \{a_i, b_i : i < \omega\}$ and $C = \{c_i, c'_i : i \leq t_0\}$,
- R is the reflexive-transitive closure of the union of the following sets:
  - $\{(a_{i+1}, a_i), (a_{i+1}, b_i) : i \in \omega\} \cup \{(b_{i+1}, b_i), (b_{i+2}, a_i) : i \in \omega\}$
  - $\{(c_{i+1}, c_i), (c'_{i+1}, c'_i) : i < t_0\} \cup \{(c'_i, c_i), (c'_i, a_{i+1}) : i \leq t_0\}$
  - $\{(c_{t+1}, \langle t, k, l\rangle_0), (a_{k+1}, \langle t, k, l\rangle_1), (b_{l+1}, \langle t, k, l\rangle_2) : \langle t, k, l\rangle \in \mathcal{R}each\}$
  - $\{(z_1, z_0), (z_2, z_0), (z_2, z_3) : z \in \mathcal{R}each\}$
  - $\{(x_0, x_1), (c_0, x_1), (x_2, x_1), (x_2, x_3), (x_4, x_3), (x_4, a_0), (x_5, b_0)\}$
  - $\{(e, b_i) : i \in \omega\} \cup \{(e, c'_{t_0})\}$
- $A = \{U \subseteq W : U \text{ is finite or cofinite in } AB\}$.

The underlying frame of $\mathbb{F}$ is depicted in Figure 6.2.

To prove the main theorem, we first introduce some formulas that define a point or a set of points in $\mathbb{F}$. Note that closed formulas do not work in this case.

6.3.4. Definition. Let $p, p_0, p_1, p_2 \in \mathsf{Prop}$. We define the following formulas:

- $\varphi^* := (p \wedge \Diamond(\neg p \wedge \Diamond p)) \vee \bigwedge_{i \leq 2} \Diamond(p_i \wedge \bigwedge_{i \neq j \leq 2} \Box \neg p_j)$.
- $\varphi_{x_3} := \Box\blacksquare\Box\neg\varphi^*$, $\varphi_{x_5} := \blacksquare\Box\blacksquare\Box\blacksquare\neg\varphi_{x_3}$ and $\varphi_{x_0} := \Box\neg\varphi_{x_3} \wedge \blacksquare\Box\blacksquare\neg\varphi_{x_5}$;
- $\varphi_{x_1} := \blacklozenge\varphi_{x_0} \wedge \neg\varphi_{x_0}$, $\varphi_{x_2} := \Diamond\varphi_{x_1} \wedge \Diamond\varphi_{x_3}$ and $\varphi_{x_4} := \Diamond\varphi_{x_3} \wedge \blacksquare\neg\varphi_{x_2}$
- $\varphi_{a_0} := \blacklozenge\varphi_{x_4} \wedge \Box\neg\varphi_{x_3}$, $\varphi_{b_0} := \blacklozenge\varphi_{x_5} \wedge \neg\varphi_{x_5}$ and $\varphi_{b_1} := \Diamond\varphi_{b_0} \wedge \Box\neg\varphi_{a_0} \wedge \blacksquare\neg\varphi_{x_5}$
- $\varphi_{AB} := \Box\neg\varphi_{x_1} \wedge \Box\neg\varphi_{x_5} \wedge (\Diamond\varphi_{b_0} \vee \varphi_{a_0})$

The semantical meaning of the formulas in Definition 6.3.4 is explained as follows:

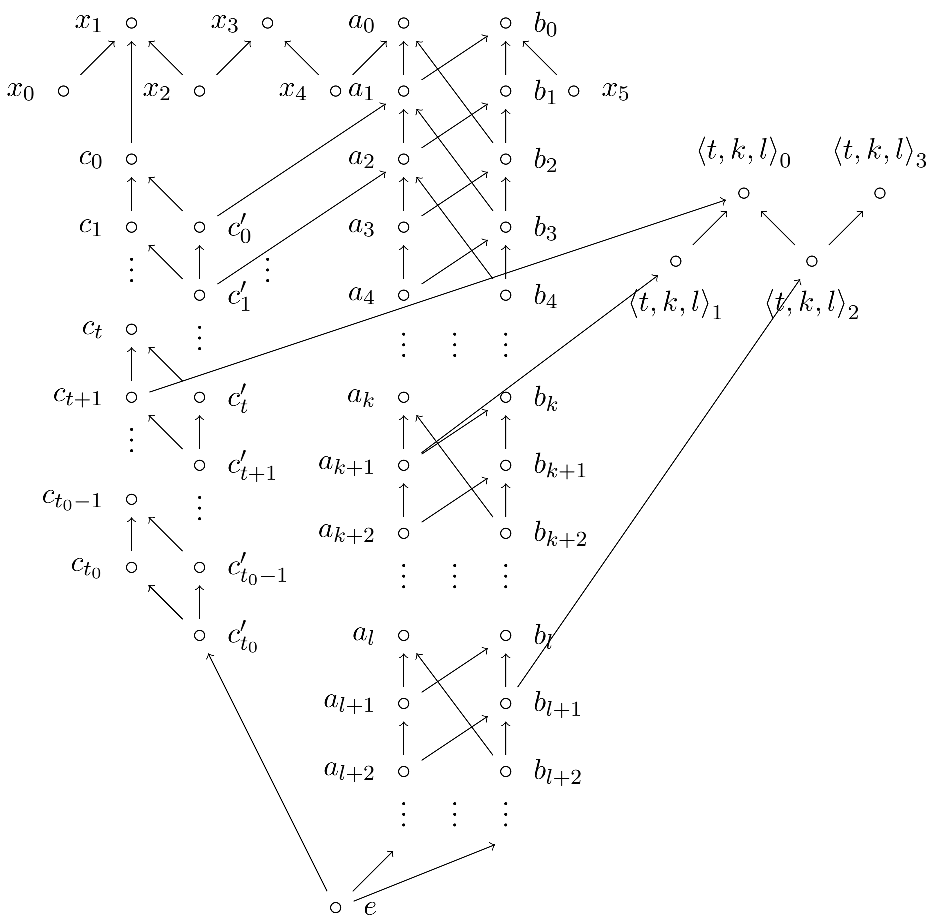

Figure 6.2: The frame $\mathbb{F}$

6.3.5. Lemma. *Let V be a valuation in $\mathbb{F}$. Then*

(1) $V(\varphi^*) \cap (\{x_i : i \leq 5\} \cup \{a_0, b_0\}) = \varnothing$.

Moreover, if $V(\varphi^*) \neq \varnothing$, then the following statements hold:

(2) for all $u \in \{x_i : i \leq 5\} \cup \{a_0, b_0, b_1\}$, $V(\varphi_u) = \{u\}$;

(3) $V(\varphi_{AB}) = AB = \{a_i, b_i : i \in \omega\}$.

Proof:
It is straightforward to verify that φ^* can be satisfied only at points w such that either $\mathsf{wid}^+(w) > 2$ or $\mathsf{dep}(w) > 2$. Thus, if φ^* is satisfiable at w, then $w \notin \{x_i : i \leq 5\} \cup \{a_0, b_0\}$. Thus, (1) holds.

For (2) and (3), suppose $V(\varphi^*) \neq \varnothing$. By (1), we see that $e \in R^{-1}[V(\varphi^*)]$ and $R^{-1}[V(\varphi^*)] \cap (\{x_i : i \leq 5\} \cup \{a_0, b_0\}) = \varnothing$. Thus, $R[R^{-1}[V(\varphi^*)]] = W \setminus \{x_0, x_2, x_3, x_4, x_5\}$ and $R^{-1}[R[R^{-1}[V(\varphi^*)]]] = W \setminus \{x_3\}$, which entails that $V(\varphi_{x_3}) = \{x_3\}$. Note that since $\{x_5\} = R^{-1}[R[R^{-1}[R[R^{-1}[x_3]]]]] \setminus R^{-1}[R[R^{-1}[R[x_3]]]]$, we obtain that $V(\varphi_{x_5}) = \{x_5\}$. The rest of this proof is standard. ☐

6.3.6. Lemma. $\mathbb{F} \not\models \neg\varphi^*$.

Proof:
Let V be a valuation in $\mathbb{F}$ such that $V(p) = \{a_0, a_2\}$. Then $\mathbb{F}, V, a_2 \models \varphi^*$, which entails $\mathbb{F} \not\models \neg\varphi^*$. ☐

Minsky machine simulation

Let $\varphi_{AB}[q'/\varphi^*]$ be the formula obtained from φ_{AB} by replacing φ^* with q'. In other words, $\varphi_{AB}[q'/\varphi^*]$ is defined in the same way as in Definition 6.3.4 but starting with $\varphi_{x_3}[q'/\varphi^*] = \Box\blacksquare\Box\neg q' \wedge \blacklozenge\Diamond\blacklozenge q'$. Recall that in the proof of Lemma 6.3.5, we only used the assumption that $V(\varphi^*) \neq \emptyset$ and the fact that $V(\varphi^*) \cap (\{x_i : i \leq 5\} \cup \{a_0, b_0\}) = \varnothing$. Thus, we obtain a similar lemma for $\varphi_{AB}[q'/\varphi^*]$.

6.3.7. Lemma. *Let V be a valuation in $\mathbb{F}$. Suppose $V(q') \neq \varnothing$ and $V(q') \cap (\{x_i : i \leq 5\} \cup \{a_0, b_0\}) = \varnothing$. Then, $V(\varphi_{AB}[q'/\varphi^*]) = \{a_i, b_i : i \in \omega\}$.*

6.3.8. Definition. Let $\overline{q} = (q, q', q'')$ be a sequence of propositional variables. We define the following formulas:

$$\begin{aligned}
\tau(\overline{q}) &\coloneqq q', \\
\tau'(\overline{q}) &\coloneqq \varphi_{AB}[q'/\varphi^*] \wedge \Diamond q' \wedge \Diamond q'' \wedge \Box\neg(\Diamond q \wedge \Diamond q'' \wedge \Box\neg q') \\
&\quad\ \wedge \nabla^{\leq 7}(q \vee q' \vee q'' \to \varphi_{AB}[q'/\varphi^*]), \\
\tau''(\overline{q}) &\coloneqq \varphi_{AB}[q'/\varphi^*] \wedge \neg q'' \wedge \Diamond(q \wedge \blacksquare\neg q'' \wedge \blacklozenge(q' \wedge \Box\neg q'' \wedge \neg q)) \\
&\quad\ \wedge \Diamond(q'' \wedge \blacksquare\neg q' \wedge \Box\neg q \wedge \blacksquare\neg q) \wedge \Box\neg q'.
\end{aligned}$$

To simplify notations, for each $\varphi \in \mathsf{Form}_t$, we write $V_{AB}(\varphi)$ for $V(\varphi)\cap\{a_i, b_i : i \in \omega\}$.

6.3.9. Lemma. *Let V be a valuation in $\mathbb{F}$. Then for all $i \in \omega$,*

(1) $\langle V(q), V(q'), V(q'')\rangle = \langle\{b_i\}, \{b_{i+1}\}, \{a_i\}\rangle$ *if and only if* $\langle V(\tau), V(\tau'), V(\tau'')\rangle = \langle\{b_{i+1}\}, \{b_{i+2}\}, \{a_{i+1}\}\rangle$.

(2) $\langle V(q), V(q'), V(q'')\rangle = \langle\{a_i\}, \{a_{i+1}\}, \{b_{i+1}\}\rangle$ *if and only if* $\langle V(\tau), V(\tau'), V(\tau'')\rangle = \langle\{a_{i+1}\}, \{a_{i+2}\}, \{b_{i+2}\}\rangle$.

Proof:
For (1), suppose $\langle V(q), V(q'), V(q'')\rangle = \langle\{b_i\}, \{b_{i+1}\}, \{a_i\}\rangle$. Then $V(\tau) = V(q') = \{b_{i+1}\}$. By Lemma 6.3.7, $V(\varphi_{AB}[q'/\varphi^*]) = \{a_i, b_i : i \in \omega\}$. It is straightforward to verify that $b_{i+2} \in V(\tau')$ and $a_{i+1} \in V(\tau'')$, and $V(\tau')\cup V(\tau'') \subseteq \{a_i, b_i : i \in \omega\}$. Note that $(R^{-1}[b_i]\cap R^{-1}[a_i]\setminus R^{-1}[b_{i+1}])\cap\{a_i, b_i : i \in \omega\} = \{a_{i+1}\}$, we have $V(\tau'') = \{a_{i+1}\}$. Because $(R^{-1}[b_{i+1}]\cap R^{-1}[a_i]\setminus R^{-1}[a_{i+1}])\cap\{a_i, b_i : i \in \omega\} = \{b_{i+2}\}$, we see that $V(\tau') = \{b_{i+2}\}$.

Suppose $\langle V(\tau), V(\tau'), V(\tau'')\rangle = \langle\{b_{i+1}\}, \{b_{i+2}\}, \{a_{i+1}\}\rangle$. Then $V(q') = V(\tau) = \{b_{i+1}\}$. By Lemma 6.3.7, $V(\varphi_{AB}[q'/\varphi^*]) = \{a_i, b_i : i \in \omega\}$. Note that since $\mathbb{F}, V, b_{i+2} \models \nabla^{\leq 7}(q \vee q' \vee q'' \to \varphi_{AB}[q'/\varphi^*])$, we have that $V(q \vee q' \vee q'') \subseteq \{a_i, b_i : i \in \omega\}$. Since $\mathbb{F}, V, a_{i+1} \models \Diamond(q \wedge \blacksquare\neg q'' \wedge \blacklozenge(q' \wedge \Box\neg q'' \wedge \neg q))$, there exists $u \in R[a_{i+1}]$ such that $\mathbb{F}, V, u \models q \wedge \blacksquare\neg q'' \wedge \blacklozenge(q' \wedge \Box\neg q'' \wedge \neg q)$, which entails $\mathbb{F}, V, b_{i+1} \models \Box\neg q'' \wedge \neg q$. Because $\mathbb{F}, V, a_{i+1} \models \Diamond(q'' \wedge \blacksquare\neg q' \wedge \Box\neg q \wedge \blacksquare\neg q) \wedge \neg q''$, there exists $v \in R[a_{i+1}]$ such that $v \neq a_{i+1}$ and $\mathbb{F}, V, v \models q'' \wedge \blacksquare\neg q' \wedge \Box\neg q \wedge \blacksquare\neg q$. Note that $\mathbb{F}, V, b_{i+1} \models \Box\neg q''$ and $v \in V(q'') \subseteq \{a_i, b_i : i \in \omega\}$. Thus, we have $v \notin R[b_{i+1}]$ and so $v = a_i$. Because $\mathbb{F}, V, a_i \models \Box\neg q \wedge \blacksquare\neg q$ and $\mathbb{F}, V, b_{i+1} \models \neg q$, we have $V(q) = V_{AB}(q) = \{b_i\}$. Finally, by $\mathbb{F}, V, b_{i+1} \models \Box\neg q''$ and $\mathbb{F}, V, b_i \models \blacksquare\neg q''$, we have $V_{AB}(q'') = \{a_i\}$.

The proof of (2) is analogous. □

A visualization of Lemma 6.3.9 is provided in Figure 6.3.

By the formulas introduced in Definition 6.3.8, we can define more formulas for characterizing subsets of W.

6.3.10. Definition. Recall that φ_{b_0}, φ_{b_1}, and φ_{a_0} are already defined. For each $i \geq 1$, we define

$$\begin{aligned}\varphi_{b_{i+1}} &:= \tau'(\varphi_{b_{i-1}}, \varphi_{b_i}, \varphi_{a_{i-1}}),\\ \varphi_{a_i} &:= \tau''(\varphi_{b_{i-1}}, \varphi_{b_i}, \varphi_{a_{i-1}}).\end{aligned}$$

Moreover, we define the following formulas:

- $\varphi_{c_i'} := \neg\varphi_{AB} \wedge \Diamond(\varphi_{a_{i+1}} \wedge \varphi_{AB}) \wedge \Box\neg(\varphi_{a_{i+2}} \wedge \varphi_{AB})$, for all $i \leq t_0$;

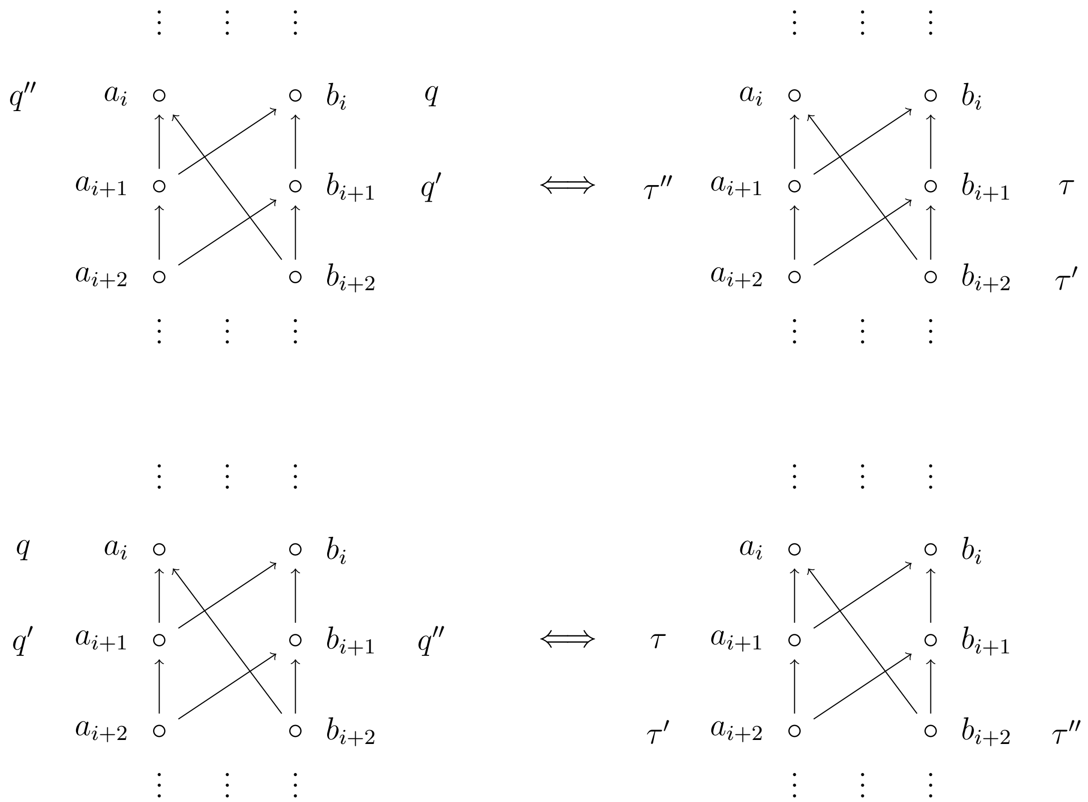

Figure 6.3: Visualization of Lemma 6.3.9

- $\varphi_{c_0} \coloneqq \blacklozenge \varphi_{c'_0} \wedge \neg\varphi_{c'_0} \wedge \Diamond \varphi_{x_1} \wedge \neg\varphi_{x_1}$;
- $\varphi_{c_i} \coloneqq \Diamond \varphi_{c_0} \wedge \blacklozenge \varphi_{c'_i} \wedge \blacksquare \neg\varphi_{c'_{i-1}} \wedge \neg\varphi_{c'_i}$, for all $1 \leq i \leq t_0$;
- $\varphi_{R0} \coloneqq \blacklozenge \varphi_{c_{t_0}} \wedge \blacklozenge \varphi_{AB}$;
- $\varphi_{R1} \coloneqq \Box\Diamond \varphi_{R0} \wedge \neg\varphi_{R0}$;
- $\varphi_{R2} \coloneqq \Diamond \varphi_{R0} \wedge \neg\varphi_{R0} \wedge \neg\varphi_{R1} \wedge \Box\neg\varphi_{x_1} \wedge \neg\varphi_{AB}$;
- $\varphi_{R3} \coloneqq \blacklozenge \varphi_{R2} \wedge \neg\varphi_{R0} \wedge \neg\varphi_{R2}$.

6.3.11. Lemma. *Let V be a valuation in $\mathbb{F}$ such that $V(\varphi^*) \neq \emptyset$. Then*

(1) for all $u \in \{a_i, b_i : i \in \omega\}$, $V(\varphi_u) = \{u\}$;

(2) for all $u \in \{c_i, c'_i : i \leq t_0\}$, $V(\varphi_u) = \{u\}$;

(3) $V(\varphi_{Ri}) = \{z_i : z \in \mathcal{R}each\}$ for all $i \leq 3$.

Proof:
(1) follows from Lemmas 6.3.5 and 6.3.9, and (2) and (3) follow from (1). $\square$

To simulate the Minsky machine $\mathcal{M}$, we need one more family of formulas:

6.3.12. Definition. For each state t of $\mathcal{M}$ and formulas $\pi, \pi', \pi'', \kappa, \kappa', \kappa'' \in \mathsf{Form}_t$, we define the formula $\sigma(t, \pi, \pi', \pi'', \kappa, \kappa', \kappa'')$ as follows:

$$\begin{aligned}\sigma(t, \pi, \pi', \pi'', \kappa, \kappa', \kappa'') :=& \varphi_{R0} \wedge \blacklozenge \varphi_{c_{t+1}} \wedge \blacksquare \neg \varphi_{c_t} \\ & \wedge \blacklozenge(\varphi_{R1} \wedge \blacklozenge(\Diamond \pi \wedge \Diamond \pi'' \wedge \Box \neg \pi') \wedge \blacksquare \neg \pi' \wedge \blacksquare \neg \pi'') \\ & \wedge \blacklozenge(\varphi_{R2} \wedge \blacklozenge \kappa' \wedge \blacksquare \neg \kappa \wedge \blacksquare \neg \kappa'').\end{aligned}$$

We abbreviate $\sigma(t, \pi, \pi', \pi'', \kappa, \kappa', \kappa'')$ as $\sigma(t, \overline{\pi}, \overline{\kappa})$. As it is shown in the following lemma, the formulas σ characterize points of the form $\langle t, k, l \rangle_0$.

6.3.13. Lemma. *Let V be a valuation in $\mathbb{F}$ such that $V(\varphi^*) \neq \varnothing$. Then*

(1) Suppose that there exists $k, l \in \omega$ such that $\langle V(\pi), V(\pi'), V(\pi'') \rangle = \langle \{b_k\}, \{b_{k+1}\}, \{a_k\} \rangle$ and $\langle V(\kappa), V(\kappa'), V(\kappa'') \rangle = \langle \{b_l\}, \{b_{l+1}\}, \{a_l\} \rangle$. Then

(a) $\langle t, k, l \rangle \in \mathcal{R}each$ implies $V(\sigma(t, \overline{\pi}, \overline{\kappa})) = \{\langle t, k, l \rangle_0\}$; and

(b) $\langle t, k, l \rangle \notin \mathcal{R}each$ implies $V(\sigma(t, \overline{\pi}, \overline{\kappa})) = \varnothing$.

(2) Suppose $V(\pi) = \{a_i\}$, $V(\pi') = \{a_{i+1}\}$ and $V(\pi'') = \{b_{i+1}\}$ for some $i \in \omega$. Then $V(\sigma(t, \overline{\pi}, \overline{\kappa})) = \varnothing$.

(3) Suppose $V(\kappa) = \{a_i\}$, $V(\kappa') = \{a_{i+1}\}$ and $V(\kappa'') = \{b_{i+1}\}$ for some $i \in \omega$. Then $V(\sigma(t, \overline{\pi}, \overline{\kappa})) = \varnothing$.

Proof:
For (1), let V be a valuation satisfying the assumptions. We first show that $w = \langle t, k, l \rangle$ whenever $w \in V(\sigma(t, \overline{\pi}, \overline{\kappa}))$. Take any $w \in V(\sigma(t, \overline{\pi}, \overline{\kappa}))$. Since $\mathbb{F}, V, w \models \varphi_{R0}$, by Lemma 6.3.11(3), w is of the form $\langle t', k', l' \rangle_0$. Because $\mathbb{F}, V, w \models \blacklozenge \varphi_{c_{t+1}} \wedge \blacksquare \neg \varphi_{c_t}$, by Lemma 6.3.11(2), we see that $c_{t+1} R w$ and $w \notin R[c_t]$, which entails $t' = t$.

By assumption, we have $V(\Diamond \pi \wedge \Diamond \pi'' \wedge \Box \neg \pi') = \{a_{k+1}\}$. Since $\mathbb{F}, V, w \models \blacklozenge(\varphi_{R1} \wedge \blacklozenge(\Diamond \pi \wedge \Diamond \pi'' \wedge \Box \neg \pi') \wedge \blacksquare \neg \pi' \wedge \blacksquare \neg \pi'')$, we see that $\mathbb{F}, V, \langle t', k', l' \rangle_1 \models \blacklozenge(\Diamond \pi \wedge \Diamond \pi'' \wedge \Box \neg \pi') \wedge \blacksquare \neg \pi''$, which entails that $\langle t', k', l' \rangle_1 \in R[a_{k+1}] \setminus R[a_k]$ and so $k' = k$.

By $\mathbb{F}, V, w \models \blacklozenge(\varphi_{R2} \wedge \blacklozenge \kappa' \wedge \blacksquare \neg \kappa \wedge \blacksquare \neg \kappa'')$, we see that $\mathbb{F}, V, \langle t', k', l' \rangle_2 \models \blacklozenge \kappa' \wedge \blacksquare \neg \kappa$, which entails that $\langle t', k', l' \rangle_2 \in R[b_{l+1}] \setminus R[b_l]$ and so $l' = l$.

Thus, $w = \langle t, k, l \rangle_0$, which implies $V(\sigma(t, \overline{\pi}, \overline{\kappa})) \subseteq W \cap \{\langle t, k, l \rangle_0\}$. It remains to check that $\mathbb{F}, V, \langle t, k, l \rangle_0 \models \sigma(t, \overline{\pi}, \overline{\kappa})$ when $\langle t, k, l \rangle \in \mathcal{R}each$. It is standard to check that (i) $\mathbb{F}, V, \langle t,k,l\rangle_0 \vDash \varphi_{R0} \wedge \blacklozenge\varphi_{c_{t+1}} \wedge \blacksquare\neg\varphi_{c_t}$; (ii) $\mathbb{F}, V, \langle t,k,l\rangle_1 \vDash \varphi_{R1} \wedge \blacklozenge(\Diamond\pi \wedge \Diamond\pi'' \wedge \Box\neg\pi') \wedge \blacksquare\neg\pi' \wedge \blacksquare\neg\pi''$ and (iii) $\mathbb{F}, V, \langle t,k,l\rangle_2 \vDash \varphi_{R2} \wedge \blacklozenge\kappa' \wedge \blacksquare\neg\kappa \wedge \blacksquare\neg\kappa''$. Thus, $\mathbb{F}, V, \langle t,k,l\rangle_0 \vDash \sigma(t,\overline{\pi},\overline{\kappa})$. Hence, (1) holds.

For (2), let V be such that $V(\pi) = \{a_i\}$, $V(\pi') = \{a_{i+1}\}$ and $V(\pi'') = \{b_{i+1}\}$ for some $i \in \omega$. Then we see that $V(\Diamond\pi \wedge \Diamond\pi'' \wedge \Box\neg\pi') = \{b_{i+2}\}$. By the construction of $\mathbb{F}$, for any $z \in \mathcal{R}each$ such that $z_1 \in R[b_{i+2}]$, we have $z_1 \in R[a_i] \subseteq R[a_{i+1}]$. By Lemma 6.3.11, $\mathbb{F}, V \vDash \varphi_{R1} \wedge \blacklozenge(\Diamond\pi \wedge \Diamond\pi'' \wedge \Box\neg\pi') \to \blacklozenge\pi'$, which entails that $V(\sigma(t,\overline{\pi},\overline{\kappa})) = \varnothing$.

For (3), let V be such that $V(\kappa) = \{a_i\}$, $V(\kappa') = \{a_{i+1}\}$ and $V(\kappa'') = \{b_{i+1}\}$ for some $i \in \omega$. Then it is clear that for all $z \in \mathcal{R}each$, if $z_2 \in R[a_{i+1}]$, then $z_2 \in R[b_{i+1}]$. By Lemma 6.3.11, $\mathbb{F}, V \vDash \varphi_{R2} \wedge \blacklozenge\kappa' \to \blacklozenge\kappa''$ and so $V(\sigma(t,\overline{\pi},\overline{\kappa})) = \varnothing$. □

Lemma 6.3.13 is for capturing the point $\langle t,k,l\rangle_0$ via the formula $\sigma(t,\overline{\tau},\overline{\kappa})$. However, as the reader might have noticed, the assumption on the valuation in Lemma 6.3.13(1) is very strong, namely, we assume that $\langle V(\pi), V(\pi'), V(\pi'')\rangle = \langle\{b_k\},\{b_{k+1}\},\{a_k\}\rangle$ and $\langle V(\kappa), V(\kappa'), V(\kappa'')\rangle = \langle\{b_l\},\{b_{l+1}\},\{a_l\}\rangle$. Thus, before introducing the axioms that simulate the instructions of $\mathcal{M}$, we need some extra formulas to characterize the valuations which meet the assumption of Lemma 6.3.13(1).

6.3.14. Definition. Let $\overline{q} = (q, q', q'')$ be a sequence of propositional variables. We define the following formulas:

$$\begin{aligned}
\psi_S(\overline{q}) :=& \nabla^{\leq 7}(q \vee q' \vee q'' \to \varphi_{AB}) \\
& \wedge \Delta^{\leq 7}(\varphi_{AB} \wedge q \wedge \Box\neg q' \wedge \blacklozenge q' \wedge \Box\neg q'' \wedge \blacksquare\neg q'') \\
& \wedge \Delta^{\leq 7}(\varphi_{AB} \wedge q' \wedge \Diamond q \wedge \blacksquare\neg q \wedge \Box\neg q'' \wedge \blacksquare\neg q'') \\
& \wedge \Delta^{\leq 7}(\varphi_{AB} \wedge q'' \wedge \Box\neg q \wedge \blacksquare\neg q \wedge \Box\neg q' \wedge \blacksquare\neg q'), \\
\psi(\overline{q}) :=& \psi_S(\overline{q}) \wedge q, \\
\psi'(\overline{q}) :=& \psi_S(\overline{q}) \wedge q', \\
\psi''(\overline{q}) :=& \psi_S(\overline{q}) \wedge q''.
\end{aligned}$$

Moreover, for each formula $\varphi(\overline{q})$, let $\varphi^+(\overline{q})$ be the formula obtained by applying the substitution $(-)^+ : [\tau(\overline{q})/q, \tau'(\overline{q})/q', \tau''(\overline{q})/q'']$ to $\varphi(\overline{q})$.

Note that φ_{AB} has no occurrence of q, q', q'', so it remains the same under $(-)^+$.

6.3.15. Lemma. *Let V be a valuation in $\mathbb{F}$ such that $V(\varphi^*) \neq \varnothing$. Then the following are equivalent:*

(1) $V(\psi_S(\overline{q})) \neq \varnothing$;

(2) $V(\psi_S(\overline{q})) = W$;

(3) there exists $i \in \omega$ such that one of the following holds:

(a) $V(q) = \{b_i\}$, $V(q') = \{b_{i+1}\}$ and $V(q'') = \{a_i\}$.

(b) $V(q) = \{a_i\}$, $V(q') = \{a_{i+1}\}$ and $V(q'') = \{b_{i+1}\}$.

Proof:
The equivalence between (1) and (2) follows from the fact that $\mathbb{F} \models \Delta^{\leq 8}p \to \Delta^{\leq 7}p$. The implication from (3) to (1) is straightforward. We now prove that (1) implies (3). Suppose $V(\psi_S(\overline{q})) \neq \varnothing$. Then $V(q \vee q' \vee q'') \subseteq \{a_i, b_i : i \in \omega\}$. By Lemma 6.3.5, there exist $u, u', u'' \in \{a_i, b_i : i \in \omega\}$ such that $\mathbb{F}, V, u \models q \wedge \Box\neg q'' \wedge \blacksquare\neg q'' \wedge \Box\neg q'$, $\mathbb{F}, V, u' \models q' \wedge \Box\neg q'' \wedge \blacksquare\neg q''$ and $\mathbb{F}, V, u'' \models q''$. Then we see that $\{u, u'\} \cap (R[u''] \cup R^{-1}[u'']) = \varnothing$ and $u' \notin R[u]$. By the construction of $\mathbb{F}$, either $u'' = a_i$ or $u'' = b_{i+1}$ for some $i \in \omega$. Suppose $u'' = a_i$. Then we see that $u = b_i$ and $u' = b_{i+1}$. Since $\mathbb{F}, V, a_i \models \Box\neg q \wedge \blacksquare\neg q$ and $\mathbb{F}, V, b_{i+1} \models \blacksquare\neg q$, we have $V(q) = \{b_i\}$. It is also straightforward to check that $V(q') = \{b_{i+1}\}$ and $V(q'') = \{a_i\}$. Thus, (a) holds. Similarly, if $u'' = b_{i+1}$, then (b) holds. Hence, (1) implies (3). $\Box$

In other words, Lemma 6.3.15 says that $\psi_S(\overline{q})$ is satisfiable in $(\mathbb{F}, V)$ if and only if $V(\overline{q})$ forms a triangle as depicted in Figure 6.4.

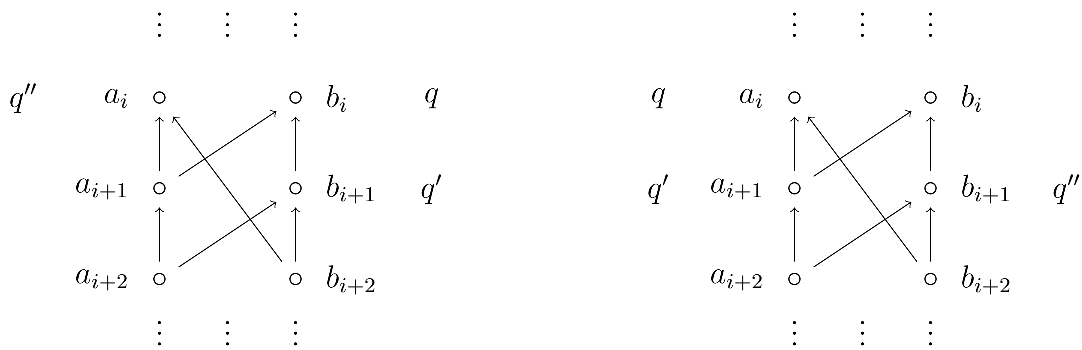

Figure 6.4: Possible cases when $\psi_S(\overline{q})$ is satisfied

6.3.16. LEMMA. *Let V be a valuation such that $V(\varphi^*) \neq \varnothing$ and one of $V(\psi_S(\overline{q}))$ and $V(\psi_S^+(\overline{q}))$ is non-empty. Then there exists $i \in \omega$ such that one of the following holds:*

(1) $V(\psi) = \{b_i\}$, $V(\psi') = V(\psi^+) = \{b_{i+1}\}$, $V(\psi'') = \{a_i\}$, $V(\psi'^+) = \{b_{i+2}\}$ and $V(\psi''^+) = \{a_{i+1}\}$.

(2) $V(\psi) = \{a_i\}$, $V(\psi') = V(\psi^+) = \{a_{i+1}\}$, $V(\psi'') = \{b_{i+1}\}$, $V(\psi'^+) = \{a_{i+2}\}$ *and* $V(\psi''^+) = \{b_{i+2}\}$.

Proof:
Suppose $V(\psi_S(\overline{q})) \neq \varnothing$. By Lemma 6.3.15, one of the following cases holds:
Case 1.1. $V(q) = \{b_i\}$, $V(q') = \{b_{i+1}\}$ and $V(q'') = \{a_i\}$ for some $i \in \omega$. By Lemma 6.3.9, we have $V(\tau) = \{b_{i+1}\}$, $V(\tau') = \{b_{i+2}\}$ and $V(\tau'') = \{a_{i+1}\}$. By Lemma 6.3.15, $V(\psi_S^+(\overline{q})) = V(\psi_S(\overline{q})) = W$, which implies (1).
Case 1.2. $V(q) = \{a_i\}$, $V(q') = \{a_{i+1}\}$ and $V(q'') = \{b_{i+1}\}$ for some $i \in \omega$. Similarly, by Lemmas 6.3.9 and 6.3.15, (2) holds.

Suppose $V(\psi_S^+(\overline{q})) = V(\psi_S(\tau, \tau', \tau'')) \neq \varnothing$. By the proof of Lemma 6.3.15, one of the following holds:
Case 2.1. $V(\tau) = \{b_j\}$, $V(\tau') = \{b_{j+1}\}$ and $V(\tau'') = \{a_j\}$ for some $j \in \omega$. By $\mathbb{F}, V, a_j \vDash \tau''$, we have $\mathbb{F}, V, a_j \vDash \Diamond q'' \wedge \neg q''$, which entails $j \neq 0$. Let $i = j - 1$. Then $V(\tau) = \{b_{i+1}\}$, $V(\tau') = \{b_{i+2}\}$ and $V(\tau'') = \{a_{i+1}\}$. By Lemma 6.3.9, $V(q) = \{b_i\}$, $V(q') = \{b_{i+1}\}$ and $V(q'') = \{a_i\}$. Thus, (1) holds.
Case 2.2. $V(\tau) = \{a_j\}$, $V(\tau') = \{a_{j+1}\}$ and $V(\tau'') = \{b_{j+1}\}$ for some $j \in \omega$. By $\mathbb{F}, V, b_{j+1} \vDash \tau''$, we have $\mathbb{F}, V, b_{j+1} \vDash \Diamond(q \wedge \blacklozenge q')$. Since $V(q') = V(\tau) = \{a_j\}$ and $a_0 \notin R^{-1}[R[b_1]]$, we see that $j \neq 0$. Let $i = j - 1$. Then $V(\tau) = \{a_{i+1}\}$, $V(\tau') = \{a_{i+2}\}$ and $V(\tau'') = \{b_{i+2}\}$. By Lemma 6.3.9, we have $V(q) = \{a_i\}$, $V(q') = \{a_{i+1}\}$ and $V(q'') = \{b_{i+1}\}$. By Lemma 6.3.15, $V(\psi_S^+(\overline{q})) = V(\psi_S(\overline{q})) = W$, which implies (2). □

For $i \in \omega$, let $[i/\overline{q}]$ be the substitution $[\varphi_{b_i}/q, \varphi_{b_{i+1}}/q', \varphi_{a_i}/q'']$. We also write $\psi(i)$ for $\psi[i/\overline{q}]$. Moreover, as usual, we abbreviate $(\psi(\overline{q}), \psi'(\overline{q}), \psi''(\overline{q}))$ as $\overline{\psi}(\overline{q})$. These notations apply to other formulas as well. For example, under our notation, $\sigma(t, \overline{\psi}(\overline{q}), \overline{\psi}(0))$ is the substitution result of

$$\sigma(t, \psi(\overline{q}), \psi'(\overline{q}), \psi''(\overline{q}), \psi(\overline{r}), \psi'(\overline{r}), \psi''(\overline{r}))\ [\varphi_{b_0}/r, \varphi_{b_1}/r', \varphi_{a_0}/r''].$$

Now we are ready to introduce the axioms characterizing the instructions in $\mathcal{M}$.

6.3.17. Definition. With each instruction I in $\mathcal{M}$ we associate a formula AxI by taking:

- If $I = t \to \langle t', 1, 0\rangle$, then

$$AxI := \varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi}(\overline{q}), \overline{\psi}(\overline{r})) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(t', \overline{\psi^+}(\overline{q}), \overline{\psi}(\overline{r})).$$

- If $I = t \to \langle t', 0, 1\rangle$, then

$$AxI := \varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi}(\overline{q}), \overline{\psi}(\overline{r})) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(t', \overline{\psi}(\overline{q}), \overline{\psi^+}(\overline{r})).$$

- If $I = t \to \langle t', -1, 0\rangle(\langle t'', 0, 0\rangle)$, then

$$\begin{aligned} AxI :=&(\varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi^+}(\overline{q}), \overline{\psi}(\overline{r})) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(t', \overline{\psi}(\overline{q}), \overline{\psi}(\overline{r}))) \\ &\wedge (\varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi}(0), \overline{\psi}(\overline{r})) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(t'', \overline{\psi}(0), \overline{\psi}(\overline{r}))). \end{aligned}$$

- If $I = t \to \langle t', 0, -1\rangle(\langle t'', 0, 0\rangle)$, then

$$\begin{aligned} AxI :=&(\varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi}(\overline{q}), \overline{\psi^+}(\overline{r})) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(t', \overline{\psi}(\overline{q}), \overline{\psi}(\overline{r}))) \\ &\wedge (\varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi}(\overline{q}), \overline{\psi}(0)) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(t'', \overline{\psi}(\overline{q}), \overline{\psi}(0))). \end{aligned}$$

Let $AxM := \bigwedge_{I \in \mathcal{M}} AxI$.

6.3.18. LEMMA. $\mathbb{F} \vDash AxM$.

Proof:
It suffices to show that $\mathbb{F} \vDash AxI$ for each instruction $I \in \mathcal{M}$. We show the cases where $I = t \to \langle t', 1, 0\rangle$ and $I = t \to \langle t', -1, 0\rangle(\langle t'', 0, 0\rangle)$.

For the case $I = t \to \langle t', 1, 0\rangle$, let V be a valuation in $\mathbb{F}$ and $w \in W$. Suppose

$$\mathbb{F}, V, w \vDash \varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi}(\overline{q}), \overline{\psi}(\overline{r})).$$

Then, $V(\varphi^*) \neq \emptyset$ and $V(\sigma(t, \overline{\psi}(\overline{q}), \overline{\psi}(\overline{r}))) \neq \emptyset$. By the construction of σ, we have $V(\overline{\psi}(\overline{q})) \neq \varnothing$ and so $V(\psi_S(\overline{q})) \neq \varnothing$. By Lemma 6.3.16, either (i) $V(\psi'(\overline{q})) = \{b_{k+1}\}$ and $V(\psi''(\overline{q})) = \{a_k\}$ or (ii) $V(\psi'(\overline{q})) = \{a_{k+1}\}$ and $V(\psi''(\overline{q})) = \{b_{k+1}\}$ for some $k \in \omega$. It follows from Lemma 6.3.13(2) that (ii) is impossible. Thus, by Lemma 6.3.16 again, we have $V(\psi(\overline{q})) = \{b_k\}$, $V(\psi'(\overline{q})) = V(\psi^+(\overline{q})) = \{b_{k+1}\}$, $V(\psi''(\overline{q})) = \{a_k\}$, $V(\psi'^+(\overline{q})) = \{b_{k+2}\}$ and $V(\psi''^+(\overline{q})) = \{a_{k+1}\}$. Similarly, by Lemmas 6.3.16 and 6.3.13(3), we see that $V(\psi(\overline{r})) = \{b_l\}$, $V(\psi'(\overline{r})) = V(\psi^+(\overline{r})) = \{b_{l+1}\}$, $V(\psi''(\overline{r})) = \{a_l\}$, $V(\psi'^+(\overline{r})) = \{b_{l+2}\}$ and $V(\psi''^+(\overline{r})) = \{a_{l+1}\}$ for some $l \in \omega$.

Thus, it follows from Lemma 6.3.13(1) that

$$V(\sigma(t, \overline{\psi}(\overline{q}), \overline{\psi}(\overline{r}))) = \{\langle t, k, l\rangle_0\}$$

This implies that $\langle t, k, l\rangle_0 \in W$, which yields $\langle t', k+1, l\rangle_0 \in W$ by applying $I = t \to \langle t', 1, 0\rangle$. Again by Lemma 6.3.13(1), we have

$$V(\sigma(t', \overline{\psi^+}(\overline{q}), \overline{\psi}(\overline{r}))) = \{\langle t', k+1, l\rangle_0\} \neq \emptyset.$$

Thus,

$$\mathbb{F}, V, w \vDash \varphi^* \wedge \Delta^{\leq 7}\sigma(t', \overline{\psi^+}(\overline{q}), \overline{\psi}(\overline{r})),$$

which entails that $\mathbb{F}, V, w \vDash AxI$.

For the case $I = t \to \langle t', -1, 0\rangle(\langle t'', 0, 0\rangle)$, let V be a valuation in $\mathbb{F}$ and $w \in W$. Then we have two subcases.

First, suppose that

$$\mathbb{F}, V, w \vDash \varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi^+}(\overline{q}), \overline{\psi}(\overline{r})).$$

Then, $V(\varphi^*) \neq \emptyset$ and $V(\sigma(t, \overline{\psi^+}(\overline{q}), \overline{\psi}(\overline{r}))) \neq \emptyset$. As in the previous case, by a similar argument using Lemmas 6.3.16 and 6.3.13, we have $V(\psi(\overline{q})) = \{b_k\}$, $V(\psi'(\overline{q})) = V(\psi^+(\overline{q})) = \{b_{k+1}\}$, $V(\psi''(\overline{q})) = \{a_k\}$, $V(\psi'^+(\overline{q})) = \{b_{k+2}\}$ and $V(\psi''^+(\overline{q})) = \{a_{k+1}\}$ for some $k \in \omega$, and $V(\psi(\overline{r})) = \{b_l\}$, $V(\psi'(\overline{r})) = V(\psi^+(\overline{r})) = \{b_{l+1}\}$, $V(\psi''(\overline{r})) = \{a_l\}$, $V(\psi'^+(\overline{r})) = \{b_{l+2}\}$ and $V(\psi''^+(\overline{r})) = \{a_{l+1}\}$ for some $l \in \omega$.

Thus, it follows from Lemma 6.3.13(1) that

$$V(\sigma(t, \overline{\psi^+}(\overline{q}), \overline{\psi}(\overline{r}))) = \{\langle t, k+1, l\rangle_0\}$$

This implies that $\langle t, k+1, l\rangle_0 \in W$, which yields $\langle t', k, l\rangle_0 \in W$ by applying I. Again by Lemma 6.3.13(1), we have

$$V(\sigma(t', \overline{\psi}(\overline{q}), \overline{\psi}(\overline{r}))) = \{\langle t', k, l\rangle_0\} \neq \emptyset.$$

Thus,

$$\mathbb{F}, V, w \vDash \varphi^* \wedge \Delta^{\leq 7}\sigma(t', \overline{\psi}(\overline{q}), \overline{\psi}(\overline{r})).$$

Next, suppose that

$$\mathbb{F}, V, w \vDash \varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi}(0), \overline{\psi}(\overline{r})).$$

Then, $V(\varphi^*) \neq \emptyset$ and $V(\sigma(t, \overline{\psi}(0), \overline{\psi}(\overline{r}))) \neq \emptyset$. As in the previous case, by a similar argument using Lemmas 6.3.16 and 6.3.13, we have $V(\psi(\overline{r})) = \{b_l\}$, $V(\psi'(\overline{r})) = V(\psi^+(\overline{r})) = \{b_{l+1}\}$, $V(\psi''(\overline{r})) = \{a_l\}$, $V(\psi'^+(\overline{r})) = \{b_{l+2}\}$ and $V(\psi''^+(\overline{r})) = \{a_{l+1}\}$ for some $l \in \omega$.

Thus, it follows from Lemma 6.3.13(1) that

$$V(\sigma(t, \overline{\psi}(0), \overline{\psi}(\overline{r}))) = \{\langle t, 0, l\rangle_0\}$$

This implies that $\langle t, 0, l\rangle \in \mathcal{R}each$. By applying instruction I, we see that $\langle t'', 0, l\rangle \in \mathcal{R}each$. Again by Lemma 6.3.13(1), we have

$$V(\sigma(t'', \overline{\psi}(0), \overline{\psi}(\overline{r}))) = \{\langle t'', 0, l\rangle_0\} \neq \emptyset.$$

Thus,

$$\mathbb{F}, V, w \vDash \varphi^* \wedge \Delta^{\leq 7}\sigma(t'', \overline{\psi}(0), \overline{\psi}(\overline{r})),$$

which entails that $\mathbb{F}, V, w \vDash AxI$. Since w and V are arbitrarily chosen, $\mathbb{F} \vDash AxI$. The other two cases follow analogously, and we conclude $\mathbb{F} \vDash AxI$. □

Now we turn to syntactic analysis of AxM. Recall that $\psi(i)$ denotes the substitution result of $\psi(q, q', q'')[\varphi_{b_i}/q, \varphi_{b_{i+1}}/q', \varphi_{a_i}/q'']$ and similar for other formulas.

6.3.19. LEMMA. *For any* $i \in \omega$*,*

$$\begin{aligned}\psi^+(i) &= \psi(i+1),\\ \psi'^+(i) &= \psi'(i+1),\\ \psi''^+(i) &= \psi''(i+1).\end{aligned}$$

Proof:
We only show the first case. Note that φ_{AB} has no occurrence of q, q', q''. By Definitions 6.3.8 and 6.3.10, we have

$$\begin{aligned}\tau(i) &= \varphi_{b_{i+1}},\\ \tau'(i) &= \varphi_{b_{i+2}},\\ \tau''(i) &= \varphi_{a_{i+1}}.\end{aligned}$$

So,

$$\begin{aligned}\psi^+(i) &= (\psi_S(\tau(\overline{q}), \tau'(\overline{q}), \tau''(\overline{q})) \wedge \tau(\overline{q}))[i/\overline{q}]\\ &= \psi_S[\tau(i), \tau'(i), \tau''(i)] \wedge \tau(i)\\ &= \psi_S(\varphi_{b_{i+1}}, \varphi_{b_{i+2}}, \varphi_{a_{i+1}}) \wedge \varphi_{b_{i+1}}\\ &= \psi(\varphi_{b_{i+1}}, \varphi_{b_{i+2}}, \varphi_{a_{i+1}})\\ &= \psi(i+1).\end{aligned}$$

The other two cases follow analogously. □

By Lemma 6.3.19, we can prove the following lemma.

6.3.20. LEMMA. *For any configuration* $\langle t, k, l\rangle \in \mathcal{R}each$*,*

$$\varphi^* \wedge \Delta^{\leq 7}\sigma(s, \overline{\psi}(n), \overline{\psi}(m)) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi}(k), \overline{\psi}(l)) \in \mathsf{S4}_t \oplus AxM.$$

Proof:
We prove by induction on the length of the computation of $\mathcal{M}$. The base case, where $\langle t, k, l\rangle = \langle s, n, m\rangle$, is clear since the corresponding formula is a tautology. Consider the computation of the form $\langle s, n, m\rangle \rightsquigarrow \langle t, k, l\rangle \to \langle \tilde{t}, \tilde{k}, \tilde{l}\rangle$, where I is the last instruction applied. By induction hypothesis, we have

$$\varphi^* \wedge \Delta^{\leq 7}\sigma(s, \overline{\psi}(n), \overline{\psi}(m)) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi}(k), \overline{\psi}(l)) \in \mathsf{S4}_t \oplus AxM.$$

So, it suffices to show

$$\varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi}(k), \overline{\psi}(l)) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(\tilde{t}, \overline{\psi}(\tilde{k}), \overline{\psi}(\tilde{l})) \in \mathsf{S4}_t \oplus AxM.$$

We distinguish cases according to the form of I.

- $I = t \to \langle t', 1, 0\rangle$. Then $\langle \tilde{t}, \tilde{k}, \tilde{l}\rangle = \langle t', k+1, l\rangle$. From AxI we have

$$\varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi}(\overline{q}), \overline{\psi}(\overline{r})) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(t', \overline{\psi^+}(\overline{q}), \overline{\psi}(\overline{r})) \in \mathsf{S4}_t \oplus AxM.$$

  Applying the substitution $[k/\overline{q}, l/\overline{r}]$, with Lemma 6.3.19, we obtain

$$\varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi}(k), \overline{\psi}(l)) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(t', \overline{\psi}(k+1), \overline{\psi}(l)) \in \mathsf{S4}_t \oplus AxM.$$

- $I = t \to \langle t', 0, 1\rangle$. This case is similar to the previous one.
- $I = t \to \langle t', -1, 0\rangle(\langle t'', 0, 0\rangle)$. We further distinguish cases depending on whether $k = 0$.

  * $k \geq 1$. Then $\langle \tilde{t}, \tilde{k}, \tilde{l}\rangle = \langle t', k-1, l\rangle$. From AxI we have

$$\varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi^+}(\overline{q}), \overline{\psi}(\overline{r})) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(t', \overline{\psi}(\overline{q}), \overline{\psi}(\overline{r})) \in \mathsf{S4}_t \oplus AxM.$$

    Applying the substitution $[k-1/\overline{q}, l/\overline{r}]$, with Lemma 6.3.19, we obtain

$$\varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi}(k), \overline{\psi}(l)) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(t', \overline{\psi}(k-1), \overline{\psi}(l)) \in \mathsf{S4}_t \oplus AxM.$$

  * $k = 0$. Then $\langle \tilde{t}, \tilde{k}, \tilde{l}\rangle = \langle t'', 0, l\rangle$. From AxI we have

$$\varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi}(0), \overline{\psi}(\overline{r})) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(t'', \overline{\psi}(0), \overline{\psi}(\overline{r})) \in \mathsf{S4}_t \oplus AxM.$$

    Applying the substitution $[l/\overline{r}]$, we obtain

$$\varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi}(0), \overline{\psi}(l)) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(t'', \overline{\psi}(0), \overline{\psi}(l)) \in \mathsf{S4}_t \oplus AxM.$$

- $I = t \to \langle t', 0, -1\rangle(\langle t'', 0, 0\rangle)$. The proof is analogous.

Thus, we conclude our induction. □

Reduction

Next, we construct the desired reduction. Similar to the construction in Section 6.2, we first define tense logics $L(t, k, l)$ for all configuration $\langle t, k, l\rangle$.

6.3.21. Definition. For each configuration $\langle t, k, l\rangle$, we define

$$\begin{aligned} L(t,k,l) := \ & \mathsf{S4}_t \oplus \mathsf{bw}_5^- \oplus \mathsf{br}_7 \oplus \mathsf{grz}^+ \oplus \mathsf{grz}^- \\ & \oplus AxM && (6.1) \\ & \oplus (\varphi^* \wedge \Delta^{\leq 7}\sigma(s, \overline{\psi}(n), \overline{\psi}(m)) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi}(k), \overline{\psi}(l))) \to \neg\varphi^* && (6.2) \\ & \oplus \varphi^* \to \psi_S(\varphi_{b_0}, \varphi_{b_1}, \varphi_{a_0}) && (6.3) \\ & \oplus \varphi^* \to (\psi_S(\overline{q}) \to \psi_S^+(\overline{q})). && (6.4) \end{aligned}$$

The intuitive meaning of the axioms of $L(t,k,l)$ is the following.

- (6.1) is an axiom for simulating the instructions in $\mathcal{M}$,
- (6.2) relates the logic $L(t,k,l)$ to the configuration $\langle t,k,l\rangle$,
- (6.3) finds three points each resembling a_0, b_0, b_1 in $\mathbb{F}$,
- (6.4) forces an infinite descending chain from the three points provided by (6.3), which would imply Kripke incompleteness.

6.3.22. Lemma. *For any* $\langle t,k,l\rangle \notin \mathcal{R}each$*, we have* $\mathbb{F} \vDash L(t,k,l)$.

Proof:
It is clear that $\mathbb{F} \vDash \mathsf{S4}_t \wedge \mathsf{grz}^+ \wedge \mathsf{br}_7$. Note that the admissible sets of $\mathbb{F}$ are finite and cofinite sets with respect to $AB = \{a_i, b_i : i \in \omega\}$. Thus, $\mathbb{F} \vDash \mathsf{grz}^-$. To show that $\mathbb{F} \vDash \mathsf{bw}_5^-$, it suffices to notice that there exists no $w \in W$ such that $R^{-1}[w]$ contains an anti-chain of length greater than 5. By Lemma 6.3.18, $\mathbb{F} \vDash AxM$. Suppose $\mathbb{F}, V, w \vDash \varphi^*$ for some valuation V on $\mathbb{F}$ and $w \in W$. Then $V(\varphi^*) \neq \emptyset$. By Lemma 6.3.5, we obtain $V(\varphi_{b_0}) = \{b_0\}$, $V(\varphi_{b_1}) = \{b_1\}$ and $V(\varphi_{a_0}) = \{a_0\}$. By Lemma 6.3.15, $V(\psi_S(\varphi_{b_0}, \varphi_{b_1}, \varphi_{a_0})) = W$ and so $\mathbb{F}, V, w \vDash \psi_S(\varphi_{b_0}, \varphi_{b_1}, \varphi_{a_0})$. Thus, $\mathbb{F} \vDash$ (6.3). Similarly, by Lemma 6.3.16, we have $\mathbb{F} \vDash$ (6.4).

Now it remains to show that $\mathbb{F} \vDash$ (6.2). Take any $\langle t,k,l\rangle \notin \mathcal{R}each$. Then the point $\langle t,k,l\rangle_0$ does not exist in $\mathbb{F}$, while $\langle s,n,m\rangle_0$ always exists since $\langle s,n,m\rangle \rightsquigarrow \langle s,n,m\rangle$. We show that the contrapositive of (6.2) is valid in $\mathbb{F}$.

Suppose $\mathbb{F}, V, w \vDash \varphi^*$ for some valuation V on $\mathbb{F}$ and $w \in W$. Then, $V(\varphi^*) \neq \emptyset$. It follows from Lemmas 6.3.5, 6.3.11, and 6.3.13 that

$$\mathbb{F}, V, \langle s,n,m\rangle \vDash \sigma(s, \overline{\psi}(n), \overline{\psi}(m))$$

and

$$V(\sigma(t, \overline{\psi}(k), \overline{\psi}(l))) = \emptyset.$$

Thus, $\mathbb{F}, V, w \nvDash \varphi^* \wedge \Delta^{\leq 7}\sigma(s, \overline{\psi}(n), \overline{\psi}(m)) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi}(k), \overline{\psi}(l))$, which entails that $\mathbb{F}, V, w \vDash$ (6.2). Since V and w are arbitrarily chosen, $\mathbb{F} \vDash$ (6.2). □

Using this lemma, we show that $L(t,k,l)$ is Kripke incomplete (Lemma 6.3.23) and undecidable (Lemma 6.3.24) for any configuration $\langle t,k,l\rangle \notin \mathcal{R}each$.

6.3.23. Lemma. *For any* $\langle t,k,l\rangle \notin \mathcal{R}each$*,* $L(t,k,l)$ *is Kripke incomplete.*

Proof:
Let $\langle t,k,l\rangle$ be a configuration such that $\langle t,k,l\rangle \notin \mathcal{R}each$. Then $\mathbb{F} \vDash L(t,k,l)$ by Lemma 6.3.22. Since $\mathbb{F} \nvDash \neg\varphi^*$ by Lemma 6.3.6, $\neg\varphi^* \notin L(t,k,l)$. So, it suffices to show $\neg\varphi^* \in \mathsf{Log}(\mathsf{Fr}(L(t,k,l)))$.

Let $\mathfrak{F} = (U, S)$ be a rooted Kripke frame such that $\mathfrak{F} \models L(t, k, l)$. Suppose that $\mathfrak{F} \not\models \neg\varphi^*$ for a contradiction. Then $\mathfrak{F}, V, w \models \varphi^*$ for some valuation V in $\mathfrak{F}$ and $w \in U$. By (6.3), $\mathfrak{F}, V, w \models \psi_S(\varphi_{b_0}, \varphi_{b_1}, \varphi_{a_0})$. Let $\mathfrak{M} = (\mathfrak{F}, V)$. Then there exists $w_0, u_0 \in U$ such that

$$\mathfrak{M}, w_0 \models \varphi_{a_0} \wedge \varphi_{AB} \wedge \Box\neg\varphi_{b_0} \wedge \blacksquare\neg\varphi_{b_0} \wedge \Box\neg\varphi_{b_1} \wedge \blacksquare\neg\varphi_{b_1},$$
$$\mathfrak{M}, u_0 \models \varphi_{b_0} \wedge \varphi_{AB} \wedge \Box\neg\varphi_{a_0} \wedge \blacksquare\neg\varphi_{a_0} \wedge \Box\neg\varphi_{b_1} \wedge \blacklozenge\varphi_{b_1}.$$

By $\mathfrak{M}, u_0 \models \blacklozenge\varphi_{b_1}$, there exists $u_1 \in S^{-1}[u_0]$ such that $\mathfrak{M}, u_1 \models \varphi_{b_1}$. Let $U_1 = \{w_0, u_0, u_1\}$. Then clearly, $S{\restriction}U_1 = \{(w_0, w_0), (u_0, u_0), (u_1, u_1), (u_1, u_0)\}$. Note that $\mathfrak{M}, w \models \nabla^{\leq 7}(\varphi_{b_1} \to \varphi_{AB})$ and $\mathfrak{F} \models \mathsf{br}_7$. Thus, $\mathfrak{M}, u_1 \models \varphi_{AB}$. Note that since q, q', q'' do not occur in any of φ^*, φ_{a_0}, φ_{b_0}, φ_{b_1} and φ_{AB} (see Definition 6.3.4), we may vary the value of q, q', q'' while keeping the value of these formulas. For two valuations V_1 and V_2, we write $V_1 \equiv V_2$ if they agree on propositional variables except for q, q', q''.

Let $V' \equiv V$ be a valuation such that $V'(q) = \{w_0\}$, $V'(q') = \{u_0\}$ and $V'(q'') = \{u_1\}$. Let $\mathfrak{M}' = (\mathfrak{F}, V')$. Since $S{\restriction}U_1 = \{(w_0, w_0), (u_0, u_0), (u_1, u_1), (u_1, u_0)\}$ and $V' \equiv V$, we see that $\mathfrak{M}', w \models \varphi^* \wedge \psi_S(\overline{q})$. By $\mathfrak{F} \models (6.4)$, we obtain that $\mathfrak{M}', w \models \psi_S^+(\overline{q})$ and so $\mathfrak{M}', w \models \psi_S(\overline{\tau})$. By the construction of ψ_S, there exist w_1 and u_2 such that $\mathfrak{M}', w_1 \models \tau''(\overline{q})$ and $\mathfrak{M}', u_2 \models \tau'(\overline{q})$. Let $U_2 = \{w_0, u_0, u_1, w_1, u_2\}$. By the definition of τ' and τ'', it is straightforward to check that (i) $w_1 \neq u_2$, (ii) $w_1, u_2 \notin U_1$, and (iii) $S{\restriction}U_2$ is the transitive-reflexive closure of $\{(u_2, u_1), (u_1, u_0), (w_1, w_0), (u_2, w_0)\}$. Thus, φ_{AB} is satisfied at w_1, u_1, u_2 in $\mathfrak{M}'$.

Thus, by repeating the argument above (e.g., consider the valuation $V'' \equiv V$ such that $V'(q) = \{w_1\}$, $V'(q') = \{u_1\}$ and $V'(q'') = \{u_2\}$ for the next step), we obtain two infinite descending chains $\{w_i : i \in \omega\}$ and $\{u_i : i \in \omega\}$ in $\mathfrak{F}$. Therefore, by Proposition 2.2.46, $\mathfrak{F} \not\models \mathsf{grz}^-$, which contradicts the assumption that $\mathfrak{F} \models L(t, k, l)$. Hence, $\neg\varphi^* \in \mathsf{Log}(\mathsf{Fr}(L(t, k, l)))$, and we conclude that $L(t, k, l)$ is Kripke incomplete. ☐

Figure 6.5 illustrates the proof of Lemma 6.3.23.

6.3.24. Lemma. *For any $\langle t, k, l\rangle \notin \mathcal{R}each$, the logic $L(t, k, l)$ is undecidable.*

Proof:
Take any $\langle t, k, l\rangle \notin \mathcal{R}each$. Then $\mathbb{F} \models L(t, k, l)$ by Lemma 6.3.22. We reduce the Minsky machine problem $\mathcal{R}each$ to the decision problem of $L(t, k, l)$. Let $\langle t', k', l'\rangle$ be an arbitrary configuration. If $\langle t', k', l'\rangle \in \mathcal{R}each$, then by Lemma 6.3.20 we see that

$$\varphi^* \wedge \Delta^{\leq 7}\sigma(s, \overline{\psi}(n), \overline{\psi}(m)) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(t', \overline{\psi}(k'), \overline{\psi}(l')) \in L(t, k, l).$$

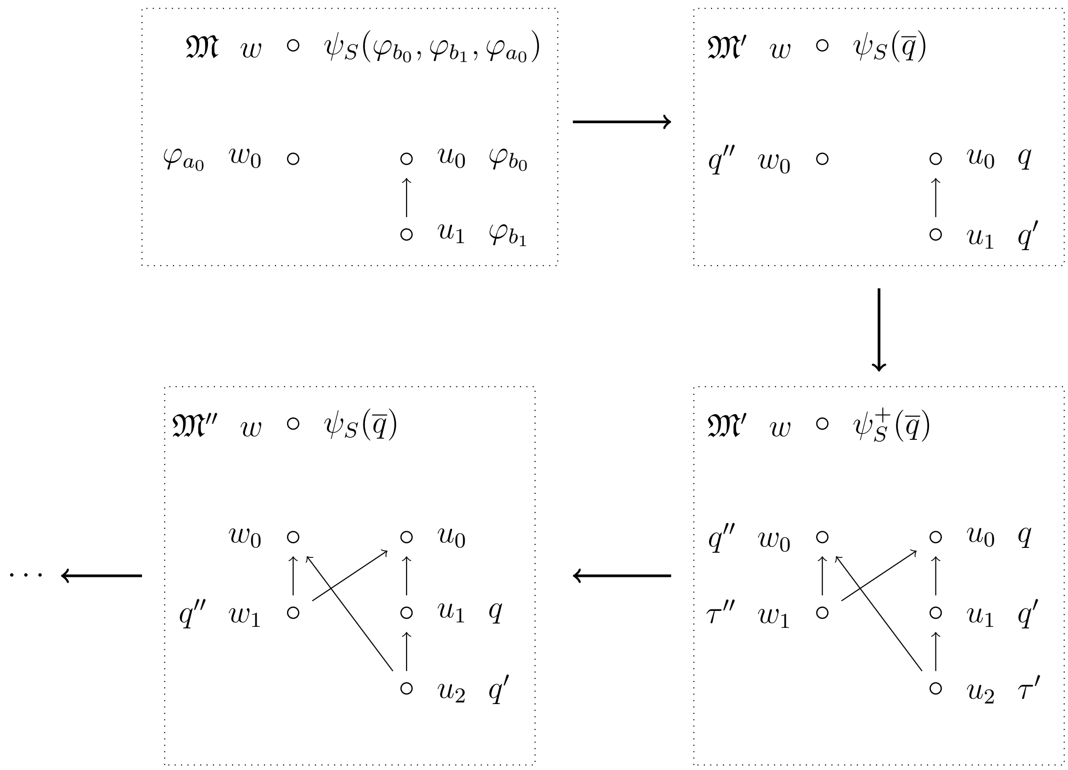

Figure 6.5: Proof sketch of Lemma 6.3.23

Suppose $\langle t', k', l' \rangle \notin \mathcal{R}each$. By Lemma 6.3.6, there is a valuation V on $\mathbb{F}$ such that $V(\varphi^*) \neq \emptyset$. Then similar to the proof of Lemma 6.3.22, it follows from Lemmas 6.3.5, 6.3.11, and 6.3.13 that

$$\mathbb{F}, V, \langle s, n, m \rangle \vDash \sigma(s, \overline{\psi}(n), \overline{\psi}(m))$$

and

$$V(\sigma(t', \overline{\psi}(k'), \overline{\psi}(l'))) = \emptyset.$$

By Lemma 6.3.22, $\mathbb{F} \vDash L(t, k, l)$, which entails that

$$\varphi^* \wedge \Delta^{\leq 7}\sigma(s, \overline{\psi}(n), \overline{\psi}(m)) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(t', \overline{\psi}(k'), \overline{\psi}(l')) \notin L(t, k, l).$$

Therefore, since $\mathcal{R}each$ is undecidable, $L(t, k, l)$ is also undecidable. □

6.3.25. Lemma. *For any $\langle t, k, l \rangle \in \mathcal{R}each$, the following holds:*

(1) $L(t, k, l) = \mathsf{S4}_t \oplus \{\mathsf{grz}^+, \mathsf{grz}^-, \mathsf{bw}_5^-, \mathsf{br}_7, \neg\varphi^*\}$.

(2) $L(t, k, l)$ *is tabular,*

(3) $L(t,k,l)$ *is locally tabular,*

(4) $L(t,k,l)$ *has the FMP,*

(5) $L(t,k,l)$ *is decidable,*

(6) $L(t,k,l)$ *is canonical,*

(7) $L(t,k,l)$ *is Kripke complete.*

Proof:
Take any $\langle t,k,l\rangle \in \mathcal{R}each$. By Lemma 6.3.20 and (6.2), $\neg\varphi^* \in L(t,k,l)$. Thus, (1) holds. For (2), by the definition of φ^*, we see that $\mathsf{bd}_2, \mathsf{bw}_2^+ \in L(t,k,l)$. Note that $\mathsf{bw}_5^-, \mathsf{br}_7, \mathsf{grz}^+ \in L(t,k,l)$. By Corollary 4.2.13, $L(t,k,l)$ is tabular. (3) - (7) follow immediately from (2). □

Finally, we are ready to prove our main theorem in this section:

6.3.26. Theorem. *The following sets are undecidable:*

(1) $\{\varphi \in \mathsf{Form}_t : \mathsf{S4}_t \oplus \varphi$ *is tabular*$\}$,

(2) $\{\varphi \in \mathsf{Form}_t : \mathsf{S4}_t \oplus \varphi$ *is locally tabular*$\}$,

(3) $\{\varphi \in \mathsf{Form}_t : \mathsf{S4}_t \oplus \varphi$ *has the FMP*$\}$,

(4) $\{\varphi \in \mathsf{Form}_t : \mathsf{S4}_t \oplus \varphi$ *is decidable*$\}$,

(5) $\{\varphi \in \mathsf{Form}_t : \mathsf{S4}_t \oplus \varphi$ *is canonical*$\}$,

(6) $\{\varphi \in \mathsf{Form}_t : \mathsf{S4}_t \oplus \varphi$ *is Kripke complete*$\}$.

Proof:
For (1), note that the reduction $\langle t,k,l\rangle \mapsto L(t,k,l)$ satisfies the following:

- If $\langle t,k,l\rangle \in \mathcal{R}each$, then $L(t,k,l)$ is tabular by Lemma 6.3.25.
- If $\langle t,k,l\rangle \notin \mathcal{R}each$, then $L(t,k,l) \notin \mathsf{KC}$ by Lemma 6.3.23, which entails that $L(t,k,l)$ is non-tabular.

Thus, we obtain a reduction from the set $\mathcal{R}each$, which is undecidable, to the set $\{\varphi \in \mathsf{Form}_t : \mathsf{S4}_t \oplus \varphi$ is tabular$\}$, which is therefore also undecidable. Similarly, (2) - (6) can be proved by applying Lemmas 6.3.25, 6.3.24 and 6.3.23. □

6.3.27. THEOREM. *There exist countably many tabular tense logics L for which the coincidence problem is undecidable, i.e., the following set is undecidable:*

$$\{\varphi \in \mathsf{Form}_t : \mathsf{S4}_t \oplus \varphi = L\}.$$

Proof:
For each $d \in \omega$, let $L_d = \mathsf{S4}_t \oplus \{\mathsf{grz}^+, \mathsf{grz}^-, \mathsf{bw}_5^-, \mathsf{br}_{d+7}, \neg\varphi^*\}$, and we define for each configuration $\langle t, k, l\rangle$ the logic $L_d(t, k, l)$ as follows:

$$\begin{aligned} L_d(t,k,l) := \; & \mathsf{S4}_t \oplus \mathsf{bw}_5^- \oplus \mathsf{br}_{d+7} \oplus \mathsf{grz}^+ \oplus \mathsf{grz}^- \\ & \oplus AxM \\ & \oplus (\varphi^* \wedge \Delta^{\leq 7}\sigma(s, \overline{\psi}(n), \overline{\psi}(m)) \to \varphi^* \wedge \Delta^{\leq 7}\sigma(t, \overline{\psi}(k), \overline{\psi}(l))) \to \neg\varphi^* \\ & \oplus \varphi^* \to \psi_S(\varphi_{b_0}, \varphi_{b_1}, \varphi_{a_0}) \\ & \oplus \varphi^* \to (\psi_S(\overline{q}) \to \psi_S^+(\overline{q})). \end{aligned}$$

Note that $L_{d_1} \neq L_{d_2}$ for all $d_1 \neq d_2$. By Corollary 4.2.13, L_d is tabular. Clearly, if $\langle t, k, l\rangle \in \mathcal{R}each$, then $L_d(t, k, l) = L_d$. Since $L_d(t, k, l) \subseteq L(t, k, l)$, by Lemma 6.3.22, $\mathbb{F} \models L_d(t, k, l)$. Similar to Lemmas 6.3.23, for all $\langle t, k, l\rangle \notin \mathcal{R}each$, $L_d(t, k, l)$ is Kripke incomplete and so $L_d(t, k, l) \neq L_d$. Thus, $\langle t, k, l\rangle \mapsto L_d(t, k, l)$ is a reduction from the set $\mathcal{R}each$ to the set $\{\varphi \in \mathsf{Form}_t : \mathsf{S4}_t \oplus \varphi = L_d\}$. Hence, $\{\varphi \in \mathsf{Form}_t : \mathsf{S4}_t \oplus \varphi = L_d\}$ is undecidable for any $d \in \omega$. □

6.3.28. REMARK. By Theorem 6.3.27, there exist countably many tabular logics L in $\mathsf{NExt}(\mathsf{S4}_t)$ such that the coincidence problem for L is undecidable. On the other hand, by Theorems 6.3.2 and 5.4.2, the sets $\{\varphi : \mathsf{S4}_t \oplus \varphi = \mathsf{Log}(\mathfrak{Cl}_n)\}$ are decidable for all $n \in \mathbb{Z}^+$. Thus, there also exist countably many tabular logics L in $\mathsf{NExt}(\mathsf{S4}_t)$ such that the coincidence problem for L is decidable. Hence, although a full characterization is still lacking, we have determined the number of tabular logics L for which the coincidence problem is (un)decidable.

6.4 Summary

The main motivation behind this chapter is to understand how the interactions of modalities affect the decidability of logical properties. We have shown that most properties, as listed in Corollary 6.2.14, are undecidable in the lattice $\mathsf{NExt}(\mathsf{K4}_t)$. Moreover, we have proved that in the lattice $\mathsf{NExt}(\mathsf{S4}_t)$, many logical properties are still undecidable, while consistency is decidable. In fact, by looking into the proofs, we see that the same result holds for the lattice $\mathsf{NExt}(\mathsf{S4}_t \oplus \{\mathsf{grz}^+, \mathsf{grz}^-, \mathsf{bw}_5^-, \mathsf{br}_7\})$. Unlike the case for $\mathsf{K4}_t$, we do not have a characterization which captures those tabular logics L such that $\{\varphi \in \mathsf{Form}_t : \mathsf{S4}_t \oplus \varphi = L\}$ is (un)decidable. We leave it for future research.

The results obtained in this chapter, together with the known ones (see Table 6.1), suggest that the interaction of modalities tends to increase the complexity of decision problems for logical properties. Moreover, another observation from Table 6.1 is that a stronger base logic tends to make properties decidable. The undecidability results for $\mathsf{NExt}(\mathsf{K4}_t)$ indicate that $\mathsf{K4}_t$ is not strong enough in this sense. We leave it for future research to analyze the decidability of logical properties in the lattice of extensions of other bimodal logics. For example, the tense logic Grz_t of posets and the product logic $\mathsf{S5} \times \mathsf{S5}$ would be natural choices.

Chapter 7

Conclusion

In this thesis, we studied lattices of tense logics. We investigated Post-completeness, tabularity and pretabularity in these lattices. We also investigated Kripke completeness and the finite model property by studying the degree of Kripke incompleteness and the degree of FMP, respectively. Moreover, we studied the decidability of logical properties in lattices of tense logics. In the course of these investigations, we obtained a range of technical results that contribute to a more detailed understanding of lattices of tense logics.

From a methodological perspective, the proofs in this thesis combined adaptations of known techniques with several new constructions tailored to the tense setting. On the one hand, many arguments built on existing methods from modal logic, such as 0-filtration, splitting techniques, and reductions via Minsky machines. On the other hand, essential modifications were required to handle the interaction between tense modalities. In particular, we systematically exploited the interactions between the future-looking and past-looking modalities to simplify constructions and obtain more refined results (for example, via reflective unfolding). Moreover, several new technical ingredients were introduced. In the study of pretabularity, a central role was played by frame constructions based on generalized Thue-Morse sequences, which provided a flexible way of generating large families of frames with controlled combinatorial properties. Together with the use of generalized Jankov formulas and local t-morphisms, these techniques formed a toolbox for analyzing the structure of lattices of tense logics and may be of independent interest for the study of other polymodal systems.

Beyond the technical results on tense logics and their lattices, this thesis contributes to a broader understanding of how the presence of multiple interacting modalities affects the structure of lattices of logics. Compared with the well-studied monomodal case, lattices of tense logics exhibit a more complex structure. In particular, our results show that classical results concerning logical properties such as Post-completeness, tabularity, and decidability may persist, fail, or take new forms in the tense setting. This suggests that tense logics provide a natural setting for understanding how interactions between modalities influence properties of polymodal logics.

In what follows, we briefly summarize the main results obtained in this thesis.

Post-completeness and Tabularity. We first obtained a series of results on tabularity and Post-completeness in tense logic. We established one new characterization of tabular tense logics, two characterization theorems for Post-completeness in tabular tense logic, and one characterization of Post-complete tense logics. The Post-numbers of tense logics such as $\mathsf{K4}_t$, $\mathsf{D4}_t$ and B_t were determined. We also proved the anti-dichotomy theorem for Post-numbers of tense logic (Theorem 3.3.10). These results give a clearer picture of the top part of the lattices of tense logics.

Pretabularity. We introduced tense logics with bounded parameters and gave a full characterization of pretabular fully bounded tense logics. We also investigated concrete tense logics with bounded parameters: full characterizations for pretabular logics extending $\mathsf{S4.3}_t$ and $\mathsf{S4BP}^{2,\omega}_{2,2}$ were provided. Finally, we studied pretabular tense logics in $\mathsf{NExt}(\mathsf{S4BP}^{2,\omega}_{2,3})$. By Theorem 4.6.28, the cardinality of $\mathsf{PTAB}(\mathsf{S4BP}^{2,\omega}_{2,3})$ is $2^{\aleph_0}$. Theorem 4.6.29 answers the open problem raised in [107].

Kripke completeness. We obtained a series of results on the degree of Kripke incompleteness in lattices of tense logics. By studying the splitting lattices of tense logics, we provided characterizations of the degree of Kripke incompleteness in $\mathsf{NExt}(\mathsf{K}_t)$, $\mathsf{NExt}(\mathsf{K4}_t)$ and $\mathsf{NExt}(\mathsf{S4}_t)$. Consequently, we generalized Blok's dichotomy theorem from $\mathsf{NExt}(\mathsf{K})$ to these lattices of tense logics. By showing that the degree of Kripke incompleteness coincides with the degree of FMP in all the lattices mentioned above, we also obtained dichotomy theorems for the degree of FMP in these lattices. These results illustrate both similarities and differences of the degree of Kripke incompleteness and the degree of FMP between the lattices of modal logics and tense logics (cf. [9, 14]).

(Un)decidability of logical properties. We generalized the undecidability result for tabularity in $\mathsf{NExt}(\mathsf{K4}_t)$ by establishing a general criterion for the undecidability of logical properties, which implies the undecidability of Kripke completeness, decidability and the FMP. Then we moved to the lattice $\mathsf{NExt}(\mathsf{S4}_t)$. We showed directly that logical properties such as Kripke completeness, tabularity and decidability are undecidable. Moreover, we proved that there exist countably many tabular logics L for which the coincidence problem is undecidable. On the other hand, we proved that in $\mathsf{NExt}(\mathsf{S4}_t)$ there exist also countably many tabular logics L for which the coincidence problem is decidable. These results indicate the complexity of lattices of tense logics.

Future work

In this final section, we outline several directions for future research. While the results obtained in this thesis advance the understanding of lattices of tense logics, they also raise a number of natural questions and open problems. We highlight some of these problems and suggest possible avenues for further investigation.

Kripke completeness, Post-completeness and tabularity. It is worth mentioning that so far all Post-complete tense logics we have studied are either tabular or Kripke incomplete. So a natural question is: do Kripke completeness and Post-completeness imply tabularity? Our conjecture is that there exist continuum many non-tabular Post-complete logics that are Kripke complete.

Pretabular logics in $\mathsf{NExt}(\mathsf{S4}_t)$ with the FMP. Pretabular logics can be viewed as boundaries of tabular logics. Moreover, pretabular logics with the FMP act as the limits of sets of tabular logics. As it is shown in [29, Theorem 12.11], every pretabular modal logic in $\mathsf{NExt}(\mathsf{K4})$ has the FMP. By Theorem 4.5.15, every pretabular tense logic in $\mathsf{NExt}(\mathsf{S4BP}^{2,\omega}_{2,2})$ has the FMP. However, if our conjecture in Remark 4.6.32 is proved to be correct, then there exists a continuum-sized family of Kripke complete pretabular tense logics lacking the FMP in $\mathsf{NExt}(\mathsf{S4BP}^{2,\omega}_{2,3})$. This raises at least two natural questions:

(1) When does $\mathsf{NExt}(L)$ contain pretabular logics lacking the FMP?

(2) How many finitely approximable pretabular logics exist in $\mathsf{NExt}(\mathsf{S4}_t)$?

Degree of FMP of tense logics. We have shown in Sections 5.3 and 5.4 that the degree of FMP and the degree of Kripke incompleteness coincide in the lattices $\mathsf{NExt}(\mathsf{K}_t)$, $\mathsf{NExt}(\mathsf{K4}_t)$ and $\mathsf{NExt}(\mathsf{S4}_t)$. By the results obtained in Chapter 5, the dichotomy theorem for the degree of FMP holds for all three of these lattices. On the other hand, Bezhanishvili et al. [9] proved the anti-dichotomy theorem for the degree of FMP for $\mathsf{NExt}(\mathsf{K4})$ and $\mathsf{NExt}(\mathsf{S4})$. This raises a natural follow-up question: Is there any tense logic L such that the anti-dichotomy theorem holds for $\mathsf{NExt}(L)$?

Decidability of interpolation property. We have proved the undecidability of Kripke completeness, finite model property and tabularity in $\mathsf{NExt}(\mathsf{S4}_t)$. However, the undecidability of the Craig interpolation property (CIP) does not follow directly from our proof. In fact, the undecidability of the CIP for $\mathsf{NExt}(\mathsf{GL})$ is established via a characterization of closed formulas, a method that does not extend to $\mathsf{NExt}(\mathsf{S4}_t)$ (see, e.g., [29]). Thus, we again have a natural question: Is the CIP decidable in $\mathsf{NExt}(\mathsf{S4}_t)$?

To conclude, this thesis provides a systematic study of lattices of tense logics through the investigation of tabularity, Post-completeness, and Kripke completeness, as well as the decidability of logical properties in these lattices. The results obtained here extend and refine known results from the monomodal setting, while also revealing new phenomena arising from the interaction between tense modalities. Although a number of questions have been resolved, many problems remain open, indicating that the study of lattices of tense logics is far from complete. We hope that the methods and results developed in this thesis will contribute to further progress in this area and stimulate future research on polymodal logics.

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Index

摘要

本文研究时态逻辑的格。时态逻辑是一类包含将来必然算子 $\Box$ 以及过去可能算子 $\blacklozenge$ 的正规双模态逻辑，其模态算子 $\Box$ 与 $\blacklozenge$ 相互伴随。对任意时态逻辑 L，其所有扩张所构成的集合 $\mathsf{NExt}(L)$ 形成一个格。K_t、$\mathsf{K4}_t$ 以及 $\mathsf{S4}_t$ 分别是经典模态逻辑 K、K4、S4 所对应的时态逻辑。本文主要研究：（1）$\mathsf{NExt}(\mathsf{K}_t)$ 上的波斯特完全性；（2）$\mathsf{NExt}(\mathsf{K}_t)$ 上的表列性质；（3）$\mathsf{NExt}(\mathsf{S4}_t)$ 上的濒表列性质；（4）$\mathsf{NExt}(\mathsf{K}_t)$、$\mathsf{NExt}(\mathsf{K4}_t)$ 以及 $\mathsf{NExt}(\mathsf{S4}_t)$ 上的克里普克不完全度；（5）$\mathsf{NExt}(\mathsf{K}_t)$、$\mathsf{NExt}(\mathsf{K4}_t)$ 以及 $\mathsf{NExt}(\mathsf{S4}_t)$ 上逻辑性质的（不）可判定性。

本论文的第一部分研究时态逻辑格中靠近顶端的逻辑，重点关注表列性质与波斯特完全性。表列时态逻辑是单个有穷时态代数的时态逻辑。我们基于一族公式 tab_n^T 给出了表列时态逻辑的一个新刻画。波斯特完全的时态逻辑是没有一致真扩张的一致时态逻辑。我们分别给出波斯特完全表列时态逻辑以及波斯特完全时态逻辑的刻画，并进一步研究时态逻辑的波斯特数，证明：对任意基数 α，若 $1 \leq \alpha \leq \aleph_0$ 或 $\alpha = 2^{\aleph_0}$，则存在波斯特数为 α 的时态逻辑。

继而，本文研究 $\mathsf{NExt}(\mathsf{S4}_t)$ 上的濒表列逻辑。该类逻辑构成表列逻辑的边界：濒表列逻辑本身不具有表列性质，而其所有真扩张都具有表列性质。首先，我们引入了有界时态逻辑，给出具有濒表列性质的完全有界时态逻辑的完全刻画；随后，分别刻画 $\mathsf{S4.3}_t$ 与 $\mathsf{S4BP}_{2,2}^{2,\omega}$ 的濒表列扩张；最后，构造连续统多个 $\mathsf{S4BP}_{2,3}^{2,\omega}$ 的濒表列扩张，从而解决了关于 $\mathsf{S4}_t$ 的濒表列扩张数量的开问题。

本论文的第二部分关注时态逻辑格的整体结构。为此，首先研究时态逻辑的克里普克不完全度。Blok 关于 $\mathsf{NExt}(\mathsf{K})$ 的二分定理是克里普克不完全度的一个经典结果：任意模态逻辑 $L \in \mathsf{NExt}(\mathsf{K})$ 的克里普克不完全度必然是 1 或 $2^{\aleph_0}$。本文将该定理推广至时态逻辑格 $\mathsf{NExt}(\mathsf{K}_t)$、$\mathsf{NExt}(\mathsf{K4}_t)$ 以及 $\mathsf{NExt}(\mathsf{S4}_t)$。我们进一步讨论并切分、迭代切分逻辑与克里普克不完全度之间的关系。

最后，本文研究时态逻辑中逻辑性质的可判定性。模态逻辑中的一个基本问题是：给定一类逻辑，判定其中某个逻辑是否具有特定性质。本文首先证明严格克里普克完全性在 $\mathsf{NExt}(\mathsf{K4}_t)$ 上是可判定的；其次，给出逻辑性质在 $\mathsf{NExt}(\mathsf{K4}_t)$ 上不可判定的一般判据；最后，证明严格克里普克完全性在 $\mathsf{NExt}(\mathsf{S4}_t)$ 上可判定，而表列性质、克里普克完全性以及可判定性则不可判定。

Abstract

This thesis investigates the lattices of tense logics. A tense logic is a normal bimodal logic equipped with a future-looking necessity modality $\Box$ and a past-looking possibility modality $\blacklozenge$, with the modalities $\Box$ and $\blacklozenge$ forming an adjoint pair. For each tense logic L, the set $\mathsf{NExt}(L)$ of all tense logics extending L forms a lattice. Tense logics K_t, $\mathsf{K4}_t$ and $\mathsf{S4}_t$ are analogues of the well-known modal logics K, $\mathsf{K4}$ and $\mathsf{S4}$, respectively. In this thesis, we study the following topics: (1) Post-completeness in the lattice $\mathsf{NExt}(\mathsf{K}_t)$; (2) tabularity in the lattice $\mathsf{NExt}(\mathsf{K}_t)$; (3) pretabularity in the lattice $\mathsf{NExt}(\mathsf{S4}_t)$; (4) the degree of Kripke incompleteness in $\mathsf{NExt}(\mathsf{K}_t)$, $\mathsf{NExt}(\mathsf{K4}_t)$ and $\mathsf{NExt}(\mathsf{S4}_t)$; and (5) (un)decidability of logical properties in $\mathsf{NExt}(\mathsf{K}_t)$, $\mathsf{NExt}(\mathsf{K4}_t)$ and $\mathsf{NExt}(\mathsf{S4}_t)$.

The first part of the thesis studies logics near the top of the lattices of tense logics, focusing on tabularity and Post-completeness. A tense logic is tabular if it is the logic of a finite tense algebra. We give a new characterization of tabular tense logics based on a family of formulas tab_n^T. A tense logic is Post-complete if it is consistent and has no consistent proper extension. We provide full characterizations of Post-complete tabular tense logics and Post-complete tense logics, respectively. Moreover, we study the Post-number of tense logics. We show that for each cardinal α, if $1 \leq \alpha \leq \aleph_0$ or $\alpha = 2^{\aleph_0}$, then there exists a tense logic with Post-number α.

Next, we study pretabular tense logics in $\mathsf{NExt}(\mathsf{S4}_t)$, which form the boundary of tabular tense logics, in the sense that a pretabular logic L is non-tabular while every proper extension of L is tabular. We introduce tense logics with bounded parameters and give a full characterization of pretabular fully bounded tense logics. Then, we give full characterizations for pretabular logics extending $\mathsf{S4.3}_t$ and $\mathsf{S4BP}^{2;\omega}_{2,2}$, respectively. Finally, we show that there are continuum many pretabular extensions of $\mathsf{S4BP}^{2;\omega}_{2,3}$, which answers the open problem about the cardinality of pretabular extensions of $\mathsf{S4}_t$.

The second part of the thesis focuses on the structure of lattices of tense logics

as a whole. To this end, we begin by studying the degree of Kripke incompleteness of tense logics. A celebrated result in this area is Blok's dichotomy theorem of the degree of Kripke incompleteness for $\mathsf{NExt}(\mathsf{K})$, which states that every logic $L \in \mathsf{NExt}(\mathsf{K})$ is of the degree of Kripke incompleteness either 1 or $2^{\aleph_0}$. We generalize this dichotomy theorem to the lattices $\mathsf{NExt}(\mathsf{K}_t)$, $\mathsf{NExt}(\mathsf{K4}_t)$ and $\mathsf{NExt}(\mathsf{S4}_t)$. Moreover, we discuss the relation between union-splittings, iterated splittings and the degree of Kripke incompleteness.

Finally, we investigate the decidability of logical properties of tense logics. A central theme of the study of modal logic is to determine whether a logic has a certain logical property. In this thesis, we prove that strict Kripke completeness is decidable in $\mathsf{NExt}(\mathsf{K4}_t)$. Also, we give a general criterion for a logical property to be undecidable in $\mathsf{NExt}(\mathsf{K4}_t)$. Finally, we show that in $\mathsf{NExt}(\mathsf{S4}_t)$, strictly Kripke completeness is decidable, while tabularity, Kripke completeness and decidability are undecidable.

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