STABLE MODAL LOGICS

GURAM BEZHANISHVILI, NICK BEZHANISHVILI, JULIA ILIN

Abstract. We develop the theory of stable modal logics, a class of modal logics introduced in [3]. We give several new characterizations of stable modal logics, and show that there are continuum many such. Since some basic modal systems such as **K4** and **S4** are not stable, for a modal logic L, we introduce the concept of an L-stable extension of L. We prove that there are continuum many **S4**-stable modal logics, and continuum many **K4**-stable modal logics between **K4** and **S4**. We axiomatize **K4**-stable and **S4**-stable modal logics by means of stable canonical formulas of [3], and discuss the connection between **S4**-stable modal logics and stable superintuitionistic logics of [2]. We conclude the paper with examples of **K4**-stable modal logics, and compare **K4**-stable modal logics to subframe and splitting transitive modal logics.

1. Introduction

Stable multi-conclusion consequence relations and stable modal logics were introduced in [3]. The defining feature of these systems is that they admit filtration and hence have the finite model property. As was shown in [3], these systems can be axiomatized by stable rules. It is the goal of this article to develop further the theory of stable modal systems. We will obtain several new characterizations of stable modal systems, and will show that there are continuum many such systems. We will also show that some basic modal logics such as **K4** and **S4** are not stable. Because of this, for a modal system $\mathcal{S}$, we will introduce the concept of a $\mathcal{S}$-stable extension of $\mathcal{S}$. We will mostly concentrate on **K4**-stable and **S4**-stable logics, for which we obtain several useful characterizations. We will also show that there are continuum many **S4**-stable logics, and continuum many **K4**-stable logics between **K4** and **S4**. We conclude the paper with many examples of **K4**-stable modal logics, and compare the class of **K4**-stable modal logics to those of subframe and splitting transitive modal logics.

The paper is organized as follows. In the next two sections we recall the definitions of stable consequence relations and stable modal logics from [3] and provide new characterizations of these. We also show that there are continuum many such systems. It turns out that many standard modal logics are not stable. This suggests to relativize the concept of stability to that of L-stability, where L is some normal modal logic. This is discussed in Section 4. In Section 5 we study **K4**-stable and **S4**-stable logics. These are special instances of L-stable logics. We obtain several useful characterizations of **K4**-stable and **S4**-stable logics, and show that they can be axiomatized by stable formulas. In Section 6 we discuss the connection between **S4**-stable logics and stable superintuitionistic logics of [2]. We show that the superintuitionistic fragment of an **S4**-stable logic is stable, and that the least modal companion of a stable superintuionistic logic is **S4**-stable. We also show that there are continuum many **S4**-stable logics and continuum many **K4**-stable logics in between **K4** and **S4**. In the final section we present many examples (and non-examples) of **K4**-stable and **S4**-stable logics, provide their axiomatizations in terms of stable formulas, and compare them to subframe and splitting logics.

2010 *Mathematics Subject Classification.* 03B45, 03B55.

Key words and phrases. Modal logic, modal consequence relation, canonical formula, canonical rule, intuitionistic logic.

2. Stable consequence relations

We will assume the reader's familiarity with modal logic. We will use [9, 11, 6, 17] as our main references for the basic theory of normal modal logics, [7] for universal algebra, [14, 12] for modal consequence relations, and [10, 3] for multi-conclusion modal consequence relations. We will use the same notation as in [3]. We will freely use the duality between modal algebras and modal spaces (descriptive frames). In fact, we will often do not distinguish between modal algebras and their duals.

Definition 2.1.
(1) *Let* $\mathfrak{A} = (A, \Diamond)$ *and* $\mathfrak{B} = (B, \Diamond)$ *be modal algebras. A Boolean homomorphism* $h : A \to B$ *is* stable *provided* $\Diamond h(a) \le h(\Diamond a)$ *for all* $a \in A$.
(2) *We call* $\mathfrak{A}$ *a* stable subalgebra *of* $\mathfrak{B}$ *if* A *is a Boolean subalgebra of* B *and the inclusion* $A \hookrightarrow B$ *is a stable embedding.*
(3) *A class* $\mathcal{K}$ *of modal algebras is* stable *provided for modal algebras* $\mathfrak{A}$ *and* $\mathfrak{B}$, *if* $\mathfrak{B} \in \mathcal{K}$ *and there is a stable embedding* $\mathfrak{A} \rightarrowtail \mathfrak{B}$, *then* $\mathfrak{A} \in \mathcal{K}$.

Definition 2.2.
(1) *Let* $\mathfrak{X} = (X, R)$ *and* $\mathfrak{Y} = (Y, R)$ *be modal spaces. A map* $f : X \to Y$ *is* stable *if it is continuous and* xRy *implies* $f(x)Rf(y)$.
(2) *A class* $\mathcal{K}$ *of modal spaces is* stable *provided for modal spaces* $\mathfrak{X}$ *and* $\mathfrak{Y}$, *if* $\mathfrak{X} \in \mathcal{K}$ *and there is an onto stable map* $f : X \twoheadrightarrow Y$, *then* $\mathfrak{Y} \in \mathcal{K}$.

By [3, Lem. 3.3], $h : \mathfrak{A} \to \mathfrak{B}$ is stable iff its dual $h_* : \mathfrak{B}_* \to \mathfrak{A}_*$ is stable. Consequently, a class $\mathcal{K}$ of modal algebras is stable iff its dual class $\mathcal{K}_*$ of modal spaces is stable.

Definition 2.3. *A normal modal multi-conclusion consequence relation* $\mathcal{S}$ *is* stable *provided the corresponding universal class* $\mathcal{U}(\mathcal{S}) := \{\mathfrak{A} \mid \mathfrak{A} \models \mathcal{S}\}$ *of modal algebras is stable.*

Stable multi-conclusion consequence relations can be axiomatized as follows. Let $\mathfrak{A} = (A, \Diamond)$ be a finite modal algebra. For every $a \in A$, let p_a be a propositional letter. The *stable (multi-conclusion) rule* $\rho(\mathfrak{A})$ is defined as Γ/Δ, where

$$\begin{aligned}\Gamma = &\{p_{a\vee b} \leftrightarrow p_a \vee p_b \mid a, b \in A\} \cup \\ &\{p_{\neg a} \leftrightarrow \neg p_a \mid a \in A\} \cup \\ &\{\Diamond p_a \to p_{\Diamond a} \mid a \in A\}\end{aligned}$$

and

$$\Delta = \{p_a \mid a \in A, a \neq 1\}.$$

By [3, Thm. 7.4], $\mathcal{S}$ is stable iff $\mathcal{S}$ is axiomatizable by stable rules, and by [3, Thm. 7.8], every stable multi-conclusion consequence relation has the finite model property. We next give a convenient characterization of stable multi-conclusion consequence relations. We recall (see, e.g., [7, Thm. V.2.20]) that a universal class $\mathcal{U}$ is generated by a class $\mathcal{K}$ iff $\mathcal{U} = \mathbf{ISP}_U(\mathcal{K})$, where $\mathbf{I}$, $\mathbf{S}$, and $\mathbf{P}_U$ are the operations of taking isomorphic copies, subalgebras, and ultraproducts. Note that in general $\mathbf{ISP}_U(\mathcal{K})$ differs from the variety generated by $\mathcal{K}$.

Theorem 2.4. *Let* $\mathcal{S}$ *be a normal multi-conclusion consequence relation. The following are equivalent.*
(1) $\mathcal{S}$ *is stable.*
(2) $\mathcal{U}(\mathcal{S})$ *is generated by a stable class of modal algebras.*
(3) $\mathcal{U}(\mathcal{S})$ *is generated by a stable class of finite modal algebras.*

Proof. The implication (3) $\Rightarrow$ (2) is trivial. For the implication (2) $\Rightarrow$ (1), suppose that $\mathcal{K}$ is a stable class of modal algebras that generates $\mathcal{U}(\mathcal{S})$. Let $\mathcal{A}$ be the set of finite nonisomorphic modal algebras not belonging to $\mathcal{K}$. We claim that $\mathcal{S}$ is axiomatizable by the stable rules $\{\rho(\mathfrak{A}) \mid \mathfrak{A} \in \mathcal{A}\}$. First we show that each member of $\mathcal{K}$ satisfies $\rho(\mathfrak{A})$ for each $\mathfrak{A} \in \mathcal{A}$. Indeed, if there are $\mathfrak{B} \in \mathcal{K}$ and $\mathfrak{A} \in \mathcal{A}$ such that $\mathfrak{B} \not\models \rho(\mathfrak{A})$, then by [3, Prop. 7.1], there is a stable embedding $\mathfrak{A} \rightarrowtail \mathfrak{B}$. Since $\mathcal{K}$ is stable, $\mathfrak{A} \in \mathcal{K}$, a contradiction. Because $\mathcal{U}(\mathcal{S})$ is generated by $\mathcal{K}$, it follows that each member of $\mathcal{U}(\mathcal{S})$ satisfies $\rho(\mathfrak{A})$ for each $\mathfrak{A} \in \mathcal{A}$. Conversely, suppose that $\mathfrak{B}$ satisfies $\rho(\mathfrak{A})$ for each $\mathfrak{A} \in \mathcal{A}$. If $\mathfrak{B} \notin \mathcal{U}(\mathcal{S})$, then there is a multi-conclusion rule Γ/Δ such that $\mathcal{S} \vdash \Gamma/\Delta$ but $\mathfrak{B} \not\models \Gamma/\Delta$. By [3, Thm. 7.8], there is a finite stable subalgebra $\mathfrak{B}'$ of $\mathfrak{B}$ such that $\mathfrak{B}' \not\models \Gamma/\Delta$. Since $\mathfrak{B}'$ is a stable subalgebra of $\mathfrak{B}$, by [3, Prop. 7.1], $\mathfrak{B} \not\models \rho(\mathfrak{B}')$. As $\mathfrak{B}$ satisfies $\rho(\mathfrak{A})$ for each $\mathfrak{A} \in \mathcal{A}$, we see that $\mathfrak{B}' \in \mathcal{K}$, so $\mathfrak{B}' \in \mathcal{U}(\mathcal{S})$. But this contradicts to $\mathfrak{B}' \not\models \Gamma/\Delta$. Therefore, $\mathfrak{B} \in \mathcal{U}(\mathcal{S})$, and so $\mathcal{S}$ is axiomatizable by the stable rules $\{\rho(\mathfrak{A}) \mid \mathfrak{A} \in \mathcal{A}\}$. Thus, $\mathcal{S}$ is stable by [3, Thm. 7.4].

Finally, for the implication (1) $\Rightarrow$ (3), suppose $\mathcal{S}$ is stable. By [3, Thm. 7.8], $\mathcal{S}$ has the finite model property, so $\mathcal{U}(\mathcal{S})$ is generated by the class $\mathcal{K}$ of its finite members. Since $\mathcal{S}$ is stable, $\mathcal{U}(\mathcal{S})$ is stable by [3, Thm. 7.4]. Thus, $\mathcal{K}$ is stable, completing the proof. □

Remark 2.5. From next section on we will be mostly concerned with stable modal logics. But all the results we obtain have obvious analogues for multi-conclusion modal consequence relations, which often have even easier proofs. Because of this, we will often not discuss these analogues explicitly.

3. Stable modal logics

Stable modal logics were introduced in [3]. They can be defined as those normal modal logics L whose corresponding variety $\mathcal{V}(L)$ is generated by a stable universal class. By [3, Thm. 7.8], stable modal logics have the finite modal property. The next theorem is an analogue of Theorem 2.4 and provides a convenient characterization of stable modal logics.

We recall that an element a of a modal algebra $\mathfrak{A}$ is an *opremum* if $a \neq 1$ and for each $b \neq 1$ there is $n \in \omega$ with $\blacksquare_n a \leq c$, where $\Box^0 a = a$, $\Box^{n+1} a = \Box\Box^n a$, and $\blacksquare_n a = \bigwedge_{k \leq n} \Box^k a$. A modal algebra $\mathfrak{A}$ is subdirectly irreducible iff it has an opremum.

An element x of a modal space $\mathfrak{X} = (X, R)$ is a *root* if $X = R^\omega[x]$ and a *topo-root* if $R^\omega[x]$ is dense in X, where $R^0[x] = \{x\}$, $R^{n+1}[x] = R[R^n[x]]$, and $R^\omega[x] = \bigcup_{n \in \omega} R^n[x]$. We call $\mathfrak{X}$ *rooted* if it has a root, and *topo-rooted* if the set of topo-roots is not co-dense (the interior is nonempty). By [16, Thm. 2], a modal algebra $\mathfrak{A}$ is subdirectly irreducible iff its dual modal space $\mathfrak{A}_*$ is topo-rooted. Therefore, if $\mathfrak{A}$ is finite, then $\mathfrak{A}$ is subdirectly irreducible iff $\mathfrak{A}_*$ is rooted [15, Thm. 3.1]. Since the topology on a finite modal spaces is discrete, we will identify finite modal spaces with finite Kripke frames.

Theorem 3.1. *Let L be a normal modal logic. The following are equivalent.*

1. *L is stable.*
2. *$\mathcal{V}(L)$ is generated by a stable class of modal algebras.*
3. *$\mathcal{V}(L)$ is generated by a stable class of finite modal algebras.*
4. *$\mathcal{V}(L)$ is generated by a stable class of finite subdirectly irreducible modal algebras.*

Proof. The implications (4) $\Rightarrow$ (3) $\Rightarrow$ (2) are trivial. For the implication (2) $\Rightarrow$ (1), suppose that $\mathcal{V}(L)$ is generated by a stable class $\mathcal{K}$. By Theorem 2.4, the universal class $\mathcal{U}(\mathcal{K})$ generated by $\mathcal{K}$ is stable. Clearly $\mathcal{U}(\mathcal{K})$ and $\mathcal{K}$ generate the same variety $\mathcal{V}(L)$, so $\mathcal{V}(L)$ is generated by a stable universal class, and hence L is stable. For the implication (1) $\Rightarrow$ (3), suppose L is stable. Then $\mathcal{V}(L)$ is generated by a stable universal class $\mathcal{K}$. By Theorem 2.4, $\mathcal{K}$ is generated by its finite members, which is a stable class since $\mathcal{K}$ is stable. Thus, $\mathcal{V}(L)$ is generated by a stable class of finite modal algebras.

Finally, for the implication (3) $\Rightarrow$ (4), suppose $\mathcal{K}$ is a stable class of finite modal algebras that generates $\mathcal{V}(L)$. Let $\mathcal{K}_{si}$ be the class of all subdirectly irreducible members of $\mathcal{K}$. It is sufficient to show that $\mathcal{K}_{si}$ generates $\mathcal{V}(L)$, and for this it is sufficient to show that $\mathcal{K}$ is contained in the variety generated by $\mathcal{K}_{si}$. Suppose $\mathfrak{A} \in \mathcal{K}$. If $\mathfrak{A}$ is subdirectly irreducible, then $\mathfrak{A} \in \mathcal{K}_{si}$, and there is nothing to prove. Otherwise $\mathfrak{A}$ is a subdirect product of its finite subdirectly irreducible homomorphic images. Therefore, to conclude that $\mathfrak{A}$ is in the variety generated by $\mathcal{K}_{si}$, it is sufficient to see that every subdirectly irreducible homomorphic image $\mathfrak{B}$ of $\mathfrak{A}$ belongs to this variety. Let $\mathfrak{B}$ be a subdirectly irreducible homomorphic image of $\mathfrak{A}$. Since $\mathfrak{A}$ is finite, so is $\mathfrak{B}$. Therefore, the dual $\mathfrak{Y} = (Y, R)$ of $\mathfrak{B}$ is rooted. Let $\mathfrak{Y}' = (Y', R')$ be obtained from $\mathfrak{Y}$ by adding a new reflexive root that sees every point of $\mathfrak{Y}$; that is, $Y' = Y \cup \{r\}$ for some $r \notin Y$ and $R' = R \cup \{(r, x) \mid x \in Y'\}$. Let $\mathfrak{X}$ be the dual of $\mathfrak{A}$. Since $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$, we see that $\mathfrak{Y}$ is a generated subframe of $\mathfrak{X}$. Since $\mathfrak{A}$ is not subdirectly irreducible, but $\mathfrak{B}$ is, $\mathfrak{X}$ is not rooted, but $\mathfrak{Y}$ is. So $\mathfrak{Y} \neq \mathfrak{X}$. Define $f : X \to Y'$ by mapping the points of Y to themselves and every other point of X to r. It is easy to see that f is an onto stable map. Therefore, there is a stable embedding from the dual algebra $\mathfrak{B}'$ of $\mathfrak{Y}'$ to $\mathfrak{A}$. Since $\mathfrak{A} \in \mathcal{K}$ and $\mathcal{K}$ is stable, we conclude that $\mathfrak{B}' \in \mathcal{K}$. As $\mathfrak{Y}'$ is finite and rooted, $\mathfrak{B}'$ is subdirectly irreducible, and hence $\mathfrak{B}' \in \mathcal{K}_{si}$. Now, $\mathfrak{Y}$ is a generated subframe of $\mathfrak{Y}'$, so $\mathfrak{B}$ is a homomorphic image of $\mathfrak{B}'$, and hence $\mathfrak{B}$ belongs to the variety generated by $\mathcal{K}_{si}$, as desired. □

As we already pointed out, we will often not distinguish between modal algebras and their duals. So if $\mathfrak{A}$ is a finite modal algebra and $\mathfrak{X}$ is its dual, then we often write $\rho(\mathfrak{X})$ instead of $\rho(\mathfrak{A})$. As usual, we denote a reflexive point by $\circ$ and an irreflexive point by $\bullet$.

We let **Form** be the inconsistent logic, **KD** $:=$ **K** $+ (\Box p \to \Diamond p)$ be the logic of serial frames, and **KT** $:=$ **K** $+ (p \to \Diamond p)$ be the logic of reflexive frames. As follows from [3, Thm. 8.3], **Form** is axiomatized by $\rho(\circ)$, **KD** is axiomatized by $\rho(\bullet)$ and $\rho(\overset{\bullet}{\underset{\circ}{\uparrow}})$, and **KT** is axiomatized by $\rho(\bullet)$ and $\rho(\bullet\!\leftrightarrow\!\circ)$. We give more examples in the next theorem.

Theorem 3.2.

(1) *For a finite modal algebra* $\mathfrak{A}$*, let* **Stable**$(\mathfrak{A})$ *be the class of modal algebras that are isomorphic to stable subalgebras of* $\mathfrak{A}$*, and let* $L($**Stable**$(\mathfrak{A}))$ *be the logic of* **Stable**$(\mathfrak{A})$*. Then* $L($**Stable**$(\mathfrak{A}))$ *is a stable modal logic.*

(2) *Every extension of* **S5** *is a stable modal logic.*

Proof. (1). Clearly **Stable**$(\mathfrak{A})$ is a stable class of finite modal algebras. Now apply Theorem 3.1.

(2). It is well known that an **S5**-algebra is subdirectly irreducible iff its dual is a cluster. It is easy to see that the class of finite clusters is a stable class. Since **S5** is the logic of this class, **S5** is a stable logic by Theorem 3.1. It is also well known that for every extension L of **S5** there is n such that L is the logic of m-clusters for $m \leq n$. This class is stable by the same reasoning. Thus, every extension of **S5** is stable. □

We next show that there are continuum many stable modal logics. In fact, we will see that there are continuum many stable logics above the logic **wK4** of weakly transitive frames, where a frame $\mathfrak{F} = (X, R)$ is *weakly transitive* provided xRy, yRz, and $x \neq z$ imply xRz. For our proof we will make use of Jankov formulas for finite **wK4**-algebras from [1]. For a finite subdirectly irreducible **wK4**-algebra $\mathfrak{A}$ let $\chi(\mathfrak{A})$ be the Jankov formula of $\mathfrak{A}$. Then for a **wK4**-algebra $\mathfrak{B}$, we have

$$\mathfrak{B} \not\models \chi(\mathfrak{A}) \text{ iff } \mathfrak{A} \text{ is a subalgebra of a homomorphic image of } \mathfrak{B} \text{ (see [1, Prop. 7.5]).}$$

Dually, if $\mathfrak{F}$ is a finite rooted weakly transitive frame and $\mathfrak{G}$ is an arbitrary weakly transitive space, then we have

$$\mathfrak{G} \not\models \chi(\mathfrak{F}) \text{ iff } \mathfrak{F} \text{ is a p-morphic image of a generated subframe of } \mathfrak{G}.$$

Theorem 3.3. *There is a continuum of weakly transitive non-transitive stable modal logics.*

Proof. For $n \geq 3$ let $\mathfrak{C}_n = (X_n, R_n)$ be the irreflexive n-point cluster depicted in Figure 1; that is, $X_n = \{x_1, \dots x_n\}$ and $R_n = \{(x_i, x_j) \in X_n \times X_n \mid i \neq j\}$.

$\mathfrak{C}_3$ $\mathfrak{C}_4$ $\mathfrak{C}_5$

FIGURE 1

Let $\mathbb{N}_{\geq 3} := \{n \in \mathbb{N} \mid n \geq 3\}$. For $I \subseteq \mathbb{N}_{\geq 3}$ set

$$\mathcal{K}_I = \{\mathfrak{X} \mid \exists n \in I \text{ such that } \mathfrak{X} \text{ is a stable image of } \mathfrak{C}_n\}.$$

It is clear that $\mathcal{K}_I$ is a stable class of modal spaces. Let L_I be the logic of $\mathcal{K}_I$. Since $\mathcal{K}_I$ is stable, by Theorem 3.1, L_I is a stable modal logic. We show that if $I \neq J$, then $L_I \neq L_J$. For this we first show that $n \in I$ iff $\chi(\mathfrak{C}_n) \notin L_I$. If $n \in I$, then $\mathfrak{C}_n \in \mathcal{K}_I$, so $\mathfrak{C}_n \models L_I$. Clearly $\mathfrak{C}_n \not\models \chi(\mathfrak{C}_n)$, which implies that $\chi(\mathfrak{C}_n) \notin L_I$. Conversely, suppose that $\chi(\mathfrak{C}_n) \notin L_I$. Since L_I is the logic of $\mathcal{K}_I$, there is $\mathfrak{X} \in \mathcal{K}_I$ such that $\mathfrak{X} \not\models \chi(\mathfrak{C}_n)$. Therefore, $\mathfrak{C}_n$ is a p-morphic image of a generated subframe of $\mathfrak{X}$. But the only generated subframe of $\mathfrak{X}$ is $\mathfrak{X}$, so $\mathcal{K}_I$ is closed under generated subframes. Also a p-morphic image of $\mathfrak{X}$ is a stable image of $\mathfrak{X}$, and $\mathcal{K}_I$ is closed under stable images. Thus, $\mathfrak{C}_n \in \mathcal{K}_I$. If $n \notin I$, then there is $m \in I$ and an onto stable map $f : \mathfrak{C}_m \twoheadrightarrow \mathfrak{C}_n$. Since $m = |\mathfrak{C}_m| > |\mathfrak{C}_n| = n$, we see that f must identify at least two points of $\mathfrak{C}_m$. Therefore, there are distinct $x, y \in \mathfrak{C}_m$ with $f(x) = f(y)$. Thus, $xR_m y$ and $f(x) \not\!R_n f(y)$, which is a contradiction because f is stable. Consequently, $n \in I$, and so $n \in I$ iff $\chi(\mathfrak{C}_n) \notin L_I$. Now, if $I \neq J$, then without loss of generality we may assume that there is $n \in I \setminus J$. Therefore, $\chi(\mathfrak{C}_n) \in L_J \setminus L_I$, and hence $L_I \neq L_J$. Since each $\mathfrak{C}_n$ is weakly transitive and non-transitive, we conclude that $\{L_I \mid I \subseteq \mathbb{N}_{\geq 3}\}$ is a continual family of weakly transitive non-transitive stable logics. □

On the other hand, we will see in Section 7 that many well-known modal logics are not stable. For example, by Theorem 7.6, none of the logics **K4**, **S4**, **KB**, and **K5** is stable, and neither are the logics **GL**, **S4.Grz**, **K4.1**, and **S4.1**. This motivates the key definition of the next section.

4. L-STABLE MODAL LOGICS

Since many well-known modal logics are not stable, we require to relativize the concept of stability as follows.

Definition 4.1.

(1) *Suppose $\mathcal{K}$ and $\mathcal{V}$ are two classes of modal algebras with $\mathcal{K} \subseteq \mathcal{V}$. We say that $\mathcal{K}$ is $\mathcal{V}$-*stable *(or* stable within $\mathcal{V}$*) provided for $\mathfrak{A}, \mathfrak{B} \in \mathcal{V}$, if $\mathfrak{B} \in \mathcal{K}$ and there is a stable embedding $\mathfrak{A} \rightarrowtail \mathfrak{B}$, then $\mathfrak{A} \in \mathcal{K}$.*

(2) *Suppose $\mathcal{S}$ is a normal modal multi-conclusion consequence relation and $\mathcal{T}$ is a normal extension of $\mathcal{S}$. We say that $\mathcal{T}$ is $\mathcal{S}$-*stable *(or* stable over $\mathcal{S}$*) provided the universal class $\mathcal{U}(\mathcal{T})$ is $\mathcal{U}(\mathcal{S})$-stable.*

(3) *Suppose L is a normal modal logic and M is a normal extension of L. We say that M is L-*stable *(or* stable over L*) provided the variety $\mathcal{V}(M)$ is generated by a universal class which is $\mathcal{V}(L)$-stable.*

Lemma 4.2. *Suppose L, M, N are normal modal logics with $L \subseteq M \subseteq N$.*

(1) *If M is stable, then M is L-stable.*

(2) *If N is L-stable, then N is M-stable.*

(3) *If $\mathcal{V}(M)$ is a $\mathcal{V}(L)$-stable class, then N is M-stable iff N is L-stable.*

Proof. (1). Since M is stable, $\mathcal{V}(M)$ is generated by a stable universal class $\mathcal{K}$. As $\mathcal{K}$ is stable, it is obviously $\mathcal{V}(L)$-stable. Thus, M is L-stable.

(2). Apply an argument similar to (1).

(3). One implication follows from (2). For the other, suppose N is M-stable. Then $\mathcal{V}(N)$ is generated by a universal class $\mathcal{K}$ which is $\mathcal{V}(M)$-stable. Since $\mathcal{V}(M)$ is $\mathcal{V}(L)$-stable, $\mathcal{K}$ is also $\mathcal{V}(L)$-stable. Therefore, N is L-stable. □

We next show that Theorem 3.1 has an obvious generalization to L-stable logics, provided L admits filtration. We recall the algebraic account of filtrations given in [3, Sec. 4].

Definition 4.3. *Suppose $\mathfrak{A} = (A, \Diamond)$ is a modal algebra, V is a valuation on $\mathfrak{A}$, and Σ is a finite set of formulas closed under subformulas. Let A' be the Boolean subalgebra of A generated by $V(\Sigma)$. Then A' is finite because Σ is finite. Set $D = \{V(\varphi) \mid \Diamond\varphi \in \Sigma\}$. Let $\Diamond'$ be a modal operator on A' and V' be a valuation on $\mathfrak{A}' = (A', \Diamond')$ satisfying*

- *The inclusion $\mathfrak{A}' \hookrightarrow \mathfrak{A}$ is a stable embedding;*
- *$V'(p) = V(p)$ for all propositional letters $p \in \Sigma$;*
- *$\Diamond' a = \Diamond a$ for all $a \in D$.*

Then $(\mathfrak{A}', V')$ is called a filtration of $(\mathfrak{A}, V)$ through Σ.

By the Filtration Lemma (see [3, Lem. 4.4]), if $(\mathfrak{A}', V')$ is a filtration of $(\mathfrak{A}, V)$ through Σ, then $V(\varphi) = V'(\varphi)$ for all $\varphi \in \Sigma$.

Definition 4.4. *A normal modal logic L* admits filtration *if for every L-algebra $\mathfrak{A}$, every valuation V on $\mathfrak{A}$, and every finite set Σ of formulas closed under subformulas, there is a filtration $(\mathfrak{A}', V')$ of $(\mathfrak{A}, V)$ through Σ such that $\mathfrak{A}'$ is an L-algebra.*

It is easy to see that if L admits filtration, then L has the finite model property. Indeed, if $L \nvdash \varphi$, then there is an L-algebra $\mathfrak{A}$ and a valuation V on $\mathfrak{A}$ such that $\mathfrak{A} \not\models \varphi$. Let Σ be the set of subformulas of φ. Since L admits filtration, there is a finite L-algebra $\mathfrak{A}'$ and a valuation V' on $\mathfrak{A}'$ such that $(\mathfrak{A}', V')$ is a filtration of $(\mathfrak{A}, V)$ through Σ. By the Filtration Lemma, $\mathfrak{A}' \not\models \varphi$. Thus, L has the finite model property.

Theorem 4.5. *Suppose L is a normal modal logic that admits filtration and M is a normal extension of L. The following are equivalent.*

1. *M is L-stable.*
2. *M is axiomatizable over L by stable rules of finite L-algebras.*
3. *$\mathcal{V}(M)$ is generated by a $\mathcal{V}(L)$-stable class of modal algebras.*
4. *$\mathcal{V}(M)$ is generated by a $\mathcal{V}(L)$-stable class of finite modal algebras.*

Proof. The proof is very similar to the proofs of Theorem 3.1 and [3, Thm. 7.4]. One simply has to use L-filtrations instead of arbitrary filtrations in the corresponding steps. As an example, we prove the implication (1) ⇒ (2). Suppose that M is L-stable. Then $\mathcal{V}(M)$ is generated by an L-stable universal class $\mathcal{K}$. Let $\mathcal{A}$ be the set of finite nonisomorphic L-algebras not belonging to $\mathcal{K}$. Let $\mathfrak{B}$ be an L-algebra. We claim that $\mathfrak{B} \in \mathcal{K}$ iff $\mathfrak{B} \models \rho(\mathfrak{A})$ for each $\mathfrak{A} \in \mathcal{A}$. Suppose $\mathfrak{B} \in \mathcal{K}$. If there is $\mathfrak{A} \in \mathcal{A}$ with $\mathfrak{B} \not\models \rho(\mathfrak{A})$, then by [3, Prop. 7.1], there is a stable embedding $\mathfrak{A} \rightarrowtail \mathfrak{B}$. Since $\mathcal{K}$ is L-stable, $\mathfrak{A} \in \mathcal{K}$, contradicting $\mathfrak{A} \in \mathcal{A}$. Therefore, $\mathfrak{B} \models \rho(\mathfrak{A})$ for each $\mathfrak{A} \in \mathcal{A}$. Conversely, suppose $\mathfrak{B}$ satisfies $\rho(\mathfrak{A})$ for each $\mathfrak{A} \in \mathcal{A}$. If $\mathfrak{B} \notin \mathcal{K}$, then since $\mathcal{K}$ is a universal class, there is a multi-conclusion rule Γ/Δ such that $\mathcal{K} \models \Gamma/\Delta$ but $\mathfrak{B} \not\models \Gamma/\Delta$. Let V be a valuation on $\mathfrak{B}$ refuting Γ/Δ, and let Σ be the set of subformulas of $\Gamma \cup \Delta$. Since L admits filtration, there is a finite L-algebra $\mathfrak{A}$ and a valuation V' on $\mathfrak{A}$ such that $(\mathfrak{A}, V')$ is a filtration of $(\mathfrak{B}, V)$ through Σ. Therefore, $\mathfrak{A}$ refutes Γ/Δ, yielding $\mathfrak{A} \notin \mathcal{K}$. Thus, $\mathfrak{A} \in \mathcal{A}$. On the other hand, since $\mathfrak{A}$ is a stable subalgebra of $\mathfrak{B}$, by [3, Prop. 7.1], $\mathfrak{B} \not\models \rho(\mathfrak{A})$. The obtained contradiction proves that $\mathfrak{B} \in \mathcal{K}$ iff $\mathfrak{B} \models \rho(\mathfrak{A})$ for each $\mathfrak{A} \in \mathcal{A}$. Since $\mathcal{K}$ generates $\mathcal{V}(M)$, we conclude that M is axiomatized over L by $\{\rho(\mathfrak{A}) \mid \mathfrak{A} \in \mathcal{A}\}$. □

Remark 4.6. If the finite frames of L are closed under the operation of adding a new reflexive root that sees every point, then a straightforward adjustment of the proof of Theorem 3.1 shows that another equivalent condition for M to be L-stable is that M is generated by an L-stable class of finite subdirectly irreducible L-algebras. Examples of such logics are **K4** and **S4**.

Remark 4.7. The definition of a normal modal multi-conclusion consequence relation $\mathcal{S}$ admitting filtration, the proof that such $\mathcal{S}$ has the finite model property, and an analogue of Theorem 4.5 are proved similarly, so we skip the details.

5. K4-stable and S4-stable logics

We next concentrate on the well-known modal logics **K4** and **S4**, and study **K4**-stable and **S4**-stable logics. Since both **K4** and **S4** admit filtration, Theorem 4.5 and Remark 4.6 hold for both logics.

Let $\mathfrak{A} = (A, \Diamond)$ be a **K4**-algebra. As usual, for $a \in A$, we set $\Diamond^+ a = a \vee \Diamond a$ and $\Box^+ a = a \wedge \Box a$. Then $\mathfrak{A}^+ = (A, \Diamond^+)$ is an **S4**-algebra. We call $\mathfrak{A}$ *well-connected* if $\Diamond^+ a \wedge \Diamond^+ b = 0$ implies $a = 0$ or $b = 0$. Equivalently, $\mathfrak{A}$ is well-connected if $\Box^+ a \vee \Box^+ b = 1$ implies $a = 1$ or $b = 1$. Each subdirectly irreducible **K4**-algebra is well-connected. To see this, suppose $\mathfrak{A}$ is subdirectly irreducible and $\Box^+ a \vee \Box^+ b = 1$. If $a, b \neq 1$, then since $\mathfrak{A}$ is subdirectly irreducible, it has an opremum $c \neq 1$, so $a, b \neq 1$ implies $\Box^+ a, \Box^+ b \leq c$, so $\Box^+ a \vee \Box^+ b \leq c \neq 1$, a contradiction. Therefore, $a = 1$ or $b = 1$, and hence $\mathfrak{A}$ is well-connected. While the converse is not true in general, it is true for finite **K4**-algebras.

For a **K4**-space $\mathfrak{X} = (X, R)$, let R^+ be the reflexive closure of R. Then $\mathfrak{X}^+ = (X, R^+)$ is an **S4**-space. Since in a **K4**-space $R^\omega = R^+$, we see that a **K4**-space is rooted iff there is $x \in X$ such that $X = R^+[x]$. It is well known that a **K4**-algebra is well-connected iff its dual **K4**-space is rooted.

Lemma 5.1. *Suppose $\mathfrak{A} = (A, \Diamond_A)$ and $\mathfrak{B} = (B, \Diamond_B)$ are* **K4***-algebras. If $\mathfrak{A}$ is well-connected and $\mathfrak{B}$ is a stable subalgebra of $\mathfrak{A}$, then $\mathfrak{B}$ is well-connected.*

Proof. Since $\mathfrak{B}$ is a stable subalgebra of $\mathfrak{A}$, we see that $\Diamond_A b \leq \Diamond_B b$ for all $b \in B$. Therefore, $\Diamond_A^+ b \leq \Diamond_B^+ b$ for all $b \in B$. Now, let $a, b \in B$ with $\Diamond_B^+ a \wedge \Diamond_B^+ b = 0$. Then $\Diamond_A^+ a \wedge \Diamond_A^+ b = 0$. As $\mathfrak{A}$ is well-connected, $a = 0$ or $b = 0$. Thus, $\mathfrak{B}$ is well-connected. □

As was shown in [3, Sec. 6.2], if $\mathfrak{A}$ is a finite subdirectly irreducible **K4**-algebra, then the stable rule $\rho(\mathfrak{A}) = \Gamma/\Delta$ can be rewritten as the *stable formula*

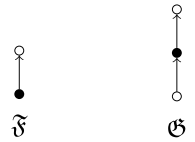
$$\gamma(\mathfrak{A}) := \bigwedge\{\Box^+\gamma \mid \gamma \in \Gamma\} \to \bigvee\{\Box^+\delta \mid \delta \in \Delta\}$$

so that for every **K4**-algebra $\mathfrak{B}$, we have $\mathfrak{B} \not\models \gamma(\mathfrak{A})$ iff there is a subdirectly irreducible homomorphic image $\mathfrak{C}$ of $\mathfrak{B}$ such that $\mathfrak{A}$ is isomorphic to a stable subalgebra of $\mathfrak{C}$. If $\mathfrak{B}$ is well-connected, then one implication of this equivalence can be strengthened.

Lemma 5.2. *Suppose $\mathfrak{A}$ is a finite subdirectly irreducible* **K4***-algebra and $\mathfrak{B}$ is a well-connected* **K4***-algebra. If $h : \mathfrak{A} \rightarrowtail \mathfrak{B}$ is a stable embedding, then $\mathfrak{B} \not\models \gamma(\mathfrak{A})$.*

Proof. Let V be a valuation on $\mathfrak{A}$ such that $V(p_a) = a$, and let $V' := h \circ V$. As in the proof of [3, Thm. 6.8], we have that $V'(\Box^+\gamma) = 1$ for all $\gamma \in \Gamma$ and $V'(\Box^+\delta) \neq 1$ for all $\delta \in \Delta$. Therefore, $V'(\bigwedge\{\Box^+\gamma \mid \gamma \in \Gamma) = 1$, and since $\mathfrak{B}$ is well-connected, $V'(\bigvee\{\Box^+\delta \mid \delta \in \Delta) \neq 1$. Thus, V' witnesses that $\mathfrak{B} \not\models \gamma(\mathfrak{A})$. □

Example 5.3. The converse of Lemma 5.2 is not true in general. Let $\mathfrak{A}$ and $\mathfrak{B}$ be the **K4**-algebras that are dual to the **K4**-frames $\mathfrak{F}$ and $\mathfrak{G}$ shown below.

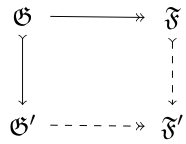


Clearly both $\mathfrak{F}, \mathfrak{G}$ are rooted and $\mathfrak{F}$ is a generated subframe of $\mathfrak{G}$. So $\mathfrak{A}$ is a subdirectly irreducible homomorphic image of $\mathfrak{B}$, and hence $\mathfrak{B} \not\models \gamma(\mathfrak{A})$. On the other hand, $\mathfrak{F}$ is not a stable image of $\mathfrak{G}$ since an onto stable map would send the root of $\mathfrak{G}$ to the root of $\mathfrak{F}$. But the root of $\mathfrak{G}$ is reflexive while the root of $\mathfrak{F}$ is irreflexive, a contradiction. Thus, there does not exist a stable embedding of $\mathfrak{A}$ into $\mathfrak{B}$.

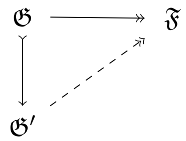
Of course, the key is that the root of $\mathfrak{F}$ is irreflexive. The next lemma shows that this is essential.

Lemma 5.4. *Let $\mathfrak{F} = (X, R)$, $\mathfrak{G} = (Y, Q)$, and $\mathfrak{G}' = (Y', Q')$ be finite* **K4***-frames such that $\mathfrak{F}$ is a stable image of $\mathfrak{G}$ and $\mathfrak{G}$ is a generated subframe of $\mathfrak{G}'$.*

(1) *There is a finite* **K4***-frame $\mathfrak{F}' = (X', R')$ such that $\mathfrak{F}$ is a generated subframe of $\mathfrak{F}'$, $\mathfrak{F}'$ is a stable image of $\mathfrak{G}'$, and the following diagram commutes.*



(2) *If in addition $\mathfrak{F}$ has a reflexive root, then $\mathfrak{F}$ is a stable image of $\mathfrak{G}'$ and the following diagram commutes.*



Proof. (1). Let $f : \mathfrak{G} \twoheadrightarrow \mathfrak{F}$ be an onto stable map. If $\mathfrak{G} = \mathfrak{G}'$, then there is nothing to show as we can take $\mathfrak{F}'$ to be $\mathfrak{F}$. Otherwise we construct $\mathfrak{F}'$ by adding a reflexive root to $\mathfrak{F}$; that is, $X' = X \cup \{r\}$ for some $r \notin X$ and $R' = R \cup \{(r, x) \mid x \in X\} \cup \{(r, r)\}$. It is easy to see that $\mathfrak{F}'$ is a **K4**-frame and that $\mathfrak{F}$ is a generated subframe of $\mathfrak{F}'$. Define $g : Y' \to X'$ so that the restriction of g to Y is f and g maps $Y' \setminus Y$ to the root r of $\mathfrak{F}'$; that is,

$$g(y) = \begin{cases} f(y) & \text{if } y \in Y \\ r & \text{otherwise.} \end{cases}$$

Since f is onto and $Y' \setminus Y \neq \emptyset$, it is clear that g is a well-defined onto map. To see that g is stable, let $x, y \in Y'$ with $x \leq y$. If $x \in Y$, then there is nothing to verify as f is stable. Otherwise $g(x) = r$ is the reflexive root, and so $g(x) R' g(y)$. Finally, it follows from the definition that the diagram commutes.

(2). Define $g : Y' \to X$ so that the restriction of g to Y is f and g maps $Y' \setminus Y$ to the reflexive root r of $\mathfrak{F}$ (provided $Y' \setminus Y \neq \varnothing$). An argument similar to the above shows that g is an onto stable map, and that the diagram commutes. $\square$

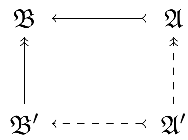
Using duality theory, we can reformulate Lemma 5.4 in algebraic terms. For a **K4**-algebra $\mathfrak{A} = (A, \Diamond)$ consider the condition:

(∗) There is an atom $a \in A$ such that $a \leq \Diamond b$ for each $b \neq 0$.

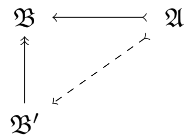
If $\mathfrak{A}$ is the dual of $\mathfrak{F}$, then it is easy to see that $\mathfrak{F}$ has a reflexive root iff $\mathfrak{A}$ satisfies (∗). Thus, Lemma 5.4 can be reformulated as follows.

Lemma 5.5. *Let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{B}'$ be finite* **K4***-algebras such that there is a stable embedding of $\mathfrak{A}$ into $\mathfrak{B}$ and $\mathfrak{B}$ is a homomorphic image of $\mathfrak{B}'$.*

(1) *There is a finite* **K4**-*algebra* $\mathfrak{A}'$ *such that* $\mathfrak{A}$ *is a homomorphic image of* $\mathfrak{A}'$, $\mathfrak{A}'$ *is isomorphic to a stable subalgebra of* $\mathfrak{B}'$, *and the following diagram commutes.*



(2) *If in addition* $\mathfrak{A}$ *satisfies* $(*)$, *then there is a stable embedding of* $\mathfrak{A}$ *into* $\mathfrak{B}'$ *and the following diagram commutes.*



We next build on Theorem 4.5 and obtain several more convenient characterizations for a logic above **K4** to be **K4**-stable. For a class $\mathcal{K}$ of **K4**-algebras, we let $\mathcal{K}_{wc}$ denote the class of well-connected members of $\mathcal{K}$.

Theorem 5.6. *Let* L *be a normal extension of* **K4**. *The following are equivalent.*

(1) L *is* **K4**-*stable.*
(2) L *is axiomatizable over* **K4** *by stable rules of finite* **K4**-*algebras.*
(3) $\mathcal{V}(L)$ *is generated by a* **K4**-*stable class of* **K4**-*algebras.*
(4) $\mathcal{V}(L)$ *is generated by a* **K4**-*stable class of finite* **K4**-*algebras.*
(5) $\mathcal{V}(L)_{wc}$ *is* **K4**-*stable.*
(6) *If* $\mathfrak{A}, \mathfrak{B}$ *are subdirectly irreducible* **K4**-*algebras,* $\mathfrak{A} \models L$, *and* $\mathfrak{B}$ *is a stable subalgebra of* $\mathfrak{A}$, *then* $\mathfrak{B} \models L$.

Moreover, each **K4**-*stable logic is axiomatizable by stable formulas.*

Proof. Since **K4** admits filtration, the equivalence of the first four conditions follows from Theorem 4.5. To prove the implication (4) ⇒ (5), we require the following claim.

Claim 5.7. *Suppose* $\mathcal{V}(L)$ *is generated by a* **K4**-*stable class* $\mathcal{K}$ *of finite* **K4**-*algebras,* $\mathfrak{A}$ *is a finite subdirectly irreducible* **K4**-*algebra, and* $\mathfrak{A} \not\models L$. *Then* $\gamma(\mathfrak{A}) \in L$.

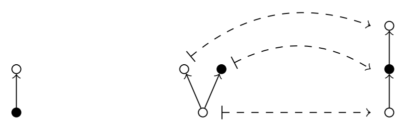
Proof. Suppose that $\gamma(\mathfrak{A}) \notin L$. Since $\mathcal{K}$ generates $\mathcal{V}(L)$, there is $\mathfrak{B} \in \mathcal{K}$ such that $\mathfrak{B} \not\models \gamma(\mathfrak{A})$. By [3, Thm. 6.8], there is a subdirectly irreducible homomorphic image $\mathfrak{C}$ of $\mathfrak{B}$ and a stable embedding of $\mathfrak{A}$ into $\mathfrak{C}$. By Lemma 5.5(1), there is a **K4**-algebra $\mathfrak{D}$ such that $\mathfrak{D}$ is isomorphic to a stable subalgebra of $\mathfrak{B}$ and $\mathfrak{A}$ is a homomorphic image of $\mathfrak{D}$. Since $\mathcal{K}$ is **K4**-stable and $\mathfrak{B} \in \mathcal{K}$, we have that $\mathfrak{D} \in \mathcal{K}$. Because $\mathcal{V}(L)$ is closed under homomorphic images, $\mathfrak{A} \in \mathcal{V}(L)$, which contradicts to $\mathfrak{A} \not\models L$. Therefore, $\gamma(\mathfrak{A}) \in L$, as desired. □

Now, suppose $\mathfrak{A}, \mathfrak{B}$ are **K4**-algebras, with $\mathfrak{B} \in \mathcal{V}(L)_{wc}$, and there is a stable embedding of $\mathfrak{A}$ into $\mathfrak{B}$. Since $\mathfrak{B}$ is well-connected, so is $\mathfrak{A}$ by Lemma 5.1. If $\mathfrak{A} \not\models L$, then $\mathfrak{A} \not\models \varphi$ for some $\varphi \in L$. As **K4** admits filtration, there is a finite **K4**-algebra $\mathfrak{C}$ such that $\mathfrak{C}$ is a stable subalgebra of $\mathfrak{A}$ and $\mathfrak{C} \not\models \varphi$. But then there is a stable embedding of $\mathfrak{C}$ into $\mathfrak{B}$. Since $\mathfrak{B}$ is well-connected, so is $\mathfrak{C}$ by Lemma 5.1. Therefore, as $\mathfrak{C}$ is finite, it is subdirectly irreducible. By Claim 5.7, $\gamma(\mathfrak{C}) \in L$. Because there is a stable embedding of $\mathfrak{C}$ into $\mathfrak{B}$, it follows by Lemma 5.2 that $\mathfrak{B} \not\models \gamma(\mathfrak{C})$, which contradicts to $\mathfrak{B} \models L$. Thus, $\mathfrak{A} \models L$, so $\mathfrak{A} \in \mathcal{V}(L)_{wc}$, and hence $\mathcal{V}(L)_{wc}$ is **K4**-stable.

The implication (5) ⇒ (6) follows from the fact that every subdirectly irreducible **K4**-algebra is well-connected. To see the implication (6) ⇒ (4), we show that L is the logic of finite subdirectly irreducible L-algebras. Indeed, if $L \not\vdash \varphi$, then there is a subdirectly irreducible L-algebra $\mathfrak{A}$ such that $\mathfrak{A} \not\models \varphi$. Since **K4** admits filtration, there is a finite **K4**-algebra $\mathfrak{C}$ such that $\mathfrak{C}$ is a stable subalgebra of $\mathfrak{A}$ and $\mathfrak{C} \not\models \varphi$. As $\mathfrak{A}$ is subdirectly irreducible, it is well-connected. Therefore, so is $\mathfrak{C}$ by Lemma 5.1. Thus, since $\mathfrak{C}$ is finite, it is subdirectly irreducible. But then (6) yields that $\mathfrak{C} \models L$. Consequently, $\mathcal{V}(L)$ is generated by the class of finite subdirectly irreducible L-algebras. Since this class is **K4**-stable, (4) follows.

Finally, we show that **K4**-stable logics are axiomatizable by stable formulas. Suppose that L is a **K4**-stable logic. Let $\mathcal{A}$ be the set of finite nonisomorphic subdirectly irreducible **K4**-algebras not belonging to $\mathcal{V}(L)$. We claim that $L = \mathbf{K4} + \{\gamma(\mathfrak{A}) : \mathfrak{A} \in \mathcal{A}\}$. The inclusion $\mathbf{K4} + \{\gamma(\mathfrak{A}) : \mathfrak{A} \in \mathcal{A}\} \subseteq L$ follows from Claim 5.7. For the reverse inclusion, let $\mathcal{V}$ be the variety corresponding to $\mathbf{K4} + \{\gamma(\mathfrak{A}) : \mathfrak{A} \in \mathcal{A}\}$. As subdirectly irreducible members of $\mathcal{V}$ generate $\mathcal{V}$, it is sufficient to show that each subdirectly irreducible member of $\mathcal{V}$ belongs to $\mathcal{V}(L)$. Let $\mathfrak{B}$ be a subdirectly irreducible member of $\mathcal{V}$. If $\mathfrak{B} \not\models L$, then since **K4** admits filtration, there is a finite **K4**-algebra $\mathfrak{B}'$ such that $\mathfrak{B}'$ is a stable subalgebra of $\mathfrak{B}$ and $\mathfrak{B}' \not\models L$. Because $\mathfrak{B}$ is subdirectly irreducible, it is well-connected. Therefore, $\mathfrak{B}'$ is well-connected by Lemma 5.1. Thus, as $\mathfrak{B}'$ is finite, it is subdirectly irreducible. So $\mathfrak{B}' \in \mathcal{A}$. Now, $\mathfrak{B} \not\models \gamma(\mathfrak{B}')$ by Lemma 5.2. Consequently, $\mathfrak{B} \notin \mathcal{V}$, a contradiction. This yields that $\mathfrak{B} \models L$, and hence $L = \mathbf{K4} + \{\gamma(\mathfrak{A}) : \mathfrak{A} \in \mathcal{A}\}$. □

Example 5.8. On the other hand, there exist logics above **K4** that are axiomatizable over **K4** by stable formulas, but are not **K4**-stable logics. To see this, consider the **K4**-frames $\mathfrak{F}$, $\mathfrak{G}$, and $\mathfrak{H}$ shown below.



$\mathfrak{F}$ $\mathfrak{H}$ $\mathfrak{G}$

We set $L = \mathbf{K4} + \gamma(\mathfrak{F})$. Clearly $\mathfrak{H}$ is the only non-singleton rooted upset of $\mathfrak{H}$ and $\mathfrak{F}$ is not a stable image of $\mathfrak{H}$ since $\mathfrak{H}$ has a reflexive root and $\mathfrak{F}$ has an irreflexive root. Therefore, $\mathfrak{H} \models \gamma(\mathfrak{F})$, and so $\mathfrak{H} \models L$. Next consider the map $\mathfrak{H} \to \mathfrak{G}$ indicated in the picture above. It is easy to see that it is a stable map from $\mathfrak{H}$ onto $\mathfrak{G}$. If L were **K4**-stable, Theorem 5.6 would yield $\mathfrak{G} \models \gamma(\mathfrak{F})$. However, $\mathfrak{G} \not\models \gamma(\mathfrak{F})$ as we already discussed in Example 5.3. Thus, L is not **K4**-stable.

Remark 5.9. It is of interest to study further the class of modal logics axiomatized by stable formulas. It is not even clear whether all such logics have the finite model property.

In Example 5.8 it was essential that the root of $\mathfrak{G}$ was irreflexive. We next show that every logic that is axiomatized over **K4** by stable formulas of finite **K4**-frames with reflexive roots are **K4**-stable. In algebraic terms we will show that every logic that is axiomatized over **K4** by stable formulas of finite **K4**-algebras that satisfy (∗) are **K4**-stable.

Proposition 5.10.
(1) *Let* $\mathfrak{A}$ *be a finite* **K4**-*algebra satisfying* (∗). *For a well-connected* **K4**-*algebra* $\mathfrak{B}$ *we have* $\mathfrak{B} \not\models \gamma(\mathfrak{A})$ *iff there is a stable embedding of* $\mathfrak{A}$ *into* $\mathfrak{B}$.
(2) *Suppose* $L = \mathbf{K4} + \{\gamma(\mathfrak{A}_i) \mid i \in I\}$, *where each* $\mathfrak{A}_i$ *satisfies* (∗). *Then* L *is* **K4**-*stable.*

Proof. (1). The right to left direction was already proven in Lemma 5.2. For the left to right direction, let $\mathfrak{B}$ be a **K4**-algebra such that $\mathfrak{B} \not\models \gamma(\mathfrak{A})$. Note that for this direction it is not needed that $\mathfrak{B}$ is well-connected. Since **K4** admits filtration, there is a finite **K4**-algebra $\mathfrak{C}$ that is a stable subalgebra of $\mathfrak{B}$ and $\mathfrak{C} \not\models \gamma(\mathfrak{A})$. By [3, Thm. 6.8], there is a subdirectly irreducible homomorphic image $\mathfrak{D}$ of $\mathfrak{C}$ and a stable embedding of $\mathfrak{A}$ into $\mathfrak{D}$. Since $\mathfrak{A}$ satisfies (∗), by Lemma 5.5(2), there is a stable embedding of $\mathfrak{A}$ into $\mathfrak{C}$, and hence a stable embedding of $\mathfrak{A}$ into $\mathfrak{B}$.

(2). It is immediate from (1) that the class of well-connected algebras of L is **K4**-stable. Now apply Theorem 5.6. □

Since every finite subdirectly irreducible **S4**-algebra satisfies $(*)$, Proposition 5.10 yields:

Corollary 5.11. *Let $\mathfrak{A}$ be a finite subdirectly irreducible* **S4**-*algebra. For every well-connected* **S4**-*algebra $\mathfrak{B}$ we have $\mathfrak{B} \not\models \gamma(\mathfrak{A})$ iff there is a stable embedding of $\mathfrak{A}$ into $\mathfrak{B}$.*

This immediately yields that all logics axiomatizable over **S4** by stable formulas of finite subdirectly irreducible **S4**-algebras are **S4**-stable. Thus, for normal extensions of **S4** we obtain the following improvement of Theorem 5.6.

Corollary 5.12. *Let L be a normal extension of* **S4**. *The following are equivalent.*

1. *L is* **S4**-*stable.*
2. *L is axiomatizable over* **S4** *by stable rules of finite* **S4**-*algebras.*
3. *L is axiomatizable over* **S4** *by stable formulas of finite subdirectly irreducible* **S4**-*algebras.*
4. *$\mathcal{V}(L)$ is generated by a* **S4**-*stable class of* **S4**-*algebras.*
5. *$\mathcal{V}(L)$ is generated by a* **S4**-*stable class of finite* **S4**-*algebras.*
6. *$\mathcal{V}(L)_{wc}$ is* **S4**-*stable.*
7. *If $\mathfrak{A}, \mathfrak{B}$ are subdirectly irreducible* **S4**-*algebras, $\mathfrak{A} \models L$, and $\mathfrak{B}$ is a stable subalgebra of $\mathfrak{A}$, then $\mathfrak{B} \models L$.*

6. Connection with stable superintuionistic logics

In this section we discuss the connection between **K4**-stable logics and **S4**-stable logics, as well as between **S4**-stable logics and stable superintuionistic logics of [2]. We conclude the section by showing that there are continuum many **S4**-stable logics, and continuum many **K4**-stable logics between **K4** and **S4**. While most of the results in the previous sections are stated in terms of algebras, from now on we will mainly work with frames, and utilize their geometric intuition.

We follow the notation of [9, Sec. 9.6]. If L is a superintuitionistic logic, then we denote by τL the least modal companion of L. Also, if M is a normal extension of **S4**, then we denote by ρM the superintuitionistic fragment of M. For an **S4**-frame $\mathfrak{F} = (X, R)$, let $\rho\mathfrak{F} = (\rho X, \rho R)$ be the skeleton $\mathfrak{F}$, which is obtained by modding out the clusters of $\mathfrak{F}$. It is well known (see, e.g., [9, Lem. 9.67]) that for every **S4**-frame $\mathfrak{F}$, we have $\mathfrak{F} \models \tau L$ iff $\rho\mathfrak{F} \models L$.

Theorem 6.1.

1. *Let $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, R)$ be finite rooted* **S4**-*frames. If $\mathfrak{G}$ is a stable image of $\mathfrak{F}$, then $\rho\mathfrak{G}$ is a stable image of $\rho\mathfrak{F}$.*
2. *If L is a stable superintuitionistic logic, then τL is* **S4**-*stable.*
3. *If $L = \mathbf{IPC} + \{\gamma(\mathfrak{G}_i) \mid i \in I\}$, then $\tau L = \mathbf{S4} + \{\gamma(\mathfrak{G}_i) \mid i \in I\}$.*

Proof. (1). Let $f : X \to Y$ be an onto stable map. Since the quotient map $\pi_Y : Y \to \rho Y$ is an onto p-morphism, the composition $\pi_Y \circ f : X \to \rho Y$ is onto and stable. Define $g : \rho X \to \rho Y$ by $g(\pi_X(x)) = \pi_Y(f(x))$. Because $\pi_Y \circ f$ is stable, g is well defined, and it is clear that g is onto and stable. Therefore, $\rho\mathfrak{G}$ is a stable image of $\rho\mathfrak{F}$.

(2). Let L be a stable superintuitionistic logic. By [2, Thm. 6.8], L has the finite model property. Therefore, so does τL (see, e.g., [9, p. 328]). Thus, τL is the logic of its finite rooted frames. We show that this class is **S4**-stable. Let $\mathfrak{F}$ be a finite rooted τL-frame and $\mathfrak{G}$ be a finite rooted **S4**-frame that is a stable image of $\mathfrak{F}$. Since $\mathfrak{F}$ is a τL-frame, $\rho\mathfrak{F}$ is an L-frame. By (1), $\rho\mathfrak{G}$ is a stable image of $\rho\mathfrak{F}$. As L is stable, $\rho\mathfrak{G} \models L$. Therefore, $\mathfrak{G} \models \tau L$, and hence the class of finite rooted τL-frames is **S4**-stable. Thus, by Corollary 5.12, τL is an **S4**-stable logic.

(3). Let $M = \mathbf{S4} + \{\gamma(\mathfrak{G}_i) \mid i \in I\}$. By (1) and Corollary 5.12, both τL and M are **S4**-stable. Therefore, to see that $\tau L = M$, it is sufficient to check that the two logics have the same finite rooted frames. Let $\mathfrak{F}$ be a finite rooted **S4**-frame. If $\mathfrak{F} \not\models \tau L$, then $\rho\mathfrak{F} \not\models L$, so $\mathfrak{G}_i$ is a stable image of $\rho\mathfrak{F}$ for some $i \in I$. Since $\rho\mathfrak{F}$ is a stable image of $\mathfrak{F}$, we conclude that $\mathfrak{G}_i$ is a stable image of $\mathfrak{F}$. Thus, $\mathfrak{F} \not\models \gamma(\mathfrak{G}_i)$, and hence $\mathfrak{F} \not\models M$. Conversely, if $\mathfrak{F} \not\models M$, then $\mathfrak{G}_i$ is a stable image of $\mathfrak{F}$ for some

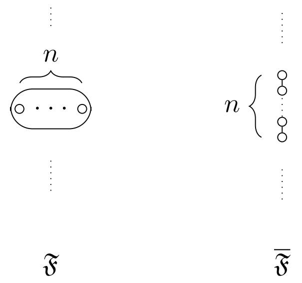


FIGURE 2

$i \in I$. From (1) it follows that $\rho\mathfrak{G}_i$ is a stable image of $\rho\mathfrak{F}$. Since $\mathfrak{G}_i$ is partially ordered, $\mathfrak{G}_i \cong \rho\mathfrak{G}_i$, implying that $\mathfrak{G}_i$ is a stable image of $\rho\mathfrak{F}$. Thus, $\rho\mathfrak{F} \not\models L$, and so $\mathfrak{F} \not\models \tau L$. □

For a finite rooted **S4**-frame $\mathfrak{F} = (X, R)$, let $\overline{\mathfrak{F}} = (X, \overline{R})$ be the partially ordered **S4**-frame that is obtained from $\mathfrak{F}$ by unraveling each n-cluster into an n-chain (see Figure 2); that is, if $X = C_1 \cup \cdots \cup C_k$ is the division of $\mathfrak{F}$ into clusters, with $C_i = \{x_{i_1}, \ldots, x_{i_{n_i}}\}$, then for all $x = x_{i_l}$ and $y = x_{j_m}$, we have

$$x\overline{R}y \quad \text{iff} \begin{cases} i = j \text{ and } l \geq m \text{ or} \\ i \neq j \text{ and } xRy. \end{cases}.$$

Note that $w_{i_{n_i}}$ is the root of the chain C_i in $\overline{\mathfrak{F}}$.

Theorem 6.2.
(1) *Let $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, R)$ be finite rooted* **S4***-frames, with $\mathfrak{G}$ being partially ordered. Then $\mathfrak{F}$ is a stable image of $\mathfrak{G}$ iff $\overline{\mathfrak{F}}$ is a stable image of $\mathfrak{G}$.*
(2) *If M is* **S4***-stable, then ρM is stable.*
(3) *If $M = \mathbf{S4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$, then $\rho M = \mathbf{IPC} + \{\gamma(\overline{\mathfrak{F}}_i) \mid i \in I\}$.*

Proof. (1). Since $\mathfrak{F}$ is easily seen to be a stable image of $\overline{\mathfrak{F}}$, the implication from right to left is obvious. Conversely, suppose that $f : \mathfrak{G} \to \mathfrak{F}$ is an onto stable map. We transform f into a stable map $\overline{f} : \mathfrak{G} \to \overline{\mathfrak{F}}$ by shuffling the values of f belonging to some cluster of $\mathfrak{F}$. Let C_i be a cluster of $\mathfrak{F}$ and let $Y' = f^{-1}(C_i)$. We view Y' as a subframe of $\mathfrak{G}$, and define $\overline{f} : Y' \to C_i$ by induction on the depth of points in Y'. The idea is to map the points of the smallest depth injectively onto the first $n_i - 1$ points of C_i and all the other points of Y' to the root $w_{i_{n_i}}$. More precisely, suppose $\{y_1, \ldots, y_m\} \subseteq Y'$ are the points of depth d and we have mapped all the points of Y' of smaller depth injectively onto $\{w_{i_1}, \ldots, w_{i_l}\}$. If $m \leq n_i - l$, then set $\overline{f}(y_h) = x_{i_{l+h}}$ for all $1 \leq h \leq m$. If $m \not\leq n_i - l$, then define $\overline{f}$ as before for all y_l with $l \leq m - (n_i - l)$ and map all the other points of Y' to $w_{i_{n_i}}$. It is straightforward to check that $\overline{f}$ is stable.

(2). Since M is **S4**-stable, it has the finite model property. Therefore, so does ρM (see, e.g., [9, p. 328]). It thus suffices to show that the finite rooted ρM-frames form a stable class. Suppose $\mathfrak{G}$ is a stable image of a finite rooted ρM-frame $\mathfrak{F}$. From $\mathfrak{F} \models \rho M$ it follows that $\mathfrak{F} \models M$. Since M is stable, $\mathfrak{G} \models M$. Consequently, $\mathfrak{G} \models \rho M$.

(3). Since M is **S4**-stable, ρM is stable by (2). Let $L = \mathbf{IPC} + \{\gamma(\overline{\mathfrak{F}}_i) \mid i \in I\}$. By [2, Thm. 6.11], L is stable. Therefore, both ρM and L have the finite model property, and hence it suffices to show that the two logics have the same finite rooted frames. Suppose $\mathfrak{G}$ is a finite rooted frame such that $\mathfrak{G} \models \rho M$. If $\mathfrak{G} \not\models L$, then there is $i \in I$ such that $\mathfrak{G} \not\models \gamma(\overline{\mathfrak{F}}_i)$. Therefore, $\overline{\mathfrak{F}}_i$ is a stable image of $\mathfrak{G}$. By (1), $\mathfrak{F}_i$ is a stable image of $\mathfrak{G}$. Thus, $\mathfrak{G} \not\models \gamma(\mathfrak{F}_i)$, and so $\mathfrak{G} \not\models M$, contradicting $\mathfrak{G} \models \rho M$. Consequently, $\mathfrak{G} \models L$. Conversely, if $\mathfrak{G} \not\models \rho M$, then $\mathfrak{G} \not\models M$, and hence $\mathfrak{G} \not\models \gamma(\mathfrak{F}_i)$ for some $i \in I$.

Therefore, $\mathfrak{F}_i$ is a stable image of $\mathfrak{G}$. By (1), $\overline{\mathfrak{F}_i}$ is a stable image of $\mathfrak{G}$. Thus, $\mathfrak{G} \not\models \gamma(\overline{\mathfrak{F}}_i)$, yielding that $\mathfrak{G} \not\models L$. □

Corollary 6.3.

(1) *A superintuitionistic logic L is stable iff τL is* **S4***-stable.*
(2) *An* **S4***-stable logic is the least modal companion of a superintuitionistic logic iff it can be axiomatized by stable formulas of finite rooted partially ordered* **S4***-frames.*

Proof. (1). It is well known that $L = \rho\tau L$ (see, e.g., [9, Thm. 9.57]). Now apply Theorems 6.1(3) and 6.2(3).

(2). Suppose M is the least modal companion of a superintuitionistic logic L. Then $M = \tau(L)$, and so $L = \rho(M)$. Since M is **S4**-stable, L is stable by Theorem 6.2(2). Therefore, by [2, Thm. 6.11], there are finite rooted partially ordered frames $\{\mathfrak{F}_i \mid i \in I\}$ such that $L = \mathbf{IPC} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$. Thus, $M = \mathbf{S4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$ by Theorem 6.1(3). Conversely, if $M = \mathbf{S4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$ for some finite rooted partially ordered **S4**-frames $\{\mathfrak{F}_i \mid i \in I\}$, then $\rho(M) = \mathbf{IPC} + \{\gamma(\overline{\mathfrak{F}}_i) \mid i \in I\}$ by Theorem 6.2(3). Since $\overline{\mathfrak{F}}_i = \mathfrak{F}_i$ for all $i \in I$, we conclude that $\tau\rho(M) = \mathbf{IPC} + \{\gamma(\overline{\mathfrak{F}}_i) \mid i \in I\} = M$, and hence M is the least modal companion of $\rho(M)$. □

Remark 6.4. On the other hand, the greatest modal companion of a stable superintuitionistic logic is not necessarily **S4**-stable. For instance, the Grzegorczyk logic **S4**.**Grz** is the greatest modal companion of the intuitionistic propositional calculus **IPC**, and we will see in Section 7 that it is not **S4**-stable.

Next we discuss connections between **S4**-stable and **K4**-stable logics. For a formula φ, let φ^+ be obtained from φ by replacing each subformula of φ of the form $\Box\psi$ by $\psi \wedge \Box\psi$. If $L = \mathbf{S4} + \Gamma$ is a normal extension of **S4**, let $L^+ = \mathbf{K4} + \Gamma^+$, where $\Gamma^+ = \{\varphi^+ \mid \varphi \in \Gamma\}$. For a binary relation R on X, let $R^+ := R \cup \{(x,x) \mid x \in X\}$ be the *reflexive closure* of R. For a **K4**-space $\mathfrak{F} = (X, R)$, define the *reflexivization* of $\mathfrak{F}$ as $\mathfrak{F}^+ = (X, R^+)$. Then $\mathfrak{F}^+$ is an **S4**-space and $\mathfrak{F} \models L^+$ iff $\mathfrak{F}^+ \models L$. Therefore, L^+ is the logic of $\{\mathfrak{F} \mid \mathfrak{F}^+ \models L\}$ (see, e.g., [9, Sec. 3.9]).

Lemma 6.5.

(1) *Let $\mathfrak{F}$ be a finite* **S4***-frame and let $\mathfrak{G}$ be a* **K4***-space. Then $\mathfrak{F}$ is a stable image of $\mathfrak{G}$ iff $\mathfrak{F}$ is a stable image of $\mathfrak{G}^+$.*
(2) *If $L = \mathbf{S4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$, where the $\mathfrak{F}_i$ are* **S4***-frames, then $L^+ = \mathbf{K4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$.*
(3) *If L is* **S4***-stable, then L^+ is* **K4***-stable.*

Proof. (1). Immediate since $\mathfrak{F}$ is reflexive.

(2). By (1) and Corollary 5.11, if $\mathfrak{G}$ is a rooted **K4**-space, then $\mathfrak{G} \models \gamma(\mathfrak{F}_i)$ iff $\mathfrak{G}^+ \models \gamma(\mathfrak{F}_i)$. Therefore, $\mathfrak{G} \models L^+$ iff $\mathfrak{G}^+ \models L$ iff $\mathfrak{G}^+ \models \{\gamma(\mathfrak{F}_i) \mid i \in I\}$ iff $\mathfrak{G} \models \{\gamma(\mathfrak{F}_i) \mid i \in I\}$. Thus, L^+ and $\mathbf{K4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$ have the same rooted **K4**-spaces, and hence the two logics coincide.

(3). If L is **S4**-stable, then L is axiomatized by stable formulas of **S4**-frames. By (2), L^+ is axiomatized by the same stable formulas. In particular, L^+ is axiomatized by stable formulas of frames with reflexive roots. Thus, L^+ is **K4**-stable by Proposition 5.10. □

For two normal modal logics L_1 and L_2, let $L_1 \vee L_2$ denote the join of these logics in the lattice of normal modal logics.

Lemma 6.6. *Let L be a normal extension of* **K4***.*

(1) *If $\mathbf{S4} \subseteq L$, then L is* **K4***-stable iff L is* **S4***-stable.*
(2) *If L is* **K4***-stable, then $\mathbf{S4} \vee L$ is* **S4***-stable.*
(3) *If $L = \mathbf{K4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$, then $\mathbf{S4} \vee L = \mathbf{S4} + \{\gamma(\mathfrak{F}_i) \mid \mathfrak{F}_i = \mathfrak{F}_i^+\}$.*
(4) *If $L = \mathbf{K4} + \{\gamma(\mathfrak{F}_i) \mid i \in I\}$, then $L \subseteq \mathbf{S4}$ iff each $\mathfrak{F}_i$ contains an irreflexive point.*

Proof. (1). Observe that **S4** is a **K4**-stable class and apply Lemma 4.2(3).

(2). By Theorem 5.6, the rooted L-spaces are **K4**-stable. Therefore, the rooted $(\mathbf{S4} \vee L)$-spaces are **S4**-stable. Thus, $\mathbf{S4} \vee L$ is **S4**-stable by Corollary 5.12.

(3). Let $\mathfrak{G}$ be a rooted **S4**-space. We have $\mathfrak{G} \models \mathbf{S4} \vee L$ iff $\mathfrak{G} \models L$ iff $\mathfrak{G} \models \gamma(\mathfrak{F}_i)$ for all $i \in I$. It is obvious that $\mathfrak{G} \models \gamma(\mathfrak{F}_i)$ for every $\mathfrak{F}_i$ that contains an irreflexive point because no such $\mathfrak{F}_i$ can be a stable image of a reflexive space. Therefore, $\mathfrak{G} \models \gamma(\mathfrak{F}_i)$ for all $i \in I$ is equivalent to $\mathfrak{G} \models \gamma(\mathfrak{F}_i)$ for all $\mathfrak{F}_i$ with $\mathfrak{F}_i = \mathfrak{F}_i^+$. Thus, $\mathbf{S4} \vee L = \mathbf{S4} + \{\gamma(\mathfrak{F}_i) \mid \mathfrak{F}_i = \mathfrak{F}_i^+\}$.

(4). First suppose that each $\mathfrak{F}_i$ contains an irreflexive point. Then $\mathfrak{F}_i \neq \mathfrak{F}_i^+$ for all $i \in I$. Therefore, (3) implies that $L \vee \mathbf{S4} = \mathbf{S4}$, and hence $L \subseteq \mathbf{S4}$. Conversely, suppose that some $\mathfrak{F}_i$ is reflexive. Since $\mathfrak{F}_i \not\models L$ and $\mathfrak{F}_i$ is an **S4**-frame, we see that $L \not\subseteq \mathbf{S4}$. □

Theorem 6.7.

(1) *There are continuum many* **K4**-*stable logics above* **S4**.

(2) *There are continuum many* **K4**-*stable logics between* **K4** *and* **S4**.

Proof. (1). By [2, Thm. 6.13], there are continuum many stable superintuitionistic logics. Since $L \neq L'$ implies $\tau L \neq \tau L'$, this together with Lemma 6.1 yields continuum many **S4**-stable logics above **S4**. By Lemma 6.6(1), these logics are also **K4**-stable. Thus, there are continuum many **K4**-stable logics above **S4**.

(2). Consider the sequence $\{\mathfrak{F}_n \mid n \in \mathbb{N}_{\geq 1}\}$ shown in Figure 3. By [2, Lem. 6.12], $\mathfrak{F}_n$ is not a stable image of $\mathfrak{F}_m$ for $n \neq m$. We slightly modify the sequence. For $n \in \mathbb{N}_{\geq 1}$, let $\mathfrak{G}_n$ be the **K4**-frame that is obtained from $\mathfrak{F}_n$ by making x_1 irreflexive. The proof of [2, Lem. 6.12] shows that $\mathfrak{G}_n$ is not a stable image of $\mathfrak{G}_m$ for $n \neq m$.

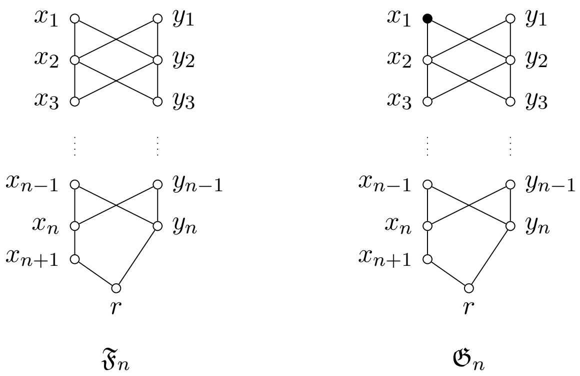


FIGURE 3

For $I \subseteq \mathbb{N}_{\geq 1}$ let $L_I = \mathbf{K4} + \{\gamma(\mathfrak{G}_n) \mid n \in I\}$. Since each $\mathfrak{G}_n$ has a reflexive root, by Proposition 5.10, every L_I is **K4**-stable. As each $\mathfrak{G}_n$ has a an irreflexive point, by Lemma 6.6(4), $L_I \subseteq \mathbf{S4}$ for every $I \subseteq \mathbb{N}_{\geq 1}$. Thus, every L_I is a **K4**-stable logic between **K4** and **S4**. Finally, if $n \in I \setminus J$, then $\gamma(\mathfrak{G}_n) \in L_J \setminus L_I$, so the cardinality of $\{L_I \mid I \subseteq \mathbb{N}_{\geq 1}\}$ is that of continuum, completing the proof. □

7. EXAMPLES OF K4-STABLE AND S4-STABLE LOGICS

In this final section we give many examples of **K4**-stable and **S4**-stable logics, and compare them to subframe and splitting transitive logics. We also show, as promised earlier, that some well-known logics are not stable.

We start by considering the following well-known logics (see, e.g., [9, p. 116]):

- $\mathbf{D4} = \mathbf{K4} + \Box p \to \Diamond p$, the logic of serial **K4**-frames;
- $\mathbf{K4B} = \mathbf{K4} + p \to \Box\Diamond p$, the logic of symmetric **K4**-frames;
- $\mathbf{K4.2} = \mathbf{K4} + \Diamond(p \wedge \Box q) \to \Box(p \vee \Diamond q)$, the logic of directed **K4**-frames;
- $\mathbf{K4.3} = \mathbf{K4} + \Box(\Box^+ p \to q) \vee \Box(\Box^+ q \to p)$, the logic of connected **K4**-frames;
- $\mathbf{K4BW}_n = \mathbf{K4} + \mathbf{bw}_n$, the logic of **K4**-frames of width $\leq n$, where

$$\mathbf{bw}_n := \bigwedge_{i=0}^{n-1} \Diamond p_i \to \bigvee_{0 \leq i \neq j \leq n} \Diamond(p_i \wedge \Diamond^+ p_j);$$

- $\mathbf{K4Alt}_n = \mathbf{K4} + \mathbf{alt}_n$, the logic of **K4**-frames such that each point has $\leq n$ alternatives, where

$$\mathbf{alt}_n := \Box p_0 \vee \Box(p_0 \to p_1) \vee \cdots \vee \Box(p_0 \wedge \cdots \wedge p_{n-1} \to p_n).$$

We also let $\mathbf{K4BTW}_n$ be the logic of **K4**-frames of top width $\leq n$, and consider the following normal extensions of **S4**:

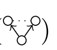
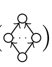
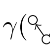
- $\mathbf{S5} = \mathbf{S4} \vee \mathbf{K4B}$;
- $\mathbf{S4.2} = \mathbf{S4} \vee \mathbf{K4.2}$;
- $\mathbf{S4.3} = \mathbf{S4} \vee \mathbf{K4.3}$;
- $\mathbf{S4BW}_n = \mathbf{S4} \vee \mathbf{K4BW}_n$;
- $\mathbf{S4BTW}_n = \mathbf{S4} \vee \mathbf{K4BTW}_n$;
- $\mathbf{S4Alt}_n = \mathbf{S4} \vee \mathbf{K4Alt}_n$.

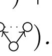
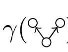
Their corresponding superintuitionistic fragments are the following logics:

- $\mathbf{CPC} = \mathbf{IPC} + p \vee \neg p$, the classical logic;
- $\mathbf{LC} = \mathbf{IPC} + (p \to q) \vee (q \to p)$, the Gödel-Dummett logic;
- $\mathbf{KC} = \mathbf{IPC} + \neg p \vee \neg\neg p$, the logic of weak excluded middle;
- $\mathbf{BW}_n = \mathbf{IPC} + \bigvee_{i=0}^{n}(p_i \to \bigvee_{j \neq i} p_j)$;
- $\mathbf{BTW}_n = \mathbf{IPC} + \bigwedge_{0 \leq i \leq j \leq n} \neg(\neg p_i \wedge \neg p_j) \to \bigvee_{i=0}^{n}(\neg p_i \to \bigvee_{j \neq i} \neg p_j)$.

In fact, $\mathbf{S5} = \tau(\mathbf{CPC})$, $\mathbf{S4.2} = \tau(\mathbf{LC})$, $\mathbf{S4.3} = \tau(\mathbf{KC})$, and more generally, $\mathbf{S4BW}_n = \tau(\mathbf{BW}_n)$ and $\mathbf{S4BTW}_n = \tau(\mathbf{BTW}_n)$ for every n. Lemma 6.1 together with the axiomatizations provided in [2, Thm. 7.5] yield:

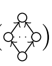
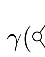
Proposition 7.1. *The logics* **S4.2** *and* **S4.3** *are* **S4**-*stable. More generally,* $\mathbf{S4BW}_n$ *and* $\mathbf{S4BTW}_n$ *are* **S4**-*stable for every* n. *These logics are axiomatized by the following stable formulas:*

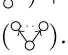
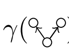
(1) $\mathbf{S4BW}_n = \mathbf{S4} + \gamma($$) + \gamma($$)$. *In particular,* $\mathbf{S4.3} = \mathbf{S4} + \gamma($$) + \gamma($$)$.

(2) $\mathbf{S4BTW}_n = \mathbf{S4} + \gamma($$)$. *In particular,* $\mathbf{S4.2} = \mathbf{S4} + \gamma($$)$.

Since $\mathbf{K4.2} = \mathbf{S4.2}^+$, $\mathbf{K4.3} = \mathbf{S4.3}^+$, and more generally, $\mathbf{K4BW}_n = (\mathbf{S4BW}_n)^+$ and $\mathbf{K4BTW}_n = (\mathbf{S4BTW}_n)^+$ for every n, Proposition 7.1 together with Lemma 6.5 yield:

Proposition 7.2. *The logics* **K4.2** *and* **K4.3** *are* **K4**-*stable. More generally,* $\mathbf{K4BW}_n$ *and* $\mathbf{K4BTW}_n$ *are* **K4**-*stable for every* n. *These logics are axiomatized by the following stable formulas:*

(1) $\mathbf{K4BW}_n = \mathbf{K4} + \gamma($$) + \gamma($$)$. *In particular,* $\mathbf{K4.3} = \mathbf{K4} + \gamma($$) + \gamma($$)$.

(2) $\mathbf{K4BTW}_n = \mathbf{K4} + \gamma($$)$. *In particular,* $\mathbf{K4.2} = \mathbf{K4} + \gamma($$)$.

It is well known (see, e.g., [9]) that all these logics under consideration have the finite model property, and hence are the logics of their finite rooted frames.

Theorem 7.3. *The logics* **D4**, **S4**, **K4B**, *and* $\mathbf{K4Alt}_n$ *are* **K4**-*stable for all* n. *They are axiomatized by the following stable formulas:*

(1) $\mathbf{D4} = \mathbf{K4} + \gamma(\bullet)$;

(2) $\mathbf{S4} = \mathbf{K4} + \gamma(\bullet) + \gamma(\overset{\circ}{\bullet})$;
(3) $\mathbf{K4B} = \mathbf{K4} + \gamma(\overset{\circ}{\circ})$;
(4) $\mathbf{K4Alt}_n = \mathbf{K4} + \gamma(\boxed{\circ\cdots\circ})$, *where* $\boxed{\circ\cdots\circ}$ *denotes the* $(n+1)$*-cluster.*

Proof. Since all the logics in the above list have the finite model property, they are logics of their finite rooted frames. Therefore, to see that these logics are **K4**-stable, it suffices to show that their finite rooted frames form a **K4**-stable class. But this is straightforward.

As an example, we show that the class of finite rooted frames for **K4B** is **K4**-stable. Let $\mathfrak{F} = (X, R)$ be a finite rooted **KB**-frame with root r. Then $\mathfrak{F}$ is transitive and symmetric. Let $f : \mathfrak{F} \to \mathfrak{G}$ be a stable map onto the finite **K4**-frame $\mathfrak{G} = (Y, R)$. We show that $\mathfrak{G}$ is symmetric. Suppose that $f(w)Rf(v)$. Since r is a root of $\mathfrak{F}$, rRw and rRv in $\mathfrak{F}$, and by symmetry, also vRr implying $f(r)Rf(w)$ and $f(v)Rf(r)$. Thus, $f(v)Rf(w)$ by transitivity of $\mathfrak{F}$.The stability of the other logics can be proven similarly. Next we verify the axiomatizations in terms of stable formulas.

(1). Let $\mathfrak{X}$ be a **K4**-space. It is sufficient to show that $\mathfrak{X} \models \Box p \to \Diamond p$ iff $\mathfrak{X} \models \gamma(\bullet)$. If $\mathfrak{X} \not\models \Box p \to \Diamond p$, then there is $x \in X$ such that $x \not R y$ for all $y \in X$. Therefore, $\{x\}$ is a closed up-set of X, and $\mathfrak{Y} = (\{x\}, \emptyset)$ is a finite rooted **K4**-frame. The unique map from $\mathfrak{Y}$ onto $\bullet$ is stable, and so we conclude that $\mathfrak{X} \not\models \gamma(\bullet)$. Conversely, suppose that $\mathfrak{X} \not\models \gamma(\bullet)$. Then there is a stable map from a topo-rooted closed up-set $\mathfrak{Y}$ of $\mathfrak{X}$ onto $\bullet$. This implies that $\mathfrak{Y}$ is a singleton with no R-successors, and hence $\mathfrak{X}$ contains a point with no R-successors. Thus, $\mathfrak{X} \not\models \Box p \to \Diamond p$.

(2). Let $\mathfrak{X}$ be a **K4**-space. It is sufficient to show that $\mathfrak{X} \models p \to \Diamond p$ iff $\mathfrak{X} \models \gamma(\bullet), \gamma(\overset{\circ}{\bullet})$. Suppose $\mathfrak{X} \not\models \gamma(\bullet)$ or $\mathfrak{X} \not\models \gamma(\overset{\circ}{\bullet})$. Then there is a topo-rooted closed up-set $\mathfrak{Y}$ of $\mathfrak{X}$ and a stable map from $\mathfrak{Y}$ onto $\bullet$ or $\overset{\circ}{\bullet}$. Observe that under a stable map a preimage of an irreflexive point has to be irreflexive. Now both of the latter frames contain an irreflexive point, so in either case $\mathfrak{Y}$ contains an irreflexive point. Therefore, so does $\mathfrak{X}$. Thus, $\mathfrak{X}$ is not reflexive, and so $\mathfrak{X} \not\models p \to \Diamond p$. For the converse, suppose that x is an irreflexive point of $\mathfrak{X}$. Consider the closed up-set $Y := R^+[x]$ of $\mathfrak{X}$, and let $\mathfrak{Y}$ be the corresponding **K4**-space. Clearly x is a unique root of $\mathfrak{Y}$. Since $x \notin R[x]$, there is a clopen subset of $\mathfrak{X}$ separating x from $R[x]$. Therefore, x is an isolated point of Y. Thus, $\mathfrak{Y}$ is topo-rooted. If $Y = \{x\}$, then the unique map from $\mathfrak{Y}$ onto $\bullet$ is stable, and so $\mathfrak{X} \not\models \gamma(\bullet)$. Otherwise mapping x to the root of $\overset{\circ}{\bullet}$ and the rest of Y to the top point of $\overset{\circ}{\bullet}$ gives rise to a stable map, and hence $\mathfrak{X} \not\models \gamma(\overset{\circ}{\bullet})$.

(3). As we already pointed out, **K4B** has the finite model property. Also, since $\mathbf{K4} + \gamma(\overset{\circ}{\circ})$ is axiomatized over **K4** by the stable formula of a finite rooted **K4**-frame with a reflexive root, it has the finite model property by Proposition 5.10. Therefore, it is sufficient to show that for any finite rooted **K4**-frame $\mathfrak{F} = (X, R)$, we have $\mathfrak{F} \models p \to \Box\Diamond p$ iff $\mathfrak{F} \models \gamma(\overset{\circ}{\circ})$. Suppose $\mathfrak{F} \not\models p \to \Box\Diamond p$. Then $\mathfrak{F}$ is not symmetric, and so there are $x, y \in X$ such that xRy but $y \not R x$. Define $f : \mathfrak{F} \to \overset{\circ}{\circ}$ by mapping $R^+[y]$ to the top node of $\overset{\circ}{\circ}$ and the rest to the root of $\overset{\circ}{\circ}$. It is easy to see that f is an onto stable map. Therefore, $\mathfrak{F} \not\models \overset{\circ}{\circ}$. Conversely, if $\mathfrak{F} \not\models \gamma(\overset{\circ}{\circ})$, then since $\mathfrak{F}$ is rooted, by Lemma 5.2, there is a stable map from $\mathfrak{F}$ onto $\overset{\circ}{\circ}$. Let x be a root of $\mathfrak{F}$ and let $y \in X$ be such that $f(y)$ is the top point of $\overset{\circ}{\circ}$. Since f is stable, xRy but $y \not R x$. Thus, $\mathfrak{F}$ is not symmetric. This yields that $\mathfrak{F} \not\models p \to \Box\Diamond p$.

(4). Observe that there is a stable map from a finite rooted **K4**-frame $\mathfrak{F}$ onto the $(n+1)$-cluster $\boxed{\circ\cdots\circ}$ iff the cardinality of $\mathfrak{F}$ is greater than n. The result follows since both $\mathbf{K4Alt}_n$ and $\mathbf{K4} + \gamma(\boxed{\circ\cdots\circ})$ have the finite model property. $\square$

As a corollary to Theorem 7.3 we obtain:

Corollary 7.4. *The logics* **S5** *and* $\mathbf{S4Alt}_n$ *are* **S4***-stable for all* n*. They are axiomatized over* **S4** *by the following stable formulas:*

(1) $\mathbf{S5} = \mathbf{S4} + \gamma(\ )$.
(2) $\mathbf{S4Alt}_n = \mathbf{S4} + \gamma(\circ \cdots \circ)$.

In the following table we summarize the axiomatizations of **K4**-stable and **S4**-stable logics obtained above.

Table 1. Axiomatizations of some **K4**-stable and **S4**-stable logics

D4	=	$\mathbf{K4} + \gamma(\bullet)$	**S4**	=	$\mathbf{K4} + \gamma(\bullet) + \gamma(\ )$
K4B	=	$\mathbf{K4} + \gamma(\ )$	**S5**	=	$\mathbf{S4} + \gamma(\ )$
K4.2	=	$\mathbf{K4} + \gamma(\ )$	**S4.2**	=	$\mathbf{S4} + \gamma(\ )$
K4.3	=	$\mathbf{K4} + \gamma(\ ) + \gamma(\ )$	**S4.3**	=	$\mathbf{S4} + \gamma(\ ) + \gamma(\ )$
$\mathbf{K4BW}_n$	=	$\mathbf{K4} + \gamma(\ ) + \gamma(\ )$	$\mathbf{S4BW}_n$	=	$\mathbf{S4} + \gamma(\ ) + \gamma(\ )$
$\mathbf{K4BTW}_n$	=	$\mathbf{K4} + \gamma(\ )$	$\mathbf{S4BTW}_n$	=	$\mathbf{S4} + \gamma(\ )$
$\mathbf{K4Alt}_n$	=	$\mathbf{K4} + \mathbf{K4} + \gamma(\circ \cdots \circ)$	$\mathbf{S4Alt}_n$	=	$\mathbf{S4} + \gamma(\circ \cdots \circ)$

Remark 7.5. As we saw in Theorem 3.2, **S5** is actually a stable logic. Therefore, by [3, Prop. 7.6], it is axiomatizable over **K** by stable rules. We show that **S5** is in fact axiomatized by the following set $\Gamma := \{\rho(\bullet), \rho(\bullet\leftrightarrow\circ), \rho(\ ), \rho(\ )\}$ of stable rules. For this we first observe that a finite rooted frame validates Γ iff it is a cluster. It is easy to see that none of the frames $\bullet$, $\bullet\leftrightarrow\circ$, , and is a stable image of a finite cluster, so every finite cluster validates Γ. Conversely, suppose that $\mathfrak{F} = (X, R)$ is a finite rooted frame that is not a cluster. There are several cases to consider. If $\mathfrak{F}$ is a singleton, then it is irreflexive, so $\bullet$ is a stable image of $\mathfrak{F}$, and hence $\mathfrak{F} \not\models \rho(\bullet)$. Suppose that $\mathfrak{F}$ has at least two points. If $\mathfrak{F}$ contains an irreflexive point x, then $\bullet\leftrightarrow\circ$ is a stable image of $\mathfrak{F}$ as mapping x to the irreflexive point of $\bullet\leftrightarrow\circ$ and the rest to the reflexive point of $\bullet\leftrightarrow\circ$ is an onto stable map. Therefore, $\mathfrak{F} \not\models \rho(\bullet\leftrightarrow\circ)$. Finally, suppose that $\mathfrak{F}$ is reflexive. If $\mathfrak{F}$ contains exactly two points x and y, then without loss of generality we may assume that xRy and $y\not{R}x$. Thus, mapping x to the root of and y to the other point of is stable and onto, and hence $\mathfrak{F} \not\models \rho(\ )$. Suppose $\mathfrak{F}$ has at least three points. We show that there is a stable map from $\mathfrak{F}$ onto . Since $\mathfrak{F}$ is not a cluster, without loss of generality we may assume that there are $x, y \in \mathfrak{F}$ with $x\not{R}y$. Then mapping x to the top node, y to the bottom right node, and all the other points to the bottom left node of provides a stable map that is also onto since $\mathfrak{F}$ contains at least three points. This yields $\mathfrak{F} \not\models \rho(\ )$.

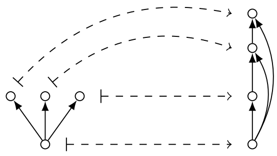
Now, let L be the logic axiomatized over **K** by Γ. Since **S5** is the logic of finite clusters and each such validates Γ, we see that $L \subseteq \mathbf{S5}$. Conversely, by Theorem 3.1, L is the logic of a stable class of finite rooted frames. Each such must be a cluster. Therefore, $\mathbf{S5} \subseteq L$, and hence **S5** is axiomatized over **K** by Γ.

Next, as promised, we show that several well-known logics are not stable. Namely, consider the following well-known logics (see, e.g., [9, p. 116]):

- $\mathbf{KB} = \mathbf{K} + p \to \Box\Diamond p$, the logic of symmetric frames;
- $\mathbf{K5} = \mathbf{K} + \Diamond\Box p \to \Box p$, the logic of Eucliedean frames;
- $\mathbf{GL} = \mathbf{K4} + \Box(\Box p \to p) \to \Box p$, the logic of dually well-founded **K4**-frames;
- $\mathbf{S4.Grz} = \mathbf{S4} + \Box(\Box(p \to \Box p) \to p) \to p$, the logic of Noetherian **S4**-frames;
- $\mathbf{K4.1} = \mathbf{K4.1} + \Box\Diamond p \to \Diamond\Box p$, the logic of **K4**-frames with degenerate final clusters;
- $\mathbf{S4.1} = \mathbf{S4} \vee \mathbf{K4.1}$, the logic of **S4**-frames with degenerate final clusters.

Theorem 7.6. *None of the logics* $\mathbf{K4}, \mathbf{S4}, \mathbf{KB}$*, and* $\mathbf{K5}$ *is stable. Neither are the logics* $\mathbf{GL}$*,* $\mathbf{S4.Grz}$*,* $\mathbf{K4.1}$*, and* $\mathbf{S4.1}$*. In fact,* $\mathbf{GL}$ *and* $\mathbf{K4.1}$ *are not* $\mathbf{K4}$*-stable and* $\mathbf{S4.Grz}$ *and* $\mathbf{S4.1}$ *are neither* $\mathbf{K4}$*-stable nor* $\mathbf{S4}$*-stable.*

Proof. We start by showing that $\mathbf{K4}$ is not stable. If $\mathbf{K4}$ were stable, then by Theorem 3.1, there would exist a stable class $\mathcal{K}$ of finite rooted $\mathbf{K4}$-frames whose logic is $\mathbf{K4}$. Consider the finite rooted frames $\mathfrak{F}, \mathfrak{G}$ and an onto stable map $\mathfrak{G} \twoheadrightarrow \mathfrak{F}$ shown below.



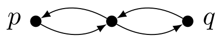
$\mathfrak{G}$ $\qquad\qquad$ $\mathfrak{F}$

Note that $\mathfrak{G}$ is transitive, but $\mathfrak{F}$ is not. Since $\mathfrak{G}$ is a $\mathbf{K4}$-frame and $\mathfrak{G} \not\models \gamma(\mathfrak{G})$, we see that $\mathbf{K4} \not\models \gamma(\mathfrak{G})$. Therefore, there is $\mathfrak{H} \in \mathcal{K}$ such that $\mathfrak{H} \not\models \gamma(\mathfrak{G})$. As $\mathfrak{G}$ has a reflexive root, by Proposition 5.10(1), $\mathfrak{G}$ is a stable image of $\mathfrak{H}$. Thus, since $\mathcal{K}$ is stable, $\mathfrak{G} \in \mathcal{K}$. The same reasoning yields $\mathfrak{F} \in \mathcal{K}$. But this is a contradiction as $\mathfrak{F}$ is not transitive. Consequently, $\mathbf{K4}$ is not a stable logic.

A similar reasoning gives that $\mathbf{S4}$ is not a stable logic. We next show that $\mathbf{KB}$ is not a stable logic. If it were, then by Theorem 3.1, there would exist a stable class $\mathcal{K}$ of finite rooted $\mathbf{KB}$-frames whose logic is $\mathbf{KB}$.

Claim 7.7. *There is* $\mathfrak{F} \in \mathcal{K}$ *containing distinct* x, y *that are not* R*-related to each other.*

Proof. Clearly the $\mathbf{KB}$-model



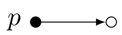
refutes $\mathbf{bw}_1 = \Diamond p \wedge \Diamond q \to \Diamond(p \wedge \Diamond^+ q) \vee \Diamond(q \wedge \Diamond^+ p)$. Therefore, $\mathbf{KB} \not\vdash \mathbf{bw_1}$. Thus, there is $\mathfrak{F} \in \mathcal{K}$ such that $\mathfrak{F} \not\models \mathbf{bw}_1$. It is easy to see that $\mathfrak{F}$ has the desired property. $\square$

For such an $\mathfrak{F} = (X, R)$ define $\mathfrak{F}' = (X, R')$, where $R' = R \cup \{(x, y)\}$. Then the identity map is a stable map from $\mathfrak{F}$ onto $\mathfrak{F}'$. Since $\mathcal{K}$ is stable, $\mathfrak{F}' \in \mathcal{K}$. But this is a contradiction as $\mathfrak{F}'$ is not symmetric. Thus, $\mathbf{KB}$ is not a stable logic.

Next we show that $\mathbf{K5}$ is not a stable logic. If $\mathbf{K5}$ were stable, then there would be a stable class $\mathcal{K}$ of finite rooted $\mathbf{K5}$-frames whose logic is $\mathbf{K5}$.

Claim 7.8. *There is* $\mathfrak{F} \in \mathcal{K}$ *containing* x, y *such that* xRy *and* $x \not R x$*.*

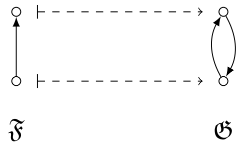
Proof. Clearly the $\mathbf{K5}$-model



refutes the formula $\varphi := p \to \Diamond p \vee \Box\bot$. Therefore, $\mathbf{K5} \not\vdash \varphi$. Thus, there is $\mathfrak{F} \in \mathcal{K}$ such that $\mathfrak{F} \not\models \varphi$. It is easy to see that $\mathfrak{F}$ has the desired property. $\square$

For such an $\mathfrak{F} = (X, R)$ define $\mathfrak{F}' = (X, R')$, where $R' = R \cup \{(y, x)\}$. Then the identity map is a stable map from $\mathfrak{F}$ onto $\mathfrak{F}'$. Since $\mathcal{K}$ is stable, $\mathfrak{F}' \in \mathcal{K}$. But this is a contradiction as $\mathfrak{F}'$ is not Euclidean because in an Euclidean frame every successor is reflexive. Thus, $\mathbf{K5}$ is not a stable logic.

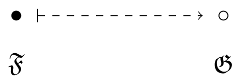
Next we show that $\mathbf{S4.Grz}$ is not a stable logic. By Lemma 4.2(1), it is sufficient to show that $\mathbf{S4.Grz}$ is not $\mathbf{S4}$-stable. It is easy to see that the map $\mathfrak{F} \twoheadrightarrow \mathfrak{G}$ between finite rooted $\mathbf{S4}$-frames depicted below is stable.



Note that $\mathfrak{F}$ is a **S4.Grz**-frame, while $\mathfrak{G}$ is not. Therefore, by Corollary 5.12, **S4.Grz** is not **S4**-stable. Thus, by Lemma 6.6(1), **S4.Grz** is not **K4**-stable.

The same argument yields that **S4.1** is not **S4**-stable. Therefore, by Lemma 6.6(1), **S4.1** is not **K4**-stable. Since **S4.1** = **S4** $\vee$ **K4.1**, Lemma 6.6(2) yields that **K4.1** is not **K4**-stable. Thus, neither **S4.1** nor **K4.1** is stable by Lemma 4.2(1).

Finally, we show that **GL** is not stable. For this it is sufficient to show that **GL** is not **K4**-stable. It is easy to see that the map depicted below is a stable map from a finite rooted **GL**-frame $\mathfrak{F}$ onto a finite rooted **K4**-frame $\mathfrak{G}$, which is not a **GL**-frame.



The rest of the argument is the same as in the case of **S4.Grz**. □

We conclude the paper by comparing **K4**-stable logics to transitive subframe, cofinal subframe, and union-splitting logics (these classes of logics are discussed in detail in [9, Sec. 10.5 and 11.3]). Table 2 provides examples that tell these classes apart.

Table 2

	transitive subframe	transitive cofinal subframe	K4-stable	S4-stable	union K4-splitting	union S4-splitting
S4.2	-	✓	✓	✓	✓	✓
S4.Grz	✓	✓	-	-	✓	✓
GL	✓	✓	-	×	-	×
τL	-	-	✓	✓	✓	✓
K4BTW$_3$	-	✓	✓	×	-	×
S4BTW$_3$	-	✓	✓	✓	-	-

"✓" means the logic belongs to the class; "-" means the logic does not belong to the class; "×"means not applicable.

- By Proposition 7.1, **S4.2** is **S4**-stable. Therefore, by Lemma 6.6(1), **S4.2** is **K4**-stable. It is well known that **S4.2** is **S4**-splitting (see, e.g., [13]). Since **S4** is a union **K4**-splitting, it follows that **S4.2** is a union **K4**-splitting. Finally, it is well known that **S4.2** is a cofinal subframe logic (see, e.g., [9, Sec. 9.4]), and it is easy to see that **S4.2** is not a subframe logic.
- By Theorem 7.6, **S4.Grz** is neither **S4**-stable nor **K4**-stable. On the other hand, it is well known that **S4.Grz** is a subframe logic (see, e.g., [9, Sec. 9.4]). Therefore, **S4.Grz** is a cofinal subframe logic. Finally, it is well known that **S4.Grz** is a union **S4**-splitting (see, e.g., [8, Exm. 1.11]). Thus, **S4.Grz** is a union **K4**-splitting.
- By Theorem 7.6, **GL** is not **K4**-stable, and it is well known that **GL** is not a union **K4**-splitting (see, e.g., [9, Exe. 9.13]). On the other hand, it is well known that **GL** is a subframe logic (see, e.g., [9, Sec. 9.4]). Thus, **GL** is a cofinal subframe logic.
- It was shown in [4] that there is a stable superintuitionistic logic L which is not a cofinal subframe logic. Therefore, neither is τL. Thus, τL is not a subframe logic. By Lemma 6.1, τL is **S4**-stable. Since L is a tabular logic, it is a union splitting superintuionistic logic (see,

e.g., [5, Thm. 3.4.27]). By [9, Cor. 9.64], τL is a union **S4**-splitting logic, hence a union **K4**-splitting logic.

- It is easy to see that neither $\mathbf{S4BTW}_3$ nor $\mathbf{K4BTW}_3$ is a subframe logic. It follows from [9, Sec. 9.4 and Cor. 9.64] that $\mathbf{S4BTW}_3$ is a cofinal subframe logic. Since $\mathbf{K4BTW}_3 = \mathbf{S4BTW}_3{}^+$, it follows that $\mathbf{K4BTW}_3$ is a cofinal subframe logic. An adaptation of the proof of [9, Prop. 9.50] shows that $\mathbf{K4BTW}_3$ is not a union **K4**-splitting logic and $\mathbf{S4BTW}_3$ is not a union **S4**-splitting logic. On the other hand, by Proposition 7.1, $\mathbf{S4BTW}_3$ is **S4**-stable, and by Proposition 7.2, $\mathbf{K4BTW}_3$ is **K4**-stable.

References

[1] G. Bezhanishvili and N. Bezhanishvili. Canonical formulas for wK4. *Rev. Symb. Log.*, 5(4):731–762, 2012.
[2] G. Bezhanishvili and N. Bezhanishvili. Locally finite reducts of Heyting algebras and canonical formulas. *Notre Dame Journal of Formal Logic*, 2016. To appear.
[3] G. Bezhanishvili, N. Bezhanishvili, and R. Iemhoff. Stable canonical rules. *J. Symbolic Logic*, 2016. To appear.
[4] G. Bezhanishvili, N. Bezhanishvili, and J. Ilin. Cofinal stable logics. Submitted; available at: *http://www.illc.uva.nl/Research/Publications/Reports/PP-2015-08.text.pdf.*
[5] N. Bezhanishvili. *Lattices of Intermediate and Cylindric Modal Logics.* PhD thesis, University of Amsterdam, 2006.
[6] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic.* Cambridge University Press, 2001.
[7] R. Burris and H. Sankappanavar. *A Course in Universal Algebra.* Springer, 1981.
[8] A. Chagrov, F. Wolter, and M. Zakharyaschev. Advanced modal logic. In *Handbook of philosophical logic, Vol. 3*, pages 83–266. Kluwer Acad. Publ., Dordrecht, 2001.
[9] A. Chagrov and M. Zakharyaschev. *Modal Logic.* The Clarendon Press, New York, 1997.
[10] E. Jeřábek. Canonical rules. *J. Symbolic Logic*, 74(4):1171–1205, 2009.
[11] M. Kracht. *Tools and Techniques in Modal Logic.* North-Holland Publishing Co., Amsterdam, 1999.
[12] M. Kracht. Modal consequence relations. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of modal logic*, pages 491–545. Elsevier, 2007.
[13] W. Rautenberg. Splitting lattices of logics. *Arch. Math. Logik Grundlag.*, 20(3-4):155–159, 1980.
[14] V. V. Rybakov. *Admissibility of Logical Inference Rules.* North-Holland Publishing Co., Amsterdam, 1997.
[15] G. Sambin. Subdirectly irreducible modal algebras and initial frames. *Studia Logica*, 62(2):269–282, 1999.
[16] Y. Venema. A dual characterization of subdirectly irreducible BAOs. *Studia Logica*, 77(1):105–115, 2004.
[17] Y. Venema. Algebras and coalgebras. In P. Blackburn, J. van Benthem, and F. Wolter, editors, *Handbook of modal logic*, pages 331–426. Elsevier, 2007.

Guram Bezhanishvili: Department of Mathematical Sciences, New Mexico State University, Las Cruces NM 88003, guram@math.nmsu.edu

Nick Bezhanishvili: Institute for Logic, Language and Computation, University of Amsterdam, P.O. Box 94242, 1090 GE Amsterdam, The Netherlands, N.Bezhanishvili@uva.nl

Julia Ilin: Institute for Logic, Language and Computation, University of Amsterdam, P.O. Box 94242, 1090 GE Amsterdam, The Netherlands, ilin.juli@gmail.com