Weak Subintuitionistic Logics

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Abstract

A system WF of subintuitionistic logic is introduced, weaker than Corsi's basic subintuitionistic system F. A derivation system with and without hypotheses is given in line with the authors' derivation system for F. A neighborhood semantics is introduced with a somewhat more complex definition than the neighborhood semantics for non-normal modal logics. Completeness is proved for WF with respect to this neighborhood semantics, and similarly for some logics between WF and F which characterize nice frame classes. The study by the authors of the conservativity of IPC over F with respect to some classes of implications is extended to WF, and shows clearly the difference in strength between the two logics. Study of translations of these weak subintuitionistic logics into non-normal modal logics turned out to be hard because of the difference between their respective neighborhood structures and leaves us with some open problems.

Keywords: subintuitionistic logic, intuitionistic logic, conservativity, neighborhood semantics, translation.

1 Introduction

Subintuitionistic logics were studied by Corsi in 1987 [2], who introduced a basic system F and by Restall in 1994 [9], who defined a similar system SJ, both with Kripke models in which no assumption of preservation is made and also not of reflexivity and transitivity. F cannot prove formulas like $A \to (B \to A)$. Corsi showed that F can be translated into the modal logic K just as IPC into S4.

A much studied extension of F, Basic logic BPC, was introduced by Visser in 1981 [11] and shown in 2015 by K. Sano and M. Ma [10] to be translatable into the modal logic WK4.

Neighborhood structures are the standard semantic tool used to study non-normal modal logic. In a neighborhood model for modal logic, each state is associated with a collection of subsets of the universe and a modal formula $\Box\varphi$ is true at a state w if the set of all states in which φ is true is a neighborhood of w.

M. Moniri and F. Shirmohammadzadeh Maleki in 2015 [8] presented a neighborhood semantics for IPC and BPC. In this paper we will introduce a system WF, weaker than F, for which we define a neighborhood semantics. The neighborhood semantics needs to be more complex than the neighborhood semantics of modal logic.

The structure of this paper is as follows. In Section 2 we introduce the logic WF, it is created by replacing some axioms of F by rules. The derivation system for WF. with and without hypotheses, is modeled on the one for F of [5]. We prove a strong completeness theorem for WF. In Section 3 we prove strong completeness theorems

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for the logics between WF and F, formed by combinations of five axioms and rules. In Section 4, we show that WF has the finite model property. In Section 5 we study conservativity of IPC over WF with respect to the simple implications introduced in [5]. There is a clear difference in strength compared to the system F: IPC was proved to be conservative over F with respect to the much more complex basic implications [5]. In Section 6 we investigate the relation between WF and modal logic. The logic WF is clearly related to the non-normal modal logic EN. But because of the difference of the models we were able to prove only $\vdash_{\mathsf{WF}} A \;\Rightarrow\; \vdash_{\mathsf{EN}} A$. The other direction remains an open problem. A similar situation arises between the basic monotonic modal logic M and our system $\mathsf{WFI_RI_L}$.

2 Soundness and Completeness

In this section we will introduce the logic WF, a logic strictly weaker than F and prove soundness and completeness of WF.

Definition 2.1
A pair $F = \langle W, g, NB\rangle$ is called a **Neighborhood Frame** of subintuitionistic logic if W is a non-empty set and NB is a neighborhood function from W into $P((P(W))^2)$ such that

1. $\forall w \in W,\ \forall X, Y \in P(W),\ (X \subseteq Y \;\Rightarrow\; (X, Y) \in NB(w))$;
2. $NB(g) = \left\{(X, Y) \in (P(W))^2 \mid X \subseteq Y\right\}$.

Here g is called **omniscient** (i.e. has the property 2).

We use the existence of omniscient worlds in the proofs of soundness and of characterization of properties of frames.

Definition 2.2
A **Neighborhood Model** of subintuitionistic logic is a tuple $M = \langle W, g, NB, V\rangle$, where $\langle W, g, NB\rangle$ is a neighborhood frame of subintuitionistic logic and $V : P \to 2^W$ a valuation function on the set of propositional variables P.

Definition 2.3
(Truth in a Neighborhood Model) Let $M = \langle W, g, NB, V\rangle$ be a model and $w \in W$. Truth of a propositional formula in a world w is defined inductively as follows.

1. $M, w \Vdash p \quad \Leftrightarrow \quad w \in V(p)$;
2. $M, w \Vdash A \wedge B \quad \Leftrightarrow \quad M, w \Vdash A$ and $M, w \Vdash B$;
3. $M, w \Vdash A \vee B \quad \Leftrightarrow \quad M, w \Vdash A$ or $M, w \Vdash B$;
4. $M, w \Vdash A \to B \quad \Leftrightarrow \quad \left((A)^M, (B)^M\right) \in NB(w)$;
5. $M, w \nVdash \bot$,

where $(A)^M$ denotes the truth set of A. If $X \subseteq W$ is such that $X = (A)^M$ then we call X **definable**. We will liberalize Definition 2.1 to require the conditions 1 and 2 to apply only to definable X, Y (see e.g. [3],[7]).

Definition 2.4
A formula A is **true in a model** $M = \langle W, g, NB, V\rangle$, $M \Vdash A$ if for all $w \in W$, $M, w \Vdash A$ and if all models force A, we write $\Vdash A$ and call A **valid**. A formula A is **valid on a frame** $F = \langle W, g, NB\rangle$, $F \Vdash A$ if A is true in every model based on that frame.

In the following definition of neighborhood models we use the more standard neighborhood function.

Definition 2.5
An **N-Neighborhood Frame** is a triple $F = \langle W, g, N\rangle$. N is a neighborhood function from W into 2^{2^W}, $g \in W$, and for each $w \in W$ we have $W \in N(w)$ and $N(g) = \{W\}$ (g is called **omniscient**). An **N-Neighborhood Model** is a quadruple $M = \langle W, g, N, V\rangle$ with $V : P \to 2^W$ a valuation function on the set of propositional variables P.

Definition 2.6
(Truth in an N-Neighborhood Model) Let $M = \langle W, N, V\rangle$ be a model and $w \in W$. Truth of a propositional formula in a world w is defined inductively as in Definition 2.3 with the following clause for $\to$:

$$M, w \Vdash A \to B \Leftrightarrow \{v \mid v \Vdash A \ \Rightarrow \ v \Vdash B\} \in N(w).$$

It maybe easier to think of $\{v \mid v \Vdash A \ \Rightarrow \ v \Vdash B\}$ as the set $((A)^M)^c \cup (B)^M$. Here we denote $W - (A)^M$ by $((A)^M)^c$.

Indeed, we opted for N-neighborhood frames first but were not able to prove completeness for WF with respect to these frames.

Lemma 2.7
For every N-neighborhood model $M_N = \langle W, g, N, V\rangle$, there is a pointwise equivalent neighborhood model $M_{NB} = \langle W, g, NB, V\rangle$.

Proof. The proof is straightforward by considering, for each $w \in W$,
$NB(w) = \{(X, Y) \mid (X)^c \cup Y \in N(w)\}$. ∎

In the other direction the connection is not clear.

Definition 2.8
WF is the logic given by the following axioms and rules,

1. $A \to A \vee B$
2. $B \to A \vee B$
3. $A \wedge B \to A$
4. $A \wedge B \to B$
5. $\frac{A \to B \quad A \to C}{A \to B \wedge C}$
6. $\frac{A \to C \quad B \to C}{A \vee B \to C}$
7. $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$
8. $\bot \to A$
9. $A \to A$
10. $\frac{A \quad A \to B}{B}$
11. $\frac{A \quad B}{A \wedge B}$
12. $\frac{A}{B \to A}$
13. $\frac{A \to B \quad B \to C}{A \to C}$
14. $\frac{A \leftrightarrow B \quad C \leftrightarrow D}{(A \to C) \leftrightarrow (B \to D)}$

The rules are to be applied in such a way that, if the formulas above the line are theorems of WF, then the formula below the line is a theorem as well. We return to the rules when we discuss deduction from hypotheses. We write $\vdash$ for $\vdash_{\mathsf{WF}}$.

The logic WF misses the following axioms of F (Corsi's system [2]):

$(A \to B) \wedge (A \to C) \to (A \to B \wedge C)$
$(A \to B) \wedge (C \to B) \to (A \vee C \to B)$
$(A \to B) \wedge (B \to C) \to (A \to C)$

Typical for intuitionistic logic is that often axioms and their corresponding rules are different in strength. That comes out nicely here.

First we will show that WF has the disjunction property.

Definition 2.9 ([6])
We define $|A$ by induction on A, as follows

1. Not $|p$,
2. $|A \wedge B$ iff $|A$ and $|B$,
3. $|A \vee B$ iff $|A$ or $|B$,
4. $|A \to B$ iff $\vdash A \to B$ and (if $|A$ then $|B$).

Theorem 2.10
$|A \;\Leftrightarrow\; \vdash A$.

Proof. The proof is a trivial modification of the standard one for IPC. ∎

Theorem 2.11
If $\vdash A \vee B$ then $\vdash A$ or $\vdash B$.

Proof. Assume $\vdash A \vee B$, by Theorem 2.10, $|A \vee B$. So $|A$ or $|B$. Again by Theorem 2.10, $\vdash A$ or $\vdash B$. ∎

Theorem 2.12
The formula $A \to B$ is valid if and only if for all models M,

$$(A)^M \subseteq (B)^M.$$

Proof. $\Rightarrow$: Let $\Vdash A \to B$, so for all M and for all $w \in M$, $((A)^M, (B)^M) \in NB(w)$. So, for all M, $((A)^M, (B)^M) \in NB(g)$. Hence, by Definition 2.1(2), $(A)^M \subseteq (B)^M$.

$\Leftarrow$: We should prove that for all M and for all $w \in M$, $((A)^M, (B)^M) \in NB(w)$. By assumption $(A)^M \subseteq (B)^M$, so by definition of neighborhood frames for all M and for all $w \in M$, $((A)^M, (B)^M) \in NB(w)$. That is, $\Vdash A \to B$. ∎

Corollary 2.13
$\Vdash A \leftrightarrow B$ if and only if for all models M, $(A)^M = (B)^M$.

Proof. Obvious. ∎

Lemma 2.14
(a) If $\Vdash A$ and $\Vdash A \to B$, then $\Vdash B$.
(b) If $\Vdash A \to B$ and $\Vdash A \to C$, then $\Vdash A \to B \wedge C$.
(c) If $\Vdash A \to C$ and $\Vdash B \to C$, then $\Vdash A \vee B \to C$.
(d) If $\Vdash A \to B$ and $\Vdash B \to C$, then $\Vdash A \to C$.
(e) If $\Vdash A \leftrightarrow B$ and $\Vdash C \leftrightarrow D$, then $\Vdash (A \to C) \leftrightarrow (B \to D)$.

Proof. We only prove (a) and (e). The other cases are similar.

(a) Let $\Vdash A \to B$. By Theorem 2.12, for all models M, $(A)^M \subseteq (B)^M$. By assumption, for all models M, $(A)^M = W$. So also, $(B)^M = W$. Hence, $\Vdash B$.

(e) By Corollary 2.13, it is sufficient to show that for all models M, $(A \to C)^M = (B \to D)^M$. By assumption and Corollary 2.13, we have $(A)^M = (B)^M$ *and* $(C)^M = (D)^M$. Therefore, $\{v \mid ((A)^M, (C)^M) \in NB(v)\} = \{v \mid ((B)^M, (D)^M) \in NB(v)\}$. That is, $(A \to C)^M = (B \to D)^M$. ∎

Observation 2.15
The formula $(p \to q) \wedge (p \to r) \to (p \to q \wedge r)$ is not valid in the class of all neighborhood frames.

Proof. Consider the neighborhood frame $F = \langle W, g, NB\rangle$ with

$$W = \{g, w\},\ NB(w) = \{(W, \{w\}), (W, \{g\})\} \cup \{(X, Y) \in (P(W))^2 \mid X \subseteq Y\}.$$

Also consider the valuation $V(p) = W$, $V(r) = \{g\}$, $V(q) = \{w\}$. We have,

$$\begin{aligned}
&(W, \{w\}) \in NB(w) \ \Rightarrow\ w \Vdash p \to q,\\
&(W, \{g\}) \in NB(w) \ \Rightarrow\ w \Vdash p \to r,\\
&(W, \varnothing) \notin NB(w) \ \Rightarrow\ w \nVdash p \to q \wedge r.
\end{aligned}$$

So, $((p \to q) \wedge (p \to r))^M \nsubseteq (p \to q \wedge r)^M$ and therefore,

$$(((p \to q) \wedge (p \to r))^M, (p \to q \wedge r)^M) \notin NB(g).$$

That is, $g \nVdash (p \to q) \wedge (p \to r) \to (p \to q \wedge r)$. ∎

Observation 2.16
The formula $(p \to q) \wedge (r \to q) \to (p \vee r \to q)$ is not valid in the class of all neighborhood frames.

Proof. Consider the neighborhood frame $F = \langle W, g, NB\rangle$ with

$$W = \{w, g\},\ NB(w) = \{(\{g\}, \{w\})\} \cup \{(X, Y) \in (P(W))^2 \mid X \subseteq Y\}.$$

Also consider the valuation $V(p) = V(q) = \{w\}$, $V(r) = \{g\}$. We have,

$$\begin{aligned}
&(\{w\}, \{w\}) \in NB(w) \ \Rightarrow\ w \Vdash p \to q,\\
&(\{g\}), \{w\}) \in NB(w) \ \Rightarrow\ w \Vdash r \to q,\\
&(W, \{w\})) \notin NB(w) \ \Rightarrow\ w \nVdash p \vee r \to q.
\end{aligned}$$

So $((p \to q) \wedge (r \to q))^M \nsubseteq (p \vee r \to q)^M$ and therefore,

$$(((p \to q) \wedge (r \to q))^M, (p \vee r \to q)^M) \notin NB(g).$$

That is, $g \nVdash (p \to q) \wedge (r \to q) \to (p \vee r \to q)$. ∎

Observation 2.17
The formula $(p \to q) \wedge (q \to r) \to (p \to r)$ is not valid on the class of all neighborhood frames.

Proof. Consider the neighborhood frame $F = \langle W, g, NB\rangle$ with

$$W = \{w, g\}, NB(w) = \{(\{w\}, \{g\}), (\{g\}, \varnothing)\} \cup \{(X, Y) \in (P(W))^2 \mid X \subseteq Y\}.$$

Also consider the valuation $V(p) = \{w\}$, $V(r) = \varnothing$ and $V(q) = \{g\}$. We have,

$$(\{w\}, \{g\}) \in NB(w) \;\Rightarrow\; w \Vdash p \to q,$$
$$(\{g\}, \varnothing) \in NB(w) \;\Rightarrow\; w \Vdash q \to r,$$
$$(\{w\}, \varnothing) \notin NB(w) \;\Rightarrow\; w \nVdash p \to r.$$

So, $((p \to q) \wedge (q \to r))^M \nsubseteq (p \to r)^M$ and therefore,

$$(((p \to q) \wedge (q \to r))^M, (p \to r)^M) \notin NB(g).$$

That is, $g \nVdash (p \to q) \wedge (q \to r) \to (p \to r)$. ∎

For the strong completeness theorem we will show that if $\Sigma \nvdash A$ then there exists a state Δ in the canonical model such that $\Delta \Vdash \Sigma$ and $\Delta \nVdash A$. For this purpose we need to have some definitions and propositions. Actually the definitions of theory and of derivation from hypotheses are identical to the ones for F in [5].

Definition 2.18
A set of sentences Δ is a **theory** if and only if

1. $A, B \in \Delta \;\Rightarrow\; A \wedge B \in \Delta$,
2. $\vdash A \to B \;\Rightarrow\;$ (if $A \in \Delta$, then $B \in \Delta$),
3. If $\vdash A \quad\Rightarrow\; A \in \Delta$.

Definition 2.19
(a) We define $\Gamma \vdash A$ if there is a derivation of A from Γ and theorems of WF using the rules $\frac{A \quad B}{A \wedge B}$ and $\frac{A \quad A \to B}{B}$ where in the latter case the restriction is that $A \to B$ has to be provable in WF.

(b) We define $\Gamma \Vdash A$ iff for all M, $w \in M$, if $M, w \Vdash \Gamma$ then $M, w \Vdash A$.

Proposition 2.20
Δ is a theory $\iff$ $\Delta \vdash A$ if and only if $A \in \Delta$.

Proof. $\Rightarrow$: The proof from right to left is immediate. The other direction is by induction on the length of the derivation. If A is theorem of WF, then by definition of theory $A \in \Delta$.

If $\Delta \vdash A$ and $\Delta \vdash B$, then, by induction hypothesis $A \in \Delta$ and $B \in \Delta$. So by definition of theory $A \wedge B \in \Delta$.

If $\vdash A \to B$ and $\Delta \vdash A$, then, by induction hypothesis $A \in \Delta$ and by definition of theorem $B \in \Delta$.

$\Leftarrow$: This direction is easy. We only check case 2 of the definition of theory. Let $\vdash A \to B$ and $A \in \Delta$. Then by assumption $\Delta \vdash A$, and so $\Delta \vdash B$. Again by assumption $B \in \Delta$. ∎

Theorem 2.21
(Weak Deduction Theorem) $A \vdash B$ if and only if $\vdash A \to B$.

Proof. $\Rightarrow$: By induction on the length of the proof.
If B is an axiom. Then $\vdash B$, so by rule 11, $\vdash A \to B$.
If $A \vdash A$ is covered $\vdash A \to A$.
If $A \vdash B$ and $A \vdash C$. By induction hypothesis $\vdash A \to B$ and $\vdash A \to C$, so by rule 5, $\vdash A \to B \wedge C$.

If $A \vdash B$ and $\vdash B \to C$. Then by induction hypothesis $\vdash A \to B$, so by rule 12, $\vdash A \to C$.
$\Leftarrow$: Let $\vdash A \to B$ and $\Gamma = \{A\}$ then by Definition 2.19, we have $\Gamma \vdash B$, that is $A \vdash B$. ∎

COROLLARY 2.22
(a) $A_1, ..., A_n \vdash B$ if and only if $\vdash A_1 \wedge ... \wedge A_n \to B$.
(b) $\Delta \vdash B$ if and only if, for some $A_1, ..., A_n \in \Delta$, $\vdash A_1 \wedge ... \wedge A_n \to B$.

PROOF. The proof is easy. ∎

DEFINITION 2.23
A set of sentences Δ is **prime** if and only if, if $A \vee B \in \Delta$, then $A \in \Delta$ or $B \in \Delta$.

THEOREM 2.24
If $\Sigma \nvdash D$ then there is a prime theory Δ such that $\Delta \supseteq \Sigma$, $D \notin \Delta$.

PROOF. By assumption and by definition of provability we conclude that $D \notin \Sigma$. Enumerate all formulas, with infinitely many repetitions: $B_0, B_1, ...$ and define,

$$\Delta_0 = \Sigma,$$
$$\Delta_{n+1} = \Delta_n \cup \{B_n\} \text{ if } \Delta_n, B_n \nvdash D,$$
$$\Delta_{n+1} = \Delta_n \text{ otherwise.}$$

Take Δ to be the union of all Δ_n. Clearly, $\Delta \nvdash D$. We must show that Δ is a prime theory. Assume that $F \in \Delta$, $G \in \Delta$ and $F \wedge G \notin \Delta$. Let $F = B_i$, $G = B_j$ and $F \wedge G = B_n$ with $j \geq i \geq n$. So, $\Delta_n, F \wedge G \vdash D$ and hence $\Delta_n, F, G \vdash D$. But $F, G \in \Delta_j$, so $\Delta_n, F, G \nvdash D$, a contradiction.

Now if $\vdash A \to B$ and $A \in \Delta$. Asuume $B \notin \Delta$. Let $B = B_n$, then, $\Delta_n, B_n \vdash D$ and so $\Delta_n, A \vdash D$, because from $\vdash A \to B$ and A we can derive B. But this is a contradiction, since $A \in \Delta$.

Assume that $F \vee G \in \Delta$, and $F \notin \Delta$, $G \notin \Delta$. Let $F = B_n$ and $G = B_k$, so $\Delta_n, F \vdash D$ and $\Delta_k, G \vdash D$. Then by Corollary 2.22, there exist $\bar{B}_1, ..., \bar{B}_m \in \Delta_n$ such that $\vdash \bar{B}_1 \wedge ... \wedge \bar{B}_m \wedge F \to D$ and also there exist $B_1^{'}, ..., B_{m'}^{'} \in \Delta_k$ such that $\vdash B_1^{'} \wedge ... \wedge B_{m'}^{'} \wedge G \to D$. W.l.o.g. take $n \geq k$, then $\bar{B}_1, ..., \bar{B}_m, B_1^{'}, ..., B_{m'}^{'} \in \Delta_n$. Thus by some steps and using some rules of WF we will have,

$$\vdash (\bar{B}_1 \wedge ... \wedge \bar{B}_m) \wedge (B_1^{'} \wedge ... \wedge B_{m'}^{'}) \wedge (F \vee G) \to D \tag{2.1}$$

Again by Corollary 2.22 and (2.1) we have $\Delta_n, F \vee G \vdash D$. But that cannot be true since $F \vee G \in \Delta$.

Assuming that $\vdash F$, we want to show that $F \in \Delta$. Let $F = B_n$ and $F \notin \Delta$, so $\Delta_n, F \vdash D$. From Δ_n we can derive F, so $\Delta_n \vdash D$. But this is a contradiction, hence $F \in \Delta$. So Δ is a prime theory.

Since Δ is a theory and $\Delta \nvdash D$, $D \notin \Delta$. ∎

DEFINITION 2.25
Let W_{WF} be the set of all prime theories. Given a formula A, we define the set $[\![A]\!]$ as follows,

$$[\![A]\!] = \{\Delta \mid \Delta \in W_{\mathsf{WF}},\ A \in \Delta\}.$$

Lemma 2.26
Let C and D are formulas. Then

(a) $[\![C \wedge D]\!] = [\![C]\!] \cap [\![D]\!]$.
(b) $[\![C \vee D]\!] = [\![C]\!] \cup [\![D]\!]$.
(c) $[\![C]\!] \subseteq [\![D]\!]$ iff $\vdash C \rightarrow D$.
(d) $[\![C]\!] = [\![D]\!]$ iff $\vdash C \leftrightarrow D$.

Proof. The proofs are easy. We only prove (c).

(c) Let $\nvdash C \rightarrow D$. Then by the Weak Deduction Theorem $C \nvdash D$. Let $\Sigma = \{C\}$, then by Theorem 2.24, there exist a prime theory Γ such that, $C \in \Gamma$ and $D \notin \Gamma$. That is $[\![C]\!] \nsubseteq [\![D]\!]$.

Now let $\vdash C \rightarrow D$. Assume $\Gamma \in W_{\mathsf{WF}}$, $C \in \Gamma$. Then by definition of theory $D \in \Gamma$. So $[\![C]\!] \subseteq [\![D]\!]$. ∎

When constructing a canonical model the states of the world will be prime theories, i.e. elements of W_{WF}. Consider the function $NB_{\mathsf{WF}} : W_{\mathsf{WF}} \rightarrow P((P(W_{\mathsf{WF}}))^2)$ such that for each $\Gamma \in W_{\mathsf{WF}}$,

$$NB_{\mathsf{WF}}(\Gamma) = \{([\![A]\!], [\![B]\!]) \mid A \rightarrow B \in \Gamma\}.$$

In the completeness proof we need to be sure that, if $([\![A]\!], [\![B]\!]) \in NB_{\mathsf{WF}}(\Gamma)$, then $A \rightarrow B \in \Gamma$. This does not follow directly from the definition. It only follows directly that C, D exist such that $C \rightarrow D \in \Gamma$, and $[\![C]\!] = [\![A]\!]$ and $[\![D]\!] = [\![B]\!]$. In the following lemma we obtain what is needed to make the argument go through.

Lemma 2.27
If $NB_{\mathsf{WF}} : W_{\mathsf{WF}} \rightarrow P((P(W_{\mathsf{WF}}))^2)$ is a function such that for each $\Gamma \in W_{\mathsf{WF}}$, $NB_{\mathsf{WF}}(\Gamma) = \{([\![A]\!], [\![B]\!]) | A \rightarrow B \in \Gamma\}$. Then $([\![A]\!], [\![B]\!]) \in NB_{\mathsf{WF}}(\Gamma)$ implies $A \rightarrow B \in \Gamma$.

Proof. Assume $([\![A]\!], [\![B]\!]) \in NB_{\mathsf{WF}}(\Gamma)$. Then for some C, D, $[\![A]\!] = [\![C]\!]$, $[\![B]\!] = [\![D]\!]$, $C \rightarrow D \in \Gamma$. By Lemma 2.26, we have $\vdash A \leftrightarrow C$ and $\vdash B \leftrightarrow D$. Then by rule 14 we will have $\vdash (A \rightarrow B) \leftrightarrow (C \rightarrow D)$. By assumption, $C \rightarrow D \in \Gamma$. Hence, by definition of prime theory we conclude that, $A \rightarrow B \in \Gamma$. ∎

Now we want to define the canonical model for WF.

Definition 2.28
The **Canonical model** $M^{\mathsf{WF}} = \langle W_{\mathsf{WF}}, g, NB_{\mathsf{WF}}, V\rangle$ of WF is defined by:

1. g is the set of theorems of WF,
2. For each $\Gamma \in W$ and all formulas A and B,

$$NB_{\mathsf{WF}}(\Gamma) = \{([\![A]\!], [\![B]\!]) \mid A \rightarrow B \in \Gamma\},$$

3. If $p \in At$, then $V(p) = [\![p]\!] = \{\Gamma \mid \Gamma \in W_{\mathsf{WF}} \textit{ and } p \in \Gamma\}$.

By Theorem 2.11, WF has the disjunction property and therefore is a prime theory. It is easy to see that in the canonical model WF is omniscient. Also for all $\Gamma \in W_{\mathsf{WF}}$, if $[\![A]\!] \subseteq [\![B]\!]$, then $([\![A]\!], [\![B]\!]) \in NB_{\mathsf{WF}}(\Gamma)$. Since if $[\![A]\!] \subseteq [\![B]\!]$, then by Lemma 2.26, $\vdash A \rightarrow B$ and so $A \rightarrow B \in \Gamma$.

THEOREM 2.29
(Truth Lemma) For any formula E, if M^{WF} is the canonical model of WF, then

$$(E)^{M^{\mathsf{WF}}} = [\![E]\!].$$

PROOF. By induction on E. The atomic case holds by the definition of canonical model.

($E := A \wedge B$) Let $\Gamma \in W_{\mathsf{WF}}$ and $\Gamma \Vdash A \wedge B$ then $\Gamma \Vdash A$ and $\Gamma \Vdash B$. By the induction hypothesis $A \in \Gamma$ and $B \in \Gamma$. Γ is a theory so $A \wedge B \in \Gamma$.

Now let $A \wedge B \in \Gamma$. We have $\vdash A \wedge B \to A$ and $\vdash A \wedge B \to B$, hence by definition of theory we conclude that $A \in \Gamma$ and $B \in \Gamma$. By induction hypothesis $\Gamma \Vdash A$ and $\Gamma \Vdash B$ so $\Gamma \Vdash A \wedge B$.

($E := A \vee B$) Let $\Gamma \in W_{\mathsf{WF}}$ and $\Gamma \Vdash A \vee B$. Then $\Gamma \Vdash A$ or $\Gamma \Vdash B$. By the induction hypothesis $A \in \Gamma$ or $B \in \Gamma$. We have $\vdash A \to A \vee B$ and $\vdash B \to A \vee B$, so by the definition of theory we conclude that $A \vee B \in \Gamma$.

Now let $A \vee B \in \Gamma$. Γ is a prime so $A \in \Gamma$ or $B \in \Gamma$. By induction hypothesis we conclude that $\Gamma \Vdash A$ or $\Gamma \Vdash B$. That is $\Gamma \Vdash A \vee B$.

($E := A \to B$) Let $\Gamma \in W_{\mathsf{WF}}$, then,

$$\begin{array}{rcl} \Gamma \Vdash A \to B & \Longleftrightarrow & ((A)^{M^{\mathsf{WF}}}, (B)^{M^{\mathsf{WF}}}) \in NB_{\mathsf{WF}}(\Gamma) \\ \text{(by induction hypothesis)} & \Longleftrightarrow & ([\![A]\!], [\![B]\!]) \in NB_{\mathsf{WF}}(\Gamma) \\ \text{(by Lemma 2.27)} & \Longleftrightarrow & A \to B \in \Gamma. \end{array}$$

∎

THEOREM 2.30
The logic WF is sound and strongly complete with respect to the class of neighborhood frames.

PROOF. Soundness is straightforward (in fact already shown in earlier lemmas).

Let $\Sigma \nvdash A$, then by Theorem 2.24, there is a prime theory $\Delta \supseteq \Sigma$ such that $A \notin \Delta$. So, in the canonical model we have $M^{\mathsf{WF}}, \Delta \Vdash \Sigma$, $M^{\mathsf{WF}}, \Delta \nVdash A$. That is, $\Sigma \nVdash A$. ∎

3 Completeness for Logics between WF and F

In this section we consider some formulas which characterize special classes of frames. We form the logics axiomatized by some of these formulas and prove their completeness.

DEFINITION 3.1
For every neighborhood frame $F = \langle W, g, NB \rangle$, we list some relevant properties as follows. Here X, Y, Z are definable subsets of $P(W)$.

1. F is closed under **intersection** if and only if for all $w \in W$, if $(X, Y) \in NB(w)$, $(X, Z) \in NB(w)$ then $(X, Y \cap Z) \in NB(w)$.
2. F is closed under **union** if and only if for all $w \in W$, if $(X, Y) \in NB(w)$, $(Z, Y) \in NB(w)$ then $(X \cup Z, Y) \in NB(w)$.
3. F satisfies **transitivity** if and only if for all $w \in W$, if $(X, Y) \in NB(w)$, $(Y, Z) \in NB(w)$ then $(X, Z) \in NB(w)$.
4. F is closed under **upset** if and only if for all $w \in W$, if $(X, Y) \in NB(w)$ and $Y \subseteq Z$ then $(X, Z) \in NB(w)$.
5. F is closed under **downset** if and only if for all $w \in W$, if $(X, Y) \in NB(w)$ and $Z \subseteq X$ then $(Z, Y) \in NB(w)$.

Lemma 3.2
The formula $(p \to q) \wedge (p \to r) \to (p \to q \wedge r)$ characterizes the class of frames $F = \langle W, g, NB \rangle$ satisfying closure under intersection.

Proof. Let F be closed under intersection and $M = \langle W, g, NB, V \rangle$ be any model based on F. We have to prove for all $w \in W$,

$$(((p \to q) \wedge (p \to r))^M, (p \to q \wedge r)^M) \in NB(w).$$

For this purpose it is sufficient to show that, $((p \to q) \wedge (p \to r))^M \subseteq (p \to q \wedge r)^M$. Let $w \in W$, $w \Vdash p \to q$ and $w \Vdash p \to r$ then,

$$(V(p), V(q)) \in NB(w) \tag{3.1}$$

$$(V(p), V(r)) \in NB(w) \tag{3.2}$$

The frame is closed under intersection, so by (3.1) and (3.2), $(V(p), V(q) \cap V(r)) \in NB(w)$. So, $w \Vdash p \to q \wedge r$. Hence, by definition of neighborhood frames for all $w \in W$,

$$(((p \to q) \wedge (p \to r))^M, (p \to q \wedge r)^M) \in NB(w).$$

For the other direction, we use contraposition. Suppose that the class is not closed under intersection. Then there is a frame F and $w \in F$ such that $(X, Y) \in NB(w)$ and $(X, Z) \in NB(w)$ but $(X, Y \cap Z) \notin NB(w)$. Consider the valuation V such that, $V(p) = X$, $V(r) = Z$ and $V(q) = Y$. Then,

$$\begin{aligned}
&(V(p), V(q)) \in NB(w) \;\Rightarrow\; w \Vdash p \to q,\\
&(V(p), V(r)) \in NB(w) \;\Rightarrow\; w \Vdash p \to r,\\
&(V(p), V(q \wedge r)) \notin NB(w) \;\Rightarrow\; w \nVdash p \to q \wedge r.
\end{aligned}$$

So $((p \to q) \wedge (p \to r))^M \nsubseteq (p \to q \wedge r)^M$. Then by definition of neighborhood frames $g \nVdash (p \to q) \wedge (p \to r) \to (p \to q \wedge r)$. Therefore $F \nVdash (p \to q) \wedge (p \to r) \to (p \to q \wedge r)$. ■

Lemma 3.3
The formula $(p \to q) \wedge (r \to q) \to (p \vee r \to q)$ characterizes the class of frames $F = \langle W, g, NB \rangle$ satisfying closure under union.

Proof. Let F be closed under union and $M = \langle W, g, NB, V \rangle$ be any model based on F. We have to prove for all $w \in W$,

$$(((p \to q) \wedge (r \to q))^M, (p \vee r \to q)^M) \in NB(w).$$

For this purpose it is sufficient to show that, $((p \to q) \wedge (r \to q))^M \subseteq (p \vee r \to q)^M$. Let $w \in W$, $w \Vdash p \to q$ and $w \Vdash r \to q$ then,

$$(V(p), V(q)) \in NB(w) \tag{3.3}$$

$$(V(r), V(q)) \in NB(w) \tag{3.4}$$

The frame is closed under union, so by (3.3) and (3.4), $(V(p) \cup V(r), V(q)) \in NB(w)$. So, $w \Vdash p \vee r \to q$. Hence by definition of neighborhood frames for all $w \in W$,

$$(((p \to q) \wedge (r \to q))^M, (p \vee r \to q)^M) \in NB(w).$$

For the other direction, we use contraposition. Suppose that the class is not closed under union. Then there is a frame F and $w \in F$ such that $(X, Y) \in NB(w)$ and $(Z, Y) \in NB(w)$ but $(X \cup Z, Y) \notin NB(w)$. Consider the valuation V such that, $V(p) = X$, $V(r) = Z$ and $V(q) = Y$. Then,

$$\begin{aligned}
(V(p), V(q)) \in NB(w) &\Rightarrow w \Vdash p \to q,\\
(V(r), V(q)) \in NB(w) &\Rightarrow w \Vdash r \to q,\\
(V(p \vee r), V(q)) \notin NB(w) &\Rightarrow w \nVdash p \vee r \to q.
\end{aligned}$$

So $((p \to q) \wedge (r \to q))^M \nsubseteq (p \vee r \to q)^M)$. Then by definition of neighborhood frames $g \nVdash (p \to q) \wedge (r \to q) \to (p \vee r \to q)$. Therefore $F \nVdash (p \to q) \wedge (r \to q) \to (p \vee r \to q)$. ∎

It is very notable that, if we use N-neighborhoods instead of neighborhoods, the formulas $(p \to q) \wedge (p \to r) \to (p \to q \wedge r)$ and $(p \to q) \wedge (r \to q) \to (p \vee r \to q)$ characterize the same class of frames. This is clearly undesirable and strong evidence that our definition of neighborhood is the right one.

LEMMA 3.4
The formula $(p \to q) \wedge (q \to r) \to (p \to r)$ characterizes the class of frames $F = \langle W, g, NB \rangle$ satisfying transitivity.

PROOF. Let F satisfy transitivity and $M = \langle W, g, NB, V \rangle$ be any model based on F. We have to prove for all $w \in W$,

$$(((p \to q) \wedge (q \to r))^M, (p \to r)^M) \in NB(w).$$

For this purpose it is sufficient to show that, $((p \to q) \wedge (q \to r))^M \subseteq (p \to r)^M$. Let $w \in W$, $w \Vdash p \to q$ and $w \Vdash q \to r$ then,

$$(V(p), V(q)) \in NB(w) \tag{3.5}$$

$$(V(q), V(r)) \in NB(w) \tag{3.6}$$

The frame satisfies transitivity, so by (3.5) and (3.6), $(V(p), V(r)) \in NB(w)$. So, $w \Vdash p \to r$. Hence, by definition of neighborhood frames for all $w \in W$,

$$(((p \to q) \wedge (q \to r))^M, (p \to r)^M) \in NB(w).$$

For the other direction, we use contraposition. Suppose that the class does not satisfy transitivy. Then there is a frame F and $w \in F$ such that $(X, Y) \in NB(w)$ and $(Y, Z) \in NB(w)$ but $(X, Z) \notin NB(w)$. Consider the valuation V such that, $V(p) = X$, $V(r) = Z$ and $V(q) = Y$. Then,

$$\begin{aligned}
(V(p), V(q)) \in NB(w) &\Rightarrow w \Vdash p \to q,\\
(V(q), V(r)) \in NB(w) &\Rightarrow w \Vdash q \to r,\\
(V(p), V(r)) \notin NB(w) &\Rightarrow w \nVdash p \to r.
\end{aligned}$$

So $((p \to q) \wedge (q \to r))^M \nsubseteq (p \to r)^M)$. Then by the definition of neighborhood frames $g \nVdash (p \to q) \wedge (q \to r) \to (p \to r)$. Therefore $F \nVdash (p \to q) \wedge (q \to r) \to (p \to r)$. ∎

LEMMA 3.5
The rule $\frac{p\to q}{(q\to r)\to(p\to r)}$ characterizes the class of frames $F = \langle W, g, NB\rangle$ satisfying closure under downset.

PROOF. Let for all $M = \langle W, g, NB, V\rangle$ on a frame F, which is closed under downset, $M \Vdash p \to q$. We have to prove for all M on F, $M \Vdash (q \to r) \to (p \to r)$. For this purpose we show that for all M on F, $(q \to r)^M \subseteq (p \to r)^M$. Let $w \in M$ and $w \Vdash q \to r$ then, $(V(q), V(r)) \in NB(w)$. Also by assumption $V(p) \subseteq V(q)$. The frame is closed under downset so, $(V(p), V(r)) \in NB(w)$, therefore $w \Vdash p \to r$. So $(q \to r)^M \subseteq (p \to r)^M$, i.e. by Theorem 2.12, for all M on F, $M \Vdash (q \to r) \to (p \to r)$.

For the other direction, we use contraposition. Suppose that the class is not closed under downset. That is there is a frame F and $w \in F$ such that $(X, Y) \in NB(w)$, $Z \subseteq X$ but $(Z, Y) \notin NB(w)$. Consider the valuation V such that, $V(p) = Z$, $V(r) = Y$ and $V(q) = X$. So $(V(q), V(r)) \in NB(w)$, $(V(p), V(r)) \notin NB(w)$ and $V(p) \subseteq V(q)$, then $M, w \Vdash q \to r$, $M, w \nVdash p \to r$ and $M \Vdash p \to q$. Hence $(q \to r)^M \nsubseteq (p \to r)^M$ so, $M \nVdash (q \to r) \to (p \to r)$. ∎

LEMMA 3.6
The rule $\frac{p\to q}{(r\to p)\to(r\to q)}$ characterizes the class of frames $F = \langle W, g, NB\rangle$ satisfying closure under upset.

PROOF. Let for all $M = \langle W, g, NB, V\rangle$ on a frame F, which is closed under upset, $M \Vdash p \to q$. We have to prove for all M on F, $M \Vdash (r \to p) \to (r \to q)$. For this purpose we show for all M on F, $(r \to p)^M \subseteq (r \to q)^M$. Let $w \in M$ and $w \Vdash r \to p$. Then, $(V(r), V(p)) \in NB(w)$. Also by assumption $V(p) \subseteq V(q)$. The frame is closed under downset so, $(V(r), V(q)) \in NB(w)$, therefore $w \Vdash r \to q$. So $(r \to p)^M \subseteq (r \to q)^M$, i.e. by Theorem 2.12, for all M on F, $M \Vdash (r \to p) \to (r \to q)$.

For the other direction, we use contraposition. Suppose that the class is not closed under upset. That is there is a frame F and $w \in F$ such that $(X, Y) \in NB(w)$, $Y \subseteq Z$ but $(X, Z) \notin NB(w)$. Consider the valuation V such that, $V(p) = Y$, $V(r) = X$ and $V(q) = Z$. So $(V(r), V(p)) \in NB(w)$, $(V(r), V(q)) \notin NB(w)$ and $V(p) \subseteq V(q)$, then $M, w \Vdash r \to p$, $M, w \nVdash r \to q$ and $M \Vdash p \to q$. Hence $(r \to p)^M \nsubseteq (r \to q)^M$ so, $M \nVdash (r \to p) \to (r \to q)$. ∎

In this section we will be interested in the following axiom schemas and rules.

(**C**) $(A \to B) \land (A \to C) \to (A \to B \land C)$
(**D**) $(A \to B) \land (C \to B) \to (A \lor C \to B)$
(**I**) $(A \to B) \land (B \to C) \to (A \to C)$
($\mathbf{I}_L$) $\frac{A\to B}{(C\to A)\to(C\to B)}$
($\mathbf{I}_R$) $\frac{A\to B}{(B\to C)\to(A\to C)}$

LEMMA 3.7
(a) If $\mathsf{WFC} \subseteq L$, then the canonical model of logic L is closed under intersection.
(b) If $\mathsf{WFD} \subseteq L$, then the canonical model of logic L is closed under union.
(c) If $\mathsf{WFI} \subseteq L$, then the canonical model of logic L satisfies transitivity.
(d) If $\mathsf{WFI_L} \subseteq L$, then the canonical model of logic L is closed under upset.
(e) If $\mathsf{WFI_R} \subseteq L$, then the canonical model of logic L is closed under downset.

PROOF. We only prove (a) and (e). The other cases are similar.

(a) Suppose that in the canonical model of logic L, $(X,Y) \in NB(\Gamma)$ and $(X,Z) \in NB(\Gamma)$. By definition of NB in the canonical model there exist formulas A,B and C such that $(X,Y) = ([\![A]\!],[\![B]\!])$ and $(X,Z) = ([\![A]\!],[\![C]\!])$, where $A \to B \in \Gamma$ and $A \to C \in \Gamma$. Hence $(A \to B) \wedge (A \to C) \in \Gamma$ and so using $(\mathbf{C})$, $A \to B \wedge C \in \Gamma$. Hence $([\![A]\!],[\![B \wedge C]\!]) \in NB(\Gamma)$. Therefore since $[\![B]\!] \cap [\![C]\!] = [\![B \wedge C]\!]$, $(X, Y \cap Z) \in NB(w)$. So NB is closed under intersection.

(e) Suppose that in the the canonical model of $\mathsf{WFI_R}$, $(X,Y) \in NB(\Gamma)$ and $Z \subseteq X$. Then there exist formulas A,B and C, such that $(X,Y) = ([\![A]\!],[\![B]\!])$, $Z = [\![C]\!]$ and $[\![C]\!] \subseteq [\![A]\!]$. By definition of neighborhood in the canonical model and using Lemma 2.26, we conclude that $A \to B \in \Gamma$ and $\vdash C \to A$, and so using $(\mathbf{I}_R)$ and modus ponens, $C \to B \in \Gamma$. Hence $([\![C]\!],[\![B]\!]) = (Z,Y) \in NB(\Gamma)$. Therefore NB is closed under downset. ∎

THEOREM 3.8
If $\Gamma \subseteq \{\mathbf{C}, \mathbf{D}, \mathbf{I}, \mathbf{I_L}, \mathbf{I_R}\}$, then $\mathsf{WF\Gamma}$ is sound and strongly complete with respect to the class of neighborhood frames with all the properties defined by the schemes in Γ.

PROOF. Immediate by Lemma 3.7. ∎

The logic F is the smallest set of formulas closed under instances of WF, **C, D** and **I**. Theorem 3.8 shows that F is sound and complete with respect neighborhood models closed under intersection and union, and satisfying transitivity. A rooted subintuitionistic Kripke model for F is a quadruple $M = \langle W, g, R, V\rangle$ with R lacking the properties of reflexivity, transitivity and preservation of rooted intuitionistic Kripke models [5]. The following theorem gives another proof of completeness for F with respect neighborhood models closed under intersection, union and satisfying transitivity, using the completeness of F with respect to Kripke models ([2, 9, 5]).

THEOREM 3.9
For every rooted subintuitionistic Kripke model $M_k = \langle W, g, R, V\rangle$, there is a pointwise equivalent neighborhood model $M_n = \langle W, g, NB, V\rangle$, closed under intersection and union, and satisfying transitivity.

PROOF. Let $M_k = \langle W, g, R, V\rangle$ be a rooted subintuitionistic kipke model and $w \in W$. For each $w \in W$, we define $R(w) = \{u \in W \mid wRu\}$ and

$$NB(w) = \{(X,Y) \mid R(w) \subseteq (W - X) \cup (Y)\}.$$

We show that $M_n = \langle W, g, NB, V\rangle$ is a neighborhood model closed under intersection and union, and satisfying transitivity. We know that $R(g) = W$, so

$$NB(g) = \{(X,Y) \mid W \subseteq (W - X) \cup (Y)\} = \{(X,Y) \mid X \subseteq Y\}.$$

It is easy to show that with this definition NB is closed under intersection, union and satisfies transitivity. We only show that NB is closed under intersection. Let (X,Y) and (X,Z) be in $NB(w)$. We want to show that $(X, Y \cap Z) \in NB(w)$. So by definition of NB,

$$R(w) \subseteq (W - X) \cup (Y), \tag{3.7}$$

$$R(w) \subseteq (W - X) \cup (Z). \tag{3.8}$$

By (3.7) and (3.8), $R(w) \subseteq (W-X)\cup(Y\cap Z)$. So, by definition of NB, $(X, Y\cap Z) \in NB(w)$. Now we prove that M_k and M_n are pointwise equivalent. The proof is by induction on the complexity of formulas. We only consider the implication case. Let $M_k, w \Vdash A \to B$. We want to prove that $M_n, w \Vdash A \to B$ that is $((A)^{M_n}, (B)^{M_n}) \in N(w)$. For this purpose it is sufficient to show that $R(w) \subseteq (W - (A)^{M_n}) \cup (B)^{M_n}$. Let $v \in R(w)$ and $M_n, v \Vdash A$. Then wRv and by induction hypothesis $M_k, v \Vdash A$. Then, by assumption $M_k, v \Vdash B$. So, by induction hypothesis, $M_n, v \Vdash B$. That is $v \in (B)^{M_n}$ and so $R(w) \subseteq (W - (A)^{M_n}) \cup (B)^{M_n}$. Hence $((A)^{M_n}, (B)^{M_n}) \in N(w)$.

Now assume $M_n, w \Vdash A \to B$. We want to prove that $M_k, w \Vdash A \to B$. Let wRv, $M_k, v \Vdash A$. Then, by induction hypothesis, $M_n, v \Vdash A$. Also, by assumption, $v \in R(w)$ and $R(w) \subseteq (W - (A)^M) \cup (B)^M$. So $M_n, v \Vdash B$. Again by induction hypothesis $M_k, v \Vdash B$. That is $M_k, w \Vdash A \to B$. ∎

4 Finite model property

In this section we will show that WF has the finite model property. We are going to prove this result by means of finite theories in so-called adequate sets.

Definition 4.1
A set of formulas Σ is **adequate** if for all formulas A, B, if $A \vee B \in \Sigma$ then so are A and B, if $A \wedge B \in \Sigma$ then so are A and B, and if $A \to B \in \Sigma$ then so are A and B.

Lemma 4.2
Let Σ be a set of formulas and let D be a formula such that $\Sigma \cup \{D\} \subseteq \Phi$ and $\Sigma \nvdash D$. Then there is a prime theory $\Delta \subseteq \Phi$ such that

1. $\Sigma \subseteq \Delta$,
2. $A, B \in \Delta$ and $A \wedge B \in \Phi \;\Rightarrow\; A \wedge B \in \Delta$,
3. If $A, B \in \Phi$ and $\vdash A \to B \;\Rightarrow\;$ (if $A \in \Delta$, then $B \in \Delta$),
4. If $A \in \Phi$ and $\vdash A \;\Rightarrow\; A \in \Delta$,
5. If $A \vee B \in \Delta \;\Rightarrow\; A \in \Delta$ *or* $B \in \Delta$.

Proof. Everything is similar to Theorem 2.24, except that we consider an enumeration $B_1, B_2, ..., B_n$ of the elements of Φ. ∎

Now we make the model $\mathsf{M}^\Phi = \langle W^\Phi, g, NB^\Phi, V^\Phi\rangle$ as follows,

1. $W^\Phi := \{\Delta \subseteq \Phi \mid \Delta$ satisfies conditions 2 to 5 in Lemma 4.2$\}$,
2. $g := \mathsf{WF}_\Phi = \{A \mid A \in \Phi \textit{ and } \vdash A\}$,
3. For each $\Delta \in W^\Phi$, $NB^\Phi(\Delta) = \big\{([\![A]\!]^\Phi, [\![B]\!]^\Phi) \mid A \to B \in \Delta\big\}$ where $[\![A]\!]^\Phi = \big\{\Delta \in W^\Phi \mid A \in \Delta\big\}$,
4. $V^\Phi(p) = [\![p]\!]^\Phi = \big\{\Delta \mid \Delta \in W^\Phi$ and $p \in \Delta\big\}$.

It is easy to show that $\mathsf{WF}_\Phi \in W^\Phi$ and that WF_Φ is an omniscient world. So, M^Φ is in fact a neighborhood model.

Lemma 4.3
For every formula $E \in \Phi$, and for each $\Gamma \in M^\Phi$, $(E)^{M^\Phi} = [\![E]\!]_\Phi$.

Proof. The proof is similar to Theorem 2.29. ∎

THEOREM 4.4
(Finite Model Property) If $\nvdash A$, then there exists a finite countermodel for A.

PROOF. We consider the finite adequate set Φ such that $A \in \Phi$ and consider the model M^{Φ}. By Lemma 2.29, there exists a set $\Delta \subseteq \Phi$ such that $A \notin \Delta$ and Δ is a node of M^{Φ}. Hence $M^{\Phi}, \Delta \nVdash A$ and so M^{Φ} is a finite neighborhood model which does not force A. ∎

5 Relation of WF to F

In [5] we obtained conservativity results for IPC over F. In this section, we prove such results for WF as well. This clarifies the difference in strength between F and WF.

DEFINITION 5.1
Let us call a formula $A \to B$ with A and B containing only $\wedge$ and $\vee$ a **simple implication**.

The following is a uniform substitution theorem for WF. It states that we can always replace logically equivalent formulas by each other.

THEOREM 5.2
(Uniform Substitution) If $\vdash A \leftrightarrow B$, then $\vdash E[A/p] \leftrightarrow E[B/p]$, where p is an atom.

PROOF. The proof is easy by induction on E. We only check the disjunction and implication cases. Let $E = C \vee D$, then

1. $\vdash C[A/p] \to C[B/p]$	By induction hypothesis
2. $\vdash C[B/p] \to (C \vee D)[B/p]$	axioms 1
3. $\vdash C[A/p] \to (C \vee D)[B/p]$	Follows from 1 and 2 using rule 12
4. $\vdash D[A/p] \to D[B/p]$	By induction hypothesis
5. $\vdash D[B/p] \to (C \vee D)[B/p]$	axioms 1
6. $\vdash D[A/p] \to (C \vee D)[B/p]$	Follows from 4 and 5 using rule 12
7. $\vdash (C \vee D)[A/p] \to (C \vee D)[B/p]$	Follows from 3 and 6 using rule 6

The other direction is similar to this. So $\vdash (C \vee D)[A/p] \leftrightarrow (C \vee D)[B/p]$.
Now Let $E = C \to D$, then

1. $\vdash C[A/p] \leftrightarrow C[B/p]$	By induction hypothesis
2. $\vdash D[B/p] \leftrightarrow D[A/p]$	By induction hypothesis
3. $\vdash (C[A/p] \to D[A/p]) \leftrightarrow (C[B/p] \to D[B/p])$	Follows from rule 13

That is, $\vdash E[A/p] \leftrightarrow E[B/p]$. ∎

THEOREM 5.3
Let A be constructed by applying only $\wedge$ and $\vee$ to atoms. Then there are formulas A', A'' such that

1. $\vdash A \leftrightarrow A'$ and A' is a disjunction of conjunctions atoms.
2. $\vdash A \leftrightarrow A''$ and A'' is a conjunction of disjunctions of atoms.

Proof. The proof is straightforward. ∎

Now by the theorem just proved, a simple implication $A \to B$ can be replaced by a WF- and IPC-equivalent $A^{'} \to B^{'}$ such that $A^{'}$ is a disjunction of conjunctions and $B^{'}$ is a conjunction of disjunctions.

Lemma 5.4
For all $p_i, 1 \leq i \leq k$ and $q_j, 1 \leq j \leq m$ we have,

$$\vdash_{\mathsf{WF}} p_1 \vee ... \vee p_k \to q_1 \wedge ... \wedge q_m$$

$$\text{iff}$$

$$\vdash_{\mathsf{WF}} p_i \to q_j \ \text{ for all i, j.}$$

Proof. Easy. ∎

Definition 5.5
A formula $A \to B$ called a **very simple implication** if A is conjunction of atoms and B is disjunction of atoms.

By the previous lemma we can conclude that to show that IPC is conservative over WF with respect to simple implications it is sufficient to do so for very simple implications. We can do so now for very simple implications, and in fact even for CPC instead of F.

Theorem 5.6
If IPC (or CPC) proves a very simple implication, then WF proves it as well.

Proof. Let $A \to B$ be a very simple implication, so $A = \bigwedge_i (P_i)$ and $B = \bigvee_j (q_j)$. Assume $\nvdash_{\mathsf{WF}} A \to B$ then by the completeness theorem there exists a neighborhood model M and $w \in M$, such that $M, w \nVdash A \to B$. Hence $(A)^M \nsubseteq (B)^M$. So there exists $v \in M$ such that $M, v \Vdash A$ and $M, v \nVdash B$. Now we select this point v from M and then we make the one point IPC model $M_{\mathsf{IPC}} = \langle v, NB(v), \vDash \rangle$ such that $NB(v) = \{(v, v), (\varnothing, v)\}$ and for all propositional variable p, $M_{\mathsf{IPC}}, v \vDash p$ if and only if $M, v \Vdash p$. Clearly

$$M_{\mathsf{IPC}}, v \vDash p_i, \text{ for all } i,$$

$$M_{\mathsf{IPC}}, v \nvDash q_j, \text{ for all } j.$$

So, $((A)^{M_{\mathsf{IPC}}}, (B)^{M_{\mathsf{IPC}}}) = (v, \varnothing)$ and $(v, \varnothing) \notin NB(v)$. That is $M_{\mathsf{IPC}}, v \vDash A \to B$, so $\nvdash_{\mathsf{IPC}} A \to B$. ∎

In [5] IPC was proved to be conservative over F with respect to basic implications, formulas of the form $A \to B$ with A and B conjunction/disjunctions of simple implications. For example $(p \to q) \wedge (p \to r) \to (p \to q \wedge r)$ is a basic implication provable in F for which $\nvdash_{\mathsf{WF}} (p \to q) \wedge (p \to r) \to (p \to q \wedge r)$. This shows a clear difference in strength between F and WF.

6 WF and Modal Logic

We consider the translation $\Box$ from L, the language of propositional logic, to $L_{\Box}$, the language of modal propositional logic. It is given by:

1. $p^{\Box} = p$;
2. $(A \wedge B)^{\Box} = A^{\Box} \wedge B^{\Box}$;
3. $(A \vee B)^{\Box} = A^{\Box} \vee B^{\Box}$;
4. $(A \to B)^{\Box} = \Box(A^{\Box} \to B^{\Box})$.

Definition 6.1
A system of modal logic is classical iff it is closed under RE ($\frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$) [1].

E is the smallest classical modal logic. The logic EN extends E by adding the axiom scheme $\Box\top$. Completeness holds for EN with respect neighborhood frames that contain the unit, i.e. for all $w \in W$, $W \in N(w)$ [1]. The clause for $\Box\varphi$ in the N-neighborhood models is: $w \Vdash \Box\varphi$ iff $V(\varphi) \in N(w)$.

We will show that if $\vdash_{\mathsf{WF}} A$, then $\vdash_{\mathsf{EN}} A^{\Box}$. We conjecture that the reverse direction holds as well but we were unable to prove it because of the difference between neighborhoods and N-neighborhoods.

Proposition 6.2
If $M = \langle W, N, V\rangle$ is an intuitionistic N-neighborhood model without an omniscient world then M can be extended by adding an omniscient world to obtain a model M' such that for all formulas A and for all $w \in W$,

$$M, w \Vdash A \ \text{ iff } \ M', w \Vdash A.$$

Proof. We add a world g to W and make a new model $M' = \langle W', g, N', V'\rangle$, with $W' = W \cup \{g\}$, for all propositional letters p, $(p)^{M'} = (p)^{M}$ and for all $w \in W$ and $X \in P(W)$,

$$N'(w) = \{X \cup \{g\}\,,\ X \mid X \in N(w)\}\,,\ N'(g) = \{W'\}.$$

Then we will show that for all formulas D, $(D)^{M'} \cap W = (D)^{M}$, in other words $(D)^{M'} = (D)^{M}$ or $(D)^{M'} = (D)^{M} \cup \{g\}$. The proof is by induction on D. The case where D is a proposition letter follows by definition. Conjunction and disjunction are easy, and if $(E)^{M'} \cap W = (E)^{M}$, then $((E)^{M'})^{c} \cap W = ((E)^{M})^{c}$. So let $D = E \to F$ and $w \in W$. Then,

$$\begin{aligned} M, w \Vdash E \to F &\iff ((E)^{M})^{c} \cup (F)^{M} \in N(w)\\ \text{(by induction hypothesis)} &\iff^{*} ((E)^{M'})^{c} \cup (F)^{M'} \in N'(w)\\ &\iff M', w \Vdash' E \to F. \end{aligned}$$

The explanation of $\iff^{*}$ is that by induction hypothesis,

$$(E^{M'})^{c} = (E^{M})^{c} \text{ or } (E^{M'})^{c} = (E^{M})^{c} \cup \{g\}\,,$$

$$F^{M'} = F^{M} \text{ or } F^{M'} = F^{M} \cup \{g\}\,.$$

∎

We can interpret the N-neighborhood models in two ways. As an N-neighborhood model with $\Vdash_{\mathsf{N}}$ and as a modal model with $\Vdash_{\mathsf{EN}}$.

LEMMA 6.3
Let $M = \langle W, N, V \rangle$ be an N-neighborhood model. Then for all $w \in W$,

$$M, w \Vdash_{\mathsf{N}} A \;\text{ iff }\; M, w \Vdash_{\mathsf{M}} A^{\Box}.$$

PROOF. The proof is by induction on A. The atomic case holds by induction and the conjunction and disjunction cases are easy. We only check the implication case. So let $A = C \to D$, then

$$\begin{array}{rl} M, w \Vdash_{\mathsf{N}} C \to D & \iff \{v \mid v \nVdash_{\mathsf{N}} C\} \cup \{v \mid v \Vdash_{\mathsf{N}} D\} \in N(w) \\ \text{(by induction hypothesis)} & \iff \left\{v \mid v \nVdash_{\mathsf{M}} C^{\Box}\right\} \cup \left\{v \mid v \Vdash_{\mathsf{M}} D^{\Box}\right\} \in N(w) \\ & \iff \left\{v \mid v \Vdash_{\mathsf{M}} \neg C^{\Box}\right\} \cup \left\{v \mid v \Vdash_{\mathsf{M}} D^{\Box}\right\} \in N(w) \\ & \iff \left\{v \mid v \Vdash_{\mathsf{M}} \neg C^{\Box} \vee D^{\Box}\right\} \in N(w) \\ & \iff M, w \Vdash_{\mathsf{M}} \Box(\neg C^{\Box} \vee D^{\Box}) \\ & \iff M, w \Vdash_{\mathsf{M}} (C \to D)^{\Box}. \end{array}$$

∎

THEOREM 6.4
For all formulas A, if $\vdash_{\mathsf{WF}} A$, then $\vdash_{\mathsf{EN}} A^{\Box}$.

PROOF. Assume $\nvdash_{\mathsf{EN}} A^{\Box}$, then by the completeness theorem there exists $M = \langle W, N, V \rangle$ and $w \in W$, such that $M, w \nVdash_{\mathsf{EN}} A^{\Box}$. Then, by Lemma 6.3, $M, w \nVdash_{\mathsf{N}} A$. To transfer M into a proper N-neighborhood model for WF it needs to have an omniscient world. In case $M = \langle W, N, V \rangle$ does not have one, by Proposition 6.2, there exists M' with an omniscient world such that $M', w \nVdash_{\mathsf{N}} A$. By Lemma 2.7 there is then a pointwise equivalent neighborhood model $M'_{NB} = \langle W, g, NB, V \rangle$, such that $M'_{NB}, w \nVdash A$. So, by soundness, $\nvdash_{\mathsf{WF}} A$. ∎

DEFINITION 6.5
An N-neighborhood frame $F = \langle W, g, N \rangle$, is closed under **superset** if and only if

$$\forall w \in W,\ \forall X, Y \in P(W):\ X \subseteq Y,\ X \in N(w) \ \Rightarrow\ Y \in N(w).$$

DEFINITION 6.6
A system of modal logic is monotone iff it is closed under RM ($\frac{A \to B}{\Box A \to \Box B}$) [4].

M is the smallest monotonic modal logic. Completeness holds for M with respect monotonic neighborhood frames, i.e. in $F = \langle W, N \rangle$, N is closed under superset [4].

LEMMA 6.7
For every N-neighborhood model $M_N = \langle W, g, N, V \rangle$ closed under superset there is a pointwise equivalent neighborhood model $M_{NB} = \langle W, g, NB, V \rangle$ closed under upset and downset.

PROOF. The proof is easy if, by considering, as in Lemma 2.7, for each $w \in W$,
$$NB(w) = \{(X, Y) \mid (X)^c \cup Y \in N(w)\}.$$
∎

THEOREM 6.8
For all formulas A, if $\vdash_{\mathsf{WFI_R I_L}} A$, then $\vdash_{\mathsf{M}} A^{\Box}$.

PROOF. Assume $\nvdash_{\mathsf{M}} A^{\Box}$. Then by the completeness theorem there exist $M = \langle W, N, V \rangle$, closed under superset, and $w \in W$ such that $M, w \nVdash_{\mathsf{M}} A^{\Box}$. So, by Lemma 6.3, $M, w \nVdash_{\mathsf{N}} A$. In case $M = \langle W, N, V \rangle$ does not have an omniscient world, then by

Proposition 6.2, there exists M' with an omniscient world such that $M', w \nVdash_{\mathsf{N}} A$. Moreover, from the proof of Lemma 2.7, it is clear that, if M is closed under superset, M' is as well. By Lemma 6.7 then, there is a pointwise equivalent neighborhood model $M'_{NB} = \langle W, g, NB, V \rangle$, closed under upset and downset, such that $M'_{NB}, w \nVdash A$. So by soundness, $\nvdash_{\mathsf{WFI_R I_L}} A$. ■

Also in this case we were not able to prove the converse.

7 Conclusion

In this article we constructed a neighborhood semantics for weak subintuitionistic logics extending a basic logic WF. It uses pairs of subsets of the set of worlds instead of just subsets. This definitely seems the right choice, especially in view of the results of Section 3 where the various obvious extensions of WF can be neatly separated by frame properties. And in that way it does become clear what the right logics to be studies in this area are. It does make the connection to non-normal modal logic less clear than one could hope for because of the different semantics, and this lead to some open problems, for example the relationship between WF and EN. The relationship between the models used in modal logic and subintuitionistic logic does need clearing up. The conservativity result in Section 5 makes clear what kind of implications can be expected to be provable in WF and separates WF from F, for which we obtained a conservativity result before. It is not clear how this relates to the complexity of the decision problem of the logics. This may be an object for further study. Finally, it seems worthwhile to study interpolation for WF and other logics.

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