TOPOLOGICAL AND LOGICAL EXPLORATIONS OF KRULL DIMENSION

GURAM BEZHANISHVILI, NICK BEZHANISHVILI, JOEL LUCERO-BRYAN, JAN VAN MILL

ABSTRACT. Krull dimension measures the depth of the spectrum $\mathsf{Spec}(R)$ of a commutative ring R. Since $\mathsf{Spec}(R)$ is a spectral space, Krull dimension can be defined for spectral spaces. Utilizing Stone duality, it can also be defined for distributive lattices. For an arbitrary topological space, the notion of Krull dimension is less useful. Isbell [23] remedied this by introducing the concept of graduated dimension. In this paper we propose an alternate concept, that of localic Krull dimension of a topological space, which has its roots in modal logic. This is done by investigating the concept of Krull dimension for closure algebras and Heyting algebras, which formalize the notions of powerset and open set algebras of topological spaces [30, 31, 35]. We compare localic Krull dimension to other well-known dimension functions, and show that it can detect topological differences between topological spaces that Krull dimension is unable to detect. We also investigate applications of localic Krull dimension to modal logic. We prove that for a T_1-space to have a finite localic Krull dimension can be described by an appropriate generalization of the well-known concept of a nodec space. These considerations yield topological completeness and incompleteness results in modal logic that we examine in detail.

1. INTRODUCTION

For a commutative ring R, let $\mathsf{Spec}(R)$ be the set of prime ideals of R topologized with the Zariski topology. As usual, we refer to $\mathsf{Spec}(R)$ as the spectrum of R. The Krull dimension of R is defined as the supremum of the lengths of chains in $(\mathsf{Spec}(R), \subseteq)$. This notion is of fundamental importance in commutative algebra and algebraic geometry (see, for example, [14, Ch. 8]). Since $\mathsf{Spec}(R)$ is a spectral space, where the inclusion on prime ideals is the specialization order of the Zariski topology, we can define the Krull dimension of a spectral space X as the supremum of the lengths of chains in the specialization order of X. By Stone duality [36], spectral spaces are dual to distributive lattices, which paves the way to defining the Krull dimension of a distributive lattice L as the supremum of the lengths of chains in $(\mathsf{Spec}(L), \subseteq)$, where $\mathsf{Spec}(L)$ is the Stone dual of L. For different characterizations of the Krull dimension of distributive lattices see [4, 18, 19, 7, 8] and the references therein. If a distributive lattice L is a frame or locale (that is, a topological space without points), then a convenient characterization of the Krull dimension of L can be found in [2, Thm. 6.9].

If we define the Krull dimension of an arbitrary topological space X by means of the specialization order of X, then to quote Isbell [23], the result is "spectacularly wrong for the most popular spaces, vanishing for all non-empty Hausdorff spaces; but it seems to be the only dimension of interest for the Zariski spaces of algebraic geometry." Isbell remedied this by proposing the definition of graduated dimension. In this article we propose a different approach. We define the localic Krull dimension of a topological space X as the Krull dimension of the locale $\Omega(X)$ of all open subsets of X. As we will see, this definition turns out to be more refined. For example, every nonempty Stone space has Krull dimension and

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graduated dimension 0. On the other hand, for each n (including ∞), there is a Stone space X such that the localic Krull dimension of X is n. Thus, localic Krull dimension provides more refined classification of Stone spaces, and this extends to spectral spaces and beyond.

Instead of working with the locale $\Omega(X)$, we could work with the powerset algebra $\wp(X)$ equipped with the closure operator or its dual interior operator. Then the locale $\Omega(X)$ can be realized as the fixed points of the interior operator. This line of research was developed by McKinsey and Tarski in their classic paper [30]. It gives rise to the concept of a closure algebra, which is a pair $\mathfrak{A} = (A, \mathbf{C})$, where A is a Boolean algebra and $\mathbf{C} : A \to A$ is a unary function satisfying Kuratowski's axioms for closure.

In this article we introduce and develop the theory of Krull dimension for closure algebras. We prove that the Krull dimension of a closure algebra coincides with the Krull dimension of its open elements. Thus, the localic Krull dimension of a topological space X can alternatively be defined as the Krull dimension of the closure algebra $(\wp(X), \mathbf{C})$.

This approach has a number of logical applications. Closure algebras serve as algebraic models of Lewis' well-known modal logic $\mathbf{S4}$ (see, e.g., [35]). For each $n \geq 1$, there is a modal formula φ_n that is satisfied in a closure algebra $\mathfrak{A}$ iff the Krull dimension of $\mathfrak{A}$ is $\leq n$. Therefore, for each $n \geq 1$, adding φ_n to $\mathbf{S4}$ yields the modal logic $\mathbf{S4}_n$, which is the logic of all closure algebras of Krull dimension $\leq n$. This yields that $\mathbf{S4}_n$ is the logic of all topological spaces whose localic Krull dimension is $\leq n$.

We generalize the well-known concept of a nodec space to that of an n-discrete space, and prove that if X is a T_1-space, then the localic Krull dimension of X is $\leq n$ iff X is n-discrete. As was shown in [1], the modal logic of nodec spaces is the well-known Zeman logic $\mathbf{S4.Z}$. For each $n \geq 1$, we generalize the Zeman logic $\mathbf{S4.Z}$ to the n-Zeman logic $\mathbf{S4.Z}_n$, and show that $\mathbf{S4.Z}_n$ is a proper extension of $\mathbf{S4}_n$. From this we derive that $\mathbf{S4}_n$ is topologically incomplete for any class of T_1-spaces. In fact, we show that no logic in the interval $[\mathbf{S4}_n, \mathbf{S4.Z}_n)$ is the logic of any class of T_1-spaces. On the other hand, for each $n \geq 1$, we construct a countable crowded ω-resolvable Tychonoff space Z_n of localic Krull dimension n such that $\mathbf{S4.Z}_n$ is the logic of Z_n.

The article is organized as follows. Section 2 recalls the basic definitions, notation, and terminology used for closure algebras and Heyting algebras, as well as some connections between these classes of algebras. We also review Esakia duality for these classes of algebras. In Section 3 we define the Krull dimension of a closure algebra in two ways, externally and internally. The main result of Section 3 shows that the two definitions are equivalent, and also demonstrates multiple equivalent conditions characterizing Krull dimension of a closure algebra. In Section 4 we define external and internal Krull dimension of Heyting algebras, and show that they are equivalent. We connect the Krull dimension of Heyting algebras with that of closure algebras, and exhibit equivalent conditions characterizing the Krull dimension of a Heyting algebra.

In Section 5 we introduce the localic Krull dimension of an arbitrary topological space X, and provide equivalent conditions characterizing finite (nonnegative) localic Krull dimension of X, some of which require us to develop a topological version of the Jankov-Fine formulas [24, 20]. We also compare localic Krull dimension to other well-known dimension functions. In Section 6 we generalize the notion of a discrete closure algebra to that of an n-discrete closure algebra, and characterize dually when a closure algebra is n-discrete. We also generalize the well-known Zeman formula to a family of formulas we call the n-Zeman formulas zem_n, and show that zem_n characterizes n-discrete closure algebras. We conclude the section by showing that each logic $\mathbf{S4.Z}_n$ obtained by adding zem_n to $\mathbf{S4}$ has the finite model property. In Section 7 we introduce n-discrete topological spaces. We prove that being n-discrete is a topological property that is defined by zem_n, and show that $\mathbf{S4.Z}_n$ is the logic of n-discrete spaces. In addition, we prove that a T_1-space is n-discrete iff its localic Krull dimension is $\leq n$, thus yielding topological incompleteness results. The section culminates by showing that each logic $\mathbf{S4.Z}_n$ is the logic of a single countable crowded ω-resolvable Tychonoff space of localic Krull dimension n.

2. Closure algebras and Heyting algebras

Definition 2.1. A *closure operator* on a Boolean algebra A is a unary function $\mathbf{C} : A \to A$ satisfying Kuratowski's axioms:

- $a \leq \mathbf{C}(a)$;
- $\mathbf{CC}(a) \leq \mathbf{C}(a)$;
- $\mathbf{C}(a \vee b) = \mathbf{C}(a) \vee \mathbf{C}(b)$;
- $\mathbf{C}(0) = 0$.

A *closure algebra* is a pair $\mathfrak{A} = (A, \mathbf{C})$, where A is a Boolean algebra and $\mathbf{C}$ is a closure operator on A. As usual, if $\mathbf{C}$ is a closure operator on A, then its dual *interior operator* is defined by $\mathbf{I}(a) = -\mathbf{C}(-a)$ for each $a \in A$.

A typical example of a closure algebra is the powerset algebra of a topological space X; that is, the pair $\mathfrak{A}_X = (\wp(X), \mathbf{C}_X)$, where $\wp(X)$ is the powerset of X and $\mathbf{C}_X$ is closure in X. By the McKinsey-Tarski representation theorem [30], every closure algebra is represented as a subalgebra of $\mathfrak{A}_X$ for some topological space X.

There is another representation of closure algebras due to Jónsson-Tarski [25] and Kripke [28], which is central to modal logic. Recall that a *Kripke frame* is a pair $\mathfrak{F} = (W, R)$, where W is a set and R is a binary relation on W. Kripke frames provide relational semantics of modal logic [3, 6]. Those Kripke frames where the relation is reflexive and transitive provide relational semantics of Lewis' well-known modal system $\mathbf{S4}$. This is why Kripke frames with reflexive and transitive relations are often referred to as $\mathbf{S4}$-*frames*. Given an $\mathbf{S4}$-frame $\mathfrak{F} = (W, R)$ and $A \subseteq W$, let

$$R^{-1}[A] = \{w \in W \mid \exists a \in A \text{ with } wRa\}.$$

Then the powerset algebra $\mathfrak{A}_{\mathfrak{F}} = (\wp(W), \mathbf{C}_R)$ is a closure algebra, where $\mathbf{C}_R(A) = R^{-1}[A]$. Moreover, every closure algebra is represented as a subalgebra of $\mathfrak{A}_{\mathfrak{F}}$ for some $\mathbf{S4}$-frame $\mathfrak{F}$.

There is a close connection between the McKinsey-Tarski and Kripke representations. Suppose $\mathfrak{F} = (W, R)$ is an $\mathbf{S4}$-frame. Call $U \subseteq W$ and R-*upset* if $w \in U$ and wRv imply $v \in U$ (R-*downsets* are defined dually). Let τ_R be the collection of all R-upsets of $\mathfrak{F}$. Then τ_R is a topology on W in which every $w \in W$ has a least open neighborhood

$$R[w] := \{v \in W \mid wRv\}.$$

Such topological spaces are called *Alexandroff spaces*, and can alternatively be described as the topological spaces in which intersections of arbitrary families of opens are open. Thus, $\mathbf{S4}$-frames correspond to Alexandroff spaces.

In [16] Esakia put together Stone duality for Boolean algebras with Kripke representation of closure algebras to obtain a full duality for closure algebras. By Esakia duality, the category of closure algebras is dually equivalent to the category of Esakia spaces (aka descriptive $\mathbf{S4}$-frames).

Definition 2.2. A *Stone space* is a zero-dimensional compact Hausdorff space, and an *Esakia space* is an $\mathbf{S4}$-frame $\mathfrak{F} = (W, R)$ such that W is equipped with a Stone topology satisfying

- $R[w]$ is closed;
- U clopen implies $R^{-1}[U]$ is clopen.

The dual Esakia space of a closure algebra $\mathfrak{A}$ is the pair $\mathfrak{A}_* = (W, R)$, where W is the Stone space of A and

$$wRv \text{ iff } (\forall a \in \mathfrak{A})(a \in v \Rightarrow \mathbf{C}(a) \in w).$$

The dual closure algebra of an Esakia space $\mathfrak{F} = (W, R)$ is the closure algebra $\mathfrak{F}^* = (\mathsf{Clop}(W), \mathbf{C}_R)$, where $\mathsf{Clop}(W)$ is the Boolean algebra of clopen subsets of W and

$$\mathbf{C}_R(U) = R^{-1}[U] \text{ and hence } \mathbf{I}_R(U) = W \setminus R^{-1}[W \setminus U].$$

Then $\beta : \mathfrak{A} \to \mathfrak{A}_*{}^*$ and $\varepsilon : \mathcal{E} \to \mathcal{E}^*{}_*$ are isomorphisms, where

$$\beta(a) = \{w \in W \mid a \in w\} \text{ and } \varepsilon(w) = \{U \in \mathsf{Clop}(W) \mid w \in U\}.$$

In the finite case, the topology on an Esakia space becomes discrete, and we identify finite Esakia spaces with finite **S4**-frames.

Let $\mathfrak{A}$ be a closure algebra and $\mathfrak{A}_*$ be the Esakia space of $\mathfrak{A}$. As is customary, we adopt topological terminology and call $a \in \mathfrak{A}$ *closed* if $a = \mathbf{C}(a)$, *open* if $a = \mathbf{I}(a)$, *dense* if $\mathbf{C}(a) = 1$, and *nowhere dense* if $\mathbf{IC}(a) = 0$. The following is well known (and easy to see):

- a is closed iff $\beta(a)$ is a clopen R-downset in $\mathfrak{A}_*$;
- a is open iff $\beta(a)$ is a clopen R-upset in $\mathfrak{A}_*$;
- a is dense iff $\mathbf{C}_R\beta(a) = W$;
- a is nowhere dense iff $\mathbf{I}_R\mathbf{C}_R\beta(a) = \varnothing$.

The *relativization* of $\mathfrak{A}$ to $a \in \mathfrak{A}$ is the closure algebra $\mathfrak{A}_a$ whose underlying set is the interval $[0, a]$, the meet and join in $\mathfrak{A}_a$ coincide with those in $\mathfrak{A}$, the complement of $b \in \mathfrak{A}_a$ is given by $a - b$, and the closure of b is given by $\mathbf{C}_a(b) = a \wedge \mathbf{C}(b)$. It follows that the interior of $b \in \mathfrak{A}_a$ is given by $\mathbf{I}_a(b) = a \wedge \mathbf{I}(a \to b)$. If $\mathfrak{A} = \mathfrak{A}_X$ is the powerset algebra of a topological space X and $Y \subseteq X$, then the relativization of $\mathfrak{A}$ to Y is the powerset algebra $\mathfrak{A}_Y$ of the subspace Y of X.[1] The relativization $\mathfrak{A}_a$ is realized dually as the restriction of R to the clopen subspace $\beta(a)$ of $\mathfrak{A}_*$. In order to describe dually a connection between nowhere dense elements and relativizations, we recall the notion of an R-maximal point.

Definition 2.3. Let $\mathfrak{F} = (W, R)$ be an **S4**-frame, $U \subseteq W$, and $w \in U$. Then w is an *R-maximal point* of U provided wRu and $u \in U$ imply uRw. We denote the set of R-maximal points of $\mathfrak{F}$ by $\max_R(\mathfrak{F})$.

It is well known (see, e.g., [17, Sec. III.2]) that in an Esakia space $\mathfrak{F} = (W, R)$, the set $\max_R(\mathfrak{F})$ is a closed R-upset, and for each $w \in W$ there is $v \in \max_R(\mathfrak{F})$ such that wRv.

Lemma 2.4. *Let $\mathfrak{A}$ be a closure algebra and $\mathfrak{A}_*$ be its Esakia space. Suppose $a \in \mathfrak{A}$ and $d \in \mathfrak{A}_a$. Then d is nowhere dense in $\mathfrak{A}_a$ iff $\beta(d) \cap \max_Q \beta(a) = \varnothing$, where Q is the restriction of R to $\beta(a)$.*

Proof. Since $\mathfrak{A}_*$ is an Esakia space and $\beta(a)$ is clopen in $\mathfrak{A}_*$, it is well known (see, e.g., [17, Sec. III.2]) that $\mathfrak{F} = (\beta(a), Q)$ is also an Esakia space. As $\max_Q \beta(a)$ is a Q-upset of $\beta(a)$, the condition $\beta(d) \cap \max_Q \beta(a) = \varnothing$ is equivalent to $\mathbf{C}_Q[\beta(d)] \cap \max_Q \beta(a) = \varnothing$, which in turn is equivalent to $\mathbf{I}_Q\mathbf{C}_Q[\beta(d)] \cap \max_Q \beta(a) = \varnothing$. Since $\mathbf{I}_Q\mathbf{C}_Q[\beta(d)]$ is a Q-upset of $\beta(a)$, the last condition is equivalent to $\mathbf{I}_Q\mathbf{C}_Q[\beta(d)] = \varnothing$. Therefore, $\beta(d) \cap \max_Q \beta(a) = \varnothing$ iff $\beta(\mathbf{I}_a\mathbf{C}_a d) = \varnothing$, which is equivalent to d being nowhere dense in $\mathfrak{A}_a$. □

We next turn to Heyting algebras, which are closely related to closure algebras [31, 35].

[1] Despite subscript being used to denote both a relativization of a closure algebra $\mathfrak{A}$ and the powerset algebra of a space X, there is no ambiguity when $\mathfrak{A} = \mathfrak{A}_X$ because $(\mathfrak{A}_X)_Y = \mathfrak{A}_Y$.

Definition 2.5. A *Heyting algebra* is a bounded implicative lattice; that is, a bounded distributive lattice such that $\wedge$ has a residual $\to$ satisfying

$$x \le a \to b \text{ iff } a \wedge x \le b.$$

If $\mathfrak{A}$ is a closure algebra, then $\mathfrak{H}(\mathfrak{A}) := \{\mathbf{I}a \mid a \in \mathfrak{A}\}$ is a Heyting algebra. Conversely, if $\mathfrak{H}$ is a Heyting algebra, then the free Boolean extension $\mathfrak{B}(\mathfrak{H})$ of $\mathfrak{H}$ can be equipped with a closure operator $\mathbf{C}$ so that $\mathfrak{A}(\mathfrak{H}) := (\mathfrak{B}(\mathfrak{H}), \mathbf{C})$ is a closure algebra, $\mathfrak{H}$ is isomorphic to $\mathfrak{H}(\mathfrak{A}(\mathfrak{H}))$, and $\mathfrak{A}(\mathfrak{H}(\mathfrak{A}))$ is isomorphic to a subalgebra of $\mathfrak{A}$ (see, e.g., [35, Sec. IV.1 and IV.3] or [17, Sec. II.2 and II.5]).

As with closure algebras, a typical example of a Heyting algebra is the Heyting algebra $\mathfrak{H}_X$ of all opens of a topological space X, and every Heyting algebra is represented as a subalgebra of $\mathfrak{H}_X$ for some topological space X [31, 35].

Another representation of Heyting algebras is by means of R-upsets of **S4**-frames, but since R-upsets do not distinguish between points that are R-related to each other, we may restrict ourselves to those **S4**-frames that are in addition antisymmetric. This extends to Esakia duality between Heyting algebras and partially ordered Esakia spaces [16].

Define an equivalence relation on an **S4**-frame $\mathfrak{F} = (W, R)$ by setting

$$w \sim v \text{ iff } wRv \text{ and } vRw.$$

As is customary, we call equivalence classes of $\sim$ *R-clusters*.

If $\mathfrak{A}$ is a closure algebra and $\mathfrak{A}_*$ is the dual of $\mathfrak{A}$, then the dual $\mathfrak{H}(\mathfrak{A})_*$ of $\mathfrak{H}(\mathfrak{A})$ is obtained by modding out $\mathfrak{A}_*$ by the equivalence relation $\sim$. Conversely, if $\mathfrak{H}$ is a Heyting algebra, then the dual $\mathfrak{A}(\mathfrak{H})_*$ of $\mathfrak{A}(\mathfrak{H})$ is isomorphic to the dual $\mathfrak{H}_*$ of $\mathfrak{H}$ (see, e.g., [17, Sec. III.4]).

We conclude this preliminary section by a brief discussion of relativizations of Heyting algebras. Let $\mathfrak{H}$ be a Heyting algebra and $a \in \mathfrak{H}$. The *relativization* of $\mathfrak{H}$ with respect to a is the Heyting algebra $\mathfrak{H}_a$ whose underlying set is the interval $[a, 1]$ and the meet, join, and implication in $\mathfrak{H}_a$ coincide with those in $\mathfrak{H}$. If $\mathfrak{H} = \mathfrak{H}_X$ is the Heyting algebra of all opens of a topological space X and U is an open subset of X, then the relativization of $\mathfrak{H}$ with respect to U is isomorphic to the Heyting algebra of all opens of the subspace $X \setminus U$.

3. Krull dimension of closure algebras

In this section we define the Krull dimension of a closure algebra $\mathfrak{A}$ via chains in the Esakia space of $\mathfrak{A}$. We relate this definition to the concept of depth of an Esakia space, which plays an important role in modal logic. The main result of the section is a pointfree characterization of the Krull dimension of $\mathfrak{A}$ without accessing the Esakia space of $\mathfrak{A}$.

For an **S4**-frame $\mathfrak{F} = (W, R)$, we write $w\vec{R}v$ provided wRv and $v\not\!Rw$. We call a finite sequence $\{w_i \in W \mid i < n\}$ an *R-chain* provided $w_i\vec{R}w_{i+1}$ for all i, and define the *length* of the R-chain $\{w_i \in W \mid i < n\}$ to be $n-1$. Note that we allow the empty R-chain which has length -1.

Definition 3.1. Let $\mathfrak{A}$ be a closure algebra. Define the *Krull dimension* $\operatorname{kdim}(\mathfrak{A})$ of $\mathfrak{A}$ as the supremum of the lengths of R-chains in $\mathfrak{A}_*$. If the supremum is not finite, then we write $\operatorname{kdim}(\mathfrak{A}) = \infty$.

The definition of the length of an R-chain that we have adopted has its roots in algebra. Modal logicians have used a similar concept of *depth* of a frame $\mathfrak{F} = (W, R)$. But in modal logic the length of an R-chain $\{w_i \in W \mid i < n\}$ is typically defined to be n. This notion of length is always one more than the notion of length in algebra. The difference is whether we count the number of R-links in the R-chain (as algebraists do) or the number of points in the R-chain (as modal logicians do). Therefore, the Krull dimension of $\mathfrak{A}$ is one less than the depth of $\mathfrak{A}_*$ (provided the Krull dimension of $\mathfrak{A}$ is finite). Thus, $\mathrm{kdim}(\mathfrak{A}) = n$ iff $\mathrm{depth}(\mathfrak{A}_*) = n + 1$ for $n \in \omega$. It is expressible by a modal formula whether $\mathrm{depth}(\mathfrak{A}_*)$ is bounded by n.

Definition 3.2. For $n \geq 1$, consider the formulas:

$$\begin{aligned} \mathsf{bd}_1 &= \Diamond\Box p_1 \to p_1, \\ \mathsf{bd}_{n+1} &= \Diamond\left(\Box p_{n+1} \wedge \neg \mathsf{bd}_n\right) \to p_{n+1}. \end{aligned}$$

The modal language is interpreted in a closure algebra $\mathfrak{A} = (A, \mathbf{C})$ by assigning to each propositional letter an element of $\mathfrak{A}$, letting the classical connectives disjunction and negation be the Boolean join and complement in $\mathfrak{A}$, and letting the modal diamond be the closure $\mathbf{C}$. For a modal formula φ, we say φ is *satisfiable* in $\mathfrak{A}$ provided φ is interpreted as 1 for some assignment of the propositional letters. We say φ is *valid* in $\mathfrak{A}$, written $\mathfrak{A} \vDash \varphi$, whenever φ is 1 under all assignments of the letters. The following lemma is well known (see, e.g., [6, Prop. 3.44]).

Lemma 3.3. *Let $\mathfrak{A}$ be a nontrivial closure algebra and $n \geq 1$. Then* $\mathrm{depth}(\mathfrak{A}_*) \leq n$ *iff* $\mathfrak{A} \vDash \mathsf{bd}_n$.

We next describe when $\mathrm{kdim}(\mathfrak{A}) \leq 0$. Recall that $\mathfrak{A}$ is *trivial* if $0 = 1$, it is *discrete* if $\mathbf{C}$ is the identity function, and it is an **S5**-*algebra* (or *monadic algebra*) if $a \leq \mathbf{IC}(a)$ for all $a \in A$. If $\mathfrak{A}_*$ is the Esakia space of $\mathfrak{A}$ then it is well known that $\mathfrak{A}$ is trivial iff $\mathfrak{A}_* = \varnothing$, that $\mathfrak{A}$ is discrete iff R is the identity, and $\mathfrak{A}$ is an **S5**-algebra iff R is an equivalence relation.

Lemma 3.4. *Let $\mathfrak{A}$ be a closure algebra.*

(1) $\mathrm{kdim}(\mathfrak{A}) = -1$ *iff $\mathfrak{A}$ is the trivial algebra.*
(2) $\mathrm{kdim}(\mathfrak{A}) \leq 0$ *iff $\mathfrak{A}$ is an* **S5**-*algebra.*
(3) $\mathrm{kdim}(\mathfrak{A}) = 0$ *iff $\mathfrak{A}$ is a nontrivial* **S5**-*algebra.*
(4) *If $\mathfrak{A}$ is discrete, then* $\mathrm{kdim}(\mathfrak{A}) \leq 0$.

Proof. (1) Suppose $\mathfrak{A}$ is trivial. Then $\mathfrak{A}_* = \varnothing$, so the only R-chain in $\mathfrak{A}_*$ is the empty chain whose length is -1. Therefore, $\mathrm{kdim}(\mathfrak{A}) = -1$. Conversely, if $\mathrm{kdim}(\mathfrak{A}) = -1$, then every R-chain in $\mathfrak{A}_*$ has length -1, and hence is the empty chain. Thus, $\mathfrak{A}_* = \varnothing$, and so $\mathfrak{A}$ is the trivial algebra.

(2) Suppose $\mathfrak{A}$ is an **S5**-algebra. Then R is an equivalence relation, so there are no $w, v \in \mathfrak{A}_*$ with $w\vec{R}v$. Therefore, every R-chain in $\mathfrak{A}_*$ has length ≤ 0. Thus, $\mathrm{kdim}(\mathfrak{A}) \leq 0$. Conversely, suppose $\mathrm{kdim}(\mathfrak{A}) \leq 0$. Then every R-chain in $\mathfrak{A}_*$ has length ≤ 0. Therefore, if xRy, then it cannot be the case that $y\not{R}x$. Thus, R is symmetric, and so $\mathfrak{A}$ is an **S5**-algebra.

(3) This follows from (1) and (2).

(4) This follows from (2) since every discrete algebra is an **S5**-algebra. $\square$

Remark 3.5.

(1) Since not every **S5**-algebra is discrete, the converse of Lemma 3.4(4) does not hold.
(2) Suppose $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}_X$ for some topological space X. If $\mathfrak{A}$ consists of clopen subsets of X, then $\mathfrak{A}$ is discrete, and hence $\mathrm{kdim}(\mathfrak{A}) \leq 0$.

By Lemma 3.4, whether the Krull dimension of $\mathfrak{A}$ is ≤ 0 can be determined internally in $\mathfrak{A}$, without accessing $\mathfrak{A}_*$. The goal of the remainder of this section is to develop a pointfree description of the Krull dimension of $\mathfrak{A}$ that does not require the Esakia space of $\mathfrak{A}$. In fact, we will prove that $\mathrm{kdim}(\mathfrak{A})$ can be defined recursively as follows.

Definition 3.6. The *Krull dimension* $\mathrm{kdim}(\mathfrak{A})$ of a closure algebra $\mathfrak{A}$ can be defined as follows:

$$\begin{array}{lll}
(1) & \mathrm{kdim}(\mathfrak{A}) = -1 & \text{if } \ \mathfrak{A} \text{ is the trivial algebra,} \\
(2) & \mathrm{kdim}(\mathfrak{A}) \le n & \text{if } \ \mathrm{kdim}(\mathfrak{A}_d) \le n-1 \text{ for every nowhere dense } d \in \mathfrak{A}, \\
(3) & \mathrm{kdim}(\mathfrak{A}) = n & \text{if } \ \mathrm{kdim}(\mathfrak{A}) \le n \text{ and } \mathrm{kdim}(\mathfrak{A}) \not\le n-1, \\
(4) & \mathrm{kdim}(\mathfrak{A}) = \infty & \text{if } \ \mathrm{kdim}(\mathfrak{A}) \not\le n \text{ for any } n = -1, 0, 1, 2, \dots.
\end{array}$$

To show that Definitions 3.1 and 3.6 are equivalent requires some preparation. For now we refer to Definition 3.1 as the external Krull dimension and to Definition 3.6 as the internal Krull dimension of $\mathfrak{A}$.

Lemma 3.7. *Let $\mathfrak{A}$ be a closure algebra, $a \in \mathfrak{A}$, and $d \in \mathfrak{A}_a$. If d is nowhere dense in $\mathfrak{A}_a$, then d is nowhere dense in $\mathfrak{A}$.*

Proof. Set $u = \mathbf{IC}(d)$. Then u is open, so

$$\mathbf{I}_a(a \wedge u) = a \wedge \mathbf{I}(a \to (a \wedge u)) = a \wedge \mathbf{I}(a \to u) \ge a \wedge \mathbf{I}(u) = a \wedge u.$$

Therefore, $a \wedge u$ is open in $\mathfrak{A}_a$. Moreover, $a \wedge u \le a \wedge \mathbf{C}(d) = \mathbf{C}_a(d)$. Thus, $a \wedge u \le \mathbf{I}_a\mathbf{C}_a(d)$. Since d is nowhere dense in $\mathfrak{A}_a$, we obtain $a \wedge u = 0$. This yields $d \wedge u = 0$ as $d \le a$. Therefore, $d \le -u$. Since $-u$ is closed, we obtain $\mathbf{C}(d) \le -u$, giving $\mathbf{C}(d) \wedge u = 0$. Thus, $u = 0$, and hence d is nowhere dense in $\mathfrak{A}$. $\square$

Definition 3.8. For $a_1, \dots, a_{n+2} \in \mathfrak{A}$, define:

$$\begin{array}{lclclcl}
d_1 & = & \mathbf{CI}a_1 - a_1 & \text{and} & e_1 & = & \mathbf{C}(\mathbf{I}a_2 \wedge d_1), \\
\vdots & & & & \vdots & & \\
d_{n+1} & = & \mathbf{C}(\mathbf{I}a_{n+1} \wedge d_n) - a_{n+1} & \text{and} & e_{n+1} & = & \mathbf{C}(\mathbf{I}a_{n+2} \wedge d_{n+1}).
\end{array}$$

Clearly $d_{n+2} = \mathbf{C}(\mathbf{I}a_{n+2} \wedge d_{n+1}) - a_{n+2} = e_{n+1} - a_{n+2}$, so $d_{n+2} \le e_{n+1}$. It is also straightforward to see that if we interpret p_i as a_i, then since $\Box$ is interpreted as $\mathbf{I}$ and $\Diamond$ as $\mathbf{C}$, the formula $\neg\mathsf{bd}_n$ is interpreted as d_n, and the antecedent of bd_{n+1} as e_n.

Lemma 3.9. *Let $\mathfrak{A}$ be a closure algebra, $a_1, \dots, a_{n+2} \in \mathfrak{A}$, and d_i, e_i be defined as in Definition 3.8.*

(1) *d_1 and e_1 are nowhere dense in $\mathfrak{A}$.*
(2) *$\mathbf{I}a_{n+2} \wedge d_{n+1}$ is nowhere dense in $\mathfrak{A}_{e_n}$.*
(3) *e_{n+1} and d_{n+2} are nowhere dense in $\mathfrak{A}_{e_n}$.*

Proof. (1) We have

$$\mathbf{IC}(d_1) = \mathbf{IC}(\mathbf{CI}a_1 - a_1) \le \mathbf{I}(\mathbf{CI}a_1 - \mathbf{I}a_1) = \mathbf{ICI}a_1 - \mathbf{CI}a_1 \le \mathbf{CI}a_1 - \mathbf{CI}a_1 = 0.$$

Therefore, d_1 is nowhere dense in $\mathfrak{A}$. This yields that $\mathbf{I}a_2 \wedge d_1$ is nowhere dense in $\mathfrak{A}$. Thus, $e_1 = \mathbf{C}(\mathbf{I}a_2 \wedge d_1)$ is nowhere dense in $\mathfrak{A}$.

(2) Since e_n is closed in $\mathfrak{A}$, we have $\mathbf{C}_{e_n}(a) = \mathbf{C}(a)$ for all $a \le e_n$. To see that $\mathbf{I}a_{n+2} \wedge d_{n+1}$ is nowhere dense in $\mathfrak{A}_{e_n}$, let u be open in $\mathfrak{A}_{e_n}$ with $u \le \mathbf{C}(\mathbf{I}a_{n+2} \wedge d_{n+1})$. We set $u' = u \wedge \mathbf{I}a_{n+1}$. Then u' is open in $\mathfrak{A}_{e_n}$ and $u' \le a_{n+1}$, so

$$u' \wedge \mathbf{I}a_{n+2} \wedge d_{n+1} = u' \wedge \mathbf{I}a_{n+2} \wedge (e_n - a_{n+1}) \le u' \wedge (e_n - a_{n+1}) = u' - a_{n+1} = 0.$$

Therefore, $u' \wedge \mathbf{C}(\mathbf{I}a_{n+2} \wedge d_{n+1}) = 0$. This together with $u' \le u \le \mathbf{C}(\mathbf{I}a_{n+2} \wedge d_{n+1})$ yields that $u' = 0$. Thus, $u \wedge \mathbf{I}a_{n+1} = 0$, and so $u \wedge \mathbf{I}a_{n+1} \wedge d_n = 0$. But $\mathbf{I}a_{n+1} \wedge d_n$ is dense in $\mathfrak{A}_{e_n}$, giving that $u = 0$. Consequently, $\mathbf{I}a_{n+2} \wedge d_{n+1}$ is nowhere dense in $\mathfrak{A}_{e_n}$.

(3) By (2), $\mathbf{I}a_{n+2} \wedge d_{n+1}$ is nowhere dense in $\mathfrak{A}_{e_n}$. Therefore, $e_{n+1} = \mathbf{C}(\mathbf{I}a_{n+2} \wedge d_{n+1})$ is nowhere dense in $\mathfrak{A}_{e_n}$. Thus, $d_{n+2} = e_{n+1} - a_{n+2}$ is nowhere dense in $\mathfrak{A}_{e_n}$. $\square$

The next lemma concerns the internal Krull dimension of a closure algebra.

Lemma 3.10. *Let $\mathfrak{A}$ be a closure algebra.*

(1) *For* $a \in \mathfrak{A}$, *we have* $\mathrm{kdim}(\mathfrak{A}_a) \le \mathrm{kdim}(\mathfrak{A})$.
(2) $\mathrm{kdim}(\mathfrak{A}) \le n$ *iff* $\mathrm{kdim}(\mathfrak{A}_d) \le n-1$ *for every closed nowhere dense* $d \in \mathfrak{A}$.

Proof. (1) If $\mathrm{kdim}(\mathfrak{A}) = \infty$, then there is nothing to prove. Suppose $\mathrm{kdim}(\mathfrak{A}) = n$. Let $d \in \mathfrak{A}_a$ be nowhere dense in $\mathfrak{A}_a$. By Lemma 3.7, d is nowhere dense in $\mathfrak{A}$. Since $\mathrm{kdim}(\mathfrak{A}) = n$, we see that $\mathrm{kdim}(\mathfrak{A}_d) \le n-1$. Because $(\mathfrak{A}_a)_d = \mathfrak{A}_d$, we conclude that $\mathrm{kdim}(\mathfrak{A}_a) \le n$. Thus, $\mathrm{kdim}(\mathfrak{A}_a) \le \mathrm{kdim}(\mathfrak{A})$.

(2) One implication is trivial. For the other, let d be nowhere dense in $\mathfrak{A}$. Then $\mathbf{C}(d)$ is closed and nowhere dense in $\mathfrak{A}$. Therefore, $\mathrm{kdim}(\mathfrak{A}_{\mathbf{C}(d)}) \le n-1$. Thus, (1) yields

$$\mathrm{kdim}(\mathfrak{A}_d) = \mathrm{kdim}((\mathfrak{A}_{\mathbf{C}(d)})_d) \le \mathrm{kdim}(\mathfrak{A}_{\mathbf{C}(d)}) \le n-1.$$

Consequently, $\mathrm{kdim}(\mathfrak{A}) \le n$. □

We next recall the notion of an Esakia morphism between Esakia spaces.

Definition 3.11. Suppose $\mathfrak{F} = (W, R)$ and $\mathfrak{G} = (V, Q)$ are Esakia spaces.
(1) A map $f : W \to V$ is a *p-morphism* provided $\mathbf{C}_R f^{-1}(v) = f^{-1}(\mathbf{C}_Q\{v\})$ for all $v \in V$.
(2) An *Esakia morphism* is a continuous p-morphism $f : W \to V$.

It is well known (see, e.g., [17, Sec. IV.3]) that Esakia morphisms correspond dually to closure algebra homomorphisms; that is, $h : \mathfrak{A} \to \mathfrak{B}$ is a closure algebra homomorphism iff $h_* : \mathfrak{B}_* \to \mathfrak{A}_*$ is an Esakia morphism, where $h_*(w) = h^{-1}(w)$. Moreover, h is 1-1 (resp. onto) iff h_* is onto (resp. 1-1).

The modal language is interpreted in an Esakia space $\mathfrak{F}$ by interpreting the modal formulas in the dual closure algebra $\mathfrak{F}^*$. Consequently, a modal formula φ is satisfiable (resp. valid, written $\mathfrak{F} \vDash \varphi$) in $\mathfrak{F}$ exactly when φ is satisfiable (resp. valid) in $\mathfrak{F}^*$.

We call $\mathfrak{F}$ *rooted* if there is $r \in W$ with $W = R[r]$. We refer to r as a root of $\mathfrak{F}$. In general, r is not unique. Let $\mathfrak{F} = (W, R)$ be a finite rooted **S4**-frame. It is well known [24, 20] that with $\mathfrak{F}$ we can associate the *Jankov-Fine formula* $\chi_{\mathfrak{F}}$, which satisfies the following property:

$\chi_{\mathfrak{F}}$ is satisfiable in an Esakia space $\mathfrak{G}$ iff there is an Esakia space $\mathfrak{E}$
and Esakia morphisms $\mathfrak{F} \xleftarrow{f} \mathfrak{E} \xrightarrow{g} \mathfrak{G}$ such that f is onto and g is 1-1.

Let $\mathfrak{F}_n = (W_n, R)$ be the n-element chain, where $W_n = \{w_0, \dots, w_{n-1}\}$ and $w_i R w_j$ iff $j \le i$; see Figure 1.

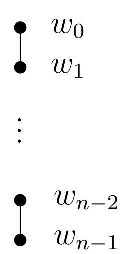


FIGURE 1. The n-element chain.

We are ready to characterize the internal Krull dimension of a closure algebra.

Theorem 3.12. *Let* $\mathfrak{A}$ *be a nontrivial closure algebra and* $n \ge 1$. *The following are equivalent:*
(1) $\mathrm{kdim}(\mathfrak{A}) \le n-1$.
(2) *There does not exist a sequence* $e_0, \dots, e_n$ *of nonzero closed elements of* $\mathfrak{A}$ *such that* $e_0 = 1$ *and* e_{i+1} *is nowhere dense in* $\mathfrak{A}_{e_i}$ *for each* $i \in \{0, \dots, n-1\}$.

(3) $\mathfrak{A} \vDash \mathsf{bd}_n$.
(4) $\operatorname{depth}(\mathfrak{A}_*) \leq n$.
(5) $\mathfrak{A} \vDash \neg\chi_{\mathfrak{F}_{n+1}}$.
(6) *$\mathfrak{F}_{n+1}^*$ is not isomorphic to a subalgebra of a homomorphic image of $\mathfrak{A}$.*
(7) *There do not exist an Esakia space $\mathfrak{G}$ and Esakia morphisms $\mathfrak{F}_{n+1} \xleftarrow{f} \mathfrak{G} \xrightarrow{g} \mathfrak{A}_*$ such that f is onto and g is 1-1.*
(8) *$\mathfrak{F}_{n+1}^*$ is not isomorphic to a subalgebra of $\mathfrak{A}$.*
(9) *$\mathfrak{F}_{n+1}$ is not an image of $\mathfrak{A}_*$ under an onto Esakia morphism.*

Proof. (1)⇒(2): Induction on n. Let $n = 1$. Since $\mathfrak{A}$ is nontrivial, $\operatorname{kdim}(\mathfrak{A}) \leq 0$ yields $\operatorname{kdim}(\mathfrak{A}) = 0$. Therefore, for any nowhere dense d in $\mathfrak{A}$, we have $\operatorname{kdim}(\mathfrak{A}_d) = -1$, so $\mathfrak{A}_d$ is trivial, and hence $d = 0$. Thus, $\mathfrak{A}$ has no nonzero closed nowhere dense elements, as required. Next let $n > 1$ and $\operatorname{kdim}(\mathfrak{A}) \leq n-1$. Suppose there is a sequence $e_0, \dots, e_n$ of nonzero closed elements of $\mathfrak{A}$ such that $e_0 = 1$ and e_{i+1} is nowhere dense in $\mathfrak{A}_{e_i}$ for each $i \in \{0, \dots, n-1\}$. Then $e_1, \dots, e_n$ is a sequence of nonzero closed elements of $\mathfrak{A}_{e_1}$ such that e_{i+1} is nowhere dense in $\mathfrak{A}_{e_i}$ for each $i \in \{1, \dots, n-1\}$. By the induction hypothesis, applied to $\mathfrak{A}_{e_1}$, we have $\operatorname{kdim}(\mathfrak{A}_{e_1}) > n-1$. Since e_1 is nowhere dense in $\mathfrak{A}$ with $\operatorname{kdim}(\mathfrak{A}_{e_1}) > n-1$, we conclude that $\operatorname{kdim}(\mathfrak{A}) > n$. This contradicts (1).

(2)⇒(3): If $\mathfrak{A} \nvDash \mathsf{bd}_n$, then there exist $a_1, \dots, a_n \in \mathfrak{A}$ such that $d_n \neq 0$, where d_n is defined as in Definition 3.8. Put $e_0 = 1$ and $a_{n+1} = 1$. Let $e_1, \dots, e_n$ be defined as in Definition 3.8. Then

$$e_n = \mathbf{C}(\mathbf{I}a_{n+1} \wedge d_n) = \mathbf{C}(\mathbf{I}1 \wedge d_n) = \mathbf{C}(d_n) \geq d_n \neq 0.$$

Therefore, $e_0, \dots, e_n$ is a sequence of nonzero closed elements in $\mathfrak{A}$ such that $e_0 = 1$ and, by Lemma 3.9, e_{i+1} is nowhere dense in $\mathfrak{A}_{e_i}$ for each $i \in \{0, \dots, n-1\}$.

(3)⇒(1): Suppose that $\operatorname{kdim}(\mathfrak{A}) > n-1$. We define a decreasing sequence $b_0, \dots, b_n$ of closed elements in $\mathfrak{A}$ such that b_{i+1} is nowhere dense in $\mathfrak{A}_{b_i}$ and $\operatorname{kdim}(\mathfrak{A}_{b_{i+1}}) > (n-1)-(i+1)$. Set $b_0 = 1$. If b_i is already defined with $\operatorname{kdim}(\mathfrak{A}_{b_i}) > (n-1)-i$, then by Lemma 3.10, there is a closed nowhere dense b_{i+1} of $\mathfrak{A}_{b_i}$ such that $\operatorname{kdim}(\mathfrak{A}_{b_{i+1}}) > (n-1)-(i+1)$. Noting that $\operatorname{kdim}(\mathfrak{A}_{b_n}) > (n-1)-n = -1$, it follows that $\mathfrak{A}_{b_n}$ is not trivial, and hence $b_n \neq 0$.

Let $a_i = -b_i$ and let d_i be defined as in Definition 3.8. We show that for each i we have $b_i = d_i$. If $i = 1$, then since b_1 is nowhere dense in $\mathfrak{A}$, we have

$$b_1 = 1 \wedge b_1 = (-\mathbf{IC}b_1) \wedge b_1 = \mathbf{CI}(-b_1) \wedge -(-b_1) = \mathbf{CI}a_1 - a_1 = d_1.$$

Next suppose that $b_i = d_i$, and show that $b_{i+1} = d_{i+1}$. Since a_{i+1} is open in $\mathfrak{A}$, b_{i+1} is nowhere dense in $\mathfrak{A}_{b_i}$, and b_i is closed in $\mathfrak{A}$, we have

$$\begin{aligned} b_{i+1} &= b_i \wedge b_{i+1} = \mathbf{C}(b_i - b_{i+1}) \wedge b_{i+1} = \mathbf{C}(b_i - b_{i+1}) - (-b_{i+1}) \\ &= \mathbf{C}(a_{i+1} \wedge b_i) - a_{i+1} = \mathbf{C}(\mathbf{I}a_{i+1} \wedge d_i) - a_{i+1} = d_{i+1}. \end{aligned}$$

Thus, $d_n = b_n \neq 0$. Since $\neg\mathsf{bd}_n$ is interpreted in $\mathfrak{A}$ as d_n, we conclude that $\mathfrak{A}$ refutes bd_n.

(3)⇔(4)⇔(8): This is well known; see, e.g., [29, Lem. 2].

(5)⇔(7): This is the Jankov-Fine Theorem.

(6)⇔(7): This follows from Esakia duality.

(6)⇒(8): This is obvious.

(8)⇔(9): This follows from Esakia duality.

(4)⇒(7): This is obvious since onto Esakia morphisms do not increase the depth. ☐

As an immediate consequence, we obtain:

Corollary 3.13. *The internal and external Krull dimensions of a closure algebra coincide, and so Definitions 3.1 and 3.6 are equivalent.*

4. Krull dimension of Heyting algebras

We next study Krull dimension of Heyting algebras. As with closure algebras, we first define Krull dimension of Heyting algebras externally and then provide an equivalent internal definition of it. We also show that Krull dimensions of a closure algebra $\mathfrak{A}$ and the associated Heyting algebra $\mathfrak{H}(\mathfrak{A})$ of opens of $\mathfrak{A}$ coincide.

Definition 4.1. Let $\mathfrak{H}$ be a Heyting algebra. Define the *Krull dimension* $\mathrm{kdim}(\mathfrak{H})$ of $\mathfrak{H}$ as the supremum of the lengths of R-chains in $\mathfrak{H}_*$. If the supremum is not finite, then we write $\mathrm{kdim}(\mathfrak{H}) = \infty$.

Lemma 4.2.
(1) *If* $\mathfrak{A}$ *is a closure algebra, then* $\mathrm{kdim}(\mathfrak{A}) = \mathrm{kdim}(\mathfrak{H}(\mathfrak{A}))$.
(2) *If* $\mathfrak{H}$ *is a Heyting algebra, then* $\mathrm{kdim}(\mathfrak{H}) = \mathrm{kdim}(\mathfrak{A}(\mathfrak{H}))$.

Proof. (1) Since $\mathfrak{H}(\mathfrak{A})_*$ is obtained from $\mathfrak{A}_*$ by modding out R-clusters, we see that the corresponding R-chains in $\mathfrak{A}_*$ and $\mathfrak{H}(\mathfrak{A})_*$ have the same length. Thus, $\mathrm{kdim}(\mathfrak{A}) = \mathrm{kdim}(\mathfrak{H}(\mathfrak{A}))$.

(2) This is obvious since $\mathfrak{H}_*$ is isomorphic to $(\mathfrak{A}(\mathfrak{H}))_*$. ◻

As with closure algebras, the concept of Krull dimension of a Heyting algebra $\mathfrak{H}$ is closely related to that of the depth of $\mathfrak{H}$. It is well known that whether the depth of $\mathfrak{H}$ is $\leq n$ is described by the following formulas in the language of intuitionistic logic.

Definition 4.3. For $n \geq 1$, consider the formulas:

$$\begin{aligned} \mathsf{ibd}_1 &= p_1 \vee \neg p_1, \\ \mathsf{ibd}_{n+1} &= p_{n+1} \vee (p_{n+1} \to \mathsf{ibd}_n). \end{aligned}$$

The intuitionistic language is interpreted in a Heyting algebra $\mathfrak{H}$ by assigning to propositional letters elements of $\mathfrak{H}$ and by interpreting conjunction, disjunction, implication, and negation as the corresponding operations of $\mathfrak{H}$. The next lemma is well known (see, e.g., [6, Prop. 2.38]).

Lemma 4.4. *Let* $\mathfrak{H}$ *be a nontrivial Heyting algebra and* $n \geq 1$*. Then* $\mathrm{depth}(\mathfrak{H}_*) \leq n$ *iff* $\mathfrak{H} \models \mathsf{ibd}_n$.

To characterize the Krull dimension of a Heyting algebra internally, we require some preparation. We call an element a of a Heyting algebra $\mathfrak{H}$ *dense* if $\neg a = 0$.

Lemma 4.5. *Let* $\mathfrak{H}$ *be a Heyting algebra,* $a \in \mathfrak{H}$*, and* $b \in \mathfrak{H}_a$*. If* b *is dense in* $\mathfrak{H}_a$*, then* b *is dense in* $\mathfrak{H}$.

Proof. Since b is dense in $\mathfrak{H}_a$ and a is the bottom of $\mathfrak{H}_a$, we have $b \to a = a$. Therefore, $\neg b = b \to 0 \leq b \to a = a$. On the other hand, $a \leq b$ implies $\neg b \leq \neg a$. Thus, $\neg b \leq a \wedge \neg a = 0$, and hence b is dense in $\mathfrak{H}$. ◻

Lemma 4.6. *Let* $\mathfrak{A}$ *be a closure algebra and let* $a \in \mathfrak{A}$ *be open. Then* a *is dense in* $\mathfrak{H}(\mathfrak{A})$ *iff* $-a$ *is nowhere dense in* $\mathfrak{A}$.

Proof. Since a is open, $-a$ is closed. Therefore,

$$\begin{aligned} a \text{ is dense in } \mathfrak{H}(\mathfrak{A}) \quad &\text{iff} \quad \neg a = 0 \\ &\text{iff} \quad \mathbf{I}(-a) = 0 \\ &\text{iff} \quad \mathbf{IC}(-a) = 0 \\ &\text{iff} \quad -a \text{ is nowhere dense.} \end{aligned}$$

◻

We are ready to give an internal recursive definition of the Krull dimension of a Heyting algebra.

Definition 4.7. The *Krull dimension* $\mathrm{kdim}(\mathfrak{H})$ of a Heyting algebra $\mathfrak{H}$ can be defined as follows:

$$\begin{array}{lll} \mathrm{kdim}(\mathfrak{H}) = -1 & \text{if} & \mathfrak{H} \text{ is the trivial algebra,} \\ \mathrm{kdim}(\mathfrak{H}) \le n & \text{if} & \mathrm{kdim}(\mathfrak{H}_b) \le n-1 \text{ for every dense } b \in \mathfrak{H}, \\ \mathrm{kdim}(\mathfrak{H}) = n & \text{if} & \mathrm{kdim}(\mathfrak{H}) \le n \text{ and } \mathrm{kdim}(\mathfrak{H}) \not\le n-1, \\ \mathrm{kdim}(\mathfrak{H}) = \infty & \text{if} & \mathrm{kdim}(\mathfrak{H}) \not\le n \text{ for any } n = -1, 0, 1, 2, \dots. \end{array}$$

The next two results concern the internal definition of the Krull dimension.

Lemma 4.8. *Let $\mathfrak{H}$ be a Heyting algebra and let $a \in \mathfrak{H}$. Then* $\mathrm{kdim}(\mathfrak{H}_a) \le \mathrm{kdim}(\mathfrak{H})$.

Proof. If $\mathrm{kdim}(\mathfrak{H}) = \infty$, then there is nothing to prove. Suppose $\mathrm{kdim}(\mathfrak{H}) = n$. Let $b \in \mathfrak{H}_a$ be dense in $\mathfrak{H}_a$. By Lemma 4.5, b is dense in $\mathfrak{H}$. Since $\mathrm{kdim}(\mathfrak{H}) = n$, we see that $\mathrm{kdim}(\mathfrak{H}_b) \le n-1$. Because $(\mathfrak{H}_a)_b = \mathfrak{H}_b$, we conclude that $\mathrm{kdim}(\mathfrak{H}_a) \le n$. Thus, $\mathrm{kdim}(\mathfrak{H}_a) \le \mathrm{kdim}(\mathfrak{H})$. ☐

Theorem 4.9.

(1) *If $\mathfrak{A}$ is a closure algebra, then* $\mathrm{kdim}(\mathfrak{A}) = \mathrm{kdim}(\mathfrak{H}(\mathfrak{A}))$.
(2) *If $\mathfrak{H}$ is a Heyting algebra, then* $\mathrm{kdim}(\mathfrak{H}) = \mathrm{kdim}(\mathfrak{A}(\mathfrak{H}))$.

Proof. (1) By Theorem 3.12, $\mathrm{kdim}(\mathfrak{A}) \ge n$ iff there is a sequence $e_0, \dots, e_n$ of nonzero closed elements of $\mathfrak{A}$ such that $e_0 = 1$ and e_{i+1} is nowhere dense in $\mathfrak{A}_{e_i}$ for each $i \in \{0, \dots, n-1\}$. By [2, Thm. 6.9], $\mathrm{kdim}(\mathfrak{H}(\mathfrak{A})) \ge n$ iff there is a sequence $1 > b_1 > \cdots > b_n > 0$ in $\mathfrak{H}(\mathfrak{A})$ such that b_{i-1} is dense in $\mathfrak{H}(\mathfrak{A})_{b_i}$ for each $i \in \{1, \dots, n\}$. The two conditions are equivalent by Lemma 4.6. The result follows.

(2) Since $\mathfrak{H}$ is isomorphic to $\mathfrak{H}(\mathfrak{A}(\mathfrak{H}))$, we have $\mathrm{kdim}(\mathfrak{H}) = \mathrm{kdim}(\mathfrak{H}(\mathfrak{A}(\mathfrak{H})))$. By (1), $\mathrm{kdim}(\mathfrak{H}(\mathfrak{A}(\mathfrak{H}))) = \mathrm{kdim}(\mathfrak{A}(\mathfrak{H}))$. Thus, $\mathrm{kdim}(\mathfrak{H}) = \mathrm{kdim}(\mathfrak{A}(\mathfrak{H}))$. ☐

As a consequence we obtain:

Corollary 4.10. *The external and internal definitions of the Krull dimension of a Heyting algebra coincide, so Definitions 4.1 and 4.7 are equivalent.*

Proof. Apply Corollary 3.13, Lemma 4.2, and Theorem 4.9. ☐

Let $\mathfrak{L}_n$ be the $(n+1)$-element linear Heyting algebra. Then $(\mathfrak{L}_n)_*$ is isomorphic to the n-element chain $\mathfrak{F}_n$ shown in Figure 1. Let $\chi(\mathfrak{L}_n)$ be the Jankov formula of $\mathfrak{L}_n$. Another immediate consequence of our results is the following:

Corollary 4.11. *Let $\mathfrak{H}$ be a nontrivial Heyting algebra and $n \ge 1$. The following are equivalent:*

(1) $\mathrm{kdim}(\mathfrak{H}) \le n-1$.
(2) *There does not exist a sequence $1 > b_1 > \cdots > b_n > 0$ in $\mathfrak{H}$ such that b_{i-1} is dense in $\mathfrak{H}_{b_i}$ for each $i \in \{1, \dots, n\}$.*
(3) $\mathfrak{H} \vDash \mathsf{ibd}_n$.
(4) $\mathrm{depth}(\mathfrak{H}_*) \le n$.
(5) $\mathfrak{H} \vDash \neg\chi(\mathfrak{L}_{n+1})$.
(6) *$\mathfrak{L}_{n+1}$ is not isomorphic to a subalgebra of a homomorphic image of $\mathfrak{H}$.*
(7) *$\mathfrak{L}_{n+1}$ is not isomorphic to a subalgebra of $\mathfrak{H}$.*

5. Localic Krull dimension of topological spaces

As we pointed out in the introduction, it is inadequate to define the Krull dimension of a topological space X as the supremum of the lengths of chains in the specialization order of X. The previous two sections suggest that a more adequate definition would result by working with the Krull dimension of either $\mathfrak{A}_X$ or $\mathfrak{H}_X$. Since $\mathfrak{H}_X$ is the Heyting algebra of opens of $\mathfrak{A}_X$, by Lemma 4.2 (or Theorem 4.9), $\mathrm{kdim}(\mathfrak{A}_X) = \mathrm{kdim}(\mathfrak{H}_X)$. Because $\mathfrak{H}_X$ is the locale of opens of X, we refer to this new concept as the localic Krull dimension of X.

Definition 5.1. Define the *localic Krull dimension* $\mathrm{ldim}(X)$ of a topological space X as the Krull dimension of $\mathfrak{H}_X$ (equivalently, as the Krull dimension of $\mathfrak{A}_X$); that is, $\mathrm{ldim}(X) = \mathrm{kdim}(\mathfrak{H}_X) = \mathrm{kdim}(\mathfrak{A}_X)$.

Remark 5.2. It is immediate from the results of the previous section that the localic Krull dimension of a topological space X can be defined recursively as follows:

$$\begin{array}{lll}
\mathrm{ldim}(X) = -1 & \text{if} & X = \varnothing, \\
\mathrm{ldim}(X) \leq n & \text{if} & \mathrm{ldim}(D) \leq n-1 \text{ for every nowhere dense subset } D \text{ of } X, \\
\mathrm{ldim}(X) = n & \text{if} & \mathrm{ldim}(X) \leq n \text{ and } \mathrm{ldim}(X) \not\leq n-1, \\
\mathrm{ldim}(X) = \infty & \text{if} & \mathrm{ldim}(X) \not\leq n \text{ for any } n = -1, 0, 1, 2, \ldots.
\end{array}$$

Lemma 5.3. *If Y is a subspace of X, then* $\mathrm{ldim}(Y) \leq \mathrm{ldim}(X)$.

Proof. By Lemma 3.10(1), $\mathrm{ldim}(Y) = \mathrm{kdim}(\mathfrak{A}_Y) \leq \mathrm{kdim}(\mathfrak{A}_X) = \mathrm{ldim}(X)$. □

Lemma 5.4. $\mathrm{ldim}(X) \leq n$ *iff for every closed nowhere dense subset D of X we have* $\mathrm{ldim}(D) \leq n-1$.

Proof. Apply Lemma 3.10(2). □

To obtain an analogue of Theorem 3.12 for localic Krull dimension, we require an analogue of the Jankov-Fine theorem for topological spaces. Let $\mathfrak{F} = (W, R)$ be a finite rooted **S4**-frame and choose any enumeration of $W = \{w_i \mid i < n\}$ in which w_0 is a root of $\mathfrak{F}$. We recall [20] that the Jankov-Fine formula $\chi_{\mathfrak{F}}$ associated with $\mathfrak{F}$ is the conjunction of the following formulas:

(1) p_0,
(2) $\Box(p_0 \vee \cdots \vee p_{n-1})$,
(3) $\Box(p_i \to \neg p_j)$ for distinct $i, j < n$,
(4) $\Box(p_i \to \Diamond p_j)$ whenever $w_i R w_j$, and
(5) $\Box(p_i \to \neg\Diamond p_j)$ whenever $w_i \not{R} w_j$.

The modal language is interpreted in a topological space X by interpreting it in the powerset algebra $\mathfrak{A}_X$. Consequently, a modal formula φ is satisfiable (resp. valid, written $X \vDash \varphi$) in X exactly when φ is satisfiable (resp. valid) in $\mathfrak{A}_X$. If φ is satisfiable at $x \in X$, then we write $x \vDash \varphi$. An *interior map* between topological spaces X, Y is a continuous open map $f : X \to Y$. We call Y an *interior image* of X if there is an onto interior map $f : X \to Y$. The next lemma generalizes [20, Lem. 1] to topological spaces.

Lemma 5.5. *Let X be a topological space. Then $\chi_{\mathfrak{F}}$ is satisfiable in X iff $\mathfrak{F}$ is an interior image of an open subspace of X.*

Proof. First suppose that $\mathfrak{F}$ is an interior image of an open subspace U of X, say via $f : U \to \mathfrak{F}$. Let p_i be interpreted as $A_i := f^{-1}(w_i)$ when $i < n$ and as $A_i := \varnothing$ when $i \geq n$. Since $A_0 = f^{-1}(w_0) \neq \varnothing$, there is $x \in U$ with $x \vDash p_0$. We show that $x \vDash \chi_{\mathfrak{F}}$. As $A_0 \cup \cdots \cup A_{n-1} = U$ and $x \in U$, we see that $x \vDash \Box(p_0 \vee \cdots \vee p_{n-1})$. Suppose $i \neq j$. Because $A_i \cap A_j = \varnothing$, we see that $x \vDash \Box(p_i \to \neg p_j)$. Suppose $w_i R w_j$. Then $w_i \in \mathbf{C}_R\{w_j\}$, so since f is interior,

$A_i = f^{-1}(w_i) \subseteq f^{-1}\mathbf{C}_R\{w_j\} = \mathbf{C}_U f^{-1}(w_j) = \mathbf{C}A_j$, where $\mathbf{C}_U$ denotes closure in the subspace U. Therefore, $x \vDash \Box(p_i \to \Diamond p_j)$. Finally, suppose $w_i \not R w_j$. Then $\{w_i\} \cap \mathbf{C}_R\{w_j\} = \varnothing$. As f is interior, this yields $f^{-1}(w_i) \cap \mathbf{C}_U f^{-1}(w_j) = \varnothing$. Thus, $A_i \cap \mathbf{C}_U A_j = \varnothing$, which gives $x \vDash \Box(p_i \to \neg\Diamond p_j)$. Consequently, $\chi_{\mathfrak{F}}$ is satisfiable at x in X.

Conversely suppose that $\chi_{\mathfrak{F}}$ is satisfied at some $x \in X$ by interpreting p_i as $A_i \subseteq X$. Set

$$\begin{aligned} U \;=\; & \mathbf{I}\left(\bigcup_{i<n} A_i\right) \cap \bigcap_{0\le i\ne j<n} \mathbf{I}\left((X\setminus A_i)\cup(X\setminus A_j)\right) \\ & \cap \bigcap_{w_i R w_j} \mathbf{I}\left((X\setminus A_i)\cup \mathbf{C}A_j\right) \cap \bigcap_{w_i \not R w_j} \mathbf{I}\left((X\setminus A_i)\cup(X\setminus \mathbf{C}A_j)\right) \end{aligned}$$

Then U is open and nonempty since $x \in A_0 \cap U$. Define $f : U \to \mathfrak{F}$ by setting $f(y) = w_i$ provided $y \in A_i$ (for $i < n$). To see that f is well defined, let $y \in A_i \cap A_j$. Then $y \notin X \setminus \mathbf{C}(A_i \cap A_j) = \mathbf{I}((X \setminus A_i) \cup (X \setminus A_j))$. Therefore, it follows from the definition of U that $i = j$, and so f is well defined.

To see that f is onto, since w_0 is a root of $\mathfrak{F}$, we have $w_0 R w_j$, and so $U \subseteq (X \setminus A_0) \cup \mathbf{C}A_j$ for all $j < n$. Recalling that $x \in A_0 \cap U$, we get $x \in \mathbf{C}A_j$ for each $j < n$. As U is open and contains x, we have $U \cap A_j \neq \varnothing$ for each $j < n$. Thus, f is onto.

Finally, to see that f is interior, it is sufficient to show that $f^{-1}(\mathbf{C}_R\{w_j\}) = \mathbf{C}_U f^{-1}(w_j)$ for each $j < n$. Suppose $y \in f^{-1}(\mathbf{C}_R\{w_j\})$. Then $f(y) R w_j$. Assuming $f(y) = w_i$, we have $y \in A_i$ and $y \in (X \setminus A_i) \cup \mathbf{C}A_j$, giving $y \in \mathbf{C}A_j$. So $y \in \mathbf{C}_U A_j = \mathbf{C}_U f^{-1}(w_j)$. Conversely, suppose $y \notin f^{-1}(\mathbf{C}_R\{w_j\})$. Then $f(y) \not R w_j$. Assuming $f(y) = w_i$, we have $y \in A_i$ and $y \in (X \setminus A_i) \cup (X \setminus \mathbf{C}A_j)$, yielding $y \in X \setminus \mathbf{C}A_j$. Thus, $y \notin \mathbf{C}A_j$, and hence $y \notin \mathbf{C}_U A_j = \mathbf{C}_U f^{-1}(w_j)$. Consequently, f is interior, and hence $\mathfrak{F}$ is an interior image of an open subspace of X. $\Box$

The next theorem is an analogue of Theorem 3.12 for localic Krull dimension, and is the main result of this section.

Theorem 5.6. *Let $X \neq \varnothing$ and $n \geq 1$. Let $\mathfrak{F}_{n+1}$ be the $(n+1)$-element chain. The following are equivalent:*

1. $\mathrm{ldim}(X) \leq n - 1$.
2. *There does not exist a sequence $E_0, \dots, E_n$ of nonempty closed subsets of X such that $E_0 = X$ and E_{i+1} is nowhere dense in E_i for each $i \in \{0, \dots, n-1\}$.*
3. $X \vDash \mathsf{bd}_n$.
4. $X \vDash \neg\chi_{\mathfrak{F}_{n+1}}$.
5. $\mathfrak{F}_{n+1}$ *is not an interior image of any open subspace of X.*
6. $\mathfrak{F}_{n+1}$ *is not an interior image of X.*

Proof. (1)⇔(2)⇔(3): This follows from the equivalence of Items (1), (2), and (3) of Theorem 3.12, Definition 5.1, the correspondence between relativizations and subspaces, and the fact that X and $\mathfrak{A}_X$ validate exactly the same modal formulas.

(4)⇔(5): We have $X \vDash \neg\chi_{\mathfrak{F}_{n+1}}$ iff $\chi_{\mathfrak{F}_{n+1}}$ is not satisfiable in X. This, by Lemma 5.5, is equivalent to $\mathfrak{F}_{n+1}$ not being an interior image of any open subspace of X.

(5)⇒(6): This is obvious.

(6)⇒(2): Suppose there is a sequence $E_0, \dots, E_n$ of nonempty closed subsets of X such that $E_0 = X$ and E_{i+1} is nowhere dense in E_i for each $i \in \{0, \dots, n-1\}$. We show that $\mathfrak{F}_{n+1}$ is an interior image of X. Let $E_{n+1} = \varnothing$. Define $f : X \to W_{n+1}$ by $f(x) = w_i$ if $x \in E_i \setminus E_{i+1}$ for $i \leq n$. Clearly f is well-defined and onto since $\{E_i \setminus E_{i-1} \mid i \leq n\}$ is a partition of X. Moreover, $\mathbf{C}(E_i \setminus E_{i+1}) = E_i$ since E_i is closed in X and E_{i+1} is nowhere dense in E_i for

$i \le n$. Thus,

$$f^{-1}(\mathbf{C}_R\{w_i\}) = f^{-1}(\{w_i, \dots, w_n\}) = \bigcup_{j=i}^{n} (E_j \setminus E_{j+1}) = E_i = \mathbf{C}(E_i \setminus E_{i+1}) = \mathbf{C}f^{-1}(w_i).$$

Consequently, f is an onto interior map, and hence $\mathfrak{F}_{n+1}$ is an interior image of X.

(1)⇒(5): Suppose there is an open subspace Y of X and an onto interior map $f : Y \to \mathfrak{F}_{n+1}$. For each $i < n+1$, put $E_i = f^{-1}(\mathbf{C}_R\{w_i\}) \neq \varnothing$. Since f is interior, each E_i is closed in Y. Therefore, for $i < n$, we have

$$\begin{aligned}\mathbf{I}_{E_i}\mathbf{C}_{E_i}(E_{i+1}) &= \mathbf{I}_{E_i}(E_{i+1}) = E_i \setminus \mathbf{C}_{E_i}(E_i \setminus E_{i+1}) = E_i \setminus \mathbf{C}_Y(E_i \setminus E_{i+1}) \\ &= E_i \setminus \mathbf{C}_Y f^{-1}(w_i) = E_i \setminus f^{-1}(\mathbf{C}_R\{w_i\}) = E_i \setminus E_i = \varnothing.\end{aligned}$$

So E_{i+1} is nowhere dense in E_i. Moreover, $E_0 = f^{-1}(\mathbf{C}_R\{w_0\}) = Y$. Thus, there is a sequence $E_0, \dots, E_n$ of nonempty closed subsets of Y such that $E_0 = Y$ and E_{i+1} is nowhere dense in E_i for each $i \in \{0, \dots, n-1\}$. Since Items (1) and (2) are equivalent, $\mathrm{ldim}(Y) > n-1$. By Lemma 5.3, $\mathrm{ldim}(X) \ge \mathrm{ldim}(Y) > n-1$. □

We next compare localic Krull dimension to other well-known dimension functions. We recall that if X is a regular space, then the Menger-Urysohn dimension of X is denoted by $\mathrm{ind}(X)$, if X is a Tychonoff space, then the Čech-Lebesgue dimension of X is denoted by $\dim(X)$, and if X is a normal space, then the Brouwer-Čech dimension of X is denoted by $\mathrm{Ind}(X)$ (see, e.g., [15, Ch. 7] for a detailed account of these three dimension functions). Also, for a spectral space X, let $\mathrm{kdim}(X)$ denote the Krull dimension of X, and for a T_0-space X, let $\mathrm{gdim}(X)$ denote Isbell's graduated dimension of X [23].

Lemma 5.7. *Let X be a topological space.*

1. *If X is a spectral space, then* $\mathrm{kdim}(X) \le \mathrm{ldim}(X)$.
2. *If X is a T_0-space, then* $\mathrm{gdim}(X) \le \mathrm{ldim}(X)$.
3. *If X is a regular space, then* $\mathrm{ind}(X) \le \mathrm{ldim}(X)$.
4. *If X is a normal space, then* $\mathrm{Ind}(X) \le \mathrm{ldim}(X)$ *and* $\dim(X) \le \mathrm{ldim}(X)$.

Proof. (1) The Krull dimension of a spectral space X can be defined as the supremum of the lengths of chains in the specialization order R of X. Since $\varepsilon : X \to (\mathfrak{A}_X)_*$ is an R-embedding, the supremum of the lengths of chains in the specialization order of X can be no larger than the supremum of the lengths of chains in $(\mathfrak{A}_X)_*$. The result follows.

(2) Recall that Isbell's graduated dimension of a T_0-space X is the least n such that some lattice basis of $\mathfrak{H}_X$ is a directed union of finite topologies of Krull dimension n. Suppose the Isbell dimension of X is n. The lattice of all opens $\mathfrak{H}_X$ is a directed union of finite topologies τ_i since the variety of distributive lattices is locally finite. Because the Krull dimension of each τ_i is $\ge n$, we see that $\mathrm{ldim}(X) \ge n$, as desired.

(3) Induction on $n \ge -1$. The base case is clear since $\mathrm{ind}(X) = -1$ iff $X = \varnothing$, which happens iff $\mathrm{ldim}(X) = -1$. For the inductive step, suppose $\mathrm{ldim}(X) = n$. If Y is closed and nowhere dense in X, then $\mathrm{ldim}(Y) \le n-1$. By the inductive hypothesis, $\mathrm{ind}(Y) \le n-1$. Because the boundary of an open set is (closed and) nowhere dense in X, it follows that the boundary B of any open subset of X has $\mathrm{ind}(B) \le n-1$. Thus, $\mathrm{ind}(X) \le n$.

(4) Let X be normal. Replacing each occurrence of ind in the proof of (3) by Ind yields $\mathrm{Ind}(X) \le \mathrm{ldim}(X)$. By [15, Thm. 7.2.8], $\dim(X) \le \mathrm{Ind}(X) \le \mathrm{ldim}(X)$. □

Remark 5.8. It remains open whether $\dim(X) \le \mathrm{ldim}(X)$ for any Tychonoff space X.

We next calculate the localic Krull dimension of some well-known spaces.

Example 5.9.

(1) It follows from the celebrated McKinsey-Tarski theorem [30] that every finite rooted **S4**-frame is an interior image of any separable crowded metrizable space. Let $\mathbb{R}$, $\mathcal{C}$, and $\mathbb{Q}$ denote the real line, the Cantor discontinuum, and the rational line, respectively. It follows from Theorem 5.6 that each of $\mathbb{R}, \mathcal{C}, \mathbb{Q}$ has infinite localic Krull dimension.
(2) Let $n \geq 1$. It is well known that the n-element chain is an interior image of the ordinal ω^n, and that the $(n+1)$-element chain is not an interior image of ω^n. By Theorem 5.6, $\operatorname{ldim}(\omega^n) = n - 1$.
(3) A reasoning similar to (2) yields that $\operatorname{ldim}(\omega^n + 1) = n$ and $\operatorname{ldim}(\omega^\omega + 1) = \infty$. Since these ordinals are compact, and hence Stone spaces, we obtain the examples alluded to in the introduction.

For T_1-spaces there is an alternate description of localic Krull dimension, which is based on an appropriate generalization of the concept of a nodec space. In the next section we first generalize the concept of a discrete closure algebra to that of an n-discrete closure algebra, and show that n-discrete closure algebras can be characterized by an appropriate generalization of the well-known Zeman formula in modal logic. We then use these results in Section 7 to give an alternate description of localic Krull dimension for T_1-spaces.

6. n-discrete algebras and n-Zeman formulas

Definition 6.1. Let $\mathfrak{A}$ be a closure algebra.
(1) Call $\mathfrak{A}$ 0-*discrete* if $\mathfrak{A}$ is discrete.
(2) For $n \geq 1$, call $\mathfrak{A}$ n-*discrete* if $\mathfrak{A}_a$ is $(n-1)$-discrete for each nowhere dense $a \in \mathfrak{A}$.

Let $\mathfrak{F} = (W, R)$ be an Esakia space. Set

$$M_1 = \max_R(W) \text{ and } M_{n+1} = \max_R\left(W \setminus \bigcup_{i=1}^{n} M_i\right) \text{ for } n \geq 1.$$

Note that M_1 is always closed, and if $n > 1$, then M_n could be empty.

We will freely use the well-known fact that if U, V are disjoint closed subsets of W such that U is an R-upset and V is an R-downset, then there is a clopen R-upset containing U and disjoint from V.

Definition 6.2. Let $\mathfrak{F} = (W, R)$ be an Esakia space and let $U \subseteq W$.
(1) We call $w \in U$ an *R-minimal point of* U if $u \in U$ and uRw imply wRu.
(2) We call w a *strictly minimal point of* U if $u \in U$ and uRw imply $u = w$.

Clearly every strictly minimal point is R-minimal, but the converse is not true in general. We let $\min_R(\mathfrak{F})$ denote the set of R-minimal points and $\min(\mathfrak{F})$ the set of strictly minimal points of $\mathfrak{F}$.

Theorem 6.3. *A closure algebra* $\mathfrak{A}$ *is n-discrete iff* $\operatorname{depth}(\mathfrak{A}_*) \leq n+1$ *and* $M_{n+1} \subseteq \min(\mathfrak{A}_*)$.

Proof. Suppose that $\mathfrak{A}$ is n-discrete. Let $\mathfrak{A}_* = (W, R)$. First we show that $\operatorname{depth}(\mathfrak{A}_*) \leq n+1$. If not, then there exist $w_0, \dots, w_{n+1} \in W$ such that $w_{i+1} \vec{R} w_i$ for $i \leq n$. We build inductively a decreasing sequence of clopen R-downsets $A_0, \dots, A_{n+1}$ of $\mathfrak{A}_*$ such that $w_i \notin A_{i+1}$, $w_{i+1} \in A_{i+1}$, and $A_{i+1} \cap \max_R(A_i) = \varnothing$. Let $A_0 = W$. Suppose A_i is already constructed. Since $w_{i+1} \vec{R} w_i$, we have $w_{i+1} \notin R[\max_R(A_i) \cup \{w_i\}]$. Therefore, there is a clopen R-downset A_{i+1} such that $A_{i+1} \subseteq A_i$, $w_{i+1} \in A_{i+1}$, and $A_{i+1} \cap R[\max_R(A_i) \cup \{w_i\}] = \varnothing$. Let $a_0, \dots, a_{n+1} \in \mathfrak{A}$ be such that $\beta(a_i) = A_i$ for $i \leq n+1$. Since $A_{i+1} \cap \max_R(A_i) = \varnothing$, Lemma 2.4 yields that a_{i+1} is nowhere dense in $\mathfrak{A}_{a_i}$. Because $\mathfrak{A}$ is n-discrete, $\mathfrak{A}_{a_i}$ is $(n-i)$-discrete for each $i \leq n$.

So $\mathfrak{A}_{a_n}$ is 0-discrete, and hence discrete. As a_{n+1} is nowhere dense in $\mathfrak{A}_{a_n}$, we must have $a_{n+1} = 0$. But this is a contradiction since $w_{n+1} \in A_{n+1} = \beta(a_{n+1})$. Thus, $\mathrm{depth}(\mathfrak{A}_*) \leq n+1$.

Next we show that $M_{n+1} \subseteq \min(\mathfrak{A}_*)$. If not, then there are distinct $w, v \in M_{n+1}$ such that wRv and vRw. Let $w_0, \dots, w_{n-1} \in W$ be such that $w_i \in M_{i+1}$ for $i \leq n-1$ and $w_{i+1} \vec{R} w_i$ for $i < n-1$. Set $w_n := w$ and $w_{n+1} := v$. Then $w_n \vec{R} w_{n-1}$. Let $A_0, \dots, A_n$ be the clopen R-downsets of $\mathfrak{A}_*$ constructed above for the sequence $w_0, \dots, w_n$. Since $w_n \neq w_{n+1}$ there is clopen A_{n+1} of $\mathfrak{A}_*$ such that $w_n \notin A_{n+1}$, $w_{n+1} \in A_{n+1}$, and A_{n+1} can be chosen so that $A_{n+1} \cap \max_R(A_n) = \varnothing$. Let $a_0, \dots, a_{n+1} \in \mathfrak{A}$ be such that $\beta(a_i) = A_i$. The same argument as above yields that a_{i+1} is nowhere dense in $\mathfrak{A}_{a_i}$ for $i \leq n$. Since $\mathfrak{A}$ is n-discrete, $\mathfrak{A}_{a_n}$ is discrete, so $a_{n+1} = 0$, contradicting to $w_{n+1} \in A_{n+1} = \beta(a_{n+1})$. Thus, $M_{n+1} \subseteq \min(\mathfrak{A}_*)$, as desired.

Conversely, suppose that $\mathfrak{A}$ is not n-discrete and $\mathrm{depth}(\mathfrak{A}_*) \leq n+1$. We show that $M_{n+1} \not\subseteq \min(\mathfrak{A}_*)$. Since $\mathfrak{A}$ is not n-discrete, there is a sequence of closed elements $a_0, \dots, a_n \in \mathfrak{A}$ such that $a_0 = 1$, a_{i+1} is nowhere dense in $\mathfrak{A}_{a_i}$ for $i < n$, and $\mathfrak{A}_{a_n}$ is not discrete. Let $A_i := \beta(a_i)$ for $i \leq n$. Then each A_i is a clopen R-downset and Lemma 2.4 gives $A_{i+1} \cap \max_R(A_i) = \varnothing$ for $i < n$. As $\mathfrak{A}_{a_n}$ is not discrete, there is $a \in \mathfrak{A}_{a_n}$ such that $a \neq \mathbf{C}a$. Therefore, there is $w \in \mathbf{C}_R\beta(a) \setminus \beta(a)$. Thus, there is $v \in \beta(a)$ such that wRv. Clearly w, v are distinct. We build $w_0, \dots, w_{n+1}$ as follows. Set $w_{n+1} := w$ and $w_n := v$. As $a \leq a_n$, we see that $w_n \in A_n$. Suppose w_i has already been chosen in A_i for $1 \leq i \leq n$. Since $A_i \subseteq A_{i-1}$, there is $w_{i-1} \in \max_R(A_{i-1})$ such that $w_i R w_{i-1}$. As a_i is nowhere dense in $\mathfrak{A}_{a_{i-1}}$, we have $w_i \notin \max_R(A_{i-1})$, so $w_i \vec{R} w_{i-1}$. Therefore,

$$w_{n+1} R w_n \vec{R} w_{n-1} \vec{R} \cdots \vec{R} w_1 \vec{R} w_0.$$

Since $\mathrm{depth}(\mathfrak{A}_*) \leq n+1$, we must have $w_n R w_{n+1}$. Thus, $w_n, w_{n+1} \in M_{n+1}$, and hence $M_{n+1} \not\subseteq \min(\mathfrak{A}_*)$. $\square$

In order to axiomatize n-discrete closure algebras, we generalize the Zeman formula

$$\mathsf{zem} = \Box\Diamond\Box p \to (p \to \Box p)$$

as follows.

Definition 6.4. For $n \geq 1$, let

$$\mathsf{zem}_n = \Box(\Box(\Box p_{n+1} \to \mathsf{bd}_n) \to p_{n+1}) \to (p_{n+1} \to \Box p_{n+1}).$$

We call zem_n the *n-Zeman formula*.

As we saw in Section 3, if we interpret p_i as $a_i \in \mathfrak{A}$, then $\neg\mathsf{bd}_n$ is interpreted as d_n. Therefore, $\Box(\Box p_{n+1} \to \mathsf{bd}_n)$ is interpreted as $-\mathbf{C}(\mathbf{I}a_{n+1} \cap d_n) = -e_n$.

Lemma 6.5. *Suppose $\{b_n \mid n \in \omega\}$ is a family of closed elements of $\mathfrak{A}$ such that $b_0 = 1$ and b_{n+1} is nowhere dense in $\mathfrak{A}_{b_n}$. For $n \geq 1$, set $a_n = -b_n$. Then $e_n = d_n = b_n$, where e_n and d_n are defined as in Definition 3.8.*

Proof. We show by induction on $n \geq 1$ that $d_n = b_n$. If $n = 1$, then as b_1 is closed and nowhere dense in $\mathfrak{A}$, we see that a_1 is open and dense in $\mathfrak{A}$. So

$$d_1 = \mathbf{CI}a_1 - a_1 = \mathbf{C}a_1 - a_1 = 1 - a_1 = b_1.$$

For the inductive step, notice that $b_n - b_{n+1}$ is dense in $\mathfrak{A}_{b_n}$ because b_{n+1} is nowhere dense in $\mathfrak{A}_{b_n}$. Therefore, $\mathbf{C}(b_n - b_{n+1}) = b_n$. Since a_{n+1} is open in $\mathfrak{A}$, we have

$$\begin{aligned} d_{n+1} &= \mathbf{C}(\mathbf{I}a_{n+1} \wedge d_n) - a_{n+1} = \mathbf{C}(a_{n+1} \wedge b_n) - a_{n+1} \\ &= \mathbf{C}(b_n - b_{n+1}) - a_{n+1} = b_n - a_{n+1} = b_n \wedge b_{n+1} = b_{n+1}. \end{aligned}$$

Finally,

$$e_n = \mathbf{C}(\mathbf{I}a_{n+1} \wedge d_n) = \mathbf{C}(a_{n+1} \wedge b_n) = \mathbf{C}(b_n - b_{n+1}) = b_n.$$

□

Suppose that $\mathfrak{A}$ is a closure algebra and $\mathfrak{A}_* = (W, R)$ is its Esakia space. For $a_1, \dots, a_{n+1} \in \mathfrak{A}$, let d_n and e_n be defined as in Definition 3.8. Set $D_n = \beta(d_n)$ and $E_n = \beta(e_n)$. If we interpret p_i as $\beta(a_i)$, then D_n is the interpretation of $\neg\mathsf{bd}_n$ and $W \setminus E_n$ is the interpretation of $\Box(\Box p_{n+1} \to \mathsf{bd}_n)$.

Lemma 6.6. *Suppose $\mathfrak{A}$ is a closure algebra and $\mathfrak{A}_* = (W, R)$ is the Esakia space of $\mathfrak{A}$. Let $n \geq 1$, $a_1, \dots, a_n \in \mathfrak{A}$, d_n is as in Definition 3.8, and $D_n = \beta(d_n)$. Then $(\bigcup_{i=1}^n M_i) \cap D_n = \varnothing$.*

Proof. Set $A_i = \beta(a_i)$. First suppose that $n = 1$ and $w \in M_1 \cap D_1$. Then $w \in M_1$ and $w \in D_1 = \mathbf{C}_R\mathbf{I}_R(A_1) \setminus A_1$. Therefore, there is $v \in \mathbf{I}_R(A_1)$ with wRv. Since $w \in M_1$ and wRv, we see that vRw. Thus, as $v \in \mathbf{I}_R(A_1)$, we have $w \in A_1$. The obtained contradiction proves that $M_1 \cap D_1 = \varnothing$.

Next suppose that $(\bigcup_{i=1}^n M_i) \cap D_n = \varnothing$ and $w \in \left(\bigcup_{i=1}^{n+1} M_i\right) \cap D_{n+1}$. Then $w \in \bigcup_{i=1}^{n+1} M_i$ and $w \in \mathbf{C}_R(\mathbf{I}_R A_{n+1} \cap D_n) \setminus A_{n+1}$. Therefore, there is $v \in \mathbf{I}_R A_{n+1} \cap D_n$ with wRv. From $v \in D_n$ it follows that $v \notin \bigcup_{i=1}^n M_i$, so $v \in M_{n+1}$. But then $w \in M_{n+1}$, so vRw. This together with $v \in \mathbf{I}_R(A_{n+1})$ yields $w \in A_{n+1}$, a contradiction. Thus, $\left(\bigcup_{i=1}^{n+1} M_i\right) \cap D_{n+1} = \varnothing$. □

Therefore, Lemma 6.6 yields that $\bigcup_{i=1}^n M_i$ is always contained in the interpretation of bd_n.

Theorem 6.7. *Let $\mathfrak{A}$ be a nondiscrete closure algebra, $\mathfrak{A}_* = (W, R)$ be its Esakia space, and $n \geq 1$. Then $\mathfrak{A}$ is n-discrete iff $\mathfrak{A} \vDash \mathsf{zem}_n$.*

Proof. By Theorem 6.3, $\mathfrak{A}$ is n-discrete iff $\mathrm{depth}(\mathfrak{A}_*) \leq n+1$ and $M_{n+1} \subseteq \min(\mathfrak{A}_*)$. Thus, it is sufficient to prove that $\mathfrak{A} \vDash \mathsf{zem}_n$ iff $\mathrm{depth}(\mathfrak{A}_*) \leq n+1$ and $M_{n+1} \subseteq \min(\mathfrak{A}_*)$.

Suppose that $\mathfrak{A} \vDash \mathsf{zem}_n$. First we show that $\mathrm{depth}(\mathfrak{A}_*) \leq n+1$. If $\mathrm{depth}(\mathfrak{A}_*) > n+1$, then there are $w_0, \dots, w_{n+1} \in X$ such that $w_{i+1}\vec{R}w_i$ for each $i \leq n$. We build $B_0, \dots, B_{n+1} \subseteq X$ as follows. Set $B_0 = X$. Suppose B_i is already built. Since $w_{i+1}\vec{R}w_i$, we have $w_{i+1} \notin R[\max_R(B_i) \cup \{w_i\}]$. Therefore, as $\mathfrak{A}_*$ is an Esakia space, there is an R-downset $B_{i+1} \subseteq W$ such that $B_{i+1} \subseteq B_i$, $w_{i+1} \in B_{i+1}$, and $B_{i+1} \cap R[\max_R(B_i) \cup \{w_i\}] = \varnothing$. For $i = 1, \dots, n$, let $A_i = X \setminus B_i$ and let $A_{n+1} = X \setminus (B_n \setminus B_{n+1})$. Let b_i be such that $\beta(b_i) = B_i$, and set $A_i = \beta(a_i)$, $D_i = \beta(d_i)$, and $E_i = \beta(e_i)$.

We claim that if we interpret p_i as A_i, then $\mathfrak{A}_* \nvDash \mathsf{zem}_n$. For this it is sufficient to show that $\mathbf{I}_R(E_n \cup A_{n+1}) \cap A_{n+1} \not\subseteq \mathbf{I}_R(A_{n+1})$. Clearly $w_{n+1} \in A_{n+1}$. By Lemma 6.5, $e_n = b_n$, and hence $E_n = B_n$. Since

$$\begin{aligned} w_{n+1} \in X &= \mathbf{I}_R(X) = \mathbf{I}_R(A_{n+1} \cup (X \setminus A_{n+1})) = \mathbf{I}_R(A_{n+1} \cup (B_n \setminus B_{n+1})) \\ &\subseteq \mathbf{I}_R(A_{n+1} \cup \mathbf{C}_R(B_n \setminus B_{n+1})) = \mathbf{I}_R(A_{n+1} \cup B_n) = \mathbf{I}_R(A_{n+1} \cup E_n), \end{aligned}$$

we have that $w_{n+1} \in \mathbf{I}_R(E_n \cup A_{n+1}) \cap A_{n+1}$. On the other hand, since $w_{n+1}Rw_n$ and $w_n \notin A_{n+1}$, we see that $w_{n+1} \notin \mathbf{I}_R(A_{n+1})$. Thus, $\mathbf{I}_R(E_n \cup A_{n+1}) \cap A_{n+1} \not\subseteq \mathbf{I}_R(A_{n+1})$, and hence $\mathfrak{A}_* \nvDash \mathsf{zem}_n$. The obtained contradiction proves that $\mathrm{depth}(\mathfrak{A}_*) \leq n+1$.

Next we show that $M_{n+1} \subseteq \min(\mathfrak{A}_*)$. If not, then there are distinct $w, v \in M_{n+1}$ such that wRv and vRw. Since $w \in M_{n+1}$, there are $w_0, \dots, w_{n-1} \in X$ such that $w_i \in M_{i+1}$, $w_{i+1}Rw_i$, and wRw_{n-1}. Let $w_n := w$ and $w_{n+1} := v$. Build $B_0, \dots, B_n \subseteq W$ as above. Since $w_n \neq w_{n+1}$, there is $B_{n+1} \subseteq W$ such that $w_n \notin B_{n+1}$ and $w_{n+1} \in B_{n+1}$, and B_{n+1} can be selected so that $\max_R(B_n) \cap B_{n+1} = \varnothing$. For $i = 1, \dots, n$, let $A_i = W \setminus B_i$ and let $A_{n+1} = W \setminus (B_n \setminus B_{n+1})$. The same argument as above yields that $\mathbf{I}_R(E_n \cup A_{n+1}) \cap A_{n+1} \not\subseteq \mathbf{I}_R(A_{n+1})$. Therefore, $\mathfrak{A}_* \nvDash \mathsf{zem}_n$, a contradiction. This proves that if $\mathfrak{A} \vDash \mathsf{zem}_n$, then $\mathrm{depth}(\mathfrak{A}_*) \leq n+1$ and $M_{n+1} \subseteq \min(\mathfrak{A}_*)$.

Conversely, suppose that $\mathrm{depth}(\mathfrak{A}_*) \leq n+1$ and $M_{n+1} \subseteq \min(\mathfrak{A}_*)$. Let $p_1, \dots, p_{n+1}$ be interpreted as $a_1, \dots, a_{n+1} \in \mathfrak{A}$. Let $A_i = \beta(a_i)$, $D_i = \beta(d_i)$, and $E_i = \beta(e_i)$. Then in $\mathfrak{A}_*$ the formula bd_n is interpreted as $W \setminus D_n$ and $\Box(\Box p_{n+1} \to \mathsf{bd}_n)$ is interpreted as $W \setminus E_n$. Therefore, to see that $\mathfrak{A}$ satisfies zem_n, it is sufficient to show that $\mathbf{I}_R(E_n \cup A_{n+1}) \cap A_{n+1} \subseteq \mathbf{I}_R(A_{n+1})$. Suppose $w \in \mathbf{I}_R(E_n \cup A_{n+1}) \cap A_{n+1}$ and wRv. Then $w \in A_{n+1}$ and $v \in E_n \cup A_{n+1}$. If $v \in E_n$, then $v \in \mathbf{C}_R(\mathbf{I}_R A_{n+1} \cap D_n)$. So there is $u \in \mathbf{I}_R A_{n+1} \cap D_n$ such that vRu. By Lemma 6.6, $u \notin \bigcup_{i=1}^n M_i$. Thus, $u \in M_{n+1} \subseteq \min(\mathfrak{A}_*)$, yielding $w = v = u$. Consequently, $v \in A_{n+1}$, and hence $w \in \mathbf{I}_R(A_{n+1})$, as desired. $\square$

Let $\mathbf{S4}_n := \mathbf{S4} + \mathsf{bd}_n$ and $\mathbf{S4.Z}_n := \mathbf{S4} + \mathsf{zem}_n$. It is well known (see, e.g., [6, Thm. 8.85]) that every normal extension of $\mathbf{S4}_n$ has the finite model property.

Theorem 6.8.
(1) $\mathbf{S4}_{n+1} \subsetneq \mathbf{S4.Z}_n$.
(2) $\mathbf{S4.Z}_n$ *has the finite model property.*

Proof. (1) Suppose $\mathfrak{A} \vDash \mathbf{S4.Z}_n$. By Theorem 6.7, $\mathrm{depth}(\mathfrak{A}_*) \leq n+1$. Therefore, by Theorem 3.12, $\mathfrak{A} \vDash \mathbf{S4}_{n+1}$. Thus, $\mathbf{S4}_{n+1} \subseteq \mathbf{S4.Z}_n$. To see that the inclusion is proper, consider the finite $\mathbf{S4}$-frame $\mathfrak{F}$ depicted in Figure 2. Since $\mathrm{depth}(\mathfrak{F}) = n+1$, we see that $\mathfrak{F}$ satisfies $\mathbf{S4}_{n+1}$. On the other hand, as $M_{n+1} = \{r_1, r_2\} \not\subseteq \varnothing = \min(\mathfrak{F})$, the frame $\mathfrak{F}$ does not satisfy $\mathbf{S4.Z}_n$.

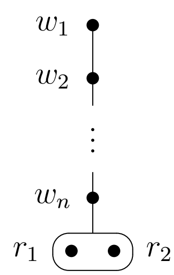


FIGURE 2. An $\mathbf{S4}_{n+1}$-frame $\mathfrak{F}$ that is not an $\mathbf{S4.Z}_n$-frame.

(2) Follows from (1) since every normal extension of $\mathbf{S4}_n$ has the finite model property. $\square$

7. n-DISCRETE SPACES AND TOPOLOGICAL COMPLETENESS OF $\mathbf{S4.Z}_n$

In this section we use the results of the previous section to generalize the concept of a discrete space to that of an n-discrete space. We show that a T_1-space X is 1-discrete iff X is nodec, and more generally, X is n-discrete iff $\mathrm{ldim}(X) \leq n$. This yields a number of topological incompleteness results in modal logic. The main result of the section is the construction of a countable crowded ω-resolvable Tychonoff space Z_n of localic Krull dimension n such that $\mathbf{S4.Z}_n$ is the logic of Z_n.

Definition 7.1. A nonempty topological space X is called *n-discrete* provided $\mathfrak{A}_X$ is n-discrete.

Remark 7.2. It follows that X is 0-discrete iff X is discrete, and for $n \geq 1$, X is n-discrete iff every nowhere dense subset of X is $(n-1)$-discrete.

Remark 7.3. Let X be a nonempty nondiscrete space. By Theorem 6.7, X is n-discrete iff $X \vDash \mathsf{zem}_n$.

Remark 7.4. Let X be a nonempty Alexandroff space and let R be the specialization order of X. Then $\mathfrak{F} := (X, R)$ is an $\mathbf{S4}$-frame. Since X and $\mathfrak{F}$ satisfy the same modal formulas, by Remark 7.3, X is n-discrete iff $\mathfrak{F} \vDash \mathsf{zem}_n$. Moreover, a slightly simplified version of the proof of Theorem 6.7 yields that $\mathfrak{F} \vDash \mathsf{zem}_n$ iff $\mathrm{depth}(\mathfrak{F}) \leq n+1$ and $M_{n+1} \subseteq \min(\mathfrak{F})$. This, in particular, implies that each $\mathbf{S4.Z}_n$ is a canonical logic. In fact, since by Theorem 6.8(2), $\mathbf{S4.Z}_n$ has the finite model property, $\mathbf{S4.Z}_n$ is the logic of finite n-discrete Alexandroff spaces.

Recall that a space is *nodec* if every nowhere dense set is closed. It is well known (see, e.g., [10]) that a space is nodec iff every nowhere dense set is closed and discrete. As follows from [1], the Zeman formula zem defines the class of nodec spaces, and $\mathbf{S4.Z} = \mathbf{S4} + \mathsf{zem}$ is the logic of the class of nodec spaces.

Lemma 7.5. *Let X be a T_1-space.*

(1) $\mathrm{ldim}(X) \leq 0$ *iff X is discrete.*
(2) $\mathrm{ldim}(X) \leq 1$ *iff X is nodec.*

Proof. (1) If X is discrete, then the only nowhere dense subset of X is $\varnothing$. Therefore, $\mathrm{ldim}(X) \leq 0$. Conversely, if X is not discrete, then there is $x \in X$ such that $\{x\}$ is not open, so $\mathbf{I}\{x\} = \varnothing$. Since X is T_1, we see that $\mathbf{IC}\{x\} = \mathbf{I}\{x\} = \varnothing$, so $\{x\}$ is nowhere dense. Thus, $\mathrm{ldim}(X) > 0$.

(2) Suppose X is nodec. Let D be an arbitrary nowhere dense subspace of X. Then D is closed and discrete. Since X is T_1, so is D. Therefore, by (1), $\mathrm{ldim}(D) \leq 0$. Thus, $\mathrm{ldim}(X) \leq 1$. Conversely, if X is not nodec, then there is a nowhere dense subspace D of X which is not closed. Therefore, $\mathbf{C}D$ is a nowhere dense subspace of X which is not discrete. By (1), $\mathrm{ldim}(\mathbf{C}D) > 0$. Thus, $\mathrm{ldim}(X) > 1$. ☐

Lemma 7.6. $\mathbf{S4.Z} = \mathbf{S4.Z}_1$.

Proof. By Remark 7.4 and Lemma 7.5, $\mathbf{S4.Z}_1$ is the logic nodec spaces. By [1, Thm. 4.6], the same is true of $\mathbf{S4.Z}$. Thus, $\mathbf{S4.Z} = \mathbf{S4.Z}_1$. ☐

Theorem 7.7. *Let X be a T_1-space and $n \in \omega$. Then* $\mathrm{ldim}(X) \leq n$ *iff X is n-discrete.*

Proof. By induction on n. The case $n = 0$ is Lemma 7.5(1). Suppose for every T_1-space Y, we have Y is n-discrete iff $\mathrm{ldim}(Y) \leq n$. We show that X is $(n+1)$-discrete iff $\mathrm{ldim}(X) \leq n+1$. We have $\mathrm{ldim}(X) \leq n+1$ iff $\mathrm{ldim}(Y) \leq n$ for every nowhere dense subspace Y of X. Since a subspace of a T_1-space is a T_1-space, by inductive hypothesis, this is equivalent to every nowhere dense subspace Y of X being n-discrete. But this is equivalent to X being $(n+1)$-discrete. ☐

Corollary 7.8. *For $n \geq 1$, no logic in the interval* $[\mathbf{S4}_{n+1}, \mathbf{S4.Z}_n)$ *is the logic of any class of T_1-spaces.*

Proof. Suppose $L \in [\mathbf{S4}_{n+1}, \mathbf{S4.Z}_n)$ and $\mathcal{K}$ is a class of T_1-spaces. If L is the logic of $\mathcal{K}$, then for each $X \in \mathcal{K}$, we have $X \vDash L$. Therefore, since $\mathbf{S4}_{n+1} \subseteq L$, we have $X \vDash \mathsf{bd}_{n+1}$. By Theorem 5.6, $\mathrm{ldim}(X) \leq n$. As X is T_1, by Theorem 7.7, X is n-discrete. By Remark 7.3, $X \vDash \mathsf{zem}_n$. Thus, $\mathbf{S4.Z}_n \subseteq L$, a contradiction. Consequently, L is not the logic of any class of T_1-spaces. ☐

On the other hand, for each $n \geq 1$ we construct a countable crowded ω-resolvable Tychonoff space Z_n of localic Krull dimension n such that $\mathbf{S4.Z}_n$ is the logic of Z_n. The basic building block for the construction is a countable crowded ω-resolvable Tychonoff nodec space Y such that the remainder $Y^* = \beta Y \setminus Y$ contains a subspace homeomorphic to $\beta\omega$ which consists entirely of remote points of Y. In Section 7.1 we explain why such a building block Y exists, in Section 7.2 we build the spaces Z_n from Y, and in Section 7.3 we prove that $\mathbf{S4.Z}_n$ is the logic of Z_n.

7.1. **The basic building block.** Let X be a topological space. We recall (see Juhász [26, 27]) that a *π-base* of X is a collection $\mathcal{B}$ of nonempty open subsets of X such that every nonempty open subset of X contains a member of $\mathcal{B}$. The *π-weight* $\pi(X)$ of X is the smallest cardinality of such a family. We will be interested in Tychonoff spaces of countable π-weight.

For a compact Hausdorff space X, let EX be the *Gleason cover* of X [21, 34]. It is well known that EX is constructed as the Stone space of the Boolean algebra of regular open subsets of X, and hence EX is an extremally disconnected compact Hausdorff space, where we recall that a space is *extremally disconnected* if the closure of each open set is open.

If $\nabla \in EX$, then $\bigcap\{\mathbf{C}_X(U) \mid U \in \nabla\}$ is a singleton of X, which we denote by $p_X(\nabla)$. This defines a map $p_X : EX \to X$. It is well known that p_X is an *irreducible map*; that is, p_X is an onto continuous map such that for every proper closed subset F of EX, the image $p_X(F)$ is a proper closed subset of X. Since p_X is evidently closed, this yields that $F \subseteq EX$ is nowhere dense iff $p_X(F) \subseteq X$ is nowhere dense, and that $\pi(X) = \pi(EX)$.

Let Z be a subspace of X. A point $x \in X \setminus Z$ is *remote* from Z provided $x \notin \mathbf{C}_X(D)$ for every nowhere dense subset D of Z. Observe that if x is remote from Z, then x is remote from every subspace of Z. The following simple lemma was used in [32, 12] for constructing various examples.

Lemma 7.9. *For a T_1-space X, if every $x \in X$ is remote from $X \setminus \{x\}$, then X is nodec.*

Proof. Let D be a nowhere dense subset of X and $x \notin D$. Since X is a T_1-space, D is a nowhere dense subset of $X \setminus \{x\}$. Therefore, as x is remote from $X \setminus \{x\}$, we see that $x \notin \mathbf{C}(D)$. Thus, X is nodec. □

Suppose X is a Tychonoff space. A *remote point* of X is a point $p \in \beta X \setminus X$ that is remote from X. In the context of Čech-Stone compactifications, remote points are very well studied in the literature. In particular, we have:

Theorem 7.10. [5, 9] *If X is a nonpseudocompact Tychonoff space with countable π-weight, then the remainder $X^* := \beta X \setminus X$ contains a point that is remote from X.*

Here we recall that a Tychonoff space X is *pseudocompact* if every continuous real-valued function on X is bounded. This result was generalized to products of such spaces in [11].

Let $\mathbb{I}$ be the closed unit interval and let $E\mathbb{I}$ be the Gleason cover of $\mathbb{I}$. For $t \in \mathbb{I}$, let $X = E\mathbb{I} \setminus p_{\mathbb{I}}^{-1}(\{t\})$. Since X is a dense subspace of $E\mathbb{I}$, it is C^*-embedded in $E\mathbb{I}$ (see, e.g., [37, Prop. 10.47]), meaning that every bounded continuous real-valued function on X extends to $E\mathbb{I}$. Therefore, by [37, Thm. 1.46], $\beta X = E\mathbb{I}$. It is also clear that X is a nonpseudocompact Tychonoff space with countable π-weight. Thus, by Theorem 7.10, there is a point $x_t \in p_{\mathbb{I}}^{-1}(\{t\})$ that is remote from X.

Let D be any countable dense subset of $\mathbb{I}$ (e.g., $D = \mathbb{I} \cap \mathbb{Q}$). We set

$$Y := \{x_t \mid t \in D\}.$$

Lemma 7.11. [32, 12] *Y is a countable crowded extremally disconnected ω-resolvable nodec space that is of countable π-weight.*

Here we recall (see, e.g., [13]) that a partition $\mathcal{P}$ of a space X is *dense* if each $D \in \mathcal{P}$ is dense in X, and that X is *κ-resolvable* if it has a dense partition of size κ. We now isolate the crucial property of Y that makes our construction in Section 7.2 work.

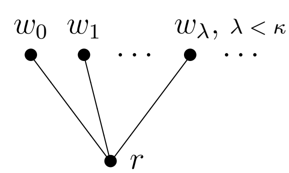
Proposition 7.12. *Y has a compact set of remote points that is homeomorphic to $\beta\omega$.*

Proof. Since Y is countable, we can pick a nonempty closed G_δ-subset S of βY such that $Y \cap S = \varnothing$. Put $T = \beta Y \setminus S$. By [37, Thm. 1.49], $\beta T = \beta Y$ and $T^* = S$. By [9, Thm. 11.1], we can choose a countably infinite discrete set D consisting entirely of remote points of T every limit point of which is also a remote point of T. Observe that every point from D is remote from Y since Y is a subspace of T. We show that D is C^*-embedded in βY by utilizing a technique of [33]. Since $D \subseteq T^* = S$ and S is closed, $\mathbf{C}D \subseteq S$. Because $Y \subseteq \beta Y \setminus S$, we see that $\mathbf{C}(D) \cap Y \subseteq \mathbf{C}(D) \setminus S = \varnothing$. Therefore, D is closed in the subspace $D \cup Y$, which is normal since it is countable. By the Tietze Extension Theorem (see, e.g.,[15, Thm. 2.1.8]), D is C^*-embedded in $D \cup Y$, and so D is C^*-embedded in βY. This, by [37, Thm. 1.46], yields that $\mathbf{C}(D) = \beta D$, and hence Y has a compact set of remote points that is homeomorphic to $\beta\omega$. □

7.2. **The spaces Z_n.** Let $\mathfrak{F} = (W, R)$ be a rooted **S4**-frame. We call $\mathfrak{F}$ a *tree* if R is a partial order and $(\forall w, u, v \in W)(u, v \in \mathbf{C}_R\{w\} \Rightarrow uRv$ or $vRu)$. We will always denote the root of a tree $\mathfrak{F}$ by r, the R-maximal points of $\mathfrak{F}$ by $\max(\mathfrak{F})$, and call v a *child* of w provided $w\vec{R}v$ and from $wRuRv$ it follows that $w = u$ or $u = v$. For $n \geq 1$, let $\mathfrak{T}_n$ denote the tree of depth n in which all non-R-maximal points have ω children.

We call a cluster in $\mathfrak{F}$ *trivial* if it is a singleton, and *proper* otherwise. The *skeleton* of $\mathfrak{F}$ is the partially ordered **S4**-frame obtained by modding out the clusters of $\mathfrak{F}$. We call $\mathfrak{F}$ a *quasi-tree* if the skeleton of $\mathfrak{F}$ is a tree. A cluster of a quasi-tree $\mathfrak{F}$ is *maximal* if all its points are R-maximal, and it is the *root cluster* if it contains a root of $\mathfrak{F}$.

Let $\mathcal{P}$ be a partition of a space X. We call $\mathcal{P}$ *clopen* provided each $A \in \mathcal{P}$ is clopen in X. For a cardinal κ, we consider the *κ-fork* depicted in Figure 3.



κ-fork

Figure 3. The κ-fork.

Lemma 7.13. *The κ-fork is an interior image of a space X iff there are a closed nowhere dense subset N of X and a clopen partition $\mathcal{P} = \{A_\lambda \mid \lambda < \kappa\}$ of the subspace $X \setminus N$ such that $\mathbf{C}A = A \cup N$ for each $A \in \mathcal{P}$.*

Proof. Let $\mathfrak{F} = (W, R)$ be the κ-fork. First suppose that $f : X \to W$ is an onto interior map. Let $N = f^{-1}(r)$ and $A_\lambda = f^{-1}(w_\lambda)$. Then

$$\mathbf{C}N = \mathbf{C}f^{-1}(r) = f^{-1}\mathbf{C}_R\{r\} = f^{-1}(r) = N$$

and

$$\mathbf{IC}N = \mathbf{I}N = \mathbf{I}f^{-1}(r) = f^{-1}\mathbf{I}_R\{r\} = f^{-1}(\varnothing) = \varnothing.$$

Therefore, N is closed and nowhere dense in X. Clearly $\mathcal{P} = \{A_\lambda \mid \lambda < \kappa\}$ is a partition of $X \setminus N$. Moreover, since each $\{w_\lambda\}$ is simultaneously an R-upset and an R-downset in the subframe $W \setminus \{r\}$, each A_λ is clopen in $X \setminus N$. Finally,

$$\mathbf{C}A_\lambda = \mathbf{C}f^{-1}(w_\lambda) = f^{-1}(\mathbf{C}_R\{w_\lambda\}) = f^{-1}(\{w_\lambda, r\}) = A_\lambda \cup N.$$

Next suppose that there are a closed nowhere dense subset N of X and a clopen partition $\mathcal{P} = \{A_\lambda \mid \lambda < \kappa\}$ of the subspace $X \setminus N$ such that $\mathbf{C}A = A \cup N$ for each $A \in \mathcal{P}$. Define

$f : X \to W$ by setting

$$f(x) = \begin{cases} r & \text{if } x \in N \\ w_\lambda & \text{if } x \in A_\lambda \end{cases}$$

It is clear that f is a well-defined onto map. Moreover,

$$f^{-1}(\mathbf{C}_R\{r\}) = f^{-1}(r) = N = \mathbf{C}N = \mathbf{C}f^{-1}(r)$$

and

$$f^{-1}(\mathbf{C}_R\{w_\lambda\}) = f^{-1}(\{w_\lambda, r\}) = A_\lambda \cup N = \mathbf{C}A_\lambda = \mathbf{C}f^{-1}(w_\lambda).$$

Thus, f is interior. $\square$

We assume the reader is familiar with the construction of attaching spaces or adjunction space (see, e.g., [22, pp. 12–14] or [38, pp. 65–66]). Given an indexed family of spaces X_i and subspaces $Y_i \subseteq X_i$, along with continuous maps $f_i : Y_i \to Z$, one can form an *adjunction space* which is a quotient of the topological sum $\bigoplus_{i\in I} X_i$ in which the only nontrivial equivalence classes are

$$\{(y_i, y_j) \mid i, j \in I,\ y_i \in Y_i,\ y_j \in Y_j,\ f_i(y_i) = f_j(y_j)\}.$$

When Z is a singleton, the adjunction space is often referred to as the *wedge sum*.

Given an equivalence relation $\equiv$ on a set X, let $[x]$ be the equivalence class of $x \in X$. we call $U \subseteq X$ *saturated* provided that $x \in U$ implies $[x] \subseteq U$. Recall that open (resp. closed) sets in a quotient space $X/{\equiv}$ correspond to saturated open (resp. closed) sets in X.

Using Y we recursively build the family of spaces $\{Z_n \mid n \geq 1\}$ such that each Z_n is a subspace of Z_{n+1} and there is an onto interior mapping $\alpha_n : Z_n \to \mathfrak{T}_{n+1}$.

Base case $(n = 1)$**:** Let $\{Y_n \mid n \in \omega\}$ be a pairwise disjoint family of spaces such that there is a homeomorphism $h_n : Y \to Y_n$ for each $n \in \omega$. Fix $y \in Y$ and set $y_n = h_n(y)$. Let Z_1 be the wedge sum of $\{(Y_n, y_n) \mid n \in \omega\}$. We identify each $Y_n \setminus \{y_n\}$ with its image in Z_1 and refer to the point $\{y_n \mid n \in \omega\}$ in Z_1 using the symbol y; see Figure 4. Since $\mathfrak{T}_2$ is the ω-fork and $\{y\}$ is a closed nowhere dense subset of Z_1 such that $\{Y_n \setminus \{y_n\} \mid n \in \omega\}$ is a clopen partition of $Z_1 \setminus \{y\}$ satisfying $y \in \mathbf{C}_{Z_1}(Y_n \setminus \{y_n\})$, it follows from Lemma 7.13 that there is an onto interior mapping $\alpha_1 : Z_1 \to \mathfrak{T}_2$ such that $\alpha_1^{-1}(r) = \{y\}$.

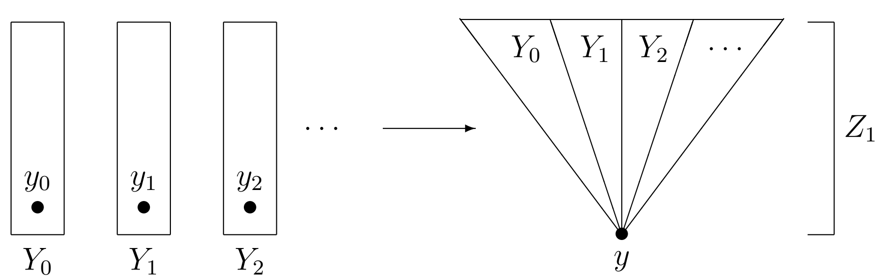


FIGURE 4. Realizing Z_1 as a wedge sum of the Y_i.

Recursive step $(n \geq 1)$**:** Suppose Z_n with the above properties is already built. Identify $\mathfrak{T}_{n+1}$ with the subframe $\mathfrak{T}_{n+2} \setminus \max(\mathfrak{T}_{n+2})$. Enumerate $\max(\mathfrak{T}_{n+1})$ as $\{w_i \mid i \in \omega\}$. Label points in $\max(\mathfrak{T}_{n+2})$ as $w_{i,j}$ where $w_{i,j}$ is the j^{th} child of w_i. Let $\alpha_n : Z_n \to \mathfrak{T}_{n+1}$ be an onto interior map such that $(\alpha_n)^{-1}(r) = \{y\}$ where y is the point in the base case defining Z_1. Set $X_i = (\alpha_n)^{-1}(\mathbf{C}_R\{w_i\})$; see Figure 5.

Since X_i is countable, there is a continuous bijection $f : \omega \to X_i$ which extends to a continuous onto map $g : \beta\omega \to \beta X_i$. Up to homeomorphism, $\beta\omega$ is a subspace of βY such that each point in $\beta\omega$ is a remote point of Y. Consider the quotient space Q_i of βY obtained by the equivalence relation whose only nontrivial equivalence classes are the fibers

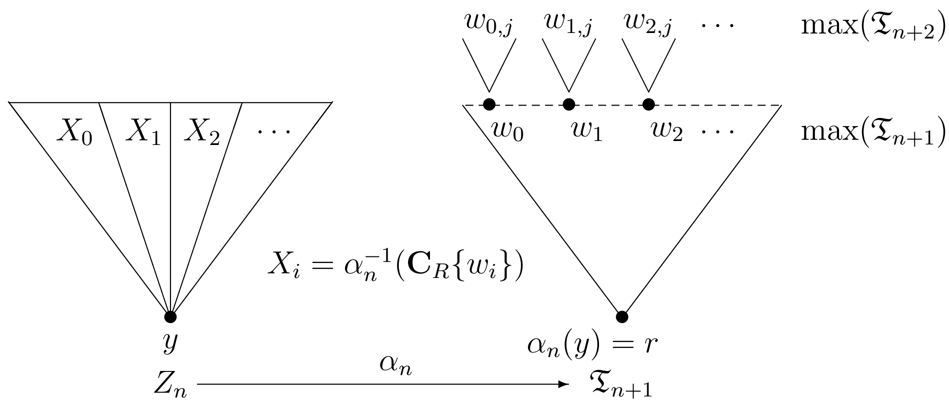


FIGURE 5. Mapping Z_n onto $\mathfrak{T}_{n+1}$ viewed as a subframe of $\mathfrak{T}_{n+2}$.

of g, namely $g^{-1}(x)$ for each $x \in \beta X_i$. By [15, Thm. 2.4.13] the quotient mapping of βY onto Q_i is closed. Intuitively, Q_i is obtained from βY by replacing the copy of $\beta\omega$ that 'is remote from Y' by βX_i. We identify Y, βX_i, and X_i with their respective images in Q_i, see Figure 6. For a nowhere dense subset N of Y, we have $\mathbf{C}_{\beta Y}(N) \cap \beta\omega = \varnothing$, so $\mathbf{C}_{\beta Y}(N)$ is saturated, and hence $\mathbf{C}_{Q_i}(N) \cap \beta X_i = \varnothing$.

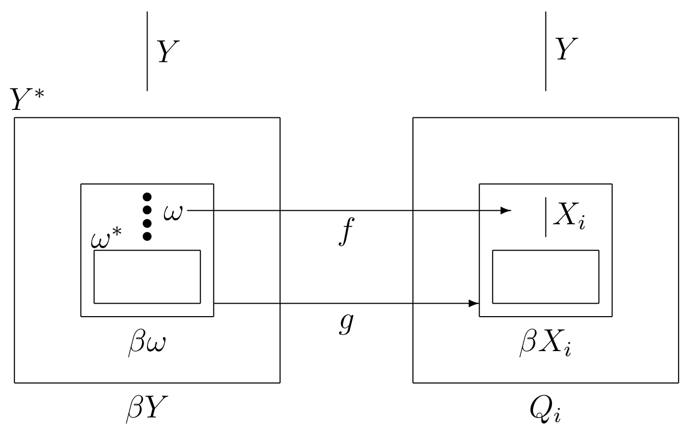


FIGURE 6. Identifying Y, βX_i, and X_i in the quotient Q_i of βY.

Viewing $Y \cup X_i$ as a subspace of Q_i, the subsets Y and X_i are complements of each other, Y is dense, and X_i is closed and nowhere dense. Let A_i be the adjunction space of ω copies of $Y \cup X_i$ glued through the identity map on the copies of X_i. That is, up to homeomorphism, A_i is the quotient of the topological sum $\bigcup_{m\in\omega}(Y \cup X_i) \times \{m\}$ under the equivalence relation whose nontrivial equivalence classes are $\{(x, m) \mid m \in \omega\}$ for each $x \in X_i$; see Figure 7.

To facilitate defining $\alpha_{n+1} : Z_{n+1} \to \mathfrak{T}_{n+2}$ we denote the ω copies of Y in A_i by $Y_{i,j}$ where $j \in \omega$. We also identify X_i with its homeomorphic copy in A_i. The quotient mapping from $\bigoplus_{j\in\omega} Y_{i,j} \cup X_i$ onto A_i is closed. Thus, in A_i we have that $\bigcup_{j\in\omega} Y_{i,j}$ and X_i are complements of each other, $\bigcup_{j\in\omega} Y_{i,j}$ is dense, and X_i is closed and nowhere dense.

We define Z_{n+1} as the adjunction space of the A_i for $i \in \omega$ through the following gluing. For each A_i consider the inclusion mapping $I_i : X_i \to Z_n$. Glue through the equivalence relation whose nontrivial equivalence classes are $\{(x_i, x_j) \mid x_i \in X_i,\ x_j \in X_j,\ I_i(x_i) = I_j(x_j)\}$. Intuitively the gluing is through identifying points in X_i and X_j that are equal in Z_n; see Figure 8. Identify the $Y_{i,j}$, X_i, and Z_n with their images in Z_{n+1}. Observe that $Y_{i,j}$ is open in $Y_{i,j} \cup X_i$ and saturated in $\bigoplus_{j\in\omega}(Y_{i,j} \cup X_i)$, hence open in A_i. Similarly, $Y_{i,j}$ is saturated in

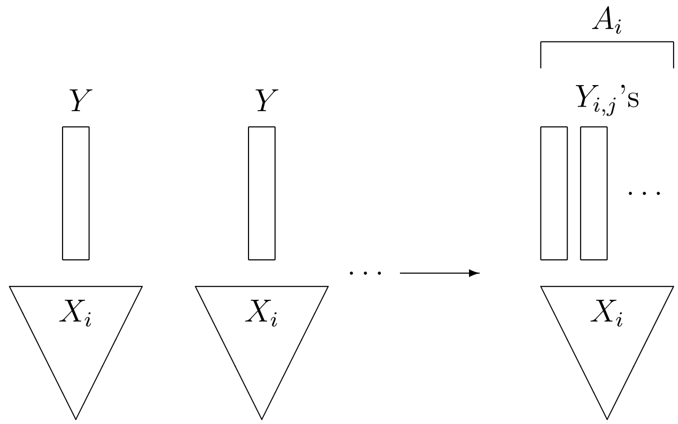


FIGURE 7. The adjunction space A_i obtained by gluing ω copies of $Y \cup X_i$ through X_i.

$\bigoplus_{i\in\omega} A_i$, and so open in Z_{n+1}. Thus, in Z_{n+1} we have that $\bigcup_{i,j\in\omega} Y_{i,j}$ and Z_n are complements of each other, $\bigcup_{i,j\in\omega} Y_{i,j}$ is dense and open, and Z_n is closed and nowhere dense.

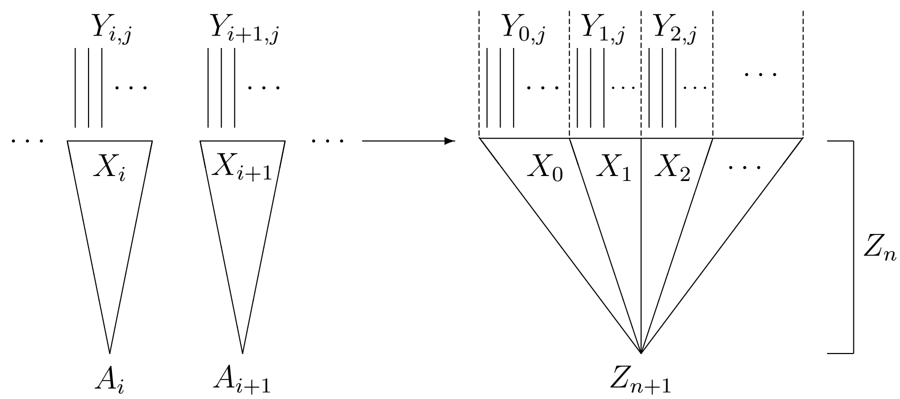


FIGURE 8. Attaching the A_i to obtain Z_{n+1}.

We now extend $\alpha_n : Z_n \to \mathfrak{T}_{n+1}$ to $\alpha_{n+1} : Z_{n+1} \to \mathfrak{T}_{n+2}$ by setting $\alpha_{n+1}(z) = w_{i,j}$ for each $z \in Y_{i,j}$. Let $w \in \mathfrak{T}_{n+2}$. If $w = w_{i,j} \in \max(\mathfrak{T}_{n+2})$, then

$$\begin{aligned}\alpha_{n+1}^{-1}(\mathbf{C}_R\{w_{i,j}\}) &= \alpha_{n+1}^{-1}\left(\{w_{i,j}\} \cup \mathbf{C}_R\{w_i\}\right) = \alpha_{n+1}^{-1}(w_{i,j}) \cup \alpha_n^{-1}(\mathbf{C}_R\{w_i\}) \\ &= Y_{i,j} \cup X_i = \mathbf{C}_{Z_{n+1}}(Y_{i,j}) = \mathbf{C}_{Z_{n+1}}\alpha_{n+1}^{-1}(w_{i,j}).\end{aligned}$$

Otherwise $w \in \mathfrak{T}_{n+1}$, so since α_n is interior and Z_n is closed in Z_{n+1}, we have

$$\alpha_{n+1}^{-1}(\mathbf{C}_R\{w\}) = \alpha_n^{-1}(\mathbf{C}_R\{w\}) = \mathbf{C}_{Z_n}\alpha_n^{-1}(w) = \mathbf{C}_{Z_{n+1}}\alpha_{n+1}^{-1}(w).$$

Thus, α_{n+1} is interior and $\alpha_{n+1}^{-1}(r) = \{y\}$.

Lemma 7.14. *Let $X = \bigoplus_{i\in\omega} Y_i$. For $n \in \omega$, if $0 \le \mathrm{ldim}(Y_i) \le n$ for each i, then $\mathrm{ldim}(X) \le n$.*

Proof. Induction on n. **Base case ($n = 0$):** $\mathrm{ldim}(Y_i) = 0$. Let N be nowhere dense in X. Then $N_i = N \cap Y_i$ is nowhere dense in Y_i. Therefore, $\mathrm{ldim}(N_i) = -1$, and so $N_i = \varnothing$. Thus, $N = \varnothing$. From this it follows that $\mathrm{ldim}(N) = -1$, and hence $\mathrm{ldim}(X) = 0$.

Inductive step ($n \ge 0$): Suppose for any family of spaces $\{Y_i' \mid i \in \omega\}$, if $0 \le \mathrm{ldim}(Y_i') \le n$ for each i, then $\mathrm{ldim}(\bigoplus_{i\in\omega} Y_i') \le n$. Assume $0 \le \mathrm{ldim}(Y_i) \le n+1$ for each $i \in \omega$. Let N be nowhere dense in X. Then $Y_i' = N \cap Y_i$ is nowhere dense in Y_i. Therefore, $\mathrm{ldim}(Y_i') \le n$. By the inductive hypothesis, $\mathrm{ldim}(N) \le n$. Thus, $\mathrm{ldim}(X) \le n+1$. □

Lemma 7.15. *For $n \geq 1$, $\mathrm{ldim}(Z_n) = n$.*

Proof. Since $\mathfrak{T}_{n+1}$ is an interior image of Z_n, the $(n+1)$-element chain is an interior image of Z_n. By Theorem 5.6, $\mathrm{ldim}(Z_n) \geq n$. We show that $\mathrm{ldim}(Z_n) \leq n$ by induction on $n \geq 1$.

Base case $(n = 1)$**:** Let N be nowhere dense in Z_1. Set $N_i = N \cap Y_i$ for each $i \in \omega$. Then N_i is nowhere dense in Z_1. Noting Y_i is a closed subspace of Z_1 homeomorphic to Y (which is a crowded T_1-space), it follows that N_i is nowhere dense in Y_i. Because Y is nodec, Y_i is nodec, and so N_i is closed in Y_i. Let N' be the union of the N_i in the topological sum of the Y_i which is the preimage of the adjunction space Z_1. Then N' is closed in the sum. Since N' is the preimage of N, we see that N is closed in Z_1. Therefore, Z_1 is nodec. Because Z_1 is a T_1-space, it follows from Lemma 7.5(2) that $\mathrm{ldim}(Z_1) \leq 1$.

Inductive step $(n \geq 1)$**:** Assume $\mathrm{ldim}(Z_n) = n$. Since Z_{n+1} was constructed in three stages, our proof is also in three stages. First we show that $\mathrm{ldim}(Y \cup X_i) \leq n+1$, next that $\mathrm{ldim}(A_i) \leq n+1$, and finally that $\mathrm{ldim}(Z_{n+1}) \leq n+1$.

Stage 1: Since $\mathrm{ldim}(Z_n) = n$ and each $X_i \subseteq Z_n$, by Lemma 5.3, $\mathrm{ldim}(X_i) \leq n$. Also, the $(n+1)$-element chain is an interior image of X_i, giving that $\mathrm{ldim}(X_i) \geq n$. Thus, $\mathrm{ldim}(X_i) = n$.

Let N be nowhere dense in $Y \cup X_i$, and set $M = N \cap Y$. Then M is nowhere dense in $Y \cup X_i$. Let U be an open subset of Y contained in $\mathbf{C}_Y M$. Since Y is open in $Y \cup X_i$, we have that U is open in $Y \cup X_i$ and is contained in $\mathbf{C}_Y M \subseteq \mathbf{C}M$. Because M is nowhere dense in $Y \cup X_i$, we obtain $U = \varnothing$, and so M is nowhere dense in Y. Since Y is nodec, M is closed and discrete in Y. By the construction of $Y \cup X_i$, each $x \in X_i$ is the image of a set of points each remote from Y, and hence $\mathbf{C}M \cap X_i = \varnothing$. Thus, $\mathbf{C}M \subseteq Y$, from which it follows that $\mathbf{C}_Y M = \mathbf{C}M$. Therefore, since M is closed in Y, it is closed in $Y \cup X_i$. Consequently, M is closed in N. In fact, M is clopen in N since Y is open and $M = N \cap Y$. Therefore, N is the disjoint union of M and $N \cap X_i$. As M is discrete, $\mathrm{ldim}(M) \leq 0$. Also, since $N \cap X_i$ is a subspace of X_i, we have $\mathrm{ldim}(N \cap X_i) \leq \mathrm{ldim}(X_i) = n$. By Lemma 7.14, $\mathrm{ldim}(N) \leq n$. Thus, $\mathrm{ldim}(Y \cup X_i) \leq n+1$.

Stage 2: Let N be nowhere dense in A_i. Set $N_j = N \cap Y_{i,j}$. Recalling that $Y_{i,j} \cup X_i$ is homeomorphic to $Y \cup X_i$, by replacing M by N_j and $Y \cup X_i$ by $Y_{i,j} \cup X_i$ in the proof of Stage 1, we see that N_j is closed in $Y_{i,j} \cup X_i$ and $N_j \cap X_i = \varnothing$ for all $j \in \omega$. Therefore, $\bigcup_{j\in\omega} N_j$ is closed in the topological sum $\bigoplus_{j\in\omega}(Y_{i,j} \cup X_i)$. Since $\bigcup_{j\in\omega} N_j$ is also saturated in $\bigoplus_{j\in\omega}(Y_{i,j} \cup X_i)$, it is closed in A_i, and hence closed in N. Also, $\bigcup_{j\in\omega} N_j = N \cap \bigcup_{j\in\omega} Y_{i,j}$ is open in N since $\bigcup_{j\in\omega} Y_{i,j}$ is open in A_i. Therefore, N is the disjoint union of $N \cap X_i$ and $\bigcup_{j\in\omega} N_j$. By Lemma 7.14, $\mathrm{ldim}\left(\bigcup_{j\in\omega} N_j\right) \leq 1 \leq n$ since $\mathrm{ldim}(N_j) \leq \mathrm{ldim}(Y_{i,j}) \leq 1$. Also $\mathrm{ldim}(N \cap X_i) \leq \mathrm{ldim}(X_i) = n$, so utilizing Lemma 7.14 again yields $\mathrm{ldim}(N) \leq n$. Thus, $\mathrm{ldim}(A_i) \leq n+1$.

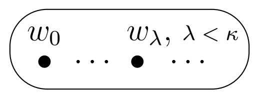
Stage 3: Let N be nowhere dense in Z_{n+1}. Set $N_i = (N \cap A_i) \setminus X_i$. By recognizing that N_i is realized within the discussion of Stage 2 as $\bigcup_{j\in\omega} N_j$, we see that each N_i is closed in A_i, and hence $\bigcup_{i\in\omega} N_i$ is closed in $\bigoplus_{i\in\omega} A_i$. Moreover, $\bigcup_{i\in\omega} N_i$ is saturated, and so $\bigcup_{i\in\omega} N_i$ is closed in Z_{n+1}. Therefore, $\bigcup_{i\in\omega} N_i$ is also closed in N. But $\bigcup_{i\in\omega} N_i = N \cap (Z_{n+1} \setminus Z_n)$, so $\bigcup_{i\in\omega} N_i$ is open in N. Thus, N is the disjoint union of $N \cap Z_n$ and $\bigcup_{i\in\omega} N_i$. Since $\mathrm{ldim}(N_i) \leq \mathrm{ldim}(A_i \setminus X_i) = \mathrm{ldim}\left(\bigoplus_{j\in\omega} Y_{i,j}\right) \leq 1$, Lemma 7.14 yields that $\mathrm{ldim}\left(\bigcup_{i\in\omega} N_i\right) \leq 1 \leq n$. Also $\mathrm{ldim}(N \cap Z_n) \leq \mathrm{ldim}(Z_n) = n$, so by Lemma 7.14, $\mathrm{ldim}(N) \leq n$. Consequently, $\mathrm{ldim}(Z_{n+1}) \leq n+1$. $\square$

7.3. **Completeness.** Since $\mathbf{S4.Z}_n$ has the finite model property, $\mathbf{S4.Z}_n$ is the logic of finite uniquely rooted $\mathbf{S4}$-frames $\mathfrak{F}$ of depth $\leq n+1$. Since each such $\mathfrak{F}$ can be unraveled into a uniquely rooted finite quasi-tree $\mathfrak{T}$ whose depth is $\leq n+1$, we see that $\mathbf{S4.Z}_n$ is the logic of uniquely rooted finite quasi-trees $\mathfrak{T}$ of depth $\leq n+1$.

Let $\mathcal{Q}_n$ be the quasi-tree whose skeleton is $\mathfrak{T}_n$ and in which the root cluster is the only trivial cluster and all other clusters are countably infinite. Clearly identifying the clusters yields an onto p-morphism $p_n : \mathcal{Q}_n \to \mathfrak{T}_n$. Because every uniquely rooted finite quasi-tree of depth $\leq n+1$ is an interior image of $\mathcal{Q}_{n+1}$, we see that $\mathbf{S4.Z}_n$ is the logic of $\mathcal{Q}_{n+1}$. Since we will utilize this fact, we state is as a lemma.

Lemma 7.16. $\mathbf{S4.Z}_n$ *is the logic of* $\mathcal{Q}_{n+1}$.

Since $\mathrm{ldim}(Z_n) = n$ and Z_n is T_1, we see that $Z_n \models \mathbf{S4.Z}_n$. Therefore, to show that $\mathbf{S4.Z}_n$ is the logic of Z_n, in view of Lemma 7.16, it is sufficient to prove that $\mathcal{Q}_{n+1}$ is an interior image of Z_n. The idea of the proof is to 'fatten' the mapping $\alpha_n : Z_n \to \mathfrak{T}_{n+1}$ to a mapping $Z_n \to \mathcal{Q}_{n+1}$. Let $\mathfrak{C}_\kappa$ be the κ-*cluster* as depicted in Figure 9.



$\mathfrak{C}_\kappa$

FIGURE 9. The κ-cluster.

Lemma 7.17. *A space X is κ-resolvable iff $\mathfrak{C}_\kappa$ is an interior image of X.*

Proof. First suppose that X is κ-resolvable. Then there is a dense partition $\{D_\lambda : \lambda < \kappa\}$ of X. Define $f : X \to \mathfrak{C}_\kappa$ by $f(x) = w_\lambda$ if $x \in D_\lambda$. Clearly f is a well-defined onto map. Moreover, for each $\lambda < \kappa$, we have:

$$\mathbf{C}f^{-1}(w_\lambda) = \mathbf{C}(D_\lambda) = X = f^{-1}(\{w_\lambda : \lambda < \kappa\}) = f^{-1}(\mathbf{C}_R\{w_\lambda\}).$$

Thus, f is an interior map.

Conversely, let $f : X \to \mathfrak{C}_\kappa$ be an onto interior map. Then $\{f^{-1}(w_\lambda) : \lambda < \kappa\}$ is a partition of X such that

$$\mathbf{C}f^{-1}(w_\lambda) = f^{-1}(\mathbf{C}_R\{w_\lambda\}) = f^{-1}(\{w_\lambda : \lambda < \kappa\}) = X.$$

Thus, $\{f^{-1}(w_\lambda) : \lambda < \kappa\}$ is a dense partition of X, and hence X is κ-resolvable. $\square$

Theorem 7.18. *For each $n \geq 1$, $\mathbf{S4.Z}_n$ is the logic of Z_n.*

Proof. As we already pointed out, in view of Lemma 7.16, it is sufficient to show that $\mathcal{Q}_{n+1}$ is an interior image of Z_n. The proof is by induction on n.

Let $n = 1$. Let $\mathfrak{C}_i$ be the maximal cluster in $\mathcal{Q}_2$ whose p_2-image is $w_i \in \max(\mathfrak{T}_2)$ (here we are using the enumeration of $\max(\mathfrak{T}_2)$ as it appears in the recursive step of defining the Z_n). So $\mathfrak{C}_i = p_2^{-1}(w_i)$. Since each $Y_i \setminus \{y_i\}$ is an open subspace of Y_i, Y_i is homeomorphic to Y, and Y is ω-resolvable, we see that each $Y_i \setminus \{y_i\}$ is ω-resolvable. As $Y_i \setminus \{y_i\}$ is homeomorphic to the subspace $Y_i \setminus \{y\}$ of Z_1, by Lemma 7.17, there is an onto interior map $f_i : Y_i \setminus \{y\} \to \mathfrak{C}_i$. Define $f : Z_1 \to \mathcal{Q}_2$ by

$$f(z) = \begin{cases} f_i(z) & \text{if } z \in Y_i \setminus \{y\} \\ r & \text{if } z = y \end{cases}$$

Since $\{Y_i \setminus \{y\} \mid i \in \omega\} \cup \{y\}$ is a partition of Z_1 and each f_i is onto, f is a well-defined onto map. Let $w \in \mathcal{Q}_2$. Suppose $w \in \mathfrak{C}_i$ for some $i \in \omega$. Then

$$\begin{aligned} f^{-1}(\mathbf{C}_R\{w\}) &= f^{-1}(\mathfrak{C}_i \cup \{r\}) = f_i^{-1}(\mathfrak{C}_i) \cup \{y\} \\ &= (Y_i \setminus \{y\}) \cup \{y\} = \mathbf{C}_{Z_1}(Y_i \setminus \{y\}) = \mathbf{C}_{Z_1} f^{-1}(w). \end{aligned}$$

Otherwise w is the root, and so

$$f^{-1}(\mathbf{C}_R\{w\}) = f^{-1}(w) = \{y\} = \mathbf{C}_{Z_1}\{y\} = \mathbf{C}_{Z_1} f^{-1}(w).$$

Thus, $f : Z_1 \to \mathcal{Q}_2$ is an onto interior map.

Let $n \geq 1$. Suppose $g : Z_n \to \mathcal{Q}_{n+1}$ is an onto interior map. Identify $\mathcal{Q}_{n+1}$ with the subframe $\mathcal{Q}_{n+2} \setminus \max_R(\mathcal{Q}_{n+2})$. Let $w_{i,j} \in \max(\mathfrak{T}_{n+2})$ be the j^{th} child of $w_i \in \max(\mathfrak{T}_{n+1})$ (as in the recursive step of building the Z_n). Let $\mathfrak{C}_{i,j}$ be the maximal cluster in $\mathcal{Q}_{n+2}$ whose p_{n+2}-image is $w_{i,j}$. So $\mathfrak{C}_{i,j} = p_{n+2}^{-1}(w_{i,j})$. Also, let $\mathfrak{C}_i$ be the maximal cluster in $\mathcal{Q}_{n+1}$ whose p_{n+2}-image is $w_i \in \max(\mathfrak{T}_{n+1})$. So $\mathfrak{C}_i = p_{n+2}^{-1}(w_i)$. Since each subspace $Y_{i,j}$ of Z_{n+1} is homeomorphic to Y, we see that $Y_{i,j}$ is ω-resolvable. By Lemma 7.17, there is an onto interior map $f_{i,j} : Y_{i,j} \to \mathfrak{C}_{i,j}$. Define $f : Z_{n+1} \to \mathcal{Q}_{n+2}$ by

$$f(z) = \begin{cases} f_{i,j}(z) & \text{if } z \in Y_{i,j} \\ g(z) & \text{if } z \in Z_n \end{cases}$$

Since $\{Y_{i,j} \mid i,j \in \omega\} \cup \{Z_n\}$ is a partition of Z_{n+1} and the $f_{i,j}$ and g are onto, f is a well-defined onto map. Let $w \in \mathcal{Q}_{n+2}$. Suppose $w \in \mathfrak{C}_{i,j}$ for some $i,j \in \omega$. Because Z_n is closed in Z_{n+1}, both g and $f_{i,j}$ are interior maps, and $g^{-1}\mathbf{C}_R(\mathfrak{C}_i) = X_i$, we have

$$\begin{aligned} f^{-1}(\mathbf{C}_R\{w\}) &= f^{-1}(\mathfrak{C}_{i,j} \cup \mathbf{C}_R(\mathfrak{C}_i)) = f_{i,j}^{-1}(\mathfrak{C}_{i,j}) \cup g^{-1}\mathbf{C}_R(\mathfrak{C}_i) = Y_{i,j} \cup X_i \\ &= \mathbf{C}_{Z_{n+1}}(Y_{i,j}) = \mathbf{C}_{Z_{n+1}}(\mathbf{C}_{Y_{i,j}}(f_{i,j}^{-1}(w)) = \mathbf{C}_{Z_{n+1}} f^{-1}(w). \end{aligned}$$

Otherwise $w \in \mathcal{Q}_{n+1}$, and so

$$f^{-1}(\mathbf{C}_R\{w\}) = g^{-1}(\mathbf{C}_R\{w\}) = \mathbf{C}_{Z_n} g^{-1}(w) = \mathbf{C}_{Z_{n+1}} f^{-1}(w).$$

Thus, $f : Z_{n+1} \to \mathcal{Q}_{n+2}$ is an onto interior map. $\square$

As an immediate consequence, we obtain:

Corollary 7.19. *For each $n \geq 1$, $\mathbf{S4.Z}_n$ is the logic of a countable crowded ω-resolvable Tychonoff space of localic Krull dimension n.*

Moreover, since $\mathbf{S4.Z} = \mathbf{S4.Z}_1$, we obtain the following topological completeness for the Zeman logic:

Corollary 7.20. **S4.Z** *is the logic of a countable crowded ω-resolvable Tychonoff nodec space.*

That **S4.Z** is the logic of nodec spaces was shown in [1, Thm. 4.6], but the proof required the use of Alexandroff nodec spaces. The above corollary strengthens this result considerably by providing a topologically "nice" nodec space whose logic is **S4.Z**.

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Guram Bezhanishvili: Department of Mathematical Sciences, New Mexico State University, Las Cruces NM 88003, USA, guram@math.nmsu.edu

Nick Bezhanishvili: Institute for Logic, Language and Computation, University of Amsterdam, 1090 GE Amsterdam, The Netherlands, N.Bezhanishvili@uva.nl

Joel Lucero-Bryan: Department of Applied Mathematics and Sciences, Khalifa University, Abu Dhabi, UAE, joel.lucero-bryan@kustar.ac.ae

Jan van Mill: Korteweg-de Vries Institute for Mathematics, University of Amsterdam, 1098 XG Amsterdam, The Netherlands, j.vanMill@uva.nl