Subminimal Negation

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Abstract

Minimal Logic, i.e. intuitionistic logic without the ex falso principle, is investigated in its original form with a negation symbol instead of a symbol denoting the contradiction. A Kripke semantics is developed for minimal logic and its sublogics with a still weaker negation by introducing a function on the upward closed sets of the models. The basic logic is a logic in which the negation has no properties but the one of being a unary operator. A number of extensions is studied of which the most important ones are contraposition logic and negative ex falso, a weak form of the ex falso principle. Completeness is proved and the created semantics is further studied. The negative translation of classical logic into intutionistic logic is made part of a chain of translations by introducing translations from minimal logic into contraposition logic and intuitionistic logic into minimal logic, the latter having been discovered in the correspondence between Johansson and Heyting. Finally, as a bridge to the work of Franco Montagna a start is made of a study of linear models of these logics.

Dedicated to the memory of Franco Montagna

1 Introduction

In this paper, we study minimal logic in its two equivalent formulations. Given a countable set of propositional variables, the formulation used nowadays is based on the propositional language of the positive fragment of intuitionistic logic, i.e., $\mathcal{L}^+ = \{\wedge, \vee, \rightarrow\}$, with an additional propositional variable f, representing *falsum*. In this setting, negation of φ is defined as $\varphi \rightarrow f$ and denoted by $\neg\varphi$. The significant difference between minimal and intuitionistic logic is that the former does not consider the *ex falso quodlibet* axiom as a valid axiom. If IPC^+ denotes the positive fragment of intuitionistic logic, minimal logic has the same axioms as IPC^+ and hence, f does not have the same properties as the intuitionistic $\bot$. We write MPC_f for this formalization of minimal logic.

The other formulation of minimal logic makes use of the language $\mathcal{L}^+ \cup \{\neg\}$, where the unary symbol $\neg$ represents negation. Thus, we denote with $\mathsf{MPC}_\neg$ the system axiomatized by the IPC^+ axioms and the additional axiom $(p \to q) \wedge (p \to \neg q) \to \neg p$. This version of minimal logic is the one originally proposed by Johansson [5], and even before, by Kolmogorov [6]. Completeness with respect to our Kripke-style semantics is proven for both versions of minimal logic.

We study a *weak* form of negation, considering sub-systems of minimal logic while fixing the IPC^+ axioms. Our basic system is N, in which negation has no properties but the one of being a 'function'. We define a semantics of negation by means of an auxiliary persistent function N. A canonical model is defined in order to prove completeness. Among the extensions of N studied here, the one axiomatized by $(p \to q) \to (\neg q \to \neg p)$ and denoted as CoPC is the most striking. We succeed in interpreting minimal logic in CoPC. To some extent, we connect with Franco Montagna's work, by considering the extensions of these logics by way of LC: $(p \to q) \vee (q \to p)$. Such extensions represent weakenings of the Gödel-Dummett logic. We conclude stating some remarks and ideas for further research.

The *ex falso quodlibet* or, as it is called in paraconsistent settings, the *law of explosion* [1], is the logical law expressing that any statement can be proven from a contradiction (or a falsehood). Classical logic (CPC), intuitionistic logic and many other systems consider *ex falso* to be valid. However, there hasn't always been widespread agreement about this. Some supporters of an intuitionistic standpoint, like the early Kolmogorov [6], rejected *ex falso*. According to him, *ex falso* asserts something about a consequence of something 'impossible' and hence, it is unacceptable. But, since Heyting's formalization of intuitionistic logic [4], it has been assumed as an axiom for such a system. In paraconsistent logic, it is necessary to reject *ex falso*, in order to allow for inconsistent theories and 'accept' contradictions[1]. We present in this paper minimal logic, CoPC and its subsystems as paraconsistent variations of intuitionistic logic.

[1]Kolmogorov and Johansson's minimal intuitionistic logic is introduced as MIL in the "big manifesto" paper on paraconsistency, in [1].

2 Intuitionistic Logic

The propositional language of IPC consists of a set P of propositional variables $\{p_0, p_1, p_2, \dots\}$, the propositional constants $\bot, \top$ and the set of binary connectives $\mathcal{L}^+(P)$. For any formula φ, its negation $\neg\varphi$ is defined as $\varphi \to \bot$ [10]. In practice, it is often more convenient to conceive formulas as containing both $\neg$ and $\bot$, and to add $\top$. We take the axioms of IPC as in [10].

2.1 Kripke Semantics for Intuitionistic Logic

Definition 1. A *propositional Kripke frame* of IPC is a pair $\mathfrak{F} = (W, R)$, where W is a non-empty set of possible worlds and R is a partial order.
For $w \in W$, $R(w)$ denotes the upward closed set generated by w. Note that for every $v \in W$, wRv iff $v \in R(w)$.
A *propositional Kripke model* is a triple $\mathfrak{M} = (W, R, V)$, where (W, R) is a Kripke frame and V is a valuation $V : P \to \mathcal{P}(W)$ such that, for any $p \in P$, $V(p)$ is persistent, *i.e., for all* $w, v \in W$, *if* $w \in V(q)$ *and* $v \in R(w)$ *then* $v \in V(q)$.

- $w \vDash p \Leftrightarrow w \in V(p)$
- $w \nvDash \bot$
- $w \vDash \varphi \wedge \psi \Leftrightarrow w \vDash \varphi$ *and* $w \vDash \psi$
- $w \vDash \varphi \vee \psi \Leftrightarrow w \vDash \varphi$ *or* $w \vDash \psi$
- $w \vDash \varphi \to \psi \Leftrightarrow \forall v((wRv$ *and* $v \vDash \varphi) \Rightarrow v \vDash \psi)$

Defining $\neg\varphi$ as $\varphi \to \bot$, we get $w \vDash \neg\varphi \Leftrightarrow \forall v(wRv \Rightarrow v \nvDash \varphi)$. We write $V(\varphi)$ for $\{w | w \vDash \varphi\}$. We may emphasize a valuation V by writing $\vDash_V$ for $\vDash$, and sometimes we may stress the particular model and write $\vDash_{\mathfrak{M}}$.

Lemma 2.1.

1. (**Persistency**) *If* wRv *and* $w \models \varphi$, *then* $v \models \varphi$
2. (**Locality**) *If* $V \restriction R(w) = V' \restriction R(w)$, *then* $w \vDash_V \varphi$ *iff* $w \vDash_{V'} \varphi$

Proof. Straightforward induction on the structure of φ. □

Theorem 2.2. (**Soundness and Completeness of** IPC)
Given a set of IPC *formulas* Γ, *then* $\Gamma \vdash_{\mathsf{IPC}} \varphi$ *if and only if* φ *is valid in all Kripke models of* Γ *for* IPC.

The proof goes via a canonical model, defined as follows.

Definition 2. The *canonical model* for IPC is the triple $\mathcal{M} = \langle \mathcal{W}, \mathcal{R}, \mathcal{V} \rangle$, where

- $\mathcal{W} := \{\Delta | \Delta$ is a consistent theory with the *disjunction property*: $\forall \varphi, \psi\, (\varphi \vee \psi \in \Delta \Rightarrow \varphi \in \Delta$ or $\psi \in \Delta)\}$,
- $\mathcal{R} := \subseteq$,
- Valuation $\mathcal{V}$: $\Delta \in \mathcal{V}(p) \Leftrightarrow p \in \Delta$.

3 Minimal Logic

3.1 Minimal Logic as MPC_f

The propositional language $\mathcal{L}_f(P)$ consists of the language of IPC^+ to which a propositional variable f representing '*falsum*' is added. Negation $\neg\varphi$ is defined as $\varphi \to f$. The axioms for minimal logic with f are just the axioms of IPC^+.

Definition 3. A *propositional Kripke frame* of MPC_f is a triple $\mathfrak{F} = (W, R, F)$, where W is a non-empty set of possible worlds, R is a partial order and $F \subseteq W$ is an upward closed set, intended to be $\{w \in W \,|\, w \vDash f\}$.

Kripke models are defined in the obvious way, adding the new clause:

$$w \vDash f \Leftrightarrow w \in F.$$

For negation we get $w \vDash \neg\varphi \Leftrightarrow \forall v((wRv \text{ and } v \vDash \varphi) \Rightarrow v \vDash f)$. Observe that the semantics of f is essentially the same as for the other propositional variables. Soundness of these models for MPC_f is straightforward. Completeness was proved before by Odintsov and Rybakov in [9]. The canonical model is the quadruple $\mathcal{M}_f = (\mathcal{W}, \mathcal{R}, \mathcal{F}, \mathcal{V})$ defined as for intuitionistic logic, with the additional definition of $\mathcal{F}$, as the set of theories with the disjunction property containing f:

$$\Delta \in \mathcal{F} \Leftrightarrow f \in \Delta.$$

We drop the condition that the considered theories have to be consistent sets (i.e., we allow theories containing f). The proof is a trivial modification of the one for intuitionistic logic.

The following proposition, known to Johansson, is easy to prove.

Proposition 3.1. *Given an arbitrary formula φ,*

$$\mathsf{MPC}_f \vdash f \leftrightarrow (\neg\varphi \wedge \neg\neg\varphi),$$

where $\neg\varphi$ is expressed as $\varphi \to f$.

It follows that the notion of contradiction expressed by f in MPC_f will be available in $\mathsf{MPC}_\neg$ as $\neg p \wedge \neg\neg p$.

3.2 Minimal Logic as $\mathsf{MPC}_\neg$

In this second framework, the propositional language $\mathcal{L}_\neg(P)$ is just the language of intuitionistic logic, i.e., $\mathcal{L}_\neg = \{\wedge, \vee, \to, \neg\}$. This formulation is axiomatized by the axioms of IPC^+, with the additional axiom

$$(p \to q) \wedge (p \to \neg q) \to \neg p.$$

This axiom expresses that the negation of φ holds, whenever φ leads to a contradiction. It does not give any further indication of *what* a contradiction is. If a formula φ proves $\neg\psi$ and ψ, then $\neg\varphi$ holds. And, the other way around, if $\neg\varphi$ holds, then φ proves a contradiction (namely, φ and $\neg\varphi$).

The considered axiom was explicitly used by Johansson in his original article [5]. However, it was previously introduced by Kolmogorov in the article that has been included in the book "*From Frege to Gödel: a source book in mathematical logic*", a collection by Jean van Heijenoort [12]. Kolmogorov says: "The usual principle of contradiction: *A judgment cannot be true and false*, cannot be formulated in terms of an arbitrary judgment, implication, and negation. Our principle contains something else: namely, from it, together with the first axiom of implication, there follows the principle of *reductio ad absurdum*."

From the axiom the principles of Negative ex Falso and Absorption of Negation, as we will call them, readily follow.

Lemma 3.2.

1. $\mathsf{MPC}_\neg \vdash p \wedge \neg p \to \neg q$,
2. $\mathsf{MPC}_\neg \vdash (p \to \neg p) \to \neg p$.

Proof. (1) is trivial. For the proof of (2), see Proposition 5.2. □

Kripke frames and models are defined as in the case of MPC_f by means of the upward closed set F, using the following clause for negation:

$$w \vDash \neg\varphi \Leftrightarrow \forall v((wRv \text{ and } v \vDash \varphi) \Rightarrow v \in F).$$

Soundness of these models is again a trivial matter. The canonical model for $\mathsf{MPC}_\neg$ is the quadruple $\mathcal{M}_\neg = (\mathcal{W}, \mathcal{R}, \mathcal{F}, \mathcal{V})$ as before, with the new clause for $\mathcal{F}$

$$\Delta \in \mathcal{F} \Leftrightarrow \text{ for some formula } \varphi, \text{ both } \varphi \text{ and } \neg\varphi \text{ are in } \Delta.$$

Again, we leave out the condition that the members of the canonical model have to be consistent sets.

Lemma 3.3. *For every $\Delta \in \mathcal{W}$, $\Delta \in \mathcal{F}$ if and only if $\neg\psi \in \Delta$ for all ψ.*

Proof. The right-to-left direction of the statement is trivial. We focus on the other direction. Assume Δ to be in $\mathcal{F}$, and consider an arbitrary formula ψ. The definition of $\mathcal{F}$ gives us the existence of a contradiction in Δ, i.e., there is a formula φ in Δ, whose negation is also an element of Δ. The formulas φ and $\neg\varphi$ both being logical consequences of Δ, imply $\Delta \vdash \varphi \wedge \neg\varphi$. Lemma 3.2-(1) leads us to $\Delta \vdash \neg\psi$, via an application of modus ponens. The set Δ is a theory, and hence $\neg\psi \in \Delta$. □

Completeness is proved as for intuitionistic logic. It is sufficient to prove that for any theory in the canonical model, membership relation and truth relation coincide. We prove the induction step concerning the negation $\neg\varphi$.

Proof. The left-to-right goes by contraposition. Assume $\neg\varphi \notin \Delta$, for $\Delta \in \mathcal{W}$. This gives us $\Delta \nvdash \neg\varphi$. By Lemma 3.2-(2), $\Delta \vdash (\varphi \to \neg\varphi) \to \neg\varphi$. Thus, $\Delta \nvdash \varphi \to \neg\varphi$. This is equivalent to saying that the formula $\neg\varphi$ is not a logical consequence of the set $\Delta \cup \{\varphi\}$. From the standard Lindenbaum type lemma, we get the existence of a theory $\Gamma \in \mathcal{W}$, extending $\Delta \cup \{\varphi\}$ and not containing $\neg\varphi$. Apply now Lemma 3.3, to get that Γ is not an element of $\mathcal{F}$. Moreover, $\Gamma \vDash \varphi$ by induction hypothesis. The last two results are equivalent to $\Gamma \nvDash \neg\varphi$. The canonical model $\mathcal{M}_\neg$ being persistent, we conclude $\Delta \nvDash \neg\varphi$.

For the right-to-left direction, we proceed directly. Suppose $\neg\varphi \in \Delta$, and consider an arbitrary $\subseteq$-successor Γ of Δ. Assume $\Gamma \vDash \varphi$. The induction hypothesis gives us $\varphi \in \Gamma$. We assumed $\neg\varphi$ to be an element of Δ, and hence, of Γ. Both φ and $\neg\varphi$ being in Γ, we conclude $\Gamma \in \mathcal{F}$. Therefore, $\Delta \vDash \neg\varphi$ as desired. □

4 Basic Subminimal Logic: N

The propositional language coincides with the one for minimal logic with negation $\neg$. The semantics of negation is defined in terms of an auxiliary persistent function N. Different axioms attribute different properties to such a function. The aim of the Kripke semantics is that a negated formula $\neg\varphi$ is true in a world if and only if that world is in the image of $V(\varphi)$ under N.

The basic logic N is axiomatized by $(p \leftrightarrow q) \rightarrow (\neg p \leftrightarrow \neg q)$ (N).

Definition 4. A *propositional Kripke frame* is a triple $\mathfrak{F} = (W, R, N)$, where W is a non-empty set of possible worlds, R is a partial order on W and N is a function $N : \mathcal{U}(W) \rightarrow \mathcal{U}(W)$, where $\mathcal{U}(W)$ is the set of all upward closed subsets of W.

Kripke models are defined in the usual way, by adding a persistent valuation V to the frames. In order to have a correct semantics for N, we require the function N to have the following properties:

P1: $w \in N(U) \Leftrightarrow w \in N(U \cap R(w))$, with $R(w)$ the upward closed set generated by w.

P2: If $w \in N(U)$ then, for all v such that wRv, $v \in N(U)$.

Property **P1** expresses *locality*, i.e., the value of a formula in a world w depends only on the value of such a formula in all worlds accessible from w. The second property, **P2**, expresses *persistence* of negation '$\neg$'. Observe that it is not necessary to explicitly state **P2** as a property, because it already follows from the fact that N maps upward closed sets to upward closed sets. We add it as an explicit requirement because it will be necessary to check it when building particular models. Note also that **P1** expresses the validity of the axiom N, which can therefore be considered the axiom for the basic logic of a unary operator. The truth relation is defined as before, substituting the negation clause, for each formula φ, by

$$w \vDash \neg\varphi \Leftrightarrow w \in N(V(\varphi)).$$

An important unsurprising consequence of **P1** is that generated submodels preserve valuations.

Definition 5. Given a frame $\mathfrak{F} = (W, R, F)$ and a world $w \in W$, the subframe $\mathfrak{F}_w$ *generated by* w is defined on the set of worlds $R(w)$, with the function $N_w(U) = N(U) \cap R(w)$, for every upward closed set U.

Similarly, $\mathfrak{M}_w$ is defined on the basis of the model $\mathfrak{M}$.

Lemma 4.1. *Given $v \in R(w)$, then: $v \vDash_{\mathfrak{M}_w} \varphi$ if and only if $v \vDash_{\mathfrak{M}} \varphi$.*

Proof. We only unfold the induction step of the proof concerning the negation. Indeed, $v \vDash_{\mathfrak{M}_w} \varphi$ is equivalent to $v \in N_w(V_w(\varphi))$, which means $v \in N(V_w(\varphi)) \cap R(w)$, and it is equivalent to $v \in N(V(\varphi) \cap R(w)) \cap R(w)$ (by induction hypothesis). By **P1**, this is equivalent to $v \in N(V(\varphi) \cap R(v)) \cap R(w)$ which, again by **P2**, is just $v \in N(V(\varphi))$, as desired. $\square$

Soundness is a trivial matter. For proving completeness via a canonical model, we need to give an appropriate definition of N in such a model. The canonical model for N is $\mathcal{M}_N = (\mathcal{W}, \mathcal{R}, \mathcal{N}, \mathcal{V})$ is defined as in the minimal logic case, substituting the $\mathcal{F}$ clause with:

$$\mathcal{N}(U) := \{\Delta \in \mathcal{W} \mid \exists \varphi [U \cap \mathcal{R}(\Delta) = [\![\varphi]\!] \cap \mathcal{R}(\Delta) \text{ and } \neg\varphi \in \Delta]\},$$

for every $U \in \mathcal{U}(\mathcal{W})$, and where $[\![\varphi]\!] := \{\Gamma \in \mathcal{W} \mid \varphi \in \Gamma\}$. Again, the condition that the theories in the canonical model need to be consistent is left out. It still remains to be proven that such a canonical model is indeed a model on an N Kripke frame. Hence, we verify $\mathcal{N}$ to have properties **P1** and **P2**.

Lemma 4.2. *$\mathcal{N}$ satisfies* **P1** *and* **P2***.*

Proof. The proof goes as follows.

P1: To show: $\Delta \in \mathcal{N}(U)$ if and only if $\Delta \in \mathcal{N}(U \cap \mathcal{R}(\Delta))$.
Note that $\Delta \in \mathcal{N}(U)$ means $U \cap \mathcal{R}(\Delta) = [\![\varphi]\!] \cap \mathcal{R}(\Delta)$ and $\neg\varphi \in \Delta$, for some φ. This is equivalent to: $(U \cap \mathcal{R}(\Delta)) \cap \mathcal{R}(\Delta) = [\![\varphi]\!] \cap \mathcal{R}(\Delta)$ and $\neg\varphi \in \Delta$ for the same φ, by associativity of $\cap$. The latter means exactly $\Delta \in \mathcal{N}(U \cap \mathcal{R}(\Delta))$, and hence we proved the desired equivalence.

P2: To show: if $\Delta \in \mathcal{N}(U)$ and $\Delta \subseteq \Delta'$ hold, then $\Delta' \in \mathcal{N}(U)$.
Assume the antecedent and note that this means $U \cap \mathcal{R}(\Delta) = [\![\varphi]\!] \cap \mathcal{R}(\Delta)$ and $\neg\varphi \in \Delta$, for some φ. By the inclusion $\Delta \subseteq \Delta'$, we get $\neg\varphi \in \Delta'$. Moreover, $\Delta \subseteq \Delta'$ if and only if $\mathcal{R}(\Delta') = \mathcal{R}(\Delta) \cap \mathcal{R}(\Delta')$. This, by associativity of $\cap$, implies $U \cap \mathcal{R}(\Delta') = [\![\varphi]\!] \cap \mathcal{R}(\Delta')$. Therefore, $\Delta' \in \mathcal{N}(U)$.

$\square$

Theorem 4.3. *The basic logic of unary operator* N *is complete with respect to the class of Kripke models defined above.*

Proof. By contraposition we prove: if $\Gamma \nvdash_{\mathsf{N}} \varphi$, then $\Delta \nvDash \varphi$, for some Δ containing Γ in the canonical model. First we show by induction on φ that, for any Δ in the canonical model, $\Delta \vDash \varphi \Leftrightarrow \varphi \in \Delta$. We only treat the negation case. We need to prove that $\Delta \vDash \neg\varphi \Leftrightarrow \neg\varphi \in \Delta$.

($\Rightarrow$) Assume $\Delta \vDash \neg\varphi$. So, $\Delta \in \mathcal{N}([\![\varphi]\!])$. By definition, there is a formula ψ such that $[\![\psi]\!] \cap \mathcal{R}(\Delta) = [\![\varphi]\!] \cap \mathcal{R}(\Delta)$ and $\neg\psi \in \Delta$. Then, for all extensions Γ of Δ, $\varphi \in \Gamma$ if and only if $\psi \in \Gamma$. As in IPC, $\varphi \leftrightarrow \psi \in \Delta$. By the axiom N, $\neg\varphi \leftrightarrow \neg\psi \in \Delta$ as well. So, it follows that $\neg\varphi \in \Delta$.

($\Leftarrow$) Assume $\neg\varphi \in \Delta$. Then $\exists\psi$ ($[\![\varphi]\!] \cap \mathcal{R}(\Delta) = [\![\psi]\!] \cap \mathcal{R}(\Delta)$ and $\neg\psi \in \Delta$), namely $\psi := \varphi$. Hence, $\Delta \in \mathcal{N}([\![\varphi]\!])$ and, by induction hypothesis, $\Delta \in \mathcal{N}(\mathcal{V}(\varphi))$, and hence $\Delta \vDash \neg\varphi$. □

It is worthwhile remarking here that the axiom N is exactly what is needed to prove the substitution theorem, $\vdash_{\mathsf{N}} (\varphi_1 \leftrightarrow \varphi_2) \rightarrow (\psi[\varphi_1/p] \leftrightarrow \psi[\varphi_2/p])$.

5 Extensions of N

We present some extensions of the basic logic N. Each of the additional axioms will enrich the semantic function N with a different property.

5.1 Axioms of Negation

Consider the following axioms.

1. *Absorption of negation:* $(p \rightarrow \neg p) \rightarrow \neg p$
2. *Contraposition:* $(p \rightarrow q) \rightarrow (\neg q \rightarrow \neg p)$
3. *Negative ex Falso:* $(p \wedge \neg p) \rightarrow \neg q$
4. *Double negation:* $p \rightarrow \neg\neg p$
5. *Distributive law:* $\neg(p \wedge q) \rightarrow (\neg p \vee \neg q)$

The contraposition axiom seems to express a very basic property of negation. Earlier, contraposition has been studied as a rule, instead of as an axiom ([3]). Studying it as an axiom is quite natural: the deduction theorem remains in force, and the axiom N is a theorem in the contraposition system.

In Section 7.3, we will give a semantic proof of the fact that absorption of negation does not follow from contraposition. We already saw in Lemma 3.2-(1) that Negative ex Falso follows from Contraposition.

Remark 5.1. Note that the contraposition instance that we are considering, denoted as CoPC, is the one valid in intuitionistic logic, while the instance $(\neg q \to \neg p) \to (p \to q)$ is not. Moreover, from the latter, the law of explosion follows. Thus, a logic in which $(\neg q \to \neg p) \to (p \to q)$ is accepted is no longer paraconsistent.

In what follows, we denote axiom 1 as An, and axiom 3 as NeF. We prove that minimal logic can also be axiomatized by CoPC + An. We study the logic CoPC, axiomatized by contraposition, and we will see later on that minimal logic and CoPC are closely related systems.

Proposition 5.2. *Minimal logic* $\mathsf{MPC}_\neg$ *can be equivalently axiomatized by* CoPC + An. *In other words,* $\mathsf{MPC} = \mathsf{CoPC} + \text{An}$.

Proof. We first show that $(p \to q) \wedge (p \to \neg q) \to \neg p$ is a theorem of $\mathsf{CoPC} + \text{An}$.

From CoPC we have $(p \to \neg q) \wedge (p \to q) \to (\neg q \to \neg p)$
By transitivity we obtain $(p \to \neg q) \wedge (p \to q) \to (p \to \neg p)$
Because of An we have $(p \to \neg q) \wedge (p \to q) \to \neg p$

Next, we prove CoPC and An in $\mathsf{MPC}_\neg$.

- In MPC we prove CoPC.

	$\vdash_{\mathsf{MPC}}$	$(p \to \neg q) \wedge (p \to q) \to \neg p$
	$\vdash_{\mathsf{MPC}}$	$\neg q \wedge (p \to q) \to \neg p$
By commutativity of $\wedge$ we obtain	$\vdash_{\mathsf{MPC}}$	$(p \to q) \wedge \neg q \to \neg p$
Thus follows	$\vdash_{\mathsf{MPC}}$	$(p \to q) \to (\neg q \to \neg p)$

- In MPC we prove An.

	$\vdash_{\mathsf{MPC}}$	$(p \to \neg q) \wedge (p \to q) \to \neg p$
changing q into p we obtain	$\vdash_{\mathsf{MPC}}$	$(p \to \neg p) \wedge (p \to p) \to \neg p$
	$\vdash_{\mathsf{MPC}}$	$(p \to \neg p) \to \neg p.$

□

In a similar way, it can be shown that minimal logic is equivalent to N+NeF+An.

5.2 Contraposition Logic: CoPC

The Kripke-style semantics for this system is exactly the same as in N. An additional requirement for the function N needs to be specified. Indeed, the semantic function N needs to satisfy **P1**, **P2** and *anti-monotonicity:*

$\mathbf{P}_{\mathsf{CoPC}}$: For all U, $U' \in \mathcal{U}(W)$, if $U \subseteq U'$, then $N(U') \subseteq N(U)$

Such a property is basically the 'functional' equivalent of what CoPC expresses. Contraposition logic is complete with respect to the N Kripke frames satisfying $\mathbf{P}_{\mathsf{CoPC}}$. The proof, via canonical model, requires a different definition of the function $\mathcal{N}$ in the canonical model, as follows:

$$\mathcal{N}(U) := \{\Delta \in \mathcal{W} \mid \forall\varphi : [\![\varphi]\!] \cap \mathcal{R}(\Delta) \subseteq U \text{ implies } \neg\varphi \in \Delta\},$$

for every $U \in \mathcal{U}(\mathcal{W})$.

5.3 Negative ex Falso: NeF

The Kripke semantics is just the same as for the basic logic N, with the additional requirement for the function N

$\mathbf{P}_{\mathsf{NeF}}$: For all U, $U' \in \mathcal{U}(W)$, $U \cap N(U) \subseteq N(U')$

Negative ex falso characterizes exactly the N frames which satisfy $\mathbf{P}_{\mathsf{NeF}}$. The logical system NeF is complete with respect to that class of frames. Similarly to the previous case, we need to define the function $\mathcal{N}$ in the canonical model in such a way that also $\mathbf{P}_{\mathsf{NeF}}$ is satisfied. The definition is the following:

$$\mathcal{N}(U) := \{\Delta \mid \exists\varphi(U \cap \mathcal{R}(\Delta) = [\![\varphi]\!] \cap \mathcal{R}(\Delta) \text{ and } \neg\varphi \in \Delta) \text{ or } \forall\varphi(\neg\varphi \in \Delta)\},$$

for every $U \in \mathcal{U}(\mathcal{W})$.

For both contraposition logic and negative ex falso logic, the finite model property holds. For the proof, theories within an adequate set have been used.

6 Relation between CoPC and Minimal Logic

We begin this section by giving an example of a derivation in CoPC.

Proposition 6.1.

$$\mathsf{CoPC} \vdash \neg\neg\neg p \to \neg p$$

Proof. The following is a Hilbert-style derivation in CoPC.

$$
\begin{array}{lll}
\text{By NeF} & \vdash & (p \wedge \neg p) \to \neg\neg p. \\
\text{by IPC}^+ & \vdash & p \to (\neg p \to \neg\neg p) \\
\text{by CoPC} & \vdash & p \to (\neg\neg\neg p \to \neg\neg p) \\
\text{by IPC}^+ & \vdash & \neg\neg\neg p \to (p \to \neg\neg p) \\
\text{by CoPC} & \vdash & \neg\neg\neg p \to (\neg\neg\neg p \to \neg p) \\
\text{by IPC}^+ & \vdash & \neg\neg\neg p \to \neg p.
\end{array}
$$

□

From this we get that we don't need more than 3 negations in CoPC.[2]

Corollary 6.2. CoPC $\vdash \neg\neg\neg\neg p \leftrightarrow \neg\neg p$.

Proof. The two directions of the proof go as follows.
(⇒) Substitute $\neg p$ for p in Proposition 6.1.
(⇐) Apply CoPC to Proposition 6.1. □

6.1 Translating MPC into CoPC

In the first part of this section we present a translation of minimal logic into contraposition logic. Presenting later a translation of intuitionistic logic into minimal logic, we get a 'chain' of interpretations between contraposition logic and classical logic.

Recall that the "negative" translation from classical logic into intuitionistic logic ensures that IPC has at least the same *expressive power* and *consistency strength* of classical logic [10]. A similar thing happens with Gödel's translation of IPC into the modal logic S4. Here, we establish a similar translation from minimal logic into CoPC.

Consider $\sim \varphi := \varphi \to \neg\varphi$. We define a translation such that $(\neg\varphi)^{\sim} :=\sim \varphi^{\sim}$, while every other connective is left unchanged (i.e., $(\varphi \circ \psi)^{\sim} := \varphi^{\sim} \circ \psi^{\sim}$, for $\circ \in \{\wedge, \vee, \to\}$, and also every atom stays the same).

Theorem 6.3. *The considered translation is sound and truthful, i.e.,*

$$\mathsf{MPC} \vdash \varphi \Leftrightarrow \mathsf{CoPC} \vdash \varphi^{\sim}.$$

Proof. The proof goes by induction on the depth of a derivation. It suffices to check the axioms in which '¬' occurs. First, we need to show that

$$\mathsf{CoPC} \vdash (p \to q) \wedge (p \to \sim q) \to \sim p,$$

i.e., CoPC $\vdash ((p \to q) \wedge (p \to (q \to \neg q))) \to (p \to \neg p)$. Indeed, using only the positive fragment of intuitionistic logic, we get $(p \to \neg q)$ from $(p \to q)$

[2]We thank Lex Hendriks for these observations.

and $p \to (q \to \neg q)$. Now, from $(p \to q)$ and $(p \to \neg q)$ we get $(p \to \neg p)$, just by means of IPC^+ and negative ex falso. Observe that the right-to-left direction follows from the fact that $\mathsf{MPC} \vdash \varphi \leftrightarrow \varphi^\sim$. □

It is worth to be noticed that the considered translation works also for the negative ex falso logic (instead of CoPC), and even for a weakening of NeF, axiomatized by $(p \wedge \neg p) \to (q \to \neg q)$.

6.2 A Translation of Intuitionistic Logic into MPC

In the chain of interpretations

$$\mathsf{CPC} - \mathsf{IPC} - \mathsf{MPC} - \mathsf{CoPC},$$

a translation of intuitionistic logic into minimal logic was missing. We have found one, in a letter from Johansson to Heyting from 1935[3]. In the margin, Heyting scribbled: "My $A \to B$ is Johansson's $A \to B \vee f$". Johansson, on the same track, discovered on the way that the alternative $(A \to f) \vee (B \to f)$ does *not* work. Indeed, if one defines hj for implication $(\varphi \to \psi)^{hj} := \varphi^{hj} \to (\psi^{hj} \vee f)$, and leaves all the other connectives untouched, the result is a translation of intuitionistic logic into minimal logic. Nonetheless, the proof is not quite as straightforward as one might expect. The translation of the axioms is dealt with quite easily, but with modus ponens the following happens: suppose $\mathsf{MPC} \vdash \varphi^{hj}$ and $\mathsf{MPC} \vdash (\varphi \to \psi)^{hj}$. The latter means $\mathsf{MPC} \vdash \varphi^{hj} \to (\psi^{hj} \vee f)$, which leads to $\mathsf{MPC} \vdash \psi^{hj} \vee f$. This is not good enough though, because we need to get an MPC derivation of ψ^{hj}. However, here the so-called *disjunction property* of minimal logic comes to the rescue. Indeed, whenever $\mathsf{MPC} \vdash A \vee B$, we have $\mathsf{MPC} \vdash A$ or $\mathsf{MPC} \vdash B$ ([5]). So, we have a derivation $\mathsf{MPC} \vdash \psi^{hj}$ or $\mathsf{MPC} \vdash f$. Clearly, the latter is not the case, and hence we can conclude $\mathsf{MPC} \vdash \psi^{hj}$ as desired. This argument can be found in one of the letters from Johansson to Heyting already. Moreover, Johansson argued that the considered translation can be extended to first-order logic, by means of $(\forall x \varphi)^{hj} = \forall x(\varphi(x)^{hj} \vee f)$.

7 Linear Frames

In this section we want to analyze the frames of our systems in which the LC-axiom, i.e., $(p \to q) \vee (q \to p)$, is valid. For each logic, the class of frames satisfying the considered formula corresponds to the class of *upwards linear frames.*

[3]This correspondence has been studied with Tim van der Molen.

7.1 Linear Frames in Minimal Logic

In this section we use $n(w)$ to denote $N(R(w))$. The fact that we are dealing with linear frames make our lives easier. The reason why such a class of frames is interesting, is that, in a finite linear frame, every upward closed set is the set of successors of some world w, and hence it is completely determined by its root. Here, we want to emphasize how the shape of the set $n(w)$ in a linear frame of MPC depends on whether the world w makes f true, or not. Indeed:

- If $w \notin F$, $n(w) = F$.
- If $w \in F$, $n(w)$ is the whole set, i.e., $n(w) = W$.

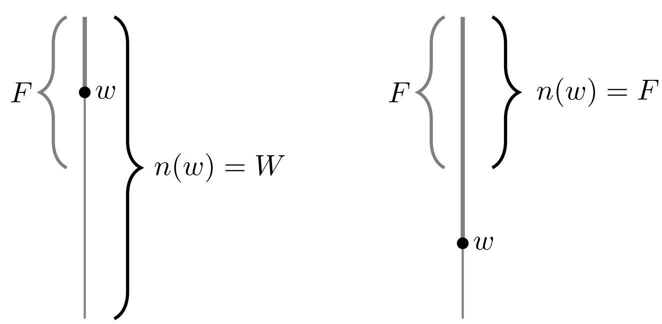

Figure 1: Conditions for Minimal Logic

7.2 Linear Frames in Subminimal Systems

In order to have a picture of the linear frames in the basic logic N, we need to understand how the locality condition gets implemented in this particular case. The condition

$$\mathbf{P1}: \forall w \in W, U \in \mathcal{U}(W) : w \in N(U) \Leftrightarrow w \in N(U \cap R(w)),$$

turns out to be equivalent, in this setting, to:

$$\forall w, v \in W : w \in N(R(v)) \Leftrightarrow w \in N(R(v) \cap R(w)).$$

Hence, we get that if v is a successor of w, i.e., wRv, locality imposes no restrictions, because we get $w \in n(v) \Leftrightarrow w \in n(v)$[4]. On the other hand,

[4]Similarly, there are no restrictions by locality on $N(U)$ for $U = \emptyset$.

if v is a predecessor of w, $w \in n(v)$ if and only if $w \in n(w)$. The set $\{w \in W | w \in n(w)\}$ plays therefore an important role and represents a weakened form of F. We shall denote this set in this section therefore as F. Indeed it has some of the properties of the F of MPC, since in any N model, $w \in F \Leftrightarrow (w \vDash p \Rightarrow w \vDash \neg p)$.

For the case of N, we can then state the conditions as:

- If $w \notin F$, then $n(w) = F$,
- If $w \in F$, then $n(w) \supseteq R(w)$.

The first condition is the same for all the systems between N and MPC. The second condition varies with the strength of the logic. In the case of NeF, the second condition is influenced by the properties of F and the axiom $p \wedge \neg p \rightarrow \neg q$, and becomes:

- If $w \in F$, then $n(w) \supseteq F$.

In the case of CoPC, such a second condition remains in force, together with the condition that: $wRv \Rightarrow n(w) \subseteq n(v) \subseteq N(\emptyset)$.

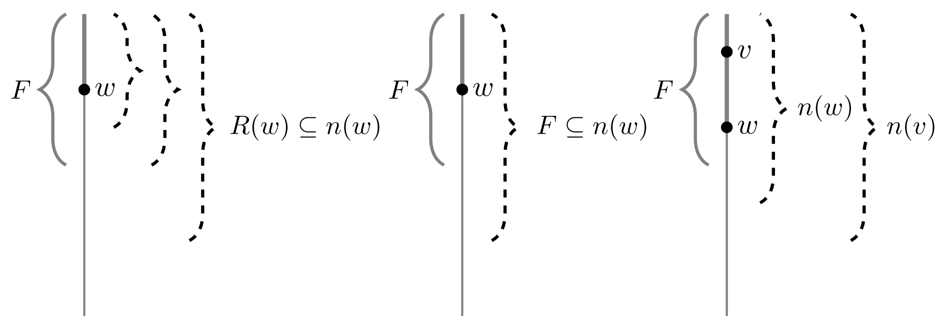

Figure 2: Condition two for N, NeF and CoPC

7.3 Counterexamples

In the last part of this section we give two examples to show how the different axioms we are considering are logically related to each other.

Proposition 7.1. *Absorption of negation* An *is not a theorem in* CoPC.

Proof. (Figure 3) The idea is that we consider a linear finite CoPC frame in which the set F is a proper subset of W and, for every upward closed set U, $N(U) = F$. In this way, by assigning a valuation $V(p) \subseteq F$ for some propositional variable p, we get that every world $v \notin F$ does not force $\neg p$, while it forces the implication $p \to \neg p$. Observe that a frame in which $N(U) = F$ for every U is indeed a CoPC frame. The only thing we need to check is the locality condition, given that the other two properties trivially hold. Also locality is quite trivial, given that $w \in N(U)$ if and only if $w \in F$, which again would be equivalent to $w \in N(U \cap R(w))$. □

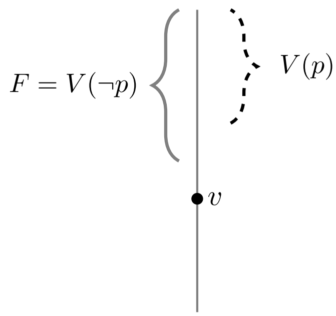

Figure 3: CoPC counterexample to absorption of negation

Proposition 7.2. *Contraposition* CoPC *is not a theorem of* N.

Proof. (Figure 4) For obtaining an N model in which CoPC does not hold, it is enough to consider an arbitrary finite linear frame such that $n(w) = R(w)$ for every world. For the sake of simplicity, let $\bar{w}$ be the greatest world in the frame, and assign a valuation such that $V(p) = \{\bar{w}\}$ and $V(q) = R(v)$, where $v \neq \bar{w}$, for some propositional variables p, q. Indeed, the world v forces the implication $p \to q$. On the other hand though, $\neg q$ is true in v, while $\neg p$ is not. Therefore, CoPC is not valid on the considered frame. Note again that the function N defined as we did is persistent. Moreover, whenever $w \in N(R(v))$ for some v, this means that $R(w) \subseteq R(v)$ and hence, $w \in N(R(v) \cap R(w))$ amounts to $w \in N(R(w)) = n(w)$, which is true by definition. For the other direction, again, saying that $w \in N(R(v) \cap R(w))$ for some v implies that $R(w) \subseteq R(v) \cap R(w)$ which indeed means $R(w) \subseteq R(v)$. The definition on N implies $w \in n(v) = N(R(v))$, as desired. □

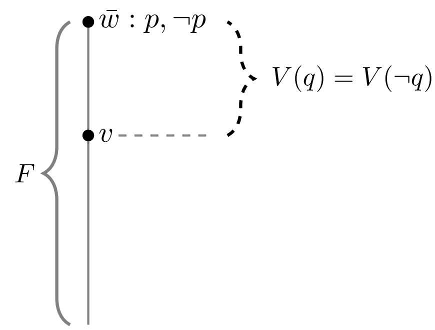

Figure 4: N counterexample to contraposition

8 Conclusions and Further Research

The main purpose of this paper was to explore and analyze minimal logic with negation as a primitive and its subminimal subsystems with a weaker negation. We concentrate mainly on a basic logic N where the negation is just a unary operator without additional properties, and on two of its extensions: contraposition logic and negative ex falso. The semantics of negation is defined in terms of a persistent function N on the set of upward closed sets of a Kripke model. Completeness can be proved by means of canonical models.

We show that Contraposition Logic can interpret minimal logic by means of a sound translation, and complete the chain of translations from contraposition logic to classical logic by presenting a translation of intuitionistic logic into minimal logic appearing in the correspondence between Johansson and Heyting in 1935.

For future work a first step is to make a natural generalization of the canonical models studied in this paper to models allowing the function N that interprets negation to be partial (compare to neighborhood models of modal logic [2, 7]). This produces more natural and general canonical models. A further introduction of generalized and descriptive models seems then indicated. The corresponding algebras for a study of duality are bottomless Heyting algebras (see e.g. [11]).

The above mentioned translations are effective for first order logic as well, and in general there are many interesting questions about first order logic.

It is also already clear that the systems are very suitable for introduction of cut-free sequent systems to prove properties like interpolation.

The study of the models of weak Gödel-Dummett logic which provides a bridge to the work of Franco Montagna can be extended by looking at the behavior of the logics on the models (0,1] and [0,1]. Here also the algebras and the proof theory [8] seem well-worth studying.

Finally, the structure of the lattice of all logics between the basic logic N and minimal logic is intriguing. Certainly it will contain infinitely many logics.

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