Multilateral calculi for free-choice logics

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written by

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Abstract

This dissertation is a proof-theoretical investigation on some variants of free-choice logics from the standpoint of inferential expressivism.

The phenomenon of free-choice inference has been recently studied in the context of team semantics, starting with [Alo22]. Advancements in this tradition have mostly relied on model-theoretical approaches, and, while those have been successful in offering predictions for natural-language phenomena such as free-choice permissions ([Kam13]) and epistemic contradictions ([Yal07]), the development of proof-systems connected to these logics has been slower and not always fully satisfactory. In particular, axiomatizations for logics not admitting the empty set as a licit team have not yet been investigated. This work offers axiomatizations for two such logics $\mathcal{BSL}_m^{\mathbb{V}}$ and $\mathcal{BSL}_m$, in the style of multilateral calculi developed in the context of Inferential expressivism ([IS23]).

After studying some model-theoretical properties, a natural deduction system for $\mathcal{BSL}_m^{\mathbb{V}}$ is outlined, and an interpretation of the logic from the perspective of inferential expressivism is given. In particular, the system is shown to comply with proof-theoretic harmony in the form of local soundness and completeness. As a result, specific constants employs $\mathcal{BSL}_m^{\mathbb{V}}$, to model phenomena as epistemic contradictions and free-choice permissions are shown to qualify as meaning-determining connectives, from the standpoint of an inferentialist theory of meaning. The system is then shown to be sound and complete, and a normal form result together with a weak subformula property is offered. Last, a labelled sequent system for the sub-logic $\mathcal{BSL}_m$ is outlined, together with results of soundness, completeness, and proof-search termination.

"Infinite hopes - and fears - may both be yours. Be sure that, whatever else you get, you will not get justice"
"Are the Gods not just?"
"Oh no, child. What would become of us if they were?"

Contents

Chapter 1

Introduction

This work investigates multilateral proof-systems for state based logics not admitting the empty set as a licit state. Most of the dissertation is devoted to the development of a natural deduction axiomatization for the system "$\mathcal{BSL}_m^{\mathbb{V}}$", which features two distinct kinds of disjunctions, as well as to the study of the model theoretical properties of $\mathcal{BSL}_m^{\mathbb{V}}$ itself. The remainder of the work proves normalization and weak subformula property for our natural deduction systems, and then presents a terminating labelled sequent system for the sub-logic "$\mathcal{BSL}_m$", which features one kind of disjunction only. These systems are multilateral in that they are developed from the standpoint of inferential expressivism, with the calculus manipulating *signed formulae*, which comprise a logical formula together with a specific force marker. State-based logics typically employ a range of non-classical logical constants to successfully represent a wealth of natural language phenomena linked with disjunctive sentences and modal expressions. By providing a suitable axiomatization of the logic, and by showing it complies with a suitable notion of proof-theoretic "harmony", this work aims to show that such constants can be taken as meaning-bearing for our workaday linguistic practice, under the perspective of an inferential expressivist theory of meaning.

In this chapter, we present the historical and conceptual roots of the issues motivating the research of this dissertation. Moreover, we briefly introduce state-based modal logics as well as inferential expressivism, which are the two main methodological matrices behind the present work.

1.1 History and motivations

In the context of natural language, a number of curious phenomena is known to be linked with both disjunctive expression as well as with epistemic modals. The word "curious" here means that those expression exhibit conversational behaviours which are hardly captured by the ordinary meaning of classical logical constants, or by the interaction between connectives which is prescribed by ordinary classical (modal) logic. In [Yal07], Seth Yalcin observed that the following sentences[1] are "odd, contradictory-sounding, and generally unassertable at a context":[2]

[1]In the present chapter, we use "sentence" to refer to *utterance-types* in natural language, as opposed to individual utterance-token, which are specific events in time and space. When introducing our signed calculus in section 2.2, we shall use "sentence" to refer to combinations of a logical formula and a force marker, as those are meant to schematically represent the structure of utterances in natural language (see section 2.2.1 below).

[2]It is generally agreed upon that the default interpretation of the respective modal expression in 1., 2., 3., 4. is epistemic. Clearly, nothing hinges on this claim a long as there is *some* expression articulating epistemic possibility in our language.

1. It is raining and it might not be raining.
2. It is raining and possibly it is not raining.
3. It is not raining and it might be raining.
4. It is not raining and possibly it is raining.

Moreover, Yalcin claims the problem lies not with any specific word employed in the examples above, but with the logical form of epistemic modals *in general* as well as with their interaction with the negation of their non-modal content. In short, the crux of the problem putatively lies with the following configuration of logical constants:

$$\phi \wedge \blacklozenge \neg\phi \qquad \neg\phi \wedge \blacklozenge \phi$$

Where we use "$\blacklozenge$" as the formal counterpart to any natural language expression for epistemic possibility. Expressions of the forms displayed above are sometimes referred to as *Yalcinean sentences* (see e.g., [IS23]). It was observed early on that Yalcinean sentences bear some resemblance to *Moorean sentences* of the kind:

"It is raining and I do not know it is raining"

While Moorean sentences also appear to be in general un-utterable, their defectiveness has historically been deemed pragmatic rather than semantic, based on its cancellability under embedding. In fact:

"Assume it is raining and I do not know it is raining"

is a perfectly well-sounding sentence. Yalcinean sentences, on the contrary, carry over their defectiveness to embedded contexts. In particular, sentences as:

5. Assume it is raining and it might not be raining.
6. It might occur that it either rains or it might not be raining.
7. Either, it is raining and it might not be raining, or it is sunny and it might not be sunny.

All appear just as bad-sounding and unassertable as their cognates in 1-4 do. Accordingly, the problem lying with Yalcinean sentences seems to be not pragmatic, but semantic in nature.
Incurvati and Schlöder ([IS19],[AIS23],[IS23]) offer a general account of Yalcinean sentences[3] along the lines of *inferential expressivism*. The basic tenet of inferential expressivism is that utterances of a speaker in conversation *express* attitudes the speaker themselves hold with respect to certain sentential contents. In turn, different attitudes contribute in distinct ways to the conversation in that they differently manipulate the shared body of information the conversation takes place against. On Incurvati an Schlöder's account, Yalcinean sentences 1.-5. turn out semantically contradictory in that their utterance commits the speaker to mutually incoherent attitudes which cannot be jointly complied with.[4] However, their rendition of 7.'s defectiveness is not semantic, but pragmatic. In this sense, disjunctions of Yalcinean sentences are not contradictory in that they raise mutually incoherent commitments, but they are pragmatically infelicitous in that uttering them breaks defeasible principles usually employed by parties in conversation to supplement other speaker's explicit contributions to the conversational environment.

[3]Different names have been attributed in the literature to embedded Yalcinean sentences based on the operator they embed under. For ease of readability we will employ "Yalcinean sentences" for both 1.-4. and their embedded occurrences.
[4]Various authors from different traditions have argued for a general correspondence between conversational attitudes and rights/obligations, see e.g.,[Bra09], [Mar19], [Kle24].

A claim to the semantic defectiveness of 6. and 7. has been advanced by Mandelkern ([Man19]), stipulating, for 7, the principle that disjunctions of contradictory sentences should themselves turn out contradictory. One glaring issue with this attempt, however, is raised by noticing that disjunctive sentences with *just one* epistemically defective disjunct sound just as bad as 7. itself:

7'. Either it is raining and it might not be, or the sun is shining.

Mandelkern's new principle does not predict 7' to be semantically contradictory. Indeed, on the classical reading of logical disjunction, any disjunctive sentence will commit the speaker to whatever *both* disjuncts individually also commit the speaker to. Thus, if one believes 7' to be contradictory in the same vein as 7, there are reasons not to rest content with Mandelkern's account.
Contrary to the behaviour of classical disjunction, the semantic defectiveness of sentences like 7'. seems to suggest that some disjunctive sentences in natural language commit the speaker to the assertibility of each of the disjuncts individually. A similar behaviour of natural language disjunction has been brought to the fore by Kamp ([Kam13]), in terms of the problem of the so-called "free-choice permissions". In particular, the following pair of inferences appear valid in natural language, whenever disjunctive expressions are embedded under *deontic* modals:

- From "John may go by bus or by boat" infer "John may go by bus";
- From "John may go by bus or by boat" infer "John may go by boat".

A thorough investigation by Zimmermann ([Zim00]) identifies the root of the problem in the default interpretation of natural language disjunctions as conjunctions of *epistemic* possibilities. Consequently, the validity of the *free-choice* inference pattern is extended to *epistemic* modals as well.

- From "It might be rainy or sunny" infer "It might be rainy";
- From "It might be rainy or sunny" infer "It might be sunny".

Furthermore, there is ample evidence for the fact that natural language expressions for epistemic modals should distribute and collect over disjunctions. To witness, natural language seems to license the following pair of inferences:

- From "It might be rainy or sunny" infer "It might be rainy or it might be sunny";
- From "It might be rainy or it might be sunny" infer "It might be rainy or sunny".

All in all, natural language putatively warrants the following family of *free-choice inferences*:

free-choice inferences:

$$\frac{\blacklozenge(\phi \vee \psi)}{\blacklozenge\phi} \qquad \frac{\blacklozenge(\phi \vee \psi)}{\blacklozenge\psi} \qquad \frac{\blacklozenge\phi \vee \blacklozenge\psi}{\blacklozenge\phi} \qquad \frac{\blacklozenge\phi \vee \blacklozenge\psi}{\blacklozenge\psi}$$

The phenomena of free-choice has more recently been investigated through the tools of *state-based (modal) logics*, or, equivalently, *team logics* ([Alo22]), showing that team-based approaches are capable, under the right setting, of successfully predicting a wide variety of natural language phenomena including Yalcinean sentences 1-7' as well as the free-choice inferences given above.

Notwithstanding their predictive success, some issues have risen with respect to team-based analyses of these phenomena. To begin with, most of the work in this tradition has relied on the construction of

models yielding the correct predictions upon calculation of semantic values for suitable formalizations of natural language sentences. At the same time, the development of proof-systems and axiomatization for these logics have been slower and often met with less immediate success in term of elegance and need for *ad hoc* rule stipulations. Though some notable improvement have appeared recently (see e.g., [AAY24],[AHY25]), most proposals up to the present day fail to comply with any suitable criterion of *harmony*. The word "harmony" refers here to a family of criteria devised in the tradition of proof-theoretic semantics to single out meaning-determining logical constants from defective connectives such a Prior's infamous "Tonk". Even if no conclusive agreement has been reached upon what makes a connective harmonious (but see section 1.3.1 below), the lack of a proper axiomatization complying with at least *some* of the candidates for harmony raises the question of whether the team-logicians' analysis conclusively captures the inferential practice it was meant to clarify.
In fact, unlike in the case of ordinary propositional and classical logic, in the tradition of state-based logics the semantic value of a formula is given not by a single valuation function, but by a family of valuations, which constitutes a *team* (or, equivalently, a *state*). On one common interpretation, (see e.g., [Alo22],[AAY24]) a team in a model is meant to represent a *state of information* available for a speaker to assert or reject a certain formula. Accordingly, the intended natural language interpretation of the formal relation of model-theoretic satisfaction adopted in team logics (the dual notions of "support" and "anti-support") is that of *assertibility* and *rejectability* of a certain sentential content by a speaker. To this extent, the ultimate aim of the model-theoretical machinery deployed by the team-logician is to provide information about an alleged speaker's inferential practice of assertion and rejection in the context of ordinary discourse.[5] But how is the interpretation of some formulae onto models relevant for the putative meaning of logical constants appearing in ordinary inferential practice? While many would be ready to accept that interpretation of natural language into suitable models would suffice for this purpose, a long tradition beginning with Michael Dummett ([Dum91], [Dum75a], but see also [Dum00] for a discussion of algebraic vs. proof-theoretical semantics) has defended *use in deductive practice* to be properly captured not by models, but by a suitable proof-system, specifying the use of each connective by means of appropriate rules. Since the requirement of harmony is meant to capture precisely when those rules are meaning-determining for the connectives they define, it seems that the lack of an harmonious axiomatization stands in the way of the team-logicians' intent of providing data about the assertibility and rejectability of logical constant in inferential practice.

In this work, we tackle the issues discussed above in a systematic manner. In chapter 2, we investigate $\mathcal{BSL}_m^{\mathbb{V}}$, a specific representative of state-based logic validating free-choice inferences, and licensing the semantic contradictoriness of all the Yalcinean sentences seen above. We study some model-theoretical properties of $\mathcal{BSL}_m^{\mathbb{V}}$, and we offer an harmonious, sound and complete axiomatization motivating it from the perspective of inferential expressivism. This achieves the three goals of:

1. Expanding the range of phenomena amenable to an inferential-expressivist explanation to free-choice inferences and other patterns specific to *neglect-zero* reasoning (see section 1.2 below);
2. Grounding the team-logician's claim to the relevance of their model theory for ordinary deductive practice and discourse, from the standpoint of an inferentialist theory of meaning;
3. Advancing the axiomatizations of state-based modal logics by laying down an harmonious, and

[5] A slightly different interpretation is given in [CR11]. This interpretation does, however, preserve the intent of representing a speaker's conversational practice. We shall come back to issues tied to inquisitive disjunction in section 2.2.2.

arguably less *ad hoc* system for $\mathcal{BSL}_m^{\mathbb{V}}$, a system so far not properly axiomatized.

In chapter 3, we put the harmony of our system to practice, by showing it leads to a normal form theorem, and to a weak subformula property theorem. Last, in chapter 4 we outline a labelled sequent system for the sub-logic $\mathcal{BSL}_m$, and show soundness, completeness, and termination. This further contributes to goal 3. above, opening the path to the study of axiomatization of state-based modal logic without the empty team as an admissible state.
Overall, this work presents a uniform account of Yalcinean sentences as semantic contradictions, rather than pragmatic failures. This is done by expanding [IS23]'s approach to the domain of neglect-zero reasoning, and to free choice inferences in particular. In doing this, we tie together proof-theoretic and model-theoretic approaches to the issue Yalcinean sentences, and we offer a proof-theoretic, inferential-expressivist, treatment of free-choice phenomena. In the next two subsections, we go over the history and development of both state-based modal logics and inferential expressivism, and we offer some remark on the criterion of harmony mentioned above.

1.2 State-based logics

State-based modal logics are a family of logics born out of Väänänen's seminal work on *dependence logics* ([Vää07]), which has been widely utilized to model general phenomena of dependence in mathematics as well as in real-life situations. As anticipated already, the hallmark of state-based modal logic is the semantic evaluation of formulae not against one single valuation function, but against a family thereof, which is called a "team" (hence the equivalent nomenclature of "team logics"); whenever a certain formula is true in a given team it is said that the team "supports" the formula.

The application of team logics to language-specific phenomena began with [Cia08], who exploited the expressivity of their semantics to bring substantial advancements to the study of the logical structure of questions. The role of state-based modal logics in the study of free-choice inferences as well as Yalcinean sentences begins with Aloni ([Alo22]), who enriches the semantic structure of the logics by complementing the support-conditions of formulas over teams with dual "anti-support" conditions; in general, teams are meant to represent a speaker's information state, and whereas the support conditions determine which formulas are *assertible* at that state, the anti-support conditions dually determine what the speaker is entitled to *reject* or *deny* at a state. In the chapters to come, we will exploit this feature to align our logic of choice $\mathcal{BSL}_m^{\mathbb{V}}$ with the attitudes of assertion and rejection as treated in the context of inferential expressivism.
Under Aloni's reading, the origin of free-choice phenomena lies with what she christens "neglect-zero" tendency in human cognition, meaning that human speakers tend to cognitively disregard, when reconstructing the meaning of a sentence, empty scenarios, or situations that would render the sentence true in a trivial sense. To model free-choice phenomena, therefore, team-logicians often choose to enrich the logical vocabulary with a special propositional atom NE which is satisfied only by nonempty states or teams. As much as NE proves a useful device in a model-theoretical setting, from the perspective of an inferentialist approach to meaning it not entirely clear how the NE atom should be treated. NE does not translate to any specific natural-language expression, and allowing for it to be embedded under the ordinary logical constant would fair poorly with the desirable goal of faithfully representing in our system all and only inferences which have a manifest counterpart in conversational practice (see e.g. [Dum75b]). Given the aim of this work, we therefore do without the usage of an

explicit non-emptiness atom NE in the object-language, and simply stipulate that no team in our logics of choice $\mathcal{BSL}_m$ and $\mathcal{BSL}_m^{\mathbb{V}}$ is allowed to be empty. The effects of this choice are investigated in chapter 2.

Whereas the model theory underlying most state based logics is well known (see [Hel+14], [YV17] and [AK25] for the case of Aloni's original system), the investigation of proof systems for those logics has advanced at a slower rate, with significant progress made only recently in e.g., [AHY25],[AAY24]. In terms of natural deduction,[6] most systems up to today still make use of several *ad-hoc* rules to mimic important semantic properties in the calculus, and the proof-theoretical properties of the calculi themselves are seldom investigated. In our study of $\mathcal{BSL}_m^{\mathbb{V}}$, we offer a system that spares the use of *ad hoc* rules, and autonomously derives the relevant semantical properties of the models. Furthermore, the exclusion of the empty team from the semantics causes the *empty team property* (see definition 15) to systematically fail in $\mathcal{BSL}_m$ as well as $\mathcal{BSL}_m^{\mathbb{V}}$. From a proof-theoretical standpoint, this presents unique challenges, as it entails that our logic are no longer extensions of classical propositional modal logic. To the best of our knowledge, this is the first work where an axiomatization is offered for a team logic not admitting the empty set as a licit team.

We now give turn to a brief overview of proof-theoretic semantics, and, more specifically, of inferential expressivism, which constitutes the second main root of the present work.

1.3 Inferential expressivism

The project of a systematic theory of meaning carrying the flag of a "meaning-as-use" conception first emerged in Michael Dummett's *The logical basis of metaphysics* ([Dum91]). There, Dummett suggest to codify the informal notion of "use" by means of a *two-aspect model of meaning*: the use of an expression φ is rendered by fixing the conditions under which φ is appropriately assertible (the so called "I-principles"), and the further expressions any speaker having uttered φ is entitled to assert by virtue of the fact that they have asserted φ ("E-principles"). Roughly speaking, whereas I-principles are meant to capture the sufficient grounds for the assertion of a certain sentence, the E-principles fix just what the assertion of a certain sentence allows a speaker to do in the context of conversational practice. Since most of Dummett's efforts have been directed to singling out the meaning of logical constants, it has become custom, with and after Dummett himself, to specify I-principles and E-principles for a certain expression by means of Introduction and Elimination rules in the style of Natural deduction calculi.[7] A simple yet illustrative case is provided by the rules governing the use of conjunction (where we indicate alternative conclusions with a "/"):

$$\frac{A \qquad B}{A \wedge B}\,{\scriptstyle \wedge I} \qquad\qquad \frac{A \wedge B}{A / B}\,{\scriptstyle \wedge E}$$

stating that a speaker is entitled to the assertion of a conjunction whenever they are entitled to the assertion of both conjuncts, and likewise, the assertion of a conjunction gives the utterer the right of asserting each conjunct individually. Since meaning is fixed by use, and use is fixed by Introduction-Elimination rules, the display above exhaustively determine the meaning (and thus the

[6] For a proof-theoretical study of state-based logics under a sequent settings see e.g., [LS25], [Bar+26], [AIY25].

[7] For systematic developments of use-based theories of meaning relying on other devices such as sequent calculi see e.g., [Ten10], [Res05], [Rea00]. For the majority of this work, we will only be concerned with the more traditional style of natural deduction, when we turn to labelled sequents in chapter 4, we shall no longer be concerned with issues of harmony specifically.

role in discursive or inferential practice) borne by the logical constant "$\wedge$".
Things being this way, it seems that all need to be done to specify the meaning of a logical symbol is to provide it with an arbitrary pair of introduction and elimination rules. However, such optimistic expectations about the meaning of logical constants were vanquished by Prior's infamous "Tonk" example ([Pri60]):[8]

$$\frac{A}{A\ Tonk\ B}\ {\scriptstyle Tonk\ I} \qquad\qquad \frac{A\ Tonk\ B}{B}\ {\scriptstyle Tonk\ E}$$

It is easy to see that any language featuring $Tonk$ among its connectives will turn to be an (inferentially) trivial one, as $Tonk$ allows any sentence of the language to be lawfully derived from any other simply by introducing and eliminating $Tonk$, while choosing the desired formula for B above.
Prior's example brings highlight the insufficiency of the naively optimistic conception of meaning outlined before. Unless one is ready to admit trivializing connectives such as $Tonk$ as meaningful, it is clear that further restrictions are required on what kind of I-principles and E-principles qualify a logical constant as part of a non-defective inferential practice. The quest for proof-theoretic *harmony* is precisely the quest for singling out non-defective logical constants by providing necessary and sufficient conditions for their identifications. Before dealing with harmony extensively, however, we shall briefly recount the road from Dummett's proposal to inferential expressivism.

In the works of Dummett and Prawitz (see e.g.,[Dum75a], [Pra77]), the project of a general inferential, and thus, in light of the above, *proof-theoretical* semantics was weaved together with the advocacy for the adoption of intuitionistic rather than classical logic. The crux of the problem lied with the classical rules for negation, where the presence of the classical rule of *reductio ad absurdum* broke the criteria for harmony proposed by Dummett and Prawitz themselves. As a consequence, they argued that a non-defective inferential practice would not sanction the use of *reductio ad absurdum*, vouching for the adoption of intuitionistic logic instead.
It was not until Ian Rumfitt's [Rum00] that the project of a proof-theoretic semantics freed itself from the aspiration to logical revisionism. Rumfitt's starting point was a proposal by Smiley, who in [Smi96] argued that the traditional approach to specifying the meaning of logical constants based on the model-theoretic relation of entailment ("$\models$") left the putative meaning of classical connective underspecified, as there was no intelligible way of excluding unintended interpretations which systematically left the "$\models$" relation unaltered. Smiley's suggestion was to enrich our usual deductive machinery by considering rules for both asserted and rejected constants, and proving that if a calculus is required to be sound and complete with respect to both these kinds of rules, the underspecification problems mentioned above do not arise. Rumfitt expanded on Smiley's insight by introducing a classical calculus for *signed formulae*, where the symbol "+" would signify a formula is assertible, and the symbol "-" would signify a formula to be rejectable. Most importantly, Rumfitt showed his *bilateral* calculus to posses the same derivability relation as classical propositional logic, while he employed new rules for negation which now complied with Dummett's requirement for harmonious logical constants. Proof-theoretical *bilateralism*, born out of Smiley and Rumfitt's work, thus redeemed classical connectives as harmonious, and the playing field between classicists and logical revisionists was at last evened.
From a conceptual standpoint, the central realization behind proof-theoretical bilateralism was that

[8] A number of "Tonkish" connectives has sprung out from Prior's original example, usually as counterexamples to the sufficiency of this or that conception of harmony to fully determine meaningful logical constants. See [VS23], [Sim25] for some examples.

different attitudes a speaker may entertain with respect to sentential contents are not logically immaterial, a position famously popularized by Frege ([Fre19], [Fre79]). In particular, even if the global consequence relation of a logic may not be sensitive to the embedding of different attitudes of the speaker within the syntax of the object-language, their presence is still important for the upholding of more fine-grained properties, which are local in the sense of depending on specific rule-patterns exhibited in inference. Accordingly, bilateralists endorse as their central tenet the equal considerations of the distinct attitudes of *assertion* and *rejection* as symmetrical, primitive logical attitudes, realized in conversation by their homonymous speech acts:

> Affirmative answers [...] enjoy no priority over negative ones. The acts of answering a propositional question affirmatively and of answering it negatively - the acts of accepting its content and of rejecting it - are conceived to be on all fours. ([Rum00] p. 797)

An immediate consequence of this is that Dummett's two-aspect theory of meaning must now be extended so as to cover rejection of logical constants as well: to specify the use of an expression φ is to specify the conditions under which φ is appropriately assertible *and* the conditions under which it is appropriately rejectable. Likewise, to fix what expression a speaker is entitled to utter in virtue of having uttered φ now includes both the expressions they are licensed to reject by having rejects φ as well as the expressions they are entitled to assert by having asserted φ.
Another important step taken by bilateralists was to recognize the existence of *linguistical realizations of the speaker's attitudes*, which may feature as words or terms and embed in subsentential position. To put it simply, speaker may refer to the attitudes they themselves entertain in speech, and they can do so by resorting to specific words or terms which appear as components of the content of an uttered sentence. The immediate implication of this is that to grasp the meaning of a sentence in conversation now involves the grasp not only of the inferential possibilities borne by the words in terms of assertion, but a grasp of the different attitudes entertained by the speaker and their interaction with the subsentential components as well. To witness, let us briefly consider Rumfitt's new rules for the negation operator:

$$\frac{{}^{+}A}{{}^{-}\neg A} \qquad\qquad \frac{{}^{-}A}{{}^{+}\neg A}$$

Where double lines indicate interderivability, introduction rules go top-to-bot, and elimination rules bot-to-top. These rule define the inferential profile (and thus the meaning) of negation as embedding an act of rejection of A within an utterance with assertive force. Likewise, negation embeds an act of A-assertion within an utterance of rejective force. To this extent, negation serves as the linguistical realization of a rejection act when occurring in an assertive sentence, and as the linguistical realization of an assertion when occurring in a rejective sentence. We shall encounter some more cases of linguistical realizations in the chapters to come.

As much as the bilateralist programme was met with success, the original Dummettian goal of a *generalized* inferential, proof-theoretical semantics, beyond just the scope of logical constants was never brought within reach. In fact, while it is rather natural to single out the inferential profile of logical connectives, sailing from this safe docks into the seas of language as a whole has traditionally represented a substantial hurdle for the inferentialist programme, one showing little promise of being overcome in a convincing manner:

if you want an explicit theory of how some particular linguistic construction contributes to the meanings of sentences in which it occurs, the inferentialist is unlikely to have one. ([Wil10] pp. 22, 23.)

Inferential expressivism takes a substantial step towards the solution of this problem. The programme of inferential expressivism was initiated in [IS23] as a refinement of the bilateralist programme outlined above. For once, inferential expressivism renders explicit the *expressivist* commitments already implicit in the bilateral tradition: if the meaning of a term is determined by its specific inferential profile, as captured by harmonious introduction and elimination rules, such profile is now (whenever appropriate) explicitly specified in terms of the different attitudes the term commits speaker to entertaining.
In particular, whereas bilateralism contemplates two distinct stances of assertion and rejection the speaker may entertain with respect to a content, inferential expressivism is *multilateral* in that it admits a possibly open ended number of attitudes which a speaker may express. Each attitude comes with a specific number of expressions acting as its own linguistic realizations, as well as suitable structural rules coordinating the given attitude with the other available ones. This strategy allows for the expansion of the inferentialist programme far beyond the scope of logical constants. A simple example is given by expressions of moral praise (e.g., "Good") or blemish (e.g., "Bad"), which obtain now a definite inferential profile as linguistic realizations of approval-disapproval (here marked with ☺, ☹ following [IS23]):

$$\frac{\smile A}{\overline{{}^{+}Good(A)}} \qquad \frac{\frown A}{\overline{{}^{+}Bad(A)}}$$

For our purposes, the most relevant new attitudes studied by inferential expressivists are those of *weak assertion* ("$\oplus$") and *weak rejection* ("$\ominus$"), which stand for the speaker's "refraining from rejecting" and "refraining from asserting" a certain content in conversation. While we will deal with the specifics of these attitudes further down the line (section 2.2.1), it is worth noting now that the natural language expressions "might" and "might not" are typically taken by inferential expressivists to be linguistic realizations of weak assertion and rejections respectively. Weak assertion and rejection thus play a crucial role in delivering the right predictions as well as meaning-explanations, for Yalcinean sentences, and we'll make extensive use of them in the chapters to come.

One more feature distinguishing inferential expressivism from most inference-based theories of meaning is their proposed reading of the meaning of "inference" itself. On one commonly endorsed view an *inference* is an operation (or a practice[9]) of passing from a collection of sentences called *premisses* to (typically) one further sentence named *conclusion*. Schematically, with premisses an conclusion marked by their initial letters, we signal an inference step with an horizontal line:

$$\frac{P_1 \qquad \ldots \qquad P_n}{C}$$

Clearly, not every juxtaposition of premisses and conclusion would classify as an inference,[10] and

[9]The notion of inference as an operation of the mind is already found in the Scholastics ([AN96]), together with judgment, ordering, and apprehension. From a linguistic standpoint, such operation becomes explicit as the practice of sequential utterance of premisses and conclusion.

[10]The issue of what constitutes a lawful inference is not reducible to the related problem of harmony. In particular, the latter purports to find necessary criteria for the meaningfulness of logical constants, often in terms of certain features if their introduction and elimination rules. As not all inferences concern logical constants, it is clear that harmony is a much more restricted problem than the nature of inference itself. Further, whenever the choice is made to define harmony via features of introduction-elimination rules, those rules are typically already understood to be inferences themselves. In this sense, the problem of harmony is logically posterior to the problem of nature of inference.

historically, the requirement has been advanced that inferences preserve *evidence* from premisses to conclusions:

> For it [viz. the practice of deductive inference] to be legitimate, the process of recognising the premisses as true must already have accomplished whatever is needed for the recognition of the truth of the conclusion ([Dum74]. p. 297)

On the basis of this requirement, Dickie ([Dic10]) has argued that bilateral inferentialist theories of meaning could not provide an intelligible criterion for "inference" without falling back into non-harmonious rules for negation. In particular, Dickie notices that rejection is "messy", in that natural language licenses rejecting a sentence both on the basis of evidence for its falsity, and on the basis of presupposition failures, or lack of suitable semantic data:

1. A: "Homer wrote 'the Republic'"
   B: "No! Homer existed, but he did not write 'the Republic'"
2. A: "Homer wrote 'the Iliad'"
   B: "No! In fact, Homer did not exist"
3. A: "Homer is a permitted author"
   B: "No! The committee is yet to deliberate on the matter"

Item 1 constitutes a rejection based on evidence for the falsity of "Homer wrote the Republic", in that it is backed by evidence of Homer existing, and by evidence of the fact that he did *not* write "the Republic". On the contrary, the rejection in 2 is licensed by failure of presupposition, and the rejection in 3 by the fact that truth conditions for answers to the posed questions are currently unknown. Evidence licensing rejections in natural language is inherently ambiguous, and so, Dickie argues, there is no clear sense in which a rejective sentence can be said to have standard or canonical evidence conditions the same way as assertive sentences do. As a consequence, Bilateral proof-systems cannot plainly rely on an evidence-preservation notion of inference, and are thus called to provide a different characterization.

The answers in 2, 3 above constitute the kind of data motivating the addition of *weak* rejections (assertions) in the context of Inferential expressivism. The "no!" therein expresses not rejection proper, but as a refusal, on part of the interlocutor, to allow the first speaker's assertion of e.g "Homer wrote the Iliad". Accordingly, inferential expressivist also accept that inference licensed by weak assertions or rejections will not be evidence-preserving, and are thus compelled by Dickie's challenge to explain inferences in some other terms. Partially inspired by [Bra94][11], inferential expressivist take inference to be a practice preserving not evidence, but *commitment*. On this view, the use of different speech acts in conversation commits the speaker to entertaining certain attitudes (we'll see the details of this in section 2.2.1), and a move from premisses to conclusion constitutes an inference step whenever a speaker committed to the attitudes jointly expressed by the premisses also ought to be committed to the attitude expressed in the conclusion.

One central feature of the notion of commitment is its *virtual* character. Commitments ultimately constitute rights and obligations, and to fulfil an obligation is to perform a certain action not at any time but precisely *when the issue is brought to relevance.* Accordingly, for a speaker to be committed

[11]According to Brandom, the notion of good inferential practice is prior to that of truth on an explanatory level. The most basic notion of a "good" inference practice, is one where parties explicitly fulfil the duties their utterances implicitly commit them to.

to certain attitudes does not require them to express those attitudes at any time, but precisely when their obligation is brought to salience:

> Someone expressing an attitude need not immediately draw all the associated inferences; in fact, they cannot do so, due to cognitive limitations. Nonetheless [...] if it is pointed out to them that from an attitude they have expressed some further attitude expression follows, they must express this further attitude or admit to a mistake. ([IS23] pp. 71, 72.)

The turn to commitment thus allows Inferential expressivists to answer Dickie's challenge, as well as solving neighboring problems such as logical omniscience. While referring the reader to [IS23] for further details, in this last introductory section we give a brief overview of the different conceptions of Harmony on the market and justify our choice of *local soundness and completeness*.

1.3.1 Harmony

The term "harmony" was introduced by M. Dummett to mark the obtaining of a non-defective linguistic practice, both at the level of a language *tout court*, and at the level of the introduction and elimination principles codifying the use of individual expressions or logical constants. The former, language-wide, conception went under the name of "global harmony", but it was the latter, "local harmony", which proved historically more important in the context of Proof-theoretic semantics. In the following, we shall concern ourselves exclusively with local harmony, namely, harmony at the level of individual logical constants.

A criterion of local harmony purports to identify precisely the necessary conditions for the meaningfulness of a logical connective, and, consequently, also to rule out Tonk-like connectives as inferentially meaningless.

From an historical standpoint three main reasons were advanced for the alleged infelicity of Prior's *Tonk*. For once, Nuel Belnap ([Bel18]) argued the problem with $Tonk$ lied in that it provides a non-conservative extension of any non-trivial logic it could be added to.[12] Therefore, a connective is harmonious whenever its addition to a given logic results in a conservative extension. Belnap's proposal is peculiar in that it does not treat harmony as a property of connectives, but rather conceives it as a property of a connective together with a specific background logic. Even if Belnap's insight found some followers ([Dic16]), most of the debate around $Tonk$ agreed in identifying the issue of harmony to be one of *balance* between introduction and elimination rules. This means that all and only that should be inferred from a logical constant is what could also have been inferred from the grounds required for asserting the constant itself. This analysis is enshrined in the so-called "inversion principle":[13]

> In eliminating a symbol, the formula, whose terminal symbol we are dealing with, may be used only in the sense afforded it by the introduction of that symbol ([Gen64] p. 295)

The leading idea behind endorsing the inversion principle is that inference associated with logical constants should bring about no new "atomic" fact about the world ([Ste11]). Logic is thus *innocent* in that there is no primitive worldly fact whose truth is concealed behind the performance of specific logical deductions. According to this conception, $Tonk$ in not a logical constant because, by eliminating

[12] We say a logic $\mathcal{L}'$ is a conservative extension of a logic $\mathcal{L}$ if the every sentence-forming operator of $\mathcal{L}$ is also a sentence-forming operator of $\mathcal{L}'$, and the two logics share the same derivability relation. We say a logic is trivial if its derivability relation has no underivable sentences

[13] The terminology comes from Lorentzen [Lor13].

it, a speaker is allowed to licitly conclude sentences they would not have been able to conclude *without* inferring them from $Tonk$.
Although the general soundness of the inversion principle is widely agreed upon, there is no unanimity on the details of its concrete implementation. Defenders of the *General elimination* harmony, for example,argue that the inversion principle is best enforced by requiring all elimination rules of the system under study to take a specific form, the form of "general rules", which mimicks the specification of connectives as inductive types (see [Von01] for details).
On the contrary, supporters of the *Local intrinsic* conception of harmony argue that the notion is best cashed out in terms of local reduction and expansion procedures involving the constant at stake. In particular, a constant is said to be harmonious when it is both *locally sound* as well as *locally complete*. The notion of local soundness was already enforced by Dummett, as a mean to ensure the elimination rule for a given constant are not *too strong*, i.e., did not allow for deriving sentences which could not be derived without eliminating the constant itself. Letting # stand for an arbitrary connective in a logic $\mathcal{L}$ we thus say:

Definition 1 (**Local soundness**). A logical constant # is *locally sound* if and only if every instance of #-introduction followed immediately by an instance of #-elimination can be contracted into the derivation of some premiss of $\#I$.

Using calligraphic letters for derivations, we show the virtuous example of classical "$\wedge$", below, together with the vicious one of $Tonk$, which fails to meet this requirement.

$$\dfrac{\dfrac{\begin{matrix}\mathcal{A}\\ A\end{matrix} \quad \begin{matrix}\mathcal{B}\\ B\end{matrix}}{A \wedge B}\,\wedge I}{\begin{matrix}A\\ \mathcal{C}\end{matrix}}\,\wedge E \;\rightsquigarrow\; \begin{matrix}\mathcal{A}\\ A\\ \mathcal{C}\end{matrix} \qquad\qquad \dfrac{\dfrac{\begin{matrix}\mathcal{A}\\ A\end{matrix}}{A\,Tonk\,B}\,Tonk\ I}{\begin{matrix}B\\ \mathcal{B}\end{matrix}}\,Tonk\ E \;\rightsquigarrow\; ?$$

To avoid the emergence of Tonk variants, Pfenning and Davies ([PD01]) propose to complement the requirement above with the expansion procedure given by *local completeness*:

Definition 2 (**Local completeness**). A logical constant # is *locally complete* if and only if every instance of a formula where # appears as the main connective can be expanded into a derivation comprising an instance of $\#E$ and an instance of all the #-introduction rules, each concluding the original formula.

Converse to the case of local soundness, local completeness is meant to prevent an elimination rule being *too weak*: everything that can be inferred from a sentence should be also inferrable by eliminating the main connective of that same sentence. One easy example is given again by classical conjunction:

$$\begin{matrix}\mathcal{A}\\ A \wedge B\\ \mathcal{B}\end{matrix} \;\rightsquigarrow\; \dfrac{\dfrac{\begin{matrix}\mathcal{A}\\ A \wedge B\end{matrix}}{A}\,\wedge E \qquad \dfrac{\begin{matrix}\mathcal{A}\\ A \wedge B\end{matrix}}{B}\,\wedge E}{\begin{matrix}A \wedge B\\ \mathcal{B}\end{matrix}}\,\wedge I$$

Local soundness and completeness jointly ensure that elimination rules are neither too strong nor too weak with respect to their introduction counterparts, thus ensuring the harmony of the connective at stake.
At last, a further conception of harmony is developed by Tennant, who characterize this notion as the obtaining of a certain *deductive equilibrium* between the introduction and elimination rules (see e.g.,

[Ten20]).

As most of Inferential expressivism has been investigated on the backdrop of the local intrinsic notion of Harmony, in the following we will identify harmony with the joint properties of local soundness and completeness as well. It is interesting to point out that the specification of the inversion principle has led to the formulation of some additional criteria specifically tied with logical bilateralism ([VS23], [Kur21]). Furthermore, a brand new conception, aimed balancing the derivations of asserted and rejected versions of the same formula, has recently been proposed in [Sim25]. Although those advancements are certainly compelling, we shall not consider them in the context of this work. For once, the contribution just mentioned deal with harmony in a bilateral context; the multilateral framework of inferential expressivism comes with its own specific challenges in terms of restrictions and rule-admissibility. It is not always the case that results from the bilateral setting can be transferred plainly to the multilateral one, consequently, there is no guarantee that bilateral harmony criteria should be enforced for multilateral system like ours. Secondly, most of the proof-techniques involved in applying the new conceptions make abundant use of the bilateral rule of *Smileian reductio*:

$$\dfrac{\begin{array}{c}[{}^{+}A]\\ \mathcal{D}\\ \bot\end{array}}{{}^{-}A}\ {}^{+}SR \qquad\qquad \dfrac{\begin{array}{c}[{}^{-}A]\\ \mathcal{D}\\ \bot\end{array}}{{}^{+}A}\ {}^{-}SR$$

Rules of *reductio* are widely endorsed in bilateral logics (starting from [Rum00]), and suitable variations of them are often endorsed in multilateral contexts too (see e.g. [IS22] and [IS23][14]).
However, as argued in [AIS23], those who take Yalcinean sentences to be semantically contradictory should also be ready to give up the unrestricted validity of SR rules in natural language, for those two together lead to a modal collapse in the presence of Rumfitt's rules for negation, effectively licensing the inference of ${}^{+}A$ from ${}^{+} \blacklozenge\ A$:

$$\dfrac{\dfrac{\dfrac{\dfrac{[{}^{+}\neg A] \qquad {}^{+}\blacklozenge\ A}{\neg A \wedge \blacklozenge A}\ {}^{+}\wedge I}{\bot}\ \text{Y.S.}}{{}^{-}\neg A}\ {}^{+}SR}{{}^{+}A}\ {}^{-}\neg E$$

As Yalcinean sentences are central to the system studied in this dissertation, the calculi here presented shall not warrant the rules of *reductio*. Consequently, it both remains unclear whether the novel conceptions of harmony can reasonably be enforced in our systems, as well as whether they should be deemed appropriate when modeling natural language inferences.

We now turn to the study of our system $\mathcal{BSL}_m^{\mathbb{V}}$.

[14] While endorsing rules of *reductio*, Incurvati and Schlöder employ specific restrictions to prevent the modal collapse outlined above from occurring. Those restrictions, however, interact poorly with free-choice phenomena as they still license forms of *reversal* (the meta-rule $\dfrac{{}^{+}A}{{}^{+}B} \Leftrightarrow \dfrac{{}^{-}B}{{}^{-}A}$, see section 2.1.4.1 below). With free-choice in the picture, reversal runs the risk of allowing spurious inferences such as, $\dfrac{{}^{+}\blacklozenge\ A \vee \blacklozenge B}{{}^{+}\blacklozenge\ B} \Rightarrow \dfrac{{}^{+}\neg \blacklozenge\ B}{{}^{+}\neg \blacklozenge\ A \wedge \neg \blacklozenge\ B}$. Different solutions are thus called for in the present case.

Chapter 2

Natural deduction for $\mathcal{BSL}_m^{\mathbb{V}}$

2.1 Syntax, semantics, and basic properties

In this section, we introduce the syntax and semantics of $\mathcal{BSL}_m^{\mathbb{V}}$. We then show that many basic properties of state-based modal logics also hold for the logics of our interest.

2.1.1 Syntax

Definition 3 (Syntax). Fix a countable set of propositional variables Prop. The set of formulae of the Bilateral State Logic (enriched with "might" and global disjunction "$\mathbb{V}$") $\mathcal{BSL}_m^{\mathbb{V}}$ is generated by the following grammar, with $p_i \in \mathsf{Prop}$:

$$p_i \mid \bot \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \mathbb{V} \psi \mid \blacklozenge \phi$$

In the following, we will use Greek letters as variables ranging over formulae of $\mathcal{BSL}_m^{\mathbb{V}}$, and we will use the letter p together with an alphabetical subscript, as a variable ranging over propositional atoms in Prop. We will use the term "literal" to denote either a propositional atom, or a propositional atom prefixed with negation. Last, we will refer to the connective "$\vee$" as "tensor disjunction" or "split disjunction", and to the connective "$\mathbb{V}$" as "global disjunction" or "inquisitive disjunction". The unary connective "$\blacklozenge$" is referred to as "might".

2.1.2 Semantics

The semantics for the logics above is given through discrete Kripke models:

Definition 4 (Models). Let $\mathsf{N} \subseteq \mathsf{Prop}$, a discrete Kripke model $\mathfrak{M}$ over N is a triple $\langle W, Id_W, V\rangle$ where:

1. W is a nonempty set;
2. $Id_W : W \to W$ is the identity map sending every element of W to itself;
3. $V : \mathsf{N} \to \mathcal{P}(W)$ is the valuation function.

Elements of W are referred to as "points", or "worlds". Given a model $\mathfrak{M} = \langle W, Id_W, V\rangle$ we will sometimes refer to the first, second, and third projection of the triple as $W_{\mathfrak{M}}$, $Id_{\mathfrak{M}}$, $V_{\mathfrak{M}}$ respectively.

Contrary to the custom in modal logic, where formulae are evaluated with respect to points of a model, in the present setting formulae are evaluated over collection of points, which are called "states" or "teams".

Definition 5 (**Teams**)**.** Given a discrete Kripke model $\mathfrak{M}$ over N, a Team is an element of $\mathcal{P}(W_{\mathfrak{M}})$ distinct from $\varnothing$.

While variables over points are written as lower-case latin letters, upper-case latin letters represent variables over teams. It is a peculiar feature of our models that the empty set does not constitute a team. This distinguishes $\mathcal{BSL}_m^{\mathbb{V}}$ (as well as its non-inquisitive fragment studied in chapter 4) from most state-based logics, and it will have a profound impact both on the predictions borne by the logic and on the properties of the logic itself.
The semantics of $\mathcal{BSL}_m^{\mathbb{V}}$ is spelt out in terms of two fundamental relations of "Support" ($\vDash$) and "Anti-support" ($\Dashv$) between teams and formulae of the language. Intuitively, while teams represent the speaker's information states, support and anti-support stand for the speaker's expressive possibilities with respect to a certain sentential content. In particular, we write $\mathfrak{M}, T \vDash \phi$ to express that the formula ϕ is assertible for the speaker given the information state T, and dually we write $\mathfrak{M}, T \Dashv \phi$ to express that ϕ is rejectable by the speaker given the information state T. We now give the semantic clauses for $\mathcal{BSL}_m^{\mathbb{V}}$:

Definition 6 (**Semantics**)**.** Let $\mathfrak{M}$ be a model over N, $\phi \in \mathcal{BSL}_m^{\mathbb{V}}$ and $T \subseteq W_{\mathfrak{M}}$ a team. The relations of support and anti-support are defined recursively as of below:

$$
\begin{array}{lll}
\mathfrak{M}, T \vDash p_i & \iff & w \in V(p_i), \text{ for all } w \in T \\
\mathfrak{M}, T \Dashv p_i & \iff & w \notin V(p_i), \text{ for all } w \in T \\
\mathfrak{M}, T \vDash \bot & & \text{never} \\
\mathfrak{M}, T \Dashv \bot & & \text{always} \\
\mathfrak{M}, T \vDash \neg\phi & \iff & \mathfrak{M}, T \Dashv \phi \\
\mathfrak{M}, T \Dashv \neg\phi & \iff & \mathfrak{M}, T \vDash \phi \\
\mathfrak{M}, T \vDash \phi \wedge \psi & \iff & \mathfrak{M}, T \vDash \phi \text{ and } \mathfrak{M}, T \vDash \psi \\
\mathfrak{M}, T \Dashv \phi \wedge \psi & \iff & \text{there exist teams } S, P \subseteq T \text{ with: } S \cup P = T,\ \mathfrak{M}, S \Dashv \phi, \text{ and } \mathfrak{M}, P \Dashv \psi \\
\mathfrak{M}, T \vDash \phi \vee \psi & \iff & \text{there exist teams } S, P \subseteq T \text{ with: } S \cup P = T,\ \mathfrak{M}, S \vDash \phi, \text{ and } \mathfrak{M}, P \vDash \psi \\
\mathfrak{M}, T \Dashv \phi \vee \psi & \iff & \mathfrak{M}, T \Dashv \phi \text{ and } \mathfrak{M}, T \Dashv \psi \\
\mathfrak{M}, T \vDash \phi \mathbin{\mathbb{V}} \psi & \iff & \mathfrak{M}, T \vDash \phi \text{ or } \mathfrak{M}, T \vDash \psi \\
\mathfrak{M}, T \Dashv \phi \mathbin{\mathbb{V}} \psi & \iff & \mathfrak{M}, T \Dashv \phi \text{ and } \mathfrak{M}, T \Dashv \psi \\
\mathfrak{M}, T \vDash \blacklozenge \phi & \iff & \text{there exists } S \subseteq T \text{ with: } \mathfrak{M}, S \vDash \phi \\
\mathfrak{M}, T \Dashv \blacklozenge \phi & \iff & \mathfrak{M}, T \Dashv \phi
\end{array}
$$

One additional unary modality $\blacksquare$ is defined as $\blacksquare \phi = \neg \blacklozenge \neg\phi$, yielding the following semantic clauses:

$$
\begin{array}{lll}
\mathfrak{M}, T \vDash \blacksquare \phi & \iff & \mathfrak{M}, T \vDash \phi \\
\mathfrak{M}, T \Dashv \blacksquare \phi & \iff & \text{there exists a team } S \subseteq T \text{ with: } \mathfrak{M}, S \Dashv \phi
\end{array}
$$

When the model is clear from context, we sometimes write $T \vDash \phi$ and $T \mathrel{=\!\!|} \phi$ in place of $\mathfrak{M}, T \vDash \phi$ and $\mathfrak{M}, T \mathrel{=\!\!|} \phi$.

Definition 7 (**Consequence, Equivalence**)**.** Given two $\mathcal{BSL}_m^{\mathbb{V}}$ formulae ϕ, ψ, we say:

1. ψ is a (semantic) consequence of ϕ (notation: $\phi \vDash \psi$) whenever for all models $\mathfrak{M}$ and states T, S, $\mathfrak{M}, T \vDash \phi$ implies $\mathfrak{M}, S \vDash \psi$. In this case, we also say ϕ entails ψ.
   Semantic consequence is extended to set of formulae in the usual way, i.e., by setting:
   $$\{\phi_1, ..., \phi_n\} \vDash \psi \iff \bigwedge\{\phi_1, .., \phi_n\} \vDash \psi;$$
2. ϕ, ψ are "equivalent" (notation: $\phi \equiv \psi$) whenever $\phi \vDash \psi$ and $\phi \vDash \psi$;
3. ϕ, ψ are "strongly equivalent" (notation: $\phi \boxminus \psi$) whenever $\phi \equiv \psi$ and $\neg\phi \equiv \neg\psi$.

It is important to notice the distinction between equivalence and strong equivalence. An instructive example is the following: let us define the nullary connective *verum* as follows: $\top :=_{def} \neg\bot$. Then, for any $\phi \in \mathcal{BSL}_m^{\mathbb{V}}$ we have $\blacklozenge\phi \equiv \phi \vee \top$, while $\blacklozenge\phi \boxminus \phi \vee \top$ does not hold. Indeed, notice that any nonempty team supports $\top$ so that for any model and team $\mathfrak{M}$ T, for an arbitrary $S \subseteq T$ we have:

$$T \vDash \blacklozenge\phi \iff (S \vDash \phi \text{ and } T \vDash \top) \iff T \vDash \phi \vee \top$$

As clearly $S \cup T = T$. However, $T \mathrel{=\!\!|} \blacklozenge\phi$ if and only if $T \mathrel{=\!\!|} \phi$, while $T \mathrel{=\!\!|} \phi \vee \top$ implies $S \mathrel{=\!\!|} \top$ for some S a nonempty subset of T. From the semantics of negation one readily sees:

$$S \mathrel{=\!\!|} \top \iff S \mathrel{=\!\!|} \neg\bot \iff S \vDash \bot$$

establishing $S \mathrel{=\!\!|} \top$ never occurs. By setting $\phi = p_i$ and choosing a model $\mathfrak{M}$ with $V_{\mathfrak{M}}(p_i) = W_{\mathfrak{M}}$ one then obtains a counterexample to $\blacklozenge\phi \boxminus \phi \vee \top$.

As a sanity check, we should ensure the natural language inferences we desired to capture are effectively licensed by the semantics. In terms of interpreting natural language-sentences in our logic, we render disjunctions in epistemic or deontic contexts with the split disjunction "$\vee$". The global disjunction "$\mathbb{V}$" acts as a "question forming" operator instead (see [Cia08] and section 2.2.2.1 below) We give here a wider selection of inference related to free-choice and epistemic modal, all of which our systems should validate:

i) *Modal disjunction*: from "it either rains or it is windy", infer "it might rain and it might be windy"
$$(\phi \vee \psi) \vDash \blacklozenge\phi \wedge \blacklozenge\psi$$
ii) *Negative conjunction*: from "it isn't both rainy and windy" infer "it might not be rainy, and, it might not windy"
$$\neg(\phi \wedge \psi) \vDash \blacklozenge\neg\phi \wedge \blacklozenge\neg\psi$$
iii) *Narrow scope free-choice*: from "it might be rainy or windy" infer "it might be rainy and it might be windy"
$$\blacklozenge(\phi \vee \psi) \vDash \blacklozenge\phi \wedge \blacklozenge\psi$$

iv) *Wide scope free-choice*: from "it might be rainy or it might be windy" infer "it might be rainy and it might be windy"

$$\blacklozenge \phi \vee \blacklozenge \psi \models \blacklozenge \phi \wedge \blacklozenge \psi$$

v) *Double negation free-choice*: from "it is false that it can't be rainy or windy" infer "it might be rainy or it might be windy"

$$\neg\neg \blacklozenge (\phi \vee \psi) \models \blacklozenge \phi \wedge \blacklozenge \psi$$

vi) *Negative narrow scope free-choice*: from "it might not be rainy and windy" infer "it might not be rainy and it might not be windy"

$$\blacklozenge \neg(\phi \wedge \psi) \models \blacklozenge \neg\phi \wedge \blacklozenge \neg\psi$$

vii) *Negative wide scope free-choice*: from "it is not the case that it both might be rainy and it might be windy" infer "it might not be rainy and it might not be windy"

$$\neg(\blacklozenge \phi \wedge \blacklozenge \psi) \models \blacklozenge \neg\phi \wedge \blacklozenge \neg\psi$$

viii) *Negative disjunction*: from "it is not the case that it either is rainy or windy" infer "it is not the case that it is rainy and it is not the case that it is windy"

$$\neg(\phi \vee \psi) \models \neg\phi \wedge \neg\psi$$

ix) *Dual prohibition*: from "it is not false that it might either be rainy or windy" infer "it is false that is it may be rainy, and it is false that is it may be windy"

$$\neg \blacklozenge (\phi \vee \psi) \models \neg \blacklozenge \phi \wedge \neg \blacklozenge \psi$$

x) *Epistemic contradiction*: "it does not rain and it might be raining" is contradictory.

$$\blacklozenge \phi \wedge \neg\phi \models \bot$$

xi) *Non-factivity*: you should not infer "it rains" from "it might be raining"

$$\blacklozenge \phi \not\models \phi$$

xii) *Disjunctive epistemic contradiction*: "it either rains and it might not be raining, or it is windy" is contradictory.

$$(\blacklozenge \phi \wedge \neg\phi) \vee \psi \models \bot$$

xiii) *Modal epistemic contradiction*: "it might be that it does not rain and it may be raining" is contradictory

$$\blacklozenge (\neg\phi \wedge \blacklozenge \phi) \models \bot$$

Notice that item xi) ensures we avoid the risk of modal collapse foreshadowed at the end of section 1.3.1, while items x), xii), xiii) cover the cases of the Yalcinean sentences outlined before.

Proposition 8. Every $\mathcal{BSL}_m^{\mathbb{V}}$ formula to the left of the support relation symbol in i) - xiii) entails the formula to the right of the same support relation symbol.

Proof. Items i)-ix) are immediate by inspection of the semantic clauses.
Item x) is proven simultaneously for the support and anti-support relation by induction on the subformulae of ϕ. Let $\mathfrak{M}$ be a model and T a team over $\mathcal{P}(W_{\mathfrak{M}})$, if ϕ is a propositional atom p_i, then by the semantics of "$\neg$", we have $T \vDash \neg p_i$ if and only if every $w \in T \notin V_{\mathfrak{M}}(p_i)$, while $T \vDash \blacklozenge p_i$ if and only if there exists a nonempty team $S \subseteq T$ with $S \vDash p_i$, meaning that $v \in V_{\mathfrak{M}}(p_i)$ for all $v \in S$. Given $S \subseteq T$, this contradicts the statement that $T \vDash \neg p_i$. This situation therefore never occurs, and the result vacuously follows. The base case for the anti-support relation is symmetric.
For the inductive step, we detail the case for split disjunction as all the other connectives are treated similarly, with the only exception of "$\neg$" where the case for support follows directly from the inductive hypothesis for anti-support, and the case for anti-support follows directly from the inductive hypothesis for support.
Let now $\phi = \mu \vee \nu$. By the semantic clauses we have $T \vDash \neg(\mu \vee \nu)$ if and only if $T \dashv \mu$ and $T \dashv \nu$, whereas $T \vDash \blacklozenge(\mu \vee \nu)$ if and only if $S \vDash \mu \vee \nu$ for some $S \subseteq T$. By the semantic clauses for split disjunction, the latter implies $P \vDash \mu$ for some $P \subseteq S$, which in turn yields $T \vDash \blacklozenge \mu$. Since $T \dashv \mu$ holds, this implies a contradiction by induction hypothesis together with the semantics of negation. Therefore, this situation never occurs and the desired vacuously follows. The anti-support case is symmetric to the one for support.
Item xi) is established with a counterexample, e.g., by considering $\mathsf{N} = \{p_1\}$ and $\mathfrak{M} = \langle \{w_1, w_2\}, Id_{\{w_1,w_2\}}, V\rangle$ with $V(p_1) = \{w_1\}$, where we clearly have $\mathfrak{M}, \{w_1, w_2\} \vDash \blacklozenge p_1$ as witnessed by the subteam $\{w_1\}$ together with $\mathfrak{M}, \{w_1, w_2\} \nvDash p_1$ since $w_1 \notin V(p_1)$.
For item xii) consider a model and a team $\mathfrak{M}, T$ such that $\mathfrak{M}, T \vDash (\neg\phi \wedge \blacklozenge \phi) \vee \psi$. Then, there exists a nonempty team S subsets of T with $\mathfrak{M}, S \vDash \neg\phi \wedge \blacklozenge \phi$. By item x), this would imply $\mathfrak{M}, S \vDash \bot$ which can never occur, so we conclude that $\mathfrak{M}, T \vDash (\neg\phi \wedge \blacklozenge \phi) \vee \psi$ never occurs as well and the result follows vacuously.
Item xiii) follows from x) together with the semantics of $\blacklozenge$. ■

2.1.3 Some elementary properties

We now turn to investigate some usual properties of team-based logics in the present setting.

2.1.3.1 Substitution

We begin by introducing some notation. In the following, we write $\phi[p_i := \psi]$ to denote "global substitution" of p_i with ψ. That is, the formula resulting from the substitution of *every instance* of p_i with ψ in ϕ.
The notion of global substitution is recursively defined as customary (see e.g., [Van04]), and likewise lifted from propositional atoms to subformulae.
We write $\phi(\zeta := \psi)$ to denote "local substitution" of ζ with ψ, i.e., the substitution of *one specific instance* of p_i with ψ in ϕ. Formally, local substitution relies on an enumeration of subformulae of ϕ: we assign to each symbol occurring in ϕ a positive integer starting from the leftmost symbol and proceeding to the right adding 1 to the previous number for each symbol encountered. Letting $\phi = (p_1 \wedge p_2) \mathbin{\vee\!\!\!\vee} \neg p_3$, we have:

$$({}^1 p_1^2 \wedge^3 p_2^4)^5 \mathbin{\vee\!\!\!\vee}^6 \neg^7 p_3^8$$

Where the numbering is indicated with a superscript. We then refer to a subfomula ζ of ϕ with the pair $\langle \zeta, n \rangle$ where n is the number assigned to the main connective of ζ. Local substitution of ζ with ψ in ϕ is then defined as $\phi[\langle \zeta, n \rangle := \psi]$. In most cases, however, which subformula is being substituted is going to be clear from context and we will thus adopt the lighter notation $\phi(\zeta := \psi)$.
When no risk of confusion arises, we shall sometimes write $\phi[p_i]$ and $\phi[\psi]$ to denote ϕ before and after global substitution of p_i with ψ. Likewise, we shall write $\phi(\zeta)$ and $\phi(\psi)$ to denote ψ before and after the local substitution of ζ with ϕ.

As with most inquisitive and team-based logics, the semantic consequence relation in $\mathcal{BSL}_m^{\mathbb{V}}$ is not closed under uniform substitution. For example, we have:

$$\models p_i \mathbb{V} \neg p_i \mathbb{V} (p_i \vee \neg p_i) \text{ whereas } \not\models (p_j \wedge p_k) \mathbb{V} \neg(p_j \wedge p_k) \mathbb{V} ((p_j \wedge p_k) \vee \neg(p_j \wedge p_k))$$

As witnessed by any $\mathfrak{M}$ with $V_{\mathfrak{M}}(p_k) = W_{\mathfrak{M}}$, $V_{\mathfrak{M}}(p_i) = V_{\mathfrak{M}}(p_j) = \varnothing$.
Similarly, equivalence is not preserved under uniform substitution: for equivalent formulae $\phi \equiv \psi$, we do not in general have $\zeta[p_i := \phi] \equiv \zeta[\psi]$.
The crux lies in the behavior of bilateral negation, which causes bivalence to fail for the totality of formulae in $\mathcal{BSL}_m^{\mathbb{V}}$. Indeed, whenever the replaced formula does not occur under the scope of negation, preservation of equivalence under uniform substitution is recovered.

Proposition 9. For any ζ, ψ, ϕ $\mathcal{BSL}_m^{\mathbb{V}}$-formulae, if $\phi \equiv \psi$, and $p_i \in \mathsf{N}$ does not lie under the scope of $\neg$ in ζ, then $\zeta[p_i := \phi] \equiv \zeta[p_i := \psi]$.

Proof. By induction on ζ. The case for $\zeta = p_j$ is twofold depending on whether $i = j$ or not. If $i = j$, then $\zeta[p_i := \psi] = \psi \equiv \phi = \zeta[p_i := \phi]$, if not, then we have $\zeta[p_i := \psi] = p_j \equiv p_j = \zeta[p_i := \phi]$.
The inductive cases reduce to checking that each connective is congruent with respect to logical equivalence. We only give details for the case of split disjunction as the other are similar. If $\zeta = \mu \vee \nu$ then we have $\mu \vee \nu[p_i := \phi] = \mu[p_i := \phi] \vee \nu[p_i := \phi]$. By inductive hypothesis we know $\mu[p_i := \phi] \equiv \mu[p_i := \psi]$ and $\nu[p_i := \phi] \equiv \nu[p_i := \psi]$, so it remains to check that split disjunction is congruent with respect to logical equivalence. Let $\mathfrak{M}, X$ be an arbitrary model and team, then we have:

$$\begin{aligned}
\mathfrak{M}, X \models \mu \vee \nu[p_i := \phi] &\iff \mathfrak{M}, X \models \mu[p_i := \phi] \vee \nu[p_i := \phi] \\
&\iff X = Y_1 \cup Y_2 \text{ with } \mathfrak{M}, Y_1 \models \mu[p_i := \phi],\ \mathfrak{M}, Y_2 \models \nu[p_i := \phi] \\
&\iff \mathfrak{M}, Y_1 \models \mu[p_i := \psi],\ \mathfrak{M}, Y_2 \models \nu[p_i := \psi] \\
&\iff \mathfrak{M}, X \models \mu[p_i := \psi] \vee \nu[p_i := \psi] \\
&\iff \mathfrak{M}, X \models \mu \vee \nu[p_i := \psi]
\end{aligned}$$

■

On the contrary, strong equivalence is always preserved under uniform substitution:

Proposition 10. For any ζ, $\psi, 1, \phi$ $\mathcal{BSL}_m^{\mathbb{V}}$-formulae, if $\phi \boxminus \psi$, then $\zeta[p_i := \phi] \boxminus \zeta[p_i := \psi]$.

Proof. By induction on ζ. As above, the base case is twofold depending on whether $i = j$ for $\zeta = p_j$. If $i = j$, then $\zeta[p_i := \psi] = \psi \boxminus \phi = \zeta[p_i := \phi]$, if not, then the statement reduces to $p_j \boxminus p_j$ which is

straightforward to verify.
If $\zeta = \neg\mu$ then we have, for $\mathfrak{M}, X$ arbitrary:

$$\begin{aligned}
\mathfrak{M}, X \models \neg\mu[p_i := \phi] &\iff \mathfrak{M}, X \models \neg(\mu[p_i := \phi]) \\
&\iff \mathfrak{M}, X \dashv \mu[p_i := \phi] \\
&\overset{*}{\iff} \mathfrak{M}, X \dashv \mu[p_i := \psi] \\
&\iff \mathfrak{M}, X \models \neg(\mu[p_i := \psi]) \\
&\iff \mathfrak{M}, X \models \neg\mu[p_i := \psi]
\end{aligned}$$

Where at $*$ we have used the inductive hypothesis $\mu[p_i := \phi] \boxminus \mu[p_i := \psi]$ which in particular implies $\mathfrak{M}, X \dashv \mu[p_i := \phi] \iff \mathfrak{M}, X \dashv \mu[p_i := \psi]$. The remaining inductive cases reduce to checking that all connectives behaves as congruences with respect to strong logical equivalence, all of them are similar to the case for split disjunction given in proposition 9, making use of the new inductive hypothesis in the same way as $*$ in the case for negation. They are thus omitted. ■

2.1.4 Distributivity and negation normal form

The following laws hold in $\mathcal{BSL}_m^{\mathbb{V}}$, all of which can easily be checked by unraveling the semantic clauses:

Fact 11 (**Distributivity**). The following hold in $\mathcal{BSL}_m^{\mathbb{V}}$:

1. $\blacklozenge(\phi \mathbin{\mathbb{V}} \psi) \equiv \blacklozenge\phi \mathbin{\mathbb{V}} \blacklozenge\psi$;
2. $\blacklozenge(\phi \wedge \psi) \models \blacklozenge\phi \wedge \blacklozenge\psi$;
3. $\phi \vee (\psi \mathbin{\mathbb{V}} \zeta) \equiv (\phi \vee \psi) \mathbin{\mathbb{V}} (\phi \vee \zeta)$;
4. $\phi \wedge (\psi \mathbin{\mathbb{V}} \zeta) \equiv (\phi \wedge \psi) \mathbin{\mathbb{V}} (\phi \wedge \zeta)$;
5. $\blacklozenge(\phi \vee \psi) \equiv \blacklozenge\phi \vee \blacklozenge\psi$.
6. $(\phi \wedge \psi) \vee (\phi \wedge \zeta) \models \phi \wedge (\psi \vee \zeta)$, while $\phi \wedge (\psi \vee \zeta) \not\models (\phi \wedge \psi) \vee (\phi \wedge \zeta)$;
7. $(\phi \vee (\psi \wedge \zeta)) \models (\phi \vee \psi) \wedge (\phi \vee \zeta)$, while $(\phi \vee \psi) \wedge (\phi \vee \zeta) \not\models (\phi \vee (\psi \wedge \zeta))$.

A counterexample to 6, 7. can be found setting $\phi = (p_i \vee p_j)$, $\zeta = p_k$, and $\psi = p_y$. For any $\mathfrak{M}$ with $V_{\mathfrak{M}}(p_i) \cup V_{\mathfrak{M}}(p_j) = W_{\mathfrak{M}}$, $V_{\mathfrak{M}}(p_i) \cap V_{\mathfrak{M}}(p_j) = \varnothing$, and both $V_{\mathfrak{M}}(p_k) = V_{\mathfrak{M}}(p_i)$, $V_{\mathfrak{M}}(p_j) = V_{\mathfrak{M}}(p_y)$, we have $\mathfrak{M}, W_{\mathfrak{M}} \models (p_i \vee p_j) \wedge (p_k \vee p_i)$ together with $\mathfrak{M}, W_{\mathfrak{M}} \not\models (p_i \wedge p_y) \vee (p_j \wedge p_k)$.
By setting $\phi = p_i$ the same model also witnesses a counterexample to 7.

As customary in the context of team-based logics, each formula of $\mathcal{BSL}_m^{\mathbb{V}}$ is strongly equivalent to a formula in negation normal form, where with the latter expression we refer to the fact the the unary connective "$\neg$" appears only in front of propositional atoms, and never in front of complex formulae. Semantically, this is an easy corollary to the validity of double negation elimination as well as De Morgan laws. Both of those facts are easily established by inspecting the semantic clauses:

Fact 12 (**Double negation elimination**). For $\phi \in \mathcal{BSL}_m^{\mathbb{V}}$, it holds that $\phi \boxminus \neg\neg\phi$.

Fact 13 (**De Morgan laws**). For $\phi, \psi \in \mathcal{BSL}_m^{\mathbb{V}}$, the following holds:

1. $\neg(\phi \wedge \psi) \boxminus \neg\phi \vee \neg\psi$;
2. $\neg(\phi \vee \psi) \boxminus \neg\phi \wedge \neg\psi$;
3. $\neg(\phi \mathbin{\mathbb{V}} \psi) \equiv \neg\phi \wedge \neg\psi$.

We then obtain negation normal form as an easy corollary:

Corollary 14 (Negation normal form). Every formula ϕ of $\mathcal{BSL}_m^{\mathbb{V}}$ is equivalent to one in negation normal form.

Proof. By induction on the complexity of ϕ. The base case is immediate as all propositional atoms are in negation normal form. Since whenever μ, ν are in negation normal form, so is $\mu \# \nu$ for $\# \in \{\wedge, \vee, \mathbb{V}\}$ the inductive cases for inquisitive disjunction, conjunction, and split disjunction follow directly by induction hypothesis. The case for might is identical to those of the binary connectives.
The case for $\phi = \neg\psi$ is fourfold: if ψ is an atom the result follows immediately, if ψ is again a negative formula then we apply fact 12 and then the inductive hypothesis. If ψ is either a conjunction or a disjunction, then the desired follows by fact 13 together with the induction hypothesis applied to the immediate subformulae of ψ. Finally, if ψ is a modal formula then the desired follows by fact 12, and the inductive hypothesis, together with the definition: $\blacksquare = \neg \blacklozenge \neg$. ■

2.1.4.1 Reversal

After [Rum00], we identify reversal with the following property holding in classical logic, where we use capital Greek letters to denote finite sets of $\mathcal{BSL}_m^{\mathbb{V}}$-formulae:

$$\Gamma, \phi \vDash \psi \iff \Gamma, \neg\psi \vDash \neg\phi$$

It is easy to see that this property systematically fails in our system. To witness an easy case, let $\mathfrak{M}$ be such that $W_{\mathfrak{M}} = \{w\}$ and $V_{\mathfrak{M}}(p_i) = \{w\}$ and $V_{\mathfrak{M}}(p_j) = \varnothing$ then we clearly have:

$$\mathfrak{M}, W_{\mathfrak{M}} \dashv p_j \text{ while } \mathfrak{M}, W_{\mathfrak{M}} \not\dashv p_i \vee p_j$$

which provides a counterexample to the elementary reversal entailment:

$$p_i \wedge p_j \vDash p_j \iff \neg p_j \vDash \neg(p_i \wedge p_j)$$

In the presence of free-choice the failure of reversal is virtuous, indeed, unrestricted reversal would license the following vicious meta-inference:

From "It may be raining or it may be sunny", infer: "It may be raining"

$$\iff$$

From "It can't be raining", infer: "It can't be raining and it can't be sunny"

A proof theoretic counterpart to reversal is often classically derivable using the bilateral rules of *Smileian reductio*. As a consequence, this fact will bear some weight on our choice of axioms in section 2.2.

2.1.4.2 Closures and empty team

The following closure properties prove central in the study of many team-based logics. Since our system does not support the empty team, we split the definitions between *positive* and *negative* cases, referring as to whether the property hods with respect to the relation of support or anti-support:

Definition 15 (Closures). Let $\phi \in \mathcal{BSL}_m^{\mathbb{V}}$, we say that ϕ is:

1. **Positive union-closed** if and only if, for any $\mathfrak{M}$ and any family of teams $\{T_i\}_{i\in\mathbb{I}}$ with $T_i \subseteq W_{\mathfrak{M}}$:

$$\mathfrak{M}, T_i \vDash \phi \text{ for all } i \in \mathbb{I} \text{ implies } \mathfrak{M}, \bigcup_{i\in\mathbb{I}} T_i \vDash \phi.$$

2. **Negative union-closed** if and only if, for any $\mathfrak{M}$ and any family of teams $\{T_i\}_{i\in\mathbb{I}}$ with $T_i \subseteq W_{\mathfrak{M}}$:

$$\mathfrak{M}, T_i \dashv \phi \text{ for all } i \in \mathbb{I} \text{ implies } \mathfrak{M}, \bigcup_{i\in\mathbb{I}} T_i \dashv \phi.$$

3. **Positive down-closed** if and only if for any nonempty $T, S \subseteq W_{\mathfrak{M}}$ in a model $\mathfrak{M}$ such that $S \subseteq T$ $\mathfrak{M}, T \vDash \phi$ implies $\mathfrak{M}, S \vDash \phi$;
4. **Negative down-closed** if and only if for any nonempty $T, S \subseteq W_{\mathfrak{M}}$ in a model $\mathfrak{M}$ such that $S \subseteq T$, $\mathfrak{M}, T \dashv \phi$ implies $\mathfrak{M}, S \dashv \phi$;
5. **Positive flat** if and only if $\mathfrak{M}, T \vDash \phi \iff$ for all $w \in T$, $\mathfrak{M}, \{w\} \vDash \phi$;
6. **Negative flat** if and only if $\mathfrak{M}, T \dashv \phi \iff$ for all $w \in T$, $\mathfrak{M}, \{w\} \dashv \phi$;
7. Has the **empty team property** if and only if $\mathfrak{M}, \varnothing \vDash \phi$.

It is easy to see that formulae containing an instance of inquisitive disjunction may, in general, fail to be union-closed. For example, given $\mathfrak{M}, T \vDash p_i \wedge \neg p_j$ and $\mathfrak{M}, S \vDash p_j \wedge \neg p_i$, we have that both S, T support $p_i \vee\!\!\!\vee\, p_j$, but clearly $S \cup T$ does not. More generally, we then obtain the following:

Proposition 16 (Union closure). All formulae ζ in the $\vee\!\!\!\vee$-free fragment of $\mathcal{BSL}_m^{\vee\!\!\!\vee}$ are positive and negative union closed.

Proof. We prove both statements together by simultaneous induction on ζ over support and anti-support relation, where the base case follows by definition of the support and anti-support relation for propositional variables.

For $\zeta = \neg\psi$, $\mathfrak{M}, T_i \vDash \neg\psi$ if and only if $\mathfrak{M}, T_i \dashv \psi$. By induction hypothesis ψ is union-closed, so we get $\bigcup_{i\in\mathbb{I}} T_i \dashv \phi$ and thus also $\bigcup_{i\in\mathbb{I}} T_i \vDash \neg\phi$. The case for anti-support is symmetrical.

For $\zeta = \phi \vee \psi$ by the definition of the support relation, for every T_i we have $T_i^1 \cup T_i^2 = T_i$ with $\mathfrak{M}, T_i^1 \vDash \phi$ and $\mathfrak{M}, T_i^2 \vDash \psi$. Since both ϕ, ψ are union-closed by hypothesis, we have $\mathfrak{M}, \bigcup_{i\in\mathbb{I}} T_i^1 \vDash \phi$ and $\mathfrak{M}, \bigcup_{i\in\mathbb{I}} T_i^2 \vDash \psi$, Since $\bigcup_{i\in\mathbb{I}} T_i^1 \cup \bigcup_{i\in\mathbb{I}} T_i^2 = \bigcup_{i\in\mathbb{I}} T_i$, the desired then follows. The case for anti-support of $\phi \wedge \psi$ is symmetrical to this case.

For $\zeta = \phi \wedge \psi$ we have $\mathfrak{M}, T_i \vDash \phi$ and $\mathfrak{M}, T_i \vDash \psi$, as both formulae are union-closed by hypothesis, the union of the T_i also supports both, and the desired follows. The cases for anti-support of both $\phi \vee\!\!\!\vee\, \psi$ and $\phi \vee \psi$ are symmetrical to this case.

Last, if $\zeta = \blacklozenge\phi$, then we have for every i a nonempty $S_i \subseteq T_i$ with $\mathfrak{M}, S_i \vDash \phi$, as ϕ is union closed, $\mathfrak{M}, \bigcup_{i\in\mathbb{I}} S_i \vDash \phi$, since $\bigcup_{i\in\mathbb{I}} S_i \subseteq \bigcup_{i\in\mathbb{I}} T_i$, this yields the desired. The case for anti-support of $\blacklozenge\phi$ follows immediately from the induction hypothesis.

■

Since, $\mathcal{BSL}_m^{\vee\!\!\!\vee}$ admits no empty model or team, the behaviour of properties 3 - 7 in the present context differs significantly from the usual setting. For once, the following is immediate from definition 5:

Fact 17 (Empty team property). No formula of $\mathcal{BSL}_m^{\vee\!\!\!\vee}$ has the empty team property.

Similarly, because $\varnothing$ is not a team in any model of $\mathcal{BSL}_m^{\vee\!\!\!\vee}$, whenever $\mathfrak{M}, T \vDash \mu \vee \nu$ for some μ, ν formulae of $\mathcal{BSL}_m^{\vee\!\!\!\vee}$, it is not in general true that all $S \subseteq T$ also supports $\mu \vee \nu$, as $\mu \vee \nu$ is going to be supported only by those subteams that are the union of two further *nonempty* subteams each supporting μ and ν individually. It is easy to see that a similar situation also holds with respect to formulae prefixed with "might". Generalizing, we can state the following:

Proposition 18 (**Down-closure**)**.** Every formula ζ in the class D^+ given by the following grammar:

$$p_i \mid \neg p_i \mid \phi \wedge \psi \mid \blacksquare\, \phi \mid \phi \vee\!\!\!\vee \psi$$

is positively down-closed. Every formula ζ in the class D^- given by the following grammar:

$$p_i \mid \neg p_i \mid \phi \vee \psi \mid \blacklozenge\, \phi \mid \phi \vee\!\!\!\vee \psi$$

Is negatively down-closed.

Proof. By simultaneous induction on the definition of ζ.

If ζ is a propositional variable, then the statement follows by the definition of the $\vDash$ relation for both the support and anti-support case.

If ζ is the negation of a propositional variable $\neg p_i$, then $\mathfrak{M}, T \vDash \neg p_i$ if and only if $\mathfrak{M}, T_i \dashv p_i$, the statement then follows by the second induction hypothesis $p_i \in \mathsf{D}^-$. The case for anti-support is similar, making use of $p_i \in \mathsf{D}^-$ instead.

If ζ is the conjunction of $\phi \in \mathsf{D}^+$ and $\phi \in \mathsf{D}^+$, then the desired follows by the positive induction hypothesis. The case for $\zeta = \phi \vee \psi$ with both disjuncts in $\phi \in \mathsf{D}^-$ is similar.

If $\zeta = \blacksquare\, \phi$ for $\phi \in \mathsf{D}^+$, then the desired follows by induction hypothesis, and the case for $\zeta = \blacklozenge\, \phi$ with $\phi \in \mathsf{D}^-$ is symmetrical. The case for inquisitive disjunction is immediate by induction hypothesis for both D^+ and D^-. ■

Down-closed formulae (both positive and negative) will play a crucial role in the development of our proof-system, as we will see they encode Incurvati and Schlöder's notion of evidence-preserving premisses in the context of team-semantics (see section 2.2.2).
The class of flat formulae is easily obtained from the previous ones:

Proposition 19 (**Flatness**)**.** Every formula ζ in the class F^+ given by the following grammar:

$$p_i \mid \neg p_i \mid \phi \wedge \psi \mid \blacksquare\, \phi$$

is positive flat. Every formula ζ in the class F^- given by the following grammar:

$$p_i \mid \neg p_i \mid \phi \vee \psi \mid \blacklozenge\, \phi \mid \phi \vee\!\!\!\vee \psi$$

Is negative flat.

Proof. Both statement follow from the observation that a formula ζ is positive flat if and only if it is positive down-closed and union closed. Likewise, ζ is negative flat if and only if it is negative down-closed and union-closed. We show the first statement as the second one is symmetric. For the left-to-right direction, assume ζ is positive flat, and $\mathfrak{M}, T \vDash \zeta$ for some $\mathfrak{M}$, T. By (positive) flatness, we have $\mathfrak{M}, \{w\} \vDash \zeta$ for all $w \in T$, yielding $\mathfrak{M}, S \vDash \zeta$ for any $S \subseteq T$ because we know $\mathfrak{M}, S \vDash \zeta$ if and only if $\mathfrak{M}, \{v\} \vDash \zeta$ for all $v \in V$. This shows ζ is positive down-closed. To see it also is union closed, let $T_i \vDash \zeta$ for a family of teams $\{T\}_{i \in \mathbb{I}}$ over some model $\mathfrak{M}$. As ζ is positive flat, every singleton subset of any T_i supports ζ, thus implying the desired as $\bigcup_{i\in\mathbb{I}} T_i = \bigcup_{i\in\mathbb{I}}\bigcup_{w\in T_i}\{w\}$. For the converse direction, that $\mathfrak{M}, T \vDash \zeta$ implies $\mathfrak{M}, \{w\} \vDash \zeta$ for every $w \in T$ is immediate from the assumption that ζ is down-closed. That $\mathfrak{M}.\{w\} \vDash \zeta$ for every $w \in T$ implies $\mathfrak{M}, T \vDash \zeta$ is immediate from the assumption that ζ is union closed together with the fact that $T = \bigcup_{w\in T}\{w\}$. ∎

In the presence of the empty team property, it is sometimes possible to extend the support conditions from singleton teams to points in a model, by defining satisfaction at a point through the usual clauses for possible-worlds semantics for modal logic and replacing tensor disjunction with classical disjunction ([AAY24]). In the present case, however, such correspondence cannot be obtained. Considering a model $\mathfrak{M}$ and an arbitrary $w \in W_{\mathfrak{M}}$, we see that w satisfies the classical disjunction of two formulae ϕ and ψ precisely when it satisfies either one of the two. On the contrary, the only instance where $\mathfrak{M}, \{w\}$ supports $\phi \vee \psi$ is when $\{w\}$ supports both formulae individually: since the subteams witnessing the split disjunction must be nonempty, the fact that $\{w\}$ is a singleton implies that they must coincide. The following proposition establishes the collapse of split disjunctions into conjunctions over singleton teams:

Proposition 20. For $\mathfrak{M}$ a model and $w \in W_{\mathfrak{M}}$, let $^{\&} : \mathcal{BSL}_m^{\mathbb{V}} \to \mathcal{BSL}_m^{\mathbb{V}}$ be defined as:

$$\zeta^{\&} = \begin{cases} \phi \wedge \psi & \text{if } \zeta = \phi \vee \psi \\ \zeta & \text{otherwise} \end{cases}$$

Then we have: $\mathfrak{M}, \{w\} \vDash \zeta \iff \mathfrak{M}, w \vDash \zeta^{\&}$.

Proof. A routine induction on the structure of ζ, all the cases follow from unraveling the semantic clauses. ∎

We now turn to present "Hintikka formulae", or "characteristic formulae" for our system, which will prove crucial in showing completeness furhter down the line.

2.1.5 Bisimulation and characteristic formulae

As in many other team-based logics, the structure of $\mathcal{BSL}_m^{\mathbb{V}}$-models can be characterized syntactically up to a suitable notion of equivalence. In this section, we introduce the notion of "bisimulation" and discuss some of its properties which will prove useful in showing the soundness and completeness of our axiomatization.

Although the syntax of $\mathcal{BSL}_m^{\mathbb{V}}$ features the modal operator "might", to ensure two teams support the same formulae it is sufficient to require that for each point in either team, there exist a point in the other agreeing on all propositional valuations. Accordingly, our notion of bisimulation is adapted from that of "0-level" bisimulation given in [AAY24].
From this point onward, we fix a finite set of propositional variables $\mathsf{N} \subseteq \mathsf{Prop}$ of arbitrary cardinality. We define a "pointed model" to be a pair $\langle \mathfrak{M}, w\rangle$ with $w \in V_{\mathfrak{M}}$. angled brackets are omitted when they are clear from context. Bisimulation for pointed models id defined as follows:

Definition 21 (Point-bisimulation). For models $\mathfrak{M}$, $\mathfrak{N}$ and points w, v over an index N we say $\langle \mathfrak{M}, w\rangle$ and $\langle \mathfrak{N}, v\rangle$ are bisimilar (notation: $\langle \mathfrak{M}, w\rangle \leftrightharpoons \langle \mathfrak{N}, v\rangle$) if and only if:

$$\mathfrak{M}, w \vDash p_i \iff \mathfrak{N}, v \vDash p_i \text{ for all } i \in \mathsf{N}$$

The definition is lifted from points to teams as of below:

Definition 22 (**Team-bisimulation**)**.** For models $\mathfrak{M}$, $\mathfrak{N}$ and teams T, S over an index N we have: $\mathfrak{M}, T \leftrightarrows \mathfrak{N}, S$ if and only if both of the following hold:

1. Forth condition: for all $w \in T$ there is some $v \in S$ with $\langle \mathfrak{M}, w\rangle \leftrightarrows \langle \mathfrak{N}, v\rangle$,
2. Back condition: For all $v \in S$ there is some $w \in T$ with $\langle \mathfrak{N}, v\rangle \leftrightarrows \langle \mathfrak{M}, w\rangle$.

Notably, team-bisimulation is closed under subteams:

Lemma 23. For models and teams $\mathfrak{M}$, $\mathfrak{N}$, T, S, if $\mathfrak{M}, T \leftrightarrows \mathfrak{N}, S$ then for any pair of teams $T_1, T_2 \subseteq T$ with $T_1 \cup T_2 = T$ there are teams $S_1, S_2 \subseteq S$ with $S_1 \cup S_2 = S$ and $S_i \leftrightarrows T_i$ for $i \in \{1, 2\}$.

Proof. Let $\mathfrak{M}, T \leftrightarrows \mathfrak{N}, S$ and T_1, T_2 be as described above. Further, let $S_1 = \{v \in S \mid \exists w \in T_1[\mathfrak{M}, w \leftrightarrows \mathfrak{N}, v]\}$, and $S_2 = \{v \in S \mid \exists w \in T_2[\mathfrak{M}, w \leftrightarrows \mathfrak{N}, v]\}$. Since $T \leftrightarrows S$, for every $w \in T_1$ there exists some $v_w \in S$ with $w \leftrightarrows v_w$, and, by construction we have $\{v_w\}_{w \in T_1} \subseteq S_1$. This shows the forth condition of $T_1 \leftrightarrows S_1$. Since the back condition of the bisimulation relation between S_1 and T_1 is satisfied by definition of S_1, we conclude $T_1 \leftrightarrows S_1$, and likewise we get $T_2 \leftrightarrows S_2$ by the same reasoning. It remains to ensure that $S_1 \cup S_2 = S$. Let $s \in S$ be arbitrary, as $T \leftrightarrows S$, by the back condition there exists a $t \in T$ with $t \leftrightarrows s$. As $T_1 \cup T_2 = T$, s is an element of either S_1 or S_2, which then implies $S \subseteq S_1 \cup S_2$ which shows the desired. ■

As in the case of ordinary modal logic, bisimulation implies language-equivalence both at the level of states and at the level of teams. Whenever two teams are bisimilar, there is no formula of $\mathcal{BSL}_m^{\mathbb{V}}$ that is supported by one and not by the other.

Proposition 24. For models $\mathfrak{M}$, $\mathfrak{N}$ and states w, v, $\langle \mathfrak{M}, w\rangle \leftrightarrows \langle \mathfrak{N}, v\rangle$ implies, for any $\zeta \in \mathcal{BSL}_m^{\mathbb{V}}$:

$$\mathfrak{M}, \{w\} \models \zeta \iff \mathfrak{N}, \{v\} \models \zeta \text{ and } \mathfrak{M}, \{w\} \mathrel{=\!|} \zeta \iff \mathfrak{N}, \{v\} \mathrel{=\!|} \zeta$$

Proof. By simultaneous induction on the structure of ζ. If ζ a propositional variable, then the result follows by the definition of bisimulation. The case for anti-support is similar. The support and anti-support cases for negation follow by induction hypothesis, as do the support case for conjunction and the anti-support case for split disjunction.
If $\zeta = \phi \vee \psi$ by the semantic clauses of "$\vee$" we have that $\mathfrak{M}, \{w\} \models \phi$ and $\mathfrak{M}, \{w\} \models \psi$, which gives the result by induction hypothesis. The anti-support case for conjunction is similar. The case support case for $\zeta = \blacklozenge\, \phi$ is similar to the one for disjunction, and the anti-support case follows directly from the inductive hypothesis.
Last, both the support and anti-support cases for inquisitive disjunction follow immediately from the inductive hypothesis. ■

The statement above generalizes straightforwardly to language equivalence for teams:

Proposition 25. For models $\mathfrak{M}$, $\mathfrak{N}$ and teams T, S, $\mathfrak{M}, T \leftrightarrows \mathfrak{N}$, implies for any $\zeta \in \mathcal{BSL}_m^{\mathbb{V}}$:

$$\mathfrak{M}, T \models \zeta \iff \mathfrak{N}, S \models \zeta \text{ and } \mathfrak{M}, T \mathrel{=\!|} \zeta \iff \mathfrak{N}, S \mathrel{=\!|} \zeta$$

Proof. Let T, S be teams over $\mathfrak{M}, \mathfrak{N}$ such that $\mathfrak{M}, T \leftrightarrows \mathfrak{N}, S$. We prove the statement by simultaneous induction on ζ. For the base case, let $\zeta = p_i$ so that we have $\mathfrak{M}, T \models p_i \iff \forall w \in T[w \in V_{\mathfrak{M}}(p_i)]$

and thus also $\mathfrak{M}, \{w\} \vDash p_i$ by the semantic clauses. By the back condition of team-bisimulation and proposition 24, we get $\forall v \in S[\mathfrak{N}, \{v\} \vDash p_i]$ which in turn implies $v \in V_{\mathfrak{N}}(p_i)$ for all $v \in S$. We then obtain $\mathfrak{N}, S \vDash p_i$ by the semantic clauses. The converse direction follows by the same reasoning using the forth condition of bisimulation instead.
For the anti-support case, we have $\mathfrak{M}, T \dashv p_i \iff \forall w \in T[w \notin V_{\mathfrak{M}}(p_i)] \iff \mathfrak{M}, \{w\} \dashv p_i$. By the back condition and proposition 24, as before this implies $\forall v \in S[\mathfrak{N}, \{v\} \dashv p_i]$ which in turn implies $v \notin V_{\mathfrak{N}}(p_i)$ for all $v \in S$. We then obtain $\mathfrak{N}, S \dashv p_i$ by the semantic clauses. The converse direction is symmetrical as before.
The inductive case is split into subcases depending on the main connective for ζ. Since it can be easily verified that bisimulation is an equivalence relation, and thus in particular symmetric it suffices to show the the left-to-right direction for both claims.
If $\zeta = \neg\phi$ then the desired follows immediately by applying the induction hypothesis for anti-support. The case for anti-support of negative sentences is symmetrical using the induction hypothesis for support instead.
If $\zeta = \phi \wedge \psi$ then $\mathfrak{M}, T \vDash \phi$ and $\mathfrak{M}, T \vDash \psi$, by induction hypothesis we have $\mathfrak{M}, T \vDash \phi \iff \mathfrak{N}, S \vDash \phi$ and $\mathfrak{M}, T \vDash \psi \iff \mathfrak{N}, S \vDash \psi$ given $T \leftrightarrows S$ which then gives the desired. The case for anti-support of both global and split disjunction is symmetrical.
If $\zeta = \phi \vee \psi$ then $\mathfrak{M}, T_1 \vDash \phi$ and $\mathfrak{M}, T_2 \vDash \psi$, with $T_1 \cup T_2 = T$. By lemma 23 there exist S_1, S_2 with $S_i \leftrightarrows T_i$ and such that $S_1 \cup S_2 = S$, then, by induction hypothesis it follows that $\mathfrak{N}, S_1 \vDash \phi$ and $\mathfrak{N}, S_2 \vDash \psi$ which implies $\mathfrak{N}, S \vDash \phi \vee \psi$. The case for anti-support of conjunction is symmetrical.
If $\zeta = \blacklozenge\phi$, then $\mathfrak{M}, T_1 \vDash \phi$ for some nonempty $T_1 \subseteq T$. By setting $T_1 = T \setminus T_1$, the result follows by lemma 23 and the disjunctive case. The case for anti-support follows immediately from the inductive hypothesis.
Last, if $\zeta = \phi \mathbin{\vee\!\!\!\vee} \psi$ then $\mathfrak{M}, T \vDash \phi \mathbin{\vee\!\!\!\vee} \psi$ if and only if either $\mathfrak{M}, T \vDash \phi$ or $\mathfrak{M}, T \vDash \psi$. Without loss of generality, assume the former, as $T \leftrightarrows S$ we get $\mathfrak{N}, S \vDash \phi$ by induction hypothesis which then implies the desired by the semantic clauses. ■

We now introduce the notion of *Hintikka* or *characteristic formulae* for $\mathcal{BSL}_m^{\mathbin{\vee\!\!\!\vee}}$. Intuitively, if a team consists in a collection of points in a model, which is nothing but a collection of valuations over a a finite set of propositional variables N, then the characteristic formula of a team characterizes uniquely just which valuations this team contains. Since a team may contain multiple copies of the same valuation, but those are irrelevant for which formulae the team supports, characteristic formulae should capture not identity between teams *simpliciter*, but identity up to bisimulation. In light of proposition 24 this provides us with a tool to decide whether two teams are distinguishable from the perspective of what is expressible in $\mathcal{BSL}_m^{\mathbin{\vee\!\!\!\vee}}$.

Definition 26 (**Hintikka formulae - points**). Let $\mathfrak{M}$ be model over N, and let $w \in W_{\mathfrak{M}}$ The *characteristic formula*, or *Hintikka formula* of w, χ_w^{N} is defined as follows:

$$\chi_w^{\mathsf{N}} =_{def.} \bigwedge\{p_i \mid i \in \mathsf{N},\, w \in V_{\mathfrak{M}}(p_i)\} \wedge \bigwedge\{\neg p_i \mid i \in \mathsf{N},\, w \notin V_{\mathfrak{M}}(p_i)\}$$

Notice that N being finite ensures Hintikka formulae for points are always well-defined.
The definition is lifted from points to states in the expected way:

Definition 27 (**Hintikka formulae - teams**). Let $\mathfrak{M}$ be model over N, and $T \subseteq W_{\mathfrak{M}}$ a team. The *characteristic formula*, or *Hintikka formula* of T, $\theta^{\mathsf{N}}_{\mathfrak{M},T}$, is defined as follows:

$$\theta^{\mathsf{N}}_{\mathfrak{M},T} =_{def.} \bigvee_{w\in T} \chi^{\mathsf{N}}_{w}$$

Intuitively, Hintikka formulae for points describe the image of a propositional valuation $v : \mathsf{N} \to \mathsf{Bool}$ where the value 0 is marked with a prefixed negation. Hintikka formulae for teams express that a team is the union of a certain family of propositional valuations. It is important to notice that, since we assumed N to be finite, there are at most 2^{N} distinct valuations, and thus also at most 2^{N} distinct point-formulae χ^{N}_{w}. This guarantees that Hintikka formulae are finite and thus well-defined for any team T.
Let us now introduce the notion of a Property, which we denote with capital typescript letters:

Definition 28 (Property). Let N be a finite collection of propositional variables, property $\mathtt{P}$ is a family of pairs $\langle \mathfrak{M}, T\rangle$ where $\mathfrak{M}$ is as $\mathcal{BSL}^{\mathbb{V}}_{m}$-model over N and $T \subseteq W_{\mathfrak{M}}$ is a team.

Characteristic formulas for properties are defined as follows:

Definition 29 (Hintikka formulae - property). Let $\mathtt{P}$ be a property. The *characteristic formula*, or *Hintikka formula* of $\mathtt{P}$ over N, $\pi^{\mathsf{N}}_{\mathtt{P}}$, is thus defined:

$$\pi^{\mathsf{N}}_{\mathtt{P}} =_{def.} \mathop{\mathbb{V}}_{\langle\mathfrak{M},T\rangle\in\mathtt{P}} \theta^{\mathsf{N}}_{\mathfrak{M},T}$$

As for the case of team-properties before, notice that the number of distinct possible collections of valuations (up to extensionality) is bounded upwards by $2^{2^{\mathsf{N}}}$. Since N is finite, hintikka formulae for properties are ensured to be well-defined for any property $\mathtt{P}$.

Going forward, we will often drop both the collection of propositional variables N, and the specific model $\mathfrak{M}$ at stake when those are clear from context. We will thus often write:

$$\chi_w \qquad\qquad \theta_T = \bigvee_{w\in T} \chi_w \qquad\qquad \pi_{\mathtt{P}} = \mathop{\mathbb{V}}_{T\in\mathtt{P}} \theta_T$$

To keep our notation light. We now show Hintikka formulae effectively characterize teams up to bisimulation:

Proposition 30. Let $\mathfrak{M}$, $\mathfrak{N}$ be models over N with respective points w, v, then we have:

$$\mathfrak{M}, \{w\} \vDash \chi_v \iff \mathfrak{M}, w \leftrightharpoons \mathfrak{N}, v.$$

Proof. The right-to-left direction follows immediately from the definition of point-bisimulation and the construction of Hintikka formulae for points. To see that, if w supports χ_v then w and v are point-bisimilar, consider that $\mathfrak{M}, \{w\} \vDash \chi_v$ implies $v \in V_{\mathfrak{N}}(p_i) \iff w \in V_{\mathfrak{M}}(p_i)$ for every $i \in \mathsf{N}$. Since by assumption $\mathfrak{M}$, $\mathfrak{N}$ range over N, then they must agree on the value of all propositional variables, thus implying state-bisimilarity. ∎

Proposition 31. Let $\mathfrak{M}$, $\mathfrak{N}$ be models over N, for arbitrary teams $T \subseteq W_{\mathfrak{M}}$, $S \subseteq W_{\mathfrak{N}}$ we have:

$$\mathfrak{M}, T \vDash \theta_S \iff \mathfrak{M}, T \leftrightharpoons \mathfrak{N}, S.$$

Proof. We show both directions, which then yield the desired result.

$\Longrightarrow$ Let $\mathfrak{M}, T \vDash \theta_S = \bigvee_{v \in S} \chi_v$, since by hypothesis $T \neq \varnothing$, there exists $\{W_v\}_{v \in S}$ such that, for each v, $\mathfrak{M}, W_w \vDash \chi_v$ and $\bigcup_{v \in S} W_w = T$. Let now $x \in T$ be arbitrary, by assumption we have $x \in W_v$ for some $v \in S$, since χ_v is positive flat, from $\mathfrak{M}, W_v \vDash \chi_v$ it follows $\mathfrak{M}, \{x\} \vDash \chi_v$, Finally, by proposition 30, we obtain $x \risingdotseq v$. As x was arbitrary, this shows the forth condition.
Let now $y \in S$ be arbitrary. By hypothesis, there is some $W_y \subseteq T$ with $\mathfrak{M}, W_y \vDash \chi_y$, since W_y must be nonempty, we obtain $y \risingdotseq w$ for any $w \in W_y$ by the same reasoning as above. This shows the back condition and thus also that $T \risingdotseq S$.

$\Longleftarrow$ Let $T \risingdotseq S$, and let $y \in S$ be arbitrary. By the back condition of the bisimulation relation, there exists some $x \in T$ with $x \risingdotseq y$, which implies $\mathfrak{M}, \{x\} \vDash \chi_y$ by proposition 30. Since y was arbitrary, there exists some $R \subseteq T$ with $\mathfrak{M}, R \vDash \theta_S$ given by:

$$R = \{w \in T \mid \exists v \in S[\mathfrak{M}, \{w\} \vDash \chi_v]\}$$

However, by the forth condition of the bisimulation relation, we readily see that $T \subseteq R$, which then yields $\mathfrak{M}, T \vDash \theta_S$.

■

Closing out on this section, we first show an expressive completeness result for $\mathcal{BSL}_m^{\mathbb{V}}$, and then the finite model property for our logic. Recall that, given a collection of propositional variables N, a Property $\mathtt{P}$ is a family of teams over N. We say that a logic $\mathcal{L}$ is expressively complete for a class of properties $\mathbb{C}$ just in case the collection of teams supporting any $\mathcal{L}$-formula is precisely $\mathbb{C}$.
Let us specify some notation first: given a formula $\phi \in \mathcal{BSL}_m^{\mathbb{V}}$, we write $[\![\phi]\!]$ to denote the property thus given:

$$\{\langle \mathfrak{M}, X\rangle \mid X \subseteq W_{\mathfrak{M}} \text{ and } \mathfrak{M}, X \vDash \phi\}$$

Definition 32 (**Expressive completeness**)**.** Let $\mathbb{C}$ be a class of properties over N. A language $\mathcal{L}$ is expressively complete for $\mathbb{C}$ if and only if:

$$\mathbb{C} = [\![\mathcal{L}]\!].$$

Say that a property $\mathtt{P}$ is *closed under bisimulation* whenever, for any $\langle \mathfrak{M}, X\rangle \in \mathtt{P}$, $\langle \mathfrak{N}, Y\rangle \risingdotseq \langle \mathfrak{M}, X\rangle$ implies $\langle \mathfrak{N}, Y\rangle \in \mathtt{P}$. It is easy to see that $\mathcal{BSL}_m^{\mathbb{V}}$ is expressively complete for the class of all properties not containing the empty team and closed under bisimulation. Let:

$$\mathbb{U} = \{\mathtt{P} \subseteq \mathcal{P}(\mathsf{N} \to \mathtt{Bool}) \mid \varnothing \notin \mathtt{P} \text{ and } \mathtt{P} \text{ is closed under bisimulation}\}$$

Then the following holds:

Theorem 33 (**Expressive completeness**)**.** $\mathcal{BSL}_m^{\mathbb{V}}$ is expressively complete for $\mathbb{U}$.

Proof. We show both inclusions, which then imply $\mathbb{U} = [\![\mathcal{BSL}_m^{\mathbb{V}}]\!]$.

$\subseteq$: Let $\mathtt{P}$ be a collection of teams over N closed under bisimulation and such that $\varnothing \notin \mathtt{P}$, we show $\mathtt{P} = [\![\psi]\!]$ for some $\psi \in \mathcal{BSL}_m^{\mathbb{V}}$. Recall that for any point w in a team T and a model $\mathfrak{M}$ the characteristic formula of w, χ_w, is given by the conjunction of all the literals supported by w in $\mathfrak{M}$. Consider now the formula:

$$\pi_{\mathtt{P}} = \bigvee\!\!\!\bigvee_{S \in \mathtt{P}} \theta_S = \bigvee\!\!\!\bigvee_{S \in \mathtt{P}} \bigvee_{v \in S} \chi_v$$

As said already, up to logical equivalence, the size of $\mathtt{P}$ is bounded (upwards) by $|\mathcal{P}(\mathsf{N} \to \mathtt{Bool})|$. Similarly, the number of disjuncts in each θ_s is bounded upwards by $|\mathsf{N} \to \mathtt{Bool}|$. Since we assumed N to be finite, each θ_S has finitely many disjuncts, and the property $\mathtt{P}$ is finite as well, thus, the formula $\pi_{\mathtt{P}}$ is well-defined.

Let now $\langle \mathfrak{M}, X \rangle \in \mathtt{P}$ be arbitrary. Since bisimulation is reflexive, we have $X \leftrightarrows X$, which then implies $\mathfrak{M}, X \vDash \theta_X$ by proposition 31, thus yielding $\mathfrak{M}, X \vDash \pi_{\mathtt{P}}$ as $\langle \mathfrak{M}, X \rangle \in \mathtt{P}$ by assumption. For the reverse implication, let $\langle \mathfrak{N}, Y \rangle$ be arbitrary in $[\![\pi_{\mathtt{P}}]\!]$. Since $\mathfrak{N}, Y \vDash \pi_{\mathtt{P}}$, $\mathfrak{N}, Y \vDash \theta_K$ for some $\langle \mathfrak{L}, K \rangle \in \mathtt{P}$. By proposition 31, this implies $\mathfrak{N}, Y \leftrightarrows \mathfrak{L}, K$, as is $\mathtt{P}$ closed under bisimulation, we can then conclude $\langle \mathfrak{N}, Y \rangle \in \mathtt{P}$. With $\mathtt{P}$ arbitrary, this shows the left-to-right inclusion.

$\supseteq$: Let ϕ be an arbitrary $\mathcal{BSL}_m^{\mathbb{V}}$-formula, and consider the property $[\![\phi]\!]$. As the empty team supports no $\mathcal{BSL}_m^{\mathbb{V}}$-formula, it is immediate that $\varnothing \notin [\![\phi]\!]$. Furthermore, proposition 25 immediately implies that $[\![\phi]\!]$ is closed under bisimulation. We thus conclude $[\![\phi]\!] \in \mathtt{U}$ which shows the left-to-right inclusion.

$\blacksquare$

We now turn to showing $\mathcal{BSL}_m^{\mathbb{V}}$ enjoys the finite model property. We have seen that, over a fixed finite N, only $2^{2^{\mathsf{N}}}$ many distinct teams exists, and clearly, all of those teams must be finite whenever N also is. The task of proving the finite model property for $\mathcal{BSL}_m^{\mathbb{V}}$ therefore reduces to the simpler task of finding, given a team T, a finite representative of T's own bisimulation class. To do so, we first define the *disjoint union* of a property $\mathtt{P}$:

Definition 34 (disjoint union of a property). For $\mathbb{I}$ a collection of indices, the disjoint union of a property $\mathtt{P} = \{\langle \mathfrak{M}_i, T_i \rangle\}_{i \in \mathbb{I}}$ is defined as:

$$\biguplus \mathtt{P} = \langle \biguplus_{i \in \mathbb{I}} \mathfrak{M}_i, \biguplus_{i \in \mathbb{I}} T_i \rangle$$

Where $W_{\biguplus_{i \in \mathbb{I}} \mathfrak{M}_i} = \bigcup_{i \in \mathbb{I}} (W_{\mathfrak{M}_i} \times \{i\})$, $V_{\biguplus_{i \in \mathbb{I}} \mathfrak{M}_i}(p_k) = \bigcup_{i \in \mathbb{I}} (V_{\mathfrak{M}_i}(p_k) \times \{i\})$, $Id_{\biguplus_{i \in \mathbb{I}} \mathfrak{M}_i}$ is the identity map, and $\biguplus_{i \in \mathbb{I}} T_i = \bigcup_{i \in \mathbb{I}} (T_i \times \{i\})$.

It is a known result in modal logic that any Kripke model is point-bisimilar to its own disjoint union:

Proposition 35. Let $\{\mathfrak{M}_i\}_{i \in \mathbb{I}}$ be a family of models, for $w \in W_{\mathfrak{M}_j}$ we have:

$$\langle \mathfrak{M}_j, w \rangle \leftrightarrows \langle \biguplus_{i \in \mathbb{I}} \mathfrak{M}_i, w \rangle$$

Proof. Without loss of generality, assume $w = \langle v, j \rangle$; by the construction of $\biguplus_{i \in \mathbb{I}} \mathfrak{M}_i$ we have immediately $\langle v, j \rangle \in V_{\biguplus_{i \in \mathbb{I}} \mathfrak{M}_i}(p_k) \iff w \in V_{\mathfrak{M}_j}$. $\blacksquare$

which implies the following immediately:

Corollary 36. Let $\{\mathfrak{M}_i\}_{i \in \mathbb{I}}$ be a family of models, for $T \subseteq \mathfrak{M}_j$ a team, we have , $\mathfrak{M}_j, T \leftrightarrows \biguplus_{i \in \mathbb{I}} \mathfrak{M}_i, T$.

We are now in the position to show finite model property for $\mathcal{BSL}_m^{\mathbb{V}}$. Let $\mathsf{Sub}(\phi)$ denote the collection of subformulae of ϕ, defined in the usual way, furthermore, let $\mathsf{Var}(\phi) = \mathsf{Prop} \cap \mathsf{Sub}(\phi)$. Clearly, $\mathsf{Var}(\phi)$ is finite for any ϕ as $\mathsf{Sub}(\phi)$ is. We then have.

Theorem 37 (**Finite model property**). any $\mathcal{BSL}_m^{\mathbb{V}}$ formula ϕ, if for any model and teams $\mathfrak{M}$, T with over N with $\mathsf{N} = \mathsf{Var}(\phi)$ we have $\mathfrak{M}, T \nvDash \phi$, then there are also finite $\mathfrak{N}$, S with $\mathfrak{N}, S \nvDash \phi$.

Proof. Let ϕ be arbitrary with $\mathfrak{M}, T \nvDash \phi$, and consider θ_T. Let M be an enumeration of the disjuncts of θ_T, so that we have $\theta_T = \bigvee_{m\in\mathsf{M}} \chi_m$. Define the family of teams: $\mathtt{W} = \{\mathfrak{W}, W_{\mathfrak{W}}\}_{m\in\mathsf{M}}$ as follows:

$$\mathfrak{W}_m = \langle \{v_m\}, Id_{\{v_m\}}, V_v^{\chi_m} \rangle$$

With $V_v^{\chi_m}(p_i) = \varnothing \iff \neg p_i \in \mathsf{Sub}(\chi_m)$ for any $p_i \in \mathsf{N}$. It is easy to see that $V_v^{\chi_w}$ is well defined by the construction of Hintikka formulae for points. Notice that the number of disjuncts of θ_T is bounded upwards (up to bisimulation) by $|\mathsf{N} \to \mathsf{Bool}|$, so we can assume $\mathtt{W}$ to be finite whenever N is.
One readily verifies that for each $w \in T$, there is some $m^w \in \mathsf{M}$ with $\mathfrak{W}_m, \{v_m\} \vDash \chi_w$ which by proposition 30 implies $\mathfrak{M}, m \leftrightarroweq \mathfrak{W}_{m^w}, v_{m^w}$. By corollary 36, we also have $\mathfrak{W}_{m^w}, v_{m^w} \leftrightarroweq \biguplus \mathtt{W}, v_{m^w}$, which shows for every $w \in T$ there is some $v \in \{v_m\}_{m\in\mathsf{M}}$ with $w \leftrightarroweq v$. Conversely, for every $v \in \{v_m\}_{m\in\mathsf{M}}$ by construction we have $\biguplus \mathtt{W}, w \vDash \chi_m$ for some $m \in \mathsf{M}$ and therefore $v \leftrightarroweq w$ for some $w \in T$. We then obtain $\biguplus \mathtt{W}, \{v_m\}_{m\in\mathsf{M}} \leftrightarroweq \mathfrak{M}, T$. By proposition 25, this entails $\biguplus \mathtt{W}, \{v_m\}_{m\in\mathsf{M}} \nvDash \phi$ which yields the desired as $\mathtt{W}$ is finite by construction. ■

At last, decidability of our logic then follows as an easy corollary:

Corollary 38 (**Decidability**). $\mathcal{BSL}_m^{\mathbb{V}}$ is decidable.

Proof. Let ϕ, $\psi \in \mathcal{BSL}_m^{\mathbb{V}}$ and set $\mathsf{N} = \mathsf{Var}(\phi) \cup \mathsf{Var}(\psi)$. By theorem 37, if $\phi \vDash \psi$ over all finite models, then $\phi \vDash \psi$. As there are only finitely many distinct Hintikka formulas for teams over N, there are only finitely many finite models up to bisimulation. ■

2.2 The system NFM_{ne}

In this section, we provide an axiomatization of $\mathcal{BSL}_m^{\mathbb{V}}$ by presenting the system NFM_{ne}. While N is the usual Gentzen prefix for natural-deduction systems, F and M stand for *free-choice* and *multilateral* respectively. Last, the suffix *ne* flags *non-emptiness*, as our system encodes reasoning under neglect-zero assumptions.
In the context of proof-theoretic semantics, different kinds of multilateral calculi have been proposed, with the general common intent of providing a suitable *inferential* theory of meaning for natural language expressions (see e.g., [WA23],[Fra19]). Within inferential expressivism specifically, the most prominent calculus is "Epistemic Multilateral Logic" (EML, in [IS23], with an earlier version in [IS22]) which supports formally Incurvati and Schlöder account of Yalcinean sentences outlined in section 1.1. With the system NFM_{ne}, our goal is to extend Incurvati and Schlöder's EML to cover free-choice phenomena as well as reasoning under neglect-zero assumptions.
To recount, the central tenet of inferential expressivism is that a well-formed sentences of a language *expresses* a certain attitude the speaker holds with regard to the the content of the sentence itself. Furthermore, to hold an attitude with respect to some content is to *use* this content in a certain way

to achieve something in a linguistic interaction. As a consequence, what it means to hold an attitude with any *determinate* content is explained in terms of what other attitudes with different contents this content-and-attitude commits the speaker to having. More concisely:

> The semantic value of a term is given by the inferential relations that an attitude expression towards a sentence containing the term bears to other attitude expressions in virtue of the fact that the sentence contains the term. ([IS23] p. 71)

Clearly, as the meaning of sentences is not determined by interpreting its content over a model, but by specifying what the sentence allows the speaker to do in terms of linguistic interactions, the minimal meaning-bearing piece of language must be one that can reasonably be taken up by a speaker to do something with. According to Michael Dummett ([Dum91],[Dum81])[1], the minimal structure a piece of language must possess to be susceptible of linguistic use comprises what we have called a *content* which is used in a certain way, and a *signature* or "force marker", which specifies which attitude is entertained with respect to the content. That is, it specifies what is being done with the content in the conversational context.

We refer to this minimal unit as a "sentence", or equivalently, as "signed formula". In the following, we fix a set of signatures Sig and we will write:

$$^{\bullet}\phi$$

where $\bullet \in \mathsf{Sig}$ and $\phi \in \mathcal{BSL}_m^{\mathbb{V}}$, to express the piece of language obtained by entertaining the attitude specified by $\bullet$ with respect to the content ϕ. Furthermore, we refer to "$\bullet$" as the *signature*, and to "ϕ" as the *content* of the sentence $^{\bullet}\phi$. We assume all $\mathcal{BSL}_m^{\mathbb{V}}$-formulae to constitute contents of sentences, if not further qualified.

We said the signature of a sentence marks just what the content is being used for in the context of conversation. Clearly, one further ingredient needed for a satisfactory meaning-explanation along inferentialist lines is an account of what it is, in general, to contribute in some way to a linguistic or conversational interaction. Here, we follow Incurvati and Schlöder by adopting a Stalnakerian ([Sta02]) model of conversation. According to this view, conversational interaction takes place against a *positive common ground* of beliefs[2] which is assumed to be shared by all participants. Similarly, we assume the presence of a *negative common ground* of disbeliefs which is likewise shared by all participants. We remain agnostic with respect to the notions of *belief* and *disbelief*, if not for the fact that we take both of those to be mental (or intentional) states of the participants, and that we assume it to be incoherent, for any thinking individual, to believe and disbelieve the same content at the same time. This caveat puts a first constraint on both the positive and the negative common grounds: whatever content appears in the one shall not also appear in the other. Further constraints will be later introduced through the rules of our system.

We turn now to the effective exposition of the system, beginning by outlining our signatures of choice, and explaining their function with respect to both common grounds.

[1]See also [Mar17] for further historical roots of this idea.

[2]for reasons linked with the structure of reflexive intentions (see [Gri91]) Stalnaker takes common grounds to comprise, besides beliefs, *acceptances*. Where acceptance is a broader notion than the former, corresponding roughly to provisional endorsement of a content for some conversational purpose. In the following, we'll stick to the notion of Belief, but all is said can be easily rephrased in terms of acceptance.

2.2.1 Signatures

Our collection of signatures is given below:

$$\mathsf{Sig} = \{+, -, \oplus, \ominus\}$$

We refer to a formula prefixed with "+" as "asserted", and to contents prefixed with "−" as "rejected". Further, contents prefixed with "⊕" are said to be "weakly asserted", and those prefixed with "⊖" are "weakly rejected". Accordingly, the elements of Sig stand for the attitudes of (from left to right) "assertion", "rejection", "weak assertion", and "weak rejection". We also call the collection $\mathsf{S} = \{+, -\}$ "strong signatures", and the collection $\mathsf{W} = \{\oplus, \ominus\}$ "weak signatures". Moreover, we say a formula is *positively signed* if it is prefixed with + or ⊕, and we say it si *negatively signed* if it is prefixed with − or ⊖.

The symbols $\bullet$, $\circ$ are used as variables over Sig. At times, it will prove useful to also have specific variables for S and W, for which we employ ${}^{\bullet}{}_{s}\phi$, ${}^{\circ}{}_{s}\phi$ and ${}^{\bullet}{}_{w}\phi$, ${}^{\circ}{}_{w}\phi$, respectively. We take filled and empty variables to stand for distinct signatures; if they range over distinct sets, then we assume (if not otherwise specified in context, for example by saying that $\circ \in \mathsf{S}$) one of the two to be either assertion or weak assertion, and the other to be either rejection or weak rejection. For example, the pair ${}^{\bullet}{}_{s}\phi$, ${}^{\circ}{}_{w}\psi$ stands for one of the following combinations: ${}^{+}\phi$, ${}^{\ominus}\psi$ or ${}^{-}\phi$, ${}^{\oplus}\psi$, or ${}^{+}\psi$, ${}^{\ominus}\phi$ or ${}^{-}\psi$, ${}^{\oplus}\phi$.

What does it mean, according to our Stalnakerian model of conversation, to entertain either of the attitudes just introduced? Regarding assertion, we take the conversational effect of asserting a certain content ϕ to be a proposal of updating the positive common ground of belief with the content ϕ. In this sense, to assert something is to propose a certain update on the common ground of shared beliefs. Conversely, to reject something is to propose an update on the negative common ground of shared disbeliefs. It should be noticed that, although closely related, this is a distinct act that to propose an update of the positive common ground with the content $\neg\phi$.

If assertion and rejection fulfil the purpose of adding contents to either common ground, the story is not the same for attitudes expressed by weak signatures. Following Incurvati and Schlöder, we take weak assertion and rejection to express refrain, on part of the speaker, from rejecting a certain content. Therefore, to weakly assert a content ϕ is to block a successful rejection of ϕ, without committing oneself to the assertion of ϕ. Similarly to weakly reject a content ϕ is to block a successful assertion of ϕ, without thereby committing oneself to rejecting ϕ firsthand. As discussed in the introduction, the kind of linguistic data motivating the adoption of weak assertion and rejection is that evidence-unspecific assertions and denials:

A:"X will win the election!" B: "No, X or Y will win"

A: "X will not win the election!" B: "Perhaps she'll win"

(Adapted from ([IS23] pp. 87, 105.)

From the standpoint of the Stalnakerian common grounds, we can thus understand weak rejection as preventing the update of the positive common ground with the content proposed by a participant in the conversation. Likewise, we understand weak assertion as preventing the update of the negative common ground with the content proposed by a participant in the conversation. As Stalnaker ([Sta78]) puts it:"If an assertion is rejected, the context remains the same as it was."

Together with the *dialogical* interpretation here given, a parallel *epistemic*, or *intentional* interpretation of the signatures can be given. We saw how the dialogical interpretation explains the conversational effect of the speaker's attitude in terms of their contribution to two common grounds of beliefs and disbeliefs. Since beliefs and disbeliefs are taken to be mental, or intentional states of the speaker, it is also possible to understand the different attitudes as conversational correspondents of the speaker's own mental states. Given the explanations above, it is only natural to take assertion as the correspondent to the mental state of *belief*, and rejection as the correspondent to the mental state of *disbelief*. The two weak signatures of weak assertion and weak rejection can then be understood as believing to be plausible, which we refer to as *permitting*, and believing to be plausibly false, which we may christen *doubting*. The rationale behind the interpretation is as follows: if a speaker is refrains from asserting a content ϕ, and that is, is committed to preventing updates of the common ground of beliefs with ϕ, this must be because they have a ground to *doubt* ϕ, which means they conceive of at least one plausible situation where ϕ would be rejected. Similarly, if a speaker is committed to preventing updates of the negative common ground with ϕ, this must be because they have a ground to *permit* ϕ, namely they can conceive of at least one plausible situation where ϕ would be accepted instead.

2.2.2 Rules

We now turn to outlining the rules of our system NFM_{ne}. In the following, we adopt the convention that calligraphic latin letters refer to derivation trees in NFM_{ne}, while capital Greek letters refer to finite sets of sentences. First, we introduce some basic notions:

Definition 39 (Derivation-tree). Given a finite set of signed formulae Γ and a signed formula ${}^\bullet\phi$ with $\phi \in \mathcal{BSL}_m^{\mathbb{V}}$ a NFM_{ne} derivation-tree $\mathcal{D}$ of ${}^\bullet\phi$ from Γ (notation: $\mathcal{D} : \Gamma \vdash {}^\bullet\phi$) is defined as follows:

1. If ${}^\bullet\phi \in \Gamma$, then:

$$\frac{}{{}^\bullet\phi}$$

   is a derivation tree of ${}^\bullet\phi$ from Γ.
2. If $\Gamma = \varnothing$ and ${}^\bullet\phi =^+ \pi_{[\![\top]\!]}$ then:

$$\frac{}{{}^+\pi_{[\![\top]\!]}}\, Ne.$$

   is a derivation tree of ${}^\bullet\phi$ from Γ;
3. If $\mathcal{C}_n, ..., \mathcal{C}_n$ are derivation trees of, respectively, ${}^\bullet\psi_1$,...,${}^\bullet\psi_n$ from Γ_1,..., Γ_n with $\Gamma = \Gamma_1 \cup ... \cup \Gamma_n$, then:

$$\frac{\begin{matrix}\mathcal{C}_1 \\ {}^\bullet\psi_1\end{matrix} \quad ... \quad \begin{matrix}\mathcal{C}_n \\ {}^\bullet\psi_n\end{matrix}}{{}^\bullet\phi}\, \mathrm{R}$$

   is a derivation tree of ${}^\bullet\phi$ from Γ, provided "R" is one of the rules listed in section 2.2.2.1.

Definition 40 (Height). Given a derivation tree $\Gamma \vdash {}^\bullet\phi$ for NFM_{ne} $\mathcal{D}$, the Height of $\mathcal{D}$ (notation: $H(\mathcal{D})$) is recursively defined as follows:

1. If $\mathcal{D}$ is of the form $\frac{}{{}^\bullet\phi}$ with ${}^\bullet\phi \in \Gamma$, then $H(\mathcal{D}) = 0$;

2. If $\mathcal{D}$ is of the form $\overline{{}^{+}\pi_{[\![\top]\!]}}$, then $H(\mathcal{D}) = 0$;
3. If $\mathcal{D}$ is of the form $\dfrac{\overset{\mathcal{C}_1}{{}^{\bullet}\psi_1} \quad \dots \quad \overset{\mathcal{C}_n}{{}^{\bullet}\psi_n}}{{}^{\bullet}\phi}\,\mathrm{R}$, then $H(\mathcal{D}) = \max(H(\mathcal{C}_1), ..., H(\mathcal{C}_n)) + 1$.

Definition 41 (**Open assumptions**)**.** Given a derivation $\mathcal{D}$ of ${}^{\bullet}\phi$ from Γ the set of "open assumptions" of $\mathcal{D}$ (notation:$\mathsf{Open}(\mathcal{D})$) is recursively defined as follows:

1. If $H(\mathcal{D}) = 0$ then $\mathsf{Open}(\mathcal{D}) = \Gamma$;
2. If $H(\mathcal{D}) = m + 1$ then $\mathcal{D}$ is as follows:

$$\dfrac{\overset{\mathcal{C}_1}{{}^{\bullet}\psi_1} \quad \dots \quad \overset{\mathcal{C}_n}{{}^{\bullet}\psi_n}}{{}^{\bullet}\phi}\,\mathrm{R}$$

The set of open assumptions $\mathsf{Open}(\mathcal{D}) = (\mathsf{Open}(\mathcal{C}_1) \cup ... \cup \mathsf{Open}(\mathcal{C}_n)) \setminus SQ$ where SQ is the set of sentences appearing within square brackets in R.

In the following, we are going to employ two distinct kinds of restrictions on subderivation, which we mark as "$*$" and "$\star$" respectively. Intuitively, both of those restrictions state that sentences which fail to be down-closed (see definition 42 just below) may be used to weaken the conclusion of a (sub)derivation by introducing a disjunction, but may not be used to strengthen the conclusion with a conjunction, or to derive contradictions. Another way to read the restrictions is as articulating the requirement that contents which are *merely permitted* (*doubted*) may not be further specified by other merely permitted (doubted) contents. What is the rationale behind this requirement? One possible reading is that contents which are merely permitted or doubted are not *evidence specific*, in the sense that they appear in conversation as weak assertions or rejections, which need not to be supported by full belief or disbelief in the uttered content. But what sentences should be considered evidence-specific? Incurvati and Schlöder mark formulae prefixed with $\oplus$, $\ominus$, as well as those obtained by elimination of "might" as evidence-unspecific. We shall follow them in counting signed formulae, together with contents embedded under "might" as evidence-unspecific. Furthermore, one effect of our neglect-zero reasoning assumption in reasoning is that disjunctive formulas as well are interpreted as lists of epistemic possibilities as well.[3] Since epistemic possibilities are evidence-unspecific, this provides a reason to doubt disjunctive formulae should be taken as evidence specific as well. These considerations suggests to heuristically identify the "evidence-specific" formulas of $\mathcal{BSL}_m^{\mathbb{V}}$ with the collection of down-closed sentences D^+ (or D^- for anti-support). As a result, both of our restrictions specify that the subderivation may use non-evidence specific premisses if not to weaken the derivation by introducing a disjunction.

Definition 42 (**Down-closed sentences**)**.** A *sentence* ${}^{\bullet}\phi$ is down-closed if and only if either of the following hold:

1. $\bullet = +$ and, if ϕ is not negation-prefixed, then $\phi \in \mathsf{D}^+$, and if $\phi = \neg\psi$ then $\psi \in \mathsf{D}^-$;
2. $\bullet = -$ and, if ϕ is not negation-prefixed, then $\phi \in \mathsf{D}^-$, and if $\phi = \neg\psi$ then $\psi \in \mathsf{D}^+$,

Remark 43. In the following, we often speak of a signed formula ${}^{\bullet}\phi$ occurring in $\mathsf{Open}(\mathcal{D})$ (with $\mathcal{D}$ given derivation) being down-closed or not. Formally, this amounts to the following requirement:

[3]See the proposal in [Zim00] and [Geu05]. The connection with neglect-zero is argued for in [Alo22].

1. Either ${}^{\bullet}\phi$ is a down-closed sentence and ${}^{\bullet}\phi \in \mathsf{Open}(\mathcal{D})$; or,
2. If an instance of ${}^{\bullet}\phi$ appears as a premiss in $\mathcal{D}$, ${}^{\bullet}\phi$ is a down-closed sentence and there exists a subderivation $\mathcal{E}$ of $\mathcal{D}$ such that $\mathcal{E} : E \vdash {}^{\bullet}\phi$ with $E \subseteq \mathsf{Open}(\mathcal{D})$. In particular, this means ${}^{\bullet}\phi$ cannot depend on undischarged open assumption induced by rules within $\mathcal{D}$.

The rationale of this construction is to avoid attributing to sentences of $\mathcal{D}$ the wrong status. For example, if $\mathcal{D}$ shall not use signed formulae with a non-down-closed content, then the usage of ${}^{+} \blacklozenge\, p_i$ is illicit, while the usage of ${}^{+}p_j \wedge p_k$ is lawful. However, there is no way to distinguish between those formulae when they appear conjoined in the open assumptions of $\mathcal{D}$, which happens for example when we have:

$$\mathcal{D} :{}^{+} \blacklozenge\, p_i \wedge (p_j \wedge p_k) \vdash {}^{\bullet}\psi$$

The remark above allows us to treat all formulae appearing as premisses in a derivation as if they were part of the assumption set, which lightens the notation in the context of proofs. As a result e.g., the usage of ${}^{+}p_j \wedge p_k$ in $\mathcal{D}$ in the derivation above is licit, while the usage of ${}^{+} \blacklozenge\, p_i \wedge p_j$ is not. It is easy to see that this brings about no change in the derivability and support relations.[4] With this formal proviso in place, we will in practice simply write ${}^{\bullet}\phi \in \mathsf{Open}(\mathcal{D})$ is (not) down-closed, understanding that intermediate subderivations may be ignored for this purpose.

The formal definition[5] of the restrictions is given below (rules referred to in the following definitions are found in boxed in section 2.2.2.1):

Definition 44 (**Restrictions**)**.** The symbols "$*$" and "$\star$" qualify derivation trees occurring as premisses of a rule application R. Let SQ be the collection of sentences within squared brackets occurring in the rule R. The restrictions are then defined as follows:

1. A $*$-marked derivation tree $\mathcal{D}^*$ is such that every signed formula ${}^{\bullet}\phi$ in $\mathsf{Open}(\mathcal{D}) \setminus SQ$ is either down-closed, or either:
   (a) There are non-downclosed formulae occurring in $\mathsf{Open}(\mathcal{D}) \setminus SQ$ ${}^{\bullet}\delta_1, ..., {}^{\bullet}\delta_n$. For $1 \leq i \leq n$, there is a derivation $\mathcal{C}_i$ from ${}^{\bullet}\delta_i$ to ${}^{\bullet}\psi_i$ where the latter occurs as a premiss of the rules ${}^{+} \vee I$, or ${}^{-} \wedge I$ in $\mathcal{D}$. Letting:

[4] In particular, in the context of the soundness proof below (theorem 48), this creates no differences in the base case. In the inductive case, it is sufficient to apply the inductive hypothesis one additional time to the subderivation $\mathcal{E}$ itself.

[5] For reference, and ease of readability, we give a brief heuristic overview of the formal definitions laid out in the text:

1. $*$ is mainly enforced on rules of Weak inference (see below). In a derivation marked with asterisk, only down-closed sentences may appear as premisses, non-downclosed sentences can only be used to weaken the conclusion with a split disjunction (or conjunction under rejection). The splitting of the derivation into several equivalent $\mathcal{D}_1 ... \mathcal{D}_n$ is needed to accommodate the normalization process (chapter 3).
2. $\star$ is enforced on pairs of derivation induced by the elimination of split disjunction under assertion, and conjunction under rejection. The $\star$ restriction states that, if we are to use the rules above to derive a sentence which is neither a split disjunction under assertion, nor a conjunction under rejection, then such sentence must be union closed (This corresponds to the case where one infers an evidence-preserving specific sentence because it holds in all conceivable epistemic possibilities). Moreover, one cannot use non-downclosed premisses in the derivations if not to weaken the conclusion with split disjunctions under assertions or conjunctions under rejection. However, one may use a down-closed sentence embedded under "might" to derive a contradiction, provided it contradicts both disjuncts (conjuncts) equally.
   If instead the derivations operate on each disjunct (conjunct) separately, then the restriction $*$ is enforced on both of them, furthermore, the sentence concluding the two subderivations must preserve the shape of the leftmost premiss, meaning it has to either be a conjunction under rejection or a disjunction under assertion. In this case, there is no need for the conclusion to be union closed. No restrictions are enforced when it comes to associativity of the connectives. Once again, the division of $\mathcal{C}$, $\mathcal{D}$ in two subderivations each is meant to allow permutation conversions in the normalization process (section 3.2.3).

$$\mathcal{D} = \begin{array}{c} \mathcal{D}_0 \\ \vdots \\ \mathcal{D}_n \end{array}$$

Where $\mathcal{D}_i$ is the tree starting from the application of $^+ \vee I$ or $^- \wedge I$ with $^\bullet\psi_i$ as a premiss, then we have that elements of SQ as well as sentences in $\mathcal{C}_i$ do not appear in $\mathcal{D}_i$ if not as a premiss of a $\downarrow A$, $\downarrow R$ rule for $i \neq 0$.

(b) for $^\bullet\psi$ as in (a), $\bullet \in \mathsf{S}$, ψ is $\blacklozenge$-free, and $^\bullet\psi$ occurs as a premiss of an EC rule.

2. A pair of $\star$-marked derivations $\mathcal{D}^\star$, $\mathcal{C}^\star$ are such that we have:

$$\mathcal{D} = \begin{array}{c} \mathcal{D}_1 \\ ^\bullet\phi/\bot \\ \mathcal{D}_2 \end{array} \text{ and } \mathcal{C} = \begin{array}{c} \mathcal{C}_1 \\ ^\bullet\phi/\bot \\ \mathcal{C}_2 \end{array}$$

Where, if $\bullet \in \{+, \oplus\}$ then $\phi \in \mathsf{U}^+$ and if $\bullet \in \{-, \ominus\}$ then $\phi \in \mathsf{U}^-$. Furthermore it either holds:

(a) $\mathsf{Open}(\mathcal{D}_1) \setminus SQ$, $\mathsf{Open}(\mathcal{D}_1) \setminus SQ$ contain only down-closed sentences, and $\mathsf{Open}(\mathcal{C}_2) = \mathsf{Open}(\mathcal{D}_2)$, and elements of SQ do not appear in $\mathcal{C}_2$, $\mathcal{D}_2$ if not as premisses of $\downarrow A, \downarrow R$;

(b) $\mathsf{Open}(\mathcal{C}_1) \setminus SQ$, $\mathsf{Open}(\mathcal{D}_1) \setminus SQ$ contain each a non-down-closed sentence (or a conjunction thereof) $^\bullet\psi_1$ $^\bullet\psi_2$ and there are derivations $\mathcal{E}_1$ $\mathcal{E}_2$ from $^\bullet\psi_i$ concluding a premiss of either $^+ \vee I$, or $^- \wedge I$ to then conclude $^\bullet\phi$ in the last rule application of both $\mathcal{C}_1$ and $\mathcal{D}_1$. Further, $\mathsf{Open}(\mathcal{C}_2) = \mathsf{Open}(\mathcal{D}_2)$ and no element of either SQ or $\mathcal{E}_1$, $\mathcal{E}_2$ appears in $\mathcal{C}_2$, $\mathcal{D}_2$ if not as a premiss of a $\downarrow A$, $\downarrow R$ rule.

(c) $\mathsf{Open}(\mathcal{C}_1) \setminus SQ$, $\mathsf{Open}(\mathcal{D}_1) \setminus SQ$ contain each a non-down-closed sentence $^\bullet\zeta$ with a derivation $\mathcal{E}$ from $^\bullet\zeta$ to a non-down-closed $^\bullet\psi$ such that: $\bullet = \oplus$ and $\psi \in \mathsf{D}^+$ or $\bullet = +$ and $\psi = \blacklozenge\mu$ with $\mu \in \mathsf{D}^+$, or $\bullet = \ominus$ and $\psi \in \mathsf{D}^-$. Further, there are derivations $\mathcal{F}_1$ $\mathcal{F}_2$ from $^\bullet\psi$ concluding a premiss of either an EC rule to then conclude $\bot$ in the last rule application of both $\mathcal{C}_1$ and $\mathcal{D}_1$. Last, $\mathsf{Open}(\mathcal{C}_2) = \mathsf{Open}(\mathcal{D}_2)$ and and no element of SQ or $\mathcal{E}$ appears in $\mathcal{C}_2$, $\mathcal{D}_2$ if not as a premiss of a $\downarrow A$, $\downarrow R$ rule.

(d) The leftmost premiss of the Rule $^\bullet\zeta$ is of the shape $^+(\kappa \vee \nu) \vee \rho$ (or $^-(\kappa \wedge \nu) \wedge \rho$)) and $\mathcal{D}_1 :^+ \kappa \vee \nu, {}^+(\kappa \vee \nu) \vee \rho \vdash {}^\bullet\psi$ and $\mathcal{C}_1 :^+ \rho, {}^+(\kappa \vee \nu) \vee \rho \vdash {}^\bullet\psi$ with $^\bullet\psi =^+ \kappa \vee (\nu \vee \rho)$ ($\mathcal{D}_1 :^- \kappa \wedge \nu, {}^-(\kappa \wedge \nu) \wedge \rho \vdash {}^\bullet\psi$ and $\mathcal{C}_1 :^- \rho, {}^-(\kappa \wedge \nu) \wedge \rho \vdash {}^\bullet\psi$ with $^\bullet\psi =^- \kappa \wedge (\nu \wedge \rho)$), and $\mathsf{Open}(\mathcal{C}_2) = \mathsf{Open}(\mathcal{D}_2)$. Furthermore, elements of SQ do not appear in $\mathcal{C}_2$, $\mathcal{D}_2$ if not as premisses of $\downarrow A$, $\downarrow R$.

Otherwise, if $^\bullet\phi$ is not a down-closed sentence, we either have the same situation as (d), or else the rule closing $\mathcal{D}_2$, $\mathcal{C}_2$ is $^+ \vee E$ with leftmost premiss $^+\mu \vee \nu$, or $^-\mu \wedge \nu$. Further $\mathcal{C}_1$, $\mathcal{D}_1$ respect the $*$ (asterisk) restriction, and there are subderivations of $\mathcal{C}_1$, $\mathcal{D}_1$ respectively: $\mathcal{J} : \mathsf{Open}(\mathcal{C}_1), {}^+\mu \vdash {}^\circ\lambda$ and $\mathcal{K} : \mathsf{Open}(\mathcal{D}_1), {}^+\nu \vdash {}^\circ\kappa$ (or $\mathcal{J} : \mathsf{Open}(\mathcal{C}_1), {}^-\mu \vdash {}^\circ\lambda$ and $\mathcal{K} : \mathsf{Open}(\mathcal{D}_1), {}^-\nu \vdash {}^\circ\kappa$) with $\circ \in \mathsf{S}$. Last, if $\circ = +$, then $^\bullet\phi =^+ \lambda \vee \kappa$, and if $\circ = -$, then $^\bullet\phi =^- \lambda \wedge \kappa$.
As before, elements of SQ do not appear in $\mathcal{C}_2$, $\mathcal{D}_2$ if not as premisses of $\downarrow A$, $\downarrow R$, and we have $\mathsf{Open}(\mathcal{C}_2) = \mathsf{Open}(\mathcal{D}_2)$

Restrictions over derivations are written in in-line notation as $\Gamma * \vdash {}^\bullet\phi$, $\Gamma \star \vdash {}^\bullet\phi$ or jointly as $\Gamma \star * \vdash {}^\bullet\phi$. When marked derivations appear a sub-derivation within a more general tree, the marks must be taken to mean that the subderivation preserve compliance with the restrictions: if the derivations up to the premisses respects $\star$, $*$, then extending it with a marked subderivation yields a tree complying with $\star$, $*$.

Last, as customary in Inferential expressivism, we take inferences to be *commitment-preserving* rather than *evidence-preserving*. In turn, we assume commitment to be a "virtual" notion, meaning that any speaker committed to the premisses of an inference must take up the attitude and content expressed in the conclusion only when and if the issue is raised.

2.2.2.1 Primitive rules

We now turn to the *structural rules* of NFM_{ne}. Those are rules that do not determine the meaning of any term of the language in particular, rather, they define the lawful interactions between signatures in conversation. In other words, they coordinate the different attitudes a speaker may entertain with respect to some content on the backdrop of some shared common grounds of beliefs and disbeliefs. In other words, structural rule encode features of dialogical interaction *in general*, independently of the specific kind of contents uttered uttered in the situation.
We said that for a speaker to assert a content ϕ is for them to propose the addition of ϕ to the common background of shared beliefs. As the same speaker shares by assumption the negative common ground of shared disbelief, and it is incoherent for any individual to both believe and disbelieve the same content ϕ, any speaker committed to the assertion of ϕ must likewise be committed to weakly assert of ϕ i.e,. to preventing any addition of ϕ to the negative common ground. By a similar reasoning, any speaker committed to the rejection of a content must also be committed to the weak rejection of it. We are therefore justified in endorsing the following two *downscale* rules:

Downscale (AR) rules:

$$\frac{{}^{+}\phi}{{}^{\oplus}\phi}\downarrow A. \qquad\qquad \frac{{}^{-}\phi}{{}^{\ominus}\phi}\downarrow R$$

The dialogical function of weak sentences is codified by the rules of *Weak Inference.* We refer to those two jointly as WI rules.

Weak inference (WI) rules:

$$\frac{{}^{\oplus}\phi \qquad \begin{array}{c}[{}^{+}\phi]\\ \mathcal{D}^{*}\\ {}^{+}\psi/\bot\end{array}}{{}^{\oplus}\psi/\bot}\,W.I._{1} \qquad \frac{{}^{\oplus}\phi \qquad \begin{array}{c}[{}^{+}\phi]\\ \mathcal{D}^{*}\\ {}^{\oplus}\psi/\bot\end{array}}{{}^{\oplus}\psi/\bot}\,W.I._{2}$$

$$\frac{{}^{\ominus}\phi \qquad \begin{array}{c}[{}^{-}\phi]\\ \mathcal{D}^{*}\\ {}^{-}\psi/\bot\end{array}}{{}^{\ominus}\psi/\bot}\,W.I._{3} \qquad \frac{{}^{\ominus}\phi \qquad \begin{array}{c}[{}^{-}\phi]\\ \mathcal{D}^{*}\\ {}^{\ominus}\psi/\bot\end{array}}{{}^{\ominus}\psi/\bot}\,W.I._{4}$$

The rules of weak inference jointly state that the attitudes of weak assertion and rejection are closed under derivability. If a speaker refrains from asserting a certain content ϕ, that is, refuses to update the positive common ground with ϕ, then they must also refrain from adding to the positive common ground any further content which they would assert only provided they also asserted ϕ. On the epistemic interpretation, the rules are justified by considering that whenever a speaker conceives ϕ as plausible, they must also conceive as plausible anything that believing ϕ commits them to either believe or permit. Similarly, whenever a speaker doubts ϕ, they should also doubt anything that rejecting ϕ

commits them to reject or doubt.
The restriction $*$ is enforced on all four rules, requiring that strengthening of ϕ occurs only via evidence-specific sentences. To witness, assuming "perhaps" is a linguistic marker for weak assertions as in [IS19], from the utterances of "Perhaps X will win the elections" and "Perhaps Y will win the elections" we shall not conclude "Perhaps X will win the election and Y will win the election". This kind of vicious inferences are here blocked by the enforcement of $*$.

The following two rules of *Epistemic contradiction* internalize the semantic inconsistency of Yalcinean sentences into NFM_{ne}. We sometimes refer to these two rules as EC rules.

Epistemic contradiction (EC) **rules:**

$$\frac{{}^{+}\phi \qquad {}^{\ominus}\phi}{\bot}\ EC_1 \qquad\qquad \frac{{}^{-}\phi \qquad {}^{\oplus}\phi}{\bot}\ EC_2$$

The rules state that it is incoherent for a speaker to both asserting and refraining from asserting a certain content. Likewise, it is incoherent for a speaker to both deny and refraining from denying a certain content. As customary in the inferentialist tradition, we take the $\bot$ symbol to work as a punctuation symbol, indicating that a dead end of the conversational interaction has been reached (see e.g.,[Ten99]). Accordingly, the symbol $\bot$ is the only non-signed symbol occurring in NFM_{ne}. For ease of readability, we will sometimes treat $\bot$ as a signed formula, and we shall assume that for elimination and introduction rules manipulating signed formulae ${}^{\bullet}\phi_1 ... {}^{\bullet}\phi_n$, $\bot$ can figure as some ${}^{\bullet}\phi_i$ for $1 \leq i \leq n$. Since restriction affect differently derivations concluding $\bot$ rather than other signed formulae, we shall make the $\bot$ explicit instead in the case of rules employing marked derivations as premisses as in the case of weak inferences above. The working of the rules is otherwise unaffected.

The next rule of *Non-emptiness*, bridges together our inferentialist treatment of NFM_{ne} with the semantic approaches to natural language studied in the context of team semantics. According to ([Alo22]) the perceived validity of inferences as those listed in proposition 8, is linked with the tendency in human cognition to disregard "zero-model configurations" i.e., not to reason about situations which would validate vacuously the premisses of an inference. Given the epistemic interpretation of the signatures, we can incorporate this statement by requiring that the belief (or disbelief) basis shared by our speakers never contains zero-models, or empty configurations. This is tantamount to requiring that no speaker's assertion of a content ϕ is the linguistic realization of them believing ϕ because it is vacuously true. Similarly, no speaker shall refrain from denying ϕ only because they conceive it to be plausible in some never-occurring situation.
The nonemptiness rule achieves this by stating that the positive common ground shared by all participants is not empty: in whatever conversational interaction, any participant is unconditionally committed to asserting that the current common grounds of shared beliefs realizes *some* nonempty configuration. Formally, this is done by postulating the unconditional assertion of the Hintikka formula for the property $[\![\top]\!]$ as a structural rule of NFM_{ne}.

Non-emptiness rule:

$$\frac{}{{}^{+}\pi_{[\![\top]\!]}}\ Ne.$$

Last, we have *Ex Falso Quodlibet*, which we follow [Ste09] in classifying as a structural rule.

Ex Falso Quodlibet:

$$\frac{\bot}{{}^{\bullet}\phi}\ \bot E$$

The rule of *Ex falso* specifies how to handle the specific conversational dead-end marked by the $\bot$ punctuation mark.[6]

This exhausts the stock of structural rules of NFM_{ne}. when comparing those to other bilateral and multilateral natural deduction systems, notable absents are the *reductio* rules.[7] It is easy to see that, by employing the rules for *reductio*, it is possible to derive a proof-theoretic version of the reversal entailment, which was seen in section 2.1.4.1 to be invalid over $\mathcal{BSL}_m^{\mathbb{V}}$ models. In most multilateral systems, the rule of reductio permits to only specify rules for asserted connectives, while retaining those for rejected connectives as derived. Since reductio is not admissible in NFM_{ne}, we will specify rule for both asserted and rejected connectives in what follows. We now turn to the rules for logical constants.

Rules for asserted logical constants are given below, where we use "/" to indicate that from the premisses one may infer either of the conclusion separated by the stroke. If "/" occurs among the premisses instead, the speaker is licensed in inferring the conclusion from either the premisses separated by the stroke. In terms of interpretation, recall we retain split disjunction "$\vee$" to render natural language disjunction, especially in modal contexts, while the global disjunction "$\mathbb{V}$" functions as a question-forming device.[8]

Asserted constants rule:

$$\frac{{}^{+}\phi \qquad {}^{+}\psi}{{}^{+}\phi \wedge \psi}\ {}^{+}\wedge I \qquad\qquad \frac{{}^{+}\phi \wedge \psi}{{}^{+}\phi / {}^{+}\psi}\ {}^{+}\wedge E$$

$$\frac{{}^{+}\phi \qquad {}^{\oplus}\psi}{{}^{+}\phi \vee \psi}\ {}^{+}\vee I \qquad\qquad \frac{{}^{+}\phi \vee \psi \qquad \begin{matrix}[{}^{+}\phi] \\ \mathcal{D}^{\star} \\ {}^{\bullet}\zeta/\bot\end{matrix} \qquad \begin{matrix}[{}^{+}\psi] \\ \mathcal{E}^{\star} \\ {}^{\bullet}\zeta/\bot\end{matrix}}{{}^{\bullet}\zeta/\bot}\ {}^{+}\vee E$$

$$\frac{{}^{+}\phi / {}^{+}\psi}{{}^{+}\phi \mathbin{\mathbb{V}} \psi}\ {}^{+}\mathbb{V} I \qquad\qquad \frac{{}^{+}\phi \mathbin{\mathbb{V}} \psi \qquad \begin{matrix}[{}^{+}\phi] \\ \mathcal{D} \\ {}^{\bullet}\zeta/\bot\end{matrix} \qquad \begin{matrix}[{}^{+}\psi] \\ \mathcal{E} \\ {}^{\bullet}\zeta/\bot\end{matrix}}{{}^{\bullet}\zeta/\bot}\ {}^{+}\mathbb{V} E$$

The rules for rejected constants are dual:

Rejected constants rules:

[6] The rule marker $\bot E$ is kept for notational consistency, but it should not be taken as flagging $\bot$ as some specific logical constant.

[7] The rules of *reductio* are given schematically as $\dfrac{\begin{matrix}{}^{\bullet}{}_{s}\phi \\ \mathcal{D} \\ \bot\end{matrix}}{{}^{\circ}{}_{w}\phi}$ and $\dfrac{\begin{matrix}{}^{\bullet}{}_{w}\phi \\ \mathcal{D} \\ \bot\end{matrix}}{{}^{\circ}{}_{s}\phi}$.

[8] we'll come back to this with the rule Rw. in a few pages.

$$\dfrac{{}^-\phi \qquad {}^\ominus\psi}{{}^-\phi\wedge\psi}\,{}^-\wedge I \qquad\qquad \dfrac{{}^-\phi\wedge\psi \qquad \begin{array}{c}[{}^-\phi]\\ \mathcal{D}^\star\\ {}^\bullet\zeta/\bot\end{array} \qquad \begin{array}{c}[{}^-\psi]\\ \mathcal{E}^\star\\ {}^\bullet\zeta/\bot\end{array}}{{}^\bullet\zeta/\bot}\,{}^-\wedge E$$

$$\dfrac{{}^-\phi \qquad {}^-\psi}{{}^-\phi\vee\psi}\,{}^-\vee I \qquad\qquad \dfrac{{}^-\phi\vee\psi}{{}^-\phi/{}^-\psi}\,{}^-\vee E$$

$$\dfrac{{}^-\phi \qquad {}^-\psi}{{}^-\phi\mathbin{\mathbb{V}}\psi}\,{}^-\mathbin{\mathbb{V}} I \qquad\qquad \dfrac{{}^-\phi\mathbin{\mathbb{V}}\psi}{{}^-\phi/{}^-\psi}\,{}^-\mathbin{\mathbb{V}} E$$

Further, two special sets of rules are required to handle disjunctions in our system. The rules for *free-choice* (collectively FC) state that a speaker refraining from rejecting a disjunctive sentence is also committed to refraining from rejecting each disjunct individually. This is a consequence of our neglect-zero assumptions: if no configuration is admissibly empty, then every rejection of a disjunctive sentence implies the rejection of each of the disjunct individually.

free-choice (FC) rules:

$$\dfrac{{}^\oplus\phi\vee\psi}{{}^\oplus\phi/{}^\oplus\psi}\,{}^+FC \qquad\qquad \dfrac{{}^\ominus\phi\wedge\psi}{{}^\ominus\phi/{}^\ominus\psi}\,{}^-FC$$

The interaction between free-choice rules and ordinary rules for logical constants deserves some comment. Under neglect-zero assumptions, disjunctions are interpreted as lists of epistemic possibilities and, as a result, the rules for asserted split disjunction and rejected conjunction must respect the $\star$ restriction. The difference with the $*$ restriction is accounted for by the fact that lists of possibilities brought about in this way are exhaustive, in the sense that they cover the whole epistemic space. The presence of $\star$ proves crucial in blocking vicious derivations in our system. For example, the following derivation does not go through because of the illicit use of non-downclosed premisses:

$$\dfrac{{}^+p_i\vee\neg p_i \qquad \dfrac{[{}^+p_i] \qquad \dfrac{\dfrac{\dfrac{{}^+p_i\vee\neg p_i}{{}^\oplus p_i\vee\neg p_i}\,\downarrow A}{{}^\oplus\neg p_i}\,{}^+FC \qquad \dfrac{[{}^+\neg p_i]}{{}^-p_i}\,\mathsf{Coord}}{{}^\ominus p_i}\,\mathsf{WI}}{\bot}\,\mathsf{EC} \qquad \dfrac{\dfrac{[{}^+\neg p_i]}{{}^-p_i}\,\mathsf{Coord} \qquad \dfrac{\dfrac{{}^+p_i\vee\neg p_i}{{}^\oplus p_i\vee\neg p_i}\,\downarrow A}{{}^\oplus p_i}\,{}^+FC}{\bot}\,\mathsf{EC}}{\bot}\,{}^+\vee E$$

At the same time, the interaction between free-choice and constants rules is virtuous in that it helps predicting a number of interesting natural language phenomena. To witness (beyond the cases in proposition 8), the following paradoxical inference ([Ros44]) does not go through, unless both disjuncts are least permitted in the context:

From: "You ought to mail the letter" infer "You ought to mail the letter or burn it".

Letting ϕ stand for "mail the letter", and ψ for "burn the letter", we then have, in inline notation and with derived rules:

$$\begin{array}{lcl} {}^+\blacksquare\,\phi \vdash {}^+\phi & \qquad\qquad & {}^+\blacksquare\,E \\ \qquad\not\vdash^+\phi\vee\psi & & {}^+\vee I \\ \qquad\vdash^+\blacksquare\,\phi\vee\psi & & {}^+\blacksquare\,I \end{array}$$

Where the second step is blocked by the unavailability of ${}^{\oplus}\psi$ in the context. Notably, "must" does not distribute over split disjunctions, implying ${}^{+}\blacksquare\ \phi \vee \psi \nvdash {}^{+}\blacksquare\ \phi \vee \blacksquare\ \psi$. On the contrary, assuming "burn the letter" is actually prohibited, licensing the inference above would lead to a contradiction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{{}^{+}\blacksquare\ \phi}{{}^{+}\blacksquare\ \phi \vee \psi}\,?}{{}^{+}\phi \vee \psi}\,{}^{+}\blacksquare\ E}{{}^{\oplus}\phi \vee \psi}\downarrow A}{{}^{\oplus}\phi}\,{}^{+}FC \qquad \dfrac{{}^{-}\blacklozenge\ \psi}{{}^{-}\psi}\,{}^{-}\blacklozenge\ E}{\bot}\,\mathsf{EC}$$

Let us return to the rules outlining the inferences licensed by the negation operator. The double line indicates that one is licensed in inferring both the conclusion from the premisses, as well as the premisses from the conclusion, effectively comprising two rules into one. We display the name of the top-to-bottom rule on the right, and the name of the bottom-to-top rule on the left. To ease the notation, we will sometimes refer to the collection of the rules listed just below as Coord rules.

Negation (Coord) rules:

$$-\neg \Rightarrow + \,\frac{{}^{+}\phi}{{}^{-}\neg\phi}\, + \Rightarrow -\neg \qquad +\neg \Rightarrow - \,\frac{{}^{-}\phi}{{}^{+}\neg\phi}\, - \Rightarrow +\neg\ ,$$

The Coord rules for strong signatures state that whenever a speaker is committed to asserting (rejecting) a certain content, they must also be committed to rejecting (asserting) its negation. This amounts to requiring that whenever a speaker disbelieves ϕ, then they should also believe $\neg\phi$ and viceversa.

We then have rules governing the behavior of "might". The first pair characterize "might" as a linguistic realization of the attitude of weak assertion, while the second pair states that "might" is immaterial under rejection. Accordingly a speaker asserting $\blacklozenge\phi$ is also committed to prevent any addition of ϕ to the negative common grounds. This is in line with our epistemic interpretation given above: if weak assertion is the linguistic realization of the intentional state of permitting (conceiving to be plausible), then to believe that it might be that ϕ should commit a speaker to conceive ϕ as plausible and viceversa. The behavior of "might" under rejection can be explained as follows: to reject $\blacklozenge\phi$ is to reject that ϕ is plausible, which plainly amounts to rejecting ϕ itself. Conversely, any speaker disbelieving ϕ should also disbelieve that ϕ is plausibly true.

"Might" rules:

$${}^{+}\blacklozenge\ E\,\frac{{}^{\oplus}\phi}{{}^{+}\blacklozenge\ \phi}\,{}^{+}\blacklozenge\ I \qquad {}^{-}\blacklozenge\ E\,\frac{{}^{-}\phi}{{}^{-}\blacklozenge\ \phi}\,{}^{-}\blacklozenge\ I$$

The last rule of our concern is *Rewrite* (for short: *Rw.*), which states that instances of inquisitive disjunction can always be treated as the main connective of the signed formulae they appear in. As apparent from the rule ${}^{+}\ \mathbin{\mathbb{V}}\ E$ and ${}^{-}\ \mathbin{\mathbb{V}}\ E$, the inferential profile of inquisitive disjunction is quite different when occurring under a positive signature from when occurring under a negative one. To specify the behavior of $\mathbin{\mathbb{V}}$ beyond its occurrence as main connective of a formula ϕ, then, we need to specify precisely which subformulae of ϕ the speaker is possibly committed to asserting when they assert ϕ itself.

Definition 45 (**Asserted position**). Let ${}^{+}\phi$ be a sentence, and $\psi \in \mathsf{Sub}(\phi)$. We say an instance (ψ) of ψ occurs in asserted position in ϕ if and only if $(\psi) \in \mathsf{Sub}^{+}({}^{+}\phi)$. Likewise (ψ) occurs in rejected position if and only if $(\psi) \in \mathsf{Sub}^{-}({}^{+}\psi)$. The collections $\mathsf{Sub}^{+}({}^{+}\phi)$ and $\mathsf{Sub}^{-}({}^{+}\phi)$ are specified recursively on the structure of ϕ:

1. $\mathsf{Sub}^{+}({}^{+}p_i) = \{p_i\}$ and $\mathsf{Sub}^{-}({}^{+}p_i) = \varnothing$, for $p_i \in \mathsf{N}$;
2. $\mathsf{Sub}^{+}({}^{+}\mu \wedge \nu) = \mathsf{Sub}^{+}({}^{+}\mu) \cup \mathsf{Sub}^{+}({}^{+}\nu) \cup \{{}^{+}\mu \wedge \nu\}$, and $\mathsf{Sub}^{-}({}^{+}\mu \wedge \nu) = \mathsf{Sub}^{-}({}^{+}\mu) \cup \mathsf{Sub}^{-}({}^{+}\nu)$;
3. $\mathsf{Sub}^{+}({}^{+}\mu \vee \nu) = \mathsf{Sub}^{+}({}^{+}\mu) \cup \mathsf{Sub}^{+}({}^{+}\nu) \cup \{{}^{+}\mu \vee \nu\}$, and $\mathsf{Sub}^{-}({}^{+}\mu \vee \nu) = \mathsf{Sub}^{-}({}^{+}\mu) \cup \mathsf{Sub}^{-}({}^{+}\nu)$;
4. $\mathsf{Sub}^{+}({}^{+}\mu \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \nu) = \mathsf{Sub}^{+}({}^{+}\mu) \cup \mathsf{Sub}^{+}({}^{+}\nu) \cup \{{}^{+}\mu \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \nu\}$, and $\mathsf{Sub}^{-}({}^{+}\mu \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \nu) = \mathsf{Sub}^{-}({}^{+}\mu) \cup \mathsf{Sub}^{-}({}^{+}\nu)$[9];
5. $\mathsf{Sub}^{+}({}^{+}\blacklozenge\, \mu) = \mathsf{Sub}^{+}({}^{+}\mu) \cup \{{}^{+}\blacklozenge\, \mu\}$, and $\mathsf{Sub}^{-}({}^{+}\blacklozenge\, \mu) = \mathsf{Sub}^{-}({}^{+}\mu)$;
6. $\mathsf{Sub}^{+}({}^{+}\neg\mu) = \mathsf{Sub}^{-}({}^{+}\mu) \cup \{{}^{+}\neg\mu\}$, and $\mathsf{Sub}^{-}({}^{+}\neg\mu) = \mathsf{Sub}^{+}({}^{+}\mu) \cup \{\mu\}$.

Let now ${}^{+}\phi$ be a sentence, let $\alpha, \beta \in \mathcal{BSL}_m^{\mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}}}$ and $(\alpha \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \beta)$ be an instance of inquisitive disjunction occurring in asserted position within ${}^{+}\phi$. The rule of $Rw.$ is stated as follows:

Rewrite rule:

$$\dfrac{{}^{+}\phi(\alpha \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \beta) \qquad \begin{matrix}[{}^{+}\phi(\alpha \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \beta := \alpha)] \\ \mathcal{D} \\ {}^{\bullet}\psi\end{matrix} \qquad \begin{matrix}[{}^{+}\phi(\alpha \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \beta := \beta)] \\ \mathcal{E} \\ {}^{\bullet}\psi\end{matrix}}{{}^{\bullet}\psi}\, Rw. \quad \alpha \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \beta \in \mathsf{Sub}^{+}(\phi)$$

Some comments on the rule above are in order. From a technical standpoint, Rewrite, together with its dual of "Inscription" displayed further down, incorporate a *deep-inference* mechanism in our calculus, allowing to manipulate instances of global disjunction even when appearing in subsentential position. From the perspective of what is manifest in natural language, this rule encodes the original function of inquisitive disjunction as a *question forming operator*. According to a prominent view in inquisitive logic, question do not directly contribute to the assertive content of an utterance, rather, they correspond their content is determined by *space of possible utterances* which is the space of assertions the inquirer deems admissible responses.[10] As a result, to assert a (global) disjunctive sentence is not yet to be committed to the belief of any of the disjuncts individually, but to be committed to proposing an update of the common ground with *some* out of a certain family of determinate contents[11] From a conceptual standpoint, the rule of $Rw.$ witnesses inquisitive disjunction (and thus question-forming) as an operator which distributes over assertion with "or". We say an operator # distributes over assertions with ⌑ whenever, letting ⌑ be a suitable connective in the metalanguage, we have:

$$^{+}B(\#(A_1, .., A_n)) \Longrightarrow^{+} B(A_1)\text{⌑}...\text{⌑}^{+}B(A_n)$$

Provided $\#(A_1...A_n)$ occurs in asserted position. To witness, if a speaker A utters "It looks like rain today, and I wonder whether I should bring a raincoat or an umbrella with me", then, we contend, their interlocutor B should be licensed in concluding A is either committed to the belief that it is going to rain today, and that they should bring a raincoat, or to the belief that it is going to rain today, and that they should bring an umbrella. While this is not the place for a thorough study of questions

[9] The inclusion of inquisitive disjunction itself among the asserted positions of a sentence is needed for technical reasons. This should not be taken to indicate that a speaker asserting $\mu \mathbin{\rotatebox[origin=c]{-90}{$\geqslant$}} \nu$ is committed to both inquisitive disjuncts μ and ν.

[10] See [Cia15] for an exhaustive treatment.

[11] A dedicated study of this phenomenon in a multilateral setting, trough the use of a new attitude of "question acceptance" is articulated in [Rem26].

and distributivity, the rule of $Rw.$ accounts for the structural role of question forming operator $\vee\!\!\vee$ by licensing inferences based on the nested occurrence of questions like the one above.[12]

2.2.2.2 Derived rules

We outline now the derived rules of the system, which range from logical constants for weak signatures to general properties like contraction and weakening of disjunction. In the sections to come, we make free use of derived rules as they sometimes substantially shorten the derivations. Most of the derivations for the rules below are elementary and are thus omitted, we do, however, make use of additional symbols to flag special properties of the derived ruled which are due to the $*, \star$ restrictions in their respective derivation trees.

The rules for weakly signed logical constants can easily be obtained making use of Weak Inferences:

Weakly asserted constant rules:

$$\dagger\ \frac{{}^{+}\phi \qquad {}^{\oplus}\psi}{{}^{\oplus}\phi \wedge \psi}\ {}^{\oplus}\wedge I \qquad\qquad \frac{{}^{\oplus}\phi \wedge \psi}{{}^{\oplus}\phi / {}^{\oplus}\psi}\ {}^{\oplus}\wedge E$$

$$\frac{{}^{\oplus}\phi \qquad {}^{\oplus}\psi}{{}^{\oplus}\phi \vee \psi}\ {}^{\oplus}\vee I \qquad\qquad \frac{{}^{\oplus}\phi \vee \psi \qquad \begin{matrix}[{}^{+}\phi]\\ \mathcal{D}^{*}\\ {}^{+}\zeta\end{matrix} \quad / \quad \begin{matrix}[{}^{+}\psi]\\ \mathcal{E}^{*}\\ {}^{+}\zeta\end{matrix}}{{}^{\oplus}\zeta}\ {}^{\oplus}\vee E$$

$$\frac{{}^{\oplus}\phi / {}^{\oplus}\psi}{{}^{\oplus}\phi \vee\!\!\vee \psi}\ {}^{\oplus}\vee\!\!\vee I \qquad\qquad *\ \frac{{}^{\oplus}\phi \vee\!\!\vee \psi \qquad \begin{matrix}[{}^{+}\phi]\\ \mathcal{D}^{*}\\ {}^{+}\zeta\end{matrix} \qquad \begin{matrix}[{}^{+}\psi]\\ \mathcal{E}^{*}\\ {}^{+}\zeta\end{matrix}}{{}^{\oplus}\zeta}\ {}^{\oplus}\vee\!\!\vee E$$

The $\dagger$ symbol on ${}^{\oplus}\wedge I$ signals the requirement that ϕ be down-closed. This follows from the derivation tree for the rule which makes use of weak inference on the leftmost premiss ${}^{\oplus}\psi$, together with the obvious fact that the conclusion of the rule should be positively signed and conjunctive rather than disjunctive. Again, the rules for weak rejection are dual:

Weakly rejected constants rules:

$$\frac{{}^{\ominus}\phi \qquad {}^{\ominus}\psi}{{}^{\ominus}\phi \wedge \psi}\ {}^{\ominus}\wedge I \qquad\qquad \frac{{}^{\ominus}\phi \wedge \psi \qquad \begin{matrix}[{}^{-}\phi]\\ \mathcal{D}^{*}\\ {}^{-}\zeta\end{matrix} \quad / \quad \begin{matrix}[{}^{-}\psi]\\ \mathcal{E}^{*}\\ {}^{-}\zeta\end{matrix}}{{}^{\ominus}\zeta}\ {}^{\ominus}\wedge E\ ,$$

$$\dagger\ \frac{{}^{-}\phi \qquad {}^{\ominus}\psi}{{}^{\ominus}\phi \vee \psi}\ {}^{\ominus}\vee I \qquad\qquad \frac{{}^{\ominus}\phi \vee \psi}{{}^{\ominus}\phi / {}^{\ominus}\psi}\ {}^{\ominus}\vee E$$

$$\dagger\ \frac{{}^{-}\phi \qquad {}^{\ominus}\psi}{{}^{\ominus}\phi \vee\!\!\vee \psi}\ {}^{\ominus}\vee\!\!\vee I \qquad\qquad \frac{{}^{\ominus}\phi \vee\!\!\vee \psi}{{}^{\ominus}\phi / {}^{\ominus}\psi}\ {}^{\ominus}\vee\!\!\vee E$$

As before, the $\dagger$ restriction simply amounts to the requirement that $\phi \in \mathsf{D}^{-}$.

The Coord rules for weak signatures state that if a speaker refrains from asserting (rejecting) a content

[12] Similar considerations motivate [CR11] (p. 90). to adopt a notion of semantic support as *knowing how*. The considerations here given show the traditional interpretation of support as assertibility not to be deficient from this perspective.

ϕ, then they also must refrain from rejecting (asserting) the negation of ϕ and viceversa. For suppose a certain speaker both refrains from asserting ϕ and does not refrain from asserting $\neg\phi$, then, whenever an update of the positive common ground with $\neg\phi$ is proposed, then they must accept $\neg\phi$. and therefore be committed to rejecting ϕ by the $+\neg \Rightarrow -$ rule. Since it is incoherent for a speaker to reject and refrain from rejecting the same content, the weak Coord rules are justified in virtue of the strong ones.

Weak negation rules:

$$\ominus\neg \Rightarrow \oplus \; \frac{{}^{\oplus}\phi}{{}^{\ominus}\neg\phi} \; \oplus \Rightarrow \ominus\neg \qquad\qquad \oplus\neg \Rightarrow \ominus \; \frac{{}^{\ominus}\phi}{{}^{\oplus}\neg\phi} \; \ominus \Rightarrow \oplus\neg \; ,$$

We then have some basic properties of might as well as split disjunction. The rules of *Idempotence*, *Weakening*, *Contraction* and *Monotonicity* are given below. For the latter three, the dual rules for rejected assertion are also derivable, they are however omitted for the sake of conciseness:

Additional property rules:

$$\frac{{}^{\oplus}\blacklozenge\,\phi}{{}^{\oplus}\phi}\;\blacklozenge\,Id. \qquad\qquad \frac{{}^{+}\phi}{{}^{+}\phi \vee \phi}\;{}^{+}\vee\, Wkn.$$

$$\ddagger\;\frac{{}^{+}\phi \vee \phi}{{}^{+}\phi}\;{}^{+}\vee\, Cntr. \qquad\qquad \frac{{}^{+}\phi \vee \psi \qquad \begin{matrix}[{}^{+}\psi]\\ \mathcal{D}^{*}\\ {}^{+}\zeta\end{matrix}}{{}^{+}\phi \vee \zeta}\;{}^{+}\vee\, Mon.$$

The "‡" symbol on ${}^{+}\vee\, Cntr.$ signals that the rule is applicable only if the formula ϕ is positive union-closed. It is indeed easy to see that the derivation tree for ${}^{+}\vee\, Cntr.$ makes use of ${}^{+}\vee\, E$ in particular with $\star$ condition 2. (a). This is as intended, indeed without this restriction in place from the sentence ${}^{+}p_1 \vee p_2$ we would be able to infer ${}^{+}p_1 \mathbin{\vee\!\!\!\vee} p_2$ by applying ${}^{+}\vee\, E$ and then ${}^{+}\mathbin{\vee\!\!\!\vee}\, I$ on both disjuncts. However it follows easily from the semantics of $\mathcal{BSL}_m^{\mathbin{\vee\!\!\!\vee}}$ that a team may very well support $p_1 \vee p_2$ without supporting neither p_1 nor p_2 individually, so this inference would not be sound.
One rule allowing the introduction of global disjunction is ${}^{+}\vee\, Mon.$, in this case, however, the risk of modal collapse is averted by keeping one of the two split-disjuncts fixed between premisses and conclusions. Since it is less straightforward, the tree for the rule is given below.

$$\dfrac{{}^{+}\phi \vee \psi \qquad \dfrac{[{}^{+}\phi] \qquad \dfrac{\dfrac{\dfrac{{}^{+}\phi \vee \psi}{{}^{\oplus}\phi \vee \psi}\;{\downarrow}A}{{}^{\oplus}\psi}\;{}^{+}FC \qquad \begin{matrix}[{}^{+}\psi]\\ \mathcal{D}\\ {}^{+}\zeta\end{matrix}}{{}^{\oplus}\zeta}\;\mathsf{WI}}{{}^{+}\phi \vee \zeta}\;{}^{+}\vee I \qquad \dfrac{\begin{matrix}[{}^{+}\psi]\\ \mathcal{D}\\ {}^{+}\zeta\end{matrix} \qquad \dfrac{\dfrac{{}^{+}\phi \vee \psi}{{}^{\oplus}\phi \vee \psi}\;{\downarrow}A}{{}^{\oplus}\phi}\;{}^{+}FC}{{}^{+}\phi \vee \zeta}\;{}^{+}\vee I}{{}^{+}\phi \vee \zeta}\;{}^{+}\vee E$$

It is easy to check the $\star$ requirement is complied with on the assumption that $\mathcal{D}$ complies with $*$.

2.2.3 Local soundness and completeness

In section 1.3.1, we argued for a notion of *harmony* as local soundness (definition 1) and completeness (definition 2). We show below the connectives of our calculus effectively comply with those two requirements (soundness is shown to the left, and completeness to the right). For all constants except

$\blacklozenge$, we specify only the assertion cases as those for denial are symmetric.
The case of conjunction is standard:

$$\begin{array}{c}\mathcal{A}\qquad\mathcal{B}\\ \dfrac{\dfrac{{}^{+}\phi \qquad {}^{+}\psi}{{}^{+}\phi\wedge\psi}\,{\scriptstyle +\wedge I}}{{}^{+}\phi}\,{\scriptstyle +\wedge E}\\ \mathcal{C}\end{array} \rightsquigarrow \begin{array}{c}\mathcal{A}\\ {}^{+}\phi\\ \mathcal{C}\end{array} \qquad \begin{array}{c}\mathcal{A}\\ {}^{+}\phi\wedge\psi\\ \mathcal{B}\end{array} \rightsquigarrow \begin{array}{c} \dfrac{\begin{array}{c}\mathcal{A}\\ \dfrac{{}^{+}\phi\wedge\psi}{{}^{+}\phi}\,{\scriptstyle +\wedge E}\end{array} \qquad \begin{array}{c}\mathcal{A}\\ \dfrac{{}^{+}\phi\wedge\psi}{{}^{+}\psi}\,{\scriptstyle +\wedge E}\end{array}}{{}^{+}\phi\wedge\psi}\,{\scriptstyle +\wedge I}\\ \mathcal{B}\end{array}$$

Disjunction makes essential use of the $\star$ restrictions:

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\phi\end{array} \qquad \begin{array}{c}\mathcal{B}\\ {}^{+}\psi\end{array}}{{}^{+}\phi\vee\psi}\,{\scriptstyle +\vee I} \qquad \begin{array}{c}[{}^{+}\phi]\\ \mathcal{C}\\ {}^{\bullet}\mu\end{array} \qquad \begin{array}{c}[{}^{+}\psi]\\ \mathcal{D}\\ {}^{\bullet}\mu\end{array}}{{}^{\bullet}\mu}\,{\scriptstyle +\vee E}\\ \mathcal{E}\end{array} \rightsquigarrow \begin{array}{c}\mathcal{A}\\ {}^{+}\phi\\ \mathcal{C}\\ {}^{\bullet}\mu\\ \mathcal{E}\end{array}$$

$$\begin{array}{c}\mathcal{A}\\ {}^{+}\phi\vee\psi\\ \mathcal{B}\end{array} \rightsquigarrow \begin{array}{c}\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\phi\vee\psi\end{array} \qquad \dfrac{[{}^{+}\phi] \qquad \begin{array}{c}\mathcal{A}\\ \dfrac{\dfrac{{}^{+}\phi\vee\psi}{{}^{\oplus}\phi\vee\psi}\,{\scriptstyle \downarrow A}}{{}^{\oplus}\psi}\,{\scriptstyle +FC}\end{array}}{{}^{+}\phi\vee\psi}\,{\scriptstyle +\vee I} \qquad \dfrac{[{}^{+}\psi] \qquad \begin{array}{c}\mathcal{A}\\ \dfrac{\dfrac{{}^{+}\phi\vee\psi}{{}^{\oplus}\phi\vee\psi}\,{\scriptstyle \downarrow A}}{{}^{\oplus}\phi}\,{\scriptstyle +FC}\end{array}}{{}^{+}\phi\vee\psi}\,{\scriptstyle +\vee I}}{{}^{+}\phi\vee\psi}\,{\scriptstyle +\vee E}\\ \mathcal{B}\end{array}$$

The harmony for "might" and negation is immediate by the rules, we display the former case for illustration:

$$\begin{array}{c}\mathcal{A}\\ \dfrac{\dfrac{{}^{\oplus}\phi}{{}^{+}\blacklozenge\,\phi}\,{\scriptstyle +\blacklozenge I}}{{}^{\oplus}\phi}\,{\scriptstyle +\blacklozenge E}\\ \mathcal{B}\end{array} \rightsquigarrow \begin{array}{c}\mathcal{A}\\ {}^{\oplus}\phi\\ \mathcal{B}\end{array} \qquad \begin{array}{c}\mathcal{A}\\ {}^{+}\blacklozenge\,\phi\\ \mathcal{B}\end{array} \rightsquigarrow \begin{array}{c}\mathcal{A}\\ \dfrac{\dfrac{{}^{+}\blacklozenge\,\phi}{{}^{\oplus}\phi}\,{\scriptstyle +\blacklozenge E}}{{}^{+}\blacklozenge\,\phi}\,{\scriptstyle +\blacklozenge I}\\ \mathcal{B}\end{array}$$

Harmony of *Rw.*, by contrast, is not immediate and is shown via admissibility of the following rule of *Inscription*, which acts as an "introduction" counterpart to the "elimination" rule of *Rw.*.[13]

Inscription rule:

$$\dfrac{{}^{+}\phi(\alpha)}{{}^{+}\phi(\alpha \mathbin{\vee\!\!\!\vee} \beta)}\,{\scriptstyle Inscr.}$$

This rule adds nothing to the deductive power of the calculus, but it makes Rewrite harmonious in a straightforward fashion.
Intuitively, whenever we would use Inscription in a derivation, we can always eliminate the premiss of *Inscr.* until we get to the relevant instance (α), then introduce the inquisitive disjunction, and build the premiss back up. The following is easy to see:

Proposition 46. The rule *Inscr.* is derivable in NFM_{ne}.

[13] Another option would be to regard *Rw.* as a structural rule, which would lift the need for Rewrite to be shown harmonious. Under this perspective, Rewrite simply spells out a general property of an attitude of "questioning", or "inquiring". Since we do not employ a separate attitude for handling questions, this option is not pursued here, but see [Rem26] for a development in this direction.

Proof. By induction on the subformulae of the unique premiss ${}^{+}\phi(\alpha)$. If ${}^{+}\phi(\alpha)$ is atomic then we have two cases depending on whether $p_i = \alpha$ or not: if so, then we replace the instance of *Inscr.* with an instance of ${}^{+}\mathbin{\vee\!\!\vee} I$, otherwise we have a derivation of ${}^{+}\phi$ already by assumption. It is easy to see that both cases respect the $\star$ restriction as well as the $*$ one.
For any binary connective $\#$, we can without loss of generality assume that (α) occurs in μ where $\phi = \mu\#\nu$. The inductive cases then follow by inductive hypothesis, eliminating $\mu\#\nu$ to obtain the minor premiss when needed to conclude ${}^{+}\mu(\alpha \mathbin{\vee\!\!\vee} \beta)\#\nu$. The case for $\# = \vee$ makes use of the assumption that the derivation given by the inductive hypothesis respects $*$ (notice ν is kept fixed so inserting $\mathbin{\vee\!\!\vee}$ is licit). The case for $\phi = \blacklozenge\mu$ is obtained by eliminating $\blacklozenge$, applying ${}^{+}WI_1$ and then the induction hypothesis, using the assumption that the derivation thus given respects $\star$. Last, if $\phi = \neg\mu$ then notice that for (α) to occur in asserted position within $\neg\mu$, it must occur in rejected position within μ. The case for negation thus reduces to the others, as it suffices to eliminate the outermost connective of ${}^{-}\mu$, apply $- \Rightarrow +\neg$ to the suitable subformula and then use the inductive hypothesis. Last, use the appropriate introduction rules to conclude ${}^{+}\mu(\alpha)$. We give the case for $\mu = \gamma \vee \delta$ as an example:

$$
\begin{prooftree}
\AxiomC{${}^{+}\neg(\gamma(\alpha) \vee \delta)$}
\RightLabel{Coord}
\UnaryInfC{${}^{-}\gamma(\alpha) \vee \delta$}
\RightLabel{${}^{-}\vee E$}
\UnaryInfC{${}^{-}\gamma(\alpha)$}
\RightLabel{Coord}
\UnaryInfC{${}^{+}\neg\gamma(\alpha)$}
\RightLabel{I.H.}
\UnaryInfC{${}^{+}\neg\gamma(\alpha \mathbin{\vee\!\!\vee} \beta)$}
\RightLabel{Coord}
\UnaryInfC{${}^{-}\gamma(\alpha)$}
\AxiomC{${}^{+}\neg(\gamma(\alpha) \vee \delta)$}
\RightLabel{Coord}
\UnaryInfC{${}^{-}\gamma(\alpha) \vee \delta$}
\RightLabel{${}^{-}\vee E$}
\UnaryInfC{${}^{-}\delta$}
\RightLabel{${}^{-}\vee I$}
\BinaryInfC{${}^{-}\gamma(\alpha \mathbin{\vee\!\!\vee} \beta) \vee \delta$}
\RightLabel{Coord}
\UnaryInfC{${}^{+}\neg(\gamma(\alpha \mathbin{\vee\!\!\vee} \beta) \vee \delta)$}
\end{prooftree}
$$

■

With the *Inscr.* rule in place, local soundness and completeness of *Rw.* are straightforward to check:

$$
\begin{prooftree}
\AxiomC{$\mathcal{A}$}
\noLine
\UnaryInfC{${}^{+}\phi(\alpha \mathbin{\vee\!\!\vee} \beta)$}
\AxiomC{$[{}^{+}\phi(\alpha)]$}
\RightLabel{\textit{Inscr.}}
\UnaryInfC{${}^{+}\phi(\alpha \mathbin{\vee\!\!\vee} \beta)$}
\AxiomC{$[{}^{+}\phi(\beta)]$}
\RightLabel{\textit{Inscr.}}
\UnaryInfC{${}^{+}\phi(\alpha \mathbin{\vee\!\!\vee} \beta)$}
\RightLabel{\textit{Rw.}}
\TrinaryInfC{${}^{+}\phi(\alpha \mathbin{\vee\!\!\vee} \beta)$}
\noLine
\UnaryInfC{$\mathcal{B}$}
\end{prooftree}
\quad \rightsquigarrow \quad
\begin{array}{c} \mathcal{A} \\ {}^{+}\phi(\alpha \mathbin{\vee\!\!\vee} \beta) \\ \mathcal{B} \end{array}
$$

2.3 Soundness

We now prove soundness of NFM_{ne} with respect to $\mathcal{BSL}_m^{\mathbin{\vee\!\!\vee}}$-models. Since formulae of $\mathcal{BSL}_m^{\mathbin{\vee\!\!\vee}}$ are not signed, we need to translate between two systems. Conceptually, we proceed as follows: in section 2.2.1, we took the attitudes of assertion and rejection to fulfil the conversational function of updating the positive and negative common grounds, respectively. Given a $\mathcal{BSL}_m^{\mathbin{\vee\!\!\vee}}$ model $\mathfrak{M}$, we may consider the team given by $W_{\mathfrak{M}}$ to represent the positive common ground of shared beliefs under the relation of support. Symmetrically, we take $W_{\mathfrak{M}}$ to represent the negative common ground of shared disbeliefs under the relation of anti-support. In this sense, given a model $\mathfrak{M}$, we represent the positive and negative common grounds a:

$$\{\phi \in \mathcal{BSL}_m^{\mathbin{\vee\!\!\vee}} \mid \mathfrak{M}, W_{\mathfrak{M}} \vDash \phi\}, \text{ and } \{\phi \in \mathcal{BSL}_m^{\mathbin{\vee\!\!\vee}} \mid \mathfrak{M}, W_{\mathfrak{M}} \dashv \phi\}$$

Thus, to assert a content ϕ is to claim it is supported in the largest team available in a model, and to reject a sentence ψ is to claim it is anti-supported in the largest team available in a model.

The attitudes of weak assertion and weak rejection were interpreted as, respectively, refraining from rejecting and refraining from asserting. In turn both those conversational functions were considered as linguistic representations of the intentional states of permitting (conceiving as plausible) and doubting (conceiving as plausibly false). Accordingly, we may interpret the attitude of weak assertion and weak rejection as *support at a subteam* and *anti-support at a subteam*, respectively. To weakly assert a sentence ϕ is then to refrain from rejecting it on the ground of ϕ being supported at some nonempty subteam of $W_{\mathfrak{M}}$, likewise, to weakly reject a sentence is to refrain from asserting it on the ground of ϕ being anti-supported at some nonempty subteam of $W_{\mathfrak{M}}$.

Formally, we make use of a map $\iota : \mathsf{NFM}_{ne} \to \mathcal{BSL}_m^{\mathbb{V}}$ defined as follows:

$$\iota({}^{\bullet}\phi) = \begin{cases} \phi & \text{if } \bullet = + \\ \neg\phi & \text{if } \bullet = - \\ \blacklozenge\phi & \text{if } \bullet = \oplus \\ \blacklozenge\neg\phi & \text{if } \bullet = \ominus \end{cases}$$

Before showing soundness for the system, we show an (easy) intermediary lemma which proves useful for the case of *Rewrite*:

Lemma 47. Let $\mathfrak{M}$, X be a $\mathcal{BSL}_m^{\mathbb{V}}$ model and teams, and let ϕ, α, $\beta \in \mathcal{BSL}_m^{\mathbb{V}}$.
If $(\alpha \mathbin{\mathbb{V}} \beta)$ occurs in asserted position in ${}^{+}\phi$, then $\mathfrak{M}, X \vDash \phi(\alpha \mathbin{\mathbb{V}} \beta)$ implies either $\mathfrak{M}, X \vDash \phi(\alpha)$ or $\mathfrak{M}, X \vDash \phi(\beta)$. Moreover, if $(\alpha \mathbin{\mathbb{V}} \beta)$ occurs in rejected position in ϕ, then $\mathfrak{M}, X \mathrel{=\!\!\mid} \phi(\alpha \mathbin{\mathbb{V}} \beta)$ implies either $\mathfrak{M}, X \mathrel{=\!\!\mid} \phi(\alpha)$ or $\mathfrak{M}, X \mathrel{=\!\!\mid} \phi(\beta)$.

Proof. By induction on the subformulae of ϕ, simultaneously for both the support and anti-support conditions. We give the proof for the case $\mathfrak{M}, X \vDash \phi(\alpha \mathbin{\mathbb{V}} \beta)$ implies $\mathfrak{M}, X \vDash \phi(\alpha)$, as the case with β is symmetrical.
For the base case we have $\phi = p_i$ for $p_i \in \mathsf{N}$. From the definition of asserted and rejected position (definition 45), we have $\alpha \mathbin{\mathbb{V}} \beta \notin \mathsf{Sub}^{+}({}^{+}p_i)$ and $\alpha \mathbin{\mathbb{V}} \beta \notin \mathsf{Sub}^{-}({}^{+}p_i)$, so the statement holds vacuously.
If $\phi = \mu \vee \nu$, then without loss of generality we may assume $(\alpha \mathbin{\mathbb{V}} \beta)$ occurs in μ. We then have $\mathfrak{M}, X \vDash \mu(\alpha \mathbin{\mathbb{V}} \beta) \vee \nu$, which implies $\mathfrak{M}, Y \vDash \mu(\alpha \mathbin{\mathbb{V}} \beta)$ for some nonempty $Y \subseteq X$. By definition of asserted position, we have $(\alpha \mathbin{\mathbb{V}} \beta) \in \mathsf{Sub}^{+}({}^{+}\mu \vee \nu) \iff (\alpha \mathbin{\mathbb{V}} \beta) \in \{\mu \vee \nu\} \cup \mathsf{Sub}^{+}({}^{+}\mu) \cup \mathsf{Sub}^{+}({}^{+}\nu)$. Since the main connective is "$\vee$", and we assumed $(\alpha \mathbin{\mathbb{V}} \beta)$ to occur in μ, this implies $(\alpha \mathbin{\mathbb{V}} \beta) \in \mathsf{Sub}^{+}({}^{+}\mu)$. Therefore, by induction hypothesis we get $\mathfrak{M}, Y \vDash \mu(\alpha)$ and thus also $\mathfrak{M}, X \vDash \mu(\alpha) \vee \nu$ which is the same as $\mathfrak{M}, X \vDash (\mu \vee \nu)(\alpha)$. The case for anti-support is symmetric to the support case for conjunction using the defintion of Sub^{-} instead.

If $\phi = \mu \wedge \nu$ then by the same considerations as before we have $\alpha \vee\!\!\!\vee \beta \in \mathsf{Sub}^+({}^+\mu)$ which then implies:

$$\begin{aligned}
\mathfrak{M}, X \vDash (\mu \wedge \nu)(\alpha \vee\!\!\!\vee \beta) &\iff \mathfrak{M}, X \vDash \mu(\alpha \vee\!\!\!\vee \beta) \wedge \nu && \text{Rewriting}\\
&\iff \mathfrak{M}, X \vDash \mu(\alpha \vee\!\!\!\vee \beta) \text{ and } \mathfrak{M}, X \vDash \nu && \text{Semantic clauses}\\
&\implies \mathfrak{M}, X \vDash \mu(\alpha) \text{ and } \mathfrak{M}, X \vDash \nu && \text{Inductive hypothesis}\\
&\iff \mathfrak{M}, X \vDash \mu(\alpha) \wedge \nu && \text{Semantic clauses}\\
&\iff \mathfrak{M}, X \vDash (\mu \wedge \nu)(\alpha) && \text{Rewriting}
\end{aligned}$$

The case for anti-support is symmetrical to the support case for split disjunction using the definition of Sub^- instead.
If $\phi = \mu \vee\!\!\!\vee \nu$ then we have without loss of generality $(\mu \vee\!\!\!\vee \nu)(\alpha \vee\!\!\!\vee \beta) = \mu(\alpha \vee\!\!\!\vee \beta) \vee\!\!\!\vee \nu$ or $(\mu \vee\!\!\!\vee \nu)(\alpha \vee\!\!\!\vee \beta) = \alpha \vee\!\!\!\vee \beta$. In the first case we know $\alpha \vee\!\!\!\vee \beta \in \mathsf{Sub}^+({}^+\mu)$. Thus we get $\mathfrak{M}, X \vDash \mu(\alpha \vee\!\!\!\vee \beta) \vee\!\!\!\vee \nu$ if and only if $\mathfrak{M}, X \vDash \mu(\alpha \vee\!\!\!\vee \beta)$, in which case the result follows by inductive hypothesis and the semantics of inquisitive disjunction, or $\mathfrak{M}, X \vDash \nu$, in which case the result follows directly from the semantics of inquisitive disjunction. In the second case, we know X supports $\alpha \vee\!\!\!\vee \beta$ if and only if it either supports α or it supports β, which is the same as supporting $(\mu \vee\!\!\!\vee \nu)(\alpha)$ or $(\mu \vee\!\!\!\vee \nu)(\beta)$ respectively. In the case for anti-support, we also have either $(\mu \vee\!\!\!\vee \nu)(\alpha \vee\!\!\!\vee \beta) = \mu(\alpha \vee\!\!\!\vee \beta) \vee\!\!\!\vee \nu$ or $(\mu \vee\!\!\!\vee \nu)(\alpha \vee\!\!\!\vee \beta) = \alpha \vee\!\!\!\vee \beta$. Both cases are symmetrical to the support case for conjunction.
If $\phi = \blacklozenge \mu$, then the support case follows by inductive hypothesis applied to a suitable subteam $Y \subseteq X$, and the anti-support case follows immediately from the inductive hypothesis.
If $\phi = \neg\mu$ then as before we have $(\neg\mu)(\alpha \vee\!\!\!\vee \beta) = \neg(\mu(\alpha \vee\!\!\!\vee \beta))$. Further, because the main connective of ϕ is $\neg$, by definition 45 it must be that $\alpha \vee\!\!\!\vee \beta \in \mathsf{Sub}^-(\mu)$. We then have $\mathfrak{M}, X \vDash \neg\mu(\alpha \vee\!\!\!\vee \beta) \iff \mathfrak{M}, X \Dashv \mu(\alpha \vee\!\!\!\vee \beta)$ thus yielding $\mathfrak{M}, X \Dashv \mu(\alpha)$ by induction hypothesis and therefore $\mathfrak{M}, X \vDash \neg\mu(\alpha)$ as well. For the anti-support case, given that $\alpha \vee\!\!\!\vee \beta$ occurs in rejected position we may have either $\alpha \vee\!\!\!\vee \beta \in \mathsf{Sub}^+(\mu)$ or $\alpha \vee\!\!\!\vee \beta = \mu$. In the first case, we have

$$\begin{aligned}
\mathfrak{M}, X \Dashv (\neg\mu)(\alpha \vee\!\!\!\vee \beta) &\iff \mathfrak{M}, X \Dashv \neg(\mu(\alpha \vee\!\!\!\vee \beta)) && \text{Rewriting}\\
&\iff \mathfrak{M}, X \vDash \mu(\alpha \vee\!\!\!\vee \beta) && \text{Semantic clauses}\\
&\implies \mathfrak{M}, X \vDash \mu(\alpha) && \text{Inductive hypothesis}\\
&\iff \mathfrak{M}, X \Dashv \neg\mu(\alpha) && \text{Semantic clauses}\\
&\iff \mathfrak{M}, X \Dashv (\neg\mu)(\alpha) && \text{Rewriting}
\end{aligned}$$

In the second case we have $\mathfrak{M}, X \Dashv \neg(\alpha \vee\!\!\!\vee \beta) \iff \mathfrak{M}, X \vDash \alpha \vee\!\!\!\vee \beta$ which in either case yields $\mathfrak{M}, X \Dashv \neg\mu(\alpha)$ or $\mathfrak{M}, X \Dashv \neg\mu(\beta)$ by the semantics of negation. ■

We are now in the position to show soundness for our system.

Theorem 48 (**Soundness**)**.** Let Γ be a finite set of sentences with $\mathsf{N} = \bigcup_{\bullet\mu\in\Gamma} \mathsf{Var}(\mu)$, and ϕ, ψ be arbitrary in $\mathcal{BSL}_m^{\vee\!\!\!\vee}$; if there is a $\mathcal{D}$ deriving $\Gamma \vdash {}^\bullet\phi$ then $\iota[\Gamma] \vDash \iota({}^\bullet\phi)$.

Proof. Let $\mathfrak{M}$, T be an arbitrary model and team, we proceed by induction on $H(\mathcal{D})$. To ease the notation, for a derivation $\mathcal{D} : \Gamma \vdash {}^\bullet\phi$, we assume in the following the sets $\Gamma = \mathsf{Open}(\mathcal{D})$. It is straightforward that this makes no difference in the proof.

If $H(\mathcal{D}) = 0$, then either ${}^{\bullet}\phi ={}^{\bullet} \mu$ for some ${}^{\bullet}\mu \in \Gamma$, or ${}^{\bullet}\phi ={}^{+} \pi_{[\![\top]\!]}$.
In the first case, letting $\mathfrak{M}, T \models \iota[\Gamma]$ we have $\mathfrak{M}, T \models \iota({}^{\bullet}\phi)$ by assumption as $\iota[\Gamma] = \bigwedge\{\iota({}^{\bullet}\mu) \mid \mu \in \Gamma\}$.
In the second case, as bisimulation is reflexive we have $T \leftrightarrows T$ and thus $\mathfrak{M}, T \models \theta_T$ by proposition 31.
As $\mathfrak{M}, T \models \top$, we get $\langle \mathfrak{M}, T\rangle \in [\![\top]\!]$ and thus $\mathfrak{M}, T \models \iota({}^{+}\pi_{[\![\top]\!]})$ by the semantics of $\mathbb{V}$.
Going forward, we will drop the ι notation whenever no issue of misunderstanding can reasonably arise.
The inductive step is split in several subcases depending on the last rule applied in the derivation. Let then $H(\mathcal{D}) = m + 1$ and consider the last rule applied in $\mathcal{D}$:

${}^{+}\wedge I$ If the last rule applied is conjunction introduction for assertion, then we have $\Gamma \vdash {}^{+}\mu$ and $\Gamma \vdash^{+} \nu$. Assuming $\mathfrak{M}, T \models \Gamma$, we obtain $\mathfrak{M}, T \models \mu$ and $\mathfrak{M}, T \models \nu$ by induction hypothesis. The desired then follows by the support conditions for conjunctive formulae.

${}^{+}\vee I$ If the last rule applied is split disjunction introduction for assertion, then we have $\Gamma \vdash {}^{+}\mu$ and $\Gamma \vdash {}^{\oplus}\nu$. Assuming $\mathfrak{M}, T \models \Gamma$, we obtain $\mathfrak{M}, T \models \mu$ and $\mathfrak{M}, T \models \blacklozenge\nu$ by induction hypothesis. Let $S \subseteq T$ be the witness for $\mathfrak{M}, T \models \blacklozenge\nu$, because we know $S, T \neq \varnothing$, and clearly $T = S \cup T$, S and T are witnesses for $\mathfrak{M}, T \models \mu \vee \nu$.

${}^{+}\wedge E$ Given $\Gamma \vdash {}^{+}\mu \wedge \nu$, we have here $\mathfrak{M}, T \models \mu \wedge \nu$ by induction, hypothesis, which implies the desired by the semantics of conjunction.

${}^{+}\vee E$ If the last rule applied is split disjunction elimination for assertion, then we have the following derivations:

$$\mathcal{A} : \Gamma \vdash {}^{+}\phi \vee \psi$$
$$\mathcal{B} : \Delta',{}^{+} \phi\star \vdash {}^{\bullet}\zeta$$
$$\mathcal{C} : \Theta',{}^{+} \psi\star \vdash {}^{\bullet}\zeta$$

where ζ is union-closed.
Assuming $\mathfrak{M}, T \models \Gamma \cup \Delta' \cup \Theta'$, we then obtain $\mathfrak{M}, T \models \phi \vee \psi$ by inductive hypothesis, meaning there are $S, P \subseteq T$ with $S \cup P = T$ and $\mathfrak{M}, S \models \phi$, and $\mathfrak{M}, P, \models \psi$. We have now different cases corresponding to the clauses in the definition of the $\star$ restriction.
First, it is easy to see that for any sentence ${}^{\bullet}\phi \in \Delta'$, Θ', ${}^{\bullet}\phi$ is down-closed as defined in definition 42, if and only if $\phi \in \mathsf{D}^{+}$ whenever $\bullet = +$ and $\phi \in \mathsf{D}^{-}$ whenever $\bullet = -$. In the following, we will therefore only speak of formulae being down-closed without further qualification.

If all sentences in the open assumption of $\mathcal{B}_1$, $\mathcal{C}_1$ are downward-closed, with:

$$\mathcal{B} = \begin{array}{c}\mathcal{B}_1 \\ \mathcal{B}_2\end{array} \text{ and } \mathcal{C} = \begin{array}{c}\mathcal{C}_1 \\ \mathcal{C}_2\end{array}$$

Then, given $\mathcal{B}_1 : \mathsf{Open}(\mathcal{B}_1) \vdash {}^{\bullet}\mu$ and $\mathcal{C}_1 : \mathsf{Open}(\mathcal{C}_1) \vdash {}^{\bullet}\mu$ we know $\mathsf{Open}(\mathcal{B}_1) \subseteq \Delta' \cup \{{}^{+}\phi\}$ and $\mathsf{Open}(\mathcal{C}_1) \subseteq \Delta' \cup \{{}^{+}\psi\}$, which then implies $\mathfrak{M}, S \models \iota[\mathsf{Open}(\mathcal{B}_1)]$ and $\mathfrak{M}, P \models \iota[\mathsf{Open}(\mathcal{C}_1)]$. In turn, this implies by induction hypothesis $\mathfrak{M}, S \models \iota({}^{\bullet}\mu)$ and $\mathfrak{M}, P \models \iota({}^{\bullet}\mu)$. By union closure, we get $\mathfrak{M}, T \models \iota({}^{\bullet}\mu)$, which then implies $\mathfrak{M}, T \models \iota({}^{\bullet}\zeta)$ by applying the induction hypothesis again to B_2 or $\mathcal{C}_2$ since by assumption none of those derivations contains ${}^{+}\phi$, ${}^{+}\psi$ if not as a premiss of a downscale rule (notice, in particular $\iota({}^{\oplus}\phi) = \blacklozenge\phi$ which is supported by T as witnessed by the subteam S, similar considerations apply to ${}^{\oplus}\psi$ and P).

If some formulae ${}^{\bullet}\delta$, ${}^{\bullet}\theta$ appearing in Δ', Θ' are not down-closed, then by the $\star$ condition there are

derivations $\mathcal{M}$, $\mathcal{N}$ deriving, for $M \subseteq \Delta$, $M, {}^{\bullet}\delta \vdash {}^{\bullet}\epsilon$ and, for $N \subseteq \Theta$, $N, {}^{\bullet}\theta \vdash {}^{\bullet}\eta$ where $\iota(\epsilon)$, $\iota(\eta)$ figure as premisses for either of the rules ${}^{+}\vee I$ or ${}^{-}\wedge I$ in the last step of a a pair of subderivations $\mathcal{B}_1$, $\mathcal{C}_1$ with $\mathcal{B} = \begin{matrix}\mathcal{B}_1\\ \mathcal{B}_2\end{matrix}$ and $\mathcal{C} = \begin{matrix}\mathcal{C}_1\\ \mathcal{C}_2\end{matrix}$ and $\mathsf{Open}(\mathcal{B}_2) = \mathsf{Open}(\mathcal{C}_2)$, where $\mathsf{Open}(\mathcal{B}_1) = \Delta \subseteq \Delta'$ and $\mathsf{Open}(\mathcal{C}_1) = \Theta \subseteq \Theta'$. We give the details for the ${}^{+} \vee I$ case as the other one is symmetric.
Let us say, without loss of generality that ${}^{+}\mu \vee \nu$ is the conclusion of both $\mathcal{B}_1$, $\mathcal{C}_1$. Given the premisses of the ${}^{+} \vee I$-rule we then have one the the following pair of derivations:

(1)	$\mathcal{D} : \Delta \setminus \{{}^{\bullet}\delta\}{}^{+}\phi\star \vdash {}^{+}\mu$	$\mathcal{E} : \Theta \setminus \{{}^{\bullet}\theta\}{}^{+}\psi\star \vdash {}^{+}\mu$
(2)	$\mathcal{D} : \Delta \setminus \{{}^{\bullet}\delta\}{}^{+}\phi\star \vdash {}^{\oplus}\nu$	$\mathcal{E} : \Theta \setminus \{{}^{\bullet}\theta\}{}^{+}\psi\star \vdash {}^{\oplus}\nu$
(3)	$\mathcal{D} : \Delta \setminus \{{}^{\bullet}\delta\}{}^{+}\phi\star \vdash {}^{+}\mu$	$\mathcal{E} : \Theta \setminus \{{}^{\bullet}\theta\}{}^{+}\psi\star \vdash {}^{\oplus}\nu$
(4)	$\mathcal{D} : \Delta \setminus \{{}^{\bullet}\delta\}{}^{+}\phi\star \vdash {}^{+}\mu$	$\mathcal{E} : \Theta \setminus \{{}^{\bullet}\theta\}{}^{+}\psi\star \vdash {}^{+}\nu$
(5)	$\mathcal{D} : \Delta \setminus \{{}^{\bullet}\delta\}{}^{+}\phi\star \vdash {}^{\oplus}\mu$	$\mathcal{E} : \Theta \setminus \{{}^{\bullet}\theta\}{}^{+}\psi\star \vdash {}^{\oplus}\nu$

Where $\mathcal{B}_1$ is obtained from $\mathcal{D}$, $\mathcal{M}$ with and application of ${}^{+} \vee I$, and $\mathcal{C}_1$ is obtained from $\mathcal{E}$, $\mathcal{N}$ with an application of ${}^{+} \vee I$.

If $\mathcal{D}$, $\mathcal{E}$ are as listed in (1), then notice that, by assumption $\Theta \setminus \{\theta\}$ and $\Delta \setminus \{\delta\}$ only contain down-closed formulae, which in turn implies $\mathfrak{M}, S \models \Delta \setminus \{\delta\}$ and $\mathfrak{M}, P \models \Theta \setminus \{\theta\}$. By induction hypothesis, then, we obtain $\mathfrak{M}, S \models \mu$ and $\mathfrak{M}, P \models \mu$ which implies $\mathfrak{M}, T \models \mu$ by union-closure. Further, by the structure of ${}^{+} \vee I$ we must have ${}^{\bullet}\epsilon ={}^{\bullet} \eta ={}^{\oplus} \nu$, and because $\mathfrak{M}, T \models \Delta \cup \Theta$ we have $\mathfrak{M}, T \models \blacklozenge\nu$ by induction hypothesis applied to $\mathcal{M}$ (or $\mathcal{N}$). In turn, this implies $\mathfrak{M}, T \models \mu \vee \nu$ as desired. Now, because ${}^{+}\phi$, ${}^{+}\psi$ do not appear in $\mathcal{C}_2$ if not as premisses of $\downarrow A$, $\downarrow R$, we have

$$\mathsf{Open}(\mathcal{C}_2) \setminus \{{}^{+}\phi, {}^{+}\psi\}, {}^{\oplus}\phi, {}^{\oplus}\psi, {}^{+}\mu \vee \nu \vdash {}^{\bullet}\zeta$$

together with $\mathfrak{M}, T \models \mu \vee \nu$ and $\mathfrak{M}, T \models \Delta' \cup \Theta'$, we then get $\mathfrak{M}, T \models \iota({}^{\bullet}\zeta)$ with one more application of the inductive hypothesis to either $\mathcal{C}_2$ or $\mathcal{B}_2$.
If the subderivations are as listed in (2), then by downward closure of $\Theta \setminus \{\theta\}$ and $\Delta \setminus \{\delta\}$ together with the induction hypothesis we obtain $\mathfrak{M}, S \models \blacklozenge\nu$ and $\mathfrak{M}, P \models \blacklozenge\nu$ which in turn implies $\mathfrak{M}, T \models \blacklozenge\nu$ by the semantics of $\blacklozenge$ and the assumption $T = S \cup P$. Since we now have ${}^{\bullet}\epsilon ={}^{\bullet} \eta ={}^{+} \mu$, from $\mathfrak{M}, T, \models \Delta \cup \Theta$ we get $\mathfrak{M}, T \models \mu$ by applying the inductive hypothesis to $\mathcal{M}$ the desired follows. We then obtain $\mathfrak{M}, T \models \iota({}^{\bullet}\zeta)$ in the same way as in case (1).
If the subderivations are as listed in (3), then similarly to the preceding cases, we have $\mathfrak{M}, S \models \mu$ and $\mathfrak{M}, P \models \blacklozenge\nu$ by inductive hypothesis. By the structure of the ${}^{+} \vee I$ rule, we also know that ${}^{\bullet}\epsilon ={}^{\oplus} \nu$ and ${}^{\bullet}\eta ={}^{+} \mu$, by a further application of the inductive hypothesis on $\mathcal{N}$, then, it follows in particular that $\mathfrak{M}, T \models \mu$, so we obtain $\mathfrak{M}, T \models \mu \vee \nu$ by taking T and S as witnesses. We then obtain $\mathfrak{M}, T \models \iota({}^{\bullet}\zeta)$ in the same way as in case (1).
If the subderivations are as listed in (4), then we have $\mathfrak{M}, S \models \mu$ and $\mathfrak{M}, P \models \nu$ by the same reasoning as above. The $\mathfrak{M}, T \models \mu \vee \nu$ then follows by union-closure and the fact that $S \cup P = T$, whereas $\mathfrak{M}, T \models \iota({}^{\bullet}\zeta)$ is obtained in the same way as in case (1)
Last, if the subderivations are as listed in (5) then clearly ${}^{\bullet}\epsilon ={}^{+} \nu$ and ${}^{\bullet}\eta ={}^{+} \mu$, so that the statement follows by the fact that $\mathfrak{M}, T \models \Gamma \cup \Delta' \cup \Theta'$ applying the inductive hypothesis to $\mathcal{M}$, $\mathcal{N}$.

Alternatively, it may occur that ${}^{\bullet}\delta = {}^{\bullet}\theta$ where there is an $\mathcal{M}$ witnessing $M, {}^{\bullet}\delta \vdash {}^{\bullet}\epsilon$. We have $\mathfrak{M}, T, \models \iota({}^{\bullet}\epsilon)$ by the same reasoning as before, and there are several similar cases depending on the shape of ${}^{\bullet}\epsilon$.

If ${}^{\bullet}\epsilon ={}^{+} \blacklozenge\eta$ for η down-closed, then we have $\mathfrak{M}, T \models \blacklozenge\eta$. Let R be the witnessing team for η, because R is non-empty, and η is down-closed, from $S \cup P = T$ we conclude $R \cap S \neq \varnothing$. By assumption the last rule occurring in $\mathcal{B}_1$ is an EC rule, and there is a derivations $\mathcal{F}$ deriving from ${}^{+}\blacklozenge\ \eta$ a premiss ${}^{\bullet}\kappa$ of this rule. Likewise $\Delta \setminus \{{}^{\bullet}\delta\}$ by assumption contains only down-closed formulae, letting ${}^{\bullet}\gamma$ be the other premiss for EC, and $\mathcal{D}$ the derivation of ${}^{\bullet}\gamma$ from $\Delta \setminus \{{}^{\bullet}\delta\}$, we have $\mathfrak{M}, S \cap R \models \iota({}^{\bullet}\gamma)$ by applying the induction hypothesis to $\mathcal{D}$. As η is down-closed, we also have $\mathfrak{M}, S \cap R \models \eta$ which implies $S \cap R$ also supports $\blacklozenge\eta$ which is the same as $\iota({}^{+}\blacklozenge\eta)$. By applying the induction hypothesis to $\mathcal{B}_1$ we would then obtain $\mathfrak{M}, S \cap R \models \bot$. Since this can never happen, this case also never occurs and the desired follows vacuously.

The cases for ${}^{\bullet}\epsilon ={}^{\oplus}\ \eta$ and ${}^{\bullet}\epsilon ={}^{\ominus}\ \eta$ are identical, noticing $\iota({}^{\oplus}\eta) = \blacklozenge\eta$, $\iota({}^{\ominus}\eta) = \blacklozenge\neg\eta$ and that $S \cap R \models \neg\eta$ because in the latter case η is required to be negatively down-closed.

If ${}^{+}\phi \vee \psi ={}^{+} \gamma \vee (\delta \vee \kappa)$ then we may have:

$$\mathcal{B} = \begin{array}{c}\mathcal{B}_1\\ \mathcal{B}_2\end{array} \text{ and } \mathcal{C} = \begin{array}{c}\mathcal{C}_1\\ \mathcal{C}_2\end{array}$$

With $\mathcal{B}_1$, $\mathcal{C}_1$ deriving ${}^{+}(\gamma \vee \delta) \vee \kappa$ from ${}^{+}\phi \vee \psi$ together with each disjunct individually. Here, from $\mathfrak{M}, T \models \phi \vee \psi$ we obtain $\mathfrak{M}, T \models (\gamma \vee \delta) \vee \kappa$ by the semantic clauses. Since the restrictions ensure that ${}^{+}\gamma$, ${}^{+}\delta$, ${}^{+}\kappa$ do not appear in $\mathcal{B}_2$, $\mathcal{C}_2$, the desired follows by applying the inductive hypothesis to those two derivations.

Last, if $\mathcal{C}, \mathcal{D}$ conclude a sentence which fails to be union-closed, then we know there are subderivations of $\mathcal{B}_1, \mathcal{C}_1, \mathcal{J}, \mathcal{K}$ which conclude, respectively, ${}^{\circ}\lambda$ from $\mathsf{Open}(\mathcal{B}_1) \cup \{{}^{+}\phi\}$, and ${}^{\circ}\kappa$ from $\mathsf{Open}(\mathcal{C}_1) \cup \{{}^{+}\psi\}$. Furthermore $\mathcal{B}_1$, $\mathcal{C}_1$ each respect $\star$, and each conclude ${}^{+}\lambda \vee \kappa$ if $\circ = +$ or ${}^{-}\lambda \wedge \kappa$ if $\circ = -$. This case then reduce to the case for Weak inference. In particular, we know $\mathfrak{M}, S \models \phi$ and $\mathfrak{M}, P \models \psi$, since $\mathcal{B}_1$ respects $*$, $\mathcal{J}$ respects it as well, so we obtain $\mathfrak{M}, S \models \iota({}^{\circ}\lambda)$ by the case for weak inference applied to $\mathcal{K}$. By the same reasoning applied to $\mathcal{K}$, we also get $\mathfrak{M}, S \models \iota({}^{\circ}\kappa)$, which then implies either $\mathfrak{M}, T \models \lambda \vee \kappa$ or $\mathfrak{M}, T \models \neg(\lambda \wedge \kappa)$ depending on the value of $\circ$. Since as before ${}^{+}\phi, {}^{+}\ \psi$ do not appear in $\mathcal{B}_2$, $\mathcal{C}_{\in}$, applying the inductive hypothesis to either of those derivations yields $\mathfrak{M}, T \models \iota({}^{\bullet}\zeta)$.

${}^{+}\varovee I$ Given $\Gamma \vdash {}^{+}\mu$, we then have $\mathfrak{M}, T \models \mu$ by induction hypothesis, which implies the desired by the semantics of inquisitive disjunction. The case for $\Gamma \vdash {}^{+}\nu$ is symmetrical.

${}^{+}\varovee E$ If the last rule applied is inquisitive disjunction elimination for assertion, then we have three derivations: $\mathcal{A} : \Gamma \vdash {}^{+}\mu \varovee \nu$, $\mathcal{B} : \Delta, {}^{+}\mu \vdash {}^{\bullet}\epsilon$, and $\mathcal{C} : \Theta, {}^{+}\nu \vdash {}^{\bullet}\epsilon$.

Let $\mathfrak{M}, T \models \Gamma \cup \Delta \cup \Theta$, then, in particular, $\mathfrak{M}, T \models \Gamma$ so that by the inductive hypothesis and $\mathcal{A}$ we obtain $\mathfrak{M}, T \models \mu \varovee \nu$. By the semantic clauses for inquisitive disjunction, we know that either $\mathfrak{M}, T \models \mu$ or $\mathfrak{M}, T \models \nu$. In the first case, it follows that $\mathfrak{M}, T \models \Delta \cup \{\mu\}$, so that by inductive hypothesis and $\mathcal{B}$ it follows that $\mathfrak{M}, T \models \iota({}^{\bullet}\epsilon)$. The second case is symmetrical using $\mathcal{C}$.

${}^{-}\wedge I$ If the last rule applied is conjunction introduction for rejection, then we have $\Gamma \vdash {}^{-}\mu$ and $\Gamma \vdash {}^{\ominus}\nu$. Assuming $\mathfrak{M}, T \models \Gamma$, we obtain $\mathfrak{M}, T \models \neg\mu$ and $\mathfrak{M}, T \models \blacklozenge\neg\nu$ by induction hypothesis, with the latter witnessed by a nonempty $S \subseteq T$. By the semantic clauses for negation, we get $\mathfrak{M}, T \mathrel{=\!\!\mid} \mu$

and $\mathfrak{M}, S \dashv \nu$, witnessing $\mathfrak{M}, T \dashv \mu \wedge \nu$ and thus also $\mathfrak{M}, T \vDash \neg(\mu \wedge \nu)$.

$^- \vee I$ This case is similar to conjunction introduction for assertion, using the semantic clauses of negation as in $^- \wedge I$.

$^- \wedge E$ This case is symmetric to $^+ \vee E$.

$^- \vee E$ This case is similar to conjunction elimination for assertion, using the semantic clauses of negation as in $^- \wedge I$.

$^- \veebar I$ If the last rule applied is inquisitive disjunction introduction for rejection, then we have $\Gamma \vdash^n {}^-\phi$ and $\Gamma \vdash^n {}^-\psi$. By induction hypothesis, it then follows that $\mathfrak{M}, T \vDash \neg\phi$ and $\mathfrak{M}, T \vDash \neg\psi$, which alos implies $\mathfrak{M}, T \dashv \phi$ and $\mathfrak{M}, T \dashv \psi$. The desired then follows by the semantic clauses for inquisitive disjunction.

$^- \veebar E$ If the last rule applied is inquisitive disjunction elimination for rejection, then we have by inductive hypothesis that $\mathfrak{M}, T \vDash \neg(\mu \veebar \nu)$ and thus also $\mathfrak{M}, T \dashv \mu$ and $\mathfrak{M}, T \dashv \nu$, implying the desired via the semantic clauses for negation.

$\downarrow A$ In this case we have $\Gamma \vDash {}^+\mu$, and so by inductive hypothesis also $\mathfrak{M}, T \vDash \mu$. Since $T \subseteq T$, we obtain immediately $\mathfrak{M}, T \vDash \blacklozenge \mu$.

$\downarrow R$ We have $\Gamma \vDash {}^-\mu$, and so by inductive hypothesis also $\mathfrak{M}, T \vDash \neg\mu$ and thus in particular $\mathfrak{M}, T \dashv \mu$. Since $T \subseteq T$, we obtain immediately $\mathfrak{M}, T \vDash \blacklozenge \neg\mu$.

$\blacklozenge$ rules The cases for the rules: $^+ \blacklozenge I, ^+ \blacklozenge E, ^- \blacklozenge I, ^- \blacklozenge E$ all follow immediately by induction hypothesis. In fact, we have: $\iota(^\oplus\phi) = \blacklozenge\phi = \iota(^+\blacklozenge\phi)$ and $\iota(^-\phi) = \neg\phi \equiv \neg\blacklozenge\ \phi = \iota(^-\blacklozenge\phi)$.

$\bot E$ If the last rule applied is *Ex Falso*, then the desired vacuously follows, as no $\mathcal{BSL}_m^{\veebar}$-model supports $\bot$.

Coord The case for the negation rules follow immediately by inductive hypothesis together with the semantic clauses for negation.

$W.I._{1,2}$ If the last rule applied is *Weak inference 1*, then by assumption we have two derivations as of below:

$$\mathcal{A} : \Gamma \vdash {}^\oplus\mu$$
$$\mathcal{B} : \Delta, {}^+\mu * \vdash {}^+\nu$$

Let now $\mathfrak{M}, T \vDash \Gamma \cup \Delta$. By inductive hypothesis together with $\mathcal{A}$, it follows that $\mathfrak{M}, T \vDash \blacklozenge \mu$, meaning there is some nonempty $S \subseteq T$ with $\mathfrak{M}, S \vDash \mu$. Since $\mathcal{B}$ complies with the $*$ restriction, there are two possible cases.

If all formulae in Δ are down-closed, then given $S \subseteq T$ and $\mathfrak{M}, T \vDash \Delta$, this implies $\mathfrak{M}, S \vDash \Delta \cup \{\mu\}$, so we obtain $\mathfrak{M}, S \vDash \psi$ by inductive hypothesis together with $\mathcal{B}$. In turn, this implies $\mathfrak{M}, T \vDash \blacklozenge\nu$ which is the desired.

If Δ, contains non-downclosed formulae, then without loss of generality let us assume $^\bullet\delta$ is the unique non-downclosed formula appearing in Δ.[14] Let there be a derivation $\mathcal{C} : D, {}^\bullet\delta \vdash {}^\bullet\epsilon$ where $D \subseteq \Delta$ and $^\bullet\epsilon$ appears as a premiss of either $^+ \vee I$ or $^- \wedge I$ in $\mathcal{B}$. Letting $\mathcal{D}$ be the subderivation of $\mathcal{B}$ concluding the other premiss of $^+ \vee I$ ($^- \wedge I$), $^\bullet\eta$. $\mathcal{B}$ is then of the shape:

[14] If there are multiple non-downclosed formulae $^\bullet\delta_1, .., {}^\bullet\delta_n$ then the desired by applying the two cases below n times starting from the topmost $^\bullet\delta_1$ appearing in $\mathcal{B}$.

$$\dfrac{\begin{array}{c}\Delta \setminus \{^{\bullet}\delta\} \\ \mathcal{D} \\ ^{\bullet}\eta\end{array} \qquad \begin{array}{c}D, \{^{\bullet}\delta\} \\ \mathcal{C} \\ ^{\bullet}\epsilon\end{array}}{^{+}\eta \vee \epsilon}$$
$$\mathcal{E}$$

We detail the case where the last rule applied is $^{+} \vee I$ as the other case is symmetrical .
We have again two cases, depending on whether $^{\bullet}\epsilon$ is strongly or weakly asserted.
If $^{\circ}\epsilon$ is the strongly asserted premiss of $^{+} \vee I$, then we have the following derivation:

$$\mathcal{D} : \Delta \setminus \{\delta\},^{+} \phi * \vdash {}^{\oplus}\eta$$

By hypothesis, all open assumptions in $\mathcal{D}$ are down-closed, so applying the inductive hypothesis to $\mathcal{D}$ we obtain $\mathfrak{M}, S \vDash \blacklozenge\, \eta$ from $\mathfrak{M}, S \vDash (\Delta \setminus \{\iota(\epsilon)\}) \cup \{\phi\}$. As we know $\mathfrak{M}, T \vDash \Delta \cup \Gamma$ we also get $\mathfrak{M}, T \vDash \epsilon$ and, because $\mathfrak{M}, T \vDash \blacklozenge\, \eta$, we conclude $\mathfrak{M}, T \vDash \epsilon \vee \eta$. Since by assumption $^{+}\mu$ does not appear in $\mathcal{E}$ if not as a premiss of $\downarrow A$, $\downarrow R$, we have:

$$\mathsf{Open}(\mathcal{E}) \setminus \{^{+}\mu\},^{+} \epsilon \vee \eta,^{\oplus} \mu * \vdash \nu$$

we can apply the induction hypothesis again to obtain $\mathfrak{M}, T \vDash \nu$ and thus also $\mathfrak{M}, T \vDash \blacklozenge\, \nu$.

If δ is the weakly asserted premiss, then, similar to before, we obtain the following derivation:

$$\mathcal{D} : \Delta \setminus \{\delta\},^{+} \phi * \vdash {}^{+}\eta$$

Since now $\Delta \setminus \{^{\bullet}\delta\}$ is a set of down-closed formulae, by induction hypothesis together with $\mathcal{D}$ it follows that: $\mathfrak{M}, S \vDash \eta$. Further, by applying the inductive hypothesis to $\mathcal{C}$, we get $\mathfrak{M}, T \vDash \blacklozenge\, \epsilon$, which must have a nonempty witness $P \subseteq T$. We therefore obtain $\mathfrak{M}, S \cup P \vDash \eta \vee \epsilon$. Since by assumption $^{+}\mu$ does not appear in $\mathcal{E}$ if not as a premiss of $\downarrow A$, $\downarrow R$, we have:

$$\mathsf{Open}(\mathcal{E}) \setminus \{^{+}\mu\},^{+} \epsilon \vee \eta,^{\oplus} \mu * \vdash \nu$$

since clearly $\mathfrak{M}, S \cup P \vDash \blacklozenge\, \nu$ we can apply the induction hypothesis again to obtain $\mathfrak{M}, S \cup P \vDash \nu$ which also yields $\mathfrak{M}, T \vDash \blacklozenge\, \nu$.

Alternatively, we may have some $C \subseteq \Delta$ and a derivation $\mathcal{C} : {}^{\bullet}\delta, C \vdash {}^{\bullet}\gamma$ where γ is a premiss of an EC rule and $\bullet \in \mathsf{S}$. In this case we have $\mathfrak{M}, T \vDash \iota(^{\bullet}\gamma)$ by the same reasoning as above, which in light of $\bullet \in \mathsf{S}$ entails either $\mathfrak{M}, T \vDash \gamma$ or $\mathfrak{M}, T \mathrel{=\!\!|} \gamma$. In the first case, by the structure of the EC rules we know the other premiss of the rule must be $^{\ominus}\gamma$, so that we get $\mathfrak{M}, S \mathrel{=\!\!|} \gamma$ by applying the inductive hypothesis to $\mathcal{D}$. However, by proposition 8 we know this implies $\mathfrak{M}, T \vDash \bot$ and thus this situation never occurs.

The cases where *Weak Inference 2* is used are treated in the same way as those for *Weak Inference 1*.

$W.I._{3,4}$ This case is similar to the previous one, making here use of the semantic clauses for negation.

FC The case for both free-choice rules is immediate from the induction hypothesis and the fact that no $\mathcal{BSL}_m^{\mathbb{V}}$-team is empty.

EC The rules EC_1 and EC_2 are vacuously sound by proposition 8, ensuring that $\mathfrak{M}, T \models \phi \wedge \blacklozenge \neg\phi$, as well as $\mathfrak{M}, T \dashv \phi \vee \neg \blacklozenge \phi$ never occur.

Rw. Let $^{+}\phi(\alpha \veebar \beta)$ be such that $\alpha \veebar \beta$ occurs in asserted position. We have three derivations:

$$\mathcal{A} : \Gamma \models^{+} \phi(\alpha \veebar \beta)$$
$$\mathcal{B} : \Delta,^{+} \phi(\alpha) \models {}^{\bullet}\mu$$
$$\mathcal{C} : \Theta,^{+} \phi(\beta) \models {}^{\bullet}\mu$$

Given $\mathfrak{M}.T \models \Gamma \cup \Delta \cup \Theta$, by inductive hypothesis applied to $\mathcal{A}$ we obtain $\mathfrak{M}, T \models \phi(\alpha \veebar \beta)$. By lemma 47, it the follows that $\mathfrak{M}, T \models \phi(\alpha)$ or $\mathfrak{M}, T \models \phi(\beta)$. In the first case, we obtain the desired applying the inductive hypothesis to $\mathcal{B}$, in the second, we obtained the desired applying it to $\mathcal{C}$ instead.

■

2.4 Completeness

We now turn to proving completeness of NFM_{ne} with respect to $\mathcal{BSL}_m^{\veebar}$-models. To do so, we first show that any sentence of NFM_{ne} is interderivable with the asserted Hintikka formula for some property. Next, we use this to argue that whenever a given team T supports a finite set of formulae $\Gamma = \{\gamma_1, .., \gamma_n\}$, T will be bisimilar to some team T' lying at the intersection of all the properties corresponding to the formulae in Γ. At last, for any ϕ such that $\Gamma \models \phi$, this implies a contradiction with the assumption that $\Gamma \nvdash^{+} \phi$.

First, we need show that a number of properties holding for $\mathcal{BSL}_m^{\veebar}$-formulae also hold for sentences of NFM_{ne}. We begin with some basic facts about negation:

Lemma 49 (**Double negation rules**)**.** The rules ${}^{\bullet}DNE$ and ${}^{\bullet}DNI$ are derivable, for $\bullet \in \{+, -, \oplus, \ominus\}$:

$$\frac{{}^{\bullet}\phi}{{}^{\bullet}\neg\neg\phi}\,{}^{\bullet}DNE\ , \quad \frac{{}^{\bullet}\neg\neg\phi}{{}^{\bullet}\phi}\,{}^{\bullet}DNI$$

Proof. We show the case for $\bullet = +$, the case for rejection is symmetrical. As for the weak speech acts, it is easy to verify that the trees comply with the $*$ constraint, so that the desired follows from WI together with the derivations for the strong cases.
The derivability of rules ^{+}DNI and ^{+}DNE is witnessed together by the tree:

$$\cfrac{\cfrac{{}^{+}\phi}{{}^{-}\neg\phi}\;{+}\Rightarrow\neg{-}}{{}^{+}\neg\neg\phi}\;{-}\Rightarrow\neg{+}$$

■

We now turn to De Morgan laws. We give the specifics for the sole case of strong assertion, as before, the case for strong rejection is symmetric and the cases for weak signatures follow from weak inference. We refer to De Morgan rules as ${}^{\bullet}DM_1$, ${}^{\bullet}DM_2$ respectively.

Lemma 50 (**De Morgan rules**)**.** For $\phi, \psi \in \mathcal{BSL}_m^{\veebar}$, the following are interderivable:

$^{+}DM_1$ $\ {}^{+}\neg(\phi \wedge \psi) \dashv\star\ast\vdash^{+} \neg\phi \vee \neg\psi$;

$^{+}DM_2$: $^{+}\neg(\phi \vee \psi) \dashv\star\ast\vdash^{+} \neg\phi \wedge \neg\psi$.

Proof.

$^{+}DM_1$: The right-to-left direction is witnessed by the following tree:

$$
\begin{prooftree}
\AxiomC{$^{+}\neg(\phi \wedge \psi)$}
\RightLabel{Coord.}
\UnaryInfC{$^{-}\phi \wedge \psi$}
\AxiomC{$[^{-}\phi]$}
\RightLabel{Coord.}
\UnaryInfC{$^{+}\neg\phi$}
\AxiomC{$^{+}\neg(\phi \wedge \psi)$}
\RightLabel{Coord.}
\UnaryInfC{$^{-}\phi \wedge \psi$}
\RightLabel{$\downarrow R$}
\UnaryInfC{$^{\ominus}\phi \wedge \psi$}
\RightLabel{$^{-}FC.$}
\UnaryInfC{$^{\ominus}\psi$}
\RightLabel{Coord.}
\UnaryInfC{$^{\oplus}\neg\psi$}
\RightLabel{$^{+}\vee I$}
\BinaryInfC{$^{+}\neg\phi \vee \neg\psi$}
\AxiomC{$[^{-}\psi]$}
\RightLabel{Coord.}
\UnaryInfC{$^{+}\neg\psi$}
\AxiomC{$^{-}\phi \wedge \psi$}
\RightLabel{$\downarrow R$}
\UnaryInfC{$^{\ominus}\phi \wedge \psi$}
\RightLabel{$^{-}FC.$}
\UnaryInfC{$^{\ominus}\phi$}
\RightLabel{Coord.}
\UnaryInfC{$^{\oplus}\neg\phi$}
\RightLabel{$^{+}\vee I$}
\BinaryInfC{$^{+}\neg\phi \vee \neg\psi$}
\RightLabel{$^{-}\wedge E$}
\TrinaryInfC{$^{+}\neg\phi \vee \neg\psi$}
\end{prooftree}
$$

The converse direction is witnessed by the following derivation:

$$
\begin{prooftree}
\AxiomC{$^{+}\neg\phi \vee \neg\psi$}
\AxiomC{$[^{+}\neg\phi]$}
\RightLabel{Coord.}
\UnaryInfC{$^{-}\phi$}
\AxiomC{$^{+}\neg\phi \vee \neg\psi$}
\RightLabel{$\downarrow A$}
\UnaryInfC{$^{\oplus}\neg\phi \vee \neg\psi$}
\RightLabel{$^{+}FC.$}
\UnaryInfC{$^{\oplus}\neg\psi$}
\RightLabel{Coord.}
\UnaryInfC{$^{\ominus}\psi$}
\RightLabel{$^{-}\wedge I$}
\BinaryInfC{$^{-}\ \phi \wedge \psi$}
\AxiomC{$[^{+}\neg\psi]$}
\RightLabel{Coord.}
\UnaryInfC{$^{-}\psi$}
\AxiomC{$^{+}\neg\phi \vee \neg\psi$}
\RightLabel{$\downarrow A$}
\UnaryInfC{$^{\oplus}\neg\phi \vee \neg\psi$}
\RightLabel{$^{+}FC.$}
\UnaryInfC{$^{\oplus}\neg\phi$}
\RightLabel{Coord.}
\UnaryInfC{$^{\ominus}\phi$}
\RightLabel{$^{-}\wedge I$}
\BinaryInfC{$^{-}\ \phi \wedge \psi$}
\RightLabel{$^{+}\vee E$}
\TrinaryInfC{$^{-}\phi \wedge \psi$}
\RightLabel{Coord.}
\UnaryInfC{$^{+}\neg(\phi \wedge \psi)$}
\end{prooftree}
$$

$^{+}DM_2$: For the left-to-right direction we have the following witness:

$$
\begin{prooftree}
\AxiomC{$^{+}\neg(\phi \vee \psi)$}
\RightLabel{Coord.}
\UnaryInfC{$^{-}\phi \vee \psi$}
\RightLabel{$^{-}\vee E$}
\UnaryInfC{$^{-}\phi$}
\RightLabel{Coord.}
\UnaryInfC{$^{+}\neg\phi$}
\AxiomC{$^{+}\neg(\phi \vee \psi)$}
\RightLabel{Coord.}
\UnaryInfC{$^{-}\phi \vee \psi$}
\RightLabel{$^{-}\vee E$}
\UnaryInfC{$^{-}\psi$}
\RightLabel{Coord.}
\UnaryInfC{$^{+}\neg\psi$}
\RightLabel{$^{+}\wedge I$}
\BinaryInfC{$^{+}\neg\phi \wedge \neg\psi$}
\end{prooftree}
$$

The left-to-right direction is as of below:

$$
\begin{prooftree}
\AxiomC{$^{+}\neg\phi \wedge \neg\psi$}
\RightLabel{$^{+}\wedge E$}
\UnaryInfC{$^{+}\neg\phi$}
\RightLabel{Coord.}
\UnaryInfC{$^{-}\phi$}
\AxiomC{$^{+}\neg\phi \wedge \neg\psi$}
\RightLabel{$^{+}\wedge E$}
\UnaryInfC{$^{+}\neg\psi$}
\RightLabel{Coord.}
\UnaryInfC{$^{-}\psi$}
\RightLabel{$^{-}\vee I$}
\BinaryInfC{$^{-}\phi \vee \psi$}
\RightLabel{Coord.}
\UnaryInfC{$^{+}\neg(\phi \vee \psi)$}
\end{prooftree}
$$

It is easy to check that all the trees above comply with the restrictions, which then shows the desired. ■

We thus obtain the following corollary:

Corollary 51. Every sentence $^{\bullet}\phi$ of NFM_{ne} is provably equivalent to one in negation normal form.

Proof. By induction on the structure of $^{\bullet}\phi$, all cases are beside the one for negation are immediate. The cases for negation follow by induction hypothesis together with lemma 49 and lemma 50. ■

Last a handful of properties about the interaction between logical constants with inquisitive disjunction is needed for completeness:

Lemma 52. The following rules are derivable:

1. $^{+}\blacklozenge\ (\phi \mathbin{\vee\!\!\!\vee} \psi) \dashv\star\ast\vdash^{+} \blacklozenge\phi \mathbin{\vee\!\!\!\vee} \blacklozenge\psi$.
2. $^{+}\blacklozenge\ (\phi \wedge \psi) \star\ast\vdash^{+} \blacklozenge\phi \wedge \blacklozenge\psi$.
3. $^{+}\phi \vee (\psi \mathbin{\vee\!\!\!\vee} \zeta) \dashv\star\ast\vdash^{+} (\phi \vee \psi) \mathbin{\vee\!\!\!\vee} (\phi \vee \zeta)$.
4. $^{+}\phi \wedge (\psi \mathbin{\vee\!\!\!\vee} \zeta) \dashv\star\ast\vdash^{+} (\phi \wedge \psi) \mathbin{\vee\!\!\!\vee} (\phi \wedge \zeta)$
5. $^{+}\blacklozenge\ (\phi \vee \psi) \dashv\star\ast\vdash^{+} \blacklozenge\phi \vee \blacklozenge\psi$.

Proof.

1. The left-to-right direction is witnessed by the following tree:

$$\dfrac{^{+}\blacklozenge\ (\phi \mathbin{\vee\!\!\!\vee} \psi) \qquad \dfrac{[^{+}\blacklozenge\ \phi]}{^{+}\blacklozenge\ \phi \mathbin{\vee\!\!\!\vee} \blacklozenge\psi}\,{}^{+}\mathbin{\vee\!\!\!\vee} I \qquad \dfrac{[^{+}\blacklozenge\ \psi]}{^{+}\blacklozenge\ \phi \mathbin{\vee\!\!\!\vee} \blacklozenge\psi}\,{}^{+}\mathbin{\vee\!\!\!\vee} I}{^{+}\blacklozenge\ \phi \mathbin{\vee\!\!\!\vee} \blacklozenge\psi}\,Rw.$$

The right-to-left direction is easy by noticing that $^{+}\blacklozenge\ \phi$ derives $^{+}\blacklozenge\ (\phi \mathbin{\vee\!\!\!\vee} \psi)$ by applying Weak inference and then $^{+}\mathbin{\vee\!\!\!\vee} I$, and the same goes for $^{+}\psi$. The statement then follows by a further application of $^{+}\mathbin{\vee\!\!\!\vee} E$.

2. The witnessing tree is as of below:

$$\dfrac{\dfrac{\dfrac{\dfrac{^{+}\blacklozenge\ \phi \wedge \psi}{^{\oplus}\phi \wedge \psi}\,{}^{+}\blacklozenge E \qquad \dfrac{[^{+}\phi \wedge \psi]}{^{+}\phi}\,{}^{+}\wedge E}{^{\oplus}\phi}\,W.I._1}{^{+}\blacklozenge\ \phi}\,{}^{+}\blacklozenge I \qquad \dfrac{\dfrac{\dfrac{^{+}\blacklozenge\ \phi \wedge \psi}{^{\oplus}\phi \wedge \psi}\,{}^{+}\blacklozenge E \qquad \dfrac{[^{+}\phi \wedge \psi]}{^{+}\psi}\,{}^{+}\wedge E}{^{\oplus}\psi}\,W.I._1}{^{+}\blacklozenge\ \psi}\,{}^{+}\blacklozenge I}{^{+}\blacklozenge\ \phi \wedge \blacklozenge\psi}\,{}^{+}\wedge I$$

3. The left-to-right direction is given as below:

$$\dfrac{^{+}\phi \vee (\psi \mathbin{\vee\!\!\!\vee} \zeta) \qquad \dfrac{[^{+}\phi \vee \psi]}{^{+}(\phi \vee \psi) \mathbin{\vee\!\!\!\vee} (\phi \vee \zeta)}\,{}^{+}\mathbin{\vee\!\!\!\vee} I \qquad \dfrac{[^{+}\phi \vee \zeta]}{^{+}(\phi \vee \psi) \mathbin{\vee\!\!\!\vee} (\phi \vee \zeta)}\,{}^{+}\mathbin{\vee\!\!\!\vee} I}{^{+}(\phi \vee \psi) \mathbin{\vee\!\!\!\vee} (\phi \vee \zeta)}\,Rw.$$

The right-to-left direction follows easily by an application of $^{+}\mathbin{\vee\!\!\!\vee} E$ together with $^{+}\vee Mon$.

4. Both directions follow easily using inquisitive disjunction elimination and the conjunction rules for assertion.
5. The left-to-right direction is derived as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{^{+}\blacklozenge\ (\phi \vee \psi)}{^{\oplus}\phi \vee \psi}\,{}^{+}\blacklozenge E}{^{\oplus}\phi}\,{}^{+}FC}{^{+}\blacklozenge\ \phi}\,{}^{+}\blacklozenge I \qquad \dfrac{\dfrac{\dfrac{\dfrac{^{+}\blacklozenge\ (\phi \vee \psi)}{^{\oplus}\phi \vee \psi}\,{}^{+}\blacklozenge E}{^{\oplus}\psi}\,{}^{+}FC}{^{+}\blacklozenge\ \psi}\,{}^{+}\blacklozenge I}{^{\oplus}\blacklozenge\ \psi}\,\downarrow A}{^{+}\blacklozenge\ \phi \vee \blacklozenge\psi}\,{}^{+}\vee I$$

The other direction is as below:

$$
\begin{prooftree}
\AxiomC{$^{+}\blacklozenge\,\phi \vee \blacklozenge\,\psi$}
\RightLabel{$\downarrow A$}
\UnaryInfC{$^{\oplus}\blacklozenge\,\phi \vee \blacklozenge\,\psi$}
\RightLabel{^{+}FC}
\UnaryInfC{$^{\oplus}\blacklozenge\,\phi$}
\RightLabel{$\blacklozenge\, Id.$}
\UnaryInfC{$^{\oplus}\phi$}
\AxiomC{$^{+}\blacklozenge\,\phi \vee \blacklozenge\,\psi$}
\RightLabel{$\downarrow A$}
\UnaryInfC{$^{\oplus}\blacklozenge\,\phi \vee \blacklozenge\,\psi$}
\RightLabel{^{+}FC}
\UnaryInfC{$^{\oplus}\blacklozenge\,\psi$}
\RightLabel{$\blacklozenge\, Id.$}
\UnaryInfC{$^{\oplus}\psi$}
\RightLabel{$^{\oplus}\vee I$}
\BinaryInfC{$^{\oplus}\phi \vee \psi$}
\RightLabel{$^{+}\blacklozenge\, I$}
\UnaryInfC{$^{+}\blacklozenge\,(\phi \vee \psi)$}
\end{prooftree}
$$

■

Next, we show proof-theoretic analogues of proposition 24 and proposition 25.

Lemma 53. Let $\mathfrak{M}, \mathfrak{N}$ be models over N and w, v arbitrary states therein, then:

$$\mathfrak{M}, w \leftrightharpoons \mathfrak{N}, v \iff {}^{+}\chi_w \dashv\star\ast\vdash^{+} \chi_v$$

Proof.

- $\Longrightarrow$ Let $\mathfrak{M}, w \leftrightharpoons \mathfrak{N}, v$, then by proposition 24, we know $\mathfrak{M}, w \equiv \mathfrak{N}, v$, which in particular implies w and v agree on all literals over N. By construction of χ_w and χ_v (definition 26), then, it follows that $\chi_w = \chi_v$ and thus each direction of the desired follows by $^{+}\wedge I$ and $^{+}\wedge E$. Since all characteristic formulae for states are positively downclosed, it is easy to see that the restrictions are complied with.
- $\Longleftarrow$ Let ${}^{+}\chi_w \dashv\star\ast\vdash^{+} \chi_v$, by soundness (theorem 48), this implies $\chi_w \equiv \chi_w$. Since clearly we have both $\mathfrak{M}, w \vDash \chi_w$ and $\mathfrak{M}, v \vDash \chi_v$, we then obtain the desired by proposition 30.

■

This result generalizes to team-formulae in a straightforward manner:

Lemma 54. Let $\mathfrak{M}$, $\mathfrak{N}$ be models over N and T, S arbitrary teams therein, then:

$$\mathfrak{M}, T \leftrightharpoons \mathfrak{N}, S \iff {}^{+}\theta_T \dashv\star\ast\vdash^{+} \theta_S$$

Proof.

- $\Longrightarrow$ Let $\mathfrak{M}, T \leftrightharpoons \mathfrak{N}, S$, then by the forth condition of the bisimulation relation, for all $w \in T$ there exists a $v \in S$ with $w \leftrightharpoons v$. By lemma 53, this in turn implies $\chi_w^{+} \dashv\star\ast\vdash^{+} \chi_v$, particular, by the proof of lemma 53, we readily see that indeed $\chi_v = \chi_w$. The left-to-right derivation then follows by applying $^{+}\vee Wkn.$ and $^{+}\vee Cntr.$ an appropriate number of times. Notice that, as S is bounded upwards by $|\mathcal{P}(\mathsf{N} \to \mathsf{Bool})|$, the derivation procedure is finite, furthermore, it is imemdiate to check that Hintikka formulae for states are union-closed, so application of $^{+}\vee Cntr.$ are licit. The left-to-right derivation is similar using the back condition instead.
- $\Longleftarrow$ Let ${}^{+}\theta_T \dashv\star\ast\vdash^{+} \theta_S$. By soundness (theorem 48) we have $\theta_T \equiv \theta_S$. As $\mathfrak{M}, T \vDash \theta_T$, this implies $\mathfrak{M}, T \vDash \theta_S$, hence yielding the desired by proposition 31.

■

Conversely, if two teams are not bisimilar, then the conjunction of their characteristic formulae is contradictory:

Lemma 55. Let $\mathfrak{M}$, $\mathfrak{N}$ be models over N and T, S arbitrary teams such that $\mathfrak{M}, T \not\leftrightarrows \mathfrak{N}, S$, then we have: ${}^{+}\theta_T, {}^{+}\theta_S \star * \vdash \bot$.

Proof. Let $\mathfrak{M}$, $\mathfrak{N}$, S and T be as illustrated above. Notice that since $T \not\leftrightarrows S$ and both $S, T \neq \varnothing$, there is some $w \in T$ with $w \not\leftrightarrows v$ for any $v \in S$.
Consider then χ_w and χ_u for an arbitrary $u \in S$, as $w \not\leftrightarrows u$, then there is some $p_j \in \mathsf{N}$ for which (without loss of generality) $w \in V_{\mathfrak{M}}(p_j)$ and $v \notin V_{\mathfrak{N}}(p_j)$. It is easy to see that ${}^{+}\chi_w \vdash^{+} p_j$, while ${}^{+}\chi_u \vdash^{+} \neg p_j$ via ${}^{+} \wedge E$, so that we obtain ${}^{+}\chi_w, {}^{+}\chi_u \vdash \bot$ as witnessed by the tree below:

$$\dfrac{\dfrac{\dfrac{{}^{+}\chi_w}{{}^{+}p_j}\,{}^{+}\wedge E}{{}^{\oplus}p_j}\downarrow A \qquad \dfrac{\dfrac{{}^{+}\chi_u}{{}^{+}\neg p_j}\,{}^{+}\wedge E}{{}^{-}p_j}\,\mathsf{Coord}}{\bot}\,EC_1$$

Notice that the restrictions are complied with as characteristic formulae for points are down-closed by construction. Since u was arbitrary, we then obtain for every $v \in S$:

$$\mathcal{A} :{}^{+} \chi_w, {}^{+}\chi_v \star * \vdash \bot$$

Since χ_v is flat for every v, therefore the following stronger statement also holds:

$$\mathcal{B} :{}^{\oplus} \chi_w, {}^{+}\chi_u \star * \vdash \bot$$

As witnessed by the derivation tree below;

$$\dfrac{\dfrac{{}^{\oplus}\chi_w \qquad \dfrac{[{}^{+}\chi_w]}{{}^{+}p_j}\,{}^{+}\wedge E}{{}^{\oplus}p_j}\,W.I._1 \qquad \dfrac{\dfrac{{}^{+}\chi_u}{{}^{+}\neg p_j}\,{}^{+}\wedge E}{{}^{-}p_j}\,\mathsf{Coord}}{\bot}\,EC_1$$

With $\mathcal{B}$, we then obtain the following derivation $\mathcal{D}$, complying in particular with the $\star, *$ restrictions:

$$\dfrac{{}^{+}\bigvee_{v\in S}\chi_v \qquad \dfrac{[{}^{\oplus}\chi_a] \qquad {}^{+}\chi_w}{\bot}\,\mathcal{B} \qquad \dots \qquad \dfrac{[{}^{\oplus}\chi_z] \qquad {}^{+}\chi_w}{\bot}\,\mathcal{B}}{\bot}\,{}^{+}\vee E$$

We can use $\mathcal{D}$ to construct one further derivation:

$$\dfrac{\dfrac{\dfrac{{}^{+}\bigvee_{m\in T}\chi_m}{{}^{\oplus}\bigvee_{m\in T}\chi_m}\downarrow A}{{}^{\oplus}\chi_m}\,{}^{+}FC \qquad {}^{+}\bigvee_{v\in S}\chi_v}{\bot}\,\mathcal{D}$$

At last, we therefore obtain $\theta_T, \theta_S \star * \vdash \bot$ which was the desired.

■

We are now in the position to show that any asserted formula ${}^{+}\phi$ is provably equivalent to the assertion of the characteristic formula of some property.

Lemma 56 (Provable equivalence to normal form). For any $\phi \in \mathcal{BSL}_m^{\mathbb{V}}$, there exists a property $\mathtt{C}$ such that :

$${}^{+}\phi \dashv \star * \vdash^{+} \pi_{\mathtt{C}}$$

Proof. Recall that $\pi_{\mathtt{C}} = \bigvee\!\!\!\bigvee_{X\in\mathtt{C}} \theta_X$. We proceed by induction on the structure of ϕ. Notice that, in light of corollary 51, we can reduce the case of negation to that of negative literals.

Literals: We ought to show, for $p_i \in \mathsf{N}$, $p_i \dashv\star\ast\vdash \pi_{[\![p_i]\!]}$, and $\neg p_i \dashv\star\ast\vdash \pi_{[\![\neg p_i]\!]}$. We detail the case of the positive literals, as that for the negative ones is symmetric.

Let $\mathtt{C} = [\![p_i]\!]$, and notice that both positive and negative literals are down-closed. For the right-to-left direction, by assumption we have $\mathfrak{M}, T \models p_i$ for all $\langle\mathfrak{M}, T\rangle \in [\![p_i]\!]$, so the result follows by first applying ${}^{+}\mathbin{\vee\!\!\!\vee} E$ and then ${}^{+}\vee E$ an appropriate number of times. It is easy to see that the subderivation respect the $\star$ restriction as each ${}^{+}\chi_w$ for $w \in X$ derives ${}^{+}p_i$ simply by eliminating conjunctions. A fragment of the derivation is given below:

$$\dfrac{{}^{+}\bigvee\!\!\!\bigvee_{X\in\mathtt{C}}\theta_X \qquad \dfrac{[{}^{+}\bigvee_{w\in A}\chi_v] \qquad \dfrac{[{}^{+}\chi_a]}{{}^{+}p_i}\,{}^{+}\wedge E \quad \dots \quad \dfrac{[{}^{+}\chi_z]}{{}^{+}p_i}\,{}^{+}\wedge E}{{}^{+}p_i}\,{}^{+}\vee E \qquad \dots}{{}^{+}p_i}\,{}^{+}\mathbin{\vee\!\!\!\vee} E$$

The left to right direction follows from the $Ne.$ rule, together with ${}^{+}\mathbin{\vee\!\!\!\vee} E$. Pick any nonempty team and model $\langle\mathfrak{M}, Y\rangle$ over N, and consider θ_Y. We have different cases depending on whether $\mathfrak{M}, Y \models p_i$, $\mathfrak{M}, X, \models \neg p_i$ or $\mathfrak{M}, X \models p_i \vee \neg p_i$.
In the first case, we know that $\langle\mathfrak{M}, Y\rangle \in [\![p_i]\!]$, so we obtain ${}^{+}\pi_{[\![p_i]\!]}$ simply by ${}^{+}\mathbin{\vee\!\!\!\vee} I$:

$$\dfrac{[{}^{+}\theta_Y]}{{}^{+}\bigvee\!\!\!\bigvee_{X\in[\![p_i]\!]}\theta_X}\,{}^{+}\mathbin{\vee\!\!\!\vee} I$$

In the second case, we know that $\langle\mathfrak{M}, Y\rangle \in [\![\neg p_i]\!]$ which implies that for every $w \in Y$, $\chi_w \vdash \neg p_i$ with ${}^{+}\wedge E$, so we get $\theta_Y, p_i \vdash \bot$ yielding the desired via *Ex Falso*:

$$\dfrac{\dfrac{{}^{+}p_i \qquad \dfrac{\dfrac{\dfrac{\dfrac{[{}^{+}\theta_Y]}{{}^{\oplus}\theta_Y}\,\downarrow A}{{}^{\oplus}\chi_w}\,{}^{+}FC}{{}^{\oplus}\neg p_i}\,{}^{\oplus}\wedge E}{{}^{\ominus}p_i}\,\mathsf{Coord}}{\bot}\,EC_1}{{}^{+}\pi_{[\![p_i]\!]}}\,\bot E$$

Last, if $\mathfrak{M}, Y \models p_i \vee \neg p_i$ then we have $\chi_z \vdash \neg p_i$ for at least one $z \in X$, and we obtain ${}^{+}\pi_{[\![p_i]\!]}$ in the same way as the previous case, choosing the appropriate χ_z. Since θ_Y was arbitrary, we then conclude ${}^{+}\pi_{[\![p_i]\!]}$ by an application of ${}^{+}\mathbin{\vee\!\!\!\vee} E$:

$$\dfrac{\dfrac{}{{}^{+}\pi_{[\![\top]\!]}}\,Ne. \qquad \begin{matrix}[{}^{+}\theta_A], {}^{+}p_i \\ \mathcal{D}_1 \\ {}^{+}\pi_{[\![p_i]\!]}\end{matrix} \qquad \dots \qquad \begin{matrix}[{}^{+}\theta_Z], {}^{+}p_i \\ \mathcal{D}_n \\ {}^{+}\pi_{[\![p_i]\!]}\end{matrix}}{{}^{+}\pi_{[\![p_i]\!]}}\,{}^{+}\mathbin{\vee\!\!\!\vee} E$$

$\wedge$: Let $\phi = \mu \wedge \nu$, by induction hypothesis we have:

$$ {}^{+}\mu \dashv\star\ast\vdash^{+} \pi_{\mathtt{C}} \tag{2.1}$$
$$ {}^{+}\nu \dashv\star\ast\vdash^{+} \pi_{\mathtt{D}} \tag{2.2}$$

We then obtain:[15]

$$\begin{aligned} {}^+\mu \wedge \nu &\dashv\vdash^+ \bigvee\!\!\!\bigvee_{X\in \mathtt{C}} \theta_X \wedge \bigvee\!\!\!\bigvee_{Y\in \mathtt{D}} \theta_Y && (2.1),\ (2.2),\ {}^+\wedge E,\ {}^+\wedge I \\ &\dashv\vdash^+ \bigvee\!\!\!\bigvee_{X\in \mathtt{C}} \bigvee\!\!\!\bigvee_{Y\in \mathtt{D}} \theta_X \wedge \theta_Y && \text{lemma 52 (4)} \end{aligned}$$

Let now $\mathtt{E} = \{Z \subseteq \mathsf{N} \to \mathsf{Bool} \mid \exists C \in \mathtt{C}, D \in \mathtt{D}[C \leftrightarrows Z \leftrightarrows D]\}$. Since every pair of model and teams in $\mathtt{C}, \mathtt{D}$ are either bisimilar to each other or not, we can rewrite the above as below, where with $[X]$ for X a team we mean the equivalence class of which X is a representative:

$$^+(\bigvee\!\!\!\bigvee_{X\in \mathtt{C}\cap\mathtt{E}} \bigvee\!\!\!\bigvee_{Y\in \mathtt{D}\cap[X]} \theta_X \wedge \theta_Y) \mathbin{\mathbb{V}} (\bigvee\!\!\!\bigvee_{X\in \mathtt{C}\cap\mathtt{E}} \bigvee\!\!\!\bigvee_{Y\in \mathtt{D}\setminus[X]} \theta_X \wedge \theta_Y)) \mathbin{\mathbb{V}} (\bigvee\!\!\!\bigvee_{X\in \mathtt{C}\setminus\mathtt{E}} \bigvee\!\!\!\bigvee_{Y\in \mathtt{D}} \theta_X \wedge \theta_Y))$$

Moreover, by lemma 55, we know that all pairs in the second and third disjunct derive falsum, so that by using ${}^+\mathbin{\mathbb{V}} E$ together with $\bot E$ we obtain:

$$\begin{aligned} \ldots &\dashv\vdash^+ \bigvee\!\!\!\bigvee_{X\in \mathtt{C}\cap\mathtt{E}} \bigvee\!\!\!\bigvee_{Y\in \mathtt{D}\cap[X]} \theta_X \wedge \theta_Y && {}^+\mathbin{\mathbb{V}} E,\ \bot E \\ &\dashv\vdash^+ \bigvee\!\!\!\bigvee_{E\in \mathtt{E}} \theta_E && {}^+\mathbin{\mathbb{V}} E,\ \text{lemma 54.} \end{aligned}$$

Which is the same as ${}^+\pi_{\mathtt{E}}$.

$\mathbb{V}$: Let $\phi = \mu \mathbin{\mathbb{V}} \nu$. By induction hypothesis we have the following derivations;

$$^+\mu \dashv\star * \vdash^+ \pi_{\mathtt{C}} \tag{2.3}$$

$$^+\nu \dashv\star * \vdash^+ \pi_{\mathtt{D}} \tag{2.4}$$

Then, we obtain ${}^+\mu \vee \nu \dashv\star\vdash \pi_{\mathtt{C}\cup\mathtt{D}}$ using, for both of the directions, ${}^+\mathbin{\mathbb{V}} E$ together with (2.3) / (2.4) and ${}^+\mathbin{\mathbb{V}} I$.

$\vee$: Let $\phi = \mu \vee \nu$. By induction hypothesis we have the usual derivations;

$$^+\mu \dashv\star * \vdash^+ \pi_{\mathtt{C}} \tag{2.5}$$

$$^+\nu \dashv\star * \vdash^+ \pi_{\mathtt{D}} \tag{2.6}$$

Using ${}^+\vee Mon$ we then obtain:

$$\begin{aligned} {}^+\phi \vee \psi &\dashv\vdash^+ \bigvee\!\!\!\bigvee_{X\in \mathtt{C}} \theta_X \vee \bigvee\!\!\!\bigvee_{Y\in \mathtt{D}} \theta_Y && (2.5),\ (2.6),\ {}^+\vee Mon \\ &\dashv\vdash^+ \bigvee\!\!\!\bigvee_{X\in \mathtt{C}} \bigvee\!\!\!\bigvee_{Y\in \mathtt{D}} \theta_X \vee \theta_Y && \text{lemma 52 (3)} \\ &\dashv\vdash^+ \bigvee\!\!\!\bigvee_{X\in \mathtt{C}} \bigvee\!\!\!\bigvee_{Y\in \mathtt{D}} \theta_{X\uplus Y} && \text{Rewriting} \end{aligned}$$

Which is the same as $\pi_{\mathtt{C}\uplus\mathtt{D}}$.

$\blacklozenge$: Let $\phi = \blacklozenge\mu$. By induction hypothesis we have:

$$^+\mu \dashv\star * \vdash^+ \pi_{\mathtt{C}} \tag{2.7}$$

[15]We omit $\star, *$ in inline derivations for reasons of space.

which entails (given the $\star$ restriction):

$$\begin{aligned}
{}^{+}\blacklozenge\,\phi &\dashv\vdash^{+} \blacklozenge \bigvee\!\!\!\!\bigvee_{X\in\mathsf{C}} \theta_X && (2.7),\ W.I._1,\ {}^{+}\blacklozenge\ I \\
&\dashv\vdash^{+} \bigvee\!\!\!\!\bigvee_{X\in\mathsf{C}} \blacklozenge\,\theta_X && \text{lemma 52 (1)} \\
&\dashv\vdash^{+} \bigvee\!\!\!\!\bigvee_{X\in\mathsf{C}} \bigvee_{m\in X} \blacklozenge\,\chi_m && \text{lemma 52 (5)}
\end{aligned}$$

Consider now ${}^{+}\blacklozenge\ \chi_m$, we show ${}^{+}\blacklozenge\ \chi_m \dashv\star\ast\vdash^{+} \pi_{[\![\blacklozenge\chi_m]\!]}$, which then implies the desired in light of lemma 52 (3).

The right-to-left direction follows easily by noticing that any $\langle\mathfrak{M}, Y\rangle \in [\![\blacklozenge\,\chi_m]\!]$ is such that $\mathfrak{M}, Z \vDash \chi_m$ for some nonempty $Z \subseteq Y$. Thus, it suffices to apply ${}^{+}\vee\!\!\!\vee\, E$ and ${}^{+}FC$ an appropriate number of times.

$$\dfrac{{}^{+}\bigvee\!\!\!\!\bigvee_{Y\in[\![\blacklozenge\chi_m]\!]}\theta_Y \qquad \dfrac{\dfrac{\dfrac{[{}^{+}\theta_A]}{{}^{\oplus}\theta_A}\downarrow A}{{}^{\oplus}\chi_m}{}^{+}FC}{{}^{+}\blacklozenge\ \chi_m}{}^{+}\blacklozenge\ I \qquad \cdots}{{}^{+}\blacklozenge\ \chi_m}{}^{+}\vee\!\!\!\vee\, E$$

The left-to-right direction follows from $Ne.$: Considering any nonempty model and team $\mathfrak{M}$, Y over N it is easy to see that if $\langle\mathfrak{M}, Y\rangle \notin [\![\blacklozenge\,\chi_m]\!]$ then for every $v \in Y$, χ_v and χ_m must disagree over the valuation of at least one literal p_k, so that we get ${}^{+}\theta_Y, \chi_m \vdash \bot$ by the proof of lemma 55, and then the desired by $\bot E$:

$$\dfrac{\dfrac{[\bigvee_{v\in Y}\chi_v] \qquad \dfrac{\dfrac{\dfrac{{}^{+}\blacklozenge\ \chi_m}{{}^{\oplus}\blacklozenge\ \chi_m}{}^{+}\blacklozenge\ E}{{}^{\oplus}p_k}{}^{+}FC \qquad \dfrac{\dfrac{\dfrac{[{}^{+}\chi_a]}{{}^{+}\neg p_k}{}^{+}\wedge E}{{}^{-}p_k}\mathsf{Coord}}{{}^{\ominus}p_k}\downarrow R}{\bot}EC_1 \qquad \cdots}{\bot}{}^{+}\vee E}{{}^{+}\pi_{[\![\blacklozenge\chi_m]\!]}}\bot E$$

Since there is a unique formula from which falsum is derived in all the disjuncts of θ_Y, one readily sees $\star$ is complied with.

If, on the other hand $\langle\mathfrak{M}, Y\rangle \in [\![\blacklozenge\,\chi_m]\!]$, then the result follows simply by ${}^{+}\vee\!\!\!\vee\, I$. At last, we derive ${}^{+}\pi_{[\![\blacklozenge\chi_m]\!]}$ by applying ${}^{+}\vee\!\!\!\vee\, E$ as in the case of literals.

$$\dfrac{\dfrac{}{{}^{+}\pi_{[\![\top]\!]}}Ne. \qquad \begin{matrix}[{}^{+}\theta_A],\ {}^{+}\blacklozenge\ \chi_m \\ \mathcal{D}_1 \\ {}^{+}\pi_{[\![\blacklozenge\chi_m]\!]}\end{matrix} \qquad \cdots \qquad \begin{matrix}[{}^{+}\theta_Z],\ {}^{+}\blacklozenge\ \chi_m \\ \mathcal{D}_n \\ {}^{+}\pi_{[\![\blacklozenge\chi_m]\!]}\end{matrix}}{{}^{+}\pi_{[\![\blacklozenge\chi_m]\!]}}{}^{+}\vee\!\!\!\vee\, E$$

■

One further fact is needed for completeness, stating that whenever a certain sentence does not follow from a sequence of inquisitive disjunctions, then we can always isolate a sequence of individual disjuncts from which the sentence also does not follow:

Lemma 57 (Inverse global disjunction property).

1. If $\Gamma,^+ \bigvee_{i\in\mathbb{I}} \phi_i \nvdash^+ \psi$ then $\Gamma,^+ \phi_i \nvdash^+ \psi$ for some $i \in \mathbb{I}$.
2. If $\{^+\bigvee_{j\in\mathbb{J}_i} \phi_j \mid i \in \mathbb{I}\} \nvdash^+ \psi$ then $\{^+\phi_i \mid i \in \mathbb{I}\} \nvdash^+ \psi$ for some collection of ϕ_i with, for every i, $\phi_i \in \{\phi_j \mid j \in \mathbb{J}_i\}$.

Proof. Let $\mathbb{I}$, $\mathbb{J}$ be collection of indices, then it holds:

1. If $\Gamma,^+ \phi_i \vdash^+ \psi$ held for all $i \in \mathbb{I}$, then $\Gamma,^+ \bigvee_{i\in\mathbb{I}} \phi_i \vdash^+ \psi$ would also follow by inquisitive disjunction elimination for assertion.
2. Without loss of generality, let $\mathbb{I} = m \in \mathbb{N}$. Let $\Gamma_m = \{^+\bigvee_{j\in\mathbb{J}_i} \phi_j \mid i < m\}$ (notice that, for $m = 1$, we simply have $\Gamma_m = \varnothing$). By item 1, we know $\Gamma_m,^+ \bigvee_{j\in\mathbb{J}_m} \phi_{jm} \nvdash^+ \psi$ implies $\Gamma_m,^+ \phi_{j'm} \nvdash^+ \psi$ for some $j'm \in \mathbb{J}_m$. The result then follows by iterating the construction $m-1$ times, and taking Γ_{m-n} to be $\{^+\bigvee_{j\in\mathbb{J}_i} \phi_j \mid i < (m-n)\} \cup \bigcup\{\phi_{j'(m-s)} \mid 0 \leq s \leq n\}$.

■

We can at last show completeness for our system. Formally, we employ the same $\iota : \mathsf{NFM}_{ne} \to \mathcal{BSL}_m^{\mathbb{V}}$ as before, where in practice the ι notation is omitted when clear from context.
Notice that, thanks to the Coord and $\blacklozenge$ rules, it is sufficient to show the result for asserted formulae.

Theorem 58 (Completeness). For any Γ, $^+\phi$, $^+\psi \in \mathsf{NFM}_{ne}$, if $\iota[\Gamma], \phi \vDash \psi$, then $\Gamma,^+ \phi \vdash^+ \psi$.

Proof. We prove the contrapositive, assume $\Gamma,^+ \phi \nvdash^+ \psi$. Without loss of generality, let $\Gamma = \{^+\gamma_0, .., ^+\gamma_n\}$ for $n \in \mathbb{N}$. By lemma 56, we know

$$^+\phi \dashv\star\!\ast\!\vdash^+ \pi_{\mathtt{C}},\ ^+\psi \dashv\star\!\ast\!\vdash^+ \pi_{\mathtt{D}} \text{ and, for } i \leq n\ , \gamma_i \dashv\star\!\ast\!\vdash^+ \pi_{\mathtt{G}_i}$$

Which then implies $^+\pi_{\mathtt{G}_0}, ...,^+ \pi_{\mathtt{G}_n},^+ \pi_{\mathtt{C}} \nvdash^+ \psi$. By lemma 57, there exist a $\Delta = \{^+\theta_{G_0}, ...,^+ \theta_{Gn},^+ \theta_C\}$ such that $\Delta \nvdash^+ \psi$.
Notice this also implies Δ is semantically consistent. Indeed, if $\Delta \vDash \bot$ then, since all $\mathcal{BSL}_m^{\mathbb{V}}$-teams are consistent, we know that for any $^+\delta_j \in \Delta$, $^+\delta_j \nvdash \bot$ (otherwise we would obtain $Z \vDash \bot$ for some $Z \vDash^+ \delta_j$ by soundness). Therefore, it must be that $C \not\leftrightharpoons G_i$ for some $i \leq n$. However, by lemma 55, this implies $^+\theta_C,^+ \theta_{G_i} \vdash \bot$, so we would also obtain $^+\psi$ by *Ex Falso*, a contradiction to our initial assumption that $\Delta \nvdash^+ \psi$.
Thus, Δ is semantically consistent. Assume now, for arbitrary $\langle \mathfrak{M}, X\rangle$, $\mathfrak{M}, X \vDash \Delta$. As:

$$\mathfrak{M}, X \vDash \Delta \iff \mathfrak{M}, X \vDash \theta_{G_0} \wedge ... \wedge \theta_C$$

By proposition 31, it follows in particular that $X \leftrightharpoons C$. Assume now for contradiction $\mathfrak{M}, X \vDash \psi$, by Soundness (theorem 48), from $^+\psi \dashv\vdash^+ \pi_{\mathtt{D}}$ we obtain $\psi \equiv \pi_{\mathtt{D}}$ which then entails $\mathfrak{M}, X \vDash \theta_D$ for some $D \in \mathtt{D}$. In turn, this implies $X \leftrightharpoons D$ by proposition 31. Since bisimulation is an equivalence relation, from the above it readily follows that $C \leftrightharpoons D$, thus implying $^+\theta_C \dashv\star\!\ast\!\vdash^+ \theta_D$ by lemma 54. In particular, this also yields:

$$\begin{aligned} ^+\theta_C &\vdash^+ \theta_D && \text{lemma 54} \\ &\vdash^+ \pi_{\mathtt{D}} && ^+ \mathbb{V} I \\ &\vdash^+ \psi && \text{hypothesis} \end{aligned}$$

This is a contradiction to our assumption that $\Delta \nvdash^+ \psi$, and so we conclude $\mathfrak{M}, X \nvDash \psi$. ■

Chapter 3

Properties and normal form of NFM_{ne}

In this chapter, we investigate some proof-theoretical properties of NFM_{ne}. First, we put our theoretical criteria of harmony to practice, by showing they motivate a number of rewriting procedures ("conversion steps") over derivation trees which preserve assumptions and conclusion, and which can be licensed on the basis of the "inversion principle" discussed at section 1.3.1.
Next, we show those lead to a normal form theorem for NFM_{ne}, and then show that the system enjoys a weak version of the subformula property. As for the latter, the greatest challenges come from the rule *Rw*., as the assumptions induced by the major premiss of the rule need not be a subformula of the major premiss itself. For this reason, we show that a normal derivation $\mathcal{D}$ is such that premisses of *Rw*. rules always appear before major premisses of any other rule, so that the normal form and weak subformula property for the *Rw*.-free fragment of the calculus can then be extended to the full NFM_{ne}.

In the immediate following, we lay down some basic notation and definitions, and we then outline the different reduction steps, together with their respective motivations.

3.1 Preliminaries

We will often speak, in what follows, of major and minor premisses of a rule. This notion is defined like so:

Definition 59 (Major premiss, Minor premiss). For a two-premiss rule R., the major premiss is the premiss bearing the sign of the constant the rule itself is marked with (if any). In the case of *Rw*., the major premiss is the signed formula where the designated instance of global disjunction appears. If the rule is a structural rule, different from weak inference, we refer to the premiss bearing a strong signature as the major premiss, and to to the one bearing a weak signature as the minor premiss.
For rules of Weak inference, the major premiss is the weakly signed formula which also appears under strong signature within brackets on top of the other premiss.

We also sometimes speak of induced assumptions:

Definition 60 (Induced assumptions). For a rule instance of a rule R. in a derivation $\mathcal{D}$, the induced assumptions of R. is defined according to the shape of R.
If R $\in \{{}^{+}\vee E,\ {}^{-}\wedge E,\ {}^{+}\veebar E\}$, then the induced assumptions are $[{}^{\bullet}\phi]$, $[{}^{\bullet}\psi]$ where the major premiss of R is either ${}^{\bullet}\phi \wedge \psi$, ${}^{\bullet}\phi \vee \psi$, or ${}^{\bullet}\phi \veebar \psi$.

If R is $Rw.$ over ${}^{+}\phi(\alpha \vee\!\!\vee \beta)$, then the induced assumptions are $[{}^{+}\phi(\alpha)]$ and $[{}^{+}\phi(\beta)]$. If R is a weak inference rule, then the induced assumption of R is $[{}^{\bullet}{}_{s}\phi]$ where the major premiss of R is ${}^{\bullet}{}_{w}\phi$.

Since most of the conversions discussed in the following behave similarly for rules ${}^{+}\vee E$, ${}^{-}\wedge E$, ${}^{+}\vee\!\!\vee E$, in this chapter we refer to them collectively as Disj rules. We also make use of the collective naming conventions outlined in section 2.2.2, so Coord are rules of the shape $\bullet \Rightarrow \circ\neg$, AR are rules $\downarrow A$, $\downarrow R$, and WI are rules of weak inference.
One important notion is that of a *Segment*. Intuitively, given a derivation $\mathcal{D}$, a segment is a sequence of sentences ${}^{\bullet}\alpha_1, ..., {}^{\bullet}\alpha_n$ where each α is some formula of $\mathcal{BSL}_m^{\vee\!\!\vee}$ identified up to prefixed negation. The rationale is that, unlike in the classical case, where repetitions of formulae are induced only by the minor premisses of $\vee E$ rules:

$$\dfrac{A \vee B \qquad \begin{matrix}[A]\\ \mathcal{D}\\ C\end{matrix} \qquad \begin{matrix}[B]\\ \mathcal{E}\\ C\end{matrix}}{C}\ \vee E$$

in our case repetitions of formulae are induced by Coord, AR, WI, and Disj rules. In particular, we need to ensure that the following sequences (in orange) are captured by our notion of segments.

$$\dfrac{{}^{\bullet}\phi \vee \psi \qquad \begin{matrix}[{}^{\bullet}\phi]\\ \mathcal{D}\\ {}^{\bullet}\mu\end{matrix} \qquad \begin{matrix}[{}^{\bullet}\psi]\\ \mathcal{E}\\ {}^{\bullet}\mu\end{matrix}}{{}^{\bullet}\mu}\ {}^{+}\vee E \qquad \dfrac{\dfrac{{}^{+}\phi}{{}^{-}\neg\phi}\ \mathsf{Coord}}{{}^{+}\phi}\ \mathsf{Coord}$$

$$\dfrac{{}^{\bullet}{}_{w}\phi \qquad \begin{matrix}[{}^{\bullet}{}_{s}\phi]\\ \mathcal{D}\\ {}^{\bullet}\mu\end{matrix}}{{}^{\bullet}{}_{w}\mu}\ \mathsf{WI} \qquad \dfrac{{}^{\bullet}{}_{w}\phi \qquad \begin{matrix}[{}^{\bullet}{}_{s}\phi]\\ \mathcal{D}\\ {}^{\bullet}\mu\end{matrix}}{{}^{\bullet}{}_{w}\mu}\ \mathsf{WI}$$

Definition 61 (**Segments**)**.** For $\phi \in \mathcal{BSL}_m^{\vee\!\!\vee}$, a segment σ is a finite sequence $\langle {}^{\bullet}\alpha_1, ..., {}^{\bullet}\alpha_n\rangle$ such that, for $\alpha_i = \phi$, $\alpha_{i+1} \in \{\phi, \neg\phi\}$; if $\phi = \neg\psi$, then $\alpha_{i+1} \in \{\phi, \neg\phi, \psi\}$. Further, the following holds:

1. ${}^{\bullet}\alpha_1$ is not the conclusion of a Disj, nor of an AR rule, nor of an Coord rule, nor of a WI rule. Moreover ${}^{\bullet}\alpha_1$ is not the induced assumption of a WI rule.
2. The following all hold:
   (a) If ${}^{\bullet}\alpha_i$ is the conclusion of a minor premiss of a Disj rule, or the conclusion of a minor premiss of a WI rule, then ${}^{\bullet}\alpha_{i+1}$ is the sentence immediately below;
   (b) If ${}^{\bullet}\alpha_i$ is the premiss of a Coord or of a AR rule, then ${}^{\bullet}\alpha_{i+1}$ is the sentence immediately below;
   (c) If ${}^{\bullet}\alpha_i$ is the major premiss of a WI rule, then ${}^{\bullet}\alpha_{i+1}$ is the induced assumption of the same instance of WI.
3. ${}^{\bullet}\alpha_n$ is not the premiss of a Coord or AR rule, nor the major premiss of a WI rule, nor the conclusion of a minor premiss of a Disj or WI rule.

In the context of derivation trees, we indicate a segment by $\sigma = \langle {}^{\bullet}\alpha_1, ..., {}^{\bullet}\alpha_n\rangle$ by picking one representative of the α's, and enclosing it withing angled brackets, marking them with the name of the segment. One example is given below, with $\sigma = \langle {}^{+}\phi \vee \psi, {}^{-}\neg(\phi \vee \psi), {}^{+}\phi \vee \psi, {}^{\oplus}\phi \vee \psi\rangle$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}}{^{+}\phi \vee \psi}}{^{-}\neg(\phi \vee \psi)}}{^{+}\phi \vee \psi}}{^{\oplus}\phi \vee \psi}}{^{\oplus}\phi}\,\mathsf{FC} \quad \text{becomes:} \quad \dfrac{\dfrac{\mathcal{D}}{\langle ^{\oplus}\phi \vee \psi \rangle_{\sigma}}}{^{\oplus}\phi}\,\mathsf{FC}$$

The choice of the representative will be forced by the rule applied to $^{\bullet}\alpha_n$ (in the case FC), when the rule applied to the formula above $^{\bullet}\alpha_1$ is relevant as well, we mention both formulas within brackets. In general, we do not assume the segment to end with the last representative displayed.

Segments can be exploited to establish a ordering among derivations of NFM_{ne}.

Definition 62 (Weight, length, level). For $\sigma = \langle ^{\bullet}\alpha_1, .., ^{\bullet}\alpha_n \rangle$ a segment, the *weight* of σ (notation: $W(\sigma)$) is the logical complexity of $pos(\alpha_i)$ where:

$$pos(\phi) = \begin{cases} \psi & \text{if } \phi = \neg^1 ... \neg^n \psi \\ \phi & \text{otherwise} \end{cases}$$

The *length* of σ (notation: $L(\sigma)$) is n where $\sigma = \langle ^{\bullet}\alpha_1, .., ^{\bullet}\alpha_n \rangle$.
Last, the *level* of σ (notation $Lev(\sigma)$) is the number of sequences $\langle ^{\bullet}\alpha_i, ^{\bullet}\alpha_{i+1} \rangle$ occurring in σ where $^{\bullet}\alpha_i$ is the conclusion of the minor premiss of a WI rule and $^{\bullet}\alpha_{i+1}$ is the conclusion of the same WI instance.[1]

One readily sees that $W(\sigma)$ is well defined as the complexity of $pos(\alpha_i)$ is identical for all α's. The rationale for the definition, is that we want to avoid cases where negation causes segments with formulae of intuitively different logical complexity to have the same weight. For example:

$$\sigma = \langle ^{+}\alpha \wedge \beta \rangle \text{ and } \lambda = \langle ^{-}\neg\alpha, ^{+}\alpha \rangle$$

Should have different weights. An even more telling case is illustrated in section 3.2.2.4, where the normalization procedure converts σ into λ:

$$\sigma = \langle ^{+}\neg\phi, ^{-}\phi, ^{+}\neg\phi \rangle \;\; \lambda = \langle ^{+}\neg\phi \rangle.$$

where the latter must not be heavier than the former.
We can order segments lexicographically using the notions defined above. Given two segments σ, λ say:

$$\begin{aligned} \sigma <_S \lambda \iff & W(\sigma) < W(\lambda) \text{ or,} \\ & W(\sigma) = W(\lambda) \text{ and } L(\sigma) < L(\lambda) \text{ or,} \\ & W(\sigma) = W(\lambda) \text{ and } L(\sigma) = L(\lambda) \text{ and } Lev(\sigma) < Lev(\lambda) \end{aligned}$$

The notion of *cut segment* is meant to capture redundant sequences of formulae in the context of a derivation.

Definition 63 (Cut segments). Let $\sigma = \langle ^{\bullet}\alpha_1, .., ^{\bullet}\alpha_n \rangle$. We say σ is a "cut segment" if and only if any of the following holds:

[1] For the sake of the normalization proof, we are counting sequences of signed formulae which are induced by Weak inference rules as one degree higher than identical sequences of formulae otherwise induced (e.g., by AR rules). A similar strategy is employed for bilateral normalizations in e.g., [Kur21].

1. $^\bullet\alpha_1$ is the conclusion of $\bot E$ and $^\bullet\alpha_n$ is either: the conclusion of an AR rule, or the premiss of $\bot E$, or the premiss of an EC rule or the induced assumption of an WI rule, or the conclusion of a WI rule.
2. $^\bullet\alpha_1$ is the conclusion of an introduction rule and $^\bullet\alpha_n$ is the premiss of an elimination rule (including free-choice).
3. $^\bullet\alpha_i$ is the premiss of a Coord rule introducing (eliminating) negation and $^\bullet\alpha_{i+j}$ is the conclusion different coordination rule eliminating (introducing) negation, and $i - j \neq 1$.
4. $^\bullet\alpha_1$ is the conclusion of a introduction rule and $^\bullet\alpha_n$ is the major premiss of an EC rule.
5. $^\bullet\alpha_i$ is the induced premiss of a WI rule, and $n \neq 2$, except the case where $^\bullet\alpha_1$ is the major premiss of a WI rule and $^\bullet\alpha_3 ... ^\bullet\alpha_n$ are all the conclusion of a Coord rule introducing (eliminating) negation, or $n = m+1$, $^\bullet\alpha_3 ... ^\bullet\alpha_m$ are the conclusion of the a Coord rule introducing (eliminating) and $^\bullet\alpha_m$ is the conclusion of a WI rule.[2]
6. $^\bullet\alpha_i$ is the conclusion of a Disj rule and $^\bullet\alpha_{i+j}$ is either the premiss of an elimination rule or the major premiss of a WI rule.

Definition 64 (Rank of a tree). For $\mathcal{D}$ a derivation tree, the rank of $\mathcal{D}$ (notation: $R(\mathcal{D})$) is a triple $\langle x, y, z\rangle$ where $x \in \mathbb{N}$ is the maximum of the weight of all cut segments appearing in $\mathcal{D}$, and $y \in \mathbb{N}$ is the sum of the lengths of all segments in $\mathcal{D}$ of weight x. Last z is the sum of the level of cut segments of weight x in $\mathcal{D}$

The rank of a derivation straightforwardly induces a lexicographic ordering $\leq_R$ among derivation trees. We exploit the lexicographic ordering of derivation trees as an induction parameter to show normalization of our system. For ease of exposition, we first outline the individual steps of normalization in the next section.

3.2 Conversions

In the following, we detail the reduction steps employed in the proof of normalization. The idea of normalization for natural deduction system stems from Prawitz ([Pra71]), with the goal of showing that there exists a restricted class of derivation trees, the so called "normal derivations", enjoying some specific structural properties, and that any derivation tree whatsoever could be turned into a normal tree preserving the same assumptions and conclusion.
Given an arbitrary derivation tree $\mathcal{D}$, $\mathcal{D}$ is transformed into a normal tree $\mathcal{D}'$ in a step-wise fashion, where each of the transformations is called a "conversion step", which we mark graphically with the curly arrow "$\rightsquigarrow$". At the most general level, the process of normalization corresponds to pruning a derivation tree from "redundant" subderivations that could be dispensed with. In practice, we follow Prawitz in distinguishing between "simplification", "detour", and "permutation" conversions. Each of those is treated separately, together with an explanation of its proof-theoretical significance.
Before turning to the illustration of each of those classes, however, we show that derivations in NFM_{ne} can always be turned into derivations where the $\bot E$ rule concludes signed propositional atoms only.

Definition 65 (Falsum Rank). Given a derivation tree $\mathcal{D}$ the Falsum rank of $\mathcal{D}$ (notation: $FR(\mathcal{D})$) is the maximum of the logical complexity of all sentences concluded by $\bot E$ in $\mathcal{D}$.

[2] These provisos ensure that all instances of weak Coord rules, as well as derived rules for connectives, do not qualify as cut segments.

Proposition 66 (Ex falso concludes atoms). Let $\mathcal{D} : \Gamma \vdash {}^{\bullet}\phi$ be a derivation; there exists a derivation $\mathcal{C} : \Gamma \vdash {}^{\bullet}\phi$ such that for every ${}^{\bullet}\psi$ the conclusion of an instance of $\bot E$ rule in $\mathcal{C}$, then $\phi = p_i$ for $p_i \in \mathsf{N}$.

Proof. Let $\mathcal{D}$ be as defined above. We proceed by induction on $FR(\mathcal{D})$. The base case is obvious as the claim holds vacuously and $\mathcal{D} = \mathcal{C}$. For the inductive case, let $\mathcal{D}$ be a derivation with Falsum rank $n+1$, notice that there are only finitely many instances of formulae concluded by *Ex falso* with complexity $n+1$ in $\mathcal{D}$. Thus, it suffices to show that any such instance of the *Ex falso* rule can be turned into an instance of *Ex falso* concluding a formula of complexity n, with the same open assumptions and conclusions of before. We start from the topmost instance of *Ex Falso* concluding a formula of complexity $n+1$ so as not to multiply instances of $\bot E$ concluding a formula of highest complexity. The claim then follows by induction hypothesis after a finite number of such transformations.
Every transformation has the following form: let $\#$ be any connective, and ${}^{\bullet}\gamma_1 ... {}^{\bullet}\gamma_n$ be the premisses of the ${}^{\bullet}\#I$ rule with conclusion ${}^{\bullet}\#\phi$ the reduction is then as below:

$$\dfrac{\begin{matrix}\mathcal{A}\\ \bot\end{matrix}}{{}^{\bullet}\#\phi}\,\bot E \atop \mathcal{B} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\mathcal{A}\\ \bot\end{matrix}}{{}^{\bullet}\gamma_1}\,\bot E \quad \cdots \quad \dfrac{\begin{matrix}\mathcal{A}\\ \bot\end{matrix}}{{}^{\bullet}\gamma_n}\,\bot E}{{}^{\bullet}\#\phi}\,{}^{\bullet}\#I \atop \mathcal{B}$$

By inspection of the rules of the system, it is straightforward that ${}^{\bullet}\gamma_1, ..., {}^{\bullet}\gamma_n$ have strictly lower complexity that ${}^{\bullet}\#\phi$, which then implies the desired. ■

We now turn to the illustration of the different conversion cases, noticing that, as derivation trees are finite by design, we can assume to always be operating on the topmost and rightmost segment of highest weight available:

Remark 67 (Normalization process). When performing one of the conversion step outlined below on a derivation tree $\mathcal{D}$ of rank $\langle m, n\rangle$, we may assume without loss of generality to have chosen a segment σ such that:

1. the *weight* of σ is m, and;
2. for all $\lambda = \langle {}^{\bullet}\beta_1, ..., {}^{\bullet}\beta_n\rangle$ with ${}^{\bullet}\beta_i$ appearing above σ, or above the minor premisses of any elimination rule of which σ is the major premiss, we have $W(\lambda) < W(\sigma)$.

It is easy to see this is possible as derivation trees are finite.

3.2.1 Simplifications

The first class of conversions is that of simplifications. Given a derivation tree $\mathcal{D}$, simplification conversions correspond to subderivations appearing as minor premisses of a rule R, which do not make use of the induced assumption of R itself. In all these cases, we can do away without applying R itself, and use our subderivation instead.
Simplification conversions for NFM_{ne} are as of below:

$$\dfrac{\begin{matrix}\mathcal{E}\\ {}^{+}\phi \vee \psi\end{matrix} \quad \begin{matrix}\mathcal{D}\\ \langle {}^{\bullet}\mu\rangle_{\sigma}\end{matrix} \quad \begin{matrix}[{}^{+}\psi]\\ \mathcal{C}\\ \langle {}^{\bullet}\mu\rangle_{\lambda}\end{matrix}}{\langle {}^{\bullet}\mu\rangle_{\sigma,\lambda}}\,{}^{+}\vee E \atop \mathcal{F} \quad \rightsquigarrow \quad \begin{matrix}\mathcal{D}\\ \langle {}^{\bullet}\mu\rangle_{\sigma}\\ \mathcal{F}\end{matrix}$$

$$
\dfrac{\begin{array}{c}\mathcal{E}\\ {}^{-}\phi \wedge \psi\end{array} \qquad \begin{array}{c}\mathcal{D}\\ \langle{}^{\bullet}\mu\rangle_{\sigma}\end{array} \qquad \begin{array}{c}[{}^{-}\psi]\\ \mathcal{C}\\ \langle{}^{\bullet}\mu\rangle_{\lambda}\end{array}}{\begin{array}{c}\langle{}^{\bullet}\mu\rangle_{\sigma,\lambda}\\ \mathcal{F}\end{array}}\;{}^{-}\wedge E \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{D}\\ \langle{}^{\bullet}\mu\rangle_{\sigma}\\ \mathcal{F}\end{array}
$$

$$
\dfrac{\begin{array}{c}\mathcal{E}\\ {}^{+}\phi \vee\!\!\!\vee \psi\end{array} \qquad \begin{array}{c}\mathcal{D}\\ \langle{}^{\bullet}\mu\rangle_{\sigma}\end{array} \qquad \begin{array}{c}[{}^{+}\psi]\\ \mathcal{C}\\ \langle{}^{\bullet}\mu\rangle_{\lambda}\end{array}}{\begin{array}{c}\langle{}^{\bullet}\mu\rangle_{\sigma,\lambda}\\ \mathcal{F}\end{array}}\;{}^{+}\vee\!\!\!\vee E \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{D}\\ \langle{}^{\bullet}\mu\rangle_{\sigma}\\ \mathcal{F}\end{array}
$$

In all three cases above, it is straightforward to verify that the length of σ is reduced as a result of the conversion step. A kindred situation is generated by the rule of $Rw.$ whenever we are rewriting over a $\vee\!\!\!\vee$-free sentence. Assuming the relevant instance of global disjunction is $\alpha \vee\!\!\!\vee \beta$, we have $\phi(\alpha) = \phi(\alpha \vee\!\!\!\vee \beta := \alpha) = \phi$, and similarly $\phi(\beta) = \phi(\alpha \vee\!\!\!\vee \beta := \beta) = \phi$. Then we can simplify as follows:

$$
\dfrac{\begin{array}{c}\mathcal{E}\\ {}^{+}\phi\end{array} \qquad \begin{array}{c}[{}^{+}\phi(\alpha)]\\ \mathcal{D}\\ \langle{}^{\bullet}\mu\rangle_{\sigma}\end{array} \qquad \begin{array}{c}[{}^{+}\phi(\beta)]\\ \mathcal{C}\\ \langle{}^{\bullet}\mu\rangle_{\lambda}\end{array}}{\begin{array}{c}\langle{}^{\bullet}\mu\rangle_{\sigma,\lambda}\\ \mathcal{F}\end{array}}\;Rw. \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{E}\\ {}^{+}\phi\\ \mathcal{D}\\ \langle{}^{\bullet}\mu\rangle_{\sigma}\\ \mathcal{F}\end{array}
$$

Also reducing the length of σ. Last, similar steps are induced by the rules of weak inference:

$$
\dfrac{\begin{array}{c}\mathcal{C}\\ {}^{\bullet}{}_{w}\phi\end{array} \qquad \begin{array}{c}\mathcal{D}\\ \langle{}^{\bullet}{}_{s}\mu\rangle_{\sigma}\end{array}}{\begin{array}{c}\langle{}^{\bullet}{}_{w}\mu\rangle_{\sigma}\\ \mathcal{E}\end{array}}\;W.I._{1,3} \rightsquigarrow \dfrac{\begin{array}{c}\mathcal{D}\\ \langle{}^{\bullet}{}_{s}\mu\rangle_{\sigma}\end{array}}{\begin{array}{c}{}^{\bullet}{}_{w}\mu\\ \mathcal{E}\end{array}}\;\mathsf{AR} \qquad \dfrac{\begin{array}{c}\mathcal{C}\\ {}^{\bullet}{}_{w}\phi\end{array} \qquad \begin{array}{c}\mathcal{D}\\ \langle{}^{\bullet}{}_{w}\mu\rangle_{\sigma}\end{array}}{\begin{array}{c}\langle{}^{\bullet}{}_{w}\mu\rangle_{\sigma}\\ \mathcal{E}\end{array}}\;W.I._{2,4} \rightsquigarrow \begin{array}{c}\mathcal{D}\\ \langle{}^{\bullet}{}_{w}\mu\rangle_{\sigma}\\ \mathcal{E}\end{array}
$$

As Weak inferences only conclude weakly signed formulae, in the case of $W.I._{1}$, $W.I._{3}$ the conclusion of the subderivation needs to be adjusted using the AR rule. Notice that by our definition of cut segments, no new cut segments is created in the process (see remark 68 just below). Further, while the cases $W.I._{2,4}$ reduce the length of the segment, the case for $W.I._{1,3}$ reduces the level of σ, as it replaces a pair of identical *formulae* induces by weak inference with one such pair induced by downscaling rules.

Remark 68 (**Downscaling with AR**)**.** The process simplification and detour conversion involves, for rules $W.I._{1}$ and $W.I._{3}$ the downscaling of the conclusion via an application of AR rules. By inspecting the definition of cut segments as well as the conversions here specified, one sees that the addition of an instance of AR rule to a segment $\lambda = \langle{}^{\bullet}\beta_1, ..., {}^{\bullet}\beta_n\rangle$ induces a new cut segment only if ${}^{\bullet}\beta_1$ is the conclusion of some introduction rule and ${}^{\bullet}\beta_n$ the premiss of an elimination rule, or ${}^{\bullet}\beta_1$ was the conclusion of an *Ex falso* rule. In both cases, we readily see λ is a cut segment before the conversion step is operated, and thus no new cut segment is brought about by the conversion step.

3.2.2 Detours

In section 1.3.1, we argued for the so called "inversion principle" to mark the correct balance between introduction and elimination rules within a natural deduction system. This principle, recall, states that a speaker is licensed in inferring from a compound sentence ${}^{\bullet}\#\phi$ only sentences which they could also have inferred from the grounds from which ${}^{\bullet}\#\phi$ itself was inferred, thus guaranteeing that neither introduction nor elimination of logical constants alters the inferential profile of the common background

of belief against which the conversation is carried out. On function of the of class of detour conversions is to internalize the inversion principle into NFM_{ne}, transposing it from our theorizing about deductive systems into being a lawful conversion step from one deductive tree to another.
Another function of detour conversions, is to establish a discipline with respect to what should be concluded from the use of the $\bot$ punctuation mark.

We first outline detour conversions involving the $\bot$ sign, then, we move onto conversions concerning weak inferences and logical constants.

3.2.2.1 *Ex Falso* and Epistemic contradictions

One way to induce a detour simplification, is to use the rule $\bot E$ immediately followed by a rule which is not an introduction rule. We can read those conversions as stating that whatever is inferred to from the occurrence of the conversational dead-end marked by $\bot$ should be an atomic sentence. Notice that this is already partially achieved with proposition 66, but we cover here the new cases of AR rules. We now give the desired reductions, marking the relevant sequence of rules with a "$\Rightarrow$":

$\bot E \Rightarrow \mathsf{AR}$ We give the case for assertion, the one with rejection being symmetric:

$$\begin{array}{c}\mathcal{D}\\ \dfrac{\dfrac{\bot}{\langle {}^{+}\phi\rangle_\sigma}\,\bot E}{\langle {}^{\oplus}\phi\rangle_\sigma}\,\downarrow A\\ \mathcal{C}\end{array} \rightsquigarrow \begin{array}{c}\mathcal{D}\\ \dfrac{\bot}{\langle {}^{\oplus}\phi\rangle_\sigma}\,\bot E\\ \mathcal{C}\end{array}$$

Where the length of the segment decreases in the reduction.

$\bot E \Rightarrow \bot E$ Iterations of the *Ex falso* rules are eliminated as below, reducing the length of the $\bot$ segment.

$$\begin{array}{c}\mathcal{D}\\ \dfrac{\dfrac{\bot}{\bot}\,\bot E}{{}^{\bullet}\phi}\,\bot E\\ \mathcal{C}\end{array} \rightsquigarrow \begin{array}{c}\mathcal{D}\\ \dfrac{\bot}{{}^{\bullet}\phi}\,\bot E\\ \mathcal{C}\end{array}$$

$\bot E \Rightarrow \mathsf{EC}$ Conclusions of *Ex falso* which are used to introduce $\bot$ are also manifestly redundant:

$$\begin{array}{c}\dfrac{\begin{array}{c}\mathcal{D}\\ \dfrac{\bot}{\langle {}^{\bullet}{}_{s}\psi\rangle_\sigma}\,\bot E\end{array} \qquad \begin{array}{c}\mathcal{C}\\ \langle {}^{\circ}{}_{w}\neg\psi\rangle_\lambda\end{array}}{\bot}\,\mathsf{EC}\\ \mathcal{F}\end{array} \rightsquigarrow \begin{array}{c}\mathcal{D}\\ \bot\\ \mathcal{F}\end{array}$$

Where the length of both segments, as well as their complexity, are reduced as they are removed from the tree.

$\bot E \Rightarrow \mathsf{WI}$ We have the following reduction which lowers the length of the segment σ in both cases, and of the segment λ in the case for weak signatures. We detail the positive cases as those for rejection and weak rejection are symmetric:

$$\begin{array}{c}\dfrac{\begin{array}{c}\mathcal{A}\\ \dfrac{\bot}{\langle {}^{\oplus}\phi\rangle_\sigma}\,\bot E\end{array} \qquad \begin{array}{c}\langle[{}^{+}\phi]\rangle_\sigma\\ \mathcal{B}\\ \langle {}^{+}\psi\rangle_\lambda\end{array}}{\langle {}^{\oplus}\psi\rangle_\lambda}\,W.I._1\\ \mathcal{C}\end{array} \rightsquigarrow \begin{array}{c}\mathcal{A}\\ \dfrac{\bot}{\langle {}^{+}\phi\rangle_\sigma}\,\bot E\\ \mathcal{B}\\ \dfrac{\langle {}^{+}\psi\rangle_\lambda}{\langle {}^{\oplus}\psi\rangle_\lambda}\,\downarrow A\\ \mathcal{C}\end{array} \qquad \begin{array}{c}\dfrac{\begin{array}{c}\mathcal{A}\\ \dfrac{\bot}{\langle {}^{\oplus}\phi\rangle_\sigma}\,\bot E\end{array} \qquad \begin{array}{c}\langle[{}^{+}\phi]\rangle_\sigma\\ \mathcal{B}\\ \langle {}^{\oplus}\psi\rangle_\lambda\end{array}}{\langle {}^{\oplus}\psi\rangle_\lambda}\,W.I._3\\ \mathcal{C}\end{array} \rightsquigarrow \begin{array}{c}\mathcal{A}\\ \dfrac{\bot}{\langle {}^{+}\phi\rangle_\sigma}\,\bot E\\ \mathcal{B}\\ \langle {}^{\oplus}\psi\rangle_\lambda\\ \mathcal{C}\end{array}$$

Here the length of σ is diminished, and the level of λ as well. Notice that here we replace every instance of the induced assumption $[{}^{+}\phi]$ in $\mathcal{B}$ with the derivation $\begin{array}{c}\mathcal{A}\\ \dfrac{\bot}{{}^{+}\phi}\end{array}$. Thus, we may create multiple copies of whatever segment κ occurs in $\mathcal{A}$ above σ. In light of the reduction strategy outlined in remark 67, we may however assume no $\kappa >_S \sigma$ occurs in $\mathcal{A}$ and thus the rank of $\mathcal{D}$ decreases after the conversion is applied.

Alternatively, we may have a segment that begins with the conclusion of $\bot E$ and closes with the conclusion of WI. Conversion then occurs as follows:

$$\begin{array}{c}\mathcal{A}\\ \dfrac{\dfrac{\bot}{\langle {}^{\bullet}\phi\rangle_\sigma}\,\bot E}{\langle {}^{\bullet}{}_{w}\phi\rangle_\sigma}\,\mathsf{WI}\\ \mathcal{B}\end{array} \rightsquigarrow \begin{array}{c}\mathcal{A}\\ \dfrac{\dfrac{\bot}{\bot}\,\mathsf{WI}}{\langle {}^{\bullet}{}_{w}\phi\rangle_\sigma}\,\bot E\\ \mathcal{B}\end{array}$$

Where the length of the segment decreases.

We now turn to the cases involving rules of epistemic contradictions. We can read the detours presented below as enforcing the requirement that contradictions should coordinate only sentences that are atomic, while compound sentences are handled by structural rules ([Rum08]).
If the minor premiss of EC is likewise introduced, then the conversions are as follows, where we make use of derived rules to ease the depiction.

$\wedge$: We give the case for assertion, the one for negation being similar to that of asserted split disjunction:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\phi\end{array}\quad\begin{array}{c}\mathcal{B}\\ {}^{+}\psi\end{array}}{\langle {}^{+}\phi\wedge\psi\rangle_\sigma}\,{}^{+}\wedge I\qquad \dfrac{\begin{array}{c}\mathcal{C}\\ {}^{\ominus}\phi\end{array}\quad\begin{array}{c}\mathcal{D}\\ {}^{\ominus}\psi\end{array}}{\langle {}^{\ominus}\phi\wedge\psi\rangle_\lambda}\,{}^{\ominus}\wedge I}{\begin{array}{c}\bot\\ \mathcal{E}\end{array}}\,\mathsf{EC} \rightsquigarrow \dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\phi\end{array}\quad\begin{array}{c}\mathcal{C}\\ {}^{\ominus}\phi\end{array}}{\begin{array}{c}\bot\\ \mathcal{E}\end{array}}\,\mathsf{EC}$$

With the complexity of both segments decreasing.

$\vee$: We give the positive case as the negative one is similar to the above

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\phi\end{array}\quad\begin{array}{c}\mathcal{B}\\ {}^{\oplus}\psi\end{array}}{\langle {}^{+}\phi\vee\psi\rangle_\sigma}\,{}^{+}\vee I\qquad \dfrac{\begin{array}{c}\mathcal{C}\\ {}^{-}\phi\end{array}\quad\begin{array}{c}\mathcal{D}\\ {}^{\ominus}\psi\end{array}}{\langle {}^{\ominus}\phi\vee\psi\rangle_\lambda}\,{}^{\ominus}\vee I}{\begin{array}{c}\bot\\ \mathcal{E}\end{array}}\,\mathsf{EC} \rightsquigarrow \dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\phi\end{array}\quad\begin{array}{c}\mathcal{C}\\ \dfrac{{}^{-}\phi}{{}^{\ominus}\phi}\downarrow R\end{array}}{\begin{array}{c}\bot\\ \mathcal{E}\end{array}}\,\mathsf{EC}$$

Reducing the complexity of the segments. The cases for the other combinations of strong and weakly asserted premisses are similar and spare the addition of the AR rule.

$\mathbb{V}$: The positive and negative case are here similar, choosing the correct premiss introducing the witness formula of σ

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\phi\end{array}}{\langle {}^{+}\phi\mathbin{\mathbb{V}}\psi\rangle_\sigma}\,{}^{+}\mathbb{V} I\qquad \dfrac{\begin{array}{c}\mathcal{C}\\ {}^{-}\phi\end{array}\quad\begin{array}{c}\mathcal{D}\\ {}^{\ominus}\psi\end{array}}{\langle {}^{\ominus}\phi\mathbin{\mathbb{V}}\psi\rangle_\lambda}\,{}^{\ominus}\mathbb{V} I}{\begin{array}{c}\bot\\ \mathcal{E}\end{array}}\,\mathsf{EC} \rightsquigarrow \dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\phi\end{array}\quad\begin{array}{c}\mathcal{C}\\ \dfrac{{}^{-}\phi}{{}^{\ominus}\phi}\downarrow R\end{array}}{\begin{array}{c}\bot\\ \mathcal{E}\end{array}}\,\mathsf{EC}$$

The complexity of the segment decreases here as well. The other cases are similar save the application of AR.

In all the above cases, if the witness formula for λ is not the topmost formula of the segment, and is obtained via an AR rule along λ instead, then the situation is similar, applying the same AR rule to the premisses of the strongly signed version of the witness formula.
Similarly, if the λ segment comprises a conclusion of a WI rule, then in the rightmost derivation we apply WI to the relevant premiss, possibly replacing the instance of $\downarrow A$, $\downarrow R$ whenever possible. For example, in the ${}^{\ominus}\mathbin{\vee\!\!\vee} I$ case:

$$\cfrac{\cfrac{\begin{matrix}\mathcal{A}\\ {}^{+}\phi\end{matrix}}{\langle {}^{+}\phi \mathbin{\vee\!\!\vee} \psi\rangle_\sigma}\,{}^{+}\mathbin{\vee\!\!\vee} I \qquad \cfrac{\cfrac{\begin{matrix}\mathcal{C}\\ {}^{-}\phi\end{matrix}\quad\begin{matrix}\mathcal{D}\\ {}^{\ominus}\psi\end{matrix}}{\langle {}^{\ominus}\phi \mathbin{\vee\!\!\vee} \psi\rangle_\lambda}\,{}^{\ominus}\mathbin{\vee\!\!\vee} I}{\langle {}^{\ominus}\phi \mathbin{\vee\!\!\vee} \psi\rangle_\lambda}\,W.I._4}{\begin{matrix}\bot\\ \mathcal{E}\end{matrix}}\,\mathsf{EC} \quad\leadsto\quad \cfrac{\begin{matrix}\mathcal{A}\\ {}^{+}\phi\end{matrix}\quad \cfrac{\begin{matrix}\mathcal{C}\\ {}^{-}\phi\end{matrix}}{{}^{\ominus}\phi}\,W.I._4}{\begin{matrix}\bot\\ \mathcal{E}\end{matrix}}\,\mathsf{EC}$$

If the minor premiss is not introduced, the complexity of the segment can still be decreased by eliminating it instead. Notice that, by assumption, the minor premiss of EC is in this case not introduced (up to segment composition), moreover light of proposition 66, *Ex falso* only concludes literals, so no new cut is created when the right premiss is concluded by $\bot E$ and it is immediate that whenever $\mathcal{C}$ terminates with an elimination rule, no new cut segment is created as well. The only remaining cases are those where the minor premiss is obtained via AR or WI, if this happens, we perform the relevant conversions on the right premiss as part of the very same conversion step outlined below, as a result, those reductions creates cut segments of at worst lesser weight than σ.[3]

$$\cfrac{\cfrac{\begin{matrix}\mathcal{A}\\ {}^{+}\phi\end{matrix}\quad\begin{matrix}\mathcal{B}\\ {}^{+}\psi\end{matrix}}{\langle {}^{+}\phi \wedge \psi\rangle_\sigma}\,{}^{+}\wedge I \qquad \begin{matrix}\mathcal{C}\\ \langle {}^{\ominus}\phi \wedge \psi\rangle_\lambda\end{matrix}}{\begin{matrix}\bot\\ \mathcal{E}\end{matrix}}\,\mathsf{EC} \quad\leadsto\quad \cfrac{\begin{matrix}\mathcal{A}\\ {}^{+}\phi\end{matrix}\quad \cfrac{\begin{matrix}\mathcal{C}\\ \langle {}^{\ominus}\phi \wedge \psi\rangle_\lambda\end{matrix}}{{}^{\ominus}\phi}\,\mathsf{FC}}{\begin{matrix}\bot\\ \mathcal{E}\end{matrix}}\,\mathsf{EC}$$

$$\cfrac{\cfrac{\begin{matrix}\mathcal{A}\\ {}^{+}\phi\end{matrix}\quad\begin{matrix}\mathcal{B}\\ {}^{\oplus}\psi\end{matrix}}{\langle {}^{+}\phi \vee \psi\rangle_\sigma}\,{}^{+}\vee I \qquad \begin{matrix}\mathcal{C}\\ \langle {}^{\ominus}\phi \vee \psi\rangle_\lambda\end{matrix}}{\begin{matrix}\bot\\ \mathcal{E}\end{matrix}}\,\mathsf{EC} \quad\leadsto\quad \cfrac{\begin{matrix}\mathcal{A}\\ {}^{+}\phi\end{matrix}\quad \cfrac{\begin{matrix}\mathcal{C}\\ \langle {}^{\ominus}\phi \vee \psi\rangle_\lambda\end{matrix}}{{}^{\ominus}\phi}\,{}^{\ominus}\vee E}{\begin{matrix}\bot\\ \mathcal{E}\end{matrix}}\,\mathsf{EC}$$

$$\cfrac{\cfrac{\begin{matrix}\mathcal{A}\\ {}^{+}\phi\end{matrix}}{\langle {}^{+}\phi \mathbin{\vee\!\!\vee} \psi\rangle_\sigma}\,{}^{+}\mathbin{\vee\!\!\vee} I \qquad \begin{matrix}\mathcal{C}\\ \langle {}^{\ominus}\phi \mathbin{\vee\!\!\vee} \psi\rangle_\lambda\end{matrix}}{\begin{matrix}\bot\\ \mathcal{E}\end{matrix}}\,\mathsf{EC} \quad\leadsto\quad \cfrac{\begin{matrix}\mathcal{A}\\ {}^{+}\phi\end{matrix}\quad \cfrac{\begin{matrix}\mathcal{C}\\ \langle {}^{\ominus}\phi \mathbin{\vee\!\!\vee} \psi\rangle_\lambda\end{matrix}}{{}^{\ominus}\phi}\,{}^{\ominus}\mathbin{\vee\!\!\vee} E}{\begin{matrix}\bot\\ \mathcal{E}\end{matrix}}\,\mathsf{EC}$$

The cases for denial are symmetrical, with the exception of the inquisitive disjunction one. There we have different cases depending on whether the minor premiss is obtained via $\downarrow A$ or not. In the latter case, we have:

$$\cfrac{\cfrac{\begin{matrix}\mathcal{A}\\ {}^{-}\phi\end{matrix}\quad\begin{matrix}\mathcal{B}\\ {}^{-}\psi\end{matrix}}{\langle {}^{-}\phi \mathbin{\vee\!\!\vee} \psi\rangle_\sigma}\,{}^{-}\mathbin{\vee\!\!\vee} I \qquad \begin{matrix}\mathcal{C}\\ \langle {}^{\oplus}\phi \mathbin{\vee\!\!\vee} \psi\rangle_\lambda\end{matrix}}{\begin{matrix}\bot\\ \mathcal{E}\end{matrix}}\,\mathsf{EC} \quad\leadsto\quad \cfrac{\begin{matrix}\mathcal{C}\\ \langle {}^{\oplus}\phi \mathbin{\vee\!\!\vee} \psi\rangle_\lambda\end{matrix}\quad \cfrac{\begin{matrix}\mathcal{A}\\ {}^{-}\phi\end{matrix}\quad [{}^{\oplus}\phi]}{\bot}\,\mathsf{EC}\quad \cfrac{\begin{matrix}\mathcal{B}\\ {}^{-}\psi\end{matrix}\quad [{}^{\oplus}\psi]}{\bot}\,\mathsf{EC}}{\begin{matrix}\bot\\ \mathcal{E}\end{matrix}}\,{}^{\oplus}\mathbin{\vee\!\!\vee} E$$

[3] Although foreign to classical and intuitionistic logic, this situation is often found in bilateral systems, see e.g. [Kur21].

If the representative of λ is obtained via $\downarrow A$, then we can't convert as above because the shorthand ${}^{\oplus} \Vvee E$ is obtained through Weak inference and this may create a new cut on λ with the same complexity as σ. However, as we still have that ${}^{+}\phi \Vvee \psi$ is not introduced, we can simply retain ${}^{+}\phi \Vvee \psi$ and apply ${}^{+} \Vvee E$ as above. If λ is obtained via Weak inference, we permute ${}^{\oplus} \Vvee E$ above weak inference and perform the further required conversions as a part of this very same conversion step. Both of the options reduce the complexity of the cut segment.

The situation is similar whenever premisses of EC are obtained via Coord rules. Here again, λ is not made into a cut segment since it is not initiated by introduction rules by assumption, if λ is obtained via downscaling rules or weak inference rules, we apply multiple conversions in one step the same way as before. The case where λ is initiated by an introduction rule is akin to those above where premisses of EC were not prefixed by negation. The case for conjunction is given here below:

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{A}\\ {}^{+}\phi\end{matrix} \qquad \begin{matrix}\mathcal{B}\\ {}^{+}\psi\end{matrix}}{\langle {}^{+}\phi \wedge \psi\rangle_\sigma}\,{}^{+}\wedge I}{\langle {}^{-}\neg(\phi \wedge \psi)\rangle_\sigma}\,\mathsf{Coord} \qquad \begin{matrix}\mathcal{C}\\ \langle {}^{\oplus}\neg(\phi \wedge \psi)\rangle_\lambda\end{matrix}}{\begin{matrix}\bot\\ \mathcal{E}\end{matrix}}\,\mathsf{EC} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\mathcal{A}\\ {}^{+}\phi\end{matrix} \qquad \dfrac{\dfrac{\begin{matrix}\mathcal{C}\\ \langle {}^{\oplus}\neg(\phi \wedge \psi)\rangle_\lambda\end{matrix}}{\langle {}^{\ominus}\phi \wedge \psi\rangle_\lambda}\,\mathsf{Coord}}{{}^{\ominus}\phi}\,{}^{\ominus}\wedge E}{\begin{matrix}\bot\\ \mathcal{E}\end{matrix}}\,\mathsf{EC}$$

3.2.2.2 Weak inference

Some detour conversions are induced by the family of WI rules. One glaringly redundant case is the following, which reduces the length of the segment:

$$\dfrac{\begin{matrix}\mathcal{A}\\ {}^{\bullet}{}_{w}\phi\end{matrix} \qquad \dfrac{\begin{matrix}[{}^{\bullet}{}_{s}\phi]\\ \mathcal{B}\\ \langle {}^{\bullet}{}_{s}\psi\rangle_\sigma\end{matrix}}{\langle {}^{\bullet}{}_{w}\psi\rangle_\sigma}\,\mathsf{AR}}{\langle {}^{\bullet}{}_{w}\psi\rangle_\sigma}\,W.I._{2,4} \quad \rightsquigarrow \quad \dfrac{\begin{matrix}\mathcal{A}\\ {}^{\bullet}{}_{w}\phi\end{matrix} \qquad \begin{matrix}[{}^{\bullet}{}_{s}\phi]\\ \mathcal{B}\\ \langle {}^{\bullet}{}_{s}\psi\rangle_\sigma\end{matrix}}{\langle {}^{\bullet}{}_{w}\psi\rangle_\sigma}\,W.I._{1,3}$$

The following case exemplifies the rationale behind the inversion principle, everything that is licitly inferrable from a sentence is inferrable whatever is a sufficient ground for inferring that sentence. According to our structural rules, a strongly signed sentence is always a sufficient ground to infer the same weakly signed sentence.
As usual, we illustrate the positive case:

$$\begin{matrix}\dfrac{\dfrac{\begin{matrix}\mathcal{A}\\ \langle {}^{+}\phi\rangle_\sigma\end{matrix}}{\langle {}^{\oplus}\phi\rangle_\sigma}\,\downarrow A \qquad \begin{matrix}[\langle {}^{+}\phi\rangle_\sigma]\\ \mathcal{B}\\ \langle {}^{\bullet}\psi\rangle_\lambda\end{matrix}}{\langle {}^{\bullet}{}_{w}\psi\rangle_\lambda}\,\mathsf{WI}\\ \mathcal{C}\end{matrix} \quad \rightsquigarrow \quad \begin{matrix}\mathcal{A}\\ {}^{+}\phi\\ \mathcal{B}\\ \dfrac{\langle {}^{\bullet}\psi\rangle_\lambda}{\langle {}^{\bullet}{}_{w}\psi\rangle_\lambda}\,\downarrow A\\ \mathcal{C}\end{matrix}$$

The use of $\downarrow A$ may be spared with depending on the exact Weak Inference rule at stake.

$$\dfrac{\begin{matrix}\mathcal{C}\\ \langle {}^{\oplus}\phi\rangle_\sigma\end{matrix} \qquad \begin{matrix}\dfrac{[\langle {}^{+}\phi\rangle_\sigma]}{\langle {}^{\oplus}\phi\rangle_\sigma}\,\downarrow A\\ \mathcal{D}\\ \langle {}^{+}\psi\rangle_\lambda\end{matrix}}{\langle {}^{\oplus}\psi\rangle_\lambda}\,W.I._{1} \quad \rightsquigarrow \quad \begin{matrix}\mathcal{C}\\ {}^{\oplus}\phi\\ \mathcal{D}\\ \dfrac{\langle {}^{+}\psi\rangle_\lambda}{\langle {}^{\oplus}\psi\rangle_\lambda}\,\downarrow A\end{matrix}$$

In both these cases, notice we are not multiplying segments κ of higher $<_S$ order than σ by our assumption remark 67. Furthermore, the addition of an AR rule does not create new cut segments (remark 68). Last, the length of σ and the level of λ are reduced in all cases. Depending on the WI rule employed, the length of λ is reduced as well.

At last, we have the detours for introduction-elimination sequences, which are reminiscent of the detours in classical and intuitionistic (unilateral) logic:

3.2.2.3 Introduction - elimination

We divide the cases between whether we have "strong" or "weak" segments, *id est*, whether the topmost and bottom-most formulae of the sequents are the premiss and conclusion respectively of an AR rule (weak case) or not (strong case). As usual we give only the positive/asserted version as the denied ones are dual with the exception of inquisitive disjunction.

Strong: We have the Prawitzian detours inherited from classical logic:

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{cc}\mathcal{A} & \mathcal{B}\\ {}^{+}\phi & {}^{+}\psi\end{array}}{\langle {}^{+}\phi\wedge\psi\rangle_\sigma}\;{}^{+}\wedge I}{{}^{+}\phi}\;{}^{+}\wedge E\\ \mathcal{C}\end{array} \rightsquigarrow \begin{array}{c}\mathcal{A}\\ {}^{+}\phi\\ \mathcal{C}\end{array} \qquad \begin{array}{c}\dfrac{\dfrac{\begin{array}{cc}\mathcal{A} & \mathcal{B}\\ {}^{+}\phi & {}^{\oplus}\psi\end{array}}{\langle {}^{+}\phi\vee\psi\rangle_\sigma}\;{}^{+}\vee I \quad \begin{array}{c}[{}^{+}\phi]\\ \mathcal{C}\\ {}^{\bullet}\mu\end{array} \quad \begin{array}{c}[{}^{+}\psi]\\ \mathcal{D}\\ {}^{\bullet}\mu\end{array}}{{}^{\bullet}\mu}\;{}^{+}\vee E\\ \mathcal{D}\end{array} \rightsquigarrow \begin{array}{c}\mathcal{A}\\ {}^{+}\phi\\ \mathcal{C}\\ {}^{\bullet}\mu\\ \mathcal{D}\end{array}$$

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\phi\end{array}}{\langle {}^{+}\phi\vee\!\!\vee\psi\rangle_\sigma}\;{}^{+}\vee\!\!\vee I \quad \begin{array}{c}[{}^{+}\phi]\\ \mathcal{C}\\ {}^{\bullet}\mu\end{array} \quad \begin{array}{c}[{}^{+}\psi]\\ \mathcal{D}\\ {}^{\bullet}\mu\end{array}}{{}^{\bullet}\mu}\;{}^{+}\vee E\\ \mathcal{D}\end{array} \rightsquigarrow \begin{array}{c}\mathcal{A}\\ {}^{+}\phi\\ \mathcal{C}\\ {}^{\bullet}\mu\\ \mathcal{D}\end{array}$$

And two extra cases for might and for the rejected inquisitive disjunction:

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{\oplus}\phi\end{array}}{\langle {}^{+}\blacklozenge\phi\rangle_\sigma}\;{}^{+}\blacklozenge I}{{}^{\oplus}\phi}\;{}^{+}\blacklozenge E\\ \mathcal{B}\end{array} \rightsquigarrow \begin{array}{c}\mathcal{A}\\ {}^{\oplus}\phi\\ \mathcal{B}\end{array} \qquad \begin{array}{c}\dfrac{\dfrac{\begin{array}{cc}\mathcal{A} & \mathcal{B}\\ {}^{-}\phi & {}^{-}\psi\end{array}}{\langle {}^{-}\phi\vee\!\!\vee\psi\rangle_\sigma}\;{}^{-}\vee\!\!\vee I}{{}^{-}\phi}\;{}^{-}\vee\!\!\vee E\\ \mathcal{C}\end{array} \rightsquigarrow \begin{array}{c}\mathcal{A}\\ {}^{-}\phi\\ \mathcal{C}\end{array}$$

In all of the previous cases the logical complexity of the segment decreases. Notice, further, that since the only introduction rules for weakly signed formulae are AR formulae, all the detours for segments that only contain weakly signed formulae are covered as permutations.

Weak: We consider here segments that comprise both weakly and strongly signed premisses. Those are the detours involving FC rules, which we give below:

$$\begin{array}{c}\dfrac{\dfrac{\dfrac{\begin{array}{cc}\mathcal{A} & \mathcal{B}\\ {}^{+}\phi & {}^{\oplus}\psi\end{array}}{\langle {}^{+}\phi\vee\psi\rangle_\sigma}\;{}^{+}\vee I}{\langle {}^{\oplus}\phi\vee\psi\rangle_\sigma}\;\downarrow A}{{}^{\oplus}\phi}\;{}^{+}FC\\ \mathcal{C}\end{array} \rightsquigarrow \begin{array}{c}\mathcal{A}\\ \dfrac{{}^{+}\phi}{{}^{\oplus}\phi}\;\downarrow A\\ \mathcal{C}\end{array} \qquad \begin{array}{c}\dfrac{\dfrac{\dfrac{\begin{array}{cc}\mathcal{A} & \mathcal{B}\\ {}^{+}\phi & {}^{\oplus}\psi\end{array}}{\langle {}^{+}\phi\vee\psi\rangle_\sigma}\;{}^{+}\vee I}{\langle {}^{\oplus}\phi\vee\psi\rangle_\sigma}\;\downarrow A}{{}^{\oplus}\psi}\;{}^{+}FC\\ \mathcal{C}\end{array} \rightsquigarrow \begin{array}{c}\mathcal{A}\\ {}^{\oplus}\psi\\ \mathcal{C}\end{array}$$

Where in all the cases the logical complexity of the segment decreases. Detours for negatively signed sentences are symmetric.

3.2.2.4 Coordination

One further class of detour conversions are induced by Coord rules:

$$\begin{array}{c}\mathcal{A}\\ \dfrac{\dfrac{\langle^\bullet\phi\rangle_\sigma}{\langle^\circ\neg\phi\rangle_\sigma}\,\textsf{Coord}}{\langle^\bullet\phi\rangle_\sigma}\,\textsf{Coord}\\ \mathcal{B}\end{array} \rightsquigarrow \begin{array}{c}\mathcal{A}\\ \langle^\bullet\phi\rangle_\sigma\\ \mathcal{B}\end{array} \qquad \begin{array}{c}\mathcal{A}\\ \dfrac{\dfrac{\langle^\bullet\neg\phi\rangle_\sigma}{\langle^\circ\phi\rangle_\sigma}\,\textsf{Coord}}{\langle^\bullet\neg\phi\rangle_\sigma}\,\textsf{Coord}\\ \mathcal{B}\end{array} \rightsquigarrow \begin{array}{c}\mathcal{A}\\ \langle^\bullet\neg\phi\rangle_\sigma\\ \mathcal{B}\end{array}$$

Reducing the length of segment in a straightforward manner. If the segment σ contains a the conclusion of a minor premiss of a (or more) Disj rule, then we perform the normalization as above, and apply the Disj rule immediately afterwards, which reduced the length of the segment as before:[4]

$$\begin{array}{c}\dfrac{\begin{array}{c}\mathcal{A}\\ {}^+\phi\vee\psi\end{array} \quad \begin{array}{c}[{}^+\phi]\\ \mathcal{B}\\ \dfrac{\langle^\bullet\neg\mu\rangle_\sigma}{\langle^\circ\mu\rangle_\sigma}\,\textsf{Coord}\end{array} \quad \begin{array}{c}[{}^+\psi]\\ \mathcal{C}\\ \dfrac{\langle^\bullet\neg\mu\rangle_\kappa}{\langle^\circ\mu_\kappa\rangle}\,\textsf{Coord}\end{array}}{\dfrac{\langle^\circ\mu\rangle_{\sigma,\kappa}}{\langle^\bullet\neg\mu\rangle_{\sigma,\kappa}}\,\textsf{Coord}}\,{}^+\vee E\\ \mathcal{E}\end{array} \rightsquigarrow \begin{array}{c}\dfrac{\begin{array}{c}\mathcal{A}\\ {}^+\phi\vee\psi\end{array} \quad \begin{array}{c}[{}^+\phi]\\ \mathcal{B}\\ \langle^\bullet\mu\rangle_\sigma\end{array} \quad \begin{array}{c}[{}^+\psi]\\ \mathcal{C}\\ \langle^\bullet\mu\rangle_\kappa\end{array}}{\langle^\bullet\neg\mu\rangle_{\sigma,\kappa}}\,{}^+\vee E\\ \mathcal{E}\end{array}$$

3.2.3 Permutations

Permutation conversions avoid cut segment amenable to detour conversion being covertly separated by instances of rules having subderivations as premisses. In this sense, permutation conversions manipulate the structure of derivations as to make all detour conversions explicit. In the classical case, the only rule generating permutation conversion is disjunction elimination. In our setting, we must take care of the instances arising from Disj rules, and from WI rules as well.

Disj rules induce permutation conversions whenever their conclusion is also the premiss of an elimination rule, or the premiss of a WI (up to segment composition). We ${}^+\vee E$ as an example, the other two cases being similar:

$$\begin{array}{c}\dfrac{\dfrac{\begin{array}{c}\mathcal{A}\\ {}^+\phi\vee\psi\end{array} \quad \begin{array}{c}[{}^+\phi]\\ \mathcal{B}\\ \langle^\bullet\zeta\rangle_\sigma\end{array} \quad \begin{array}{c}[{}^+\psi]\\ \mathcal{C}\\ \langle^\bullet\zeta\rangle_\kappa\end{array}}{\langle^\bullet\zeta\rangle_{\sigma,\kappa}}\,{}^+\vee E \qquad \mathcal{D}}{\langle^\bullet\mu\rangle_\lambda}\,\text{Elim/WI}\\ \mathcal{E}\end{array} \rightsquigarrow$$

$$\begin{array}{c}\dfrac{\begin{array}{c}\mathcal{A}\\ {}^+\phi\vee\psi\end{array} \quad \begin{array}{c}[{}^+\phi]\\ \mathcal{B}\\ \dfrac{\langle^\bullet\zeta\rangle_\sigma \quad \mathcal{D}}{\langle^\bullet\mu\rangle_\lambda}\,\text{Elim/WI}\end{array} \quad \begin{array}{c}[{}^+\psi]\\ \mathcal{C}\\ \dfrac{\langle^\bullet\zeta\rangle_\kappa \quad \mathcal{D}}{\langle^\circ\mu\rangle_\lambda}\,\text{Elim/WI}\end{array}}{\langle^\circ\mu\rangle_\lambda}\,{}^+\vee E\\ \mathcal{E}\end{array}$$

Notice that, as both $\mathcal{B}$, $\mathcal{C}$ conclude the same formula respecting the $\star$ restriction in the leftmost tree, the $\star$ restriction is likewise complied with in the rightmost tree, as $\textsf{Open}(\mathcal{D}) = \textsf{Open}(\mathcal{D})$ always

[4] Due to the nature of the *weight* measure on segment here adopted, single instance of Coord rules across Disj rules cannot be treated as permutations. If the coordination rules are paired as in the example here given, however, they treated directly as detours instead.

holds. The conversion reduces the length of the segments σ, κ, the λ segment is made longer, but under the remark 67 strategy we can assume the weight of λ is strictly lower than the weight of σ, so that the overall rank of the derivation decreases after the conversion.

For WI rules, there are different permutations depending on the elimination rule applied. The first case is given by the iteration of WI rules. We illustrate the positive case as before.

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{\oplus}\phi\end{array} \quad \begin{array}{c}[{}^{+}\phi]\\ \mathcal{B}\\ \langle {}^{+}\psi\rangle_\sigma\end{array}}{\langle {}^{\oplus}\psi\rangle_\sigma}\,W.I._1 \qquad \begin{array}{c}[\langle {}^{+}\psi\rangle_\sigma]\\ \mathcal{C}\\ {}^{\bullet}\zeta\end{array}}{{}^{\bullet}{}_{w}\zeta}\,\mathsf{WI} \\ \mathcal{C}\end{array} \quad \rightsquigarrow \quad \begin{array}{c} \dfrac{\begin{array}{c}\mathcal{A}\\ \langle {}^{\oplus}\phi\rangle_\sigma\end{array} \qquad \begin{array}{c}[\langle {}^{+}\phi\rangle_\sigma]\\ \mathcal{B}\\ {}^{+}\psi\\ \mathcal{C}\\ {}^{\bullet}\zeta\end{array}}{{}^{\bullet}{}_{w}\zeta}\,\mathsf{WI} \\ \mathcal{C}\end{array}$$

As before, we readily see that the $*$ requirement is complied on the right since it was complied with by assumption on $\mathcal{B}$, $\mathcal{C}$ individually. Furthermore, in light of remark 67, we do not duplicate maximal cut segments in $\mathcal{B}$ in the process; the length of the segment is here reduced.
The case for $W.I._2$ is similar:

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{\oplus}\phi\end{array} \quad \begin{array}{c}[{}^{+}\phi]\\ \mathcal{B}\\ \langle {}^{\oplus}\psi\rangle_\sigma\end{array}}{\langle {}^{\oplus}\psi\rangle_\sigma}\,W.I._1 \qquad \begin{array}{c}[\langle {}^{+}\psi\rangle_\sigma]\\ \mathcal{C}\\ \langle {}^{\bullet}\zeta\rangle_\lambda\end{array}}{\langle {}^{\bullet}{}_{w}\zeta\rangle_\lambda}\,\mathsf{WI} \\ \mathcal{C}\end{array} \quad \rightsquigarrow \quad \begin{array}{c} \dfrac{\begin{array}{c}\mathcal{A}\\ \langle {}^{\oplus}\phi\rangle_\sigma\end{array} \qquad \dfrac{\begin{array}{c}[\langle {}^{+}\phi\rangle_\sigma]\\ \mathcal{B}\\ {}^{\oplus}\psi\end{array} \qquad \begin{array}{c}[{}^{+}\psi]\\ \mathcal{C}\\ \langle {}^{\bullet}{}_{w}\zeta\rangle_\lambda\end{array}}{\langle {}^{\bullet}{}_{w}\zeta\rangle_\lambda}\,\mathsf{WI}}{\langle {}^{\oplus}\zeta\rangle_\lambda}\,W.I._3 \\ \mathcal{C}\end{array}$$

As in the case for disjunctive rules, while the σ segment is reduced, the λ segment is extended in the conversion process. Assumption remark 67 ensures conversions are performed in a way such that the overall rank of the derivation does not increase.

One last kind of permutation conversions are induced by free-choice rules. We illustrate the positive cases below:

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{\oplus}\phi\end{array} \quad \begin{array}{c}[{}^{+}\phi]\\ \mathcal{B}\\ \langle {}^{+}\mu\vee\nu\rangle_\sigma\end{array}}{\langle {}^{\oplus}\mu\vee\nu\rangle_\sigma}\,W.I._1}{{}^{\oplus}\nu}\,{}^{+}FC \\ \mathcal{D}\end{array} \quad \rightsquigarrow \quad \begin{array}{c} \dfrac{\begin{array}{c}\mathcal{A}\\ {}^{\oplus}\phi\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}[{}^{+}\phi]\\ \mathcal{B}\\ \langle {}^{+}\mu\vee\nu\rangle_\sigma\end{array}}{\langle {}^{\oplus}\mu\vee\nu\rangle_\sigma}\,\downarrow A}{{}^{\oplus}\nu}\,{}^{+}FC}{{}^{\oplus}\nu}\,W.I._1 \\ \mathcal{D}\end{array}$$

Reducing the level of the segment.

$$\begin{array}{c} \dfrac{\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{\oplus}\phi\end{array} \quad \begin{array}{c}[{}^{+}\phi]\\ \mathcal{B}\\ \langle {}^{+}\mu\vee\nu\rangle_\sigma\end{array}}{\langle {}^{\oplus}\mu\vee\nu\rangle_\sigma}\,W.I._2}{{}^{\oplus}\nu}\,{}^{+}FC \\ \mathcal{D}\end{array} \quad \rightsquigarrow \quad \begin{array}{c} \dfrac{\begin{array}{c}\mathcal{A}\\ {}^{\oplus}\phi\end{array} \qquad \dfrac{\begin{array}{c}[{}^{+}\phi]\\ \mathcal{B}\\ \langle {}^{\oplus}\mu\vee\nu\rangle_\sigma\end{array}}{{}^{\oplus}\nu}\,{}^{+}FC}{{}^{\oplus}\nu}\,W.I._1 \\ \mathcal{D}\end{array}$$

Reducing the complexity of the segment.

We now turn to investigate the behaviour of the Rewrite rule specifically.

3.3 Constraining *Rw.*

In the previous paragraph we concerned ourselves only marginally with the rule of Rewriting ($Rw.$) which appear to be susceptible only to simplification conversions. The reason is that, unlike WI or Disj rules in our system, the induced premisses of $Rw.$ need not be subformulae of the major premiss of the rule itself. This is easily seen for example in the case below:

$$\dfrac{{}^{+}\alpha \wedge (\beta \vee\!\!\!\vee \gamma) \qquad \begin{matrix}[{}^{+}\alpha \wedge \beta] \\ \mathcal{A} \\ {}^{\bullet}\phi\end{matrix} \qquad \begin{matrix}[{}^{+}\alpha \wedge \gamma] \\ \mathcal{B} \\ {}^{\bullet}\phi\end{matrix}}{{}^{\bullet}\phi}\ Rw.$$

As our ultimate goal is to regiment derivations so that subformulae are preserved whenever possible from premisses to conclusion, the unconstrained presence of $Rw.$ rules would categorically prevent us from achieving this. In this paragraph, we thus aim to show that any derivation tree $\mathcal{D}$ can be turned into a derivation tree $\mathcal{C}$ where major premisses of $Rw.$ always occur before major premisses of any other rule except for Coord and ${}^{+} \blacklozenge\ I$.[5] We first need to introduce the notion of a *Thread*:

Definition 69. In a derivation $\mathcal{D}$, a "thread" is a sequence of signed $\mathcal{BSL}_m^{\vee\!\!\!\vee}$ formulae $\langle {}^{\bullet}\alpha_1, .., {}^{\bullet}\alpha_n \rangle$ such that:

1. ${}^{\bullet}\alpha_1$ is either an open assumption of $\mathcal{D}$, or ${}^{+}\pi_{[\![\top]\!]}$;
2. If ${}^{\bullet}\alpha_i$ is not the major premiss of a Disj rule, nor of a WI rule,, nor of $Rw.$, then ${}^{\bullet}\alpha_{i+1}$ is the signed formula immediately below it;
   If ${}^{\bullet}\alpha_i$ is the major premiss of $Rw.$ or of a Disj rule, then ${}^{\bullet}\alpha_{i+1}$ is one of the assumptions induced by the $Rw.$ or Disj rule ${}^{\bullet}\alpha_i$;
   If ${}^{\bullet}\alpha_i$ is the major premiss of a WI rule, then ${}^{\bullet}\alpha_{i+1}$ is the unique assumption induced by the WI rule;
3. ${}^{\circ}\alpha_n$ is either the conclusion of $\mathcal{D}$ or the minor premiss of an EC rule.

Intuitively, a thread is a sequence of sentences beginning with the topmost signed formula in the branch of a derivation, and proceeding downwards, where it either ends at minor premisses of epistemic contradictions, or it picks a new topmost formula if it is induced by a WI, Disj rule. It is straightforward to verify that any sentence appearing in a derivation must also appear in some thread:

Definition 70. Given a derivation $\mathcal{D}$ and a thread $\tau = \langle {}^{\bullet}\beta_1, ..., {}^{\circ}\beta_n \rangle$ therein, τ is of order 0 if ${}^{\circ}\beta_n$ is the conclusion of $\mathcal{D}$, and is of order $m + 1$ if ${}^{\circ}\beta_n$ is the minor premiss of and EC rule application with major premiss belonging to a thread of order m.

Proposition 71. Let $\mathcal{D}$ be a derivation. $\mathcal{D}$ has at least one thread of order 0, further, any signed formula appearing in $\mathcal{D}$ belongs to some thread.

Proof. A routine induction on the height of $\mathcal{D}$, each step follows by induction hypothesis and applying the definition of thread to the bottom-most formula in the tree. ■

A first step in showing the desired is given by the lemma below, which states that for any individual instance of a premiss of $Rw.$ over a subformula $(\alpha \vee\!\!\!\vee \beta)$, this instance can be permuted as so to either apply to the first appearance of $(\alpha \vee\!\!\!\vee \beta)$ in a thread, or to the induced assumption of some WI, Disj rule instance:

[5] Redundant instanced of Coord are then eliminated as a part of the normalization process.

Lemma 72. For $\mathcal{D}$ a derivation $\Gamma \vdash {}^{\bullet}\psi$, and ${}^{+}\phi(\alpha \vee\!\!\vee \beta)$ a major premiss of *Rw.* therein, there exists a $\mathcal{D}' : \Gamma \vdash {}^{\bullet}\psi$ where ${}^{+}\phi(\alpha \vee\!\!\vee \beta)$ replaced by a major premiss of *Rw.* which is either:

1. the conclusion of a ${}^{+} \vee\!\!\vee I$ rule;
2. the topmost formula of some thread τ;
3. the induced assumption of a $\mathsf{Disj}, \mathsf{WI}$ rule.
4. a formula in a thread τ preceded only by premisses of *Rw.* and ${}^{+} \blacklozenge\ I$ rules;

Proof. Let $\mathcal{D}$ be a derivation. If $\mathcal{D}$ contains no instances of *Rw.* after applying all simplification conversions, then we are done. Let ${}^{+}\phi(\alpha \vee\!\!\vee \beta)$ be a major premiss of *Rw.* in $\mathcal{D}$. Without loss of generality we have a situation as the one below, where both formulae appear along a thread τ and ${}^{+}\zeta(\alpha \vee\!\!\vee \beta)$ is the first instance of an asserted formula containing an asserted instance of $\alpha \vee\!\!\vee \beta$ along τ:

$$\begin{array}{c} \mathcal{A} \\ {}^{+}\zeta(\alpha \vee\!\!\vee \beta) \\ \mathcal{B} \\ \dfrac{{}^{+}\phi(\alpha \vee\!\!\vee \beta)}{{}^{\bullet}\psi}\ Rw. \\ \mathcal{C} \end{array}$$

We show by induction on the height of $\mathcal{B}$ that *Rw.* can be permuted up as to apply to ${}^{+}\zeta(\alpha \vee\!\!\vee \beta)$ (or to one of the induced assumption of the rule ${}^{+}\zeta(\alpha \vee\!\!\vee \beta)$ appears as a major premiss of, if any) .

$0:$ In this case we have $\zeta = \phi$ and thus we are done.

$m+1:$ We have different cases depending on the last rule applied in $\mathcal{B}$. We give a selection of interesting cases as most of them follow the same schema, without loss of generality we shall assume the relevant $(\alpha \vee\!\!\vee \beta)$ occurs in the leftmost formula when $\phi = \mu\#\nu$ for any connective $\#$.

${}^{+} \vee I$: We are here in the following situation:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\zeta(\alpha \vee\!\!\vee \beta)\\ \mathcal{X}\\ {}^{+}\mu(\alpha \vee\!\!\vee \beta)\end{array} \qquad \begin{array}{c}\mathcal{A}\\ {}^{+}\zeta(\alpha \vee\!\!\vee \beta)\\ \mathcal{Y}\\ {}^{\oplus}\nu\end{array}}{{}^{+}\mu(\alpha \vee\!\!\vee \beta) \vee \nu}\ {}^{+}\vee I \qquad \begin{array}{c}[{}^{+}\mu(\alpha) \vee \nu]\\ \mathcal{C}\\ {}^{\bullet}\psi\end{array} \qquad \begin{array}{c}[{}^{+}\mu(\beta) \vee \nu]\\ \mathcal{D}\\ {}^{\bullet}\psi\end{array}}{\begin{array}{c}{}^{\bullet}\psi\\ \mathcal{E}\end{array}}\ Rw.$$

By inductive hypothesis we have , $\mathcal{J} :{}^{+} \zeta(\alpha) \vdash^{+} \mu(\alpha)$, $\mathcal{K} :{}^{+} \zeta(\alpha) \vdash^{\oplus} \nu$ and $\mathcal{L} :{}^{+} \zeta(\beta) \vdash^{+} \mu(\beta)$, $\mathcal{M} :{}^{+} \zeta(\beta) \vdash^{\oplus} \nu$. We then construct the following tree:

$$\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\zeta(\alpha \vee\!\!\vee \beta)\end{array} \qquad \begin{array}{c}\dfrac{\begin{array}{c}[{}^{+}\zeta(\alpha)]\\ \mathcal{J}\\ {}^{+}\mu(\alpha)\end{array} \quad \begin{array}{c}[{}^{+}\zeta(\alpha)]\\ \mathcal{K}\\ {}^{\oplus}\nu\end{array}}{{}^{+}\mu(\alpha) \vee \nu}\ {}^{+}\vee I\\ \mathcal{C}\\ {}^{\bullet}\psi\end{array} \qquad \begin{array}{c}\dfrac{\begin{array}{c}[{}^{+}\zeta(\beta)]\\ \mathcal{L}\\ {}^{+}\mu(\beta)\end{array} \quad \begin{array}{c}[{}^{+}\zeta(\beta)]\\ \mathcal{M}\\ {}^{\oplus}\nu\end{array}}{{}^{+}\mu(\beta) \vee \nu}\ {}^{+}\vee I\\ \mathcal{D}\\ {}^{\bullet}\psi\end{array}}{\begin{array}{c}{}^{\bullet}\psi\\ \mathcal{E}\end{array}}\ Rw.$$

All the other introduction rule function similarly.

${}^{+} \wedge E$: We are in the following situation:

$$\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\zeta(\alpha \mathbin{\vee\!\!\!\vee} \beta)\\ \mathcal{X}\\ \dfrac{{}^{+}\mu(\alpha \mathbin{\vee\!\!\!\vee} \beta)\wedge\nu}{{}^{+}\mu(\alpha \mathbin{\vee\!\!\!\vee} \beta)}\,{\scriptstyle {}^{+}\wedge E}\end{array}\qquad \begin{array}{c}[{}^{+}\mu(\alpha)]\\ \mathcal{C}\\ {}^{\bullet}\psi\end{array}\qquad \begin{array}{c}[{}^{+}\mu(\beta)]\\ \mathcal{D}\\ {}^{\bullet}\psi\end{array}}{\begin{array}{c}{}^{\bullet}\psi\\ \mathcal{E}\end{array}}\,{\scriptstyle Rw.}$$

By inductive hypothesis we have $, \mathcal{J} :{}^{+} \zeta(\alpha) \vdash^{+} \mu(\alpha)\wedge\nu$, and $, \mathcal{L} :{}^{+} \zeta(\beta) \vdash^{+} \mu(\beta)\wedge\nu$, and we construct the following derivation:

$$\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\zeta(\alpha \mathbin{\vee\!\!\!\vee} \beta)\end{array}\qquad \begin{array}{c}[{}^{+}\zeta(\alpha)]\\ \mathcal{J}\\ \dfrac{{}^{+}\mu(\alpha)\wedge\nu}{{}^{+}\mu(\alpha)}\,{\scriptstyle {}^{+}\wedge E}\\ \mathcal{C}\\ {}^{\bullet}\psi\end{array}\qquad \begin{array}{c}[{}^{+}\zeta(\beta)]\\ \mathcal{L}\\ \dfrac{{}^{+}\mu(\beta)\wedge\nu}{{}^{+}\mu(\beta)}\,{\scriptstyle {}^{+}\wedge E}\\ \mathcal{D}\\ {}^{\bullet}\psi\end{array}}{\begin{array}{c}{}^{\bullet}\psi\\ \mathcal{E}\end{array}}\,{\scriptstyle Rw.}$$

Disj: We detail the case for ${}^{+}\vee E$, the other cases are similar. The derivation is as follows:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\zeta(\alpha \mathbin{\vee\!\!\!\vee} \beta)\\ \mathcal{C}\\ {}^{+}\mu\vee\nu\end{array}\qquad \begin{array}{c}[{}^{+}\mu]\\ \mathcal{D}\\ {}^{+}\phi(\alpha \mathbin{\vee\!\!\!\vee} \beta)\end{array}\qquad \begin{array}{c}[{}^{+}\nu]\\ \mathcal{E}\\ {}^{+}\phi(\alpha \mathbin{\vee\!\!\!\vee} \beta)\end{array}}{{}^{+}\phi(\alpha \mathbin{\vee\!\!\!\vee} \beta)}\,{\scriptstyle {}^{+}\vee E}\qquad \begin{array}{c}[\phi(\alpha)]\\ \mathcal{F}\\ {}^{\bullet}\psi\end{array}\qquad \begin{array}{c}[\phi(\beta)]\\ \mathcal{G}\\ {}^{\bullet}\psi\end{array}}{\begin{array}{c}{}^{\bullet}\psi\\ \mathcal{H}\end{array}}\,{\scriptstyle Rw.}$$

We operate a permutation conversion to move the instance of Rewrite above the ${}^{+}\vee E$.

$$\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\zeta(\alpha \mathbin{\vee\!\!\!\vee} \beta)\\ \mathcal{C}\\ {}^{+}\mu\vee\nu\end{array}\qquad \dfrac{\begin{array}{c}[{}^{+}\mu]\\ \mathcal{D}\\ {}^{+}\phi(\alpha \mathbin{\vee\!\!\!\vee} \beta)\end{array}\quad \begin{array}{c}[\phi(\alpha)]\\ \mathcal{F}\\ {}^{\bullet}\psi\end{array}\quad \begin{array}{c}[\phi(\beta)]\\ \mathcal{G}\\ {}^{\bullet}\psi\end{array}}{{}^{\bullet}\psi}\,{\scriptstyle Rw.}\qquad \dfrac{\begin{array}{c}[{}^{+}\nu]\\ \mathcal{E}\\ {}^{+}\phi(\alpha \mathbin{\vee\!\!\!\vee} \beta)\end{array}\quad \begin{array}{c}[\phi(\alpha)]\\ \mathcal{F}\\ {}^{\bullet}\psi\end{array}\quad \begin{array}{c}[\phi(\beta)]\\ \mathcal{G}\\ {}^{\bullet}\psi\end{array}}{{}^{\bullet}\psi}\,{\scriptstyle Rw.}}{\begin{array}{c}{}^{\bullet}\psi\\ \mathcal{H}\end{array}}\,{\scriptstyle {}^{+}\vee E}$$

Notice that $\mathcal{D}$, $\mathcal{E}$ conclude the same sentence while complying with the $\star$ restriction by assumption. Therefore, $\star$ is not breached by performing the permutation.

After the permutation above, this case is then reduced to the others, depending on the last rule in $\mathcal{D}, \mathcal{E}$ respectively.

${}^{+}\blacklozenge\, I$: We cover might introduction explicitly since this case subsumes that of Weak Inference:

$$\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\zeta(\alpha \mathbin{\vee\!\!\!\vee} \beta)\\ \mathcal{X}\\ \dfrac{{}^{\oplus}\mu(\alpha \mathbin{\vee\!\!\!\vee} \beta)}{{}^{+}\blacklozenge\ \mu(\alpha \mathbin{\vee\!\!\!\vee} \beta)}\,{\scriptstyle {}^{+}\blacklozenge\, I}\end{array}\qquad \begin{array}{c}[{}^{+}\blacklozenge\ \mu(\alpha)]\\ \mathcal{C}\\ {}^{\bullet}\psi\end{array}\qquad \begin{array}{c}[{}^{+}\blacklozenge\ \mu(\beta)]\\ \mathcal{D}\\ {}^{\bullet}\psi\end{array}}{\begin{array}{c}{}^{\bullet}\psi\\ \mathcal{E}\end{array}}\,{\scriptstyle Rw.}$$

By inductive hypothesis we have: $, \mathcal{J} :{}^{+} \zeta(\alpha) \vdash^{\oplus} \mu(\alpha)\wedge\nu$, and $, \mathcal{L} :{}^{+} \zeta(\beta) \vdash^{\oplus} \mu(\beta)\wedge\nu$ thus allowing for the construction of the following tree:

$$
\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\zeta(\alpha \mathbin{\vee\!\!\vee} \beta)\end{array} \qquad \begin{array}{c}[{}^{+}\zeta(\alpha)]\\ \mathcal{J}\\ \dfrac{{}^{\oplus}\mu(\alpha)}{{}^{+}\blacklozenge\,\mu(\alpha)}\,{\scriptstyle {}^{+}\blacklozenge I}\\ \mathcal{C}\\ {}^{\bullet}\psi\end{array} \qquad \begin{array}{c}[{}^{+}\zeta(\beta)]\\ \mathcal{L}\\ \dfrac{{}^{\oplus}\mu(\beta)\wedge\nu}{{}^{+}\blacklozenge\,\mu(\beta)}\,{\scriptstyle {}^{+}\blacklozenge I}\\ \mathcal{D}\\ {}^{\bullet}\psi\end{array}}{\begin{array}{c}{}^{\bullet}\psi\\ \mathcal{E}\end{array}}\,{\scriptstyle Rw.}
$$

By repeatedly applying the reduction above, the claim then follows.

■

By repeatedly applying the previous propositions, a stronger statement follows, declaring that for any derivation where an instance of Rewrite appears, the *Rw.* rule in question may always be applied at the very beginning of the thread it appears in, preserving premisses and conclusion:

Proposition 73. For $\mathcal{D}$ a derivation $\Gamma \vdash {}^{\bullet}\psi$, there exists a $\mathcal{D}' : \Gamma \vdash {}^{\bullet}\psi$ where major premisses of *Rw.*. always precede premisses of any other rule except for ${}^{+}\blacklozenge\, I$.

Proof. Let $\mathcal{D}$ be a derivation. If there are no instances of premisses of *Rw.* rule in $\mathcal{D}$, then we are done. If there is at least one instance ${}^{+}\phi(\alpha \mathbin{\vee\!\!\vee} \beta)$, then begin by the leftmost such instance ${}^{+}\rho_1$ (if multiple instances occur in the same formula, we simply pick one).[6] First, operate all the relevant simplification conversion. By lemma 72, and our choice of the topmost leftmost instance, we can assume ${}^{+}\phi(\alpha \mathbin{\vee\!\!\vee} \beta)$ occurs as the conclusion of a ${}^{+}\mathbin{\vee\!\!\vee} I$ rule, or at the beginning of a thread, or as the induced assumption of a Disj, WI rule. In the first case, perform the following detour conversion and move to a new relevant instance if present.

$$
\dfrac{\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\alpha\end{array}}{{}^{+}\alpha \mathbin{\vee\!\!\vee} \beta}\,{\scriptstyle {}^{+}\mathbin{\vee\!\!\vee} I} \qquad \begin{array}{c}[{}^{+}\alpha]\\ \mathcal{B}\\ {}^{\bullet}\psi\end{array} \qquad \begin{array}{c}[{}^{+}\beta]\\ \mathcal{C}\\ {}^{\bullet}\psi\end{array}}{\begin{array}{c}{}^{\bullet}\psi\\ \mathcal{D}\end{array}}\,{\scriptstyle Rw.} \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{A}\\ {}^{+}\alpha\\ \mathcal{B}\\ {}^{\bullet}\psi\\ \mathcal{D}\end{array}
$$

In the second case, we move to the next topmost leftmost instance right away. In the third case we are in one of the following three situations:

1.

$$
\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\mu(\alpha \mathbin{\vee\!\!\vee} \beta)\vee\nu\end{array} \qquad \dfrac{[{}^{+}\mu(\alpha \mathbin{\vee\!\!\vee} \beta)] \qquad \begin{array}{c}[\mu(\alpha)]\\ \mathcal{B}\\ {}^{\bullet}\psi\end{array} \qquad \begin{array}{c}[\mu(\beta)]\\ \mathcal{C}\\ {}^{\bullet}\psi\end{array}}{{}^{\bullet}\psi}\,{\scriptstyle Rw.} \qquad \begin{array}{c}[{}^{+}\nu]\\ \mathcal{D}\\ {}^{\bullet}\psi\end{array}}{\begin{array}{c}{}^{\bullet}\psi\\ \mathcal{E}\end{array}}\,{\scriptstyle {}^{+}\vee E}
$$

Which we permute as follows:

$$
\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{+}\mu(\alpha \mathbin{\vee\!\!\vee} \beta)\vee\nu\end{array} \qquad \dfrac{[{}^{+}\mu(\alpha)\vee\nu] \qquad \begin{array}{c}[{}^{+}\mu(\alpha)]\\ \mathcal{B}\\ {}^{\bullet}\psi\end{array} \qquad \begin{array}{c}[{}^{+}\nu]\\ \mathcal{D}\\ {}^{\bullet}\psi\end{array}}{{}^{\bullet}\psi}\,{\scriptstyle {}^{+}\vee E} \qquad \dfrac{[{}^{+}\mu(\beta)\vee\nu] \qquad \begin{array}{c}[{}^{+}\mu(\beta)]\\ \mathcal{C}\\ {}^{\bullet}\psi\end{array} \qquad \begin{array}{c}[{}^{+}\nu]\\ \mathcal{D}\\ {}^{\bullet}\psi\end{array}}{{}^{\bullet}\psi}\,{\scriptstyle {}^{+}\vee E}}{\begin{array}{c}{}^{\bullet}\psi\\ \mathcal{E}\end{array}}\,{\scriptstyle Rw.}
$$

[6] Since the proof of proposition 77 duplicates the derivation $\mathcal{A} :{}^{+} \zeta(\alpha \mathbin{\vee\!\!\vee} \beta) \vdash^{+} \phi(\alpha \mathbin{\vee\!\!\vee} \beta)$, it is important we begin by the topmost instance in the thread.

2. The case for inquisitive disjunction is similar to the one above.
3. We are here in the following situation:

$$\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{\oplus}\phi(\alpha \mathbin{\vee\mkern-12mu\vee} \beta)\end{array} \qquad \dfrac{[{}^{+}\phi(\alpha \mathbin{\vee\mkern-12mu\vee} \beta)] \qquad \begin{array}{c}[{}^{+}\phi(\alpha)]\\ \mathcal{B}\\ {}^{\bullet}\psi\end{array} \qquad \begin{array}{c}{}^{+}\phi(\beta)\\ \mathcal{C}\\ {}^{\bullet}\psi\end{array}}{{}^{\bullet}\psi}\,Rw.}{\begin{array}{c}{}^{\bullet}{}_{w}\psi\\ \mathcal{E}\end{array}}\,\mathsf{WI}$$

Which we permute up as:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{A}\\ {}^{\oplus}\phi(\alpha \mathbin{\vee\mkern-12mu\vee} \beta)\end{array}}{{}^{+}\blacklozenge\ \phi(\alpha \mathbin{\vee\mkern-12mu\vee} \beta)}\,{}^{+}\blacklozenge\ I \qquad \dfrac{\dfrac{[{}^{+}\blacklozenge\ (\phi(\alpha)]}{{}^{\oplus}\phi(\alpha)}\,{}^{+}\blacklozenge\ E \qquad \begin{array}{c}[{}^{+}\phi(\alpha)]\\ \mathcal{B}\\ {}^{\bullet}\psi\end{array}}{{}^{\bullet}{}_{w}\psi}\,\mathsf{WI} \qquad \dfrac{\dfrac{[{}^{+}\blacklozenge\ (\phi(\beta)]}{{}^{\oplus}\phi(\beta)}\,{}^{+}\blacklozenge\ E \qquad \begin{array}{c}[{}^{+}\phi(\beta)]\\ \mathcal{C}\\ {}^{\bullet}\psi\end{array}}{{}^{\bullet}{}_{w}\psi}\,\mathsf{WI}}{\begin{array}{c}{}^{\bullet}{}_{w}\psi\\ \mathcal{E}\end{array}}\,Rw.$$

After applying the permutation, we apply lemma 72 and then again the permutations every times until ${}^{+}\rho_1$ is either eliminated as a detour, or it is the topmost formula of the thread or it is preceded only by ${}^{+}\blacklozenge\ I$. Since there are only finitely many premisses of a $\mathsf{Disj}, \mathsf{WI}$ rule in $\mathcal{D}$, this procedure terminates. We apply the same reduction procedure to the next topmost rightmost premiss of $Rw.$, taking care of not permuting this new instance above any other premiss of $Rw.$ which is either the topmost formula of a thread, or is preceded in a thread solely by premisses of $Rw.$ and premisses of ${}^{+}\blacklozenge\ I$. Since there are only finitely many premisses of $Rw.$ in $\mathcal{D}$, this shows the desired. ■

3.4 Normal form and subformula variants

In this section we define the notion of a "normal derivation", and we investigate the properties of this restricted class of derivation trees.

3.4.1 Normal form derivation trees

First, we define a normal form tree to be a derivation such that all of the simplifications presented up to now cannot be applied:

Definition 74 (Normal form derivation tree). Let $\mathcal{D}$ be a derivation tree for NFM_{ne}. $\mathcal{D}$ is in normal form if and only if $Fr(\mathcal{D}) = 0$, $R(\mathcal{D}) = \langle 0, 0, 0\rangle$, and all major premisses of $Rw.$ rules along any thread τ precede all premisses of any other rules except for ${}^{+}\blacklozenge\ I$.

Theorem 75. For any $\mathcal{D} : \Gamma \vdash {}^{\bullet}\phi$ a derivation in NFM_{ne}, there exists a normal derivation of NFM_{ne}, $\mathcal{C} : \Gamma \vdash {}^{\bullet}\phi$.

Proof. Let $\mathcal{D} : \Gamma \vdash {}^{\bullet}\phi$ be a derivation tree. Perform all the relevant simplification conversions. By proposition 66, we obtain a tree $\mathcal{D}_1 : \Gamma \vdash {}^{\bullet}\phi$ such that $Fr(\mathcal{D}_1) = 0$. By proposition 73, we obtain a tree $\mathcal{D}_2 : \Gamma \vdash {}^{\bullet}\phi$ such that no premiss of a rule different from ${}^{+}\blacklozenge\ I$ or $Rw.$ occurs before any major premiss of $Rw.$ along any thread. It is easy to check that this procedure does not increase the falsum rank of $\mathcal{D}_1$ so we may assume $Fr(\mathcal{D}_2) = 0$. Let then $R(\mathcal{D}_2) = \langle m, n\rangle$, if both m, n are zero then we are done. Otherwise, perform all the relevant simplification conversions, and choose the rightmost topmost cut segment of maximal weight in $\mathcal{D}$ (remark 67), perform all the relevant permutation conversions, and the relevant detour conversions. All the conversions either decrease the weight or the length, or the level of the segments. Because there are finitely many segments of maximum weight of the same length in $\mathcal{D}_2$, the claim then follows by induction hypothesis after a finite number of conversions. Notice that no conversion permutes a thread below another, and no conversion increases the falsum rank of the tree. Thus, setting $\mathcal{C}$ to be obtained from $\mathcal{D}_2$ after the application of the inductive hypothesis, we get $Fr(\mathcal{C})$, $R(\mathcal{C})$ and the claim follows. ∎

Threads of normal form derivations bear a specific structure, closely resembling the situation in classical logic. For the sake of the proofs to come, we will consider WI rule as elimination rules. Further, we speak of "coordination rules of inverse polarity", to express the consecutive iterations of $\circ \Rightarrow \bullet\neg$, $\bullet\neg \Rightarrow \circ$ rules, or of $\circ\neg \Rightarrow \bullet$, $\bullet \Rightarrow \circ\neg$ rules.

Proposition 76 (Structure of normal threads). Let $\upsilon = \langle {}^{\bullet}\beta_1, ..., {}^{\bullet}\beta_n \rangle$ be a thread in a normal derivation $\mathcal{D}$, then, there exists a series $\rho = \langle {}^{\bullet}\beta_1, .., {}^{\bullet}\beta_m \rangle$ with $m \leq n$ such that all sentences in ρ are the major premiss of a Rw. rule. For $\tau = \upsilon \setminus \rho$ it holds:

1. all premisses of elimination rules precede any premiss of an introduction rule in τ;
2. all premisses of elimination rules precede any premiss of $\bot E$ in τ;
3. there is at most one premiss of $\bot E$ in τ, and such premiss comes before all premiss of any introduction rule, and after all the premisses of elimination rules.

Proof. The existence of ρ is immediate by proposition 73, we show the three claims in their respective order.
For 1, assume for contradiction that in τ a thread in a normal derivation $\mathcal{D}$, an instance of an introduction rule **intro** precedes an elimination rule **elim**. Without loss of generality, we may assume **elim** is the first elimination rule occurring below **intro** along τ. Considering the major premiss of **elim** ${}^{\bullet}\alpha$, we readily see ${}^{\bullet}\alpha$ is not the first sentence appearing in τ as it has at least one formula above it. By proposition 66, α cannot be the conclusion of *Ex falso* unless **elim** is a WI rule, but that is in contradiction with $\mathcal{D}$ being normal (section 3.2.2.1), and so it cannot occur.
Since **elim** is the first elimination rule by assumption, the first formula above ${}^{\bullet}\alpha$ must either be: (i) the premiss of an introduction rule, or (ii) the premiss of an AR rule, or (iii) the premiss of a Coordination rule.
(i) implies a contradiction with our assumption that $\mathcal{D}$ was normal (section 3.2.2.3, section 3.2.2.2), so either (ii) or (iii) hold. If **elim** is a weak inference rule, then case (ii) is a contradiction to $\mathcal{D}$ being normal (section 3.2.2.2), meaning **elim** is not a WI rule. If the formula above ${}^{\bullet}\alpha$ is the premiss of an AR rule, and **elim** is not weak inference, then the formula-occurrence above it must either be (a) the conclusion of an introduction rule or (b) of a coordination rule (notice it can't be obtained via $\bot E$ by section 3.2.2.1). Case (ii. a) is a contradiction to our assumption that $\mathcal{D}$ was normal (section 3.2.2.3), so we must have (ii. b), the premiss of an introduction rule followed by the premiss of a coordination rule, followed by the premiss of an AR rule and then the premiss of an elimination rule. However, no elimination rules besides Weak inferences accept negation-prefixed sentences as premisses, and the only premisses of a coordination rules that can be concluded from an introduction rule are premisses of $\bullet \Rightarrow \circ\neg$ rules. Noticing that iteration of coordination rules of inverse polarity causes never occurs in a normal derivation (section 3.2.2.4), we conclude situation (ii. b) never occurs as well.
Case (iii) is similar to (ii. b) in that the only coordination rules capable of concluding premisses of

elim are $\circ\neg \Rightarrow \bullet$ rules, but premisses of such rules cannot be obtained via introduction rules, and iterations of Coord rule of opposite polarity are ruled out by normality of $\mathcal{D}$. This shows the desired.

For 2, assume for contradiction that in τ a thread in a normal derivation $\mathcal{D}$, an instance of $\bot E$ precedes an elimination rule **elim**. Consider the major premiss of **elim**, ${}^{\bullet}\alpha$. Since ${}^{\bullet}\alpha$ has at least one formula above itself, it is not an open assumption of $\mathcal{D}$, and we can assume without loss of generality that **elim** is the first instance of an elimination rule below $\bot E$ in τ. Consider the (signed) formula immediately above ${}^{\bullet}\alpha$ along τ, by proposition 66, such formula can't be $\bot$, and by Point 1 above, there can be no formula-instances concluded by an introduction rule above **elim**. Thus, the formula above ${}^{\bullet}\alpha$ must be the premiss of coordination rule. However, the presence of a segment beginning with an application of $\bot E$ and closing with the premiss of a Weak Inference rule is a contradiction to $\mathcal{D}$ being normal (section 3.2.2.1), and weak inferences are the only elimination rules applicable to propositional atoms. This shows 2.

For 3, suppose there is a premiss ${}^{\bullet}\alpha$ of an introduction rule **intro** preceding an instance of an $\bot E$ rule. Without loss of generality, we may assume **intro** to be the last introduction rule occurring before $\bot E$ in τ, moreover, we know by 1 that the sentence occurring below ${}^{\bullet}\alpha$ cannot be the major premiss of an elimination rule, nor such elimination rule can come after coordination and AR rules. Thus, the relevant $\bot$ instance must be obtained via an application of an EC rule and not an elimination rule. Since threads end at minor premisses of EC rules, the major premiss of such EC rule must occur below α possibly separated by a coordination rule. Both cases lead to a contradiction with the assumption that $\mathcal{D}$ is normal (section 3.2.2.1), thus, no instance of an introduction rule precedes an instance of an ex falso rule in a normal derivation.
Last, the assumption of there being more than one occurrence of *Ex falso* in a normal derivation, implies, in light of the the above, that either the first instance is used to conclude $\bot$, or to conclude a premiss of an EC rule, but both cases are in contradiction with the normality of $\mathcal{D}$ (section 3.2.2.1). ■

3.4.2 Weak subformula property

In this paragraph, we exlore if and to which extent the subformula property holds in NFM_{ne}. In light of proposition 76, we know that for a normal derivation $\mathcal{D}$, all threads are structured as follows, where the arrows indicate the succession of thread-parts, the ρ indicated the part composed of only premisses of *Rw*. and the question marks indicate that the relevant part may or may not be present: $^{+}\blacklozenge\, I$:

$$\rho \rightarrow \textbf{elim} \rightarrow ?\mathsf{EC} \rightarrow ?\bot E \rightarrow \textbf{intro}$$

By inspecting the rules of the system, it is easy to see that for all introduction rules, the premisses of the rules are subformulae of the conclusion, while for all the elimination rules the conclusion of the rule is a subformula of either the unique premiss, or of the conclusion of a minor premiss of the rule. Furthermore, induced assumptions of elimination rules are subformulae of the major premiss of the rule. This situation resembles the intuitionistic setting and would allow us to obtain full subformula property for our calculus. However, rules of Rewrite and coordination cause complications in this regard, indeed, while induced assumptions of *Rw*. are not subformulae of the major premiss of the rule, rules of coordination seemingly introduce and eliminate prefixed negations *ad libitum* without much possibility of being constrained. We saw in proposition 73 a strategy to deal with premisses of Rewrite, for anything occurring outside the ρ-part of a segment, the following holds:

Proposition 77 (*Rw.***-free subformulae).** Let $\mathcal{D}$ be a derivation in normal form where no premiss of a $Rw.$-rule appears, and ${}^\bullet\phi$ a signed formula therein. Then either:

i) ${}^\bullet\phi = \bot$, or;
ii) ${}^\bullet\phi$ is a subformula of some ${}^\bullet\psi \in \mathsf{Open}(\mathcal{D})$, or;
iii) ${}^\bullet\phi$ is a subformula of the conclusion of $\mathcal{D}$, or;
iv) ${}^\bullet\phi$ is a subformula of ${}^+\pi_{[\![\top]\!]}$, or;
v) ${}^\bullet\phi = {}^\bullet\neg^1 ... \neg^n\psi$ with ${}^\bullet\psi$ a subformula of $\mathsf{Open}(\mathcal{D})$ if n even and ${}^\circ\psi$ a subformula of $\mathsf{Open}(\mathcal{D})$ if n odd. Furthermore, ${}^\bullet\neg^1, ..., \neg^m\psi$ with $m \geq n$ is the last formula appearing in the elimination part of the thread.

Proof. Let ${}^\bullet\phi$ be a formula appearing in a normal derivation $\mathcal{D}$ as described above. By proposition 71 we know ${}^\bullet\phi$ appears in some thread $\tau = \langle \beta_1, ..., \beta_n \rangle$. By proposition 76, ${}^\bullet\phi$ is either $\bot$, or appears in either the introduction part or the elimination part of τ. We proceed by cases:

1. If ${}^\bullet\phi = \bot$, then the claim is satisfied.
2. If ${}^\bullet\phi$ appears in the elimination part of τ, then by inspecting the rules, we readily see that every signed formula appearing as the conclusion is a subformula of the major premiss of the rule, or, in the case of WI and Disj, a subformula of the conclusion of the minor premiss of the rule. Furthermore, induced assumtpions of WI and Disj rules are subformulae of the major premiss of the same rule-instance. Given the definition of thread, this implies that for $\phi = \beta_{m+1}$, ϕ is a subformula of β_m, in all cases except the one where ${}^\bullet\phi$ is the premiss of a $\bullet \Rightarrow \circ\neg$ rule.
   In the latter case, it is immediate that no elimination rule can be applied to ${}^\circ\neg\phi$, so it must be that either the elimination part ends with ${}^\circ\neg\phi$, or that ${}^\circ\neg\phi$ occurs as a premiss of an EC rule, or ${}^\circ\neg\phi$ is the premiss of either an AR or Coord rule. If ${}^\circ\neg\psi$ is the premiss of an AR rule, then ${}^\circ\neg\psi = {}^\circ{}_s\neg\psi$ and the thread must conclude with ${}^\circ{}_w\neg\psi$ figuring as the minor premiss of EC (notice that applying Weak inference contradicts $\mathcal{D}$ being normal by section 3.2.2.2). Alternatively, the introduction part of the thread may begin.
   If ${}^\circ\neg\psi$ is the premiss of an Coord rule, then the only rule applicable is $\bullet \Rightarrow \circ\neg$, for iterations of coordination rule of inverse polarity contradicts the normality of $\mathcal{D}$. Thus, after a finite number of applications of $\bullet \Rightarrow \circ\neg$ either we reach the premiss of an EC rule, or of an AR rule (and the previous point applies), or the introduction part of the thread begins.
3. If ${}^\bullet\phi$ occurs in the introduction part, then by inspecting the introduction rules of the system, we see that every formula appearing as a premiss of an introduction rule is a subformula of the formula concluded by that same rule. Thus, the claim holds in all cases except the one where ${}^\bullet\phi$ is the premiss of a $\bullet\neg \Rightarrow \circ$ rule. In that case, since $\mathcal{D}$ is normal, it follows that ${}^\bullet\phi$ cannot also be the conclusion of a Coord rule of inverse polarity. Since no introduction rule concludes negation-prefixed formulae, and no instance of $\bot E$ concludes negation-prefixed formulae by proposition 66, then, given that ${}^\bullet\phi = {}^\bullet\neg\psi$, ${}^\bullet\neg\psi$ must either be the first formula occurring in the introduction part of τ, so that ${}^\bullet\neg\psi$ is the conclusion of an elimination rule, or it must have been obtained from the conclusion of an elimination rule through a finite number of $\bullet\neg \Rightarrow \circ$ rule instances.

Let now ${}^{\bullet}\phi$ be arbitrary in $\mathcal{D}$. 1 above, if ${}^{\bullet}\phi$ is $\bot$, then i) holds.
If ${}^{\bullet}\phi$ occurs in the elimination part of a thread τ, then it either is a subformula of the topmost formula of τ, meaning either ii) or iv) hold, or there is a sequence of $\bullet \Rightarrow \circ\neg$ rule instances closing with the premiss of an EC rule, or with the premiss of some introduction rule. In the first case, v) holds, in the second case, it is straightforward that no $\circ\neg \Rightarrow \bullet$ rule can occur in the elimination part, lest $\mathcal{D}$ fails to be normal. Since we saw that $\circ\neg \Rightarrow \bullet$ can only occur at the beginning of the introduction part of a thread, by 3, ${}^{\bullet}\phi$ is a subformula of the conclusion of $\mathcal{D}$ and iii) holds.
Last, if ${}^{\bullet}\phi$ occurs in the introduction part of τ, by 3 above either ${}^{\bullet}\phi$ is a subformula of the conclusion of $\mathcal{D}$, and iii) holds, or there is a sequence S of $\bullet\neg \Rightarrow \circ$ rule instances beginning with the conclusion of some elimination rule in τ and concluding with ${}^{\bullet}\phi$. Clearly, no instance of $\circ \Rightarrow \bullet\neg$ rules can occur immediately before S, otherwise $\mathcal{D}$ would fail to be normal. Since $\circ \Rightarrow \bullet\neg$ can only occur at the end of the elimination part of τ, ${}^{\bullet}\phi$ is a subformula of the topmost formula in τ, and either ii) or iv) hold. ■

By adopting the notion of resolutions for inquisitive formulae from inquisitive logic, we can generalize the previous statement to the totality of NFM_{ne} derivations. Fix now an appropriate N, the $\mathsf{Res}(\mathcal{D})$ for $\mathcal{D} : \Gamma \vdash {}^{\bullet}\nu$ a derivation of NFM_{ne} is defined the set of all suitable substitutions of asserted instances of inquisitive disjunction in Γ with each of the disjuncts:

$$\bigcup_{{}^{+}\mu\in\Gamma}\ \bigcup_{\langle\alpha,\beta\rangle\in(\mathsf{Sub}(\mu)\times\mathsf{Sub}(\mu))}\ \bigcup_{{}^{+}\mu(\alpha\mathbin{\rotatebox[origin=c]{180}{$\curlywedge$}}\beta)\in\Gamma} \{{}^{\bullet}\mu(\alpha), {}^{\bullet}\mu(\beta)\}$$

It is then easy to establish the following:

Theorem 78 (**General subformulae**). For $\mathcal{D}$ a normal derivation, every sentence ${}^{\bullet}\phi$ appearing in $\mathcal{D}$ is either $\bot$, or a subformula of an element of $\mathsf{Open}(\mathcal{D}) \cup \mathsf{Res}(\mathcal{D})$, or a subformula of ${}^{+}\pi_{[\![\top]\!]}$, or ${}^{\bullet}\phi = {}^{\bullet}\neg^{1}...\neg^{n}\psi$ with ${}^{\bullet}\psi$ a subformula of $\mathsf{Open}(\mathcal{D}) \cup \mathsf{Res}(\mathcal{D})$ if n even and ${}^{\circ}\psi$ a subformula of $\mathsf{Open}(\mathcal{D}) \cup \mathsf{Res}(\mathcal{D})$ if n odd (and ${}^{\bullet}\neg^{1}, ..., \neg^{m}\psi$ with $m \geq n$ is the last formula appearing in the elimination part of the current thread), or either ${}^{\bullet}\phi$ or ${}^{\bullet} \blacklozenge\, \phi$ are a premiss of a $Rw.$ rule.

Proof. Immediate by proposition 73 and proposition 77, observing that all the induced assumptions of $Rw.$ rules are collected in $\mathsf{Res}(\mathcal{D})$, and they are preceded in a thread only by major premisses of the $Rw.$ or premisses of ${}^{+} \blacklozenge\, I$. ■

Chapter 4

Labelled sequents for $\mathcal{BSL}_m$

In this short chapter, we develop an axiomatization for $\mathcal{BSL}_m$, the non-inquisitive fragment of the logic $\mathcal{BSL}_m^{\mathbb{V}}$. This fragment presents additional difficulties in that the lack of inquisitive disjunction, paired with the lack of empty team property does not support the construction of Hintikka formulae for properties, making the logic ill-suited for the usual devices of natural deduction and sequent calculi systems.[1]

As a result, we resort to the device of *labelled sequents* to formulate our system LFM_{ne}. We first introduce the axioms, and then show soundness, completeness, and termination of the proof-search with respect to $\mathcal{BSL}_m$-models.

4.1 Axioms and basic facts

We begin by outlining the basics of our reduced logic $\mathcal{BSL}_m$. As said, $\mathcal{BSL}_m$ is the $\mathbb{V}$-free fragment of the bilateral-state-logic presented in section 2.1, which is given below in BNF form for any $p_i \in \mathsf{Prop}$.

$$p_i \mid \bot \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \blacklozenge\, \phi$$

In practice, we will always implicitly restrict ourselves to an arbitrary finite collection of variables $\mathsf{N} \subseteq \mathsf{Prop}$.

The semantics is obtained as expected, by eliminating the clauses for inquisitive disjunction from our original definition. For ease of readability, we report the restricted clauses here below, where the notion

[1] Another major issue is the inexpressibility of a (not necessarily characteristic) formula for $[\![\top]\!]$ in $\mathcal{BSL}_m$, which makes it difficult to substitute global disjunction with sets of formulae as done for example in [AAY24].

of model is inherited from definition 4:

$$\begin{array}{lll}
\mathfrak{M}, T \models p_i & \iff & w \in V(p_i), \text{ for all } w \in T \\
\mathfrak{M}, T \dashv p_i & \iff & w \notin V(p_i), \text{ for all } w \in T \\
\mathfrak{M}, T \models \bot & & \text{never} \\
\mathfrak{M}, T \dashv \bot & & \text{always} \\
\mathfrak{M}, T \models \neg\phi & \iff & \mathfrak{M}, T \dashv \phi \\
\mathfrak{M}, T \dashv \neg\phi & \iff & \mathfrak{M}, T \models \phi \\
\mathfrak{M}, T \models \phi \wedge \psi & \iff & \mathfrak{M}, T \models \phi \text{ and } \mathfrak{M}, T \models \psi \\
\mathfrak{M}, T \dashv \phi \wedge \psi & \iff & \text{there exist teams } S, P \subseteq T \text{ with: } S \cup P = T, \ \mathfrak{M}, S \dashv \phi, \text{ and } \mathfrak{M}, P \dashv \psi \\
\mathfrak{M}, T \models \phi \vee \psi & \iff & \text{there exist teams } S, P \subseteq T \text{ with: } S \cup P = T, \ \mathfrak{M}, S \models \phi, \text{ and } \mathfrak{M}, P \models \psi \\
\mathfrak{M}, T \dashv \phi \vee \psi & \iff & \mathfrak{M}, T \dashv \phi \text{ and } \mathfrak{M}, T \dashv \psi \\
\mathfrak{M}, T \models \blacklozenge \phi & \iff & \text{there exists } S \subseteq T \text{ with: } \mathfrak{M}, S \models \phi \\
\mathfrak{M}, T \dashv \blacklozenge \phi & \iff & \mathfrak{M}, T \dashv \phi
\end{array}$$

As in definition 5, $\varnothing$ fails to qualify as a licit team for $\mathcal{BSL}_m$ models, to the extent that the logic does not enjoy the empty team property as usually intended. When the empty team is explicitly excluded, the class of formulae defined by the properties of *union closure*, *downward closure*, *flatness* are easily extrapolated from the $\mathcal{BSL}_m^{\mathbb{V}}$ definition by restriction. In particular, we have:

$$\begin{array}{ll}
\text{Union-closure } (\mathsf{U}): & \mathcal{BSL}_m \\
\text{Downward closure (positive) } (\mathsf{D}^+): & p_i \mid \neg p_i \mid \phi \wedge \psi \\
\text{Downward closure (negative) } (\mathsf{D}^-): & p_i \mid \neg p_i \mid \phi \vee \psi \mid \blacklozenge \phi \\
\text{Flatness (positive) } (\mathsf{F}^+): & \mathsf{U} \cap \mathsf{D}^+ = \mathsf{D}^+ \\
\text{Flatness (negative) } (\mathsf{F}^-): & \mathsf{U} \cap \mathsf{D}^- = \mathsf{D}^-
\end{array}$$

The notions of equivalence and bisimulation are drawn from definition 7 and definition 22, with all the associated results holding for the non-inquisitive case as well.

Let us now turn to the formulation of LFM_{ne}. In the following, we draw from [LS25], [Bar+26] in presenting a terminating labelled calculus making use of *lists of premisses* which we mark with curly brackets. In particular, we extend the design of [Bar+26] to the setting of team logics excluding the empty set as a lawful state.
Central to the design of our calculus is the notion of *labels*:

Definition 79 (Labels). For $\mathcal{V}$ a countable set of variables, denoted with $u, v, w....$ A Label is a finite nonempty set $\pi \subseteq \mathcal{V}$.[2]

In the following, we will conflate singleton labels with elements of $\mathcal{V}$, so that we write, in general $u, v, w, ...$ for the labels $\{u\}$, $\{v\}$, $\{w\}, ...$, and we reserve lower-case greek letters for labels of cardinality

[2]For the purpose of completeness, notice that the finite model property of $\mathcal{BSL}_m$ is directly entailed by theorem 37. As a result, there is no loss of generality in considering *finite* labels only.

≥ 2.
Another central notion is that of an *interpretation*:

Definition 80 (**Interpretations**)**.** Given a finite set of propositional variables N, an interpretation is a map $\mathtt{int} : \mathcal{V} \to \mathsf{N} \to \mathsf{Bool}$. Interpretations are lifted from elements of $\mathcal{V}$ to labels by setting $\mathtt{int}(\pi) = \{\mathtt{int}(v), ..., \mathtt{int}(w)\}$ for $\pi = \{v, ..., w\}$.

An interpretation turns any $v \in \mathcal{V}$ into a valuation function from N to Bool. Thus, interpreted labels operate as surrogate teams and provide an alternative semantics for our calculus. In fact, for a label $\pi = \{v_1, v_2, ...\}$,we can think of the expression:

$$\mathtt{int}(\pi) \vDash \phi$$

As "the team $\bigcup_{i\in\pi} \mathtt{Int}(v_i)$ supports ϕ". We will see later on how this correspondence can be made more precise. Going forward, we will refer to semantic validity with respect to interpretations as "satisfaction", and we'll retain the notion of "support" for semantic validity over teams. The clauses for satisfaction are the same as the one for support so we do not repeat them here.
Some notation is needed to record the composition of a finite team $X \subseteq (\mathsf{N} \to \mathsf{Bool})$.

Definition 81 (**Splitting**)**.** Given a label π, the splitting of π (notation: $\mathcal{S}(\pi)$) is the set:

$$\{\langle \sigma, \tau \rangle \mid \sigma, \tau \in \mathcal{P}(\mathcal{V}) \setminus \{\varnothing\}, \pi = \sigma \cup \tau\}$$

We then define *Sequents* to be sets of signed formulae indexed by labels and separated by a "$\mapsto$" sign:

Definition 82 (**Sequent**)**.** A sequent is an expression of the shape $\Gamma \mapsto \Delta$ where both Γ, Δ are finite sets of labelled expressions $L : {}^\bullet\phi$, with L a label and ${}^\bullet\phi$ a sentence.

With the notion of sequent in place, we can turn to defining trees and branches in the usual way:

Definition 83 (LFM_{ne} **derivation**)**.** The notion of a LFM_{ne} derivation is defined recursively as follows: instances of the rules Ax_{At}, $\mathrm{Ax}_{\perp}$, Ax_{LEM}, Ax_{NC} are derivations; if $\mathcal{R}_1, ..., \mathcal{R}_n$, are derivations and $\mathcal{Q}$ is obtained from $\mathcal{R}_1, ..., \mathcal{R}_n$ by applying one of the rules of section 4.1, then $\mathcal{T}$ is a derivation.

The notion of a tree generalizes that of derivation eliminating the requirement that sequents without premisses always be obtained via axioms:

Definition 84 (LFM_{ne} **tree**)**.** Every sequent S of LFM_{ne} is a tree. If $\mathcal{R}_1, ..., \mathcal{R}_n$, are trees and $\mathcal{T}$ is obtained from $\mathcal{R}_1, ..., \mathcal{R}_n$ by applying one of the rules of section 4.1, then $\mathcal{T}$ is a derivation tree.

Definition 85 (**Branch**)**.** Let $\mathcal{T}$ be a tree for LFM_{ne}, a branch B of $\mathcal{T}$ is a (possibly infinite) sequence of sequents $\langle S_1, ... \rangle$ such that S_1 is the lowest sequent appearing in $\mathcal{T}$ and S_{i+1} is one of the premisses of S_i (if any).

Satisfaction of a sequent is defined making use of the usual map ι appropriately restricted to $\mathcal{BSL}_m$:

Definition 86 (**Satisfaction, Validity**)**.** A labelled sentence $\pi : {}^\bullet\phi$ is satisfied by an interpretation $\mathtt{Int}$ whenever $\mathtt{Int}(\pi) \vDash \iota({}^\bullet\phi)$.
A sequent $S : \Gamma \mapsto \Delta$ is valid whenever for any $\mathtt{Int}$ satisfying all the labelled formulae of Γ, $\mathtt{Int}$ satisfies some labelled formula in Δ.

We are now in the position to lay down the rules of our system.

Axioms:

$$\bullet \in \mathsf{S}\ \frac{}{v : {}^{\bullet}p_i, \Gamma \mapsto \Delta, v : {}^{\bullet}p_i}\ \mathrm{Ax}_{AT} \qquad \frac{}{\pi : \bot, \Gamma \mapsto \Delta}\ \mathrm{Ax}_{\bot}$$

$$\frac{}{\Gamma \mapsto \Delta, v :^{+} p_i, v :^{-} p_i}\ \mathrm{Ax}_{LEM} \qquad \frac{}{v :^{+} p_i, v :^{-} p_i, \Gamma \mapsto \Delta}\ \mathrm{Ax}_{NC}$$

Notice that the rule Ax_{AT} is restricted to strongly signed formulae, the rationale being that over singleton teams we have $\mathtt{Int}(v) \vDash \blacklozenge p_i \iff \mathtt{Int}(v) \vDash p_i$. The axioms of (atomic) Non-Contradiction and (atomic) Excluded Middle needed to handle the failure of the empty team property whilst allowing completeness via countermodel construction. While the law of Excluded Middle does not in general hold in $\mathcal{BSL}_m$, it is easy to see the Ax_{LEM} rule is sound for singleton specifically, simply by noticing that for any $\mathfrak{M}$ and $w \in W_{\mathfrak{M}}$ either $w \in V_{\mathfrak{M}}(p_i)$ or not. The rules for logical constants are given below, recalling that curly brackets stand for collection of premisses, the number of which is specified in the subscript to the bracket itself:

Rules:

$$(v \in \pi)\ \frac{v : {}^{\bullet}p_i, \pi : {}^{\bullet}p_i, \Gamma \mapsto \Delta}{\pi : {}^{\bullet}p, \Gamma \mapsto \Delta}\ \text{At. } L \qquad \frac{\{\Gamma \mapsto \Delta, v : {}^{\bullet}p_i\}_{v \in \pi}}{\Gamma \mapsto \Delta, \pi : {}^{\bullet}p_i}\ \text{At. } R$$

$$\frac{\pi :^{+} \phi, \pi :^{+} \psi, \Gamma \mapsto \Delta}{\pi :^{+} \phi \wedge \psi, \Gamma \mapsto \Delta}\ {}^{+}\wedge L \qquad \frac{\Gamma \mapsto \Delta, \pi :^{+} \phi \qquad \Gamma \mapsto \Delta, \pi :^{+} \psi}{\Gamma \mapsto \Delta, \pi :^{+} \phi \wedge \psi}\ {}^{+}\wedge R$$

$$\frac{\{\sigma :^{+} \phi, \tau :^{+} \psi, \Gamma \mapsto \Delta\}_{\langle \sigma, \tau \rangle \in \mathcal{S}(\pi)}}{\pi :^{+} \phi \vee \psi, \Gamma \mapsto \Delta}\ {}^{+}\vee L$$

$$(\sigma \cup \tau = \pi)\ \frac{\Gamma \mapsto \Delta, \pi :^{+} \phi \vee \psi, \sigma :^{+} \phi \qquad \Gamma \mapsto \Delta, \pi :^{+} \phi \vee \psi, \tau :^{+} \psi}{\Gamma \mapsto \Delta, \pi :^{+} \phi \vee \psi}\ {}^{+}\vee R$$

$$\frac{\{\sigma :^{+} \phi, \Gamma \mapsto \Delta\}_{\sigma \in \mathcal{P}(\pi)}}{\pi :^{\oplus} \phi, \Gamma \mapsto \Delta}\ {}^{+}SubT.L \qquad (\sigma \subseteq \pi)\ \frac{\Gamma \mapsto \Delta, \pi :^{\oplus} \phi, \sigma :^{+} \phi}{\Gamma \mapsto \Delta, \pi :^{\oplus} \phi}\ {}^{+}SubT.R$$

The Subteam rules internalize the semantics of signs $\oplus$, $\ominus$ to our calculus. Moreover, the right rules for *SubT.*, $\vee$ are cumulative in that they repeat the formula bearing the connective marked to the right of the rule from each premiss to the conclusion. When performing root-first proof-search, this allows for the same rule to be applied multiple times checking all appropriate subteams of π.

$$\frac{\pi :^{\oplus} \phi, \Gamma \mapsto \Delta}{\pi :^{+} \blacklozenge \phi, \Gamma \mapsto \Delta}\ {}^{+}\blacklozenge L \qquad \frac{\Gamma \mapsto \Delta, \pi :^{\oplus} \phi}{\Gamma \mapsto \Delta, \pi :^{+} \blacklozenge \phi}\ {}^{+}\blacklozenge R$$

$$\bullet \in \mathsf{S}\ \frac{\Gamma \mapsto \Delta, \pi : {}^{\bullet}\phi}{\Gamma \mapsto \Delta, \pi : {}^{\circ}\neg\phi}\ \mathsf{Coord}R \qquad \bullet \in \mathsf{S}\ \frac{\pi : {}^{\bullet}\phi, \Gamma \mapsto \Delta}{\pi : {}^{\circ}\neg\phi, \Gamma \mapsto \Delta}\ \mathsf{Coord}L$$

$$\frac{\Gamma \mapsto \Delta, \pi : {}^{\bullet}\phi}{\Gamma \mapsto \Delta, \pi : {}^{\bullet}{}_{w}\phi}\ {\downarrow} R \qquad \frac{\pi : {}^{\bullet}{}_{w}\phi, \Gamma \mapsto \Delta}{\pi : {}^{\bullet}\phi, \Gamma \mapsto \Delta}\ {\downarrow} L$$

The rules for negatively signed formulae are dual except for $\blacklozenge$ which gives:

$$\frac{\pi :^{-} \phi, \Gamma \mapsto \Delta}{\pi :^{-} \blacklozenge \phi, \Gamma \mapsto \Delta}\ {}^{-}\blacklozenge L \qquad \frac{\Gamma \mapsto \Delta, \pi :^{-} \phi}{\Gamma \mapsto \Delta, \pi :^{-} \blacklozenge \phi}\ {}^{-}\blacklozenge R$$

Since we are not concerned here with issues of Harmony, but solely with soundness, completeness, and termination, there is no need for either Weak Inferences or right rules of epistemic contradictions. We give the rest of the negatively signed rules below for completeness:

$$\frac{\pi :^{-} \phi, \pi :^{-} \psi, \Gamma \mapsto \Delta}{\pi :^{-} \phi \vee \psi, \Gamma \mapsto \Delta}\ {}^{+}\vee L \qquad \frac{\Gamma \mapsto \Delta, \pi :^{-} \phi \qquad \Gamma \mapsto \Delta, \pi :^{-} \psi}{\Gamma \mapsto \Delta, \pi :^{-} \phi \wedge \psi}\ {}^{+}\vee R$$

$$\frac{\{\sigma :^{-} \phi, \tau :^{-} \psi, \Gamma \mapsto \Delta\}_{\langle \sigma, \tau \rangle \in \mathcal{S}(\pi)}}{\pi :^{-} \phi \wedge \psi, \Gamma \mapsto \Delta}\ {}^{-}\wedge L$$

$$(\sigma \cup \tau = \pi)\ \frac{\Gamma \mapsto \Delta, \pi :^{-} \phi \wedge \psi, ^{-} :^{+} \phi \qquad \Gamma \mapsto \Delta, \pi :^{-} \phi \wedge \psi, \tau :^{-} \psi}{\Gamma \mapsto \Delta, \pi :^{-} \phi \wedge \psi}\ {}^{-}\wedge R$$

$$\frac{\{\sigma :^{-} \phi, \Gamma \mapsto \Delta\}_{\sigma \in \mathcal{P}(\pi)}}{\pi :^{\ominus} \phi, \Gamma \mapsto \Delta}\ {}^{-}SubT.L \qquad (\sigma \subseteq \pi)\ \frac{\Gamma \mapsto \Delta, \pi :^{\ominus} \phi, \sigma :^{-} \phi}{\Gamma \mapsto \Delta, \pi :^{\ominus} \phi}\ {}^{-}SubT.R$$

Moving between $\mathcal{BSL}_m$-models and interpretations is straightforward:

Remark 87. Given an interpretation $\mathtt{Int} : \mathcal{V} \to \mathsf{N} \to \mathsf{Bool}$, define model $\mathfrak{M}$ setting: $W_{\mathfrak{M}} = \mathcal{V}$, $Id_{\mathfrak{M}} = Id_{\mathcal{V}}$, and $V_{\mathfrak{M}}(p_i) = \{v \in \mathcal{V} \mid \mathtt{Int}(v)(p_1) = 1\}$.
Given a model $\mathfrak{M}$, define an interpretation $\mathtt{Int}$ taking $\mathcal{V} = W_{\mathfrak{M}}$, $Id_{\mathcal{V}}$ and $\mathtt{Int}(v)(p_i) = 1 \iff v \in V_{\mathfrak{M}}(p_i)$.

It is immediate to verify that any pair of model and interpretation so constructed will be bisimilar. We then observe the following:

$$\iota[\Gamma] \vDash_{\mathtt{Int}} \iota(^{\bullet}\phi) \iff \iota[\Gamma] \vDash_{\mathcal{BSL}_m} \iota(^{\bullet}\phi)$$

In light of the above, we show soundness and completeness with respect to interpretations, which in turn implies soundness and completeness with respect to $\mathcal{BSL}_m$-models.

4.2 Soundness, completeness, and termination

We now turn to showing some basic properties of LFM_{ne}. Soundness is entirely routine.

Proposition 88 (**Soundness**)**.** For any derivable sequent $S : \Gamma \mapsto \Delta$, S is valid.

Proof. By induction on the height of the derivation. We give the positive cases as the negative ones are symmetrical. We have four base cases:

- Ax_{AT} Let $\mathtt{Int}(v) \vDash \iota(^{+}p_i)$, then taking $\{\mathtt{Int}(v)\}$ as a witness it follows that $\mathtt{Int}(v) \vDash \blacklozenge p_i = \iota(^{\oplus}p_i)$.
- $\mathrm{Ax}_{\bot}$ We know no label is empty so $\mathtt{Int}(v) \vDash \bot$ never holds.
- Ax_{LEM} This case is immediate by the assumption that $\mathtt{Int}$ is well defined on v.
- Ax_{NC} This case never occurs as implied by proposition 8 together with remark 87.

The positive inductive cases are as of below:

At. L Let $\texttt{Int}(\pi) \models p_i$, since propositional atoms are downclosed, we get $\texttt{Int}(u) \models p_i$ for all u with $\texttt{Int}(u) \subseteq \texttt{Int}(\pi)$, (notice we have $\texttt{Int}(\pi) = \{\texttt{Int}(k) \mid k \in \mathcal{V} \cap \pi\}$ by construction). The desired the follows by induction hypothesis.

At. R Let $\texttt{Int}$ be such that is satisfies Γ, by induction hypothesis we obtain $\texttt{Int}(v) \models p_i$ for all $v \in \pi$, which implies the desired by union-closure of $\mathcal{BSL}_m$-formulae.

$^+ \wedge L, R$ Immediate by the semantic satisfaction clauses for $\wedge$.

$^+ \vee L$ Let $\texttt{Int}$ be such that $\texttt{Int}(\pi) \models \phi \vee \psi$, we then have some splitting of $\texttt{Int}(\pi)$ $\langle \gamma, \delta \rangle$ with $\texttt{Int}(\gamma) \models \phi$ and $\texttt{Int}(\delta) \models \psi$ and $\texttt{Int}(\gamma) \cup \texttt{Int}(\delta) = \texttt{Int}(\pi)$, the desired is given by applying the inductive hypothesis to the correct pair of $\langle \gamma, \delta \rangle$ out of the hypotheses set.

$^+ \wedge R$ Immediate by the semantic satisfaction clauses for $\vee$.

$^+ SubT.L$ Given $\texttt{Int}$ with $\texttt{Int}(\pi) \models \blacklozenge \phi$ then there exists a $\texttt{Int}(\gamma) \subseteq \texttt{Int}(\pi)$ with $\texttt{Int}(\phi)$. By construction of labels and interpretations, we know $\gamma \subseteq \pi$, which gives the desired applying the inductive hypothesis to the correct σ out of the premisses.

$^+ SubT.R$ Immediate by the semantic clauses for $\blacklozenge$.

$^+ \blacklozenge\ L, R$ immediate by the definition of ι.

$\mathsf{Coord}L, R$ Immediate by the semantic clauses for negation.

$\downarrow R, L$ Both cases are immediate by the semantics of $\blacklozenge$, taking $\{\texttt{Int}(\pi)\}$ as a witness.

The negative inductive cases are dual except for $^- \blacklozenge\ L, R$ which are immediate by the semantics of $\blacklozenge$ under $\Dashv$. ■

For completeness we first show that root-first proof-search terminates simply by applying all possible designated rules to the sequent under investigation. We then show that a refuting interpretation can be extracted from such procedure, which then gives completeness with respect to $\mathcal{BSL}_m$-models in light of remark 87.
By the structure of $\mathcal{BSL}_m$-models (or interpretations, for that matter), everything that can be inferred in a team from a weakly signed formula can also be inferred from its strongly signed variant. In this sense, the rules $\downarrow L$, $\downarrow R$ always conclude a deductively weaker sequent than the one featuring as a premiss. When performing root-first proof-search, we are interested in checking whether each sequent appearing in LFM_{ne} tree $\mathcal{T}$ eventually produces an instance of an axiom applying the rules in suitable order from conclusion to premisses. In light of the discussion above, anytime an axiom derives the conclusions of a $\downarrow L$, $\downarrow R$ rule, we can be ensured the premisses will be derived as well. Therefore, we can safely exclude the application of $\downarrow L$, $\downarrow R$ from our proof-search procedure below.

It is interesting to notice that the addition of rules of Epistemic contradiction as well as weak inference in the form, e.g., of:

$$\frac{\Gamma \mapsto \Delta, \pi : {}^{\bullet}{}_{w}\phi \qquad \{\sigma : {}^{\bullet}{}_{s}\phi, \Gamma \mapsto \Delta, \sigma : {}^{\bullet}{}_{s}\psi\}_{\sigma \in \mathcal{P}(\pi)}}{\Gamma \mapsto \Delta, \pi : {}^{\bullet}{}_{w}\psi}\ WI$$

$$\frac{\Gamma \mapsto \Delta, \pi : {}^{\bullet}{}_{s}\phi \qquad \Gamma \mapsto \Delta, \pi : {}^{\circ}{}_{w}\phi}{\Gamma \mapsto \Delta, \pi : \bot}\ EC$$

to our calculus, would not break the proof-search procedure, while the soundness of these rules is easy to verify. Given the fact that $\mathcal{BSL}_m$ is a sub-logic of $\mathcal{BSL}_m^{\mathbb{V}}$, we obtain the finite model property for the former from theorem 37. With this, we get that the consequence relation of $\mathcal{BSL}_m$ is decidable, which means there are only finitely many distinct formulas over a fixed finite N up to logical equivalence. We have in particular:

$$\phi \equiv \psi \iff (\mathfrak{M}, X \vDash \phi \iff \mathfrak{M}, X \vDash \psi) \iff [\![\phi]\!] = [\![\psi]\!]$$

But as there are only $|\mathcal{P}(\mathsf{N} \to \mathsf{Bool})|$ distinct teams, there are also finitely many distinct $\equiv$-classes. The same applies to $\boxminus$ taking the intersection with the negation-prefixed formulas. Therefore, weak inference as well as epistemic contradiction need only be applied finitely many times for any labelled sentence appearing in a sequent $\Gamma \mapsto \Delta$ to cover the totality of the search space.
Let now the collection of designated rules DR be given by the rules listed in section 4.1, excluding $\downarrow R$, $\downarrow L$.

To define a suitable countermodel from a root-first proof-search procedure, it is necessary to define safeguards to signal whether a certain rule has already been applied to a given formula. The first step in this direction is to establish a metric for whether the proof-search procedure ends in success or not.

Definition 89 (**Open branch, closed branch**). Let $\mathcal{T}$ be a tree in LFM_{ne}, and let B be a branch in $\mathcal{T}$. We say B is "closed" if the first sequent appearing at the top of B is marked with one of the four Ax. rules. We say B is "open" otherwise.

The idea is that, given a certain sequent S, one iteratively applies all the applicable rules from conclusion to premisses. If every branch of the tree thus generated is at some point closed by an axiom, then the process will also have constructed a derivation $\mathcal{D}$ witnessing the derivability of S. If this does not happen, then there must be some sequent Q such that it cannot be derived through any axiom. If the search procedure terminates, then we'll have obtained a tree $\mathcal{T}$ with at least one open branch B, signaling that S cannot be derived.

The notion of a saturated branch marks the occurrence of an open branch where all the applicable rules have been applied already.

Definition 90 (**Saturated branch**). Let $\mathcal{T}$ be a tree in LFM_{ne}, and let B be a branch therein. Set: $\Gamma^B = \bigcup_{S \in B} \Gamma_S$ as well as $\Delta^B = \bigcup_{S \in B} \Delta_S$. We say B is "saturated" is and only if B is open and all of the following hold for any label π:

1. $\pi : {}^{\bullet}{}_s p_i \in \Gamma^B$ implies $v : {}^{\bullet}{}_s p_i \in \Gamma^B$ for all $v \in \pi$;
2. $\pi :^{+} \phi \wedge \psi \in \Gamma^B$ implies $\pi :^{+} \phi \in \Gamma^B$ and $\pi :^{+} \psi \in \Gamma^B$;
3. $\pi :^{+} \phi \vee \psi \in \Gamma^B$ implies $\sigma :^{+} \phi \in \Gamma^B$ or $\tau :^{+} \psi \in \Gamma^B$ for some $\langle \sigma, \tau \rangle \in \mathcal{S}(\pi)$;
4. $\pi :^{-} \phi \vee \psi \in \Gamma^B$ implies $\pi :^{-} \phi \in \Gamma^B$ and $\pi :^{-} \psi \in \Gamma^B$;
5. $\pi :^{-} \phi \wedge \psi \in \Gamma^B$ implies $\sigma :^{-} \phi \in \Gamma^B$ or $\tau :^{-} \psi \in \Gamma^B$ for some $\langle \sigma, \tau \rangle \in \mathcal{S}(\pi)$;
6. $\pi : {}^{\bullet}{}_w \phi \in \Gamma^B$ implies $\sigma : {}^{\bullet}\phi \in \Gamma^B$ for some nonempty $\sigma \in \mathcal{P}(\pi)$;
7. $\pi :^{+} \blacklozenge \phi \in \Gamma^B$ implies $\pi :^{\oplus} \phi \in \Gamma^B$;
8. $\pi : {}^{\bullet}\neg\phi \in \Gamma^B$ implies $\pi : {}^{\circ}\phi \in \Gamma^B$;
9. $\pi :^{-} \blacklozenge \phi \in \Gamma^B$ implies $\pi :^{-} \phi \in \Gamma^B$;
10. $\pi : {}^{\bullet}p_i \in \Delta^B$ implies $v : {}^{\bullet}p_i \in \Delta^B$ for some $v \in \pi$;
11. $\pi :^{+} \phi \wedge \psi \in \Delta^B$ implies $\pi :^{+} \phi \in \Delta^B$ or $\pi :^{+} \phi \in \Delta^B$;
12. $\pi :^{+} \phi \vee \psi \in \Delta^B$ implies $\sigma :^{+} \phi \in \Delta^B$ or $\tau :^{+} \phi \in \Delta^B$ for all $\langle \sigma, \tau \rangle \in \mathcal{S}(\pi)$;
13. $\pi :^{-} \phi \vee \psi \in \Delta^B$ implies $\pi :^{-} \phi \in \Delta^B$ or $\pi :^{-} \phi \in \Delta^B$;

14. $\pi :^{-} \phi \wedge \psi \in \Delta^B$ implies $\sigma :^{-} \phi \in \Delta^B$ or $\tau :^{-} \phi \in \Delta^B$ for all $\langle \sigma, \tau \rangle \in \mathcal{S}(\pi)$;
15. $\pi : {}^{\bullet}{}_{w}\phi \in \Delta^B$ implies $\sigma : {}^{\bullet}\phi \in \Delta^B$ for all nonempty $\sigma \in \mathcal{P}(\pi)$;
16. $\pi :^{+} \blacklozenge \phi \in \Delta^B$ implies $\pi :^{\oplus} \phi \in \Delta^B$;
17. $\pi : {}^{\bullet}\neg\phi \in \Delta^B$ implies $\pi : {}^{\circ}\phi \in \Delta^B$;
18. $\pi :^{-} \blacklozenge \phi \in \Delta^B$ implies $\pi :^{-} \phi \in \Delta^B$.

When arguing for termination of our system, it will prove useful to consider weak signatures as contributing in some way to the complexity of sentences. The following measure captures the idea that, while assertion and rejection are treated as equal citizens under the flag of multilateralism, the attitudes of weak assertion (refraining from rejecting) as well of weak rejection (refraining from asserting) depend conceptually on the former pair.

Definition 91 (sentential complexity). For ${}^{\bullet}\phi$ a sentence, the sentential complexity of ${}^{\bullet}\phi$ is a pair $\langle x, y \rangle$ where x is the logical complexity of ϕ, while $y = 1$ if $\bullet \in \mathsf{W}$, and $y = 0$ if $\bullet \in \mathsf{S}$.

Let $\leq_C$ be the lexicographical ordering of sentences induced by the definition above. Further, let a $\pi_{\vee R}$-sequence ($\pi_{\wedge R}$-sequence) be the collection of root-first applications of ${}^{+} \vee R$ (${}^{-} \wedge R$) to a single labelled formula $\pi : {}^{\bullet}\phi$ in a tree $\mathcal{T}$. Likewise, let a $\pi_{SubT.R}$-sequence ($\pi_{\ominus R}$) be the collection of root-first applications of ${}^{+}SubT.R$ (${}^{-}SubT.R$) to a single labelled formula $\pi : {}^{\bullet}\phi$. We are then in the position of showing termination for LFM_{ne} root-first proof-search:

Proposition 92 (Proof-search termination). Let S be a sequent of LFM_{ne}. The tree generated by applying all licit rules DR to any labelled formula of S whenever applicable, never choosing for any $\pi_{\vee R}$-sequence (or any $\pi_{\wedge R}$-sequence) the same splitting $\langle \sigma, \tau \rangle$ more than once, and never choosing the same $\sigma \subseteq \pi$ more than once for any $\pi_{+SubT.R}$-sequence (π_{-SubTR}-sequence), is finite.

Proof. All formulae in the premisses of the designated rules DR are subformulae of some formula in the conclusion. Further, there are only finitely many splitting of a finite label π, and there are only finitely many nonempty sub-labels σ. Given the choice discipline outlined above, there are only finitely many rules applicable to each branch. ■

It is easy to see that, as the proof search terminates, each branch of any derivation tree either closes or becomes saturated. Let B be a saturated branch in a tree $\mathcal{T}$, we then define the interpretation `Count` as follows for any $v \in \mathcal{V}$:

$$\texttt{Count}(v)(p_i) = \begin{cases} 0 & \text{if } v :^{-} p_i \in \Gamma^B \\ 1 & \text{if } v :^{+} p_i \in \Gamma^B \\ 1 & \text{if } v :^{-} p_i \in \Delta^B \text{ and } v :^{-} p_i \notin \Gamma^B \\ 0 & \text{if } v :^{+} p_i \in \Delta^B \text{ and } v :^{+} p_i \notin \Gamma^B \\ 0 & \text{otherwise} \end{cases}$$

Consider that, by assumption B is saturated and thus open, so we know that we don't have both $v :^{-} p_i \in \Delta^B$ and $v :^{+} p_i \in \Delta^B$, otherwise the branch would have been closed by Ax_{LEM}. Similarly, we don't have both $v :^{-} p_i \in \Gamma^B$ and $v :^{+} p_i \in \Gamma^B$, otherwise the branch would have been closed by Ax $_{NC}$, so we can conclude `Count` is well defined.

Proposition 93 (Countermodel construction). Given a tree $\mathcal{T}$ in LFM_{ne}, and B a saturated branch therein, for any $\pi : {}^\bullet\phi \in \Gamma^B$ we have $\mathtt{Count}(\pi) \vDash \iota({}^\bullet\phi)$. Furthermore, for any $\pi : {}^\bullet\phi \in \Delta^B$ we have $\mathtt{Count}(\pi) \nvDash \iota({}^\bullet\phi)$

Proof. By induction on the sentential complexity of ${}^\bullet\phi$. We first show the claim for elements of $\mathcal{V}$.
If $v :^+ p_i \in \Gamma^B$, then we have $\mathtt{Count}(v) \vDash p_i$ by construction of $\mathtt{Count}$, and the same goes for $v :^- p_i$ which yields $\mathtt{Count}(v) \vDash \neg p_i$ be the semantics clauses for negation. If $v :^+ p_i \in \Delta^B$ then notice that as B is open we don't have $v :^+ p_i \in \Gamma^B$, so the desired follows by construction of $\mathtt{Count}$. Similarly, if $v :^- p_i \in \Delta^B$ then we know $v :^- p_i \notin \Gamma^B$ by the fact that B is open, which then implies $\mathtt{Count}(v) \vDash p_i$ and so $\mathtt{Count}(v) \nvDash \neg p_i$ by consistency.
If $|\pi| \neq 1$, then from $\pi :^+ p_i \in \Gamma^B$ we get $v :^+ p_i \in \Gamma^B$ for all $v \in \pi$ by the saturation conditions. By the above, (and definition 80) this implies $\mathtt{Count}(v) \vDash p_i$ for all $\mathtt{Count}(v) \in \mathtt{Count}(\pi)$, thus giving $\mathtt{Count}(\pi) \vDash p_i$ by union closure. The case for $\pi :^- p_i \in \Gamma^B$ is symmetrical.
If $\pi :^+ p_i \in \Delta^B$ then by saturation we have $v :^+ p_i \in \Gamma^B$ for some $v \in \pi$ thus implying $\mathtt{Count}(v) \vDash \neg p_i$ by the above and so in turn $\mathtt{Count}(\pi) \nvDash p_i$. The case for $\pi :^- p_i \in \Gamma^B$ is symmetric.
The inductive cases are as of below (notice weakly signed formula appear here only due to the lexicographic $\leq_C$ ordering on sentences).

If $\pi : {}^\bullet{}_w\phi \in \Gamma^B$ then by saturation we have $\sigma : {}^\bullet{}_s\phi \in \Gamma^B$, which gives $\mathtt{Count}(\sigma) \vDash \iota({}^\bullet{}_s\phi)$ by induction hypothesis, implying the desired as $\sigma \subseteq \pi$ by assumption. Similarly, $\pi : {}^\bullet{}_w\phi \in \Delta^B$ by saturation $\sigma : {}^\bullet{}_w\phi \in \Delta^B$ for all nonempty $\sigma \subseteq \pi$, meaning $\mathtt{Count}(\sigma) \nvDash \iota({}^\bullet{}_s\phi)$ by induction hypothesis, thus implying $\mathtt{Count}(\pi) \nvDash \iota({}^\bullet{}_w\phi)$.
If $\pi :^+ \phi \wedge \psi \in \Gamma^B$ then π marked with both conjuncts appears in Γ^B by saturation, which implies the desired by inductive hypothesis. If $\pi :^+ \phi \wedge \psi \in \Delta^B$ then π marked with one of the two conjuncts appears in Δ^B, implying the desired by induction hypothesis. The cases for $\pi :^- \phi \vee \psi$ are similar.
If $\pi :^+ \phi \vee \psi \in \Gamma^B$ then by saturation condition there is a splitting $\langle \sigma, \tau \rangle$ with $\sigma :^+ \phi$, $\tau :^+ \psi \in \Gamma^B$, yielding $\mathtt{Count}(\sigma) \vDash \phi$ and $\mathtt{Count}(\tau) \vDash \psi$, as we know by construction of the labels that $\mathtt{Count}(\pi) = \mathtt{Count}(\sigma) \cup \mathtt{Count}(\tau)$, this then implies the desired. If $\pi :^+ \phi \vee \psi \in \Delta^B$, then by saturation all splitting of π must appear in Δ^B as well, each marked with ${}^+\phi$ and ${}^+\psi$ on the first and second projection respectively. By induction hypothesis, this implies that all pair of subteams of $\mathtt{Count}(\pi)$ fail to support ϕ, ψ whenever their union equals π, which in turn yields $\mathtt{Count}(\pi) \nvDash \phi \vee \psi$. The cases for $\pi :^- \phi \wedge \psi$ is symmetric.
The case $\pi :^+ \blacklozenge\, \phi$ reduce to the cases of $\pi :^\oplus \phi$ by the saturation conditions. Further, cases for $\pi :^- \blacklozenge\, \phi$ follow immediately by inductive hypothesis together with the saturation condition.
Last, if $\pi : {}^\bullet\neg\phi \in \Gamma^B$ then by saturation $\pi : {}^\circ\phi \in \Gamma^B$ and the desired follows by the semantic clauses together with inductive hypothesis. The case $\pi : {}^\bullet\neg\phi \in \Delta^B$ is similar. ∎

At last, we obtain completeness for LFM_{ne}:

Theorem 94 (Completeness). LFM_{ne} is complete with respect to $\mathcal{BSL}_m$-models.

Proof. Say $S : \Gamma \mapsto \Delta$ is an unprovable sequent of LFM_{ne}, then by proposition 92 there exists a finite tree $\mathcal{T}$ with at least one open saturated branch B where S appears at the root. By proposition 93 there exists an interpretation $\mathtt{Count}$ such that it satisfies all labelled formulae of Γ and does not satisfy any of the labelled formulae of Δ. By remark 87, there is a $\mathcal{BSL}_m$-model bisimilar to $\mathtt{Count}$, so we conclude $\Gamma \mapsto \Delta$ is not valid over $\mathcal{BSL}_m$-models. ∎

Chapter 5

Conclusions

In this work, we have investigated two proof systems for state based modal logics through the tools of multilateral calculi developed in the context of Inferential expressivism. Our main goal was to provide an inferential-expressivist investigation of the linguistic phenomena captured by state-based modal logics under neglect-zero reasoning. To this aim, the choice of $\mathcal{BSL}_m^{\mathbb{V}}$ and $\mathcal{BSL}_m$ was motivated by a number of interesting natural language phenomena they encode, including free-choice inferences and contradictoriness of Yalcinean sentences under embedding. Furthermore, those logics incorporate the neglect-zero assumption discussed in [Alo22] directly into the model-theory, without relying on syntactical devices such as the NE atoms as it often occurs in other representative of their family.
In chapter 1 we introduced the theoretical background for the rest of the work, discussing the phenomena of epistemic contradictions (Yalcinean sentences) in natural language as well as free-choice inferences, together with their connection to the neglect-zero assumption in discursive sentence-interpretation. Furthermore, we introduced the basics of state-based modal logics and of Inferential Expressivism, which constitute the two main conceptual matrices of the present work.
In chapter 2 we introduced the system $\mathcal{BSL}_m^{\mathbb{V}}$ and we studied some of its model-theoretical properties. We then outline the natural deduction system NFM_{ne}, exploiting it to offer an inferential-expressivist interpretation of the inference licensed semantically by $\mathcal{BSL}_m^{\mathbb{V}}$. We then showed NFM_{ne} to be (proof-theoretically) locally sound and complete, as well as sound and complete with respect to $\mathcal{BSL}_m^{\mathbb{V}}$-models.
In chapter 2 we put the harmony of NFM_{ne} in practice showing it leads to a normal form result, as well as to some specific version of subformula property. Last, in chapter 4 we turned to the sub-logic $\mathcal{BSL}_m$, providing an axiomatization through the labelled sequent system LFM_{ne}. As no adequate conception of harmony is available for labelled sequent systems, we closed by showing soundness, completeness and proof-search termination for LFM_{ne}.
Overall, this work brought together the the proof-theoretic, inferentialist approach of [IS23] and the model-theoretic approach of [Alo22] to the problems of both epistemic contradictions and free-choice inferences. For once, we have shown that, from an inferential expressivist perspective, all Yalcinean sentences, (embedded and un-embedded alike) can be seen as licensing contradictory inference patterns when reasoning under neglect-zero assumptions. This also contributed to the development of the inferential expressivist programme: by giving an appropriate axiomatization and interpretation of $\mathcal{BSL}_m^{\mathbb{V}}$, we have shown inferential expressivism possesses the resources to explain reasoning and conversation under neglect-zero assumptions, if nothing in the case of free-choice inferences.
Last, team logics which exclude the empty set as a licit state have not so far been properly axiomatized. By axiomatizing $\mathcal{BSL}_m^{\mathbb{V}}$ as well as $\mathcal{BSL}_m$, this work partially filled this gap, hopefully paving the way

for the study of different logics in this category.

In terms of further research, multiple directions are open. From a model-theoretical standpoint, in this work we focused our efforts on the logic $\mathcal{BSL}_m^{\mathbb{V}}$. A further study of model theoretical properties of $\mathcal{BSL}_m$ would be an interesting development of the present research, especially if aimed at normal form and expressibility results. Equally, a more detailed proof-theoretical study of $\mathcal{BSL}_m$ would be desirable, a complete natural deduction or non-labelled system for this logic is presently absent, and some limitative results would be interesting to obtain if the aforementioned systems simply are not expressive enough to properly axiomatize the logic.
From a philosophical standpoint, a thorough study of how pragmatic phenomena interact with harmony requirements would be of great interest. While the present work shows that constants induced by inferences under neglect-zero reasoning still qualifies as meaning-determining from an inferential standpoint, we still are not in proximity of fully understanding this interaction. In particular, epistemic contradictions make a case for the failure of the rule of *Smileian reductio* in natural language (though some weaker variants may hold in different context). While our systems do not sanction any form of the reductio rule, and we thus settled for a notion of harmony as local soundness and completeness, a systematic study of what is reasonable to require of harmony under different settings or pragmatic assumptions would prove the most desirable. None of the logics here studied incorporates any form of implication in the object-language. This often occurs in the context of team logics, however, the addition of a conditional to the framework developed here would be interesting, especially if paired with a system of *hypothetical speech acts* as done in [IS23]. Ultimately, this would provide us with better understanding of the relationship between inferential expressivism and team semantics.

At last, the work at hand has run its course. On the issues drawn above, I hope I'll be able to speak in the future.

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