A Study of Proof-theoretic Harmony in Multilateral Sequent Calculus

MSc Thesis (*Afstudeerscriptie*)

written by

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Le è bastato un salto, da mezzanotte in sù
verso una luna a due facce
perchè la terra ne ha una
e a lei ne servono di più

Lucio Corsi, *La Lepre*

Abstract

Bilateralist proof systems, in which the speech acts of assertion and rejection are taken to be both primitive and formulae are presented in assertive and rejective form, have enjoyed a diverse and conspicuous development in the past decades. Not only natural deduction, but sequent calculus formulations have been proposed, and generalisations to more speech acts, giving rise to *multilateral systems*, have been formulated. Within the field of proof-theoretic semantics, where such proof systems and their underlying philosophical positions are being developed the most, questions arise about how to formulate criterion of why these proof systems can confer meaning, known as *harmony* criteria. Moreover, an explicit treatment of how to generalise harmony to the multilateralist setting requires further development. In this thesis, I present an account of proof-theoretic harmony to fit a twofold purpose, namely (i) to understand whether coordination principles constitute a bi- and multilateral generalisation of the cut rule and (ii) provide a generalisation of the current literature on harmony suitable to explain what *correct use* amounts to in multilateral contexts. Turns out that an answer to the more technically-sounding question (i) is highly sensitive to what we take harmony to be, and I defend the view that, if we endorse a broadly Dummettian conception of harmony, for harmony to be an explanatory notion about how meaning arises from use, we should not conflate unilateral, bilateral and multilateral notions of harmony in one single general notion. I will then show how admissibility results of cut and coordination principles enforce these different dimensions of harmony. This, I will argue, means that we should keep a non-trivial distinction between cut and the coordination principles. I will conclude by formulating an account of multilateral harmony and its relation to epistemic modality. This will be carried out in a suitably developed sequent calculus, based on the logics BML and EML presented in (Incurvati and Schlöder, 2023b).

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I cannot but conclude (to the relief of the by now exhausted reader) by expressing my deepest gratitude to my parents, Paola and Gabriele, for their unwavering support in all my major life choices, and for making this fantastic university experience possible. Needless to say that I could have done none of this without them.

Contents

Introduction

There is a sense in which this is a thesis on philosophically-oriented proof theory, and a sense in which this is a thesis on proof-theoretic semantics. It is a proof theory thesis insofar as a major component of it concerns the development of proof calculi of which all the important metatheorems are proven and contextualised. Rather than carrying out the task expected of a semantics, like defining explicit clauses and establishing soundness and completeness results, all such results that can be found in this thesis are relative to proof calculi, and there is a more substantial focus on admissibility of rules and structural proof theory than semantics per se.

Conversely, this is a thesis in proof-theoretic semantics since the present proof calculi, as well as the main questions that arise about them, must be contextualised within an inferentialist semantical endeavour. Arguably, logical bilateralism and multilateralism are not remotely as much of concern in pure proof theory communities than in proof-theoretic semantics ones, albeit the overlap between the two is significantly increasing. The problem of proof-theoretic harmony, in particular, has no meaningful *raison d'etre* if we do not charge our proof rules with semantic connotation about the connectives that they govern. While proof theorists strive to establish consistency of deductive systems, proof-theoretic *semanticists* are concerned with explaining how correct language use comes about. Surely correct language use should be consistent language use, or at the very least non-trivial and non-arbitrary language use. However, the task of understanding harmony is not simply a task of imposing non-triviality, i.e., establish conditions that imply consistency, but also, and most importantly, an endeavour to provide insight on how *why* we can say something and what we can *do* with it interact. The overlap of techniques used to establish the former and the latter is so significant that the same results, most notably cut-elimination, have a rich bearing on both tasks. However, it is important not to oversiplify matters to the point that the proof-theoretic and the semantic objectives of studying harmony are made to coincide, since while the proof theory tells us that our semantic framing works, our semantic framework needs to explain why there is any meaning to be read into the proof theory.

This thesis therefore aims to sharpen and advance the work that has been done so far about proof theory and proof-theoretic harmony multilateral logic, using as a starting point the philosophical system of Inferential Expressivism (Incurvati and Schlöder, 2023b). In their paper 'Inferential Deflationism', Luca Incurvati and Julian Schlöder remark that the principle of *Smileian Reductio* plays in bilateral natural deduction a role analogous to *cut* in sequent calculus (Incurvati and Schlöder, 2023a). Along similar lines, Ryan Simonelli claims, with respect to his bilateralist sequent calculus, that the admissibility of the coordination principle of *Rejection* is a "bilateralist generalisation" of the admissibility of cut (Simonelli, 2024a). However, a deeper analysis of how cut and the coordination principles interact in bi- and multilateral sequent calculi is missing in the literature. The aim of the present thesis is therefore twofold: I will (i) provide such analysis and (ii) show how it can be applied to refine criteria of proof-theoretic harmony that are suitable for multilateral logics which contain, in addition to assertion and rejection, speech act of *weak rejection* and *weak assertion*, based on Incurvati and Schlöder's *Basic Multilateral Logic* (BML) and *Epistemic Multilateral Logic* (EML). In the first chapter I will rehearse the philosophical preliminaries required for our discussion, while in Chapter 2 I will present a sequent calculus for Basic Multilateral Logic, G2BML, based on a translation schema proposed by (Francez, 2025), and outline its proof-theoretic properties. Then, I will focus on the coordination principles, to highlight their connection to cut and their role for a generalised conception of bilateral harmony. There I will argue for how the relationship between unilateral and multilateral harmony should be understood, and I will provide a defence for a pragmatist understanding of the rules, which I argue to be vindicated by the presence of weak rejections. In Chapter 3 I will then consider the modal case of *Epistemic Multilateral Logic*, where the expressivist resources are furthermore enhanced by introducing the speech act of *weak assertion*. I will present a cut-free sequent calculus for the logic, G2EML, thus providing harmonious connective clauses for epistemic modality that do not appeal to possible worlds or any other modal semantic framework. The presence of weak assertion and the relative coordination principles will then provide a natural setting to investigate a multilateral generalisation of the uni- and bilateral conception of harmony presented in Chapter 2.

Chapter 1

Technical and Philosophical Background

In this chapter, we present the core concepts of the thesis, introducing how and why the system under scrutiny came about, and giving an overview of what the problem of harmony amounts to. We will accordingly outline the notions of inferentialism, harmony, expressivism, bilateralism and multilateralism, which will be used throughout the thesis. The calculi outlined in the present thesis are meant to be used as a background system for providing a *proof-theoretic semantics* for a version of classical logic augmented with speech act force markers (Incurvati and Schlöder, 2023b). We briefly sketch in this section the main tenets of inferentialism, expressivism, and how combining the two is possible by adopting a bilateral logic in the style of (Smiley, 1996; I. Rumfitt, 2000). We will conclude the section by showing how some challenges waged against bilateralism motivate an extension to a *multilateral* logic (Incurvati and Schlöder, 2017).

1.1 Bilateralism, Multilateralism and Inferential Expressivism

Inferentialism is a semantic theory for which the meaning of a sign is given by its use in the inferences involving it. In logic this translates into the programme of proof-theoretic semantics, which aims to provide an account of the meanings of the logical constants in terms of inference rules governing them. There are various ways in which proof-theoretic semantics can be carried out, a discussion of which would lead us astray from the main point of this thesis. I therefore defer the discussion to (Schroeder-Heister, 2023; Wansing, 2000; Gheorghiu and Pym, 2025) and focus on the core of the view, namely that inference rules, in some form or another, can be seen as genuine *definitions* of connectives. A more explicit position about the most suitable such form for our aims will be argued for in chapter 2. What is (almost) unanimously accepted, however, is that a "meaning as use" paradigm does not licence *any* use, and we should therefore impose some restrictions to define what features "use" must have to confer meaning without being arbitrary. This is the so-called problem of *harmony*, the main target of the present inquiry. In Dummett's words, "[w]hat is demanded of a meaning theory is that it give an acceptable explanation of what a mastery of a whole language consists in" (Dummett, 1991, p. 213).

Such explanation is formulated as requiring some balance between the grounds for asserting a statement and the consequences that can be drawn from it (Dummett, 1991, p. 247). In logic, this means that introduction and elimination rules cannot be too strong (or conversely, too weak) with respect to one another. A paradigmatic example of disharmonious rules is Prior's tonk:

$$\frac{A}{A\mathsf{tonk}B}\ \mathsf{tonk}I \qquad \frac{A\mathsf{tonk}B}{B}\ \mathsf{tonk}E$$

It is easy to see that applying $\mathsf{tonk}E$ after $\mathsf{tonk}I$ allows us to infer any B from any A. Conversely, consider the following connective, called tunk (Simonelli, 2025; Francez, 2015):

$$\frac{A \quad B}{A\mathsf{tunk}B}\ \mathsf{tunk}I \qquad \frac{A\ \mathsf{tunk}\ B \quad \begin{array}{c}[A]\\ \vdots \\ \alpha\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ \alpha\end{array}}{\alpha}\ \mathsf{tunk}E$$

Here the language is not trivialised by a connective where its introduction rules are relatively too strong, but it suffers from the impairment of having relatively too weak elimination rules, which do not allow us to draw all the conclusions one could draw from the premises of the introduction rule. It is now widely accepted, following Dummett and Prawitz, that harmonious rules should allow for *reductions*, i.e., detours of introduction followed by eliminations of the same connective should be eliminated, and *expansions*, i.e., one should be able to recover the grounds from which an elimination rule is applicable by constructing an elimination-introduction detour (Simonelli, 2025; Dummett, 1991; Naibo and Takahashi, 2026)[1].

In the case of conjunction, this looks as follows:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ A\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ B\end{array}}{A \wedge B}\ \wedge I}{A}\ \wedge E \quad \rightsquigarrow_{red} \quad \begin{array}{c}\mathcal{D}_1\\ A\end{array}$$

$$\begin{array}{c}\mathcal{D}\\ A \wedge B\end{array} \quad \rightsquigarrow_{exp} \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A \wedge B\end{array}}{A}\ \wedge E_1 \quad \dfrac{\begin{array}{c}\mathcal{D}\\ A \wedge B\end{array}}{B}\ \wedge E_2}{A \wedge B}\ \wedge I$$

Where reductions rule out *tonk*ish connectives, and expansions rule out *tunk*ish ones. This allows us to make clear an important difference between what we are looking for while doing proof theory, and what we are looking for while doing proof-theoretic *semantics*. When it comes to proof theory per se, a reduction procedure amounts to bringing the derivation in normal form, enforcing the subformula property and hence consistency of the system (Negri and von Plato, 2001; Dummett, 1991). As pointed

[1] The reader familiar with the λ-calculus will notice a parallel with β-reduction and η-expansion. This is fully worked out in (Naibo and Takahashi, 2026).

out by Dummett, on the other hand, consistency is necessary but not sufficient for harmony (Dummett, 1991, p. 246), which is a primarily semantic concern. Indeed, harmony is not just a requirement that the connective rule-clauses do not generate inconsistencies, but is a condition for when a connective is justifiable and meaningful. That is, it imposes that the conditions of acquisition and manifestation of such connective, i.e., the grasp of the conditions for and the consequences of using such connective, are not prone to arbitrariness. tonk is gibberish because it licences too many consequences given the conditions for asserting it, while tunk is nonsense because it does not allow us to infer consequences that are legitimate according to the conditions that licence its assertion. In the context of proof-theoretic semantics therefore, proof theoretic results of the kind of reduction and expansion jointly work as a criterion for justifying why a given connective is a meaningful component of a language. Thus reductions, enforcing consistency, are only one part of a bigger picture that requires expansions as well. Moreover, the converse is also true: (Dummett, 1991) points out that a proof of normalisation is more complex than one of harmony (if we only consider the reduction part), since reduction alone does not include the rearrangement of the proof in canonical normal form, but merely comprises the reduction of detours, a process that became known as *reduction of local peaks* (Dummett, 1991; Steinberger, 2013). Reductions and expansions individually are chiefly proof-theoretic and computational techniques (Naibo and Takahashi, 2026; nLab authors, 2026), while together they can be thought of inferentialist semantic proofs of a justificatory nature (cf. Dummett, 1991).

Due to disharmonious negation rules, Dummett famously argued that inferentialism sanctions intuitionistic logic, whose negation rules exhibit no unbalance (Dummett, 1991). In response, Ian Rumfitt adopted a position first proposed by Timothy Smiley, namely *logical bilateralism*, in which the speech act of rejection, *contra* Fregean orthodoxy, is not reducible to that of asserting a negation, but is to be considered primitive (Smiley, 1996). Then, the resulting rules for negation are as follows, where formulae are signed by assertions (+) or rejections (−):

$$\frac{-A}{+\neg A}\,{+}\neg I \qquad \frac{+\neg A}{-A}\,{+}\neg E \qquad \frac{+A}{-\neg A}\,{-}\neg I \qquad \frac{-\neg A}{+A}\,{-}\neg E$$

which satisfy Dummett's harmony constraints (I. Rumfitt, 2000). Moreover, the system has rules that do not regiment any specific connective, but the interaction between speech acts. These are called *coordination principles*, and include *Rejection*, which states that asserting and rejecting a formula[2] is incoherent, and *Smileian Reductio*, which allows us to discharge an absurdity:

$$\frac{+A \qquad -A}{\bot}\,Rej \qquad \frac{\begin{matrix}[+A]\\ \vdots \\ \bot\end{matrix}}{-A}\,SR_1 \qquad \frac{\begin{matrix}[-A]\\ \vdots \\ \bot\end{matrix}}{+A}\,SR_2$$

Incurvati and Schlöder find in the bilateralist approach to inferentialism the framework in which to combine the semantic insight of inferentialism with that of another non-denotationalist meaning theory,

[2]The question of whether the coordination principles can be applied to atomic formulae or to arbitrary ones is subtle, and will be dealt with in section 3.

namely expressivism. Rather succinctly, expressivism explains meanings as expressions of propositional attitudes. A paradigmatic example is explaining negation as a device for expressing denial, as in Huw Price's seminal (Price, 1990). Expressivism however suffers from the so-called Frege-Geach problem, concerning speech acts in embedded contexts (Incurvati and Schlöder, 2023b). Consider the following instance of *modus ponens*:

$$\frac{\text{If not } p\text{, then not } q \qquad \text{not } p}{\text{not } q}$$

If we were to indicate the rejection of p as "not p", we would require that someone stating "If not p, then not q" must abide to rejecting p, which need not be the case. Indeed, one can well utter the conditional 'If not p, then not q" while still believing that p. In this embedded context, therefore, for this conditional claim to be intelligible, *not* must *modify* p and not merely express an attitude (Incurvati and Schlöder, 2023b). We can weaken our assumption that *not* is a force-marker expressing rejection by stating that this must hold in *unembedded* contexts, such as, for example, the second premiss. However, this comes at the cost of invalidating *modus ponens*, since the antecedent of the first premiss and the second premiss would end up being semantically different, the former modifying content and the latter modifying force. For the above to be a valid *modus ponens* instance, therefore, *not* must modify the content of p in both premises, at the expense of interpreting it as a speech act of rejection in the second. Thus, negation cannot be an indicator of rejection (Incurvati and Schlöder, 2023b). However, the expressivist insight can be salvaged by inferential means through bilateralism. Indeed, having a logic that includes speech acts of assertion and rejection which sign all logical formulae, allows us to treat "not" as a compositional operator modifying p, while still *inferentially* explaining it in terms of the speech act of rejection (Incurvati and Schlöder, 2023b). Smiley-Rumfitt bilateralism, therefore, can be seen as an *Inferential Expressivist* semantic theory (Ibid.)[3].

Bilateralism, however, as pointed out by Imogen Dickie in (Dickie, 2010), has problems if we assume, along Dummett and Prawitz, that valid inference should preserve evidence (Incurvati and Schlöder, 2023b). According to the received view, an inference is valid if the evidence that warrants the assertibility of the premises warrants that of the conclusions. However, rejections can be *evidentially unspecific*, i.e., less informative and not necessarily grounded on the same basis as a negative assertion. Consider for example a scenario, due to (Incurvati and Schlöder, 2023b), in which A claims that Homer wrote the Iliad, and B rejects the claim by affirming that Homer did not exist. This, rather than being a rejection on grounds of evidence that Homer did not write the Iliad, is a claim that Homer did not exist in the first place, and thus (if so) there cannot be any evidence for Homer writing the Iliad. To sharpen the focus on why this is problematic for bilateralism, let us apply *Smileian reductio* to this scenario (Incurvati and Schlöder, 2023b): If we assume Homer did not exist, it is absurd to hold that he wrote the Iliad. By Smileian reductio,

[3]Conversely, the benefit of inferential expressivism for *inferentialism* concerns the limited applicability objection that the significance of inferentialist meaning theories is limited to the logical constants. Incurvati and Schlöder, however, provide many examples of broader applications of inferentialism once augmented with speech acts, such as metaethical expressivism, truth-theoretic deflationism and epistemic modals. A detailed discussion is deferred to (Incurvati and Schlöder, 2023b).

then, we infer that Homer did not write the Iliad. However, the evidence supporting "Homer did not exist" and "Homer did not write the Iliad" need not coincide. The bilateralist therefore finds themself in a predicament, namely that unless they impose that all rejections must be evidentially specific (i.e., strong), an evidence-preserving account of inference cannot include them (Dickie, 2010; Incurvati and Schlöder, 2023b). However, restricting rejections to strong ones would invalidate instances of Smileian Reductio such as the one presented in the example above. Indeed, the problem runs even deeper: not only for Smileian reductio to be valid we need to accept the possibility of weak rejections, which cannot be premises or conclusions in an evidence preserving proof system, but we also need Smileian reductio to validate *reductio ad absurdum* (Incurvati and Schlöder, 2023a), and thus full classicality for bilateral arguments.

The solution proposed by Incurvati and Schlöder (Incurvati and Schlöder, 2017; Incurvati and Schlöder, 2023b) is to distinguish between two kinds of inferences, and accordingly adopt two different rejection signs. The key insight is that possibly weak rejections can serve as suitable premises and conclusions of a *commitment*, rather than evidence-preserving proof theory, and that strong rejections correspond to the particular case in which rejections preserve evidence instead. That is, inference rules regiment what new commitments one is bound by against the backdrop of commitments expressed in the premises (Incurvati and Schlöder, 2017). Commitment is defined in terms of a conversational common ground, to which one can propose or reject an update. Thus *asserting* p amounts to adding p to the common ground, *strongly rejecting* p to adding $\neg p$ to the common ground and *weakly rejecting* p to opposing to adding p to the common ground (Incurvati and Schlöder, 2017).

The coordination principles, as well as all the connective-defining rules, can be accordingly rephrased in terms of conversational commitment. Rejection, for example, can be read as committing and not committing to A being incoherent, and Smileian reductio as (without loss of generality) the normative requirement of not committing to A were such commitment incoherent, and vice versa. If we sign a weakly rejected formula with the force marker "⊖", we obtain the following version of the coordination principles:

$$\frac{+A \qquad \ominus A}{\bot}\,Rej \qquad\qquad \begin{array}{c}[+A]\\ \vdots\\ \dfrac{\bot}{\ominus A}\,SR_1\end{array} \qquad\qquad \begin{array}{c}[\ominus A]\\ \vdots\\ \dfrac{\bot}{+A}\,SR_2\end{array}$$

Furthermore, if we impose that no formulae signed with "⊖" occur in the premises, we can obtain, in our commitment-preserving proof theory, the aforementioned special case of evidence-preserving inference, in which rejections are strong. This is because the absence of weakly rejected premisses suffices to rule out evidential unspecificity (Incurvati and Schlöder, 2023b). We therefore have as a result the following coordination principles:

$$
\frac{+A \qquad -A}{\bot}\,Rej^{\sim}
\qquad
\frac{+A \qquad \ominus A}{\bot}\,Rej^{\lightning}
\qquad
\frac{\begin{matrix}[+A]\\ \vdots^{*}\\ \bot\end{matrix}}{-A}\,SR_1^{\star}
\qquad
\frac{\begin{matrix}[-A]\\ \vdots^{*}\\ \bot\end{matrix}}{+A}\,SR_2^{\star}
\qquad
\frac{\begin{matrix}[+A]\\ \vdots\\ \bot\end{matrix}}{\ominus A}\,SR_1
\qquad
\frac{\begin{matrix}[\ominus A]\\ \vdots\\ \bot\end{matrix}}{+A}\,SR_2
$$

Where $\vdots^{*}$ denotes an inference with no weakly rejected premises.

The resulting logic, *Basic Multilateral Logic* (BML) has introduction and elimination rules for all connectives in all speech act forms, and can be found in the figure below. For the present discussion, it is convenient to present BML in sequent calculus form, which I will do in the upcoming chapter. A philosophical discussion of the significance of the main admissibility theorems for proof-theoretic harmony will follow.

Figure 1: The natural deduction calculus $\mathsf{N}_{\mathsf{BML}}$ (Incurvati and Schlöder, 2023b)

Assertion

$$\frac{-A}{+\neg A}\,(+\neg I) \qquad \frac{+\neg A}{-A}\,(+\neg E)$$

$$\frac{+A \qquad +B}{+A \wedge B}\,(+\wedge I) \qquad \frac{+A \wedge B}{+A}\,(+\wedge E_1) \qquad \frac{+A \wedge B}{+B}\,(+\wedge E_2)$$

$$\frac{+A}{+A \vee B}\,(+\vee I_1) \qquad \frac{+B}{+A \vee B}\,(+\vee I_2) \qquad \frac{+A \vee B \qquad \begin{array}{c}[+A]\\ \vdots^{*}\\ \alpha\end{array} \qquad \begin{array}{c}[+B]\\ \vdots^{*}\\ \alpha\end{array}}{\alpha}\,(+\vee E)$$

$$\frac{\begin{array}{c}[+A]\\ \vdots^{*}\\ +B\end{array}}{+A \supset B}\,(+\supset I) \qquad \frac{+A \supset B \qquad +A}{+B}\,(+\supset E)$$

Strong Rejection

$$\frac{+A}{-\neg A}\,(-\neg I) \qquad \frac{-\neg A}{+A}\,(-\neg E)$$

$$\frac{-A}{-A \wedge B}\,(-\wedge I_1) \qquad \frac{-B}{-A \wedge B}\,(-\wedge I_2) \qquad \frac{-A \wedge B \qquad \begin{array}{c}[-A]\\ \vdots^{*}\\ \alpha\end{array} \qquad \begin{array}{c}[-B]\\ \vdots^{*}\\ \alpha\end{array}}{\alpha}\,(-\wedge E)$$

$$\frac{-A \qquad -B}{-A \vee B}\,(-\vee I) \qquad \frac{-A \vee B}{-A}\,(-\vee E_1) \qquad \frac{-A \vee B}{-B}\,(-\vee E_2)$$

$$\frac{+A \qquad -B}{-A \supset B}\,(-\supset I) \qquad \frac{-A \supset B}{+A}\,(-\supset E_1) \qquad \frac{-A \supset B}{-B}\,(-\supset E_2)$$

Weak Rejection

$$\frac{+A}{\ominus \neg A}\,(\ominus\neg I) \qquad \frac{\begin{array}{c}\vdots^{*}\\ \ominus\neg A\end{array}}{+A}\,(\ominus\neg E)$$

$$\frac{\ominus A}{\ominus A \wedge B}\,(\ominus\wedge I_1) \qquad \frac{\ominus B}{\ominus A \wedge B}\,(\ominus\wedge I_2) \qquad \frac{\ominus A \wedge B \qquad \begin{array}{c}[\ominus A]\\ \vdots\\ \alpha\end{array} \qquad \begin{array}{c}[\ominus B]\\ \vdots\\ \alpha\end{array}}{\alpha}\,(\ominus\wedge E)$$

$$\frac{\begin{array}{c}\vdots^{*}\\ \ominus A\end{array} \qquad \begin{array}{c}\vdots^{*}\\ \ominus B\end{array}}{\ominus A \vee B}\,(\ominus\vee I) \qquad \frac{\ominus A \vee B}{\ominus A}\,(\ominus\vee E_1) \qquad \frac{\ominus A \vee B}{\ominus B}\,(\ominus\vee E_2)$$

$$\frac{+A \qquad \ominus B}{\ominus A \supset B}\,(\ominus\supset I) \qquad \frac{\begin{array}{c}\vdots^{*}\\ \ominus A \supset B\end{array}}{+A}\,(\ominus\supset E_1) \qquad \frac{\ominus A \supset B}{\ominus B}\,(\ominus\supset E_2)$$

Chapter 2

Basic Multilateral Logic: Rules, Results and Bilateral Harmony

We now introduce the proof system, based on a translation schema for multilateral sequent calculi proposed by (Francez, 2025) and adapted to BML. The resulting system is a G2-style (Troelstra and Schwichtenberg, 2000) single conclusion calculus, which we accordingly call G2BML. The reasons behind this can already be found in (I. Rumfitt, 2000; Dummett, 2002; Steinberger, 2011b), and I briefly summarise them here. In Gentzen-style sequent calculus, the difference between classical and intuitionistic logic is that, while intuitionistic logic requires sequents with a single conclusion of the form $\Gamma \Rightarrow A$, classical logic allows multiple conclusions like $\Gamma \Rightarrow A, \Delta$ (Troelstra and Schwichtenberg, 2000). Rumfitt, however, refused to adopt a multiple-conclusion sequent calculus to inferentially vindicate classical logic on the grounds that no linguistic practice makes use of multiple-conclusion inferences, and therefore such a calculus cannot serve as a representation of linguistic mastery (I. Rumfitt, 2000; Dummett, 2002). Steinberger, moreover, points out that, in proof-theoretic *semantics*, multiple conclusions introduce a vicious circularity that would undermine the project of giving an inferentialist meaning to the logical constants.

Indeed, a multiple-conclusion sequent of the form $\Gamma \Rightarrow \Delta$ is canonically understood as $\bigwedge \Gamma \rightarrow \bigvee \Delta$. However, whereas committing to $\bigwedge \Gamma$ does not require any previous understanding of conjunction, since one need only commit to each conjunct individually, a different matter is for $\bigvee \Delta$[1]. Indeed, to commit to $\bigvee \Delta$ one needs to commit to *at least one* disjunct, once all disjuncts have been uttered. But this would require a previous understanding of disjunction, implying that proof-theoretic semantics would not be able to serve as a non-circular meaning theory (Steinberger, 2011b). As it will become evident after the soundness and completeness proof with respect to Incurvati and Schlöder's natural deduction system, G2BML will therefore have the proof-theoretic advantage of being classical, and the semantic advantage of being single-conclusion. Steinberger's circularity argument, moreover, imposes careful revision on the interpretation of the sequents beyond the single-conclusion debate. Indeed, the *formula* interpretation

[1] The question, presented in (Ceragioli, 2022), of whether the same problem applies to multiple assumptions is beyond the scope of this thesis.

$\bigwedge \Gamma \rightarrow \bigvee \Delta$ seems to suffer from semantic circularity problems also in terms of "$\rightarrow$". However, the discussion on commitment-preservation in the previous chapter allows us to give a reading of sequents in terms of consequential commitment.

Indeed, we can interpret our sequent $\Gamma \Rightarrow \alpha$ as a constraint on legitimate commitment: if I commit to all signed formulae $\gamma \in \Gamma$, then I must commit to α. We can take commitment as our most general notion our proof theory must preserve, and consider evidence as a particular case thereof, so that then, if there are only formulae signed with $+$ or $-$ in Γ, we can say that evidence for Γ supports commitment (or even evidence) for α. We now pin down some notions that will allows us to make formal sense of what it means for a sequent to contain signed formulae. We accordingly outline the preliminary definitions for $\mathsf{G2BML}$, as based on Francez's translation schema. Let $S = \{+, -, \ominus\}$ be the set of signs, representing our speech-act force markers of *assertion, strong rejection* and *weak rejection* respectively. Further, let the grammar of *Classical Propositional Logic* (CPL) be the following:

$$A ::= \bot \mid p \mid \neg A \mid A \wedge A \mid A \vee A \mid A \supset A$$

Definition 1 (Signed formula)**.** *Let A be a formula in the language of classical propositional logic (CPL). A BML signed formula has the form $\mathbf{s}A$ for $\mathbf{s} \in S$. The language of BML is therefore defined as $\mathcal{L}_{\mathsf{BML}} := \{\mathbf{s}A \mid A \in \mathcal{L}_{\mathsf{CPL}}, \mathbf{s} \in \{+, -, \ominus\}\}$. For Γ a multiset of CPL formulae, let $\Gamma^{\mathbf{s}} := \{\mathbf{s}A \mid A \in \Gamma\}$*

Since we will only be dealing with signed formulae throughout, the superscript in $\Gamma^{\mathbf{s}}$ will be dropped, and from now onwards all capital Greek letters will denote signed-formula contexts. Notice that this implies that formulae of the form "$\mathbf{s}A \vee \mathbf{s}B$" are ungrammatical in $\mathcal{L}_{\mathsf{BML}}$, as we should expect given the non-embeddability of the speech acts, which always take the scope of the whole formula (e.g., $\ominus A \vee B$).

From now onwards, we use A, B, C to denote CPL formulae, and α, β, γ to denote BML arbitrary formulae, which can thus be signed formulae or $\bot$.

Definition 2 (Formula signed sequent)**.** *A formula signed sequent has the form $\Gamma \Rightarrow \alpha$ such that Γ is a multiset of signed formulae and α is a signed formula or $\bot$.*

Definition 3 (Signed sequent calculus)**.** *A signed sequent calculus is a non-empty collection of rules and axioms, such that rules have the shape*

$$\frac{s_1 \cdots s_n}{s}$$

where $s, s_1, \ldots, s_n$ are signed formula sequents.

2.1 The Rules

The following is the complete calculus for G2BML. Throughout, rules marked with (∗) indicate the restriction that no formulae signed with "⊖" occur in Γ following Incurvati and Schlöder, 2023b.

Initial Sequents

$$\frac{}{\Gamma, \mathbf{s}p \Rightarrow \mathbf{s}p}\ Ax \qquad \frac{}{\Gamma, -p \Rightarrow \ominus p}\ Ax \qquad \frac{}{\Gamma, \bot \Rightarrow \alpha}\ \bot_L$$

$$\frac{}{\Gamma, +p, -p \Rightarrow \bot}\ Inc. \qquad \frac{}{\Gamma, +p, \ominus p \Rightarrow \bot}\ Inc.$$

Assertion

$$\frac{\Gamma \Rightarrow -A}{\Gamma \Rightarrow +\neg A}\ (+\neg R) \qquad \frac{\Gamma, -A \Rightarrow \alpha}{\Gamma, +\neg A \Rightarrow \alpha}\ (+\neg L)$$

$$\frac{\Gamma \Rightarrow +A \qquad \Gamma \Rightarrow +B}{\Gamma \Rightarrow +A \wedge B}\ (+\wedge R) \qquad \frac{\Gamma, +A \Rightarrow \alpha}{\Gamma, +A \wedge B \Rightarrow \alpha}\ (+\wedge L_1) \qquad \frac{\Gamma, +B \Rightarrow \alpha}{\Gamma, +A \wedge B \Rightarrow \alpha}\ (+\wedge L_2)$$

$$\frac{\Gamma \Rightarrow +A}{\Gamma \Rightarrow +A \vee B}\ (+\vee R_1) \qquad \frac{\Gamma \Rightarrow +B}{\Gamma \Rightarrow +A \vee B}\ (+\vee R_2) \qquad (*)\ \frac{\Gamma, +A \Rightarrow \alpha \qquad \Gamma, +B \Rightarrow \alpha}{\Gamma, +A \vee B \Rightarrow \alpha}\ (+\vee L)$$

$$(*)\ \frac{\Gamma, +A \Rightarrow +B}{\Gamma \Rightarrow +A \supset B}\ (+\supset R) \qquad (*)\ \frac{\Gamma \Rightarrow +A \qquad +B, \Gamma \Rightarrow \alpha}{\Gamma, +A \supset B \Rightarrow \alpha}\ (+\supset L)$$

Strong Rejection

$$\frac{\Gamma \Rightarrow +A}{\Gamma \Rightarrow -\neg A}\ (-\neg R) \qquad \frac{\Gamma, +A \Rightarrow \alpha}{\Gamma, -\neg A \Rightarrow \alpha}\ (-\neg L)$$

$$\frac{\Gamma \Rightarrow -A}{\Gamma \Rightarrow -A \wedge B}\ (-\wedge R_1) \qquad \frac{\Gamma \Rightarrow -B}{\Gamma \Rightarrow -A \wedge B}\ (-\wedge R_2) \qquad (*)\ \frac{\Gamma, -A \Rightarrow \alpha \qquad \Gamma, -B \Rightarrow \alpha}{\Gamma, -A \wedge B \Rightarrow \alpha}\ (-\wedge L)$$

$$\frac{\Gamma \Rightarrow -A \qquad \Gamma \Rightarrow -B}{\Gamma \Rightarrow -A \vee B}\ (-\vee R) \qquad \frac{\Gamma, -A \Rightarrow \alpha}{\Gamma, -A \vee B \Rightarrow \alpha}\ (-\vee L_1) \qquad \frac{\Gamma, -B \Rightarrow \alpha}{\Gamma, -A \vee B \Rightarrow \alpha}\ (-\vee L_2)$$

$$\frac{\Gamma \Rightarrow +A \qquad \Gamma \Rightarrow -B}{\Gamma \Rightarrow -A \supset B}\ (-\supset R) \qquad \frac{\Gamma, +A \Rightarrow \alpha}{\Gamma, -A \supset B \Rightarrow \alpha}\ (-\supset L_1) \qquad \frac{\Gamma, -B \Rightarrow \alpha}{\Gamma, -A \supset B \Rightarrow \alpha}\ (-\supset L_2)$$

Weak Rejection

$$\frac{\Gamma \Rightarrow +A}{\Gamma \Rightarrow \ominus\neg A}\,(\ominus\neg R) \qquad (*)\,\frac{\Gamma, +A \Rightarrow \alpha}{\Gamma, \ominus\neg A \Rightarrow \alpha}\,(\ominus\neg L)$$

$$\frac{\Gamma \Rightarrow \ominus A}{\Gamma \Rightarrow \ominus A \wedge B}\,(\ominus \wedge R_1) \qquad \frac{\Gamma \Rightarrow \ominus B}{\Gamma \Rightarrow \ominus A \wedge B}\,(\ominus \wedge R_2) \qquad \frac{\Gamma, \ominus A \Rightarrow \alpha \qquad \Gamma, \ominus B \Rightarrow \alpha}{\Gamma, \ominus A \wedge B \Rightarrow \alpha}\,(\ominus \wedge L)$$

$$(*)\,\frac{\Gamma \Rightarrow \ominus A \qquad \Gamma \Rightarrow \ominus B}{\Gamma \Rightarrow \ominus A \vee B}\,(\ominus \vee R) \qquad \frac{\Gamma, \ominus A \Rightarrow \alpha}{\Gamma, \ominus A \vee B \Rightarrow \alpha}\,(\ominus \vee L_1) \qquad \frac{\Gamma, \ominus B \Rightarrow \alpha}{\Gamma, \ominus A \vee B \Rightarrow \alpha}\,(\ominus \vee L_2)$$

$$\frac{\Gamma \Rightarrow +A \qquad \Gamma \Rightarrow \ominus B}{\Gamma \Rightarrow \ominus A \supset B}\,(\ominus \supset R) \qquad (*)\,\frac{\Gamma, +A \Rightarrow \alpha}{\Gamma, \ominus A \supset B \Rightarrow \alpha}\,(\ominus \supset L_1) \qquad \frac{\Gamma, \ominus B \Rightarrow \alpha}{\Gamma, \ominus A \supset B \Rightarrow \alpha}\,(\ominus \supset L_2)$$

Notice that we have an axiom $\overline{\Gamma, -p \Rightarrow \ominus p}^{Ax}$. This is added to the system as it captures an important tenet of inferential expressivism, namely the incompatobility of assent and dissent: if we commit to dissenting with p (strong rejection), we must also dissent with assenting to p (weak rejection). Moreover, this is required, as it will be shown later, for the cut admissibility proof to go through in the cases pertaining to *Smileian reductio*.

Structural Rules and Coordination Principles

We define the structural rules of *weakening, contraction* and *cut*. We will later show *weakening* and *cut* to be admissible, while, given that not all rules are invertible (notably, conjunction and disjunction), contraction will have to be taken as primitive. This does not constitute a significant disadvantage from the viewpoint of proof-theoretic harmony, and is customary in G2 calculi (Troelstra and Schwichtenberg, 2000), while it constitutes a great advantage in terms of simplicity of the translation proof with the natural deduction. We leave the development of an invertible calculus with admissible contraction to future work.

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma, \gamma \Rightarrow \alpha}\,Wk \qquad \frac{\Gamma, \gamma, \gamma \Rightarrow \alpha}{\Gamma, \gamma \Rightarrow \alpha}\,Ctr \qquad \frac{\Gamma \Rightarrow \alpha \qquad \alpha, \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \beta}\,Cut$$

In what follows, it will be convenient to use some schematic shorthands, by defining polarity-changing maps based on the following diagram of opposition:

$$\begin{tikzcd}
+ \arrow[r, leftrightarrow, "\text{Contrary } (SR^{\star})"] \arrow[dr, leftrightarrow, "\text{Contradictory } (SR)"'] & - \arrow[d, leftrightarrow, dashed, "\text{Subaltern}"] \\
& \ominus
\end{tikzcd}$$

Definition 4 (Polarity-changing maps). *Let* $_{+/-} \in \{+,-\} \subseteq S$, $_{+/\ominus} \in \{+,\ominus\} \subseteq S$, $_{-/\ominus} \in \{-,\ominus\} \subseteq S$. *We then define the following maps:*

$$\begin{aligned} ()^{\sim} : \{+,-\} \to \{+,-\} &:= \quad {}_{+/-}A \mapsto \textit{contrary of } {}_{+/-}A \\ ()^{\natural} : \{+,\ominus\} \to \{+,\ominus\} &:= \quad {}_{+/\ominus}A \mapsto \textit{contradictory of } {}_{+/\ominus}A \\ ()^{\flat} : \{-,\ominus\} \to \{-,\ominus\} &:= \quad {}_{-/\ominus}A \mapsto \textit{subaltern of } {}_{-/\ominus}A \end{aligned}$$

Accordingly, let, for $\alpha = \mathbf{s}A$, $\alpha^{\circ} = \mathbf{s}^{\circ}A$, with $\circ \in \{\sim, \natural, \flat\}$. This allows us to propose a compact formulation of the coordination principles of *Smileian reductio* and *Rejection*. The former we take to be primitive on the atoms with admissible generalisation to arbitrary formulae, the latter will be proven admissible.

$$(*)\ \frac{\Gamma, {}_{+/-}{}^{\sim}p \Rightarrow \bot}{\Gamma \Rightarrow {}_{+/-}p}\ SR^{\star}\text{-in} \qquad (*)\ \frac{\Gamma, \Rightarrow {}_{+/-}{}^{\sim}p}{\Gamma, {}_{+/-}p \Rightarrow \bot}\ SR^{\star}\text{-out}$$

$$\frac{\Gamma, {}_{+/\ominus}{}^{\natural}p \Rightarrow \bot}{\Gamma \Rightarrow {}_{+/\ominus}p}\ SR\text{-in} \qquad \frac{\Gamma, \Rightarrow {}_{+/\ominus}{}^{\natural}p}{\Gamma, {}_{+/\ominus}p \Rightarrow \bot}\ SR\text{-out}$$

$$\frac{\Gamma \Rightarrow {}_{+/-}{}^{\sim}A \qquad \Delta \Rightarrow {}_{+/-}A}{\Gamma, \Delta \Rightarrow \bot}\ Rej^{\sim} \qquad \frac{\Gamma \Rightarrow {}_{+/\ominus}{}^{\natural}A \qquad \Delta \Rightarrow {}_{+/\ominus}A}{\Gamma, \Delta \Rightarrow \bot}\ Rej^{\natural}$$

Notice that the rules for *Smileian reductio* are split in the sequent calculus in an "-in" and an "-out" version. This strategy is also devised by Ryan Simonelli in (Simonelli, 2024a; Simonelli, forthcoming), it is sensible for a sequent calculus, has natural deduction analogues, and we will see it is required for our admissibility results to go through. The reason why we take *Smileian reductio* to be primitive on the atoms is that, to achieve the full classicality results proven for the natural deduction system in (Incurvati and Schlöder, 2023b), the rule is necessary for the derivation of the asserted law of excluded middle while still retaining single conclusions. This is witnessed by the following derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{-p, +p \Rightarrow \bot}\ Inc.}{-p, -\neg p \Rightarrow \bot}\ -\neg L}{-p, -p \vee \neg p \Rightarrow \bot}\ -\vee L_2}{-p \vee \neg p, -p \vee \neg p \Rightarrow \bot}\ -\vee L_1}{-p \vee \neg p \Rightarrow \bot}\ ctr}{\Rightarrow +p \vee \neg p}\ SR^{\star}\text{-in}\ (*)$$

Notice that the asserted excluded middle follows by application of the rules of negation, disjunction and, most importantly, *Smileian reductio* from the axiom of incoherence, which states that it is absurd to commit to both p and its rejection. This derivation has the upshot of not only showing that, but also showing *why* the asserted excluded middle is a theorem of BML, and the role played by the notion of incompatibility hardwired in the axioms and the rule of *Smileian reductio*. Therefore, thus formulated, this proof is also epistemologically meaningful with respect to the issue of bilateral classicality. It is furthermore interesting to point out that the translation schema proposed by (Francez, 2025) does not properly capture how the coordination principles should be treated. Indeed, whereas signed formulae and signed-formula sequents are defined, no specific treatment of the coordination principles is proposed. Considering, as we will see, the variety of non-equivalent ways in which they can be treated, this constitutes an argument for questioning the philosophical import of the bi-interpretability result he proves about signed formula multilateralism and the different kind of multilateralism proposed by (Wansing and Ayhan, 2023), where speech acts are replaced by different derivability relations (e.g., proofs and refutations). We now provide soundness and completeness results, followed by admissibility results suitably supplied with a discussion on proof-theoretic harmony.

2.2 Soundness and Completeness

We now present the properties of the calculus G2BML and the corresponding proofs. Throughout, we denote with $\mathsf{N}_{\mathsf{BML}}$ the natural deduction system of BML as defined in (Incurvati and Schlöder, 2023b). First, G2BML is indeed a proper translation of $\mathsf{N}_{\mathsf{BML}}$, and the two proof systems are thus equivalent. We accordingly define our induction measure, namely the depth of a tree, in the canonical way, following (Troelstra and Schwichtenberg, 2000).

Definition 5 (Depth of a proof tree)**.** *The* depth *of a proof tree (natural deduction and sequent calculus) is the maximum of the length of the branches of the tree, where by "length" we mean the number of nodes minus one.*

Theorem 2.2.1 (Completeness)**.** *If* $\mathsf{N}_{\mathsf{BML}} \vdash \Gamma \Rightarrow \alpha$*, then* G2BML + *Cut* $\vdash \Gamma \Rightarrow \alpha$*.*

Proof. By induction on the depth of the natural deduction tree. The base case corresponds to a single node $\overline{\mathbf{s}A}$, which we translate as the axiom $\overline{\mathbf{s}A \Rightarrow \mathbf{s}A}^{Ax}$. For the induction step, we consider several cases corresponding to the signed connectives of the language, mapping in the canonical way introduction and eliminations rules to right and left rules respectively. Since the rules of the asserted fragment of BML, with the exception of negation [2] have exactly the same shape of those of the intuitionistic propositional calculus G2ip (because of the single conclusions), we leave the assertive cases of $\wedge, \vee, \supset$ to the reader.

Negation
Without loss of generality (the rejected rules being symmetric)

$$\dfrac{\begin{array}{c}[\Gamma]\\ \vdots\\ -A\end{array}}{+\neg A}\,(+\neg I) \qquad \Longrightarrow \qquad \dfrac{\begin{array}{c}\mathcal{IH}\\ \Gamma \Rightarrow -A\end{array}}{\Gamma \Rightarrow +\neg A}\,(+\neg R)$$

[2] Indeed, the negation rule thus defined allows to derive classicality of the calculus (Incurvati and Schlöder, 2023b) while maintaining the well-behaved features of intuitionistic proof systems (I. Rumfitt, 2000; Dummett, 1991)

$$\begin{array}{c}[\Gamma]\\ \vdots\\ \dfrac{\ominus A}{+\neg A}\,(+\neg I)\end{array} \quad\Longrightarrow\quad \begin{array}{c}\mathcal{IH}\\ \dfrac{\Gamma \Rightarrow \ominus A}{\Gamma \Rightarrow +\neg A}\,(+\neg R)\end{array}$$

$$\begin{array}{c}[\Gamma]\\ \vdots{*}\\ \dfrac{+\neg A}{-A}\,(+\neg E)\end{array} \quad\Longrightarrow\quad \dfrac{\begin{array}{c}\mathcal{IH}\\ \Gamma \Rightarrow +\neg A\end{array} \qquad (*)\,\dfrac{\dfrac{}{\Gamma, -A \Rightarrow -A}\,\text{Ax}}{+\neg A, \Gamma \Rightarrow -A}\,+\neg L}{\Gamma \Rightarrow -A}\,\text{Cut}$$

$$\begin{array}{c}[\Gamma]\\ \vdots{*}\\ \dfrac{+\neg A}{\ominus A}\,(+\neg E)\end{array} \quad\Longrightarrow\quad \dfrac{\begin{array}{c}\mathcal{IH}\\ \Gamma \Rightarrow +\neg A\end{array} \qquad (*)\,\dfrac{\dfrac{}{\Gamma, \ominus A \Rightarrow \ominus A}\,\text{Ax}}{+\neg A, \Gamma \Rightarrow \ominus A}\,+\neg L}{\Gamma \Rightarrow \ominus A}\,\text{Cut}$$

Where "$*$" next to $\vdots$ or a sequent rule application denotes an inference where no formulae signed with $\ominus$ occur in the premises Γ. We will use this notation throughout the proof.

We now check the cases of $\wedge, \vee, \supset$ for strong rejection.

Conjunction

$$\begin{array}{c}[\Gamma]\\ \vdots\\ \dfrac{-A_i}{-A_1 \wedge A_2}\,(-\wedge I_i)\end{array} \quad\Longrightarrow\quad \begin{array}{c}\mathcal{IH}\\ \dfrac{\Gamma \Rightarrow -A_i}{\Gamma \Rightarrow -A_1 \wedge A_2}\,(-\wedge R_i)\end{array}$$

$$\dfrac{\begin{array}{c}[\Gamma]\\ \vdots\\ -A \wedge B\end{array} \qquad \begin{array}{c}[\Gamma, -A]\\ \vdots{*}\\ \alpha\end{array} \qquad \begin{array}{c}[\Gamma, -B]\\ \vdots{*}\\ \alpha\end{array}}{\alpha}\,(-\wedge E) \quad\Longrightarrow\quad \dfrac{\begin{array}{c}\mathcal{IH}\\ \Gamma \Rightarrow -A \wedge B\end{array} \qquad (*)\,\dfrac{\begin{array}{c}\mathcal{IH}\\ \Gamma, -B \Rightarrow \alpha\end{array} \qquad \begin{array}{c}\mathcal{IH}\\ \Gamma, -A \Rightarrow \alpha\end{array}}{-A \wedge B, \Gamma \Rightarrow \alpha}\,(-\wedge L)}{\Gamma \Rightarrow \alpha}\,\text{Cut}$$

Disjunction

$$\dfrac{\begin{array}{c}[\Gamma]\\ \vdots\\ -A\end{array} \qquad \begin{array}{c}[\Gamma]\\ \vdots\\ -B\end{array}}{-A \vee B}\,(-\vee I) \quad\Longrightarrow\quad \dfrac{\begin{array}{c}\mathcal{IH}\\ \Gamma \Rightarrow -A\end{array} \qquad \begin{array}{c}\mathcal{IH}\\ \Gamma \Rightarrow -B\end{array}}{\Gamma \Rightarrow -A \vee B}\,(-\vee R)$$

$$\begin{array}{c}[\Gamma]\\ \vdots\\ \dfrac{-A_1 \vee A_2}{-A_i}\,(-\vee E_i)\end{array} \quad\Longrightarrow\quad \dfrac{\begin{array}{c}\mathcal{IH}\\ \Gamma \Rightarrow -A_1 \vee A_2\end{array} \qquad \dfrac{\dfrac{}{\Gamma, -A_i \Rightarrow -A_i}\,\text{Ax}}{-A_1 \vee A_2, \Gamma \Rightarrow -A_i}\,(-\vee L_i)}{\Gamma \Rightarrow -A_i}\,\text{Cut}$$

Implication

$$\dfrac{\begin{matrix}[\Gamma]\\ \vdots \\ +A\end{matrix} \qquad \begin{matrix}[\Gamma]\\ \vdots\\ -B\end{matrix}}{-A\supset B}\,(-\supset I) \qquad\Longrightarrow\qquad \dfrac{\begin{matrix}\mathcal{IH}\\ \Gamma\Rightarrow +A\end{matrix}\qquad \begin{matrix}\mathcal{IH}\\ \Gamma\Rightarrow -B\end{matrix}}{\Gamma\Rightarrow -A\supset B}\,(-{\supset}R)$$

Without loss of generality, we check the case of $-\supset E_1 \implies -\supset L_1$, the other rule being symmetric.

$$\dfrac{\begin{matrix}[\Gamma]\\ \vdots \\ -A\supset B\end{matrix}}{+A}\,(-\supset E_1) \qquad\Longrightarrow\qquad \dfrac{\begin{matrix}\mathcal{IH}\\ \Gamma\Rightarrow -A\supset B\end{matrix}\qquad \dfrac{\overline{\Gamma, +A\Rightarrow +A}^{\,\mathrm{Ax}}}{-A\supset B, \Gamma\Rightarrow +A}\,(-\supset L_1)}{\Gamma\Rightarrow +A}\,\mathrm{Cut}$$

This completes the direction for the strong rejective fragment of BML. We now conlcude by checking the cases for weak rejection of the constants $\wedge, \vee, \supset$.

Conjunction

$$\dfrac{\begin{matrix}[\Gamma]\\ \vdots \\ \ominus A_i\end{matrix}}{\ominus A_1\wedge A_2}\,(\ominus\wedge I_i) \qquad\Longrightarrow\qquad \dfrac{\begin{matrix}\mathcal{IH}\\ \Gamma\Rightarrow \ominus A_i\end{matrix}}{\Gamma\Rightarrow \ominus A_1\wedge A_2}\,(\ominus\wedge R_i)$$

$$\dfrac{\begin{matrix}[\Gamma]\\ \vdots \\ \ominus A\wedge B\end{matrix} \qquad \begin{matrix}[\Gamma,\ominus A]\\ \vdots \\ \alpha\end{matrix} \qquad \begin{matrix}[\Gamma,\ominus B]\\ \vdots \\ \alpha\end{matrix}}{\alpha}\,(\ominus\wedge E) \qquad\Longrightarrow\qquad \dfrac{\begin{matrix}\mathcal{IH}\\ \Gamma\Rightarrow \ominus A\wedge B\end{matrix}\qquad \dfrac{\begin{matrix}\mathcal{IH}\\ \Gamma,\ominus B\Rightarrow\alpha\end{matrix}\qquad\begin{matrix}\mathcal{IH}\\ \Gamma,\ominus A\Rightarrow\alpha\end{matrix}}{\ominus A\wedge B, \Gamma\Rightarrow \alpha}\,(\ominus{\wedge}L)}{\Gamma\Rightarrow \alpha}\,\mathrm{Cut}$$

Disjunction

$$\dfrac{\begin{matrix}[\Gamma]\\ \vdots{*} \\ \ominus A\end{matrix} \qquad \begin{matrix}[\Gamma]\\ \vdots{*}\\ \ominus B\end{matrix}}{\ominus A\vee B}\,(\ominus\vee I) \qquad\Longrightarrow\qquad (*)\,\dfrac{\begin{matrix}\mathcal{IH}\\ \Gamma\Rightarrow \ominus A\end{matrix}\qquad \begin{matrix}\mathcal{IH}\\ \Gamma\Rightarrow \ominus B\end{matrix}}{\Gamma\Rightarrow \ominus A\vee B}\,(\ominus{\vee}R)$$

$$\dfrac{\begin{matrix}[\Gamma]\\ \vdots \\ \ominus A_1\vee A_2\end{matrix}}{\ominus A_i}\,(\ominus\vee E_i) \qquad\Longrightarrow\qquad \dfrac{\begin{matrix}\mathcal{IH}\\ \Gamma\Rightarrow \ominus A_1\vee A_2\end{matrix}\qquad \dfrac{\overline{\Gamma, \ominus A_i\Rightarrow \ominus A_i}^{\,\mathrm{Ax}}}{\ominus A_1\vee A_2, \Gamma\Rightarrow \ominus A_i}\,(\ominus\vee L_i)}{\Gamma\Rightarrow \ominus A_i}\,\mathrm{Cut}$$

Implication

$$\dfrac{\begin{array}{c}[\Gamma]\\ \vdots\\ +A\end{array} \quad \begin{array}{c}[\Gamma]\\ \vdots\\ \ominus B\end{array}}{\ominus A \supset B}\,(\ominus \supset I) \quad \Longrightarrow \quad \dfrac{\begin{array}{c}\mathcal{IH}\\ \Gamma \Rightarrow +A\end{array} \quad \begin{array}{c}\mathcal{IH}\\ \Gamma \Rightarrow \ominus B\end{array}}{\Gamma \Rightarrow \ominus A \supset B}\,(\ominus{\supset}R)$$

Without loss of generality, we check the case of $\ominus \supset E_1 \implies \ominus \supset L_1$, the other rule being symmetric, with the only difference that no restrictions are applied on the premises.

$$\dfrac{\begin{array}{c}[\Gamma]\\ \vdots *\\ \ominus A \supset B\end{array}}{+A}\,(\ominus \supset E_1) \quad \Longrightarrow \quad \dfrac{\begin{array}{c}\mathcal{IH}\\ \Gamma \Rightarrow \ominus A \supset B\end{array} \quad (*)\,\dfrac{\overline{\Gamma, +A \Rightarrow +A}^{\,\text{Ax}}}{\ominus A \supset B, \Gamma \Rightarrow +A}\,(\ominus \supset L_1)}{\Gamma \Rightarrow +A}\,\text{Cut}$$

Coordination Principles Without loss of generality, the remaining cases being symmetric.

$$\dfrac{\begin{array}{c}[\Gamma, +A]\\ \vdots *\\ \bot\end{array}}{-A}\,(SR^{\star}) \quad \Longrightarrow \quad (*)\,\dfrac{\begin{array}{c}\mathcal{IH}\\ \Gamma, +A \Rightarrow \bot\end{array}}{\Gamma \Rightarrow -A}\,SR^{\star}$$

$$\dfrac{\begin{array}{c}[\Gamma, +A]\\ \vdots\\ \bot\end{array}}{\ominus A}\,(SR) \quad \Longrightarrow \quad \dfrac{\begin{array}{c}\mathcal{IH}\\ \Gamma, +A \Rightarrow \bot\end{array}}{\Gamma \Rightarrow \ominus A}\,SR$$

$$\dfrac{\begin{array}{c}[\Gamma]\\ \vdots\\ +A\end{array} \quad \begin{array}{c}[\Delta]\\ \vdots\\ -A\end{array}}{\bot}\,Rej \quad \Longrightarrow \quad \dfrac{\begin{array}{c}\mathcal{IH}\\ \Gamma \Rightarrow +A\end{array} \quad \begin{array}{c}\mathcal{IH}\\ \Delta \Rightarrow -A\end{array}}{\Gamma, \Delta \Rightarrow \bot}\,Rej^{\sim}$$

$$\dfrac{\begin{array}{c}[\Gamma]\\ \vdots\\ +A\end{array} \quad \begin{array}{c}[\Delta]\\ \vdots\\ \ominus A\end{array}}{\bot}\,Rej \quad \Longrightarrow \quad \dfrac{\begin{array}{c}\mathcal{IH}\\ \Gamma \Rightarrow +A\end{array} \quad \begin{array}{c}\mathcal{IH}\\ \Delta \Rightarrow \ominus A\end{array}}{\Gamma, \Delta \Rightarrow \alpha}\,Rej^{\lightning}$$

This is possible thanks to *Smileian reductio* being generalisable to arbitrary formulae, as we will prove later in the chapter. Furthermore, the structural rules can be translated the canonical way: *weakening* corresponds to vacuous discharge, *contraction* corresponds to multiple assumptions involving the same formula, and cut to a detour via an introduction and elimination of an arbitrary rule. This concludes the induction, and therefore we obtain that if $\mathsf{N}_{\mathsf{BML}} \vdash \Gamma \Rightarrow \alpha$, then $\mathsf{G2BML} + \text{Cut} \vdash \Gamma \Rightarrow \alpha$. □

Theorem 2.2.2 (Soundness)**.** *If* $\mathsf{G2BML} \vdash \Gamma \Rightarrow \alpha$*, then* $\mathsf{N}_{\mathsf{BML}} \vdash \Gamma \Rightarrow \alpha$*.*

Proof. By induction on the depth of the sequent calculus tree. The base cases are the axiom $\overline{\Gamma, \mathbf{s}A \Rightarrow \mathbf{s}A}^{Ax}$ for an arbitrary $\mathbf{s} \in \{+, -, \ominus\}$, which we translate as the one-node derivation $\overline{\mathbf{s}A}$, and the initial sequent of incoherence, which we translate as explosion applied to atoms. For the induction step, we consider several cases corresponding to the signed connectives of the language, mapping in the canonical way right and left rules to introduction and elimination rules respectively. Where possible, we will make use of schematic notation for signs and connectives, using the polarity change maps defined in Section 1.1.4. Since the rules of the asserted fragment of BML, with the exception of negation have exactly the same shape of those of the intuitionistic propositional calculus, we leave the assertive cases of $\wedge, \vee, \supset$ to the reader (a reference for which can be found in (Troelstra and Schwichtenberg, 2000)).

As for the other direction we present representative cases of negation without loss of generality.

Negation

$$\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma \Rightarrow -A\end{matrix}}{\Gamma \Rightarrow +\neg A}\,(+\neg R) \qquad\Longrightarrow\qquad \dfrac{\begin{matrix}[\Gamma]\\ \mathcal{IH}\\ -A\end{matrix}}{+\neg A}\,(+\neg I)$$

$$\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma, -A \Rightarrow \alpha\end{matrix}}{\Gamma, +\neg A \Rightarrow \alpha}\,(\mathbf{s}\neg L) \qquad\Longrightarrow\qquad \begin{matrix}[\mathcal{IH}]\\ \dfrac{+\neg A}{[-A]}\,+\neg E\\ \mathcal{D}'\\ \alpha\end{matrix}$$

$$\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma \Rightarrow \ominus A\end{matrix}}{\Gamma \Rightarrow +\neg A}\,(+\neg R) \qquad\Longrightarrow\qquad \dfrac{\begin{matrix}[\Gamma]\\ \mathcal{IH}\\ \ominus A\end{matrix}}{+\neg A}\,(+\neg I)$$

$$(*)\,\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma, \ominus A \Rightarrow \alpha\end{matrix}}{\Gamma, +\neg A \Rightarrow \alpha}\,(+\neg L) \qquad\Longrightarrow\qquad \begin{matrix}[\mathcal{IH}]\\ \dfrac{+\neg A}{[\ominus A]}\,+\neg E\\ \mathcal{D}^*\\ \alpha\end{matrix}$$

Conjunction

$$\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma \Rightarrow -A_i\end{matrix}}{\Gamma \Rightarrow -A_1 \wedge A_2}\,(- \wedge R_i) \qquad\Longrightarrow\qquad \dfrac{\begin{matrix}[\Gamma]\\ \mathcal{IH}\\ -A_i\end{matrix}}{-A_1 \wedge A_2}\,(- \wedge I_i)$$

$$(*)\,\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma, -A \Rightarrow \alpha\end{matrix} \qquad \begin{matrix}\mathcal{D}_1\\ \Gamma, -B \Rightarrow \alpha\end{matrix}}{\Gamma, -A \wedge B \Rightarrow \alpha}\,(-\wedge L) \qquad\Longrightarrow\qquad \dfrac{\begin{matrix}\mathcal{IH}\\ -A \wedge B\end{matrix} \qquad \begin{matrix}[\Gamma, -A]\\ \mathcal{IH}*\\ \alpha\end{matrix} \qquad \begin{matrix}[\Gamma, -B]\\ \mathcal{IH}*\\ \alpha\end{matrix}}{\alpha}\,(-\wedge E)$$

Disjunction

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma \Rightarrow -A\end{matrix}\qquad \begin{matrix}\mathcal{D}_1\\ \Gamma\Rightarrow -B\end{matrix}}{\Gamma\Rightarrow -A\vee B}\,(-\vee R) \quad\Longrightarrow\quad \dfrac{\begin{matrix}[\Gamma]\\ \mathcal{IH}\\ -A\end{matrix}\qquad\begin{matrix}[\Gamma]\\ \mathcal{IH}\\ -B\end{matrix}}{-A\vee B}\,(-\vee I)$$

$$\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma, -A_i \Rightarrow \alpha\end{matrix}}{\Gamma, -A_1\vee A_2\Rightarrow \alpha}\,(-\vee L_i) \quad\Longrightarrow\quad \begin{matrix}\dfrac{\begin{matrix}[\mathcal{IH}]\\ -A_1\vee A_2\end{matrix}}{[-A_i]}\,(-\vee E_i)\\ \mathcal{D}'\\ \alpha\end{matrix}$$

Implication

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma \Rightarrow +A\end{matrix}\qquad \begin{matrix}\mathcal{D}_1\\ \Gamma\Rightarrow -B\end{matrix}}{\Gamma\Rightarrow -A\supset B}\,(-\supset R) \quad\Longrightarrow\quad \dfrac{\begin{matrix}[\Gamma]\\ \mathcal{IH}\\ +A\end{matrix}\qquad\begin{matrix}[\Gamma]\\ \mathcal{IH}\\ -B\end{matrix}}{-A\supset B}\,(-\supset I)$$

Without loss of generality, we check the case of $-\supset L_1 \Rightarrow -\supset E_1$, the other rule being symmetric.

$$\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma, +A \Rightarrow \alpha\end{matrix}}{\Gamma, -A\supset B\Rightarrow \alpha}\,(-\supset L_1) \quad\Longrightarrow\quad \begin{matrix}\dfrac{\begin{matrix}[\mathcal{IH}]\\ -A\supset B\end{matrix}}{[+A]}\,(-\supset E_1)\\ \mathcal{D}'\\ \alpha\end{matrix}$$

We now check the cases of the weakly rejected $\wedge, \vee, \supset$

Conjunction

$$\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma \Rightarrow \ominus A_i\end{matrix}}{\Gamma\Rightarrow \ominus A_1\wedge A_2}\,(\ominus\wedge R_i) \quad\Longrightarrow\quad \dfrac{\begin{matrix}[\Gamma]\\ \mathcal{IH}\\ \ominus A_i\end{matrix}}{\ominus A_1\wedge A_2}\,(\ominus\wedge I_i)$$

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma, \ominus A \Rightarrow \alpha\end{matrix}\qquad \begin{matrix}\mathcal{D}_1\\ \Gamma, \ominus B\Rightarrow \alpha\end{matrix}}{\Gamma, \ominus A\wedge B\Rightarrow \alpha}\,(\ominus\wedge L) \quad\Longrightarrow\quad \dfrac{\begin{matrix}\mathcal{IH}\\ \ominus A\wedge B\end{matrix}\qquad\begin{matrix}[\Gamma, \ominus A]\\ \mathcal{IH}\\ \alpha\end{matrix}\qquad\begin{matrix}[\Gamma, \ominus B]\\ \mathcal{IH}\\ \alpha\end{matrix}}{\alpha}\,(\ominus\wedge E)$$

Disjunction

$$(*)\,\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma \Rightarrow \ominus A\end{matrix}\qquad \begin{matrix}\mathcal{D}_1\\ \Gamma\Rightarrow \ominus B\end{matrix}}{\Gamma\Rightarrow \ominus A\vee B}\,(\ominus\vee R) \quad\Longrightarrow\quad \dfrac{\begin{matrix}[\Gamma]\\ \mathcal{IH}*\\ \ominus A\end{matrix}\qquad\begin{matrix}[\Gamma]\\ \mathcal{IH}*\\ \ominus B\end{matrix}}{\ominus A\vee B}\,(\ominus\vee I)$$

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma, \ominus A_i \Rightarrow \alpha\end{array}}{\Gamma, \ominus A_1 \vee A_2 \Rightarrow \alpha}\,(\ominus \vee L_i) \quad\Longrightarrow\quad \begin{array}{c}[\mathcal{IH}]\\ \dfrac{\ominus A_1 \vee A_2}{[\ominus A_i]}\,(\ominus\vee E_i)\\ \mathcal{D}'\\ \alpha\end{array}$$

Implication

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow +A\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \Gamma \Rightarrow \ominus B\end{array}}{\Gamma \Rightarrow \ominus A \supset B}\,(\ominus\supset R) \quad\Longrightarrow\quad \dfrac{\begin{array}{c}[\Gamma]\\ \mathcal{IH}\\ +A\end{array} \qquad \begin{array}{c}[\Gamma]\\ \mathcal{IH}\\ \ominus B\end{array}}{\ominus A \supset B}\,(\ominus \supset I)$$

Without loss of generality, we check the case of $\ominus \supset L_1 \Rightarrow \ominus \supset E_1$, the other rule being symmetric with the only difference that in this case restrictions on the premises are required.

$$(*)\,\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma, +A \Rightarrow \alpha\end{array}}{\Gamma, \ominus A \supset B \Rightarrow \alpha}\,(\ominus \supset L_1) \quad\Longrightarrow\quad \begin{array}{c}[\mathcal{IH}*]\\ \dfrac{\ominus A \supset B}{[+A]}\,(\ominus\supset E_1)\\ \mathcal{D}'\\ \alpha\end{array}$$

Coordination Principles

We show some cases without loss of generality, the other ones being symmetric.

$$(*)\,\dfrac{\Gamma, +p \Rightarrow \bot}{\Gamma \Rightarrow -p}\,SR^{\star}\text{-in} \quad\Longrightarrow\quad \begin{array}{c}[\Gamma, +p]\\ \mathcal{IH}*\\ \dfrac{\bot}{-p}\,(SR^{\star})\end{array}$$

$$(*)\,\dfrac{\Gamma \Rightarrow -p}{\Gamma, +p \Rightarrow \bot}\,SR^{\star} - out \quad\Longrightarrow\quad \dfrac{\begin{array}{c}[\Gamma]\\ \mathcal{IH}\\ -p\end{array} \qquad +p}{\bot}\,(Rej)$$

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow +A\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \Delta \Rightarrow -A\end{array}}{\Gamma, \Delta \Rightarrow \bot}\,Rej^{\sim} \quad\Longrightarrow\quad \dfrac{\begin{array}{c}[\Gamma]\\ \mathcal{IH}\\ +A\end{array} \qquad \begin{array}{c}[\Delta]\\ \mathcal{IH}\\ -A\end{array}}{\bot}\,Rej$$

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow +A\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \Delta \Rightarrow \ominus A\end{array}}{\Gamma, \Delta \Rightarrow \bot}\,Rej^{\lightning} \quad\Longrightarrow\quad \dfrac{\begin{array}{c}[\Gamma]\\ \mathcal{IH}\\ +A\end{array} \qquad \begin{array}{c}[\Delta]\\ \mathcal{IH}\\ \ominus A\end{array}}{\bot}\,Rej$$

Furthermore, the structural rules can be translated the canonical way: *weakening* corresponds to vacuous discharge, *contraction* corresponds to multiple assumptions involving the same formula, and cut to a detour via an introduction and elimination of an arbitrary rule.

□

Corollary 2.2.1. $\mathsf{G2BML} + Cut \vdash \Gamma \Rightarrow \alpha$ *iff* $\mathsf{N_{BML}} \vdash \Gamma \Rightarrow \alpha$.

Corollary 2.2.2. $\Gamma \vdash^{\mathsf{G2BML}+Cut} +A$ *iff* $\Gamma \vdash^{\mathsf{CPL}} A$.

Corollary 2.2.3. $\mathsf{G2BML} + Cut$ *is sound and complete w.r.t. to ω-pointed models (Incurvati and Schlöder, 2023b)*

The second corollary follows from the classicality result of (Incurvati and Schlöder, 2017), which showed classicality of the natural deduction system of BML with respect to the inferences of the logic. Notice that however this need not apply to the meta-inference level, as shown in (Incurvati and Schlöder, 2023b; Kortenbach, Incurvati, and Schlöder, 2025). The third one, on the other hand, follows from soundness and completeness of $\mathsf{N_{BML}}$ with respect to so-called ω-pointed models, proven in (Incurvati and Schlöder, 2017). We now define the following notion, which will be used throughout as induction measure.

Definition 6 (Weight of $\mathcal{L}_{\mathsf{BML}}$ formula)**.** *Weight of formulae of* BML *is defined the canonical way as follows, for* $\mathbf{a}, \mathbf{b}, \mathbf{s} \in \{{+/-}, {+/\ominus}\}$ *and "$\circ$" an arbitrary binary connective:*

$$\begin{aligned}
w(\bot) &:= 0\\
w(\ominus p) &:= 1\\
w(\mathbf{a}p) &:= 2 \text{ for } \mathbf{a} \in \{+, -\}\\
w(\mathbf{s}\neg A) &:= w(\mathbf{s}^\star A) + 1, \text{ for } \star \in \{\sim, \lightning\}\\
w(\mathbf{s}A \circ B) &:= w(\mathbf{a}A) + w(\mathbf{b}B) + 1
\end{aligned}$$

Theorem 2.2.3 (Generalised initial sequents)**.** $\Gamma, \alpha \Rightarrow \alpha$ *and* $\Gamma, -A \Rightarrow \ominus A$ *are derivable for α an arbitrary signed formula and Γ an arbitrary context of signed formulae.*

Proof. By induction on formula weight. The proof is the same as the one for intuitionistic logic in (Negri and von Plato, 2001), with more cases corresponding to different speech act markers. However, such cases are analogous to the dual connective in assertive form. We therefore only consider two representative cases unique to the multilateral setting of BML.

$\alpha = \mathbf{s}\neg A$. We have that $w(\mathbf{s}^\star A) \leq w(\mathbf{s}\neg A) \leq n$, for $\star \in \{\sim, \lightning\}$. The left label $(**)$ denotes possible restrictions in case $\mathbf{s}\neg L \equiv \ominus\neg L$. Then,

$$\cfrac{\cfrac{\overset{\mathcal{IH}}{\mathbf{s}^{\star}A,\Gamma \Rightarrow \mathbf{s}^{\star}A}}{\mathbf{s}\neg A,\Gamma \Rightarrow \mathbf{s}^{\star}A}\ \mathbf{s}\neg L}{\mathbf{s}\neg A,\Gamma \Rightarrow \mathbf{s}\neg A}\ (\mathbf{s}\neg R) \quad (**)$$

$\alpha = {-/\ominus}A \supset B$. We have that $w(+A) \leq w({-/\ominus}A \supset B) \leq n$ and $w({-/\ominus}B) \leq w({-/\ominus}A \supset B) \leq n$. Then,

$$\cfrac{\cfrac{\overset{\mathcal{IH}}{+A,\Gamma \Rightarrow +A}}{{-/\ominus}A \supset B,\Gamma \Rightarrow +A}\ {-/\ominus}\supset L \qquad \cfrac{\overset{\mathcal{IH}}{{-/\ominus}B,\Gamma \Rightarrow {-/\ominus}B}}{{-/\ominus}A \supset B,\Gamma \Rightarrow {-/\ominus}B}\ {-/\ominus}\supset L}{{-/\ominus}A \supset B,\Gamma \Rightarrow {-/\ominus}A \supset B}\ {-/\ominus}\supset R$$

We conclude by considering a representative case of multilateral initial sequent generalisation.

$-A = -\neg B$. We have, for $\mathbf{a} \in \{-,\ominus\}$ that $w(\mathbf{a}B) \leq w(\mathbf{a}\neg B) \leq n$. Then, for Γ arbitrary, we get the following, without loss of generality:

$$\cfrac{\cfrac{\overset{\mathcal{IH}}{+B,\Gamma \Rightarrow +B}}{-\neg B,\Gamma \Rightarrow +B}\ (-\neg L)}{-\neg B,\Gamma \Rightarrow \ominus\neg B}\ (\ominus\neg R)$$

This completes our induction. Therefore, $\Gamma,\alpha \Rightarrow \alpha$ and $\Gamma,-A \Rightarrow \ominus A$ are derivable in BML for α an arbitrary signed formula and Γ an arbitrary context of signed formulae.

□

2.3 Conceptions of Harmony and Admissibility results

In this section we present admissibility results of a number of rules and coordination principles, in the context of the discourse of proof-theoretic harmony. We endorse a broadly Dummettian background conception of harmony, according to which harmony should provide an explanation of why rules are justified as meaning-conferring (cf. Dummett, 1991).

Theorem 2.3.1 (Height-preserving Weakening)**.** *Let $\vdash_n \Gamma \Rightarrow \alpha$ denote a derivation of height n of the sequent $\Gamma \Rightarrow \alpha$. If $\vdash_n \Gamma \Rightarrow \alpha$, then $\vdash_n \Gamma,\gamma \Rightarrow \alpha$.*

Proof. By induction on n. The base case is the cas in which $\Gamma \Rightarrow \alpha$ is an axiom, and thus α is and atom and $\Gamma,\gamma \Rightarrow \alpha$ is an axiom as well, with $\gamma = \alpha$. For the inductive step, we assume by *induction hypothesis* that $\vdash_m \Gamma \Rightarrow \alpha \implies \vdash_m \Gamma,\gamma \Rightarrow \alpha$, and we suppose $\vdash_{m+1} \Gamma \Rightarrow \alpha$. We need to consider all the schematic rule cases corresponding to the possible shape of α, but since the proof will go through in the same way for all cases, we pick a representative one to illustrate the proof method. Suppose we have the following derivation of length $m+1$:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \Rightarrow -A\end{array}}{\Gamma \Rightarrow +\neg A}\ (+\neg R)$$

Then we have $\vdash_m \Gamma \Rightarrow -A$. By IH, then, we obtain $\vdash_m \Gamma, \gamma \Rightarrow -A$. Hence, by applying the rule, this gives us $\vdash_{m+1} \Gamma, \gamma \Rightarrow +\neg A$. The reader can easily verify that this method works for all other cases. Notice that for two-premiss rules the IH needs to be applied twice.

□

2.3.1 Cut Admissibility and the Coordination Principles

To prove Cut admissibility for G2BML we require a lexicographic induction for which we define the following induction measure the obvious way, according to (Negri and von Plato, 2001).

Definition 7 (Cut height)**.** *Let the height of a derivation $\mathcal{D}$ be the length of its longest branch minus one. Then the cut height is the sum of the lengths of the derivations of the premisses of Cut.*

Theorem 2.3.2. (Cut admissibility). *The rule of cut*

$$\dfrac{\Gamma \Rightarrow \alpha \qquad \alpha, \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \beta}\ Cut$$

Is admissible in G2BML.

Proof. We proceed by lexicographic induction on the weight of the cut formula and cut height. The strategy is the standard one for cut-admissibility proofs (Negri and von Plato, 2001), and consists of permuting applications of *cut* higher-up in the proof tree, so that both cut height and the weight of the cut formula are reduced. If this can be done inductively, then we can permute *cut* away until we reach an axiom, obtaining a cut-free proof of the same sequent.
The first base case we consider consists of an application of cut with an axiom as premiss. We distinguish a number of subcases, namely when the axiom is the left premiss and when the axiom is in the right premiss. If the axiom is in the left premiss, we then have $\Gamma \Rightarrow \alpha \equiv \Gamma', \alpha \Rightarrow \alpha$ and thus the right premiss becomes $\Gamma', \alpha, \alpha \Rightarrow \beta$, which by contraction yields $\Gamma', \alpha \Rightarrow \beta \equiv \Gamma \Rightarrow \beta$. If the axiom occurs in the right premiss we have two sub-cases. If $\alpha = \beta$, then the left premiss is $\Gamma \Rightarrow \beta$ and so the conclusion follows immediately. If on the other hand $\beta \in \Gamma$, then the conclusion $\Gamma \Rightarrow \beta$ consists of a weakening of the axiom $\Delta, \beta \Rightarrow \beta$ for some $\Delta \subseteq \Gamma$. Moreover, suppose we have the following instance:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta \Rightarrow -p\end{array} \qquad \dfrac{}{-p, \Gamma \Rightarrow \ominus p}\,Ax}{\Gamma, \Delta \Rightarrow \ominus p}\ cut$$

If $-p$ is not principal, then there is a rule R such that $\dfrac{\begin{matrix}\mathcal{D}' \\ \Delta' \Rightarrow -p\end{matrix}}{\Delta \Rightarrow -p}\,R$. Therefore, we can obtain the conclusion by an application of R, obtaining the following derivation of lower cut height:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}' \\ \Delta' \Rightarrow -p\end{matrix} \qquad \dfrac{}{-p, \Gamma' \Rightarrow \ominus p}\,Ax}{\Gamma, \Delta' \Rightarrow \ominus p}\,cut}{\Gamma, \Delta \Rightarrow \ominus p}\,R$$

If $\Delta \Rightarrow -p$ is obtained by $SR^{\star}$-in, *cut* can be eliminated entirely as follows:

$$\dfrac{(*)\,\dfrac{\begin{matrix}\mathcal{D} \\ \Delta, +p \Rightarrow \bot\end{matrix}}{\Delta \Rightarrow -p}\,SR^{\star}\text{-in} \qquad \dfrac{}{-p, \Gamma \Rightarrow \ominus p}\,Ax}{\Gamma, \Delta \Rightarrow \ominus p}\,cut \qquad \Longrightarrow \qquad \dfrac{\dfrac{\begin{matrix}\mathcal{D} \\ \Delta, +p \Rightarrow \bot\end{matrix}}{\Delta \Rightarrow \ominus p}\,SR\text{-in}}{\Gamma, \Delta \Rightarrow \ominus p}\,Wk$$

We also consider cases in which either premiss is an instance of the *Incoherence* initial sequent. If *Incoherence* occurs as a left premiss, then the conclusion of a putative application of *cut* follows by $\bot_L$. If it occurs on the right, on the other hand, if the left premiss is an axiom we get that the conclusion of cut is still an instance of *Incoherence*.

Further, we have cases concerning *Smileian reductio* and the initial sequents of incoherence, $\Gamma, +p, -p \Rightarrow \bot$ and $\Gamma, +p, \ominus p \Rightarrow \bot$. Suppose we have the following derivation, without loss of generality:

$$\dfrac{(*)\,\dfrac{\begin{matrix}\mathcal{D}_0 \\ \Gamma, +p \Rightarrow \bot\end{matrix}}{\Gamma \Rightarrow -p}\,SR^{\star}\text{-in} \qquad \dfrac{}{\Delta, +p, -p \Rightarrow \bot}\,Inc.}{\Gamma, +p, \Delta \Rightarrow \bot}\,cut$$

This can be replaced by obtaining the conclusion using an application of $SR^{\star}$-out and (possibly repeated) weakening:

$$\dfrac{(*)\,\dfrac{\begin{matrix}\mathcal{D} \\ \Gamma \Rightarrow -p\end{matrix}}{+p, \Gamma \Rightarrow \bot}\,SR^{\star}\text{-out}}{+p, \Gamma, \Delta \Rightarrow \bot}\,Wk$$

Otherwise, suppose we have applications of $SR^{\star}$ on both premises (the case of SR is analogous, modulo restrictions on the contexts):

$$\dfrac{(*)\,\dfrac{\begin{matrix}\mathcal{D}_0 \\ \Gamma, +p \Rightarrow \bot\end{matrix}}{\Gamma \Rightarrow -p}\,SR^{\star}\text{-in} \qquad (*)\,\dfrac{\begin{matrix}\mathcal{D}_1 \\ \Delta \Rightarrow +p\end{matrix}}{-p, \Delta \Rightarrow \bot}\,SR^{\star}\text{-out}}{\Gamma, \Delta \Rightarrow \bot}\,cut$$

In this case, we can reduce cut height as follows:

$$\dfrac{\begin{matrix}\mathcal{D}_2\\ \Delta \Rightarrow +p\end{matrix} \qquad \begin{matrix}\mathcal{D}_0\\ \Gamma, +p \Rightarrow \bot\end{matrix}}{\Gamma, \Delta \Rightarrow \bot}\ cut$$

If, on the other hand, we have restrictions on cut only on one side, consider the representative case below, where the derivation is of cuth height $m + n + 1 + 1$:

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ \dfrac{\Gamma, \ominus p \Rightarrow \bot}{\Gamma \Rightarrow +p}\ SR\text{-}in\end{matrix} \quad (*)\ \begin{matrix}\mathcal{D}_1\\ \dfrac{\Delta \Rightarrow -p}{+p, \Delta \Rightarrow \bot}\ SR^\star - out\end{matrix}}{\Gamma, \Delta \Rightarrow \bot}\ cut$$

Here we cannot immediately apply cut to the premises as in the previous case, but we can nonetheless reduce cut height to $m + n + 1$ this way (notice that the newly introduced cut can be reduced according to one of the previous base cases):

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \Delta \Rightarrow -p\end{matrix} \qquad \dfrac{}{-p, \Delta \Rightarrow \ominus p}\ Ax}{\Delta \Rightarrow \ominus p}\ cut \qquad \begin{matrix}D_0\\ \ominus p, \Gamma \Rightarrow \bot\end{matrix}}{\Gamma, \Delta \Rightarrow \bot}\ cut$$

For the induction step we consider some subcases according to whether or not the cut formula α is principal in either premiss of *cut*. We consider some representative schematic cases

The cut formula α is not principal in the left premiss, with $\mathbf{a}, \mathbf{b}, \mathbf{s} \in \{+/-, +/\ominus\}$, $\star \in \{\sim, ↯\}$

<u>$\mathbf{s}\neg L$, with $\Gamma = \Gamma', \mathbf{s}\neg A$.</u> We then have the following derivation with cut height $n + m + 1$:

$$\dfrac{\begin{matrix}\mathcal{D}_0\\ \mathbf{s}\neg L\ \dfrac{\Gamma', \mathbf{s}^\star A \Rightarrow \alpha}{\Gamma', \mathbf{s}\neg A \Rightarrow \alpha}\ (**)\end{matrix} \qquad \begin{matrix}\mathcal{D}_1\\ \alpha, \Delta \Rightarrow \beta\end{matrix}}{\Delta, \Gamma', \mathbf{s}\neg A \Rightarrow \beta}\ Cut$$

Which becomes the following, where cut is applied to formulae of lower weight and is of cut height $n + m$:

$$(**)\ \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma', \mathbf{s}^\star A \Rightarrow \alpha\end{array} \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \alpha, \Delta \Rightarrow \beta\end{array}}{\alpha, \Delta, \Gamma', \mathbf{s}^\star A \Rightarrow \beta}\,Wk}{\Delta, \Gamma', \mathbf{s}^\star A \Rightarrow \beta}\,Cut}{\Delta, \Gamma', \mathbf{s}\neg A \Rightarrow \beta}\,\mathbf{s}\neg L$$

Where $(**)$ denotes that restrictions would be needed in the premises if $\mathbf{s} = \ominus$.

$\underline{\mathbf{s} \circ L \text{ with } \Gamma = \Gamma', \mathbf{s}A \circ B}$. We then have the following derivation with cut height $\max(m, n) + k + 1 + 1$:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma', \mathbf{a}A \Rightarrow \alpha\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \Gamma', \mathbf{b}B \Rightarrow \alpha\end{array}}{\Gamma', \mathbf{s}A \circ B \Rightarrow \alpha}\,\mathbf{s} \circ L \qquad \begin{array}{c}\mathcal{D}_2\\ \alpha, \Delta \Rightarrow \beta\end{array}}{\Delta, \Gamma', \mathbf{s}A \circ B \Rightarrow \beta}\,Cut$$

Which then becomes the following, with lower cut height and lower weight of cut formula:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma', \mathbf{a}A \Rightarrow \alpha\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \alpha, \Delta \Rightarrow \beta\end{array}}{\Delta, \Gamma', \mathbf{a}A \Rightarrow \beta}\,Cut \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma', \mathbf{b}B \Rightarrow \beta\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \alpha, \Delta \Rightarrow \beta\end{array}}{\Delta, \Gamma', \mathbf{b}B \Rightarrow \beta}\,Cut}{\Delta, \Gamma', \mathbf{s}A \circ B \Rightarrow \beta}\,\mathbf{s} \circ L$$

$\underline{\mathbf{s}^\star \circ L \text{ with } \Gamma = \Gamma', \mathbf{s}^\star A \circ B}$. We then have the following derivation with cut height $m + n + 1$ without loss of generality:

$$\dfrac{\mathbf{s}^\star \circ L_1\ \dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma', \mathbf{a}^\star A \Rightarrow \alpha\end{array}}{\Gamma', \mathbf{s}^\star A \circ B \Rightarrow \alpha}\,(**) \qquad \begin{array}{c}\mathcal{D}_1\\ \alpha, \Delta \Rightarrow \beta\end{array}}{\Delta, \Gamma', \mathbf{s}^\star A \circ B \Rightarrow \beta}\,Cut$$

Which becomes the following, where cut is applied to formulae of lower weight and is of cut height $n + m$:

$$(**)\ \dfrac{\dfrac{\begin{array}{c}\mathcal{IH}\\ \Gamma', \mathbf{a}^\star A \Rightarrow \alpha\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \alpha, \Delta \Rightarrow \beta\end{array}}{\Delta, \Gamma', \mathbf{a}^\star A \Rightarrow \beta}\,Cut}{\Delta, \Gamma', \mathbf{s}^\star A \circ B \Rightarrow \beta}\,\mathbf{s}^\star \circ L_1$$

Where $(**)$ denotes that restrictions would be needed in the premises if $\mathbf{s} = \ominus$. The case of $+ \supset L$ is analogous to that of intuitionistic calculus G2i and is therefore left to the reader. Moreover, as the reader may easily verify, the case in which the cut formula α is only principal in the left premiss is symmetric to the above.

The cut formula α is principal in both premises, with $\mathbf{a}, \mathbf{b}, \mathbf{s} \in \{{+/-}, {+/\ominus}\}$, $\star \in \{\sim, \lightning\}$

$\underline{\alpha = \mathbf{s}\neg A}$. We then have the following derivation with cut height $m + n + 1$:

$$\dfrac{(**)\ \dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow \mathbf{s}^\star A\end{array}}{\Gamma \Rightarrow \mathbf{s}\neg A}\,\mathbf{s}\neg R \qquad (**)\ \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta, \mathbf{s}^\star A \Rightarrow \beta\end{array}}{\Delta, \mathbf{s}\neg A \Rightarrow \beta}\,\mathbf{s}\neg L}{\Delta, \Gamma \Rightarrow \beta}\,Cut$$

Which becomes the following, with both cut height and weight reduced:

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow \mathbf{s}^\star A\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \Delta, \mathbf{s}^\star A \Rightarrow \beta\end{array}}{\Gamma, \Delta \Rightarrow \beta}\,Cut$$

$\underline{\alpha = \mathbf{s}A \circ B}$. We then have the following derivation with cut height $m + \max(n + k) + 1 + 1$, without loss of generality:

$$\dfrac{\mathbf{s} \circ R_1\,\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow \mathbf{a}A\end{array}}{\Gamma \Rightarrow \mathbf{s}A \circ B} \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta, \mathbf{a}A \Rightarrow \beta\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Delta, \mathbf{b}B \Rightarrow \beta\end{array}}{\mathbf{s}A \circ B, \Delta \Rightarrow \beta}\,\mathbf{s} \circ L}{\Delta, \Gamma \Rightarrow \beta}\,Cut$$

Which becomes the following, with both cut height and weight reduced:

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow \mathbf{a}A\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \Delta, \mathbf{a}A \Rightarrow \beta\end{array}}{\Delta, \Gamma \Rightarrow \beta}\,Cut$$

$\underline{\alpha = \mathbf{s}^\star A \circ B}$. We then have the following derivation with cut height $\max(m + n) + k + 1 + 1$, without loss of generality:

$$\dfrac{\mathbf{s}^\star \circ R\,\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow \mathbf{a}^\star A\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \Gamma \Rightarrow \mathbf{b}^\star B\end{array}}{\Gamma \Rightarrow \mathbf{s}^\star A \circ B} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \Delta, \mathbf{a}^\star A \Rightarrow \beta\end{array}}{\Delta, \mathbf{s}^\star A \circ B \Rightarrow \beta}\,\mathbf{s}^\star \circ L_1}{\Delta, \Gamma \Rightarrow \beta}\,Cut$$

Which becomes the following, with both cut height and weight reduced:

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow \mathbf{a}^\star A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \Delta, \mathbf{a}^\star A \Rightarrow \beta\end{array}}{\Delta, \Gamma \Rightarrow \beta}\,Cut$$

This step concludes our induction. Therefore, the rule of *Cut* is admissible in $\mathsf{G2BML}$.

□

Cut admissibility allows us to derive the following corollaries:

Corollary 2.3.1. $\mathsf{G2BML} \vdash \Gamma \Rightarrow \alpha$ *iff* $\mathsf{N_{BML}} \vdash \Gamma \Rightarrow \alpha$.

Corollary 2.3.2. $\Gamma \vdash^{\mathsf{G2BML}} +A$ *iff* $\Gamma \vdash^{\mathsf{CPL}} A$.

Corollary 2.3.3. $\mathsf{G2BML}$ *is sound and complete wrt* ω*-pointed models*

Corollary 2.3.4. $\mathsf{N_{BML}}$ *has the normalisation property.*

This means that classicality and soundness and completeness (both with respect to $\mathsf{N_{BML}}$ and ω-pointed models) are cut-free, and that $\mathsf{N_{BML}}$ has the normalisation property. Moreover, and most importantly, this implies that, for all signs, the rules are conservative with respect to the calculus, as no new information arises were it not already present, in line with the Dummettian conception of harmony outlined in (Dummett, 1991; Steinberger, 2011a). Therefore, we have a suitable reduction procedure to enforce harmony for all speech acts of BML, which, jointly with the expansion procedure represented by de derivability of generalised initial sequents of the form $\Gamma, \alpha \Rightarrow \alpha$ and $\Gamma, -A \Rightarrow \ominus A$, yields that all the rules of $\mathsf{G2BML}$ are unilaterally harmonious. However, when dealing with more than one speech act, reductions and expansions thus defined are not enough to preserve harmony. In the context of unilateral harmony, we saw what the problematic nature of *tonk* and *tunk* amounts to. When expanding to a bilateral context, however, unilateral harmony, i.e., harmony between introduction and elimination rules for a fixed speech act, is not enough to safeguard global harmony. Indeed, there is a prolific literature on connectives that cause either *tonk*ish or *tunk*ish behaviour purely by applying introduction *or* elimination rules and the coordination principles (see for example Francez, 2014; Gabbay, 2017; del Valle-Inclan and Schlöder, 2023; Simonelli, 2025). An example, borrowed from (del Valle-Inclan and Schlöder, 2023), is the connective bink, defined by the inference rules below:

$$\frac{-A \qquad +A}{+A\mathsf{bink}B}\,(+\mathsf{bink}I) \qquad \frac{+\mathsf{bink}A}{+A}\,(+\mathsf{bink}E_1) \qquad \frac{+\mathsf{bink}A}{-A}\,(+\mathsf{bink}E_2)$$

$$\frac{-A}{-\mathsf{bink}A}\,(-\mathsf{bink}I) \qquad \frac{-\mathsf{bink}A}{-A}\,(-\mathsf{bink}E)$$

Considering assertion and rejection individually, the rules for bink are clearly harmonious, as one is the inverse of the other and hence expansion and reduction are easy to obtain. However, in presence of the coordination principles, assuming $+\mathsf{bink}A$ generates triviality:

$$\dfrac{\dfrac{\dfrac{\dfrac{[+\mathsf{bink}A]^1}{+A}\,+\mathsf{bink}E_2 \qquad \dfrac{[+\mathsf{bink}A]^1}{-A}\,+\mathsf{bink}E_2}{\perp}\,Rej}{-\mathsf{bink}A}\,SR^{\star}}{-A}\,-\mathsf{bink}E$$

del Valle-Inclan and Schlöder, 2023 contend that in all disharmonious cases they examine, the problem is that the coordination principles are applied to complex formulae and cannot be applied to atomic ones, and they identify the culprit for this particular instance in *Smileian reductio*. Their solution is therefore to restrict its application only to signed atoms. They take this to be a solution to a problem of Rumfitt's conception of the coordination principles outlined in (Rumfitt, 2008). Let a coordination principle be *preserved* if, upon stipulation of its atomic rule, its instances of applications of the principle to arbitrary sentences are derivable. Rumfitt contends that while preservation of *rejection* is required for the connectives of different polarities to have a coherent meaning, *Smileian reductio* is not required for enforcing harmony, and is on the contrary a rule to enforce bilateral classicality (alongside with negation rules), and should thus be applicable to sentences of any complexity (del Valle-Inclan and Schlöder, 2023; Rumfitt, 2008). Schlöder and del Valle-Inclan, on the contrary, argue that Rumfitt has not provided an explanation of how rules of negation and *Smileian reductio* are in harmony with one another: if bilateralism is justified as a way to inferentially vindicate classical logic by laying down harmonious rules for a classically-behaving negation, the fact that *Smileian reductio* is also required for classicality means that a harmony criterion should encompass it as well (del Valle-Inclan and Schlöder, 2023). They accordingly take the case of bink to show that for the purpose of bilateral harmony, *Smileian reductio* should be preserved in the same way *rejection* must be.

We contend, however, in the context of our sequent calculus, that for bink to be problematic, the ineliminability of *rejection* suffices. A similar conclusion was reached by (Simonelli, 2024a), who argues that the criterion for bilateral harmony proposed by Schlöder and del Valle-Inclan is overtly restrictive. He puts forward a bilateral version of reduction and expansion which, in the sequent calculus he presents in (Simonelli, 2024a), corresponds respectively to the eliminability of *rejection* and the derivability of generalised initial sequents. His argument is that the eliminability of *rejection* constitutes a form of conservativity analogous to that of the eliminability of *cut*: no new inconsistency between speech acts should arise were it not already present in the assumptions. Indeed, while cut-elimination ensures conservativity by ruling out unbalanced introduction and elimination rules, rejection-elimination ensures that deriving a bilateral contradiction is not *too easy* (Simonelli, 2024a; Simonelli, 2025). While the eliminability of cut ensures conservativity of the connective rules, eliminability of rejection ensures that otherwise harmonious rules do not prove too much in the presence of coordination principles. This is therefore in line with one of the most widely-accepted tenets in the literature on harmony, namely that conservativity, enforced by reductions, is a necessary condition for connectives to be harmonious (cf. Dummett, 1991, pp. 217-218). Moreover, it has the advantage that it enforces the coordination principle of rejection to apply to simpler formulae without restricting its applicability to atoms, provided that we add incoherence $\overline{\Gamma, -/\ominus p, +p \Rightarrow \perp}^{\,Inc.}$ as an axiom (as it will be evident in the base case of the admissibility proof). Taking incoherence as an axiom rather than formulating rejection for atoms enforces the same principle, namely that is is incoherent to assert and (possibly weakly) reject a proposition is incoherent, while allowing for greater economy of the proof system and have a simple and elegant proof-method to enforce bilateral conservativity [3].

Indeed, the sequent setting allows us to use admissibility results as harmony criterion in a very natural way. As a sanity check, we present a sequent version of the bink connective, showing that indeed *rejection* is not eliminable in the derivation of triviality:

$$\frac{\Gamma \Rightarrow +A \qquad \Gamma \Rightarrow -A}{\Gamma \Rightarrow +\mathsf{bink}A}\,(+\mathsf{bink}R) \qquad \frac{\Gamma, +A \Rightarrow \alpha}{\Gamma, +\mathsf{bink}A \Rightarrow \alpha}\,(+\mathsf{bink}L_1) \qquad \frac{\Gamma, -A \Rightarrow \alpha}{\Gamma, +\mathsf{bink}A \Rightarrow \alpha}\,(+\mathsf{bink}L_2)$$

$$\frac{\Gamma \Rightarrow -A}{\Gamma \Rightarrow -\mathsf{bink}A}\,(-\mathsf{bink}R) \qquad \frac{\Gamma, -A \Rightarrow \alpha}{\Gamma, -\mathsf{bink}A \Rightarrow \alpha}\,-\mathsf{bink}L$$

bink is pathological as assuming it implies that arbitrary sentences are rejected because of the interaction between *rejection* and the right rule for −bink, and in the derivation we can see that rejection is indeed not eliminable:

$$(*)\,\cfrac{\cfrac{\cfrac{\mathcal{D}}{\Gamma \Rightarrow +\mathsf{bink}A}}{\Gamma, -A \Rightarrow +\mathsf{bink}A}\,Wk \qquad \cfrac{\cfrac{}{\Gamma, -A \Rightarrow -A}\,Init.}{\Gamma, -A \Rightarrow -\mathsf{bink}A}\,-\mathsf{bink}R}{\cfrac{\Gamma, -A \Rightarrow \bot}{\Gamma \Rightarrow +A}\,SR^{\star}\text{-in}}\,Rej^{\sim}$$

And we obtain an arbitrary signed formula by means of *Smileian reductio* not because *Smileian reductio* must be eliminable to ensure conservativity, but because non-conservative incoherence was already introduced by means of a non-eliminable application of rejection (highlighted in orange). Analogously, derivability of generalised initial sequents serves the same purpose as in the unilateral case (Simonelli, 2024a).

It is however important to pause and highlight the still non-trivial role of *Smileian reductio* in the preservation of harmony. Indeed, taking the rule as primitive for the purpose of classicality, means that it should not upset our cut-admissibility proof. Recall the relevant passage:

$$\cfrac{(*)\,\cfrac{\cfrac{\mathcal{D}_0}{\Gamma, +p \Rightarrow \bot}}{\Gamma \Rightarrow -p}\,SR^{\star}\text{-}in \qquad (*)\,\cfrac{\cfrac{\mathcal{D}_1}{\Delta \Rightarrow +p}}{-p, \Delta \Rightarrow \bot}\,SR^{\star}\text{-}out}{\Gamma, \Delta \Rightarrow \bot}\,cut \quad \Longrightarrow \quad \cfrac{\cfrac{\mathcal{D}_2}{\Delta \Rightarrow +p} \qquad \cfrac{\mathcal{D}_0}{\Gamma, +p \Rightarrow \bot}}{\Gamma, \Delta \Rightarrow \bot}\,cut$$

[3] An initial sequent of incoherence is also presented in (Simonelli, forthcoming), albeit for the different purpose of showing the commonalities between different brands of bilateralism. The need of the incoherence axiom for proof-theoretic harmony in a sequent setting is conversely a contribution of this thesis.

Were SR and $SR^\star$ defined differently, this may have not been necessarily the case. For example, we need to stipulate "-in" and "-out" versions of the rule. Moreover, consider the following equally plausible formulation:

$$(*)\ \dfrac{\Gamma, {+/-}^{\sim}p \Rightarrow \alpha \qquad \Gamma, {+/-}^{\sim}p \Rightarrow \alpha^{\sim}}{\Gamma \Rightarrow {+/-}p}\ SR^\star\text{-in} \qquad (*)\ \dfrac{\Gamma, \Rightarrow {+/-}^{\sim}p}{\Gamma, {+/-}^{\sim}p \Rightarrow \beta}\ SR^\star\text{-out}$$

$$\dfrac{\Gamma, {+/\ominus}^{\lightning}p \Rightarrow \alpha \qquad \Gamma, {+/\ominus}^{\lightning}p \Rightarrow \alpha^{\lightning}}{\Gamma \Rightarrow {+/\ominus}p}\ SR\text{-in} \qquad \dfrac{\Gamma, \Rightarrow {+/\ominus}^{\lightning}p}{\Gamma, {+/\ominus}p \Rightarrow \beta}\ SR\text{-out}$$

Where the *-in* rule emulates a *rejection-reductio* step while the *-out* rule emulates some version of *ex falso*. However, such formulation would break cut admissibility:

$$\dfrac{(*)\ \dfrac{\begin{matrix}\mathcal{D}_0 \\ \Gamma, +p \Rightarrow \alpha\end{matrix} \qquad \begin{matrix}\mathcal{D}_1 \\ \Gamma, +p \Rightarrow \alpha^{\sim}\end{matrix}}{\Gamma \Rightarrow -p}\ SR^\star - in \qquad (*)\ \dfrac{\begin{matrix}\mathcal{D}_2 \\ \Delta \Rightarrow +p\end{matrix}}{-p, \Delta \Rightarrow \beta}\ SR^\star - out}{\Gamma, \Delta \Rightarrow \beta}\ cut$$

As there is no way to reduce cut height for this derivation by using any inductive hypothesis (the candidate reduced cut premises are coloured, to show the impossibility of the reduction due to different conclusions). This non-example is a concrete example for the remark made in (del Valle-Inclan and Schlöder, 2023) about the necessity of enforcing harmony for *Smileian reductio*, as it makes clear how whether or not a system is harmonious is sensitive to the formulation of the rule. Indeed, laying down *Smileian reductio* to obtain full classicality, as Schlöder and del Valle-Inclan point out, imposes that harmony restrictions must apply to it as well. However, we hold alongside Simonelli that the conservativity argument presented above is convincing enough to adjust the conclusion of (del Valle-Inclan and Schlöder, 2023): the way *Smileian reductio* must be harmonious is not different from the way any connective rule must be, and the formal requirement for harmony to hold is to impose conservativity by means of admissibility of *cut* for unilateral and *rejection* for bilateral harmony.

We now conclude the discussion on the tension between harmony and (object-language) classicality surrounding *Smileian reductio* by showing that the generalisation of the *-in* and *-out* version of the rule to arbitrary formulae is admissible, satisfying the desideratum of (Rumfitt, 2008) while nonetheless having addressed the concerns of (del Valle-Inclan and Schlöder, 2023). As a preliminary, it is required to define the following induction measure:

Definition 8 (*Smileian reductio* height)**.** *Let the height of a derivation $\mathcal{D}$ be the length of its longest branch minus one. Then the* Smileian reductio *height is the length of the derivations of the premisses of $SR^{(\star)}$-in or $SR^{(\star)}$-out .*

Theorem 2.3.3 (Generalised *Smileian Reductio*)**.** *The following rules*

$$(*)\ \frac{\Gamma, {}_{+/-}{\sim}A \Rightarrow \bot}{\Gamma \Rightarrow {}_{+/-}A}\ SR^{\star}\text{-}in \qquad (*)\ \frac{\Gamma, \Rightarrow {}_{+/-}{\sim}A}{\Gamma, {}_{+/-}A \Rightarrow \bot}\ SR^{\star}\text{-}out$$

$$\frac{\Gamma, {}_{+/\ominus}{\not{}}A \Rightarrow \bot}{\Gamma \Rightarrow {}_{+/\ominus}A}\ SR\text{-}in \qquad \frac{\Gamma, \Rightarrow {}_{+/\ominus}{\not{}}A}{\Gamma, {}_{+/\ominus}A \Rightarrow \bot}\ SR\text{-}out$$

Are admissible in G2BML *for all formulae* $A \in \mathcal{L}_{\mathsf{CPL}}$.

Proof. We prove both results by induction on *Smileian reductio* height. In both *-in* and *-out* instances (with and without context restrictions), the base cases are trivial since the rule is primitive on atoms. For the inductive step we proceed similarly to *cut* admissibility, by showing that each application of Smileian reductio can be reduced to a lower *reductio* height by applying the rule to formulae of (possibly weakly) lower weight. We begin by considering some representative cases for SR-in, being the case of $SR^{\star}$-in analogous modulo context restrictions.

$\underline{{}_{+/\ominus}A = \ominus\neg B}$. then the derivation on the left can be transformed into the one on the right of lower *reductio* height:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma, -B \Rightarrow \bot\end{matrix}}{\Gamma, +\neg B \Rightarrow \bot}\ +\neg L}{\Gamma \Rightarrow \ominus\neg B}\ SR\text{-}in \quad \Longrightarrow \quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma, -B \Rightarrow \bot\end{matrix}}{\Gamma \Rightarrow +B}\ SR\text{-}in}{\Gamma \Rightarrow \ominus\neg B}\ \ominus\neg R$$

$\underline{{}_{+/\ominus}A = +B \wedge C}$. then the derivation on the left can be transformed into the one on the right of lower *reductio* height:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma, \ominus B \Rightarrow \bot\end{matrix} \quad \begin{matrix}\mathcal{D}_1\\ \Gamma, \ominus C \Rightarrow \bot\end{matrix}}{\Gamma, \ominus B \wedge C \Rightarrow \bot}\ \ominus\wedge L}{\Gamma \Rightarrow +B \wedge C}\ SR\text{-}in \quad \Longrightarrow \quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma, \ominus B \Rightarrow \bot\end{matrix}}{\Gamma \Rightarrow +B}\ SR\text{-}in \quad \dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma, \ominus C \Rightarrow \bot\end{matrix}}{\Gamma \Rightarrow +C}\ SR\text{-}in}{\Gamma \Rightarrow +B \wedge C}\ +\wedge R$$

The case of disjunction is symmetric.

$\underline{{}_{+/\ominus}A = +B \supset C}$. then the derivation on the left can be transformed into the one on the right of lower *reductio* height:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma, \ominus C \Rightarrow \bot\end{matrix}}{\Gamma, \ominus B \supset C \Rightarrow \bot}\ \ominus\supset L_2}{\Gamma \Rightarrow +B \supset C}\ SR\text{-}in \quad \Longrightarrow \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma, \ominus C \Rightarrow \bot\end{matrix}}{\Gamma \Rightarrow +C}\ SR\text{-}in}{\Gamma, +B \Rightarrow +C}\ Wk}{\Gamma \Rightarrow +B \supset C}\ +\supset R$$

$\underline{{}_{+/\ominus}A = \ominus B \supset C}$. then the derivation on the left can be transformed into the one on the right of lower *reductio* height:

$$
\dfrac{\dfrac{\overset{\mathcal{D}_0}{\Gamma \Rightarrow +B} \qquad \overset{\mathcal{D}_1}{\Gamma, +C \Rightarrow \bot}}{\Gamma, +B \supset C \Rightarrow \bot}\, + \supset L}{\Gamma \Rightarrow \ominus B \supset C}\, SR\text{-}in \quad \Longrightarrow \quad \dfrac{\overset{\mathcal{D}_0}{\Gamma \Rightarrow +B} \qquad \dfrac{\overset{\mathcal{D}_1}{\Gamma, +C \Rightarrow \bot}}{\Gamma \Rightarrow \ominus C}\, SR\text{-}in}{\Gamma \Rightarrow \ominus B \supset C}\, \ominus \supset R
$$

We now consider representative cases of SR-out, being the case of $SR^\star$-out analogous modulo context restrictions. The other cases are similar and therefore omitted.

$\underline{{}_{+/\ominus}A = +\neg B}$. then the derivation on the left can be transformed into the one on the right of lower *reductio* height:

$$
\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma \Rightarrow +B}}{\Gamma \Rightarrow \ominus\neg B}\, \ominus\neg R}{\Gamma, +\neg B \Rightarrow \bot}\, SR\text{-}out \quad \Longrightarrow \quad \dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma \Rightarrow +B}}{\ominus B, \Gamma \Rightarrow \bot}\, SR\text{-}out}{\Gamma, +\neg B \Rightarrow \bot}\, +\neg L
$$

$\underline{{}_{+/\ominus}A = +B \wedge C}$. then the derivation on the left can be transformed into the one on the right of lower *reductio* height:

$$
\dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma \Rightarrow \ominus C}}{\Gamma \Rightarrow \ominus B \wedge C}\, \ominus \wedge R}{\Gamma, +B \wedge C \Rightarrow \bot}\, SR\text{-}out \quad \Longrightarrow \quad \dfrac{\dfrac{\overset{\mathcal{D}}{\Gamma \Rightarrow \ominus C}}{\Gamma, +C \Rightarrow \bot}\, SR\text{-}out}{\Gamma, +B \wedge C \Rightarrow \bot}\, + \wedge L
$$

$\underline{{}_{+/\ominus}A = +B \supset C}$. then the derivation on the left can be transformed into the one on the right of lower *reductio* height:

$$
\dfrac{\dfrac{\overset{\mathcal{D}_0}{\Gamma \Rightarrow +B} \qquad \overset{\mathcal{D}_1}{\Gamma \Rightarrow \ominus C}}{\Gamma \Rightarrow \ominus B \supset C}\, \ominus \supset R}{\Gamma, +B \supset C \Rightarrow \bot}\, SR\text{-}out \quad \Longrightarrow \quad \dfrac{\overset{\mathcal{D}_0}{\Gamma \Rightarrow +B} \qquad \dfrac{\overset{\mathcal{D}_1}{\Gamma \Rightarrow \ominus C}}{\Gamma, +C \Rightarrow \bot}\, SR\text{-}out}{\Gamma, +B \supset C \Rightarrow \bot}\, + \supset L
$$

This concludes our induction, witnessing that the atomic rules for *Smileian reductio* can be generalised to arbitrary formulae.

□

From the fact that eliminability of *rejection* ensures a bilateral notion of conservativity of the calculus, (Simonelli, 2024a) claims that *rejection* constitutes a generalised bilateral version of *cut*. If we were to follow his insight, we would then settle the questions *(i) do the coordination principles constitute a bilateral generalisation of cut? (ii) Is such generalisation due to* SR *or* Rejection*?* by answering *yes* to *(i)* and *Rejection* to *(ii)*. We however contend that this conclusion is incorrect, and while it is undeniable that the objectives of proving the admissibility of *cut* and *rejection* both concern conservativity, the way the two notions are conflated in (Simonelli, 2024a) reflects a problematic interpretation of the relationship between unilateral and bilateral harmony. To properly spell out this criticism, before outlining Simonelli's proposal, we distinguish between what constitutes a *conception* of harmony and a *criterion* of harmony. By "conception", we mean a philosophical position that aims to explain, as described in (Dummett, 1991), how correct language use comes about, what aspects thereof should be in balance, and what constitutes such balance. A *criterion* of harmony, on the other hand, is a collection of formal constraints on the calculus such that the principles defended in the conception are respected. As we will show, Simonelli's proposal is successful as a criterion, but problematic as a conception, insofar as it rules out pathological connectives but nonetheless relies of a formulation of the rules that is unsatisfactory from a semantic viewpoint. The contribution of the present thesis to the issue will be a revision of his criterion in order to fit a more suitable conception of harmony, capable of explaining correct language use and to do so beyond the bilateral setting, including multilateralism as well.

In his formulation of harmonious rules in sequent form, Simonelli, 2024a chooses to present connective rules only in introduction form, on the grounds that, since Gentzen, it is orthodoxy to consider the introduction rules to be meaning-conferring, as they represent the assertibility conditions for a complex formula. This is indeed what Dummett calls *verificationist* principles (henceforth v-principles), as opposed to *pragmatist* principles (henceforth p-principles), which take the consequences one draws from a given connective, assuming its construction to be valid, to be the proper indicator of use, and hence meaning (Dummett, 1991). An alternative popular terminology in the inferentialist literature, which I will use interchangeably throughout, is respectively *categorical* and *hypothetical* reading of rules (Schroeder-Heister, 2012). I contend two points that challenge Simonelli's approach, namely *(i)* in a multilateral setting, the presence of weak rejections vindicates p-principles as more adequate meaning clauses, and *(ii)* regardless on one's approach to meaning, formulating rules and harmony conditions for introduction (resp., elimination) rules only is incomplete.

We begin by addressing (i). In unilateral proof theoretic semantics, the proof system *per se* does not dictate which between introduction and elimination rules should be taken to define meaning. However, as we saw in Chapter 1, the evidential unspecificity of rejection requires revision on the underlying conception of proof, since the presence of rejections that can be weak is incompatible in an *evidence*-preserving calculus if we want *Smileian reductio* to hold. This was taken by (Incurvati and Schlöder, 2023b) to motivate the distinction between strong and weak rejection, and hence adopt a multilateral *commitment*-preserving proof theory. However, the consequences of the resulting conception of proof for the choice between v-principles and p-principles as definitions for the connectives have not been further investigated. When valid inferences are taken to constrain our commitments to the ones that follow were we to commit to the premisses, we are following a hypothetical, consequentialist reading of the logical rules, which vindicates a pragmatist, rather than a verificationist, conception of the rules. Evidence-first proof theory prioritises expressibility conditions, favuoring a proof-theoretic semantics based on v-principles. On the other hand, commitment-first proof theory does not require evidence to be explicit, as we may have premisses that are weakly rejected on the basis of unknown evidence. Therefore, if we were to take v-principles as meaning conferring, we would do so with possibly unknown conditions for an utterance. This would be in clear contradiction with the idea underlying a categorical reading of the rules. Conversely, taking

p-principles as self-justifying clauses allows for unspecific expression conditions, and takes what we can do upon acceptance to be constitutive of the content of an attitude expression (cf. Dummett, 1991, p. 280). The presence of possibly weak rejections, therefore, vindicates a pragmatist meaning theory.

However, regardless of the choice one may make between p- and ν-principles, we contend that the rule-clauses should be presented in both introduction and eliminaition form, and we thus turn to argument (ii). Dummett's lesson is that the purpose of harmony is to establish whether a set of practices amounts to correct use, i.e., whether a set of rules constitutes a justifiable meaning theory (Dummett, 1991). That is, acquisition and manifestation, conditions and consequences, introductions and eliminations should be such that they are in balance with respect to the inferences or practices that they licence. That is, p-principles and ν-principles cannot be given independently of one another (Steinberger, 2011a). Simonelli only considers ν-principles, and restates harmony in terms of positive and negative introduction rules, rather than introduction and elimination rules, using coordination principles to derive the resulting commitments, like elimination rules are used to derive consequences in unilateral accounts (Simonelli, 2025). In this setup, harmony still holds, since bilateral harmony is enforced and unilateral harmony holds trivially. That is, Simonelli's proposal is successful as a harmony *criterion*, provided that the *conception* only requires ruling out *tonk*ish and *tunk*ish connectives.

But should we be content with trivial unilateral harmony? After all, from a chiefly proof-theoretic perspective, all the boxes are ticked. However, we contend that from a more general *language use* perspective, Simonelli's approach lacks in explanatory power of how linguistic mastery comes about, and has problems passing the Ockham's razor test in explaining how one draws conclusions from the grounds of the premises. Indeed, given the warrants for an attitude expression[4] (introduction rule), it is quite unnatural to suppose that for one to draw coherent conclusions they must infer that they would get a contradiction were they to oppose it, and then conclude what follows from the putative incoherence by Smileian reductio. On the other hand, committing to what follows from the premises were they correct is a much simpler and intuitive way of understanding consequential commitment given expressibility grounds. Moreover, in the presence of multiple speech acts, conditions and consequences for assertions and rejections are not necessarily the same: a consequence of an assertion and a condition for a rejection need not coincide (and, often times, do not) but nonetheless we want to keep all such aspects of language in harmony with one another. Enforcing harmony between conditions for assertions and rejections would thus not imply harmony between conditions and consequences for a given speech act. Therefore, being the purpose of harmony to provide an account of what a correct mastery of a language amounts to (Dummett, 1991), keeping both introduction and elimination rules is the most explanatory powerful option even in the presence of alternative devices in the bilateralist and multilateralist setting. From this it follows that *rejection* cannot properly be considered as a *generalisation* of *cut*, but merely as a counterpart. Luckily this objection does not impose revisions to Simonelli's approach beyond including more cases in our metatheoretical proofs, as it can be seen in Theorem 2.3.2 and 3.2.4. However, it is interesting to notice that the apparently chiefly technical question "*is* rejection *a bilateralist generalisation*

[4]We use *expressibility*, instead of the Dummettian *assertibility conditions*, as we are in a multilateral context were formulae can be rejected, and we accordingly have more possible attitude expressions than mere assertion.

of cut?" is addressed on purely philosophical grounds, namely those of the suitable *conception* of harmony, as opposing answers, namely the present and that of (Simonelli, 2024a) are both adequate from the viewpoint of proof theory alone.

Wet now show that the coordination principle of rejection, with respect to both strong and weak rejection, is admissible, which will yield that the rules of G2BML are not only unilaterally, but also bilaterally harmonious. In particular, it will yield bilateral harmony between the rules signed by + and those signed by both − and ⊖. A proof for the admissibility of rejection in a bilateralist calculus can already be found in (Simonelli, 2024a). However, his system is meaningfully different from G2BML, for the conceptual reasons outlined above, and for the absence of weak rejections. We begin by defining the following measure:

Definition 9 (Rejection height)**.** *Let the height of a derivation $\mathcal{D}$ be the length of its longest branch minus one. Then the Rejection height is the sum of the lengths of the derivations of the premisses of Rej.*

The strategy is analogous to that for the admissibility of *Cut*, since it involves a permutation argument so that applications of the Rejection rules are pushed upwards in the tree until they reach an initial sequent.

Theorem 2.3.4. (Admissibility of Rejection). *The following rules*

$$\dfrac{\Gamma \Rightarrow {}_{+/-}{}^{\sim}A \qquad \Delta \Rightarrow {}_{+/-}A}{\Gamma, \Delta \Rightarrow \bot}\; Rej^{\sim} \qquad\qquad \dfrac{\Gamma \Rightarrow {}_{+/\ominus}{}^{\natural}A \qquad \Delta \Rightarrow {}_{+/\ominus}A}{\Gamma, \Delta \Rightarrow \bot}\; Rej^{\natural}$$

are admissible in G2BML.

Proof. By lexicographic induction on formula weight and rejection height. We consider only $Rej^{\sim}$ as the proof for $Rej^{\natural}$ is analogous, modulo restrictions on the contexts.

For the base case, we need to consider a number of subcases. Suppose, without loss of generality that the left premiss is an axiom (the case in which the right premiss is an axiom is symmetric). We then have that $\Gamma = \Gamma', +p$ and $\Gamma \Rightarrow +A \equiv +p, \Gamma' \Rightarrow +p$. Then the derivation looks as follows:

$$\dfrac{\dfrac{}{+p, \Gamma' \Rightarrow +p}Ax \qquad \begin{array}{c}\mathcal{D}\\ \Delta \Rightarrow -p\end{array}}{+p, \Gamma', \Delta \Rightarrow \bot}\; Rej^{\sim}$$

If $-p$ must not be principal, and thus there must be some rule R such that $\dfrac{\begin{array}{c}\mathcal{D}'\\ \Delta' \Rightarrow -p\end{array}}{\Delta \Rightarrow -p}R$. Therefore, we can obtain the conclusion by an application of R, obtaining the following derivation of lower rejection height:

$$\dfrac{\dfrac{}{+p,\Gamma' \Rightarrow +p}Ax \qquad \begin{matrix}\mathcal{D}' \\ \Delta' \Rightarrow -p\end{matrix}}{\dfrac{+p,\Gamma',\Delta' \Rightarrow \bot}{+p,\Gamma',\Delta \Rightarrow \bot}R} Rej^{\sim}$$

If, on the other hand, $\Delta \Rightarrow -p$ was obtained by $SR^{\star}$-in, we have the following instance of $Rej^{\sim}$

$$\dfrac{\dfrac{}{+p,\Gamma' \Rightarrow +p}Ax \qquad (*)\ \dfrac{\begin{matrix}\mathcal{D} \\ \Delta,+p \Rightarrow \bot\end{matrix}}{\Delta \Rightarrow -p}SR^{\star}\text{-in}}{+p,\Gamma',\Delta \Rightarrow \bot} Rej^{\sim}$$

which we can reduce to *cut* as follows:

$$\dfrac{\dfrac{}{+p,\Gamma' \Rightarrow +p}Ax \qquad \begin{matrix}\mathcal{D} \\ \Delta,+p \Rightarrow \bot\end{matrix}}{+p,\Gamma',\Delta \Rightarrow \bot} cut$$

If both premises are axioms of the form $\dfrac{}{+p,\Gamma' \Rightarrow +p}Ax$ and $\dfrac{}{-p,\Delta' \Rightarrow -p}Ax$ on the other hand, we have that the conclusion of the $Rej^{\sim}$ rule can be obtained via *Cut* and the initial sequent of *Incoherence*:

$$\dfrac{\dfrac{}{+p,\Gamma' \Rightarrow +p}Ax \qquad \dfrac{}{+p,-p,\Delta' \Rightarrow \bot}Inc.}{+p,\Gamma',-p,\Delta' \Rightarrow \bot} Cut$$

Where $Rej^{\sim}$ is eliminated entirely. If either Γ or Δ contain $\bot$, then the conclusion can just be obtained by $\bot_L$ and we are done. If the premises are applications of *Smileian* reductio, on the other hand, we have the following, without loss of generality (modulo context restrictions):

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0 \\ \Gamma,\ominus p \Rightarrow \bot\end{matrix}}{\Gamma \Rightarrow +p}SR\text{-in} \qquad \dfrac{\begin{matrix}\mathcal{D}_1 \\ \Delta,+p \Rightarrow \bot\end{matrix}}{\Delta \Rightarrow \ominus p}SR\text{-in}}{\Gamma,\Delta \Rightarrow \bot} Rej^{\lightning}$$

Which we can transform in a derivation where Rej is eliminated entirely by means of *cut*[5]:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0 \\ \Gamma,\ominus p \Rightarrow \bot\end{matrix}}{\Gamma \Rightarrow +p}SR\text{-in} \qquad \begin{matrix}\mathcal{D}_1 \\ +p,\Delta \Rightarrow \bot\end{matrix}}{\Gamma,\Delta \Rightarrow \bot} cut$$

For the inductive step, let the *rejection formula* be $\mathbf{s}A$ for $\mathbf{s}$ arbitrary. This allows us to have a uniform case distinction despite the differences in signs of A on the two premises. We accordingly consider the case in which the rejection formula is not principal in the left premiss (the right case being symmetric) and the one in which it is principal in both.

[5] Again, notice that the shape of *Smileian reductio* needs to be such that bilateral harmony is preserved.

The rejection formula sA is not principal in the left premiss

$\underline{\Gamma = \Gamma', +\neg A}$, without loss of generality. We then have the following derivation of rejection height $m + n + 1$:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma', -A \Rightarrow -B\end{matrix}}{\Gamma', +\neg A \Rightarrow -B}\,{+\neg L} \qquad \begin{matrix}\mathcal{D}_1\\ \Delta \Rightarrow +B\end{matrix}}{\Delta, \Gamma', +\neg A \Rightarrow \bot}\,Rej^{\sim}$$

Which becomes the following, where *rejection* is applied to formulae of lower weight and is of *rejection* height $n + m$:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma', -A \Rightarrow -B\end{matrix} \qquad \begin{matrix}\mathcal{D}_1\\ \Delta \Rightarrow +B\end{matrix}}{\Delta, \Gamma', -A \Rightarrow \bot}\,Rej^{\sim}}{\Delta, \Gamma', +\neg A \Rightarrow \bot}\,{+\neg L}$$

$\underline{\Gamma = \Gamma', \mathbf{s}A \circ B, \text{ for } \circ \in \{\wedge, \vee\}}$. We then have the following tree, of *rejection* height $\max(n+m)+k+1+1$:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma', \mathbf{a}A \Rightarrow -C\end{matrix} \qquad \begin{matrix}\mathcal{D}_1\\ \Gamma', \mathbf{b}B \Rightarrow -C\end{matrix}}{\Gamma', \mathbf{s}A \circ B \Rightarrow -C}\,\mathbf{s} \circ L \qquad \begin{matrix}\mathcal{D}_2\\ \Delta \Rightarrow +C\end{matrix}}{\Delta, \Gamma', \mathbf{s}A \circ B \Rightarrow \bot}\,Rej^{\sim}$$

Which then becomes the following, with rejection height $\max(m, k)+\max(n, k) \leq \max(n+m)+k+1+1$ and lower weight of rejection formula:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma', \mathbf{a}A \Rightarrow -C\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ \Delta \Rightarrow +C\end{matrix}}{\Delta, \Gamma', \mathbf{a}A \Rightarrow \bot}\,Rej^{\sim} \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ \Gamma', \mathbf{b}B \Rightarrow -C\end{matrix} \qquad \begin{matrix}\mathcal{D}_2\\ \Delta \Rightarrow +C\end{matrix}}{\Delta, \Gamma', \mathbf{b}B \Rightarrow \bot}\,Rej^{\sim}}{\Delta, \Gamma', \mathbf{s}A \circ B \Rightarrow \bot}\,\mathbf{s} \circ L$$

$\underline{\Gamma = \Gamma', \mathbf{s}^{\sim}A \circ B, \text{ for } \circ \in \{\wedge, \vee\}}$. We then have the following derivation of *rejection* height $m + n + 1$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma', \mathbf{a}^{\sim}A \Rightarrow -C\end{matrix}}{\Gamma', \mathbf{s}^{\sim}A \circ B \Rightarrow C}\,\mathbf{s}^{\sim} \circ L_1 \qquad \begin{matrix}\mathcal{D}_1\\ \Delta \Rightarrow +C\end{matrix}}{\Delta, \Gamma', \mathbf{s}^{\sim}A \circ B \Rightarrow \bot}\,Rej^{\sim}$$

Which becomes the following, where *rejection* is applied to formulae of lower weight and is of *rejection* height $n + m$:

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{IH} & \mathcal{D}_1 \\ \Gamma', \mathbf{a}^{\sim}A \Rightarrow -C & \Delta \Rightarrow +C\end{array}}{\Delta, \Gamma', \mathbf{a}^{\sim}A \Rightarrow \bot}\ Rej^{\sim}}{\Delta, \Gamma', \mathbf{s}^{\sim}A \circ B \Rightarrow \bot}\ \mathbf{s}^{\sim} \circ L_1$$

The case of $^{-}/_{\ominus} \supset$ is analogous.

$\underline{\Gamma = \Gamma', +A \supset B}$. We then have the following tree, of *rejection* height $\max(n + m) + k + 1 + 1$:

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_0 & \mathcal{D}_1 \\ \Gamma \Rightarrow +A & +B, \Gamma' \Rightarrow -C\end{array}}{\Gamma', +A \supset B \Rightarrow -C}\ (+ \supset L) \quad \begin{array}{c}\mathcal{D}_2 \\ \Delta \Rightarrow +C\end{array}}{\Delta, \Gamma', +A \supset B \Rightarrow \bot}\ Rej^{\sim}$$

Which becomes the following, of *rejection* height and weight reduced:

$$\dfrac{Wk\ \dfrac{\begin{array}{c}\mathcal{D}_0 \\ \Gamma' \Rightarrow +A\end{array}}{\Gamma', \Delta \Rightarrow +A} \quad \dfrac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ +B, \Gamma' \Rightarrow -C & \Delta \Rightarrow +C\end{array}}{+B, \Delta, \Gamma' \Rightarrow \bot}\ Rej^{\sim}}{\Delta, \Gamma', +A \supset B \Rightarrow \bot}\ (+ \supset L)$$

sA is principal in both premises

$\underline{\mathbf{s}A = \mathbf{s}\neg B}$. We then have the following tree of *rejection* height $m + n + 1$:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_0 \\ \Gamma \Rightarrow -B\end{array}}{\Gamma \Rightarrow +\neg B}\ (+\neg R) \quad \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Delta \Rightarrow +B\end{array}}{\Delta \Rightarrow -\neg B}\ (-\neg R)}{\Delta, \Gamma \Rightarrow \bot}\ Rej^{\sim}$$

Which becomes the following, with both *rejection* height and weight reduced:

$$\dfrac{\begin{array}{cc}\mathcal{D}_0 & \mathcal{D}_1 \\ \Gamma \Rightarrow -B & \Delta \Rightarrow +B\end{array}}{\Gamma, \Delta \Rightarrow \bot}\ Rej^{\sim}$$

$\underline{\mathbf{s}A = \mathbf{s}B \wedge C}$. We then have the following derivation with *rejection* height $m + \max(n + k) + 1 + 1$, without loss of generality ($(- \wedge R_2)$ and the disjunction case are both analogous):

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_0 & \mathcal{D}_2 \\ \Gamma \Rightarrow +B & \Gamma \Rightarrow +C\end{array}}{\Gamma \Rightarrow +B \wedge C}\ (+ \wedge R) \quad \dfrac{\begin{array}{c}\mathcal{D}_2 \\ \Gamma \Rightarrow -B\end{array}}{\Gamma \Rightarrow -B \wedge C}\ (- \wedge R_1)}{\Delta, \Gamma \Rightarrow \bot}\ Rej^{\sim}$$

Which becomes the following, with both rejection height and formula weight reduced:

$$\dfrac{\dfrac{\mathcal{D}_0}{\Gamma \Rightarrow +B} \quad \dfrac{\mathcal{D}_2}{\Delta \Rightarrow -B}}{\Delta, \Gamma \Rightarrow \bot} \, Rej^{\sim}$$

$\underline{\mathbf{s}A = \mathbf{s}B \supset C}$. We then have the following derivation with *rejection* height $\max(m+n)+k+1+1$, without loss of generality:

$$\dfrac{(- \supset R)\,\dfrac{\dfrac{\mathcal{D}_0}{\Delta \Rightarrow +A} \quad \dfrac{\mathcal{D}_1}{\Delta \Rightarrow -B}}{\Delta \Rightarrow -A \supset B} \quad \dfrac{\dfrac{\mathcal{D}_2}{\Gamma, +A \Rightarrow +B}}{\Gamma \Rightarrow +A \supset B}\,(+ \supset R)}{\Delta, \Gamma \Rightarrow \bot} \, Rej^{\sim}$$

Which becomes the following, with both *rejection* height and formula weight reduced:

$$\dfrac{\dfrac{\mathcal{D}_1}{\Delta \Rightarrow +A} \quad \dfrac{\dfrac{\mathcal{D}_0}{\Gamma, +A \Rightarrow +B} \quad \dfrac{\mathcal{D}_2}{\Delta \Rightarrow -B}}{+A, \Delta, \Gamma \Rightarrow \bot}\, Rej^{\sim}}{\Delta, \Gamma \Rightarrow \bot} \, Cut$$

Since the remaining cases, and the case for $Rej^{\lightning}$ are analogous, this completes our induction, witnessing the admissibility of the coordination principle of *Rejection*.

□

We saw how neither of the coordination principles are reducible to a generalisation of *cut*, if we understand harmony in broadly dummettian terms as a notion that should explain why meaning justifiably arises from proof rules. However, it is interesting to point out that in some instances of the reduction of *rejection height* the *cut* rule was indispensable. This we can take to be evidence for the fact that unilateral harmony, represented by the admissibility of *cut*, is a necessary condition for bilateral harmony to hold. Furthermore, it is a core feature of inferential expressivism to hold that inferences and attitude expressions are conceptually and operationally interrelated, and the use of coordination principles in inferences, as well as the need of structural rules to admit them in the system, is evidence for such interdependence as well.

In this chapter, we have therefore outlined a sequent calculus for *Basic Multilateral Logic*, defended a conception of bilateral harmony suitable for the presence of strong and weak rejections and enforced it by showing the admissibility of *cut* and *rejection*. However, we did not say anything about genuinely *multi*lateral harmony, which should also encompass harmony between $-$ and $\ominus$−rules. We will discuss this issue in the following chapter, where the extended framework of *Epistemic Multilateral Logic*, comprising of the speech act of *weak assertion* ($\oplus$), provides a more suitable setting for the discussion.

Chapter 3

Epistemic Multilateral Logic: Modalities, Weak Assertion and Multilateral Harmony

In this chapter we study an extension of BML developed in (Incurvati and Schlöder, 2019; Incurvati and Schlöder, 2022; Incurvati and Schlöder, 2023b) to explain epistemic modal vocabulary, *Epistemic Multilateral Logic* (EML). While it was originally presented to solve linguistic puzzles about epistemic modals and in doing so address the limited applicability objection to inferentialism, here we use it to continue on our task of understanding multilateral proof-theoretic harmony in a sequent calculus setting. Moreover, we will discuss the relevance of the multilateralist approach to modal proof theory and proof-theoretic semantics in general, covering its strengths, limitations, promises and prospects. Proofs of soundness, completeness and admissibility theorems for structural rules and coordination principles will be presented.

3.1 Background: Weak Assertion and Epistemic Modals

In sentences like "it might rain" or "the murderer must be the butler", modal expressions such as *might* and *must* are used epistemically, and thus referred to as *epistemic modals*. Among the diverse proposals presented in the literature to provide a semantics to epistemic modal discourse, here we focus on the inferentialist ones, and in particular on the multilateralist one of Incurvati and Schlöder 2019; 2022; 2023. We begin by rehearsing the arguments behind an inferential expressivist account of epistemic modality, and we then proceed with presenting the calculus of EML and its properties. In doing so, we follow the steps of Incurvati and Schlöder, 2023b's general multilarteralist method, applied to weak assertion and epistemic modality. To inferentially explain the meaning of a linguistic item in terms of an attitude expression, we must (i) identify the linguistic realization of this expression (ii) expand the coordination principles so as to specify how this attitude expression interacts with other attitude expressions (iii) determine which new inferences preserve evidence to appropriately restrict the coordination principles.

3.1.1 From Yalcin Sentences to EML

Consider the sentence "it is raining and it might not be raining". This example is one of many infelicitous utterances presented in (Yalcin, 2007), and which have been taken as evidence that classical logic should be revised to account for epistemic modality (Incurvati and Schlöder, 2019). The linguistic data presented by Yalcin is the following, witnessing that the epistemic modal *might* has an embedding behaviour that is different from the self-reported ignorance of its contrary (as customary, "♯" denotes infelicituos utterances):

(I) ♯ a. It is raining and I don't know that it is raining.
b. Suppose it is raining and I don't know that it is raining...
c. If it is raining and I don't know that it is raining...
♯ d. it is raining and it might not be raining.
♯ e. Suppose it is raining and it might not be raining...
♯ f. It it is raining and it might not be raining ...

To understand the problematicity of the use of *might* in (Id,e,f), it is enlightening to compare it with (Ia,b,c), where (Ia) constitutes an instance of Moore's paradox. (Ia) is considered infelicitous on pragmatic grounds: if one states that it is raining, it pragmatically follows that that person must know that it is raining. The pragmatic nature of this entailment is corroborated by its suspension under *suppose* and *if*, i.e., it does not carry through in embedded contexts. On the contrary, the infelicity of (Id,e,f) persists even if embedded under *suppose* and *if*, ruling out a pragmatic explanation according to which (Ie,f) should be acceptable (Incurvati and Schlöder, 2019). A satisfactory theory of epistemic modals must therefore explain the infelicity of (Id,e,f), without causing a *modal collapse*, i.e., without trivialising the modality by obtaining inferences of the sort "if *might* p then p" (Ibid.). Indeed, this happens if we accept classical logic and take p and *might not* p to be contradictory. Say that a state S *rules out* p if updating S with p gives us an absurd state (Russell and Hawthorne, 2016). Then, we can formulate the following desiderata for a theory of epistemic modality, following (Russell and Hawthorne, 2016; Incurvati and Schlöder, 2019):

(MIGHT) If S does not rule out *might* p, it does not rule out p.
(NOT) Upon updating S with *not* p, the resulting S' rules out p.

However, suppose by contradiction that a state S, when updated with *not* p, yields some S' that does not rule out *might* p. By (MIGHT), then, p is not ruled out either. As this contradicts (NOT), we conclude that updating a state S with *not* p, yields a state S' that does rule out *might* p. Thus, updating S with *not* p and then *might* p yields an absurd state. Dynamic logic, as in Russell and Hawthorne, 2016's argument, has the feature that update is commutative, i.e., updating S with φ and then with ψ yields the same S' as updating S with ψ and then φ. Moreover, "if φ then ψ" is defined as any state updated with φ ruling out *not* ψ. So, updating S with *might* p and then *not* p yields the same absurd state. And therefore, updating S with *might* p does rule out *not* p by definition. Moreover, by entailment in dynamic logic, we obtain that if *might* p, then *not* p. Therefore, if we were to have a notion of update and retain classical logic and commutativity, the two desiderata (MIGHT) and (NOT) result being incompatible (Incurvati and Schlöder, 2019). We now outline how extending our multilateral calculus with the speech act of *weak assertion* allows us to achieve our desiderata while avoiding modal collapse. The key insight, as for the case of rejection and negation in BML, is to understand the interactions between force markers pertaining to epistemic possibility, such as *perhaps*, and modal operators, such as *might* ($\Diamond$).

Presenting linguistic evidence about embedding behaviour and expressive role, (Incurvati and Schlöder, 2019) conclude that while *perhaps* only modifies force, *might* modifies content and exhibits the embeddability behaviour of logical operators such as negation. On the other hand, they show that in a fixed context, the same inferences can be drawn from "*perhaps* it φs" and "it *might* φ". As for negation and rejection, we have a force marker and an embeddable operator that are interchangeable in in non-embedded context, while they differ in their linguistic function. That is, *perhaps* and *might* are not reducible to one another, but they are nonetheless interderivable. The speech act witnessed by *perhaps* is that of *weak assertion*, which we denote with $\oplus$. Similarly to $+, -, \ominus$, the force marker $\oplus$ is defined against the backdrop of a conversational common ground, so that committing to p amounts to updating the pool of shared facts in a dialogue with p. While $+p$ signals assent to adding p to the dialogue's common ground, $-p$ represents assent to updating the common ground with $\neg p$. As for the weak speech acts, on the other hand, we have that $\ominus p$ witnesses a dissent with committing to p, and similarly we can define $\oplus p$ as the dissent with adding $\neg p$ to the shared commitments. Within this extended framework, following a method analogous to BML, we present the rule-clauses defining epistemic modality. Alongside the connective rules and coordination principles presented below, this constitutes the logic EML, or *Epistemic Multilateral Logic*.

3.1.2 Epistemic Multilateral Logic: Rules and Main Results

In EML, the modality *might* ($\Diamond$) is defined in inferential terms, as its assertion is interderivable with the weak assertion of the formula in the scope of the modality. That is, asserting "it may rain" implies "perhaps is raining" and vice versa, while maintaining their difference *qua* force marker and embeddable operator. Moreover, the rejected versions are reconstructed as rejecting A as rejecting the possibility of A, both in strong and weak form, and the weakly asserted modality follows as a weaker case of the strongly asserted one. The natural deduction rules for the introduction and elimination of the epistemic $\Diamond$, under all four speech acts, therefore look as follows (Incurvati and Schlöder, 2019):

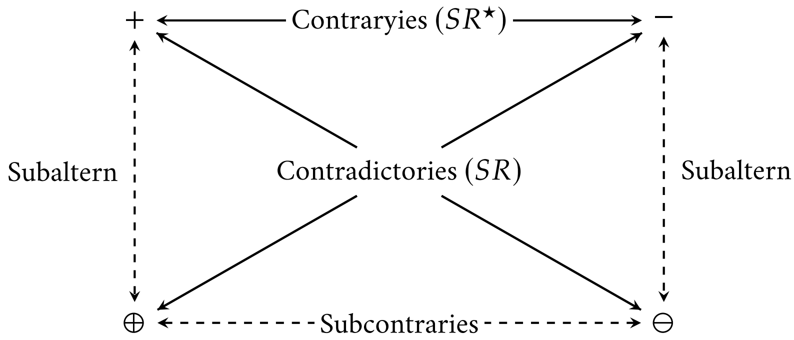
$$\frac{\oplus A}{+\Diamond A}\,+\Diamond I \qquad \frac{+\Diamond A}{\oplus A}\,+\Diamond E \qquad \frac{-A}{-\Diamond A}\,-\Diamond I \qquad \frac{-\Diamond A}{-A}\,-\Diamond E$$

$$\frac{\oplus A}{\oplus\Diamond A}\,\oplus\Diamond I \qquad \frac{\oplus\Diamond A}{\oplus A}\,\oplus\Diamond E \qquad \frac{-A}{\ominus\Diamond A}\,\ominus\Diamond I \qquad \frac{\ominus\Diamond A}{-A}\,\ominus\Diamond E$$

Moreover, having the further speech act of weak assertion implies that the following can be added to the rules of negation, with $\ominus\neg I$ and $\ominus\neg E$ being more general than their BML counterparts:

$$\frac{\ominus A}{\oplus \neg A}\oplus\neg I \qquad \frac{\oplus \neg A}{\ominus A}\oplus\neg E \qquad \frac{\oplus A}{\ominus \neg A}\ominus\neg I \qquad \frac{\ominus \neg A}{\oplus A}\ominus\neg E$$

These, in addition to the conjunction rules for strong assertion of BML, constitute the primitive connective rules of EML in (Incurvati and Schlöder, 2019). We further update the coordination principles of Smileian reductio and add two further coordination principles. For this purpose, we update the polarity-changing maps and the diagram of opposition:



The maps are updated as follows:

Definition 10 (Polarity-changing maps, quadrilateral)**.** *Let* ${}_{+/-} \in \{+,-\} \subseteq S$, ${}_{+/\ominus} \in \{+,\ominus\} \subseteq S$, ${}_{-/\ominus} \in \{-,\ominus\} \subseteq S$, ${}_{\oplus/\ominus} \in \{\oplus,\ominus\} \subseteq S$. *We then define the following maps:*

$$\begin{array}{lll} ()^{\sim} : S \to S & := & {}_{+/-}A \mapsto \textit{contrary of } {}_{+/-}A, \quad {}_{\oplus/\ominus}A \mapsto \textit{subcontrary of } {}_{\oplus/\ominus}A \\ ()^{\not\,} : \{+,\ominus\} \to \{+,\ominus\} & := & {}_{+/\ominus}A \mapsto \textit{contradictory of } {}_{+/\ominus}A \\ ()^{\flat} : \{-,\ominus\} \to \{-,\ominus\} & := & {}_{-/\ominus}A \mapsto \textit{subaltern of } {}_{-/\ominus}A \end{array}$$

Accordingly, for $\alpha = \mathbf{s}A$, $\alpha^\circ = \mathbf{s}^\circ A$ *for* $\circ \in \{+,-,\oplus,\ominus\}$.

Furthermore, it will be later be convenient to also have a map $()^{\mathbf{1}} : S \to S$ such that $\mathbf{s} \mapsto \mathbf{s}$. In addition to the coordination principles for BML, in (Incurvati and Schlöder, 2023b) we can also find versions of Smileian reductio and Rejection for ⊕, where the latter is referred to as *Weak Assertion* (WA):

$$\frac{\oplus A \qquad -A}{\bot}WA \qquad \begin{array}{c}[\oplus A]\\ \vdots \\ \dfrac{\bot}{-A}SR_3\end{array} \qquad \begin{array}{c}[-A]\\ \vdots \\ \dfrac{\bot}{\oplus A}SR_4\end{array}$$

We further have distinctively multilateral coordination principles pertaining to subalternity relations between spech acts. The first, SA_i, characterises strong speech acts as particular cases of weak ones, while the second, *Weak Inference*, ensures that commitment-preserving inference is closed under evidence-preserving inference. In particular, weak assertion is closed under strongly asserted implication, allowing for inferences of the sort *"if α then β. perhaps α, then perhaps β"*. Below, we represent them schematically for expository convenience:

$$\frac{{}^{+/-}A}{{}^{\oplus/\ominus}A}\,(SA_i) \qquad \dfrac{{}^{+/-}{}^{\flat}A \qquad \begin{array}{c}{}^{+/-}A\\ \vdots *\\ \alpha\end{array}}{\alpha^{\flat}}\,WI_i$$

Where α can be $\perp$, or, for an arbitrary B, $+B$ in the case of WI_1 and $-B$ in the case of WI_2. Evidence-preservation, denoted by "$\vdots *$", now restricts derivations not only to those without premises signed with $\ominus$, but also to ones without premises signed with $\oplus$ and inferences without applications of $(+\Diamond E)$ and $(\oplus\Diamond E)$. Indeed, as shown in (Incurvati and Schlöder, 2019; Incurvati and Schlöder, 2022), applications of $(+\Diamond E)$ and $(\oplus\Diamond E)$ face specificity problems analogous as having premisses signed with $\ominus$. Recall Dickey's unspecificity problem for weak rejection, i.e., that the grounds for a correct weak rejection of A are ambiguous between evidence against A and absence of evidence for A *tout court* (Incurvati and Schlöder, 2023b; Incurvati and Schlöder, 2017). The importance for this issue in the justification of *Weak Inference*, as argued in (Incurvati and Schlöder, 2019; Incurvati and Schlöder, 2022) pertains to the legitimacy of the subderivation from ${}^{+/-}A$ to α. Assume we have evidence that licenses $\oplus A$, and that for the sake of argument we suppose that A, thus having $+A$ as assumption of our subderivation. Notice that in the context in which we have evidence for $\oplus A$, we may or may not have evidence for $+A$, and so we should not use premises or inference steps that sanction rejections due to unavailable evidence. This, as it is by now established, rules out premisses signed with $\ominus$. However our calculus, by means of rules for $\mathbf{s}\Diamond$ and $\mathbf{s}\neg$, allows $\oplus$ to substitute with $\ominus$ or $+\Diamond$. This means that we should also restrict evidence-preservation to inferences without weakly asserted premises and applications of (weakly) asserted elimination rules for $\Diamond$ (Incurvati and Schlöder, 2022). This concludes the outline of the primitive connectives for EML, as we take $\Box, \supset, \vee$ to be defined the usual way.

The distinction that follows between commitment- and evidence-preserving inferences is of particular relevance for the definition of the derived rules of EML, as well as the solution to the epistemic puzzles presented above. An interesting case worth mentioning is that of the (derived) rules for $\mathbf{s}\Box$, where a necessitation-like rule is valid even if A is not a theorem:

$$\frac{+A}{+\Box A}\,+\Box I \qquad \frac{+\Box A}{+A}\,+\Box E \qquad \frac{\ominus A}{-\Box A}\,-\Box I \qquad \frac{-\Box A}{\ominus A}\,-\Box E$$

$$\frac{+A}{\oplus\Box A}\,\oplus\Box I \qquad \frac{\oplus\Box A}{+A}\,\oplus\Box E \qquad \frac{\ominus A}{\ominus\Box A}\,\ominus\Box I \qquad \frac{\ominus\Box A}{\ominus A}\,\ominus\Box E$$

Rules of this sort, in unilateral modal logic, would lead to modal collapse. However, as shown in (Incurvati and Schlöder, 2023b), the derivation of $+\Diamond A \supset A$ would require applications of $(+\Diamond E)$, and the rules for $+\Box$ are derived from those of $+\Diamond$ using *Weak Inference*, which rules out applications of $(+\Diamond E)$. Below we present the rules for the full calculus, which expands that of BML. The proofs are deferred to (Incurvati and Schlöder, 2022).

Figure 1: The natural deduction calculus $\mathsf{N_{EML}}$ (Incurvati and Schlöder, 2023b)

$$\frac{\ominus A}{\oplus\neg A}\ \oplus\neg I \qquad \frac{\oplus\neg A}{\ominus A}\ \oplus\neg E \qquad \frac{\oplus A}{\ominus\neg A}\ \ominus\neg I \qquad \frac{\ominus\neg A}{\oplus A}\ \ominus\neg E$$

$$\frac{\oplus A}{+\Diamond A}\ +\Diamond I \qquad \frac{+\Diamond A}{\oplus A}\ +\Diamond E \qquad \frac{-A}{-\Diamond A}\ -\Diamond I \qquad \frac{-\Diamond A}{-A}\ -\Diamond E$$

$$\frac{\oplus A}{\oplus\Diamond A}\ \oplus\Diamond I \qquad \frac{\oplus\Diamond A}{\oplus A}\ \oplus\Diamond E \qquad \frac{-A}{\ominus\Diamond A}\ \ominus\Diamond I \qquad \frac{\ominus\Diamond A}{-A}\ \ominus\Diamond E$$

$$\frac{+A}{+\Box A}\ +\Box I \qquad \frac{+\Box A}{+A}\ +\Box E \qquad \frac{\ominus A}{-\Box A}\ -\Box I \qquad \frac{-\Box A}{\ominus A}\ -\Box E$$

$$\frac{+A}{\oplus\Box A}\ \oplus\Box I \qquad \frac{\oplus\Box A}{+A}\ \oplus\Box E \qquad \frac{\ominus A}{\ominus\Box A}\ \ominus\Box I \qquad \frac{\ominus\Box A}{\ominus A}\ \ominus\Box E$$

$$\frac{\oplus A \qquad +B}{\oplus A\wedge B}\ (\oplus\wedge I_1) \qquad \frac{+A \qquad \oplus B}{\oplus A\wedge B}\ (\oplus\wedge I_2) \qquad \frac{\oplus A\wedge B}{\oplus A}\ (\oplus\wedge E_1) \qquad \frac{\oplus A\wedge B}{\oplus B}\ (\oplus\wedge E_2)$$

$$\frac{\oplus A}{\oplus A\vee B}\ (\oplus\vee I_1) \qquad \frac{\oplus B}{\oplus A\vee B}\ (\oplus\vee I_2) \qquad \frac{\oplus A\vee B \qquad \begin{array}{c}[\oplus A]\\ \vdots^{*}\\ \alpha\end{array} \qquad \begin{array}{c}[\oplus B]\\ \vdots^{*}\\ \alpha\end{array}}{\alpha}\ (\oplus\vee E)$$

$$\frac{\begin{array}{c}[+A]\\ \vdots^{*}\\ \oplus B\end{array}}{\oplus A\supset B}\ (\oplus\supset I) \qquad \frac{\oplus A\supset B \qquad +A}{\oplus B}\ (\oplus\supset E)$$

EML clearly satisfies the desiderata (MIGHT) and (NOT), the former being encapsulated by the rules for $\oplus\Diamond$ and the latter being a straightforward application of the coordination principle of rejection. Moreover, we can derive a contradiction from the Yalcin sentence "p and it *might not* p", which we formalise as $+p\wedge\Diamond\neg p$ (Incurvati and Schlöder, 2019):

$$\dfrac{\dfrac{+p\wedge\Diamond\neg p}{+p}\,(+\wedge E) \qquad \dfrac{\dfrac{\dfrac{\dfrac{+p\wedge\Diamond\neg p}{+\Diamond\neg p}\,(+\wedge E)}{\oplus\neg p}\,(+\Diamond E)}{\ominus p}\,(\oplus\neg E)}{}}{\perp}\,(Rej)$$

Inferential expressivism, extended with rules for modality in terms of weak assertion, therefore satisfies the key desiderata of a theory of epistemic modality, in light of the problems presented in (Yalcin, 2007). We conclude this section by presenting another noteworthy result about EML. *Epistemic Multilateral Logic* is sound and complete with respect to **S5** under global consequence[1], modulo a suitable translation (Incurvati and Schlöder, 2022). Let α a signed formula of EML and $\tau : \mathcal{L}_{\mathsf{EML}} \rightarrow \mathcal{L}_{\mathsf{ML}}$, where $\mathcal{L}_{\mathsf{ML}}$ denotes the language of modal logic. In particular, EML *extends* **S5** as its strongly asserted fragment, translated by τ, corresponds to **S5** consequence:

[1] By global consequence we mean that if the premises are satisfied at all worlds, so is the conclusion, as opposed to the local notion of consequence of standard modal logics, where at all worlds, if the premises are satisfied so is the conclusion.

Theorem 3.1.1 (Incurvati and Schlöder, 2022, Theorem 4.9, p.524). *$\Gamma \vdash^{\mathbf{S5}} A$, iff $\Gamma \vdash^{\mathsf{EML}} +A$, modulo the following translation:*

$$\tau(\alpha) = \begin{cases} \Box A & \text{if } \alpha = +A, \\ \Diamond\neg A & \text{if } \alpha = \ominus A, \\ \Diamond A & \text{if } \alpha = \oplus A. \end{cases}$$

As (Incurvati and Schlöder, 2022) point out, τ is merely a technical translation, as the accessibility relation of **S5** frames is not meant as the intended semantics for the force markers. However, the result remains of interest from a proof-theoretic semantics point of view, as it allows for investigation on the extent to which an inferential expressivist semantics can cover the different modalities of the modal cube. We now present a sequent calculus translation for EML, and prove soundness and completeness with respect to $\mathsf{N}_{\mathsf{EML}}$, as well as admissibility results for structural rules and coordination principles.

3.2 The Sequent Calculus G2EML

3.2.1 Proof Rules and Coordination Principles

As in the case of G2BML, we present a G2-style single-conclusion sequent calculus for EML. We begin by defining the grammar, analogously to the case of BML. Let the language of modal logic $\mathcal{L}_{\mathsf{ML}}$ be defined accordingly to the following syntax:

$$A ::= \bot \mid p \mid \neg A \mid A \wedge A \mid A \vee A \mid A \supset A \mid \Diamond A$$

where $\Box$ is defined the canonical way. The, the language of EML is defined as follows:

Definition 11 (Signed formula of EML). *Let A be a formula in$\mathcal{L}_{\mathsf{ML}}$). An EML signed formula has the form $\mathbf{s}A$ for $\mathbf{s} \in \{+, -, \oplus, \ominus\}$ and $A \in \mathcal{L}_{\mathsf{ML}}$. The language of EML is therefore defined as $\mathcal{L}_{\mathsf{EML}} := \{\mathbf{s}A \mid A \in \mathcal{L}_{\mathsf{ML}}, \mathbf{s} \in \{+, -, \oplus, \ominus\}\}$. For Γ a multiset of $\mathcal{L}_{\mathsf{ML}}$ formulae, let $\Gamma^{\mathbf{s}} := \{\mathbf{s}A \mid A \in \Gamma\}$*

As for G2BML, we drop the superscript and denote contexts of signed formulae with capital Greek letters. We accordingly add the following connective rules and initial sequents to the proof system of G2BML:

$$\frac{}{\Gamma, \oplus p \Rightarrow \oplus p}\,\text{Ax} \qquad \frac{}{\Gamma, {}_{+/-}p \Rightarrow {}_{+/-}{}^{\flat}p}\,\text{Ax} \qquad \frac{}{\Gamma, \oplus p, -p \Rightarrow \bot}\,\text{Inc.}$$

$$\frac{\Gamma \Rightarrow {}_{\oplus/\ominus}{}^{\sim}A}{\Gamma \Rightarrow {}_{\oplus/\ominus}\neg A}\,({}_{\oplus/\ominus}\neg R) \qquad \frac{\Gamma, {}_{\oplus/\ominus}{}^{\sim}A \Rightarrow \alpha}{\Gamma, {}_{\oplus/\ominus}\neg A \Rightarrow \alpha}\,({}_{\oplus/\ominus}\neg L)$$

$$\frac{\Gamma \Rightarrow +A \qquad \Gamma \Rightarrow \oplus B}{\Gamma \Rightarrow \oplus A \wedge B}\,(\oplus \wedge R_1) \qquad \frac{\Gamma \Rightarrow \oplus A \qquad \Gamma \Rightarrow +B}{\Gamma \Rightarrow \oplus A \wedge B}\,(\oplus \wedge R_2)$$

$$\frac{\Gamma, \oplus A \Rightarrow \alpha}{\Gamma, \oplus A \wedge B \Rightarrow \alpha}\,(\oplus \wedge L_1) \qquad \frac{\Gamma, \oplus B \Rightarrow \alpha}{\Gamma, \oplus A \wedge B \Rightarrow \alpha}\,(\oplus \wedge L_2)$$

$$\frac{\Gamma \Rightarrow \oplus A}{\Gamma \Rightarrow \oplus A \vee B}\,(\oplus \vee R_1) \qquad \frac{\Gamma \Rightarrow \oplus B}{\Gamma \Rightarrow \oplus A \vee B}\,(\oplus \vee R_2) \qquad (*)\,\frac{\Gamma, \oplus A \Rightarrow \alpha \qquad \Gamma, \oplus B \Rightarrow \alpha}{\Gamma, \oplus A \vee B \Rightarrow \alpha}\,(\oplus \vee L)$$

$$(*)\,\frac{\Gamma, \oplus A \Rightarrow \oplus B}{\Gamma \Rightarrow \oplus A \supset B}\,(\oplus \supset R) \qquad (*)\,\frac{\Gamma \Rightarrow \oplus A \qquad \oplus B, \Gamma \Rightarrow \alpha}{\Gamma, \oplus A \supset B \Rightarrow \alpha}\,(\oplus \supset L)$$

With rules for the modalities being thus defined:

$$\frac{\Gamma \Rightarrow \oplus A}{\Gamma \Rightarrow +\Diamond A}\,+\Diamond R \qquad \frac{\Gamma, \oplus A \Rightarrow \alpha}{\Gamma, +\Diamond A \Rightarrow \alpha}\,+\Diamond L \qquad \frac{\Gamma \Rightarrow -A}{\Gamma \Rightarrow -\Diamond A}\,-\Diamond R \qquad \frac{\Gamma, -A \Rightarrow \alpha}{\Gamma, -\Diamond A \Rightarrow \alpha}\,-\Diamond L$$

$$\frac{\Gamma \Rightarrow \oplus A}{\Gamma \Rightarrow \oplus\Diamond A}\,\oplus\Diamond I \qquad \frac{\Gamma, \oplus A \Rightarrow \alpha}{\Gamma, \oplus\Diamond A \Rightarrow \alpha}\,\oplus\Diamond L \qquad \frac{\Gamma \Rightarrow -A}{\Gamma \Rightarrow \ominus\Diamond A}\,\ominus\Diamond R \qquad \frac{\Gamma, -A \Rightarrow \alpha}{\Gamma, \ominus\Diamond A \Rightarrow \alpha}\,\ominus\Diamond L$$

$$\frac{\Gamma \Rightarrow +A}{\Gamma \Rightarrow +\Box A}\,+\Box R \qquad \frac{\Gamma, +A \Rightarrow \alpha}{\Gamma, +\Box A \Rightarrow \alpha}\,+\Box L \qquad \frac{\Gamma \Rightarrow \ominus A}{\Gamma \Rightarrow -\Box A}\,-\Box R \qquad \frac{\Gamma, \ominus A \Rightarrow \alpha}{\Gamma, -\Box A \Rightarrow \alpha}\,-\Box L$$

$$\frac{\Gamma \Rightarrow +A}{\Gamma \Rightarrow \oplus\Box A}\,\oplus\Box R \qquad \frac{\Gamma, +A \Rightarrow \alpha}{\Gamma, \oplus\Box A \Rightarrow \alpha}\,\oplus\Box L \qquad \frac{\Gamma \Rightarrow \ominus A}{\Gamma \Rightarrow \ominus\Box A}\,\ominus\Box R \qquad \frac{\Gamma, \ominus A \Rightarrow \alpha}{\Gamma, \ominus\Box A \Rightarrow \alpha}\,\ominus\Box L$$

Extending the observation made for G2BML, axioms can take shape $\overline{\Gamma, +p \Rightarrow \oplus p}^{Ax}$ and $\overline{\Gamma, -p \Rightarrow \ominus p}^{Ax}$ in addition to the obvious ones. This extends to weak assertion the primitive assumption that assent and dissent must me incompatible, i.e., if we commit to assenting to p, we must commit not to dissent from p, and conversely, if we dissent with p we must commit not to assent to p. Indeed, as argued in (Incurvati and Schlöder, 2019, p. 748), it is a tenet of inferential expressivism to take the incompatibility of assent and dissent as primitive, to explain the inconsistency of statements such as "p and $\neg p$", rather than the other way around. This cornerstone of multilateralist thought is represented by taking $\overline{\Gamma, +p \Rightarrow \oplus p}^{Ax}$ and $\overline{\Gamma, -p \Rightarrow \ominus p}^{Ax}$ as axioms.

The structural rules, being formulated for arbitrary signed formulae, are the same as in G2BML. Notice that, given the updated definition of the polarity-changing maps, also the updated Smileian reductio and Rejection principles are covered by the schemas already presented as part of G2BML. Thus, these are the sequent versions of Subalternity and Weak inference:

$$\frac{\Gamma \Rightarrow {}_{+/-}A}{\Gamma \Rightarrow {}_{+/-}{}^{\flat}A}\,(SA_i) \qquad (*)\,\frac{\Gamma \Rightarrow \oplus A \qquad \Delta, +A \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \alpha^{\flat}}\,WI_1 \qquad (*)\,\frac{\Gamma \Rightarrow \ominus A \qquad \Delta, -A \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \alpha^{\flat}}\,WI_2$$

Where α can be $\bot$, or, for an arbitrary $B \in \mathcal{L}_{\mathsf{ML}}$, $+B$ in the case of WI_1 and $-B$ in the case of WI_2. Evidence preservation, denoted with $(*)$, now restricts derivations not only to those without premises signed with $\ominus$, but also to ones without premises signed with $\oplus$ and inferences without applications of $(+\Diamond L)$ and $(\oplus\Diamond L)$, in accordance with the remarks of (Incurvati and Schlöder, 2019; Incurvati and Schlöder, 2022).

We now prove soundness and completeness with respect to $\mathsf{N_{EML}}$. The connective cases are analogous to the strong assertion rules, and SA_i is immediate, and thus we only check modalities and Weak Inference rules. As most cases follow the same proof shape, only the representative cases will be considered.

Theorem 3.2.1 (Soundness and Completeness)**.** $\vdash_{\mathsf{N_{EML}}} \Gamma \Rightarrow \alpha$ *iff* $\vdash_{\mathsf{G2EML}+Cut} \Gamma \Rightarrow \alpha$

Proof. We prove both directions by induction on the depth of the natural deduction and sequent calculus tree respectively.

($\Rightarrow$) We consider the representative cases of $+\Diamond I, +\Diamond E$ and WI_1, the others being similar (or analogous to the respective BML counterpart).

$$\begin{array}{c}[\Gamma]\\ \vdots\\ \dfrac{\oplus A}{+\Diamond A}\,(+\neg I)\end{array} \qquad\Longrightarrow\qquad \begin{array}{c}\mathcal{IH}\\ \dfrac{\Gamma \Rightarrow \oplus A}{\Gamma \Rightarrow +\Diamond A}\,(+\neg R)\end{array}$$

$$\begin{array}{c}[\Gamma]\\ \vdots\\ \dfrac{+\Diamond A}{\oplus A}\,(+\Diamond E)\end{array} \qquad\Longrightarrow\qquad \dfrac{\begin{array}{c}\mathcal{IH}\\ \Gamma \Rightarrow +\Diamond A\end{array} \qquad (*)\,\dfrac{\dfrac{}{\Gamma, \oplus A \Rightarrow \oplus A}\,\text{Init.}}{+\Diamond A, \Gamma \Rightarrow \oplus A}\,+\Diamond L}{\Gamma \Rightarrow \oplus A}\,\text{Cut}$$

$$\dfrac{\begin{array}{c}[\Gamma]\\ \vdots\\ \oplus A\end{array} \qquad \begin{array}{c}[\Gamma, +A]\\ \vdots *\\ \alpha\end{array}}{\alpha^{\flat}}\,WI_1 \qquad\Longrightarrow\qquad (*)\,\dfrac{\begin{array}{c}\mathcal{IH}\\ \Gamma \Rightarrow \oplus A\end{array} \qquad \begin{array}{c}\mathcal{IH}\\ \Gamma, +A \Rightarrow \alpha\end{array}}{\Gamma \Rightarrow \alpha^{\flat}}\,WI_1$$

($\Leftarrow$) We consider the representative cases of $+\Diamond I, +\Diamond E$ and WI_1, the others being symmetric.

$$\begin{array}{c}\mathcal{D}\\ \dfrac{\Gamma \Rightarrow \oplus A}{\Gamma \Rightarrow +\Diamond A}\,(+\Diamond R)\end{array} \qquad\Longrightarrow\qquad \begin{array}{c}[\Gamma]\\ \mathcal{IH}\\ \dfrac{\oplus A}{+\Diamond A}\,(+\Diamond I)\end{array}$$

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma, \oplus A \Rightarrow \alpha\end{array}}{\Gamma, +\Diamond A \Rightarrow \alpha}\,(+\Diamond L) \quad \Longrightarrow \quad \begin{array}{c}[\mathcal{IH}]\\ \dfrac{+\Diamond A}{[\oplus A]}\,+\Diamond E\\ \mathcal{D}'\\ \alpha\end{array}$$

$$(*)\ \dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow \oplus A\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \Gamma, +A \Rightarrow \alpha\end{array}}{\Gamma \Rightarrow \alpha^\flat}\,WI_1 \quad \Longrightarrow \quad \dfrac{\begin{array}{c}[\Gamma]\\ \mathcal{IH}\\ \oplus A\end{array} \qquad \begin{array}{c}[\Gamma, +A]\\ \mathcal{IH}*\\ \alpha\end{array}}{\alpha^\flat}\,WI_1$$

This concludes our induction, witnessing that G2EML is a proper translation of $\mathsf{N}_{\mathsf{EML}}$

□

This yields the following corollary:

Corollary 3.2.1. G2EML + *Cut is sound and complete with respect to* **S5***, modulo* τ*.*

This follows the result of (Incurvati and Schlöder, 2022) that the translation of EML into the language of modal logic is sound and complete with respect to the modal logic **S5**.

Given our generalised axiom forms, we can prove the following multilateral generalisation of the derivability of generalised initial sequents for G2EML. To do so, we update our definition of formula weight:

Definition 12 (Weight of $\mathcal{L}_{\mathsf{EML}}$ formula)**.** *Weight of formulae of* BML *is defined the canonical way as follows, for* $\mathbf{a}, \mathbf{b}, \mathbf{s} \in \{+, -, \oplus, \ominus\}$ *and "$\circ$" an arbitrary binary connective:*

$$\begin{aligned}
w(\bot) &:= 0\\
w({\scriptstyle +/-}^\flat p) &:= 1\\
w({\scriptstyle +/-} p) &:= 2\\
w(\mathbf{s}\neg A) &:= w(\mathbf{s}^{\sim} A) + 1\\
w(\mathbf{s}\nabla A) &:= w(\mathbf{s}A) + 1 \textit{ for } \nabla \in \{\Box, \Diamond\}\\
w(\mathbf{s}A \circ B) &:= w(\mathbf{a}A) + w(\mathbf{b}B) + 1
\end{aligned}$$

With a properly defined induction measure, we can now prove the following theorem:

Theorem 3.2.2 (Multilateral Initial Sequents)**.** $\Gamma, \mathbf{s}A \Rightarrow \mathbf{s}^{\natural}A$ *is derivable in* G2EML *for* $\mathbf{s} \in S, \natural \in \{\flat, \mathbf{1}\}$*, A an arbitrary well-formed formula of* $\mathcal{L}_{\mathsf{ML}}$ *and* Γ *an arbitrary context of signed formulae.*

Proof. By induction on the weight of $\mathbf{s}A$. The base case is witnessed by the axioms $\overline{\Gamma, {\scriptstyle +/-}p \Rightarrow {\scriptstyle +/-}p}^{\,Ax}$ (together with their weak counterparts) and $\overline{\Gamma, {\scriptstyle +/-}p \Rightarrow {\scriptstyle +/-}^\flat p}^{\,Ax}$. For the inductive step, we consider some representative cases, the others being analogous to the following, or to the case of BML.

$_{-}A = +A_1 \wedge A_2$. We have, for $\mathbf{a} \in \{+, \oplus\}, i \in \{1, 2\}$ that $w(\mathbf{a}A_i) \leq w(\mathbf{a}A_1 \wedge A_2) \leq n$. Then, for Γ arbitrary, we get the following, without loss of generality:

$$\dfrac{\dfrac{\dfrac{\mathcal{IH}}{+A_1, \Gamma \Rightarrow +A_1}}{+A_1 \wedge A_2, \Gamma \Rightarrow +A_1}\,(\oplus \wedge L) \qquad \dfrac{\dfrac{\mathcal{IH}}{\oplus A_2, \Gamma \Rightarrow \oplus A_2}}{+A_1 \wedge A_2, \Gamma \Rightarrow \oplus A2}\,(\oplus \wedge L)}{+A \wedge A_2, \Gamma \Rightarrow \oplus A_1 \wedge A_2}\,(\oplus \wedge R)$$

$+/{-}A = +\Diamond B$. We have, for $\mathbf{a} \in \{+, \oplus\}$ that $w(\mathbf{a}B) \leq w(\mathbf{a}\Diamond B) \leq n$. Then, for Γ arbitrary, we get the following, without loss of generality:

$$\dfrac{\dfrac{\dfrac{\mathcal{IH}}{\oplus B, \Gamma \Rightarrow \oplus B}}{+\Diamond B, \Gamma \Rightarrow \oplus B}\,(+\Diamond L)}{+\Diamond B, \Gamma \Rightarrow \oplus\Diamond B}\,(\oplus\Diamond R)$$

$+/{-}A = +\neg B$. We have, for $\mathbf{a} \in \{+, \oplus\}$ that $w(\mathbf{a}B) \leq w(\mathbf{a}\neg B) \leq n$. Then, for Γ arbitrary, we get the following, without loss of generality:

$$\dfrac{\dfrac{\dfrac{\mathcal{IH}}{-B, \Gamma \Rightarrow \ominus B}}{+\neg B, \Gamma \Rightarrow \ominus B}\,(+\neg L)}{+\neg B, \Gamma \Rightarrow \oplus\neg B}\,(\oplus\neg R)$$

$+/{-}A = +\Box B$. We have, for $\mathbf{a} \in \{+, \oplus\}$ that $w(\mathbf{a}B) \leq w(\mathbf{a}\Box B) \leq n$. Then, for Γ arbitrary, we get the following, without loss of generality:

$$\dfrac{\dfrac{\dfrac{\mathcal{IH}}{+B, \Gamma \Rightarrow +B}}{+\Box B, \Gamma \Rightarrow +B}\,(+\Box L)}{+\Box B, \Gamma \Rightarrow \oplus\Box B}\,(\oplus\Box R)$$

This concludes our induction, and shows that $\Gamma, \mathbf{s}A \Rightarrow \mathbf{s}^{\natural}A$ is indeed derivable in G2EML for $\natural \in \{\flat, \mathbf{1}\}$, A an arbitrary well-formed formula of $\mathcal{L}_{\mathsf{CPL}}$ and Γ an arbitrary context of signed formulae.

$\square$

3.2.2 Admissibility Theorems

We can also extend the results of the admissibility of the structural rules proven for G2BML to G2EML. Given the similarity between the rules for the modalities and those of negation, the proofs are immediate. However, the case of cut admissibility is of exceptional interest, and therefore we will report a representative case. The proof is the canonical one, by lexicographic induction on weight of the cut formula and cut height, and we will present the further case of the asserted $\Diamond$ modality (the boolean cases being analogous to the strongly asserted fragment). The cases of $\Box$ and $\Diamond$ signed with other force markers are analogous.

Theorem 3.2.3 (Cut admissibility)**.** *The rule of cut*

$$\dfrac{\Gamma \Rightarrow \alpha \qquad \alpha, \Delta \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \beta}\, Cut$$

Is admissible in G2EML*.*

Proof. The base case is analogous to the case of G2BML, with the additional cases involving $\oplus$ treated the same way. We report here some representative examples. Suppose we have the following instance:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta \Rightarrow +p\end{array} \qquad \dfrac{}{+p, \Gamma \Rightarrow \oplus p}\,Ax}{\Gamma, \Delta \Rightarrow \oplus p}\, cut$$

If $-p$ is not principal, then there is a rule R such that $\dfrac{\begin{array}{c}\mathcal{D}'\\ \Delta' \Rightarrow +p\end{array}}{\Delta \Rightarrow +p}\,R$. Therefore, we can obtain the conclusion by an application of R, obtaining the following derivation of lower cut height:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}'\\ \Delta' \Rightarrow +p\end{array} \qquad \dfrac{}{+p, \Gamma' \Rightarrow \oplus p}\,Ax}{\Gamma, \Delta' \Rightarrow \oplus p}\, cut}{\Gamma, \Delta \Rightarrow \oplus p}\, R$$

If $\Delta \Rightarrow +p$ is obtained by $SR^{\star}$-in (wlog), *cut* can be eliminated entirely as follows:

$$(*)\ \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta, -p \Rightarrow \bot\end{array}}{\Delta \Rightarrow +p}\,SR^{\star}\text{-in} \qquad \dfrac{}{+p, \Gamma \Rightarrow \oplus p}\,Ax}{\Gamma, \Delta \Rightarrow \ominus p}\,cut \qquad \Longrightarrow \qquad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Delta, -p \Rightarrow \bot\end{array}}{\Delta \Rightarrow \oplus p}\,SR\text{-in}}{\Gamma, \Delta \Rightarrow \oplus p}\,Wk$$

We also consider cases in which either premiss is an instance of the *Incoherence* initial sequent. If *Incoherence* occurs as a left premiss, then the conclusion of a putative application of *cut* follows by $\bot_L$. If it occurs on the right, on the other hand, if the left premiss is an axiom we get that the conclusion of cut is still an instance of *Incoherence*.

Further, we have cases concerning *Smileian reductio* and the initial sequents of incoherence, $\Gamma, +p, -p \Rightarrow \bot$, $\Gamma, +p, \ominus p \Rightarrow \bot$ and $\Gamma, \oplus p, -p \Rightarrow \bot$. Suppose we have the following derivation, without loss of generality:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma, \oplus p \Rightarrow \bot\end{matrix}}{\Gamma \Rightarrow -p}\,SR\text{-in} \qquad \dfrac{}{\Delta, \oplus p, -p \Rightarrow \bot}\,Inc.}{\Gamma, \oplus p, \Delta \Rightarrow \bot}\,cut$$

This can be replaced by obtaining the conclusion using an application of $SR^{\star}$-out and (possibly repeated) weakening:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}'\\ \Gamma \Rightarrow -p\end{matrix}}{\oplus p, \Gamma \Rightarrow \bot}\,SR\text{-out}}{\oplus p, \Gamma, \Delta \Rightarrow \bot}\,Wk$$

The reduction of cut height with $SR^{(\star)}$-in and $SR^{(\star)}$-out with symmetric restrictions is analogous to the one of G2BML. If, on the other hand, we have restrictions on *reductio* only on one side, we also consider the case of weak assertion. We examine the representative case below, where the derivation is of cuth height $m + n + 1 + 1$:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma, \oplus p \Rightarrow \bot\end{matrix}}{\Gamma \Rightarrow -p}\,SR\text{-}in \qquad (*)\,\dfrac{\begin{matrix}\mathcal{D}_1\\ \Delta \Rightarrow +p\end{matrix}}{-p, \Delta \Rightarrow \bot}\,SR^{\star}\text{-out}}{\Gamma, \Delta \Rightarrow \bot}\,cut$$

Here we cannot immediately apply cut to the premises as in the previous case, but we can nonetheless reduce cut height to $m + n + 1$ this way (notice that the newly introduced cut can be reduced according to one of the previous base cases):

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \Delta \Rightarrow +p\end{matrix} \qquad \dfrac{}{+p, \Delta \Rightarrow \oplus p}\,Ax}{\Delta \Rightarrow \oplus p}\,cut \qquad \begin{matrix}D_0\\ \oplus p, \Gamma \Rightarrow \bot\end{matrix}}{\Gamma, \Delta \Rightarrow \bot}\,cut$$

For the inductive step, we consider the following cases:

The cut formula α is not principal in the left premiss

<u>$+\Diamond L$, with $\Gamma = \Gamma', +\Diamond A$.</u> We then have the following derivation with cut height $n + m + 1$:

$$
\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma', \oplus A \Rightarrow \alpha\end{matrix}}{\Gamma', +\Diamond A \Rightarrow \alpha}{+\Diamond L} \qquad \begin{matrix}\mathcal{D}_1\\ \alpha, \Delta \Rightarrow \beta\end{matrix}}{\Delta, \Gamma', +\Diamond A \Rightarrow \beta}\,Cut
$$

Which becomes the following, where cut is applied to formulae of lower weight and is of cut height $n + m$:

$$
\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma', \oplus A \Rightarrow \alpha\end{matrix} \qquad \begin{matrix}\mathcal{D}_1\\ \alpha, \Delta \Rightarrow \beta\end{matrix}}{\Delta, \Gamma', \oplus A \Rightarrow \beta}\,Cut}{\Delta, \Gamma', +\Diamond A \Rightarrow \beta}\,+\Diamond L
$$

The case in which $\alpha$is not principal in the right premiss is symmetric and therefore omitted. We now turn to the final case:

The cut formula α is principal in both premises.

$\underline{\alpha = +\Diamond A}$. We then have the following derivation with cut height $m + n + 1$:

$$
\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma \Rightarrow \oplus A\end{matrix}}{\Gamma \Rightarrow +\Diamond A}\,+\Diamond R \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ \Delta, \oplus A \Rightarrow \beta\end{matrix}}{\Delta, +\Diamond A \Rightarrow \beta}\,+\Diamond L}{\Delta, \Gamma \Rightarrow \beta}\,Cut
$$

Which becomes the following, with both cut height and weight reduced:

$$
\dfrac{\begin{matrix}\mathcal{IH}\\ \Gamma \Rightarrow \oplus A\end{matrix} \qquad \begin{matrix}\mathcal{IH}\\ \Delta, \oplus A \Rightarrow \beta\end{matrix}}{\Gamma, \Delta \Rightarrow \beta}\,Cut
$$

□

This yields as a corollary that soundness and completeness, with respect to both $\mathsf{N}_{\mathsf{EML}}$ and **S5**, is cut-free. In (Incurvati and Schlöder, 2022), Incurvati and Schlöder prove that the strong assertive fragment of EML corresponds to **S5** consequence modulo the translation τ, as mentioned in the previous paragraph. Then it follows from our soundness and completeness result between G2EML and $\mathsf{N}_{\mathsf{EML}}$ and our cut admissibility result, that, modulo τ, $\Gamma \vdash^{\mathbf{S5}} A$ iff $\Gamma \vdash^{\mathsf{G2EML}} +A$. As a sanity check for our cut-admissibility proof, we use this to show a cut-free derivation in G2EML of the asserted B axiom of **S5** modal logic, $+p \supset \Box\Diamond p$. This is particularly interesting as in the sequent calculus for **S5**, the B axiom is known to be non-derivable without *cut*. Indeed,

$$\dfrac{\dfrac{\dfrac{\overline{+p \Rightarrow \oplus p}^{\,Ax}}{+p \Rightarrow +\Diamond p}\,+\Diamond R}{+p \Rightarrow +\Box\Diamond p}\,+\Box R}{\Rightarrow +p \supset \Box\Diamond p}\,+\supset R$$

Notice that the axiom $\overline{\Gamma, +p \Rightarrow \oplus p}^{\,Ax}$ is required for the proof search to terminate. From the viewpoint of proof-theoretic semantics for modal logics, this makes G2EML particularly appealing. Indeed, it is hard to have proof theories for modal logic which do not refer more or less explicitly to possible world semantics, or nonetheless some kind of model-theoretic semantics (see Poggiolesi and Restall, 2012 for an overview). Even the proposal advanced by Poggiolesi and Restall in (2012) of using tree-hypersequents, which avoid explicit reference to Kripke frames, suffers from similar issues. Indeed, tree-hypersequents are still implicitly dependent on notions of "zones" and "accessibility", where if not possible worlds, something suspiciously similar is assumed to explain language use with respect to modal operators. G2EML, on the contrary, not only avoids any reference to models, but also manages to explain the meaning of (epistemic) modal vocabulary on conceptually independent terms, fully based on conversational commitment, while retaining the proof-theoretic advantage of being cut-free.

The updated version of the coordination principle of rejection is also admissible.

Theorem 3.2.4. (Admissibility of Rejection). *The following rules*

$$\dfrac{\Gamma \Rightarrow {}_{+/-}{\sim}A \qquad \Delta \Rightarrow {}_{+/-}A}{\Gamma, \Delta \Rightarrow \bot}\,Rej^{\sim} \qquad\qquad \dfrac{\Gamma \Rightarrow {}_{+/\ominus}{\lightning}A \qquad \Delta \Rightarrow {}_{+/\ominus}A}{\Gamma, \Delta \Rightarrow \bot}\,Rej^{\lightning}$$

are admissible in G2BML.

Proof. The base cases involving initial sequents and *Smileian reductio* are analogous to their G2BML counterpart, with the inconsistency between $\oplus$ and $-$ being dealt with in the same way as the one between $+$ and $\ominus$. We therefore only consider two inductive steps of interest, which are representative of the instances of *rejection* involving modality.

The rejection formulae is not principal in the left premiss, with $\Gamma = \Gamma', +\Diamond A$

Without loss of Generality, consider the following:

$$\dfrac{+\Diamond L\,\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma', \oplus A \Rightarrow -B\end{matrix}}{\Gamma', +\Diamond A \Rightarrow -B} \qquad \begin{matrix}\mathcal{D}_1\\ \Delta \Rightarrow +B\end{matrix}}{\Delta, \Gamma', +\Diamond A \Rightarrow \bot}\,Rej^{\sim}$$

Which becomes the following, where *rejection* is applied to formulae of lower weight and is of *rejection* height $n + m$:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0 \\ \Gamma', \oplus A \Rightarrow -B\end{matrix} \qquad \begin{matrix}\mathcal{D}_1 \\ \Delta \Rightarrow +B\end{matrix}}{\Delta, \Gamma', \oplus A \Rightarrow \bot}\, Rej^{\sim}}{\Delta, \Gamma', +\Diamond A \Rightarrow \bot}\, +\Diamond L$$

The rejection formula is principal in both premisses

Without loss of generality, consider the following tree of *rejection height* $m + n + 1 + 1$:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0 \\ \Gamma \Rightarrow \oplus B\end{matrix}}{\Gamma \Rightarrow +\Diamond B}\,(+\Diamond R) \qquad \dfrac{\begin{matrix}\mathcal{D}_1 \\ \Delta \Rightarrow -B\end{matrix}}{\Delta \Rightarrow \ominus\Diamond B}\,(\ominus\Diamond R)}{\Delta, \Gamma \Rightarrow \bot}\, Rej^{\lightning}$$

Which becomes the following, with both *rejection* height and weight reduced:

$$\dfrac{\begin{matrix}\mathcal{D}_0 \\ \Gamma \Rightarrow \oplus B\end{matrix} \qquad \begin{matrix}\mathcal{D}_1 \\ \Delta \Rightarrow -B\end{matrix}}{\Gamma, \Delta \Rightarrow \bot}\, Rej^{\lightning}$$

This concludes our induction, and we can therefore conclude that *rejection* is admissible in G2EML.

□

Further, we extend the admissibility of generalised Smileian reductio to the relevant G2EML cases.

Theorem 3.2.5 (Generalised *Smileian Reductio*)**.** *The following rules*

$$(*)\,\dfrac{\Gamma, \mathbf{s}^{\sim}A \Rightarrow \bot}{\Gamma \Rightarrow \mathbf{s}A}\, SR^{\star}\text{-}in \qquad (*)\,\dfrac{\Gamma, \Rightarrow \mathbf{s}^{\sim}A}{\Gamma, \mathbf{s}A \Rightarrow \bot}\, SR^{\star}\text{-}out$$

$$\dfrac{\Gamma, \mathbf{s}^{\lightning}A \Rightarrow \bot}{\Gamma \Rightarrow \mathbf{s}A}\, SR\text{-}in \qquad \dfrac{\Gamma, \Rightarrow \mathbf{s}^{\lightning}A}{\Gamma, \mathbf{s}A \Rightarrow \bot}\, SR\text{-}out$$

Are admissible in G2EML *for all formulae* $A \in \mathcal{L}_{\mathsf{ML}}$.

Proof. We prove both results by induction on *Smileian reductio* height. In both *-in* and *-out* instances (with and without context restrictions), the base cases are trivial since the rule is primitive on atoms. For the inductive step we proceed similarly to *cut* admissibility, by showing that each application of Smileian reductio can be reduced to a lower *reductio* height.

We begin by considering some representative cases for SR-in, being the case of $SR^{\star}$-in analogous modulo context restrictions. Since the boolean operators are the same as in G2BML (with the weakly asserted rules behaving like they strongly asserted counterparts), we consider only representative modality cases.

$\underline{\mathbf{s}A = +\Diamond B}$. then the derivation on the left can be transformed into the one on the right of lower *reductio* height:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma, -B \Rightarrow \bot\end{array}}{\Gamma, \ominus\Diamond B \Rightarrow \bot}\,\ominus\Diamond L}{\Gamma \Rightarrow +\Diamond B}\,SR\text{-}in \quad\Longrightarrow\quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma, -B \Rightarrow \bot\end{array}}{\Gamma \Rightarrow \oplus B}\,SR\text{-}in}{\Gamma \Rightarrow +\Diamond B}\,+\Diamond R$$

We now consider representative modal cases for SR-*out*, being the boolean cases analogous to G2BML and the case of $SR^{\star}$-*out* similar modulo context restrictions.

$\underline{\mathbf{s}A = +\Diamond B}$. then the derivation on the left can be transformed into the one on the right of lower *reductio* height:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \Rightarrow -B\end{array}}{\Gamma \Rightarrow \ominus\Diamond B}\,\ominus\Diamond R}{\Gamma, +\Diamond B \Rightarrow \bot}\,SR\text{-}out \quad\Longrightarrow\quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \Rightarrow -B\end{array}}{\oplus B, \Gamma \Rightarrow \bot}\,SR\text{-}out}{\Gamma, +\Diamond B \Rightarrow \bot}\,+\Diamond L$$

This concludes our induction, witnessing that the atomic rules for *Smileian reductio* can be generalised to arbitrary formulae in G2EML.

□

We now begin to show admissibility of the new coordination principles of G2EML by showing that the rules SA_i are admissible. Interestingly, the use of *cut* is required, in this case jointly with multilateral initial sequents.

Theorem 3.2.6 (Admissibility of SA_i)**.** *The rule*

$$\dfrac{\Gamma \Rightarrow {}_{+/-}A}{\Gamma \Rightarrow {}_{+/-}{}^{\flat}A}\,(SA_i)$$

is admissible in G2EML.

Proof. We show it is admissible by applying cut to ${}_{+/-}A$ and using multilateral initial sequents. Suppose that the premiss $\Gamma \Rightarrow {}_{+/-}A$ is derivable. Then, the conclusion can be obtained this way:

$$\dfrac{\begin{array}{c}\mathcal{D}\\ \Gamma \Rightarrow {}_{+/-}A\end{array} \qquad \dfrac{}{{}_{+/-}A \Rightarrow {}_{+/-}{}^{\flat}A}\,Init.}{\Gamma \Rightarrow {}_{+/-}{}^{\flat}A}\,Cut$$

□

Moreover, *Weak Inference* is admissible, as witnessed by the following theorem:

Theorem 3.2.7 (Admissibility of Weak Inference)**.** *The following rules*

$$(*)\,\frac{\Gamma \Rightarrow \oplus A \qquad \Delta, +A \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \alpha^{\flat}}\,WI_1 \qquad\qquad (*)\,\frac{\Gamma \Rightarrow \ominus A \qquad \Delta, -A \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \alpha^{\flat}}\,WI_2$$

are admissible in G2BML*, for α an arbitrary ${\scriptstyle +/-}B$ or $\bot$ and where (*) indicates that no premises are signed with ${\scriptstyle \oplus/\ominus}$ and the derivation of α does not include applications of $+\Diamond E$ or $\oplus\Diamond E$.*

The proof strategy is similar to the one for the rejection rule, where each application of *Weak Inference* is permuted upwards in the proof tree. Accordingly, we define the notion of *Weak Inference height* the canonical way.

Definition 13 (Weak Inference height)**.** *Let the height of a derivation $\mathcal{D}$ be the length of its longest branch minus one. Then the* Weak Inference height *is the sum of the lengths of the derivations of the premisses of WI_1 or WI_2. We call* weak inference formula *the formula $\mathbf{s}A$ for $\mathbf{s}$ arbitrary in the premisses of weak inference.*

We can now prove our admissibility theorem, using the following schematic version:

$$\frac{\Gamma \Rightarrow {\scriptstyle +/-}^{\flat}A \qquad \Delta, {\scriptstyle +/-}A \Rightarrow \alpha}{\Gamma, \Delta \Rightarrow \alpha^{\flat}}\,WI_i$$

Proof. By lexicographic induction on the weight of $\mathbf{s}A$ and *Weak Inference height*. The base case is the axiom case or an instance of $\bot_L$. Suppose, without loss of generality, that the left premiss is an axiom (the case in which the right premiss is an axiom is symmetric). We then have, for $\natural \in \{\flat, \mathbf{1}\}$ that $\Gamma = \Gamma', {\scriptstyle +/-}^{\natural}p$ and $\Gamma \Rightarrow {\scriptstyle +/-}^{\flat}A \equiv {\scriptstyle +/-}^{\natural}p, \Gamma' \Rightarrow {\scriptstyle +/-}^{\flat}p$. Then the derivation looks as follows:

$$\frac{\dfrac{}{{\scriptstyle +/-}^{\natural}p, \Gamma' \Rightarrow {\scriptstyle +/-}^{\flat}p}\,Ax \qquad \begin{matrix}\mathcal{D}\\ \Delta, {\scriptstyle +/-}p \Rightarrow \alpha\end{matrix}}{{\scriptstyle +/-}^{\natural}p, \Gamma', \Delta \Rightarrow \alpha^{\flat}}\,WI_i$$

If $\Delta, {\scriptstyle +/-}p \Rightarrow \alpha$ is not an axiom then, suppose ${\scriptstyle +/-}p$ is principal, and thus there must be some rule R such that $\dfrac{\begin{matrix}\mathcal{D}'\\ \Delta', {\scriptstyle +/-}p \Rightarrow \beta\end{matrix}}{\Delta, {\scriptstyle +/-}p \Rightarrow \beta}\,R$. Therefore, we can obtain the conclusion by an application of R, resulting in the following derivation of lower rejection height:

$$\dfrac{\dfrac{\dfrac{}{{\scriptstyle +/-}^{\natural}p, \Gamma' \Rightarrow {\scriptstyle +/-}^{\flat}p}\,Ax \qquad \begin{matrix}\mathcal{D}'\\ \Delta', {\scriptstyle +/-}p \Rightarrow \alpha\end{matrix}}{{\scriptstyle +/-}^{\natural}p, \Gamma', \Delta' \Rightarrow \alpha^{\flat}}\,WI_i}{{\scriptstyle +/-}^{\natural}p, \Gamma', \Delta \Rightarrow \alpha^{\flat}}\,R$$

If both premises are axioms, they must be of the form $\dfrac{}{{\scriptstyle +/-}^{\natural}p, \Gamma' \Rightarrow {\scriptstyle +/-}^{\flat}p}\,Ax$ and $\dfrac{}{{\scriptstyle +/-}p, \Delta' \Rightarrow {\scriptstyle +/-}p}\,Ax$. We then have that the conclusion of the WI rule is already an axiom, i.e., $\dfrac{}{{\scriptstyle +/-}^{\natural}p, {\scriptstyle +/-}^{\flat}p, \Gamma', \Delta' \Rightarrow {\scriptstyle +/-}^{\flat}p}\,Ax$ If either Γ or Δ contain $\bot$, then the conclusion can just be obtained by $\bot_L$ and we are done.

We conclude the base case by considering the case of *Smileian reductio*. We present some representative cases, the others being analogous modulo context restrictions. Suppose the left premiss constitutes an application of $SR - in$, while the right premiss is an initial sequent of Incoherence. Then, without loss of generality:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma, +p \Rightarrow \bot\end{matrix}}{\Gamma \Rightarrow \ominus p}\, SR\text{-in} \qquad \dfrac{}{\Delta, +p, -p \Rightarrow \bot}\, Inc.}{\Gamma, +p, \Delta \Rightarrow \bot}\, WI_2$$

This can be replaced by obtaining the conclusion using an application of SR-out, and (possibly repeated) applications of weakening:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Gamma \Rightarrow \ominus p\end{matrix}}{+p, \Gamma \Rightarrow \bot}\, SR\text{-out}}{+p, \Gamma, \Delta \Rightarrow \bot}\, Wk$$

If we have an application of SR-out on the right and a multilateral initial sequent on the left, without loss of generality:

$$\dfrac{\dfrac{}{+p, \Gamma' \Rightarrow \oplus p}\, Ax \qquad \dfrac{\begin{matrix}\mathcal{D}\\ \Delta \Rightarrow \ominus p\end{matrix}}{\Delta, +p \Rightarrow \bot}\, SR\text{-out}}{+p, \Gamma', \Delta \Rightarrow \bot}\, WI_i$$

We can eliminate *Weak inference entirely* by means of SR-out and (possibly repeated) weakening:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \Delta \Rightarrow \ominus p\end{matrix}}{+p, \Delta \Rightarrow \bot}\, SR\text{-out}}{+p, \Gamma, \Delta \Rightarrow \bot}\, Wk$$

Otherwise, consider the following derivation with applications of SR in both premisses:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_0\\ \Gamma, +p \Rightarrow \bot\end{matrix}}{\Gamma \Rightarrow \ominus p}\, SR\text{-in} \qquad \dfrac{\begin{matrix}\mathcal{D}_1\\ \Delta \Rightarrow \oplus p\end{matrix}}{-p, \Delta \Rightarrow \bot}\, SR\text{-out}}{\Gamma, \Delta \Rightarrow \bot}\, WI_2$$

In this case, we can reduce WI-height as follows:

$$\dfrac{\begin{matrix}\mathcal{D}_1\\ \Delta \Rightarrow \oplus p\end{matrix} \qquad \begin{matrix}\mathcal{D}_0\\ \Gamma, +p \Rightarrow \bot\end{matrix}}{\Gamma, \Delta \Rightarrow \bot}\, WI_1$$

As a last case, if we have asymmetric restrictions on the two premisses, we can reduce *weak inference* to *cut*, as in the following example:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma,+p\Rightarrow\bot\end{array}}{\Gamma\Rightarrow\ominus p}\,SR\text{-in}\quad(*)\,\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta\Rightarrow +p\end{array}}{-p,\Delta\Rightarrow\bot}\,SR^{\star}\text{-out}}{\Gamma,\Delta\Rightarrow\bot}\,WI_2\quad\Longrightarrow\quad\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta\Rightarrow +p\end{array}\quad\begin{array}{c}\mathcal{D}_0\\ \Gamma,+p\Rightarrow\bot\end{array}}{\Gamma,\Delta\Rightarrow\bot}\,cut$$

For the inductive step, we proceed analogously to the case of *rejection*. We accordingly consider the case in which the rejection formula is not principal in the left premiss (the right case being symmetric) and the one in which it is principal in both. We will only be considering the rules for $\oplus A\wedge B$; ${}_{\oplus/\ominus}\neg A$; $\mathbf{s}\Diamond A$, the other cases being analogous.

The weak inference formula sA is not principal in the left premiss

$\underline{\Gamma=\Gamma',\ominus\neg B}$, without loss of generality (the other case being symmetric). We then have the following derivation of *weak inference* height $m+n+1=m+2$:

$$\dfrac{(\ominus\neg L)\,\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma',\oplus B\Rightarrow\oplus A\end{array}}{\Gamma',\ominus\neg B\Rightarrow\oplus A}\quad\begin{array}{c}\mathcal{D}_1\\ \Delta,+A\Rightarrow\alpha\end{array}}{\Delta,\Gamma',\ominus\neg B\Rightarrow\alpha^{\flat}}\,WI_1$$

Which becomes the following, where *weak inference* is applied to formulae of lower weight and is of *weak inference* height $n+m=m+1$:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma',\oplus B\Rightarrow\oplus A\end{array}\quad\begin{array}{c}\mathcal{D}_1\\ \Delta,+A\Rightarrow\alpha\end{array}}{\Delta,\Gamma',\oplus B\Rightarrow\alpha^{\flat}}\,WI_1}{\Delta,\Gamma',\ominus\neg B\Rightarrow\alpha^{\flat}}\,\ominus\neg L$$

$\underline{\Gamma=\Gamma',\oplus A\wedge B}$. We then have the following tree, of *weak inference* height $m+n+1$:

$$\dfrac{(\oplus\wedge L)\,\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma',\oplus B\Rightarrow\oplus A\end{array}}{\Gamma',\oplus A\wedge B\Rightarrow\oplus A}\quad\begin{array}{c}\mathcal{D}_1\\ \Delta,+A\Rightarrow\alpha\end{array}}{\Delta,\Gamma',\oplus A\wedge B\Rightarrow\alpha^{\flat}}\,WI_1$$

Which then becomes the following, with weak inference height $m+n$ and lower weight of weak inferece formula:

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_0 & \mathcal{D} \\ \Gamma', \oplus B \Rightarrow \oplus A & \Delta, +A \Rightarrow \alpha\end{array}}{\Delta, \Gamma', \oplus B \Rightarrow \alpha^\flat}\ WI_1}{\Delta, \Gamma', \oplus A \wedge B \Rightarrow \alpha^\flat}\ \oplus \wedge L$$

$\underline{\Gamma = \Gamma', \oplus\Diamond B}$, without loss of generality (the other cases being symmetric). We then have the following derivation of *weak inference* height $m + n + 1$:

$$\dfrac{(\oplus\Diamond L)\ \dfrac{\begin{array}{c}\mathcal{D}_0 \\ \Gamma', \oplus B \Rightarrow \oplus A\end{array}}{\Gamma', \oplus\Diamond B \Rightarrow \oplus A} \quad \begin{array}{c}\mathcal{D}_1 \\ \Delta, +A \Rightarrow \alpha\end{array}}{\Delta, \Gamma', \oplus\Diamond A \Rightarrow \alpha^\flat}\ WI_1$$

Which becomes the following, where *weak inference* is applied to formulae of lower weight and is of *weak inference* height $n + m$:

$$\dfrac{\dfrac{\begin{array}{cc}\mathcal{D}_0 & \mathcal{D}_1 \\ \Gamma', \oplus B \Rightarrow \oplus A & \Delta, +A \Rightarrow \alpha\end{array}}{\Delta, \Gamma', \oplus B \Rightarrow \alpha}\ WI_1}{\Delta, \Gamma', \oplus\Diamond A \Rightarrow \alpha^\flat}\ \oplus\Diamond L$$

sA is principal in both premises

$\underline{\mathbf{s}A = {}_{+/-}{}^{\natural}\neg B}$. We then have the following tree of *weak inference* height $m + n + 1 + 1$:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_0 \\ \Gamma \Rightarrow \ominus B\end{array}}{\Gamma \Rightarrow \oplus\neg B}\ (\oplus\neg R) \quad \dfrac{\begin{array}{c}\mathcal{D}_1 \\ \Delta, -B \Rightarrow \alpha\end{array}}{\Delta, +\neg B \Rightarrow \alpha}\ (+\neg L)}{\Delta, \Gamma \Rightarrow \alpha^\flat}\ WI_1$$

Which becomes the following, with both weak inference height and weight reduced:

$$\dfrac{\begin{array}{cc}\mathcal{D}_0 & \mathcal{D}_1 \\ \Gamma \Rightarrow \ominus B & \Delta, -B \Rightarrow \alpha\end{array}}{\Gamma, \Delta \Rightarrow \alpha^\flat}\ WI_2$$

$\underline{\mathbf{s}A = {}_{+/-}{}^{\natural}B \wedge C}$. We then have the following derivation with cut height $m + \mathsf{max}(n + k) + 1 + 1$, without loss of generality (($- \wedge R_2$) and the disjunction case are both analogous):

$$
\dfrac{\dfrac{\dfrac{\mathcal{D}_0}{\Gamma \Rightarrow \oplus B} \qquad \dfrac{\mathcal{D}_2}{\Gamma \Rightarrow +C}}{\Gamma \Rightarrow \oplus B \wedge C}\,(\oplus \wedge R) \qquad \dfrac{\dfrac{\mathcal{D}_2}{\Delta, +B \Rightarrow \alpha}}{\Delta, +B \wedge C \Rightarrow \alpha}\,(+ \wedge L)}{\Delta, \Delta \Rightarrow \alpha^{\flat}}\,WI_1
$$

Which becomes the following, with both rejection height and formula weight reduced:

$$
\dfrac{\dfrac{\mathcal{D}_0}{\Gamma \Rightarrow \oplus B} \qquad \dfrac{\mathcal{D}_2}{\Delta, +B \Rightarrow \alpha}}{\Delta, \Gamma \Rightarrow \alpha^{\flat}}\,WI_1
$$

<u>$\mathbf{s}A = {}_{+/-}{}^{\natural}\Diamond B$</u>. We then have the following tree of *weak inference* height $m + n + 1 + 1$:

$$
\dfrac{\dfrac{\dfrac{\mathcal{D}_0}{\Gamma \Rightarrow \oplus B}}{\Gamma \Rightarrow \oplus \Diamond B}\,(\oplus \Diamond R) \qquad \dfrac{\dfrac{\mathcal{D}_1}{\Delta, \oplus B \Rightarrow \alpha}}{\Delta, +\Diamond B \Rightarrow \alpha}\,(+\Diamond L)}{\Delta, \Gamma \Rightarrow \alpha^{\flat}}\,WI_1
$$

Which becomes the following, with both weak inference height and weight reduced:

$$
\dfrac{\dfrac{\dfrac{\mathcal{D}_0}{\Gamma \Rightarrow \oplus B} \qquad \dfrac{\mathcal{D}_1}{\oplus B, \Delta \Rightarrow \alpha}}{\Gamma, \Delta \Rightarrow \alpha}\,Cut \qquad \dfrac{}{\Delta, \alpha \Rightarrow \alpha^{\flat}}\,Init.}{\Gamma, \Delta \Rightarrow \alpha^{\flat}}\,Cut
$$

Since the remaining cases are analogous, this completes our induction, witnessing the admissibility of the coordination principle of *Weak Inference*.

□

3.3 Meaning and Harmony in a Multilateral Modal Context

In the previous chapters, we saw how balance between conditions and consequences is a desideratum of unilateral harmony, and balance between assertions and rejections is a desideratum for bilateral harmony. The question therefore naturally arises about what distinctively multilateral notions should be in balance for harmony to be preserved in the presence of subaltern speech acts. What should not licence too much or too little with respect to what? We begin this investigation by defining the connectives monk and mank, to show what can go wrong in otherwise harmonious systems, allowing us to formulate a suitable extension of bilateral harmony for the multilateral setting. Let monk be the connective whose meaning is defined by the following clauses:

$$
\dfrac{\Gamma \Rightarrow +A_1 \qquad \Gamma \Rightarrow +A_2}{\Gamma \Rightarrow +A_1 \mathsf{monk} A_2}\,(+\mathsf{monk}R) \qquad\qquad \dfrac{\Gamma, +A_i \Rightarrow \alpha}{\Gamma, +A_1 \mathsf{monk} A_2 \Rightarrow \alpha}\,(+\mathsf{monk}L_i)
$$

$$\frac{\Gamma \Rightarrow -A_i}{\Gamma \Rightarrow -A_1 \mathsf{monk} A_2}\ (-\mathsf{monk}R_i) \qquad \frac{\Gamma, -A_1 \Rightarrow \alpha \qquad \Gamma, -A_2 \Rightarrow \alpha}{\Gamma, -A_1 \mathsf{monk} A_2 \Rightarrow \alpha}\ (-\mathsf{monk}L)$$

$$\frac{\Gamma \Rightarrow \oplus A_i}{\Gamma \Rightarrow \oplus A_1 \mathsf{monk} A_2}\ (\oplus\mathsf{monk}R_i) \qquad \frac{\Gamma, \oplus A_1 \Rightarrow \alpha \qquad \Gamma, \oplus A_2 \Rightarrow \alpha}{\Gamma, \oplus A_1 \mathsf{monk} A_2 \Rightarrow \alpha}\ (\oplus\mathsf{monk}L)$$

$$\frac{\Gamma \Rightarrow \ominus A_1 \qquad \Gamma \Rightarrow \ominus A_2}{\Gamma \Rightarrow \ominus A_1 \mathsf{monk} A_2}\ (\ominus\mathsf{monk}R) \qquad \frac{\Gamma, \ominus A_i \Rightarrow \alpha}{\Gamma, \ominus A_1 \mathsf{monk} A_2 \Rightarrow \alpha}\ (\ominus\mathsf{monk}L_i)$$

The connective monk is defined in such a way that each speech act is unilaterally and bilaterally harmonious. The proof of this is left to the reader, but for the purpose of the present discussion it is useful to notice that as far as introduction and elimination rules, and rules for contraries and subcontraries are concerned, monk is defined as familiarly harmonious connectives. However, if we add monk to the calculus G2EML, triviality ensues, as we are able to infer an arbitrary $\oplus q$ from any assertion $+p$:

$$\frac{\dfrac{\overline{+p \Rightarrow \oplus p}\ Ax}{+p \Rightarrow \oplus p \mathsf{monk} q}\ \oplus\mathsf{monk}R_1 \qquad \dfrac{\overline{+p, +q \Rightarrow +q}\ Ax}{+p, +p \mathsf{monk} q \Rightarrow +q}\ +\mathsf{monk}L}{+p \Rightarrow \oplus q}\ WI_1$$

According to our understanding of the force markers in terms of conversational commitment, this means that if we accept monk as a rule, committing to any proposition p forbids us to rule out any proposition q. Therefore, in the same way we want connectives to introduce no new information (see Steinberger, 2011a), we also want them to introduce no new commitments, also with respect to weaker speech acts. Two more details are worth mentioning. The first is that the application of WI_1 is necessary to conclude $+p \Rightarrow \oplus q$, and hence not eliminable. The second is that the rules for +monk and $\oplus$monk have very different shapes, the former being conjunctive and the latter disjunctive (similarly to tonk, with the difference that the introduction and elimination rules for each speech act are harmonious).

However, the coordination principle of *weak inference* is designed to impose a parallelism between strongly and weakly signed inferences, which breaks in the case of monk: entailment should be preserved from "if α, then β" to "if *perhaps* α, then *perhaps* β", but this does not seem to be the case for monk. From this we can conclude that the admissibility of *weak inference* is required for pathological connectives of this sort to be ruled out.

This allows us also to address the question raised at the end of chapter 2: while G2BML is bilaterally harmonious with respect to contraries and contradictories, is it harmonious along all sides of the square of opposition? To answer, we need to check if WI with respect to string and weak rejection is admissible in G2BML. We consider the critical case of negation in the admissibility proof, which requires changes of sign and accordingly constitutes, given the lack of a speech act of $\oplus$, the only possibly non-analogous step for G2BML:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow +B\end{array}}{\Gamma \Rightarrow \ominus\neg B}\,(\ominus\neg R) \qquad (*)\,\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Delta, +B \Rightarrow \alpha\end{array}}{\Delta, -\neg B \Rightarrow \alpha}\,(+\neg L)}{\Delta, \Gamma \Rightarrow \alpha^\flat}\,WI \qquad \Longrightarrow \qquad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Gamma \Rightarrow +B\end{array} \qquad \begin{array}{c}\mathcal{D}_1\\ \Delta, +B \Rightarrow \alpha\end{array}}{\Gamma, \Delta \Rightarrow \alpha}\,cut}{\Gamma, \Delta \Rightarrow \alpha^\flat}\,SA$$

As the reader can notice, the admissibility of WI in G2BML requires the admissibility of SA, which in turn can be derived by *cut* and the initial sequent $-A, \Gamma \Rightarrow \ominus A$. Given that adding the latter to the system is well-motivated and required for the success of cut admissibility, and its generalisation is analogous to a particular case of that of G2EML, multilateral harmony is easily recoverable in G2BML as well.

This however begs the question of what *kind* of balance this enforces between strong and weak speech acts, and, most importantly, what aspects of language need to be in balance for harmony of subalterns to obtain. We accordingly have a straightforward candidate *criterion*, and we need to understand what a suitable conception of harmony would be. Further, consider the connective mank, which we define according to the following rule-clauses:

$$\dfrac{\Gamma \Rightarrow \oplus A}{\Gamma \Rightarrow +\mathsf{mank}A}\,(+\mathsf{mank}R) \qquad \dfrac{\Gamma, \oplus A \Rightarrow \alpha}{\Gamma, +\mathsf{mank}A \Rightarrow \alpha}\,(+\mathsf{mank}L)$$

$$\dfrac{\Gamma \Rightarrow +A}{\Gamma \Rightarrow \oplus\mathsf{mank}A}\,(\oplus\mathsf{mank}R) \qquad \dfrac{\Gamma, +A \Rightarrow \alpha}{\Gamma, \oplus\mathsf{mank}A \Rightarrow \alpha}\,(\oplus\mathsf{mank}L)$$

$$\dfrac{\Gamma \Rightarrow \ominus A}{\Gamma \Rightarrow -\mathsf{mank}A}\,(-\mathsf{mank}R) \qquad \dfrac{\Gamma, \ominus A \Rightarrow \alpha}{\Gamma, -\mathsf{mank}A \Rightarrow \alpha}\,(-\mathsf{mank}L)$$

$$\dfrac{\Gamma \Rightarrow -A}{\Gamma \Rightarrow \ominus\mathsf{mank}A}\,(\ominus\mathsf{mank}R) \qquad \dfrac{\Gamma, -A \Rightarrow \alpha}{\Gamma, \ominus\mathsf{mank}A \Rightarrow \alpha}\,(\ominus\mathsf{mank}L)$$

mank is unilaterally and bilaterally harmonious, as the reader can easily verify by noticing hat $\oplus$mank is defined analogously to the harmonious $\oplus\Box$ rules, $+$mank in the same way as $+\Diamond$ and similarly the rejected versions. However, the relative generalisation of initial sequents is not derivable:

$$\dfrac{\dfrac{\overset{?}{\oplus A \Rightarrow +A}}{+\mathsf{mank}A, \Gamma \Rightarrow +A}\,(+\mathsf{mank}L)}{+\mathsf{mank}A, \Gamma \Rightarrow \oplus\mathsf{mank}A}\,(\oplus\mathsf{mank}R) \qquad \dfrac{\dfrac{\overset{?}{\ominus A \Rightarrow -A}}{-\mathsf{mank}A, \Gamma \Rightarrow -A}\,(-\mathsf{mank}L)}{-\mathsf{mank}A, \Gamma \Rightarrow \ominus\mathsf{mank}A}\,(\ominus\mathsf{mank}R)$$

What's more, for the proof to go through we would need generalised initial sequents of the form $\dfrac{}{\Gamma, {}_{+/-}{}^\flat A \Rightarrow {}_{+/-}A}\,init.$. An initial seqient of this sort, which states that dissenting from dissenting (conv. consenting) implies assenting (conv., dissenting) would clearly be unacceptable and would threaten an expressive collapse somewhat *akin to* a modal collapse[2]. Indeed, *perhaps* p should not generally (axiomat-

[2]This would not necessarily constitute a modal collapse properly intended, of the kind "if *might* p then p", since mank is not necessarily a modality expressing "*might*". However, the analogy helps highlighting the problematicity of assuming

ically, even!) imply p. Allowing for an initial sequent of this sort and a connective like mank would thus make the diagram of opposition collapse to a segment $+ \leftrightarrow -$. Again, the derivability of generalised multilateral initial sequents is a clear candidate criterion to enforce that no mankish behaviour occurs in the logic, but it nonetheless leaves open the question of how to explain this imbalance between subalterns.

Recall that unilaterally disharmonious connectives display a problematic unbalance between conditions and consequences of an expression, while bilaterally disharmonious connectives are problematic with respect to assertions and rejections. We contend that the kind of imbalance connectives like monk and mank exhibit is an imbalance between *evidence* and *commitment*. This is clearer in the case of mank, which violates the intuition that evidence should be stronger than commitment by requiring, for its generalised multilateral initial sequent to be derivable, that the opposite should also be the case. The technical machinery of $\mathsf{G2EML}$, by ruling such connectives out, not only is harmonious with respect to mankish connectives, but also makes this relationship between evidence and commitment transparent, and formally enforces this core intuition of inferential expressivism. However, even though we want evidence to be a stricter requirement than commitment, monk shows that it should not be *too* much stronger either. Indeed, we should want evidence for p to licence commitment to p, but nothing more: committing to assenting with p should not commit us to the possibility of an arbitrary q, or even everything that would follow from q were it warranted by evidence, as the conclusion $+p \Rightarrow \oplus q$ seems to suggest. While evidence must be stronger than commitment, it seems like it should be minimally so.

The distinctive modal flavour of the disharmony displayed by monk may hint at the fact that multilateral harmony is a sort of modal harmony, which would also explain the modal underpinnings of the collapse issues displayed with the mank example. Indeed, $+p \Rightarrow \oplus q$ can also be read as evidence for p committing us to all sorts of irrelevant possibilities, witnessed by the interpretation of "$\oplus$" as the force marker *perhaps*. Conversely, in the mank case, the disharmony generated by requiring $\oplus A \Rightarrow +A$ can be explained from this viewpoint, as already pointed out, in terms of a modal collapse. While conceiving of multilateral harmony as modal harmony can explain the data presented by the pathological multilateral connectives, it nonetheless faces problems of conceptual priority and generalisability. Indeed, we should want to explain the possibility-actuality imbalance in the same terms in which we explain what the vocabulary related to possibility and actuality means. That is, if we want to explain the meaning of epistemic modals in terms of conversational commitment, the basic notion of conversational commitment should also be preferred to explain how it is that our proof rules assign meaning to epistemic modal symbols. Recall the reasoning pattern represented by weak inference: if I am justified in committing to *perhaps* A, and evidence of A would provide evidence for B, then I am justified in committing to *perhaps* B. Moreover, *perhaps* α was explained as dissent from adding $\neg\alpha$ to the conversational common ground, and so we want assent to a positive statement α not to normatively require dissent to too many negative statements $\neg\beta$. Indeed, we should impose dissent to no other negative statements than $\neg\alpha$ and $\neg\gamma$ for γs that

something like $\overline{\Gamma, {}_{+/-}{}^{\flat}A \Rightarrow {}_{+/-}A}^{\,init.}$.

follow from α. Evidence licences commitment, but not any commitment, and commitment licences only the commitments that would follow given hypothetical evidence. Therefore, the admissibility of weak inference can be seen as a way to ensure conservativity with respect to commitment: no new commitments should be introduced were they not already warranted by evidence with respect to previous commitments. In a slogan, evidence should not overgenerate commitments.

The proponent of a modalised conception of harmony, at this point, is naturally entitled to a reply: " in your evidence-and-commitment talk, which you are trying to sell as conceptually prior to modal talk, you still cannot get rid of hypothetical jargon. What you take to be an issue of evidence and commitment, is actually an issue of possibility or actuality of evidence. Therefore, multilateral harmony is modal after all". However, possibilities (and hence modal notions) are not indispensable to explain the hypothetical character of the treatment of evidence and commitment in the present account of multilateral harmony. It is merely conversational shorthand for the unspecificity of whether evidence is available, and, as we have seen in Chapter 1, this is addressed by postulating commitment as the basic general notion and evidence as a stronger counterpart. Thus, the hypothetical nature of multilateral harmony can be explained chiefly in terms of evidence and commitment.

We can therefore now outline a comprehensive picture of the dimensions of language at play when formulating each level of harmony, from uni- to multilateral. Unilateral harmony consists of a balance between conditions and consequences of an attitude expression, for the same attitude expression. If we add further expressive devices, we need to make sure that no new commitments or incompatibilities arise in the presence of the coordination principles. We thus need to enforce *bi*lateral harmony between opposing speech acts and *multi*lateral harmony with respect to evidence and commitment licensing the available speech acts. To do so, conservativity must hold in the system by means of admissibility of the relevant coordination principles, as well as the generalisability of the initial sequents which axiomatise coherent commitment in the extended logic. This provides a general and uniform treatment of harmony for multilateral logic, which supplies the general multilateralist method of identifying linguistic realisation of speech acts and devising commitment- and evidence-preserving coordination principles between them.

Chapter 4

Conclusion and Outlook

In this thesis, we have presented a number of contributions to multilateral proof theory and proof-theoretic semantics. On the proof theory front, we developed G2-style sequent calculi for *Basic Multilateral Logic* (BML) and *Epistemic Multilateral Logic* (EML), for which the structural rule of *cut* and the coordination principles of *rejection* and *weak inference* were proven admissible. This was argued to have bearing on philosophical issues in proof-theoretic *semantics*, in particular with respect to the issue of proof-theoretic harmony. Indeed, we argued for a conception of harmony which should explain what constitutes balance between the aspects of language involved in the formulation of a multilateral proof-theoretic semantics: conditions and consequences, assertions and rejections and evidence and commitment. All these dimensions are argued to be interrelated but not e entirely reducible to one another, which in turn imposes certain choices on how coordination principles and structural rules are related. It is argued that the admissibility of *cut*, *rejection* and *weak inference* is a necessary condition for uni-, bi- and multilateral harmony, alongside (multilateral) generalised initial sequents.

However, these admissibility theorems cannot be seen as progressive generalisations and must be kept explicitly distinct, in the same way our conception of harmony aims to make explicit that harmony between conditions and consequences is not the same as that between assertions and rejections, and similarly between evidence and commitment. Conversely, accounts that hold that *rejection* is a generalisation of *cut*, on the other hand, mistakenly conflate conditions for rejections with consequences of assertions, and, albeit proof-theoretically valid, the following admissibility results betray a non-explanatory notion of harmony which is limited to ruling out *tonk*ish and *tunk*ish connectives. Accordingly, this philosophical position is reflected in the proof theory, where the rules are defined in such a way that this is non-trivially the case. From the discussion above, therefore, it follows that *Basic Multilateral Logic* (BML) and *Epistemic Multilateral Logic* (EML) are harmonious along all edges of the multilateral square of opposition. From a methodological standpoint, moreover, we have supplied the general multilateralist method of (Incurvati and Schlöder, 2023b) with a general method to enforce harmony of the rules upon the discovery of speech acts and corresponding coordination principles.

4.1 Directions for future work

We now conclude by outlining some possible directions for future work. From a technical standpoint, the G2 calculi developed here have the advantage of being easily translatable from and into the natural deduction presented by Incurvati and Schlöder in (2017; 2019; 2022; 2023), but at the cost of having primitive contraction and non-invertible rules. This does not hinder the harmony results we presented, but it would nonetheless be more elegant to present a calculus where no such compromises are needed. We furthermore believe that the framework of *deep inference* can provide interesting insights on multilateral proof theory. Deep inference allows to manipulate connectives at any depth of the formula, similarly to how multilateral rules manipulate connectives inside the scope of the force markers. Interestingly, deep inference allows for the definition of a host of diverse reduction procedures on top of *cut*, and it may be worth it to frame the coordination principles from this angle. Moreover, no account of possible roles of deep inference in proof-theoretic semantics is present in the literature.

Throughout the thesis, we often entertained the idea that while notions like normalisation show that a proof-system is consistent, harmony should show *why* a proof-system is meaning-conferring. However, a formal treatment of this claim beyond the usual Dummettian lines is missing, and by this we mean that it would be fascinating to address harmony from the standpoint of the formal theories on mathematical explanations and proofs-why developed by Francesca Poggiolesi. Seeing whether there can be an explicit formal distinction between a harmony proof and a normalisation proof along these lines would provide an interesting insight on the relationship between proof theory and proof-theoretic semantics. Moreover, it would be worth attempting to explore modalities other than the epistemic ones, to investigate the extent to which inferential expressivism can explain the modalities of the modal cube, and, especially with respect to the intensional paradoxes, it would be extremely interesting to supply the proof-theoretic account of paradoxicality of (Tennant, 2024) with multilateral generalisations.

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