A topos for monotone modified realizability using arrow algebras

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

We use arrow algebras to construct a topos for monotone modified realizability. We also define a category of assemblies that characterizes the $\neg\neg$-separated objects in the topos. We discuss common logical principles, including independence of premise, the axiom of choice, and the existence of the Fan recursor, in our topos.

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Chapter 1

Introduction

The goal of the thesis is to apply the theory of arrow algebra to construct a topos for monotone modified realizability.

In [Gu18], a category related to monotone modified realizability was already introduced. The author obtained a category of assemblies to interpret logic, discussed several logical principles, and concluded with Fan recursors. However, a topos was not constructed in that work, and this is the exact starting point and main goal of this thesis. We first construct a topos using arrow algebra, and then work backwards to define a category of assemblies that resembles the one in [Gu18]. We will also see how this category of assemblies is related to the topos. Finally, we discuss several common logical axioms that hold in our construction.

To give some background, arrow algebras ([BB26]) are a tool for building triposes ([Pit81]), and triposes are used to construct toposes. A topos is a category with rich structure and nice properties in which one can interpret logic.

Realizability, on the other hand, has the general idea of assigning witnesses, or realizers, to logical statements to give them a constructive interpretation. In its earliest forms, realizers were simply natural numbers assigned to arithmetic formulas. Later, "modified realizability" was introduced, where the idea is to distinguish between "potential realizers" and "actual realizers".

Realizability can be treated in a syntactic or proof-theoretic manner, and used to prove independence results for certain logical axioms. But the idea of realizability has also been introduced into category theory. The general pattern is to have a "category of assemblies", where the objects are a set X, together with a collection of "realizers" for elements in X. One can use one's creativity to modify the "realizer" part to reflect various ideas of realizability.

Regarding the term "monotone modified realizability", there are discussions of the syntactic or proof-theoretic aspects of it in [Koh08], but here we understand it as "majorizability" as in [Bez85]. The idea is to have a model where every element is bounded by a "majorant". Instead of requiring every function to be continuous, this approach allows us to assume that every function has a majorant, while still validating various important logical principles.

The structure of the thesis is simple and clear, as indicated in the abstract. In Chapter 2, we review some known results on partial combinatory algebra and arrow algebra that are essential for the later discussion. In Chapter 3, we discuss various arrow algebras that will eventually provide the candidate for building the arrow topos. After that, we define a category of majorizability assemblies in Chapter 4, which is understood as a characterization of $\neg\neg$-separated objects in the arrow topos. Finally, in Chapter 5, we discuss a few common logical principles and conclude with the Fan recursor.

Chapter 2

Preliminaries

Arrow algebras ([BB26]) are a tool for building triposes, and triposes are used to build toposes. Many examples of arrow algebras arise from partial combinatory algebras. A partial combinatory algebra behaves similarly to the set of natural numbers, and it provides a setting for doing recursion theory.

In this chapter we review some known results on partial combinatory algebras and arrow algebras that are essential for later discussion.

2.1 Partial combinatory algebra

The partial combinatory algebra (PCA) we will be working with is assumed to be discretely ordered and absolute, in the sense of [Zoe22]. We still simply call them PCAs. We also refer to [Oos08] for discussions on various PCAs. The definition of PCA below is modified in the sense that it is not identical to the definition in the references, for practical purposes.

Definition 1 (partial combinatory algebra)**.** *A partial combinatory algebra is a tuple $(P, \cdot)$, where $\cdot : P \times P \rightharpoonup P$ is a partial application. We usually abbreviate $a \cdot b$ as ab, and we assume this application associates to the left, i.e., we write abc to mean $(ab)c$. Also, there exist $\underline{k}, \underline{s} \in P$ such that for any $a, b, c \in P$, we have $\underline{k}ab \preceq a$, $\underline{s}ab\downarrow$, and $\underline{s}abc \preceq ac(bc)$.*

Here, for an expression e representing a possible element in P, we write $e\downarrow$ to mean that e is actually defined, thus it is indeed an element of P. For expressions e, e' representing possible elements in P, we write $e \preceq e'$ to mean that if $e'\downarrow$, then $e\downarrow$ and $e = e'$.

We know that $\underline{k}ab$ and $\underline{s}ab$ are always defined. $\underline{s}abc$ might be undefined, but whenever $ac(bc)$ is defined, we have that $\underline{s}abc$ is defined and $\underline{s}abc = ac(bc)$.

Example 2. *One of our favorite examples of a partial combinatory algebra is $\mathcal{K}_1$, where the underlying set is the set of natural numbers $\mathbb{N}$, and for $n, m \in \mathbb{N}$, the application $n \cdot m$ is defined as $\varphi_n(m)$, where φ_n is the partial recursive function with code n.*

It turns out that we can perform "λ-abstraction" in a PCA. This is guaranteed by an important theorem about PCAs. We begin by formally defining terms:

Definition 3. *Let P be a PCA. We define a set $E(P)$ of terms that are purely syntactic. Every $a \in P$ is regarded as a term. Additionally, assume we have a fixed infinite set of variables, which are also regarded as terms. Furthermore, if t and s are terms, then so is ts.*

For a term $t \in E(P)$, we write $t(x_1, \ldots, x_n)$ to mean that the variables in t are among $x_1, \ldots, x_n$. Given $a_1, \ldots, a_n \in P$, we write $t(a_1, \ldots, a_n)$ to mean the expression resulting from replacing each x_i with a_i in t.

If a term t has no variables, then we identify t with the corresponding expression in P, which may be defined or undefined.

We then define "abstraction" in a PCA, which is inductively defined as follows:

Definition 4. *For every term $t \in E(P)$ and for any variable x, we define another term $\langle x \rangle t$ inductively on the structure of t. More precisely, if t is an element in P or a variable that is not x, then $\langle x \rangle t := \underline{k}t$. Also, $\langle x \rangle x := \underline{skk}$. Finally, $\langle x \rangle t_1 t_2 := \underline{s}(\langle x \rangle t_1)(\langle x \rangle t)$.*

This definition of abstraction behaves well in the following sense:

Proposition 5 (See the comments in Definition 1.1.4 in [Oos08])**.** *For any term $t \in E(P)$, if the variables of t are among $x, x_1, \ldots, x_n$, then the variables of $\langle x \rangle t$ are among $x_1, \ldots, x_n$. Also, for any $a, a_1, \ldots, a_n \in P$, we have:*

- $(\langle x \rangle t)(a_1, \ldots, a_n) \downarrow$.
- $(\langle x \rangle t)(a_1, \ldots, a_n)a \preceq t(a, a_1, \ldots, a_n)$.

Remark 6. *For a term $t(x_1, \ldots, x_n, \ldots, x_{n+m})$, we write $\langle x_1 \ldots x_n \rangle t$ as an abbreviation for successive abstractions $\langle x_1 \rangle (\langle x_2 \rangle (\ldots (\langle x_n \rangle t) \ldots))$.*

As a corollary of the proposition above, we see that for any term $t(x_1, \ldots, x_n, x_{n+1})$, if we set $a := \langle x_1 \ldots x_n x_{n+1} \rangle t$, then a has no variables. For every $a_1, \ldots, a_n, a_{n+1} \in P$, we have $aa_1 \ldots a_n \downarrow$, $aa_1 \ldots a_n a_{n+1} \preceq t(a_1, \ldots, a_n, a_{n+1})$.

With this useful "abstraction" and its associated application property, we can do many meaningful constructions.

Example 7. *Define $\underline{i} := \langle u \rangle u$. Then for any $a \in P$, we have $\underline{i}a \preceq a$.*

Define $\overline{\underline{k}} := \langle uv \rangle v$. Then for any $a, b \in P$, we have $\overline{\underline{k}}ab \preceq b$.

Define $\underline{p} := \langle xyz \rangle zxy$, $\underline{p_0} := \langle x \rangle x\underline{k}$, and $\underline{p_1} := \langle x \rangle x\overline{\underline{k}}$. For $a, b \in P$, we understand $\underline{p}ab$ as the pairing of a and b. Notice that $\underline{p}ab$ is always defined. We have $\underline{p_0}(\underline{p}ab) \preceq a$, and $\underline{p_1}(\underline{p}ab) \preceq b$.

In a PCA, we can define numerals and do recursion theory.

Definition 8. *Define $\overline{\underline{0}} := \underline{i}$, and $\overline{\underline{n+1}} := \underline{p}\overline{\underline{k}}\overline{\underline{n}}$.*

One can then show that we can carry out many recursion-theoretic constructions in a PCA. For instance, there exists $\underline{S} \in P$ such that $\underline{S}\overline{\underline{n}} \preceq \overline{\underline{n+1}}$. We can also have a recursion operator to implement primitive recursion. Indeed, all primitive recursive functions can be represented in a PCA. Later we will merely claim that we have an element in P representing a certain primitive recursive function without proof.

Remark 9. *After defining the numerals and so on, we might wonder when our definitions become trivial; it might be the case that all the numerals are identical. By [Zoe22], we understand that in our case, as long as $\underline{k}$ and $\overline{\underline{k}}$ are not identical, then for any $n, m \in \mathbb{N}$, if $n \neq m$, then $\overline{\underline{n}} \neq \overline{\underline{m}}$.*

A PCA where $\underline{k}$ and $\overline{\underline{k}}$ are equal is said to be trivial, see Proposition 1.3.1 in [Oos08].

One useful property of PCAs is that we can encode finite sequences similarly to the case of natural numbers; this will be important later when we want to discuss majorizability.

Definition 10. *Define $J^1(a) = a$, and $J^{n+1}(a_1, a_2, \ldots, a_{n+1}) = \underline{p}a_1 J^n(a_2, \ldots, a_{n+1})$.*

If $u_0, \ldots, u_{n-1}$ is a finite sequence of elements in P, then we use $[u_0, \ldots, u_{n_1}]$ to denote the code of this sequence. If $n = 0$, then it is the empty sequence.

We define $[] \coloneqq \underline{p}\overline{0}\overline{0}$, and for $n \geq 1$: $[u_0, \ldots, u_{n-1}] \coloneqq \underline{p}\overline{n}J^n(u_0, \ldots, u_{n-1})$.

Remark 11. *Define $\mathsf{lh} \coloneqq \underline{p_0}$. Then for any sequence $u_0, \ldots, u_{n-1}$, we know $\mathsf{lh}[u_0, \ldots, u_{n-1}] \preceq \overline{n}$.*

If our PCA P is NOT semi-trivial, then we know that the coding of finite sequences is well-defined, in the sense that, using lh, we can uniquely determine the length of a sequence from its code.

Remark 12. *There exists $\underline{c}$ in the PCA such that for any f and x in the PCA, if $x = [a_1, \ldots, a_n]$, and for any $1 \leq i \leq n$, $fa_n \downarrow$ and is a code of finite sequence, then $\underline{c}fx \downarrow$, and $\underline{c}fx$ is the code of the concatenation of $fa_1, \ldots, fa_n$. We usually use $\sqcup$ to represent concatenation of finite sequences. Thus the property of $\underline{c}$ can be written as $\underline{c}fx = fa_1 \sqcup fa_2 \sqcup \cdots \sqcup fa_n$.*

2.2 Arrow algebra

An arrow algebra is a useful tool for defining triposes. The idea is that instead of proving all the ingredients of a tripos and verifying that the given ingredients indeed form a tripos, it suffices to provide a somewhat simpler structure, namely an arrow algebra; the tripos structure then follows naturally. We refer to [BB26] and [Bri23] for detailed discussions on arrow algebras.

Definition 13 (arrow algebra)**.** *An arrow algebra is a tuple $(A, \preceq, \to, S)$ where $(A, \preceq)$ is a complete poset, $\to: A \times A \to A$ is a function that is anti-monotone in the first argument and monotone in the second argument. The tuple $(A, \preceq, \to)$ is also called an arrow structure.*

$S \subseteq A$ is called a separator. S is required to be upwards closed, closed under modus ponens, i.e., if $a \to b \in S$ and $a \in S$, then $b \in S$. Moreover, three special elements $\mathbf{k}$, $\mathbf{s}$, $\mathbf{a}$ should belong to S. Here, $\mathbf{k} \coloneqq \bigcurlywedge_{a,b} a \to b \to a$, $\mathbf{s} \coloneqq \bigcurlywedge_{a,b,c} (a \to b \to c) \to (a \to b) \to a \to c$, and $\mathbf{a} \coloneqq \bigcurlywedge_{x,(y_i)_{i\in I},(z_i)_{i\in I}} (\bigcurlywedge_{i\in I} x \to y_i \to z_i) \to x \to \bigcurlywedge_{i\in I} y_i \to z_i$.

In practice, the arrows we work with usually have the following nice property.

Definition 14 (compatible with joins)**.** *We say A is compatible with joins if for every $a \in A$ and for every $B \subseteq A$ we have: $(\bigcurlyvee_{b\in B} b) \to a = \bigcurlywedge_{b\in B} b \to a$.*

Indeed, to show that A is compatible with joins, it suffices to verify $\bigcurlywedge_{b\in B} b \to a \preceq (\bigcurlyvee_{b\in B} b) \to a$.

Every arrow algebra induces a **Set**-indexed preorder, and the theory of arrow algebras ensures that this **Set**-indexed preorder is a tripos.

Definition 15 (arrow indexed pre-order)**.** *Define an arrow* **Set***-indexed preorder P.*

For every set I, define $\mathsf{P}I \coloneqq A^I$, the set of functions from I to A. For every set I, for every $\varphi, \psi : I \to A$, define $\varphi \vdash \psi$ if $\bigcurlywedge_{i\in I} \varphi(i) \to \psi(i) \in S$. This makes each $\mathsf{P}I$ into a preorder.

For every function $f : I \to J$, define $\mathsf{P}f : \mathsf{P}J \to \mathsf{P}I$ by pre-composition: for every $\varphi : J \to A$, $\mathsf{P}f(\varphi) \coloneqq \varphi.f : I \to A$.

Notice that P is functorial: $\mathsf{Pid}_I = \mathsf{id}_{\mathsf{P}I}$, and $\mathsf{P}(g.f) = \mathsf{P}f.\mathsf{P}g$.

Theorem 16 ([BB26], Theorem 5.10)**.** *The* **Set***-indexed preorder* P *defined in the above definition is a tripos.*

If the arrow algebra is nice in the sense that it is compatible with joins, which is usually the case in practice, then we obtain a simple expression for $\exists f$.

Definition 17. *If A is compatible with joins, then in the corresponding P, for any $f : I \to J$, define $\exists f : \mathsf{P}I \to \mathsf{P}J$ to be such that $(\exists f(\varphi))(j) := \bigcurlyvee_{i, f(i)=j} \varphi(i)$. This $\exists f$ is a left adjoint to $\mathsf{P}f$, i.e., for any $R \in \mathsf{P}I$ and $S \in \mathsf{P}J$, we have $\exists f(R) \vdash S$ if and only if $R \vdash \mathsf{P}f(S)$.*

A nucleus on an arrow algebra is a quick way of obtaining a new arrow algebra and the corresponding new tripos.

Definition 18 (nucleus)**.** *Let $(A, \preceq, \to, S)$ be an arrow algebra. A map $j : A \to A$ is called a nucleus if it satisfies (1) $a \preceq b$ implies that $ja \preceq jb$ for all $a, b \in A$, (2) $\bigcurlywedge_a a \to ja$ is in S, and (3) $\bigcurlywedge_{a,b} (a \to jb) \to (ja \to jb)$ is in S.*

Remark 19 (Lemma 3.13, Lemma 6.4 in [BB26])**.** *A nucleus $j : A \to A$ has the following nice properties: the following elements all belong to the separator: (a) $\bigcurlywedge_a jja \to ja$, (b) $\bigcurlywedge_{a,b} (a \to b) \to (ja \to jb)$, (c) $\bigcurlywedge_{a,b} j(a \to b) \to (ja \to jb)$, (d) $\bigcurlywedge_{a,b} j(a \times b) \to ja \times jb$, and (e) $\bigcurlywedge_{a,b} ja \times jb \to j(a \times b)$.*

We are now ready to obtain a new arrow algebra from a nucleus.

Proposition 20 (Proposition 3.15 in [BB26])**.** *Let $(A, \preceq, \to, S)$ be an arrow algebra and let $j : A \to A$ be a nucleus. Consider the tuple $(A, \preceq, \to_j, S_j)$ where $a \to_j b := a \to jb$, and $a \in S_j$ iff $ja \in S$. Then $(A, \preceq, \to_j, S_j)$ is again an arrow algebra, which we denote by A_j.*

Write P for the arrow tripos of A, and write P_j for the arrow tripos corresponding to the arrow algebra A_j. In practice, the expression of P_j is somewhat implicit. We will consider a more explicitly expressed **Set**-indexed preorder which is denoted by P^j, and P_j and P^j are equivalent. Therefore, given an arrow algebra and a nucleus j, we obtain a handy expression for another tripos P^j.

Definition 21. *Define a* **Set***-indexed preorder P^j as follows: For every set X, $\mathsf{P}^j X := \{\alpha \in \mathsf{P}X \mid j.\alpha \vdash \alpha\}$. Thus $\mathsf{P}^j X$ is a subset of $\mathsf{P}X$, and the order on $\mathsf{P}^j X$ is defined to be the restriction of the order on $\mathsf{P}X$. For every function $f : X \to Y$, $\mathsf{P}^j f : \mathsf{P}^j Y \to \mathsf{P}^j X$ is given by pre-composition with f. This is well-defined, because if $\alpha \in \mathsf{P}^j Y$, then $\bigcurlywedge_{y \in Y} j(\alpha(y)) \to \alpha(y) \in S$, and consequently, $\bigcurlywedge_{x \in X} j(\alpha(f(x))) \to \alpha(f(x)) \in X$, which means that $j(\mathsf{P}^j f(\alpha)) \vdash \mathsf{P}^j f(\alpha)$. Also, we know that $\mathsf{P}^j f$ preserves preorders, because $\mathsf{P}f$ does. Notice that P^j is functorial, and we conclude that P^j is a* **Set***-indexed preorder.*

Proposition 22 (Proposition 6.6 in [BB26])**.** *As* **Set***-indexed preorders, P_j and P^j are equivalent. Since P_j is a* **Set***-tripos, P^j is also a* **Set***-tripos.*

Remark 23 (Remark 6.8 in [BB26])**.** *The logical operations on the tripos P^j are explicitly related to those on P:*

At each X, $\mathsf{P}^j X$ is a Heyting algebra. The top element, binary meet, and implication on $\mathsf{P}^j X$ are the same as those in $\mathsf{P}X$. The bottom element and binary join in $\mathsf{P}^j X$ are the corresponding ones in $\mathsf{P}X$ after applying j.

Suppose $\forall f : \mathsf{P}X \to \mathsf{P}Y$ *is the right adjoint of* $\mathsf{P}f$*. If* $\alpha \in \mathsf{P}^j X$*, then* $\forall f(\alpha) \in \mathsf{P}^j Y$*. Therefore, we may view* $\forall f$ *also as a map* $\forall f : \mathsf{P}^j X \to \mathsf{P}^j Y$*. Clearly, this map preserves preorders, and is the right adjoint of* $\mathsf{P}^j f$*.*

Suppose $\exists f : \mathsf{P}X \to \mathsf{P}Y$ *is the left adjoint of* $\mathsf{P}f$*, then consider the map* $j.\exists f : \mathsf{P}^j X \to \mathsf{P}^j Y$*.* $j.\exists f$ *preserves preorders, and is the left adjoint of* $\mathsf{P}^j f$*.*

Chapter 3

Majorizability arrow algebras

In this chapter, we construct the arrow algebra that we will be working with. The prototype for majorizability is $MAJ(P)$, which captures the idea of majorizability. Since we will be working in a PCA and dealing with codes of functions, it is unwise to distinguish different codes of the same function. Therefore, we add equivalence relations and make it into another arrow algebra, which will be called $EMAJ(P)$ (the letter E stands for equivalence relation). Moreover, in the spirit of modified realizability, we will consider tuples consisting of “actual realizers” and “potential realizers”. The potential realizer part is understood as an element in $EMAJ(P)$, and the actual realizer part is understood as a subset of the collection of potential realizers. We will make this idea precise, and get an arrow algebra which will be called $dEMAJ(P)$ (the letter d stands for the domain of majorizability relations). Finally, as in [Gu18] where a maximum function on the collection of potential realizers is required, we adopt this idea and implement this feature by considering a nucleus j on the arrow algebra $dEMAJ(P)$.

We will first give a recap of three known arrow algebras which are the ingredients of our own arrow algebras in the first section. Then we consider the arrow algebras $EMAJ(P)$ and $dEMAJ(P)$ respectively. Finally, we consider the nucleus j on the arrow algebra $dEMAJ(P)$.

We fix a PCA P that is not trivial (see Remark 9).

3.1 Basic examples of arrow algebras

We review the results about three basic examples of arrow algebras related to a PCA P: $Pow(P)$, $PER(P)$, and $MAJ(P)$. They are all discussed in [Bri23].

Definition 24 ($Pow(P)$)**.** *$Pow(P)$ is the collection of subsets of P, ordered by inclusion. For $A, B \subseteq P$, define $A \to B \coloneqq \{r \in P : \forall a \in A(ra\downarrow, ra \in B)\}$. Define a separator $S_{Pow(P)} \subseteq Pow(P)$ to be the collection of all non-empty subsets of P.*

Definition 25 ($PER(P)$)**.** *$PER(P)$ is the collection of symmetric and transitive relations on P, ordered by inclusion. For $E_1, E_2 \in PER(P)$, define $E_1 \to E_2 \coloneqq \{(r, r') \mid \forall (a, a') \in E_1, ra\downarrow, r'a'\downarrow, (ra, r'a') \in E_2\}$. Define a separator $S_{PER(P)} \subseteq PER(P)$ to be the collection of all non-empty relations.*

Definition 26 ($MAJ(P)$)**.** *$MAJ(P)$ is the collection of $R \subseteq P \times P$ where R is transitive, and for any $x, y \in P$, if $(x, y) \in R$, then $(y, y) \in R$. $MAJ(P)$ is ordered by inclusion. For $R, S \in MAJ(P)$, define $R \to S \coloneqq \{(r, r^*) : \forall (y, y^*) \in R(ry\downarrow, r^*y\downarrow, r^*y^*\downarrow, (ry, r^*y^*) \in S, (r^*y, r^*y^*) \in S)\}$. Define a separator $S_{MAJ(P)} \coloneqq \{R \in MAJ(P) \mid \exists x, x' \in P((x, x') \in R)\}$.*

Proposition 27. *$Pow(P)$, $PER(P)$ and $MAJ(P)$ are arrow algebras. Also, they are compatible with joins.*

Indeed, those arrow algebras are also *quasi-implicative* in the sense of Remark 2.2 in [Miq20], although this observation is not needed in the later discussion.

The theory of arrow algebra ensures that for any arrow algebra A, the corresponding **Set**-indexed preorder is a tripos. However, in general the construction of logical operations, for instance, binary meets, joins, etc., is quite complicated and is not useful in practice. However, in the case of $Pow(P)$, $PER(P)$, and $MAJ(P)$, we can actually have much simpler explicit expressions of those logical structures on the **Set**-indexed preorders. This is mainly due to the rich structure on PCA, where we can do coding to achieve features like pairs and so on.

In the following, I will provide explicit expressions for logical operations related to $MAJ(P)$. Those for $Pow(P)$ and $PER(P)$ are indeed very similar to those for $MAJ(P)$. Also, we will only provide those logical operations on the set $MAJ(P)$, i.e. $\mathsf{P}1$, where P is the **Set**-tripos corresponding to the arrow algebra $MAJ(P)$. The explicit expressions of logical operations on each $\mathsf{P}X$ for arbitrary set X can be defined in exactly the same way; this is because, in some sense, the logical operations we are about to define are uniform, in the sense that they also have a uniform "realizer".

Proposition 28. *In $MAJ(P)$, recall that we equip $MAJ(P)$ with the logical order defined as follows: $R \vdash S$ if and only if $R \to S \in S_{MAJ(P)}$.*

- *Write $\perp$ for $\emptyset$. For any $R \in MAJ(P)$, we have $\perp \to R = P \times P$.*

- *Write $\top$ for $P \times P$. For any $R \in MAJ(P)$, if $R \neq \perp$, then $R \to \perp = \perp$.*

- *For $R, S \in MAJ(P)$, define $R \times S := \{(\underline{p}ab, \underline{p}a^*b^*) \mid (a, a^*) \in R, (b, b^*) \in S\}$. We have $R \times S \in MAJ(P)$.*

  We have $(\underline{p_0}, \underline{p_0}) \in (R \times S) \to R$ and $(\underline{p_1}, \underline{p_1}) \in (R \times S) \to S$. Also, for any $T \in MAJ(P)$, if $(r, r^*) \in T \to R$ and $(s, s^*) \in T \to S$, then $(\langle x \rangle \underline{p}(rx)(sx), \langle x \rangle \underline{p}(r^*x)(s^*x)) \in T \to (R \times S)$. Thus $R \times S$ is indeed the logical binary meet in $MAJ(P)$.

- *For $R, S \in MAJ(P)$, define $R + S := \{(\underline{p}\underline{k}a, \underline{p}\underline{k}a^*) \mid (a, a^*) \in R\} \cup \{(\underline{p}\underline{\bar{k}}b, \underline{p}\underline{\bar{k}}b^*) \mid (b, b^*) \in S\}$. We have $R + S \in MAJ(P)$.*

  We have $(\langle x \rangle \underline{p}\underline{k}x, \langle x \rangle \underline{p}\underline{k}x) \in R \to R + S$, and $(\langle x \rangle \underline{p}\underline{\bar{k}}x, \langle x \rangle \underline{p}\underline{\bar{k}}x) \in S \to R + S$. Also, for any $T \in MAJ(P)$, if $(r, r^*) \in R \to T$ and $(s, s^*) \in S \to T$, then $(\langle x \rangle \underline{p_0}xrs(\underline{p_1}x), \langle x \rangle \underline{p_0}xr^*s^*(\underline{p_1}x)) \in R + S \to T$. Thus $R + S$ is indeed the logical binary join in $MAJ(P)$.

- *For any $R, S, T \in MAJ(P)$: if $(r, r^*) \in (R \times S) \to T$, then $(\langle uv \rangle r(\underline{p}uv), \langle uv \rangle r^*(\underline{p}uv)) \in R \to S \to T$. Also, if $(r, r^*) \in R \to S \to T$, then $(\langle x \rangle r(\underline{p_0}x)(\underline{p_1}x), \langle x \rangle r^*(\underline{p_0}x)(\underline{p_1}x)) \in R \times S \to T$. Therefore, $R \times S \vdash T$ if and only if $R \vdash S \to T$.*

3.2 The arrow algebra $EMAJ(P)$

In this section, we aim to construct an arrow algebra where the elements are tuples of the form (R, E), where $R \in MAJ(P)$, $E \in PER(P)$, and such that R and E are compatible. The motivation for adding an equivalence relation E to the (domain of) the majorizability relation R is that, since we are dealing with codes, which are elements in a PCA, it would be unwise to distinguish between different codes representing the same function.

We will verify that this indeed yields an arrow algebra, and show that it is compatible with joins.

Definition 29 (dom(R) and $R \restriction X$)**.** *Let $R \in MAJ(P)$. Define $dom(R) \coloneqq \{a \in P \mid \exists a^* \in P((a, a^*) \in R)\}$. We know that $(a, a^*) \in R$ implies that $(a^*, a^*) \in R$.*

For any $X \subseteq P$, define $R \restriction X \coloneqq \{(a, a^*) \in R \mid a, a^* \in X\}$. We know $R \restriction X \in MAJ(P)$. Notice that $R \restriction X$ is monotone in both arguments: if $R \subseteq R'$ and $X \subseteq X'$, then $R \restriction X \subseteq R' \restriction X'$.

Definition 30 (dom(E) and $E \restriction X$)**.** *Let $E \in PER(P)$. Define $dom(E) \coloneqq \{a \in P \mid (a, a) \in E\}$. We know that $(a, a') \in E$ implies that $a, a' \in dom(E)$.*

For any set $X \subseteq P$, define $E \restriction X \coloneqq \{(a, a') \in E \mid a, a' \in X\}$. We know $E \restriction X \in PER(P)$. Notice that $E \restriction X$ is monotone in both arguments: if $E \subseteq E'$ and $X \subseteq X'$, then $E \restriction X \subseteq E' \restriction X'$.

Definition 31 ($EMAJ(P)$)**.** *Let $EMAJ(P)$ be the collection of all tuples of the form (R, E), where*

- *$R \in MAJ(P)$.*
- *$E \in PER(P)$.*
- *$dom(R) = dom(E)$.*
- *If $(a, b) \in R$, $(a, a'), (b, b') \in E$, then $(a', b') \in R$.*

For $(R_1, E_1), (R_2, E_2) \in EMAJ(P)$, define $(R_1, E_1) \preceq (R_2, E_2)$ if $R_1 \subseteq R_2$ and $E_1 \subseteq E_2$.
For $(R_1, E_1), (R_2, E_2) \in EMAJ(P)$, define implication to be:

$$\begin{aligned} & (R_1, E_1) \to (R_2, E_2) \\ \coloneqq \quad & ((R_1 \to R_2) \restriction dom(E_1 \to E_2), (E_1 \to E_2) \restriction dom((R_1 \to R_2) \restriction dom(E_1 \to E_2))) \end{aligned}$$

Remark 32. *To see that this implication is well-defined, i.e., $(R_1, E_1) \to (R_2, E_2)$ is in $EMAJ(P)$, we verify that if $(r, r^*) \in (R_1 \to R_2) \restriction dom(E_1 \to E_2)$, $(r, r'), (r^*, r'^*) \in (E_1 \to E_2) \restriction dom((R_1 \to R_2) \restriction dom(E_1 \to E_2))$, then $(r', r'^*) \in (R_1 \to R_2) \restriction dom(E_1 \to E_2)$.*

Firstly, we show that $(r', r'^*) \in R_1 \to R_2$. For any $(a, a^*) \in R_1$, since $(r, r^*) \in R_1 \to R_2$, we know $(ra, r^*a^*), (r^*a, r^*a^*) \in R_2$. Also, we know $a, a^* \in dom(R_1) = dom(E_1)$, thus $(a, a), (a^*, a^*) \in E_1$. Since $(r, r'), (r^*, r'^*) \in E_1 \to E_2$, we know $(ra, r'a), (r^*a, r'^*a), (r^*a^*, r'^*a^*) \in E_2$. Therefore, we know $(r'a, r'^*a^*), (r'^*a, r'^*a^*) \in R_2$.

Secondly, we show that $r', r'^* \in dom(E_1 \to E_2)$. We know $(r, r'), (r^*, r'^*) \in E_1 \to E_2$, thus $(r', r'), (r'^*, r'^*) \in E_1 \to E_2$, i.e., $r', r'^* \in dom(E_1 \to E_2)$.

We conclude that the implication in $EMAJ(P)$ is well-defined.

Lemma 33. *The partial order on $EMAJ(P)$ is complete.*

Proof. Suppose $(R_i, E_i)_{i \in I}$ is a family of elements in $EMAJ(P)$.

We claim that $\bigcurlywedge_{i \in I} (R_i, E_i) = (\bigcap_{i \in I} R_i, (\bigcap_{i \in I} E_i) \restriction \mathrm{dom}(\bigcap_{i \in I} R_i))$.

We know $\bigcap_{i \in I} R_i \in MAJ(P)$.

We know $\bigcap_{i \in I} E_i \in PER(P)$, thus $(\bigcap_{i \in I} E_i) \restriction \mathrm{dom}(\bigcap_{i \in I} R_i) \in PER(P)$.

We know $\mathrm{dom}(\bigcap_{i \in I} R_i) \subseteq \mathrm{dom}(\bigcap_{i \in I} E_i)$, thus $\mathrm{dom}((\bigcap_{i \in I} E_i) \restriction \mathrm{dom}(\bigcap_{i \in I} R_i)) = \mathrm{dom}(\bigcap_{i \in I} R_i)$.

If $(a,b) \in \bigcap_{i\in I} R_i$, and $(a,a'),(b,b') \in (\bigcap_{i\in I} E_i) \restriction \mathrm{dom}(\bigcap_{i\in I} R_i)$, then for any $i \in I$, $(a,b) \in R_i$, $(a,a'),(b,b') \in E_i$, and therefore, $(a',b') \in R_i$. Therefore, $(a',b') \in \bigcap_{i\in I} R_i$.

We conclude that $(\bigcap_{i\in I} R_i, (\bigcap_{i\in I} E_i) \restriction \mathrm{dom}(\bigcap_{i\in I} R_i)) \in EMAJ(P)$.

Also, suppose $(R',E') \in EMAJ(P)$ such that for any $i \in I$, we have $(R',E') \preceq (R_i,E_i)$. Then we know $R' \subseteq \bigcap_{i\in I} R_i$, $E' \subseteq \bigcap_{i\in I} E_i$. Also, we have $\mathrm{dom}(E') = \mathrm{dom}(R') \subseteq \mathrm{dom}(\bigcap_{i\in I} R_i)$. Thus, $E' = E' \restriction \mathrm{dom}(\bigcap_{i\in I} R_i) \subseteq (\bigcap_{i\in I} E_i) \restriction \mathrm{dom}(\bigcap_{i\in I} R_i)$.

We conclude that $(R',E') \preceq (\bigcap_{i\in I} R_i, (\bigcap_{i\in I} E_i) \restriction \mathrm{dom}(\bigcap_{i\in I} R_i))$. □

Lemma 34. *The implication in $EMAJ(P)$ is anti-monotone in the first component and monotone in the second component.*

Proof. Suppose $(R_1',E_1') \preceq (R_1,E_1)$ and $(R_2,E_2) \preceq (R_2',E_2')$.

We know $R_1 \to R_2 \subseteq R_1' \to R_2'$ and $E_1 \to E_2 \subseteq E_1' \to E_2'$.

Then, using the fact that $R \restriction X$ and $E \restriction X$ are monotone in every argument, we see that $(R_1,E_1) \to (R_2,E_2) \preceq (R_1',E_1') \to (R_2',E_2')$. □

Corollary 35. *$EMAJ(P)$ is an arrow structure.*

Lemma 36. *Suppose $(R_1,E_1) \in EMAJ(P)$. Suppose $R \in MAJ(P)$ and $E \in PER(P)$. Suppose $(R \restriction dom(E), E \restriction dom(R \restriction dom(E)))$ is in $EMAJ(P)$, then the following two are the same:*

- $(R_1,E_1) \to (R \restriction dom(E), E \restriction dom(R \restriction dom(E)))$.
- $((R_1 \to R) \restriction dom(E_1 \to E), (E_1 \to E) \restriction dom((R_1 \to R) \restriction dom(E_1 \to E)))$

Proof. Write $(R_2,E_2) \coloneqq (R_1,E_1) \to (R \restriction \mathrm{dom}(E), E \restriction \mathrm{dom}(R \restriction \mathrm{dom}(E)))$.

By definition of implication in $EMAJ(P)$, we know:

- $R_2 \coloneqq (R_1 \to (R \restriction \mathrm{dom}(E))) \restriction \mathrm{dom}(E_1 \to (E \restriction \mathrm{dom}(R \restriction \mathrm{dom}(E))))$.
- $E_2 \coloneqq (E_1 \to E \restriction \mathrm{dom}(R \restriction \mathrm{dom}(E))) \restriction \mathrm{dom}(R_2)$.

We have to show two things:

(1) $R_2 = (R_1 \to R) \restriction \mathrm{dom}(E_1 \to E)$.

(2) $E_2 = (E_1 \to E) \restriction \mathrm{dom}((R_1 \to R) \restriction \mathrm{dom}(E_1 \to E))$.

So we now prove (1) and (2):

(1) Since $R \restriction \mathrm{dom}(E) \subseteq R$ and $E \restriction \mathrm{dom}(R \restriction \mathrm{dom}(E)) \subseteq E$, we know $R_2 \subseteq (R_1 \to R) \restriction \mathrm{dom}(E_1 \to E)$. Thus it suffices to prove the converse.

Suppose $(r,r^*) \in (R_1 \to R) \restriction \mathrm{dom}(E_1 \to E)$. We show that $(r,r^*) \in R_2$.

By assumption, we know

- $(r,r^*) \in R_1 \to R$.
- $(r,r),(r^*,r^*) \in E_1 \to E$.

We show that $(r, r^*) \in R_1 \to R \restriction \mathrm{dom}(E)$. For any $(a, a^*) \in R_1$, we know $(ra, r^*a^*), (r^*a, r^*a^*) \in R$. We know $(a, a), (a^*, a^*) \in E_1$, thus $(ra, ra), (r^*a, r^*a), (r^*a^*, r^*a^*) \in E$. So $(ra, r^*a^*), (r^*a, r^*a^*) \in R \restriction \mathrm{dom}(E)$. Therefore, $(r, r^*) \in R_1 \to R \restriction \mathrm{dom}(E)$.

We show that $r, r^* \in \mathrm{dom}(E_1 \to (E \restriction \mathrm{dom}(R \restriction \mathrm{dom}(E))))$. For any $(a, a') \in E_1$, we know $(ra, ra'), (r^*a, r^*a') \in E$. We still have to show that $ra, ra', r^*a, r^*a' \in \mathrm{dom}(R \restriction \mathrm{dom}(E))$. Without loss of generality, we show that $ra \in \mathrm{dom}(R \restriction \mathrm{dom}(E))$. We know $a \in \mathrm{dom}(R)$, so there exists a^* such that $(a, a^*) \in R_1$. We have shown that $(r, r^*) \in R_1 \to R \restriction \mathrm{dom}(E)$, so $(ra, r^*a^*) \in R \restriction \mathrm{dom}(E)$, thus $ra \in \mathrm{dom}(R \restriction \mathrm{dom}(E))$. We conclude that $(r, r), (r^*, r^*) \in E_1 \to E \restriction \mathrm{dom}(R \restriction \mathrm{dom}(E))$, and thus $r, r^* \in \mathrm{dom}(E_1 \to E \restriction \mathrm{dom}(R \restriction \mathrm{dom}(E)))$.

We conclude that $(r, r^*) \in R_2$. Therefore, $(R_1 \to R) \restriction \mathrm{dom}(E_1 \to E) \subseteq R_2$.

(2) Since $E \restriction \mathrm{dom}(R \restriction \mathrm{dom}(E)) \subseteq E$ and $R_2 = (R_1 \to R) \restriction \mathrm{dom}(E_1 \to E)$, we know $E_2 \subseteq (E_1 \to E) \restriction \mathrm{dom}(R_2) = (E_1 \to E) \restriction \mathrm{dom}((R_1 \to R) \restriction \mathrm{dom}(E_1 \to E))$. So it suffices to prove the converse.

Suppose $(r, s) \in (E_1 \to E) \restriction \mathrm{dom}((R_1 \to R) \restriction \mathrm{dom}(E_1 \to E))$. We show that $(r, s) \in E_2$.

By assumption, we know:

- $(r, s) \in E_1 \to E$.
- There exist r^*, s^* such that $(r, r^*), (s, s^*) \in (R_1 \to R) \restriction \mathrm{dom}(E_1 \to E)$.

The second bullet above tells us that $r, s \in (R_1 \to R) \restriction \mathrm{dom}(E_1 \to E)$. Since we have shown that $R_2 = (R_1 \to R) \restriction \mathrm{dom}(E_1 \to E)$, it remains to show that $(r, s) \in E_1 \to E \restriction \mathrm{dom}(R \restriction \mathrm{dom}(E))$.

For $(a, a') \in E_1$, we know $(ra, sa') \in E$. We still have to show that $ra, sa' \in \mathrm{dom}(R \restriction \mathrm{dom}(E))$. Without loss of generality, we show that $ra \in \mathrm{dom}(R \restriction \mathrm{dom}(E))$. We know from the second bullet above that there exists r^* such that $(r, r^*) \in (R_1 \to R) \restriction (E_1 \to E) = R_2$. In particular, $(r, r^*) \in R_1 \to R \restriction \mathrm{dom}(E)$. We also know that there exists a^* such that $(a, a^*) \in R_1$. So $(ra, r^*a^*) \in R \restriction \mathrm{dom}(E)$, and thus $ra \in \mathrm{dom}(R \restriction \mathrm{dom}(E))$.

We conclude that $(r, s) \in E_1 \to E \restriction \mathrm{dom}(R \restriction \mathrm{dom}(E))$. Therefore, $(E_1 \to E) \restriction \mathrm{dom}((R_1 \to R) \restriction \mathrm{dom}(E_1 \to E)) \subseteq E_2$. $\square$

Remark 37. *The proof above does not rely on the assumption that* $(R \restriction dom(E), E \restriction dom(R \restriction dom(E)))$ *is in* $EMAJ(P)$. *We include it there only to ensure that we can make use of the definition of implication in* $EMAJ(P)$. *Otherwise we would have to unfold the definition of that implication if* $(R \restriction dom(E), E \restriction dom(R \restriction dom(E)))$ *is not an element of* $EMAJ(P)$.

Corollary 38. *Given* $(R_1, E_1), \ldots, (R_n, E_n) \in EMAJ(P)$ $(n \geq 2)$, *we have:*

$$\begin{aligned} & (R_n, E_n) \to (R_{n-1}, E_{n-1}) \to \cdots \to (R_1, E_1) \\ = \; & ((R_n \to \cdots \to R_1) \restriction dom(E_n \to \cdots \to E_1) \\ & , (E_n \to \cdots \to E_1) \restriction dom((R_n \to \cdots \to R_1) \restriction dom(E_n \to \cdots \to E_1))). \end{aligned}$$

Proof. By induction on n. The base case where $n = 2$ is by definition. For the induction step, we use the previous lemma. $\square$

Remark 39. *It is convenient to denote tuples of the form* $(X \restriction dom(Y), Y \restriction dom(X \restriction dom(Y)))$ *as* $N(X, Y)$, *where* $X \in MAJ(P)$ *and* $Y \in PER(P)$.

Then we know that for $(R_1, E_1), (R_2, E_2) \in EMAJ(P)$, *we have:*

$$(R_1, E_1) \to (R_2, E_2) = N(R_1 \to R_2, E_1 \to E_2).$$

Also, the previous lemma can be restated as: given $(R_1, E_1) \in EMAJ(P)$, $R \in MAJ(P)$, and $E \in PER(P)$, if $N(R, E) \in EMAJ(P)$, then

$$(R_1, E_1) \to N(R, E) = N(R_1 \to R, E_1 \to E)$$

So far, we have shown that $EMAJ(P)$ is an arrow structure. We still have to show that $EMAJ(P)$ can be made into an arrow algebra.

Definition 40 (separator)**.** *Define a separator on $EMAJ(P)$ by:*

$$S_{EMAJ(P)} := \{(R, E) \in EMAJ(P) \mid R \neq \emptyset\}.$$

Lemma 41. *$S_{EMAJ(P)}$ is upward closed.*

Proof. Suppose $(R_1, E_1) \in S_{EMAJ(P)}$ and $(R_1, E_1) \preceq (R_2, E_2)$. Then we know $R_1 \neq \emptyset$ and $R_1 \subseteq R_2$. Then $R_2 \neq \emptyset$, and therefore $(R_2, E_2) \in S_{EMAJ(P)}$. □

Lemma 42. *$S_{EMAJ(P)}$ is closed under modus ponens.*

Proof. Suppose $(R_1, E_1), (R_1, E_1) \to (R_2, E_2) \in S_{EMAJ(P)}$. Then in particular, $R_1 \neq \emptyset$ and $R_1 \to R_2 \neq \emptyset$. Thus $R_2 \neq \emptyset$, and $(R_2, E_2) \in EMAJ(P)$. □

Before proving that **k**, **s**, and **a** are all in the separator $S_{EMAJ(P)}$, let us consider the following lemma, which simplifies the verification that an element is in $S_{EMAJ(P)}$.

Lemma 43. *Suppose $R \in MAJ(P)$, and $E \in PER(P)$. Suppose $N(R, E) \in EMAJ(P)$. Then the following two are equivalent:*

- *$N(R, E) \in S_{EMAJ(P)}$.*
- *There exists (a, a^*) such that:*
  - *$(a, a^*) \in R$.*
  - *$(a, a), (a^*, a^*) \in E$.*

Therefore, we may say that (a, a^*) witnesses that $N(R, E) \in S_{EMAJ(P)}$.

Proof. Unfolding the definitions, we know $N(R, E) := (R \restriction \mathrm{dom}(E), E \restriction \mathrm{dom}(R \restriction E))$.

Therefore, $N(R, E) \in S_{EMAJ(P)}$ iff $R \restriction \mathrm{dom}(E)$ is non-empty. This is equivalent to saying that there exists (a, a^*) such that $(a, a^*) \in R \restriction \mathrm{dom}(E)$, which is equivalent to:

- $(a, a^*) \in R$.
- $(a, a), (a^*, a^*) \in E$.

We may conclude that $N(R, E) \in S_{EMAJ(P)}$ iff there exists (a, a^*) such that $(a, a^*) \in R$, and $(a, a), (a^*, a^*) \in E$. □

Therefore, to show that some $N(R, E) \in EMAJ(P)$ is in the separator, it suffices to provide a witness (a, a^*) satisfying $(a, a^*) \in R$, and $(a, a), (a^*, a^*) \in E$.

Remark 44. *Let I be an index set. Suppose for each $i \in I$, $R_i \in MAJ(P)$ and $E_i \in PER(P)$. Then*

$$\bigcap_{i\in I}(R_i \restriction dom(E_i)) = (\bigcap_{i\in I} R_i) \restriction dom(\bigcap_{i\in I} E_i).$$

Similarly, we have:

$$\begin{aligned} & (\bigcap_{i\in I}(E_i \restriction dom(R_i \restriction dom(E_i)))) \restriction dom(\bigcap_{i\in I}(R_i \restriction dom(E_i))) \\ = \; & (\bigcap_{i\in I} E_i) \restriction dom(\bigcap_{i\in I}(R_i \restriction dom(E_i))) \end{aligned}$$

Therefore, if for any $i \in I$, $(R_i \restriction dom(E_i), E_i \restriction dom(R_i \restriction dom(E_i))) \in EMAJ(P)$, then

$$\begin{aligned} & \curlywedge_{i\in I}(R_i \restriction dom(E_i), E_i \restriction dom(R_i \restriction dom(E_i))) \\ = \; & (\bigcap_{i\in I}(R_i \restriction dom(E_i)) \\ & , (\bigcap_{i\in I}(E_i \restriction dom(R_i \restriction dom(E_i)))) \restriction dom(\bigcap_{i\in I}(R_i \restriction dom(E_i)))) \\ = \; & ((\bigcap_{i\in I} R_i) \restriction dom(\bigcap_{i\in I} E_i) \\ & , (\bigcap_{i\in I} E_i) \restriction dom((\bigcap_{i\in I} R_i) \restriction dom(\bigcap_{i\in I} E_i))) \end{aligned}$$

To summarize, let I be an index set, and suppose for any $i \in I$, $R_i \in MAJ(P)$, $E_i \in PER(P)$, and $N(R_i, E_i) \in EMAJ(P)$, then:

$$\curlywedge_{i\in I} N(R_i, E_i) = N(\bigcap_{i\in I} R_i, \bigcap_{i\in I} E_i).$$

Therefore, given an I-indexed collection of pairs (R_i, E_i) satisfying $N(R_i, E_i) \in EMAJ(P)$, to show that $\curlywedge_{i\in I} N(R_i, E_i)$ is in the separator $S_{EMAJ(P)}$, it suffices to provide a witness (a, a^*) such that $(a, a^*) \in \bigcap_{i\in I} R_i$, and $(a, a), (a^*, a^*) \in \bigcap_{i\in I} E_i$.

Lemma 45. $\mathbf{k}$ *is in $S_{EMAJ(P)}$.*

Proof. We claim that $(\underline{k}, \underline{k})$ witnesses that $\mathbf{k} \in S_{EMAJ(P)}$. Indeed, for any $(R_1, E_1), (R_2, E_2) \in EMAJ(P)$, we have:

$$(R_1, E_1) \to (R_2, E_2) \to (R_1, E_1) = N(R_1 \to R_2 \to R_1, E_1 \to E_2 \to E_1).$$

Moreover, we know that $(\underline{k}, \underline{k}) \in R_1 \to R_2 \to R_1$ and $(\underline{k}, \underline{k}) \in E_1 \to E_2 \to E_1$. □

Lemma 46. $\mathbf{s}$ *is in $S_{EMAJ(P)}$.*

Proof. We claim that $(\underline{s}, \underline{s})$ witnesses that $\mathbf{s} \in S_{EMAJ(P)}$. Indeed, for any $(R_1, E_1), (R_2, E_2), (R_3, E_3) \in EMAJ(P)$, we have:

$$\begin{aligned} & ((R_1, E_1) \to (R_2, E_2) \to (R_3, E_3)) \to ((R_1, E_1) \to (R_2, E_2)) \to (R_1, E_1) \to (R_3, E_3) \\ = \; & N(R_1 \to R_2 \to R_3, E_1 \to E_2 \to E_3) \to N(R_1 \to R_2, E_1 \to E_2) \to N(R_1 \to R_3, E_1 \to E_3) \\ = \; & N((R_1 \to R_2 \to R_3) \restriction (\dots) \to (R_1 \to R_2) \restriction (\dots) \to (R_1 \to R_3) \\ & , (E_1 \to E_2 \to E_3) \restriction (\dots) \to (E_1 \to E_2) \restriction (\dots) \to (E_1 \to E_3)) \end{aligned}$$

Notice that $(R_1 \to R_2 \to R_3) \upharpoonright (\dots) \to (R_1 \to R_2) \upharpoonright (\dots) \to (R_1 \to R_3) \supseteq (R_1 \to R_2 \to R_3) \to (R_1 \to R_2) \to (R_1 \to R_3)$, and $(E_1 \to E_2 \to E_3) \upharpoonright (\dots) \to (E_1 \to E_2) \upharpoonright (\dots) \to (E_1 \to E_3) \supseteq (E_1 \to E_2 \to E_3) \to (E_1 \to E_2) \to (E_1 \to E_3)$.

Recall also that $(\underline{s}, \underline{s})$ is in both $(R_1 \to R_2 \to R_3) \to (R_1 \to R_2) \to (R_1 \to R_3)$ and $(E_1 \to E_2 \to E_3) \to (E_1 \to E_2) \to (E_1 \to E_3)$. □

Lemma 47. $\mathbf{a} \in S_{EMAJ(P)}$.

Proof. We claim that $(\langle uvw\rangle uvw, \langle uvw\rangle uvw)$ witnesses that $\mathbf{a} \in S_{EMAJ(P)}$. Indeed, for any (R, E_R), $(S_i, E_{S_i})_{i\in I}$, and $(T_i, E_{T_i})_{i\in I}$ in $EMAJ(P)$, we have:

$$
\begin{aligned}
& (\bigcurlywedge_{i\in I} (R, E_R) \to (S_i, E_{S_i}) \to (T_i, E_{T_i})) \to (R, E_R) \to (\bigcurlywedge_{i\in I} (S_i, E_{S_i}) \to (T_i, E_{T_i})) \\
= \; & N(\bigcap_{i\in I} R \to S_i \to T_i, \bigcap_{i\in I} E_R \to E_{S_i} \to E_{T_i}) \to (R, E_R) \to N(\bigcap_{i\in I} S_i \to T_i, \bigcap_{i\in I} E_{S_i} \to E_{T_i}) \\
= \; & N(\bigcap_{i\in I} R \to S_i \to T_i, \bigcap_{i\in I} E_R \to E_{S_i} \to E_{T_i}) \to N(R \to \bigcap_{i\in I} S_i \to T_i, E_R \to \bigcap_{i\in I} E_{S_i} \to E_{T_i}) \\
= \; & N((\bigcap_{i\in I} R \to S_i \to T_i) \upharpoonright (\dots) \to R \to \bigcap_{i\in I} S_i \to T_i \\
& , (\bigcap_{i\in I} E_R \to E_{S_i} \to E_{T_i}) \upharpoonright (\dots) \to E_R \to \bigcap_{i\in I} E_{S_i} \to E_{T_i})
\end{aligned}
$$

Notice that $(\bigcap_{i\in I} R \to S_i \to T_i) \upharpoonright (\dots) \to R \to \bigcap_{i\in I} S_i \to T_i \supseteq (\bigcap_{i\in I} R \to S_i \to T_i) \to R \to \bigcap_{i\in I} S_i \to T_i$, and $(\bigcap_{i\in I} E_R \to E_{S_i} \to E_{T_i}) \upharpoonright (\dots) \to E_R \to \bigcap_{i\in I} E_{S_i} \to E_{T_i} \supseteq (\bigcap_{i\in I} E_R \to E_{S_i} \to E_{T_i}) \upharpoonright\to E_R \to \bigcap_{i\in I} E_{S_i} \to E_{T_i}$

Also recall that $(\langle uvw\rangle uvw, \langle uvw\rangle uvw)$ is in both $(\bigcap_{i\in I} R \to S_i \to T_i) \to R \to \bigcap_{i\in I} S_i \to T_i$ and $(\bigcap_{i\in I} E_R \to E_{S_i} \to E_{T_i}) \to E_R \to \bigcap_{i\in I} E_{S_i} \to E_{T_i}$. □

We conclude that

Corollary 48. *$EMAJ(P)$ is an arrow algebra.*

We still want to show that $EMAJ(P)$ is compatible with joins. But before doing so, let us work out an explicit expression for supremum in $EMAJ(P)$.

Definition 49 (Making $R \in MAJ(P)$ compatible with $E \in PER(P)$)**.** *Suppose $R \in MAJ(P)$ and $E \in PER(P)$, and suppose $dom(R) = dom(E)$. Then we define*

$$R \upharpoonright E \coloneqq \{(a, b) \in P \times P \mid \exists (a_1, b_1), \dots, (a_n, b_n) \in R, \; s.t. \; (a, a_1) \in E, (b_i, a_{i+1}) \in E \text{ where } 1 \le i \le n-1, (b_n, b) \in E\}.$$

Indeed, in this definition we do not really need R to be transitive

Remark 50. *The following hold for $R \upharpoonright E$ as defined in the previous definition:*

- $R \subseteq R \upharpoonright E$.
- $R \upharpoonright E \in MAJ(P)$.
- $dom(R \upharpoonright E) = dom(R) = dom(E)$.
- $(R \upharpoonright E, E) \in EMAJ(P)$.

- *If* $(R_1, E_1) \in EMAJ(P)$*,* $R \subseteq R_1$*, and* $E \subseteq E_1$*, then* $(R \upharpoonleft E, E) \preceq (R_1, E_1)$*.*

Remark 51. *Let* I *be an index set, and suppose* $E_i \in PER(P)$ *for all* $i \in I$*. We know the supremum of the* E_i *is* $\curlyvee_{i\in I} E_i$*, the transitive closure of* $\bigcup_{i\in I} E_i$*.*

Similarly, for $R_i \in MAJ(P)$*, the supremum* $\curlyvee_{i\in I} R_i$ *is the transitive closure of* $\bigcup_{i\in I} R_i$*.*

Lemma 52 (supremum in $EMAJ(P)$)**.** *Let* I *be an index set. Suppose that* $(R_i, E_i) \in EMAJ(P)$ *for all* $i \in I$*. Then*

$$\curlyvee_{i\in I} (R_i, E_i) = ((\curlyvee_{i\in I} R_i) \upharpoonleft (\curlyvee_{i\in I} E_i), \curlyvee_{i\in I} E_i).$$

Proof. Notice that $\mathrm{dom}(\curlyvee_{i\in I} R_i) = \bigcup_{i\in I} \mathrm{dom}(R_i) = \bigcup_{i\in I} \mathrm{dom}(E_i) = \mathrm{dom}(\curlyvee_{i\in I} E_i)$.

From a previous lemma, we know that $((\curlyvee_{i\in I} R_i) \upharpoonleft (\curlyvee_{i\in I} E_i), \curlyvee_{i\in I} E_i) \in EMAJ(P)$.

Also, clearly, for all $i \in I$, $(R_i, E_i) \preceq ((\curlyvee_{i\in I} R_i) \upharpoonleft (\curlyvee_{i\in I} E_i), \curlyvee_{i\in I} E_i)$.

Suppose $(R', E') \in EMAJ(P)$ and for all $i \in I$, $(R_i, E_i) \preceq (R', E')$. Then we have $\curlyvee_{i\in I} R_i \subseteq R'$ and $\curlyvee_{i\in I} E_i \subseteq E'$, and thus, by a previous remark, we have $((\curlyvee_{i\in I} R_i) \upharpoonleft (\curlyvee_{i\in I} E_i), \curlyvee_{i\in I} E_i) \preceq (R', E')$. □

Lemma 53. *Suppose* $R \in MAJ(P)$*,* $E \in PER(P)$*, and dom*$(R) =$ *dom*(E)*. Suppose* $(R_1, E_1) \in EMAJ(P)$*. Then the following two are equal:*

- $(R \upharpoonleft E \to R_1) \restriction$ *dom*$(E \to E_1)$*.*
- $(R \to R_1) \restriction$ *dom*$(E \to E_1)$*.*

Proof. Since $R \subseteq R \upharpoonleft E$ and implication is anti-monotone, we know $R \upharpoonleft E \to R_1 \subseteq R \to R_1$. Thus $(R \upharpoonleft E \to R_1) \restriction \mathrm{dom}(E \to E_1) \subseteq (R \to R_1) \restriction \mathrm{dom}(E \to E_1)$. We prove the converse.

Suppose $(r, r^*) \in (R \to R_1) \restriction \mathrm{dom}(E \to E_1)$. We show that $(r, r^*) \in (R \upharpoonleft E \to R_1) \restriction \mathrm{dom}(E \to E_1)$. It suffices to show that $(r, r^*) \in R \upharpoonleft E \to R_1$.

For any $(a, b) \in R \upharpoonleft E$, there exist $(a_1, b_1), \ldots, (a_n, b_n) \in R$ such that $(a, a_1) \in E$, $(b_i, a_{i+1}) \in E$ for all $1 \le i \le n-1$, and $(b_n, b) \in E$. We know $(ra_i, r^*b_i) \in R_1$ and $(r^*a_i, r^*b_i) \in R_1$ for $1 \le i \le n$.

Also, since $(r, r), (r^*, r^*) \in E \to E_1$, we know:

- $(ra, ra_1), (r^*a, r^*a_1) \in E_1$.
- $(r^*b_i, r^*a_{i+1}) \in E_1$ for all $1 \le i \le n-1$.
- $(r^*b_n, r^*b) \in E$.

Since $(R_1, E_1) \in EMAJ(P)$, we have $(ra, r^*b), (r^*a, r^*b) \in R_1 \upharpoonleft E_1 = R_1$. Thus $(r, r^*) \in R \upharpoonleft E \to R_1$. □

Lemma 54. *Suppose* $R \in MAJ(P)$*,* $E \in PER(P)$*, and dom*$(R) =$ *dom*(E)*. Suppose* $(R_1, E_1) \in EMAJ(P)$*. Then* $N(R \to R_1, E \to E_1) \in EMAJ(P)$*, and*

$$(R \upharpoonleft E, E) \to (R_1, E_1) = N(R \to R_1, E \to E_1).$$

Proof. We know $(R \upharpoonleft E, E) \to (R_1, E_1) = N(R \upharpoonleft E \to R_1, E \to E_1) = N(R \to R_1, E \to E_1)$. □

Remark 55. *It may be convenient to introduce one more notation. Suppose $R \in MAJ(P)$, $E \in PER(P)$, and $dom(R) = dom(E)$; then define:*

$$M(R, E) \coloneqq (R \upharpoonright E, E).$$

We know that $M(R, E) \in EMAJ(P)$. Also, for any $(R_1, E_1) \in EMAJ(P)$, the previous lemma states that:

$$M(R, E) \to (R_1, E_1) = N(R \to R_1, E \to E_1).$$

Moreover, if $(R_i, E_i)_{i \in I}$ is an I-indexed family of elements in $EMAJ(P)$, then we have:

$$\curlyvee_{i \in I} (R_i, E_i) = M(\curlyvee_{i \in I} R_i, \curlyvee_{i \in I} E_i).$$

Proposition 56 (compatible with joins)**.** *$EMAJ(P)$ is compatible with joins.*

Proof. Let I be an index set. Suppose that $(R_i, E_i) \in EMAJ(P)$ for every $i \in I$, and suppose $(R', E') \in EMAJ(P)$. Then we have:

$$\begin{aligned}
& (\curlyvee_{i \in I} (R_i, E_i)) \to (R', E') \\
= \;& M(\curlyvee_{i \in I} R_i, \curlyvee_{i \in I} E_i) \to (R', E') \\
= \;& N((\curlyvee_{i \in I} R_i) \to R', (\curlyvee_{i \in I} E_i) \to E') \\
= \;& N(\bigcap_{i \in I} (R_i \to R'), \bigcap_{i \in I} (E_i \to E)) \\
= \;& \curlywedge_{i \in I} N(R_i \to R', E_i \to E') \\
= \;& \curlywedge_{i \in I} (R_i, E_i) \to (R', E')
\end{aligned}$$

We are using the fact that $MAJ(P)$ and $PER(P)$ are both compatible with joins. □

3.3 The arrow algebra $dEMAJ(P)$

In this section, we construct an arrow algebra where the elements are tuples of the form (A, R, E), where $(R, E) \in EMAJ(P)$, $A \subseteq \mathrm{dom}(R) = \mathrm{dom}(E)$, and such that A and (R, E) are compatible in some sense. Roughly speaking, we are adding a subset of the domain of (R, E).

We call this arrow algebra $dEMAJ(P)$, and show that it is compatible with joins. We first provide a brief recap of the implication for subsets of P:

Remark 57. *Recall that for $X, Y \subseteq P$, we define:*

$$X \to Y \coloneqq \{r \in P \mid \forall a \in X, ra \downarrow, ra \in Y\}.$$

Definition 58 ($dEMAJ(P)$)**.** *Let $dEMAJ(P)$ be the collection of all tuples of the form (A, R, E) where*

- *$(R, E) \in EMAJ(P)$.*
- *$A \subseteq P$.*

- *$A \subseteq dom(R) = dom(E)$.*

- *If $a \in A$ and $(a, a') \in E$, then $a' \in A$.*

For $(A_1, R_1, E_1), (A_2, R_2, E_2) \in dEMAJ(P)$, define $(A_1, R_1, E_1) \preceq (A_2, R_2, E_2)$ if $A_1 \subseteq A_2$ and $(R_1, E_1) \preceq (R_2, E_2)$.

For $(A_1, R_1, E_1), (A_2, R_2, E_2) \in dEMAJ(P)$, define implication to be:

$$\begin{aligned} & (A_1, R_1, E_1) \to (A_2, R_2, E_2) \\ = \quad & ((A_1 \to A_2) \cap dom((R_1 \to R_2) \restriction dom(E_1 \to E_2)) \\ & , (R_1 \to R_2) \restriction dom(E_1 \to E_2) \\ & , (E_1 \to E_2) \restriction dom((R_1 \to R_2) \restriction dom(E_1 \to E_2))) \end{aligned}$$

Remark 59. *To see that this implication is well-defined, i.e. $(A_1, R_1, E_1) \to (A_2, R_2, E_2)$ is in $dEMAJ(P)$, we verify that if $r \in (A_1 \to A_2) \cap dom((R_1 \to R_2) \restriction dom(E_1 \to E_2))$ and $(r, r') \in (E_1 \to E_2) \restriction dom((R_1 \to R_2) \restriction dom(E_1 \to E_2))$, then $r' \in (A_1 \to A_2) \cap dom((R_1 \to R_2) \restriction dom(E_1 \to E_2))$.*

We first show that $r' \in A_1 \to A_2$. For any $a \in A_1$, we know $ra \in A_2$, and we have to show that $r'a \in A_2$. Since $a \in A_1 \subseteq dom(E_1)$, we know $(a, a) \in E_1$. Since $(r, r') \in E_1 \to E_2$, we have $(ra, r'a) \in E_2$. Therefore, $r'a \in A_2$.

We then show that $r' \in dom((R_1 \to R_2) \restriction dom(E_1 \to E_2))$, and this follows directly from our assumption that $(r, r') \in (E_1 \to E_2) \restriction dom((R_1 \to R_2) \restriction dom(E_1 \to E_2))$.

Lemma 60. *The partial order on $dEMAJ(P)$ is complete.*

Proof. Let I be an index set, and suppose $(A_i, R_i, E_i) \in dEMAJ(P)$ for all $i \in I$. We claim that:

$$\bigwedge_{i \in I} (A_i, R_i, E_i) = ((\bigcap_{i \in I} A_i) \cap \mathrm{dom}(\bigcap_{i \in I} R_i), \bigcap_{i \in I} R_i, (\bigcap_{i \in I} E_i) \restriction \mathrm{dom}(\bigcap_{i \in I} R_i)).$$

We first need to show that $((\bigcap_{i \in I} A_i) \cap \mathrm{dom}(\bigcap_{i \in I} R_i), \bigcap_{i \in I} R_i, (\bigcap_{i \in I} E_i) \restriction \mathrm{dom}(\bigcap_{i \in I} R_i))$ is indeed an element of $dEMAJ(P)$. We verify that if $a \in (\bigcap_{i \in I} A_i) \cap \mathrm{dom}(\bigcap_{i \in I} R_i)$ and $(a, a') \in (\bigcap_{i \in I} E_i) \restriction \mathrm{dom}(\bigcap_{i \in I} R_i)$, then $a' \in (\bigcap_{i \in I} A_i) \cap \mathrm{dom}(\bigcap_{i \in I} R_i)$. It suffices to show that $a' \in \bigcap_{i \in I} A_i$, and this follows directly from the fact that $a \in \bigcap_{i \in I} A_i$ and $(a, a') \in \bigcap_{i \in I} E_i$.

Then clearly, for any $i_0 \in I$, we have $((\bigcap_{i \in I} A_i) \cap \mathrm{dom}(\bigcap_{i \in I} R_i), \bigcap_{i \in I} R_i, (\bigcap_{i \in I} E_i) \restriction \mathrm{dom}(\bigcap_{i \in I} R_i)) \preceq (A_{i_0}, R_{i_0}, E_{i_0})$.

Suppose $(A', R', E') \preceq (A_i, R_i, E_i)$ for all $i \in I$, then we know $A' \subseteq \bigcap_{i \in I} A_i$, $R' \subseteq \bigcap_{i \in I} R_i$, and $E' \subseteq \bigcap_{i \in I} E_i$. Then $A' \subseteq \mathrm{dom}(R') \subseteq \mathrm{dom}(\bigcap_{i \in I} R_i)$, and thus $A' \subseteq (\bigcap_{i \in I} A_i) \cap \mathrm{dom}(\bigcap_{i \in I} R_i)$. Also, $\mathrm{dom}(E') = \mathrm{dom}(R') \subseteq \mathrm{dom}(\bigcap_{i \in I} R_i)$, and thus $E' = E' \restriction \mathrm{dom}(E') \subseteq (\bigcap_{i \in I} E_i) \restriction \mathrm{dom}(\bigcap_{i \in I} R_i)$. $\square$

Lemma 61. *The implication in the definition of $dEMAJ(P)$ is anti-monotone in the first component, and monotone in the second component.*

Proof. Suppose $(A'_1, R'_1, E'_1) \preceq (A_1, R_1, E_1)$ and $(A_2, R_2, E_2) \preceq (A'_2, R'_2, E'_2)$. Then we know

- $A_1 \to A_2 \subseteq A'_1 \to A'_2$.

- $R_1 \to R_2 \subseteq R'_1 \to R'_2$.

- $E_1 \to E_2 \subseteq E_1' \to E_2'$.

The result then follows from the fact that $R \restriction \mathrm{dom}(E)$ and $E \restriction \mathrm{dom}(R)$ are both monotone in every component. $\square$

Corollary 62. *$dEMAJ(P)$ is an arrow structure.*

Lemma 63. *Suppose $(A_1, R_1, E_1) \in dEMAJ(P)$. Suppose $A \subseteq P$, $R \in MAJ(P)$, and $E \in PER(P)$. If $(A \cap dom(R \restriction dom(E)), R \restriction dom(E), E \restriction dom(R \restriction dom(E)))$ is in $dEMAJ(P)$, then the following two are the same:*

- $(A_1, R_1, E_1) \to (A \cap dom(R \restriction dom(E)), R \restriction dom(E), E \restriction dom(R \restriction dom(E)))$.
- $((A_1 \to A) \cap dom((R_1 \to R) \restriction dom(E_1 \to E)), (R_1 \to R) \restriction dom(E_1 \to E), (E_1 \to E) \restriction dom((R_1 \to R) \restriction dom(E_1 \to E)))$

Proof. From a similar result in $EMAJ(P)$, we know it suffices to show that the following two are the same:

(1) $(A_1 \to A \cap \mathrm{dom}(R \restriction \mathrm{dom}(E))) \cap \mathrm{dom}((R_1 \to R) \restriction \mathrm{dom}(E_1 \to E))$.

(2) $(A_1 \to A) \cap \mathrm{dom}((R_1 \to R) \restriction \mathrm{dom}(E_1 \to E))$.

We know that (1) is contained in (2), since $A \cap \mathrm{dom}(R \restriction \mathrm{dom}(E)) \subseteq A$. We prove the other direction.

Suppose r is an element of (2). We show that r is also an element of (1). We know it suffices to show that $r \in A_1 \to A \cap \mathrm{dom}(R \restriction \mathrm{dom}(E))$.

For any $a \in A_1$, since r is in (2), we know $r \in A_1 \to A$, thus $ra \in A$.

Also, we know $r \in \mathrm{dom}((R_1 \to R) \restriction \mathrm{dom}(E_1 \to E))$, so there exists r^* such that $(r, r^*) \in (R_1 \to R) \restriction \mathrm{dom}(E_1 \to E)$. Since $a \in A_1 \subseteq \mathrm{dom}(R_1)$, we know there exists a^* such that $(a, a^*) \in R_1$. Therefore, $(ra, r^*a^*) \in R$. Moreover, we know $(r, r), (r^*, r^*) \in E_1 \to E$, and $(a, a), (a^*, a^*) \in E_1$, thus $(ra, ra), (r^*a^*, r^*a^*) \in E$. Therefore, $(ra, r^*a^*) \in R \restriction \mathrm{dom}(E)$, so $ra \in \mathrm{dom}(R \restriction \mathrm{dom}(E))$. $\square$

Remark 64. *As in the section about $EMAJ(P)$, for $A \subseteq P$, $R \in MAJ(P)$, and $E \in PER(P)$, we define*

$$N(A, R, E) := (A \cap dom(R \restriction dom(E)), R \restriction dom(E), E \restriction dom(R \restriction dom(E))).$$

We then know that implication in $dEMAJ(P)$ can be rewritten as:

$$(A_1, R_1, E_1) \to (A_2, R_2, E_2) = N(A_1 \to A_2, R_1 \to R_2, E_1 \to E_2).$$

Also, the above lemma can be understood as: for $A \subseteq P$, $R \in MAJ(P)$, and $E \in PER(P)$, if $N(A, R, E)$ is in $dEMAJ(P)$, then for any $(A_1, R_1, E_1) \in dEMAJ(P)$, we have:

$$(A_1, R_1, E_1) \to N(A, R, E) = N(A_1 \to A, R_1 \to R, E_1 \to E).$$

Corollary 65. *Given $(A_1, R_1, E_1), \ldots, (A_n, R_n, E_n) \in dEMAJ(P)$ $(n \geq 2)$, we have:*

$$
\begin{aligned}
& (A_n, R_n, E_n) \to (A_{n-1}, R_{n-1}, E_{n-1}) \to \cdots \to (A_1, R_1, E_1) \\
= \quad & N(A_n \to \cdots \to A_1, R_n \to \cdots \to R_1, E_n \to \cdots \to E_1)
\end{aligned}
$$

We now define a separator on $dEMAJ(P)$.

Definition 66 (separator)**.** *Define a separator on $dEMAJ(P)$ as:*

$$S_{dEMAJ(P)} \coloneqq \{(A, R, E) \in dEMAJ(P) \mid A \neq \emptyset\}.$$

Lemma 67. *$S_{dEMAJ(P)}$ is upwards closed.*

Proof. Suppose $(A_1, R_1, E_1) \in S_{dEMAJ(P)}$ and $(A_1, R_1, E_1) \preceq (A_2, R_2, E_2)$. Then we know $A_1 \neq \emptyset$ and $A_1 \subseteq A_2$. Then $A_2 \neq \emptyset$, and therefore $(A_2, R_2, E_2) \in S_{dEMAJ(P)}$. □

Lemma 68. *$S_{dEMAJ(P)}$ is closed under modus ponens.*

Proof. Suppose $(A_1, R_1, E_1), (A_1, R_1, E_1) \to (A_2, R_2, E_2) \in S_{dEMAJ(P)}$. Then in particular, $A_1 \neq \emptyset$ and $A_1 \to A_2 \neq \emptyset$. Thus $A_2 \neq \emptyset$, and $(A_2, R_2, E_2) \in dEMAJ(P)$. □

Again, let us list some useful facts about proving whether an element is in the separator:

Lemma 69. *Suppose $A \subseteq P$, $R \in MAJ(P)$, and $E \in PER(P)$. Suppose $N(A, R, E) \in dEMAJ(P)$. Then the following two are equivalent:*

- *$N(A, R, E) \in S_{dEMAJ(P)}$.*
- *There exists (a, a^*) such that:*
  - *$a \in A$.*
  - *$(a, a^*) \in R$.*
  - *$(a, a), (a^*, a^*) \in E$.*

Therefore, we may say that (a, a^*) witnesses that $N(A, R, E) \in S_{dEMAJ(P)}$.

Proof. Unfolding definitions, we know $N(A, R, E) \coloneqq (A \cap \mathrm{dom}(R \restriction \mathrm{dom}(E)), R \restriction \mathrm{dom}(E), E \restriction \mathrm{dom}(R \restriction E))$. Therefore, $N(A, R, E) \in S_{dEMAJ(P)}$ iff $A \cap \mathrm{dom}(R \restriction \mathrm{dom}(E))$ is non-empty. This is equivalent to saying that there exists (a, a^*) such that:

- $a \in A$.
- $(a, a^*) \in R \restriction \mathrm{dom}(E)$.

which is equivalent to:

- $a \in A$.
- $(a, a^*) \in R$.
- $(a, a), (a^*, a^*) \in E$.

We may conclude that $N(A, R, E) \in S_{dEMAJ(P)}$ iff there exists (a, a^*) such that $a \in A$, $(a, a^*) \in R$, and $(a, a), (a^*, a^*) \in E$. □

Remark 70. *Let I be an index set. Suppose for each $i \in I$, $A_i \subseteq P$, $R_i \in MAJ(P)$, and $E_i \in PER(P)$.*

From a remark in the section on $EMAJ(P)$, we know

$$\bigcap_{i\in I}(R_i \upharpoonright dom(E_i)) = (\bigcap_{i\in I} R_i) \upharpoonright dom(\bigcap_{i\in I} E_i).$$

Also, we have

$$\begin{aligned} & \bigcap_{i\in I}(A_i \cap dom(R_i \upharpoonright dom(E_i))) \cap dom(\bigcap_{i\in I}(R_i \upharpoonright dom(E_i))) \\ = \; & (\bigcap_{i\in I} A_i) \cap (\bigcap_{i\in I} dom(R_i \upharpoonright dom(E_i))) \cap dom(\bigcap_{i\in I}(R_i \upharpoonright dom(E_i))) \\ = \; & (\bigcap_{i\in I} A_i) \cap dom(\bigcap_{i\in I}(R_i \upharpoonright dom(E_i))) \\ = \; & (\bigcap_{i\in I} A_i) \cap dom((\bigcap_{i\in I} R_i) \upharpoonright dom(\bigcap_{i\in I} E_i)) \end{aligned}$$

From this and a similar remark about $EMAJ(P)$, we understand that if for every $i \in I$, $N(A_i, R_i, E_i) \in dEMAJ(P)$, then

$$\bigcurlywedge_{i\in I} N(A_i, R_i, E_i) = N(\bigcap_{i\in I} A_i, \bigcap_{i\in I} R_i, \bigcap_{i\in I} E_i).$$

Recall that we have $\underline{k} \in A_1 \to A_2 \to A_1$ for any $A_1, A_2 \subseteq P$, $\underline{s} \in (A_1 \to A_2 \to A_3) \to (A_1 \to A_2) \to (A_1 \to A_3)$, and $\langle uvw\rangle uvw \in (\bigcap_{i\in I} A \to B_i \to C_i) \to A \to \bigcap_{i\in I} B_i \to C_i$. Therefore, as in the case of $EMAJ(P)$, we understand that $(\underline{k}, \underline{k})$ witnesses that $\mathbf{k} \in S_{dEMAJ(P)}$, $(\underline{s}, \underline{s})$ witnesses that $\mathbf{s} \in S_{dEMAJ(P)}$, and $(\langle uvw\rangle uvw, \langle uvw\rangle uvw)$ witnesses that $\mathbf{a} \in S_{dEMAJ(P)}$. We conclude that:

Corollary 71. *$dEMAJ(P)$ is an arrow algebra.*

We still want to show that $dEMAJ(P)$ is compatible with joins. But before doing so, we have to work out an explicit expression for the supremum in $dEMAJ(P)$.

Definition 72 (making a subset of dom(E) compatible with E)**.** *Suppose $E \in PER(P)$ and $A \subseteq dom(E)$. Then we define:*

$$A \upharpoonleft E := \{a \in P \mid \exists b \in A, (a, b) \in E\}$$

Remark 73. *The set $A \upharpoonleft E$ from the definition above has the following properties:*

- *$A \subseteq A \upharpoonleft E \subseteq dom(E)$.*
- *If $a \in A \upharpoonleft E$ and $(a, b) \in E$, then $b \in A \upharpoonleft E$.*
- *Suppose $A' \subseteq dom(E')$ is compatible with $E' \in PER(P)$, i.e., if $a \in A'$ and $(a, b) \in E'$, then $b \in A'$.*

  If $A \subseteq A'$ and $E \subseteq E'$, then $A \upharpoonleft E \subseteq A'$.

Lemma 74 (supremum in $dEMAJ(P)$)**.** *Let I be an index set. Suppose that for each $i \in I$, $(A_i, R_i, E_i) \in dEMAJ(P)$. Then*

$$\bigcurlyvee_{i\in I}(A_i, R_i, E_i) = ((\bigcup_{i\in I} A_i) \upharpoonleft (\bigcurlyvee_{i\in I} E_i), (\bigcurlyvee_{i\in I} R_i) \upharpoonleft (\bigcurlyvee_{i\in I} E_i), \bigcurlyvee_{i\in I} E_i).$$

Proof. $\bigcup_{i\in I} A_i \subseteq \mathrm{dom}(\bigcup_{i\in I} E_i) = \mathrm{dom}(\curlyvee_{i\in I} E_i)$, hence $(\bigcup_{i\in I} A_i) \upharpoonright (\curlyvee_{i\in I} E_i) \subseteq \mathrm{dom}(\curlyvee_{i\in I} E_i)$. Note that $(\bigcup_{i\in I} A_i) \upharpoonright (\curlyvee_{i\in I} E_i)$ is compatible with $\curlyvee_{i\in I} E_i$. Also, $((\curlyvee_{i\in I} R_i) \upharpoonright (\curlyvee_{i\in I} E_i), \curlyvee_{i\in I} E_i) \in EMAJ(P)$. Therefore, $((\bigcup_{i\in I} A_i) \upharpoonright (\curlyvee_{i\in I} E_i), (\curlyvee_{i\in I} R_i) \upharpoonright (\curlyvee_{i\in I} E_i), \curlyvee_{i\in I} E_i)$ is in $dEMAJ(P)$.

Clearly, for each $i \in I$, $(A_i, R_i, E_i) \preceq ((\bigcup_{i\in I} A_i) \upharpoonright (\curlyvee_{i\in I} E_i), (\curlyvee_{i\in I} R_i) \upharpoonright (\curlyvee_{i\in I} E_i), \curlyvee_{i\in I} E_i)$.

Suppose $(A', R', E') \in dEMAJ(P)$ and that for each $i \in I$, $(A_i, R_i, E_i) \preceq (A', R', E')$. We have $\bigcup_i A_i \subseteq A'$ and $\curlyvee_{i\in I} E_i \subseteq E'$; hence, by a previous remark, $(\bigcup_{i\in I} A_i) \upharpoonright (\curlyvee_{i\in I} E_i) \subseteq A'$. Furthermore, by a similar result for $EMAJ(P)$, $((\curlyvee_{i\in I} R_i) \upharpoonright (\curlyvee_{i\in I} E_i), \curlyvee_{i\in I} E_i) \preceq (R', E')$. Hence $((\bigcup_{i\in I} A_i) \upharpoonright (\curlyvee_{i\in I} E_i), (\curlyvee_{i\in I} R_i) \upharpoonright (\curlyvee_{i\in I} E_i), \curlyvee_{i\in I} E_i) \preceq (A', R', E')$. □

Lemma 75. *Suppose $A \subseteq P$, $R \in MAJ(P)$, $E \in PER(P)$. Suppose $A \subseteq dom(R) = dom(E)$. Suppose $(A_1, R_1, E_1) \in dEMAJ(P)$. Then the following two sets are equal:*

- $(A \upharpoonright E \to A_1) \cap dom((R \to R_1) \restriction dom(E \to E_1))$.
- $(A \to A_1) \cap dom((R \to R_1) \restriction dom(E \to E_1))$.

Proof. One inclusion is clear. We show the other direction.

Suppose $r \in (A \to A_1) \cap \mathrm{dom}((R \to R_1) \restriction \mathrm{dom}(E \to E_1))$. We show that $r \in (A \upharpoonright E \to A_1) \cap \mathrm{dom}((R \to R_1) \restriction \mathrm{dom}(E \to E_1))$. It suffices to show that $r \in A \upharpoonright E \to A_1$.

Suppose $a \in A \upharpoonright E$; then there exists $b \in A$ such that $(a, b) \in E$. Since $r \in A \to A_1$, we know $rb \in A_1$. Since $r \in \mathrm{dom}((R \to R_1) \restriction \mathrm{dom}(E \to E_1))$, there exists r^* such that $(r, r^*) \in (R \to R_1) \restriction \mathrm{dom}(E \to E_1)$; thus $(r, r) \in E \to E_1$. Therefore, $(ra, rb) \in E_1$, and thus $ra \in A_1$. □

Lemma 76. *Suppose $A \subseteq P$, $R \in MAJ(P)$, $E \in PER(P)$. Suppose $A \subseteq dom(R) = dom(E)$. Suppose $(A_1, R_1, E_1) \in dEMAJ(P)$. Then $N(A \to A_1, R \to R_1, E \to E_1) \in dEMAJ(P)$, and*

$$(A \upharpoonright E, R \upharpoonright E, E) \to (A_1, R_1, E_1) = N(A \to A_1, R \to R_1, E \to E_1).$$

Proof. We know $(A \upharpoonright E, R \upharpoonright E, E) \to (A_1, R_1, E_1) = N(A \upharpoonright E \to A_1, R \upharpoonright E \to R_1, E \to E_1) = N(A \to A_1, R \to R_1, E \to E_1)$. □

Remark 77. *Mimicking the case of $EMAJ(P)$, we introduce the following notation. Suppose $A \subseteq P$, $R \in MAJ(P)$, $E \in PER(P)$, and $A \subseteq dom(R) = dom(E)$. Then define:*

$$M(A, R, E) \coloneqq (A \upharpoonright E, R \upharpoonright E, E).$$

We note that $M(A, R, E) \in dEMAJ(P)$. Also, for any $(A_1, R_1, E_1) \in EMAJ(P)$, the above lemma says that:

$$M(A, R, E) \to (A_1, R_1, E_1) = N(A \to A_1, R \to R_1, E \to E_1).$$

Moreover, let $(A_i, R_i, E_i)_{i\in I}$ be an I-indexed family of elements in $dEMAJ(P)$. We know:

$$\curlyvee_{i\in I} (A_i, R_i, E_i) = M(\bigcup_{i\in I} A_i, \curlyvee_{i\in I} R_i, \curlyvee_{i\in I} E_i).$$

Therefore, similarly to the case of $EMAJ(P)$, and recalling that the implication defined for subsets of P is compatible with joins, we conclude that:

Proposition 78 (compatible with joins). *$dEMAJ(P)$ is compatible with joins.*

3.4 The nucleus $j(A, R, E) = (A^{\mathbf{1}}, R^*, E^*)$ on $dEMAJ(P)$

In this section we define a nucleus on the arrow algebra $dEMAJ(P)$, and verify that it is indeed a nucleus. We consider a nucleus because we hope to have something acting as a maximum function on the collection of "potential" realizers. By a maximum function, we mean that given finitely many "potential realizers", we can calculate a "maximum" of those potential realizers. Making this idea precise is the job of the nucleus.

Remark 79. *Recall some basic notation for coding finite sequences in a PCA P:*

- *Suppose $a_1, \ldots, a_n \in P$. We write $[a_1, \ldots, a_n]$ for the code of the finite sequence of $a_1, \ldots, a_n$.*
- *Suppose x and y are both codes of finite sequences. Then we write $x \sqcup y$ for the code of the concatenation of x and y.*
- *Suppose $a \in P$, and $x \in P$ is a code of a finite sequence of elements in P, then we write $a \in x$ if a is an element in that finite sequence.*

Definition 80 (nucleus)**.** *Define $j : dEMAJ(P) \to dEMAJ(P)$ to be such that:*

$$j(A, R, E) = (A^{\mathbf{1}}, R^*, E^*)$$

Here,

- *$R^* \subseteq P \times P$. $(x, y) \in R^*$ iff x and y are both codes of finite sequences of elements in $dom(R)$, and they satisfy $\forall a \in x \sqcup y.\exists a^* \in y.((a, a^*) \in R)$.*

  We have that $R^* \in MAJ(P)$.

- *$E^* \subseteq P \times P$. $(x, x') \in E^*$ iff $x, x' \in dom(R^*)$, $\forall a \in x.\exists b \in x'.(a, b) \in E$ and $\forall a \in x', \exists b \in x.(a, b) \in E$.*

  We know that $E^* \in PER(P)$, and $dom(E^*) = dom(R^*)$.

- *$A^{\mathbf{1}} \subseteq P$. $x \in A^{\mathbf{1}}$ iff $x \in dom(E^*)$, and $\exists a \in A.([a], x) \in E^*$.*

  We know that x is not the code of the empty sequence, and if $x = [a_1, \ldots, a_n]$, then $a_1, \ldots, a_n \in A$.

Remark 81. *We have to verify that $j(A, R, E) = (A^{\mathbf{1}}, R^*, E^*)$ is well-defined, i.e., $(A^{\mathbf{1}}, R^*, E^*)$ is in $dEMAJ(P)$.*

Firstly, we have to show that $(R^*, E^*) \in EMAJ(P)$. We note that it remains to show that R^* and E^* are compatible. Suppose $(x, y) \in R^*$ and $(x, x_1), (y, y_1) \in E^*$. We have to show that $(x_1, y_1) \in R^*$. But this follows from the definition of E^*, and some back-and-forth argument.

Secondly, we have to show that $A^{\mathbf{1}}$ and E^* are compatible. Suppose $x \in A^{\mathbf{1}}$ and $(x, x_1) \in E^*$. Then we have to show that $x_1 \in A^{\mathbf{1}}$. But this is straightforward: there exists $a \in A$ such that $([a], x) \in E^*$, so $([a], x_1) \in E^*$.

We conclude that for $(A, R, E) \in dEMAJ(P)$, $j(A, R, E) = (A^{\mathbf{1}}, R^*, E^*)$ is also in $dEMAJ(P)$.

Lemma 82. *If $(A_1, R_1, E_1) \preceq (A_2, R_2, E_2)$, then $j(A_1, R_1, E_1) \preceq j(A_2, R_2, E_2)$.*

Proof. If $(x, y) \in R_1^*$, then x, y are both codes of finite sequences of elements from $\mathrm{dom}(R_1) \subseteq \mathrm{dom}(R_2)$, and for any $a \in x \sqcup y$, there exists $b \in y$ such that $(a, b) \in R_1 \subseteq R_2$. Therefore, $(x, y) \in R_2^*$, and thus $R_1^* \subseteq R_2^*$.

If $(x, x') \in E_1^*$, then $x, x' \subseteq \mathrm{dom}(R_1^*) \subseteq \mathrm{dom}(R_2^*)$. For any $a \in x$, there exists $b \in x'$ such that $(a, b) \in E_1 \subseteq E_2$. For any $a \in x'$, there exists $b \in x$ such that $(a, b) \in E_1 \subseteq E_2$. Therefore, $(x, x') \in E_2^*$, and thus $E_1^* \subseteq E_2^*$.

If $x \in A_1^{\mathbf{1}}$, then there exists $a \in A_1 \subseteq A_2$, such that $([a], x) \in E_1^* \subseteq E_2^*$. Therefore, $x \in A_2^{\mathbf{1}}$, and thus $A_1^{\mathbf{1}} \subseteq A_2^{\mathbf{1}}$.

We conclude that $j(A_1, R_1, E_1) \preceq j(A_2, R_2, E_2)$. $\square$

Lemma 83. $\curlywedge_{(A,R,E)\in dEMAJ(P)} (A, R, E) \to j(A, R, E)$ *is in* $S_{dEMAJ(P)}$.

Proof. We claim that $(\langle u\rangle[u], \langle u\rangle[u])$ witnesses that $\curlywedge_{(A,R,E)\in dEMAJ(P)} (A, R, E) \to j(A, R, E)$ is in $S_{dEMAJ(P)}$. Notice that for $(A, R, E) \in dEMAJ(P)$, we have $\langle u\rangle[u] \in A \to A^{\mathbf{1}}$, $(\langle u\rangle[u], \langle u\rangle[u]) \in R \to R^*$, and $(\langle u\rangle[u], \langle u\rangle[u]) \in E \to E^*$. $\square$

Remark 84. *Suppose* $(A, R, E) \in dEMAJ(P)$. *The following are true:*

- *Suppose* $x_1, \ldots, x_n$ *and* $y_1, \ldots, y_m$ *are codes of finite sequences of elements in dom*(R). *If for every* $z \in \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$, *there exists* $z^* \in \{y_1, \ldots, y_m\}$ *such that* $(z, z^*) \in R^*$, *then* $(x_1 \sqcup \cdots \sqcup x_n, y_1 \sqcup \cdots \sqcup y_m) \in R^*$.

- *Suppose* $x_1, \ldots, x_n$ *and* $y_1, \ldots, y_m$ *are codes of finite sequences of elements in dom*(E). *If for every* x_i *there exists* y_j *such that* $(x_i, y_j) \in E^*$, *and for every* y_i *there exists* x_j *such that* $(y_i, x_j) \in E^*$, *then* $(x_1 \sqcup \cdots \sqcup x_n, y_1 \sqcup \cdots \sqcup y_m) \in E^*$.

Lemma 85. $\curlywedge_{(A_1,R_1,E_1),(A_2,R_2,E_2)\in dEMAJ(P)} ((A_1, R_1, E_1) \to j(A_2, R_2, E_2)) \to (j(A_1, R_1, E_1) \to j(A_2, R_2, E_2))$ *is in* $S_{dEMAJ(P)}$.

Proof. Recall that there is an element $\underline{c} \in P$ such that for any $f \in P$ and any code of a finite sequence $x = [a_1, \ldots, a_n] \in P$, if $fa_1, \ldots, fa_n$ are all defined and each is a code of a finite sequence, then

$$\underline{c}fx = fa_1 \sqcup \cdots \sqcup fa_n.$$

We claim that $(\underline{c}, \underline{c})$ witnesses that $\curlywedge_{(A_1,R_1,E_1),(A_2,R_2,E_2)\in dEMAJ(P)} ((A_1, R_1, E_1) \to j(A_2, R_2, E_2)) \to (j(A_1, R_1, E_1) \to j(A_2, R_2, E_2))$ is in $S_{dEMAJ(P)}$.

For any $(A_1, R_1, E_1), (A_2, R_2, E_2) \in dEMAJ(P)$, we have:

$$(A_1, R_1, E_1) \to j(A_2, R_2, E_2) = N(A_1 \to A_2^{\mathbf{1}}, R_1 \to R_2^*, E_1 \to E_2^*).$$

and

$$j(A_1, R_1, E_1) \to j(A_2, R_2, E_2) = N(A_1^{\mathbf{1}} \to A_2^{\mathbf{1}}, R_1^* \to R_2^*, E_1^* \to E_2^*).$$

We understand that we have to verify the following three:

(1) $(\underline{c}, \underline{c}) \in (R_1 \to R_2^*) \restriction \mathrm{dom}(E_1 \to E_2^*) \to (R_1^* \to R_2^*)$.

(2) $(\underline{c}, \underline{c}) \in (E_1 \to E_2^*) \restriction \mathrm{dom}((R_1 \to R_2^*) \restriction \mathrm{dom}(E_1 \to E_2^*)) \to (E_1^* \to E_2^*)$.

(3) $\underline{c} \in (A_1 \to A_2^{\mathbf{1}}) \cap \mathrm{dom}((R_1 \to R_2^*) \restriction \mathrm{dom}(E_1 \to E_2^*)) \to (A_1^{\mathbf{1}} \to A_2^{\mathbf{1}})$.

(1) and (2) follow from the previous remark. For (3), suppose $r \in (A_1 \to A_2^{\mathbf{1}}) \cap \mathrm{dom}((R_1 \to R_2^*) \restriction \mathrm{dom}(E_1 \to E_2^*))$ and $x \in A_1^{\mathbf{1}}$. We show that $\underline{c}rx \in A_2^{\mathbf{1}}$:

There exists r^* such that $(r, r^*) \in (R_1 \to R_2^*) \restriction \mathrm{dom}(E_1 \to E_2^*)$, so $(r, r) \in E_1 \to E_2^*$.

Since $x \in A_1^{\mathbf{1}}$, there exists $a \in A_1$ such that $([a], x) \in E_1^*$.

Suppose $x = [a_1, \ldots, a_n]$ $(n \geq 1)$. We know $(a, a_1), \ldots, (a, a_n) \in E_1$, hence $(ra, ra_1), \ldots, (ra, ra_n) \in E_2^*$. By the previous remark, we know $(ra, ra_1 \sqcup \cdots \sqcup ra_n) \in E_2^*$.

Since $a \in A_1$ and $r \in A_1 \to A_2^{\mathbf{1}}$, we know $ra \in A_2^{\mathbf{1}}$, and thus $\underline{c}rx = ra_1 \sqcup \cdots \sqcup ra_n \in A_2^{\mathbf{1}}$. $\square$

Corollary 86. *$j(A, R, E) = (A^{\mathbf{1}}, R^*, E^*)$ is a nucleus on $dEMAJ(P)$.*

Chapter 4

Assemblies

In this chapter we define a category of majorizability assemblies which corresponds to the arrow algebra $dEMAJ(P)$ and the nucleus $j(A, R, E) = (A^{\mathbf{1}}, R^*, E^*)$ on it.

In [Gu18], Tao Gu defined a category MType, which is similar to $EMAJ(P)$ equipped with a nucleus; this nucleus is intended to behave similarly to the nucleus j discussed in the Section 3.4. Objects in MType are subsets of natural numbers equipped with an equivalence relation and a majorizability relation, satisfying various properties. For instance, Tao Gu requires majorants to be computable, which does not seem very necessary. Moreover, Tao Gu requires every object in MType to be equipped with a max function, but this assumption is too strong. In our case, the nucleus j discussed in the previous chapter captures the behavior of max: we can "majorize" a finite sequence of elements, but we do not require that we recover the original element when using it to majorize a sequence of length 1.

Tao Gu also defined a category of assemblies MMAsm, to which our construction below is similar. However, Tao Gu did not go further and construct a topos in which his category MMAsm is embedded. In our setting, we start with the arrow algebra $dEMAJ(P)_j$, which automatically gives rise to a tripos and thus a topos. We then mimic Tao Gu's idea of defining assemblies to obtain a category of majorizability assemblies.

We will show that the category of majorizability assemblies can be embedded into the arrow topos $RT(dEMAJ(P)_j)$, and is equivalent to the full subcategory of $\neg\neg$-separated objects. The embedding and the proof of the equivalence essentially borrows idea from the Chapter 4 of [Vri17].

After that, we provide explicit expressions for finite limits, exponentials, and the natural number object in our category of majorizability assemblies.

4.1 Some useful facts

Remark 87. *Recall that for $A_1, A_2 \subseteq P$, we define $A_1 \times A_2 := \{\underline{p}ab \mid a \in A_1, b \in A_2\}$.*

For $R_1, R_2 \in MAJ(P)$, we define $R_1 \times R_2 := \{(\underline{p}ab, \underline{p}a^*b^*) \mid (a, a^*) \in R_1, (b, b^*) \in R_2\}$. Then $R_1 \times R_2 \in MAJ(P)$.

For $E_1, E_2 \in PER(P)$, we define $E_1 \times E_2 := \{(\underline{p}ab, \underline{p}a'b') \mid (a, a') \in E_1, (b, b') \in E_2\}$. Then $E_1 \times E_2 \in PER(P)$.

Definition 88. *Suppose $(A_1, R_1, E_1), (A_2, R_2, E_2) \in dEMAJ(P)$. We define $(A_1, R_1, E_1) \times (A_2, R_2, E_2)$ to be:*

$$(A_1, R_1, E_1) \times (A_2, R_2, E_2) := (A_1 \times A_2, R_1 \times R_2, E_1 \times E_2).$$

One can verify that $(A_1, R_1, E_1) \times (A_2, R_2, E_2) \in dEMAJ(P)$.

Remark 89. *We write* P *for the tripos corresponding to the arrow algebra* $dEMAJ(P)$. *We know that for any set* X, $\mathsf{P}X$ *is the set of (set-theoretic) functions from* X *to* $dEMAJ(P)$.

For any $f : X \to Y$, *we know* $\mathsf{P}f : \mathsf{P}Y \to \mathsf{P}X$ *is pre-composition with* f.

We usually write $F(x) = (F_0(x), F_1(x), F_2(x))$. *So* $F_0(x) \subseteq P$, $F_1(x) \in MAJ(P)$, $F_2(x) \in PER(P)$.

For $F, G \in \mathsf{P}X$, *we define* $F \vdash G$ *iff* $\bigcurlywedge_{x \in X} F(x) \to G(x)$ *is in* $S_{dEMAJ(P)}$. *By some previous results (see Remark 64 and Remark 70), we know that:*

$$\bigcurlywedge_{x \in X} F(x) \to G(x) = N\Big(\bigcap_{x \in X} F_0(x) \to G_0(x), \bigcap_{x \in X} F_1(x) \to G_1(x), \bigcap_{x \in X} F_2(x) \to G_2(x)\Big)$$

Therefore, $\bigcurlywedge_{x \in X} F(x) \to G(x) \in S_{dEMAJ(P)}$ *iff there exists* (r, r^*) *such that:*

- $r \in \bigcap_x F_0(x) \to G_0(x)$.
- $(r, r^*) \in \bigcap_x F_1(x) \to G_1(x)$.
- $(r, r), (r^*, r^*) \in \bigcap_x F_2(x) \to G_2(x)$.

More naturally, there exists (r, r^*) *such that for any* x:

- $r \in F_0(x) \to G_0(x)$.
- $(r, r^*) \in F_1(x) \to G_1(x)$.
- $(r, r), (r^*, r^*) \in F_2(x) \to G_2(x)$.

In this case, we say that (r, r^*) *witnesses that* $F \vdash G$.

Remark 90. *The preorder* $\mathsf{P}X$ *carries a Heyting prealgebra structure. For instance, suppose* $F, G \in \mathsf{P}X$, *then we define* $F \times G : X \to dEMAJ(P)$ *as follows:*

$$F \times G : x \mapsto F(x) \times G(x).$$

We have that:

- $(\underline{p_0}, \underline{p_0})$ *witnesses that* $F \times G \vdash F$.
- $(\underline{p_0}, \underline{p_0})$ *witnesses that* $F \times G \vdash G$.
- *If* (r, r^*) *witnesses that* $H \vdash F$, *and if* (s, s^*) *witnesses that* $H \vdash G$, *then* $(\langle u \rangle \underline{p}(ru)(su), \langle u \rangle \underline{p}(r^*u)(s^*u))$ *witnesses that* $H \vdash F \times G$.

As in Proposition 28, we have explicit expressions for binary joins and the other operations in the Heyting prealgebra PX, *which we do not write out here.*

Remark 91. *We write* P^j *for the tripos corresponding to the nucleus* $j(A, R, E) = (A^{\mathbf{1}}, R^*, E^*)$. *We know that for any set* X, P^jX *is a subset of* $\mathsf{P}X$:

$$\mathsf{P}^j X \coloneqq \{S : X \to dEMAJ(P) \,|\, jS \vdash S\}.$$

The preorder on $\mathsf{P}^j X$ *is induced by the order on* $\mathsf{P}X$. *For any* $f : X \to Y$, *we know* $\mathsf{P}^j f : \mathsf{P}^j Y \to \mathsf{P}^j X$ *is pre-composition with* f. *Thus* $\mathsf{P}^j f$ *is the restriction of* $\mathsf{P}f$ *to* $\mathsf{P}^j Y$.

The Heyting prealgebra structures on P^j *are also induced from those on* P, *in the following sense:*

- $\top$, $\times$, *and* $\forall f$ *are the same as those from* P.
- *Suppose* $\exists f : \mathsf{P}X \to \mathsf{P}Y$ *is the left adjoint to* $\mathsf{P}f : \mathsf{P}Y \to \mathsf{P}X$.

  Then consider $\exists^j f : \mathsf{P}^j X \to \mathsf{P}^j Y$ *defined by* $F \mapsto j.\exists f(F)$. *We know* $\exists^j$ *is the left adjoint to* $\mathsf{P}^j f : \mathsf{P}^j Y \to \mathsf{P}^j X$.
- *We know* $\bot$ *in* $\mathsf{P}X$ *is such that* $\bot : x \mapsto (\emptyset, \emptyset, \emptyset)$. *Then the bottom element* $\bot^j$ *for* $\mathsf{P}^j X$ *is such that* $\bot^j : x \mapsto j(\emptyset, \emptyset, \emptyset)$, *i.e.,* $x \mapsto (\emptyset, \emptyset^*, \emptyset^*)$.

  Here, notice that $\emptyset^{\mathbf{1}} = \emptyset$. *Also,* $\emptyset^* \coloneqq \{([\,], [\,])\}$.

4.2 Defining the category of majorizability assemblies; Embedding

We define a category, called the category of majorizability assemblies. We show that we can embed this category of majorizability assemblies into the arrow topos corresponding to the tripos P^j.

Definition 92 (majorizability assembly)**.** *A majorizability assembly (or simply m-assembly) is a tuple* $(X, \alpha_X(x))$, *where:*

- X *is a set.*
- $\alpha_X \in \mathsf{P}^j X$. *Hence we require* $j.\alpha_X \vdash \alpha_X$.
- *For any* $x \in X$, $\alpha_X(x) \in S_{dEMAJ(P)}$. *In other words, for any* $x \in X$, $\alpha_{X0}(x) \neq \emptyset$. *Recall that we write* $\alpha_X(x) = (\alpha_{X0}(x), \alpha_{X1}(x), \alpha_{X2}(x))$.

  The set $\alpha_{X0}(x)$ *may be understood as the set of "actual realizers" of* x.
- *The pair* $(\alpha_{X1}(x), \alpha_{X2}(x))$ *is constant; i.e., there exists* $(R, \sim) \in EMAJ(P)$ *such that for any* $x \in X$, $(\alpha_{X1}(x), \alpha_{X2}(x)) = (R, \sim)$.

  We may understand $dom(R) = dom(E)$ *as the set of "potential realizers". The set carries an equivalence relation* $\sim$ *and a majorizability relation, and the two relations are compatible.*

  Therefore, we usually write $\alpha_X(x) = (\alpha_{X0}(x), R, \sim)$.

Definition 93 (morphisms of m-assemblies)**.** *Suppose* (X, α_X) *and* (Y, α_Y) *are m-assemblies. A morphism of m-assemblies from* (X, α_X) *to* (Y, α_Y) *is a set-theoretic function* $f : X \to Y$ *such that:*

$$\alpha_X \vdash \mathsf{P}^j f(\alpha_Y),$$

or, more sloppily, $\alpha_X(x) \vdash \alpha_Y(f(x))$.

If $\alpha_X \vdash \mathsf{P}^j f(\alpha_Y)$ *is witnessed by* (r, r^*), *then we also say that* $f : (X, \alpha_X) \to (Y, \alpha_Y)$ *is witnessed by* (r, r^*).

Remark 94. *To see that the two definitions together define a category, notice that:*

- *For any m-assembly* (X, α_X), $\mathsf{id}_X : \alpha_X(x) \to \alpha_X(x)$ *is witnessed by* $(\underline{i}, \underline{i})$.

- *Suppose* $f : (X, \alpha_X) \to (Y, \alpha_Y)$ *and* $g : (Y, \alpha_Y) \to (Z, \alpha_Z)$ *are morphisms of m-assemblies. We have* $\alpha_X \vdash \mathsf{P}^j f(\alpha_Y)$ *and* $\alpha_Y \vdash \mathsf{P}^j g(\alpha_Z)$, *thus* $\alpha_X \vdash \mathsf{P}^j f.\mathsf{P}^j g(\alpha_Z) \dashv\vdash \mathsf{P}^j(g.f)(\alpha_Z)$. *Therefore,* $g.f : X \to Z$ *is also a morphism of m-assemblies.*

We now try to embed the category of m-assemblies into the realizability topos $RT(dEMAJ(P)_j)$ (from now on we may simply write it as $RT(\mathsf{P}^j)$).

Definition 95 (embed objects)**.** *Let* (X, α_X) *be a m-assembly. Suppose* $\alpha_X(x) = (\alpha_{X0}(x), R, \sim)$ *for some* $(R, \sim) \in EMAJ(P)$. *We define* $E_X : X \times X \to dEMAJ(P)$ *as follows:*

$$E_X(x, x') := \begin{cases} \alpha_X(x) & \text{if } x = x' \\ (\emptyset, R, \sim) & \text{if } x \neq x' \end{cases}$$

Suppose $j\alpha_X \vdash \alpha_X$ *is witnessed by* (r, r^*). *Then we claim that* (r, r^*) *also witnesses that* $jE_X \vdash E_X$*:*

- *If* $x = x'$, *then the situation is the same as in the case* $j\alpha_X \vdash \alpha_X$.

- *If* $x \neq x'$, *then notice that* $\emptyset^{\mathbf{1}} = \emptyset$, *and we always have* $(r, r^*) \in \emptyset^{\mathbf{1}} \to \emptyset$.

  Also, $(r, r^*) \in R^* \to R$ *and* $(r, r), (r^*, r^*) \in \sim^* \to \sim$ *follow from the fact that* (r, r^*) *witnesses that* $j.\alpha_X \vdash \alpha_X$.

We conclude that $E_X \in \mathsf{P}^j(X \times X)$.

We now verify that (X, E_X) is an object in $RT(\mathsf{P}^j)$:

Lemma 96. E_X *is symmetric.*

Proof. We must show that $\models_{x,x'} E_X(x, x') \to E_X(x', x)$. This is witnessed by $(\underline{i}, \underline{i})$. □

Lemma 97. E_X *is transitive.*

Proof. We must show that $\models_{x,x',x''} E_X(x, x') \to E_X(x', x'') \to E_X(x, x'')$. This is witnessed by $(\langle uv \rangle u, \langle uv \rangle u)$. □

Corollary 98. (X, E_X) *is an object in* $RT(\mathsf{P}^j)$.

Definition 99 (embed morphisms)**.** *Let* $f : (X, \alpha_X) \to (Y, \alpha_Y)$ *be a morphism of m-assemblies. Suppose* $\alpha_X(x) = (\alpha_{X0}(x), R_X, \sim_X)$, *and* $\alpha_Y(y) = (\alpha_{Y0}(y), R_Y, \sim_Y)$. *We define* $F : X \times Y \to dEMAJ(P)$ *as follows:*

$$F(x, y) = \begin{cases} \alpha_X(x) \times \alpha_Y(y) & \text{if } f(x) = y \\ (\emptyset, R_X \times R_Y, \sim_X \times \sim_Y) & \text{if } f(x) \neq y \end{cases}$$

Now define $F_X, F_Y : X \times Y \to dEMAJ(P)$ *as follows:*

$$F_X(x, y) := \begin{cases} \alpha_X(x) & \text{if } f(x) = y \\ (\emptyset, R_X, \sim_X) & \text{if } f(x) \neq y \end{cases}$$

and

$$F_Y(x,y) \coloneqq \begin{cases} \alpha_Y(y) & \text{if } f(x) = y \\ (\emptyset, R_Y, \sim_Y) & \text{if } f(x) \neq y \end{cases}$$

We know $jF_X \vdash F_X$*,* $jF_Y \vdash F_Y$*, and* $F = F_X \times F_Y$*. Therefore, we have* $jF \vdash jF_X \times jF_Y \vdash F_X \times F_Y = F$*, so* $F \in \mathsf{P}^j(X \times Y)$*.*

We verify that F is a functional relation on (X, E_X) and (Y, E_Y):

Lemma 100. *F is relational.*

Proof. We have to show that $\models_{x,x',y,y'} F(x,y) \to E_X(x,x') \to E_Y(y,y') \to F(x',y')$. This is witnessed by $(\langle uvw\rangle u, \langle uvw\rangle u)$. □

Lemma 101. *F is strict.*

Proof. We have to show that $\models_{x,y} F(x,y) \to E_X(x,x) \times E_Y(y,y)$. This is witnessed by $(\underline{i}, \underline{i})$. □

Lemma 102. *F is single-valued.*

Proof. We have to show that $\models_{x,y,y'} F(x,y) \to F(x,y') \to E_Y(y,y')$. This is witnessed by $(\langle uv\rangle \underline{p_1} u, \langle uv\rangle \underline{p_1} u)$. □

Lemma 103. *F is total.*

Proof. We must show that $\models_x E_X(x,x) \to \exists^j y.F(x,y)$, i.e., $E_X(x,x) \vdash \exists^j y.F(x,y)$.

Since $\exists^j(\mathsf{id}_X, f).\mathsf{P}^j(\mathsf{id}_X, f)(F) \vdash F$, we have $\exists^j \pi_X.\exists^j(\mathsf{id}_X, f).\mathsf{P}^j(\mathsf{id}_X, f)(F) \vdash \exists^j \pi_X(F)$. We know $\exists^j \pi_X.\exists^j(\mathsf{id}_X, f).\mathsf{P}^j(\mathsf{id}_X, f)(F) \dashv\vdash \exists^j(\pi_X.(\mathsf{id}_X, f)).\mathsf{P}^j(\mathsf{id}_X, f)(F) \dashv\vdash \exists^j \mathsf{id}_X.\mathsf{P}^j(\mathsf{id}_X, f)(F) \dashv\vdash \mathsf{P}^j(\mathsf{id}_X, f)(F)$. Therefore, we have $\mathsf{P}^j(\mathsf{id}_X, f)(F) \vdash \exists^j \pi_X(F)$, i.e., $F(x, f(x)) \vdash \exists^j y.F(x,y)$.

We see that in order to show $E_X(x,x) \vdash \exists^j y.F(x,y)$, it suffices to show that $E_X(x,x) \vdash F(x, f(x))$.

Recall that $\alpha_X \in \mathsf{P}^j(X)$, $\alpha_Y \in \mathsf{P}^j(Y)$, and $\alpha_X \vdash \alpha_Y(f(x))$. Also, we know $F(x, f(x)) = \alpha_X(x) \times \alpha_Y(f(x))$, and $E_X(x,x) = \alpha_X(x)$. Therefore, we have $E_X(x,x) = \alpha_X(x) \vdash \alpha_X(x) \times \alpha_Y(f(x)) = F(x, f(x))$. □

Corollary 104. *F is a functional relation on (X, E_X) and (Y, E_Y).*

We still need to show that the construction is functorial:

Lemma 105. *Let (X, α_X) be an m-assembly. Write f for $\mathsf{id}_X : (X, \alpha_X) \to (X, \alpha_X)$. Then we have $F \dashv\vdash E_X$.*

Proof. We show that $E_X \vdash F$, i.e. $E_X(x,x') \vdash F(x,x')$. This is witnessed by $(\langle u\rangle \underline{p} uu, \langle u\rangle \underline{p} uu)$. □

Lemma 106. *Suppose $f : (X, \alpha_X) \to (Y, \alpha_Y)$ and $g : (Y, \alpha_Y) \to (Z, \alpha_Z)$ are morphisms of m-assemblies. Consider $h \coloneqq g.f$. We know $h : (X, \alpha_X) \to (Z, \alpha_Z)$ is a morphism of m-assemblies. Then we have $H \dashv\vdash \exists^j y.(F(x,y) \times G(y,z))$.*

Proof. We show that $H(x,z) \vdash \exists^j y.(F(x,y) \times G(y,z))$.

We know that $F(x,f(x)) \times G(f(x),z) \vdash \exists^j y.(F(x,y) \times G(y,z))$, so it suffices to show that $H(x,z) \vdash F(x,f(x)) \times G(f(x),z)$, which is equivalent to showing that $H(x,z) \vdash F(x,f(x))$ and $H(x,z) \vdash G(f(x),z)$.

We know $H(x,z) \vdash E_X(x,x) \vdash F(x,f(x))$ (for the second $\vdash$, see the proof of the fact that F is total).

Suppose $\alpha_Y(y) = (\alpha_{Y0}(y), R_Y, \sim_Y)$ and $\alpha_Z(z) = (\alpha_{Z0}(z), R_Z, \sim_Z)$. Unfolding the definitions, we see that

$$G(f(x),z) = \begin{cases} \alpha_Y(f(x)) \times \alpha_Z(z) & \text{if } h(x) = z \\ (\emptyset, R_Y \times R_Z, \sim_Y \times \sim_Z) & \text{if } h(x) \neq z \end{cases}$$

To show $H(x,z) \vdash G(f(x),z)$, it suffices to show that $\alpha_X(x) \times \alpha_Z(h(z)) \vdash \alpha_Y(f(x)) \times \alpha_Z(h(x))$. But the latter follows from the fact that $\alpha_X(x) \vdash \alpha_Y(f(x))$. $\square$

To summarize:

Corollary 107. *We have a functor from the category of m-assemblies to* $RT(\mathsf{P}^j)$*, which is defined as follows:*

- *If* (X, α_X) *is an m-assembly, suppose* $\alpha_X(x) = (\alpha_{X0}(x), R_X, \sim_X)$*. Define*

$$E_X(x,x') \coloneqq \begin{cases} \alpha_X(x) & \text{if } x = x' \\ (\emptyset, R, \sim) & \text{if } x \neq x' \end{cases}$$

  Then (X, E_X) *is an object in* $RT(\mathsf{P}^j)$*.*

- *If* $f : (X, \alpha_X) \to (Y, \alpha_Y)$ *is a morphism of m-assemblies, suppose* $\alpha_X(x) = (\alpha_{X0}(x), R_X, \sim_X)$ *and* $\alpha_Y(y) = (\alpha_{Y0}(y), R_Y, \sim_Y)$*. Define*

$$F(x,y) = \begin{cases} \alpha_X(x) \times \alpha_Y(y) & \text{if } f(x) = y \\ (\emptyset, R_X \times R_Y, \sim_X \times \sim_Y) & \text{if } f(x) \neq y \end{cases}$$

  Then F *is a functional relation on* (X, E_X) *and* (Y, E_Y)*.*

To show that this functor is indeed an embedding, we need to show that it is faithful and full.

Lemma 108 (faithful)**.** *Suppose* $f, g : (X, \alpha_X) \to (Y, \alpha_Y)$ *are two morphisms of m-assemblies. If* $F \dashv\vdash G$*, then* $f = g$*. Therefore, the functor is faithful.*

Proof. By assumption, we have $F(x,y) \vdash G(x,y)$. Suppose it is witnessed by (r, r^*). We then know that for any x and y, we have that $r \in F_0(x,y) \to G_0(x,y)$. In particular, for any x, $r \in F_0(x,f(x)) \to G_0(x,f(x))$. We know $F_0(x,f(x)) = \alpha_X(x) \times \alpha_Y(f(x))$ is non-empty, so $G_0(x,f(x))$ is also non-empty, which implies that $g(x) = f(x)$. $\square$

Lemma 109 (full)**.** *Suppose* (X, α_X) *and* (Y, α_Y) *are two m-assemblies. Suppose* $G(x,y)$ *is a functional relation on* (X, E_X) *and* (Y, E_Y)*. Then there exists a morphism* $f : (X, \alpha_X) \to (Y, \alpha_Y)$ *of m-assemblies, such that* $F \dashv\vdash G$*. Therefore, the functor is full.*

Proof. We know G is single-valued, i.e., $\models_{x,y,y'} G(x,y) \to G(x,y') \to E_Y(y,y')$. In particular, there exists $a \in P$ such that for any x, y, y', we have that $a \in G_0(x,y) \to G_0(x,y') \to E_{Y0}(y,y')$. Therefore, if $G_0(x,y)$ and $G_0(x,y')$ are both non-empty, then $E_{Y0}(y,y')$ is also non-empty, which would imply that $y = y'$. To summarize, for any x, there is at most one y such that $G_0(x,y) \neq \emptyset$.

We know G is total, i.e., $\models_x E_X(x,x) \to \exists^j y.G(x,y)$. In particular, there exists $b \in P$ such that for any x, $b \in E_{X0}(x,x) \to ((\bigcup_y G_0(x,y)) \uparrow (\curlyvee_y G_2(x,y)))^{\mathbf{1}}$ (see Definition 17 and Lemma 74 for explicit expression of existential quantifier, and see Definition 80 for the nucleus). For any x, $E_{X0}(x,x) = \alpha_{X0}(x) \neq \emptyset$, so $((\bigcup_y G_0(x,y)) \uparrow (\curlyvee_y G_2(x,y)))^{\mathbf{1}}$ is also non-empty; hence $\bigcup_y G_0(x,y) \neq \emptyset$. Thus there exists y such that $G_0(x,y) \neq \emptyset$.

We conclude from the two above observations that for any $x \in X$, there exists a unique y such that $G_0(x,y) \neq \emptyset$. Therefore, we obtain a set-theoretic function $f : X \to Y$ such that for any $x \in X$, $G_0(x, f(x)) \neq \emptyset$.

We wish to show that f is a morphism of m-assemblies: we must verify that $\alpha_X \vdash \mathsf{P}^j f(\alpha_Y)$.

Since G is strict, in particular we have $G(x,y) \vdash E_Y(y,y)$. So there exists (r, r^*) such that for any x, y,

- $r \in G_0(x,y) \to \alpha_Y(y)$.
- $(r, r^*) \in G_1(x,y) \to R_Y$.
- $(r,r), (r^*, r^*) \in G_2(x,y) \to \sim_Y$.

Since $G_0(x,y) \neq \emptyset$ iff $y = f(x)$, we know that for any x, y, the following also hold:

- $r \in G_0(x,y) \to \alpha_Y(f(x))$.
- $(r, r^*) \in G_1(x,y) \to R_Y$.
- $(r,r), (r^*, r^*) \in G_2(x,y) \to \sim_Y$.

Therefore, (r, r^*) witnesses that $G(x,y) \vdash \alpha_Y(f(x)) = \mathsf{P}^j \pi_X . \mathsf{P}^j f(\alpha_Y)$. Therefore, by adjunction, we have $\exists^j \pi_X(G(x,y)) \vdash \mathsf{P}^j f(\alpha_Y)$, i.e. $\exists^j y.G(x,y) \vdash \mathsf{P}^j f(\alpha_Y)$. Since G is total, we have $\alpha_X(x) = E_X(x,x) \vdash \exists^j y.G(x,y) \vdash \mathsf{P}^j f(\alpha_Y)$. We conclude that $f : (X, \alpha_X) \to (Y, \alpha_Y)$ is a morphism of m-assemblies.

It remains to show that $F \dashv\vdash G$. We verify that $G \vdash F$. Since G is strict, we know there exists (s, s^*) witnessing $G(x,y) \vdash E_X(x,x) \times E_Y(y,y)$. We claim that (s, s^*) also witnesses that $G(x,y) \vdash F(x,y)$: Notice that $F(x,y)$ and $E_X(x,x) \times \alpha E_Y(y,y)$ differ only on actual realizers, so we only have to worry about whether $s \in G_0(x,y) \to F_0(x,y)$ is true. For any x, y, if $G_0(x,y) = \emptyset$, then the condition holds trivially. If $G_0(x,y) \neq \emptyset$, then we know $y = f(x)$, and thus $F_0(x,y) = E_{X0}(x,x) \times E_{Y0}(y,y)$. □

Corollary 110. *The functor from the category of m-assemblies to $RT(\mathsf{P}^j)$ is an embedding.*

4.3 $\neg\neg$-separated objects

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In this section, we show that the $\neg\neg$-separated objects in $RT(\mathsf{P}^j)$ are exactly the essential images of m-assemblies.

We first show that every image of an m-assembly is $\neg\neg$-separated.

Proposition 111. *Let (X, α_X) be an m-assembly. Then (X, E_X) is $\neg\neg$-separated.*

Proof. We calculate:

$$(\neg^j E_X)(x,x') = \begin{cases} N(\emptyset, R_X \to \emptyset^*, \sim_X \to \emptyset^*) & \text{if } x = x' \\ N(P, R_X \to \emptyset^*, \sim_X \to \emptyset^*) & \text{if } x \neq x' \end{cases}$$

Write $\neg^j R_X$ for $R_X \to \emptyset^*$ and $\neg^j \sim_X$ for $\sim_X \to \emptyset^*$, we know

$$(\neg^j E_X)(x,x') = \begin{cases} (\emptyset, (\neg^j R_X) \restriction \mathrm{dom}(\neg^j \sim_X), (\neg^j \sim_X) \restriction \mathrm{dom}((\neg^j R_X) \restriction \mathrm{dom}(\neg^j \sim_X))) & \text{if } x = x' \\ (!, (\neg^j R_X) \restriction \mathrm{dom}(\neg^j \sim_X), (\neg^j \sim_X) \restriction \mathrm{dom}((\neg^j R_X) \restriction \mathrm{dom}(\neg^j \sim_X))) & \text{if } x \neq x' \end{cases}$$

where $! \coloneqq \mathrm{dom}((\neg^j R_X) \restriction \mathrm{dom}(\neg^j \sim_X))$. Notice that ! is non-empty: recall that $\emptyset^* = \{([\,],[\,])\}$ (see Remark 91), so we have $(\langle u\rangle[\,], \langle u\rangle[\,]) \in R_X \to \emptyset^*$ and $(\langle u\rangle[\,], \langle u\rangle[\,]) \in \sim_X \to \emptyset^*$, so $\langle u\rangle[\,] \in !$.

Then for $(\neg^j\neg^j E_X)(x,x')$, we are interested in the case where $x \neq x'$. In this case, we know:

$$(\neg^j\neg^j E_X)_0(x,x') = \emptyset,$$

which follows from the fact that ! is non-empty.

As a result, we understand that $\models_{x,x'} E_X(x,x) \to E_X(x',x') \to (\neg^j\neg^j E_X)(x,x') \to E_X(x,x')$ is witnessed by $(\langle uvw\rangle u, \langle uvw\rangle u)$. □

We now consider the converse. We start with an arbitrary object (X, E) in $RT(\mathsf{P}^j)$, construct an m-assembly (Y, α_Y) and a surjective functional relation F from (X, E) to (Y, E_Y). Then we show that if (X, E) is $\neg\neg$-separated, then F is also injective.

Definition 112. *Let (X, E) be an object in $RT(\mathsf{P}^j)$.*

Define $X_E \coloneqq \{x \in X \mid E_0(x,x) \neq \emptyset\}$.

Define a binary relation $\sim_E$ on X_E: $x \sim_E x'$ iff $E_0(x,x') \neq \emptyset$. Then $\sim_E$ is an equivalence relation on X_E.

Define $Y \coloneqq X_E / \sim_E$.

Define $\sim_Y^0 \coloneqq \bigcurlyvee_{x,x' \in X} E_2(x,x')$. Then $\sim_Y^0 \in PER(P)$.

Define $R_Y^0 \coloneqq (\bigcurlyvee_{x,x' \in X} E_1(x,x')) \restriction \sim_Y^0$. Then $(R_Y^0, \sim_Y^0) \in EMAJ(P)$.

For any $y \in Y$, define $\alpha_{Y0}^0(y) \coloneqq (\bigcup_{x,x' \in y} E_0(x,x')) \restriction \sim_Y^0$. Then $\alpha_{Y0}^0(y) \neq \emptyset$ and $(\alpha_{Y0}^0(y), R_Y^0, \sim_Y^0) \in dEMAJ(P)$.

Define $\alpha_Y^0(y) \coloneqq (\alpha_{Y0}^0(y), R_Y^0, \sim_Y^0)$. Then $\alpha_Y^0 \in \mathsf{P}Y$.

Define $\alpha_Y \coloneqq j.\alpha_Y^0$. Then $\alpha_Y \in \mathsf{P}^j Y$ and $\alpha_{Y0}(y) \neq \emptyset$ for any $y \in Y$. Therefore, (Y, α_Y) is an m-assembly.

Define $F : X \times Y \to dEMAJ(P)$ as follows:

$$F(x,y) \coloneqq \begin{cases} E(x,x) \times \alpha_Y(y) & \text{if } x \in y \\ (\emptyset, E_1(x,x) \times R_Y, E_2(x,x) \times \sim_Y) & \text{if } x \notin y \end{cases}$$

To see that $F \in \mathsf{P}^j(X \times Y)$, consider the following F_X and F_Y:

$$F_X(x,y) \coloneqq \begin{cases} E(x,x) & \text{if } x \in y \\ (\emptyset, E_1(x,x), E_2(x,x)) & \text{if } x \notin y \end{cases}$$

and

$$F_Y(x,y) \coloneqq \begin{cases} \alpha_Y(y) & \text{if } x \in y \\ (\emptyset, R_Y, \sim_Y) & \text{if } x \notin y \end{cases}$$

Then we know that $F = F_X \times F_Y$, $jF_X \vdash F_X$, and $jF_Y \vdash F_Y$, so $jF \vdash jF_X \times jF_Y \vdash F_X \times F_Y = F$. We conclude that $F \in \mathsf{P}^j(X \times Y)$.

In the following, we verify that F is a surjective functional relation on (X, E) and (Y, E_Y).

Lemma 113. *F is relational.*

Proof. We have to show that $\models_{x,x',y,y'} F(x,y) \to E(x,x') \to E_Y(y,y') \to F(x',y')$.

We note that $F(x,y) \times E(x,x') \times E_Y(y,y') \vdash E(x',x') \times E_Y(y',y')$. Therefore, $\models_{x,x',y,y'} F(x,y) \to E(x,x') \to E_Y(y,y') \to E_X(x',x') \times E_Y(y,y')$.

Suppose $\models_{x,x',y,y'} F(x,y) \to E(x,x') \to E_Y(y,y') \to E_X(x',x') \times E_Y(y,y')$ is witnessed by (r, r^*). Then we know that (r, r^*) also witnesses $\models_{x,x',y,y'} F(x,y) \to E(x,x') \to E_Y(y,y') \to F(x',y')$. $\square$

Lemma 114. *F is strict.*

Proof. We have to show that $\models_{x,y} F(x,y) \to E(x,x) \times E_Y(y,y)$. This is witnessed by $(\underline{i}, \underline{i})$. $\square$

Lemma 115. *F is single-valued.*

Proof. We have to show that $\models_{x,y,y'} F(x,y) \to F(x,y') \to E_Y(y,y')$. This is witnessed by $(\langle uv \rangle \underline{p_1} u, \langle uv \rangle \underline{p_1} u)$. $\square$

Lemma 116. *F is total.*

Proof. We have to show that $\models_x E(x,x) \to \exists^j y.F(x,y)$.

If $X_E = \emptyset$, then $\exists y.F(x,y) = (\emptyset, \emptyset, \emptyset)$, and $\exists^j y.F(x,y) = (\emptyset, \emptyset^*, \emptyset^*)$. Then we see that $(\langle u \rangle[\,], \langle u \rangle[\,])$ witnesses that $E(x,x) \vdash \exists^j y.F(x,y)$.

In what follows, we assume $X_E \neq \emptyset$, so $Y \neq \emptyset$. Fix $y_0 \in Y$. Define a set-theoretic function $f : X \to Y$ as follows:

$$f(x) \coloneqq \begin{cases} y_0 & \text{if } x \notin X_E \\ [x] & \text{if } x \in X_E \end{cases}$$

Here, $[x] \in Y$ is the equivalence class (with respect to the equivalence relation $\sim_E$) containing x.

We know that $F(x, f(x)) \vdash \exists^j y.F(x,y)$. Therefore, to show $E(x,x) \vdash \exists^j y.F(x,y)$, it suffices to show that $E(x,x) \vdash F(x, f(x))$.

Notice that by the construction of $\alpha_Y^0(y)$, we know that $(\underline{i}, \underline{i})$ witnesses that $E(x,x) \vdash \alpha_Y^0(f(x))$; thus, $E(x,x) \vdash j.\alpha_Y^0(f(x)) = \alpha_Y(f(x))$.

Also, notice that $F(x, f(x)) = E(x,x) \times \alpha_Y(f(x))$. Therefore, we have $E(x,x) \vdash E(x,x) \times \alpha_Y(f(x)) = F(x, f(x))$. $\square$

Lemma 117. *F is surjective.*

Proof. We have to show that $\models_y E_Y(y,y) \to \exists^j x.F(x,y)$, i.e. we have to show that $\alpha_Y(y) \vdash \exists^j x.F(x,y)$.

Since $\alpha_Y(y) = j.\alpha_Y^0(y)$, and $\exists^j x.F(x,y) = j.\exists x.F(x,y)$, we understand that it suffices to show that $\alpha_Y^0(y) \vdash \exists x.F(x,y)$.

Recall from Remark 77 that for $A \subseteq P$, $R \in MAJ(P)$, $\sim \in PER(P)$ satisfying $A \subseteq \mathrm{dom}(R) = \mathrm{dom}(\sim)$, we define

$$M(A,R,\sim) \coloneqq (A \upharpoonleft \sim, R \upharpoonleft \sim, \sim).$$

We know

- If $M(A,R,\sim) \in dEMAJ(P)$, and $(A_1,R_1,\sim_1) \in dEMAJ(P)$, then

$$M(A,R,\sim) \to (A_1,R_1,\sim_1) = N(A \to A_1, R \to R_1, \sim \to \sim_1)$$

- $\alpha_Y^0(y) = M(\bigcup_{x,x' \in y} E_0(x,x'), \curlyvee_{x,x' \in X} E_1(x,x'), \curlyvee_{x,x' \in X} E_2(x,x'))$

Therefore,

$$\begin{aligned} & \alpha_Y^0(y) \to \exists x.F(x,y) \\ = \; & ((\bigcup_{x,x' \in y} E_0(x,x')) \to (\exists x.F(x,y))_0 \\ & ,(\curlyvee_{x,x' \in X} E_1(x,x')) \to (\exists x.F(x,y))_1 \\ & ,(\curlyvee_{x,x'} \in X) E_2(x,x') \to (\exists x.F(x,y))_2) \end{aligned}$$

Suppose (r,r^*) witnesses that $E(x,x') \vdash E(x',x')$, i.e. for any x,x' we have:

- $r \in E_0(x,x') \to E_0(x',x')$.
- $(r,r^*) \in E_1(x,x') \to E_1(x',x')$.
- $(r,r),(r^*,r^*) \in E_2(x,x') \to E_2(x',x')$.

We claim that $\alpha_Y^0(y) \vdash \exists x.F(x,y)$ is witnessed by $(\langle u \rangle (ru)[u], \langle u \rangle (r^* u)[u])$. For any $y \in Y$:

- $(\bigcup_{x,x' \in y} E_0(x,x')) \to (\exists x.F(x,y))_0 = \bigcap_{x,x' \in y} (E_0(x,x') \to (\exists x.F(x,y))_0) \succeq \bigcap_{x,x' \in y} (E_0(x,x') \to F_0(x',y)) = \bigcap_{x,x' \in y} (E_0(x,x') \to E_0(x',x') \times \alpha_{Y0}(y))$. Notice that at the first "=", we used the fact that the arrow algebra $dEMAJ(P)$ is compatible with joins.

  For any $x,x' \in y$, and for any $a \in E_0(x,x')$, we have $ra \in E_0(x',x')$, $a \in \alpha_{Y0}^0(y)$, thus $[a] \in (\alpha_{Y0}^0(y))^{\mathbf{1}} = \alpha_{Y0}(y)$.

- $(\curlyvee_{x,x' \in X} E_1(x,x')) \to (\exists x.F(x,y))_1 = \bigcap_{x,x' \in X} (E_1(x,x') \to (\exists x.F(x,y))_1) \succeq \bigcap_{x,x' \in X} (E_1(x,x') \to F_1(x',y)) = \bigcap_{x,x' \in X} (E_1(x,x') \to E_1(x',x') \times \alpha_{Y1}(y))$.

  For any $x,x' \in X$, and for any $(a,a^*) \in E_1(x,x')$, we know $(ra,r^*a^*),(r^*a,r^*a^*) \in E_1(x',x')$, and $(a,a^*) \in \alpha_{Y1}^0(y)$, thus $([a],[a^*]) \in (\alpha_{Y1}^0(y))^* = \alpha_{Y1}(y)$.

- $(\bigcurlyvee_{x,x'\in X} E_2(x,x')) \to (\exists x.F(x,y))_2 = \bigcap_{x,x'\in X}(E_2(x,x') \to (\exists x.F(x,y))_2) \succeq \bigcap_{x,x'\in X}(E_2(x,x') \to F_2(x',y)) = \bigcap_{x,x'\in X}(E_2(x,x') \to E_2(x',x') \times \alpha_{Y2}(y))$.

  For any $x,x' \in X$, and for any $(a,a') \in E_2(x,x')$, we know $(ra,ra'),(r^*a,r^*a') \in E_2(x',x')$, and $(a,a') \in \alpha^0_{Y2}(y)$, thus $([a],[a']) \in (\alpha^0_{Y2}(y))^* = \alpha_{Y2}(y)$.

We conclude that $\alpha^0_Y(y) \vdash \exists x.F(x,y)$, thus $\alpha_Y(y) \vdash \exists^j y.F(x,y)$. So F is surjective. $\square$

We have proved that for any object (X,E), the corresponding F is a surjective functional relation on $(X,E),(Y,E_Y)$. We now show that if (X,E) is $\neg\neg$-separated, then F is also injective.

Lemma 118. *If (X,E) is $\neg\neg$-separated, then F is injective.*

Proof. We calculate:

$$(\neg^j E)(x,x') = \begin{cases} N(\emptyset, E_1(x,x) \to \emptyset^*, E_2(x,x) \to \emptyset^*) & \text{if } E_0(x,x') \neq \emptyset \\ N(P, E_1(x,x) \to \emptyset^*, E_2(x,x') \to \emptyset^*) & \text{if } E_0(x,x') = \emptyset \end{cases}$$

and

$$(\neg^j\neg^j E)(x,x') = \begin{cases} N(P, (\neg^j E)_1(x,x') \to \emptyset^*, (\neg^j E)_2(x,x') \to \emptyset^*) & \text{if } E_0(x,x') \neq \emptyset \\ N(\emptyset, (\neg^j E)_1(x,x') \to \emptyset^*, (\neg^j E)_2(x,x') \to \emptyset^*) & \text{if } E_0(x,x') = \emptyset \end{cases}$$

Consider $a \coloneqq \langle u\rangle[\,]$. We know that for any x,x':

- $(a,a) \in (\neg^j\neg^j E)(x,x')_1$.
- $(a,a) \in (\neg^j\neg^j E)(x,x')_2$.
- If $E_0(x,x') \neq \emptyset$, then $a \in (\neg^j\neg^j E)(x,x')_0$.

We understand that $(\langle uv\rangle a, \langle uv\rangle a)$ witnesses that $\models_{x,x',y} F(x,y) \to F(x',y) \to (\neg^j\neg^j E)(x,x')$.

Therefore, we have $F(x,y), F(x',y) \models_{x,x',y} (\neg^j\neg^j E)(x,x')$. We also have $F(x,y), F(x',y) \models_{x,x',y} E(x,x)$ and $F(x,y), F(x',y) \models_{x,x',y} E(x',x')$.

By assumption, (X,E) is $\neg\neg$-separated, so $\models_{x,x',y} E(x,x) \to E(x',x') \to (\neg^j\neg^j E)(x,x') \to E(x,x')$. We conclude that $\models_{x,x',y} F(x,y) \to F(x',y) \to E(x,x')$, i.e. F is injective. $\square$

We summarize what we have proved in this section in the following proposition:

Proposition 119. *An object (X,E) in $RT(\mathsf{P}^j)$ is $\neg\neg$-separated iff there exists an m-assembly (Y,α_Y) such that (X,E) and (Y,E_Y) are isomorphic.*

4.4 Finite limits, exponentials, and natural numbers object

In this section we consider explicit descriptions of finite limits, exponentials, and the natural numbers object in our category of m-assemblies.

Lemma 120 (terminal object)**.** *$(1 \coloneqq \{*\}, \alpha_1 : * \mapsto (P, P\times P, P\times P))$ is the terminal object in the category of m-assemblies.*

Proof. Clearly, $(1 \coloneqq \{*\}, \alpha_1 : * \mapsto (P, P \times P, P \times P))$ is an m-assembly. For any m-assembly (X, α_X), there is a unique set-theoretic function $! : X \to *$, which is a morphism of m-assemblies since it is witnessed by $(\underline{i}, \underline{i})$. $\square$

Lemma 121 (binary product)**.** *Suppose (X, α_X) and (Y, α_Y) are m-assemblies. Consider $\alpha_{X\times Y} : (x, y) \mapsto \alpha_X(x) \times \alpha_Y(y)$. Then $(X \times Y, \alpha_{X\times Y})$ with the projection maps $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ is the binary product of (X, α_X) and (Y, α_Y) in the category of m-assemblies.*

Proof. We note that $\alpha_{X\times Y}(x, y) \dashv\vdash \mathsf{P}^j\pi_X(\alpha_X) \times \mathsf{P}^j\pi_Y(\alpha_Y)$. So $j\alpha_{X\times Y} \vdash j.\mathsf{P}^j\pi_X(\alpha_X) \times j.\mathsf{P}^j\pi_Y(\alpha_Y) \vdash \mathsf{P}^j\pi_X(j.\alpha_X) \times \mathsf{P}^j\pi_Y(j.\alpha_Y) \vdash \mathsf{P}^j\pi_X(\alpha_X) \times \mathsf{P}^j\pi_Y(\alpha_Y) \dashv\vdash \alpha_{X\times Y}$. Therefore, $(X \times Y, \alpha_{X\times Y})$ is an m-assembly.

Also, the set-theoretic functions $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ are both morphisms of m-assemblies, since they are witnessed by $(\underline{p_0}, \underline{p_0})$ and $(\underline{p_1}, \underline{p_1})$ respectively.

To verify the universal property: suppose $f : (Z, \alpha_Z) \to (X, \alpha_X)$ and $g : (Z, \alpha_Z) \to (Y, \alpha_Y)$ are morphisms of m-assemblies. Then we have a unique set-theoretic function $(f, g) : Z \to X \times Y$ making the triangles commute. It remains to show that (f, g) is a morphism of m-assemblies:

Since f and g are morphisms of m-assemblies, we know $\alpha_Z \vdash \mathsf{P}^j f(\alpha_X) \dashv\vdash \mathsf{P}^j(f, g).\mathsf{P}^j\pi_X(\alpha_X)$ and $\alpha_Z \vdash \mathsf{P}^j g(\alpha_Y) \dashv\vdash \mathsf{P}^j(f, g).\mathsf{P}^j\pi_Y(\alpha_Y)$. Therefore, $\alpha_Z \vdash \mathsf{P}^j(f, g).\mathsf{P}^j\pi_X(\alpha_X) \times \mathsf{P}^j(f, g).\mathsf{P}^j\pi_Y(\alpha_Y) \dashv\vdash \mathsf{P}^j(f, g)(\mathsf{P}^j\pi_X(\alpha_X) \times \mathsf{P}^j\pi_Y(\alpha_Y)) \dashv\vdash \mathsf{P}^j(f, g)(\alpha_{X\times Y})$. $\square$

Lemma 122 (equalizer)**.** *Suppose $f, g : (X, \alpha_X) \to (Y, \alpha_Y)$ are two morphisms of m-assemblies. Consider $X_{fg} \coloneqq \{x \in X \mid f(x) = g(x)\}$, and let $e : X_{fg} \to X$ be the inclusion function. Define $\alpha_{X_{fg}} \coloneqq \mathsf{P}^j e(\alpha_X)$. Then $(X_{fg}, \alpha_{X_{fg}})$ with the inclusion map $e : X_{fg} \to X$ is the equalizer for f and g.*

Proof. Clearly, $(X_{fg}, \alpha_{X_{fg}})$ is an m-assembly, and e is a morphism of m-assemblies.

To verify the universal property: suppose $h : (Z, \alpha_Z) \to (X, \alpha_X)$ is a morphism of m-assemblies satisfying $f.h = g.h$. Then there exists a unique set-theoretic function $k : Z \to X_{fg}$ such that $e.k = h$. It remains to show that k is a morphism of m-assemblies:

Since h is a morphism of m-assemblies, we have $\alpha_Z \vdash \mathsf{P}^j h(\alpha_X) \dashv\vdash \mathsf{P}^j k.\mathsf{P}^j e(\alpha_X) = \mathsf{P}^j(\alpha_{X_{fg}})$. $\square$

Remark 123. *Let $f : X \to Y$ be a function, and suppose $A(x), B(x) \in \mathsf{P}X$, $C(y) \in \mathsf{P}Y$. Then the following two statements are equivalent:*

(1) $C(y) \vdash \bigcurlywedge_{x:f(x)=y} A(x) \to B(x)$.

(2) $C(f(x)) \vdash A(x) \to B(x)$.

For the direction from (2) to (1), we use the fact that $\mathbf{a} \in S_{dEMAJ(P)}$. The direction from (1) to (2) is straightforward. It may be useful to note the following equality:

$$\bigcurlywedge_y \bigcurlywedge_{x:f(x)=y} C(y) \to A(x) \to B(x) = \bigcurlywedge_x C(f(x)) \to A(x) \to B(x).$$

We may view $\bigcurlywedge_{x:f(x)=y} A(x) \to B(x)$ as $\forall f(A \to B)$, in which case the equivalence can be understood as:

$$C \vdash \forall f(A \to B) \text{ iff } \mathsf{P}f(C) \vdash A \to B$$

Lemma 124 (exponential)**.** *Suppose (X, α_X) and (Y, α_Y) are m-assemblies.*

- *Define Y^X to be the collection of all morphisms from (X, α_X) to (Y, α_Y).*
- *Consider $A : (f, x) \mapsto \alpha_X(x)$, and $B : (f, x) \mapsto \alpha_Y(f(x))$. Define $\alpha_{Y^X} : f \mapsto \bigcurlywedge_{x \in X} A(f, x) \to B(f, x)$. In other words, $\alpha_{Y^X}(f) = \bigcurlywedge_{x \in X} \alpha_X(x) \to \alpha_Y(f(x))$.*
- *Consider $\mathsf{ev} : Y^X \times X \to Y$, $(f, x) \mapsto f(x)$.*

Then (Y^X, α_{Y^X}) with the map $\mathsf{ev} : Y^X \times X \to Y$ is the exponential $(Y, \alpha_Y)^{(X, \alpha_X)}$.

Proof. To show (Y^X, α_{Y^X}) is an m-assembly, we have to show that $j\alpha_{Y^X} \vdash \alpha_{Y^X}$. Consider the projection map $\pi_{Y^X} : Y^X \times X \to Y^X$; we may denote α_{Y^X} as $\forall\pi_{Y^X}(A \to B)$. Therefore,

$$j\alpha_{Y^X} \vdash \alpha_{Y^X} \text{ iff } \mathsf{P}\pi_{Y^X}.j.\forall\pi_{Y^X}(A \to B) \vdash A \to B.$$

We know $\mathsf{P}\pi_{Y^X}.j.\forall\pi_{Y^X}(A \to B) = j.\mathsf{P}\pi_{Y^X}.\forall\pi_{Y^X}(A \to B) \vdash j(A \to B) \vdash jA \to jB$.

But notice that $(jA \to jB) \times A \vdash jB \vdash B$: the second $\vdash$ follows from the fact that (Y, α_Y) is an m-assembly. Therefore, we conclude that $\mathsf{P}\pi_{Y^X}.j.\forall\pi_{Y^X}(A \to B) \vdash A \to B$, so (Y^X, α_{Y^X}) is an m-assembly.

To see that $\mathsf{ev} : Y^X \times X \to Y$ is a morphism of m-assemblies, unfolding definitions we see that we have to show:

$$(\mathsf{P}\pi_{Y^X}.\forall\pi_{Y^X}(A \to B)) \times A \vdash B$$

which is straightforward.

To verify the universal property: Suppose $h : (Z, \alpha_Z) \times (X, \alpha_X) \to (Y, \alpha_Y)$ is a morphism of m-assemblies.

We define a function $H : Z \to Y^X$ by $z \mapsto (x \mapsto h(z, x))$. We have to first verify that $H(z)$ is indeed in Y^X, i.e., that $H(z)$ is a morphism of m-assemblies.

Since $h : Z \times X \to Y$ is a morphism of m-assemblies, we know $\alpha_Z(z) \times \alpha_X(x) \vdash \alpha_Y(h(z, x))$, thus $\alpha_Z(z) \vdash \alpha_X(x) \to \alpha_Y(h(z, x))$. So there exists (r, r^*) such that for any z, x:

- $r \in \alpha_{Z0}(z) \to \alpha_{X0}(x) \to \alpha_{Y0}(h(z, x))$.
- $(r, r^*) \in R_Z \to R_X \to R_Y$.
- $(r, r), (r^*, r^*) \in \sim_Z \to \sim_X \to \sim_Y$.

For any $z \in Z$, we assume that $\alpha_{Z0} \neq \emptyset$, so we pick (a_z, a_z^*) such that:

- $a_z \in \alpha_{Z0}(z)$.
- $(a_z, a_z^*) \in R_Z$.
- $(a_z, a_z), (a_z^*, a_z^*) \in \sim_Z$.

We then see that:

- $ra_z \in \alpha_{X0}(x) \to \alpha_{Y0}(H(z)(x))$.

- $(ra_z, r^*a_z^*) \in R_X \to R_Y$.

- $(ra_z, ra_z), (r^*a_z^*, r^*a_z^*) \in \sim_X \to \sim_Y$.

In other words, $(ra_z, r^*a_z^*)$ witnesses that $\alpha_X(x) \vdash \alpha_Y(H(z)(x))$. Thus $H(z) \in Y^X$, and $H : Z \to Y^X$ is a well-defined set-theoretic function. We also note that H is the unique set-theoretic function such that $\mathsf{ev}.(H \times \mathsf{id}_X) = h$. It remains to verify that H is also a morphism of m-assemblies.

We have to verify that $\alpha_Z(z) \vdash \alpha_{Y^X}(H(z))$, i.e.

$$\bigwedge_z \alpha_Z(z) \to \bigwedge_x (\alpha_X(x) \to \alpha_Y(h(z,x))) \text{ is in } S_{dEMAJ(P)}.$$

This holds if and only if

$$\bigwedge_{z,x} \alpha_Z(z) \to \alpha_X(x) \to \alpha_Y(h(z,x)) \text{ is in } S_{dEMAJ(P)}.$$

But this follows from the fact that h is a morphism of m-assemblies. □

Lemma 125 (natural numbers object)**.** *Consider* $(\mathbb{N}, \alpha_\mathbb{N})$ *where:*

- $\alpha_{\mathbb{N}0}(n) := \{\underline{\overline{n}}\}$.

- $R_\mathbb{N} = \{(\underline{\overline{n}}, \underline{\overline{m}}) \mid n \leq m\}$.

- $\sim_\mathbb{N} = \{(\underline{\overline{n}}, \underline{\overline{n}}) \mid n \in \mathbb{N}\}$.

Consider $0 : 1 \to \mathbb{N}$, $* \mapsto 0$.
Consider $s : \mathbb{N} \to \mathbb{N}$, $n \mapsto n+1$.
Then $(1, \alpha_1) \xrightarrow{0} (\mathbb{N}, \alpha_\mathbb{N}) \xrightarrow{s} (\mathbb{N}, \alpha_\mathbb{N})$ *is the natural numbers object in the category of m-assemblies.*

Proof. Note that there exists $\underline{\mathbf{max}} \in P$ such that for any $n_1, \ldots, n_k \in \mathbb{N}$, we have:

$$\underline{\mathbf{max}}[\underline{\overline{n_1}}, \ldots, \underline{\overline{n_k}}] = \underline{\overline{\max(n_1, \ldots, n_k)}}$$

We understand that $(\underline{\mathbf{max}}, \underline{\mathbf{max}})$ witnesses that $j\alpha_\mathbb{N} \vdash \alpha_\mathbb{N}$. Therefore, $(\mathbb{N}, \alpha_\mathbb{N})$ is an m-assembly.

$0 : 1 \to \mathbb{N}$ is a morphism of m-assemblies since it is witnessed by $(\langle u \rangle \underline{\overline{0}}, \langle u \rangle \underline{\overline{0}})$.

$s : \mathbb{N} \to \mathbb{N}$ is a morphism of m-assemblies since it is witnessed by $(\underline{S}, \underline{S})$. Here $\underline{S} \in P$ satisfies the property that for any $n \in \mathbb{N}$, $\underline{S}\underline{\overline{n}} \preceq \underline{\overline{n+1}}$.

We now consider the universal property of the natural numbers object. Suppose $1 \xrightarrow{x_0} (X, \alpha_X) \xrightarrow{f} (X, \alpha_X)$ is a diagram in the category of m-assemblies. Then there exists a unique set-theoretic function $h : \mathbb{N} \to X$ such that:

- $h(0) = x_0(*)$.

- $h(n+1) = f(h(n))$.

It remains to show that h is a morphism of m-assemblies.

There exists $\underline{l} \in P$ such that for any $a, g \in P$, we have:

- $\underline{l}ag\underline{\overline{0}} = a$.

- $\underline{l}ag\underline{\overline{n+1}} \preceq g(\underline{l}ag\underline{\overline{n}})$.

Since $\alpha_{X0}(h(0)) \neq \emptyset$, we fix (b, b^*) such that:

- $b \in \alpha_{X0}(h(0))$.
- $(b, b^*) \in R_X$.
- $(b, b), (b^*, b^*) \in \sim_X$.

Since $f : (X, \alpha_X) \to (X, \alpha_X)$ is a morphism of m-assemblies, there exists (r_f, r_f^*) such that for any $x \in X$

- $r_f \in \alpha_{X0}(x) \to \alpha_{X0}(f(x))$.
- $(r_f, r_f^*) \in R_X \to R_X$.
- $(r_f, r_f), (r_f^*, r_f^*) \in \sim_X \to \sim_X$.

Then define:

- $p \coloneqq \underline{l} b r_f$.
- $p^* \coloneqq \underline{l} b^* r_f^*$.

One can prove by induction the following:

- For any $n \in \mathbb{N}$, $p\underline{\overline{n}} \in \alpha_{X0}(h(n))$.
- For any $n \in \mathbb{N}$, $(p\underline{\overline{n}}, p^*\underline{\overline{n}}) \in R_X$.
- For any $n \in \mathbb{N}$, $(p\underline{\overline{n}}, p\underline{\overline{n}}), (p^*\underline{\overline{n}}, p^*\underline{\overline{n}}) \in \sim_X$.

Consequently, we know:

- For any $n \in \mathbb{N}$, $[p\underline{\overline{n}}] \in (\alpha_{X0}(h(n)))^{\mathbf{1}}$.
- For any $n \in \mathbb{N}$, $([p\underline{\overline{n}}], [p^*\underline{\overline{0}}, \ldots, p^*\underline{\overline{n}}]) \in R_X^*$.
- For any $n \leq m$, $([p^*\underline{\overline{0}}, \ldots, p^*\underline{\overline{n}}], [p^*\underline{\overline{0}}, \ldots, p^*\underline{\overline{m}}]) \in R_X^*$.
- For any $n \in \mathbb{N}$, $([p\underline{\overline{n}}], [p\underline{\overline{n}}]), ([p^*\underline{\overline{0}}, \ldots, p^*\underline{\overline{n}}], [p^*\underline{\overline{0}}, \ldots, p^*\underline{\overline{n}}]) \in \sim_X^*$.

Suppose (r, r^*) witnesses that $j\alpha_X \vdash \alpha_X$, i.e., for any $x \in X$,

- $r \in (\alpha_{X0}(x))^{\mathbf{1}} \to \alpha_{X0}(x)$.
- $(r, r^*) \in R_X^* \to R_X$.
- $(r, r), (r^*, r^*) \in \sim_X^* \to \sim_X$.

Consider $t, t^* \in P$ satisfying:

- $t\underline{\overline{n}} = r[p\underline{\overline{n}}]$.
- $t^*\underline{\overline{n}} = r^*[p^*\underline{\overline{0}}, \ldots, p^*\underline{\overline{n}}]$.

From the discussion above, we see that:

- For any $n \in \mathbb{N}$, $t\overline{\underline{n}} \in \alpha_{X0}(h(n))$.
- For any $n \leq m$, $(t\overline{\underline{n}}, t^*\overline{\underline{m}}), (t^*\overline{\underline{n}}, t^*\overline{\underline{m}}) \in R_X$.
- For any $n \in \mathbb{N}$, $(t\overline{\underline{n}}, t\overline{\underline{n}}), (t^*\overline{\underline{n}}, t^*\overline{\underline{n}}) \in \sim_X$.

Therefore, (t, t^*) witnesses that $\alpha_{\mathbb{N}}(n) \vdash \alpha_X(h(n))$, so h is a morphism of m-assemblies. $\square$

Chapter 5

Logical principles

In this chapter, we verify that $\mathrm{IP}^{\neg}$ (Independence of Premise for negated formulas) and AC (Axiom of Choice) are valid in the topos for finite types. We also show that the topos has the Fan recursor. The validity of $\mathrm{IP}^{\neg}$ and AC is expected from the modified realizability aspect of the construction, and our verification follows the proofs in [Vri17] and [Gu18], adapted to our notation. The existence of the Fan recursor may follow from the majorizability aspect of the construction, which was discussed in [Gu18]. Thus, the results in this chapter are mainly obtained by following existing ideas and unfolding them in our setting. The verification process can be somewhat tedious, especially for $\mathrm{IP}^{\neg}$ and AC.

We will be working with topos logic. Since our topos is built from a tripos, there is a standard and explicit way of describing topos logic in terms of tripos logic (Section 2.3 in [Oos08]). By topos logic, we mean that we regard a topos as a model providing semantics for logical formulas. Each formula will be interpreted as a subobject of a certain object representing the free variables in that formula.

Recall that we call our tripos P^j, and we call the topos arising from the tripos $RT(\mathsf{P}^j)$. We understand that objects in the topos $RT(\mathsf{P}^j)$ are tuples (X, E) where X is a set and $E \in \mathsf{P}^j(X \times X)$ is a symmetric and transitive relation (in terms of tripos logic) on X. We also know that subobjects of (X, E) correspond to strict relations on (X, E): they are elements of P^j which are relational and strict (in terms of tripos logic). Thus, formulas are ultimately interpreted as strict relations on certain objects, and in this sense, topos logic is reduced to tripos logic. These results will be briefly reviewed in Section 5.1.3.

Since the types of variables occurring in those logical principles ($\mathrm{IP}^{\neg}$, AC) are mainly finite types, and we know that finite types are $\neg\neg$-separated, they have a simpler description in terms of m-assemblies, as we showed in the previous chapter. Also, subobjects of $\neg\neg$-separated objects are themselves $\neg\neg$-separated, so one may expect a simpler description of subobjects of those m-assemblies in the topos. This will be briefly discussed in Section 5.1.1.

One may also notice that finite types, viewed as m-assemblies, have nicer properties than general m-assemblies. For instance, recall the natural number object discussed in Lemma 125. The natural numbers may be identified with the set of "potential realizers" $\mathrm{dom}(R_{\mathbb{N}})$, which is exactly the set $\{\underline{n} \,|\, n \in \mathbb{N}\}$. Also, the set of "actual realizers" suffices to determine the natural number: we know $\alpha_{\mathbb{N}0}(n) = \{\underline{n}\}$, and clearly if $\alpha_{\mathbb{N}0}(n) \cap \alpha_{\mathbb{N}0}(m) \neq \emptyset$ then $n = m$. It turns out that all finite types satisfy similar properties, and they will be discussed in Section 5.1.2. These properties are also expected; see Section 4.1 in [Gu18].

After discussing those useful facts, we verify that $\mathrm{IP}^{\neg}$ and AC are valid in the topos for finite types. The proofs largely consist of adapting existing arguments to our setting and unfolding the relevant definitions. After that, we work out an explicit expression of the List object in the topos, to handle the types of finite sequences that will appear in the discussion of the Fan recursor. Originally, we also hoped to discuss the Bar recursor, but eventually we decided to leave it for future work and mention it only briefly in the Conclusion chapter.

5.1 Useful facts

We discuss some simple facts about m-assemblies and finite types which will be very useful later when verifying those logical principles.

5.1.1 Canonical representations of strict relations on m-assemblies

In a tripos topos, the collection of subobjects of any object is equivalent to the collection of strict relations on that object. Every m-assembly (X, α_X) can be embedded into the realizability topos $RT(\mathsf{P}^j)$. Recall that we denote the image of this embedding by (X, E_X), where

$$E_X(x, x') \coloneqq \begin{cases} \alpha_X(x) & \text{if } x = x' \\ (\emptyset, R_X, \sim_X) & \text{if } x \neq x' \end{cases}$$

A strict relation on (X, E_X) is an element $S(x)$ in $\mathsf{P}^j X$ which is both relational and strict, i.e.,

- $\models_{x,x'} S(x) \to E_X(x, x') \to S(x')$.
- $\models_x S(x) \to E_X(x, x)$.

In general, $S_1(x)$ and $S_2(x)$ are not constant: they may vary with x. But indeed, every strict relation $S(x)$ on (X, E_X) is equivalent to a *canonically presented* one. In other words, there exists another strict relation $\widetilde{S}(x)$ on (X, E_X) such that $S(x) \dashv\vdash \widetilde{S}(x)$, where $\widetilde{S}_1$ and $\widetilde{(S)}_2$ are both constant.

Every strict relation on (X, E_X) corresponds to a subobject of (X, E_X), and since a subobject of a $\neg\neg$-separated object is again $\neg\neg$-separated, we know there exists an m-assembly (Y, α_Y) and a morphism of m-assemblies $f : (Y, \alpha_Y) \to (X, \alpha_X)$ such that $F(x, y) : (Y, E_Y) \to (X, E_X)$ is a mono in the arrow topos and defines the same subobject as $S(x)$. We know the strict relation defined by $F(x, y)$, written as $\widetilde{S}(x)$, is $\exists^j y.F(x, y) = j.\exists y.F(x, y)$. If $F_1(x, y), F_2(x, y)$ are constant, then $(\exists y.F(x, y))_1, (\exists y.F(x, y))_2$ are constant (see Lemma 74 and Definition 17 for an explicit expression of the existential quantifier $\exists$), thus $(\exists^j y.F(x, y))_1, (\exists^j y.F(x, y))_2$ are constant (see Definition 80 for the definition of the nucleus j).

We conclude that the strict relation $\widetilde{S}$ is equivalent to $S(x)$, and $\widetilde{S}_1, \widetilde{S}_2$ are constant.

5.1.2 Elementary properties of finite types

We prove that finite types in the category of m-assemblies satisfy some elementary and useful properties. These properties are all proved by induction. The induction step is to show that $(X, \alpha_X)^{(Y, \alpha_Y)}$ satisfies them, assuming that (X, α_X) and (Y, α_Y) do.

We mainly focus on the following three properties of an m-assembly (X, α_X):

(0) For any $x, x' \in X$, if $\alpha_{X0}(x) \cap \alpha_{X0}(x') \neq \emptyset$, then $x = x'$.

(1) For any $a \in \mathrm{dom}(R_X)$, there exists $x \in X$ such that $a \in \alpha_{X0}(x)$.

(2) For any $x \in X$, for any $a, a' \in \alpha_{X0}(x)$, $(a, a') \in \sim_X$.

From now on, we may simply say property (i) to refer to the property described above. For instance, we say (X, α_X) satisfies property (0) iff for any $x, x' \in X$, if $\alpha_{X0}(x) \cap \alpha_{X0}(x') \neq \emptyset$, then $x = x'$.

In the following, we first prove by induction that every finite type satisfies property (0). Then we show that every finite type satisfies (1) and (2) by induction on finite types, where the induction hypothesis is that (1) and (2) are both satisfied.

Recall that we have an explicit expression for the natural numbers object in the category of m-assemblies. It is $(\mathbb{N}, \alpha_{\mathbb{N}})$, where $\alpha_{\mathbb{N}0}(n) = \{\underline{\overline{n}}\}$, $R_{\mathbb{N}} = \{(\underline{\overline{n}}, \underline{\overline{m}}) \mid n \leq m\}$, and $\sim_{\mathbb{N}} = \{(\underline{\overline{n}}, \underline{\overline{n}}) \mid n \in \mathbb{N}\}$. For this explicit expression, one can verify that $(\mathbb{N}, \alpha_{\mathbb{N}})$ has properties (0), (1), and (2).

Remark 126. *It is straightforward to see that property (0) is stable under isomorphism. However, the following examples suggest that properties (1) and (2) may not be stable under isomorphism.*

We will consider three m-assemblies where the underlying set is always a singleton, while the realizer structures differ. They are (given in the order $\alpha_{(-)0}, R_{(-)}, \sim_{(-)}$):

(a) $\{\underline{\overline{0}}\}$, $\{(\underline{\overline{0}}, \underline{\overline{0}})\}$, *and* $\{(\underline{\overline{0}}, \underline{\overline{0}})\}$.

(b) $\{\underline{\overline{0}}\}$, $\{(\underline{\overline{0}}, \underline{\overline{0}}), (\underline{\overline{0}}, \underline{\overline{1}}), (\underline{\overline{1}}, \underline{\overline{1}})\}$, *and* $\{(\underline{\overline{0}}, \underline{\overline{0}}), (\underline{\overline{1}}, \underline{\overline{1}})\}$.

(c) $\{\underline{\overline{0}}, \underline{\overline{1}}\}$, $\{(\underline{\overline{0}}, \underline{\overline{0}}), (\underline{\overline{0}}, \underline{\overline{1}}), (\underline{\overline{1}}, \underline{\overline{1}})\}$, *and* $\{(\underline{\overline{0}}, \underline{\overline{0}}), (\underline{\overline{1}}, \underline{\overline{1}})\}$.

We note that examples (a), (b), and (c) are isomorphic. Example (a) satisfies both properties (1) and (2). Example (b) does not satisfy property (1), since $\underline{\overline{1}}$ is in dom(R_b) but not in $\alpha_{b0}(*)$. Example (c) does not satisfy property (2), since $(\underline{\overline{0}}, \underline{\overline{1}})$ is not in $\sim_c$.

Lemma 127. *Suppose (X, α_X) and (Y, α_Y) are m-assemblies. If (Y, α_Y) has property (0), then $(Y, \alpha_Y)^{(X, \alpha_X)}$ also has property (0).*

Here, we take $(Y, \alpha_Y)^{(X, \alpha_X)}$ to be the explicit expression discussed in Lemma 124; because those properties are not stable under isomorphism, it is important to choose a suitable explicit expression.

Proof. Recall that the exponential $(Y, \alpha_Y)^{(X, \alpha_X)}$ is the m-assembly (Y^X, α_{Y^X}), where Y^X is the set of all morphisms from (X, α_X) to (Y, α_Y), and for any $f : (X, \alpha_X) \to (Y, \alpha_Y)$, $\alpha_{Y^X}(f) = \bigcurlywedge_{x \in X} \alpha_X(x) \to \alpha_Y(f(x))$.

Now, suppose $f, g \in Y^X$ satisfy $\alpha_{Y^X 0}(f) \cap \alpha_{Y^X 0}(g) \neq \emptyset$. Then there exists r such that $r \in \alpha_{Y^X 0}(f)$ and $r \in \alpha_{Y^X 0}(g)$. Unfolding the definitions, we know:

- $r \in \bigcap_{x \in X} \alpha_{X0}(x) \to \alpha_{Y0}(f(x))$.
- $r \in \bigcap_{x \in X} \alpha_{X0}(x) \to \alpha_{Y0}(g(x))$.

Therefore, for any $x \in X$, since $\alpha_{X0}(x) \neq \emptyset$, fix $a \in \alpha_{X0}(x)$. Then we know $ra \in \alpha_{Y0}(f(x)) \cap \alpha_{Y0}(g(x))$. By assumption, (Y, α_Y) satisfies property (0), so $f(x) = g(x)$. Hence $f = g$. $\square$

Lemma 128. *Suppose (X, α_X) and (Y, α_Y) are m-assemblies, and suppose they both have properties (1) and (2). Then $(Y, \alpha_Y)^{(X, \alpha_X)}$ also has properties (1) and (2).*

Here, we take $(Y, \alpha_Y)^{(X, \alpha_X)}$ to be the explicit expression discussed in Lemma 124.

Proof. We first show that $(Y, \alpha_Y)^{(X,\alpha_X)}$ has property (1).

Suppose $r \in \mathrm{dom}(R_{YX})$. Then there exists r^* such that $(r, r^*) \in R_{YX}$. Recall that $\alpha_{YX}(f) = N(\bigcap_{x\in X} \alpha_{X0}(x) \to \alpha_{Y0}(f(x)), \bigcap_{x\in X} R_X \to R_Y, \bigcap_{x\in X} \sim_X \to \sim_Y)$. In particular, we know:

- $(r, r^*) \in \bigcap_{x\in X} R_X \to R_Y$.
- $(r, r), (r^*, r^*) \in \bigcap_{x\in X} \sim_X \to \sim_Y$.

For any $x \in X$ and any $a \in \alpha_{X0}(x)$, we know there exists a^* such that $(a, a^*) \in R_X$, hence $(ra, r^*a^*) \in R_Y$. Therefore, $ra \in \mathrm{dom}(R_Y)$. By the assumption that (Y, α_Y) satisfies property (1), there exists $y_a \in Y$ such that $ra \in \alpha_{Y0}(y_a)$. Moreover, for any $a' \in \alpha_{X0}(x)$, by the assumption that (X, α_X) satisfies property (2), we know $(a, a') \in \sim_X$, hence $(ra, ra') \in \sim_Y$, and thus $ra' \in \alpha_{Y0}(y_a)$.

This implies that there exists a set-theoretic function $f : X \to Y$ such that for all $x \in X$ and all $a \in \alpha_{X0}(x)$, we have $ra \in \alpha_{Y0}(f(x))$.

Notice that (r, r^*) witnesses that f is a morphism from (X, α_X) to (Y, α_Y), and thus $r \in \alpha_{Y^X 0}(f)$. We conclude that $(Y, \alpha_Y)^{(X,\alpha_X)}$ has property (1).

We now show that $(Y, \alpha_Y)^{(X,\alpha_X)}$ has property (2).

Suppose $f \in Y^X$ and $r, r' \in \alpha_{Y^X 0}(f)$. We must show that $(r, r') \in \sim_{Y^X}$. We note that it remains to show that $(r, r') \in \bigcap_{x\in X} \sim_X \to \sim_Y$. For any $x \in X$ and any $(a, a') \in \sim_X$, we show $(ra, r'a') \in \sim_Y$.

We know $a \in \mathrm{dom}(R_X)$, and by the assumption that (X, α_X) satisfies property (1), there exists $x_a \in X$ such that $a \in \alpha_{X0}(x_a)$. Since $(a, a') \in \sim_X$, we know $a' \in \alpha_{X0}(x_a)$.

Since $r, r' \in \alpha_{Y^X 0}(f)$, we know $r, r' \in \bigcap_{x\in X} \alpha_{X0}(x) \to \alpha_{Y0}(f(x))$. In particular, $r, r' \in \alpha_{X0}(x_a) \to \alpha_{Y0}(f(x_a))$. Therefore, $ra, ra', r'a, r'a' \in \alpha_{Y0}(f(x_a))$. By the assumption that (Y, α_Y) has property (2), we know $(ra, r'a') \in \sim_Y$. We conclude that $(Y, \alpha_Y)^{(X,\alpha_X)}$ has property (2). $\square$

5.1.3 Topos logic reduced to tripos logic

In this section, we give a brief overview of the reduction of topos logic to tripos logic. We refer to Section 2.3 in [Oos08] for a more complete discussion of this topic.

Suppose (X, E) is an object in $RT(\mathsf{P}^j)$. We know that the collection of subobjects of (X, E) is equivalent to the collection of strict relations on (X, E). Suppose $S(x)$ and $T(x)$ are two strict relations on (X, E):

- The conjunction $S(x)$ and $T(x)$ (representing the meet of the corresponding subobjects) can be taken to be the strict relation $S \times T$, where $\times$ denotes the binary meet in $\mathsf{P}^j X$.
- The disjunction $S(x)$ or $T(x)$ can be taken to be $S \vee T$, where $\vee$ denotes the binary join in P^j.
- The implication $S(x)$ implies $T(x)$ can be taken to be $E(x, x) \times (S(x) \to T(x))$.
- The element $\perp^j$ in $\mathsf{P}^j X$ is the bottom element in the lattice of subobjects of (X, E).
- $E(x, x)$ is the top element in the lattice of subobjects of (X, E).

Suppose (X, E_X) and (Y, E_Y) are two objects in $RT(\mathsf{P}^j)$ (here we do not assume that they come from m-assemblies), and $F(x, y)$ is a functional relation on $(X, E_X), (Y, E_Y)$. Suppose $T(y)$ is a strict relation on (Y, E_Y). Suppose $S(x, y)$ is a functional relation on $(X, E_X) \times (Y, E_Y)$, and suppose $p_X : (X, E_X) \times (Y, E_Y) \to (X, E_X)$ is the projection morphism. Then

- $F^*(T)$, as a subobject of (X, E_X), can be taken to be $\exists y.(F(x,y) \wedge T(y))$.
- $\forall p_X(S(x,y))$, as a subobject of (X, E_X), can be taken to be $E_X(x,x) \times \forall y.(E_Y(y,y) \to S(x,y))$, which is an element in $\mathsf{P}^j X$.
- $\exists p_X(S(x,y))$, as a subobject of (X, E_X), can be taken to be $\exists^j y.S(x,y)$, which is an element in $\mathsf{P}^j X$.

We then consider a useful result, which implies that when taking existential quantifiers for canonically presented strict relations on m-assemblies, we usually only need to take $\exists$ instead of $\exists^j$, since taking $\exists$ already ensures that the result is in P^j.

Lemma 129. *Suppose $f : X \to Y$ is a surjective function, and suppose $S(x) \in \mathsf{P}^j X$. If S_1 and S_2 are constants (i.e. there exists $(R, \sim) \in EMAJ(P)$ such that for all $x \in X$, $(S_1(x), S_2(x)) = (R, \sim)$), then $\exists f(S)(y) \in \mathsf{P}^j Y$.*

Proof. $(\exists f(S))(y) = \curlyvee_{x:f(x)=y} S(x) = M(\bigcup_{x:f(x)=y} S_0(x), \curlyvee_{x:f(x)=y} S_1(x), \curlyvee_{x:f(x)=y} S_2(x)) = M(\bigcup_{x:f(x)=u} S_0(x), S_1, S_2) = (\bigcup_{x:f(x)=y} S_0(x), S_1, S_2)$. Note that in the third equality we used the assumption that f is surjective, and in the fourth equality we used the fact that $\bigcup_{x:f(x)=y} S_0(x)$ and S_1 are already compatible with S_2.

We have to show that $j.(\exists f(S)) \vdash \exists f(S)$.

Suppose (r, r^*) witnesses that $j.S \vdash S$, i.e. for any $x \in X$,

- $r \in S_0^{\mathbf{1}}(x) \to S_0(x)$.
- $(r, r^*) \in S_1^* \to S_1$.
- $(r, r), (r^*, r^*) \in S_2^* \to S_2$.

We claim that (r, r^*) also witnesses that $j.(\exists f(S)) \vdash \exists f(S)$. For any $y \in Y$:

We show that $r \in (\bigcup_{x:f(x)=y} S_0(x))^{\mathbf{1}} \to (\bigcup_{x:f(x)=y} S_0(x))$. Notice that $(\bigcup_{x:f(x)=y} S_0(x))^{\mathbf{1}} = \bigcup_{x:f(x)=y} S_0^{\mathbf{1}}(x)$, so $(\bigcup_{x:f(x)=y} S_0(x))^{\mathbf{1}} \to (\bigcup_{x:f(x)=y} S_0(x)) = \bigcap_{x:f(x)=y} (S_0^{\mathbf{1}}(x) \to \bigcup_{z:f(z)=y} S_0(z)) \supseteq \bigcap_{x:f(x)=y} (S_0^{\mathbf{1}}(x) \to S_0(x))$, and we know $r \in \bigcap_{x:f(x)=y} (S_0^{\mathbf{1}}(x) \to S_0(x))$.

Also, there exists $x \in X$ such that $f(x) = y$, so $(r, r^*) \in S_1^* \to S_1$, and $(r, r), (r^*, r^*) \in S_2^* \to S_2$. $\square$

Corollary 130. *Suppose (X, α_X) and (Y, α_Y) are m-assemblies, where $Y \neq \emptyset$. Suppose $S(x,y)$ is a canonically presented strict relation on $(X \times Y, E_{X \times Y})$. Then $\exists y.S(x,y) \in \mathsf{P}^j(X)$ is a strict relation on (X, E_X), which is exactly $\exists p_X(S(x,y))$ as a subobject of (X, E_X), where $p_X : (X \times Y, E_{X \times Y}) \to (X, E_X)$ is the projection morphism.*

Proof. Since Y is non-empty, we apply the previous lemma, together with the facts about reducing topos logic to tripos logic. $\square$

5.2 IP$^\neg$ is valid for finite types in our topos

In this section, we show that IP$^\neg$ holds in our topos for objects arising from m-assemblies and satisfying certain nice properties. In particular, IP$^\neg$ holds for finite types. We first list several technical lemmas.

Lemma 131. *Suppose I is a non-empty index set, $A_i \subseteq P$ for any $i \in I$. Suppose $R \in MAJ(P)$, $E \in PER(P)$. If for any $i \in I$, $(A_i, R, E) \in dEMAJ(P)$, then*

$$\curlyvee_{i\in I}(A_i, R, E) = (\bigcup_{i\in I} A_i, R, E).$$

Proof. We have

$$\begin{aligned}
& \curlyvee_{i\in I}(A_i, R, E) \\
= \; & M(\bigcup_{i\in I} A_i, \curlyvee_{i\in I} R, \curlyvee_{i\in I} E) \\
= \; & M(\bigcup_{i\in I} A_i, R, E) \\
= \; & (\bigcup_{i\in I} A_i, R, E)
\end{aligned}$$

□

Lemma 132. *Suppose I is a non-empty index set, $A_i \subseteq P$ for each $i \in I$. Suppose $R \in MAJ(P)$, $E \in PER(P)$. If for each $i \in I$, $N(A_i, R, E) \in dEMAJ(P)$, then*

$$\curlyvee_{i\in I} N(A_i, R, E) = N(\bigcup_{i\in I} A_i, R, E).$$

Proof. We have

$$\begin{aligned}
& \curlyvee_{i\in I} N(A_i, R, E) \\
= \; & \curlyvee_{i\in I}(A_i \cap \mathrm{dom}(R \restriction \mathrm{dom}(E)), R \restriction \mathrm{dom}(E), E \restriction \mathrm{dom}(R \restriction \mathrm{dom}(E))) \\
= \; & (\bigcup_{i\in I}(A_i \cap \mathrm{dom}(R \restriction \mathrm{dom}(E))), R \restriction \mathrm{dom}(E), E \restriction \mathrm{dom}(R \restriction \mathrm{dom}(E))) \\
= \; & ((\bigcup_{i\in I} A_i) \cap \mathrm{dom}(R \restriction \mathrm{dom}(E)), R \restriction \mathrm{dom}(E), E \restriction \mathrm{dom}(R \restriction \mathrm{dom}(E))) \\
= \; & N(\bigcup_{i\in I} A_i, R, E)
\end{aligned}$$

We used the assumption that I is non-empty in the second equality. □

Lemma 133. *Suppose A_1, B_1, A_2, B_2 are subsets of P, then we have:*

$$(A_1 \cap B_1) \times (A_2 \cap B_2) = (A_1 \times A_2) \cap (B_1 \times B_2).$$

Proof. For any $\underline{p}ab$, we know $\underline{p}ab \in$ LHS iff $a \in A_1 \cap B_1$ and $b \in A_2 \cap B_2$, iff $\underline{p}ab \in A_1 \times B_1$ and $\underline{p}ab \in A_2 \times B_2$, iff $\underline{p} \in$ RHS. □

Lemma 134. *Suppose $R_1, R_2 \in MAJ(P)$, then $dom(R_1) \times dom(R_2) = dom(R_1 \times R_2)$.*

Proof. For any $\underline{p}ab$, $\underline{p}ab \in$ LHS iff $a \in \mathrm{dom}(R_1)$ and $b \in \mathrm{dom}(R_2)$, iff there exists a^*, b^* such that $(a, a^*) \in R_1$ and $(b, b^*) \in R_2$, iff there exists $\underline{p}a^*b^*$ such that $(\underline{p}ab, \underline{p}a^*b^*) \in R_1 \times R_2$, iff $\underline{p}ab \in$ RHS. □

Lemma 135. *Suppose $A_1, A_2 \subseteq P$, $R_1, R_2 \in MAJ(P)$, $E_1, E_2 \in PER(P)$. Then we have:*

$$N(A_1, R_1, E_1) \times N(A_2, R_2, E_2) = N(A_1 \times A_2, R_1 \times R_2, E_1 \times E_2).$$

Proof. The proof is about verifying three things.

Firstly, we show that $(R_1 \upharpoonright \mathrm{dom}(E_1)) \times (R_2 \upharpoonright \mathrm{dom}(E_2)) = (R_1 \times R_2) \upharpoonright \mathrm{dom}(E_1 \times E_2)$. For any $(\underline{p}ab, \underline{p}a^*b^*)$, $(\underline{p}ab, \underline{p}a^*b^*) \in$ LHS iff $(a, a^*) \in R_1 \upharpoonright \mathrm{dom}(E_1)$ and $(b, b^*) \in R_2 \upharpoonright \mathrm{dom}(E_2)$, iff $(a, a^*) \in R_1$, $(a, a), (a^*, a^*) \in E_1$, $(b, b^*) \in R_2$, $(b, b), (b^*, b^*) \in E_2$, iff $(\underline{p}ab, \underline{p}a^*b^*) \in R_1 \times R_2$ and $(\underline{p}ab, \underline{p}ab), (\underline{p}a^*b^*, \underline{p}a^*b^*) \in E_1 \times E_2$, iff $(\underline{p}ab, \underline{p}a^*b^*) \in$ RHS.

Secondly, we show that $(E_1 \upharpoonright \mathrm{dom}(R_1 \upharpoonright \mathrm{dom}(E_1))) \times (E_2 \upharpoonright \mathrm{dom}(R_2 \upharpoonright \mathrm{dom}(E_2))) = (E_1 \times E_2) \upharpoonright \mathrm{dom}((R_1 \times R_2) \upharpoonright \mathrm{dom}(E_1 \times E_2))$. For any $(\underline{p}ab, \underline{p}a'b')$, $(\underline{p}ab, \underline{p}a'b') \in$ LHS iff $(a, a') \in (E_1 \upharpoonright \mathrm{dom}(R_1 \upharpoonright \mathrm{dom}(E_1)))$ and $(b, b') \in (E_2 \upharpoonright \mathrm{dom}(R_2 \upharpoonright \mathrm{dom}(E_2)))$, iff $(a, a') \in E_1$, $a, a' \in \mathrm{dom}(R_1 \upharpoonright \mathrm{dom}(E_1))$, $(b, b') \in E_2$, $b, b' \in \mathrm{dom}(R_1 \upharpoonright \mathrm{dom}(E_1))$, iff $(\underline{p}ab, \underline{p}a'b') \in E_1 \times E_2$, $\underline{p}ab, \underline{p}a'b' \in \mathrm{dom}(R_1 \upharpoonright \mathrm{dom}(E_1)) \times \mathrm{dom}(R_2 \upharpoonright \mathrm{dom}(E_2)) = \mathrm{dom}((R_1 \upharpoonright \mathrm{dom}(E_1)) \times (R_2 \upharpoonright \mathrm{dom}(E_2))) = \mathrm{dom}((R_1 \times R_2) \upharpoonright \mathrm{dom}(E_1 \times E_2))$, iff $(\underline{p}ab, \underline{p}a'b') \in$ RHS.

Finally, we show that $(A_1 \cap \mathrm{dom}(R_1 \upharpoonright \mathrm{dom}(E_1))) \times (A_2 \cap \mathrm{dom}(R_2 \upharpoonright \mathrm{dom}(E_2))) = (A_1 \times A_2) \cap \mathrm{dom}((R_1 \times R_2) \upharpoonright \mathrm{dom}(E_1 \times E_2))$, but this follows from the results proved above, together with the previous two lemmas. $\square$

Proposition 136 (IP$^\neg$)**.** *Suppose (X, α_X) and (Y, α_Y) are both m-assemblies. Suppose (Y, α_Y) satisfies property (1), i.e., for any $a \in dom(R_Y)$, there exists $y \in Y$ such that $a \in \alpha_{Y0}(y)$. Also, suppose $Y \neq \emptyset$.*

Then for any strict relation $S(x)$ on (X, E_X), for any strict relation $T(x, y)$ on $(X, E_X) \times (Y, E_Y)$, the following sentence is valid in $RT(\mathsf{P}^j)$:

$$\forall x.(((\neg S)(x) \to \exists y.T(x, y)) \to \exists y.((\neg S)(x) \to T(x, y)))$$

Proof. We can assume $S(x)$ and $T(x, y)$ are both canonically presented, i.e. the $EMAJ(P)$-part in $S(x)$ and $T(x, y)$ are constants.

Translate topos logic into tripos logic, write $A(x) := ((S(x) \to \bot^j) \to \exists y.T(x, y))$, and $B(x) := \exists y.(E_Y(y, y) \times ((S(x) \to \bot^j) \to T(x, y)))$. We understand that the goal is to show that in tripos logic, we have:

$$E_X(x, x) \vdash E_X(x, x) \times (A(x) \to B(x)).$$

This is equivalent to showing that:

$$\models_x E_X(x, x) \to A(x) \to B(x).$$

We will show that indeed we have $\models_x A(x) \to B(x)$, i.e. $A(x) \vdash B(x)$, which suffices to prove the proposition. Let us restate the definitions of $A(x)$ and $B(x)$ (this time everything is in terms of the tripos P^j):

- $A(x) = (\neg^j S)(x) \to \exists y.T(x, y)$. Here $\neg^j S := S \to \bot^j$.
- $B(x) = \exists y.(E_Y(y, y) \times ((\neg^j S)(x) \to T(x, y)))$.

The goal is to show $A(x) \vdash B(x)$. Why do we expect this to be true? Roughly speaking, we have to transform a "realizer" of $A(x)$ into a "realizer" of $B(x)$. In $A(x)$, given something in $(\neg^j S)(x)$, we obtain something in $\exists y.T(x, y)$, but we do not know which y; in $B(x)$, however, we have to first specify which y it is. This seems to be a problem, but it can be addressed, because at the level of potential realizers there is not much difference between different y, and also the actual realizers of $\neg^j S$ have only two cases: either it is empty, or it coincides with the set of potential realizers. In this sense, things can be resolved, and the proof is mainly about applying this intuition and unfolding definitions.

We calculate:

$$(\neg^j S)(x) = S(x) \to \bot^* = N(S_0(x) \to \emptyset, S_1 \to \emptyset^*, S_2 \to \emptyset^*).$$

We know $(\neg^j S)_1 = (S_1 \to \emptyset^*) \restriction \mathrm{dom}(S_2 \to \emptyset^*)$, and $(\neg^j S)_2 = (S_2 \to \emptyset^*) \restriction \mathrm{dom}((\neg^j S)_1)$, so

$$(\neg^j S)(x) = \begin{cases} (\emptyset, (\neg^j S)_1, (\neg^j S)_2) & \text{if } S_0(x) \neq \emptyset \\ (\mathrm{dom}((\neg^j S)_1), (\neg^j S)_1, (\neg^j S)_2) & \text{if } S_0(x) = \emptyset \end{cases}$$

Consider $\Diamond := \langle u \rangle[\,]$. We know that:

- $(\Diamond, \Diamond) \in S_1 \to \emptyset^*$.
- $(\Diamond, \Diamond) \in S_2 \to \emptyset^*$.

Therefore, we know

- $\Diamond \in \mathrm{dom}((\neg^j S)_1)$.
- $(\Diamond, \Diamond) \in (\neg^j S)_1$.
- $(\Diamond, \Diamond) \in (\neg^j S)_2$.

We continue the calculation:

$$\begin{array}{rl} & \exists y. T(x, y) \\ = & (\bigcup_y T_0(x, y), T_1, T_2) \end{array}$$

Therefore,

$$\begin{array}{rl} & A(x) \\ = & (\neg^j S)(x) \to \exists y. T(x, y) \\ = & (((\neg^j S)_0 \to \bigcup_y T_0(x, y)) \cap \mathrm{dom}(((\neg^j S)_1 \to T_1) \restriction \mathrm{dom}((\neg^j S)_2 \to T_2)) \\ & , ((\neg^j S)_1 \to T_1) \restriction \mathrm{dom}((\neg^j S)_2 \to T_2) \\ & , ((\neg^j S)_2 \to T_2) \restriction \mathrm{dom}(((\neg^j S)_1 \to T_1) \restriction \mathrm{dom}((\neg^j S)_2 \to T_2))) \end{array}$$

We continue:

$$\begin{array}{rl} & (\neg^j S)(x) \to T(x, y) \\ = & N((\neg^j S)_0(x) \to T_0(x, y), (\neg^j S)_1 \to T_1, (\neg^j S)_2 \to T_2) \end{array}$$

$$\begin{array}{rl} & E_Y(y, y) \times ((\neg^j S)(x) \to T(x, y)) \\ = & N(\alpha_{Y0}(y) \times ((\neg^j S)_0(x) \to T_0(x, y)) \\ & , R_Y \times ((\neg^j S)_1 \to T_1) \\ & , \sim_Y \times ((\neg^j S)_2 \to T_2)) \end{array}$$

Therefore,

$$
\begin{aligned}
& B(x) \\
= \; & \exists y.(E_Y(y,y) \times ((\neg^j S)(x) \to T(x,y))) \\
= \; & N(\bigcup_y(\alpha_{Y0}(y) \times ((\neg^j S)_0(x) \to T_0(x,y))) \\
& , R_Y \times ((\neg^j S)_1 \to T_1) \\
& , \sim_Y \times((\neg^j S)_2 \to T_2))
\end{aligned}
$$

Suppose (r_Y, r_Y^*) witnesses that $T(x,y) \vdash E_Y(y,y)$, i.e. for any x, y,

- $r_Y \in T_0(x,y) \to \alpha_{Y0}(y)$.
- $(r_Y, r_Y^*) \in T_1 \to R_Y$.
- $(r_Y, r_Y), (r_Y^*, r_Y^*) \in T_2 \to\sim_X$.

Then we claim that $(t, t^*) := (\langle u \rangle \underline{p}(r_Y(u\lozenge))(\langle v \rangle u\lozenge), \langle u \rangle \underline{p}(r_Y^*(u\lozenge))(\langle v \rangle u\lozenge))$ witnesses that $A(x) \vdash B(x)$. Unfolding the definitions, we need to verify that for any $x \in X$,

- $t \in (((\neg^j S)_0 \to \bigcup_y T_0(x,y)) \cap \mathrm{dom}(((\neg^j S)_1 \to T_1) \restriction \mathrm{dom}((\neg^j S)_2 \to T_2))) \to \bigcup_y (\alpha_{Y0}(y) \times ((\neg^j S)_0(x) \to T_0(x,y)))$.
- $(t, t^*) \in (((\neg^j S)_1 \to T_1) \restriction \mathrm{dom}((\neg^j S)_2 \to T_2)) \to (R_Y \times ((\neg^j S)_1 \to T_1))$.
- $(t, t), (t^*, t^*) \in (((\neg^j S)_2 \to T_2) \restriction \mathrm{dom}(((\neg^j S)_1 \to T_1) \restriction \mathrm{dom}((\neg^j S)_2 \to T_2))) \to (\sim_Y \times((\neg^j S)_2 \to T_2))$.

We prove the following facts, which are sufficient to confirm our claim.

Fact 1: Suppose $(r, r^*) \in ((\neg^j S)_1 \to T_1) \restriction \mathrm{dom}((\neg^j S)_2 \to T_2)$, then $(tr, t^*r^*), (t^*r, t^*r^*) \in R_Y \times ((\neg^j S)_1 \to T_1)$.

Proof of Fact 1: Unfolding the definitions, it suffices to show that:

- $(r_Y(r\lozenge), r_Y^*(r^*\lozenge)), (r_Y^*(r\lozenge), r_Y^*(r^*\lozenge)) \in R_Y$.
- $(\langle v \rangle r\lozenge, \langle v \rangle r^*\lozenge) \in (\neg^j S)_1 \to T_1$.

Recall that $(\lozenge, \lozenge) \in (\neg^j S)_1$. Since $(r, r^*) \in (\neg^j S)_1 \to T_1$, we know $(r\lozenge, r^*\lozenge), (r^*\lozenge, r^*\lozenge) \in T_1$. Since $(r_Y, r_Y^*) \in T_1 \to R_Y$, we know $(r_Y(r\lozenge), r_Y^*(r^*\lozenge)), (r_Y^*(r\lozenge), r_Y^*(r^*\lozenge)) \in R_Y$.

Also, for any $(a, a^*) \in (\neg^j S)_1$, we have $((\langle v \rangle r\lozenge)a, (\langle v \rangle r^*\lozenge)a^*) = (r\lozenge, r^*\lozenge) \in T_1$, and $((\langle v \rangle r^*\lozenge)a, (\langle v \rangle r^*\lozenge)a^*) = (r^*\lozenge, r^*\lozenge) \in T_1$, thus $(\langle v \rangle r\lozenge, \langle v \rangle r^*\lozenge) \in (\neg^j S)_1 \to T_1$. This proves **Fact 1**.

Fact 2: Suppose $(r, r') \in ((\neg^j S)_2 \to T_2) \restriction \mathrm{dom}(((\neg^j S)_1 \to T_1) \restriction \mathrm{dom}((\neg^j S)_2 \to T_2))$, then $(tr, tr'), (t^*r, t^*r') \in\sim_Y \times((\neg^j S)_2 \to T_2)$.

Proof of Fact 2: Unfolding the definitions, it suffices to show that:

- $(r_Y(r\lozenge), r_Y(r'\lozenge)), (r_Y^*(r\lozenge), r_Y^*(r'\lozenge)) \in\sim_Y$.
- $(\langle v \rangle r\lozenge, \langle v \rangle r'\lozenge) \in (\neg^j S)_2 \to T_2$.

Recall that $(\Diamond, \Diamond) \in (\neg^j S)_2$. Since $(r, r') \in (\neg^j S)_2 \to T_2$, we know $(r\Diamond, r'\Diamond) \in T_2$. Therefore, for any $(a, a') \in (\neg^j S)_2$, we have $((\langle v\rangle r\Diamond)a, (\langle v\rangle r'\Diamond)a') = (r\Diamond, r'\Diamond) \in T_2$, thus $(\langle v\rangle r\Diamond, \langle v\rangle r'\Diamond) \in (\neg^j S)_2 \to T_2$.

Also, since $(r_Y, r_Y), (r_Y^*, r_Y^*) \in T_2 \to \sim_Y$, we know $(r_Y(r\Diamond), r_Y(r'\Diamond)), (r_Y^*(r\Diamond), r_Y^*(r'\Diamond)) \in \sim_Y$. This proves **Fact 2**.

Fact 3: Suppose $r \in ((\neg^j S)_0 \to \bigcup_y T_0(x, y)) \cap \mathrm{dom}(((\neg^j S)_1 \to T_1) \upharpoonright \mathrm{dom}((\neg^j S)_2 \to T_2))$, then $tr \in \bigcup_y (\alpha_{Y0}(y) \times ((\neg^j S)_0(x) \to T_0(x, y)))$.

Proof of Fact 3: We know $r \in ((\neg^j S)_0 \to \bigcup_y T_0(x, y))$, and there exists r^* such that $(r, r^*) \in ((\neg^j S)_1 \to T_1) \upharpoonright \mathrm{dom}((\neg^j S)_2 \to T_2)$.

There are two cases for $S_0(x)$.

Case 1, $S_0(x) \neq \emptyset$, then $(\neg^j S)_0 = \emptyset$. By **Fact 1**, we know $(tr, t^*r^*) \in R_Y \times ((\neg^j S)_1 \to T_1)$, thus $tr = \underline{p}(r_Y(r\Diamond))(\langle v\rangle r\Diamond) \in \mathrm{dom}(R_Y \times ((\neg^j S)_1 \to T_1)) = \mathrm{dom}(R_Y) \times \mathrm{dom}((\neg^j S)_1 \to T_1)$. Therefore, $r_Y(r\Diamond) \in \mathrm{dom}(R_Y)$. By assumption, the m-assembly (Y, α_Y) satisfies property (1), so for $r_Y(r\Diamond) \in \mathrm{dom}(R_Y)$, there exists $y_x \in Y$ such that $r_Y(r\Diamond) \in \alpha_{Y0}(y_x)$. At the same time, since $(\neg^j S)_0 = \emptyset$, we know $(\neg^j S)_0(x) \to T_0(x, y_x) = P$, thus $\langle v\rangle r\Diamond \in (\neg^j S)_0(x) \to T_0(x, y_x)$. We conclude that $tr \in \alpha_{Y0}(y_x) \times ((\neg^j S)_0(x) \to T_0(x, y_x)) \subseteq \bigcup_y (\alpha_{Y0}(y) \times ((\neg^j S)_0(x) \to T_0(x, y)))$.

Case 2, $S_0(x) = \emptyset$, then $(\neg^j S)_0 = \mathrm{dom}((\neg^j S)_1)$. Recall that $\Diamond \in \mathrm{dom}((\neg^j S)_1)$. Since $r \in ((\neg^j S)_0 \to \bigcup_y T_0(x, y))$, we know $r\Diamond \in \bigcup_u T_0(x, y)$. So there exists $y_x \in Y$ such that $r\Diamond \in T_0(x, y_x)$. Since $r_Y \in T_0(x, y_x) \to \alpha_{Y0}(y_x)$, we know $r_Y(r\Diamond) \in \alpha_{Y0}(y_x)$. Also, for any $a \in (\neg^j S)_0(x)$, $(\langle v\rangle r\Diamond)a = r\Diamond \in T_0(x, y_x)$, thus $\langle v\rangle r\Diamond \in (\neg^j S)_0(x) \to T_0(x, y_x)$. We conclude that $tr \in \alpha_{Y0}(y_x) \times ((\neg^j S)_0(x) \to T_0(x, y_x)) \subseteq \bigcup_y (\alpha_{Y0}(y) \times ((\neg^j S)_0(x) \to T_0(x, y)))$.

This proves **Fact 3**.

We conclude that (t, t^*) witnesses that $A(x) \vdash B(x)$, and the proposition is proved. $\square$

Corollary 137. *Suppose (X, α_X) and (Y, α_Y) are finite types in the category of m-assemblies.*

Then for any strict relation $S(x)$ on (X, E_X), for any strict relation $T(x, y)$ on $(X, E_X) \times (Y, E_Y)$, the following sentence is valid in $RT(\mathsf{P}^j)$:

$$\forall x.(((\neg S)(x) \to \exists y.T(x, y)) \to \exists y.((\neg S)(x) \to T(x, y)))$$

5.3 AC is valid for finite types in our category of m-assemblies

In this section, we show that AC holds in our topos for objects arising from m-assemblies and satisfying certain nice properties. In particular, AC holds for finite types.

Proposition 138 (AC)**.** *Suppose $(X, \alpha_X), (Y, \alpha_Y), (Z, \alpha_Z)$ are m-assemblies. Suppose $X, Y \neq \emptyset$. Suppose (X, α_X) satisfies property (2), i.e. for any $x \in X$, for any $a, a' \in \alpha_{X0}(x)$, $(a, a') \in \sim_X$.*

Suppose $S(x, y, z)$ is a strict relation on $(X, E_X) \times (Y, E_Y) \times (Z, E_Z)$. Then the following sentence is validated in the topos $RT(\mathsf{P}^j)$:

$$\forall z.((\forall x.\exists y.S(x, y, z)) \to (\exists f.\forall x.S(x, fx, z))).$$

Proof. Translate topos logic into tripos logic, write $A(z) := \forall x.(E_X(x, x) \to \exists y.S(x, y, z))$, and $B(z) := \exists f.(E_{Y^X}(f, f) \times \forall x.(E_X(x, x) \to S(x, f(x), z)))$. Then we see that it is equivalent to showing that $E_Z(z,z) \vdash E_Z(z,z) \times (A(z) \to B(z))$, i.e., $E_Z(z,z) \vdash A(z) \to B(z)$. We know it suffices to show that $A(z) \vdash B(z)$.

Let us reiterate the definition of $A(z)$ and $B(z)$:

- $A(z) \coloneqq \forall x.(E_X(x,x) \to \exists y.S(x,y,z))$.
- $B(z) \coloneqq \exists f.(E_{Y^X}(f,f) \times \forall x.(E_X(x,x) \to S(x,f(x),z)))$.

The goal is to show $A(z) \vdash B(z)$. Roughly speaking, we have to transform a realizer of $A(z)$ into a realizer of $B(z)$. A realizer of $A(z)$ already behaves like a function: given something in $E_X(x,x)$, we obtain something in $S(x,y,z)$ for some y. This turns out to suffice to obtain a realizer of a function that behaves exactly as required by $B(z)$. The rest of the proof is about applying this intuition and unfolding definitions.

We calculate:

$$\begin{aligned}
& \exists y.S(x,y,z) \\
= \quad & (\bigcup_y S_0(x,y,z), S_1, S_2)
\end{aligned}$$

$$\begin{aligned}
& E_X(x,x) \to \exists y.S(x,y,z) \\
= \quad & N(\alpha_{X0}(x) \to \bigcup_y S_0(x,y,z) \\
& , R_X \to S_1 \\
& , \sim_X \to S_2)
\end{aligned}$$

Therefore,

$$\begin{aligned}
& A(z) \\
= \quad & \forall x.(E_X(x,x) \to \exists y.S(x,y,z)) \\
= \quad & ((\bigcap_x (\alpha_{X0}(x) \to \bigcup_y S_0(x,y,z))) \cap \mathrm{dom}((R_X \to S_1) \upharpoonright \mathrm{dom}(\sim_X \to S_2)) \\
& , (R_X \to S_1) \upharpoonright \mathrm{dom}(\sim_X \to S_2) \\
& , (\sim_X \to S_2) \upharpoonright \mathrm{dom}((R_X \to S_1) \upharpoonright \mathrm{dom}(\sim_X \to S_2)))
\end{aligned}$$

We keep calculating:

$$\begin{aligned}
& E_X(x,x) \to S(x,f(x),z) \\
= \quad & N(\alpha_{X0}(x) \to S_0(x,f(x),z) \\
& , R_X \to S_1 \\
& , \sim_X \to S_2)
\end{aligned}$$

$$\begin{aligned}
& \forall x.(E_X(x,x) \to S(x,f(x),z)) \\
= \quad & N(\bigcap_x (\alpha_{X0}(x) \to S_0(x,f(x),z) \\
& , R_X \to S_1) \\
& , \sim_X \to S_2)
\end{aligned}$$

We know that for any $f \in Y^X$, $\alpha_{Y^X}(f) = \bigcurlywedge_x (\alpha_X(x) \to \alpha_Y(f(x))) = \bigcurlywedge_x N(\alpha_{X0}(x) \to \alpha_{Y0}(f(x)), R_X \to R_Y, \sim_X \to \sim_Y) = N(\bigcap_x \alpha_{X0}(x) \to \alpha_{Y0}(f(x)), R_X \to R_Y, \sim_X \to \sim_Y)$. Therefore, by a previous lemma about the interaction between N-action and taking product, we know:

$$
\begin{aligned}
& E_{Y^X}(f,f) \times \forall x.(E_X(x,x) \to S(x,f(x),z)) \\
= \quad & N((\bigcap_x \alpha_{X0}(x) \to \alpha_{Y0}(f(x))) \times (\bigcap_x \alpha_{X0}(x) \to S_0(x,f(x),z)) \\
& ,(R_X \to R_Y) \times (R_X \to S_1) \\
& ,(\sim_X \to \sim_Y) \times (\sim_X \to S_2))
\end{aligned}
$$

Since X and Y are both non-empty, we know Y^X is also non-empty. Therefore,

$$
\begin{aligned}
& B(z) \\
= \quad & \exists f.(E_{Y^X}(f,f) \times \forall x.(E_X(x,x) \to S(x,f(x),z))) \\
= \quad & N(\bigcup_f((\bigcap_x \alpha_{X0}(x) \to \alpha_{Y0}(f(x))) \times (\bigcap_x \alpha_{X0}(x) \to S_0(x,f(x),z))) \\
& ,(R_X \to R_Y) \times (R_X \to S_1) \\
& ,(\sim_X \to \sim_Y) \times (\sim_X \to S_2))
\end{aligned}
$$

Suppose (r_Y, r_Y^*) witnesses that $S(x,y,z) \vdash E_Y(y,y)$. Then for any x,y,z,

- $r_Y \in S_0(x,y,z) \to \alpha_{Y0}(y)$.
- $(r_Y, r_Y^*) \in S_1 \to R_Y$.
- $(r_Y, r_Y), (r_Y^*, r_Y^*) \in S_2 \to \sim_Y$.

Then we claim that $(t, t^*) := (\langle u \rangle \underline{p}(\langle v \rangle r_Y(uv))u, \langle u \rangle \underline{p}(\langle v \rangle r_Y^*(uv))u)$ witnesses that $A(z) \vdash B(z)$. Unfolding definitions, we see that we need to verify that for any $z \in Z$,

- $t \in ((\bigcap_x(\alpha_{X0}(x) \to \bigcup_y S_0(x,y,z))) \cap \mathrm{dom}((R_X \to S_1) \upharpoonright \mathrm{dom}(\sim_X \to S_2))) \to \bigcup_f((\bigcap_x \alpha_{X0}(x) \to \alpha_{Y0}(f(x))) \times (\bigcap_x \alpha_{X0}(x) \to S_0(x,f(x),z)))$.
- $(t, t^*) \in ((R_X \to S_1) \upharpoonright \mathrm{dom}(\sim_X \to S_2)) \to ((R_X \to R_Y) \times (R_X \to S_1))$.
- $(t,t), (t^*, t^*) \in ((\sim_X \to S_2) \upharpoonright \mathrm{dom}((R_X \to S_1) \upharpoonright \mathrm{dom}(\sim_X \to S_2))) \to ((\sim_X \to \sim_Y) \times (\sim_X \to S_2))$.

We prove the following facts, which will suffice to confirm our claim.

Fact 1: Suppose $(r, r^*) \in (R_X \to S_1) \upharpoonright \mathrm{dom}(\sim_X \to S_2)$, then $(tr, t^*r^*), (t^*r, t^*r^*) \in (R_X \to R_Y) \times (R_X \to S_1)$.

Proof of Fact 1: Unfolding definitions, it suffices to show that:

- $(\langle v \rangle r_Y(rv), \langle v \rangle r_Y^*(r^*v)), (\langle v \rangle r_Y^*(rv), \langle v \rangle r_Y^*(r^*v)) \in R_X \to R_Y$.
- $(r, r^*) \in R_X \to S_1$.

We know the second bullet directly from our assumption on (r, r^*). For the first bullet, suppose $(a, a^*) \in R_X$, then $(ra, r^*a^*) \in S_1$. Since $(r_Y, r_Y^*) \in S_1 \to R_Y$, we have $((\langle v \rangle r_Y(rv))a, \langle v \rangle r_Y^*(r^*v)a^*) = (r_Y(ra), r_Y^*(r^*a^*)) \in R_Y$, and $((\langle v \rangle r_Y^*(rv))a, \langle v \rangle r_Y^*(r^*v)a^*) = (r_Y^*(ra), r_Y^*(r^*a^*)) \in R_Y$. Therefore, we have $(\langle v \rangle r_Y(rv), \langle v \rangle r_Y^*(r^*v)), (\langle v \rangle r_Y^*(rv), \langle v \rangle r_Y^*(r^*v)) \in R_X \to R_Y$. This proves **Fact 1**.

Fact 2: Suppose $(r, r') \in (\sim_X \to S_2) \upharpoonright \mathrm{dom}((R_X \to S_1) \upharpoonright \mathrm{dom}(\sim_X \to S_2))$, then $(tr, tr'), (t^*r, t^*r') \in (\sim_X \to \sim_Y) \times (\sim_X \to S_2)$.

Proof of Fact 2: Unfolding definitions, it suffices to show that:

- $(\langle v \rangle r_Y(rv), \langle v \rangle r_Y(r'v)), (\langle v \rangle r_Y^*(rv), \langle v \rangle r_Y^*(r'v)) \in \sim_X \to \sim_Y$.

- $(r, r') \in \sim_X \to S_2$.

The second bullet follows directly from our assumption on (r, r'). For the first bullet, suppose $(a, a') \in \sim_X$, then $(ra, r'a') \in S_2$. Since $(r_Y, r_Y), (r_Y^*, r_Y^*) \in S_2 \to \sim_Y$, we know $(((\langle v\rangle r_Y(rv))a, (\langle v\rangle r_Y(r'v))a') = (r_Y(ra), r_Y(r'a')) \in \sim_Y$, and $(((\langle v\rangle r_Y^*(rv))a, (\langle v\rangle r_Y^*(r'v))a') = (r_Y^*(ra), r_Y^*(r'a')) \in \sim_Y$. Therefore, $(\langle v\rangle r_Y(rv), \langle v\rangle r_Y(r'v)), (\langle v\rangle r_Y^*(rv), \langle v\rangle r_Y^*(r'v)) \in \sim_X \to \sim_Y$. This proves **Fact 2**.

Fact 3: Suppose $r \in (\bigcap_x (\alpha_{X0}(x) \to \bigcup_y S_0(x, y, z))) \cap \mathrm{dom}((R_X \to S_1) \upharpoonright \mathrm{dom}(\sim_X \to S_2))$, then $tr \in \bigcup_f ((\bigcap_x \alpha_{X0}(x) \to \alpha_{Y0}(f(x))) \times (\bigcap_x \alpha_{X0}(x) \to S_0(x, f(x), z)))$.

Proof of Fact 3: By assumption, there exists r^* such that:

- $r \in (\bigcap_x (\alpha_{X0}(x) \to \bigcup_y S_0(x, y, z)))$.
- $(r, r^*) \in R_X \to S_1$.
- $(r, r), (r^*, r^*) \in \sim_X \to S_2$.

For any x, for any $a \in \alpha_{X0}(x)$, $ra \in \bigcup_y S_0(x, y, z)$, so there exists y_a such that $ra \in S_0(x, y_a, z)$. By the assumption on (X, α_X), we know that for any $a' \in \alpha_{X0}(x')$, $(a, a') \in \sim_X$. Since $(r, r) \in \sim_X \to S_2$, we have $(ra, ra') \in S_2$, thus $ra' \in S_0(x, y_a, z)$.

This implies there exists a set-theoretic function $f_r : X \to Y$, such that for any $x \in X$, for any $a \in \alpha_{X0}(a)$, $ra \in S_0(x, f_r(x), z)$.

We claim that $f_r \in Y^X$, i.e., f_r is a morphism of m-assemblies from (X, α_X) to (Y, α_Y). Indeed, it is witnessed by $(\langle v\rangle r_Y(rv), \langle v\rangle r_Y^*(r^*v))$: for any $x \in X$

- $r \in \alpha_{X0}(x) \to S_0(x, f_r(x), z)$, and $r_Y \in S_0(x, f_r(x), z) \to \alpha_{Y0}(f_r(x))$, thus, $\langle v\rangle r_Y(rv) \in \alpha_{X0}(x) \to \alpha_{Y0}(f_r(x))$.
- Since $(r, r^*) \in (R_X \to S_1) \upharpoonright \mathrm{dom}(\sim_X \to S_2)$, by **Fact 1**, we know $(\langle v\rangle r_Y(rv), \langle v\rangle r_Y^*(r^*v)) \in R_X \to R_Y$.
- Since $(r, r), (r^*, r^*) \in (\sim_X \to S_2) \upharpoonright \mathrm{dom}((R_X \to S_1) \upharpoonright \mathrm{dom}(\sim_X \to S_2))$, by **Fact 2**, we know $(\langle v\rangle r_Y(rv), \langle v\rangle r_Y(rv)), (\langle v\rangle r_Y^*(rv), \langle v\rangle r_Y^*(r'v)) \in \sim_X \to \sim_Y$.

Therefore, to show $tr \in \bigcup_f ((\bigcap_x \alpha_{X0}(x) \to \alpha_{Y0}(f(x))) \times (\bigcap_x \alpha_{X0}(x) \to S_0(x, f(x), z)))$, it suffices to show that $tr \in (\bigcap_x \alpha_{X0}(x) \to \alpha_{Y0}(f_r(x))) \times (\bigcap_x \alpha_{X0}(x) \to S_0(x, f_r(x), z))$.

Unfolding definitions, it suffices to show that:

- $\langle v\rangle r_Y(rv) \in \bigcap_x \alpha_{X0}(x) \to \alpha_{Y0}(f_r(x))$.
- $r \in \bigcap_x \alpha_{X0}(x) \to S_0(x, f_r(x), z)$.

But we already know that both bullets hold. This proves **Fact 3**.

We conclude that (t, t^*) witnesses that $A(z) \vdash B(z)$, and the proposition is proved. □

Corollary 139. *Suppose $(X, \alpha_X), (Y, \alpha_Y)$ are finite types in the category of m-assemblies. Suppose $X, Y \neq \emptyset$. Suppose (Z, α_Z) is an m-assembly.*

Suppose $S(x, y, z)$ is a strict relation on $(X, E_X) \times (Y, E_Y) \times (Z, E_Z)$. Then the following sentence is validated in the topos $RT(\mathsf{P}^j)$:

$$\forall z.((\forall x.\exists y.S(x, y, z)) \to (\exists f.\forall x.S(x, fx, z))).$$

5.4 Types of finite sequences

In this section, we work out an explicit expression for types of finite sequences. This will be useful in the discussion of Fan recursors, because we need explicit expressions of finite sequences of 0 and 1. Therefore, we will also briefly mention the type $2 = 1 + 1$ at the end of this section.

We identify these types of finite sequences as List. A general result in [Ber06] tells us that for any object X in the topos, if $1 + X$ is $\neg\neg$-separated, then $\mathrm{List}(X)$ is $\neg\neg$-separated. We also understand that if X is $\neg\neg$-separated, then so is $1 + X$. Therefore, we conclude that for any finite type X in the topos, $\mathrm{List}(X)$ is $\neg\neg$-separated. As a result, it suffices to verify the List universal property in the category of m-assemblies.

The idea of providing an explicit expression of the List object, or finite sequences, is straightforward. For example, we may expect the underlying set to be the set of finite sequences, and expect the realizers to be codes of finite sequences. However, verifying that this is well-defined and satisfies the correct universal property can be more complicated.

Definition 140. *Let (X, α_X) be an m-assembly. We define List(X) in the category of m-assemblies, written as (X^*, α_{X^*}):*

- *X^* is the set of all finite sequences of elements from X.*
- *For every finite sequence $s = [x_1, \dots, x_n]$ ($n = 0$, then s is the empty sequence), define $\alpha_{X^*0}(s)$ to be the set of codes of the form $[a_1, \dots, a_n]$, where for any $1 \leq i \leq n$, we have that $a_i \in \alpha_{X0}(x_i)$.*
- *Define R_{X^*} to be the collection of pairs of the form $([a_1, \dots, a_n], [b_1, \dots, b_m])$ $(a_1, \dots, a_n, b_1, \dots, b_m$ are in dom$(R_X))$, where $n \leq m$, and for any $1 \leq i \leq n$, we have that $(a_i, b_i) \in R_X$, and we require that for $1 \leq j \leq m$, $(b_j, b_j) \in R_X$. Notice that $R_{X^*} \in MAJ(P)$.*
- *Define $\sim_{X^*}$ to be the collection of pairs of the form $([a_1, \dots, a_n], [a'_1, \dots, a'_n])$ $(a_1, \dots, a_n, a'_1, \dots, a'_n$ are in dom$(\sim_X))$, where for any $1 \leq i \leq n$, we have that $(a_i, a'_i) \in \sim_X$. Notice that $\sim_{X^*} \in PER(P)$.*
- *Define $\alpha_{X^*}(s) \coloneqq (\alpha_{X^*0}(s), R_{X^*}, \sim_{X^*})$. Notice that $(R_{X^*}, \sim_{X^*}) \in EMAJ(P)$, and for any finite sequence $s \in X^*$, we know $\alpha_{X^*}(s) \in dEMAJ(P)$, therefore, $\alpha_{X^*} \in \mathsf{P}X^*$.*

Remark 141. *Recall that there exists $\mathsf{lh} \in P$ such that for any code of a finite sequence $x = [a_1, \dots, a_n]$, we have that $\mathsf{lh}x \preceq \overline{n}$.*

By abuse of notation, if y is a finite sequence (or a code of a finite sequence), we also write $\mathsf{lh}(y)$ for the length of y, which is a natural number.

To show that the above definition indeed defines an m-assembly, it remains to show that $\alpha_{X^*} \in \mathsf{P}^j X$, i.e., $j\alpha_{X^*} \vdash \alpha_{X^*}$.

Lemma 142. *$j\alpha_{X^*} \vdash \alpha_{X^*}$.*

Proof. We claim that there exists an element $\underline{c} \in P$ with the following property:

For any $f \in P$, and for any $x = [x_1, \dots, x_n] \in P$ where $x_1, \dots, x_n$ are codes of finite sequences:

- If $x = [\,]$, then $\underline{c}fx \preceq [\,]$.

- Let $x_{i_1}, \ldots, x_{i_m}$ be the collection of codes of non-empty sequences, and $1 \leq i_1 < i_2 < \cdots < i_m \leq n$, then $\underline{c}fx \preceq \underline{c}fx'$.

- Suppose $x_1, \ldots, x_n$ are all codes of non-empty sequences, and suppose $x_i = [a_i] \sqcup y_i$, where y_i is a code of another sequence, then $\underline{c}fx \preceq [f[a_1, \ldots, a_n]] \sqcup (\underline{c}f[y_1, \ldots, y_n])$.

For instance, $\underline{c}f[[a_0], [b_0, b_1], [c_0, c_1, c_2]] = [f[a_0, b_0, c_0], f[b_1, c_1], f[c_2]]$. The intuition behind considering this $\underline{c}$ is that, in order to show $j\alpha_{X^*} \vdash \alpha_{X^*}$, we have to solve the problem of taking a "maximum" over finitely many finite sequences. These finite sequences may have different lengths. So the idea of $\underline{c}$ is to take the "maximum" component-wise (here f in the definition of $\underline{c}$ is intended to represent a "maximum" operation); but when computing the i-th maximum, some of the finite sequences may have length smaller than i, so we omit those shorter sequences and only take the maximum over those that are long enough. So $\underline{c}$ implements this construction in a uniform way.

Since (X, α_X) is an m-assembly, $j\alpha_X \vdash \alpha_X$ is witnessed by some (r, r^*), i.e., we have that for any $x \in X$:

- $r \in \alpha^{\mathbf{1}}_{X0}(x) \to \alpha_{X0}(x)$.

- $(r, r^*) \in R^*_X \to R_X$.

- $(r, r), (r^*, r^*) \in \sim^*_X \to \sim_X$.

Then we claim that $(\underline{c}r, \underline{c}r^*)$ witnesses that $j\alpha_{X^*} \vdash \alpha_{X^*}$, i.e., we have to show that for any $s \in X^*$

- $\underline{c}r \in \alpha^{\mathbf{1}}_{X^*0}(s) \to \alpha_{X^*}(s)$.

- $(\underline{c}r, \underline{c}r^*) \in R^*_{X^*} \to R_{X^*}$.

- $(\underline{c}r, \underline{c}r), (\underline{c}r^*, \underline{c}r^*) \in \sim^*_{X^*} \to \sim_{X^*}$.

We will verify each of the three statements in turn.

First bullet: To show that $\underline{c}r \in \alpha^{\mathbf{1}}_{X^*0}(s) \to \alpha_{X^*}(s)$, suppose that $s = [x_1, \ldots, x_n]$. For any $y \in \alpha^{\mathbf{1}}_{X^*0}(s)$, we want to show that $\underline{c}ry \in \alpha_{X^*0}(s)$.

We understand that y is a code of a finite sequence of elements in $\alpha_{X^*0}(s)$, say $y = [y_1, \ldots, y_m]$, where $y_j \in \alpha_{X^*0}(s)$, for $1 \leq j \leq m$. Also, there exists $[a_1, \ldots, a_n] \in \alpha^{\mathbf{1}}_{X^*0}(s)$ such that $([[a_1, \ldots, a_n]], y) \in \sim^*_{X^*}$. We understand that for $1 \leq j \leq m$, we have that $([a_1, \ldots, a_n], y_i) \in \sim_{X^*}$. This implies that for each $1 \leq j \leq m$, y_j has length n, and for each $1 \leq i \leq n$, we know $(a_i, y_{j,i}) \in \sim_X$. Therefore, for any $1 \leq i \leq n$, we have $([a_i], [y_{1,i}, \ldots, y_{m,i}]) \in \sim^*_X$, thus $[y_{1,i}, \ldots, y_{m,i}] \in \alpha^{\mathbf{1}}_{X0}(x_i)$.

What is $\underline{c}ry$? By the property of $\underline{c}$ described above, we know $\underline{c}ry \preceq [r[y_{1,1}, \ldots, y_{m,1}], \ldots, r[y_{1,n}, \ldots, y_{m,n}]]$. Since for each $1 \leq i \leq n$, we have $[y_{1,i}, \ldots, y_{m,i}] \in \alpha^{\mathbf{1}}_{X0}(x_i)$, and since $r \in \alpha^{\mathbf{1}}_{X0}(x_i) \to \alpha_{X0}(x_i)$, it follows that $r[y_{1,i}, \ldots, y_{m,i}] \in \alpha_{X0}(x_i)$, so $\underline{c}ry \in \alpha_{X^*0}(s)$.

Second bullet: To show that $(\underline{c}r, \underline{c}r^*) \in R^*_{X^*} \to R_{X^*}$, suppose $(y, z) \in R^*_{X^*}$; we show that $(\underline{c}ry, \underline{c}r^*z), (\underline{c}r^*y, \underline{c}r^*z) \in R_{X^*}$.

Suppose $y = [y_1, \ldots, y_n]$ and $z = [z_1, \ldots, z_m]$. We understand that for any $t \in y \sqcup z$, there exists $t^* \in z$ such that $(t, t^*) \in R_{X^*}$. We understand that $(t, t^*) \in R_{X^*}$ says that the length of t^* is greater than or equal to the length of t, and for any $1 \leq i \leq \mathsf{lh}(t)$, $(t_i, t^*_i) \in R_X$, and for $1 \leq j \leq \mathsf{lh}(t^*)$, $(t^*_j, t^*_j) \in R_X$.

Now, to show that $(\underline{c}ry, \underline{c}r^*z) \in R_{X^*}$, we first have to show that the length of $\underline{c}ry$ is no greater than the length of $\underline{c}r^*z$, but notice that the length of $\underline{c}ry$ is equal to the longest length of elements in y, which is definitely no longer than the longest length of elements in z, which is also the length of $\underline{c}r^*z$.

Then we show that for any $1 \leq i \leq \mathsf{lh}(\underline{c}ry)$, $((\underline{c}ry)_i, (\underline{c}r^*z)_i) \in R_X$. We know $(\underline{c}r^*y)_i$ is $r[y_{j_1,i}, \ldots, y_{j_{N_i},i}]$, where $y_{j_1}, \ldots, y_{j_{N_i}}$ $(N_i > 0)$ are exactly those elements in y with length no smaller than i, and $1 \leq j_1 < \cdots < j_{N_i} \leq n$.

Similarly, $(\underline{c}r^*z)_i$ is of the form $r^*[z_{k_1,i}, \ldots, z_{k_{M_i},i}]$, where $z_{k_1}, \ldots, z_{k_{M_i}}$ $(M_i > 0)$ are exactly those elements in z with length no smaller than i, and $1 \leq k_1 < \cdots < k_{M_i} \leq m$.

We show that $((\underline{c}ry)_i, (\underline{c}r^*z)_i) \in R_X$, i.e., $(r[y_{j_1,i}, \ldots, y_{j_{N_i},i}], r^*[z_{k_1,i}, \ldots, z_{k_{M_i},i}]) \in R_X$. Since $(r, r^*) \in R_X^* \to R_X$, it suffices to show that $([y_{j_1,i}, \ldots, y_{j_{N_i},i}], [z_{k_1,i}, \ldots, z_{k_{M_i},i}]) \in R_X^*$.

For any $w \in [y_{j_1,i}, \ldots, y_{j_{N_i},i}] \sqcup [z_{k_1,i}, \ldots, z_{k_{M_i},i}]$, there exists $t \in [y_{j_1}, \ldots, y_{j_{N_i}}] \sqcup [z_{k_1}, \ldots, z_{k_{M_i}}]$ such that $t_i = w$. We know there exists $t^* \in z$ such that $(t, t^*) \in R_{X^*}$. In particular, the length of t is no greater than the length of t^*, thus we understand that t^* is in $[z_{k_1}, \ldots, z_{k_{M_i}}]$. Also, we understand that $(t_i, t_i^*) \in R_X$, i.e., $(w, t_i^*) \in R_X$, where $t_i^* \in [z_{k_1,i}, \ldots, z_{k_{M_i},i}]$.

We still have to show that for any $1 \leq j \leq \mathsf{lh}(\underline{c}r^*z)$, $((\underline{c}r^*z)_j, (\underline{c}r^*z)_j) \in R_X$. We know $(\underline{c}r^*z)_j$ is of the form $r^*[z_{k_1,j}, \ldots, z_{k_{M_j},j}]$, where $z_{k_1}, \ldots, z_{k_{M_j}}$ $(M_j > 0)$ are exactly those elements in z with length no smaller than j, and $1 \leq k_1 < \cdots < k_{M_j} \leq m$. Since $(r^*, r^*) \in R_X^* \to R_X$, to show $((\underline{c}r^*z)_j, (\underline{c}r^*z)_j) \in R_X$ it suffices to show that $([z_{k_1,j}, \ldots, z_{k_{M_j},j}], [z_{k_1,j}, \ldots, z_{k_{M_j},j}]) \in R_X^*$. For any $w \in [z_{k_1,j}, \ldots, z_{k_{M_j},j}] \sqcup [z_{k_1,j}, \ldots, z_{k_{M_j},j}]$, there exists $t \in [z_{k_1}, \ldots, z_{k_{M_j}}]$ such that $t_j = w$. We know there exists $t^* \in z$ such that $(t, t^*) \in R_{X^*}$. In particular, then length of t is no greater than the length of t^*, thus we understand that $t^* \in [z_{k_1}, \ldots, z_{k_{M_j}}]$. Also, we understand that $(t_j, t_j^*) \in R_X$, i.e. $(w, t_j^*) \in R_X$, where $t_j^* \in [z_{k_1,j}, \ldots, z_{k_{M_j},j}]$.

We conclude that $(\underline{c}ry, \underline{c}r^*z) \in R_{X^*}$. Similarly, we can prove that $(\underline{c}r^*y, \underline{c}r^*z) \in R_{X^*}$.

We conclude that $(\underline{c}r, \underline{c}r^*) \in R_{X^*}^* \to R_{X^*}$.

Third bullet: To show that $(\underline{c}r, \underline{c}r), (\underline{c}r^*, \underline{c}r^*) \in \sim_{X^*}^* \to \sim_{X^*}$, we show that $(\underline{c}r, \underline{c}r) \in \sim_{X^*}^* \to \sim_{X^*}$ as an example, since the other case is similar.

For any $(y, y') \in \sim_{X^*}^*$, we show that $(\underline{c}ry, \underline{c}ry') \in \sim_{X^*}$.

Suppose $y = [y_1, \ldots, y_n]$ and $y' = [y'_1, \ldots, y'_m]$. We understand that for any $t \in y$, there exists $t' \in y'$ such that $(t, t') \in \sim_{X^*}$ and vice versa. We understand that $(t, t') \in \sim_{X^*}$ means that t and t' have the same length, and for any $1 \leq i \leq \mathsf{lh}(t)$, we have that $(t_i, t'_i) \in \sim_X$.

Now, we want to show that $(\underline{c}ry, \underline{c}ry') \in \sim_{X^*}$. We require that $\underline{c}ry$ and $\underline{c}ry'$ have the same length, which is ensured since the lengths of the two sequences are equal to the length of the longest element in y and y', respectively, and these longest lengths are indeed the same.

We also require that for any $1 \leq i \leq \mathsf{lh}(\underline{c}ry)$, we have that $((\underline{c}ry)_i, (\underline{c}ry')_i) \in \sim_X$.

We understand that $(\underline{c}ry)_i$ is $r[y_{j_1,i}, \ldots, y_{j_{N_i},i}]$, where $y_{j_1}, \ldots, y_{j_{N_i}}$ $(N_i > 0)$ are exactly those elements in y with length no smaller than i, and $1 \leq j_1 < \cdots < j_{N_i} \leq n$.

Similarly, $(\underline{c}ry')_i$ is of the form $r[y'_{k_1,i}, \ldots, y'_{k_{M_i},i}]$, where $y'_{k_1}, \ldots, y'_{k_{M_i}}$ $(M_i > 0)$ are exactly those elements in z with length no smaller than i, and $1 \leq k_1 < \cdots < k_{M_i} \leq m$.

To show that $((\underline{c}ry)_i, (\underline{c}ry')_i) \in \sim_X$, i.e., $(r[y_{j_1,i}, \ldots, y_{j_i,i}], r[y'_{k_1,i}, \ldots, y'_{k_{M_i},i}])$, since $(r, r) \in \sim_X^* \to \sim_X$, it suffices to show that $([y_{j_1,i}, \ldots, y_{j_i,i}], [y'_{k_1,i}, \ldots, y'_{k_{M_i},i}]) \in \sim_X^*$.

For any $w \in [y_{j_1,i}, \ldots, y_{j_i,i}]$, there exists $t \in [y_{j_{N_i}}, \ldots, y_{j_{N_i}}]$ such that $t_i = w$. There exists $t' \in y'$ such that $(t, t') \in \sim_{X^*}$. So in particular, the length of t' is the same as the length of t, thus we understand that t' is among $y'_{k_1}, \ldots, y'_{k_{M_i}}$. We have that $(t_i, t'_i) \in \sim_X$, thus $(w, t'_i) \in \sim_X$ where $t'_i \in [y'_{k_1,i}, \ldots, y'_{k_{M_i},i}]$.

The other direction in showing that $([y_{j_1,i},\ldots,y_{j_i,i}],[y'_{k_1,i},\ldots,y'_{k_{M_i},i}]) \in \sim_X^*$ can be shown in a similar way.

We conclude that $(\underline{c}ry, \underline{c}ry') \in \sim_{X^*}$.

Therefore, we conclude that $(\underline{c}r, \underline{c}r), (\underline{c}r^*, \underline{c}r^*) \in \sim_{X^*}^* \rightarrow \sim_{X^*}$.

Then, we may finally conclude that $(\underline{c}r, \underline{c}r^*)$ witnesses that $j\alpha_{X^*} \vdash \alpha_{X^*}$. □

Definition 143. *We can then define two morphisms that complete the definition of List(X).*

The first map, which is denoted by $[\,] : 1 \rightarrow X^*$, sends the unique element $*$ to $[\,]$ in X^*.

The second map, which is denoted by $cons : X \times X^* \rightarrow X^*$, sends a tuple (x, s) to $[x] \sqcup s$.

Remark 144. *We have to show that the two maps defined above are indeed morphisms of m-assemblies.*

$[\,] : 1 \rightarrow X^*$ is witnessed by $(\langle u\rangle[\,], \langle u\rangle[\,])$.

To show that $cons : X \times X^* \rightarrow X^*$ is a morphism of m-assemblies, we have to show that $\alpha_X(x) \times \alpha_{X^*}(s) \vdash \alpha_{X^*}([x] \sqcup s)$, which is equivalent to showing that $\alpha_X(x) \vdash \alpha_{X^*}(s) \rightarrow \alpha_{X^*}([x] \sqcup s)$.

Let $\underline{c} \in P$ be such that for any codes of finite sequences y and z, we have that $\underline{c}yz \preceq y \sqcup z$.

We claim that $\alpha_X(x) \vdash \alpha_{X^*}(s) \rightarrow \alpha_{X^*}([x] \sqcup s)$ is witnessed by $(\langle uv\rangle\underline{c}[u]v, \langle uv\rangle\underline{c}[u]v)$.

Now, we are ready to show that X^* with the two morphisms $[\,] : 1 \rightarrow X^*$ and cons $: X \times X^* \rightarrow X^*$ acts as the List(X) in the category of m-assemblies. This is proved in the following proposition.

Proposition 145. *X^* with the two morphisms $[\,] : 1 \rightarrow X^*$ and $cons : X \times X^* \rightarrow X^*$ acts as the List(X) in the category of m-assemblies. More precisely, for any other m-assembly (Y, α_Y), for any morphisms $F : 1 \rightarrow Y$ and $G : X \times Y \rightarrow Y$, there exists a unique morphism $m : X^* \rightarrow Y$ satisfying:*

- *$m.[\,] = F$.*
- *$m.cons(x, s) = G(x, m(s))$, for any $(x, s) \in X \times X^*$.*

Proof. By set-theoretic "List" induction, we know that such an m exists as a set-theoretic function. It remains to show that m is indeed a morphism of m-assemblies, i.e., $\alpha_{X^*}(s) \vdash \alpha_Y(m(s))$.

Fix $r_F \in \alpha_{Y0}(F(*))$, and fix $(r_F, r_F^*) \in R_Y$.

$G : X \times Y \rightarrow Y$ is a morphism of m-assemblies, so we have that $\alpha_X(x) \vdash \alpha_Y(y) \rightarrow \alpha_Y(G(x, y))$. Fix (r_G, r_G^*) which witnesses this, i.e., for any x, y we have that:

- $r_G \in \alpha_{X0}(x) \rightarrow \alpha_{Y0}(y) \rightarrow \alpha_{Y0}(G(x, y))$.
- $(r_G, r_G^*) \in R_X \rightarrow R_Y \rightarrow R_X$.
- $(r_G, r_G), (r_G^*, r_G^*) \in \sim_X \rightarrow \sim_Y \rightarrow \sim_X$.

Now, consider $r \in P$ to be such that:

- $r[\,] \preceq r_F$.
- $r([a] \sqcup y) \preceq r_G a(ry)$.

Similarly, consider $r^* \in P$ to be such that:

- $r^*[\,] \preceq r_F^*$.

- $r^*([a] \sqcup y) \preceq r^*_G a(r^* y)$.

We may summarize that the tuple (r, r^*) satisfies the following property:

- For any $s \in X^*$, $r \in \alpha_{X^*0}(s) \to \alpha_{Y0}(m(s))$.
- For any $y = [a_1, \ldots, a_n]$ and $z = [b_1, \ldots, b_n]$, if for any $1 \leq i \leq n$ we have that $(a_i, b_i) \in R_X$, then $(ry, r^*z), (r^*y, r^*z) \in R_Y$.
- For any $(y, y') \in \sim_{X^*}$, we have that $(ry, ry'), (r^*y, r^*y) \in \sim_Y$.

Since (Y, α_Y) is an m-assembly, there exists (r_Y, r_Y^*) that witnesses that $j\alpha_Y \vdash \alpha_Y$, i.e., for any $y \in Y$ we have:

- $r_Y \in \alpha^{\mathbf{1}}_{Y0}(y) \to \alpha_{Y0}(y)$.
- $(r_Y, r_Y^*) \in R_Y^* \to R_Y$.
- $(r_Y, r_Y), (r_Y^*, r_Y^*) \in \sim_Y^* \to \sim_Y$.

Now, consider $\tilde{r} \in P$ to be such that for any y we have:

$$\tilde{r}y \preceq r_Y[ry].$$

Also, consider an element $\underline{c} \in P$ satisfying:

- $\underline{c}[\,] \preceq [r^*[\,]]$.
- $\underline{c}(y \sqcup [a]) \preceq (\underline{c}y) \sqcup [r^*(y \sqcup [a])]$.

So for instance, we have that $\underline{c}[a_0, a_1, a_2] = [r^*[], r^*[a_0], r^*[a_0, a_1], r^*[a_0, a_1, a_2]]$. The idea behind $\underline{c}$ is to collect the results of applying r^* to all initial segments of a sequence, so that the resulting object is a majorant in a "monotone" sense. Roughly speaking, it is not only a majorant for finite sequences of the same length, but also a majorant for finite sequences of shorter lengths. This idea was already used in the construction of the explicit expression of the natural number object in Lemma 125.

Then we consider $\tilde{r}^* \in P$ satisfying:

$$\tilde{r}^* y \preceq r_Y^*(\underline{c}y).$$

We claim that $(\tilde{r}, \tilde{r}^*)$ witnesses that $\alpha_{X^*}(s) \vdash \alpha_Y(m(s))$. Thus we have to show that for any $s \in X^*$, we have that:

- $\tilde{r} \in \alpha_{X^*0}(s) \to \alpha_{Y0}(m(s))$.
- $(\tilde{r}, \tilde{r}^*) \in R_{X^*} \to R_Y$.
- $(\tilde{r}, \tilde{r}), (\tilde{r}^*, \tilde{r}^*) \in \sim_{X^*} \to \sim_Y$.

First bullet: To show that $\tilde{r} \in \alpha_{X^*0}(s) \to \alpha_Y(m(s))$, notice that for any $y \in \alpha_{X^*0}(s)$, $ry \in \alpha_{Y0}(m(s))$, thus $[ry] \in \alpha^{\mathbf{1}}_{Y0}(m(s))$, and then $\tilde{r}y = r_Y[ry] \in \alpha_{Y0}(m(s))$.

Second bullet: To show that $(\tilde{r}, \tilde{r}^*) \in R_{X^*} \to R_Y$, we show that for any $(y, z) \in R_{X^*}$, we have that $(\tilde{r}y, \tilde{r}^*z), (\tilde{r}^*y, \tilde{r}^*z) \in R_Y$.

We then understand that $([ry], \underline{c}z), (\underline{c}y, \underline{c}z) \in R^*_Y$, thus $(r_Y[ry], r^*_Y(\underline{c}z)), (r^*_Y(\underline{c}y), r^*_Y(\underline{c}z)) \in R_Y$.

Third bullet: To show that $(\tilde{r}, \tilde{r}), (\tilde{r}^*, \tilde{r}^*) \in \sim_{X^*} \to \sim_Y$, we have to show that for any $(y, y') \in \sim_{X^*}$, we have that $(\tilde{r}y, \tilde{r}y'), (\tilde{r}^*y, \tilde{r}^*y') \in \sim_Y$.

It suffices to notice that $([ry], [ry']) \in \sim^*_Y$ and $(\underline{c}y, \underline{c}y') \in \sim^*_Y$, thus $(r_Y[ry], r_Y[ry']) \in \sim_Y$ and $(r^*_Y(\underline{c}y), r^*_Y(\underline{c}y')) \in \sim_Y$. $\square$

Remark 146. *Notice that if X satisfies properties (0), (1), and (2) (they are discussed in Subsection 5.1.2), then so does X^*.*

Let us work out an explicit expression for $1 + 1$ in the category of m-assemblies. We know $1 + 1$ taken in the topos is $\neg\neg$-separated, thus it is also the coproduct in the category of m-assemblies. Of course we can proceed as in the case of a general binary coproduct: first take the naive coproduct, then apply j. But the resulting expression is somewhat heavy. For $2 = 1 + 1$, it turns out that we have a much simpler description for 2.

Lemma 147. *Consider* $2 \coloneqq (\{0, 1\}, \alpha_2)$, *where:*

- $\alpha_{2,0}(0) \coloneqq \{\underline{\overline{0}}\}$.
- $\alpha_{2,0}(1) \coloneqq \{\underline{\overline{1}}\}$.
- $R_2 \coloneqq \{(\underline{\overline{0}}, \underline{\overline{0}}), (\underline{\overline{0}}, \underline{\overline{1}}), (\underline{\overline{1}}, \underline{\overline{1}})\}$.
- $\sim_2 \coloneqq \{(\underline{\overline{0}}, \underline{\overline{0}}), (\underline{\overline{1}}, \underline{\overline{1}})\}$

We claim that 2 is an m-assembly.

Proof. We know $\alpha_{2,0}(0)$ and $\alpha_{2,0}(1)$ are both non-empty. Also, we know $\alpha_2 \in \mathsf{P}2$. Moreover, we claim that $(\underline{\mathbf{max}}, \underline{\mathbf{max}})$ witnesses that $j\alpha_2 \vdash \alpha_2$. Here, $\underline{\mathbf{max}} \in P$ satisfies the property that for any $n_1, \ldots, n_k \in \mathbb{N}$, $\underline{\mathbf{max}}[\underline{\overline{n_1}}, \ldots, \underline{\overline{n_k}}] = \underline{\overline{\max(n_1, \ldots, n_k)}}$ $\square$

Remark 148. *Notice that 2 satisfies properties (0), (1), and (2).*

Lemma 149. *Consider functions $\nu_0 : 1 \to 2, * \mapsto 0$, $\nu_1 : 1 \to 2, * \mapsto 1$. We claim that ν_0 and ν_1 are both morphisms of m-assemblies, and 2 together with ν_0 and ν_1 is the binary coproduct $1 + 1$ in the category of m-assemblies.*

Proof. ν_0 is witnessed by $(\langle u \rangle \underline{\overline{0}}, \langle u \rangle \underline{\overline{0}})$, and ν_1 is witnessed by $(\langle u \rangle \underline{\overline{1}}, \langle u \rangle \underline{\overline{1}})$. Thus they are both morphisms of m-assemblies.

To verify the universal property: Suppose (X, α_X) is an m-assembly, $x_0, x_1 : 1 \to (X, \alpha_X)$ are two morphisms of m-assemblies. There is a unique set-theoretic function $f : 2 \to X$ such that $f(0) = x_0(*)$ and $f(1) = x_1(*)$. It remains to show that f is a morphism of m-assemblies.

Suppose (r, r^*) witnesses that $j.\alpha_X \vdash \alpha_X$: for any x:

- $r \in \alpha^{\mathbf{1}}_{X0}(x) \to \alpha_{X0}(x)$.

- $(r, r^*) \in R_X^* \to R_X$.

- $(r, r), (r^*, r^*) \in \sim_X^* \to \sim_X$.

Pick $a \in \alpha_{X0}(f(0))$. Fix a^* such that $(a, a^*) \in R_X$. We know that $(a, a), (a^*, a^*) \in \sim_X$.

Pick $b \in \alpha_{X0}(f(1))$. Fix b^* such that $(b, b^*) \in R_X$. We know that $(b, b), (b^*, b^*) \in \sim_X$.

Then we claim that $(\langle u\rangle \underline{Z} u(r[a])(r[b]), \langle u\rangle r^*[a^*, b^*])$ witnesses that $\alpha_2(x) \vdash \alpha_X(f(x))$. Here $\underline{Z} \in P$ is such that $\underline{Z}\overline{0} \preceq \underline{k}$ and $\underline{Z}\overline{n+1} \preceq \overline{k}$. □

Remark 150. *Therefore, $2 = (\{0, 1\}, \alpha_2)$ is the coproduct in the category of m-assemblies, and we know it is also the coproduct in the topos $RT(\mathsf{P}^j)$.*

We notice that R_2 is asymmetric in the sense that we have $(\underline{\overline{0}}, \underline{\overline{1}}) \in R_2$, but we do not have $(\underline{\overline{1}}, \underline{\overline{0}}) \in R_2$. This is somewhat unusual for the coproduct $1 + 1$, since intuitively, $\underline{\overline{0}}$ and $\underline{\overline{1}}$ should be symmetric in some sense. So in our situation, R_2 has a mysterious "polarization".

But probably it is the effect of the nucleus j. On the level of $\alpha_{(-),0}$, there is no majorizability and no asymmetry. Majorizability only occurs in R_-, where one has some freedom in designing this polarization, or one could just do as in the case of binary coproducts in general: first take a naive coproduct, then apply j. If we do so for 2, then in dom(R_2), we obtain elements such as $[\,], [0], [1], [0, 1]$; and we have $([\,], [0]), ([\,], [0]), ([\,], [1]), ([0], [0, 1]), ([1], [0, 1]) \in R_2$. Thus redundancy occurs, even though this time we do have symmetry.

5.5 Fan recursor

In this section we show that our topos has a Fan recursor. We follow the definition of Fan recursor given in [Gu18]. We first define a morphism $\mathsf{FR} : \mathbb{N}^{2^{\mathbb{N}}} \times 2^* \to \mathbb{N}$. We then show that it satisfies the defining property of a Fan recursor. The existence of such a Fan recursor ensures that the Fan Theorem is valid in our topos as well ([Gu18], Chapter 6).

The Fan Theorem states that every decidable binary tree in which every branch is finite has a uniform bound on the length of its branches. Here we view a decidable tree as a morphism $U : 2^* \to 2$ (2^* is the object representing the collection of finite sequences of 0 and 1) satisfying the property of being closed under initial segments. See for example [Oos08] (Theorem 3.2.25), where it is shown that the Fan Theorem does not hold in the effective topos, due to the existence of Kleene tree.

Remark 151. *In the previous section, we worked out an explicit expression for List(X) for any m-assembly, as well as an explicit expression for $2 = 1 + 1$. Therefore, we understand that 2^* has the following explicit description:*

- *2^* is the set of finite sequences of 0 and 1.*

- *For any such sequence s in 2^*, $\alpha_{2^*0}(s)$ has only one element, which is the corresponding code of a finite sequence of $\underline{\overline{0}}$ and $\underline{\overline{1}}$.*

- *R_{2^*} is the collection of pairs of the form (x, y), where x and y are codes of finite sequences of $\underline{\overline{0}}$ and $\underline{\overline{1}}$, the length of y is not smaller than the length of x, and for any $1 \leq i \leq n$, we have $(x_i, y_i) \in R_2$.*

- *$\sim_{2^*}$ is exactly the smallest equivalence relation on dom(R_{2^*}): two codes are equivalent if and only if they are the same.*

Also, the m-assembly 2^* (in the current explicit expression) satisfies the nice properties (0), (1), and (2) discussed in Section 5.1.2. Indeed, all m-assemblies considered in this section (for instance, $\mathbb{N}$, 2, $2^{\mathbb{N}}$, and $\mathbb{N}^{2^{\mathbb{N}}}$) satisfy properties (0), (1), and (2).

If an m-assembly (X, α_X) satisfies the nice properties (0), (1), and (2), then we may identify the underlying set X with the set of potential realizers dom(R_X) up to $\sim_X$. We may also freely mix the external and internal viewpoints; for instance, given an element $x \in X$, we may simply say that another element $x^* \in X$ is a majorant of x if, for the corresponding realizers r_x and r_{x^*}, we have $(r_x, r_{x^*}) \in R_X$.

Remark 152. *We can construct the following morphisms of m-assemblies (we omit the proofs that these maps are indeed morphisms of m-assemblies; they can be carried out using coding in the PCA):*

- $\hat{}: 2^* \to 2^{\mathbb{N}}$*, which sends a finite sequence s of 0 and 1 to the function $g : \mathbb{N} \to 2$ such that for any $0 \leq i \leq \mathsf{lh}(s) - 1$, $g(i) = s_i$, and for $i \geq \mathsf{lh}(s)$, we have $g(i) = 0$. The main reason that $\hat{}$ is a morphism of m-assemblies is that the constant 1 function is a majorant for every $g \in 2^{\mathbb{N}}$.*

- $*0 : 2^* \to 2^*$*, which sends a finite sequence s of 0 and 1 to the sequence $s \sqcup [0]$.*

- $*1 : 2^* \to 2^*$*, which sends a finite sequence s of 0 and 1 to the sequence $s \sqcup [1]$.*

Definition 153 (Fan recursor)**.** *Define a map $\mathsf{FR} : \mathbb{N}^{2^{\mathbb{N}}} \times 2^* \to \mathbb{N}$ satisfying the property that for any $G \in \mathbb{N}^{2^{\mathbb{N}}}$ and for any $s \in 2^*$:*

If $G(\hat{s}) < \mathsf{lh}(s)$, then $\mathsf{FR}(G, s) = 0$.

If $G(\hat{s}) \geq \mathsf{lh}(s)$, then $\mathsf{FR}(G, s) = 1 + \max(\mathsf{FR}(G, s * 0), \mathsf{FR}(G, s * 1))$.

Remark 154. *We explain how the existence of Fan recursor gives a uniform bound for a decidable tree where every branch is finite.*

Let $U : 2^* \to 2$ be a decidable tree. If every branch of U is finite, then by definition ([Oos08], Section 3.2.7) we have $\forall f : \mathbb{N} \to 2, \exists n, [f(0), \ldots, f(n-1)] \notin U$. By the Axiom of Choice (Section 5.3), there exists a functional $G : 2^{\mathbb{N}} \to \mathbb{N}$ such that $\forall f : \mathbb{N} \to 2, [f(0), \ldots, f(G(f))] \notin U$.

We then understand that for any $s : 2^*$, if $s \in U$, then it must be the case that $G(\hat{s}) \geq \mathsf{lh}(s)$.

We prove by induction that for any $s : 2^*$, if $s \in U$, then $\mathsf{FR}(G, s) + \mathsf{lh}(s) \leq \mathsf{FR}(G, [\,])$. The case where $s = [\,]$ is clear. In the induction step, we consider $s*0$ and $s*1$; we consider the case of $s*0$ as an example. If $s*0 \in U$, then $s \in U$, therefore $G(\hat{s}) \geq \mathsf{lh}(s)$, so $\mathsf{FR}(G, s) = 1+\max(\mathsf{FR}(G, s*0), \mathsf{FR}(G, s*1))$. By the induction hypothesis, we have that $\mathsf{FR}(G, s) + \mathsf{lh}(s) \leq \mathsf{FR}(G, [\,])$, thus $\mathsf{FR}(G, s * 0) + \mathsf{lh}(s * 0) = 1 + \mathsf{FR}(G, s * 0) + \mathsf{lh}(s) \leq \mathsf{FR}(G, s) + \mathsf{lh}(s) \leq \mathsf{FR}(G, [\,])$.

Therefore, since $\mathsf{FR}(G, s) \geq 0$, we conclude that for any $s : 2^*$, if $s \in U$, then $\mathsf{lh}(s) \leq \mathsf{FR}(G, [\,])$. In this sense, $\mathsf{FR}(G, [\,])$ provides a uniform upper bound for the lengths of branches in U.

Remark 155. *To show that FR defined above is a morphism of m-assemblies, we first have to show that this FR is a well-defined set-theoretic function. This means that for FR to eventually obtain a value at (G, s), we require that eventually for all extensions t of s, we have $G(\hat{t}) < \mathsf{lh}(t)$.*

We will show that this is indeed the case.

Fix an arbitrary $G \in \mathbb{N}^{2^{\mathbb{N}}}$. Then there exists an element $G^* \in \mathbb{N}^{2^{\mathbb{N}}}$ such that G^* is a majorant of G.

In $2^{\mathbb{N}}$, we know every element is majorized by the constant 1 function. We write f for this constant 1 function.

Consequently, for any $s \in 2^*$, $\hat{s}$ is majorized by f, so $G(\hat{s})$ is majorized by $G^*(f)$, i.e., $G(\hat{s}) \leq G^*(f)$.

This means that the values of $G : 2^{\mathbb{N}} \to \mathbb{N}$ are bounded by the value $G^*(f)$. Therefore, for any $s \in 2^*$, if $\mathsf{lh}(s) > G^*(f)$, then $\mathsf{lh}(s) > G(\hat{s})$, so $\mathsf{FR}(G, s) = 0$ is defined. Then there are only finitely many $s \in 2^*$ with $\mathsf{lh}(s) \leq G^*(f)$, so by a "downwards" induction with finitely many steps, we understand that $\mathsf{FR}(G, s)$ is defined for any $s \in 2^*$.

We conclude that $\mathsf{FR} : \mathbb{N}^{2^{\mathbb{N}}} \times 2^* \to \mathbb{N}$ is a well-defined set-theoretic function.

We may summarize the argument above in a more precise lemma.

Lemma 156. *Suppose $G \in \mathbb{N}^{2^{\mathbb{N}}}$. If for some $M \geq 0$ we have $G(g) \leq M$ for all $g \in 2^{\mathbb{N}}$, then for any $s \in 2^*$ we have*

$$\mathsf{FR}(G, s) \leq M + 1.$$

Proof. Notice that if $\mathsf{lh}(s) \geq M + 1$, then $\mathsf{FR}(G, s) = 0$.

By induction, we can show that for any $0 \leq n \leq M + 1$, if $\mathsf{lh}(s) = M + 1 - n$, then $\mathsf{FR}(G, s) \leq n$.

Then we may conclude that for any $s \in 2^*$ we have $\mathsf{FR}(G, s) \leq M + 1$. □

We still have to show that FR is also a morphism of m-assemblies.

Proposition 157. $\mathsf{FR} : \mathbb{N}^{2^{\mathbb{N}}} \times 2^* \to \mathbb{N}$ *is a morphism of m-assemblies.*

Proof. We have to show that $\alpha_{\mathbb{N}^{2^{\mathbb{N}}}}(G) \times \alpha_{2^*}(s) \vdash \alpha_{\mathbb{N}}(\mathsf{FR}(G, s))$, or equivalently $\alpha_{\mathbb{N}^{2^{\mathbb{N}}}}(G) \vdash \alpha_{2^*}(s) \to \alpha_{\mathbb{N}}(\mathsf{FR}(G, s))$.

So we have to find a pair $(r_{\mathsf{FR}}, r^*_{\mathsf{FR}})$ witnessing this, i.e. for any $G \in \mathbb{N}^{2^{\mathbb{N}}}$ and $s \in 2^*$:

- $r_{\mathsf{FR}} \in \alpha_{\mathbb{N}^{2^{\mathbb{N}}}0}(G) \to \alpha_{2^*0}(s) \to \alpha_{\mathbb{N}0}(\mathsf{FR}(G, s))$.
- $(r_{\mathsf{FR}}, r^*_{\mathsf{FR}}) \in R_{\mathbb{N}^{2^{\mathbb{N}}}} \to R_{2^*} \to R_{\mathbb{N}}$.
- $(r_{\mathsf{FR}}, r_{\mathsf{FR}}), (r^*_{\mathsf{FR}}, r^*_{\mathsf{FR}}) \in \sim_{\mathbb{N}^{2^{\mathbb{N}}}} \to \sim_{2^*} \to \sim_{\mathbb{N}}$.

The construction of r_{FR} is to follow the "external" definition of FR, and we use the ingredients in the PCA to construct r_{FR} as an "internal" version of FR. Notice that externally FR is defined with a "recursor" flavor; this implies that we have to achieve this in the PCA using similar ideas when dealing with primitive recursors, for instance using the idea of fix-point operations. The construction of r^*_{FR} is more relaxed: it suffices to construct a majorant of r_{FR}, and we notice that the constant 1 function is a majorant of every function in $2^{\mathbb{N}}$.

Let $\underline{l} \in P$ be a "realizer" for the $<$-function, i.e. for any $n, m \in \mathbb{N}$, we have that $\underline{l\overline{n}\overline{m}} \preceq \underline{k}$ if $n < m$ and $\underline{l\overline{n}\overline{m}} \preceq \overline{k}$ if $n \geq m$.

Let $r_H \in P$ be a "realizer" for the hat function $\hat{}: 2^* \to 2^{\mathbb{N}}$, i.e. for any code of a finite sequence s of $\overline{0}$ and $\overline{1}$ (we use s to represent both the external sequence itself and its code as an element of P), and for any $n \in \mathbb{N}$, we have that:

- If $0 \leq n \leq \mathsf{lh}(s) - 1$, then $r_H s \overline{n} \preceq \overline{s_i}$.
- If $n \geq \mathsf{lh}(s)$, then $r_H s \overline{n} \preceq \overline{0}$.

Let $r_{*0} \in P$ and $r_{*1} \in P$ be the "realizers" for the functions $*0 : 2^* \to 2^*$ and $*1 : 2^* \to 2^*$, i.e. for any code of a finite sequence s of $\overline{0}$ and $\overline{1}$ (we use s to represent both the external sequence itself and its code as an element in P),

- $r_{*0}s \preceq s \sqcup [\underline{0}]$.
- $r_{*1}s \preceq s \sqcup [\underline{1}]$.

Also recall some recursion theory notation in PCA:

- Let $\underline{S} \in P$ be such that for any $n \in \mathbb{N}$, we have $\underline{S}\overline{n} \preceq \overline{n+1}$.
- Let $\underline{\max} \in P$ be such that for any $n, m \in \mathbb{N}$, we have $\underline{\max}\overline{n}\overline{m} \preceq \overline{\max(n,m)}$.

Then define t to be the following:

$$t := \langle ruv\rangle \underline{l}(u(r_H v))(\mathsf{lh}v)(\underline{k}\overline{0})(\langle w\rangle \underline{S}(\underline{\max}(rru(r_{*0}v))(rru(r_{*1}v))))\underline{k}.$$

The intuition for t is as follows: among the three inputs r, u, and v, r is like a placeholder for iteration, u is supposed to represent an element in $\mathbb{N}^{2^{\mathbb{N}}}$, and v is supposed to represent an element in 2^*. The term $\underline{l}(u(r_H v))(\mathsf{lh}v)$ determines whether $u(\hat{v}) < \mathsf{lh}(v)$ or not; if so, it becomes $\underline{k}$, and if not, it becomes $\overline{\underline{k}}$. Then it is followed by what the recursor is supposed to do in the two cases. If $u(\hat{v}) < \mathsf{lh}(v)$, then it should output $\overline{0}$; if $u(\hat{v}) \geq \mathsf{lh}(v)$, then it should perform iteration and give results like $1 + \max((\dots, v * 0), (\dots, v * 1))$.

The reason for introducing a dummy variable w in the part $(\langle w\rangle \underline{S}(\underline{\max}(rru(r_{*0}v))(rru(r_{*1}v))))$ is to ensure that this term is always defined during the computation. It has no other role, and therefore to fit this dummy w, we must introduce a dummy element, which we choose to be $\underline{k}$ here (and it could in fact be any element in the PCA). Correspondingly, in the base case, instead of writing merely $\overline{0}$, we must include $\underline{k}$ to obtain $\underline{k}\overline{0}$.

The calculation below will illustrate the role of these seemingly redundant parts. The idea is inspired by the construction of the primitive recursor in a PCA, in Proposition 1.3.5 of [Oos08].

We define r_{FR} to be tt. We verify that r_{FR} has the desired property. For any $G \in \mathbb{N}^{2^{\mathbb{N}}}$, and for any $s \in 2^*$, we fix $r_G \in \alpha_{{\mathbb{N}^{2^{\mathbb{N}}}}_0}(G)$ and $r_s \in \alpha_{2^*0}(s)$, and we show that $r_{\mathsf{FR}}r_G r_s \in \alpha_{\mathbb{N}}(\mathsf{FR}(G,s))$.

We understand that $r_G(r_H r_s) \preceq \overline{G(\hat{s})}$, and $\mathsf{lh}r_s \preceq \overline{\mathsf{lh}(s)}$.

If $G(\hat{s}) < \mathsf{lh}(s)$, then $\underline{l}(r_G(r_H r_s))(\mathsf{lh}r_s) = \underline{k}$, thus:

$$\begin{aligned}
& r_{\mathsf{FR}}r_G r_s \\
= \;& ttr_G r_s \\
\preceq \;& \underline{l}(r_G(r_H r_s))(\mathsf{lh}r_s)(\underline{k}\overline{0})(\langle w\rangle \underline{S}(\underline{max}(ttr_G(r_{*0}r_s))(ttr_G(r_{*1}r_s))))\underline{k} \\
= \;& \underline{k}(\underline{k}\overline{0})(\langle w\rangle \underline{S}(\underline{max}(ttr_G(r_{*0}r_s))(ttr_G(r_{*1}r_s))))\underline{k} \\
\preceq \;& \underline{k}\overline{0}\underline{k} \\
= \;& \overline{0}.
\end{aligned}$$

At the same time, $\mathsf{FR}(G,s) = 0$, thus we have $r_{\mathsf{FR}}r_G r_s \in \alpha_{\mathbb{N}0}(\mathsf{FR}(G,s))$.

If $G(\hat{s}) \geq \mathsf{lh}(s)$, then $\underline{l}(r_G(r_H r_s))(\mathsf{lh}r_s) = \overline{\underline{k}}$, thus:

$$\begin{aligned}
& r_{\mathsf{FR}}r_G r_s \\
= \;& ttr_G r_s \\
\preceq \;& \underline{l}(r_G(r_H r_s))(\mathsf{lh}r_s)(\underline{k}\overline{0})(\langle w\rangle \underline{S}(\underline{max}(ttr_G(r_{*0}r_s))(ttr_G(r_{*1}r_s))))\underline{k} \\
= \;& \overline{\underline{k}}(\underline{k}\overline{0})(\langle w\rangle \underline{S}(\underline{max}(ttr_G(r_{*0}r_s))(ttr_G(r_{*1}r_s))))\underline{k} \\
\preceq \;& (\langle w\rangle \underline{S}(\underline{max}(ttr_G(r_{*0}r_s))(ttr_G(r_{*1}r_s))))\underline{k} \\
= \;& (\langle w\rangle \underline{S}(\underline{max}(r_{\mathsf{FR}}r_G(r_{*0}r_s))(r_{\mathsf{FR}}r_G(r_{*1}r_s))))\underline{k} \\
\preceq \;& \underline{S}(\underline{max}(r_{\mathsf{FR}}r_G(r_{*0}r_s))(r_{\mathsf{FR}}r_G(r_{*1}r_s)))
\end{aligned}$$

Notice that we have $r_{*0}r_s \in \alpha_{2^*0}(s*0)$ and $r_{*1}r_s \in \alpha_{2^*0}(s*1)$. As in the proof showing that FR is total, we know there is an upper bound for the value $G(\hat{s})$, thus for $t \in 2^*$ with $\mathsf{lh}(t)$ large enough, it is always the case that $G(\hat{t}) < \mathsf{lh}(t)$.

Therefore, by a suitable induction we obtain the induction hypothesis that $r_{\mathsf{FR}}r_G(r_{*0}r_s) \in \alpha_{\mathbb{N}0}(\mathsf{FR}(G, s*0))$ and $r_{\mathsf{FR}}r_G(r_{*1}r_s) \in \alpha_{\mathbb{N}0}(\mathsf{FR}(G, s*1))$ (which is equivalent to $r_G(r_{*0}r_s) \preceq \overline{\mathsf{FR}(G, s*0)}$ and $r_G(r_{*1}r_s) \preceq \overline{\mathsf{FR}(G, s*1)}$).

Thus $\underline{S}(\underline{\max}(r_{\mathsf{FR}}r_G(r_{*0}r_s))(r_{\mathsf{FR}}r_G(r_{*1}r_s))) \preceq \overline{1+\max(\mathsf{FR}(G, s*0), \mathsf{FR}(G, s*1))} \preceq \overline{\mathsf{FR}(G, s)}$, so $r_{\mathsf{FR}}r_Gr_s \in \alpha_{\mathbb{N}0}(\mathsf{FR}(G, s))$.

We conclude that $r_{\mathsf{FR}} \in \alpha_{\mathbb{N}^{2^{\mathbb{N}}}0}(G) \to \alpha_{2^*0}(s) \to \alpha_{\mathbb{N}0}(\mathsf{FR}(G, s))$.

We still have to provide a majorant r^*_{FR} for r_{FR}. The idea is straightforward from an external point of view and was already discussed when showing that FR is a well-defined set-theoretic function.

Let $r_f \in \langle u \rangle \underline{1}$ be the code of the constant function f that maps everything to 1. Then define $r^*_{\mathsf{FR}} \coloneqq \langle uv \rangle \underline{S}(ur_f)$. We show that $(r_{\mathsf{FR}}, r^*_{\mathsf{FR}}) \in R_{\mathbb{N}^{2^{\mathbb{N}}}} \to R_{2^*} \to R_{\mathbb{N}}$.

Suppose $(r_G, r_{G^*}) \in R_{\mathbb{N}^{2^{\mathbb{N}}}}$; we show that $(r_{\mathsf{FR}}r_G, r^*_{\mathsf{FR}}r_{G^*}), (r^*_{\mathsf{FR}}r_G, r^*_{\mathsf{FR}}r_{G^*}) \in R_{2^*} \to R_{\mathbb{N}}$. For any $(r_s, r_{s^*}) \in R_{2^*}$, we have that:

- $r_{\mathsf{FR}}r_Gr_s = \overline{\mathsf{FR}(G, s)}$, $r^*_{\mathsf{FR}}r_{G^*}r_{s^*} = r^*_{\mathsf{FR}}r_{G^*}r_s = \overline{G^*(f)+1}$. Since for any $g \in 2^{\mathbb{N}}$, we know $G(g) \leq G^*(f)$, by the previous lemma (Lemma 156) we know $\mathsf{FR}(G, s) \leq G^*(f)+1$. We conclude that $(r_{\mathsf{FR}}r_Gr_s, r^*_{\mathsf{FR}}r_{G^*}r_{s^*}), (r^*_{\mathsf{FR}}r_{G^*}r_s, r^*_{\mathsf{FR}}r_{G^*}r_{s^*}) \in R_{\mathbb{N}}$.

- $r^*_{\mathsf{FR}}r_{G^*}r_{s^*} = r^*_{\mathsf{FR}}r_{G^*}r_s = \overline{G^*(f)+1}$, and $r^*_{\mathsf{FR}}r_Gr_s \in \overline{G(f)+1}$. Notice that $(r_f, r_f) \in R_{2^{\mathbb{N}}}$, and $(r_G, r_{G^*}) \in R_{\mathbb{N}^{2^{\mathbb{N}}}}$; thus in particular, we have that $(r_Gr_f, r_{G^*}r_f) \in R_{\mathbb{N}}$, i.e. $G(f) \leq G^*(f)$. We conclude that $(r^*_{\mathsf{FR}}r_Gr_s, r^*_{\mathsf{FR}}r_{G^*}r_{s^*}), (r^*_{\mathsf{FR}}r_{G^*}r_s, r^*_{\mathsf{FR}}r_{G^*}r_{s^*}) \in R_{\mathbb{N}}$.

We conclude that $(r_{\mathsf{FR}}, r^*_{\mathsf{FR}}) \in R_{\mathbb{N}^{2^{\mathbb{N}}}} \to R_{2^*} \to R_{\mathbb{N}}$.

The last thing we have to verify is that $(r_{\mathsf{FR}}, r_{\mathsf{FR}}), (r^*_{\mathsf{FR}}, r^*_{\mathsf{FR}}) \in \sim_{\mathbb{N}^{2^{\mathbb{N}}}} \to \sim_{2^*} \to \sim_{\mathbb{N}}$. But everything can be reduced to external arguments, because everything in the domains of $\sim_{\mathbb{N}^{2^{\mathbb{N}}}}$, $\sim_{2^*}$, and $\sim_{\mathbb{N}}$ corresponds to an actual realizer of the corresponding functionals, finite sequences, and natural numbers. Also, two such elements are equivalent if and only if they represent the same functional, finite sequence, and natural number. So everything reduces to a calculation involving FR and FR^*.

We conclude that $\mathsf{FR} : \mathbb{N}^{2^{\mathbb{N}}} \times 2^* \to \mathbb{N}$ is a morphism of m-assemblies. $\square$

Remark 158. *We still have to show that this* $\mathsf{FR} : \mathbb{N}^{2^{\mathbb{N}}} \times 2^* \to \mathbb{N}$ *also has the correct internal property, but notice that the external behavior of* FR *already ensures the internal definition of the potential Fan recursor.*

For instance, we want to show internally that for any $G \in \mathbb{N}^{2^{\mathbb{N}}}$ *and* $s \in 2^*$, *if* $G(\hat{s}) < \mathsf{lh}(s)$, *then* $\mathsf{FR}(G, s) = 0$, *and if* $G(\hat{s}) \geq \mathsf{lh}(s)$, *then* $\mathsf{FR}(G, s) = 1 + \max(\mathsf{FR}(G, s*0), \mathsf{FR}(G, s*1))$. *Internally, we have to compare several subobjects of the object* $\mathbb{N}^{2^{\mathbb{N}}} \times 2^*$:

- *Write the subobject corresponding to* $G(s) < \mathsf{lh}(s)$ *as* S_1.
- *Write the subobject corresponding to* $G(s) \geq \mathsf{lh}(s)$ *as* S_2.
- *Write the subobject corresponding to* $\mathsf{FR}(G, s) = 0$ *as* T_1.
- *Write the subobject corresponding to* $\mathsf{FR}(G, s) = 1 + \max(\mathsf{FR}(G, s*0), \mathsf{FR}(G, s*1))$ *as* T_2.

We want to show that as subobjects of $\mathbb{N}^{2^{\mathbb{N}}} \times 2^*$, *we have*

- $S_1 \leq T_1$.
- $S_2 \leq T_2$.

But all those four subobjects arise from equalizers of certain morphisms of m-assemblies. And we know that those equalizers correspond to subsets of $\mathbb{N}^{2^{\mathbb{N}}} \times 2^*$ *(Lemma 122), and to compare subobjects of this form (those regular monos), it suffices to compare inclusion of subsets, but the inclusion of those subsets is determined externally, by the defining property of those morphisms, for instance,* FR, *and so on.*

We conclude that our topos has the Fan recursor.

Chapter 6

Conclusion

We list several problems to be addressed in the future.

Bar induction and Bar recursor The bar induction and Bar recursor are discussed in [Bez85]. The proof of the Bar recursor there makes use of bar induction, whose proof is by contradiction. Therefore, at least that specific proof does not work in our topos in general (in particular, it does not hold for $\mathcal{K}_1$). However, this issue may be addressed by choosing other PCAs with the nice property of "sequentiality", which says that for any sequence of elements $(r_n)_{n\in\mathbb{N}}$ in the PCA, there exists an element α in the PCA such that for any $n \in \mathbb{N}$, we have $\alpha \cdot \underline{\overline{n}} \preceq r_n$. For example, $\mathcal{K}_2$ satisfies this property.

So if we start with a PCA with the property of "sequentiality", then the bar induction issue may be addressed externally, and then, following Bezem's construction and proof, by translating the external arguments there, we may also obtain a Bar recursor in our topos.

Variants of $dEMAJ(P)$ In Chapter 3, we introduced the arrow algebra $dEMAJ(P)$, whose elements are tuples (A, R, E), where $R \subseteq P \times P$ (P is a fixed PCA) is a majorizability relation, $E \subseteq P \times P$ is an equivalence relation on $\mathrm{dom}(R) = \mathrm{dom}(E)$ and is compatible with R; A is a subset of $\mathrm{dom}(R)$ that is closed under the equivalence relation E. Recall that the idea is to regard $\mathrm{dom}(R) = \mathrm{dom}(E)$ as a set of potential realizers, and $A \subseteq \mathrm{dom}(R) = \mathrm{dom}(E)$ as a set of actual realizers.

Then we may ask what happens if we modify these definitions; for instance, instead of using A as a subset of $\mathrm{dom}(R)$, we consider A as a subset of R satisfying various compatibility properties.

The constructions in Chapter 4 seem to be quite robust, so we believe that in these possible variants of $dEMAJ(P)$ we may still obtain the same basic results.

MAJ and CT Since our topos believes in the Axiom of Choice, by Theorem 6.8 in [TD88], we know that Church's Thesis (CT) is not valid in the topos, even though externally every function from $\mathbb{N}$ to $\mathbb{N}$ has a realizer; any such function would be computable if the PCA were $\mathcal{K}_1$.

Let MAJ be the statement saying that every functional has a majorant. We know that this holds externally, but we do not know whether it holds internally in the topos. A similar question was also asked in Chapter 6 of [Gu18]. We tend to believe that MAJ is not internally true.

However, one might still expect $\neg\neg$CT and $\neg\neg$MAJ to hold internally. The reason is that adding negations destroys constructive information and makes statements more likely to be valid. Moreover, if this is indeed the case, then we can at least conclude that our topos is not Boolean, since otherwise CT would be true internally as well.

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