NON-WELLFOUNDED PROOF SYSTEMS FOR PROVABILITY LOGICS AND THEIR METATHEORY

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written by

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Abstract

Many provability logics enjoy fixpoint theorems. This provides a natural motivation for studying non-wellfounded proof systems for such logics, as non-wellfounded proof systems are well-suited for defining proof systems for modal logics with fixpoints. Nevertheless, comparatively few provability logics have been equipped with non-wellfounded proof systems of their own. Moreover, even for those logics for which such systems are already available, non-trivial questions remain open, such as the existence of explicit cut reduction procedures. Against this background, this thesis investigates the non-wellfounded proof theory of the following provability logics: Gödel-Löb logic (**GL**), strong Löb's logic (**iSL**$_\Box$), and Kuznetsov-Muravitsky logic (**KM**).

For **GL**, building on existing one-sided non-wellfounded systems for this logic, this thesis provides both an infinitary system and a cyclic system based on two-sided multiset sequents. These systems are formulated in **G3**-style. The main contribution of the investigation of **GL** is a direct cut reduction procedure for the cyclic system developed in the thesis. For **iSL**$_\Box$, novel non-wellfounded systems in both **G3**- and **G4**-styles are introduced. These systems are formulated using set sequents, and are shown to be sound and cut-free complete. Moreover, the **G4**-style systems admit terminating backward proof search. The existence of such systems for **iSL**$_\Box$ establishes that **iSL**$_\Box$ is a cyclic companion of intuitionistic doxastic logic. Additionally, the thesis also identifies the technical difficulties involved in adopting set sequents in the intuitionistic setting and discusses possible ways of addressing them. Lastly, for **KM**, sound **G4**-style systems are proposed.

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Contents

Chapter 1

Introduction

In this opening chapter, we provide an introduction to provability logics and non-wellfounded proof systems, and motivate the research questions pursued in this thesis. We begin with a brief introduction to classical and intuitionistic provability logics in section 1.1. Then, in section 1.2, we introduce non-wellfounded proof theory and explain its connection with provability logics. This section also provides a literature review of recent developments in the non-wellfounded proof theory of provability logics. Section 1.3 then formally presents the research questions of this thesis, while section 1.4 offers an overview of the structure of this thesis and summarizes its contributions.

1.1 Provability logics: classical and intuitionistic

'Provability logics' refers to a family of modal logics. Among them is the famous Gödel-Löb logic **GL**. It is proven to reflect properties of the provability predicate that can be derived by Peano arithmetic thanks to Solovay's completeness theorem (Cf. [Sol76]). The crucial Löb's axiom $\Box(\Box A \to A) \to \Box A$ directly reflects Löb's theorem proven in [Löb55]. **GL** has attracted great mathematical and philosophical interest and has been intensively studied since the early seventies. Basic model-theoretic properties of this logic include: it is the logic of transitive and conversely well-founded frames (Cf. [Seg71]); additionally, it is proven to be not strongly complete (Cf. [Boo95]). On the proof-theoretic side, the cut elimination property of a Gentzen-style sequent calculus for **GL** is proved by Sambin and Valentini in [Val83].

There are other logics that are also named after 'provability logic'. Indeed, since **GL** builds on top of classical logic, it is natural to envision the 'intuitionistic counterparts' of **GL**. In the intuitionistic setting, one might think 'the' natural provability logic is the provability logic of Heyting arithmetic. Nevertheless, finding an appropriate axiomatization of the provability logic for Heyting arithmetic has long been an open problem. Recently, Mojtahedi has provided an axiomatization in [Moj22], which is currently under review. So, unlike **GL**, intuitionistic provability logics are usually obtained by simply adding Löb's axiom and (perhaps other axioms) to *intuitionistic modal logics*. It is notable that in intuitionistic modal logic, $\Box$ and $\Diamond$ are no longer duals. Many intuitionistic provability logics are nomomodal, for they have only the box operator in the language. For example, adding Löb's axiom to monomodal intuitionistic normal modal logic **K** yields a logic called **iGL**. Extending **iGL** with the completeness axiom $A \to \Box A$ gives **iGLC** (Cf. [VZ19]) or $\mathbf{iSL}_{\Box}$, where '**S**' stands for 'strong'. Further extending $\mathbf{iSL}_{\Box}$ with the axiom KM: $\Box A \to (B \vee (B \to A))$ gives us logic **KM**. Bimodal intuitionistic provability logics have also been studied, for example, in [DvM24]. Nevertheless, we will restrict our attention to monomodal intuitionistic provability logics in this thesis.

Despite being called intuitionistic 'provability' logics, one might wonder whether these logics are truly

provability logics, i.e. whether they have natural arithmetic interpretations just like **GL**. Some of them do. For example, **iGL**, $\mathbf{iSL}_{\Box}$, and **KM** all have certain arithmetical interpretation [vI21; vI20; VZ19; KM86], which justifies their names. In this thesis, we will not care so much about their arithmetical interpretation, but about their proof theory only.

On the model-theoretic side, **iGL**, $\mathbf{iSL}_{\Box}$, and **KM** are similar to **GL** because the frame classes they define are all subclasses of frames where the modal relation is conversely well-founded. For a comprehensive overview of these results, we direct the reader to [Lit14]. On the proof-theoretic side, sequent calculi for these logics have recently been proposed and studied. In [vI21], van der Giessen and Iemhoff provide sequent calculi for **iGL**. For $\mathbf{iSL}_{\Box}$, the same authors also provide two sequent calculi in [vI20], one is in **G3**-style and the other in **G4**-style. In a later paper [Shi+23], another **G4**-style sequent calculus that has better terminating properties is proposed. The most recent investigation of the structural proof theory of **KM** can be found in [FS26], where a **G4**-style sequent calculus is offered.

1.2 Non-wellfounded proofs and provability logics

In contrast with standard sequent or well-founded sequent calculi, non-wellfounded systems allow branches of infinite length that satisfy certain conditions. Very intuitively, we can understand the contrast between the well-foundedness and non-wellfoundedness here in terms of 'infinite descending subtree sequence'. For well-founded sequent calculi, proofs are finite in height, so for an arbitrary proof tree, the operation of taking a subtree cannot yield an infinite sequence. Since we often assign a smaller measure (e.g. a smaller height) to a subtree of a tree, we may see this sequence as a finite descending sequence. In contrast, for non-wellfounded calculi, this is not the case – we can take a subtree of a given tree, then take a subtree of the subtree, *ad infinitum*. Even if we can assign a reasonable measure to such non-wellfounded proofs, there is risk of an infinite descending chain.

A special kind of non-wellfounded proof systems is worth mentioning. Cyclic proof systems form a subclass of non-wellfounded proof systems that enjoy finite representation. Cyclic proofs are non-wellfounded proofs containing finitely many distinct sub-proofs. As a result, cyclic proofs can usually be represented as finite objects equipped with devices that simulate the infinite branches.

Non-wellfounded proof systems provide a way to define proof systems for modal logics with fixpoints, such as modal μ-calculus. For example, we can find tableaux versions of cyclic systems for modal μ-calculus in [NW96] and [Jun09]. Additionally, an infinitary proof system of modal μ-calculus is first initiated by Dax, Hofmann, and Lange in [DHL06] and then developed by Studer in [Stu08]. Notably, many provability logics enjoy their own fixpoint theorems. For example, Sambin shows that the classical provability logic **GL** has a fixpoint theorem in [Sam76]. Indeed, the non-wellfounded systems for **GL** designed by Shamkanov in [Sha14] is motivated by the fact that **GL** can be regarded as a fragment of the modal μ-calculus (Cf. [Ben06]). On the intuitionistic side, the fixpoint theorems for **iGL** and $\mathbf{iSL}_{\Box}$ (Cf. [Lit14]) then serve as a natural motivation for developing non-wellfounded systems for these logics.

As just mentioned, the paper [Sha14] is a trailblazer in the non-wellfounded proof theory of provability logics. There, based on a sequent calculus for **K4**, Shamkanov provided two equivalent non-wellfounded systems for **GL** using one-sided sequents by allowing non-wellfounded proofs. One of the systems is infinitary, and the other is finitary and cyclic. Recently, more research on the non-wellfounded proof theory of intuitionistic provability logics has occurred. For example, in [Iem21], Iemhoff observed that **iGL** also has a natural cyclic system which is based on a sequent calculus of intuitionistic **K4**. Based on Iemhoff's work, Sierra Miranda

provides [Sie23] an alternative proof of the equivalence between the cyclic system of **iGL** and the standard well-founded one. The non-wellfounded proof theory of other intuitionistic provability logics, however, is less studied. This directly motivates the research theme of the thesis.

Before elucidating more specific research topics of the thesis, let us also mention a few other motivations for developing non-wellfounded proof systems in general. Other than being a natural proof-theoretic resource to investigate logics with fixpoints, non-wellfounded proof systems also enjoy certain nice proof-theoretic properties. For example, in the context of inductive proofs, automated proof search has difficulty dealing with 'eureka steps', i.e. the activity of conjecture in a mathematical proof. For example, in an inductive proof, we sometimes have to be careful when choosing the inductive hypothesis – correctly choosing bounded variables in the inductive hypothesis can be a non-trivial matter. As concluded by Ireland and Bundy in [IB96], this kind of 'eureka steps' spells trouble for automated proof search. In contrast, cyclic systems provide an alternative way to formulate inductive proofs whose proof search is more controllable – the task of guessing the inductive hypothesis is 'replaced by detecting possible cycles in a proof' [BSS25]. Moreover, non-wellfounded proof systems may also enjoy a simpler cut reduction procedure compared with their well-founded correspondence. For an example, we direct the reader to [HSS25], a study on the non-wellfounded proof theory of interpretability logic. Lastly, such systems are also useful in proving interpolation results, as witnessed by studies such as [Sha14], [SS21], and [HSS25].

1.3 The aim of the thesis: a novel cut elimination proof and new systems

The overarching aim of the thesis is to study the non-wellfounded proof theory for representatives of both classical and non-classical provability logics. Some of these logics, like **GL** and **iGL**, already have non-wellfounded proof systems, while others do not. Our research questions can then naturally be divided into two groups. One is about the properties of existing proof systems, while the other is about designing novel systems for those that do not yet have non-wellfounded proof systems.

For the first group of questions, we shall focus solely on the cut elimination of the non-wellfounded systems for **GL**, and in particular the cyclic system for **GL**. As mentioned in the last section, one of the motivations for developing non-wellfounded proof systems is simpler cut reduction procedures. Let us take **GL** as an example. The cut elimination of a sequent calculus of **GL** is known to be difficult. In contrast, non-wellfounded systems for **GL** share the same set of rules with a sequent calculus for **K4**, which means that the (local) cut reduction procedure could be the same as that of the calculus for **K4**. The cut reduction for **K4** is much easier, as the modal rule it endorses behaves in a much simpler way than that of **GL**'s well-founded sequent calculus.

Shamkanov's non-wellfounded systems for **GL** do not have any form of the cut rule, but they are cut-free complete. Shamkanov proves this result by showing that these systems are equivalent to a well-founded system for **GL** where cut is admissible. Therefore, a direct cut reduction procedure for these systems is missing. For the infinitary system, cut reduction and the cut elimination theorem are indeed not so difficult to prove, thanks to the coalgebraic method developed in [SSZ25]. Nevertheless, for the cyclic system, the proof is much less straightforward.

Our interest in the cut reduction procedure of cyclic **GL** can be put into a wider context. Whilst there are some recent attempts to prove cut elimination of cyclic systems directly, the cut elimination method on cyclic proofs is in general underdeveloped. In [AK24], Afshari and Kloibhofer provided the first direct cut elimination method on a specific cyclic system for temporal logic called GKe, and later works of the same authors explore this method further. Notably, this method works for a (subset of) particular kind of cyclic

systems – those with annotations. Annotation plays an important role in this proof method. However, the cyclic system for **GL** is not annotated. Given that annotation plays an essential role in Afshari and Kloibhofer's cut elimination method for cyclic proofs, the question of how cut elimination can be done for cyclic systems such as **GL** remains open.

We suggest that the cut elimination methods for cyclic proofs deserve attention. Indeed, given a cyclic system, usually by unraveling its proofs we can get an infinitary system with the same set of rules, and the cut elimination of the former could follow from the cut elimination of the latter (which can be easier to prove). If we have cut elimination for the infinitary system, *and* we have a procedure to turn a cut-free infinitary proof into a cyclic one, the cut elimination of the cyclic system follows. For example, we will see how the cut elimination of cyclic **GL** follows from the cut elimination of an infinitary system of **GL**. However, this way of proving the cut elimination of cyclic systems does not always work – the cut elimination property of a cyclic system and its 'infinitary counterpart' may come apart. For example, in [OBT25], Oda, Brotherston, and Tatsuta proved that the cyclic system for first-order logic with inductive definitions does not have cut elimination, while the infinitary system consisting of the same set of rules enjoys cut elimination. Therefore, the cut elimination method for cyclic systems cannot be reduced to the cut elimination for infinitary systems, and hence deserves its own attention.

For the second group of questions, we aim to provide non-wellfounded systems for both $\mathbf{iSL}_\Box$ and **KM**, which currently do not have any non-wellfounded proof system to the best of our knowledge. This combination should not be surprising, for these are famous intuitionistic provability logics, with **KM** being an extension of $\mathbf{iSL}_\Box$ by adding the axiom KM. Moreover, our choice of these logics also reflects a sense of technical diversity. We will provide both **G3**- and **G4**-styles systems for $\mathbf{iSL}_\Box$ that endorse sequents with simple succedents – i,e. sequents whose succedents contain at most one formula. Our motivation for developing **G3**-style systems for $\mathbf{iSL}_\Box$ is mainly the simplicity of such systems. In the context of *well-founded* sequent calculus, systems of this style are known to be non-terminating – they do not enjoy easy terminating backwards proof search. Unsurprisingly, as it turns out, our **G3**-systems for $\mathbf{iSL}_\Box$ are also non-terminating. This non-terminating property of **G3**-style systems motivates the search for **G4**-style ones. The prototype of well-founded systems of the latter style is **LJT**, designed by Dyckhoff in [Dyc92]. We wonder whether similar systems with terminating proof search can also be defined in the non-wellfounded setting and hence decide to investigate this direction in depth. Finally, for **KM**, we will design our system based on the recent **G4**-style well-founded sequent calculus for **KM** provided by Férée and Shillito in [FS26], where sequents have multi-succedents.

1.4 Structure and contributions of the thesis

After presenting preliminary definitions in chapter 2, we will focus on the cut elimination of non-wellfounded systems of **GL** in chapters 3 to 5. Chapter 3 is dedicated to introducing two two-sided non-wellfounded proof systems for **GL** with a cut rule. One of them is infinitary, while the other is cyclic, and both endorse multiset sequents. To prove that these systems are equivalent, we will need the cut elimination of the infinitary system. Chapter 4 serves this purpose. It is primarily introductory – it presents the method of coalgebraic proof translation developed by Sierra Miranda, Studer, and Zenger in [SSZ25], which is very useful for proving the cut elimination for infinitary non-wellfounded systems that we will encounter in this thesis. After that, we prove the cut elimination of the infinitary system for **GL** using this method in chapter 5. We also provide our direct cut elimination procedure for cyclic **GL** in the same chapter.

Chapters 6 and 7 then focus on the non-wellfounded proof theory of intuitionistic provability logics. In

chapter 6, we provide two non-wellfounded systems for $\mathbf{iSL}_\Box$, one in **G3**-style and and the other in **G4**-style. We will prove the soundness and cut-free completeness of these systems, and also argue that the **G4**-style system enjoys terminating backward proof search. We would like to emphasize that these systems endorse *set* sequents rather than multiset sequents. We will explain in detail the motivation for this technical choice and elucidate the difficulties that it causes in the same chapter. In chapter 7, we will present **G4**-style calculi for **KM** and prove their soundness. Completeness is left for future research.

We summarize the main contribution of the thesis as follows.

(1) Provides a novel constructive argument for an established theorem: a direct cut reduction procedure for a cyclic system of **GL** (defined in terms of multiset sequents).

(2) Designs non-wellfounded proof systems for **GL**, $\mathbf{iSL}_\Box$ and **KM** with desired properties, as shown by the table below. The '**G3**'/ '**G4**' in the names of systems indicates that system is **G3**/**G4**-style; we assume that systems of these styles are two-sided. Additionally, the '∞' in the names indicates that the system is infinitary, while the '$\circ$' indicates that the system is cyclic. 'Extended axioms' means that the system endorses axioms of the shape $A, \Gamma \Rightarrow A$, for *any* formula A.

Logic	System	System type	Main properties
GL	$\mathbf{G3GL}^{\mathrm{cut},\infty}$	multiset sequents	Sound and cut-free complete
	$\mathbf{G3GL}^{\mathrm{cut},\circ}$	multiset sequents	Sound and cut-free complete
$\mathbf{iSL}_\Box$	$\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}$	set sequents	Sound and cut-free complete
	$\mathbf{G3iSL}_\Box^{\mathrm{cut},\circ}$	set sequents	Sound and cut-free complete
	$\mathbf{G4iSL}_\Box^{\infty}$	set sequents, extended axioms	Sound and complete; terminating backward proof search
	$\mathbf{G4iSL}_\Box^{\circ}$	set sequents, extended axioms	Sound and complete; terminating backward proof search.
KM	$\mathbf{G4KM}^{\infty}$	set sequents, multi-succedents, extended axioms	Sound; terminating backward proof search.
	$\mathbf{G4KM}^{\circ}$	set sequents, multi-succedents, extended axioms	Sound; terminating backward proof search.

Table 1.1: Summary of the proof systems developed in this work

(3) Proves that $\mathbf{iSL}_\Box$ stands in a special relation to the so-called 'intuitionistic doxastic logic', i.e. $\mathbf{iSL}_\Box$ is a cyclic companion of the latter.

(4) Demonstrates the technicalities of set sequents in classical and intuitionistic settings.

Now, let us begin our exploration of non-wellfounded proof theory of provability logics.

Chapter 2

Preliminary definitions

In this chapter, we introduce preliminary definitions. Proofs can be regarded as labeled trees, so we will start with the definition of trees. The definitions that we endorse may appear complicated, but we need them for the cut elimination proof for the non-wellfounded systems in this thesis. We will also define cyclic trees and their unraveling, which is important for our later investigation of cyclic proofs.

Definition 2.1 (Words and prefix relation)**.** *Let $\mathbb{N}$ (or ω) be the set of natural numbers, and let $\mathbb{N}^*$ be the set of finite sequences of $\mathbb{N}$. We call these finite sequences* words*. We often use letters w, u, v to denote them, and use expressions like 'wu' to denote the sequence obtained by concatenating u to the left of w. ϵ is the empty sequence in $\mathbb{N}^*$. Let $\mathbb{N}^+ := \mathbb{N}^* \setminus \{\epsilon\}$.*

We define the prefix relation $\sqsubseteq \subseteq \mathbb{N}^* \times \mathbb{N}^*$ *as follows: for any $w, w' \in \mathbb{N}^*$, $w \sqsubseteq w'$ iff there is a $u \in \mathbb{N}^*$ such that $wv = w'$. We also define the strict prefix relation $w \sqsubset w'$ iff $w \sqsubseteq w'$ and $w \neq w'$. Define $w \prec u$ iff there is some $n \in \mathbb{N}$ such that $u = w(n)$, i.e. u is an immediate $\sqsubset$-successor of w. For any w, u with $w \sqsubseteq u$, let $u - w$ be the v such that $u = wv$.* ⊣

Remark 2.2. *Let w be a word, and n be a natural number. We also use 'wn' to denote the word $w(n)$.*

Note that $\sqsubset$ is the transitive closure of $\prec$, and $\sqsubseteq$ is the reflexive and transitive closure of $\prec$.

Definition 2.3 (Sequence of words)**.** *Given words $w_0, \ldots, w_n$, $[w_0, \ldots, w_n]$ is the sequence of them. We typically use letters r, s to denote sequences of words. Let* nil *be the empty sequence of words. For any sequence of words r and any word w, let $w : r$ and $r : w$ be the sequences obtained by adding w to the left of r and to the right of r, respectively. Let the set of finite sequences of words be $\mathbb{N}^{**}$.* ⊣

We will work with *trees* that have words as their vertices.

Definition 2.4 (Relational structure)**.** *A* relational structure *is a pair (X, R), where X is a set and $R \subseteq X \times X$.* ⊣

Definition 2.5 (Tree)**.** *A* tree *is a relational structure $(N, \prec)$ such that*

(1) $\emptyset \neq N \subseteq \mathbb{N}^*$ is closed under the prefix relation $\sqsubseteq$;

(2) $\prec$ is the immediate prefix relation on $N \times N$;

(3) for any $w \in N$, there is an $n \in \mathbb{N}$ such that $wi \in N$ iff $i < n$.

*Given a set of labels L, an L-*labeled tree *is a triple $(N, \prec, l)$, where $(N, \prec)$ is a tree, and $l : N \to L$ is a* labeling function. ⊣

Definition 2.6 (Arity)**.** *Let $(N, \prec)$ be a tree, and let $w \in N$. The arity of w is defined as:* $\mathrm{arity}(w) = k$ *iff for any $i \in \mathbb{N}$, it holds that wi is in N iff $i < k$.* ⊣

Trees defined over $\mathbb{N}^*$ are useful because using them, we can define coalgebraic structures more easily. But they can also be restrictive, for the class of trees may not be closed under certain operations, e.g. taking subtrees. In some contexts, the identity of the elements in the underlying set of a tree is not important to us. So it would be helpful to define tree-like structures.

Definition 2.7 (Tree-like structure)**.** *A relational structure (T, S) with $S \subseteq T \times T$ is* tree-like *iff*

(1) $T \neq \emptyset$;

(2) there is a unique $r \in T$ such that for all $t \in T$, rS^*t, where S^* is the reflexive and transitive closure of S;

(3) for any $t \neq r \in T$, there is a unique S-predecessor;

(4) for any $t \in T$, it is not the case that tS^+t, where S^+ is the transitive closure of S;

(5) for every $t \in T$, its S-successor set $\{s \in N : tSs\}$ is finite and is well-ordered by an initial fragment of ω.

Note that we require the successor set of a given node to be well-ordered. ⊣

Proposition 2.8. *Let $(N, \prec)$ be a tree. $(N, \prec)$ is tree-like.*

Proof. We check this item by item.

(1) By definition, $N \neq \emptyset$.

(2) Since N is closed under $\sqsubseteq$, ϵ is in N. Additionally, for any $w \in N$, $\epsilon \sqsubseteq w$. ϵ is the unique such node.

(3) For any $w \in \mathbb{N}^*$ different from ϵ, there is a unique $\prec$-predecessor.

(4) There are no cycles in a tree: for any $w \in N$, if $w \sqsubset w$, then $w \neq w$ by definition, which is not possible.

(5) Moreover, for any word, its $\prec$-successor set is well-ordered by the last digits of the elements.

Therefore, $(N, \prec)$ is tree-like. □

Definition 2.9 (Root, leaf, and branch)**.** *Let $T = (N, \prec)$ be a tree or a tree-like structure. The* root *of T is the unique node in N such that it reaches any other nodes via a $\prec$-path. A node $w \in N$ is* leaf *of $(N, \prec)$ iff for any $u \in N$, it does not hold that $w \prec u$. Let $\mathcal{L}^T$ be the set of leaves of T. A* branch *of T is a non-empty and possibly infinite $\prec$-sequence $w_0 \prec w_1 \prec w_2 \prec \dots$ in T.* ⊣

The class of tree-like structures is adequate: given a tree-like structure, it is isomorphic to a tree.

Definition 2.10 (Isomorphism)**.** *Let (X, R) and (Y, S) be relational structures. We say they are* isomorphic*, in notation: $(X, R) \cong (Y, S)$, iff there is a bijective $f : X \to Y$ such that for all $x, x' \in X$, it holds that xRx' iff $f(x)Sf(x')$. The function f is called an* isomorphism*. If X and Y are further decorated by a label set L by labeling functions l_X and l_Y, an isomorphism also satisfies $l_X(x) = l_Y(f(x))$.* ⊣

Proposition 2.11. *Let (T, S) be a tree-like relational structure. Then there is a tree isomorphic to (T, S).*

Proof. We construct a tree by inductively defining the words of elements in T and hence a set of words W^T, which will constitute the nodes of the tree. Formally, we inductively define an isomorphism $\mathbf{w}$.

- $\mathbf{w}(r) = \epsilon$ is in W^T;
- if $\mathbf{w}(t) \in W^T$, then for all s_i such that tSs_i, let $\mathbf{w}(t)i \in W^T$ and $\mathbf{w}(s_i) = \mathbf{w}(t)i$;
- nothing else is in W^T.

$(W^T, \prec)$ is a tree, and we can check that $\mathbf{w}$ is an isomorphism. □

The definition above suffices for standard sequent calculi and those that allow for proofs that are infinitely high (but finitely branching). But for cyclic proof systems, it is not enough. We will work with finite trees with cycles, which we call cyclic trees.

Definition 2.12 (Cyclic tree, companion)**.** *A* cyclic tree *is a tuple $(N, \prec, f)$ such that $(N, \prec)$ is a finite tree, and $f : \mathcal{L}^{(N,\prec)} \hookrightarrow N \setminus \mathcal{L}^{(N,\prec)}$ is a partial function such that $f(w) \sqsubset w$. Given a cyclic tree $(N, \prec, f)$, we can represent it with a relational structure $(N, \prec \cup f)$. For any $w \in Dom(f)$, call $f(w)$ the* companion of w. *An L-*labeled cyclic tree *is a cyclic tree with a labeling function over L.* ⊣

Strictly speaking, cyclic trees with cycles may not be trees proper, as there may be cycles in their relational structures.

Given a cyclic tree, we can unravel it and get a tree-like structure.

Definition 2.13 (Unraveling)**.** *Let $(N, \prec \cup f, l)$ be an L-labeled cyclic tree. Its* unraveling *$(N^u, \prec^u, l^u)$ is defined as follows.*

(1) $N^u \subseteq \mathbb{N}^{**}$ is the set of finite $\prec \cup f$-paths on N; the first element of any such path is ϵ;

(2) $\prec^u$ is the immediate predecessor relation on N^u, i.e. $r \prec^u s$ iff there is a $w \in N$ such that $s = r : w$;

(3) $l^u(s) = l(w_n)$, where $s = [w_0, \ldots, w_n]$, for $w_0, \ldots, w_n \in N$.

For a cyclic tree $(N, \prec \cup f)$ without a labeling function, its unraveling is just $(N^u, \prec^u)$. Define a function last $: N^u \to N$ *that maps r to w, where $r = s : w$ for some $s \in N^u$. Define $e : N \to N^u$ that maps w to s, where* last$(s) = w$ *and for any $s' \sqsubset s$, it holds that* last$(s') \neq w$. *We call e the* natural embedding *from the original cyclic tree into its unraveling.* ⊣

Proposition 2.14. *Let $(N, \prec \cup f, l)$ be an L-labeled cyclic tree. Then $(N^u, \prec^u, l^u)$ is isomorphic to an L-labeled tree.*

Proof. First, note that $(N^u, \prec^u)$ is tree-like. The unique root is the empty sequence of words nil, and nil can reach every other element via the $\prec^u$ relation. By construction, every non-empty sequence has a unique predecessor. For any non-empty sequence s, it is equal to some $s' : w$. If $s = s'' : w$, it must hold that $s' = s''$. It is easy to see that s' and s'' must have the same length. Also, if any digits of s' and s'' are different, they cannot yield one and the same s. Additionally, there is no cycle because each node is longer than its predecessor (if there is any). For any sequence $s \in N^u$, we can well-order its successors by the last element of their last words.

Take a tree $T = (N', \prec') \cong (N^u, \prec^u)$ and consider the isomorphism $f : N^u \to N'$. Define a function $l' : N' \to L$ that maps w to $l^u \circ f^{-1}(w)$. Then $l^u(w) = l' \circ f(w)$. □

Chapter 3

Non-wellfounded systems for Gödel-Löb Logic

The following three chapters together constitute a trilogy that investigates the non-wellfounded proof systems for Gödel-Löb Logic, **GL**.

As made clear in the first chapter, the aim of this trilogy is two-fold. First, we want to introduce a method of proving cut elimination of a certain class of infinitary non-wellfounded systems, using an infinitary system for **GL** as an example. This method is developed by Sierra Miranda, Studer, and Zenger in [SSZ25]. We want to introduce this method because it will also help prove the cut elimination of an infinitary non-wellfounded system for an intuitionistic provability logic that will be introduced in later chapters. Second, we want to provide an alternative, *direct* proof of the cut elimination of a cyclic system for **GL**.

In this chapter, we introduce two-sided non-wellfounded systems for **GL** in section 3.3, one infinitary and the other finitary and cyclic. These systems are based on Shamkanov's one-sided systems in [Sha14]. Since the non-wellfounded systems for **GL** are obtained by allowing for non-wellfounded proofs in a fixed sequent calculus for **K4**, we will first introduce a sequent calculus for **K4** in section 3.2. Before that, further basic definitions are in order.

3.1 Syntax and semantics of the classical language

First, we introduce the syntax of the language that we are going to study.

Definition 3.1 (Classical modal formula)**.** *Let Φ be a countably infinite set of propositional letters. The set of* well-formed modal formulas Form of classical modal language $\mathscr{L}$ *is given by the following rule:*

$$A ::= p \in \Phi \mid \bot \mid \neg A \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A$$

Let $\Diamond A := \neg\Box\neg A$.

We use capital Latin letters like 'A, B, C' as meta-variables ranging over Form. ⊣

Definition 3.2 (Normal modal logic, **K4**, and **GL**)**.** *A normal modal logic is a set of modal formulas that contains all propositional tautologies, the axiom* K: $\Box(A \to B) \to (\Box A \to \Box B)$*, the axiom dual:* $\Diamond A \leftrightarrow \neg\Box\neg A$*, and is closed under generalisation (or necessitation) and modus ponens.*

K4 *is the smallest normal modal logic that contains the axiom 4:* $\Box A \to \Box\Box A$*, and* **GL** *is the smallest normal modal logic that contains Löb's axiom:* $\Box(\Box A \to A) \to \Box A$. ⊣

Definition 3.3 (System **HGL**). *Let* **HGL** *be the Hilbertian proof system for* **GL**. *Its axioms are: propositional tautologies, the* **K** *axiom, the dual axiom, and Löb's axiom. Its inference rules are necessitation and modus ponens. A* derivation/proof *of some formula A in* **HGL** *is a finite sequence of formulas in* Form*: $A_0, \ldots, A_n = A$ such that each A_i is either an axiom, or is obtained by some A_j using necessitation, or is obtained by some A_j and A_k using modus ponens, where $j, k < i$. The* length *of* **HGL** *proofs is defined to be the length of the sequence of formulas.*

We write $\vdash_{\mathbf{HGL}} A$ if there is a proof of A in **HGL**. *For any set of formulas Γ, we write $\Gamma \vdash_{\mathbf{HGL}} A$ if there is a finite subset Γ_0 of Γ such that $\vdash_{\mathbf{HGL}} \bigwedge \Gamma_0 \to A$.* ⊣

Note that **GL** is the set of theorems of **HGL**, i.e., formulas that are provable in **HGL**.

Next, we introduce the semantics of $\mathscr{L}$.

Definition 3.4 (Kripke frames and models). *A* Kripke frame *is a pair $\mathbb{F} = (W, R)$ where $W \neq \emptyset$ and $R \subseteq W \times W$ is a relation. A* Kripke model *is a triple $\mathbb{M} = (W, R, V)$ where (W, R) is a Kripke frame and V is function from Φ to $\wp(W)$. We call V the valuation function of a given Kripke model. Given a Kripke model $\mathbb{M} = (W, R, V)$ and an $x \in W$, the pair $(\mathbb{M}, x)$ is a* pointed model*, which we usually omit parentheses. Given a frame $\mathbb{F} = (W, R)$ and a valuation V defined for $\mathbb{F}$, we also use '$(\mathbb{F}, V)$' to denote the model (W, R, V).* ⊣

Definition 3.5 (Transitive and conversely well-founded Kripke frames/models). *A Kripke frame (W, R) (or a Kripke model (W, R, V)) is* transitive *iff R is transitive; it is* conversely well-founded *iff there is no $\{x_i\}_{i \in \omega}$ in W such that $x_i R x_{i+1}$ for all $i \in \omega$, i.e. if there are no infinite ascending R-paths.* ⊣

Definition 3.6 (Truth of modal formula). *The* truth *or* satisfaction *of a modal formula at some point of a Kripke model or at some pointed model is defined as follows. Let $\mathbb{M} = (W, R, V)$ be a Kripke model and $x \in W$. For any formula $A, B \in$ Form:*

- $\mathbb{M}, x \Vdash p$ *iff* $x \in V(p)$;
- $\mathbb{M}, x \Vdash \bot$, *never;*
- $\mathbb{M}, x \Vdash \neg A$ *iff* $\mathbb{M}, x \nVdash A$;
- $\mathbb{M}, x \Vdash A \wedge B$ *iff* $\mathbb{M}, x \Vdash A$ *and* $\mathbb{M}, x \Vdash B$;
- $\mathbb{M}, x \Vdash A \vee B$ *iff* $\mathbb{M}, x \Vdash A$ *or* $\mathbb{M}, x \Vdash B$;
- $\mathbb{M}, x \Vdash A \to B$ *iff* $\mathbb{M}, x \nVdash A$ *or* $\mathbb{M}, x \Vdash B$;
- $\mathbb{M}, x \Vdash \Box A$ *iff for all* $y \in W$ *such that* xRy, *it holds that* $\mathbb{M}, y \Vdash A$.

For $\Diamond$, we can verify that for any $A \in$ Form, $\mathbb{M}, x \Vdash \Diamond A$ iff there is a $y \in W$ such that xRy and $\mathbb{M}, x \Vdash A$. ⊣

Definition 3.7 (Validity and logic of frames). *Given a kripke frame $\mathbb{F} = (W, R)$ and an $x \in W$, a formula A is* valid at x *iff $(\mathbb{F}, V), x \Vdash A$, for any V. Formally, we write $\mathbb{F}, x \Vdash A$. It is* valid in $\mathbb{F}$, *i.e. $\mathbb{F} \Vdash A$, iff it is valid at all $x \in W$. If $\mathfrak{F}$ is a class of frames, A is* valid in $\mathfrak{F}$ *iff for all $\mathbb{F}$ in $\mathfrak{F}$, $\mathbb{F} \Vdash A$. Given a frame $\mathbb{F}$, we define* the logic of $\mathbb{F}$, *i.e. $Log(\mathbb{F})$ as $\{A \in \text{Form} : \mathbb{F} \Vdash A\}$. Similarly, given a frame class $\mathfrak{F}$, we define its logic as $Log(\mathfrak{F}) = \{A \in \text{Form} : \text{for all } \mathbb{F} \in \mathfrak{F}, \mathbb{F} \Vdash A\}$.* ⊣

Definition 3.8 (Soundness, completeness, and Kripke completeness). *Given a normal modal logic* **L** *and a frame class $\mathfrak{F}$, we say* **L** *is* sound with respect to $\mathfrak{F}$ *if $L \subseteq Log(\mathfrak{F})$, and* **L** *is* complete with respect to $\mathfrak{F}$ *if $L \supseteq Log(\mathfrak{F})$.* **L** *is* Kripke complete with respect to $\mathfrak{F}$ *if $L = Log(\mathfrak{F})$.* ⊣

The following are standard Kripke completeness results regarding **K4** and **GL**.

Proposition 3.9. **K4** $= Log(\mathfrak{F}_{tran})$, *where* $\mathfrak{F}_{tran}$ *is the class of transitive frames.*

Proposition 3.10. **GL** $= Log(\mathfrak{F}_{tcw})$, *where* $\mathfrak{F}_{tcw}$ *is the class of transitive and conversely well-founded frames.*

Finally, we introduce definitions regarding sequents before formally introducing sequent calculus.

Definition 3.11 (Multiset, multiplicity)**.** *Given a set* A*, a* multiset *on* A *is a function* $\sigma : A \to \omega$*. For any* $a \in A$*,* $\sigma(a)$ *is the* multiplicity *of* a*. If* σ *and* τ *are multisets on* A*, their* union σ, τ *is the function given by the map* $a \mapsto \sigma(a) + \tau(a)$*. We use ' , ' to mean multiset union.* ⊣

Remark 3.12. *In accordance with the literature, we will often use uppercase Greek letters like* Γ *and* Δ *to denote multisets on* Form *in the context of sequent calculus, rather than lowercase letters. Nevertheless, when the context makes it necessary to talk about the multiplicity of some object* x *'in' a multiset, say* Γ*, we will use the notation* $\#_\Gamma(x)$ *to denote the multiplicity of* x *in* Γ*.*

Definition 3.13 (Sequent)**.** *A* sequent *is of the form* $\Gamma \Rightarrow \Delta$*, where* Γ *and* Δ *are finite multisets of formulas in* Form*. We let finite multisets be functions from* Form *to* ω*, where only finitely many inputs have a positive value. Let the set of sequents be* Seq*.* ⊣

We also define the standard interpretation of sequents.

Definition 3.14 (Standard interpretation)**.** *For any sequent* $\Gamma \Rightarrow \Delta$*, its* standard interpretation $(\Gamma \Rightarrow \Delta)^{\mathcal{I}}$ *is defined as*

$$(\Gamma \Rightarrow \Delta)^{\mathcal{I}} := \bigwedge \Gamma \to \bigvee \Delta.$$

If $\Gamma = \emptyset$*, let* $\bigvee \Gamma := \bot$*. Additionally, let* $\bigwedge \Gamma := \top$ *when* $\Gamma = \emptyset$*. We say a sequent is satisfied at some pointed model or valid in some frame/frame class if its standard interpretation is. Then we may write* $\mathbb{M}, x \Vdash \Gamma \Rightarrow \Delta$*,* $\mathbb{F} \Vdash \Gamma \Rightarrow \Delta$*, etc.* ⊣

Now, let us move to the sequent calculus for **K4**.

3.2 A sequent calculus for K4

We first make clear how we represent rules and rule instances.

Definition 3.15 (Rule instance/application and rule)**.** *A* rule instance *or* rule application *is a pair* (P, C)*, where*

(1) P *is a finite (possibly empty) sequence of sequents, which are premises of the instance;*

(2) C *is a sequent, which is the conclusion of the instance.*

Any sequence of formulas of length one is a rule instance without premise. A rule r *is a non-empty set of rule instances.* ⊣

Definition 3.16 (Sequent calculus)**.** *A* sequent calculus *is a set of rules.* ⊣

Remark 3.17. *Rules and sequent calculi are defined in a language-sensitive way, as they are defined in terms of sequents. We are now working with sequents defined in terms of multisets, but later on we will work with sequents of a different kind. We assume that these definitions will naturally carry over.*

Definition 3.18 (**G3cp**$^{\mathrm{cut}}$ and **G3K4**$^{\mathrm{cut}}$)**.** *The sequent calculus* **G3cp**$^{\mathrm{cut}}$ *is given by the following set of rules.*

$$\mathsf{init}\ \frac{}{p, \Gamma \Rightarrow \Delta, p} \qquad \bot_{\mathsf{L}}\ \frac{}{\bot, \Gamma \Rightarrow \Delta} \qquad \neg_{\mathsf{L}}\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta} \qquad \neg_{\mathsf{R}}\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta} \qquad \wedge_{\mathsf{L}}\ \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}$$

$$\wedge_{\mathsf{R}}\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B} \qquad \vee_{\mathsf{L}}\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee_{\mathsf{R}}\ \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}$$

$$\rightarrow_{\mathsf{L}}\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} \qquad \rightarrow_{\mathsf{R}}\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B} \qquad \mathsf{cut}\ \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}$$

The formulas explicitly written out in the rules are designated formulas*. The designated formula in the conclusion is called the* principal formula *of the rule (application).* Γ *and* Δ *are the* contexts *for each rule. The formula* A *in the rule* cut *is called the* cut formula*. Let* **G3cp** *be the system obtained by removing* cut *from* **G3cp**$^{\mathrm{cut}}$*.*

The sequent calculus **G3K4**$^{\mathrm{cut}}$ *is defined as* **G3cp**$^{\mathrm{cut}}$ *plus the following rules, where for any set of formula* Γ*,* $\Box\Gamma$ *is defined to be the multiset* γ *such that for any* A*, if* $A = \Box B$ *for some* $B \in \Gamma$*, then* $\gamma(A) = \#_{\Gamma}(B)$*; otherwise,* $\gamma(A) = 0$*. Define also* $\boxdot\Gamma := \Gamma, \Box\Gamma$*.*

$$\mathsf{w}_{\mathsf{R}}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \Sigma} \qquad \mathsf{w}_{\mathsf{L}}\ \frac{\Gamma \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta} \qquad \mathsf{K4}\ \frac{\boxdot\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}$$

We require Σ *to be non-empty in* w_{L} *and* w_{R}*. For* w_{L} *and* w_{R}*, formulas in* Σ *are principal ones. For* K4*, every formula in the conclusion is principal.*

Call sequents that have the form of $p, \Gamma \Rightarrow \Delta, p$ *or* $\bot, \Gamma \Rightarrow \Delta$ *'axioms'. We call* K4 *'the modal rule', call* cut*,* w_{L}*, and* w_{R} *'structural rules', and the rest 'logical rules'. Also, let* **G3K4** *be the system obtained by removing* cut *from* **G3K4**$^{\mathrm{cut}}$*.* ⊣

Note that in **G3K4**$^{\mathrm{cut}}$, weakening rules are primitive. We can add context to the conclusion of the rule K4 to make weakening rules admissible, but the current design may yield a simpler cut reduction procedure.

Definition 3.19 (Proof/derivation)**.** *Let* $C \in \{\mathbf{G3cp}, \mathbf{G3cp}^{\mathrm{cut}}, \mathbf{G3K4}, \mathbf{G3K4}^{\mathrm{cut}}\}$*. A* C-proof *or* C-derivation *is a labeled tree* $T = (N, \prec, l^C)$ *such that*

(1) $(N, \prec, l^C)$ *is a finite* $(C \times \mathrm{Seq})$*-labeled tree, which means* l^C *is a function from* N *to* $C \times \mathrm{Seq}$*;*

(2) for any $w \in N$*:*

- *(a) if* w *is a leaf, then* $(\mathrm{seq}_T(w))$ *is an instance of* $\mathrm{rule}_T(w)$*, where* $\mathrm{rule}_T = p_0 \circ l^C$ *and* $\mathrm{seq}_T = p_1 \circ l^C$ *for projection functions* p_0 *and* p_1*;*
- *(b) otherwise, for* $w0, \ldots, wn$*, the* $\prec$*-successors of* w*, the following sequence*

$$(\mathrm{seq}_T(w_0), \ldots, \mathrm{seq}_T(w_n), \mathrm{seq}_T(w))$$

is an instance of $\mathrm{rule}_T(w)$*.*

We might omit the superscript of 'l^C*' if the context permits. We say that a sequent* S *is provable in* C *(in symbols:* $\vdash_C S$*) iff there is a* C*-proof whose root/end sequent is* S*.* ⊣

Definition 3.20 (Height of proof)**.** *Let $\mathcal{D}$ be a proof. The* height *of $\mathcal{D}$ is the length of the longest branch(es) in $\mathcal{D}$, which is always a positive natural number.* ⊣

In **G3cp**$^{\text{cut}}$, $\mathsf{w_L}$, $\mathsf{w_R}$ are admissible, and cut is eliminable.

Proposition 3.21 (Admissibility of weakening for **G3cp**$^{\text{cut}}$)**.** $\mathsf{w_L}$ *and* $\mathsf{w_R}$ *are admissible in* **G3cp**$^{\text{cut}}$.

Proof. By straightforward induction on the height of proofs. For a detailed proof, we direct the reader to (for example) [TS00]. □

Theorem 3.22 (Cut elimination for **G3cp**$^{\text{cut}}$)**.** *For any sequent $\Gamma \Rightarrow \Delta$, if $\vdash_{\mathbf{G3cp}^{\text{cut}}} \Gamma \Rightarrow \Delta$, then $\vdash_{\mathbf{G3cp}} \Gamma \Rightarrow \Delta$.*

Proof. This is a standard result. We again direct the reader to [TS00] for a proof. □

Cut elimination holds for **G3K4**$^{\text{cut}}$. A very brief proof sketch can be found in [Val83], but the system in [Val83] uses set sequents, not multiset sequents that we are now endorsing. Here, we examine the proof in more detail as the cut elimination of the cyclic proof system for **GL** relies on the cut elimination of **G3K4**$^{\text{cut}}$. We follow the sketch of [Val83] and first define two measures of proofs. The first is the cut degree of a proof, and the second its cut depth.

Definition 3.23 (Weight of formula and cut degree)**.** *For any formula $A \in \mathsf{Form}$, we define* the weight of *A, $w(A)$, as follows:*

- $w(\bot) = w(p) = 1$;
- $w(A \circ B) = w(A) + w(B) + 1$, *for* $\circ \in \{\wedge, \vee, \rightarrow\}$;
- $w(\star A) = w(A) + 1$, *for* $\star \in \{\neg, \Box\}$

For any sequent $S = \Gamma \Rightarrow \Delta$, let $w(S) = \sum[(\#_\Gamma(A) + \#_\Delta(A)) \cdot w(A)]$ for any formula A such that $\#_\Gamma(A) + \#_\Delta(A) > 0$. Given a proof $\mathcal{D}$ and an instance of cut *thereof, the* cut degree *of this instance is the weight of the cut formula of this instance. Let the* cut degree of *$\mathcal{D}$, $dg(\mathcal{D})$, be the maximal cut degree of instances of* cut *in $\mathcal{D}$. We let $dg(\mathcal{D}) = 0$ iff there is no* cut *application in $\mathcal{D}$.* ⊣

Definition 3.24 (Cut depth)**.** *Given a proof $\mathcal{D}$ and an application of* cut *in $\mathcal{D}$, the* left depth *of this application of* cut *in $\mathcal{D}$ is the maximal number of consecutive sequents, starting from the left premise, that contain the cut formula in their succedents. Accordingly, the* right depth *of this application of* cut *in $\mathcal{D}$ is the maximal number of consecutive sequents, starting from the right premise, that contain the cut formula in their antecedents. The* depth *of this* cut *application in $\mathcal{D}$ is the sum of its left and right depth. For any* cut *application, its cut depth is at least 2. We use $dp(\mathcal{D})$ to denote the maximal cut depth of a proof. For any proof $\mathcal{D}$, we set $dp(\mathcal{D}) = 0$ iff $\mathcal{D}$ is cut-free.* ⊣

To prove cut elimination, we need the admissibility of contraction. We prove this via invertibility.

Definition 3.25 (Invertibility)**.** *A rule is* invertible *in a sequent calculus C iff for any of its instances, if its conclusion is provable in C, then so is/are its premise(s).*

A rule is height-preserving invertible *in C iff for any instance of the rule, if the conclusion has a C-proof of height at most n, then each premise has a C-proof of height at most n.*

A rule is cut-freeness-preserving invertible *in C iff for any instance of the rule, if its conclusion has a* cut*-free C-proof, then any of its premise(s) has a* cut*-free C-proof.* ⊣

Proposition 3.26 (**G3K4**$^{\mathrm{cut}}$ invertibility)**.** *All the rules of* **G3K4**$^{\mathrm{cut}}$*, except weakening rules,* K4 *and* cut*, are height-preserving and cut-freeness-preserving invertible.*

Proof. By standard induction on the height of proofs. Consider the one-premise rules. Suppose their conclusions are derivable. For each of these conclusions, if the designated formula is principal, then trivially the corresponding premise is derivable. Otherwise, apply the IH to the premise, and then apply the original rule at the bottom. The argument for two-premise ones is similar. □

Definition 3.27 (Admissibility)**.** *A rule is* admissible *in C iff for any of its instances, if all its premises are provable in C, then so is its conclusion.* Height-preserving *and* cut-freeness-preserving admissibility *are defined in analogy with Definition 3.25.* ⊣

Proposition 3.28 (Contraction for **G3K4**$^{\mathrm{cut}}$)**.** *The following rules are height-preserving and cut-freeness-preserving admissible in* **G3K4**$^{\mathrm{cut}}$*.*

$$\mathrm{ctr_L}\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad\qquad \mathrm{ctr_R}\ \frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$$

Proof. By (simultaneous) induction on the height of proofs. The base cases are trivial, so we omit them and directly consider the inductive step.

- $A = \neg B$ for some B. Then by height-preserving invertibility of $\neg_{\mathsf{L}}$ and $\neg_{\mathsf{R}}$, $\Gamma \Rightarrow \Delta, B, B$ and $B, B, \Gamma \Rightarrow \Delta$ are height-preserving derivable. Apply the IH and then $\neg_{\mathsf{L}}$ and $\neg_{\mathsf{R}}$, we get desired proofs of $\neg B, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, \neg B$.
- $A = B \wedge C$ for some B and C. By invertibility, we have height-preserving derivations of $B, C, B, C, \Gamma \Rightarrow \Delta$, $\Gamma \Rightarrow \Delta, B, B$, and $\Gamma \Rightarrow \Delta, C, C$. Apply the IH, and then $\wedge_{\mathsf{L}}$ and $\wedge_{\mathsf{R}}$.
- The cases for $\rightarrow$ and $\vee$ are similar, using invertibility. So we consider the modal case.
- $A = \Box B$ for some B. Consider the case of $\mathrm{ctr_R}$. The last rule of any derivation of $\Gamma \Rightarrow \Delta, \Box B, \Box B$ cannot be K4. Then apply the IH directly to the premise(s). For $\mathrm{ctr_L}$, we consider two cases. Suppose we are given a proof $\mathcal{D}$ of $\Box B, \Box B, \Gamma \Rightarrow \Delta$. If both designated $\Box B$ occurrences are not principal in the last rule of $\mathcal{D}$, then apply the IH to the premise(s), for they are of lower heights. If any one of the designated $\Box B$ occurrences is principal, then apply the IH to the premise twice, first on B, and then on $\Box B$.

Then we have covered all the cases of the inductive step. □

Lemma 3.29 (Cut reduction for **G3K4**$^{\mathrm{cut}}$)**.** *For any sequent $\Gamma \Rightarrow \Delta$, if there is a* **G3K4**$^{\mathrm{cut}}$ *proof $\mathcal{D}$ of it whose rule application at the root is the unique* cut *application in $\mathcal{D}$, then there is a* **G3K4** *proof of $\Gamma \Rightarrow \Delta$.*

Proof. Suppose that there is such a **G3K4**$^{\mathrm{cut}}$ proof $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$. It has the following shape, where $\mathcal{D}_0$ and $\mathcal{D}_1$ are cut-free.

$$\mathrm{cut}\ \frac{\overset{\mathcal{D}_0}{\triangledown}\quad\overset{\mathcal{D}_1}{\triangledown}}{\Gamma_0 \Rightarrow \Delta_0, A \qquad A, \Gamma_1 \Rightarrow \Delta_1}}{\Gamma \Rightarrow \Delta}$$

Let the bottom-most rules of $\mathcal{D}_0$ and $\mathcal{D}_1$ be r_0 and r_1, respectively. We show there is a **G3K4** proof of $\Gamma \Rightarrow \Delta$ by induction on $(dg(\mathcal{D}), dp(\mathcal{D}))$. The base case where $(dg(\mathcal{D}), dp(\mathcal{D})) = (0, 0)$ is trivial.

For the inductive step, fix a pair (n, m), suppose that for any (n', m') that if $(n', m') <_{lex} (n, m)$, then for any **G3K4**$^{\mathsf{cut}}$ proof measured by (n', m'), the claim holds. The relation $<_{lex}$ is the lexicographic order on ω^2. We make the following case distinctions.

- Case 1. None of r_0 and r_1 has premise.
- Exactly one of r_0 and r_1 has no premise.

  Case 2. $\mathsf{r}_0 \in \{\mathsf{init}, \bot_{\mathsf{L}}\}, \mathsf{r}_1 \notin \{\mathsf{init}, \bot_{\mathsf{L}}\}$;

  Case 3. $\mathsf{r}_0 \notin \{\mathsf{init}, \bot_{\mathsf{L}}\}, \mathsf{r}_1 \in \{\mathsf{init}, \bot_{\mathsf{L}}\}$.
- Both of r_0 and r_1 have some premise.

  Case 4. A is not principal in r_0;

  Case 5. A is not principal in r_1;

  Case 6. A is principal in both r_0 and r_1.

Case 1. None of r_0 and r_1 has premise. If A is principal in both r_0 and r_1, then A is in Γ_0, and either A is in Δ_1 or $A = \bot$. Either way, $\Gamma \Rightarrow \Delta$ is an axiom. If A is not principal in r_0, then $\Gamma_0 \Rightarrow \Delta_0$ and hence $\Gamma \Rightarrow \Delta$ are axioms. If A is not principal in r_1, then $\Gamma_1 \Rightarrow \Delta_1$ and hence $\Gamma \Rightarrow \Delta$ is an axiom.

Case 2. Exactly one of r_0 and r_1 has no premise, $\mathsf{r}_0 \in \{\mathsf{init}, \bot_{\mathsf{L}}\}$, and $\mathsf{r}_1 \notin \{\mathsf{init}, \bot_{\mathsf{L}}\}$. If $\mathsf{r}_0 = \mathsf{init}$ and A is principal in r_0, apply weakening rules to $A, \Gamma_1 \Rightarrow \Delta_1$ if necessary. If $\mathsf{r}_0 = \mathsf{init}$ and A is not principal, then $\Gamma \Rightarrow \Delta$ is an axiom. If $\mathsf{r}_0 = \bot_{\mathsf{L}}$, then A is not principal in r_0 and $\bot \in \Gamma_0$. Consequently, $\Gamma \Rightarrow \Delta$ is an axiom.

Case 3. Exactly one of r_0 and r_1 has no premise, $\mathsf{r}_0 \notin \{\mathsf{init}, \bot_{\mathsf{L}}\}$, and $\mathsf{r}_1 \in \{\mathsf{init}, \bot_{\mathsf{L}}\}$. If A is non-principal in r_1, then $\Gamma \Rightarrow \Delta$ is an axiom. Otherwise, if $\mathsf{r}_1 = \mathsf{init}$, then A is in Δ_1. Apply weakening rules under $\mathcal{D}_0$ to get a desired proof if needed. If $\mathsf{r}_1 = \bot_{\mathsf{L}}$, then $A = \bot$. Here we use the fact that for any sequent $\Gamma \Rightarrow \Delta, \bot$, if there is a **G3K4** proof of it, then there is also a **G3K4** proof of $\Gamma \Rightarrow \Delta$. This can be shown by induction on the height of **G3K4** proofs.

Note that in the above cases, the IH is not used.

Case 4. Both r_0 and r_1 have some premise and A is not principal in r_0. If r_0 has exactly one premise, then by the shape of K4 and our assumption that A is not principal in r_0, we have that $\mathsf{r}_0 \neq \mathsf{K4}$. $\mathcal{D}_0$ has the following shape.

$$\mathsf{r}_0 \dfrac{\begin{array}{c}\mathcal{D}_0' \\ \Gamma_0' \Rightarrow \Delta_0', A\end{array}}{\Gamma_0 \Rightarrow \Delta_0, A}$$

Pushing the cut up, we get the following proof. We mark the instance of cut by the pair of its degree and depth, e.g. '$\mathsf{cut}^{n,m}$'.

$$\mathsf{r}_0 \dfrac{\mathsf{IH}(\mathsf{cut}^{n,m-1}) \dfrac{\begin{array}{c}\mathcal{D}_0' \\ \Gamma_0' \Rightarrow \Delta_0', A\end{array} \quad \begin{array}{c}\mathcal{D}_1 \\ A, \Gamma_1 \Rightarrow \Delta_1\end{array}}{\Gamma_0', \Gamma_1 \Rightarrow \Delta_0', \Delta_1}}{\Gamma_0, \Gamma_1 \Rightarrow \Gamma_0, \Gamma_1}$$

If r_0 is two-premise, then $\mathcal{D}_0$ has the following shape.

$$\mathsf{r}_0\ \dfrac{\mathcal{D}_0^0 :: \Gamma_0^0 \Rightarrow \Delta_0^0, A \qquad \mathcal{D}_0^1 :: \Gamma_0^1 \Rightarrow \Delta_0^1, A}{\Gamma_0 \Rightarrow \Delta_0, A}$$

We again push the cut up and obtain the following tree.

$$\dfrac{\dfrac{\mathcal{D}_0^0 :: \Gamma_0^0 \Rightarrow \Delta_0^0, A \qquad \mathcal{D}_1 :: A, \Gamma_1 \Rightarrow \Delta_1}{\mathsf{IH}(\mathsf{cut}^{n,m-1}) :\ \Gamma_0^0, \Gamma_1 \Rightarrow \Delta_0^0, \Delta_1} \qquad \dfrac{\mathcal{D}_0^1 :: \Gamma_0^1 \Rightarrow \Delta_0^1, A \qquad \mathcal{D}_1 :: A, \Gamma_1 \Rightarrow \Delta_1}{\mathsf{IH}(\mathsf{cut}^{n,m-1}) :\ \Gamma_0^1, \Gamma_1 \Rightarrow \Delta_0^1, \Delta_1}}{\mathsf{r}_0\ \dfrac{\Gamma_0, \Gamma_1, \Gamma_1 \Rightarrow \Delta_0, \Delta_1, \Delta_1}{\mathsf{ctr}_\mathsf{L}, \mathsf{ctr}_\mathsf{R} :\ \Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1}}$$

Note that $\mathsf{r}_0 \neq \mathsf{cut}$ by assumption.

Case 5. Both of r_0 and r_1 have some premise, and A is not principal in r_1. This case is similar to Case 4. Depending on the number of premises of r_1, $\mathcal{D}_1$ has one of the following shapes.

$$\mathsf{r}_1\ \dfrac{\mathcal{D}_1' :: A, \Gamma_1' \Rightarrow \Delta_1'}{A, \Gamma_1 \Rightarrow \Delta_1} \qquad \mathsf{r}_1\ \dfrac{\mathcal{D}_1^0 :: A, \Gamma_1^0 \Rightarrow \Delta_1^0 \qquad \mathcal{D}_1^1 :: A, \Gamma_1^1 \Rightarrow \Delta_0^1}{A, \Gamma_1 \Rightarrow \Delta_1}$$

We then have the following proof trees in these cases.

$$\dfrac{\mathcal{D}_0 :: \Gamma_0 \Rightarrow \Delta_0, A \qquad \mathcal{D}_1' :: A, \Gamma_1' \Rightarrow \Delta_1'}{\mathsf{IH}(\mathsf{cut}^{n,m-1}) :\ \Gamma_0, \Gamma_1' \Rightarrow \Delta_0, \Delta_1'}\ \mathsf{r}_1$$
$$\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1$$

$$\dfrac{\dfrac{\mathcal{D}_0 :: \Gamma_0 \Rightarrow \Delta_0, A \qquad \mathcal{D}_1^0 :: A, \Gamma_1^0 \Rightarrow \Delta_1^0}{\mathsf{IH}(\mathsf{cut}^{n,m-1}) :\ \Gamma_0, \Gamma_1^0 \Rightarrow \Delta_0, \Delta_1^0} \qquad \dfrac{\mathcal{D}_0 :: \Gamma_0 \Rightarrow \Delta_0, A \qquad \mathcal{D}_1^1 :: A, \Gamma_1^1 \Rightarrow \Delta_1^1}{\mathsf{IH}(\mathsf{cut}^{n,m-1}) :\ \Gamma_0, \Gamma_1^1 \Rightarrow \Delta_0, \Delta_1^1}}{\mathsf{r}_1\ \dfrac{}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1}}$$

Note that $\mathsf{r}_1 \neq \mathsf{cut}$.

Case 6. Both of r_0 and r_1 have some premise, and A is principal in both. We consider three subcases: exactly one of r_0 and r_1 is K4, both of them are K4, and neither is K4.

First, assume exactly one of r_0 and r_1 is K4. Then, $A = \Box B$, for some B. Additionally, the rule that is distinct from K4 must be one of the weakening rules by the shape of the principal formula. We can then directly apply

weakening rules to get a desired cut-free proof of $\Gamma \Rightarrow \Delta$.

Second, both r_0 and r_1 are K4. In this case $\mathcal{D}_0$ and $\mathcal{D}_1$ are of the following forms.

$$\mathsf{K4}\,\dfrac{\begin{array}{c}\mathcal{D}_0'\\ \Sigma, \Box\Sigma \Rightarrow B\end{array}}{\Box\Sigma \Rightarrow \Box B} \qquad \mathsf{K4}\,\dfrac{\begin{array}{c}\mathcal{D}_1'\\ B, \Theta, \Box\Theta, \Box B \Rightarrow C\end{array}}{\Box B, \Box\Theta \Rightarrow \Box C}$$

Let $\Gamma_0 = \Box\Sigma$; $\Delta_0 = \emptyset$; $A = \Box B$; $\Gamma_1 = \Box\Theta$; and $\Delta_1 = \Box C$. We first apply cut on $\Box B$ and then on B. We use m' for another sum of cut depths. Since the cut degree already decreases to $n-1$, we do not care about the specific value of m'.

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_0'\\ \Sigma, \Box\Sigma \Rightarrow B\end{array} \quad \mathsf{IH}(\mathsf{cut}^{n-1,m'}) \dfrac{\begin{array}{c}\mathcal{D}_0\\ \Box\Sigma \Rightarrow \Box B\end{array} \quad \mathsf{IH}(\mathsf{cut}^{n,m-1}) \dfrac{\begin{array}{c}\mathcal{D}_1'\\ B, \Theta, \Box B, \Box\Theta \Rightarrow C\end{array}}{\Box\Sigma, B, \Theta, \Box\Theta \Rightarrow C}}{\Sigma, \Theta, \Box\Sigma, \Box\Sigma, \Box\Theta \Rightarrow C}}{\mathsf{ctr_L}\ \dfrac{}{\boxdot\Sigma, \boxdot\Theta \Rightarrow C}}}{\mathsf{K4}\,\dfrac{}{\Box\Sigma, \Box\Theta \Rightarrow \Box C}}$$

Third, r_0 and r_1 are not K4. In this case, r_0 and r_1 are propositional or structural. If one of r_0 and r_1 is a weakening rule, since A is principal, applying weakening rules yields a cut-free proof of $\Gamma \Rightarrow \Delta$.

If r_0 is one-premise, we have the following cases.

- $\mathsf{r}_0 = \neg_\mathsf{R}, \mathsf{r}_1 = \neg_\mathsf{L}$. Then we have the following proofs, where $A = \neg B$ for some B.

$$\neg_\mathsf{R}\,\dfrac{\begin{array}{c}\mathcal{D}_0'\\ \Gamma_0, B \Rightarrow \Delta_0\end{array}}{\Gamma_0 \Rightarrow \Delta_0, \neg B} \qquad \neg_\mathsf{L}\,\dfrac{\begin{array}{c}\mathcal{D}_1'\\ \Gamma_1 \Rightarrow \Delta_1, B\end{array}}{\neg B, \Gamma_1 \Rightarrow \Delta_1}$$

  Then we have a cut-free proof of $\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1$.

$$\mathsf{IH}(\mathsf{cut}^{n-1,m'})\,\dfrac{\begin{array}{c}\mathcal{D}_1'\\ \Gamma_1 \Rightarrow \Delta_1, B\end{array} \quad \begin{array}{c}\mathcal{D}_0'\\ \Gamma_0, B \Rightarrow \Delta_0\end{array}}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1}$$

- $\mathsf{r}_0 = \vee_\mathsf{R}, \mathsf{r}_1 = \vee_\mathsf{L}$, $A = B \vee C$, for some B, C.

$$\vee_\mathsf{R}\,\dfrac{\begin{array}{c}\mathcal{D}_0'\\ \Gamma_0 \Rightarrow \Delta_0, B, C\end{array}}{\Gamma_0 \Rightarrow \Delta_0, B \vee C} \qquad \vee_\mathsf{L}\,\dfrac{\begin{array}{c}\mathcal{D}_1^0\\ B, \Gamma_1 \Rightarrow \Delta_1\end{array} \quad \begin{array}{c}\mathcal{D}_1^1\\ C, \Gamma_1 \Rightarrow \Delta_1\end{array}}{B \vee C, \Gamma_1 \Rightarrow \Delta_1}$$

'Construct' the following proof.

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_0'}{\Gamma_0 \Rightarrow \Delta_0, B, C} \quad \dfrac{\mathcal{D}_1^0}{B, \Gamma_1 \Rightarrow \Delta_1}}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1, C}\,\mathsf{IH}(\mathsf{cut}^{n-1,m'}) \quad \dfrac{\mathcal{D}_1^1}{C, \Gamma_1 \Rightarrow \Delta_1}}{\Gamma_0, \Gamma_1, \Gamma_1 \Rightarrow \Delta_0, \Delta_1, \Delta_1}\,\mathsf{IH}(\mathsf{cut}^{n-1,m''})}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1}\,\mathsf{ctr_L}, \mathsf{ctr_R}$$

- $\mathsf{r}_0 = \rightarrow_{\mathsf{R}}, \mathsf{r}_1 = \rightarrow_{\mathsf{L}}$, $A = B \rightarrow C$, for some B, C.

$$\dfrac{\dfrac{\mathcal{D}_0'}{\Gamma_0, B \Rightarrow \Delta_0, C}}{\Gamma_0 \Rightarrow \Delta_0, B \rightarrow C}\,\rightarrow_{\mathsf{R}} \quad \dfrac{\dfrac{\mathcal{D}_1^0}{\Gamma_1 \Rightarrow \Delta_1, B} \quad \dfrac{\mathcal{D}_1^1}{C, \Gamma_1 \Rightarrow \Delta_1}}{B \rightarrow C, \Gamma_1 \Rightarrow \Delta_1}\,\rightarrow_{\mathsf{L}}$$

Then, we have the following proof.

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1^0}{\Gamma_1 \Rightarrow \Delta_1, B} \quad \dfrac{\mathcal{D}_0'}{\Gamma_0, B \Rightarrow \Delta_0, C}}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1, C}\,\mathsf{IH}(\mathsf{cut}^{n-1,m'}) \quad \dfrac{\mathcal{D}_1^1}{C, \Gamma_1 \Rightarrow \Delta_1}}{\Gamma_0, \Gamma_1, \Gamma_1 \Rightarrow \Delta_0, \Delta_1, \Delta_1}\,\mathsf{IH}(\mathsf{cut}^{n-1,m''})}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1}\,\mathsf{ctr_L}, \mathsf{ctr_R}$$

If r_0 is two-premise, then $\mathsf{r}_0 = \wedge_{\mathsf{R}}, \mathsf{r}_1 = \wedge_{\mathsf{L}}$, and $A = B \wedge C$ for some B, C.

$$\dfrac{\dfrac{\mathcal{D}_0^0}{\Gamma_0 \Rightarrow \Delta_0, B} \quad \dfrac{\mathcal{D}_0^1}{\Gamma_0 \Rightarrow \Delta_0, C}}{\Gamma_0 \Rightarrow \Delta_0, B \wedge C,}\,\wedge_{\mathsf{L}} \quad \dfrac{\dfrac{\mathcal{D}_1'}{B, C, \Gamma_1 \Rightarrow \Delta_1}}{B \wedge C, \Gamma_1 \Rightarrow \Delta_1}\,\wedge_{\mathsf{R}}$$

We have the following proof.

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_0^1}{\Gamma_0 \Rightarrow \Delta_0, C} \quad \dfrac{\dfrac{\mathcal{D}_0^0}{\Gamma_0 \Rightarrow \Delta_0, B} \quad \dfrac{\mathcal{D}_1'}{B, C, \Gamma_1 \Rightarrow \Delta_1}}{C, \Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1}\,\mathsf{IH}(\mathsf{cut}^{n-1,m'})}{\Gamma_0, \Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_0, \Delta_1}\,\mathsf{IH}(\mathsf{cut}^{n-1,m''})}{\Gamma_0, \Gamma_1 \Rightarrow \Delta_0, \Delta_1}\,\mathsf{ctr_L}, \mathsf{ctr_R}$$

We have thus covered each case in the inductive step, and hence have completed the proof. □

Cut elimination of **G3K4**$^{\mathsf{cut}}$ then follows immediately.

Theorem 3.30 (Cut elimination for $\mathbf{G3K4}^{\mathrm{cut}}$)**.** *For any sequent* $\Gamma \Rightarrow \Delta$*, if* $\vdash_{\mathbf{G3K4}^{\mathrm{cut}}} \Gamma \Rightarrow \Delta$*, then* $\vdash_{\mathbf{G3K4}} \Gamma \Rightarrow \Delta$*.*

Proof. Given a $\mathbf{G3K4}^{\mathrm{cut}}$ proof $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$, we eliminate cut starting from the left-most and top-most cut application. Using the cut reduction procedure for $\mathbf{G3K4}^{\mathrm{cut}}$ (Lemma 3.29), we can replace the subtree rooted in this application with a cut-free proof of its root sequent. Since proofs are finite, $\mathcal{D}$ has finite instances of cut. Hence, this reduction procedure terminates in finite steps. □

In the next section, we present the non-wellfounded systems for **GL**.

3.3 Non-wellfounded proof systems for GL

In [Sha14], Shamkanov provided two one-sided non-wellfounded proof systems for **GL** where the cut rule is admissible. The systems are $\mathbf{GL}^{\infty}$ and $\mathbf{GL}^{\mathbf{circ}}$. $\mathbf{GL}^{\infty}$ allows for infinite branches, while $\mathbf{GL}^{\mathbf{circ}}$ is cyclic and finitary. The modal language that Shamkanov endorses is different from the one that we introduced in Definition 3.1. Additionally, these two systems share a K4 rule that has contexts, which is different from the context-free K4 that we endorse. Here, we present two-sided non-wellfounded systems for **GL** (with cut) in the language that we introduced. The infinitary one is called $\mathbf{G3GL}^{\mathrm{cut},\infty}$, and the cyclic one is called $\mathbf{G3GL}^{\mathrm{cut},\circ}$.

We first introduce $\mathbf{G3GL}^{\mathrm{cut},\infty}$ preproofs. Intuitively, a $\mathbf{G3GL}^{\mathrm{cut},\infty}$ preproof is a $\mathbf{G3K4}^{\mathrm{cut}}$ 'proof' whose branch can have an infinite length.

Definition 3.31 ($\mathbf{G3GL}^{\mathrm{cut},\infty}$ preproof)**.** *A* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ preproof *is a* $\mathbf{G3K4}^{\mathrm{cut}} \times \mathrm{Seq}$*-labeled tree* $T = (N, \prec, l)$ *such that for any* $w \in N$*:*

(1) if w *is a leaf, then* $(\mathrm{seq}_T(w))$ *is an instance of* $\mathrm{rule}_T(w)$*;*

(2) otherwise, let $w0, \ldots, wn$ *be the* $\prec$*-successors of* w*, the sequence* $(\mathrm{seq}_T(w_0), \ldots, \mathrm{seq}_T(w_n), \mathrm{seq}_T(w))$ *is an instance of* $\mathrm{rule}_T(w)$*.*

Note that $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *preproofs can be infinite.* ⊣

Definition 3.32 (Progressing branch in a $\mathbf{G3GL}^{\mathrm{cut},\infty}$ preproof)**.** *A branch of a* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *preproof is* progressing *iff it has infinitely many applications of* K4*.* ⊣

Definition 3.33 ($\mathbf{G3GL}^{\mathrm{cut},\infty}$ proof and $\mathbf{G3GL}^{\mathrm{cut},\infty}$ provability)**.** *A* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ proof *is a* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *preproof such that every infinite branch is progressing. A sequent* S is provable in $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *iff there is a* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *proof whose root is decorated with* S*. We write* $\vdash_{\mathbf{G3GL}^{\mathrm{cut},\infty}} S$ *in such cases.* ⊣

Indeed, a finite $\mathbf{G3GL}^{\mathrm{cut},\infty}$ proof is a $\mathbf{G3K4}^{\mathrm{cut}}$ proof. Definition 3.33 already yields an *infinite* non-wellfounded proof system for **GL**. Note the role of the rule K4 in defining this infinitary system. It is the only rule that has a 'progressing premise'. Other rules simply do not have any progressing premise. We can then identify the system $\mathbf{G3GL}^{\mathrm{cut},\infty}$ with a pair consisting of a set of rules (i.e. $\mathbf{G3K4}^{\mathrm{cut}}$) and a function that clarifies which rules are progressing on which premise(s).

Definition 3.34 (System $\mathbf{G3GL}^{\mathrm{cut},\infty}$ and $\mathbf{G3GL}^{\infty}$)**.** *The system* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *is defined as the pair* $(\mathbf{G3K4}^{\mathrm{cut}}, p)$*, where* p *is a binary function such that, for any* $\mathsf{r} \in \mathbf{G3K4}^{\mathrm{cut}} \setminus \{\mathsf{K4}\}$ *and any rule instance* $r \in \mathsf{r}$*,* $p_{\mathsf{r}}(r) = \emptyset$*; for* K4 *and any* $r \in \mathsf{K4}$*,* $p_{\mathsf{K4}}(r) = \{0\}$*.* $\mathbf{G3GL}^{\infty}$ *is defined as* $(\mathbf{G3K4}, p^{-})$*, where* p^{-} *is the restriction of* p *to the set* **G3K4**. ⊣

Intuitively, p takes each rule to the set of the *codes* of nodes that are progressing, i.e. the last digit of the corresponding words.

We further introduce finitary proof systems that also share the same set of rules with $\mathbf{G3K4}^{\mathrm{cut}}$ but allow for *cyclic* proofs.

Definition 3.35 ($\mathbf{G3GL}^{\mathrm{cut},\circ}$ proof and $\mathbf{G3GL}^{\mathrm{cut},\circ}$ provability)**.** *A* $\mathbf{G3GL}^{\mathrm{cut},\circ}$ proof *if a finite* $(\mathbf{G3K4}^{\mathrm{cut}} \times \mathrm{Seq}) \cup \mathrm{Seq}$*-labeled cyclic tree* $(N, \prec \cup f, l)$ *such that:*

(1) $f : \mathcal{L}^T \hookrightarrow N \setminus \mathcal{L}^T$ *is a partial function on the leaves of* T*;*

(2) if $w \in N$ *is a leaf, then:*

  (a) if $w \notin Dom(f)$*, then* $l(w) \in \mathbf{G3K4}^{\mathrm{cut}} \times \mathrm{Seq}$ *and* $(\mathrm{seq}_T(w))$ *is an instance of* $\mathrm{rule}_T(w)$*;*

  (b) if $w \in Dom(f)$*, then* $l(w) \in \mathrm{Seq}$*,* $l(w) = \mathrm{seq}_T(f(w))$*, and there is a node* $u \in N$ *such that* $f(w) \sqsubseteq u \sqsubseteq w$ *and* $\mathrm{rule}_T(w) = \mathsf{K4}$*;*

(1) if $w \in N$ *is not a leaf, then* $l(w) \in \mathbf{G3K4}^{\mathrm{cut}} \times \mathrm{Seq}$*, and for the* $\prec$*-successors of* w*:* $w0, \ldots, wn$*, it holds that the sequence* $(\mathrm{seq}_T(w_0), \ldots, \mathrm{seq}_T(w_n), \mathrm{seq}_T(w))$ *is an instance of* $\mathrm{rule}_T(w)$*, where we set* $seq_T(wi) = l(wi)$ *if* $wi \in Dom(f)$*.*

For any $w \in N$ *such that* $w \in Ran(f)$*,* w *is a* successful companion *iff there is a node* $u \in N$ *such that* $w \sqsubseteq u \sqsubseteq f^{-1}(w)$ *and* $\mathrm{rule}_T(w) = \mathsf{K4}$*. Similarly, a sequent* S is provable in $\mathbf{G3GL}^{\mathrm{cut},\circ}$ *(*$\vdash_{\mathbf{G3GL}^{\mathrm{cut},\circ}} S$*) iff there is a* $\mathbf{G3GL}^{\mathrm{cut},\circ}$ *proof of* S*.* ⊣

In words, a $\mathbf{G3GL}^{\mathrm{cut},\circ}$ proof is just like a $\mathbf{G3K4}^{\mathrm{cut}}$ proof, except that each of its leaves can (but need not) have a companion; and if so, then this leaf is decorated with a sequent only, it shares its sequent with its companion, and the path connecting them passes through a K4 instance.

Example 3.36. *Here is an example of a* $\mathbf{G3GL}^{\mathrm{cut},\circ}$ *proof. We use* $\rightsquigarrow$ *to indicate non-axiomatic leaves and their companion nodes.*

$$\dfrac{\dfrac{\dfrac{\rightsquigarrow \Box(\Box p \to p) \Rightarrow \Box p}{\Box(\Box p \to p) \Rightarrow p, \Box p}\,\mathsf{w_R} \qquad \dfrac{}{\Box(\Box p \to p), p \Rightarrow p}\,\mathsf{init}}{\Box(\Box p \to p), \Box p \to p \Rightarrow p}\,{\to_\mathsf{L}}}{\dfrac{\rightsquigarrow \Box(\Box p \to p) \Rightarrow \Box p}{\Rightarrow \Box(\Box p \to p) \to \Box p}\,{\to_\mathsf{R}}}\,\mathsf{K4}$$

The unraveling of this proof is not yet a $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *proof of the root sequent. To get a* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *proof, we need to further delete certain nodes in the unraveling. More details will be discussed below.*

Following [AW24], we formally identify a cyclic system with a tuple.

Definition 3.37 (Systems $\mathbf{G3GL}^{\mathrm{cut},\circ}$ and $\mathbf{G3GL}^{\circ}$)**.** *Let* $\mathbf{G3GL}^{\mathrm{cut},\circ}$ *be the tuple* $(\mathbf{G3GL}^{\mathrm{cut},\infty}, D)$*, where* D *is the set of* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *derivations. Let* $\mathbf{G3GL}^{\circ}$ *be the tuple* $(\mathbf{G3GL}^{\infty}, D')$*, where* D' *is the set of* $\mathbf{G3GL}^{\infty}$ *derivations.* ⊣

We will prove that $\mathbf{G3GL}^{\mathrm{cut},\infty}$ and $\mathbf{G3GL}^{\mathrm{cut},\circ}$ are equivalent: a sequent is provable in one iff it is provable in another. The direction from $\mathbf{G3GL}^{\mathrm{cut},\circ}$ to $\mathbf{G3GL}^{\mathrm{cut},\infty}$ is more straightforward. The intuition is that any cyclic proof can be unraveled to get a possibly infinitary proof. There is one technical issue we should take care

of, though. Suppose we have a proof $\mathcal{D}$ with a unique cycle, and now consider the sequences in (i.e. the nodes of) the unraveling of this cyclic tree. Let s be the shortest sequence whose last word is the non-axiomatic leaf v of $\mathcal{D}$. The successor of s should be $s : c$, where c is the companion of v in $\mathcal{D}$. Recall that v is decorated with a sequent only, and each sequence (in the unraveling) is decorated with the label of its last word (in the cyclic tree). Then, s and $s : c$, these two adjacent nodes in the unraveling, are decorated with the same sequent, but s is decorated with a sequent only. To get an infinite proof, then, we should delete this s and any other nodes alike because every node in an infinitary proof is decorated with a pair consisting of a rule and a sequent.

Lemma 3.38. *For any sequent S, $\vdash_{\mathbf{G3GL}^{\mathrm{cut},\circ}} S$ implies $\vdash_{\mathbf{G3GL}^{\mathrm{cut},\infty}} S$.*

Proof. Let $T^\circ = (N, \prec \cup f, l)$ be a $\mathbf{G3GL}^{\mathrm{cut},\circ}$ proof of $\Gamma \Rightarrow \Delta$. Consider its unraveling $T^u = (N^u, (\prec \cup f)^u, l^u)$, and let e be the natural embedding of N into N^u.

We construct a $\mathbf{G3K4}^{\mathrm{cut}}$ preproof using T^u. Define $NL := \{s \in N^u : l^u(s) \in \mathrm{Seq}\}$. Define:

- $N^{u-} := N^u \setminus NL$,
- $\prec^{u-} := (\prec \cup f)^u \restriction_{N^{u-}} \cup \{(s, s') \in (N^{u-})^2 :$ there is an $x \in NL$ such that $s(\prec \cup f)^u x(\prec \cup f)^u s'\}$,
- $l^{u-} := l^u \restriction_{N^{u-}}$.

We first argue that this structure is tree-like.

(1) $N^{u-} \neq \emptyset$ because $[\epsilon] \in N^{u-}$.

(2) $[\epsilon]$ is the unique node that reaches every node via the reflexive and transitive closure of $\prec^{u-}$. For any node $s \in N^{u-}$, it starts with ϵ, and therefore there is a $\prec^{u-}$-path from $[\epsilon]$ to s. By definition, there is no other node that reaches via $[\epsilon]$ the transitive and reflexive closure other than itself.

(3) Take an arbitrary $s \in N^{u-}$ that is distinct from $[\epsilon]$, we argue that it has a unique $\prec^{u-}$-predecessor. Take an $s \neq [\epsilon]$. Suppose $s = s' : w$, then s' is a $\prec^{u-}$-predecessor. For any s'' such that $s = s'' : w$, it must holds that $s' = s''$, otherwise s' and s'' would make $s' : w$ and $s'' : w$ distinct sequences.

(4) For an arbitrary $s \in N^{u-}$, it does not reach itself via an $\prec^{u-}$-path because it is not longer than itself.

(5) For an arbitrary $s \in N^{u-}$, its $\prec^{u-}$-successors can be well-ordered by the last digits of their last words.

Then there is a tree $T' = (N', \prec', l')$ isomorphic to $(N^{u-}, \prec^{u-}, l^{u-})$. Let the isomorphism be $i : N' \to N^{u-}$. We argue that T' is a $\mathbf{G3GL}^{\mathrm{cut},\infty}$ preproof by arguing that the labeling function behaves as it should, and this is done by 'looking back' at the original cyclic proof. If w is a leaf of T', we want to check that its sequent constitutes a rule instance. Note that $i(w)$ is a leaf in $(N^{u-}, \prec^{u-}, l^{u-})$. Additionally, $i(w)$ is in the range of the natural embedding e from the original cyclic proof to its unraveling; then, $e^{-1} \circ i(w)$ is a leaf without a companion in the original cyclic proof. Since i is an isomorphism, we have that $l(e^{-1} \circ i(w)) = l^{u-}(i(w)) = l'(w)$. Since l is the labeling function of a cyclic proof, $(\mathrm{seq}_{T'}(w))$ is an instance of $\mathrm{rule}_{T'}(w)$.

Next, consider the case where $w \in N'$ is not a leaf. Further, assume that no $(\prec \cup f)^u$-successor of $i(w)$ is in NL. Without loss of generality, suppose w has only one $\prec'$-successor $w0$. Since i is an isomorphism, we have that $\mathrm{rule}_{T'}(w) = \mathrm{rule}_{T^{u-}}(i(w)) = \mathrm{rule}_{T^\circ}(\mathrm{last}(i(w)))$, $\mathrm{seq}_{T'}(w) = \mathrm{seq}_{T^{u-}}(i(w)) = \mathrm{seq}_{T^\circ}(\mathrm{last}(i(w)))$, and $\mathrm{seq}_{T'}(w0) = \mathrm{seq}_{T^{u-}}(i(w0)) = \mathrm{seq}_{T^\circ}(\mathrm{last}(i(w0)))$. Since $\mathrm{last}(i(w)) \prec \mathrm{last}(i(w0))$ and T° is a cyclic proof, we have that $(\mathrm{seq}_{T'}(w0), \mathrm{seq}_{T'}(w))$ is an instance of $\mathrm{rule}_{T'}(w)$. Finally, if $w \in N'$ is not a leaf and $i(w)$ has a

$(\prec \cup f)^u$-successor x in NL, then x has a unique $(\prec \cup f)^u$-successor $i(w0)$, where $w0$ is the unique $\prec'$-successor of w in T'. Note that

$$\begin{aligned}(\mathrm{seq}_{T'}(w0), \mathrm{seq}_{T'}(w)) &= (\mathrm{seq}_{T^{u-}} \circ i(w0), \mathrm{seq}_{T^{u-}} \circ i(w)) \\ &= (\mathrm{seq}_{T^u}(x), \mathrm{seq}_{T^u} \circ i(w)) \\ &= (\mathrm{seq}_{T^\circ} \circ \mathrm{last}(x), \mathrm{seq}_{T^\circ} \circ \mathrm{last}(i(w))).\end{aligned}$$

is an instance of $\mathrm{rule}_{T^\circ}(\mathrm{last}(i(w))) = \mathrm{rule}_{T^{u-}}(i(w)) = \mathrm{rule}_{T'}(w)$

Lastly, we argue that T' satisfies the progressing condition. We first argue that every infinite branch in the unraveling passes through K4 infinitely many times. Take an infinite branch in T^u, it is obtained by 'unraveling' a cyclic path in the cyclic proof, which passes through a K4 instance. Consider a K4 node p on the path in the cyclic proof. By the definition of unraveling, there are infinitely many nodes on the branch we fixed whose last word is p. By definition, these nodes are decorated by K4. Since the nodes we deleted are the ones not decorated with a rule, each infinite branch in $(N^{u-}, \prec^{u-}, l^{u-})$ passes through K4 infinitely many times. Since $T' \cong (N^{u-}, \prec^{u-}, l^{u-})$, each infinite path in T' satisfies the progressing condition. □

This gives us one direction of the equivalence between $\mathbf{G3GL}^{\mathrm{cut},\circ}$ and $\mathbf{G3GL}^{\mathrm{cut},\infty}$. The other direction requires more work, but the general idea is this. Given a $\mathbf{G3GL}^{\mathrm{cut},\infty}$ proof, if it is cut-free, then we can transform it into a tree decorated by finitely many sequents, and hence on every infinite branch there must be a repeating sequent. Since K4 is the only non-decreasing rule in the cut-free calculus, there must be a K4 application between any repetitions of a sequent. The problem is the cut rule, for it introduces a new formula when we go upwards in a proof. So, for the remaining direction, we need cut elimination for $\mathbf{G3GL}^{\mathrm{cut},\infty}$. This directly leads to the theme of the next chapter: the cut elimination method via coalgebraic proof translation.

Chapter 4

An interlude: the method of coalgebraic proof translation

We will appeal to the method of coalgebraic proof translation developed by Sierra Miranda, Studer, and Zenger in [SSZ25] to prove the cut elimination of $\mathbf{G3GL}^{\mathsf{cut},\infty}$ and another system that will be introduced in later chapters. This chapter serves as an interlude that introduces this method, which means most of the definitions and theorems come from [SSZ25].

We start by clarifying the intuition behind this method in section 4.1. In a nutshell, the idea is to clean the cut applications in an infinitary proof 'step by step' in a bottom-up manner, with each step cleaning a finite fragment of a given proof. This can be viewed as a process of proof translation: we translate a proof in a system that has cut into a proof in a system that does not. To ensure that 'in the end' this translation will give us a proof, some coalgebraic machinery is required. We will introduce finite fragmented labeled trees in section 4.2 and a final coalgebra on these objects in section 4.3. These will prepare us for the 'step-by-step' translation machinery presented in section 4.4.

4.1 The intuitive idea

To understand this method at a high level, it is helpful to first introduce the notion of the 'main fragment' of a $\mathbf{G3GL}^{\mathsf{cut},\infty}$ proof, which is a *finite* fragment in a proof tree.

Definition 4.1 (Main fragment, local height, and local cut-freeness of $\mathbf{G3GL}^{\mathsf{cut},\infty}$ proofs)**.** *Let $\mathcal{D}$ be a $\mathbf{G3GL}^{\mathsf{cut},\infty}$ proof. The* main fragment *of $\mathcal{D}$ is a finite subtree rooted at the root of $\mathcal{D}$ such that each leaf is decorated with either an axiomatic rule or* $\mathsf{K4}$*, and such that other nodes are decorated by $\mathbf{G3K4}^{\mathsf{cut}} \setminus \{\mathsf{K4}\}$. The* local height *of $\mathcal{D}$ is the height of its main fragment. The smallest local height is* 1*. A $\mathbf{G3GL}^{\mathsf{cut},\infty}$ proof $\mathcal{D}$ is* locally cut-free *iff its main fragment does not contain any* cut *application.* ⊣

Recall that the height of well-founded proofs is crucial for inductive reasoning about them. In contrast, for non-wellfounded proofs, such a measure is missing. It turns out that local height serves as a substitute for the height of proofs, allowing us to reason about non-wellfounded proofs inductively. To define local height, it is essential that the local fragment is *finite* as the trees we work with are finitely branching. Here is an example that demonstrates the power of this measure. By induction on local height, we can prove invertibility results for $\mathbf{G3GL}^{\mathsf{cut},\infty}$.

Proposition 4.2 ($\mathbf{G3GL}^{\mathsf{cut},\infty}$ invertibility)**.** *The rules for negation, conjunction, disjunction, and implication are local-height-preserving and local-cut-freeness-preserving invertible in $\mathbf{G3GL}^{\mathsf{cut},\infty}$.*

Proof. By induction on the local height of proofs. The argument is similar to the one for Proposition 3.26. □

With this notion at hand, it might be natural to think of the cut elimination of $\mathbf{G3GL}^{\mathrm{cut},\infty}$ as the following process. Given an infinitary $\mathbf{G3GL}^{\mathrm{cut},\infty}$ proof, we first clean up cut applications in its main fragment, which contains only finitely many nodes. Then, we are left with a finite number of sub-proofs attached to this cut-free main fragment. We proceed to clean the cut applications in these sub-proofs. For each of these sub-proofs, finitary ones are easy to deal with, for they are just $\mathbf{G3K4}^{\mathrm{cut}}$ proofs. What matters is the infinitary ones. Each of these infinitary sub-proofs has its own main fragment, which is again a finite fragment. Again, we clean the cut applications in the main fragments of these proofs, and repeat the process. We hope this process would, 'in the limit', give us a cut-free proof.

In the process above, we treated finitary sub-proofs and infinitary sub-proofs differently. But that is not necessary – a finite proof can also have its own main fragment. We can uniformly clean the cut applications in an infinite proof tree bottom-up, *finite fragment by finite fragment.* This process can also be viewed as translating fragments (possibly) with cut into ones without. In a slogan:

To eliminate cut in a proof is to translate this proof finite fragment by finite fragment.

Sierra Miranda et al. in [SSZ25] offer a formal development of this idea. Their most important result in this context is a theorem stating that, for a particular class of sequent calculi, if we have a translation function to clean up the cut applications in the main fragment of a possibly infinite proof, then we have a translation function to clean up the cut applications in the whole proof. This is essentially the content of Theorem 4.27, which is proved in [SSZ25]. The class of sequent calculi in consideration is *local progress sequent calculi.*

Definition 4.3 (Local progress sequent calculus)**.** *A* local progress sequent calculus *is a pair* $C = (\mathsf{Rul}, p)$ *such that*

(1) Rul *is a set of rules;*

(2) p *is the* local progress function of C *such that, given a* $\mathsf{r} \in \mathsf{Rul}$ *and an instance* $r = (P_0, \ldots, P_{k-1}, C) \in \mathsf{r}$, $p_{\mathsf{r}}(r) \subseteq \{0, \ldots, k-1\}$.

The output of p *is the set of premises of* r *that progress under the rule* r. ⊣

Our definition of $\mathbf{G3GL}^{\mathrm{cut},\infty}$ in terms of $(\mathbf{G3K4}^{\mathrm{cut}}, p)$ should already make the intuition of this definition clear. Proofs of these calculi can be defined uniformly.

Definition 4.4 (Preproof, progressing node, and proof)**.** *Let* $T = (N, \prec, l)$ *and* C *be a local progressing calculus.* T *is a* C-preproof *iff*

(1) $l : N \to \mathsf{Rul}^C \times \mathsf{Seq}$*;*

(2) for every $w \in N$*, if it has* k *successors for some* k*, then* $(\mathrm{seq}_T(w0), \ldots, \mathrm{seq}_T(w(k-1)), \mathrm{seq}_T(w))$ *is an instance of* $\mathrm{rule}_T(w)$*, where* $\mathrm{seq}_T := p_0 \circ l$ *and* $\mathrm{rule}_T := p_1 \circ l$*, for projection functions* p_0 *and* p_1*.*

Let the class of all C*-preproofs be* $\mathbb{P}_0^{\infty}(C)$*. Given a* $T = (N, \prec, l) \in \mathbb{P}_0^{\infty}(C)$ *and a* $w \in N$*,* w *is* progressing in C *iff there is a* k*-ary node* u *for some* k *and an* $i \in \mathbb{N}$ *such that* $w = ui$*, where* $i \in p_{\mathrm{rule}(u)}(\mathrm{seq}_T(u0), \ldots, \mathrm{seq}_T(u(k-1)), \mathrm{seq}_T(u))$.

A tree T *is a* C-proof *iff it is a* C*-preproof and each of its infinite branches passes through infinitely many progressing nodes. Let the class of* C*-proofs be* $\mathbb{P}^{\infty}(C)$. ⊣

Indeed, preproofs (and hence proofs) of such systems can all be represented with labeled trees with fragmentations – they can be carved at their progressing nodes, just like preproofs of $\mathbf{G3GL}^{\mathrm{cut},\infty}$.

Definition 4.5 (Preproof partition relation)**.** *Let $T = (N, \prec, l)$ be a C-preproof, for some local progress calculus C. Define a relation on N as follows.*

$$w \rightsquigarrow^T v \text{ iff } v = wi \text{ for some } i \text{ and } v \text{ is not progressing.}$$

Define also $w \sim^T v$ iff there is a sequence $w_0, \ldots, w_n$ such that for all $i < n$, $w_i \rightsquigarrow^T w_{i+1}$ or $w_{i+1} \rightsquigarrow^T w_i$. ⊣

Local progress calculi enjoy a uniform method of cut elimination via proof translation, which will be given by Theorem 4.27. Of course, we will appeal to this theorem to show the cut elimination of $\mathbf{G3GL}^{\mathrm{cut},\infty}$. Aiming at introducing this crucial theorem, the following three sections will be a presentation of the relevant content from [SSZ25]. Since this theorem formalizes the slogan above, a formal definition of proofs with (possibly infinitely many) finite fragments, or *finite fragmented proofs*, is required. Since proofs are labeled trees, we start with a more general theory about *finite fragmented labeled trees* developed in [SSZ25]. To officially define finite fragmented proofs, we further need a final coalgebra on *finite fragmented labeled trees* that also yields the existence of a proof translation function that we aim for. Therefore, we shall start with finite fragmented trees, and then proceed to the final coalgebra, then finite fragmented proofs, and finally the proof translation functions and the crucial theorem.

In the remainder of this chapter, all the definitions (except for Definitions 4.12 and 4.13), all the theorems, propositions, and corollaries come from [SSZ25].

4.2 Finite fragmented labeled trees

Definition 4.6. *Let $\mathbb{T}^\infty$ be the set of all and only labeled trees.* ⊣

Note that finite labeled trees are also in $\mathbb{T}^\infty$. We assume that the label sets we care about can all be encoded by natural numbers, so $\mathbb{T}^\infty$ exists somewhere in the set universe. Otherwise, it could be a proper class. Here, our choice of the definition of trees using words is also justified. If we directly go for tree-like structures, the class of such objects may well be a proper class.

Definition 4.7 (Finite fragmented tree)**.** *A* labeled finite fragmented tree *is a pair $(T, \mathcal{F})$ such that*

(1) $T \in \mathbb{T}^\infty$;

(2) $\mathcal{F}$ is a partition of the nodes of T, where each $F \in \mathcal{F}$ satisfies the following fragmentation properties:

- *(a) F is finite;*
- *(b) there is a unique root in F;*
- *(c) for any u, v, w, if $u, w \in F$ and $u \sqsubseteq v \sqsubseteq w$, then $v \in F$.*

We denote such trees with 'π'. Let the set of labeled finite fragmented trees be $\mathbb{T}^{\mathrm{ff}}$. ⊣

We will have to reason about the fragments of labeled finite fragmented trees. These fragments do not stand alone, but are interconnected. For example, one fragment can be rooted on a leaf of another. The following notion is helpful to keep track of such information about a fragment.

Definition 4.8 (Tree with non-wellfounded leaves)**.** *Let L be a label set and $*$ an object not in L. A* tree with non-wellfounded leaves ι *is a finite $L \cup \{*\}$-labeled tree $(N, \prec, l)$ such that*

(1) $l(\epsilon) \neq *$;

(2) For any $w \in N$, if $l(w) = *$, then $w \in \mathcal{L}^\iota = \mathcal{L}^{(N,\prec)}$.

Recall that $\mathcal{L}^\iota$ is the set of leaves of ι. Given a tree with non-wellfounded leaves $\iota = (N, \prec, l)$, a non-wellfounded leaf of ι *is a node labeled with $*$. A node in ι is a* proper node *iff it is not a non-wellfounded leaf. Let $\mathcal{NW}^\iota = \{w \in N : l(w) = *\}$ and $\mathcal{PN}^\iota = N \setminus \mathcal{NW}^\iota$. Let the set of such trees be $\mathbb{T}^{\mathrm{nw}}$.* ⊣

We can think of leaves labeled with $*$ as 'cut-off points'. Trees with non-wellfounded leaves are different from cyclic trees. Cyclic trees are 'trees' that allow for cycles, possibly without labels, whereas non-wellfounded trees are *labeled* finite trees, but some of their leaves may have a special mark. The trees below are examples of a cyclic tree and a tree with a non-wellfounded leaf. We use dots to represent words and use lowercase Latin letters (and '$*$') for labels.

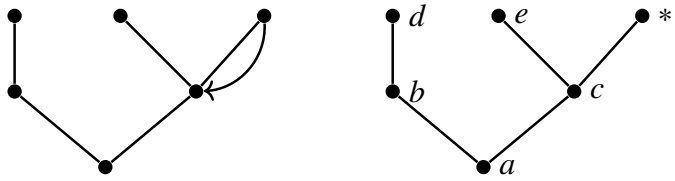


Figure 4.1: A cyclic tree and a tree with a non-wellfounded leaf

Given the notions of finite fragmented trees and trees with non-wellfounded leaves, the most useful 'unit' of a finite fragmented tree can now be introduced. Each of these units is not exactly a fragment given by the fragmentation, but something quite close – its nodes consist of those of a fragment, *plus* the roots of subtrees above that fragment. These nodes together yield a tree with non-wellfounded leaves, with the roots on top being its non-wellfounded leaves.

Definition 4.9 (Root of finite fragmented trees)**.** *Given a $\pi \in \mathbb{T}^{\mathrm{ff}}$, let $\sqrt{\pi}$ be its root; and let $\mathcal{R}^\pi = \{\sqrt{F} : F \in \mathcal{F}^\pi\}$, where $\sqrt{F}$ is the root of F, for any F satisfying the fragmentation properties.* ⊣

Definition 4.10 (Immediate root-successor relation for a finite fragmented proof)**.** *Let $\pi \in \mathbb{T}^{\mathrm{ff}}$. Define the following relation on $\mathcal{R}^\pi$:*

$$w \prec^\pi u \text{ iff } w \sqsubset u \text{ and there is no } v \in \mathcal{R}^\pi \text{ such that } w \sqsubset v \sqsubset u.$$

This is the immediate root-successor relation on the roots of π. ⊣

Definition 4.11 (Non-wellfounded fragment and subtree for finite fragmented trees)**.** *Let $\pi \in \mathbb{T}^{\mathrm{ff}}$ and $w \in \mathcal{R}^\pi$. Define the* non-wellfounded tree fragment in π at w, $\mathsf{T}_w(\pi)$, *as the following non-wellfounded tree $(N, \prec, l)$:*

(1) $N = \{v \in \mathbb{N}^* : w \sim^\pi wv\} \cup \{u \in \mathbb{N}^* : w \prec^\pi wu\}$;

(2) $l(v) = *$ if $w \prec^\pi wv$, otherwise $l(v) = l^\pi(wv)$.

$\sim^\pi$ is the equivalence relation 'being in the same $F \in \mathcal{F}^\pi$'.

Define the subtree of π at w, $\mathsf{ST}_w(\pi)$, *as the following labeled finite fragmented tree $(N, \prec, l, \mathcal{F})$:*

(1) $N = \{v \in \mathbb{N}^* : wv \in N^\pi\}$;

(2) $l(v) = l^\pi(wv)$;

(3) $v \sim u$ *iff* $wv \sim^\pi wu$.

$\mathcal{F}$ *is the partition of* N *given by* $\sim$. ⊣

The clauses that define the set of nodes above are designed as such to ensure that the words of each non-wellfounded tree fragment and subtree for finite fragmented trees are closed under the prefix relation. A subtree rooted at some vertex of a tree, say $(\epsilon, 0, 1, 2)$, is not a tree, but only a tree-like structure, as its vertex set is not closed under the prefix relation.

So much for fragmentation of labeled trees. Now, we move on to the final coalgebra as promised.

4.3 A final coalgebra on finite fragmented trees

Definition 4.12 (Coalgebra and coalgebra homomorphism)**.** *Let* C *be a category and* F *be an endofunctor on* C*. An* F-coalgebra *is a pair* (A, α)*, where* $\alpha : A \to F(A)$ *is an arrow in* C*. We call* A *the* carrier *of* (A, α)*, and* α *the* structure map *of* (A, α)*.*

A homomorphism of F-coalgebras (A, α) *and* (B, β) *is an arrow* $f : A \to B$ *such that* $\beta \circ f = F(f) \circ \alpha$*, i.e. the following diagram commutes.*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ F(A) & \xrightarrow[F(f)]{} & F(B) \end{array}$$

⊣

We will focus on the category of sets, **Set**. So endofunctors and arrows will always be set-theoretic functions.

Definition 4.13 (Finality)**.** *An* F*-coalgebra* (A, α) *is* final *iff for any* F*-coalgebra* (B, β)*, there is a unique coalgebra homomorphism from* B *to* A*.* ⊣

The final coalgebra that we need is based on the following functor.

Definition 4.14 (Functor $\mathcal{T}$)**.** *Define the functor* $\mathcal{T}$: **Set** $\to$ **Set** *as follows.*

(1) $\mathcal{T}(X) := \{(\iota, \mu) : \iota \in \mathbb{T}^{\mathrm{nw}}$ *and* $\mu : \mathcal{N}\mathcal{W}^\iota \to X\}$,

(2) $\mathcal{T}(f : X \to Y) := (\iota, \mu) \mapsto (\iota, f \circ \mu)$.

In the following, by 'coalgebra' we will usually mean $\mathcal{T}$*-coalgebra, unless specified otherwise. Given a coalgebra* (A, α) *and* $a \in A$*, let* ι_a^α *and* μ_a^α *be the components of* $\alpha(a)$*. We may omit the superscript when the context allows.* ⊣

We now have enough resources to understand the coalgebra that we need. This coalgebra is of course a $\mathcal{T}$-coalgebra, and its carrier is $\mathbb{T}^{\mathrm{ff}}$, the set of finite fragmented labeled trees. The structure map is called 'destruct'. The codomain of destruct is $\mathcal{T}(\mathbb{T}^{\mathrm{ff}}) = \{(\iota, \mu) : \iota \in \mathbb{T}^{\mathrm{nw}}$ and $\mu : \mathcal{N}\mathcal{W}^\iota \to \mathbb{T}^{\mathrm{ff}}\}$. So, given a finite fragmented labeled tree π, destruct takes it to a pair of a tree with non-wellfounded leaves ι and a map μ that records which finite fragmented labeled trees are 'cut off from' ι, and to which non-wellfounded leaf each of

the trees corresponds. Which ι does destruct take π to exactly? Its bottom-most non-wellfounded fragment $\mathsf{T}_\epsilon(\pi)$. So, destruct takes a finite fragmented labeled tree π to its bottom-most non-wellfounded fragment and subtrees of π right above this fragment. As we will see (by examining $\mathbf{G3GL}^{\mathrm{cut},\infty}$), this bottom-most non-wellfounded fragment does coincide with our previous definition of main fragment (Definition 4.1). The following is the formal definition of destruct.

Definition 4.15 (The destruct function)**.** *The function* destruct *is defined as follows:*

$$\text{destruct} : \mathbb{T}^{\mathrm{ff}} \to \mathcal{T}(\mathbb{T}^{\mathrm{ff}}) := \pi \mapsto (\mathsf{T}_\epsilon(\pi), w \mapsto \mathsf{ST}_w(\pi)).$$

We assume the w in the definition is among the non-wellfounded leaves of $\mathsf{T}_\epsilon(\pi)$. ⊣

Theorem 4.16. $(\mathbb{T}^{\mathrm{ff}}, \text{destruct})$ *is a final coalgebra.*

Corollary 4.17. *Let (A, α) be a coalgebra. Then there is a unique coalgebra homomorphism $!_\alpha : A \to \mathbb{T}^{\mathrm{ff}}$ such that $!_\alpha = \text{construct} \circ \mathcal{T}(!_\alpha) \circ \alpha$, where* construct $= \text{destruct}^{-1}$.

If we let $A = \mathbb{T}^{\mathrm{ff}}$ and $\alpha : \mathbb{T}^{\mathrm{ff}} \to \mathcal{T}(\mathbb{T}^{\mathrm{ff}})$, then $!_\alpha$ is a function from $\mathbb{T}^{\mathrm{ff}}$ to $\mathbb{T}^{\mathrm{ff}}$ and can serve as a translation function.

With the definition of $(\mathbb{T}^{\mathrm{ff}}, \text{destruct})$ ready, we can define finite fragmented proofs and proof translation for local sequent calculi.

4.4 Finite fragmented proofs and proof translation

We first present three more definitions that are necessary for the definition of finite fragmented proofs.

Definition 4.18 (α root path)**.** *Given a coalgebra (A, α) and $a \in A$, a finite sequence of words r is an α* root path *of a iff*

(1) nil *is an α root path of a;*

(2) $w : r$ is an α root path of a iff $w \in \mathcal{NW}^{\iota_a}$ and r is an α root path of $\mu_a(w)$.

Let the set of α root paths of a be $\mathcal{P}^{\alpha,a}$. ⊣

Example 4.19. *Consider the coalgebra $(\mathbb{T}^{\mathrm{ff}}, \text{destruct})$ and a $\pi \in \mathbb{T}^{\mathrm{ff}}$. Suppose π looks like the tree on the left. We use triangular shapes to represent the non-wellfounded fragments of π, and each w_i represents both a non-wellfounded leaf of the fragment below it and the root of the upper fragment. The green nodes in the right graph below correspond to a* destruct *root path $w_0 : w_k - w_0 :$* nil*. We use w/u to mark a node to indicate that it is identified with w in the tree rooted here, and identified with u in the tree with non-wellfounded leaves below it.*

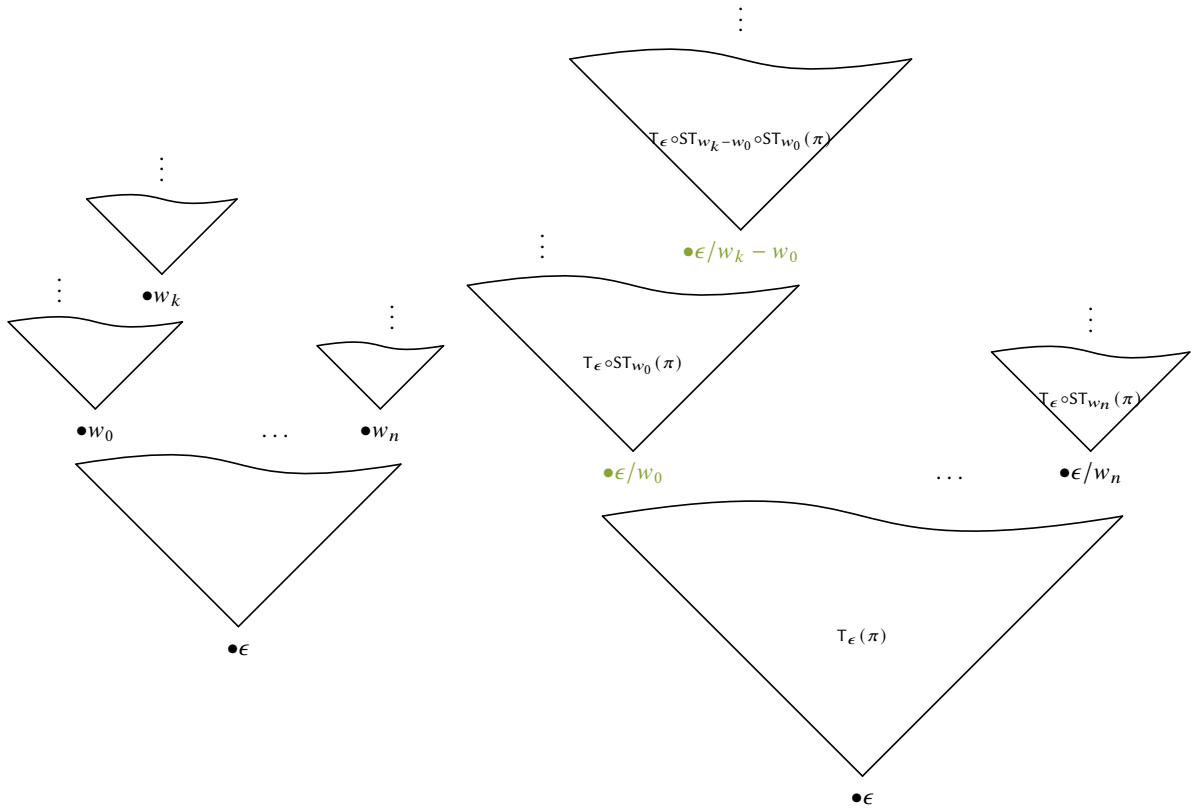


Figure 4.2: A finite fragmented tree and a root path

Note that $w_k - w_0 : \mathsf{nil}$ *is a* destruct *root path of* $\mu_\pi(w_0)$*, which is just* $\mathsf{ST}_{w_0}(\pi)$ *by the definition of* destruct*. This is because* $w_k - w_0 \in \mathcal{NW}^{\iota_{\mu_\pi(w_0)}}$*, and* nil *is a* destruct *root path of* $\mu_\pi(w_0)$*. Since* $w_k - w_0 : \mathsf{nil}$ *is a* destruct *root path of* $\mu_\pi(w_0)$ *and* $w_0 \in \mathcal{NW}^{\iota_\pi}$*, it holds that* $w_0 : w_k - w_0 : \mathsf{nil}$ *is a* destruct *root path of* π*.*

Definition 4.20 (α-subelement and α-fragment)**.** *Let* (A, α) *be a coalgebra,* $a \in A$*, and* $r \in \mathcal{P}^{\alpha,a}$*. The* α-subelement of a generated by r *is defined by recursion on* r*:*

$$\mathsf{SE}^\alpha_{\mathsf{nil}}(a) = a, \qquad \mathsf{SE}^\alpha_{w:r}(a) = \mathsf{SE}^\alpha_r(\mu^\alpha_a(w)).$$

Additionally, the α-fragment of a at r *is defined as:*

$$\mathsf{F}^\alpha_{\mathsf{nil}}(a) = \iota^\alpha_a, \qquad \mathsf{F}^\alpha_{w:r}(a) = \mathsf{F}^\alpha_r(\mu^\alpha_a(w)).$$

Call $\mathsf{F}^\alpha_{\mathsf{nil}}(a)$ *the* main α-fragment of a*.* ⊣

Example 4.21. *Consider the coalgebra* $(\mathbb{T}^{\mathrm{ff}}, \mathsf{destruct})$*, the* $\pi \in \mathbb{T}^{\mathrm{ff}}$ *and the root path given in Example 4.19. We*

omit the superscript destruct. *The following holds.*

$$\begin{aligned}\mathsf{SE}_{w_0:w_k-w_0:\mathsf{nil}}(\pi) &= \mathsf{SE}_{w_k-w_0:\mathsf{nil}}(\mu_\pi(w_0))\\ &= \mathsf{SE}_{\mathsf{nil}}(\mu_{\mu_\pi(w_0)}(w_k-w_0))\\ &= \mu_{\mu_\pi(w_0)}(w_k-w_0)\\ &= \mu_{\mathsf{ST}_{w_0}(\pi)}(w_k-w_0)\\ &= \mathsf{ST}_{w_k-w_0}\circ\mathsf{ST}_{w_0}(\pi).\end{aligned}$$

$$\begin{aligned}\mathsf{F}_{w_0:w_k-w_0:\mathsf{nil}}(\pi) &= \mathsf{F}_{w_k-w_0:\mathsf{nil}}(\mu_\pi(w_0))\\ &= \mathsf{F}_{\mathsf{nil}}(\mu_{\mu_\pi(w_0)}(w_k-w_0))\\ &= \iota_{\mu_{\mu_\pi(w_0)}(w_k-w_0)}\\ &= \iota_{\mu_{\mathsf{ST}_{w_0}(\pi)}(w_k-w_0)}\\ &= \mathsf{T}_\epsilon\circ\mathsf{ST}_{w_k-w_0}\circ\mathsf{ST}_{w_0}(\pi).\end{aligned}$$

Intuitively, for $(\mathbb{T}^{\mathrm{ff}}, \text{destruct})$*, a* $\pi \in \mathbb{T}^{\mathrm{ff}}$*, and a root path* r *of* π*,* $\mathsf{SE}_r(\pi)$ *'is' the tree rooted at the right-most member of* r*.* $\mathsf{F}_r(\pi)$ *is the main* destruct*-fragment of* $\mathsf{SE}_r(\pi)$*.*

The observations in Examples 4.19 and 4.21 can be made precise.

Definition 4.22 (Word of root path)**.** *Let* (A, α) *be a coalgebra,* $a \in A$ *and* $r \in \mathcal{P}^{\alpha,a}$*, the* word of r *is defined as follows:*

$$\mathbf{w}_{\mathsf{nil}} = \epsilon, \qquad \mathbf{w}_{w:r} = w\mathbf{w}_r.$$

For any finite fragmented tree π *and* $w \in \mathcal{R}^\pi$*, it can be shown that there is a unique sequence* $w_0 = \epsilon, w_1, \ldots, w_n = w$ *in* $\mathcal{R}^\pi$*. Define* $[w]_\pi = [w_1, w_2 - w_1, \ldots, w_n - w_{n-1}]$*.* ⊣

Proposition 4.23. *Consider* $(\mathbb{T}^{\mathrm{ff}}, \text{destruct})$*, a* $\pi \in \mathbb{T}^{\mathrm{ff}}$*, and an* $r \in \mathcal{P}^\pi$*. The following holds.*

(1) $\mathbf{w}_r \in \mathcal{R}^\pi$*,*

(2) $[\mathbf{w}_r]_\pi = r$*,*

(3) $\mathsf{F}_r(\pi) = \mathsf{T}_{\mathbf{w}_r}(\pi)$*,*

(4) $\mathsf{SE}_r(\pi) = \mathsf{ST}_{\mathbf{w}_r}(\pi)$*.*

For a proof of this proposition, we direct the reader to the proof of Lemma 2.16 in [SSZ25]. This proposition tells us that a destruct root path in some π has the shape of $[w]_\pi$ for some word w, and it formally reveals the relation between Definitions 4.11 and 4.20.

We introduced the notion of root path, subelement, and fragment not only for destruct but for an arbitrary structure map α because we want these definitions to also work for arbitrary α's, among which proof translation step, a crucial structure map yet to be defined, is a member.

The notion of finite fragmented proofs can now be cashed out.

Definition 4.24 (Proof fragment and finite fragmented proof)**.** *Let* $C = (\mathsf{Rul}, p)$ *be a local progress calculus. Let* $\iota \in \mathbb{T}^{\mathrm{nw}}$*, and suppose its label set is* $\mathsf{Rul} \times \mathsf{Seq}$*, for some set* Seq *of sequents. Let* $s : \mathcal{N}\mathcal{W}^\iota \to \mathsf{Seq}$*.* (ι, s) *is a* proof fragment in C *iff for any* $w \in \mathcal{P}\mathcal{N}^\iota$ *with arity* k*, the following hold:*

(1) $(\mathrm{seq}_\iota(w0), \ldots, \mathrm{seq}_\iota(w(k-1)), \mathrm{seq}_\iota(w)) \in \mathrm{rule}_\iota(w)$*;*

(2) for any $i < k$*,* $wi \in \mathcal{NW}^\iota$ *iff* $i \in p_{\mathrm{rule}_\iota(w)}(\mathrm{seq}_\iota(w0), \ldots, \mathrm{seq}_\iota(w(k-1)))$*.*

Let $\pi \in \mathbb{T}^{\mathrm{ff}}$*.* π *is a* finite fragmented proof in C *iff for any* $r \in \mathcal{P}^{\mathrm{destruct},\pi}$*,*

$$(\mathsf{F}_r(\pi), v \in \mathcal{NW}^{\mathsf{F}_r(\pi)} \mapsto \mathrm{seq}_{\mathsf{F}_{r:v}(\pi)}(\epsilon))$$

is a proof fragment is C*. Let the class of all finite fragmented proofs in* C *be* $\mathbb{P}^{\mathrm{ff}}(C)$*.* ⊣

Remark 4.25. *Let* $T = (N, \prec, l, \mathcal{F})$ *be a finite fragmented* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *proof. In the context of cut elimination, the following notions/objects nicely coincide.*

(1) the main fragment of a $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *proof* T *defined by Definition 4.1;*

(2) the main destruct*-fragment of* T *defined by Definition 4.20;*

(3) the bottom-most proof fragment of T *in* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *defined by Definition 4.24.*

Suppose (1) looks like the left-most tree below. Then (2) and (3) are the ones in the middle and on the right, respectively.

$$\dfrac{\dfrac{}{\Box\Sigma \Rightarrow \Box B}\,\mathsf{K4} \quad \dfrac{}{\Box B, \Box\Theta \Rightarrow \Box C}\,\mathsf{K4}}{\Box\Sigma, \Box\Theta \Rightarrow \Box C}\,\mathrm{cut} \qquad \dfrac{\dfrac{*}{\Box\Sigma \Rightarrow \Box B}\,\mathsf{K4} \quad \dfrac{*}{\Box B, \Box\Theta \Rightarrow \Box C}\,\mathsf{K4}}{\Box\Sigma, \Box\Theta \Rightarrow \Box C}\,\mathrm{cut} \qquad \dfrac{\dfrac{\Gamma \Rightarrow \Delta}{\Box\Sigma \Rightarrow \Box B}\,\mathsf{K4} \quad \dfrac{\Gamma' \Rightarrow \Delta'}{\Box B, \Box\Theta \Rightarrow \Box C}\,\mathsf{K4}}{\Box\Sigma, \Box\Theta \Rightarrow \Box C}\,\mathrm{cut}$$

Note that the leaves of (3) are not decorated with any rules. Hence, to clean cut *applications in the main fragment of a* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *proof amounts to cleaning* cut *applications in its main* destruct*-fragment, which also amounts to cleaning* cut *applications in its main* destruct*-fragment or bottom-most proof fragment.*

Given the definition of finite fragmented proof, we can then define proof translation for local progress sequent calculi.

Definition 4.26 (Proof translation for local progress sequent calculus)**.** *Let* C_0 *and* C_1 *be local progress sequent calculi. Let* $\alpha : \mathbb{T}^{\mathrm{ff}} \to \mathcal{T}(\mathbb{T}^{\mathrm{ff}})$*.* α *is a proof translation step from* C_0 *to* C_1 *iff for any* $\pi \in \mathbb{P}^{\mathrm{ff}}(C_0)$*:*

(1) $(\mathsf{F}^\alpha_{\mathsf{nil}}(\pi), w \mapsto \mathrm{seq}_{\mathsf{F}^\alpha_{[w]}(\pi)}(\epsilon))$ *is a proof fragment in* C_1*;*

(2) for any $w \in \mathcal{NW}^{\mathsf{F}^\alpha_{\mathsf{nil}}(\pi)}$*,* $\mathsf{SE}^\alpha_{[w]}(\pi) \in \mathbb{P}^{\mathrm{ff}}(C_0)$*.*

Call a function $\gamma : \mathbb{P}^{\mathrm{ff}}(C_0) \to \mathbb{P}^{\mathrm{ff}}(C_1)$ *a* finite fragmented proof translation from $C_0 \to C_1$*, and call a function* $\gamma : \mathbb{P}^{\infty}(C_0) \to \mathbb{P}^{\infty}(C_1)$ *a* non-wellfounded proof translation from C_0 to C_1*.* ⊣

Finally, we arrive at the crucial theorem.

Theorem 4.27. *Let* C_0 *and* C_1 *be local progress sequent calculi. If* α *is a proof translation step from* C_0 *to* C_1*, then the function* $T \in \mathbb{P}^{\infty}(C_0) \mapsto p_0 \circ !_\alpha(T, \sim^T)$ *is a non-wellfounded proof translation from* C_0 *to* C_1*.*

Again, for a detailed proof, we direct the reader to the proof of Theorem 3.8 in [SSZ25].

Remark 4.28. *We explain how this function ensures that the output satisfies the progressing condition on an intuitive level. Take* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *for an example. Suppose we are given a* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *proof* $\pi = (T, \sim^T)$ *and a proof translation step* α *from* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *to* $\mathbf{G3GL}^{\infty}$*. By Corollary 4.17,* $!_\alpha(\pi) = \mathrm{construct} \circ \mathcal{T}(!_\alpha) \circ \alpha(\pi)$*.* $\alpha(\pi)$ *is a pair consisting of a cut-free proof fragment* ι *and a function* μ *that assigns a* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *proof* π_i *to each of the non-wellfounded leaves* w_i *of* ι*.* $\mathcal{T}(!_\alpha)(\iota, \mu) = (\iota, !_\alpha \circ \mu)$*, so* $\mathrm{construct} \circ \mathcal{T}(!_\alpha)(\iota, \mu)$ *is a proof whose bottom-most (cut-free) proof fragment is given by* α*, and each subtree attached to this fragment is given by* $!_\alpha(\pi_i)$*. The translation on each* π_i *works similarly, starting from the bottom-most fragment. Hence, indeed, this process translates a proof fragment by fragment. Since fragments are individuated by progressing points, the result of the translation satisfies the progressing condition automatically.*

We are now ready for the cut elimination for $\mathbf{G3GL}^{\mathrm{cut},\infty}$.

Chapter 5

The cut elimination proof for G3GL$^{\text{cut},\infty}$ and G3GL$^{\text{cut},\circ}$

In this chapter, we focus on the cut elimination of **G3GL**$^{\text{cut},\infty}$ and **G3GL**$^{\text{cut},\circ}$. We first prove the cut elimination for **G3GL**$^{\text{cut},\infty}$ in section 5.1, using the coalgebraic method from [SSZ25] introduced in the previous chapter. In section 5.2, we present the direct cut elimination procedure for **G3GL**$^{\text{cut},\circ}$, which appeals to the local cut reduction procedure for **G3GL**$^{\text{cut},\infty}$. We close this chapter and our trilogy about **GL** by proving that these systems are indeed sound and complete with respect to **HGL** in section 5.3.

5.1 The cut elimination proof for G3GL$^{\text{cut},\infty}$

As standard cut elimination proofs, we need the admissibility of contraction, which usually further requires invertibility. We have already proved the latter at the beginning of the previous chapter. So we now directly proceed to contraction.

Recall that we proved invertibility results for **G3GL**$^{\text{cut},\infty}$ by induction on the local height of proofs. Unfortunately, the same strategy does not work for the admissibility of contraction. The difficulty is that now there would be a case where the root is decorated with K4; in this case, there is no way to apply the inductive hypothesis. We thus need another (much) simpler corecursive proof translation. This proof technique is similar to Shamkanov's as in [Sha14], only that we here rely on a different final coalgebra. To introduce this technique, we first introduce a particular kind of tree-like structures – full binary trees – and introduce their colorings.

Definition 5.1 (Full binary tree and coloring)**.** *The* full binary tree *is a relational structure* $(\{0,1\}^{<\omega}, \prec_l)$*, where* $\{0,1\}^{<\omega}$ *is the set of all finite sequences consisting of* $0, 1$*, and for any* $\sigma, \tau \in \{0,1\}^{<\omega}$ $\sigma \prec_l \tau$ *iff* $\tau = 0\sigma$ *or* $\tau = 1\sigma$*. Given a label set* A*, an* A*-coloring of the full binary tree is a function* $c : \{0,1\}^{<\omega} \to A$*. Let the set of* A*-colorings be* C^A*.* ⊣

Next, we introduce the coalgebra whose carrier is C^A, for some given label set A. We take Definition 5.2, Theorem 5.3, and Corollary 5.4 from [SR10].

Definition 5.2 (Functor $\mathcal{T}'$)**.** *Given a label set* A*, let* $\mathcal{T}'$ *be the endofunctor on* **Set** *such that* $\mathcal{T}'(X) = X \times A \times X$ *and* $\mathcal{T}'(f)(x_0, a, x_1) = (f(x_0), a, f(x_1))$ *for any arrow* f *in* **Set**. ⊣

Theorem 5.3. *Let* A *be a label set. The* $\mathcal{T}'$*-coalgebra* $(C^A, \langle l, o, r\rangle)$ *is a final* $\mathcal{T}'$*-coalgebra, where*

- $\langle l, o, r\rangle : C^A \to C^A \times A \times C^A$*;*

- $l(c) = \lambda x.c(0x), o(c) = c(\epsilon), r(c) = \lambda x.c(1x)$, *and* $\langle l, o, r\rangle(c) = \langle l(c), o(c), r(c)\rangle$.

Since $(C^A, \langle l, o, r\rangle)$ is a final $\mathcal{T}'$-coalgebra, the following diagram commutes for any $\mathcal{T}'$-coalgebra (X, α) for a given set A.

$$\begin{array}{ccc} U & \xrightarrow{\exists!\, h} & T_A \\ {\scriptstyle \langle l', o', r'\rangle}\downarrow & & \downarrow{\scriptstyle \langle l, o, r\rangle} \\ U \times A \times U & \xrightarrow[\mathcal{T}'(h)]{} & T_A \times A \times T_A \end{array}$$

This justifies the following form of corecursive definition.

Corollary 5.4 (Definition by $\mathcal{T}'$-corecursion)**.** *Let A be a label set, and X be a set. For any $\mathcal{T}'$-coalgebra $(X, \langle l', o', r'\rangle)$, a function $h : X \to C^A$ can be defined as follows:*

$$h(l'(x)) = l(h(x)), \quad o(h(x)) = o'(x), \quad h(r'(x)) = r(h(x)).$$

We need this form of $\mathcal{T}'$-corecursive definition to define a proof translation in the proof of the admissibility of contraction. To define such a function h, it suffices to define l', r', and o'.

Proposition 5.5 (Contraction for $\mathbf{G3GL}^{\mathrm{cut},\infty}$)**.** *For any formula A and sequent $\Gamma \Rightarrow \Delta$, if $A, A, \Gamma \Rightarrow \Delta$ is derivable in $\mathbf{G3GL}^{\mathrm{cut},\infty}$, then $A, \Gamma \Rightarrow \Delta$ is local-height-preserving and local-cut-freeness-preserving derivable in $\mathbf{G3GL}^{\mathrm{cut},\infty}$; if $\Gamma \Rightarrow \Delta, A, A$ is derivable in $\mathbf{G3GL}^{\mathrm{cut},\infty}$, then $\Gamma \Rightarrow \Delta, A$ is local-height-preserving and local-cut-free-preserving derivable in $\mathbf{G3GL}^{\mathrm{cut},\infty}$.*

Proof. By (simultaneous) main induction on the weight of A and subinduction on the local height of proofs. Suppose there are proofs $\mathcal{D}_0$ and $\mathcal{D}_1$ of $A, A, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, A, A$. Suppose further that the local heights of $\mathcal{D}_0$ and $\mathcal{D}_1$ are n_0 and n_1, and that they have cut-free main fragments. We assume that $\mathcal{D}_0$ and $\mathcal{D}_1$ are both infinite, for if any of them is finite, we can appeal to the admissibility of contraction of $\mathbf{G3K4}^{\mathrm{cut}}$.

The base case is where $A = p$ for some p and the local height is 1. Then, both $A, A, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, A, A$ are axioms because propositional variables cannot be the principal formulas of any K4 application. Consequently, $A, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, A$ are height-preserving and local-cut-freeness-preserving derivable.

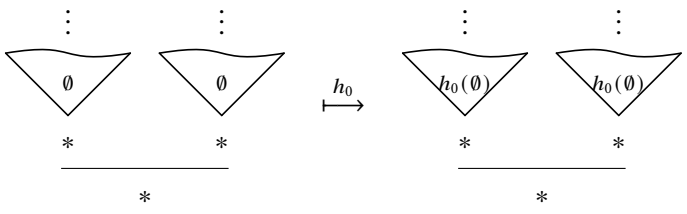
For the inductive case, first consider when $A = p$. In this case, the local heights of $\mathcal{D}_0$ and $\mathcal{D}_1$ are larger than 1 because, by the shape of the K4 rule, the bottom-most rules of $\mathcal{D}_0$ and $\mathcal{D}_1$ are not K4. Then we can apply the IH to the premises. When $A = \neg B$, there are proofs of $B, B, \Gamma \Rightarrow \Delta$ and $\Gamma \Rightarrow \Delta, B, B$. We apply the IH to these proofs, and apply rules for negation. Similarly, when the main connective of A is $\vee$, or $\wedge$, or $\rightarrow$, we appeal to the invertibility of the corresponding propositional rules and the IH. We demonstrate only the case where $A = \Box B$ for some B. For $\Gamma \Rightarrow \Delta, \Box B, \Box B$, since the bottom-most rule cannot be K4, we can either (i) appeal to the IH, or (ii) delete one instance of $\Box B$ if the bottom-most rule of $\mathcal{D}_1$ is w_R and at least one of the designated $\Box B$ instances is principal.

It remains to consider $\Box B, \Box B, \Gamma \Rightarrow \Delta$. If none of the designated instances of $\Box B$ is principal in the last rule of $\mathcal{D}_0$, then the last rule cannot be K4, and hence we can apply the IH. Otherwise, if at least one of the designated instances of $\Box B$ is principal in the last rule of $\mathcal{D}_0$, the last rule is either w_L or K4. The former case is easy, so we consider when the bottom-most rule is K4.

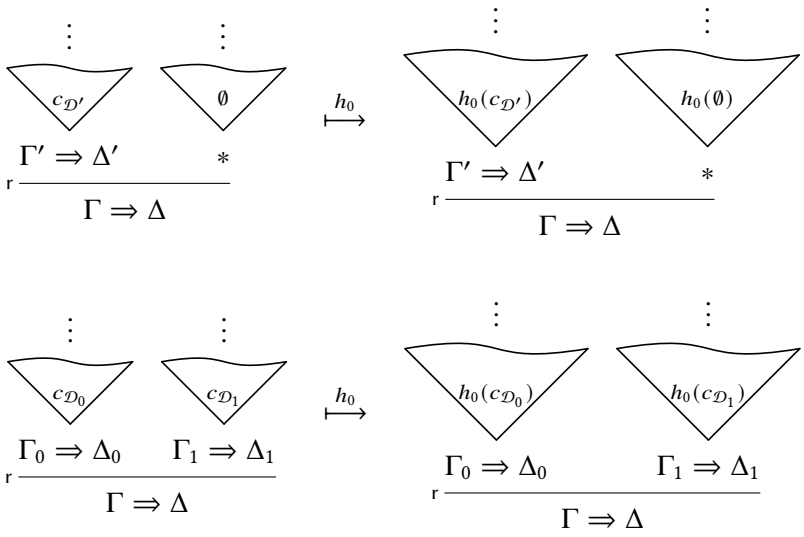
In this case, we appeal to Corollary 5.4 to define a function h_0 by $\mathcal{T}'$-corecursion that transforms $\mathcal{D}_0$ into a proof of $\Box B, \Gamma \Rightarrow \Delta$. Since all the rules of $\mathbf{G3K4}^{\mathrm{cut}}$ have at most two premises, let us work with labeled full binary trees or colorings. Let the label set be $(\mathbf{G3K4}^{\mathrm{cut}} \times \mathrm{Seq}) \cup \{*\}$, where '$*$' is a dummy label for the

nodes without any sequent or rules. Let the coloring whose range is $\{*\}$ be $\emptyset$. We assume that for one-premise rules, the right labeled subtree of the labeled binary tree is identified with $\emptyset$. To define such an h_0, it suffices to define $\langle l', o', r' \rangle$. Given any derivation $\mathcal{D}$, the corresponding coloring $c_{\mathcal{D}}$, and any sequent $\Gamma \Rightarrow \Delta$, we define $\langle l', o', r' \rangle$ and hence h_0 as follows.

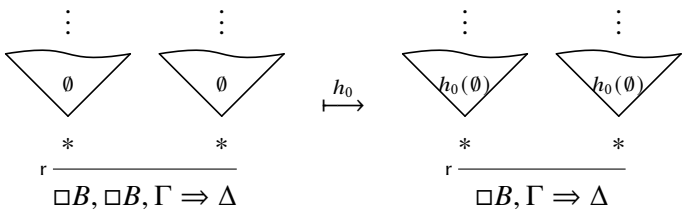
- $\langle l', o', r' \rangle(\emptyset) = \langle l'(\emptyset), o'(\emptyset), r'(\emptyset) \rangle = \langle \emptyset, *, \emptyset \rangle$. Pictorially, h_0 is defined as follows.



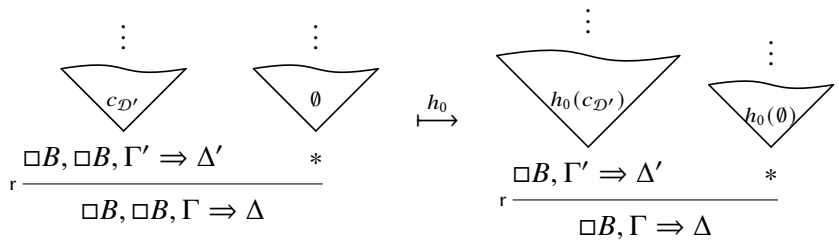
- If $\mathcal{D}$ is a derivation of $\Gamma \Rightarrow \Delta$ where $\#_\Gamma(\Box B) \leqslant 1$, then let $\langle l', o', r' \rangle(c_{\mathcal{D}}) = \langle l'(c_{\mathcal{D}}), o'(c_{\mathcal{D}}), r'(c_{\mathcal{D}}) \rangle = \langle l(c_{\mathcal{D}}), o(c_{\mathcal{D}}), r(c_{\mathcal{D}}) \rangle$. Pictorially, h_0 works as follows, depending on the number of premises of r.



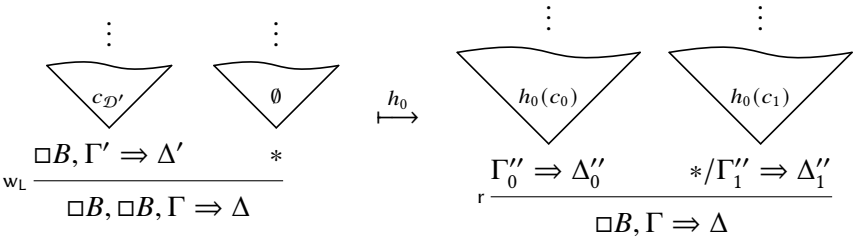
- If $\mathcal{D}$ is derivation of $\Box B, \Box B, \Gamma \Rightarrow \Delta$ whose bottom-most rule does not have any premise, then $\Box B, \Box B, \Gamma \Rightarrow \Delta$ is axiomatic. Then $\langle l', o', r' \rangle(c_{\mathcal{D}}) = \langle l'(c_{\mathcal{D}}), o'(c_{\mathcal{D}}), r'(c_{\mathcal{D}}) \rangle = \langle \emptyset, (\mathrm{r}, \Box B, \Gamma \Rightarrow \Delta), \emptyset \rangle$, where r is the corresponding axiom. Pictorially, h_0 is defined as follows in this case.



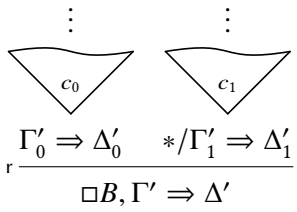
- If $\mathcal{D}$ is a derivation of $\Box B, \Box B, \Gamma \Rightarrow \Delta$ whose last rule is a one-premise rule and none of the designated instances of $\Box B$ is principal in the last rule, then let $\langle l', o', r' \rangle(c_{\mathcal{D}}) = \langle l'(c_{\mathcal{D}}), o'(c_{\mathcal{D}}), r'(c_{\mathcal{D}}) \rangle = \langle l(c_{\mathcal{D}}), (\mathrm{r}, \Box B, \Gamma \Rightarrow \Delta), \emptyset \rangle$. Then h_0 works as follows in this case.



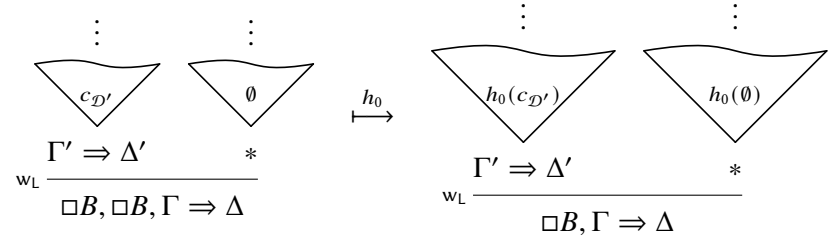
- If $\mathcal{D}$ is a derivation of $\Box B, \Box B, \Gamma \Rightarrow \Delta$ whose last rule is a one-premise rule and at least one of $\Box B$ is principal, we consider three cases. First, if the last rule is $\mathsf{w_L}$ and exactly one of the designated $\Box B$ is principal in this $\mathsf{w_L}$ application, let $\langle l', o', r' \rangle = \langle l'(c_{\mathcal{D}}), o'(c_{\mathcal{D}}), r'(c_{\mathcal{D}}) \rangle = \langle l(c_{\mathcal{D}'}), (\mathsf{r}, \Box B, \Gamma \Rightarrow \Delta), r(c_{\mathcal{D}'}) \rangle$, where r is the last rule of $\mathcal{D}'$. The function h_0 works as follows in this case, where c_0 and c_1 stand for the left and right sub-colorings of $c_{\mathcal{D}'}$.



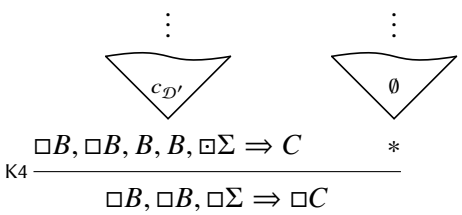
We assume that $c_{\mathcal{D}'}$ has the following shape.



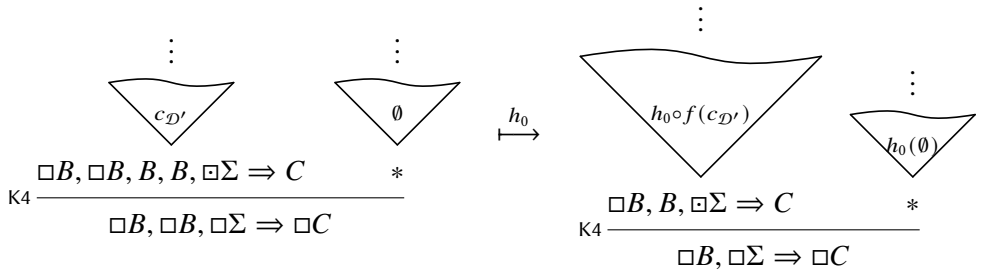
Second, if the last rule of $\mathcal{D}$ is $\mathsf{w_L}$ and both of the designated $\Box B$ are principal, we define $\langle l', o', r' \rangle(c_{\mathcal{D}}) = \langle l'(c_{\mathcal{D}}), o'(c_{\mathcal{D}}), r'(c_{\mathcal{D}}) \rangle = \langle l(c_{\mathcal{D}}), (\mathsf{w_L}, \Box B, \Gamma \Rightarrow \Delta), \emptyset \rangle$.



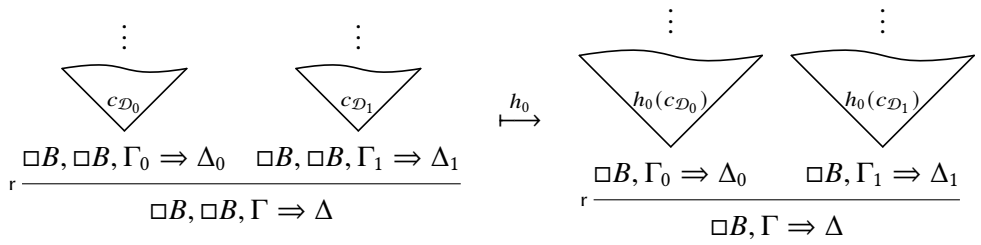
Third, if the last rule of $\mathcal{D}$ is K4, then $c_{\mathcal{D}}$ has the following shape.



Let $\langle l', o', r' \rangle = \langle l'(c_{\mathcal{D}}), o'(c_{\mathcal{D}}), r'(c_{\mathcal{D}}) \rangle = \langle f(c_{\mathcal{D}'}), (\mathsf{K4}, \Box B, \Box\Sigma \Rightarrow \Box C), \emptyset \rangle$, where $f(c_{\mathcal{D}'})$ is the coloring of a proof of $\Box B, \Box B, B, \boxdot\Sigma \Rightarrow C$, whose existence is guaranteed by the IH on formula weight. Pictorially, h_0 works as follows, where $\mathcal{D}_{f(c_{\mathcal{D}'})}$ is the proof that matches the coloring $f(c_{\mathcal{D}'})$.



- If $\mathcal{D}$ is a derivation of $\Box B, \Box B, \Gamma \Rightarrow \Delta$ whose last rule is a two-premise rule, then let $\langle l', o', r' \rangle = \langle l'(c_{\mathcal{D}}), o'(c_{\mathcal{D}}), r'(c_{\mathcal{D}}) \rangle = \langle l(c_{\mathcal{D}}), (\mathsf{r}, \Box B, \Gamma \Rightarrow \Delta), r(c_{\mathcal{D}}) \rangle$. Accordingly, h_0 works as follows.



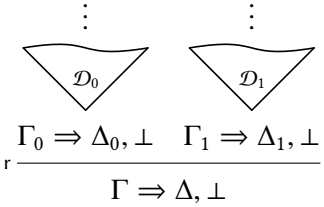
By definition, $h_0(\mathcal{D}_0)$ is a $\mathbf{G3GL}^{\mathrm{cut},\infty}$ proof, since (i) the transformation in each case does not break any rule application and (ii) h_0 does not delete any application of K4. Moreover, in each case, h_0 does not increase local height or introduce any application of cut, so h_0 preserves local-height and cut-freeness of the main fragment. □

Proposition 5.6. *If there is a* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *proof of* $\Gamma \Rightarrow \Delta, \bot$ *that has a local height at most* n *and is locally cut-free, then there is a proof of* $\Gamma \Rightarrow \Delta$ *that has a local height at most* n *and is locally cut-free.*

Proof. We prove by induction on the local height of $\mathcal{D}$ that if there is a $\mathbf{G3GL}^{\mathrm{cut},\infty}$ proof $\mathcal{D}$ of $\Gamma \Rightarrow \Delta, \bot$ whose main fragment is cut-free, then there is a $\mathbf{G3GL}^{\mathrm{cut},\infty}$ proof $\mathcal{D}'$ of $\Gamma \Rightarrow \Delta$ whose main fragment is also cut-free.

The case where r is axiomatic is trivial. By the shape of $\Gamma \Rightarrow \Delta, \bot$, the bottom-most rule r of $\mathcal{D}$ is not K4. Suppose r is a one-premise rule. If $\bot$ is not principal, then by the IH, there is a proof $\mathcal{D}'$ of $\Gamma' \Rightarrow \Delta'$ whose main fragment preserves the cut-freeness of the main fragment of $\mathcal{D}$, where $\Gamma' \Rightarrow \Delta', \bot$ is the premise of r at the bottom of $\mathcal{D}$. Then apply r at the bottom of $\mathcal{D}'$. If $\bot$ is principal, then $\mathsf{r} = \mathsf{w_L}$. Then remove the bottom-most inference step in $\mathcal{D}$.

Suppose r is two-premise. Note that the $\bot$ in the succedent of $\Gamma \Rightarrow \Delta, \bot$ is not principal. So $\mathcal{D}$ has the following shape. We use dots to indicate that $\mathcal{D}_0$ and $\mathcal{D}_1$ can be infinite.



By IH, there are $\mathbf{G3GL}^{\mathrm{cut},\infty}$ proofs $\mathcal{D}_0'$ and $\mathcal{D}_1'$ of $\Gamma_0 \Rightarrow \Delta_0$ and $\Gamma_1 \Rightarrow \Delta_1$ that preserves cut-freeness of the main fragments of $\mathcal{D}_0$ and $\mathcal{D}_1$. Combining these proofs by r, we get a desired $\mathbf{G3GL}^{\mathrm{cut},\infty}$ proof of $\Gamma \Rightarrow \Delta$. □

Now, we can prove the crucial lemma. Recall from Definition 4.24 that $\mathbb{P}^{\mathrm{ff}}(\mathbf{G3GL}^{\mathrm{cut},\infty})$ is just like the set of $\mathbf{G3GL}^{\mathrm{cut},\infty}$ proofs, except that each of its elements is implemented with an equivalence relation that gives the information of how each of them is finitely fragmented.

Lemma 5.7 (Cut reduction for $\mathbf{G3GL}^{\mathrm{cut},\infty}$)**.** *There is a function* $\mathrm{cutsup} : \mathbb{P}^{\mathrm{ff}}(\mathbf{G3GL}^{\mathrm{cut},\infty}) \to \mathbb{P}^{\mathrm{ff}}(\mathbf{G3GL}^{\mathrm{cut},\infty})$ *such that, for any* $\Gamma \Rightarrow \Delta$*, if* $\mathcal{D} \in \mathbb{P}^{\mathrm{ff}}(\mathbf{G3GL}^{\mathrm{cut},\infty})$ *is a proof of* $\Gamma \Rightarrow \Delta$*, then* $\mathrm{cutsup}(\mathcal{D}) \in \mathbb{P}^{\mathrm{ff}}(\mathbf{G3GL}^{\mathrm{cut},\infty})$ *is also a proof of* $\Gamma \Rightarrow \Delta$*, with its main fragment cut-free.*

Proof. We show this by showing the following: for any $\Gamma \Rightarrow \Delta$, if $\mathcal{D} \in \mathbb{P}^{\mathrm{ff}}(\mathbf{G3GL}^{\mathrm{cut},\infty})$ is a proof of $\Gamma \Rightarrow \Delta$ and the cut at the root of $\mathcal{D}$ is the only cut in its main fragment, then there is a $\mathcal{D}' \in \mathbb{P}^{\mathrm{ff}}(\mathbf{G3GL}^{\mathrm{cut},\infty})$ of $\Gamma \Rightarrow \Delta$ whose main fragment is cut-free. This is shown by induction on the pair (n, m) of cut degree and the sum of the local heights of $\mathcal{D}_0$ and $\mathcal{D}_1$, where $\mathcal{D}_0$ and $\mathcal{D}_1$ are given by the following picture of $\mathcal{D}$.

$$\dfrac{\dfrac{\vdots\ \mathcal{D}_0}{\Gamma_0 \Rightarrow \Delta_0, A} \quad \dfrac{\vdots\ \mathcal{D}_1}{A, \Gamma_1 \Rightarrow \Delta_1}}{\Gamma \Rightarrow \Delta}\,\mathsf{cut}$$

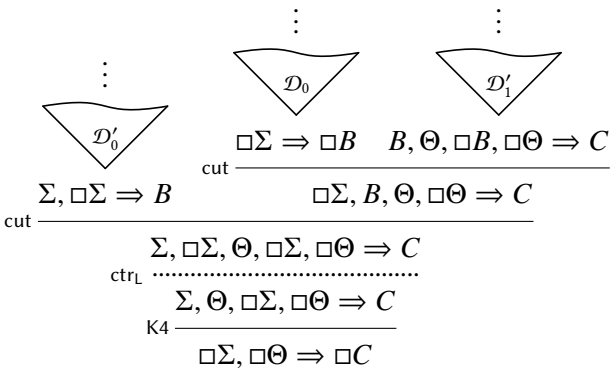
The base case $(0, 0)$ is trivial. For the inductive case, we follow the same case distinction as in Lemma 3.29. The proof is very similar, we only need to check that the IH can be applied since we have changed the measure. For the first three cases, the IH is not used, and we here appeal to Proposition 5.6. For the rest of the cases, it can be verified that the IH can indeed be applied. The case where both r_0 and r_1 are $\mathsf{K4}$ is worth mentioning. Recall that, in this case, $\mathcal{D}$ is as follows.

$$\dfrac{\dfrac{\dfrac{\vdots\ \mathcal{D}_0'}{\Sigma, \Box\Sigma \Rightarrow B}\,\mathsf{K4}}{\Box\Sigma \Rightarrow \Box B} \quad \dfrac{\dfrac{\vdots\ \mathcal{D}_1'}{B, \Theta, \Box\Theta, \Box B \Rightarrow C}\,\mathsf{K4}}{\Box B, \Box\Theta \Rightarrow \Box C}}{\Box\Sigma, \Box\Theta \Rightarrow \Box C}\,\mathsf{cut}$$

By definition, the main fragment of $\mathcal{D}$ is this.

$$\dfrac{\dfrac{}{\Box\Sigma \Rightarrow \Box B}\,\mathsf{K4} \quad \dfrac{}{\Box B, \Box\Theta \Rightarrow \Box C}\,\mathsf{K4}}{\Box\Sigma, \Box\Theta \Rightarrow \Box C}\,\mathsf{cut}$$

Then, we perform the following procedure. Recall that we can use $\mathsf{ctr}_{\mathsf{L}}$ thanks to Proposition 5.5.



Here, we do not use IH. Note that the main fragment of the proof above, i.e. the root, is cut-free.

Then, given the axiom of choice, the lemma we aim to prove can be shown by induction on the number of cut applications in the main fragment of a given proof. □

Remark 5.8. *We use the axiom of choice to simplify the proof. We conjecture that a desired function can be defined recursively, which requires us also to define functions recursively for invertibility and contraction results.*

Theorem 5.9 (Cut elimination for $\mathbf{G3GL}^{\mathrm{cut},\infty}$)**.** *For any sequent S, if $\vdash_{\mathbf{G3GL}^{\mathrm{cut},\infty}} S$, then $\vdash_{\mathbf{G3GL}^{\infty}} S$.*

Proof. We define a function $\alpha : \mathbb{P}^{\mathrm{ff}}(\mathbf{G3GL}^{\mathrm{cut},\infty}) \to \mathcal{T}(\mathbb{P}^{\mathrm{ff}}(\mathbf{G3GL}^{\infty}))$ such that

$$\alpha(T,\sim) = (\mathsf{T}_{\epsilon}(\mathrm{cutsup}(T,\sim)), w \in \mathcal{NW}^{\mathsf{T}_{\epsilon}(\mathrm{cutsup}(T,\sim))} \mapsto \mathsf{ST}_{w}(\mathrm{cutsup}(T))) = \mathrm{destruct} \circ \mathrm{cutsup}(T,\sim)$$

We show that α is a proof translation step from $\mathbf{G3GL}^{\mathrm{cut},\infty}$ to $\mathbf{G3GL}^{\infty}$. Let $(T,\sim) \in \mathbb{P}^{\mathrm{ff}}(\mathbf{G3GL}^{\mathrm{cut},\infty})$, we first show that $(\mathsf{F}^{\alpha}_{\mathsf{nil}}(T,\sim), w \mapsto \mathrm{seq}_{\mathsf{F}^{\alpha}_{[w]}(T,\sim)})$ is a proof fragment in $\mathbf{G3GL}^{\infty}$. By definition, $\mathsf{F}^{\alpha}_{\mathsf{nil}}(T,\sim) = \mathsf{T}_{\epsilon}(\mathrm{cutsup}(T,\sim))$, which is the cut-free main α-fragment of $\mathrm{cutsup}(T,\sim)$, a $\mathbf{G3GL}^{\mathrm{cut},\infty}$ proof of S. Clearly, $\mathsf{T}_{\epsilon}(\mathrm{cutsup}(T,\sim))$ is a tree with non-wellfounded leaves; redecorating each of such leaves w with $\mathrm{seq}_{\mathsf{F}^{\alpha}_{[w]}(T,\sim)}$ yields a proof fragment in $\mathbf{G3GL}^{\infty}$. Next, we argue that for any $w \in \mathcal{NW}^{\mathsf{F}^{\alpha}_{\mathsf{nil}}(T,\sim)}$, $\mathsf{SE}^{\alpha}_{[w]} \in \mathbb{P}^{\mathrm{ff}}(\mathbf{G3GL}^{\mathrm{cut},\infty})$. By definition, $\mathsf{SE}^{\alpha}_{[w]} = \mathsf{SE}^{\alpha}_{\mathsf{nil}}(\mathsf{ST}_{[w]}(\mathrm{cutsup}(T))) = \mathsf{ST}_{[w]}(\mathrm{cutsup}(T))$, which is a finite fragmented $\mathbf{G3GL}^{\mathrm{cut},\infty}$ proof. Then, by Theorem 4.27, there is a proof translation function from $\mathbf{G3GL}^{\mathrm{cut},\infty}$ to $\mathbf{G3GL}^{\infty}$ such that, given a $\mathbf{G3GL}^{\mathrm{cut},\infty}$ proof of S, it outputs a $\mathbf{G3GL}^{\infty}$ proof of the same sequent. □

Corollary 5.10 (Subformula property of $\mathbf{G3GL}^{\mathrm{cut},\infty}$)**.** *For any sequent S, if $\vdash_{\mathbf{G3GL}^{\mathrm{cut},\infty}} S$, then there is a proof of it such that any formula occurring in the proof is a subformula of some formula occurring in the sequent.*

Proof. This follows from Theorem 5.9 and the fact that for any rule of $\mathbf{G3GL}^{\mathrm{cut},\infty}$ except for cut, the formulas in the premises are subformulas of formulas in the conclusion. □

Then, we can prove that for any sequent S, if $\vdash_{\mathbf{G3GL}^{\mathrm{cut},\infty}} S$, then $\vdash_{\mathbf{G3GL}^{\mathrm{cut},\circ}} S$.

Definition 5.11 (Subformula closure)**.** *Let $\Gamma \Rightarrow \Delta$ be a sequent. Let $\mathrm{Clos}(\Gamma \Rightarrow \Delta)$ be the set that contains all and only the subformulas of any formula A such that $\#_{\Gamma}(A) + \#_{\Delta}(A) > 0$.* ⊣

Definition 5.12 (Support multiset)**.** *Given any multiset Γ of formulas in* Form*, define the* support multiset Γ^{s} *of Γ as the multiset that satisfies the following conditions*

$$\#_{\Gamma^{s}}(A) = 1 \textit{ iff } \#_{\Gamma}(A) > 0;$$

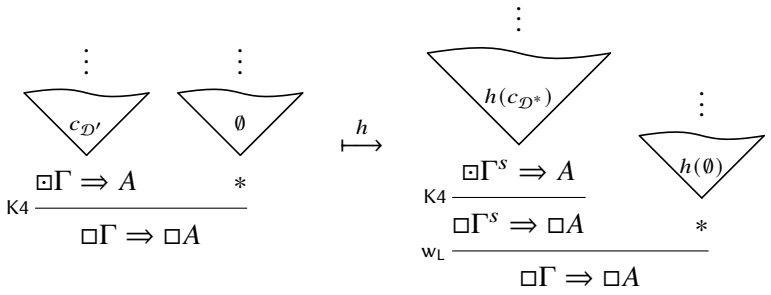
$$\#_{\Gamma^s}(A) = 0 \textit{ iff } \#_{\Gamma}(A) = 0.$$

For any sequent $\Gamma \Rightarrow \Delta$, *let* $(\Gamma \Rightarrow \Delta)^s := \Gamma^s \Rightarrow \Delta^s$. ⊣

For any sequent $\Gamma \Rightarrow \Delta$, we may call $(\Gamma \Rightarrow \Delta)^s$ the 'setification' of $\Gamma \Rightarrow \Delta$.

Proposition 5.13. *For any sequent* S, *if* $\vdash_{\mathbf{G3GL}^{\mathrm{cut},\infty}} S$, *then* $\vdash_{\mathbf{G3GL}^{\mathrm{cut},\circ}} S$.

Proof. Let S be a sequent and $\mathcal{D}$ a $\mathbf{G3GL}^{\mathrm{cut},\infty}$ proof of it. By Theorem 3.30, there is a $\mathbf{G3GL}^{\infty}$ proof $\mathcal{D}' = (N, \prec, l)$ of it. Recall that $\prec$ is the immediate successor relation on N, and l is the labeling function. We transform $\mathcal{D}'$ to a proof of S but with only finite sequents. We define a function h by corecursion in a similar way in Proposition 5.5 which turns a proof of S into a proof of the same sequent with only finitely many sequents. This time, h commutes with propositional rules. For the modal rule, we define h as follows. Let $\mathcal{D}$ be a proof whose bottom-most rule is K4. If the conclusion of $\mathcal{D}$ is S and $S = S^s$, then let $\langle l', o', l'\rangle(c_{\mathcal{D}}) = \langle l'(c_{\mathcal{D}}), o'(c_{\mathcal{D}}), l'(c_{\mathcal{D}})\rangle = (l(c_{\mathcal{D}}), S, \emptyset)$. If $S \neq S^s$, assume $S = \Box\Gamma \Rightarrow \Box A$, then the premise is $\boxdot\Gamma \Rightarrow A$. By contraction, there is a cut-free proof $\mathcal{D}''$ of $(\boxdot\Gamma \Rightarrow A)^s = \boxdot\Gamma^s \Rightarrow A$. Apply K4 under $\mathcal{D}''$, we get a proof $\mathcal{D}^*$ of $\Box\Gamma^s \Rightarrow \Box A$. Then, set $\langle l', o', l'\rangle(c_{\mathcal{D}}) = \langle l'(c_{\mathcal{D}}), o'(c_{\mathcal{D}}), l'(c_{\mathcal{D}})\rangle = (c_{\mathcal{D}^*}, (\mathsf{w_L}, \Box\Gamma^s \Rightarrow \Box A), \emptyset)$. Pictorially, h works like this.



Using h, we can transform $\mathcal{D}$ into $h(\mathcal{D})$, which has only finitely many sequents serving as the premises of K4 rule applications. We further argue that $h(\mathcal{D})$ has only finitely many sequents in total.

Let $\mathrm{Clos}(S) = \{A_0, \ldots, A_{n-1}\}$ be the subformula closure of S (we assume that $|\mathrm{Clos}(S)| = n$). Let $k = \sum_{A \in \mathrm{Clos}(S)} w(A)$. First, note that in $h(\mathcal{D})$, there are at most $2^n \cdot n$ sequents that can serve as premises of K4 applications, for we have 'setified' each premise of each K4 application. 2^n counts the number of support multisets that can be constructed from $\mathrm{Clos}(S)$ and serve as the antecedent of a setified K4 premise, while n is just the cardinality of $\mathrm{Clos}(S)$ that counts the possibilities of the succedents of K4 premises. For each premise of a K4 application P in $h(\mathcal{D})$, define

$$Prop(P) := \{S \in \mathrm{Seq} : S \text{ can be obtained from } P \text{ by applying only propositional rules other than cut}\}.$$

More formally, S is in $Prop(P)$ iff there is a finite sequence of sequents $S_0 = P, \ldots, S_m = S$ such that each S_{i+1} is obtained by applying a rule other than cut and K4 to S_i in a bottom-up manner. We argue that $Prop(P)$ is finite, for any P. Note that for any P, $w(P) \leqslant k$. Then, for any P, there are at most k^{2n} many sequents in $Prop(P)$. This is because we can represent each sequent with a vector $(\#_0, \ldots, \#_{2n-1})$ with $2n$-many entries, where each of the first group of n entries represents the number of instances of each $A_i \in \mathrm{Clos}(S)$ in the antecedent, and the second group of n entries represents those in the succedent. Note that each $\#_i$ is no larger than k, for otherwise a sequent in $Prop(P)$ would have a greater weight than P, which is impossible because propositional rules decrease sequent weight from bottom to top. So there are k^{2n} possibilities. Therefore, in total, we have at most $2^n \cdot n \cdot k^{2n} = n \cdot (2k^2)^n$ sequents decorating $h(\mathcal{D})$.

Fix an arbitrary infinite branch b in $h(\mathcal{D})$. Since it is decorated by finitely many sequents, there is at least

one sequent S' that decorates infinitely many nodes on this branch. Then, there is a lower-most node w on b decorated with the repeating sequent S'. Since there are infinitely many nodes on b that are decorated with K4, we can choose the lower-most node u that is not lower than w on b and is decorated with K4. Moreover, since S' occurs on b infinitely often, there is a lower-most node v on the path that is not lower than u and is decorated again with S'. So we have found a successful repeat. Let the set of nodes strictly above v be $b^- = \{x \in N : v \sqsubset x\}$. For each member b in the set of infinite branches B in $h(\mathcal{D})$, we define b^- similarly. Consider $(N \setminus \bigcup_{b\in B} b^-, \prec\restriction_{N\setminus\bigcup_b b^-}, l \restriction_{N\setminus\bigcup_b b^-})$. This (labeled) relational structure is a finite tree, the labeling function satisfies the condition of a $\mathbf{G3GL}^{\mathrm{cut},\circ}$ proof, and each node at a 'cut off point' has a companion. By relabeling such nodes with their sequents only (recall that the non-axiomatic leaves of a cyclic proof are decorated with a sequent *only*), we get a $\mathbf{G3GL}^{\mathrm{cut},\circ}$ proof of S. □

Theorem 5.14. *For any sequent S, $\vdash_{\mathbf{G3GL}^{\mathrm{cut},\infty}} S$ iff $\vdash_{\mathbf{G3GL}^{\mathrm{cut},\circ}} S$.*

Proof. This follows from Lemma 3.38 and Proposition 5.13. □

Corollary 5.15. $\mathbf{G3GL}^{\mathrm{cut},\circ}$ *enjoys cut elimination.*

Proof. Suppose that a sequent S is provable in $\mathbf{G3GL}^{\mathrm{cut},\circ}$, then it is provable, and hence cut-free provable, in $\mathbf{G3GL}^{\mathrm{cut},\infty}$. Say there is a $\mathbf{G3GL}^{\infty}$ proof of S, $\mathcal{D}$. By the construction in Proposition 5.13, we get a cut-free $\mathbf{G3GL}^{\mathrm{cut},\circ}$ proof of S. □

5.2 A direct cut elimination procedure for $\mathbf{G3GL}^{\mathrm{cut},\circ}$

In this section, we provide a direct cut reduction procedure for $\mathbf{G3GL}^{\mathrm{cut},\circ}$ as promised. It is direct in the sense that we do not need a detour via the cut elimination of $\mathbf{G3GL}^{\mathrm{cut},\infty}$. On a very high level: the argument has two components. The first is the cut reduction procedure that pushes up cut applications, and the second is a combinatorial argument which is inspired by the proof of Lemma 18 in [AK24].

Since [AK24] provided the first direct cut reduction proof for cyclic systems and the methods developed by the same authors for other systems are similar, it might be interesting to compare our method to theirs. The system studied in [AK24] is annotated, and the progressing condition for a cycle is to pass through a certain rule plus satisfying a condition regarding annotation. Using annotation, they manage to neatly distinguish between important cut applications and unimportant ones. Important ones are those the reduction of which might delete the rule that is necessary for a cycle. They prove cut elimination by treating important ones and unimportant ones differently.

For $\mathbf{G3GL}^{\mathrm{cut},\circ}$, the progressing condition is simpler, as we do not work with annotated sequents and there is only one progressing condition. Moreover, it turns out that all the cut applications in $\mathbf{G3GL}^{\mathrm{cut},\circ}$ can be treated uniformly.

Theorem 5.16 (Cut reduction procedure for $\mathbf{G3GL}^{\mathrm{cut},\circ}$)**.** *Let $\mathcal{D} = (N, \prec \cup f, l)$ be a $\mathbf{G3GL}^{\mathrm{cut},\circ}$ proof of $\Gamma \Rightarrow \Delta$, then there is a $\mathbf{G3GL}^{\circ}$ proof of $\Gamma \Rightarrow \Delta$.*

Proof. We will assume that operations like replacing subtrees in a tree always yield a tree-like structure, and therefore a tree.

If f is empty (i.e. there is no cycle), then $\mathcal{D}$ is a $\mathbf{G3K4}^{\mathrm{cut}}$ proof. By Theorem 3.30, there is a **G3K4** proof of $\Gamma \Rightarrow \Delta$, which is also a $\mathbf{G3GL}^{\circ}$ proof of $\Gamma \Rightarrow \Delta$.

Suppose f is not empty. Consider the infinite proof $\mathcal{D}^{\infty}$ yielded by the unraveling of $\mathcal{D}$. We 'setify' the premises of K4 applications in $\mathcal{D}^{\infty}$ and get a proof $(\mathcal{D}^{\infty})^s$ of $\Gamma \Rightarrow \Delta$. I.e. we turn each proof of a K4 premise

$\Gamma' \Rightarrow \Delta'$ in $\mathcal{D}^\infty$ into a proof of the sequent $(\Gamma' \Rightarrow \Delta')^s$. The setification is done by using the function h defined in Proposition 5.13. For each node in this tree, we define its height in the obvious way, starting from 1. Let $S = \Gamma \Rightarrow \Delta$, $|\mathsf{Clos}(S)| = n$, and $k = \sum_{A \in \mathsf{Clos}(S)} w(A)$. Then there is a proof of S such that (i) all nodes of height at most $n \cdot (2k^2)^n + 1$ are free of cut, and (ii) all premises of K4 applications are setified. The existence of such a proof is guaranteed by the following argument.

Given the cut reduction for $\mathbf{G3GL}^{\mathrm{cut},\infty}$ (Lemma 5.7), we can inductively construct an ω sequence of proofs $(\mathcal{D}_i)_{i<\omega}$ of S.

- $\mathcal{D}_0 := (\mathcal{D}^\infty)^s$.
- Given $\mathcal{D}_n$, which is a proof of S, construct a proof $\mathcal{D}_{n+1}$ of S as follows.

  (i) If $\mathcal{D}_n$ is a cut-free $\mathbf{G3GL}^{\mathrm{cut},\infty}$ proof of S, let $\mathcal{D}_{n+1} := \mathcal{D}_n$.

  (ii) Otherwise, locate the left-bottom-most cut application in $\mathcal{D}_n$; say its conclusion is some sequent S'. Let the sub-proof rooted at this designated S' be T'. By Lemma 5.7, there is a proof $\mathcal{D}'$ of S' whose main fragment is cut-free. We consider two cases.

  (a) If $\mathcal{D}'$ is finite, by the cut elimination of $\mathbf{G3K4}^{\mathrm{cut}}$, there is a cut-free $\mathcal{D}''$ of S'. We setify the premises of K4 applications in $\mathcal{D}''$ and get a cut-free proof $(\mathcal{D}'')^s$ of S'. This procedure of setification is justified because we have contraction and cut-elimination for $\mathbf{G3K4}^{\mathrm{cut}}$. Replace T' with $(\mathcal{D}'')^s$ in this case.

  (b) If $\mathcal{D}'$ is infinite, then we setify the premises of K4 applications in $\mathcal{D}'$ and get a proof $(\mathcal{D}')^s$ of S'. The setification is again done by using the function h defined in Proposition 5.13. Note that h preserves the cut-freeness of the main fragment. Replace T' with $(\mathcal{D}')^s$ in this case.

  After this operation of replacement, we get a proof $\mathcal{D}'_n$. Check whether $\mathcal{D}'_n$ is finite. If yes, by the cut elimination of $\mathbf{G3K4}^{\mathrm{cut}}$, there is a cut-free $\mathbf{G3K4}^{\mathrm{cut}}$ proof $\mathcal{D}$ of S. Set $\mathcal{D}_{n+1} := \mathcal{D}$. If not, set $\mathcal{D}_{n+1} := \mathcal{D}'_n$.

Recall that $n = |\mathsf{Clos}(S)|$ and $k = \sum_{A \in \mathsf{Clos}(S)} w(A)$. Consider the nodes of a full binary tree with height at most $n \cdot (2k^2)^n + 1$. For convenience, we use '$\mathbf{u}$' to denote this number. Then, there are $2^{\mathbf{u}} - 1$ many nodes in the full binary tree whose height is no larger than $\mathbf{u}$. We claim that there is some $m \leqslant 2^{\mathbf{u}}$ such that $\mathcal{D}_m$ is free of cut in its initial segment that is not higher than $\mathbf{u}$. We argue by considering two cases. First, if there is some $m \leqslant 2^{\mathbf{u}}$ such that $\mathcal{D}_m$ is free of cut, then proceed as in Proposition 5.13 if this $\mathcal{D}_m$ is infinitary. If $\mathcal{D}_m$ is finite, then it is already a cut-free cyclic proof.

Otherwise, for any m such that $0 < m \leqslant 2^{\mathbf{u}}$, we have that $\mathcal{D}_m$ is not cut-free. This implies that each $\mathcal{D}_m$ is infinite, and that each $\mathcal{D}_m$ is obtained by replacing a subtree in $\mathcal{D}_{m-1}$ with another proof tree. By construction, for each $\mathcal{D}_m$, it contains fewer cut applications than $\mathcal{D}_{m-1}$ on level i, where the i-th level is the lowest level in $\mathcal{D}_{m-1}$ that is not free of cut. At most, we need to perform the replacement of a subtree for *every* node of the initial segment of a full binary tree with height at most $\mathbf{u}$. That means, after $2^{\mathbf{u}}$ times of iterations, which is the number of nodes with height at most $\mathbf{u}$ plus one, the obtained proof tree is cut-free in its initial segment with height at most $\mathbf{u}$. This is because the construction never reintroduces lower applications of cut. So $\mathcal{D}_{2^{\mathbf{u}}}$ is a proof of S that is cut-free in its initial segment with height at most $\mathbf{u}$. Moreover, since $\mathcal{D}_0 = (\mathcal{D}^\infty)^s$ has only setified K4 premises and whenever we replace a subtree with another, we replace the original one with another that contains only setified K4 premises, we can prove by induction that each $\mathcal{D}_{i<\omega}$ has setified K4 premises. Hence, $\mathcal{D}_{2^{\mathbf{u}}}$ is also setified in K4 premises. In particular, its segment strictly below $\mathbf{u} + 1$ is both cut-free and setified in K4 premises.

Consequently, as shown by the combinatorial argument in Proposition 5.13, there are at most $n \cdot (2k^2)^n$ sequents in the initial fragment of $\mathcal{D}_{2^{\mathbf{u}}}$ that is not higher than $\mathbf{u} = n \cdot (2k^2)^n + 1$. Therefore, for any node ν in $\mathcal{D}_{2^{\mathbf{u}}}$ of height $\mathbf{u} = n \cdot (2k^2)^n + 1$, there is a node below it that is decorated with the same sequent as ν. For each ν, choose a lowest such node $c(\nu)$. The path between ν and $c(\nu)$ passes through an application of K4 because all rules reduce the number of connectives if read bottom-up, with only K4 (and cut) being an exception. So each node of $\mathcal{D}_{2^{\mathbf{u}}}$ with height $\mathbf{u}$ has a companion node, with a K4 application in between. Delete all the nodes strictly above this fragment, redecorate its leaves with their sequents only, and supplement the underlying structure with $\{(\nu, c(\nu)) : \nu \text{ is a leaf of the remaining fragment}\}$, we get a cut-free $\mathbf{G3GL}^{\mathrm{cut},\circ}$ proof of S. □

In this proof, we appealed to Lemma 5.7, but not to the cut elimination of the infinitary system. Crucially, the former does not depend on the $\mathcal{T}$-coalgebraic machinery introduced in section 2, while the latter does. The use of $\mathcal{T}'$-corecursive definition is not easily dispensable in the proof above because we need contraction after all.

Remark 5.17. *The reader might find the estimation of the number of sequents and the process of setification unnecessarily complicated. Indeed, if we work with set-based sequents, the estimation would be much simpler, and there would be no need for setification. Suppose we work with set-based sequents. Then, given the cardinality of the subformula closure of a sequent S, say n, there can be at most $2^n \cdot 2^n$ many distinct sequents in a cut-free proof of S. So the finiteness of sequents of a cut-free infinitary proof is immediate. Moreover, we will also have contraction a priori, and there will be no need for $\mathcal{T}'$-coalgebraic proof translation and many invertibility results.*

However, the proof above appeals to an essential property of the given multiset-based sequent calculi – that all propositional rules decrease the complexity of sequents if read bottom-up. Set-based sequent calculi do not have this feature, for the principal formula might also be in the context, and how this obstacle can be overcome turns out to be non-trivial. More about this will be discussed in the next chapter.

This cut reduction procedure for $\mathbf{G3GL}^{\mathrm{cut},\circ}$ may indicate a rough picture of how cut elimination of a certain type of cyclic systems works in general. This type of cyclic system has the following characteristics.

(a) The progressing condition is to pass through a certain rule, which we call the *progressing rule*.

(b) The rules except for the progressing rule and the cut rule are *upward-decreasing*, meaning that there is some well-founded measure m such that for any such rule with conclusion C and premise P_i, $m(P_i) < m(C)$.

(c) The cut-free fragment of the system has the *subformula property*.

(d) There is a *local cut reduction procedure* for the underlying finite system such that each cut application can eventually be 'pushed up'.

This proof also shows how cut reduction for cyclic systems can be 'simpler' than its infinitary and well-founded correspondence. For the infinitary proof system, we proved the cut elimination using both $\mathcal{T}'$-corecursive definition and $\mathcal{T}$-coalgebraic proof translation; while for the well-founded **GL**, the proof is famously highly involved.

Remark 5.18. *Note that the equivalence between* $\mathbf{G3GL}^{\mathrm{cut},\circ}$ *and* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *does not apparently follow from the cut elimination of* $\mathbf{G3GL}^{\mathrm{cut},\circ}$*. The difficulty, again, is this direction:* $\mathbf{G3GL}^{\mathrm{cut},\infty} \vdash S$ *implies* $\mathbf{G3GL}^{\mathrm{cut},\circ} \vdash S$*. Suppose there is a* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *proof of S. Since there is* cut*, it is unclear how to argue that there are only finitely many sequents decorating the infinite tree. Consequently, it remains unclear how to 'fold' this tree into a finite one*

and then apply the cut elimination of $\mathbf{G3GL}^{\mathrm{cut},\circ}$*. Then, the cut elimination of the infinitary system could be more significant in the current context: it yields the equivalence between the two systems and the cut elimination of the cyclic one.*

5.3 Soundness and completeness

We close this chapter by proving that the non-wellfounded systems we provided are indeed proof systems for **GL**.

Lemma 5.19 (Local soundness of $\mathbf{G3GL}^{\mathrm{cut},\infty}$)**.** *For any* $\mathsf{r} \in \mathbf{G3K4}^{\mathrm{cut}}$*, if all its premises are valid in the class of transitive frames, then so is its conclusion. If* r *has no premise, then its conclusion is valid in any transitive frame.*

Proof. The cases of init and $\bot_{\mathsf{L}}$ are easy. The cases of logical rules are also easy. Take $\wedge_{\mathsf{R}}$ for an example. If the conclusion is not valid, then there is some transitive frame $\mathbb{F} = (W, R)$, some $x \in W$, and some valuation V such that $(\mathbb{F}, V), x \Vdash \bigwedge \Gamma$ and $(\mathbb{F}, V), x \nVdash (\bigvee \Delta) \vee (A \wedge B)$. Then, at least one premise is refuted at the same pointed model. A similar argument also works for cut.

Consider $\mathsf{K4}$.

$$\mathsf{K4}\ \frac{\Sigma, \Box\Sigma \Rightarrow A}{\Box\Sigma \Rightarrow \Box A}$$

Assume for some transitive frame $\mathbb{F} = (W, R)$, some $x \in W$, and some V, it holds that $(\mathbb{F}, V), x \nVdash \bigwedge \Box\Sigma \to \Box A$. Then, for all $\Box D \in \Sigma$, we have $(\mathbb{F}, V), x \Vdash \Box D$. For $\Box A$ we have $(\mathbb{F}, V), x \nVdash \Box A$. Then, there is some $y \in W$ such that xRy and $(\mathbb{F}, V), y \nVdash A$. But we also have $(\mathbb{F}, V), y \Vdash \bigwedge \Box\Sigma$ and $(\mathbb{F}, V), y \Vdash \bigwedge \Sigma$ since $\mathbb{F}$ is transitive. Therefore, the premise is refuted by a pointed model as well. □

Theorem 5.20 (Semantic soundness of $\mathbf{G3GL}^{\mathrm{cut},\infty}$)**.** *Let* $\Gamma \Rightarrow \Delta \in \mathsf{Seq}$*. If* $\mathbf{G3GL}^{\mathrm{cut},\infty} \vdash \Gamma \Rightarrow \Delta$*, then* $\mathbb{F} \Vdash \Gamma \Rightarrow \Delta$ *for any transitive and conversely well-founded* $\mathbb{F}$*.*

Proof. Fix a $\Gamma \Rightarrow \Delta \in \mathsf{Seq}$. Suppose there is a $\mathbf{G3GL}^{\mathrm{cut},\infty}$ proof of $\Gamma \Rightarrow \Delta$, but there is some transitive and conversely well-founded $\mathbb{F} = (W, R)$, some $x \in W$, and some V such that $(\mathbb{F}, V), x \nVdash \Gamma \Rightarrow \Delta$. By Lemma 5.19, there exists a branch of the proof tree such that each node is decorated with a sequent whose standard interpretation is not valid in the frame class. If the branch is finite, we run into a contradiction immediately because its top-most node is decorated by init or $\bot_{\mathsf{L}}$.

If the branch is infinite, then it has infinitely many applications of $\mathsf{K4}$. Fix an infinite branch and consider an arbitrary application of $\mathsf{K4}$ on the branch. Let the sequence of sequents on this branch be $(S_n)_{n\in\omega}$. We know that each S_i is falsified in some model $\mathbb{M}, x_i$ based on a frame $\mathbb{F}$ of **GL**. Moreover, we can argue by induction that for any n, either $x_n = x_{n+1}$ or $x_n R x_{n+1}$ if (S_n, S_{n+1}) is a rule instance of $\mathsf{K4}$. Since there are infinitely many S_i being conclusions of $\mathsf{K4}$, there is an infinite R-path starting from x_0. We have a contradiction because $\mathbb{F}$ is conversely well-founded. □

Theorem 5.21 (Syntactic soundness of $\mathbf{G3GL}^{\mathrm{cut},\infty}$)**.** *Let* $\Gamma \Rightarrow \Delta \in \mathsf{Seq}$*. If* $\vdash_{\mathbf{G3GL}^{\mathrm{cut},\infty}} \Gamma \Rightarrow \Delta$*, then* $\vdash_{\mathbf{HGL}} \bigwedge \Gamma \to \bigvee \Delta$*.*

Proof. If $\mathbf{G3GL}^{\mathrm{cut},\infty} \vdash \Gamma \Rightarrow \Delta$, then by Theorem 5.20, the formula $\bigwedge \Gamma \to \bigvee \Delta$ is valid in the frames of **GL**. Then, we have $\vdash_{\mathbf{HGL}} \bigwedge \Gamma \to \bigvee \Delta$ by the completeness of **HGL**. □

Then, we are going to prove cut-free completeness. First, we show that all the axioms and the inference rules of the Hilbertian system are derivable.

Definition 5.22 (Rule derivability)**.** *A rule* r *is derivable in a proof system* C *iff for every instance* $r \in \mathsf{r}$*, if every premise is derivable in* C*, then there is a labeled tree that is just like a* C*-proof of the conclusion of* r*, except that each leaf is either axiomatic or is decorated with a premise of* r*.* ⊣

Proposition 5.23. *For any formula* A*, any finite multisets of formulas* Γ, Δ*, the following rule is derivable in* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *(and* $\mathbf{G3GL}^{\mathrm{cut},\circ}$*).*

$$\dfrac{}{A, \Gamma \Rightarrow \Delta, A}\ A \Rightarrow A$$

Proof. By induction on the complexity of A. The base cases are simple, so we only consider the inductive cases. For propositional connectives, we choose $\to$ as a representative. Suppose we have proofs of $A, \Gamma \Rightarrow \Delta, A$ and $B, \Gamma \Rightarrow \Delta, B$. Then, we can construct the following proof.

$$\dfrac{\dfrac{\overset{\text{IH}}{A, \Gamma \Rightarrow \Delta, A} \qquad \overset{\text{IH}}{B, \Gamma \Rightarrow \Delta, B}}{A \to B, A, \Gamma \Rightarrow \Delta, B}\ {\to_{\mathsf{L}}}}{A \to B, \Gamma \Rightarrow \Delta, A \to B}\ {\to_{\mathsf{R}}}$$

We also consider the modal case. By the IH, we have a proof of $A \Rightarrow A$. Apply $\mathsf{K4}$ and then weakening rules, we get a proof of $\Box A, \Gamma \Rightarrow \Delta, \Box A$ for any non-empty Γ and Δ. □

Proposition 5.24. K *and Löb's axiom are derivable in* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *and* $\mathbf{G3GL}^{\mathrm{cut},\circ}$*.*

Proof. We construct the following circular proofs, from which we can also construct infinite proofs.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A}{\Box(A \to B), \Box A, A \Rightarrow A} \qquad \dfrac{A \Rightarrow A}{\Box(A \to B), A \to B, A, B \Rightarrow B}\ {\to_{\mathsf{L}}}}{\Box(A \to B), \Box A, A \to B, A \Rightarrow B}\ \mathsf{K4}}{\Box(A \to B), \Box A \Rightarrow \Box B}\ {\to_{\mathsf{R}}}}{\Box(A \to B) \Rightarrow \Box A \to \Box B}\ {\to_{\mathsf{R}}}}{\Rightarrow \Box(A \to B) \to (\Box A \to \Box B)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\rightsquigarrow \Box(\Box A \to A) \Rightarrow \Box A \qquad \dfrac{A \Rightarrow A}{\Box(\Box A \to A), A \Rightarrow A}}{\Box(\Box A \to A), \Box A \to A \Rightarrow A}\ \mathsf{K4}}{\rightsquigarrow \Box(\Box A \to A) \Rightarrow \Box A}\ {\to_{\mathsf{R}}}}{\Box(\Box A \to A) \to \Box A}\ {\to_{\mathsf{R}}}}{\to_{\mathsf{L}}}$$

□

Moreover, necessitation and modus ponens can be simulated in both $\mathbf{G3GL}^{\mathrm{cut},\infty}$ and $\mathbf{G3GL}^{\mathrm{cut},\circ}$.

Proposition 5.25. *The following rules are derivable in* $\mathbf{G3GL}^{\mathrm{cut},\infty}$ *and* $\mathbf{G3GL}^{\mathrm{cut},\circ}$.

$$\mathsf{MP}\ \frac{\Gamma \Rightarrow A \to B \qquad \Gamma \Rightarrow A}{\Gamma \Rightarrow B} \qquad\qquad \mathsf{Nec}\ \frac{\Rightarrow A}{\Rightarrow \Box A}$$

Proof. First, consider MP. Suppose we have the following $\mathbf{G3GL}^{\mathrm{cut},\infty}$ (or $\mathbf{G3GL}^{\mathrm{cut},\circ}$) proofs of $\Gamma \Rightarrow A \to B$ and $\Gamma \Rightarrow A$.

$$\begin{array}{cc} \vdots & \vdots \\ \mathcal{D}_0 & \mathcal{D}_1 \\ \Gamma \Rightarrow A \to B & \Gamma \Rightarrow A \end{array}$$

Then, we can construct the following proof of $\Gamma \Rightarrow B$.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \mathcal{D}_1\\ \Gamma \Rightarrow A\end{array}}{\mathsf{w_L}\ \dfrac{\Gamma \Rightarrow A}{\Gamma \Rightarrow B, A}} \qquad \dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \mathcal{D}_0\\ \Gamma \Rightarrow A \to B\end{array}}{\mathsf{w_R}\ \dfrac{\Gamma \Rightarrow A \to B}{\Gamma \Rightarrow B, A \to B}}}{\mathsf{w_L}\ \dfrac{\Gamma \Rightarrow B, A \to B}{A, \Gamma \Rightarrow B, A \to B}} \qquad \dfrac{\dfrac{}{A \Rightarrow A}\ \dfrac{}{A \Rightarrow A}}{\to_{\mathsf{L}}\ \dfrac{A, \Gamma \Rightarrow B, A \qquad B, A, \Gamma \Rightarrow B}{A \to B, A, \Gamma \Rightarrow B}}}{\mathsf{cut}\ \dfrac{A, \Gamma \Rightarrow B, A \to B \qquad A \to B, A, \Gamma \Rightarrow B}{A, \Gamma \Rightarrow B}}}{\mathsf{cut}\ \dfrac{\Gamma \Rightarrow B, A \qquad A, \Gamma \Rightarrow B}{\Gamma \Rightarrow B}}$$

For Nec, we can simply apply K4 to the sequent $\Rightarrow A$. □

It then follows immediately that $\mathbf{G3GL}^{\mathrm{cut},\infty}$ and $\mathbf{G3GL}^{\mathrm{cut},\circ}$, and hence $\mathbf{G3GL}^{\infty}$ and $\mathbf{G3GL}^{\circ}$, can prove every theorem of the Hilbertian system.

Lemma 5.26 (Weak completeness of $\mathbf{G3GL}^{\mathrm{cut},\infty}$)**.** *For any formula* A*, if* $\vdash_{\mathbf{HGL}} A$*, then* $\vdash_{\mathbf{G3GL}^{\mathrm{cut},\infty}} \Rightarrow A$.

Proof. By induction on the length of **HGL** proof of A. Use the derivability of the axioms and rules to simulate a proof of A in $\mathbf{G3GL}^{\mathrm{cut},\infty}$. □

Then, we have the following.

Theorem 5.27 (Syntactic completeness of $\mathbf{G3GL}^{\mathrm{cut},\infty}$)**.** *Let* $\Gamma \Rightarrow \Delta \in \mathsf{Seq}$*. If* $\vdash_{\mathbf{HGL}} \bigwedge \Gamma \to \bigvee \Delta$*, then* $\vdash_{\mathbf{G3GL}^{\mathrm{cut},\infty}} \Gamma \Rightarrow \Delta$.

Proof. Suppose $\vdash_{\mathbf{HGL}} \bigwedge \Gamma \to \bigvee \Delta$. Then by Lemma 5.26, we have that $\vdash_{\mathbf{G3GL}^{\mathrm{cut},\infty}} \Rightarrow \bigwedge \Gamma \to \bigvee \Delta$. Then, construct the following proof.

$$\dfrac{\begin{array}{c}\mathcal{D}_0\\ \Rightarrow \bigwedge \Gamma \to \bigvee \Delta\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \Gamma \Rightarrow \bigwedge \Gamma\end{array}}{\mathsf{w_R}\ \dfrac{\Gamma \Rightarrow \bigwedge \Gamma}{\Gamma \Rightarrow \bigwedge \Gamma, \Delta}} \qquad \dfrac{\begin{array}{c}\mathcal{D}_2\\ \bigvee \Delta \Rightarrow \Delta\end{array}}{\mathsf{w_L}\ \dfrac{\bigvee \Delta \Rightarrow \Delta}{\bigvee \Delta, \Gamma \Rightarrow \Delta}}}{\to_{\mathsf{L}}\ \dfrac{\Gamma \Rightarrow \bigwedge \Gamma, \Delta \qquad \bigwedge \Gamma \to \bigvee \Delta, \Gamma \Rightarrow \Delta}{\bigwedge \Gamma \to \bigvee \Delta, \Gamma \Rightarrow \Delta}}}{\mathsf{cut}\ \dfrac{\Rightarrow \bigwedge \Gamma \to \bigvee \Delta \qquad \bigwedge \Gamma \to \bigvee \Delta, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}}$$

The existence of $\mathcal{D}_1$ and $\mathcal{D}_2$ should be obvious by the rule $A \Rightarrow A$ and propositional rules governing conjunction and disjunction. □

Then, by the equivalence between $\mathbf{G3GL}^{\mathrm{cut},\infty}$ and $\mathbf{G3GL}^{\mathrm{cut},\circ}$ and their cut elimination theorems, we have the following.

Theorem 5.28 (Syntactic completeness of $\mathbf{G3GL}^{\infty}$ and $\mathbf{G3GL}^{\circ}$)**.** *Let* $\Gamma \Rightarrow \Delta \in \mathrm{Seq}$. *If* $\vdash_{\mathbf{HGL}} \bigwedge \Gamma \to \bigvee \Delta$, *then* $\vdash_{\mathbf{G3GL}^{\infty}} \Gamma \Rightarrow \Delta$ *and* $\vdash_{\mathbf{G3GL}^{\circ}} \Gamma \Rightarrow \Delta$.

Chapter 6

Intuitionistic provability logic $\mathbf{iSL}_\Box$

We have finished our investigation of the cut elimination of non-wellfounded **GL**, and will move on to intuitionistic provability logics in the following two chapters. The current chapter focuses on the logic $\mathbf{iSL}_\Box$. Our main goal is to provide sound and cut-free complete non-wellfounded systems for $\mathbf{iSL}_\Box$ in both **G3**- and **G4**-styles. We want **G3**-style systems because the rules of such systems can be more intuitive. In contrast, the rules of **G4**-style systems can be more difficult to grasp, yet they are also desirable because they often enjoy terminating backward proof search.

Here is an overview of this chapter. After introducing basic definitions and motivating the use of set sequents in section 6.1, we introduce the **G3**-style systems in section 6.2, which is then followed by the standard soundness and completeness proofs in section 6.3. Based on these results, section 6.4 establishes that $\mathbf{iSL}_\Box$ is a 'cyclic companion' of intuitionistic doxastic logic. The remainder then dedicates to the **G4**-style systems. Section 6.5 discusses the technicalities of set-sequents in the context of backward proof search termination, which is important for the design of our (cut-free) **G4**-style systems presented in sections 6.6 and 6.7. Finally, section 6.8 proves the completeness of these systems. Now, let us start from preliminary definitions.

6.1 Syntax and semantics of monomodal intuitionistic modal language

Definition 6.1 (Intuitionistic modal formula)**.** *Let Φ be a countably infinite set of propositional letters. The set of* well-formed modal formulas Form^i of intuitioinistic modal language $\mathscr{L}^i$ *is given by the following rule:*

$$A ::= p \in \Phi \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A.$$

For any formula, let $\neg A := A \to \bot$. ⊣

We will work with the modal operator $\Box$ only. Note that in the intuitionistic setting, $\Diamond$ is normally not definable in terms of $\Box$.

Definition 6.2 (**HiPL** and **iPL**)**.** *The Hilbertian calculus* **HiPL** *consists of the following axiom schemas and the rule modus ponens.*

(1) $A \to (B \to A)$.

(2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$.

(3) $A \wedge B \to A$.

(4) $A \wedge B \to B$.

(5) $A \to (B \to A \wedge B)$.

(6) $A \to A \vee B$.

(7) $B \to A \vee B$.

(8) $(A \to B) \to ((C \to B) \to (A \vee C) \to B)$.

(9) $\bot \to A$.

A derivation *in* **HiPL** *of some formula A from some set of formulas* Γ *is a finite sequence of formulas* $\{B_i\}_{i \leqslant n}$ *(for some* $n \in \mathbb{N}$*) such that* $B_n = A$*, and each formula* B_i *is either an instance of some axiom schema (1) to (9), or is in* Γ*, or is obtained from some* B_j, B_k *with* $j, k < i \leqslant n$ *by applying modus ponens. A is called the conclusion of the derivation and* Γ *the set of premises. We write* $\Gamma \vdash_{\mathbf{HiPL}} A$ *to mean that A is derivable from* Γ *in* **HiPL**. *A* proof *in* **HiPL** *of some formula A is a derivation of A from the empty set of premises. We write* $\vdash_{\mathbf{HiPL}} A$ *to mean that A is provable in* **HiPL**. *We also define* $\Gamma \vdash_{\mathbf{HiSL}_\Box}$ *to be* $\Gamma \vdash_{\mathbf{HiSL}_\Box} \bot$*, for any* Γ. *The* length *of a derivation is defined to be the length of the sequence of formulas of this derivation.*

Define the following set.

$$\mathbf{iPL} := \{A \in \mathsf{Form}^i : \textit{There is a } \mathbf{HiPL} \textit{ proof of } A\}.$$

Any formula in **iPL** *is called an intuitionistic propositional tautology.* ⊣

Remark 6.3. *In the definition above, we distinguish between derivations and proofs. However, for other systems in this thesis, we do not distinguish between these objects.*

Additionally, we want to note the difference between $\Gamma \vdash_{\mathbf{HiPL}} A$ *and* $\Gamma \vdash_{\mathbf{HL}} A$*, where* **HL** *is a Hilbertian calculus for a normal modal logic* **L***, be it classical or intuitionistic (see the next definition). The former means that there is a derivation of A from* Γ*, where formulas in* Γ *may occur at each step. The latter means theoremhood: there is a finite subset* Γ_0 *of* Γ *such that* $\vdash_{\mathbf{HL}} \bigwedge \Gamma_0 \to A$*, i.e. the formula* $\bigwedge \Gamma_0 \to A$ *is a theorem of* **HL**.

Definition 6.4 (Intuitionistic normal (mono)modal logic)**.** *An* intuitionistic normal (mono)modal logic **L** *is a set of formulas of language* $\mathscr{L}^i$ *that contains* **iPL** *and the axiom* K*:* $\Box(A \to B) \to (\Box A \to \Box B)$*, and is closed under generalisation (or necessitation) and modus ponens.* ⊣

Definition 6.5 (The logic $\mathbf{iK}_\Box$ and $\mathbf{iSL}_\Box$)**.** *The logic* $\mathbf{iK}_\Box$ *is the smallest intuitionistic normal modal logic.*

$\mathbf{iSL}_\Box$ *is the smallest intuitionistic normal modal logic that contains the following axioms.*

- *Löb's axiom.* $\Box(\Box A \to A) \to \Box A$
- *Completeness axiom.* $A \to \Box A$

Alternatively, $\mathbf{iSL}_\Box$ *is the smallest intuitionistic modal logic that contains the strong Löb's axiom:* $(\Box A \to A) \to A$. ⊣

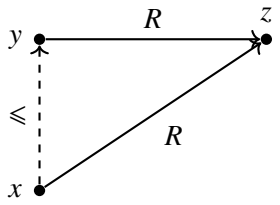
Definition 6.6 (System $\mathbf{HiSL}_\Box$)**.** *Let the Hilbertian calculus for* $\mathbf{iSL}_\Box$ *be* $\mathbf{HiSL}_\Box$*. Derivations/proofs of some formula A in* $\mathbf{HiSL}_\Box$ *are defined as Definition 3.3. The length of* $\mathbf{HiSL}_\Box$ *proofs and derivations is defined to be the length of the sequence of formulas.* ⊣

Note that $\mathbf{iSL}_\Box$ is defined to be the set of formulas that are provable from $\mathbf{HiSL}_\Box$.

Next, we present necessary definitions for the semantics of $\mathscr{L}^i$.

Definition 6.7 (Intuitionistic Kripke frames and models)**.** *An* intuitionistic Kripke frame *is a tuple* $(W, \leqslant, R)$ *where*

- $W \neq \emptyset$*;*
- $\leqslant \subseteq W \times W$ *is a partial order, i.e. reflexive, transitive, and antisymmetric;*
- $R \subseteq W \times W$*;*
- *for all* $x, y \in W$*, if* $x \leqslant yRz$*, then* xRz*. We can represent this using a diagram.*



The relation $\leqslant$ *is called the* extension relation*, and* R *is called the* accessibility relation*.*

An intuitionistic Kripke model *is a tuple* $\mathbb{M} = (W, \leqslant, R, V)$ *where* $(W, \leqslant, R)$ *is an intuitionistic Kripke frame, and* $V : \Phi \to \wp(W)$ *is a monotone function over* $\leqslant$*. That is, for any* $x, y \in W$ *and any* $p \in \Phi$*, if* $x \in V(p)$ *and* $x \leqslant y$*, then* $y \in V(p)$*.*

For any intuitionistic Kripke model $\mathbb{M} = (W, \leqslant, R, V)$ *and* $x \in W$*,* $\mathbb{M}, x$ *is a* pointed intuitionistic Kripke model*.* ⊣

Definition 6.8 (Truth of intuitionistic modal formula)**.** *The* truth *or* satisfaction *of an intuitionistic modal formula at some point of an intuitionistic Kripke model is defined as follows. Let* $\mathbb{M} = (W, \leqslant, R, V)$ *be an intuitionistic Kripke model and* $x \in W$*. For any formula* $A, B \in \mathsf{Form}^i$*:*

- $\mathbb{M}, x \Vdash p$ *iff* $x \in V(p)$*;*
- $\mathbb{M}, x \Vdash \bot$*, never;*
- $\mathbb{M}, x \Vdash A \wedge B$ *iff* $\mathbb{M}, x \Vdash A$ *and* $\mathbb{M}, x \Vdash B$*;*
- $\mathbb{M}, x \Vdash A \vee B$ *iff* $\mathbb{M}, x \Vdash A$ *or* $\mathbb{M}, x \Vdash B$*;*
- $\mathbb{M}, x \Vdash A \to B$ *iff for all* $y \in W$ *such that* $y \geqslant x$*, if* $\mathbb{M}, y \Vdash A$*, then* $\mathbb{M}, y \Vdash B$*;*
- $\mathbb{M}, x \Vdash \Box A$ *iff for all* $y \in W$ *such that* xRy*, it holds that* $\mathbb{M}, y \Vdash A$*.*

For $\neg A$*, we can verify that for any* $A \in \mathsf{Form}^i$*,* $\mathbb{M}, x \Vdash \neg A$ *iff for all* $y \geqslant x$*,* $\mathbb{M}, y \nVdash A$*.* ⊣

The following lemma is very useful, and the proof is standard.

Lemma 6.9 (Monotonicity)**.** *For any* $A \in \mathsf{Form}^i$*, any intuitionistic Kripke model* $\mathbb{M} = (W, \leq, R, V)$ *and any* $x, y \in W$*, if* $\mathbb{M}, x \Vdash A$ *and* $x \leqslant y$*, then* $\mathbb{M}, x \Vdash A$*.*

The validity of intuitionistic formulas and the logic of intuitionistic frames is defined in analogy to the classical case.

Definition 6.10 (Validity and logic of frames)**.** *Given an intuitionistic Kripke frame* $\mathbb{F} = (W, \leqslant, R)$ *and an* $x \in W$*, a formula* A *is* valid at x *iff* $(\mathbb{F}, \leqslant, R, V), x \Vdash A$*, for any* V*. Formally, we write* $\mathbb{F}, x \Vdash A$*. It is* valid in $\mathbb{F}$*, i.e.* $\mathbb{F} \Vdash A$*, iff it is valid at all* $x \in W$*. If* $\mathfrak{F}$ *is a class of intuitionistic frames,* A *is* valid in $\mathfrak{F}$ *iff for all* $\mathbb{F}$ *in* $\mathfrak{F}$*,* $\mathbb{F} \Vdash A$*.*

Given a frame $\mathbb{F}$*,* the logic of $\mathbb{F}$*,* $Log(\mathbb{F})$ *is the set* $\{A \in \mathsf{Form}^i : \mathbb{F} \Vdash A\}$*. For any frame class* $\mathfrak{F}$*, define the logic of it as* $Log(\mathfrak{F}) := \{A \in \mathsf{Form}^i : \text{For all } \mathbb{F} \in \mathfrak{F}, \mathbb{F} \Vdash A\}$*.* ⊣

Definition 6.11 (Soundness, completeness, and Kripke completeness)**.** *Given an intuitionistic normal modal logic* $\mathbf{L}$ *and a frame class* $\mathfrak{F}$*,* $\mathbf{L}$ *is* sound with respect to $\mathfrak{F}$ *iff* $\mathbf{L} \subseteq Log(\mathfrak{F})$*, and* $\mathbf{L}$ *is* complete with respect to $\mathfrak{F}$ *iff* $\mathbf{L} \supseteq Log(\mathfrak{F})$*.* L *is* Kripke complete with respect to $\mathfrak{F}$ *iff* $\mathbf{L} = Log(\mathfrak{F})$*.* ⊣

Theorem 6.12. *The logic* $\mathbf{iK}_\Box$ *is Kripke complete with respect to the class of all intuitionistic Kripke frames.*

Theorem 6.13. *The logic* $\mathbf{iSL}_\Box$ *is Kripke complete with respect to the frames* $(W, \leqslant, R)$ *where* R *relation is*

(1) transtive,

(2) conversely-well founded, i.e. there is no infinite ascending R*-path,*

(3) and $R \subseteq \leqslant$*, i.e. for any* $x, y \in W$*, if* xRy *then* $x \leqslant y$*.*

Note that the transitivity of R follows from the fact that $R \subseteq \leqslant$ and for any x, y, z in any intuitionistic frame, $x \leqslant yRz$ implies xRz.

Before moving to the calculus of $\mathbf{iSL}_\Box$, we will make clear an important technical shift. Instead of working with sequents defined in terms of finite *multisets* of formulas, we will endorse set sequents, i.e. sequents defined in terms of finite *sets* of formulas, for non-wellfounded systems for $\mathbf{iSL}_\Box$.

Definition 6.14 (Intuitionistic sequents)**.** *An* intuitionistic sequent *is of the form* $\Gamma \Rightarrow \Delta$*, where* Γ *and* Δ *are finite sets of formulas in* Form^i*, and* $|\Delta| \leqslant 1$*, i.e.* Δ *has at most one formula. Let the set of intuitionistic sequents be* Seq^i*. We use '*,*' to mean '*$\cup$*', the union operator of sets. The notation '*A, Γ*' and '*Γ, A*' are terms for the sets* $\{A\} \cup \Gamma$ *and* $\Gamma \cup \{A\}$*, respectively.* ⊣

The standard interpretation of intuitionistic modal sequents $\cdot^{\mathcal{I}}$ is defined similarly to that of classical modal sequents.

Definition 6.15 (Standard interpretation of intuitionistic sequents)**.** *For any* $\Gamma \Rightarrow \Delta \in \mathsf{Seq}^i$*, its* standard interpretation $(\Gamma \Rightarrow \Delta)^{\mathcal{I}}$ *is defined as*

$$(\Gamma \Rightarrow \Delta)^{\mathcal{I}} := \bigwedge \Gamma \to \bigvee \Delta.$$

Let $\bigvee \emptyset := \bot$ *and* $\bigwedge \emptyset := \top$*. We say a sequent is satisfied at some pointed intuitionistic model or valid in some intuitionistic frame/frame class if its standard interpretation is. Then we may write* $\mathbb{M}, x \Vdash \Gamma \Rightarrow \Delta$*, or* $\mathbb{F} \Vdash \Gamma \Rightarrow \Delta$*, etc.* ⊣

One might be curious why we switch to set sequents for $\mathbf{iSL}_\Box$, rather than stick with multiset sequents. Our motivation is two-fold.

First, if we stick with multiset sequents, we will have to prove the admissibility of contraction and hence invertibility. As is shown by the proof of Proposition 3.28, the admissibility of contraction is not easy to prove. Moreover, for non-wellfounded systems for $\mathbf{iSL}_\Box$, invertibility results will be more difficult to prove.

Second, endorsing set sequents (subject to certain restrictions) allows us to provide a direct backward proof search termination argument for systems that we care about in this work. For example, let us rewind a bit

and think of **G3GL**$^\circ$. Suppose we are given a multiset sequent $\Gamma \Rightarrow \Delta$ and start proof search for this sequent. Suppose that $|\mathrm{Clos}(\Gamma \Rightarrow \Delta)| = n$, then there can only be at most 4^n many sequents that are distinct set-wise in the proof search tree. Then, consider any node ν in a proof search tree that has height $4^n + 1$. It is decorated with a sequent $\Gamma_0 \Rightarrow \Delta_0$ that is set-wise identical with another sequent $\Gamma_1 \Rightarrow \Delta_1$ that decorates another node c below. If for any formula A, we have that that $\#_{\Gamma_0}(A) \geqslant \#_{\Gamma_1}(A)$ and that $\#_{\Delta_0}(A) \geqslant \#_{\Delta_1}(A)$, then we can apply weakening above ν to get rid of the redundant ones. But otherwise, it is not so clear what to do if we want to end this proof search branch. In contrast, if we work with set based sequents, *and* the calculus is designed in such a way that the only backwards non-decreasing rule is K4, then termination is immediate – any node of height $4^n + 1$ shares its sequent with some nodes below, and since K4 is the only rule that is non-decreasing, there must be an application of this rule between the two occurrences of the sequent.

It might be necessary to deal with some existing worries towards set sequents. For example, in [Nv11], Negri and von Plato express the following issue regarding rules based on set sequents. Consider the following rule.

$$\wedge_{\mathsf{L}} \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}$$

Suppose we read these sequents as set sequents, as defined above. Their worry is that the sequent $A \wedge A \Rightarrow C$ would not be derivable from $A \Rightarrow C$ using this very rule. Yet we think this is not really a problem, for the set $\{A\}$ is just the same as $\{A, A\}$. The idea of Negri and von Plato seems to be that we should keep track of formula *occurrences* even if we endorse set sequents, and their solution seems to accommodate this motivation. Yet we choose to be more relaxed on this problem – we will be less stringent about formula *occurrences*.

Nevertheless, despite the advantages that we discussed before, we want to emphasize that set sequents are no syntactic sugar, and Negri and von Plato are more than justified in worrying about such entities. As will be made clear, set sequents come with real prices if we stick to our pursuit of proof search termination for **G4**-style systems. In the following, we will first introduce **G3**-style non-wellfounded systems for $\mathbf{iSL}_{\Box}$, and then **G4**-style systems. As one can expect, the **G3**-style systems are obviously not 'backwards decreasing', and we do not care about backward proof search for such systems anyway. The reason we care about systems of this style is mainly their simplicity. Then, for such systems, our choice of set sequents is not so significant. However, for **G4**-style systems, we do care about backward proof search termination. There, we need to be very careful with set sequents and the design of the systems. We leave the relevant technical details until then and first introduce the simpler **G3**-style systems.

6.2 G3-style systems

Definition 6.16. *The sequent calculus* **G3ip**$^{\mathrm{cut}}$ *is given by the following set of rules.*

$$\mathsf{init}\ \frac{}{p, \Gamma \Rightarrow p} \qquad \bot_{\mathsf{L}}\ \frac{}{\bot, \Gamma \Rightarrow \Delta} \qquad \wedge_{\mathsf{L}}\ \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \wedge_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\vee_{\mathsf{L}}\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A_{i \in \{0,1\}}}{\Gamma \Rightarrow A_0 \vee A_1} \qquad \rightarrow_{\mathsf{L}}\ \frac{A \rightarrow B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta}$$

$$\rightarrow_{\mathsf{R}}\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \qquad \mathsf{cut}\ \frac{\Gamma \Rightarrow A \qquad A, \Sigma \Rightarrow \Delta}{\Gamma, \Sigma \Rightarrow \Delta}$$

Formulas that are written out explicitly are called designated formulas. *Designated formulas in the conclusions of each rule are called the* principal formula *of the rule. Sets of formulas denoted by* Γ *and* Δ *are* contexts. *Let* **G3ip** *be the system that consists of all and only the rules of* **G3ip**$^{\mathrm{cut}}$ *minus* cut.

Let **G3iDL**$^{\mathrm{cut}}$ *be the system that is obtained from adding the following rules to* **G3ip**$^{\mathrm{cut}}$*:*

$$\mathsf{w}_{\mathsf{L}}\ \frac{\Gamma \Rightarrow \Delta}{\Sigma, \Gamma \Rightarrow \Delta} \qquad \mathsf{w}_{\mathsf{R}}\ \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A} \qquad \mathsf{DL}\ \frac{\Pi, \boxdot\Gamma \Rightarrow A}{\Pi, \Box\Sigma, \Box\Gamma \Rightarrow \Box A}$$

where Π *contains no boxed formula. For* w_{L}*, we require that* $\Sigma \neq \emptyset$ *and* $\Sigma \cap \Gamma = \emptyset$*. Let* **G3iDL** *be the system that consists of all and only the rules of* **G3iDL**$^{\mathrm{cut}}$ *minus* cut. ⊣

The name 'DL' stands for 'doxastic logic', a logic that will be introduced a bit later. Let us first move on to the definition of proofs.

Definition 6.17 (**G3ip**$^{\mathrm{cut}}$, **G3ip**, **G3iDL**$^{\mathrm{cut}}$, **G3iDL** proof)**.** *Let* $C \in \{$**G3ip**$^{\mathrm{cut}}$, **G3ip**, **G3iDL**$^{\mathrm{cut}}$, **G3iDL**$\}$*. A* C-proof/derivation *is a* $C \times \mathrm{Seq}^i$*-labeled tree* $T = (N, \prec, l)$ *such that for any* $w \in N$*:*

(1) N *is finite;*

(2) if w *is a leaf, then* $\mathrm{rule}_T(w)$ *is an instance of* $\mathrm{seq}_T(w)$*;*

(2) otherwise, let $w0, \ldots, wn$ *be the* $\prec$*-successors of* w*,* $(\mathrm{seq}_T(w_0), \ldots, \mathrm{seq}_T(w_n), \mathrm{seq}_T(w))$ *is an instance of* $\mathrm{rule}_T(w)$*.*

A C*-proof of some sequent* S *is a* C*-proof whose root is decorated with the sequent* S*. A sequent* S *is provable in* C *(*$\vdash_C S$*) iff there is a* C*-proof of* S*.* ⊣

Then, we define local progress calculi for **iSL**$_\Box$ in accordance with Definition 4.3.

Definition 6.18 (Local progress calculus **G3iSL**$_\Box^{\mathrm{cut},\infty}$ and **G3iSL**$_\Box^{\infty}$)**.** *The local progress calculus* **G3iSL**$_\Box^{\mathrm{cut},\infty}$ *is defined as the pair* (**G3iDL**$^{\mathrm{cut}}$, p)*, where* p *is a function such that for any* $\mathsf{r} \in$ **G3iDL**$^{\mathrm{cut}} \setminus \{\mathsf{DL}\}$ *and any rule instance* $r \in \mathsf{r}$*, it holds that* $p_{\mathsf{r}}(r) = \emptyset$*; for* DL *and any* $r \in \mathsf{DL}$*, it holds that* $p_{\mathsf{DL}}(r) = \{0\}$*. The local progress calculus* **G3iSL**$_\Box^{\infty}$ *is defined similarly.* ⊣

Definition 4.4 already gives the definition of **G3iSL**$_\Box^{\mathrm{cut},\infty}$ and **G3iSL**$_\Box^{\infty}$ preproofs, progressing branches, and proofs. Yet we give a more explicit presentation for easier reference.

Definition 6.19 ($\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}$/$\mathbf{G3iSL}_\Box^{\infty}$ preproof)**.** *A* $\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}$ preproof *is a* $\mathbf{G3iDL}^{\mathrm{cut}} \times \mathrm{Seq}^i$*-labeled tree* $T = (N, \prec, l)$ *such that for any* $w \in N$*:*

(1) if w *is a leaf, then* $\mathrm{rule}_T(w)$ *is an instance of* $\mathrm{seq}_T(w)$*;*

(2) otherwise, let $w0, \ldots, wn$ *be the* $\prec$*-successors of* w*,* $(\mathrm{seq}_T(w_0), \ldots, \mathrm{seq}_T(w_n), \mathrm{seq}_T(w))$ *is an instance of* $\mathrm{rule}_T(w)$*.*

$\mathbf{G3iSL}_\Box^{\infty}$ *preproofs are defined similarly.* ⊣

Definition 6.20 (Progressing branch in a $\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}$/$\mathbf{G3iSL}_\Box^{\infty}$ preproof)**.** *A branch of a* $\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}$ *(or a* $\mathbf{G3iSL}_\Box^{\infty}$*) preproof is* progressing *iff it contains infinitely many applications of* DL*.* ⊣

Definition 6.21 ($\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}$ proof and $\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}$ provability)**.** *A* $\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}$ proof/derivation *is a* $\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}$ *preproof such that every infinite branch is progressing. We say that* S is provable in $\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}$ *iff there is a* $\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}$ *proof whose root is decorated with* S*. We write* $\vdash_{\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}} S$ *in such cases.* ⊣

So much for the infinitary systems for $\mathbf{iSL}_\Box$. We now move on to the cyclic system for $\mathbf{iSL}_\Box$.

Definition 6.22 ($\mathbf{G3iSL}_\Box^{\mathrm{cut},\circ}$ proof and $\mathbf{G3iSL}_\Box^{\mathrm{cut},\circ}$ provability)**.** *A* $\mathbf{G3iSL}_\Box^{\mathrm{cut},\circ}$ proof/derivation *is a finite* $(\mathbf{G3iDL}^{\mathrm{cut}} \times \mathrm{Seq}^i) \cup \mathrm{Seq}^i$*-labeled cyclic tree* $(N, \prec \cup f, l)$ *such that:*

(1) $f : \mathcal{L}^T \hookrightarrow N \setminus \mathcal{L}^T$ *a partial function on the leaves of* T

(2) if w *is a leaf, then*

(a) if $w \notin Dom(f)$*, then* $l(w) \in \mathbf{G3iDL}^{\mathrm{cut}} \times \mathrm{Seq}^i$ *and* $(\mathrm{seq}_T(w))$ *is an instance of* $\mathrm{rule}_T(w)$*;*

(b) if $w \in Dom(f)$*, then* $l(w) \in \mathrm{Seq}^i$*,* $l(w) = \mathrm{seq}_T(f(w))$*, and there is a* $u \in N$ *such that* $f(w) \sqsubseteq u \sqsubseteq w$ *and* $\mathrm{rule}_T(w) = \mathsf{DL}$*;*

(3) otherwise, $l(w) \in \mathbf{G3iDL}^{\mathrm{cut}} \times \mathrm{Seq}^i$*; for* $w0, \ldots, wn$*, the* $\prec$*-successors of* w*,*

$$(\mathrm{seq}_T(w_0), \ldots, \mathrm{seq}_T(w_n), \mathrm{seq}_T(w))$$

is an instance of $\mathrm{rule}_T(w)$*, where we set* $seq_T(wi) = l(wi)$ *if* $wi \in Dom(f)$*.*

Similarly, a sequent S is provable in $\mathbf{G3iSL}_\Box^{\mathrm{cut},\circ}$ ($\vdash_{\mathbf{G3iSL}_\Box^{\mathrm{cut},\circ}} S$) *iff there is a* $\mathbf{G3iSL}_\Box^{\mathrm{cut},\circ}$ *proof of* S*.* ⊣

Example 6.23. *Here is an example of a* $\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}$ *proof, where we derive an instance of the strong Löb's axiom.*

$$\dfrac{\dfrac{\dfrac{\rightsquigarrow \Box p \to p \Rightarrow p}{\Box p \to p \Rightarrow \Box p}\,\mathsf{DL} \qquad \dfrac{}{p \Rightarrow p}\,\mathsf{init}}{\Box p \to p \Rightarrow p}\,{\to_{\mathsf{L}}}}{\Rightarrow (\Box p \to p) \to p}\,{\to_{\mathsf{R}}}$$

Recall again from Definition 4.4 that, given a local progress calculus C, $\mathbb{P}^\infty(C)$ is the set of proofs of C.

Definition 6.24 (Systems $\mathbf{G3iSL}_\Box^{\mathrm{cut},\circ}$ and $\mathbf{G3iSL}_\Box^{\circ}$)**.** *Let* $\mathbf{G3iSL}_\Box^{\mathrm{cut},\circ}$ *be the pair* $(\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}, \mathbb{P}^\infty(\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}))$*, and* $\mathbf{G3iSL}_\Box^{\mathrm{cut},\circ}$ *be the pair* $(\mathbf{G3iSL}_\Box^{\infty}, \mathbb{P}^\infty(\mathbf{G3iSL}_\Box^{\infty}))$*.* ⊣

In the following, we aim to show that (i) $\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}$ and $\mathbf{G3iSL}_\Box^{\mathrm{cut},\circ}$ are equivalent, and (ii) their cut-free fragments are sound and complete with respect to $\mathbf{iSL}_\Box$. We will first show that the infinitary system is sound and cut-free complete. To show cut-free completeness, we show the cut elimination of the system. The equivalence of the two systems also follows from the cut elimination of the infinitary system.

6.3 Soundness and completeness of G3-style systems

We shall start with the soundness of $\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}$.

Proposition 6.25 (Local soundness of $\mathbf{G3iDL}^{\mathrm{cut}}$)**.** *For any* $\mathsf{r} \in \mathbf{G3iDL}^{\mathrm{cut}}$, r *is locally sound.*

Proof. The proof is quite similar to the proof of Proposition 6.47. Note that the change to set-based sequents does not have a great impact on the semantics. Therefore, we check only the modal rule. Suppose there is an intuitionistic frame $\mathbb{F} = (W, \leqslant, R)$ of $\mathbf{iSL}_{\Box}$ and for some $x \in V$ and V defined for $\mathbb{F}$, it holds that $(\mathbb{F}, V), x \nVdash (\Pi, \Box\Sigma, \Box\Gamma \Rightarrow \Box A)^{\mathcal{I}}$. This means $(\mathbb{F}, V), x \Vdash \bigwedge \Pi \wedge \bigwedge \Box\Sigma \wedge \bigwedge \Box\Gamma$ and $(\mathbb{F}, V), x \nVdash \Box A$. Then there is some $y \in W$ such that xRy, $(\mathbb{F}, V), y \nVdash A$, and $(\mathbb{F}, V), y \Vdash \bigwedge \Gamma$. Moreover, since $R \subseteq \leqslant$, we have $x \leqslant y$; by monotonicity it follows that $(\mathbb{F}, V), y \Vdash \bigwedge \Pi \wedge \bigwedge \Box\Gamma$. Therefore, $(\mathbb{F}, V), y \nVdash (\Pi, \boxdot\Gamma \Rightarrow A)^{\mathcal{I}}$. □

Theorem 6.26 (Soundness of $\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}$)**.** *If* $\vdash_{\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}} \Gamma \Rightarrow \Delta$, *then* $\Gamma \Rightarrow \Delta$ *is valid in every frame of* $\mathbf{iSL}_{\Box}$.

Proof. We only give a sketch of a *reductio* argument, as the details are similar to the proof of Theorem 5.20. Suppose $\vdash_{\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}} \Gamma \Rightarrow \Delta$ but it is not valid. Then there is a $\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}$ proof of $\Gamma \Rightarrow \Delta$, and an infinite branch in the proof such that every sequent on this path is falsified in one $\mathbf{iSL}_{\Box}$ frame for some valuation, and it passes through infinitely many applications of DL. Whenever we move upwards along the path and pass through such an application, we move from a state to an R-accessible state in the frame. This induces an infinite R-path in the frame, which is a contradiction. □

Then, the soundness of $\mathbf{G3iSL}_{\Box}^{\infty}$ follows immediately.

Corollary 6.27 (Semantic soundness of $\mathbf{G3iSL}_{\Box}^{\infty}$)**.** *If* $\vdash_{\mathbf{G3iSL}_{\Box}^{\infty}} \Gamma \Rightarrow \Delta$, *then* $\Gamma \Rightarrow \Delta$ *is valid in every frame of* $\mathbf{iSL}_{\Box}$.

Consequently, by the Kripke completeness of $\mathbf{HiSL}_{\Box}$, we have that $\mathbf{G3iSL}_{\Box}^{\infty}$ is sound with respect to $\mathbf{HiSL}_{\Box}$.

Corollary 6.28 (Syntactic soundness of $\mathbf{G3iSL}_{\Box}^{\infty}$)**.** $\mathbf{G3iSL}_{\Box}^{\infty}$ *is sound with respect to* $\mathbf{HiSL}_{\Box}$, *i.e. for any sequent* $\Gamma \Rightarrow \Delta$, *if* $\vdash_{\mathbf{G3iSL}_{\Box}^{\infty}} \Gamma \Rightarrow \Delta$, *then* $\vdash_{\mathbf{HiSL}_{\Box}} \bigwedge \Gamma \to \bigvee \Delta$.

Proof. If $\vdash_{\mathbf{G3iSL}_{\Box}^{\infty}} \Gamma \Rightarrow \Delta$, then by Corollary 6.27 we have that $\Gamma \Rightarrow \Delta$ is valid in the frames of $\mathbf{iSL}_{\Box}$, i.e. $\bigwedge \Gamma \to \bigvee \Delta$ is valid in frames of $\mathbf{iSL}_{\Box}$. By the completeness of $\mathbf{HiSL}_{\Box}$, $\vdash_{\mathbf{HiSL}_{\Box}} \bigwedge \Gamma \to \bigvee \Delta$. □

We now move to completeness. We will first prove that $\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}$ is complete with respect to $\mathbf{iSL}_{\Box}$. The proof will be purely syntactic, and we will be a bit more pedantic since we have changed to set-based sequents. We first show that **iPL**, K, Löb's axiom, and the completeness axiom can be derived in $\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}$.

Proposition 6.29. *The following rules are derivable in* $\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}$ *for any formula* A *and set of formulas* Γ.

$$A \Rightarrow A \ \frac{}{A, \Gamma \Rightarrow A} \qquad \mathsf{MP}\ \frac{\Gamma \Rightarrow A \to B \qquad \Gamma \Rightarrow A}{\Gamma \Rightarrow B} \qquad \mathsf{Nec}\ \frac{\Gamma \Rightarrow A}{\Gamma \Rightarrow \Box A}$$

Proof. For $A \Rightarrow A$, we proceed by induction on the number of connectives of A.

Base case. If $A = p$ for some $p \in \Phi$ or $A = \bot$, then $A, \Gamma \Rightarrow A$ is an axiom, and is hence derivable.

Inductive case. Fix an arbitrary Γ. Suppose $A = B \wedge C$. Then, we have the following proof.

$$\dfrac{\dfrac{\overset{\text{IH}}{\cdots\cdots} \quad \overset{\text{IH}}{\cdots\cdots}}{B, C, \Gamma \Rightarrow B \quad B, C, \Gamma \Rightarrow C}\,\wedge_{\mathsf{R}}}{B, C, \Gamma \Rightarrow B \wedge C}\,\wedge_{\mathsf{L}}$$
$$B \wedge C, \Gamma \Rightarrow B \wedge C$$

If $A = B \vee C$, we can construct the following proof.

$$\dfrac{\dfrac{\dfrac{\overset{\text{IH}}{\cdots\cdots}}{B, \Gamma \Rightarrow B}\,\vee_{\mathsf{R}}}{B, \Gamma \Rightarrow B \vee C} \quad \dfrac{\dfrac{\overset{\text{IH}}{\cdots\cdots}}{C, \Gamma \Rightarrow C}\,\vee_{\mathsf{R}}}{C, \Gamma \Rightarrow B \vee C}}{B \vee C, \Gamma \Rightarrow B \vee C}\,\vee_{\mathsf{L}}$$

If $A = B \to C$, then we have the following.

$$\dfrac{\dfrac{\overset{\text{IH}}{\cdots\cdots} \quad \overset{\text{IH}}{\cdots\cdots}}{B \to C, \Gamma, B \Rightarrow B \quad \Gamma, B, C \Rightarrow C}\,\to_{\mathsf{L}}}{B \to C, \Gamma, B \Rightarrow C}\,\to_{\mathsf{R}}$$
$$B \to C, \Gamma \Rightarrow B \to C$$

Finally, if $A = \Box B$, we construct the following proof. Suppose $\Gamma = \Box\Sigma, \Pi$, where Π contains no boxed formula.

$$\dfrac{\dfrac{\overset{\text{IH}}{\cdots\cdots}}{B, \Pi \Rightarrow B}\,\mathsf{w}_{\mathsf{L}}}{B, \Box B, \Pi \Rightarrow B}\,\mathsf{DL}$$
$$\Box B, \Box\Sigma, \Pi \Rightarrow \Box B$$

This concludes the inductive step.

For MP, suppose we are given proofs $\mathcal{D}$ and $\mathcal{D}'$ of $\Gamma \Rightarrow A$ and $\Gamma \Rightarrow A \to B$, respectively. We construct the following proof using the rule $A \Rightarrow A$.

$$\dfrac{\dfrac{\mathcal{D}}{\Gamma \Rightarrow A} \quad \dfrac{\dfrac{\mathcal{D}'}{\Gamma \Rightarrow A \to B} \quad \dfrac{\dfrac{\dfrac{}{A \Rightarrow A}}{A \to B, A \Rightarrow A} \quad \dfrac{\dfrac{}{A \Rightarrow A}}{A, B \Rightarrow B}\,\to_{\mathsf{L}}}{A \to B, A \Rightarrow B}\,\mathsf{cut}}{A, \Gamma \Rightarrow B}\,\mathsf{cut}}{\Gamma \Rightarrow B}$$

For necessitation, we can construct the following proof for a given proof $\mathcal{D}$ of $\Gamma \Rightarrow A$. We assume that

$\Gamma = \Pi, \Box\Sigma$, where Π does not contain any boxed formula.

$$\mathsf{DL}\,\dfrac{\mathsf{w_L}\,\dfrac{\begin{array}{c}\mathcal{D}\\ \Pi, \Box\Sigma \Rightarrow A\end{array}}{\Pi, \boxdot\Sigma \Rightarrow A}}{\Pi, \Box\Sigma \Rightarrow \Box A}$$

It is necessary to delete the application of $\mathsf{w_L}$ above if $\Sigma \subseteq \Pi \cup \Box\Sigma$. □

Then, we can show that for any axiom A of $\mathbf{HiSL}_\Box$, the sequent $\Gamma \Rightarrow A$ is derivable in $\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}$ for an arbitrary set of formulas Γ.

Proposition 6.30. *For any $A \in \mathbf{iPL}$, the sequent $\Rightarrow A$ is provable in $\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}$. Moreover, for any formulas A and B, the sequents $\Rightarrow \Box(A \to B) \to (\Box A \to \Box B)$, $\Rightarrow \Box(\Box A \to A) \to \Box A$, and $\Rightarrow A \to \Box A$ are provable in $\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}$.*

Proof. We first show that for any $A \in \mathbf{iPL}$, the sequent $\Rightarrow A$ is provable. This is shown by induction on the length of a given $\mathbf{HiSL}_\Box$ proof of A. For the base case, the length of a given proof of A is 1. Then A is an instance of an axiom scheme. We check all the cases.

(1) $A \to (B \to A)$.

$$\to_{\mathsf{R}}\,\dfrac{\to_{\mathsf{R}}\,\dfrac{A\to A\,\dfrac{}{A, B \Rightarrow A}}{A \Rightarrow B \to A}}{\Rightarrow A \to (B \to A)}$$

(2) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$.

$$\to_{\mathsf{R}}\,\dfrac{\to_{\mathsf{R}}\,\dfrac{\to_{\mathsf{R}}\,\dfrac{\to_{\mathsf{L}}\,\dfrac{A\Rightarrow A\,\dfrac{}{A \to (B \to C), A \to B, A \Rightarrow A} \qquad \to_{\mathsf{L}}\,\dfrac{A\Rightarrow A\,\dfrac{}{B \to C, A \to B, A \Rightarrow A} \qquad A\Rightarrow A\,\dfrac{}{B \to C, B, A \Rightarrow B}}{B \to C, A \to B, A \Rightarrow B} \qquad A\Rightarrow A\,\dfrac{}{C, A \to B, A \Rightarrow C}}{B \to C, A \to B, A \Rightarrow C}}{A \to (B \to C), A \to B, A \Rightarrow C}}{A \to (B \to C), A \to B \Rightarrow A \to C}}{A \to (B \to C) \Rightarrow (A \to B) \to (A \to C)}}{\Rightarrow (A \to (B \to C)) \to ((A \to B) \to (A \to C))}$$

(3) $A \wedge B \to A$.

$$\to_{\mathsf{R}}\,\dfrac{\wedge_{\mathsf{L}}\,\dfrac{A\Rightarrow A\,\dfrac{}{A, B \Rightarrow A}}{A \wedge B \Rightarrow A}}{\Rightarrow A \wedge B \to A}$$

(4) $A \wedge B \to B$. This case is similar to case (3).

(5) $A \to (B \to A \wedge B)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A, B \Rightarrow A}\,{\scriptstyle A\Rightarrow A} \quad \dfrac{}{A, B \Rightarrow B}\,{\scriptstyle A\Rightarrow A}}{A, B \Rightarrow A \wedge B}\,{\scriptstyle \wedge_R}}{A \Rightarrow B \to A \wedge B}\,{\scriptstyle \to_R}}{\Rightarrow A \to (B \to A \wedge B)}\,{\scriptstyle \to_R}$$

(6) $A \to A \vee B$.

$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}\,{\scriptstyle A\Rightarrow A}}{A \Rightarrow A \vee B}\,{\scriptstyle \vee_R}}{\Rightarrow A \to A \vee B}\,{\scriptstyle \to_R}$$

(7) $B \to A \vee B$. Similar to case (5).

(8) $(A \to B) \to ((C \to B) \to ((A \vee C) \to B))$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \to B, C \to B, A \Rightarrow A}\,{\scriptstyle A\Rightarrow A} \quad \dfrac{}{B, C \to B, A \Rightarrow B}\,{\scriptstyle A\Rightarrow A}}{A \to B, C \to B, A \Rightarrow B}\,{\scriptstyle \to_L} \quad \dfrac{\dfrac{}{A \to B, C \to B, C \Rightarrow C}\,{\scriptstyle A\Rightarrow A} \quad \dfrac{}{A \to B, B, C \Rightarrow B}\,{\scriptstyle A\Rightarrow A}}{A \to B, C \to B, C \Rightarrow B}\,{\scriptstyle \to_L}}{A \to B, C \to B, A \vee C \Rightarrow B}\,{\scriptstyle \vee_R}}{A \to B, C \to B \Rightarrow (A \vee C) \to B}\,{\scriptstyle \to_R}}{A \to B \Rightarrow (C \to B) \to ((A \vee C) \to B)}\,{\scriptstyle \to_R}}{\Rightarrow (A \to B) \to ((C \to B) \to ((A \vee C) \to B))}\,{\scriptstyle \to_R}$$

(9) $\bot \to A$.

$$\dfrac{\dfrac{\dfrac{}{\bot \Rightarrow A}\,{\scriptstyle \bot_L}}{\Rightarrow \bot \to A}\,{\scriptstyle \to_R}}{}$$

For the inductive case, the sub-case where A is an axiom is covered by the base case. If A is obtained by some previous formula B and $B \to A$, then by the IH, there are $\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}$ proofs of $\Rightarrow B$ and $\Rightarrow B \to A$. By the derivability of MP, we can get a proof of $\Rightarrow A$. Therefore, for any $A \in \mathbf{iPL}$, $\Rightarrow A$ is provable, and hence $\Gamma \Rightarrow A$ is provable for any Γ.

Now, we show that the corresponding sequents of axiom schemas with empty antecedents are provable.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A, A \to B, \Box(A \to B), \Box A \Rightarrow A}\,{\scriptstyle A\Rightarrow A} \quad \dfrac{}{B, A, \Box(A \to B), \Box A \Rightarrow B}\,{\scriptstyle A\Rightarrow A}}{A, A \to B, \Box(A \to B), \Box A \Rightarrow B}\,{\scriptstyle \to_L}}{\Box(A \to B), \Box A \Rightarrow \Box B}\,{\scriptstyle \mathsf{DL}}}{\Box(A \to B) \Rightarrow \Box A \to \Box B}\,{\scriptstyle \to_R}}{\Rightarrow \Box(A \to B) \to (\Box A \to \Box B)}\,{\scriptstyle \to_R}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\rightsquigarrow \Box A \to A \Rightarrow A}}{\Box A \to A \Rightarrow \Box A}\mathsf{DL} \quad \dfrac{}{A \Rightarrow A}A{\Rightarrow}A}{\Box A \to A, \Box(\Box A \to A) \Rightarrow \Box A}\to_{\mathsf{L}}}{\rightsquigarrow \Box A \to A \Rightarrow A}\mathsf{DL} \quad \dfrac{}{A, \Box(\Box A \to A) \Rightarrow A}A{\Rightarrow}A}{\Box A \to A, \Box(\Box A \to A) \Rightarrow A}\to_{\mathsf{L}}}{\Box(\Box A \to A) \Rightarrow \Box A}\mathsf{DL}}{\Rightarrow \Box(\Box A \to A) \to \Box A}\to_{\mathsf{R}}$$

$$\dfrac{\dfrac{\dfrac{}{A \Rightarrow A}A{\Rightarrow}A}{A \Rightarrow \Box A}\mathsf{DL}}{\Rightarrow A \to \Box A}\to_{\mathsf{R}}$$

Note that in the proof of Löb's axiom, we actually have also provided a proof of $\Rightarrow (\Box A \to A) \to A$, i.e. the sequent corresponding to the strong Löb's axiom. □

We are just one step away from the completeness of $\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}$ with respect to $\mathbf{HiSL}_{\Box}$.

Theorem 6.31 (Syntactic completeness of $\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}$)**.** *For any finite $\Gamma \subseteq \mathsf{Form}^i$ and $A \in \mathsf{Form}^i$, if $\vdash_{\mathbf{HiSL}_{\Box}} \bigwedge \Gamma \to A$, then $\vdash_{\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}} \Gamma \Rightarrow A$.*

Proof. First, note that for any theorem A of $\mathbf{HiSL}_{\Box}$, the sequent $\Rightarrow A$ is derivable in $\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}$. This is shown by routine induction on the length of $\mathbf{HiSL}_{\Box}$ proofs.

Then, if $\vdash_{\mathbf{HiSL}_{\Box}} \bigwedge \Gamma \to A$, we have that $\vdash_{\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}} \Rightarrow \bigwedge \Gamma \to A$. By a construction similar to the one in Theorem 5.27, we get the desired result. □

Given the completeness of $\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}$, we need the cut elimination of $\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}$ to prove the completeness of the cut-free system $\mathbf{G3iSL}_{\Box}^{\infty}$. To prove cut elimination, we again use the method of coalgebraic proof translation and first prove that every proof can be transformed to a proof whose main fragment is cut-free. We need the following definitions.

Definition 6.32 (Main fragment, local height, and local cut-freeness of $\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}$ proofs)**.** *Let $\mathcal{D}$ be a $\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}$ proof. The* main fragment *of $\mathcal{D}$ is a finite subtree rooted at the root of $\mathcal{D}$ such that each leaf is decorated with either an axiomatic rule or* DL*, and such that other nodes are decorated by rules in $\mathbf{G3iDL}^{\mathrm{cut}} \setminus \{\mathsf{DL}\}$. The* local height *of $\mathcal{D}$ is the height of its main fragment. The smallest local height is* 1*. A $\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}$ proof $\mathcal{D}$ is* locally cut-free *iff its main fragment does not contain any* cut *application.* ⊣

We are now going to prove the cut reduction lemma for $\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}$, which says that we can clean all cut applications in the main fragment of a given $\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}$ proof. As usual, we will need the following lemma.

Lemma 6.33. *For any Γ, if there is a $\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}$ proof of $\Gamma \Rightarrow \bot$ that is locally cut-free and has local height n, then there is a $\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty}$ proof $\mathcal{D}'$ of $\Gamma \Rightarrow$ that is also locally cut-free and has local height at most n.*

Proof. By induction on the height of main fragment, we can construct a proof that satisfies the condition. If $n = 1$, then by the shape of $\Gamma \Rightarrow \bot$ and the shape of inference rules, $\bot \in \Gamma$. So $\Gamma \Rightarrow$ is an axiom. If there is a proof whose local height is larger than 1, we use the inductive hypothesis for the cases where the last rule of

the given proof is not w_R. Note that the last rule cannot be one whose principal formula sits in the succedent of the conclusion. When the last rule is w_R, no IH is needed. □

Recall from Definition 4.24 that given a local progressing calculus $C = (\mathsf{Rul}, p)$, $\mathbb{P}^{\mathrm{ff}}(C)$ is the set of finite fragmented proofs of C. Additionally, we adapt the notion of weight of formulas and cut degree defined in Definition 3.23 to intuitionistic formulas and $\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}$ proofs. The weights of propositional letters and $\bot$ are set to be 1.

Lemma 6.34 (Cut reduction for $\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}$)**.** *There is a function*

$$\mathrm{cutsup} : \mathbb{P}^{\mathrm{ff}}(\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty}) \to \mathbb{P}^{\mathrm{ff}}(\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty})$$

such that, for any $\Gamma \Rightarrow \Delta$*, if* $\mathcal{D} \in \mathbb{P}^{\mathrm{ff}}(\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty})$ *is a proof of* $\Gamma \Rightarrow \Delta$*, then* $\mathrm{cutsup}(\mathcal{D}) \in \mathbb{P}^{\mathrm{ff}}(\mathbf{G3iSL}_\Box^{\mathrm{cut},\infty})$ *is also a proof of* $\Gamma \Rightarrow \Delta$*, with its main fragment cut-free.*

Proof. We show that for any sequent $\Gamma \Rightarrow \Delta$, if there is a proof of $\Gamma \Rightarrow \Delta$ such that the only cut application in the main fragment appears at the root, then there is a locally cut-free proof of $\Gamma \Rightarrow \Delta$. Given this claim, we can remove cut applications in the main fragment in a given proof starting from the *top* of its main fragment.

We prove this by strong induction on the lexicographic order on the pair consisting of the cut degree of the bottom-most cut application, and the sum of local heights of the proofs of the left and the right premises of the same application. Fix a sequent $\Gamma \Rightarrow \Delta$ and let the following be the proof satisfying the assumptions.

$$\mathsf{cut}\ \dfrac{\begin{array}{cc}\vdots & \vdots \\ \mathcal{D}_0 & \mathcal{D}_1 \\ \Gamma_0 \Rightarrow A & A, \Gamma_1 \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}$$

We assume that $\Gamma = \Gamma_0, \Gamma_1$. Let r_0 and r_1 be the bottom-most rules of $\mathcal{D}_0$ and $\mathcal{D}_1$, respectively. Recall that we are working with set-based sequents. If $A \in \Gamma_1$, then we can easily remove the application of cut. This can be done by applying w_L under $\mathcal{D}_1$ if $\Gamma \supsetneq \Gamma_1$, or by simply taking $\mathcal{D}_1$ if $\Gamma = \Gamma_1$. If $A \in \Gamma_0$, apply weakening under Γ_1; if $A \in \Delta$, apply weakening under $\mathcal{D}_0$. We hence assume that $A \notin \Gamma_0 \cup \Gamma_1 \cup \Delta$ in the proof below.

The smallest pair associated with a proof is $(n, m) = (1, 2)$. For any such proof $\mathcal{D}$, either both the left and right premises of the bottom-most cut application are axiomatic and the cut formula is some p or $\bot$, or $\mathsf{r}_1 = \mathsf{DL}$. Note that by the shape of the rule DL, the formula in the conclusion of this rule is a boxed formula, so r_0 cannot be DL in this case. If both the left and right premises are axiomatic, we have the following cases.

- $\mathsf{r}_0 = \mathsf{init}$. Then $A = p$ for some p, and $p \in \Gamma_0$. Apply w_L under $\mathcal{D}_1$.
- $\mathsf{r}_0 = \bot_\mathsf{L}$. Then $\bot \in \Gamma_0 \subseteq \Gamma$ and hence $\Gamma \Rightarrow \Delta$ is an axiom.
- $\mathsf{r}_1 = \mathsf{init}$. If A is not principal in r_1, then $p \in \Gamma_1 \cap \Delta$, and hence $\Gamma \Rightarrow \Delta$ is an axiom. If A is principal, then $A = p$ and $\Delta = \{p\}$ for some p. Apply w_L under $\mathcal{D}_0$.
- $\mathsf{r}_1 = \bot_\mathsf{L}$. If A is not principal, then $\bot \in \Gamma_1 \subseteq \Gamma$ and hence $\Gamma \Rightarrow \Delta$ is an axiom. Otherwise, $A = \bot$, and we have $\Gamma_0 \Rightarrow \bot$. By Lemma 6.33, there is a proof of $\Gamma_0 \Rightarrow$ whose main fragment is cut-free. Apply weakening rules under this proof to get a proof of $\Gamma \Rightarrow \Delta$.

For the other case, $\mathsf{r}_1 = \mathsf{DL}$. Then $\Gamma_1 = \Pi_1, \Box\Sigma, \Box\Gamma$ for some Π_1 that does not contain any boxed-formula, and $\Delta = \{\Box B\}$ for some B. We assume that $\Gamma_0 = \Pi_0, \Box\Theta$ where Π_0 does not contain any boxed formula.

$$\mathsf{DL}\,\dfrac{\mathsf{w_L}\,\dfrac{\mathsf{cut}\,\dfrac{\begin{array}{cc}\vdots & \vdots\\ \mathcal{D}_0 & \mathcal{D}_1'\\ \Pi_0, \Box\Theta \Rightarrow A & A, \Pi_1, \boxdot\Gamma \Rightarrow B\end{array}}{\Pi_0, \Pi_1, \Box\Theta, \boxdot\Gamma \Rightarrow B}}{\Pi_0, \Pi_1, \boxdot\Theta, \boxdot\Gamma \Rightarrow B}}{\Pi_0, \Pi_1, \Box\Theta, \Box\Sigma, \Box\Gamma \Rightarrow \Box B}$$

If $\Theta \subseteq \Pi_0 \cup \Pi_1 \cup \Box\Theta \cup \boxdot\Gamma$, then we don't need $\mathsf{w_L}$. This proof is locally cut-free, as its root is decorated with DL.

Now, consider when $(n, m) > (1, 2)$, where the pair associated with the given proof is now (n, m). Note that in the case of $(1, 2)$, we have in fact also covered all the cases where at least one of r_0 and r_1 is axiomatic. So here, we consider only those cases where none of $\mathsf{r}_0, \mathsf{r}_1$ is axiomatic. We make the following case distinctions.

Case 1. A is not principal in r_0;

Case 2. A is not principal in r_1;

Case 3. A is principal in both r_0 and r_1.

Case 1. A is not principal in r_0. If r_0 is one-premise, $\mathsf{r}_0 \neq \mathsf{DL}$ by the shape of the rule. $\mathcal{D}_0$ has the following form on the left. We transform it to the one on the right. (We only display $\mathcal{D}_0$ below.)

$$\dfrac{\begin{array}{c}\vdots\\ \mathcal{D}_0'\\ \Gamma_0' \Rightarrow A\end{array}}{\Gamma_0 \Rightarrow A}\,\mathsf{r}_0 \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots\\ \mathcal{D}_0' & \mathcal{D}_1\\ \Gamma_0' \Rightarrow A & A, \Gamma_1 \Rightarrow \Delta_1\end{array}}{\Gamma_0', \Gamma_1 \Rightarrow \Delta_1}\,\mathsf{cut}^{n,m-1}}{\Gamma \Rightarrow \Delta}\,\mathsf{r}_0$$

If r_0 is two-premise and $\mathsf{r}_0 \neq \rightarrow_{\mathsf{L}}$, we perform the following procedure.

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ \mathcal{D}_0^0 & \mathcal{D}_0^1\\ \Gamma_0^0 \Rightarrow A & \Gamma_0^1 \Rightarrow A\end{array}}{\Gamma_0 \Rightarrow A}\,\mathsf{r}_0 \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots\\ \mathcal{D}_0^0 & \mathcal{D}_1\\ \Gamma_0^0 \Rightarrow A & A, \Gamma_1 \Rightarrow \Delta\end{array}}{\Gamma_0^0, \Gamma_1 \Rightarrow \Delta}\,\mathsf{cut}^{n,m-1} \quad \dfrac{\begin{array}{cc}\vdots & \vdots\\ \mathcal{D}_0^1 & \mathcal{D}_1\\ \Gamma_0^1 \Rightarrow A & A, \Gamma_1 \Rightarrow \Delta\end{array}}{\Gamma_0^1, \Gamma_1 \Rightarrow \Delta}\,\mathsf{cut}^{n,m-1}}{\Gamma \Rightarrow \Delta}\,\mathsf{r}_0$$

If $\mathsf{r}_0 = \to_{\mathsf{L}}$, we proceed as follows.

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ \mathcal{D}_0^0 & \mathcal{D}_0^1\\ \Gamma_0', B \to C \Rightarrow B & \Gamma_0', C \Rightarrow A\end{array}}{\Gamma_0', B \to C \Rightarrow A}\,{\to_{\mathsf{L}}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \mathcal{D}_0^0\\ \Gamma_0', B \to C \Rightarrow B\end{array}}{\Gamma_0', B \to C, \Gamma_1 \Rightarrow B}\,{\mathsf{w}_{\mathsf{L}}} \qquad \dfrac{\begin{array}{cc}\vdots & \vdots\\ \mathcal{D}_0^1 & \mathcal{D}_1\\ \Gamma_0', C \Rightarrow A & A, \Gamma_1 \Rightarrow \Delta\end{array}}{\Gamma_0', C, \Gamma_1 \Rightarrow \Delta}\,{\mathsf{cut}^{n,m-1}}}{\Gamma_0', B \to C, \Gamma_1 \Rightarrow \Delta}\,{\to_{\mathsf{L}}}$$

We might need to delete the application of w_{L}.

Case 2. A is not principal in r_1. If r_1 is a one-premise rule other than DL (represented by the left tree below), we perform the following procedure. (We only display $\mathcal{D}_1$ below.)

$$\dfrac{\begin{array}{c}\vdots\\ \mathcal{D}_1'\\ A, \Gamma_1' \Rightarrow \Delta'\end{array}}{A, \Gamma_1 \Rightarrow \Delta}\,{\mathsf{r}_1} \quad \rightsquigarrow \quad \dfrac{\begin{array}{cc}\vdots & \vdots\\ \mathcal{D}_0 & \mathcal{D}_1'\\ \Gamma_0 \Rightarrow A & A, \Gamma_1' \Rightarrow \Delta_1'\end{array}}{\dfrac{\Gamma_0, \Gamma_1' \Rightarrow \Delta'}{\Gamma \Rightarrow \Delta}\,{\mathsf{r}_1}}\,{\mathsf{cut}^{n,m-1}}$$

If $\mathsf{r}_1 = \mathsf{DL}$, then we are in one of the following situations.

$$\dfrac{\begin{array}{c}\vdots\\ \mathcal{D}_1'\\ \Pi', A, \boxdot\Gamma \Rightarrow \Box C\end{array}}{\Pi', A, \Box\Sigma, \Box\Gamma \Rightarrow \Box C}\,{\mathsf{DL}} \qquad \dfrac{\begin{array}{c}\vdots\\ \mathcal{D}_1'\\ \Pi, \boxdot\Gamma \Rightarrow \Box C\end{array}}{\Pi, \Box B = A, \Box\Sigma', \Box\Gamma \Rightarrow \Box C}\,{\mathsf{DL}}$$

In the right case, simply delete $\Box B$ in the conclusion. In the left case, proceed as follows. Let $\Gamma_0 = \Pi_0 \cup \Box\Sigma_0$, where Π_0 contains no boxed formula.

$$\dfrac{\dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots\\ \mathcal{D}_0 & \mathcal{D}_1'\\ \Pi_0, \Box\Sigma_0 \Rightarrow A & \Pi', A, \boxdot\Gamma \Rightarrow C\end{array}}{\Pi_0, \Box\Sigma_0, \Pi', \boxdot\Gamma \Rightarrow C}\,{\mathsf{cut}^{n,m-1}}}{\Pi_0, \boxdot\Sigma_0, \Pi', \boxdot\Gamma \Rightarrow C}\,{\mathsf{w}_{\mathsf{L}}}}{\Pi_0, \Box\Sigma_0, \Pi', \Box\Sigma, \Box\Gamma \Rightarrow \Box C}\,{\mathsf{DL}}$$

If $\Sigma_0 \subseteq \Pi_0 \cup \Box\Sigma_0 \cup \Pi' \cup \boxdot\Gamma$, then there should be no w_{L} application.

If r_1 is two-premise, we proceed as follows.

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ \mathcal{D}_1^0 & \mathcal{D}_1\\ A,\Gamma_1^0 \Rightarrow \Delta_0 & A,\Gamma_1^1 \Rightarrow \Delta_1\end{array}}{A,\Gamma_1 \Rightarrow \Delta}\,\mathsf{r}_1 \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{array}{cc}\vdots & \vdots\\ \mathcal{D}_0 & \mathcal{D}_1^0\\ \Gamma_0 \Rightarrow A & A,\Gamma_1^0 \Rightarrow \Delta_0\end{array}}{\Gamma_0,\Gamma_1^0 \Rightarrow \Delta_0}\,\mathsf{cut}^{n,m-1} \quad \dfrac{\begin{array}{cc}\vdots & \vdots\\ \mathcal{D}_0 & \mathcal{D}_1^1\\ \Gamma_0 \Rightarrow A & A,\Gamma_1^1 \Rightarrow \Delta_1\end{array}}{\Gamma_0,\Gamma_1^1 \Rightarrow \Delta_1}\,\mathsf{cut}^{n,m-1}}{\Gamma \Rightarrow \Delta}\,\mathsf{r}_1$$

Case 3. A is principal in both r_0 and r_1. The cases where $\mathsf{r}_0 = \mathsf{w}_\mathsf{L}$ or $\mathsf{r}_1 = \mathsf{w}_\mathsf{L}$ are simple. We thus distinguish the remaining cases according to the main connective of A.

- $A = B \wedge C$. The $\mathcal{D}_0$ and $\mathcal{D}_1$ have the following shapes.

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ \mathcal{D}_0^0 & \mathcal{D}_0^1\\ \Gamma_0 \Rightarrow B & \Gamma_0 \Rightarrow C\end{array}}{\Gamma_0 \Rightarrow B \wedge C}\,\wedge_\mathsf{L} \qquad \dfrac{\begin{array}{c}\vdots\\ \mathcal{D}_1'\\ B,C,\Gamma_1 \Rightarrow \Delta\end{array}}{B \wedge C,\Gamma_1 \Rightarrow \Delta}\,\wedge_\mathsf{R}$$

$$\rightsquigarrow \quad \dfrac{\begin{array}{c}\vdots\\ \mathcal{D}_0^1\\ \Gamma_0 \Rightarrow C\end{array} \quad \dfrac{\begin{array}{cc}\vdots & \vdots\\ \mathcal{D}_0^0 & \mathcal{D}_1'\\ \Gamma_0 \Rightarrow B & B,C,\Gamma_1 \Rightarrow \Delta\end{array}}{C,\Gamma_0,\Gamma_1 \Rightarrow \Delta}\,\mathsf{cut}^{n-1,m-2}}{\Gamma \Rightarrow \Delta}\,\mathsf{cut}^{n-1,m'}$$

  We use m' for another sum of local heights. Since the cut degree already decreases to $n-1$, we do not care about the specific value of m'.

- $A = B \vee C$. Without loss of generality, we consider the case where we have a proof of $\Gamma_0 \Rightarrow B$.

$$\dfrac{\begin{array}{c}\vdots\\ \mathcal{D}_0'\\ \Gamma_0 \Rightarrow B\end{array}}{\Gamma_0 \Rightarrow B \vee C}\,\vee_\mathsf{R} \qquad \dfrac{\begin{array}{cc}\vdots & \vdots\\ \mathcal{D}_1^0 & \mathcal{D}_1^1\\ B,\Gamma_1 \Rightarrow \Delta & C,\Gamma_1 \Rightarrow \Delta\end{array}}{B \vee C,\Gamma_1 \Rightarrow \Delta}\,\vee_\mathsf{L} \quad\rightsquigarrow\quad \dfrac{\begin{array}{cc}\vdots & \vdots\\ \mathcal{D}_0' & \mathcal{D}_1^0\\ \Gamma_0 \Rightarrow B & B,\Gamma_1 \Rightarrow \Delta\end{array}}{\Gamma \Rightarrow \Delta}\,\mathsf{cut}^{n-1,m-2}$$

- $A = B \to C$.

$$\dfrac{\begin{array}{c}\vdots\\ \mathcal{D}_0'\\ \Gamma_0,B \Rightarrow C\end{array}}{\Gamma_0 \Rightarrow B \to C}\,\to_\mathsf{R} \qquad \dfrac{\begin{array}{cc}\vdots & \vdots\\ \mathcal{D}_1^0 & \mathcal{D}_1^1\\ B \to C,\Gamma_1 \Rightarrow B & C,\Gamma_1 \Rightarrow \Delta\end{array}}{B \to C,\Gamma_1 \Rightarrow \Delta}\,\to_\mathsf{L}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\vdots\ \mathcal{D}_0}{\Gamma_0 \Rightarrow B \to C} \quad \dfrac{\vdots\ \mathcal{D}_1^0}{B \to C, \Gamma_1 \Rightarrow B}}{\Gamma \Rightarrow B}\,\mathsf{cut}^{n,m-1} \quad \dfrac{\vdots\ \mathcal{D}_0'}{\Gamma_0, B \Rightarrow C}}{\Gamma \Rightarrow C}\,\mathsf{cut}^{n-1,m'} \quad \dfrac{\vdots\ \mathcal{D}_1^1}{C, \Gamma_1 \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}\,\mathsf{cut}^{n-1,m''}$$

- $A = \Box B$. $\mathsf{r}_0 = \mathsf{r}_1 = \mathsf{DL}$. Then $\mathcal{D}_0$ and $\mathcal{D}_1$ are as follows.

$$\mathsf{DL}\,\dfrac{\vdots\ \mathcal{D}_0'}{\Pi_0, \boxdot\Gamma_0 \Rightarrow B}}{\Pi_0, \Box\Sigma_0, \Box\Gamma_0 \Rightarrow \Box B} \qquad \mathsf{DL}\,\dfrac{\vdots\ \mathcal{D}_1'}{\Pi_1, \boxdot B, \boxdot\Gamma_1 \Rightarrow C}}{\Pi_1, \Box\Sigma_1, \Box B, \Box\Gamma_1 \Rightarrow \Box C}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\vdots\ \mathcal{D}_0'}{\Pi_0, \boxdot\Gamma_0 \Rightarrow B} \quad \dfrac{\vdots\ \mathcal{D}_0}{\Pi_0, \Box\Sigma_0, \Box\Gamma_0 \Rightarrow \Box B} \quad \dfrac{\vdots\ \mathcal{D}_1'}{\Pi_1, \boxdot B, \boxdot\Gamma_1 \Rightarrow C}}{\Pi_0, \Pi_1, \Box\Sigma_0, \Box\Gamma_0, B, \boxdot\Gamma_1 \Rightarrow C}\,\mathsf{cut}^{n,m-1}}{\Pi_0, \Pi_1, \Box\Sigma_0, \boxdot\Gamma_0, \boxdot\Gamma_1 \Rightarrow C}\,\mathsf{cut}^{n-1,m'}$$
$$\mathsf{w_L}\,\dfrac{\Pi_0, \Pi_1, \Box\Sigma_0, \boxdot\Gamma_0, \boxdot\Gamma_1 \Rightarrow C}{\Pi_0, \Pi_1, \boxdot\Sigma_0, \boxdot\Gamma_0, \boxdot\Gamma_1 \Rightarrow C}$$
$$\mathsf{DL}\,\dfrac{\Pi_0, \Pi_1, \boxdot\Sigma_0, \boxdot\Gamma_0, \boxdot\Gamma_1 \Rightarrow C}{\Pi_0, \Pi_1, \Box\Sigma_0, \Box\Gamma_0, \Box\Gamma_1 \Rightarrow \Box C}$$
$$\mathsf{w_L}\,\dfrac{\Pi_0, \Pi_1, \Box\Sigma_0, \Box\Gamma_0, \Box\Gamma_1 \Rightarrow \Box C}{\Pi_0, \Pi_1, \Box\Sigma_0, \Box\Sigma_1, \Box\Gamma_0, \Box\Gamma_1 \Rightarrow \Box C}$$

Again, it might be necessary to delete one or more applications of $\mathsf{w_L}$, depending on the subset relation between the sets.

We have thus covered all possible cases. Therefore, for any sequent $\Gamma \Rightarrow \Delta$, if there is a proof of $\Gamma \Rightarrow \Delta$ such that the only cut application in the main fragment appears at the root, then there is a proof of $\Gamma \Rightarrow \Delta$ that is locally cut-free.

Finally, by induction on the number of cut applications in the main fragment, for each sequent S and a proof of it, there is a proof of the same sequent that is locally cut-free. By the axiom of choice, there is a desired function. □

Theorem 6.35 (Cut elimination of $\mathbf{G3iSL}_\Box^{\mathsf{cut},\infty}$)**.** *For any sequent S, if $\vdash_{\mathbf{G3iSL}_\Box^{\mathsf{cut},\infty}} S$, then $\vdash_{\mathbf{G3iSL}_\Box^{\infty}} S$.*

Proof. Similar to the proof of Theorem 5.9. Define a function $\alpha : \mathbb{P}^{\mathrm{ff}}(\mathbf{G3iSL}_\Box^{\mathsf{cut},\infty}) \to \mathcal{T}(\mathbb{P}^{\mathrm{ff}}(\mathbf{G3iSL}_\Box^{\mathsf{cut},\infty}))$ such that

$$\alpha(T, \sim) = (\mathsf{T}_\epsilon(\mathrm{cutsup}(T, \sim)), w \in \mathcal{NW}^{\mathsf{T}_\epsilon(\mathrm{cutsup}(T,\sim))} \mapsto \mathsf{ST}_w(\mathrm{cutsup}(T))) = \mathrm{destruct} \circ \mathrm{cutsup}(T, \sim)$$

α is a proof translation step from $\mathbf{G3iSL}_\Box^{\mathsf{cut},\infty}$ to $\mathbf{G3iSL}_\Box^{\infty}$. By Theorem 4.27, there is a proof translation function from $\mathbf{G3iSL}_\Box^{\mathsf{cut},\infty}$ to $\mathbf{G3iSL}_\Box^{\infty}$ such that, given a $\mathbf{G3iSL}_\Box^{\mathsf{cut},\infty}$ proof of S, it outputs a $\mathbf{G3iSL}_\Box^{\infty}$ proof of the same sequent. □

Theorem 6.36 (Syntactic completeness of $\mathbf{G3iSL}^{\infty}_{\Box}$)**.** *For any finite set* $\Gamma \subseteq \mathsf{Form}^i$ *and any* $A \in \mathsf{Form}^i$*, if* $\vdash_{\mathbf{HiSL}_{\Box}} \bigwedge \Gamma \to A$*, then* $\vdash_{\mathbf{G3iSL}^{\infty}_{\Box}} \Gamma \Rightarrow A$*.*

Proof. Immediate from the cut elimination of $\mathbf{G3iSL}^{\mathrm{cut},\infty}_{\Box}$ (Theorem 6.35) and the completeness of $\mathbf{G3iSL}^{\mathrm{cut},\infty}_{\Box}$ with respect to $\mathbf{HiSL}_{\Box}$ (Theorem 6.31). □

Theorem 6.37 (Equivalence between $\mathbf{G3iSL}^{\mathrm{cut},\infty}_{\Box}$ and $\mathbf{G3iSL}^{\mathrm{cut},\circ}_{\Box}$)**.** *For any sequent* S*,* $\vdash_{\mathbf{G3iSL}^{\mathrm{cut},\infty}_{\Box}} S$ *iff* $\vdash_{\mathbf{G3iSL}^{\mathrm{cut},\circ}_{\Box}} S$*.*

Proof. If $\vdash_{\mathbf{G3iSL}^{\mathrm{cut},\circ}_{\Box}} S$, then by unraveling the proof and deleting redundant nodes (Cf. Lemma 3.38), we get a $\mathbf{G3iSL}^{\mathrm{cut},\infty}_{\Box}$ proof of S. For details, we refer the reader to the proof of Lemma 3.38. For the other direction, suppose $\vdash_{\mathbf{G3iSL}^{\mathrm{cut},\infty}_{\Box}} S$. Then, by the cut elimination of $\mathbf{G3iSL}^{\mathrm{cut},\infty}_{\Box}$ (Theorem 6.35), there is a $\mathbf{G3iSL}^{\infty}_{\Box}$ proof $\mathcal{D}$ of S. For every infinite branch b in $\mathcal{D}$, it has infinitely many applications of DL, and there is at least one sequent S' that appears infinitely many times on b. Therefore, we can 'prune' each infinite branch and get a $\mathbf{G3iSL}^{\mathrm{cut},\circ}_{\Box}$ proof of S. The detailed construction is similar to the proof of Proposition 5.13. □

Theorem 6.38 (Syntactic soundness and completeness of $\mathbf{G3iSL}^{\circ}_{\Box}$)**.** *For any finite* $\Gamma, \Delta \subseteq \mathsf{Form}^i$ *with* $|\Delta| \leqslant 1$*,* $\vdash_{\mathbf{HiSL}_{\Box}} \bigwedge \Gamma \to \bigvee \Delta$ *iff* $\vdash_{\mathbf{G3iSL}^{\circ}_{\Box}} \Gamma \Rightarrow \Delta$*.*

Proof. For soundness, if $\vdash_{\mathbf{G3iSL}^{\circ}_{\Box}} \Gamma \Rightarrow \Delta$, then $\vdash_{\mathbf{G3iSL}^{\mathrm{cut},\circ}_{\Box}} \Gamma \Rightarrow \Delta$. By Theorem 6.37, $\vdash_{\mathbf{G3iSL}^{\mathrm{cut},\infty}_{\Box}} \Gamma \Rightarrow \Delta$. By Theorem 6.35, we have $\vdash_{\mathbf{G3iSL}^{\infty}_{\Box}} \Gamma \Rightarrow \Delta$. Since $\mathbf{G3iSL}^{\infty}_{\Box}$ is sound with respect to $\mathbf{HiSL}_{\Box}$ (Corollary 6.28), it holds that $\vdash_{\mathbf{HiSL}_{\Box}} \bigwedge \Gamma \to \bigvee \Delta$.

For completeness, suppose $\vdash_{\mathbf{HiSL}_{\Box}} \bigwedge \Gamma \to \bigvee \Delta$. By the completeness of $\mathbf{G3iSL}^{\infty}_{\Box}$ (Theorem 6.36), we have $\vdash_{\mathbf{G3iSL}^{\infty}_{\Box}} \Gamma \Rightarrow \Delta$. Since we can 'fold' any $\mathbf{G3iSL}^{\infty}_{\Box}$ proof of $\Gamma \Rightarrow \Delta$ into a $\mathbf{G3iSL}^{\circ}_{\Box}$ proof of the same sequent, it follows that $\vdash_{\mathbf{G3iSL}^{\circ}_{\Box}} \Gamma \Rightarrow \Delta$. □

6.4 $\mathbf{iSL}_{\Box}$: a cyclic companion of iDL

Chapter 3 has shown that the logic **GL** can be obtained by allowing for cyclic proofs for a sequent calculus for **K4**. Indeed, **GL** is a *cyclic companion* of **K4**. We take this notion from [Men24].

Definition 6.39 (Cyclic companion)**.** *Let* $\mathbf{L}_0$ *and* $\mathbf{L}_1$ *be two modal logics.* $\mathbf{L}_1$ *is a* cyclic companion *of* $\mathbf{L}_0$ *iff there is a cyclic proof system* C *such that the following holds for any formula* A*:*

- $\vdash_{C} \Rightarrow A$ *iff* $A \in \mathbf{L}_1$*;*
- $\vdash_{C^-} \Rightarrow A$ *iff* $A \in \mathbf{L}_0$*.*

C^- *is the acyclic system obtained from banning cyclic proofs of* C*.* ⊣

Given the results in section 6.3, there is an analogous result for $\mathbf{iSL}_{\Box}$. It is a cyclic companion of **iDL**, the intuitionistic doxastic logic or the logic of intuitionistic beliefs, introduced by Artemov and Protopopescu in [AP16]. Originally, Artemov and Protopopescu called this logic $\mathbf{iEL}^-$, yet we name it **iDL** to make the superscript simpler.

Definition 6.40 (The logic of intuitionistic beliefs **iDL**)**.** *The logic of intuitionistic beliefs* **iDL** *is the smallest set of formulas containing the following axioms and is closed under necessitation and modus ponens.*

- *Intuitionistic propositional tautologies.*

- *Distribution.* $\Box(A \to B) \to (\Box A \to \Box B)$.
- *Co-reflection.* $A \to \Box A$.

'$\Box A$' can be read as 'It is believed that A'. Let the Hilbertian calculus of **iDL** *given by the axioms and rules above be* **HiDL**. ⊣

Clearly, Distribution and Co-reflection are identical with the axiom K and the completeness axiom in $\mathbf{iSL}_{\Box}$. Proposition 6.29 and Proposition 6.30 show that the axioms and the inference rule of **HiDL** are derivable in the system **G3iDL**. Note that in the proof of Proposition 6.30, the proof of Löb's axiom is cyclic, whilst the proofs of the remaining two are not. So $\mathbf{G3iDL}^{\mathrm{cut}}$ is complete with respect to **HiDL**. Lemma 6.34 also provides us with a cut reduction procedure for $\mathbf{G3iDL}^{\mathrm{cut}}$ proofs. Therefore, **G3iDL** is complete with respect to **HiDL**. Indeed, the rule DL is almost identical to the rule KI_1 in the sequent calculus for intuitionistic epistemic logic **iEL** provided by Krupski and Yatmanov in [KY16].

Theorem 6.41. *For any $A \in \mathsf{Form}^i$, if $A \in$ **iDL**, then $\vdash_{\mathbf{G3iDL}} \Rightarrow A$.*

Also, we can prove that the system **G3iDL** is sound with respect to **HiDL**.

Lemma 6.42. *For any finite sequence of formulas $A_0, \ldots, A_n$, it holds that*

$$\vdash_{\mathbf{HiDL}} \Box A_0 \wedge \cdots \wedge \Box A_n \to \Box(A_0 \wedge \cdots \wedge A_n).$$

Proof. By induction on n. We want to derive $\Box A_0 \wedge \cdots \wedge \Box A_n \to \Box(A_0 \wedge \cdots \wedge A_n)$, i.e. $\Box A_0 \wedge (\cdots \wedge \Box A_n) \to \Box(A_0 \wedge \cdots \wedge A_n)$. Note that $\vdash_{\mathbf{HiDL}} A_0 \to ((A_1 \wedge \cdots \wedge A_n) \to (A_0 \wedge \cdots \wedge A_n))$. Then by necessitation, distribution, and modus ponens, we have $\vdash_{\mathbf{HiDL}} \Box A_0 \to \Box((A_1 \wedge \cdots \wedge A_n) \to (A_0 \wedge \cdots \wedge A_n))$. Moreover, we have $\vdash_{\mathbf{HiDL}} \Box((A_1 \wedge \cdots \wedge A_n) \to (A_0 \wedge \cdots \wedge A_n)) \to (\Box(A_1 \wedge \cdots \wedge A_n) \to \Box(A_0 \wedge \cdots \wedge A_n))$. It then follows that $\vdash_{\mathbf{HiDL}} \Box A_0 \to (\Box(A_1 \wedge \cdots \wedge A_n) \to \Box(A_0 \wedge \cdots \wedge A_n))$. This yields $\vdash_{\mathbf{HiDL}} \Box A_0 \wedge \Box(A_1 \wedge \cdots \wedge A_n) \to \Box(A_0 \wedge \cdots \wedge A_n)$. By the IH we have that $\vdash_{\mathbf{HiDL}} \Box A_1 \wedge \cdots \wedge A_n \to \Box(A_1 \wedge \cdots \wedge A_n)$. Hence, it holds that $\vdash \Box A_0 \wedge \Box A_1 \wedge \cdots \wedge \Box A_n \to \Box(A_0 \wedge \cdots \wedge A_n)$. □

Theorem 6.43. *For any $\Gamma \Rightarrow \Delta$, if $\vdash_{\mathbf{G3iDL}} \Gamma \Rightarrow \Delta$, then $\vdash_{\mathbf{HiDL}} \bigwedge \Gamma \to \bigvee \Delta$.*

Proof. By induction on the height of **G3iDL** proofs. For the base case, the standard interpretations of **G3iDL** axioms are intuitionistic tautologies. For the inductive case, we only prove that if there is a **G3iDL** proof of some instance of the premise of DL, then the standard interpretation of the conclusion of this instance is provable in **HiDL**. The rest of the cases appeal only to propositional tautologies and modus ponens. Suppose there is a proof of $\Pi, \boxdot\Gamma \Rightarrow A$. By IH, there is a Hilbertian proof of $\bigwedge \Pi \wedge \bigwedge \boxdot\Gamma \to A$. Then, we complete the proof as follows. Therefore, we have $\vdash_{\mathbf{iDL}} \Box(\bigwedge \Pi \wedge \bigwedge \boxdot\Gamma \to A)$, which further implies $\vdash_{\mathbf{iDL}} \Box(\bigwedge \Pi \wedge \bigwedge \boxdot\Gamma) \to \Box A$. Once we show that $\vdash_{\mathbf{HiDL}} \bigwedge \Pi \wedge \bigwedge \Box\Gamma \to \Box A$, it is immediate that $\vdash_{\mathbf{HiDL}} \bigwedge \Pi \wedge \bigwedge \Box\Sigma \wedge \bigwedge \Box\Gamma \to \Box A$. So we aim to show the former. It suffices to show that $\vdash_{\mathbf{HiDL}} \bigwedge \Pi \wedge \bigwedge \Box\Gamma \to \Box(\bigwedge \Pi \wedge \bigwedge \boxdot\Gamma)$. Note that for each $B \in \Pi \cup \Box\Gamma$, we have that $\vdash_{\mathbf{HiDL}} B \to \Box B$. Then, it follows that $\vdash_{\mathbf{HiDL}} \bigwedge \Pi \wedge \bigwedge \Box\Gamma \to \bigwedge \Box\Pi \wedge \bigwedge \Box\Box\Gamma$. Then, of course $\vdash_{\mathbf{HiDL}} \bigwedge \Pi \wedge \bigwedge \Box\Gamma \to \bigwedge \Box\Pi \wedge \bigwedge \Box\Gamma \wedge \bigwedge \Box\Box\Gamma$. By Lemma 6.42, we have that $\vdash_{\mathbf{HiDL}} \bigwedge \Box\Pi \wedge \bigwedge \Box\Box\Gamma \to \Box(\bigwedge \Pi \wedge \bigwedge \Gamma \wedge \bigwedge \Box\Gamma)$. Therefore, it holds that $\vdash_{\mathbf{HiDL}} \bigwedge \Pi \wedge \bigwedge \Box\Gamma \to \Box(\bigwedge \Pi \wedge \bigwedge \boxdot\Gamma)$, and we are done. □

It then follows that for any $A \in \mathsf{Form}^i$, $\vdash_{\mathbf{G3iDL}} \Rightarrow A$ implies $A \in$ **iDL**, as $\top \in$ **iDL**.

Theorem 6.44. *For any $A \in \mathsf{Form}^i$, $\vdash_{\mathbf{G3iDL}} \Rightarrow A$ iff $A \in$ **iDL**.*

Now, by Theorem 6.38, we also have $\vdash_{\mathbf{G3iSL}^{\circ}_{\Box}} A$ iff $A \in \mathbf{iSL}_{\Box}$. Combining this with Theorem 6.44, we arrive at the conclusion that $\mathbf{iSL}_{\Box}$ is a cyclic companion of **iDL**. One might wonder what is obtained on the model-theoretic side by admitting cyclic proofs for $\mathbf{G3iDL}^{\mathrm{cut}}$. But as one might expect, we have the following result.

Theorem 6.45 (Artemov and Protopopescu, 2016)**.** **iDL** *is the logic of the class of intuitionistic frames* $(W, \leqslant, R, V)$ *where* $R \subseteq \leqslant$ *and* R *is transitive.*

In the following, we move on to the **G4**-style systems for $\mathbf{iSL}_{\Box}$. But as promised, we will first make clear relevant technicalities regarding sequents.

6.5 Technicalities regarding set sequents and backward proof search

The search for **G4**-style systems is motivated by backward proof search termination. Let us first informally introduce a system for **iPL** that is in this style. Recall **G3ip**. If we think of the sequents endorsed in this system as multiset sequents, then it is obvious that for each rule, except for $\to_{\mathsf{L}}$, the complexity of each premise is smaller than that of the conclusion. The left premise of $\to_{\mathsf{L}}$ is the only exception, making backward proof search possibly non-terminating. In contrast, **G4ip** (which is again originally based on multiset sequents) is defined in such a way that for any rule, each premise is strictly 'smaller' than the conclusion under a certain well-designed measure for sequents. This is done by replacing the $\to_{\mathsf{L}}$ rule with four other rules. Each of these rules represents a special case of the replaced $\to_{\mathsf{L}}$, where the cases are distinguished by the main connective of the antecedent of the principal formula $A \to B$.

We emphasize that the property of 'backwards decreasing' of standard propositional rules (with the possible exception $\to_{\mathsf{L}}$) obtains if we work with *multiset* sequents. If we work with *set* sequents, it is a different story. Again, take the rule $\wedge_{\mathsf{L}}$ for an example, where we use set sequents and use comma for set-theoretic union.

$$\wedge_{\mathsf{L}} \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}$$

This rule is only deceivingly backwards decreasing, for $A \wedge B$ might well be in Γ. For example, the following instances are legitimate applications of this rule.

$$\wedge_{\mathsf{L}} \frac{\wedge_{\mathsf{L}} \dfrac{A \wedge B, A, B \Rightarrow \Delta}{A \wedge B, A, B \Rightarrow \Delta}}{A \wedge B \Rightarrow \Delta}$$

If we consider the weight of sequents defined normally, the premises do not have a smaller complexity than the conclusions. Then, set sequents seem to be at odds with the very motivation of the creation of **G4**-style systems. The non-terminating property of $\mathbf{G3iSL}^{\infty}_{\Box}$ is not only caused by the left premise of $\to_{\mathsf{L}}$, but also the choice of *set* sequents.

Nevertheless, we have also made clear in section 6.1 that in the context of cyclic proof theory, set sequents may free us from proving the admissibility of contraction and give us more direct termination arguments. So, while the non-decreasing feature of rules for set-sequents is real, there is also non-trivial incentive to employ set sequents when dealing with cyclic proofs. To reconcile the non-decreasing feature of rules defined in terms of set sequents with terminating proof search, we impose a restriction on the contexts. Let us again look at

the inference steps above. To prevent such situations, we choose to delete the principal formula $A \wedge B$ in the premise. So for $\wedge_{\mathsf{L}}$, we would have the following.

$$\wedge_{\mathsf{L}} \frac{A, B, \Gamma^{-} \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}$$

The set Γ^{-} is just $\Gamma \setminus \{A \wedge B\}$, but it may contain A or/and B. This suffices for backward descent.

However, this restriction comes with two prices. The first is that the standard cut reduction procedure becomes difficult to design. For example, suppose we want to show that, for $(\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty})^{-}$ ('$-$' indicates that we restrict contexts), every proof whose only cut application in the main fragment is at the root can be turned into another proof that is locally cut-free. Suppose $\mathcal{D}_0$ and $\mathcal{D}_1$ are locally cut-free, and let r_0 be the bottom-most rule of $\mathcal{D}_0$.

$$\mathrm{cut} \frac{\begin{array}{cc} \vdots & \vdots \\ \mathcal{D}_0 & \mathcal{D}_1 \\ \Gamma_0 \Rightarrow A & A, \Gamma_1 \Rightarrow \Delta \end{array}}{\Gamma \Rightarrow \Delta}$$

Assume further that $\mathcal{D}_0$ has the following shape, where A is not principal in r_0.

$$\mathsf{r}_0 \frac{\begin{array}{cc} \vdots & \vdots \\ \mathcal{D}_0^0 & \mathcal{D}_0^1 \\ \Gamma_0^0 \Rightarrow A & \Gamma_0^1 \Rightarrow A \end{array}}{\Gamma_0 \Rightarrow A}$$

Normally, we would want to proceed as follows.

$$\mathsf{r}_0 \frac{\mathrm{cut} \dfrac{\begin{array}{cc} \vdots & \vdots \\ \mathcal{D}_0^0 & \mathcal{D}_1 \\ \Gamma_0^0 \Rightarrow A & A, \Gamma_1 \Rightarrow \Delta \end{array}}{\Gamma_0^0, \Gamma_1 \Rightarrow \Delta} \qquad \mathrm{cut} \dfrac{\begin{array}{cc} \vdots & \vdots \\ \mathcal{D}_0^1 & \mathcal{D}_1 \\ \Gamma_0^1 \Rightarrow A & A, \Gamma_1 \Rightarrow \Delta \end{array}}{\Gamma_0^1, \Gamma_1 \Rightarrow \Delta}}{\Gamma \Rightarrow \Delta}$$

However, since we require the principal formula not to be in the context of premises for certain rules, it is unclear whether we can apply r_0 as the bottom-most rule – the principal formula might be introduced to the antecedents of $\Gamma_0^0, \Gamma_1 \Rightarrow \Delta$ and $\Gamma_0^1, \Gamma_1 \Rightarrow \Delta$.

One possible solution to this problem is to read comma not as union, but as *disjoint* union. So, the expression 'Γ, Σ' presupposes that $\Gamma \cap \Sigma = \emptyset$. By doing this, we can check that indeed, in the example above, the principal formula cannot appear in Γ_1. However, if we take this route, another problem arises. For the completeness of $(\mathbf{G3iSL}_{\Box}^{\mathrm{cut},\infty})^{-}$ with respect to $\mathbf{HiSL}_{\Box}$, we want to use cut to simulate modus ponens (see Proposition 6.29). The problem is that the admissibility of modus ponens may not be proven in the standard way if we restrict contexts. Suppose we are given proofs of $\Gamma \Rightarrow A \to B$ and $\Gamma \Rightarrow A$, with $A \in \Gamma$. If the system derives that $A, \Gamma \Rightarrow A$ for any A, then we have derivations of $A, A \to B \Rightarrow A$ and $B, A \Rightarrow B$, from which we can derive

$A, A \to B \Rightarrow B$. We then want to apply cut to this sequent and $\Gamma \Rightarrow A$. Unfortunately, we can't, for the A is in Γ and hence the contexts are not disjoint. To solve this problem, one might want to consider the following cut rule, which we take from [Pfe00]. We still read comma as disjoint union.

$$\text{cut}\ \frac{\Gamma, \Sigma_0 \Rightarrow A \qquad A, \Gamma, \Sigma_1 \Rightarrow \Delta}{\Gamma, \Sigma_0, \Sigma_1 \Rightarrow \Delta}$$

While this rule allows us to simulate modus ponens, it reintroduces the previous problem when designing the cut reduction procedure.

The second price is the invertibility of the $\to_{\mathrm{R}}$ rule for **G4**-style systems with restrictions on contexts.

$$\to_{\mathrm{R}}\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

This invertibility result is crucial for **G4**-style systems because the derivability of sequents of the form $A, \Gamma \Rightarrow A$ relies on it. We call such sequents *extended axioms*, as in contrast with *atomic axioms*, where A can only range over propositional variables. Additionally, the cut reduction procedure also relies on this rule – this is another red flag indicating that we probably should not prove cut-free completeness of **G4**-style non-wellfounded systems for $\mathbf{iSL}_\Box$ via cut reduction. Now, suppose that, in the hope of deriving extended axioms, we want to prove invertibility of $\to_{\mathrm{R}}$ in a well-founded **G4**-style system by induction on the height of proofs. Also, suppose that we are in the situation where the last rule of the derivation of $\Gamma \Rightarrow A \to B$ is the one below. Γ^- is the set $\Gamma \setminus \{(A \to B) \to C\}$.

$$\to\to\ \frac{\Gamma^-, B \to C, A \Rightarrow B \qquad C, \Gamma^- \Rightarrow \Delta}{\Gamma, (A \to B) \to C \Rightarrow \Delta}$$

Then, the proof of $\Gamma \Rightarrow A \to B$ has the following shape.

$$\to\to\ \frac{\overset{\mathcal{D}_0}{\nabla}\ \Gamma^-, C, D \to E \Rightarrow D \qquad \overset{\mathcal{D}_0}{\nabla}\ \Gamma^-, E \Rightarrow A \to B}{\Gamma, (C \to D) \to E \Rightarrow A \to B}$$

By the IH on height, we have a proof of $\Gamma^-, E, A \Rightarrow B$ (let us suppose that A and E are distinct and A is not in Γ^-). We want to apply weakening to the sequent on the left to get $\Gamma^-, C, D \to E, A \Rightarrow D$ and then apply the rule $\to\to$. But we cannot do so if A, the formula that is pulled to the antecedent, is exactly the same as the principal formula $(C \to D) \to E$. If we are in a well-founded system, we could prove that for any Γ that does not contain $(C \to D) \to E$, if $E, (C \to D) \to E, \Gamma \Rightarrow \Delta$ is derivable, then $E, \Gamma \Rightarrow \Delta$ is derivable. But for non-wellfounded systems (and with restrictions on contexts!), such a proof is not so easy.

So what now? Our decision is to live with these problems when designing **G4**-style systems for $\mathbf{iSL}_\Box$. The first problem indicates that we should not prove cut-free completeness via cut elimination, whereas the second indicates that we should endorse extended axioms rather than atomic ones.

Before moving on, we also want to leave a comment on endorsing set sequents in the well-founded classical setting or the well-founded **G3** (intuitionistic) setting. First, for cut reduction, we conjecture that one could

prove standard invertibility results and appeal to them to eliminate the 'redundant' principal formula. Second, for the invertibility of $\to_{\mathsf{L}}$, if we work with set sequents, then there is no need to prove this result for we have contraction *a priori*, and the derivability of extended axioms does not rely on this result either.

Given all this background information, we are ready for the **G4**-systems for $\mathbf{iSL}_{\Box}$.

6.6 An infinitary cut-free G4-style system

Definition 6.46. *The sequent calculus* **G4ip** *is given by the following set of rules.*

$$\mathsf{init}\ \frac{}{A, \Gamma \Rightarrow A} \qquad \bot_{\mathsf{L}}\ \frac{}{\bot, \Gamma \Rightarrow \Delta} \qquad \wedge_{\mathsf{L}}\ \frac{A, B, \Gamma^{-} \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \wedge_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\vee_{\mathsf{L}}\ \frac{A, \Gamma^{-} \Rightarrow \Delta \qquad B, \Gamma^{-} \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A_{i \in \{0,1\}}}{\Gamma \Rightarrow A_0 \vee A_1} \qquad \to_{\mathsf{R}}\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B} \qquad p \to\ \frac{\Gamma^{-}, p, A \Rightarrow \Delta}{\Gamma, p, p \to A \Rightarrow \Delta}$$

$$\wedge \to\ \frac{\Gamma^{-}, A \to (B \to C) \Rightarrow \Delta}{\Gamma, (A \wedge B) \to C \Rightarrow \Delta} \qquad \vee \to\ \frac{\Gamma^{-}, A \to B, A \to C \Rightarrow \Delta}{\Gamma, (A \vee B) \to C \Rightarrow \Delta}$$

$$\to\to\ \frac{\Gamma^{-}, B \to C, A \Rightarrow B \qquad C, \Gamma^{-} \Rightarrow \Delta}{\Gamma, (A \to B) \to C \Rightarrow \Delta}$$

Designated formulas, principal formulas, and contexts of each rule are defined normally. The set Γ^{-} *in any premise is obtained from removing the principal formula from* Γ.

Let **G4iDL** *be the system that is obtained from adding the following rules to* **G4ip**:

$$\mathsf{DL}^{-}\ \frac{\Pi, \boxdot\Gamma \Rightarrow A}{\Pi, \Box\Gamma \Rightarrow \Box A} \qquad \Box \to\ \frac{\Pi, \Box A \to B, \boxdot\Gamma \Rightarrow A \qquad \Pi^{-}, \Box\Gamma, B \Rightarrow \Delta}{\Pi, \Box\Gamma, \Box A \to B \Rightarrow \Delta}$$

where Π *contains no boxed formula. Recall that for any* Γ*, we let* $\boxdot\Gamma = \Box\Gamma, \Gamma$*. For the rule* $\Box \to$*, the formula* $\Box A \to B$ *and all formulas in* $\Box\Gamma$ *are principal formulas. For the rule* DL^{-}*, only formulas in* Π *(in the conclusion) are not principal.* ⊣

Note that the systems above have restrictions on the contexts in the premises of certain rules. Other than this, the propositional rules are the standard ones.

We first introduce the infinitary **G4** system for $\mathbf{iSL}_{\Box}$. It may be surprising that a **G4**-style system is infinitary, as we want terminating backward proof search. The next section will be dedicated to addressing this discomfort, so please bear with us for a moment. This infinitary system has two progressing rules: the premise of DL^{-} and the left premise of $\Box \to$. The local soundness proof might be helpful in demonstrating why these premises are progressive.

Proposition 6.47. *The modal rules* DL^{-} *and* $\Box \to$ *are locally sound with respect to the frames of* $\mathbf{iSL}_{\Box}$.

Proof. For DL^{-}, if the conclusion is not valid, then there is some $\mathbf{iSL}_{\Box}$ model $\mathbb{M} = (W, \leqslant, R, V)$ and some $x \in W$ such that $\mathbb{M}, x \Vdash \bigwedge \Pi \wedge \bigwedge \Box\Gamma$ and $\mathbb{M}, x \nVdash \Box A$. So there is some $y \in W$ such that xRy and $\mathbb{M}, y \nVdash A$. Since xRy and $\mathbb{M}, x \Vdash \bigwedge \Box\Gamma$, we have that $\mathbb{M}, y \Vdash \bigwedge \Gamma$. Since $R \subseteq \leqslant$ and $x \leqslant y$, we also have that $\mathbb{M}, y \Vdash \bigwedge \Pi \wedge \bigwedge \Box\Gamma$.

Therefore, y falsifies the sequent $\Pi, \boxdot\Gamma \Rightarrow A$. For $\Box\rightarrow$, suppose we have a pointed model $\mathbb{M}, x$ that falsifies $\Pi, \Box\Gamma, \Box A \rightarrow B \Rightarrow \Delta$. Suppose for a contradiction that $\Pi, \boxdot\Gamma, \Box A \rightarrow B \Rightarrow A$ and $\Pi, \Box\Gamma, B \Rightarrow \Delta$ are both valid. If B is true at x, then Δ is true at x, which is a contradiction. Otherwise, both B and $\Box A$ are false at x. So there is some y such that xRy and $\mathbb{M}, y \nVdash A$. Since $\mathbb{M}, x \Vdash \bigwedge \Box\Gamma$, it follows that $\mathbb{M}, y \Vdash \bigwedge \Gamma$. Since $R \subseteq \leqslant$, we have $x \leqslant y$ and hence $\mathbb{M}, y \Vdash \bigwedge \Pi \wedge \bigwedge \Box\Gamma$ and $\mathbb{M}, y \Vdash \Box A \rightarrow B$. This contradicts the assumption that $\Pi, \boxdot\Gamma, \Box A \rightarrow B \Rightarrow A$ is valid. Therefore, one of the premises of $\Box\rightarrow$ should not be valid. □

It is clear from this proof that the left-most premises of DL^- and $\Box\rightarrow$ correspond to a 'modal jump' to a world that is accessible from the state where the conclusions are falsified. Naturally, these premises will be our progressing premises in the **G4**-style local progress calculus for $\mathbf{iSL}_\Box$.

Definition 6.48 (Local progress calculus $\mathbf{G4iSL}^\infty_\Box$)**.** *Let* $\mathbf{G4iSL}^\infty_\Box$ *be the pair* $(\mathbf{G4iDL}, p)$*, where* p *is a function such that for any* $\mathsf{r} \in \mathbf{G4iDL} \setminus \{\mathsf{DL}^-, \Box\rightarrow\}$*, it holds that* $p_{\mathsf{r}}(r) = \emptyset$*; for* DL^- *and* $\Box\rightarrow$*, any* $\mathsf{r} \in \mathsf{DL}^-$ *and any* $\mathsf{r}' \in \Box\rightarrow$*, it holds that* $p_{\mathsf{DL}^-}(r) = p_{\Box\rightarrow}(r') = \{0\}$. ⊣

Given the local soundness of the rules and the progress condition of the system, the semantic and syntactic soundness theorems follow easily.

Theorem 6.49 (Syntactic soundness of $\mathbf{G4iSL}^\infty_\Box$)**.** *For any sequent* $\Gamma \Rightarrow \Delta$*, if* $\vdash_{\mathbf{G4iSL}^\infty_\Box} \Gamma \Rightarrow \Delta$*, then* $\vdash_{\mathbf{HiSL}_\Box} \bigwedge \Gamma \rightarrow \bigvee \Delta$.

Proof. Appealing to Proposition 6.47, we can formulate a *reductio* argument for the semantic soundness of $\mathbf{G4iSL}^{\mathsf{cut},\infty}_\Box$, just like the proof of Theorem 5.20. Then, by the completeness of $\mathbf{HiSL}_\Box$, we have that $\vdash_{\mathbf{HiSL}_\Box} \bigwedge \Gamma \rightarrow \bigvee \Delta$. □

6.7 A corresponding cyclic G4-style system and backward proof search

Recall that one crucial motivation for **G4**-style systems is terminating backward proof search. One might wonder how $\Box\rightarrow$ and DL^- suit this motivation, for they have 'non-decreasing' premises. Indeed, given the following standard measure of formulas and sequents taken from [TS00], these rules are not 'upwards-decreasing'.

Definition 6.50 (**G4**-weight of formula and sequent)**.** *For each intuitionistic modal formula* A*, the* **G4** *weight of* A*,* $w^4(A)$*, is defined as follows.*

- $w^4(p) = w^4(\bot) = 2$*, where* $p \in \Phi$*;*
- $w^4(A \wedge B) = w^4(A)(1 + w^4(B))$*;*
- $w^4(A \vee B) = 1 + w^4(A) + w^4(B)$*;*
- $w^4(A \rightarrow B) = 1 + w^4(A)w^4(B)$*;*
- $w^4(\Box A) = 1 + w^4(A)$.

The **G4** *weight of a sequent* $\Gamma \Rightarrow \Delta$ *is defined as* $w^4(\Gamma \Rightarrow \Delta) = \sum_{A \in \Gamma \cup \Delta} w^4(A)$. ⊣

Remark 6.51. *In this section, the weight of formulas/sequents will always be their* **G4***-weight.*

Nevertheless, the fact that these two rules are non-decreasing is not at odds with terminating proof search. To explain why, we first need to introduce the corresponding cyclic system.

Definition 6.52 (Cyclic system $\mathbf{G4iSL}^{\circ}_{\Box}$)**.** *Let* $\mathbf{G4iSL}^{\circ}_{\Box}$ *be the pair* $(\mathbf{G4iSL}^{\infty}_{\Box}, \mathbb{P}^{\infty}(\mathbf{G4iSL}^{\infty}_{\Box}))$. ⊣

Definition 6.53 ($\mathbf{G4iSL}^{\circ}_{\Box}$ proof and $\mathbf{G4iSL}^{\circ}_{\Box}$ provability)**.** *A* $\mathbf{G4iSL}^{\circ}_{\Box}$ proof *is a finite* $(\mathbf{G4iDL} \times \mathrm{Seq}^{i}) \cup \mathrm{Seq}^{i}$*-labeled cyclic tree* $(N, \prec \cup f, l)$ *such that:*

(1) $f : \mathcal{L}^{T} \hookrightarrow N \setminus \mathcal{L}^{T}$ *a partial function on the leaves of* T*;*

(2) if w *is a leaf, then exactly one of the following holds:*

(a) $w \notin Dom(f)$, $l(w) \in \mathbf{G4iDL} \times \mathrm{Seq}^{i}$ *and* $(\mathrm{seq}_{T}(w))$ *is an instance of* $\mathrm{rule}_{T}(w)$*;*

(b) $w \in Dom(f)$, $l(w) \in \mathrm{Seq}^{i}$, $l_{T}(w) = \mathrm{seq}_{T}(f(w))$*, and there is a node* u *such that* $f(w) \sqsubset u \sqsubseteq w$*, and* u *is decorated either with the premise of a* DL^{-} *instance or with the left premise of a* $\Box\!\!\rightarrow$ *instance;*

(3) otherwise, $l(w) \in \mathbf{G4iDL} \times \mathrm{Seq}^{i}$*; for* $w0, \ldots, wn$*, the* $\prec$*-successors of* w*, the sequence*

$$(\mathrm{seq}_{T}(w_0), \ldots, \mathrm{seq}_{T}(w_n), \mathrm{seq}_{T}(w))$$

is an instance of $\mathrm{rule}_{T}(w)$*, where we set* $seq_{T}(wi) = l(wi)$ *if* $wi \in Dom(f)$*.*

A sequent S is provable in $\mathbf{G4iSL}^{\circ}_{\Box}$ *(*$\vdash_{\mathbf{G4iSL}^{\circ}_{\Box}} S$*) iff there is a* $\mathbf{G4iSL}^{\circ}_{\Box}$ *proof of* S*.* ⊣

Example 6.54. *Here is a* $\mathbf{G4iSL}^{\circ}_{\Box}$ *proof of Löb's axiom.*

$$\dfrac{\dfrac{\dfrac{\dfrac{\rightsquigarrow \Box(\Box A \to A), \Box A \to A \Rightarrow A \qquad \overset{\text{init}}{\dfrac{}{A, \Box(\Box A \to A) \Rightarrow A}}}{\Box\!\!\rightarrow \dfrac{\rightsquigarrow \Box(\Box A \to A), \Box A \to A \Rightarrow A}{\mathsf{DL}^{-} \dfrac{\Box(\Box A \to A) \Rightarrow \Box A}{\to_{\mathrm{R}} \dfrac{}{\Rightarrow \Box(\Box A \to A) \to \Box A}}}}}}$$

For this cyclic system, backward proof search is indeed terminating. To see this, we first define the **G4**-subformula closure of a sequent.

Definition 6.55 (**G4**-subformula closure)**.** *Let* $\Gamma \Rightarrow \Delta$ *be an intuitionistic sequent. Define the* **G4***-subformula closure* $\mathrm{Clos}^{4}(\Gamma \Rightarrow \Delta)$ *of* $\Gamma \Rightarrow \Delta$ *as follows.*

(1) For any $A \in \Gamma \cup \Delta$, $A \in \mathrm{Clos}^{4}(\Gamma \Rightarrow \Delta)$*.*

(2) For any $A = \Box B$*, if* $A \in \mathrm{Clos}^{4}(\Gamma \Rightarrow \Delta)$*, then* $B \in \mathrm{Clos}^{4}(\Gamma \Rightarrow \Delta)$*;*

(3) For any $A = B \circ C$ *(*$\circ \in \{\wedge, \vee\}$*), then* $B, C \in \mathrm{Clos}^{4}(\Gamma \Rightarrow \Delta)$*;*

(4) For any $A = p \to B$*, if* $A \in \mathrm{Clos}^{4}(\Gamma \Rightarrow \Delta)$*, then* $p, B \in \mathrm{Clos}^{4}(\Gamma \Rightarrow \Delta)$*;*

(5) For any $A = (B \wedge C) \to D$*, if* $A \in \mathrm{Clos}^{4}(\Gamma \Rightarrow \Delta)$*, then* $B \wedge C, D, B \to (C \to D) \in \mathrm{Clos}^{4}(\Gamma \Rightarrow \Delta)$*;*

(6) For any $A = (B \vee C) \to D$*, if* $A \in \mathrm{Clos}^{4}(\Gamma \Rightarrow \Delta)$*, then* $B \vee C, D, B \to D, C \to D \in \mathrm{Clos}^{4}(\Gamma \Rightarrow \Delta)$*;*

(7) For any $A = (B \to C) \to D$*, if* $A \in \mathrm{Clos}^{4}(\Gamma \Rightarrow \Delta)$*, then* $B, C \to D, C, D$ *are in* $\mathrm{Clos}^{4}(\Gamma \Rightarrow \Delta)$*;*

(8) For any $A = \Box B \to C$*, then* $\Box B$ *and* C *are in* $\mathrm{Clos}^{4}(\Gamma \Rightarrow \Delta)$*.*

The above are the only formulas occurring in $\mathrm{Clos}^{4}(\Gamma \Rightarrow \Delta)$*.* ⊣

Observe that for any sequent $\Gamma \Rightarrow \Delta$ and its proof search tree, the tree can only be decorated by sequents consisting of the formulas in $\mathrm{Clos}^4(\Gamma \Rightarrow \Delta)$ by the shape of the rules in **G4iDL**.

Proposition 6.56. *For any intuitionistic sequent* $\Gamma \Rightarrow \Delta$*, the set* $\mathrm{Clos}^4(\Gamma \Rightarrow \Delta)$ *is finite.*

Proof. Let $\Gamma \Rightarrow \Delta$ be a sequent. Since $\Gamma \cup \Delta$ is finite, either it is empty, or there is a formula in $\Gamma \cup \Delta$ that has the most occurrences of connectives. The empty case is trivial, so we consider the other case. Fix such a formula, and suppose it has n-many connective occurrences.

By the definition of $\mathrm{Clos}^4(\Gamma \Rightarrow \Delta)$, all formulas in $\mathrm{Clos}^4(\Gamma \Rightarrow \Delta)$ has at most n-many connectives. All propositional variables in $\mathrm{Clos}^4(\Gamma \Rightarrow \Delta)$ must already occur in $\Gamma \Rightarrow \Delta$. Therefore, $\mathrm{Clos}^4(\Gamma \Rightarrow \Delta)$ contains only a finite number of propositional variables, and possibly $\bot$. Finitely many propositional variables can only compose finitely many formulas with no more than n-many connectives. □

Theorem 6.57 (Backward $\mathbf{G4iSL}^\circ_\Box$ proof search termination)**.** *For any sequent* $\Gamma \Rightarrow \Delta$*, its backward* $\mathbf{G4iSL}^\circ_\Box$ *proof search terminates.*

Proof. Fix a sequent $\Gamma \Rightarrow \Delta$, and let $|\mathrm{Clos}^4(\Gamma \Rightarrow \Delta)| = n$. Then, any proof tree of $\Gamma \Rightarrow \Delta$ can be decorated with at most $2^n \cdot 2^n = 4^n$ distinct sequents. We argue that for any construction of a proof search tree, we can 'stop searching' when we reach the height $4^n + 1$. Let us take an arbitrary branch b in a proof search tree. Either the branch leads to an axiom, which we will find in at most 4^n-many steps; or it ends with a sequent to which no more rules can be applied, also in at most 4^n-many steps; or it is infinite. If it is infinite, consider the node w on b that has height $4^n + 1$. The sequent on w must appear on some node $u \sqsubset w$. Since all rules of $\mathbf{G4iSL}^\circ_\Box$ except for modal rules are backwards decreasing, the path from u to w must pass through an application of DL^- or the left premise of an application of $\Box\!\rightarrow$. This means that w is a leaf with a successful companion. □

The main takeaway here is that we design the system to be such that the progressing premises coincide with the non-decreasing ones. This guarantees that any branch in a proof search tree that is 'high enough' must be a progressing branch.

It should also be clear that $\mathbf{G4iSL}^\infty_\Box$ and $\mathbf{G4iSL}^\circ_\Box$ are equivalent. Every $\mathbf{G4iSL}^\circ_\Box$ proof can be unraveled to get a $\mathbf{G4iSL}^\infty_\Box$ proof, and every $\mathbf{G4iSL}^\infty_\Box$ proof can be 'pruned' at a certain height to yield a $\mathbf{G4iSL}^\circ_\Box$ proof. Thus, a proof search for the cyclic system is in effect a proof search for the infinitary system.

Granted that backward proof search for $\mathbf{G4iSL}^\circ_\Box$ (and hence in effect for $\mathbf{G4iSL}^\infty_\Box$) is terminating, one might still wonder whether it is possible to reduce the number of progressing premises for the calculus. The DL^- rule seems inevitably non-decreasing. For the system to actually have infinite branches, we want at least one rule to be non-decreasing, after all. The modal rule DL^- seems to be a more natural candidate than $\Box\!\rightarrow$, which seems to have more propositional flavor. Can we make $\Box\!\rightarrow$ decreasing? To do that, it would be nice if we could force the deletion of $\Box A \rightarrow B$ in the left premise, and not add formulas in Γ. So one might want to have the following rule.

$$\Box\!\rightarrow^* \frac{\Pi^-, \Box\Gamma \Rightarrow A \qquad \Pi^-, \Box\Gamma, B \Rightarrow \Delta}{\Pi, \Box\Gamma, \Box A \rightarrow B \Rightarrow \Delta}$$

We can check that this rule makes perfect sense semantically: it is locally sound. However, keeping the $\Box A \rightarrow B$ in the left premise is desirable for cut-free completeness. If we endorse $\Box\!\rightarrow^*$ instead, it is unclear how to get a cut-free proof of the strong Löb's axiom. The following is a failed proof search if A is not a

tautology.

$$\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow A}{\Rightarrow \Box A}\,\mathsf{DL}^{-} \qquad \dfrac{}{A \Rightarrow A}\,\mathsf{init}}{\Box A \to A \Rightarrow A}\,\Box\!\to}{\Rightarrow (\Box A \to A) \to A}\,\to_{\mathsf{R}}$$

It seems hopeless to move upwards further at the left-uppermost leaf. Additionally, it seems that the proof search must start with the rule $\to_{\mathsf{R}}$, followed by $\Box\to$. Yet if we keep the principal formula in the left premise, a proof is straightforward. It then seems that we are forced to keep $\Box A \to B$ in the left premise, and hence should endorse the following rule.

$$\Box\!\to^{**} \dfrac{\Pi, \Box\Gamma, \Box A \to B \Rightarrow \Box A \qquad \Pi^{-}, \Box\Gamma, B \Rightarrow \Delta}{\Pi, \Box\Gamma, \Box A \to B \Rightarrow \Delta}$$

Note that this rule is almost the same as the $\to_{\mathsf{L}}$ rule of **G3** systems, and the left premise is already non-decreasing. If we were to endorse this rule, then we would have a premise that is both non-decreasing and non-progressing, and termination would be out of reach. The remedy is to apply DL^{-} to the left premise and get $\Box\to$, making the left premise progressing. This results in the rule $\Box\to$ we actually endorse.

In the following, we will also see how this rule relates to a **G4**-style well-founded system for $\mathbf{iSL}_{\Box}$ introduced by van der Giessen and Iemhoff in [vI20].

6.8 Completeness of G4-style systems

Given the termination property of the cyclic system, we want it to be complete. Again, we show cut-free completeness of the infinitary system, from which the completeness of the cyclic system follows. As explained in section 6.5, it might be difficult to provide a direct cut reduction procedure because of the constraint on the context. So we need to proceed differently. Here is the plan. We first introduce two well-founded systems for $\mathbf{iSL}_{\Box}$, one is $\mathbf{G4iSL}^{mul}_{\Box}$, which is based on a multiset system introduced by van der Giessen and Iemhoff in [vI20] and is sound and complete with respect to $\mathbf{HiSL}_{\Box}$; the other is $\mathbf{G4iSL}_{\Box}$, which is newly defined. This system is based on set sequents with the constraint that the principal formula of most rules be deleted from the contexts (on the same side as the principal formula) in the premises. We will show that these systems are equivalent, and then show that every proof of the latter can be translated into a proof of the same sequent in $\mathbf{G4iSL}^{\infty}_{\Box}$.

One might also wonder why we do not prove the completeness of $\mathbf{G4iSL}^{\infty}_{\Box}$ using $\mathbf{G3iSL}^{\infty}_{\Box}$. There is a standard proof method of the completeness of a **G4** system of a logic via its completeness with respect to a complete **G3** system of the same logic. The proof of Theorem 4.3.5 in [TS00] is an example, and we briefly report it here. Suppose we have a logic **L**, a **G4**-style system $C^{\mathbf{L}}_4$ of **L**, a **G3**-style system $C^{\mathbf{L}}_3$ of **L**, and we want to show that $C^{\mathbf{L}}_3$ and $C^{\mathbf{L}}_4$ are equivalent. First, we show that the rules of C_4 are admissible in $C^{\mathbf{L}}_3$. Second, we prove by induction on the complexity of sequents that, if a sequent is derivable in $C^{\mathbf{L}}_3$, then it is derivable in $C^{\mathbf{L}}_4$. Notably, this direction also requires that each rule of the system $C^{\mathbf{L}}_4$ is backwards decreasing. This standard method might fail for $\mathbf{iSL}_{\Box}$ because (1) the **G4**-style systems have context constraints while **G3**-style systems do not; and (2) the **G4**-style systems have non-decreasing premises.

Then let us stick to our current strategy and first introduce the well-founded sequent calculus $\mathbf{G4iSL}^{mul}_{\Box}$ based on multiset sequents. It is obtained from the system introduced by van der Giessen and Iemhoff in [vI20]

by allowing for an arbitrary formula to be the designated formula of initial sequents.

Definition 6.58 (Sequent calculus $\mathbf{G4iSL}^{mul}_{\Box}$)**.** *The multiset sequent calculus* $\mathbf{G4iSL}^{mul}_{\Box}$ *is given by the following set of rules.*

$$\mathsf{init}\ \frac{}{A, \Gamma \Rightarrow A} \qquad \bot_{\mathsf{L}}\ \frac{}{\bot, \Gamma \Rightarrow \Delta} \qquad \wedge_{\mathsf{L}}\ \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \wedge_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\vee_{\mathsf{L}}\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A_{i \in \{0,1\}}}{\Gamma \Rightarrow A_0 \vee A_1} \qquad \rightarrow_{\mathsf{R}}\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \qquad p \rightarrow\ \frac{\Gamma, p, A \Rightarrow \Delta}{\Gamma, p, p \rightarrow A \Rightarrow \Delta}$$

$$\wedge \rightarrow\ \frac{\Gamma, A \rightarrow (B \rightarrow C) \Rightarrow \Delta}{\Gamma, (A \wedge B) \rightarrow C \Rightarrow \Delta} \qquad \vee \rightarrow\ \frac{\Gamma, A \rightarrow B, A \rightarrow C \Rightarrow \Delta}{\Gamma, (A \vee B) \rightarrow C \Rightarrow \Delta} \qquad \rightarrow\rightarrow\ \frac{\Gamma, B \rightarrow C, A \Rightarrow B \qquad C, \Gamma^{-} \Rightarrow \Delta}{\Gamma, (A \rightarrow B) \rightarrow C \Rightarrow \Delta}$$

$$\mathsf{SL}^{mul}\ \frac{\Pi, \boxdot\Gamma, \Box A \Rightarrow A}{\Pi, \Box\Gamma \Rightarrow \Box A} \qquad \Box \rightarrow^{mul}_{\mathsf{SL}}\ \frac{\Pi, \Box A \rightarrow B, \boxdot\Gamma, \Box A \Rightarrow A \qquad \Pi, \Box\Gamma, B \Rightarrow \Delta}{\Pi, \Box\Gamma, \Box A \rightarrow B \Rightarrow \Delta}$$

The Π *in* SL^{mul} *and* $\Box \rightarrow^{mul}_{\mathsf{SL}}$ *does not contain boxed formulas.* ⊣

Although in the original system in [vI20] the A in the rule init ranges over atomic formulas only, every multiset sequent of the form $A, \Gamma \Rightarrow A$ is derivable. It follows that the original system is equivalent to the one above. Therefore, the following results proven in [vI20] carry over to the system above.

Proposition 6.59. *The following rules are admissible in* $\mathbf{G4iSL}^{mul}_{\Box}$*, with the first one derivable.*

$$\mathsf{w}_{\mathsf{L}}\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \mathsf{ctr}_{\mathsf{L}}\ \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \mathsf{cut}\ \frac{\Gamma \Rightarrow A \qquad A, \Sigma \Rightarrow \Delta}{\Gamma, \Sigma \Rightarrow \Delta}$$

Remark 6.60. *Since the system above is well-founded, every proof has a height. Notably, we do not know whether* $\mathsf{ctr}_{\mathsf{L}}$ *is height-preserving admissible in the original system in [vI20]. Fortunately, we will not need* $\mathsf{ctr}_{\mathsf{L}}$ *to be height-preserving admissible in* $\mathbf{G4iSL}^{mul}_{\Box}$*.*

Then, we introduce the well-founded sequent calculus for $\mathbf{iSL}_{\Box}$ that is based on *set* sequents with restricted contexts.

Definition 6.61 (Sequent calculus **G4iSL$_\Box$**). *The sequent calculus* **G4iSL$_\Box$** *is given by the following set of rules.*

$$\mathsf{init}\ \frac{}{A, \Gamma \Rightarrow A} \qquad \bot_{\mathsf{L}}\ \frac{}{\bot, \Gamma \Rightarrow \Delta} \qquad \wedge_{\mathsf{L}}\ \frac{A, B, \Gamma^- \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \wedge_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\vee_{\mathsf{L}}\ \frac{A, \Gamma^- \Rightarrow \Delta \qquad B, \Gamma^- \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A_{i \in \{0,1\}}}{\Gamma \Rightarrow A_0 \vee A_1} \qquad \rightarrow_{\mathsf{R}}\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \rightarrow B} \qquad p \rightarrow\ \frac{\Gamma^-, p, A \Rightarrow \Delta}{\Gamma, p, p \rightarrow A \Rightarrow \Delta}$$

$$\wedge \rightarrow\ \frac{\Gamma^-, A \rightarrow (B \rightarrow C) \Rightarrow \Delta}{\Gamma, (A \wedge B) \rightarrow C \Rightarrow \Delta} \qquad \vee \rightarrow\ \frac{\Gamma^-, A \rightarrow B, A \rightarrow C \Rightarrow \Delta}{\Gamma, (A \vee B) \rightarrow C \Rightarrow \Delta}$$

$$\rightarrow\rightarrow\ \frac{\Gamma^-, B \rightarrow C, A \Rightarrow B \qquad C, \Gamma^- \Rightarrow \Delta}{\Gamma, (A \rightarrow B) \rightarrow C \Rightarrow \Delta} \qquad \mathsf{SL}\ \frac{\Pi, \boxdot\Gamma, \Box A \Rightarrow A}{\Pi, \Box\Gamma \Rightarrow \Box A}$$

$$\Box \rightarrow_{\mathsf{SL}}\ \frac{\Pi, \Box A \rightarrow B, \boxdot\Gamma, \Box A \Rightarrow A \qquad \Pi^-, \Box\Gamma, B \Rightarrow \Delta}{\Pi, \Box\Gamma, \Box A \rightarrow B \Rightarrow \Delta}$$

Π *does not contain any boxed formula.* ⊣

Remark 6.62. *Recall the rules* DL^- *and* $\Box\rightarrow$ *in* **G4iSL$^\infty_\Box$**. *Compared with* SL *and* $\Box\rightarrow_{\mathsf{SL}}$ *above, they can be regarded as deleting the 'diagonal formula', i.e. the formula serving as the succedent of the conclusion, in the antecedent of the left-most premise.*

We show that the systems **G4iSL$^{mul}_\Box$** and **G4iSL$_\Box$** are equivalent.

Definition 6.63 (Support set and simptiset). *Given any multiset* Γ *of formulas in* Form^i*, define the* support set $\Gamma^\flat$ *of* Γ *as*

$$\Gamma^\flat := \{A : \#_\Gamma(A) > 0\}.$$

Given a set Γ *of formulas in* Form^i*, define the* simptiset $\Gamma^\sharp$ *of* Γ *as the multiset satisfying the conditions below:*

$$\#_{\Gamma^\sharp}(A) = 1 \textit{ iff } A \in \Gamma;$$

$$\#_{\Gamma^\sharp}(A) = 0 \textit{ iff } A \notin \Gamma.$$

For any multiset sequent $\Gamma \Rightarrow \Delta$*, let* $(\Gamma \Rightarrow \Delta)^\flat := \Gamma^\flat \Rightarrow \Delta^\flat$*. For any set sequent* $\Gamma \Rightarrow \Delta$*, let* $(\Gamma \Rightarrow \Delta)^\sharp := \Gamma^\sharp \Rightarrow \Delta^\sharp$. ⊣

Lemma 6.64 (Soundness of **G4iSL$_\Box$**). *For any set sequent* $\Gamma \Rightarrow \Delta$*, if* $\vdash_{\mathbf{G4iSL}_\Box} \Gamma \Rightarrow \Delta$*, then* $\vdash_{\mathbf{G4iSL}^{mul}_\Box} (\Gamma \Rightarrow \Delta)^\sharp$.

Proof. By induction on the height of **G4iSL$_\Box$** proofs. Suppose there is a **G4iSL$_\Box$** proof $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$. The base case is trivial. For the inductive case, we make the case distinction according to the bottom-most rule of $\mathcal{D}$ and check case by case.

- The cases for propositional rules for connectives other than $\rightarrow$ are simple. Note that here, we might need to apply w_L if any of the designated formulas in the premise is in the context. For example, for $\rightarrow_\mathsf{R}$, if the principal formula is $A \rightarrow B$, and $A \in \Gamma$ where Γ is the antecedent of the conclusion, we should apply w_L to make sure that A is added to the multiset calculus derivation. Observe that in the multiset

setting, we usually need to delete formula instances (or, replace some formulas with another) when applying rules downwards.

- For $p \to$, by the IH we have $(\Gamma^-, p, A)^\sharp \Rightarrow \Delta^\sharp$ is derivable in $\mathbf{G4iSL}_\Box^{mul}$. Note that $p \to A$ does not occur in $(\Gamma^-, p, A)^\sharp$. Apply $p \to$ in the multiset system, we get a derivation of $(\Gamma^- \cup \{p, p \to A\} \setminus \{A\})^\sharp \Rightarrow \Delta^\sharp$. Therefore, if $A \notin \Gamma$, we are done: take any $C \in \Gamma \cup \{p, p \to A\}$, if $C \in \{p, p \to A\}$, then surely the multiplicity of C in $(\Gamma^- \cup \{p, p \to A\} \setminus \{A\})^\sharp$ is 1; if $C \in \Gamma \setminus \{p, p \to A\}$, since $\Gamma \setminus \{p, p \to A\} \subseteq \Gamma^-$, it holds that the multiplicity of C in $(\Gamma^- \cup \{p, p \to A\} \setminus \{A\})^\sharp$ is 1. If $A \in \Gamma$, apply $\mathsf{w_L}$.

- For $\wedge \to$, by the IH, $(\Gamma^-, A \to (B \to C))^\sharp \Rightarrow \Delta^\sharp$ is derivable in $\mathbf{G4iSL}_\Box^{mul}$. Apply the rule $\wedge \to$, we get a $\mathbf{G4iSL}_\Box^{mul}$ proof of

$$[(\Gamma^- \setminus \{A \to (B \to C)\}) \cup \{(A \wedge B) \to C\}]^\sharp \Rightarrow \Delta^\sharp.$$

  For any formula $D \in \Gamma^-$ that is distinct from $A \to (B \to C)$, it has multiplicity 1 in $[(\Gamma^- \setminus \{A \to (B \to C)\}) \cup \{(A \wedge B) \to C\}]^\sharp$. If $D = A \to (B \to C)$, apply $\mathsf{w_L}$ under the $\mathbf{G4iSL}_\Box^{mul}$ proof of $[(\Gamma^- \setminus \{A \to (B \to C)\}) \cup \{(A \wedge B) \to C\}]^\sharp \Rightarrow \Delta^\sharp$.

- The argument for $\vee \to$ is similar. We consider $\to\to$. By the IH, $(\Gamma^-, A, B \to C)^\sharp \Rightarrow B$ and $(\Gamma^-, C)^\sharp \Rightarrow \Delta^\sharp$ are derivable in $\mathbf{G4iSL}_\Box^{mul}$. Then, apply $\to\to$ under these proofs, we get a $\mathbf{G4iSL}_\Box^{mul}$ derivation of

$$[(\Gamma^- \setminus \{A, B \to C, C\}) \cup \{(A \to B) \to C\}]^\sharp \Rightarrow \Delta^\sharp.$$

  If any of $A, B \to C$, and C is in Γ, we again apply $\mathsf{w_L}$.

- Consider $\Box \to$. By the IH, $(\Pi, \boxdot\Sigma, \Box A \to B)^\sharp \Rightarrow A$ and $(\Pi^-, \Box\Sigma, B)^\sharp \Rightarrow \Delta^\sharp$ are derivable in $\mathbf{G4iSL}_\Box^{mul}$. For any $C \in (\Sigma \cap \Pi) \setminus \{\Box A \to B\}$, we apply $\mathsf{w_L}$ under the sequent $(\Pi, \boxdot\Sigma, \Box A \to B)^\sharp \Rightarrow A$ to duplicate such C's. If $B \in \Pi^-$, we also apply $\mathsf{w_L}$ to duplicate B under the sequent $(\Pi^-, \Box\Sigma, B)^\sharp \Rightarrow \Delta^\sharp$. Then, further apply $\Box \to$, we get a proof of $(\Pi, \Box\Sigma, \Box A \to B \Rightarrow \Delta)^\sharp$.

- Lastly, SL. By the IH, $(\Pi, \boxdot\Sigma, \Box A)^\sharp \Rightarrow A$ is derivable in $\mathbf{G4iSL}_\Box^{mul}$. If $\Box A \in \Box\Gamma$, then the conclusion (with simptisets) is an axiom and is of course derivable in $\mathbf{G4iSL}_\Box^{mul}$. Otherwise, for any $B \in \Sigma \cap \Pi$, we duplicate it using $\mathsf{w_L}$ under the proof of $(\Pi, \boxdot\Sigma, \Box A)^\sharp \Rightarrow A$. Then apply SL, we get a proof of $(\Pi, \Box\Sigma \Rightarrow \Box A)^\sharp$.

This concludes the induction. □

To show that the set-based calculus $\mathbf{G4iSL}_\Box$ is complete with respect to the multiset-based calculus $\mathbf{G4iSL}_\Box^{mul}$, we do not proceed by induction on the height of $\mathbf{G4iSL}_\Box^{mul}$ proofs. If we were to do that, the proof would not go through easily. Consider the following case. Suppose we have a multiset sequent $\Gamma \Rightarrow \Delta$ and one of its proofs $\mathcal{D}$, and we want to show that $(\Gamma \Rightarrow \Delta)^\flat$ is derivable in $\mathbf{G4iSL}_\Box$. Assume further that the last rule of $\mathcal{D}$ is $\wedge_\mathsf{L}$, so $\Gamma = A \wedge B, \Gamma'$. If $\#_{\Gamma'}(A \wedge B) = 0$, then we can get the desired conclusion by the IH. However, if $\#_{\Gamma'}(A \wedge B) > 0$, applying the IH to the premises gives us the derivability of sequents whose antecedent contexts contain the principal formula $A \wedge B$. Consequently, we cannot apply $\wedge_\mathsf{L}$ to get a proof of $(A \wedge B, \Gamma')^\flat \Rightarrow \Delta^\flat$. Indeed, contraction is admissible in the multiset system, so we may want to first apply contraction. Yet as made clear in Remark 6.60, we do not know whether contraction is height-preserving, and hence may not apply the IH.

An attempt to save the argument is to appeal to invertibility (provided that invertibility is available) – we first apply the IH, and then apply invertibility to get rid of the $A \wedge B$ in the antecedent. But this strategy may not work for the rule $\rightarrow\rightarrow$, for its left premise is not invertible.

Therefore, to show completeness, we instead appeal to the following well-order on sequents which is also endorsed in [vI20].

Definition 6.65 (Well-order induced by a multiset sequent)**.** *Let $S = \Gamma \Rightarrow \Delta$ be a multiset sequent. Let c be the number of boxed formulas in* $\mathrm{Clos}(\Gamma \Rightarrow \Delta)$*. For each sequent $S' = \Gamma' \Rightarrow \Delta'$, we set $w_S(S') = (c - n_{S'}, w^4(S'))$, where $n_{S'}$ is the number of boxed formulas in $(\Gamma')^\flat$.*

For any sequents $S_0 = \Gamma_0 \Rightarrow \Delta_0$ and $S_1 = \Gamma_0 \Rightarrow \Delta_0$, $S_0 \succcurlyeq_S S_1$ iff

(1) $c - n_{S_0} \geqslant 0$ and $c - n_{S_1} \geqslant 0$;

(2) $w_S(S_0) >_{lex} w_S(S_1)$, where $>_{lex}$ is the lexicographic order on pairs $(c - n, m)$ for $c - n \geqslant 0$.

We call $\succcurlyeq_S$ the well-order induced by *S on the sequents whose associated pair has a non-negative first digit.* ⊣

Remark 6.66. *Intuitively, for any two sequents S_0 and S_1 with the same weight, the one that contains more boxed formulas is bigger with respect to the order defined above. Since the first digit might be negative and hence cause an infinite descending chain, we need to restrict the order to the sequents whose associated pair has a non-negative first digit.*

Lemma 6.67 (Completeness of $\mathbf{G4iSL}_\Box$)**.** *For any multiset sequent $\Gamma \Rightarrow \Delta$, if $\vdash_{\mathbf{G4iSL}^{mul}_\Box} \Gamma \Rightarrow \Delta$, then $\vdash_{\mathbf{G4iSL}_\Box} (\Gamma \Rightarrow \Delta)^\flat$.*

Proof. Fix a multiset sequent $\Gamma \Rightarrow \Delta$. We proceed by strong induction on the well-order induced by $\Gamma \Rightarrow \Delta$. Suppose there is a $\mathbf{G4iSL}^{mul}_\Box$ proof $\mathcal{D}$ of $\Gamma \Rightarrow \Delta$. Now, if $\#_\Gamma(A) > 1$ for any formula A, since $\mathrm{ctr_L}$ is admissible in $\mathbf{G4iSL}^{mul}_\Box$, we can first use contraction on one of the repeating formulas, and then apply the IH. Therefore, in the following, we assume that $\#_\Gamma(A) \leqslant 1$ for any formula A. We make the case distinction according to the bottom-most rule of $\mathcal{D}$.

- If the bottom-most rule is axiomatic, then of course $(\Gamma \Rightarrow \Delta)^\flat$ is derivable in $\mathbf{G4iSL}_\Box$.
- We consider the cases where the bottom-most rule is $\wedge_\mathrm{R}$, $\vee_\mathrm{R}$, or $\rightarrow_\mathrm{R}$ together. This is because, in $\mathbf{G4iSL}_\Box$, there is no restriction on the contexts for these propositional rules. Simply apply the IH to the premises, and then apply the corresponding rule in the set-based system.
- $\wedge_\mathrm{L}$. Suppose $\Gamma = A \wedge B, \Gamma^*$, where $A \wedge B$ is the principal formula, and Γ^* does not contain any occurrence of $A \wedge B$. We apply the IH to get a proof of $(A, B, \Gamma^*)^\flat \Rightarrow \Delta^\flat$. By applying $\wedge_\mathrm{L}$ in the set-based system, we can get a proof of $(A \wedge B, \Gamma^*)^\flat \Rightarrow \Delta^\flat$. Note that, if A or B occurs in Γ^*, we want $(A \wedge B, \Gamma^*)^\flat$ to also retain A or B. This can be done by carefully choosing the appropriate context. Observe that the (left) propositional rules of the set-based calculus do not require us to delete formulas if we move from top to bottom when applying them.
- $\vee_\mathrm{L}$. Let $\Gamma = A \vee B, \Gamma^*$, where Γ^* has no occurrence of $A \vee B$. By the IH, $(A, \Gamma^*)^\flat \Rightarrow \Delta^\flat$ and $(B, \Gamma^*)^\flat \Rightarrow \Delta^\flat$ are derivable in $\mathbf{G4iSL}_\Box$. Since these premises satisfy the restriction on the context, we can apply $\vee_\mathrm{L}$ below and get a proof of $(A \vee B, \Gamma^*)^\flat \Rightarrow \Delta^\flat$. Again, by carefully choosing the context in the premises, we can make sure that any A such that $\#_{\Gamma^*}(A) > 0$ is in $(A \vee B, \Gamma^*)^\flat$.

- $p \to$. We assume that $\Gamma = p, p \to A, \Gamma^*$, where $p \to A$ is the principal formula, and $\#_{\Gamma^*}(p) = \#_{\Gamma^*}(p \to A) = 0$. Apply the IH, and then apply the rule $p \to$, for there is no occurrence of $p \to A$ in the left side of the premise. The arguments for $\wedge \to$ and $\vee \to$ are similar.

- We check $\to\to$. Let $\Gamma = (A \to B) \to C, \Gamma^*$, where Γ^* is free of $(A \to B) \to C$. Apply the IH to the sequents $\Gamma^*, B \to C, A \Rightarrow B$ and $C, \Gamma^* \Rightarrow \Delta$, we have that $(\Gamma^*, B \to C, A)^\flat \Rightarrow B$ and $(\Gamma^*, C)^\flat \Rightarrow \Delta$ are derivable in $\mathbf{G4iSL}_\Box$. Since Γ^* does not contain $(A \to B) \to C$, we can apply the rule $\to\to$ in the set-based calculus. If $\#_{\Gamma^*}(B \to C) = 1$ or $\#_{\Gamma^*}(A) = 1$ or $\#_{\Gamma^*}(C) = 1$, the formula is kept in the multiset premises, and hence occurs in the set premises. Again, we are not forced to delete them when applying the rule. The argument for $\Box \to^{mul}_{\mathsf{SL}}$ is similar.

- SL^{mul}. If $\#_\Gamma(\Box A) > 0$, then $\Pi, \boxdot\Gamma, \Box A \Rightarrow A$ is an axiom in the set-based calculus. Otherwise, the premise is smaller than the conclusion because the number of boxed formulas in the antecedent is larger. Apply the IH and then the rule SL in the set-based calculus in this case.

Therefore, the desired claim holds. □

We then have the following.

Theorem 6.68 (Equivalence between $\mathbf{G4iSL}_\Box$ and $\mathbf{G4iSL}^{mul}_\Box$)**.** *For any set sequent* $\Gamma \Rightarrow \Delta$*, it holds that* $\vdash_{\mathbf{G4iSL}_\Box} \Gamma \Rightarrow \Delta$ *iff* $\vdash_{\mathbf{G4iSL}^{mul}_\Box} (\Gamma \Rightarrow \Delta)^\sharp$.

Corollary 6.69. *The following rules are admissible in* $\mathbf{G4iSL}_\Box$.

$$\mathsf{w_L}\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \mathsf{cut}\ \frac{\Gamma \Rightarrow A \qquad A, \Sigma \Rightarrow \Delta}{\Gamma, \Sigma \Rightarrow \Delta} \qquad \mathsf{Löb}\ \frac{\Pi, \boxdot\Sigma, \Box A \Rightarrow A}{\Pi, \boxdot\Sigma \Rightarrow A}$$

For $\mathsf{w_L}$*, we assume that* $A \notin \Gamma$.

Proof. For $\mathsf{w_L}$, if there is a $\mathbf{G4iSL}_\Box$ proof of $\Gamma \Rightarrow \Delta$, then by Theorem 6.68, there is a $\mathbf{G4iSL}^{mul}_\Box$ proof $\Gamma^\sharp \Rightarrow \Delta^\sharp$. Since weakening is admissible in $\mathbf{G4iSL}^{mul}_\Box$, there is a $\mathbf{G4iSL}^{mul}_\Box$ proof of $A, \Gamma^\sharp \Rightarrow \Delta^\sharp$, which is just $(A, \Gamma)^\sharp \Rightarrow \Delta^\sharp$. Hence, there is a proof of $A, \Gamma \Rightarrow \Delta$ in $\mathbf{G4iSL}_\Box$.

For the cut rule, suppose $\Gamma \Rightarrow A$ and $A, \Sigma \Rightarrow \Delta$ are derivable in $\mathbf{G4iSL}^{mul}_\Box$. Then, by Theorem 6.68, $\Gamma^\sharp \Rightarrow A$ and $(A, \Sigma)^\sharp \Rightarrow \Delta^\sharp$ are derivable in $\mathbf{G4iSL}^{mul}_\Box$. Since cut is admissible in $\mathbf{G4iSL}^{mul}_\Box$, there is a derivation of the sequent $[(\Gamma \cup \Sigma) \setminus \{A\}]^\sharp \Rightarrow \Delta^\sharp$. Then, there is a derivation of $(\Gamma \cup \Sigma) \setminus \{A\} \Rightarrow \Delta$ in $\mathbf{G4iSL}_\Box$. If $A \in \Gamma \cup \Sigma$, 'apply' $\mathsf{w_L}$ under the proof of this sequent.

For Löb, suppose $\Pi, \boxdot\Sigma, \Box A \Rightarrow A$ is derivable. Using SL, we get a proof of $\Pi, \Box\Sigma \Rightarrow \Box A$. Then, 'apply' cut to $\Pi, \Box\Sigma \Rightarrow \Box A$ and $\Pi, \boxdot\Sigma, \Box A \Rightarrow A$, we get a proof of $\Pi, \boxdot\Sigma \Rightarrow A$. □

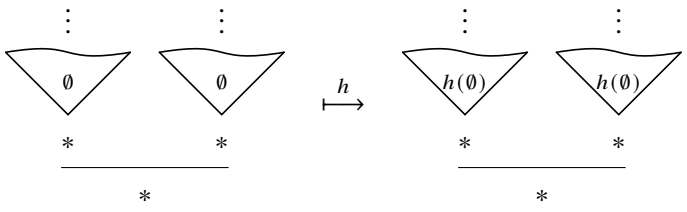
Since $\mathbf{G4iSL}^{mul}_\Box$ is complete with respect to $\mathbf{HiSL}_\Box$, the system $\mathbf{G4iSL}_\Box$ is also complete with respect to $\mathbf{HiSL}_\Box$. Then, to show the completeness of $\mathbf{G4iSL}^\infty_\Box$, it suffices to show that it is complete with respect to $\mathbf{G4iSL}_\Box$. We provide a proof translation from $\mathbf{G4iSL}_\Box$ to $\mathbf{G4iSL}^\infty_\Box$, using the $\mathcal{T}'$-corecursive definition introduced in Corollary 5.4.

Theorem 6.70 (Completeness of $\mathbf{G4iSL}^\infty_\Box$)**.** *For any set sequent* $\Gamma \Rightarrow \Delta$*, if* $\vdash_{\mathbf{G4iSL}_\Box} \Gamma \Rightarrow \Delta$*, then* $\vdash_{\mathbf{G4iSL}^\infty_\Box} \Gamma \Rightarrow \Delta$.

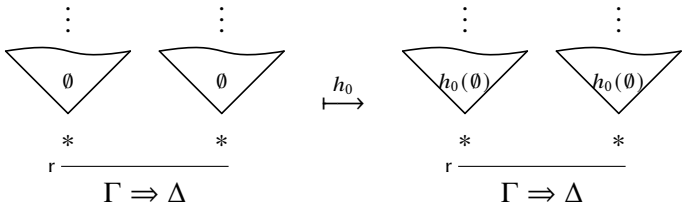
Proof. Let $\Gamma \Rightarrow \Delta$ be a set sequent that is derivable in the system $\mathbf{G4iSL}_\Box$. Let $\mathcal{D}$ be a $\mathbf{G4iSL}_\Box$ proof of $\Gamma \Rightarrow \Delta$. We define a function h by $\mathcal{T}'$-corecursion to translate $\mathcal{D}$ into a $\mathbf{G4iSL}^\infty_\Box$ proof $h(\mathcal{D})$ of $\Gamma \Rightarrow \Delta$. To do this, recall, we have to define a sequence of functions $\langle l', o', r' \rangle$. Again, given a proof $\mathcal{D}$, we let $c_\mathcal{D}$ be the corresponding

coloring on the full binary tree, with $*$ as a dummy label. Let the empty coloring be $\emptyset$. Intuitively, the function h_0 commutes with all rules other than SL and $\Box\to_{\mathsf{SL}}$. For these two rules, we appeal to the admissibility of the rule Löb in $\mathbf{G4iSL}_\Box$ (Corollary 6.69).

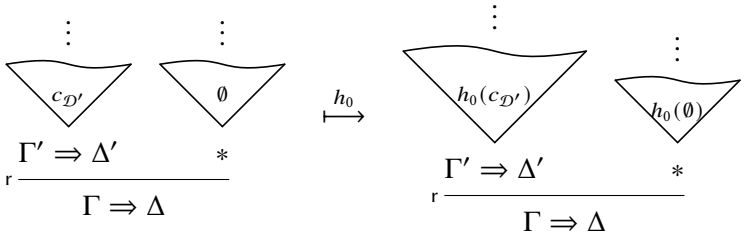
- $\langle l', o', r'\rangle(\emptyset) = \langle l'(\emptyset), o'(\emptyset), r'(\emptyset)\rangle = \langle \emptyset, *, \emptyset\rangle$. Pictorially, h_0 is defined as follows.



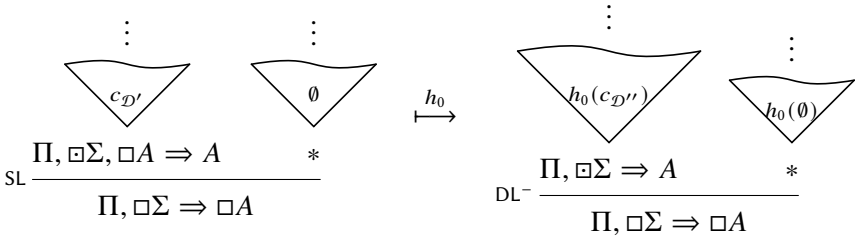
- If $\mathcal{D}$ has height 1, then $\Gamma \Rightarrow \Delta$ is axiomatic. Then $\langle l', o', r'\rangle(c_{\mathcal{D}}) = \langle l'(c_{\mathcal{D}}), o'(c_{\mathcal{D}}), r'(c_{\mathcal{D}})\rangle = \langle \emptyset, (\mathrm{r}, \Gamma \Rightarrow \Delta), \emptyset\rangle$, where r is the corresponding axiomatic rule. Pictorially, h_0 is defined as follows in this case.



- If $\mathcal{D}$ is a derivation of $\Gamma \Rightarrow \Delta$ whose last rule is one-premise and is distinct from SL, then let $\langle l', o', r'\rangle(c_{\mathcal{D}}) = \langle l'(c_{\mathcal{D}}), o'(c_{\mathcal{D}}), r'(c_{\mathcal{D}})\rangle = \langle l(c_{\mathcal{D}}), (\mathrm{r}, \Gamma \Rightarrow \Delta), \emptyset\rangle$. Then h_0 works as follows in this case.

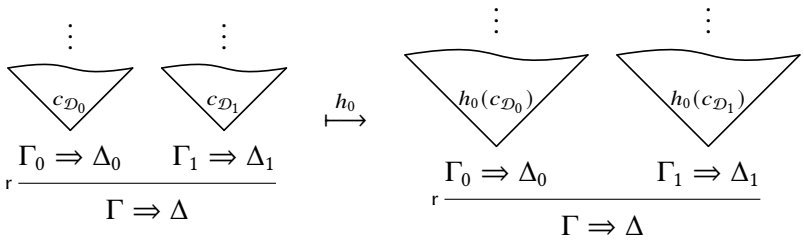


- If $\mathcal{D}$ is a derivation of $\Gamma \Rightarrow \Delta$ whose last rule is SL, then $\Gamma \Rightarrow \Delta$ has the shape of $\Pi, \Box\Sigma \Rightarrow \Box A$, for some Π free of boxed formulas. Its premise is $\Pi, \boxdot\Sigma, \Box A \Rightarrow A$. Let $\langle l', o', r'\rangle(c_{\mathcal{D}}) = \langle l'(c_{\mathcal{D}}), o'(c_{\mathcal{D}}), r'(c_{\mathcal{D}})\rangle = \langle c_{\mathcal{D}''}, (\mathsf{DL}^-, \Gamma \Rightarrow \Delta), \emptyset\rangle$, where $\mathcal{D}''$ is a $\mathbf{G4iSL}_\Box$ proof of the sequent $\Pi, \boxdot\Sigma \Rightarrow A$, whose existence is guaranteed by the admissibility of the rule Löb in $\mathbf{G4iSL}_\Box$. Then h_0 works as follows in this case.

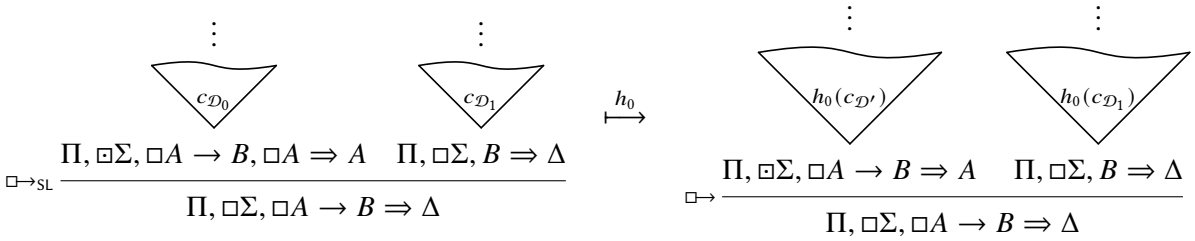


- If $\mathcal{D}$ is a derivation of $\Gamma \Rightarrow \Delta$ whose last rule is a two-premise rule other than $\Box\to$, then let $\langle l', o', r'\rangle =$

$\langle l'(c_{\mathcal{D}}), o'(c_{\mathcal{D}}), r'(c_{\mathcal{D}})\rangle = \langle l(c_{\mathcal{D}}), (\mathsf{r}, \Gamma \Rightarrow \Delta), r(c_{\mathcal{D}})\rangle$. Accordingly, h_0 works as follows.



- If $\mathcal{D}$ is a derivation of $\Gamma \Rightarrow \Delta$ whose last rule is $\Box\rightarrow_{\mathsf{SL}}$, then $\Gamma \Rightarrow \Delta$ has the shape of $\Pi, \Box\Sigma, \Box A \rightarrow B \Rightarrow \Delta$. Its left premise is $\Pi, \boxdot\Sigma, \Box A \rightarrow B, \Box A \Rightarrow A$. Let $\langle l', o', r'\rangle = \langle l'(c_{\mathcal{D}}), o'(c_{\mathcal{D}}), r'(c_{\mathcal{D}})\rangle = \langle c_{\mathcal{D}'}, (\Box\rightarrow, \Gamma \Rightarrow \Delta), r(c_{\mathcal{D}})\rangle$. $\mathcal{D}'$ is a **G4iSL**$_\Box$ proof of the sequent $\Pi, \boxdot\Sigma, \Box A \rightarrow B \Rightarrow A$, whose existence is again guaranteed by the admissibility of the rule Löb in **G4iSL**$_\Box$. Accordingly, h_0 works as follows.



Given a **G4iSL**$_\Box$ proof, we argue that $h_0(\mathcal{D})$ is indeed a **G4iSL**$^\infty_\Box$ proof. By definition, the labels of every node and its successors constitute a rule instance. We only need to check that every infinite branch is progressing. This is automatic by the design of the system – if a branch is infinite, it must pass through the premise of DL^- or the left premise of $\Box\rightarrow$ infinitely often, since these are the only non-decreasing premises. □

Therefore, the system **G4iSL**$^\infty_\Box$ is complete. By the equivalence between this system and **G4iSL**$^\circ_\Box$, we have that **G4iSL**$^\circ_\Box$ is also sound and complete.

Theorem 6.71 (Soundness and completeness of **G4iSL**$^\circ_\Box$)**.** *For any set sequent* $\Gamma \Rightarrow \Delta$*, it holds that* $\vdash_{\mathbf{G4iSL}^\circ_\Box}$ $\Gamma \Rightarrow \Delta$ *iff* $\vdash_{\mathbf{HiSL}_\Box} \bigwedge \Gamma \rightarrow \bigvee \Delta$.

Before we close this chapter, we want to leave a few comments on the trade-offs of the current design of non-wellfounded systems for **iSL**$_\Box$. We use extended axioms for these systems, which might not be very ideal. Nevertheless, these systems are indeed cut-free complete, and the backward proof search for them is terminating. A fact that might be relevant here is that, for the original version of **G4iSL**$^{mul}_\Box$ (with atomic axioms), the backward proof search terminates only modulo extended axioms. In a later paper [Shi+23], a truly terminating **G4**-style well-founded multiset sequent calculus for **iSL**$_\Box$ is proposed, and it is cut-free complete. This system endorses atomic axioms. One might wonder whether it is possible to develop non-wellfounded systems for **iSL**$_\Box$ using this terminating system.

We perceive difficulty here. The $\Box\rightarrow$ rule in that terminating well-founded system has a left premise whose antecedent does not contain the principal formula $\Box A \rightarrow B$. This feature is unfortunately unpleasant for our purpose, as demonstrated at the end of section 6.7: we want to keep $\Box A \rightarrow B$ in the antecedent of the left premise so as to have a cut-free derivation of the strong Löb's axiom in the non-wellfounded setting.

We can now close our discussion of **iSL**$_\Box$.

Chapter 7

A note on intuitionistic provability logic KM

In this penultimate chapter, we introduce **G4**-style non-wellfounded systems for Kuznetsov-Muravitsky logic **KM** and prove their soundness. This logic is again an intuitionistic normal monomodal logic, with only the operator $\Box$. Férée and Shillito provided a sequent calculus for this logic in their recent paper [FS26]. This system is **G4**-style, and endorses multi-succedent sequents. Our work in this chapter builds upon Férée and Shillito's result – we will also employ multi-succedent sequents, as they are essential for deriving an axiom of **KM**.

Definition 7.1 (The logic **KM** and system **HKM**)**.** *The logic* **KM** *is the smallest intuitionistic normal monomodal logic that contains the following axioms.*

- *Löb's axiom.* $\Box(\Box A \to A) \to \Box A$
- *Completeness axiom.* $A \to \Box A$
- *Kuznetsov-Muravitsky axiom (*KM*).* $\Box A \to (B \vee (B \to A))$

Let the Hilbertian calculus for **KM** *be* **HKM**. *Derivations/proof of some formula* A *in* **HKM** *are defined as in Definition 3.3. The length of* **HKM** *proofs is defined to be the length of the sequence of formulas.* ⊣

Theorem 7.2. **KM** *is Kripke complete with respect to the frames* $(W, \leqslant, R)$ *where* R *is transitive, conversely well-founded, and* $R = <$*, i.e.* R *coincides with the irreflexive part of* $\leqslant$*.*

Definition 7.3 (Multi-succedent intuitionistic sequents)**.** *An* multi-succedent intuitionistic sequent *is of the form* $\Gamma \Rightarrow \Delta$*, where* Γ *and* Δ *are finite sets of formulas in* Form^i*. Let the set of multi-succedent intuitionistic sequents be* Seq^{mi}*. We use '*,*' to mean '*$\cup$*', the union operator of sets. The notation '*A, Γ*' is a term for the set* $\{A\} \cup \Gamma$*.* ⊣

Definition 7.4 (Standard interpretation of multi-succedent intuitionistic sequents)**.** *For any* $\Gamma \Rightarrow \Delta \in \mathsf{Seq}^{mi}$*, its* standard interpretation $(\Gamma \Rightarrow \Delta)^{\mathcal{I}}$ *is defined similarly as in Definition 6.15.* ⊣

We can now introduce the systems for **KM**. Given the discussion of set sequents in section 6.5, we are justified in setting extended axioms as primitive again.

Definition 7.5. *The sequent calculus* $\mathbf{G4KM}_0^{\mathsf{cut}}$ *is given by the following set of rules.*

$$\mathsf{init}\ \frac{}{A, \Gamma \Rightarrow \Delta, A} \qquad \bot_{\mathsf{L}}\ \frac{}{\bot, \Gamma \Rightarrow \Delta} \qquad \wedge_{\mathsf{L}}\ \frac{A, B, \Gamma^- \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \wedge_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A, \Delta^- \qquad \Gamma \Rightarrow B, \Delta^-}{\Gamma \Rightarrow A \wedge B, \Delta}$$

$$\vee_{\mathsf{L}}\ \frac{A, \Gamma^- \Rightarrow \Delta \qquad B, \Gamma^- \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \vee_{\mathsf{R}}\ \frac{\Gamma \Rightarrow A, B, \Delta^-}{\Gamma \Rightarrow A \vee B, \Delta} \qquad \to_{\mathsf{R}}\ \frac{\Pi, \boxdot\Gamma, A \Rightarrow B \qquad \Pi, \Box\Gamma, A \Rightarrow B, \Delta^-}{\Pi, \Box\Gamma \Rightarrow A \to B, \Delta}$$

$$p \to\ \frac{\Gamma^-, p, A \Rightarrow \Delta}{\Gamma, p, p \to A \Rightarrow \Delta} \qquad \wedge \to\ \frac{\Gamma^-, A \to (B \to C) \Rightarrow \Delta}{\Gamma, (A \wedge B) \to C \Rightarrow \Delta} \qquad \vee \to\ \frac{\Gamma^-, A \to B, A \to C \Rightarrow \Delta}{\Gamma, (A \vee B) \to C \Rightarrow \Delta}$$

$$\to\to\ \frac{\Pi^-, \boxdot\Gamma, B \to C, A \Rightarrow B \qquad \Pi^-, \Box\Gamma, B \to C, A \Rightarrow B, \Delta \qquad \Pi^-, \Box\Gamma, C \Rightarrow \Delta}{\Pi, \Box\Gamma, (A \to B) \to C \Rightarrow \Delta}$$

$$\Box \to\ \frac{\Pi, \Box A \to B, \boxdot\Gamma \Rightarrow A \qquad \Pi^-, \Box\Gamma, B \Rightarrow \Delta}{\Pi, \Box\Gamma, \Box A \to B \Rightarrow \Delta} \qquad \mathsf{DL}^-\ \frac{\Pi, \boxdot\Gamma \Rightarrow A}{\Pi, \Box\Gamma \Rightarrow \Box A}$$

$$\mathsf{cut}\ \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Sigma \Rightarrow \Theta}{\Gamma, \Sigma \Rightarrow \Delta, \Theta}$$

The Π *in* $\to_{\mathsf{R}}$, $\to\to$, $\Box\to$, *and* DL^- *contains no boxed formula. Designated formulas, principal formulas, and contexts of each rule are defined as usual. The restriction on the contexts in the premises is similar to that of* $\mathbf{G3iSL}_{\Box}^{\infty}$.

Let $\mathbf{G4KM}_0$ *be* $\mathbf{G4KM}_0^{\mathsf{cut}} \setminus \{\mathsf{cut}\}$. ⊣

We take the rules $\to_{\mathsf{R}}$, $\to\to$ from Férée and Shillito's system (with minor modification), and keep the rules $\Box\to$ and DL^- in **G4iDL**.

Before moving to the non-wellfounded systems for **KM**, an explanation of the intuition behind the rules $\to_{\mathsf{R}}$ and $\to\to$ can be helpful. We do this by (1) proving the axiom KM, and (2) showing local semantic soundness of the rules.

In the **G4**-style multi-succedent system (based on multiset sequents) for **iPL** provided by Dyckhoff and Negri in [DN00], the $\to_{\mathsf{R}}$ is more standard.

$$\to_{\mathsf{R}}\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B, \Delta}$$

The context Δ is deleted in the premise to prevent a derivation of $\Rightarrow A \vee \neg A$. Nevertheless, as observed by Férée and Shillito, this rule does not allow us to derive every instance of the axiom KM. Say we want to derive $\Box p \Rightarrow q, q \to p$. After applying the more standard $\to_{\mathsf{R}}$ above, the premise should be $\Box p, q \Rightarrow p$, and there is no way to further decompose this sequent in the cut-free system. In contrast, the two-premise $\to_{\mathsf{R}}$ rule and

sequents with multi-succedent in $\mathbf{G4KM}_0^{\mathrm{cut}}$ allow us to derive $\Box A \Rightarrow B, B \to A$.

$$\dfrac{\dfrac{}{\Box A, A, B \Rightarrow A}\,\mathsf{init} \qquad \dfrac{}{\Box A, B \Rightarrow B, A}\,\mathsf{init}}{\Box A \Rightarrow B, B \to A}\,{\to_{\mathsf{R}}}$$

Next, we prove that the rules are locally sound with respect to the frames of **KM**.

Proposition 7.6 (Local soundness of $\mathbf{G4KM}_0^{\mathrm{cut}}$)**.** *For any instance of any* $\mathsf{r} \in \mathbf{G4KM}_0^{\mathrm{cut}}$*, if the premise(s) is/are sound in the frames of* **KM***, then so is the conclusion.*

Proof. We check only the rules $\to_{\mathsf{R}}$ and $\to\to$. For $\to_{\mathsf{R}}$, suppose $\Pi, \Box\Gamma \Rightarrow A \to B, \Delta$ is falsified in some pointed model $\mathbb{M}, x$, where $\mathbb{M} = (W, \leqslant, R, V)$ and $(W, \leqslant, R)$ is a **KM** frame. Then, $\mathbb{M}, x \Vdash \bigwedge \Pi$ and $\mathbb{M}, x \Vdash \bigwedge \Box\Gamma$. Additionally, $\mathbb{M}, x \nVdash (A \to B) \vee \bigvee \Delta$, which means $\mathbb{M}, x \nVdash A \to B$. Then, either $\mathbb{M}, x \Vdash A$ and $\mathbb{M}, x \nVdash B$, or there is some $y \in W$ with $x < y$ such that $\mathbb{M}, y \Vdash A$, and $\mathbb{M}, y \nVdash B$. In the first case, we have that $\mathbb{M}, x$ falsifies the sequent $\Pi, \Box\Gamma, A \Rightarrow B, \Delta^-$; in the second case, we have that $\mathbb{M}, y$ falsifies the sequent $\Pi, \boxdot\Gamma, A \Rightarrow B$. Note that since $R =<$, we have that xRy. Then, the left premise corresponds to a strict 'modal jump', whereas the right premise corresponds to staying at the current world.

For $\to\to$, fix a rule instance and suppose that the conclusion $\Pi, \Box\Gamma, (A \to B) \to C \Rightarrow \Delta$ is falsified in some pointed model $\mathbb{M}, x$, where $\mathbb{M} = (W, \leqslant, R, V)$ and $(W, \leqslant, R)$ is a **KM** frame. Then, $\mathbb{M}, x \Vdash \bigwedge \Pi \wedge \bigwedge \Box\Gamma \wedge ((A \to B) \to C)$ and $\mathbb{M}, x \nVdash \bigvee \Delta$. We have that $\mathbb{M}, x \Vdash (A \to B) \to C$. Consider the different possibilities for how this formula is true at x. We know that for any $y \geqslant x$ with $\mathbb{M}, y \Vdash A \to B$, it holds that $\mathbb{M}, y \Vdash C$. Either $x \Vdash A \to B$ or not. If $x \Vdash A \to B$, then $x \Vdash C$, which means the right premise $\Pi, \Box\Gamma, C \Rightarrow \Delta$ is falsified at $\mathbb{M}, x$. Otherwise, we have that $x \nVdash A \to B$, which means that there is some $y \geqslant x$ such that $\mathbb{M}, y \Vdash A$ but $\mathbb{M}, y \nVdash B$. Either x is as such or not. If x is as such, then the middle premise $\Pi^-, \Box\Gamma, B \to C, A \Rightarrow B, \Delta$ is falsified at $\mathbb{M}, x$. We have $\mathbb{M}, x \Vdash B \to C$ because, for any $y \geqslant x$, if $\mathbb{M}, y \nVdash C$, then $\mathbb{M}, y \nVdash A \to B$, which means there is some $z \geqslant y$ such that $\mathbb{M}, z \Vdash A$ but $\mathbb{M}, z \nVdash B$. This further implies that $\mathbb{M}, y \nVdash B$ by monotonicity. If x is not such that $\mathbb{M}, x \Vdash A$ and $\mathbb{M}, x \nVdash B$, then we have $y \neq x$ and hence $x < y$. The left premise $\Pi^-, \boxdot\Gamma, B \to C, A \Rightarrow B$ is falsified at $\mathbb{M}, y$. That $\mathbb{M}, y \Vdash B \to C$ follows from a similar argument as the one above, and that $\mathbb{M}, y \Vdash \Box\Gamma$ follows from monotonicity. Note that since $<= R$, we have that xRy in this case. Therefore, the middle and the right premises correspond to staying at the current state, while the left one corresponds to a strict 'modal jump' to a distinct state. □

The proof of local soundness should have made it clear which premises of $\to_{\mathsf{R}}$ and $\to\to$ are progressing. Then, we are ready to define local progress calculi for **KM**.

Definition 7.7 (Local progress calculi $\mathbf{G4KM}^{\mathrm{cut},\infty}$ and $\mathbf{G4KM}^{\infty}$)**.** *Let* $\mathbf{G4KM}^{\mathrm{cut},\infty}$ *be the pair* $(\mathbf{G4KM}_0^{\mathrm{cut}}, p)$*, where* p *is a function such that, for any* $\mathsf{r} \in \mathbf{G4KM}_0^{\mathrm{cut}} \setminus \{\to_{\mathsf{R}}, \to\to, \Box\to, \mathsf{DL}^-\}$ *and* $r \in \mathsf{r}$*, we have* $p_{\mathsf{r}}(r) = \emptyset$*; for any* $\mathsf{r} \in \{\to_{\mathsf{R}}, \to\to, \Box\to, \mathsf{DL}^-\}$ *and* $r \in \mathsf{r}$*, it holds that* $p_{\mathsf{r}}(r) = \{0\}$*. The local progress sequent calculus* $\mathbf{G4KM}^{\infty}$ *is defined similarly.* ⊣

Preproofs and proofs of these systems are given by the schema provided by Definition 4.4. Now, a progressing branch passes the progressing premises of some rule in the set $\{\to_{\mathsf{R}}, \to\to, \Box\to, \mathsf{DL}^-\}$ infinitely often.

Of course, the soundness of these systems follows from Proposition 7.6 and the Kripke completeness of **HKM**.

Theorem 7.8 (Syntactic soundness of $\mathbf{G4KM}^{\mathrm{cut},\infty}$ and $\mathbf{G4KM}^{\infty}$). *For any sequent* $\Gamma \Rightarrow \Delta$, *if* $\vdash_{\mathbf{G4KM}^{\mathrm{cut},\infty}} \Gamma \Rightarrow \Delta$ *or* $\vdash_{\mathbf{G4KM}^{\infty}} \Gamma \Rightarrow \Delta$, *then* $\vdash_{\mathbf{HKM}} \bigwedge \Gamma \to \bigvee \Delta$.

We also define the corresponding cyclic systems.

Definition 7.9 (Systems $\mathbf{G4KM}^{\mathrm{cut},\circ}$ and $\mathbf{G4KM}^{\circ}$). *Let* $\mathbf{G4KM}^{\mathrm{cut},\circ}$ *be the pair* $(\mathbf{G4KM}^{\mathrm{cut},\infty}, \mathbb{P}^{\infty}(\mathbf{G4KM}^{\mathrm{cut},\infty}))$, *let* $\mathbf{G4KM}^{\circ}$ *be the pair* $(\mathbf{G4KM}^{\infty}, \mathbb{P}^{\infty}(\mathbf{G4KM}^{\infty}))$. ⊣

Definition 7.10 ($\mathbf{G4KM}^{\mathrm{cut},\circ}$/$\mathbf{G4KM}^{\circ}$ proof). *A* $\mathbf{G4KM}^{\mathrm{cut},\circ}$ proof *is a finite* $(\mathbf{G4KM}_0^{\mathrm{cut}} \times \mathrm{Seq}^{mi}) \cup \mathrm{Seq}^{mi}$*-labeled cyclic tree* $(N, \prec \cup f, l)$ *such that:*

(1) $f : \mathcal{L}^T \hookrightarrow N \setminus \mathcal{L}^T$ *a partial function on the leaves of* T*;*

(2) if w *is a leaf, then:*

  (a) if $w \notin Dom(f)$, *then* $l(w) \in \mathbf{G4KM}_0^{\mathrm{cut}} \times \mathrm{Seq}^{mi}$ *and* $(\mathrm{seq}_T(w))$ *is an instance of* $\mathrm{rule}_T(w)$*;*

  (b) if $w \in Dom(f)$, *then* $l(w) \in \mathrm{Seq}^{mi}$, $l(w) = \mathrm{seq}_T(f(w))$, *and there is a node* u *such that* $f(w) \sqsubset u \sqsubseteq w$, *and* u *is decorated with the progressing premise of an instance of any rule in the set* $\{\to_{\mathrm{R}}, \to\to, \Box\to, \mathsf{DL}^{-}\}$*;*

(3) otherwise, $l(w) \in \mathbf{G4KM}_0^{\mathrm{cut}} \times \mathrm{Seq}^{mi}$*; for* $w0, \ldots, wn$, *the* $\prec$*-successors of* w, *the sequence*

$$(\mathrm{seq}_T(w_0), \ldots, \mathrm{seq}_T(w_n), \mathrm{seq}_T(w))$$

is an instance of $\mathrm{rule}_T(w)$, *where we set* $seq_T(wi) = l(wi)$ *if* $wi \in Dom(f)$.

A sequent S is provable in $\mathbf{G4KM}^{\mathrm{cut},\circ}$ ($\vdash_{\mathbf{G4KM}^{\mathrm{cut},\circ}} S$) *iff there is a* $\mathbf{G4KM}^{\mathrm{cut},\circ}$ *proof of* S. $\mathbf{G4KM}^{\circ}$ *proofs and provability are defined analogously.* ⊣

It should be clear that $\mathbf{G4KM}^{\infty}$ and $\mathbf{G4KM}^{\circ}$ are equivalent, by an argument similar to the proof of Theorem 6.37.

Theorem 7.11. *For any multi-succedent sequent* S, *it holds that* $\vdash_{\mathbf{G4KM}^{\infty}} S$ *iff* $\vdash_{\mathbf{G4KM}^{\circ}} S$.

Moreover, since every $\mathbf{G4KM}^{\mathrm{cut},\circ}$ proof can be unraveled to yield a $\mathbf{G4KM}^{\mathrm{cut},\infty}$ proof, we also have the soundness of the cyclic systems.

Theorem 7.12 (Syntactic soundness of $\mathbf{G4KM}^{\mathrm{cut},\circ}$ and $\mathbf{G4KM}^{\circ}$). *For any sequent* $\Gamma \Rightarrow \Delta$, *if* $\vdash_{\mathbf{G4KM}^{\mathrm{cut},\circ}} \Gamma \Rightarrow \Delta$ *or* $\vdash_{\mathbf{G4KM}^{\circ}} \Gamma \Rightarrow \Delta$, *then* $\Gamma \vdash_{\mathbf{HKM}} \bigvee \Delta$.

Note that all backwards non-decreasing premises again coincide with progressing premises. Hence, by an argument similar to the proof of Theorem 6.57, we have the following.

Theorem 7.13 (Backward $\mathbf{G4KM}^{\circ}$ proof search termination). *For any sequent* $\Gamma \Rightarrow \Delta$, *its backward* $\mathbf{G4KM}^{\circ}$ *proof search terminates.*

We leave the completeness of $\mathbf{G4KM}^{\infty}$ and $\mathbf{G4KM}^{\circ}$ for future research. Yet we mention how we expect the proofs to be. We plan to prove these results in a way similar to the completeness proof of $\mathbf{G4iSL}_{\Box}^{\infty}$. That is, we will not proceed via constructing a cut reduction procedure (given the difficulties demonstrated in section 6.5), but via proof translation. For this strategy to work, we might have to propose another cut-free complete sequent calculus for **KM** which is based on *multiset* sequents. This is because the calculus proposed by Férée and Shillito in [FS26] endorses a very different $\Box\to$ rule (the one in the terminating calculus for $\mathbf{iSL}_{\Box}$ presented in [Shi+23]), and this may cause difficulty in cut-free proof translation.

Chapter 8

Conclusion and future research

In this thesis, we have investigated the non-wellfounded proof theory of **GL**, $\mathbf{iSL}_\Box$, and **KM**. For **GL**, we have designed a cut reduction procedure for $\mathbf{G3GL}^{\mathrm{cut},\circ}$. For $\mathbf{iSL}_\Box$, we have presented sound and cut-free complete **G3**- and **G4**-style systems, with the latter enjoying terminating backward proof search but endorsing extended axioms. By appealing to the system $\mathbf{G3iSL}^{\circ}_{\Box}$, we managed to show that $\mathbf{iSL}_\Box$ is a cyclic companion of **iDL**. Along the way, we have also demonstrated the difficulties using set sequents to formulate **G4**-style systems, i.e. the failure of the standard cut reduction procedure and of the invertibility of the $\rightarrow_{\mathrm{R}}$ rule. Lastly, we provided **G4**-style non-wellfounded systems that are sound with respect to **KM**.

There remain many open questions for future research. We hereby list some of them.

(1) Are $\mathbf{G4KM}^{\circ}$ and $\mathbf{G4KM}^{\infty}$ complete? As already noted, there might be difficulty in proving completeness via cut elimination. Therefore, we expect to solve this problem by first exploring the relation between these systems with Férée and Shillito's well-founded system for **KM**. If necessary, we might need to introduce an alternative well-founded system for **KM**, as Férée and Shillito's system has different rules regulating the $\Box$ operator.

(2) What would a **G3**-style non-wellfounded proof system for **KM** look like?

(3) Can $\mathbf{G4iSL}^{\infty}_{\Box}$ and $\mathbf{G4iSL}^{\circ}_{\Box}$ be strengthened to only include atomic axioms as primitive? We have explained the difficulties that forced us to choose extended axioms as primitive ones. A natural question is how these obstacles can be overcome.

(4) Does our work on **G4**-systems of $\mathbf{iSL}_\Box$ already gives a **G4**-style system for **iDL**? If not, what would the rules for such a system be like?

(5) If we choose to work with set sequents in the classical or intuitionistic (**G3**) setting and impose restrictions on the contexts occurring in the rules, would this genuinely be technically innocuous? If not, what would be the potential problems? More generally, how to uniformly and elegantly formulate inference rules using set sequents in a proof-search friendly way, without sacrificing other desired proof-theoretic properties? We have touched upon these questions in passing, but a comprehensive study of them deserves a separate occasion.

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