Complexity and Tractability Islands for Combinatorial Auctions on Discrete Intervals with Gaps

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Abstract. Combinatorial auctions are mechanisms for allocating bundles of goods to agents who each have preferences over these goods. Finding an economically efficient allocation, the so-called winner determination problem, is computationally intractable in the general case, which is why it is important to identify special cases that are tractable but also sufficiently expressive for applications. We introduce a family of auction problems in which the goods on auction can be rearranged into a sequence, and each bid submitted concerns a bundle of goods corresponding to an interval on this sequence, possibly with multiple gaps of bounded length. We investigate the computational complexity of the winner determination problem for such auctions and explore the frontier between tractability and intractability in detail, identifying tractable, intractable, and fixed-parameter tractable cases.

1 INTRODUCTION

Combinatorial auctions [8] are mechanisms to allocate goods in which bidders are permitted to place bids on bundles of goods. They are widely used in practice, e.g., to auction off radio spectrum licences or the rights to serve different bus routes. The design of combinatorial auctions poses many challenges that are relevant to Artificial Intelligence (AI). This includes algorithm design based on AI techniques such as heuristic-guided search [e.g., 26], the design of expressive bidding languages building on insights from knowledge representation [e.g., 6], and the analysis of the strategic behaviour of agents participating in an auction [e.g., 28].

Due to the combinatorial structure of the bids, the *winner determination problem* (WDP), i.e., the problem of computing an allocation that maximises the revenue for the auctioneer is $\mathcal{NP}$-hard in the general case [25, 18] (e.g. Rothkopf et al. [25] observed that the problem is equivalent to a weighted set packing problem). It therefore is important to identify special cases, so-called "tractability islands", that permit efficient solutions but that are also sufficiently expressive for applications of interest. This approach was pioneered by Rothkopf et al. [25] who identified several structural restrictions on the range of permitted bids that render the WDP polynomial, and it was further refined by, amongst others, Conitzer et al. [7] and Gottlob and Greco [14].

In this paper we introduce a family of auction problems located at the frontier between tractability and intractability. Consider the example on the lefthand side of Figure 1, where six bidders each submit a bid for several connected cells on a construction ground put up for auction. If we rearrange the cells as shown on the righthand side of Figure 1, to obtain a sequence of cells (i.e., of goods on auction), we find that some of the bids (e.g., A and B) end up as intervals. By a result of Rothkopf et al. [25], the WDP is polynomial in case there is such a mapping under which all bids end up as intervals. In our example, however, some of these intervals have (small) gaps. A gap consists of consecutive positive integers missing within a bid, e.g., the bid C on the left-hand side of Figure 1 is mapped to an interval with a gap of size 2. We study the family of combinatorial auction problems that can be mapped into a linear structure such that all bids correspond to discrete intervals with a number of gaps of bounded length. Note that the example shown in Figure 1 requires one of the dimensions of the ground to be constant. This happens, for instance, in auctions for a swath of offshore waters (see Rothkopf et al. [25]).

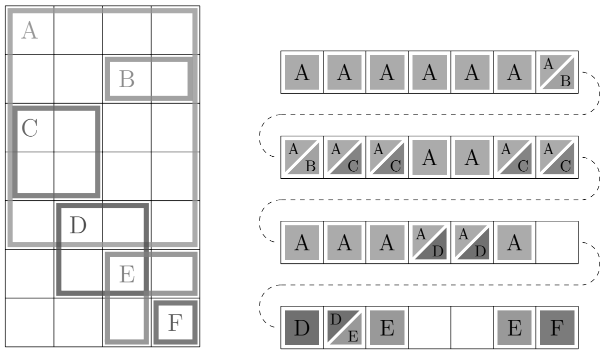

Figure 1. Bids A–F on cells of a construction ground (left), and the same auction problem mapped to a linear structure (right).

Depending on what restrictions we impose exactly on these gaps, we obtain either tractability,[5] intractability, or fixed-parameter tractability results. For the most part, we assume that the goods are represented by distinct positive integers $1, \ldots, n$, such that the bids have the structure we are presenting results for. An exception is Theorem 10, where we do not assume such on ordering to be given.

In Section 3 we show that the WDP can be solved in polynomial time if all bids correspond to intervals with multiple gaps of length at most ℓ each, for some fixed integer ℓ (as we will see, the case of $\ell = 2$ is of particular interest). This result thus significantly extends the original result of Rothkopf et al. [25]. It has immediate applications by identifying a large family of auction problems that can be solved efficiently in practice.

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[5] By tractable we mean that the problem can be solved in polynomial time with respect to the size of the input.

In Section 4 we identify several cases for which the decision variant of the WDP is $\mathcal{NP}$-complete. These are negative results, but they nevertheless are important in that they clarify the transition between tractable and intractable cases. All of our results of this kind deal with auction problems where bids concern small combinations of goods that are very close to the case of intervals with gaps covered by our positive tractability result mentioned earlier:

- Every bid is of the form $\{i, i+1\} \cup \{j, j+1\}$ with $j - i = \lfloor\sqrt{n}\rfloor$ (where n is the number of goods). Thus, every bid concerns two intervals of length 2 each, and the results applies even when we fix the distance between these two intervals to always be $\lfloor\sqrt{n}\rfloor$.
- Every bid is for an interval of length 2 together with an arbitrary third good, i.e., it is of the form $\{i, i+1, j\}$.
- Every bid is either for an interval of length 3, i.e., it is of the form $\{i, i+1, i+2\}$, or for a set of 2 arbitrary goods, i.e., it is of the form $\{i, j\}$. Thus, a single auction instance may include bids of both of these types.

We thus refine known results showing that the WDP is $\mathcal{NP}$-hard if bidders may bundle two intervals together [7] and if bids on three arbitrary goods are permitted [25]. As both 3-intervals and 2-sets *alone* have a polynomial WDP [25], our third intractability result also shows that tractable instances are not closed under taking unions.

Finally, in Section 5 we provide new insights on how parts of the input influence the computational hardness by showing that the WDP is fixed-parameter tractable with respect to the following parameters:

- the maximum length s of the section in each interval within which any subset of elements may be missing;
- the combined parameter (g, k) consisting of the maximum number g of goods in a bid and the minimum number k of bids that have to be deleted such that the remaining problem can be represented using intervals.

Thus, for auction instances where the above parameters are small constants, the WDP can again be solved efficiently. These results complement previous work on the parameterized complexity of combinatorial auctions [19]. Amongst the results of Loker and Larson [19] are that the WDP is W[1]-complete with respect to the revenue and is in $\mathcal{FPT}$ with respect to the number of distinct atomic bids. Moreover, they present complexity results for different restrictions of the so-called *bid graph*.

The remainder of this paper thus is organised as follows. Section 2, besides covering relevant background material on combinatorial auctions and parameterized complexity, formally introduces our notation and terminology. Our tractability results are presented in Section 3, our intractability results in Section 4, and our fixed-parameter tractability results in Section 5. Section 6 concludes.

2 PRELIMINARIES

In this section, we introduce relevant notation and terminology for combinatorial auctions as well as a number of specific structures we will use to describe auction instances of special interest. We also recall basic concepts from the theory of parameterized complexity.

2.1 Combinatorial auctions, WDP

By $\mathbb{N} := \{0, 1, 2, \dots\}$, we denote the set of non-negative integers. Our notation is similar to that of Rothkopf et al. [25]. A *combinatorial auction* can be described as a triple $\mathcal{C} = (A, P, b)$, where $A := \{1, \dots, n\}$ is the set of *goods* (or *assets*, or *items*) to be sold by auction (subsets of which are called *combinations*), P is the set of *permitted combinations* $C \subseteq A$ on which bids may be placed, and $b\colon P \to \mathbb{N}$ is the mapping representing the bids (which can be thought of as a list of pairs, each consisting of a combination and a price). The number $b(C)$ is the largest bid submitted for C, and $b(C) = 0$ if no bid is submitted for C. We will not model the bidders, but the set of submitted bids only. This can be thought of as each bidder being single-minded and submitting one 'atomic bid' (one desired combination), or as fewer bidders submitting a union (an OR-expression) of such atomic bids each. An *outcome* W of an auction is a set of pairwise disjoint combinations of P. The *revenue* $rev(W)$ of an outcome W is defined as $rev(W) := \sum_{C \in W} b(C)$.

The WINNER DETERMINATION problem (WDP) asks, given a combinatorial auction $\mathcal{C} = (A, P, b)$, for an outcome maximising the revenue. Its decision version asks, given a combinatorial auction $\mathcal{C} = (A, P, b)$ and a positive integer k, whether there is an outcome W such that $rev(W) \geqslant k$.

2.2 Discrete intervals, longest paths, item graphs

For $i, j \in A$, let $[i, j] := \{x \in A \mid i \leqslant x \leqslant j\}$ denote a discrete interval, i.e., $A = [1, n]$. Thus, $[i, j] = \emptyset$ if $i > j$.

We will use the problem LONGEST PATH for directed acyclic graphs (DAGs) to derive some of our tractability results. Let $G = (V, E)$ be a directed graph with edge weights given by $g\colon E \to \mathbb{N}$, and let $\pi = (v_1, v_2, \dots, v_{|\pi|})$ be a directed path in G, where $|\pi|$ is the number of vertices on the path. We define the *length* of π as the sum of its edge weights $\sum_{l=1}^{|\pi|-1} g((v_l, v_{l+1}))$. The problem LONGEST PATH then asks, given a DAG $G = (V, E)$ with edge weight function g and two vertices $v_i, v_f \in V$, for a directed path $\pi = (v_i = v_1, v_2, \dots, v_f = v_{|\pi|})$ of maximum length. This problem can be solved in linear time $O(|V| + |E|)$ [27, p. 661].

Given a combinatorial auction $\mathcal{C} = (A, P, b)$, a graph with vertex set A is called an *item graph* if the bids induce connected subgraphs [7]. A *structured* item graph has bounded treewidth.

2.3 Parameterized complexity

The computational complexity of a problem is usually studied with respect to the size of the input I of the problem. Parameterized complexity theory [10, 12, 22] additionally takes into account the size of a so-called parameter which is a certain part of the input, e.g., the size of the solution set or the maximum number of goods in a combination in a combinatorial auction. A problem is called *fixed-parameter tractable* (is in the class $\mathcal{FPT}$) with respect to a parameter k if it can be solved in time $f(k)|I|^{\mathcal{O}(1)}$, where f is a computable function and $|I|$ is the length of the encoding of I. This means that the running time of the corresponding algorithm is polynomial in the size of the input, but may be exponential or worse in terms of the parameter k; hence, for small values of k, the problem might be solvable efficiently. We will extend this definition to two-dimensional parameter spaces [23], considering a *combined* parameter of the form (k_1, k_2).

3 TRACTABLE CASES

Rothkopf et al. [25] show that WINNER DETERMINATION is solvable in quadratic time if all permitted combinations are discrete intervals. In this section, we extend their result to the case of discrete intervals with multiple gaps of length at most ℓ each, for some fixed integer ℓ.

The case of intervals with up to one missing element can be solved in time $\mathcal{O}(n^3)$ by adapting the dynamic programming algorithm from Rothkopf et al. [25]. We first consider intervals with gaps of combined length at most 2. More precisely, we show that the WDP is solvable in polynomial time for combinations

$$C_{i,j}^{M} := [i,\, j] \setminus M,\; 1 \leqslant i \leqslant j \leqslant n,$$

with $\emptyset \subseteq M \subset [i, j]$, $M \cap \{i, j\} = \emptyset$, and $0 \leqslant |M| \leqslant 2$.

Theorem 1. *Let $\mathcal{C} = (A, P, b)$ be a combinatorial auction with $P := \{C_{i,j}^{M} : 1 \leqslant i \leqslant j \leqslant n \wedge M \subseteq [i+1,\, j-1] \wedge |M| \leqslant 2\}$, where $C_{i,j}^{M} := [i,\, j] \setminus M$. Then an optimal outcome for $\mathcal{C}$ can be found in time $\mathcal{O}(n|P|) = \mathcal{O}(n^5)$, where $|P|$ is the cardinality of P.*

Proof. We construct a DAG so that all possible outcomes of the auction are represented as an edge-weighted path, where the first and the last vertex are fixed. The bids themselves will be represented as edges with positive weights.

Let us first look at how an outcome W is represented as a path π_W. Let $W := \{C_{i_1,j_1}^{M_1}, C_{i_2,j_2}^{M_2}, \ldots, C_{i_{|W|},j_{|W|}}^{M_{|W|}}\}$ be any outcome and w.l.o.g. we assume $i_1 < i_2 < \ldots < i_{|W|}$. The bids on combinations of W will be represented in the same sorted order on π_W. A vertex on π_W represents which items of the auction are assigned at this stage of the path. We will only consider paths starting in the vertex where no items are assigned and ending in the vertex where all items are assigned. Items are assigned when an edge is used to reach the next vertex of a path. We will use two kinds of edges: Edges with weight 0 assign items to the auctioneer, i. e., these items are not sold, and edges with positive weight assign a combination to the corresponding bidder—the vertex reached via the edge reflects this change. The sum of the edge weights of π_W will then correspond to the revenue of W.

The following observation on "overlapping" combinations allows us to limit the number of required vertices in the graph. We say two combinations $C_{i,j}^{M}, C_{i',j'}^{M'} \in P$ *overlap* if they are disjoint and $[i,\, j] \cap [i',\, j'] \neq \emptyset$.

Observation 2. *The union of two overlapping combinations yields again an interval with at most two missing elements.*

This is obvious if one combination is contained in the missing elements of the other. Otherwise we consider two overlapping combinations $C_{i,j}^{M}, C_{i',j'}^{M'} \in P$ with $i < i'$ (the case $i > i'$ is analogous). Since the two combinations are disjoint, we have $i' \in M$ and $j \in M'$. Thus, the missing elements of the union are a subset of $M \setminus \{i'\} \cup M' \setminus \{j\}$ with cardinality $\leqslant 2$.

The set of vertices of the graph is

$$V := \{v_i^{M} : 1 \leqslant i \leqslant n+1 \wedge M \subseteq [2,\, i-2] \wedge |M| \leqslant 2\}.$$

In the vertex v_i^{M}, all items of $[1,\, i-1] \setminus M$ are assigned. We write

$$\operatorname{asgd}\left(v_i^{M}\right) := [1,\, i-1] \setminus M$$

for the set of assigned items in a vertex $v_i^{M} \in V$. The first vertex of every path we consider is $v_1^{\emptyset}$, where the set of assigned items is empty and the last vertex is $v_{n+1}^{\emptyset}$, where every item is assigned. The reason for condition $M \subseteq [2,\, i-2]$ in the definition of V is that every set of assigned items is an interval with at most two missing *inner* elements. Let

$$\mathfrak{C} := \{C \in P : b(C) > 0\}$$

be the set of combinations with non-zero bids. We represent the bids with weighted edges. For every combination $C_{i,j}^{M} \in \mathfrak{C}$ we add a directed edge from $v_i^{\emptyset}$ to v_{j+1}^{M} with weight $b(C_{ij}^{M})$:

$$E_1 := \{(v_i^{\emptyset},\, v_{j+1}^{M}) : C_{i,j}^{M} \in \mathfrak{C}\}.$$

Furthermore, for every combination $C_{i',j'}^{M'} \in \mathfrak{C}$, we add an edge from v_i^{M} to $v_{i''}^{M''}$ with weight $b(C_{i',j'}^{M'})$, if $i' \in M$ and

$$\operatorname{asgd}\left(v_i^{M}\right) \mathbin{\dot\cup} C_{i',j'}^{M'} = \operatorname{asgd}\left(v_{i''}^{M''}\right) \tag{1}$$

holds, where $\dot\cup$ denotes the disjoint union of two sets. The weight of this edge is uniquely defined, since the union is disjoint. In order to extend the outcome with a combination $C_{i',j'}^{M'}$ it is necessary that the union in Equation (1) is disjoint, since every item can be sold only once. From Observation 2, we know that the union of two overlapping combinations is again an interval with at most 2 missing elements. So we add the following edges:

$$E_2 := \Big\{(v_i^{M},\, v_{i''}^{M''}) : M \neq \emptyset \;\wedge \Big(\exists C_{i',j'}^{M'} \in \mathfrak{C} : i' \in M \;\wedge \operatorname{asgd}\left(v_i^{M}\right) \mathbin{\dot\cup} C_{i',j'}^{M'} = \operatorname{asgd}\left(v_{i''}^{M''}\right)\Big)\Big\}.$$

The edges in the graph with weight greater than 0 are

$$E_b = E_1 \mathbin{\dot\cup} E_2.$$

There can be edges with weight 0, since not all items have to be sold. We set

$$E_0 := \Big(\{(v_i^{\emptyset},\, v_{i+1}^{\emptyset}) : 1 \leqslant i \leqslant n\} \cup \{(v_i^{M},\, v_i^{\emptyset}) : 4 \leqslant i \leqslant n+1 \wedge M \neq \emptyset\}\Big) \setminus E_b.$$

The reason for condition $i \geqslant 4$ is that for smaller i the set of assigned items cannot have missing inner elements.

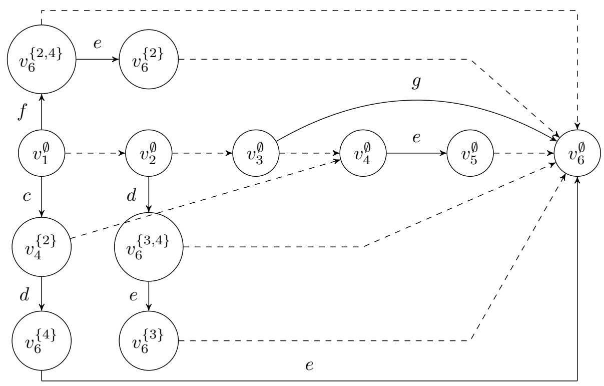

Figure 2. Winner determination for intervals with gap $\leqslant 2$: The path with maximal weight from $v_1^{\emptyset}$ to $v_6^{\emptyset}$ yields the winners of the auction with the bids $b(C_{1,3}^{\{2\}}) = c$, $b(C_{2,5}^{\{3,4\}}) = d$, $b(C_{4,4}^{\emptyset}) = e$, $b(C_{1,5}^{\{2,4\}}) = f$ and $b(C_{3,5}^{\emptyset}) = g$. The dashed edges have weight 0. The vertices not reachable from $v_1^{\emptyset}$ are omitted.

It is not hard to see that a path π with maximal weight from $v_1^{\emptyset}$ to $v_{n+1}^{\emptyset}$ in the graph $G := (V, E_b \cup E_0)$ corresponds to an optimal outcome W of the auction (see Figure 2 for an example) that can be retrieved by calculating $\mathcal{O}(n)$ differences:

$$W := \{D_p : e_p \text{ is an edge of the path } \pi \text{ with weight} > 0\},$$

where $D_p := \mathrm{asgd}(v_{i'}^{M'}) \setminus \mathrm{asgd}(v_i^M)$ for the edge $e_p = (v_i^M, v_{i'}^{M'})$. A detailed analysis yields $|P| \in \mathcal{O}(n^4)$, $|V| \in \mathcal{O}(n^3)$ and $|E_b \cup E_0| \in \mathcal{O}(n^5)$. The graph can be constructed in time $\mathcal{O}(n^5)$ and it is directed and acyclic—note that for every edge $(v_i^M, v_{i'}^{M'})$ we have $\mathrm{asgd}(v_i^M) \subsetneq \mathrm{asgd}(v_{i'}^{M'})$. Therefore, by finding the longest path in G, we can solve the WDP in $\mathcal{O}(n^5)$. This concludes the proof of Theorem 1. □

For the following tractability result, we make use of structured item graphs. We recall the corresponding definitions: For a given combinatorial auction $\mathcal{C} = (A, P, b)$, a graph with vertex set A is called an item graph if the bids induce connected subgraphs. A structured item graph has bounded treewidth. Conitzer et al. [7] show that the WDP is tractable if a structured item graph is given, i. e., they solve the WDP for this case in $\mathcal{O}(|T|^2(|B|+1)^{tw+1})$ using dynamic programming, where T is a tree decomposition with width tw of an item graph for the instance of the WDP and $|B|$ is the number of bids. Note that, however, it is $\mathcal{NP}$-complete to check whether a combinatorial auction has a structured item graph of treewidth k, even for $k = 3$ [14]. Thus, identifying natural classes of auction instances for which we actually know the treewidth of the corresponding structured item graph is important and of immediate practical interest. Next, we identify a large class of combinatorial auctions that have structured item graphs. Since we explicitly give an item graph with bounded treewidth (see Figure 3 for an example), we obtain that the WDP can be solved efficiently for this class. The permitted combinations are discrete intervals with gaps consisting of at most ℓ elements each, with ℓ being a fixed positive integer. These gaps have to be separated by at least one element that is contained in the combination.

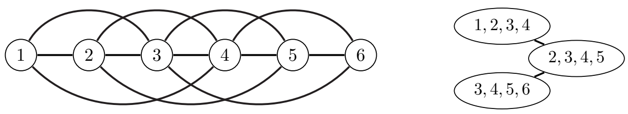

Figure 3. Item graph for bids on intervals with gaps of size at most 2 (left). The graph is chordal and the maximum clique size is 4, so the treewidth is 3 (see tree decomposition on the right). For criteria on upper and lower bounds for the treewidth of a graph see Bodlaender and Koster [4, 5].

Theorem 3. *Let $\ell \in \mathbb{N}$, $\mathcal{C} = (A, P, b)$ be a combinatorial auction with $P := \{C \subseteq A : \forall k \in [\min(C), \max(C) - \ell] : C \cap [k, k + \ell] \neq \emptyset\}$. Given m bids, an optimal outcome can be found in time $\mathcal{O}\left(n^2(m+1)^{\ell+2}\right)$.*

Proof. First, we construct an item graph for this class of permitted combinations. Consider the graph $G := (A, E)$ with

$$E := \{\{a, b\} \subseteq A : 1 \leqslant |a - b| \leqslant \ell + 1\}.$$

We show by contrapositive that G is an item graph. Let $C \subset A$ be an arbitrary set of items not inducing a connected component of G. Then there are two items $c, d \in C$ with $c < d$, $|c - d| \geqslant \ell + 2$ and $f \notin C$ for all $c < f < d$ (otherwise there is a path connecting the items of C). This implies

$$C \cap [c + 1, c + 1 + \ell] = \emptyset.$$

Hence, C is not a permitted combination.

In order to show that G is a structured item graph, we construct a tree decomposition of G with bounded width. For the underlying structure we choose the path $(X_1, X_2, \ldots, X_{n-\ell-1})$, where the bags are defined as

$$X_i := \{j \in A : i \leqslant j \leqslant i + \ell + 1\}.$$

It is easy to see that the conditions of a tree decomposition are satisfied. Every bag contains exactly $\ell + 2$ elements, the width of the decomposition is therefore $\ell + 1$. For any bag X_i, the induced subgraph $G[X_i]$ is a clique, thus $\ell + 1$ is also a lower bound for the treewidth. Consequently, the treewidth of G is $\ell + 1$.

For combinatorial auctions with structured item graphs, we can solve the WDP with the algorithm of Conitzer et al. [7, p. 214]. The running time of this algorithm for the structured item graph G with treewidth $\ell + 1$ and the tree decomposition we constructed is $\mathcal{O}(n^2(m+1)^{\ell+2})$. □

Note that Theorem 1 is a special case of Theorem 3. However, the proof of Theorem 1 will be useful for deriving Theorem 9. It also yields a better running time than Theorem 3.

4 INTRACTABLE CASES

In this section, we further explore the frontier of tractability for the WDP for intervals with gaps and consider very restricted instances for which the WDP nevertheless becomes $\mathcal{NP}$-complete. The following theorem can be obtained by a simple reduction from the WDP for 2×2 rectangles [25, Theorem 9], by mapping these rectangles to a linear structure as depicted in Figure 1.

Theorem 4. *The WDP is $\mathcal{NP}$-complete for combinatorial auctions $\mathcal{C} = (A, P, b)$ with $P := \{\{i, i+1\} \cup \{j, j+1\} : j = i + \lfloor\sqrt{n}\rfloor\}$, even if all bids have value 1.*

The WDP is equivalent to solving the maximum weighted independent set problem in a bid graph, where each vertex corresponds to a bid, weighted with the value of the bid, and there is an edge between two vertices if and only if the corresponding combinations have a non-empty intersection [cf., e. g., 19]. This connection implies, by a result of Fellows et al. [11, Theorem 1] on the parameterized complexity of k-INDEPENDENT SET for multiple-interval graphs, W[1]-hardness—with respect to the revenue of the auction—of the WDP for combinations of the form $I_1 \cup I_2$, where I_1, I_2 are discrete intervals and where all bids have value 1. Further, the work by Bar-Yehuda et al. [1, Corollary 2.3] implies $\mathcal{NP}$-hardness (and $\mathcal{APX}$-hardness) of the WDP for combinations of the form $\{i, i+1\} \cup \{j, j+1\}$ and bids with value 1. However, in Theorem 4 we restrict the distance between the two intervals of each bid, i.e., we require $j = i + \sqrt{n}$ for each combination $\{i, i+1\} \cup \{j, j+1\}$.

The WDP is also $\mathcal{NP}$-complete if bids are allowed on intervals of length 3 and on combinations with two elements.

Theorem 5. *The WDP is $\mathcal{NP}$-complete for combinatorial auctions $\mathcal{C} = (A, P, b)$ with $P := \{[i, i+2] : 1 \leqslant i \leqslant n-2\} \cup \{\{i, j\} : i, j \in A\}$, even if all bids have value 1.*

Proof. To show $\mathcal{NP}$-hardness, we use a reduction from the $\mathcal{NP}$-complete problem 2P1N-SAT, a variant of the satisfiability problem for a collection of clauses where each variable occurs exactly two times as a positive literal and one time as a negative literal [29, p. 238f]. Let $\mathcal{I}_{\text{SAT}}$ be an instance of 2P1N-SAT. Let

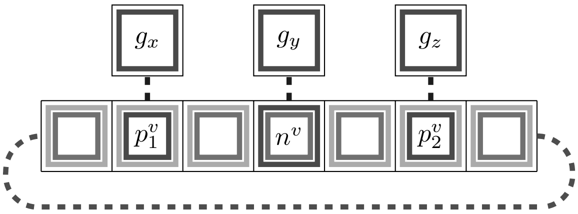

Figure 4. Construction of the bids for a variable $v \in \mathcal{V}$. The items p_1^v and p_2^v represent the occurrences of v, and n^v the occurrence of $\neg v$, respectively. Furthermore, the items g_x and g_z correspond to the clauses containing v, and g_y corresponds to the clause containing $\neg v$. If a clause contains more than one variable, the corresponding clause item occurs in the construction for multiple variables. The dashed lines indicate that the two connected items belong to the same bid. Additionally, three consecutive items of the same shade of grey represent one bid each. Altogether there are seven bids shown in the figure: Two medium grey, two light grey and three dark grey bids. The light grey and the medium grey bids simulate possible variable assignments: $v \mapsto 1$ ($v \mapsto 0$) corresponds to an outcome where the two medium (light) grey bids are accepted. In the first case the items p_1^v and p_2^v can be used to accept the bids containing g_x and g_z. In the second case the situation is analog for n^v and g_y.

$\mathcal{V} := \{v_1, v_2, \ldots, v_\mathfrak{n}\}$ denote the set of $\mathfrak{n}$ variables and $\mathcal{C} := \{c_1, c_2, \ldots, c_\mathfrak{m}\}$ the collection of $\mathfrak{m}$ clauses of $\mathcal{I}_{\text{SAT}}$. For every variable $v_i \in \mathcal{V}$, we create seven consecutive items

$$7(i-1)+1,\, 7(i-1)+2,\, \ldots,\, 7i,$$

and for every clause $c_j \in \mathcal{C}$ one item $g_j := 7\mathfrak{n} + j$. Overall, we create $n := 7\mathfrak{n} + \mathfrak{m}$ items. For better readability we define

$$\begin{aligned} p_1^{v_i} &:= 7(i-1)+2, \\ p_2^{v_i} &:= 7(i-1)+6, \\ n^{v_i} &:= 7(i-1)+4, \end{aligned}$$

where $p_1^{v_i}$ and $p_2^{v_i}$ represent the occurrences of v_i and n^{v_i} the occurrence of $\neg v_i$, respectively. Let x_i, z_i and y_i be the indices of the clauses containing the literals represented by $p_1^{v_i}$, $p_2^{v_i}$ and n^{v_i}, respectively. We create seven bids with value 1 for every variable $v_i \in \mathcal{V}$ (see Figure 4 for a visualisation):

$$\begin{aligned} B_{v_i\mapsto 1} &:= \{7(i-1)+3,\, n^{v_i},\, 7(i-1)+5\}, \\ B'_{v_i\mapsto 1} &:= \{7(i-1)+1,\, 7i\}, \\ B_{v_i\mapsto 0} &:= \{7(i-1)+1,\, p_1^{v_i},\, 7(i-1)+3\}, \\ B'_{v_i\mapsto 0} &:= \{7(i-1)+5,\, p_2^{v_i},\, 7i\}, \\ B^1_{v_i} &:= \{p_1^{v_i}, g_{x_i}\}, \\ B^2_{v_i} &:= \{p_2^{v_i}, g_{z_i}\}, \\ B_{\neg v_i} &:= \{n^{v_i}, g_{y_i}\}. \end{aligned}$$

We now show that $\mathcal{I}_{\text{SAT}}$ is satisfiable if and only if there is an outcome of the auction with revenue $k \geqslant 2\mathfrak{n} + \mathfrak{m}$.

"$\Rightarrow$": Let $\phi\colon \mathcal{V} \to \{0, 1\}$ be an assignment that satisfies all clauses. For every variable $v_i \in \mathcal{V}$ we accept the bids on $B_{v_i\mapsto\phi(v_i)}$ and $B'_{v_i\mapsto\phi(v_i)}$. This is possible since

$$B_{v_i\mapsto 1} \cap B'_{v_i\mapsto 1} = B_{v_i\mapsto 0} \cap B'_{v_i\mapsto 0} = \emptyset$$

and because these bids are on subsets of $[7(i-1)+1,\, 7i]$, i. e., items that only occur in bids created for the variable v_i. This yields a revenue of $2\mathfrak{n}$. For every clause $c_j \in \mathcal{C}$ there is at least one literal satisfied by ϕ, otherwise the clause would not be satisfied. Let l_j be any satisfied literal of the clause c_j. By construction, the item representing l_j is still available and only contained in exactly one bid involving the item g_j. Therefore, we can accept one additional bid for every clause, resulting in an outcome with revenue $2\mathfrak{n} + \mathfrak{m}$.

"$\Leftarrow$": Let W be an outcome of the auction with revenue $\geqslant 2\mathfrak{n}+\mathfrak{m}$. We first show that all clause items must have been sold. Suppose to the contrary that there is an item g_j which is not contained in an accepted bid. Then, we can accept at most $\mathfrak{m} - 1$ bids involving clause items. The remaining bids are $B_{v_i\mapsto 1}$, $B'_{v_i\mapsto 1}$, $B_{v_i\mapsto 0}$ and $B'_{v_i\mapsto 0}$ for every variable $v_i \in \mathcal{V}$. Since $(B_{v_i\mapsto 1}, B'_{v_i\mapsto 1})$ and $(B_{v_i\mapsto 0}, B'_{v_i\mapsto 0})$ are the only pairs of these bids having an empty intersection, we can accept at most $2\mathfrak{n}$ additional bids. Consequently, the revenue is at most $2\mathfrak{n} + \mathfrak{m} - 1$. This is a contradiction, thus all clause items have been sold.

We show that the accepted bids involving the clause items induce a partial mapping $\phi_p\colon \mathcal{V} \to \{0, 1\}$ such that at least one literal is satisfied in each clause. Every accepted bid involving a clause item contains exactly one other item representing a literal that determines ϕ_p for the corresponding variable, i. e., ϕ_p assigns to the variable the truth value that makes the literal satisfied. This cannot result in an inconsistent assignment, as we will show now. Suppose the result is an inconsistent assignment. Then, there is a variable $v_i \in \mathcal{V}$ such that $B_{\neg v_i}$ and at least one of the bids $B^1_{v_i}$ and $B^2_{v_i}$ are accepted. In this case only one of the bids $B_{v_i\mapsto 1}$, $B'_{v_i\mapsto 1}$, $B_{v_i\mapsto 0}$ and $B'_{v_i\mapsto 0}$ can be accepted, so the achievable revenue is at most $2\mathfrak{n}+\mathfrak{m}-1$. Again, this is a contradiction, so the assignment is consistent. Every variable not assigned by ϕ_p may be chosen arbitrary, since ϕ_p already satisfies one literal in each clause. So $\phi\colon \mathcal{V} \to \{0, 1\}$ with

$$\phi(v_i) = \begin{cases} \phi_p(v_i) & \text{if } \phi_p(v_i) \in \{0,1\}, \\ 1 & \text{otherwise}, \end{cases}$$

is an assignment satisfying all clauses of the $\mathcal{I}_{\text{SAT}}$ instance.

This construction can clearly be done in polynomial time, so the WDP is $\mathcal{NP}$-complete as claimed. □

We could have obtained a similar result by adjusting the proof of Conitzer et al. [7] showing that permitting bidders to bundle two intervals together yields a $\mathcal{NP}$-hard instance of the WDP: The same structure arises if we restrict VERTEX COVER to cubic graphs in their reduction, but then the proof still contains bids of value 1 and 2.

Since the WDP is solvable in polynomial time if either all permitted combinations are intervals or if all permitted combinations consist of at most two elements only [25], Theorem 5 implies the following corollary.

Corollary 6. *The class of tractable instances of WDP is not closed under union.*

In the next theorem, bids are allowed only on combinations consisting of two consecutive elements and an arbitrary third.

Theorem 7. *The WDP is $\mathcal{NP}$-complete for combinatorial auctions $\mathcal{C} = (A, P, b)$ with $P := \{\{i, j, k\} \subseteq A : j = i + 1\}$, even if all bids have value 1.*

In order to prove this theorem, we make use of the following lemma.

Lemma 8. *Let $G = (V, E)$ be a connected cubic graph. There is a bijection $\beta\colon E \to \{1, 2, \ldots, |E|\}$ so that for each vertex $v \in V$*

there are two incident edges that are mapped to consecutive integers. Such a bijection can be found in linear time.

Proof. Let $G = (V, E)$ be a cubic graph with vertices $v_1, v_2, \ldots, v_n$. By the handshaking lemma, every cubic graph has an even number of vertices. Hence, we have $n = 2m$ for some positive integer m. We extend G, so that every vertex has even degree: For every $i = 1, 2, \ldots, m$ we add a vertex w_i and connect it to the vertices v_{2i-1} and v_{2i} (see Figure 5).

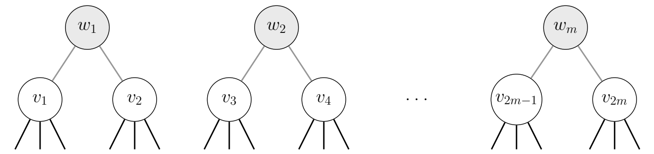

Figure 5. Extension of a cubic graph G with vertices $v_1, v_2, \ldots, v_{2m}$ to a (non-cubic) graph G', so that every vertex has even degree. The edges of G are hinted at below the vertices (every vertex is incident to three edges) and depend on the specific graph. For every $i = 1, 2, \ldots, m$ we add a vertex w_i and connect it to the vertices v_{2i-1} and v_{2i} (depicted in grey).

The resulting graph $G' = (V', E')$ is connected and every vertex has even degree, because every $v \in V$ has degree 4 in G' and every $w \in V' \setminus V$ has degree 2. Thus, G' has an Eulerian circuit and we can find one in linear time with Hierholzer's algorithm [17]. Consider an arbitrary Eulerian circuit in G'. Starting from w_1 (or any other vertex $w_i \in V' \setminus V$), we traverse the edges in the order in which they appear on the circuit. Let j be a counter initially set to 1. Each time we encounter an edge $e \in E$, we set $\beta(e) := j$ and increment j by 1. Every vertex $v \in V$ is once both entered and left via edges contained in E, since exactly one incident edge is not in E. Therefore, two of the incident edges are assigned to consecutive integers.[6] An Eulerian circuit visits every edge of a graph exactly once, so this approach yields a bijection with the desired property. With suitable data structures, the traversal of the edges can be done in $\mathcal{O}(E') = \mathcal{O}(E)$ and the test whether $e \in E$ in $\mathcal{O}(1)$. Overall, the construction needs linear time. □

Blumrosen and Nisan [3, p. 271] use a reduction from INDEPENDENT SET to prove that the WDP is $\mathcal{NP}$-complete. Since INDEPENDENT SET remains $\mathcal{NP}$-hard when restricted to 2-connected cubic planar graphs [20, p. 10 f.], we can apply Lemma 8 and prove Theorem 7 using a reduction from the restricted version of INDEPENDENT SET in an analogous manner.

We can further strengthen some of these results, using refined $\mathcal{NP}$-completeness results for several variants of the tiling problem [2, 21]. We then obtain $\mathcal{NP}$-completeness of the WDP with permitted combinations $P_1 := \{\{i, i+1, j\} \subseteq A : j = i + r \wedge r \in \{\lfloor\sqrt{n}\rfloor, \lfloor\sqrt{n}\rfloor + 1\}\}$ and $P_2 := \{[i, i+2] : 1 \leqslant i \leqslant n-2\} \cup \{\{i, i + \lfloor\sqrt{n}\rfloor\} : 1 \leqslant i \leqslant n - \lfloor\sqrt{n}\rfloor\}$, respectively, even if we only allow bids with value 1. If we replace $\lfloor\sqrt{n}\rfloor$ in the definition of P_i, $i = 1, 2$, with a constant value, the WDP can be solved in polynomial time even if we remove the restriction to bids of value 1 (see Theorem 3).

[6] At this point, it is important that we chose a vertex $w \notin V$ as a starting point. Otherwise, this would not necessarily hold for the start vertex.

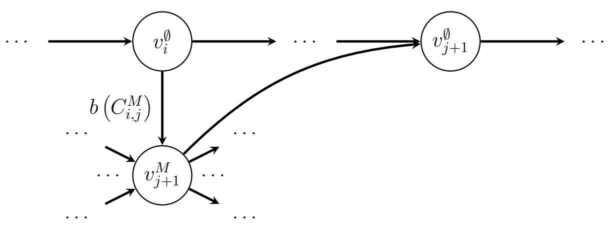

Figure 6. Representing partial assignments where the set of assigned items is not necessarily an interval. We create an edge from $v_i^{\emptyset}$ to v_{j+1}^{M} if and only if there is a bid on $C_{i,j}^{M}$. For $M = \emptyset$, this is an interval bid. If $M \neq \emptyset$, we create an edge from v_{j+1}^{M} to $v_{j+1}^{\emptyset}$, possibly with weight 0 (if this edge does not represent an existing bid). The outgoing edges from v_{j+1}^{M} for the case $M \neq \emptyset$ are explained in Figure 7. We do not have to consider the incoming edges separately, because they can be regarded as outgoing edges from other vertices.

5 FIXED-PARAMETER TRACTABLE CASES

In this section, we investigate the influence of certain parts of the input on the computational hardness of the WDP. We consider two parameters and obtain fixed-parameter tractability with respect to each of them.

Theorem 9. *For $s \in \mathbb{N}$, let $\mathcal{C} = (A, P, b)$ be a combinatorial auction with $P := \{C_{i,j}^{M} : 1 \leqslant i \leqslant j \leqslant n \wedge \exists k \in A : M \subseteq [k, k+s-1] \subseteq [i+1, j-1]\}$, where $C_{i,j}^{M} := [i, j] \setminus M$. Given m bids, an optimal outcome for $\mathcal{C}$ can be found in time. $\mathcal{O}(n^3 s^2 4^s)$, which can also be expressed in terms of the number m of submitted bids as $\mathcal{O}(n^2 m 2^s)$. Hence, the WDP is fixed-parameter tractable with respect to the length s of the section in each interval within which any subset of items may be missing. More precisely, the parameter s is defined as follows. For each combination C corresponding to a bid, let s_C denote the minimum natural number such that all missing elements of C are within an interval $[k_C,\ k_C + s_C - 1]$ with $k_C \in A$. Then, the parameter s is the maximum s_C for the given instance.*

Proof. We will only sketch the proof, since it is similar to the proof of Theorem 1. Given an instance of WDP as described, we create a graph in which the longest path between two designated vertices corresponds to an optimal outcome of the auction.

Again, in a vertex v_i^{M} all items of $[1, i-1] \setminus M$ are assigned. We start by creating a path from $v_1^{\emptyset}$ to $v_{n+1}^{\emptyset}$. An edge on this path has weight 0 if and only if there is no bid on the corresponding item. The two designated vertices are again $v_1^{\emptyset}$ and $v_{n+1}^{\emptyset}$, i. e., the longest path from $v_1^{\emptyset}$ to $v_{n+1}^{\emptyset}$ in the final graph will correspond to an optimal outcome of the auction. Now, we will create further vertices and edges such that every outcome of the auction is represented as an weighted path from $v_1^{\emptyset}$ to $v_{n+1}^{\emptyset}$ in a specific order, i. e., if we sort an arbitrary outcome by the first element of each combination, then the contained combinations appear in that order on the corresponding path. For every bid on a combination, we create vertices and edges as shown in Figures 6 and 7.

If an edge has non-zero weight, it is weighted with the maximum value that someone has bid on a combination leading to this edge. Since there might be several different combinations leading to the same edge, we label the edge also with the corresponding bid.

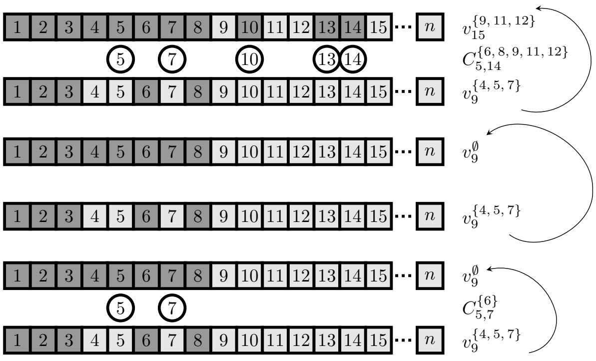

Figure 7. Possible transitions from a vertex v_i^M with $M \neq \emptyset$. We consider an example which can be easily generalised; here we have $i = 9$ and $M = \{4, 5, 7\}$. The assigned items are dark grey and the unassigned ones are light grey. The circles in one row visualise the corresponding combination with a non-zero bid on the right. On the bottom we have the case of a bid on a subset of M. Since we construct the outcomes of the auction in a specific order, we do not have to consider item 4 on this path any more. If there is a bid on $C \subset M$ with $4 \in C$, then we create a separate edge, e.g., a bid on $\{4, 5\}$ would lead to an edge to $v_9^{\{7\}}$. It can happen that several bids lead to the same successor; in this case we consider only the one with the highest value and set the weight of the edge accordingly. If there is no bid as depicted in the middle, we create an edge to $v_9^{\emptyset}$ with weight 0. This ensures that the path can be extended by combinations that do not overlap with $v_9^{\{4,5,7\}}$. Finally, there may be bids on combinations overlapping with $v_9^{\{4,5,7\}}$ that are not subsets of $M = \{4, 5, 7\}$ (shown at the top). Item 4 is again omitted.

The correctness of this construction can be shown in a similar fashion as for the proof of Theorem 1. We will briefly comment on the correctness of the case depicted on top of Figure 7. The set of vertices in the constructed graph is

$$V := \{v_i^M : 1 \leqslant i \leqslant n+1 \wedge \\ \exists k \in A : M \subseteq [k,\, k+s-1] \subseteq [2, i-2]\}.$$

Actually, fewer vertices may suffice, e.g., if V contains vertices that are not reachable from $v_1^{\emptyset}$. The important property in the top case is that the successor, say $v_{j+1}^{M'}$, reached via a combination, say $C_{i,j}^{M}$, is also an element of V, i. e., the length of the section with missing elements does not increase. Since we represent the combinations of an outcome in the order described above, we may disregard all items lower than i. Consequently, M' is a proper subset of M—due to the overlap we have $|M'| \leqslant |M| - 1$. Hence, $v_{j+1}^{M'} \in V$.

A major difference to Theorem 1 is that V and E are not of polynomial size. Here, the number of vertices is exponential in s. We have $|V| \in \mathcal{O}(n^2 2^s)$. It is not hard to see that the asymptotic number of edges is dominated by the top case in Figure 7. Let us consider an arbitrary vertex v_i^M with $M \neq \emptyset$. The section with missing elements of a combination, say $C_{i',j'}^{M'}$, as depicted in the top case is partly fixed because it has to contain $i-1$. For this reason, there are only $\mathcal{O}(s)$ possible starting points for M'. Further, there are $\mathcal{O}(s)$ possible values for i', since $i' \in M$, and $\mathcal{O}(2^s)$ possibilities for M'. Finally, there are $\mathcal{O}(n)$ possible values for j'. Thus, we can conclude $|E| \in \mathcal{O}(|V| s^2 2^s n) = \mathcal{O}(n^3 s^2 4^s)$. It is easy to see that $|E|$ is also in $\mathcal{O}(n^2 m 2^s)$, since there are $\mathcal{O}(m)$ outgoing edges for every vertex. The graph obviously is directed and acyclic, therefore computing a longest path from $v_1^{\emptyset}$ to $v_{n+1}^{\emptyset}$ that yields an optimal outcome of the auction requires time $\mathcal{O}(n^3 s^2 4^s)$ or, expressed in terms of the number m of submitted bids, time $\mathcal{O}(n^2 m 2^s)$. □

We can represent the combinations C_i of a combinatorial auction with $b(C_i) > 0$, $1 \leqslant i \leqslant m$, as a binary matrix $(c_{ij})_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n}$ with $c_{ij} = 1 \Leftrightarrow j \in C_i$ (cf. the integer program of Rothkopf et al. [25]). If this matrix has the consecutive ones property (C1P), i.e., if the columns can be permuted so that all ones appear consecutively in every row, the WDP can be solved in polynomial time [25]. Dom [9, p. viii and pp. 79–118] shows that the problem of deleting a minimum number of rows—corresponding to combinations in our case—to transform a given matrix into one with the C1P is fixed-parameter tractable with respect to the combined parameter $(\Delta,\, d)$, where Δ is the maximum number of 1-entries per row and d is the number of rows that may be deleted. Building on this result we can prove the following theorem.

Theorem 10. *The WDP is in $\mathcal{FPT}$ for combinatorial auctions $\mathcal{C} = (A, P, b)$ with unrestricted P w.r.t. the combined parameter (g, k) consisting of the maximum number g of goods in a bid and the minimum number k of bids that have to be deleted such that the remaining problem can be represented using intervals without gaps.*

The parameter k above can be thought of as a measure for the "distance" [16] of a given instance from an instance that is tractable; here the tractable instance consists of intervals and k describes the similarity to such an instance.

A result by Fomin et al. [13] implies that the WDP is in $\mathcal{FPT}$ with respect to the number of bids that need to be deleted so that the corresponding bid graph becomes an interval graph. However, Fomin et al. make the assumption that "an interval deletion set is provided as a part of the input, as it is an open question whether INTERVAL VERTEX DELETION is $\mathcal{FPT}$" [13, p. 352]; this is a restriction we do not use in Theorem 10.

6 CONCLUSION

We have introduced a new domain into the study of combinatorial auctions, consisting of all those allocation problems in which the goods can be arranged in a sequence in such a way that every bid concerns a discrete interval with multiple gaps of bounded length each. As already pointed out by Rothkopf et al. [25], even the simplest such scenario, namely the one without any gaps at all, is of some practical interest, e.g., for selling licenses for radio frequencies. Allowing for gaps increases flexibility and thereby makes this model relevant to a wider range of applications, as illustrated, for instance, by our introductory example on auctioning off the cells making up a construction ground.

For the case without any gaps at all, the problem of computing an optimal allocation was previously known to be solvable in polynomial time. We have systematically explored the extent to which this positive result can (and cannot) be generalised. If each interval has arbitrarily many gaps of at most some fixed length, there still is a polynomial algorithm. On the other hand, for several other seemingly mild extensions, we have established $\mathcal{NP}$-completeness results, thereby demonstrating just how subtle the difference between tractability and intractability can be. This complements previous work on the fine-grained complexity analysis of group decision making in AI [see, e.g., 15]. Finally, we identified two parameters that, when kept fixed, render the WDP tractable.

Our proofs employ both familiar and novel techniques. Specifically, the proof of Theorem 1 makes use of the tractability of the

LONGEST PATH problem for directed acyclic edge-weighted graphs, which is a helpful graphical representation of a dynamic programming approach.

An interesting question remains for most of the (fixed-parameter) tractable results, Theorem 10 being the exception: What is the computational complexity of recognizing instances with the considered structures if an ordering of the goods is not provided as part of the input? It may be the case that such a decision procedure is not constructive in the sense that it can be used to find an ordering with the desired properties if one exists. Then it would be interesting to know how hard it is to find such an ordering.

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REFERENCES

[1] Reuven Bar-Yehuda, Magnús M. Halldórsson, Joseph Seffi Naor, Hadas Shachnai, and Irina Shapira, 'Scheduling split intervals', *SIAM Journal on Computing*, **36**(1), 1–15, (2006).

[2] Danièle Beauquier, Maurice Nivat, Eric Rémila, and Mike Robson, 'Tiling figures of the plane with two bars', *Computational Geometry*, **5**(1), 1–25, (1995).

[3] Liad Blumrosen and Noam Nisan, 'Combinatorial auctions', In Nisan et al. [24], chapter 11, 267–300.

[4] Hans L. Bodlaender and Arie M.C.A. Koster, 'Treewidth computations I. upper bounds', *Information and Computation*, **208**(3), 259 – 275, (2010).

[5] Hans L. Bodlaender and Arie M.C.A. Koster, 'Treewidth computations II. lower bounds', *Information and Computation*, **209**(7), 1103 – 1119, (2011).

[6] Craig Boutilier and Holger H. Hoos, 'Bidding languages for combinatorial auctions', in *Proceedings of the 17th International Joint Conference on Artificial Intelligence (IJCAI-2001)*, (2001).

[7] Vincent Conitzer, Jonathan Derryberry, and Tuomas Sandholm, 'Combinatorial auctions with structured item graphs', in *Proceedings of the 19th National Conference on Artificial Intelligence (AAAI-2004)*, (2004).

[8] *Combinatorial Auctions*, eds., Peter Cramton, Yoav Shoham, and Richard Steinberg, MIT press, 2006.

[9] Michael Dom, *Recognition, Generation, and Application of Binary Matrices with the Consecutive-Ones Property*, Ph.D. dissertation, Friedrich-Schiller-Universität Jena, 2008.

[10] Rodney G. Downey and Michael R. Fellows, *Fundamentals of Parameterized Complexity*, Springer, 2013.

[11] Michael R. Fellows, Danny Hermelin, Frances Rosamond, and Stéphane Vialette, 'On the parameterized complexity of multiple-interval graph problems', *Theoretical Computer Science*, **410**(1), 53–61, (2009).

[12] Jörg Flum and Martin Grohe, *Parameterized Complexity Theory*, Texts in Theoretical Computer Science, Springer, New York, 2006.

[13] Fedor V. Fomin, Serge Gaspers, Petr A. Golovach, Karol Suchan, Stefan Szeider, Erik Jan van Leeuwen, Martin Vatshelle, and Yngve Villanger, 'k-gap interval graphs', in *LATIN 2012: Theoretical Informatics*, 350–361, Springer, (2012).

[14] Georg Gottlob and Gianluigi Greco, 'Decomposing combinatorial auctions and set packing problems', *J. ACM*, **60**(4), 24:1–24:39, (2013).

[15] Gianluigi Greco and Jérôme Lang, 'Group decision making via weighted propositional logic: Complexity and islands of tractability', in *Proceedings of the 24th International Joint Conference on Artificial Intelligence (IJCAI-2015)*, (2015).

[16] Jiong Guo, Falk Hüffner, and Rolf Niedermeier, 'A structural view on parameterizing problems: Distance from triviality', in *Proceedings of the 1st International Workshop on Parameterized and Exact Computation (IWPEC '04)*, (2004).

[17] Carl Hierholzer, 'Ueber die Möglichkeit, einen Linienzug ohne Wiederholung und ohne Unterbrechung zu umfahren', *Mathematische Annalen*, **6**(1), 30–32, (1873).

[18] Daniel Lehmann, Rudolf Müller, and Tuomas Sandholm, 'The winner determination problem', In Cramton et al. [8], chapter 12.

[19] David Loker and Kate Larson, 'Parameterizing the winner determination problem for combinatorial auctions', in *Proceedings of the 9th International Conference on Autonomous Agents and Multiagent Systems (AAMAS-2010)*, (2010).

[20] Bojan Mohar, 'Face covers and the genus problem for apex graphs', *Journal of Combinatorial Theory, Series B*, **82**(1), 102–117, (2001).

[21] Cristopher Moore and John Michael Robson, 'Hard tiling problems with simple tiles', *Discrete & Computational Geometry*, **26**(4), 573–590, (2001).

[22] Rolf Niedermeier, *Invitation to Fixed-Parameter Algorithms*, Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, 2006.

[23] Rolf Niedermeier, 'Reflections on multivariate algorithmics and problem parameterization', in *Proceedings of the 27th International Symposium on Theoretical Aspects of Computer Science (STACS-2010)*, (2010).

[24] *Algorithmic Game Theory*, eds., Noam Nisan, Tim Roughgarden, Eva Tardos, and Vijay V. Vazirani, Cambridge University Press, 2007.

[25] Michael H. Rothkopf, Aleksandar Pekeč, and Ronald M. Harstad, 'Computationally manageable combinational auctions', *Management Science*, **44**(8), 1131–1147, (1998).

[26] Tuomas Sandholm, 'Algorithm for optimal winner determination in combinatorial auctions', *Artificial Intelligence*, **135**(1), 1–54, (2002).

[27] Robert Sedgewick and Kevin Wayne, *Algorithms*, Addison-Wesley Professional, 4th edn., 2011.

[28] Makoto Yokoo, 'Characterization of strategy/false-name proof combinatorial auction protocols: Price-oriented, rationing-free protocol', in *Proceedings of the 18th International Joint Conference on Artificial Intelligence (IJCAI-2003)*, (2003).

[29] Ryo Yoshinaka, 'Higher-order matching in the linear lambda calculus in the absence of constants is NP-complete', in *Proceedings of the 16th International Conference on Term Rewriting and Applications (RTA-2005)*, Springer, (2005).