On the Efficiency of Hybrid Objective Multi-Unit Auctions

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

In this thesis, we analyze the efficiency of equilibria in multi-unit auctions (discriminatory and uniform price) in the presence of budgets where each player has their own objective that is intermediate between value and utility maximizing. We prove in both multi-unit auctions the existence of pure Nash equilibria and in the case of discriminatory multi-unit auction give a complete description of their efficiency in the presence of budgets, but also in the budgetless case. The main result of this thesis is the method of analyzing the price of anarchy in the presence of budgets. The notion of price of anarchy we use was first coined by Caragiannis & Voudouris (SAGT 2014) and recently received more attention in the autobidding survey by Aggarwal et al. (ACM 2024). By transforming a multi-unit auction with budgets into a *welfare objective auction* we manage to bound the price of anarchy (POA) of the former by bounding the POA in the latter through smoothness. The smoothness framework, introduced by Syrgkanis & Tardos (STOC 2013), utilizes the extension to different types introduced by Colini-Baldeschi et al. (arXiv, 2025). The upper bounds obtained through this method on the POA are valid for coarse correlated equilibria and for some classes even tight. The method can be extended to simultaneous first-price multi-unit auctions where players have submodular valuations. Through this, we manage to prove that the coarse correlated POA of utility maximizing first-price simultaneous auctions where bidders have additive valuations is approximately 2.18, exactly the same value Deng et al. (NeurIPS 2024) found for the price of anarchy of mixed Nash equilibria of budgetless simultaneous auctions in mixed settings.

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Contents

Chapter 1

Introduction

1.1 Background

Auctions have been present through human society for more than 2000 years. The earliest recorded auctions took place in Babylon [19]. In ancient Rome, auctions were used to sell war loot [14]. Over time, auctions evolved from an open outcry method during physical gatherings to complex digital platforms accessible world wide.

Today, auctions represent a significant portion of the global economy. In 2008, the US National Auctioneers Association reported that the auction industry had generated a gross revenue of $268.4 billion. Products auctioned include government bonds, art works and second hand items in general. Items that fall into the last category are often sold through popular platforms like eBay.

Given the economic significance of auctions, it is imperative that we study their mechanisms to ensure fairness and efficiency. In recent times, auctions have been studied through the lens of game theory and economics. In auction theory, notions that are studied include: the efficiency and fairness of mechanisms, as well as the behavior of bidders. As such, important findings can often shape the design of auction mechanisms. One example is William Vickrey's contributions to auction theory. For his work on the Vickrey auction, he earned the Nobel prize in economics. His research had a lasting impact on mechanism design, as it promotes truthful bidding as the dominant strategy [21].

The most well known instance of an auction is the first-price single-item auction. In this auction, the *auctioneer*, wants to sell a single indivisible item to a set of bidders. The auctioneer asks for all the bids simultaneously and sells the item to the highest bidder at the price of that bid. Other pricing rules exist. For example, in the second-price auction, the item is sold at the price of the second highest bid. This pricing rule has the advantage that bidding truthfully is a dominant strategy.

One possible generalization of the single-item auction, is where the auctioneer wants to sell a number of units, k, of the same indivisible item. This is called a *multi-unit auction*. Instead of being a single number, the bids are now a vector whose dimension is the same as the number of units. There are three main variants of this. The *discriminatory price auction* (DPA), *the uniform price auction* (UPA), and the *VCG auction*. The allocation rule they have in common, assigns a number of units to a player equal to the number of winning bids. Here, a winning bid is a bid that belongs to the k highest bids.

Among the three variants, two have simple pricing rules. In the *discriminatory price auction*, each player pays what they bid if that bid is ranked in the k highest bids. In the *uniform price auction*, each player pays the highest losing bid times the number of units that are assigned to them. The *VCG auction* implements the Clarke payment rule. This auction format was studied in [25] and retains the nice properties of its single-item counterpart. The equilibrium in this auction is the socially optimal outcome, and the dominant strategy of each player is bidding truthfully. The main drawback of this auction is that its payment rule is complex and difficult to implement in real-world settings. Going forward, we shall only consider multi-unit auctions under discriminatory pricing or uniform pricing rules.

Uniform pricing rule satisfies what Milgrom calls "the law of one price", [21]. That is, in a uniform price auction, the players pay the same amount for each unit they receive. In a discriminatory price auction, the price a player pays for each unit they receive will most likely differ. However, the advantage of the discriminatory price auction is that the auctioneer can not maliciously inflate the price a player pays for the units they receive. Once the player knows the number of units they are allocated, they know the price they pay for each unit.

The discriminatory price and uniform price auctions are not truthful. Players often have incentives to declare bids that deviate from their true valuations. This behavior can lead to equilibria that are not socially optimal. Since these auctions are implemented, their inefficiencies ought to be studied. The concept that formalizes this notion of inefficiency is introduced in [18] and is called *price of anarchy* (POA). The price of anarchy is the ratio between the optimal social welfare and the lowest welfare achieved by an equilibrium.

Real-world examples of multi-unit auctions include the EU Emissions Trading System auctions carbon emission permits, which companies need to legally emit CO_2 [15, 7]. Other examples include the UK or US treasury auctioning bonds to investors.

In the literature, it was until recently customary to assume that players are utility maximizers. This means that each bidder wants to maximize the objective of value minus the price they pay. However, recently the study of *autobidding* has gained traction. With autobidding, a player delegates their bidding tasks to an automated agent. With traditional bidding interfaces, the player has to manually submit their bids. In fast paced environments where the bidder partakes in multiple such auctions, this is not feasible.

Autobidding thus greatly simplifies the interaction between the bidder and the interface, as now players just have to specify their objective and possibly some constraints, taking budgets into account. Autobidding is estimated to make up for more than 80% of the online traffic in ad auctions in 2020 [12].

Value maximizing under the *return-on-spend* (RoS) constraint is a popular autobidding strategy due to its simplicity. Value maximizers just want to maximize their value and thus do not care about prices to the extent that their constraints allow them. The RoS constraint is a limit on the ratio between a bidder's payment and the value they receive. Another type of constraint is the *budget constraint* which is a hard limit on the amount a player can spend.

Until recently, the inefficiency of auctions have only been investigated for utility maximizing players. Due to the increased interest in autobidding we consider objectives beyond utility maximization, e.g., value maximizing objectives. As such, mixed settings have gained interest from the autobidding industry [10]. In this paper, they study the price of anarchy of simultaneous auctions, another generalization of single-item auctions, where both value maximizing and utility maximizing bidders are present.

In this thesis, we generalize upon this idea by allowing players not to just be value maximizing or utility maximizing, but to have an objective intermediate between value and utility maximizing, [6, 1]. Thus, each player has a parameter specifying their sensitivity to the price. This brings us to the main model of this thesis. We study discriminatory/uniform price multi-unit auctions where players have an objective that is intermediate between value and utility maximizing and are constrained by an RoS constraint and budget constraint. In this model, we answer two questions posed in the survey [1]. One about the existence of equilibria and the other about the efficiency of equilibria in the autobidding setting.

1.2 Related works

For single item auctions, the price of anarchy of pure, mixed and correlated equilibria is 1 [13]. The coarse correlated price of anarchy of single item first-price auctions is known to be $\frac{e}{e-1}$ [23]. However, for the lower bound, overbidding is necessary for one of the players in the single item auction. When considering first-price single item auctions with utility maximizers and no-overbidding, the POA becomes approximately 1.23, meaning it becomes more efficient [13]. For mixed Bayesian equilibria, the POA is $\frac{e^2}{e^2-1}$ [17].

The inefficiency of multi-unit auctions with only utility maximizers for Bayesian equilibria has been studied in [9]. They found an upper bound of $\frac{e}{e-1}$ and 3.1462 for discriminatory price and uniform price multi-unit auctions, respectively, for bidders with submodular valuations[1]. For submodular valuation functions, tight lower bounds are still absent. The best lower bound for the discriminatory price auction, to our knowledge, is 1.1204 [20] while for uniform price auctions it is 2.18 [3]. The lower bound for uniform price auctions is tight for pure Nash equilibria [3], however, for more general equilibrium concepts the separation remains.

Tangent to multi-unit auctions are simultaneous single item first-price auctions. In [5] various tight bounds on the price of anarchy of such auctions are covered. For pure Nash equilibria, it is not hard to see that the auctions are efficient. However, for mixed, correlated, coarse correlated and Bayesian equilibria the price of anarchy is $\frac{e}{e-1}$ when the valuation functions are submodular. Simultaneous first-price auctions with additive valuations are efficient for pure Nash, mixed Nash and correlated equilibria, however, for coarse correlated equilibria the POA is $\frac{e}{e-1}$.

With the advent of autobidding, value maximization, a different objective compared to utility maximization, has become increasingly popular due to its simplicity. In [10] they study the price of anarchy for mixed equilibria of simultaneous first-price auctions. For value maximizing auctions, they find a tight bound of 2, but for hybrid settings, they obtain a tight bound of 2.18, where the exact value is the result of an optimization problem.

In [6], simultaneous first-price auctions are studied where bidders are allowed to have an objective intermediate between value and utility maximizing with reserve prices, thus generalizing the model of [10]. They recover the 2.18 result from [10] through smoothness showing that the POA is 2.18 for coarse correlated equilibria. Furthermore, they prove a tight bound when there are two types of players, with one type being value maximizer and another type being intermediate between value and utility maximizing.

The efficiency of auctions where bidders have budget constraints have been analyzed in [2], [4] and [11]. More specifically, in [2] they analyze the efficiency of a modified version of simultaneous auctions. The measure of efficiency they investigate is called *liquid price of anarchy* (LPOA), which is similar to the price of anarchy, but captures the budget constraints of the players. They find an upper bound of 51.5 for mixed equilibria, but their main contribution is that they show that the LPOA of this class of auctions is upper bounded by a constant at all. Note that in [4] they analyze the efficiency of equilibria, for coarse correlated equilibria (CCE), also in terms of the liquid price of anarchy. However, the liquid price of anarchy they analyze is subtlely weaker than the liquid price of anarchy analyzed in [2].

1.3 Our contributions

In this thesis, we look at players having an intermediate objective between value and utility maximizing in discriminatory and uniform price multi-unit auctions. We extend the results found in [6] to discriminatory price multi-unit auctions and uniform price multi-unit auctions.

[1]A submodular valuation is a valuation where the marginal value per unit decreases. It captures a diminishing returns type of relationship between the number of units received and the value.

We prove three main results. The first being the existence of pure equilibria in multi-unit auctions, both discriminatory and uniform price, in the presence of budgets. The second being a complete characterization (for discriminatory price) of their price of anarchy both in the presence of budgets and without.

Lastly, we find a method to upper bound the liquid price of anarchy, presented in [4, 1], for CCEs in auctions with budgets. These bounds are obtained by transforming our auction with budgets into a *welfare objective auction* where we can analyze the price of anarchy through smoothness. Furthermore, the framework in which we produce these upper bounds is general enough that it extends to other auction settings, as opposed to the methods presented in [11], [2] and [4]. Through this method, we find tight results for the price of anarchy for CCEs for both discriminatory price multi-unit auctions and simultaneous first-price auctions with budgets.

The thesis is structured as follows. In Chapter 2, we formally define the model and the price of anarchy. Furthermore, we investigate subtleties that arise due to the RoS and budget constraints in relation to the equilibrium hierarchy, that to the best of our knowledge, have not been observed in the literature before.

Next, we shall focus solely on discriminatory auctions. In Chapter 3 we study the properties and existence of pure Nash equilibria and their inefficiencies. For budgetless multi-unit auctions, we give a complete description of the price of anarchy of pure Nash equilibria.

In Chapter 4 and 5 we investigate the price of anarchy of more general equilibrium concepts. More specifically, in Chapter 4 we cover smoothness, a technique that allows us to derive upper bounds on the price of anarchy for *coarse correlated equilibria* [24]. Furthermore, we introduce a new type of auction, *welfare objective auctions*, that will help us analyze the efficiency of equilibria in the presence of budget constraints. In Chapter 5 we compute upper bounds on the price of anarchy through the smoothness framework and show how these bounds can be lifted to auctions with budgets making use of the welfare objective auctions introduced in Chapter 4.

In Chapter 6 we cover the uniform price auction, where we summarize the corresponding results in the uniform price setting found for discriminatory auctions. We also extend these results to uniform price auctions with budgets.

Lastly, in Chapter 7, we study the incomplete information setting of Harsanyi introduced in [16]. This gives a different notion of equilibrium called the *Bayesian Nash equilibrium*, and we study the price of anarchy of this solution concept.

Chapter 2

Preliminaries

2.1 The model

The auction setting we consider contains k identical units of a single good, and $[n] = \{1, \dots, n\}$ being the set of n bidders/players. Each player $i \in [n]$ has a non-decreasing *valuation function* $v_i : \{0\} \cup [k] \to \mathbb{R}^+$ such that $v_i(0) = 0$. and a non-decreasing *bid function* $b_i : \{0\} \cup [k] \to \mathbb{R}^+$. Often, we represent v_i and b_i as a k-tuple and change back and forth between the function notation and the tuple notation. We write $\boldsymbol{v} = (v_1, \dots, v_n)$ and $\boldsymbol{b} = (b_1, \dots, b_n)$ for *valuation profile* and *bidding profile*, respectively. We call a bid b_i, of the player i, *uniform* if there exists a $\bar{b}_i \in \mathbb{R}_{\geq 0}$ such that $b_{ij} = \bar{b}_i$ for $1 \leq j \leq \ell$ where $1 \leq \ell \leq k$ and $b_{ij} = 0$ for $k \geq j > \ell$.

Given a valuation profile $\boldsymbol{v} = (v_1, \dots, v_n)$ we define a *marginal valuation* of player i as $m_i = (m_{i1}, \dots, m_{in})$ where $m_{i\ell} = v_i(\ell) - v_i(\ell - 1)$, $m_{i1} = v_i(1)$. We shall assume that the valuation functions are *submodular* which means that $v_i(x+1) - v_i(x) \geq v_i(y+1) - v_i(y)$ whenever $x < y$. In particular, by submodularity, we have that the marginal valuations will be decreasing and that $v_i(j)/j \geq v_i(\ell)/\ell$ whenever $j < \ell$. For a proof of this, we refer to [3].

Next, let $\mathcal{A}$ be the set of all possible bidding profiles. We define the *allocation rule* as a function $x : \mathcal{A} \to [k]^n$ with $\sum_{i=1}^n x_i(\boldsymbol{b}) = k$ where $x_i(\boldsymbol{b})$ is the number of goods player i attains in the bidding profile $\boldsymbol{b}$. Furthermore, we denote the map $p : \mathcal{A} \to \mathbb{R}^n_{\geq 0}$ constrained to $p_i(\boldsymbol{b}) = 0$ whenever $x_i(\boldsymbol{b}) = 0$, for each player $i \in [n]$, the *pricing rule*.

The allocation rule that we study is the following: rank all marginal bids of all players and allocate the k goods to the players associated with the k-th highest marginal bids. Here, ties are broken according to some rule. In this thesis, we shall limit ourselves to two pricing rules:

1. *Discriminatory pricing.* Players pay the marginal bids corresponding to what they receive. More specifically, given a bidding profile $\boldsymbol{b}$, for each $i \in [n]$ we have $p_i(\boldsymbol{b}) = \sum_{j=1}^{x_i(\boldsymbol{b})} b_i(j)$.

2. *Uniform pricing.* Let p be the $k+1$-th highest marginal bid, i.e., the highest losing bid. Then under uniform pricing $p_i(\boldsymbol{b}) = p \cdot x_i(\boldsymbol{b})$ for each player $i \in [n]$.

Furthermore, for ease of notation, we often write $v_i(\boldsymbol{b})$ instead of $v_i(x_i(\boldsymbol{b}))$ for the *value* player i receives and we define $u_i(\boldsymbol{b}) = v_i(\boldsymbol{b}) - p_i(\boldsymbol{b})$ to be the *utility* of player i.

Lastly, each player i has an associated objective parameter (or price/payment sensitivity parameter) $\theta_i \in [0, 1]$, budget constraint parameter $B_i \in (0, +\infty]$, and return-on-spend (RoS) constraint parameter $\tau_i \in (0, \infty)$ such that, given a bidding profile $\boldsymbol{b}_{-i}$ it is each player's objective to maximize:

$$o_i(b_i, \boldsymbol{b}_{-i}) := v_i(b_i, \boldsymbol{b}_{-i}) - \theta_i \cdot p_i(b_i, \boldsymbol{b}_{-i}),$$

subject to

$$p_i(b_i, \boldsymbol{b}_{-i}) \leq B_i \tag{2.1}$$

$$\tau_i \cdot p_i(b_i, \boldsymbol{b}_{-i}) \leq v_i(b_i, \boldsymbol{b}_{-i}). \tag{2.2}$$

We call $o_i(\boldsymbol{b})$ the *objective* that player i attains under the bidding profile $\boldsymbol{b}$. We will denote the constraints profiles by $\boldsymbol{\theta}, \boldsymbol{B}$ and $\boldsymbol{\tau}$. We denote this multi-unit auction by writing $A = (\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{B}, \boldsymbol{\tau})$, where $\boldsymbol{v}$ is the marginal valuation profile, and $\boldsymbol{\theta}$, $\boldsymbol{B}$, and $\boldsymbol{\tau}$ are the constraint profiles. We omit the allocation and pricing rules when they are clear from context.

Note that for $\theta_i = 1$, player i only desires to maximize their utility, while for $\theta_i = 0$, i desires to maximize their value. In case $B_i = \infty$, then this player would have no budget constraint.

Example 2.1. Consider the case where $n = 3$ and $k = 3$ with $\boldsymbol{\theta} = 0$ (value maximizing), $\boldsymbol{B} = (\infty, \infty, \infty)$ (no budget constraint), and $\boldsymbol{\tau} = (1, 1, 1)$, and the following marginal valuations and marginal bids:

$$v_1 = (11, 19, 25), \quad b_1 = (11, 10, 9)$$
$$v_2 = (16, 23, 23), \quad b_2 = (12, 11, 0)$$
$$v_3 = (11, 17, 21), \quad b_3 = (11, 6, 4)$$

Ties will be broken first in favor of player 2, then in favor of player 3 and lastly player 1. The highest marginal bid is 12 given by player 2. So they will certainly receive an item. For the next two items we see that it is a three way tie between all players. Since player 2 is first in the ordering, we have that one more item is allocated to player 2. Next up in the ordering is player 3, so they will receive the other item. In the case of discriminatory pricing, players 1, 2 and 3 pay $p_1 = 0$, $p_2 = 23$ and $p_3 = 11$, respectively, and attain a value of $v_1 = 0$, $v_2 = 23$ and $v_3 = 11$, respectively. However, if we apply uniform pricing then we see that $p_3 = 11$, which is the highest losing bid, $p_2 = 22$, and $p_1 = 0$.

In the example above, we see that truthful bidding is not always a dominant strategy, even for value maximizers. Had player 2 bid truthfully, they would only have received one good that would give them a value of 16, which is less than 23.

It is important to note that in the example, while player 1 over bids their value, they still satisfy the RoS constraint. So, in short, satisfying the RoS and budget constraints is not only dependent on the player bid, but may depend on the entire bidding profile. Player 1's individual bid does not satisfy the RoS constraint in Example 2.1 (as $30 \geq 25$), but since they do not receive any units, it is satisfied. So, given a bidding profile $\boldsymbol{b}_{-i}$, we define $\mathcal{B}_i(\boldsymbol{b}_{-i})$ to be the set of bids player i can make so that they satisfy the RoS and budget constraints. Furthermore, we let $\mathcal{B}$ be the set of all bidding profiles $\boldsymbol{b}$ such that $b_i \in \mathcal{B}_i(\boldsymbol{b}_{-i})$ for any player $i \in [n]$.

2.2 Solution concepts

In this section, we turn our attention to solution concepts. Note that the bids in the previous section referred to pure strategies. Due to the RoS and budget constraint, subtleties arise when we move to mixed bidding strategies. With this in mind, we first define pure equilibria and then turn our attention to more general equilibrium notions.

> **Definition 2.2.** Let $\mathcal{B}$ be the set of all bidding profiles such that all players satisfy the RoS and budget constraints. For arbitrary $\boldsymbol{\theta}, \boldsymbol{B}$ and $\boldsymbol{\tau}$ we call a bidding profile $\boldsymbol{b} \in \mathcal{B}$ a *pure Nash equilibrium* (PNE) if and only if for each player $i \in [n]$ and any bid b'_i with $b'_i \in \mathcal{B}_i(\boldsymbol{b}_{-i})$ we have:
> $$o_i(\boldsymbol{b}) \geq o_i(b'_i, \boldsymbol{b}_{-i}).$$

Henceforth, we often use the term "pure equilibrium" to refer to "pure Nash equilibrium".

Now, let us turn our attention to mixed strategies. Let χ be some distribution over some set of pure bidding profiles. Then, a natural way to define that χ satisfies the RoS and budget constraint would be to require:

$$\mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}\left[p_i(\boldsymbol{b})\right] \leq B_i,$$

$$\tau_i \cdot \mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}\left[p_i(\boldsymbol{b})\right] \leq \mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}\left[v_i(\boldsymbol{b})\right].$$

Often, to ease notation, we write $v_i(\chi)$ as opposed to $\mathbb{E}_{\boldsymbol{b}\sim\chi}\left[v_i(\boldsymbol{b})\right]$ and similarly for the objective function o_i. In other words, for more general solution concepts, we require the RoS and budget constraints to hold in expectation. In the literature, this extension is often chosen; see, e.g. [1], [10]. However, the following example will now show that pure equilibria as defined above are no longer necessarily mixed equilibria.

Example 2.3. Consider a 2-unit discriminatory auction with 2 budgetless players and $\boldsymbol{\tau} = (1,1)$. For the valuations, let $m_1 = (4,1)$ and $m_2 = (3,3)$, where $\theta_1 = 0$ and $\theta_2 = 0$ (value maximizers). Here, the ties are broken in favor of player 1. Consider the bids $b_1 = (3,0)$ and $b_2 = (3,3)$, then we note that this is a PNE defined in Definition 2.2.
However, if we allow players to deviate to mixed bids, then this will not be an equilibrium. Suppose that player 1 deviates to the following strategy. They bid $b_1 = (3,0)$ with probability $1-p$ and bid $b_1' = \left(1+\frac{1}{p}, 3\right)$ with probability $p \in (0, \frac{1}{2}]$. Under this new bidding strategy, player 1 will now also win the second item with a probability of p as $p \in (0, \frac{1}{2}]$. Hence, their expected value is $\mathbb{E}(v_1) = 4 + p$. However, note that upon winning the item, they pay $1 + \frac{1}{p}$ extra. Thus, the expected payment is $\mathbb{E}(p_1) = 3 + \left(1+\frac{1}{p}\right)p = 4 + p$ and thus the RoS constraint is satisfied. However, note that under $\boldsymbol{b} = (b_1, b_2)$ that $v_1(\boldsymbol{b}) = 4 < 4 + p$ as $p \in (0, \frac{1}{2}]$. Hence, $\boldsymbol{b} = (b_1, b_2)$ will not be a mixed equilibrium.

In short, the above example highlights the need to proceed with caution. Requiring the RoS and budget constraints to be satisfied in expectation allows for mixed strategies whose support has pure bids that violate the RoS or budget constraint. The example above shows that the RoS constraint may be violated by an unbounded amount.

So, with these considerations, despite it being a natural extension, requiring the RoS and budget constraints to satisfy in expectation has counter intuitive consequences with respect to the equilibrium hierarchy and violating the RoS or budget constraint in certain outcomes.

Two approaches could solve some parts of these issues. One would be to redefine a PNE to allow for unilateral mixed deviations. However, this would not remove outcomes where the RoS or budget constraint is severely violated. Furthermore, allowing for mixed deviations in the definition of PNEs, would give a definition that is almost never satisfied. The example above shows that for any pure bidding profile $\boldsymbol{b}$ and any value maximizers i, if $p_i(\boldsymbol{b}) < \min\left\{\frac{1}{\tau_i}v_i(\boldsymbol{b}), B_i\right\}$, then with a small probability, they will be able to bid enormous amounts, possibly yielding them extra units. More precisely, for any bidding profile where a value maximizer strictly satisfies the RoS and budget constraint, could never be a PNE in this sense. In other words, a lot of auctions will not have a PNE if mixed deviations are considered. However, as we will see later, under an appropriate tie-breaking rule, PNEs defined in Definition 2.2 will always exist for any multi-unit auction. So, instead, we choose the second approach and define a stricter version of the RoS and budget constraint for more general strategies.

Definition 2.4. Let χ_{-i} be a distribution over some set of bidding profiles. Now, let $\mathcal{S}_i(\chi_{-i})$ be the set of distributions on the bids of player i such that $\chi_i \in \mathcal{S}_i(\chi_{-i})$ iff for any $\boldsymbol{b}_{-i} \in \operatorname{supp} \chi_{-i}$ and any $b_i \in \operatorname{supp}(\chi_i)$ we have $b_i \in \mathcal{B}_i(\boldsymbol{b}_{-i})$. Next, we let $\mathcal{S}$ be the set of distributions χ such that $\chi_i \in \mathcal{S}_i(\chi_{-i})$ for any $i \in [n]$.

So, we require all outcomes of χ to satisfy the RoS and budget constraints. With this in mind, we define a mixed equilibrium as follows.

Definition 2.5. Let $\mathcal{A}_i$ be the set of pure bids player i can make and let χ_i be a distribution over $\mathcal{A}_i$. Then, $\chi = (\chi_1, \ldots, \chi_n)$ is a *mixed Nash equilibrium* (MNE) if $\chi \in \mathcal{S}$ and for any player $i \in [n]$ and any other distribution χ_i' with $\chi_i' \in \mathcal{S}_i(\chi_{-i})$ we have:

$$\mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}[o_i(\boldsymbol{b})] \geq \mathop{\mathbb{E}}_{(b_i',\boldsymbol{b}_{-i})\sim(\chi_i',\chi_{-i})}\left[o_i(b_i',\boldsymbol{b}_{-i})\right].$$

Note, that restricting in the previous definition to unilateral deviations that are pure, would lead to an equivalent definition. Similarly, we now define correlated and coarse correlated equilibria. Here again, whether we allow pure or mixed deviations will yield equivalent definitions.

Definition 2.6. Let $\chi \in \mathcal{S}$. We call χ a *correlated equilibrium* (CE) if for all players $i \in [n]$, all $\chi_i' \in \mathcal{S}_i(\chi_{-i})$ and all $b_i^* \in \mathcal{A}_i$ with $(b_i^*, \chi_{-i}) \in \mathcal{S}$ we have:

$$\mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}[o_i(\boldsymbol{b}) \mid b_i = b_i^*] \geq \mathop{\mathbb{E}}_{\substack{\boldsymbol{b}\sim\chi \\ b_i'\sim\chi_i'}}\left[o_i(b_i',\boldsymbol{b}_{-i}) \mid b_i = b_i^*\right].$$

Similarly, $\chi \in \mathcal{S}$ is a *coarse correlated equilibrium* (CCE) whenever for all players $i \in [n]$ and all $\chi_i' \in \mathcal{S}_i(\chi_{-i})$ we have:

$$\mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}[o_i(\boldsymbol{b})] \geq \mathop{\mathbb{E}}_{\substack{\boldsymbol{b}\sim\chi \\ b_i'\sim\chi_i'}}[o_i(b_i',\boldsymbol{b}_{-i})].$$

The difference between MNE and CCE is the type of distributions over bids considered. In a MNE, each player's distribution is independent from another's player. However, for a CE or CCE more general distributions over bidding profiles are considered. With these definitions, we obtain the usual hierarchy: $\mathrm{PNE}(A) \subseteq \mathrm{MNE}(A) \subseteq \mathrm{CE}(A) \subseteq \mathrm{CCE}(A)$ for any auction A. Here, for $\mathrm{EQ} \in \{\mathrm{PNE}, \mathrm{MNE}, \mathrm{CE}, \mathrm{CCE}\}$ we use $\mathrm{EQ}(A)$ to refer to the set of equilibria of auction A.

2.3 Price of Anarchy

Now that we have defined the equilibrium notions, our aim in this thesis is to study their efficiency. Nash equilibria in multi-unit auctions are in general inefficient since they do not lead to socially optimal outcomes. A prominent notion to quantify this inefficiency is the *price of anarchy*, [18]. It is the ratio between the optimum of some objective function and the lowest the objective function achieves at an equilibrium. The most prominent choice for the objective function is the *social welfare*, which, given a bidding profile $\boldsymbol{b}$ is defined as: $\mathrm{SW}(\boldsymbol{b}) := \sum_{i\in[n]} v_i(\boldsymbol{b})$. Letting $\text{SW-OPT} := \sup_{\boldsymbol{b}} \mathrm{SW}(\boldsymbol{b})$, the price of anarchy of an auction $A = (\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{B}, \boldsymbol{\tau})$ in terms of the social welfare for pure equilibria is defined as:

$$\text{SW-POA}(A) := \sup_{\boldsymbol{b}\in\mathrm{PNE}(A)} \frac{\text{SW-OPT}}{\mathrm{SW}(\boldsymbol{b})}.$$

For a class of auctions $\mathcal{C}$, we write SW-POA$(\mathcal{C}) = \sup_{A \in \mathcal{C}}$ SW-POA(A). For distributions over bidding profiles, the social welfare is extended as $\mathrm{SW}(\chi) := \mathbb{E}_{\boldsymbol{b}\sim\chi}[\mathrm{SW}(\boldsymbol{b})]$. With this, one can extend the SW-POA to more general equilibrium notions. However, the next example shows that this notion falls short in the presence of budgets as the price of anarchy of all discriminatory multi-unit auctions will be unbounded.

Example 2.7. Let $n \in \mathbb{N}_{\geq 3}$ and consider a single unit discriminatory auction with two value maximizers whose values are $v_1 = n$ and $v_2 = 1$. Let $\boldsymbol{\tau} = (1,1)$ and $\boldsymbol{B} = (0.1, 1)$. Now, we find that $\boldsymbol{b} = (0.1, 1)$ is a PNE with $\mathrm{SW}(\boldsymbol{b}) = 1$. However, we easily see that $\mathrm{OPT} = n$. Thus, for this given instance, we find that POA $\geq n$.

To capture the efficiency of equilibria in auctions with budgets more informatively two competing objective functions are most popular in literature. The first notion is *effective welfare* (see e.g., [4, 1]), while the second is *liquid welfare*, introduced in [11] and used in [2]. For pure bidding profiles, the two notions coincide and are defined as

$$\mathrm{EW}(\boldsymbol{b}) = \mathrm{LW}(\boldsymbol{b}) := \sum_{i\in[n]} \min\left\{\frac{1}{\max\{\tau_i, \theta_i\}} v_i(\boldsymbol{b}), B_i\right\},$$

while for their mixed extensions they take the form:

$$\mathrm{EW}(\chi) := \sum_{i\in[n]} \min\left\{\frac{1}{\max\{\tau_i, \theta_i\}} \mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}[v_i(\boldsymbol{b})], B_i\right\},$$
$$\mathrm{LW}(\chi) := \sum_{i\in[n]} \mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi} \min\left\{\frac{1}{\max\{\tau_i, \theta_i\}} v_i(\boldsymbol{b}), B_i\right\}.$$

Note that the expectation for the effective welfare is within the minimum, while for liquid welfare it is outside. This means that $\mathrm{EW}(\chi) \geq \mathrm{LW}(\chi)$ (as a result of Jensen's inequality). Moreover, by linearity of expectation, we have $\mathrm{LW}(\chi) = \mathbb{E}_{\boldsymbol{b}\sim\chi}[\mathrm{LW}(\boldsymbol{b})]$. Furthermore, since liquid welfare and effective welfare coincide on pure profiles, we have that EW-OPT = LW-OPT. Combining this with the fact that $\mathrm{EW}(\chi) \geq \mathrm{LW}(\chi)$, we find EW-POA $\leq$ LW-POA showing that LW-POA is a stronger notion.

However, in this thesis, we focus on the price of anarchy in terms of the effective welfare, EW-POA. The reason being that LW-POA can be unbounded for mixed equilibria. In [2], they show that for simultaneous first-price auctions the LW-POA is unbounded for mixed equilibria. Additionally, the research question on the efficiency of equilibria posed in the survey [1] also uses effective welfare (although they refer to it as liquid welfare).

Remark 2.8. The effective welfare defined in the survey [1] has a slightly different form compared to ours. They define it as:

$$\mathrm{EW}'(\chi) := \sum_{i\in[n]} \min\left\{\mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}\left[\frac{1}{\tau_i} v_i(\boldsymbol{b})\right], B_i\right\}.$$

Note the difference being $\frac{1}{\tau_i}$ instead of $\frac{1}{\max\{\tau_i,\theta_i\}}$. However, this difference matters as the following example shows that the price of anarchy defined in terms of EW$'$ is still unbounded.

Consider a single-unit discriminatory auction with two players such that $\boldsymbol{v} = (1,1)$, $\boldsymbol{\theta} = (1,0)$, $\boldsymbol{B} = \boldsymbol{\infty}$, $\boldsymbol{\tau} = (\tau_1, 1)$ where $\tau_1 < 1$ and let ties be broken in favor of player 2. Here, we easily see that $\text{OPT} = \frac{1}{\tau_1}$. However, we also see that $\boldsymbol{b} = (0,1)$ is a PNE for which $\text{EW}'(\boldsymbol{b}) = 1$. Putting it all together, we find that the LPOA bounded below by $\frac{1}{\tau_1}$, which is unbounded.

With all of the above considerations in mind, we define the price of anarchy as follows.

Definition 2.9. Let $\text{EQ} \in \{\text{PNE}, \text{MNE}, \text{CE}, \text{CCE}\}$. For an auction $A = (\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{B}, \boldsymbol{\tau})$, we let $\text{OPT}(A) := \sup_{b \in \mathcal{B}} \text{EW}(\boldsymbol{b})$ and define the *price of anarchy* of an auction as:

$$\text{EQ-POA}(A) = \sup_{\chi \in \text{EQ}(A)} \frac{\text{OPT}(A)}{\text{EW}(\chi)}.$$

For a class of auctions $\mathcal{C}$, we define the *price of anarchy* as:

$$\text{EQ-POA}(\mathcal{C}) = \sup_{A \in C} \text{EQ-POA}(A).$$

Note that for budgetless auctions, the price of anarchy coincides with the price of anarchy in terms of the social welfare.

With the new definition, we see that the POA in the example in Remark 2.8 is now bounded. This perhaps should not be a surprise, as effective welfare measures a certain "ability to pay" and when $\theta_i > \tau_i$, the player's ability to pay is restricted by θ_i and not τ_i. To see this, note that for any $i \in [n]$ and any $j \in [k]$ that $m_{ij} - \theta_i b_{ij} \geq 0$ as otherwise, player i would improve their objective by choosing $b_{ij} = 0$. So, in the case where $\theta_i > \tau_i$, we see that $v_i(\boldsymbol{b}) - \theta_i p_i(\boldsymbol{b}) < v_i(\boldsymbol{b}) - \tau_i p_i(\boldsymbol{b})$, rendering the RoS constraint obsolete.

Furthermore, under this definition of POA we prove the following proposition after which we can assume w.l.o.g. that $\tau_i = 1$ for any $i \in [n]$.

Proposition 2.10. Let $A = (\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{B}, \boldsymbol{\tau})$ be a multi-unit auction under any pricing rule, then there exists a valuation profile $\boldsymbol{v}'$ and objective parameter profile $\boldsymbol{\theta}'$ such that for $A' = (\boldsymbol{v}', \boldsymbol{\theta}', \boldsymbol{B}, \boldsymbol{1})$ we have $\text{EQ}(A) = \text{EQ}(A')$ and $\text{EQ-LPOA}(A) = \text{EQ-LPOA}(A')$.

Proof. First, we shall show that w.l.o.g. we can assume that $\tau_i \geq \theta_i$. If $\tau_i < \theta_i$, then by our discussion above, the RoS constraint will be satisfied by the player's individual rationality to have a non-negative objective. Thus, we see, w.r.t. to the player's bid strategy, that $\tau_i < \theta_i$ is equivalent to $\tau_i = \theta_i$. Furthermore, by the term $\frac{1}{\max\{\tau_i, \theta_i\}}$ in the liquid welfare, we see that the contribution to the liquid welfare of player i is unaffected under this transformation. Hence, we can assume that $\tau_i \geq \theta_i$.

Now, let $\chi \in \text{EQ}(A)$. For $i \in [n]$, since $\tau_i \geq \theta_i$, we define $\tau_i' = 1$, $v_i' = \frac{1}{\tau_i} v_i$, and $\theta_i' = \frac{\theta_i}{\tau_i}$. Since $\tau_i \geq \theta_i$, we have $\theta_i' \in [0,1]$ and is thus a well-defined objective parameter. Next, consider the auction $A' = (\boldsymbol{v}', \boldsymbol{\theta}', \boldsymbol{B}, \boldsymbol{1})$. Note that χ remains unchanged under this transformation and thus, the allocation and payments remain unchanged as well. However, since $\chi \in \text{EQ}(A)$, we have that for any $\boldsymbol{b} \in \operatorname{supp} \chi$ that $\tau_i p_i(\boldsymbol{b}) \leq v_i(\boldsymbol{b})$ which is equivalent to $\tau_i' p_i(\boldsymbol{b}) \leq v_i'(\boldsymbol{b})$. Thus, the player in A' all satisfy the RoS constraint. Similarly, since the payments did not change, the budget constraint will also be satisfied. Lastly, let $i \in [n]$ be arbitrary and consider an appropriate deviation b_i' (appropriate w.r.t. the budget and RoS constraint). Now, we see that:

$$o_i'(\chi) = v_i'(\chi) - \theta_i' p_i(\chi) = \frac{1}{\tau_i}(v_i(\chi) - \theta_i p_i(\chi)) \geq \frac{1}{\tau_i}(v_i(b_i', \chi_{-i}) - \theta_i p_i(b_i', \chi_{-i})) = o_i'(b_i', \chi_{-i}).$$

Hence, we see that $\chi \in \mathrm{EQ}(A')$. Similarly, of $\chi \in \mathrm{EQ}(A')$, by an analogous reasoning we obtain that $\chi \in \mathrm{EQ}(A)$ showing that $\mathrm{EQ}(A) = \mathrm{EQ}(A')$.

Lastly, note that for any $\chi \in \mathrm{EQ}(A) = \mathrm{EQ}(A')$ that:

$$\mathrm{EW}(\chi) = \sum_{i\in[n]} \min\left\{\frac{1}{\tau_i} v_i(\chi), B_i\right\} = \sum_{i\in[n]} \min\left\{v_i'(\chi), B_i\right\} = \sum_{i\in[n]} \min\left\{\frac{1}{\tau_i'} v_i'(\chi), B_i\right\} = \mathrm{EW}'(\chi).$$

Thus, we conclude that $\text{EQ-POA}(A) = \text{EQ-POA}(A')$, as

$$\mathrm{OPT}_A = \sup_{\boldsymbol{b}} \sum_{i\in[n]} \min\left\{\frac{1}{\tau_i} v_i(\boldsymbol{b}), B_i\right\} = \sup_{\boldsymbol{b}} \sum_{i\in[n]} \min\{v_i'(\boldsymbol{b}), B_i\} = \mathrm{OPT}_{A'}.$$

□

Henceforth, we can w.l.o.g. set $\boldsymbol{\tau} = \mathbf{1}$ effectively eliminating a parameter in our model. So, instead of referring to a multi-unit auction as $A = (\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{\tau}, \boldsymbol{B})$ we shall now instead refer to a multi-unit auction as $A = (\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{B})$. Furthermore, when $\boldsymbol{B} = \boldsymbol{\infty}$, we write $A = (\boldsymbol{v}, \boldsymbol{\theta})$. It is important to remark that this elimination of the parameter $\boldsymbol{\tau}$ would not work for the price of anarchy defined in terms of the liquid welfare.

Often, we are interested in the liquid price of anarchy of some subclass of the class of all multi-unit auctions. So, with this in mind, let $\mathcal{M}^{\text{price}}$ be the class of all multi-unit auctions under some pricing rule. We shall only choose price = disc (discriminatory pricing rule) or price = uni (uniform pricing rule). Furthermore, let $\mathcal{M}^{\text{price}}_{\boldsymbol{\theta},\boldsymbol{B}}$ be the class of multi-unit auctions under some pricing rule with objective parameter profile $\boldsymbol{\theta}$ and budget constraint profile $\boldsymbol{B}$. Lastly we let $\mathcal{M}^{\text{price}}_{\boldsymbol{\theta}}$ be the budgetless subclass of $\mathcal{M}^{\text{price}}_{\boldsymbol{\theta},\boldsymbol{B}}$.

We finish this section with a remark on the price of anarchy for uniform price multi-unit auctions. We say that a bid b_i satisfies the *no-overbidding* assumption when for all $\ell \in [k]$ we have $\sum_{j=1}^{\ell} b_{ij} \leq v_i(\ell)$. Note that this assumption implies the RoS constraint. Without this additional requirement, the price of anarchy for uniform price multi-unit auctions is unbounded, as the following example shows.

Example 2.11. Consider a single-unit uniform price auction with 2 players where $\theta_1, \theta_2 \in [0, 1]$, $v_1 = n$ and $v_2 = 1$ for some $n \in \mathbb{N}$. Consider $b_1 = 0$ and $b_2 = n + 1$, then we find that $\boldsymbol{b} = (b_1, b_2)$ is a PNE as player 1 cannot possibly obtain the item while satisfying the RoS constraint. Thus, player 1 has no incentive to change their bid. Note that player 2 receives the item at a price of 0, and thus, they have no incentive to deviate either. Thus, we find that POA $\geq n$.

If we enforce the no-overbidding assumption, player 2 cannot bid $n + 1$ rendering the above bid invalid. A bid b_i that satisfies the no-overbidding assumption we call *feasible*. Thus, in future, with regard to uniform price multi-unit auctions, we shall only consider bids that are feasible. When extending bids to mixed bids, we require the no-overbidding assumption to hold pointwise, similarly to the discussion about the RoS and budget constraints.

2.4 Simultaneous first-price auctions

Throughout this thesis, we often compare multi-unit auctions to simultaneous first-price auctions. In this section we briefly introduce them. For a more extensive introduction we refer to [5].

In a simultaneous first-price auction, we have a set of $N = [n]$ players, a set of $M = [m]$ items and each player has valuation $v_i : 2^{[m]} \to \mathbb{R}_{\geq 0}$. There are three types of valuation functions we consider. The following definition covers them.

Definition 2.12. A valuation function is called [6]

- additive if: $v_i(S) = \sum_{j \in S} v_i(j)$.
- submodular if: $v_i(S \cup \{j\}) - v_i(S) \geq v_i(T \cup \{j\}) - v_i(T)$, for all $S \subseteq T \subseteq M$.
- fractionally subadditive, or XOS, if: there exists a class $\mathcal{L} = \{(v_i^\ell(j))_{j \in M} \in \mathbb{R}^m_{\geq 0}\}$ of additive valuations such that for every $S \subseteq M$ we have $v_i(S) = \max_{\ell \in \mathcal{L}} \sum_{j \in S} v_i^\ell(j)$.

Players submit bids per item, i.e., $b_i \in \mathbb{R}^m_{\geq 0}$. Following [6], an objective function, the budget and RoS constraints and equilibrium notions may now be defined similarly as before. Furthermore, the price of anarchy in terms of effective welfare is analogously defined. The important difference with multi-unit auctions is that players now bid per item instead of a decreasing bid vector and that valuations need not be submodular. In [6], they extend the traditional smoothness framework in [24] to incorperate players with different objectives and show that the 2.18 found in [10] extends to coarse correlated equilibria and XOS valuations through this extended framework. In Chapter 4 we cover the smoothness framework and show how this extends to multi-unit auctions. Lastly, for generalized first-price (GFP) auctions a similar extension of traditional smoothness is found. We refer to Appendix B for these results.

Chapter 3

Pure Equilibria in DPAs

In this chapter we prove the existence of pure equilibria for any discriminatory price multi-unit auction under appropriate tie-breaking rules. We also study some of their properties and look at the price of anarchy of this equilibrium concept.

3.1 Existence and properties

The following theorem covers existence of PNEs in discriminatory price multi-unit auctions.

Theorem 3.1. Consider a $(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{B})$ discriminatory price multi-unit auction. Then, there exists a tie-breaking rule inducing a pure Nash equilibrium such that each player bids uniformly.

Before we prove this theorem, we first define the *average budget* of a player. We use this to construct the PNE. For $i \in [n]$, we define the *average budget* $A_i : [k] \to \mathbb{R}^+$ as $A_i(\ell) = \min\left(\frac{B_i}{\ell}, \frac{m_i(\ell)}{\theta_i}, \frac{v_i(\ell)}{\ell}\right)$, where if $\theta_i = 0$ we set $\frac{m_i(\ell)}{\theta_i} = \infty$. Viewing A_i as a k-tuple, we let $\boldsymbol{A}$ be the vector obtained from appending all tuples A_i, $i \in [n]$. Next we order $\boldsymbol{A}$ from highest to lowest to obtain $\tilde{\boldsymbol{A}}$.

Proof. First, we will construct the bidding profile and then prove it is a PNE. For each $i \in [n]$ and $j \in [k]$ we define $b_{ij} = \tilde{\boldsymbol{A}}_k$. Now, for the tie-breaking rule, ties will first be broken as follows. Let T be the set of players who are in a tie and suppose they are already allocated y_i items. Then, the tie will be broken in favour of player $\operatorname{argmax}_{i \in T} A_i(y_i + 1)$. In other words, the player who can "afford" it the most, will win the tie. If there are multiple such players, then ties may be broken according to any rule.

First, notice by construction that, given the bidding profile $\boldsymbol{b}$, both the budget and the RoS constraints are satisfied for each player. Next, we shall prove that $\boldsymbol{b}$ is a Nash equilibrium by showing that for no player $i \in [n]$ there exists a bid that improves their objective.

Let $i \in [n]$ be arbitrary and consider a deviating bid b'_i such that the budget constraint and RoS constraint are still satisfied. Furthermore, we let $x = x_i(b'_i, \boldsymbol{b}_{-i})$.

Notice if $x_i(\boldsymbol{b}) \geq x$ then we must have that $b'_{ij} \geq \tilde{A}_k$ for $1 \leq j \leq x$. So, player i receives at most as many items as before, and the price they pay for each item is at least as high as before. Thus, they cannot possibly increase their objective by deviating to a bid that would net them less items.

So, suppose now that $x > x_i(\boldsymbol{b})$. In this situation we shall prove that player i cannot have increased their objective while still satisfying the budget and RoS constraint by deriving two inequalities that contradict each other.

Notice by construction of $\boldsymbol{b}$ that $A_i(x) \leq \tilde{\boldsymbol{A}}_k$ as otherwise player i, by construction, would have been allocated at least x items under the bidding profile $\boldsymbol{b}$. This gives our first inequality:

$$A_i(x) \leq \tilde{\boldsymbol{A}}_k. \tag{3.1}$$

Now, suppose that b_i' improves player i's objective. Since player i receives more items by deviating, we see that the first x bids of b_i' must be strictly larger than $\tilde{\boldsymbol{A}}_k$. Thus, we see that $p_i(b_i', \boldsymbol{b}_{-i}) > x \cdot \tilde{\boldsymbol{A}}_k$. However, from the RoS constraint and the budget constraint, we find that:

$$\min\left(B_i, v_i(x)\right) = \min(B_i, v_i(b_i', \boldsymbol{b}_{-i})) \geq p_i(b_i', \boldsymbol{b}_{-i}) = \sum_{j=1}^{x} b_{ij}' > x \cdot \tilde{\boldsymbol{A}}_k.$$

Thus, we see that

$$\min\left(\frac{B_i}{x}, \frac{v_i(x)}{x}\right) > \tilde{\boldsymbol{A}}_k.$$

Also, we may assume w.l.o.g. that $m_i(x) - \theta_i b_{ix}' \geq 0$ as otherwise player i would be strictly better off by setting $b_{ix}' = \tilde{\boldsymbol{A}}_k$ and not attaining the x-th item. Using this, and the fact that $b_{ix}' > \tilde{\boldsymbol{A}}_k$, we see that obtain that $m_i(x) - \theta_i \tilde{\boldsymbol{A}}_k > 0$. Hence, we get that $\frac{m_i(x)}{\theta_i} > \tilde{\boldsymbol{A}}_k$ and combining this with the inequality above, we obtain that:

$$A_i(x) = \min\left(\frac{B_i}{x}, \frac{m_i(x)}{\theta_i}, \frac{v_i(x)}{x}\right) > \tilde{\boldsymbol{A}}_k. \tag{3.2}$$

Notice that the inequality in (3.1) is in contradiction with the inequality in (3.2). Hence, player i cannot increase their objective by getting more items while still satisfying the budget and RoS constraint. Hence, we conclude that $\boldsymbol{b}$ is indeed a PNE, as desired. □

An important observation about the construction in the previous proof is: if $\tilde{\boldsymbol{A}}_k > \tilde{\boldsymbol{A}}_{k+1}$, then there is only one allocation possible such that the RoS and budget constraints are satisfied. The tie-breaking rule in the proof gives exactly that allocation. Note that the proof of Theorem 3.1 works for any strategy profile $b_{ij} = b$ where $\tilde{\boldsymbol{A}}_k \geq b > \tilde{\boldsymbol{A}}_{k+1}$.

Our next goal will be to derive a useful inequality that will be helpful in determining the POA later. First, we cover a proposition about the allocations of PNEs when no value maximizers are present.

Proposition 3.2. Consider a $A = (\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{B})$ discriminatory price multi-unit auction with no value maximizer under the same tie-breaking rule as in Theorem 3.1 and let $\boldsymbol{b}^*$ be the PNE constructed in Theorem 3.1. Then for any $\boldsymbol{b} \in \mathrm{PNE}(A)$ we have that $x(\boldsymbol{b}) = x(\boldsymbol{b}^*)$. Thus, all PNEs have equal allocations.

Proof. Let $\boldsymbol{b}^*$ be the Nash equilibrium constructed in Theorem 3.1 and let $\boldsymbol{b}$ be any other Nash equilibrium. Let $\tilde{\boldsymbol{b}}^*$ be $\boldsymbol{b}^*$ with the bids ranked in decreasing order and similarly define $\tilde{\boldsymbol{b}}$. Note, since there are no value maximizers, all the winning bids of $\boldsymbol{b}$ are equal. In other words, there exists a $b \in \mathbb{R}_{\geq 0}$ such that $\tilde{\boldsymbol{b}}_i = b$ for any $1 \leq i \leq k$. By construction of $\boldsymbol{b}^*$, we must have that $\tilde{\boldsymbol{b}}_k^* \geq \tilde{\boldsymbol{b}}_k$. Without loss of generality, suppose that the inequality is strict and that the allocations are unequal. Then there exists a player i such that $y_1 := x_i(\boldsymbol{b}^*) > x_i(\boldsymbol{b}) := y_2$. Note that for this player $A_i(y_1) \geq \tilde{\boldsymbol{b}}_k^* > \tilde{\boldsymbol{b}}_k = b$ and $b_i(j) \geq \tilde{\boldsymbol{b}}_k = b$ for $1 \leq j \leq y_2$ (since otherwise they could not get y_2 number of units). Now, let $\delta > 0$ and consider the deviation $b_i' = b + \delta$ for the first $y_2 + 1$ entries and 0 for the other. By construction we find that in the profile $(b_i', \boldsymbol{b}_{-i})$ the number of units allocated to player i is $y_2 + 1$. However, since $A_i(y_1) > b$ we find that:

$$\min\left(\frac{B_i}{y_1}, \frac{m_i(y_1)}{\theta_i}, \frac{v_i(y_1)}{y_1}\right) > b + \delta,$$

for $\delta > 0$ small enough. Thus, we find $B_i > y_1(b+\delta) \geq (y_2+1)(b+\delta)$, meaning that the budget constraint is satisfied. Similarly, the RoS constraint will be satisfied. Now, note that since valuation functions are submodular, $m_i(t) > \theta_i(b+\delta)$ for $1 \leq t \leq y_1$. Hence, since $y_2+1 \leq y_1$, we obtain that:

$$\begin{aligned} o_i(b'_i, \boldsymbol{b}_{-i}) &= \sum_{j=1}^{y_2+1} m_{ij} - \theta_i(b+\delta) \\ &= m_i(y_2+1) - \theta_i b - (y_2+1)\delta + \sum_{j=1}^{y_2} m_{ij} - \theta_i b \\ &= o_i(\boldsymbol{b}) + m_i(y_2+1) - b - (y_2+1)\delta. \end{aligned}$$

Hence, since $m_i(y_2+1) > \theta b$, we again find that for δ small enough that the objective improves. In other words, $\boldsymbol{b}$ cannot be an equilibrium. Hence, by the construction of $\boldsymbol{b}^*$ we see that $\tilde{\boldsymbol{b}}_k = \tilde{\boldsymbol{b}}_k^*$, showing that the allocation of $\boldsymbol{b}$ is exactly the same as that of $\boldsymbol{b}^*$. □

The above proposition will be very useful in determining the price of anarchy, as we will see in the next section. Another useful property that results from the above is that, in discriminatory price multi-unit auctions without value maximizers, for all $i \in [n]$, we have $o_i(\boldsymbol{b}) \geq o_i(\boldsymbol{b}^*)$, if $\boldsymbol{b}^*$ is the PNE from Theorem 3.1. To see this, note that the allocation does not change, but the bids of the the players that are not value maximizers can only have increased. This property will be used in the proof of the following lemma.

Lemma 3.3. Consider a discriminatory price multi-unit auction without value maximizers. Let $\boldsymbol{b}$ be a PNE and $\boldsymbol{b}^*$ be the corresponding PNE from Theorem 3.1. Then, for any player we have:

$$o_i(\boldsymbol{b}) \geq \min\{v_i(x_i^*), B_i\} - \sum_{j=1}^{x_i^*} \beta_j(\boldsymbol{b}_{-i}^*),$$

where x_i^* is the number of units allocated to player i under the optimal allocation.

Proof. If $x_i^* = 0$, then both inequalities clearly hold as $o_i(\boldsymbol{b}) \geq 0$.

So, suppose that $x_i^* \geq 1$. Furthermore, we have that $p^* := \beta_1(\boldsymbol{b}^*) = \tilde{\boldsymbol{A}}_k$. Let $T \in [k]$ be maximal such that $v_i(T) > Tp^*$ and let $\ell = \min(T, x_i^*)$. Firstly, if such T does not exist, then the inequality follows trivially as $o_i(\boldsymbol{b}) \geq 0$. So, suppose that $\ell \geq 1$ and consider the bid $b_{ij} = p^* + \delta$ for $1 \leq j \leq \ell$ and 0 otherwise. If no $\delta > 0$ exists such that $(b'_i, b_{-i}) \in \mathcal{B}$, then we note that $\ell p^* \geq \min\{v_i(\ell), B_i\}$. By construction of T, we see that it is the budget constraint that would not be satisfied. So, similarly $x_i^* p^* \geq \min\{v_i(x_i^*), B_i\}$ as $x_i^* \geq \ell$. From this, the desired inequality follows as $\beta_1(\boldsymbol{b}_{-i}^*) = \cdots = \beta_k(\boldsymbol{b}_{-i})$ and $o_i(\boldsymbol{b}) \geq 0$. So, we suppose there exists $\delta > 0$ such that $(b'_i, \boldsymbol{b}_{-i}) \in \mathcal{B}$. Then, we have:

$$o_i(\boldsymbol{b}) \geq o_i(\boldsymbol{b}^*) \geq o_i(b'_i, \boldsymbol{b}_{-i}) = v_i(\ell) - \theta_i \ell (p^* + \delta).$$

For $j > \ell$, we find that $m_{ij} \leq p^*$. Using this, we see that:

$$o_i(\boldsymbol{b}) \geq v_i(x_i^*) - \theta_i \ell (p^* + \delta) - p^*(x_i^* - \ell) \geq v_i(x_i^*) - p^* x_i^* - \theta_i \ell \delta.$$

Note, that we can now let $\delta \to 0$ and use that $v_i(x_i^*) \geq \min\{v_i(x_i^*), B_i\}$ to find that

$$o_i(\boldsymbol{b}) \geq \min\{v_i(x_i^*), B_i\} - p^* x_i^*,$$

which is the desired inequality. □

3.2 Effective welfare POA

In this section, we shall prove that the price of anarchy of discriminatory price multi-unit auctions with no value maximizers. under discriminatory pricing is 2. Let us derive an upper bound on the price of anarchy.

Lemma 3.4. The price of anarchy of discriminatory price multi-unit auctions with budgets and no value maximizers is bounded above by 2.

Proof. Fix an instance $A = (\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{B})$ of a discriminatory price multi-unit auction and et $\boldsymbol{b}$ be a PNE. Furthermore, we define $R = \{i \in [n] \mid v_i(\boldsymbol{b}) \leq B_i\}$ and x_i^* be the number of units allocated to player i in the optimal allocation. In that case, we see:

$$\mathrm{EW}(\boldsymbol{b}) = \sum_{i \in [n]} \min\{v_i(\boldsymbol{b}), B_i\} = \sum_{i \in R} \min\{v_i(\boldsymbol{b}), B_i\} + \sum_{i \notin R} \min\{v_i(\boldsymbol{b}), B_i\}.$$

Note that for the terms in the second sum that:

$$\min\{v_i(\boldsymbol{b}), B_i\} = B_i \geq \min\{v_i(x_i^*), B_i\}.$$

So, we turn our attention to the terms in the first sum. Let $\boldsymbol{b}^*$ be the PNE from Theorem 3.1 and let p^* be the marginal bid all players bid. We obtain by Lemma 3.3 for $i \in R$:

$$\min\{v_i(\boldsymbol{b}), B_i\} = v_i(\boldsymbol{b}) \geq o_i(\boldsymbol{b}) \geq \min\{v_i(x_i^*), B_i\} - \sum_{j=1}^{x_i^*} \beta_j(\boldsymbol{b}_{-i}^*)$$

Now, summing over all players yields:

$$\begin{aligned}
\sum_{i \in [n]} \min\{v_i(\boldsymbol{b}), B_i\} &= \sum_{i \in R} \min\{v_i(\boldsymbol{b}), B_i\} + \sum_{i \notin R} \min\{v_i(\boldsymbol{b}), B_i\} \\
&\geq \mathrm{OPT} - \sum_{i \in R} \sum_{j=1}^{x_i^*} \beta_j(\boldsymbol{b}_{-i}^*) \\
&\geq \mathrm{OPT} - \sum_{i \in [n]} \sum_{j=1}^{x_i^*} \beta_j(\boldsymbol{b}^*) \\
&\geq \mathrm{OPT} - \mathrm{EW}(\boldsymbol{b}^*) \\
&= \mathrm{OPT} - \mathrm{EW}(\boldsymbol{b}).
\end{aligned}$$

Here, the last inequality follows from the fact that $\sum_{i \in [n]} \sum_{j=1}^{x_i^*} \beta_j(\boldsymbol{b})$ equals the sum of all payments, which is bounded by the effective welfare due to the budget and RoS constraint. The last equality follows since the allocations of $\boldsymbol{b}^*$ and $\boldsymbol{b}$ are equal by Proposition 3.2. Putting it all together, we see that $2 \geq \mathrm{OPT} / \mathrm{EW}(\boldsymbol{b})$ for any PNE $\boldsymbol{b}$. Hence, we conclude that:

$$\mathrm{POA}(A) = \sup_{\boldsymbol{b} \in \mathrm{PNE}(A)} \frac{\mathrm{OPT}}{\mathrm{EW}(\boldsymbol{b})} \leq 2.$$

□

We now show this upper bound is tight.

Lemma 3.5. The price of anarchy of discriminatory price multi-unit auctions with budgets is bounded below by 2.

Proof. Consider a discriminatory price multi-unit auction with $k+1$ units, 2 players with valuations:

$$m_1 = (k+1, 1, \dots, 1), \qquad m_2 = (1, \dots, 1),$$

with budgets $B_1 = k+1$ and $B_2 = k$. Furthermore, suppose that $\boldsymbol{\theta}_1 = 0$ and $\boldsymbol{\theta}_2 > 0$. Then, the tie-breaking rule from Theorem 3.1 states that all ties will be broken in favor of player 1. So, we see that: $b_1 = (1, .., 1)$ and $b_2 = (1, \dots, 1)$ will yield a PNE as player 1 will receive all items. For this PNE we have $\mathrm{EW}(\boldsymbol{b}) = k+1$. However, in the optimal allocation, we find that $\mathrm{OPT} = 2k+1$ showing that $\mathrm{POA} \geq \frac{2k+1}{k+1}$, yielding 2 in the limit. □

Note that the PNE in the above auction would still be a PNE for any objective parameter profile $\boldsymbol{\theta}$, showing that $\mathrm{POA}(\mathcal{M}_{\boldsymbol{\theta}}^{\mathrm{disc}}) \geq 2$, as well. Combining the previous two lemma's, gives the following theorem.

Theorem 3.6. The price of anarchy for the class of discriminatory multi-unit auctions with budgets and no value maximizers is 2.

For discriminatory multi-unit auctions where value maximizers are present, we find lower bounds, that do not need budgets, in the next section. In Chapter 5 we show that these lower bounds in the presence of value maximizers are indeed tight for pure, mixed, correlated and coarse correlated equilibria.

3.3 Social welfare POA

In this section we study the price of anarchy of pure equilibria in budgetless auction. We had already seen that auctions with value maximizers have a price of anarchy of 2. Thus, we see that the price of anarchy over the entire class of budgetless discriminatory price multi-unit auctions is 2. However, we will prove that $\mathrm{POA}(\mathcal{M}_{\boldsymbol{\theta}}^{\mathrm{disc}}) = 2 - \theta_{\min}$ when $\theta_{\min} > 0$. To this end, we give a lower bound.

Lemma 3.7. Consider a budgetless discriminatory price multi-unit auction $A = (\boldsymbol{v}, \boldsymbol{\theta})$ and let $\theta_{\min} = \min_{i \in [n]} \theta_i$. Then, the price of anarchy satisfies $\mathrm{POA}(\mathcal{M}_{\boldsymbol{\theta}}^{\mathrm{disc}}) \geq 2 - \theta_{\min}$.

Proof. Consider a discriminatory price multi-unit auction with two players, $k \in \mathbb{Z}^+$ items. Furthermore, without loss of generality, assume that $\theta_1 = \theta_{\min}$, consider the marginal valuation profiles $m_1 = (k - (k-1)\theta_{\min}, \theta_{\min}, \dots, \theta_{\min})$, $m_2 = (1, \dots, 1)$, and suppose that ties are broken in favor of player 1. In this case, the optimal allocation would yield a social welfare of $k + (1 - \theta_{\min})(k-1)$. However, note that $\boldsymbol{b} = (b_1, b_2)$ is a PNE when: $b_1 = (1, \dots, 1)$, $b_2 = (1, \dots, 1)$, and so, $SW(\boldsymbol{b}) = k$. Hence, we find that:

$$\mathrm{POA}(\mathcal{M}_{\boldsymbol{\theta}}^{\mathrm{disc}}) \geq \frac{k + (k-1)(1 - \theta_{\min})}{k},$$

for each $k \in \mathbb{Z}^+$. In the limit, we obtain the desired result. □

Now, we prove the main result of this section.

Theorem 3.8. For pure equilibria we have $\mathrm{POA}(\mathcal{M}_{\boldsymbol{\theta}}^{\mathrm{disc}}) = 2 - \theta_{\min}$, when $\theta_{\min} > 0$.

Proof. Let $A = (\boldsymbol{v}, \boldsymbol{\theta})$ be a discriminatory price multi-unit auction in $\mathcal{M}^{\text{disc}}_{\boldsymbol{\theta}}$ and $\boldsymbol{b} \in \text{PNE}(A)$ be arbitrary. By Lemma 3.3, we obtain for each player i a deviation b'_i such that:

$$o_i(\boldsymbol{b}) \geq o_i(b'_i, \boldsymbol{b}_{-i}) \geq v_i(x^*_i) - \sum_{j=1}^{x^*_i} \beta_j(\boldsymbol{b}^*_{-i}).$$

Summing over each player then yields:

$$\begin{aligned}
\text{SW}(\boldsymbol{b}) - \theta_{\min} \sum_{i\in[n]} p_i(\boldsymbol{b}) &\geq \sum_{i\in[n]} o_i(\boldsymbol{b}) \\
&\geq \text{OPT} - \sum_{i\in[n]} \sum_{j=1}^{x^*_i} \beta_j(\boldsymbol{b}^*_{-i}) \\
&= \text{OPT} - \sum_{i\in[n]} \sum_{j=1}^{x^*_i} \beta_j(\boldsymbol{b}^*) \\
&\geq \text{OPT} - \sum_{j=1}^{k} \beta_j(\boldsymbol{b}^*) \\
&= \text{OPT} - \sum_{i\in[n]} p_i(\boldsymbol{b}^*).
\end{aligned}$$

Note by Proposition 3.2 that the allocations under the profile $\boldsymbol{b}$ are the same as under the profile $\boldsymbol{b}^*$. This means that $\text{SW}(\boldsymbol{b}) = \text{SW}(\boldsymbol{b}^*)$. By the RoS constraint, we have that the price for each player is less than they value and thus we obtain:

$$\text{SW}(\boldsymbol{b}) - \theta_{\min} \text{SW}(\boldsymbol{b}) \geq \text{OPT} - \text{SW}(\boldsymbol{b}).$$

Since $\boldsymbol{b} \in \text{PNE}(A)$ was arbitrary, we find from the above that $\text{POA}(A) \leq 2 - \theta_{\min}$, giving the result in combination with the lower bound from Lemma 3.7. $\square$

Now, we include some remarks when budgetless discriminatory price multi-unit auctions have at least one value maximizer. To simplify our analysis, we shall analyze the case where there are only two types of players. One type is value maximizers and another type has an objective parameter $\theta \in (0, 1]$. We call such a discriminatory price multi-unit auction a *bipartite* auction. The following theorem states the price of anarchy of such class of auctions in terms of the Lambert W function. We refer to the Appendix A for some background and to [8] for an extensive treatment.

Theorem 3.9. The price of anarchy of a budgetless bipartite discriminatory price multi-unit auctions with value maximizers and θ-type players is given by:

$$\text{POA} = \begin{cases} 1 + \theta\left(1 + W(-e^{-1-\theta})\right)^{-1}, & \text{if } \theta > 1 + \frac{W(-2e^{-2})}{2} \\ 2, & \text{else.} \end{cases}$$

We will not prove this theorem now and we shall only prove the lower bound. The upper bound turns out to hold for even the coarse correlated price of anarchy as we will see in Chapter 5. This means that for this class of auctions, EQ-POA will be given by the above function. In the following lemma we prove the lower bound result using a pure equilibrium. We modify the game in [3] which, interestingly, yields the lower bound for uniform price auctions with utility maximizers. The connection between uniform price auctions with uniform objective parameter and bipartite discriminatory multi-unit auctions will be examined more closely in Chapter 6.

Lemma 3.10. The price of anarchy of budgetless discriminatory price multi-unit auctions where each player is either a value maximizer or has objective parameter $\theta \in (0, 1]$ satisfies:

$$\text{POA} \geq \begin{cases} 1 + \theta \left(1 + W(-e^{-1-\theta})\right)^{-1}, & \text{if } \theta > 1 + \frac{W(-2e^{-2})}{2} \\ 2, & \text{else.} \end{cases}$$

Proof. Let player 1 be a value maximizer and player 2 have objective parameter $\theta \in (0, 1]$. Note, that Lemma 3.7 yields a lower bound of at least 2 and covers the case where $0 < \theta \leq 1 + \frac{W(-e^{-2})}{2}$.

So, now suppose that $\theta > 1 + \frac{W(-2e^{-2})}{2}$. Consider the following auction with 2 players and k items. Player 1 is a value maximizer and player 2 has objective parameter $\theta > 1 + \frac{W(-2e^{-2})}{2}$. Let $x \in \{1, 2, \ldots, k-2\}$ and define $m_{11} = \frac{1}{\theta}\left(\frac{k-1-x}{k-1} + \sum_{i=1}^{k-1-x} \frac{i}{x+i}\right)$ and $m_{1j} = 0$ for $j \geq 2$. For player 2, we define the marginal valuation: $m_{2j} = 1$ for $1 \leq j \leq k-1$ and $m_{2k} = 0$. Now, suppose that ties are broken in favor of player 2 and consider the bids:

$$b_{1j} = \begin{cases} \frac{1}{\theta}\left(1 - \frac{x}{k-1}\right), & \text{if } j = 1 \\ \frac{1}{\theta}\left(1 - \frac{x}{k-j+1}\right), & \text{if } 2 \leq j \leq k-x \\ 0, & \text{otherwise,} \end{cases}$$

and $b_2 = (0, \ldots, 0)$. Note that the RoS constraint is satisfied by both players. Since the ties are broken in favor of player 2, we see that player 2 will be allocated x units and player 1 receives $k - x$ units.

Let us show that $\boldsymbol{b}$ is an equilibrium for this instance. As player 1 is a value maximizer who only values the first item, we see that this is an equilibrium from their perspective. For player 2, their current objective is x since they receive x units at a price of 0. If they wish to receive $y > x$ units with $k > y$ (as $m_{2k} = 0$), then they would have to submit y bids of at least $\frac{1}{\theta}\left(1 - \frac{x}{y}\right)$ and since the ties are broken in their favor, they would uniformly bid this value y times to pay the smallest possible price to receive y number of units. Let b_2' be this deviating bid and we find for the utility that:

$$o_2(b_1, b_2') = y - \theta \cdot \frac{1}{\theta}\left(1 - \frac{x}{y}\right) y = x = o_2(b_1, b_2).$$

Hence, we find that player 2 cannot possibly improve their objective. By construction, both players satisfy the RoS constraint, and we conclude that this profile is a PNE.

Now, we compute the POA of this profile. The optimal allocation is $x_1 = 1$ and $x_2 = k - 1$, giving a social welfare of $m_{11} + k - 1$. However, the social welfare of the equilibrium constructed above is $m_{11} + x$. Hence, we find that the POA for the class of bipartite auctions with one value maximizer satisfies:

$$\begin{aligned} \text{POA} &\geq \frac{\frac{1}{\theta}\left(\frac{k-1-x}{k-1} + \sum_{i=1}^{k-1-x} \frac{i}{x+i}\right) + k - 1}{\frac{1}{\theta}\left(\frac{k-1-x}{k-1} + \sum_{i=1}^{k-1-x} \frac{i}{x+i}\right) + x} \\ &= 1 + \frac{k-1-x}{\frac{1}{\theta}\left(\frac{k-1-x}{k-1} + \sum_{i=1}^{k-1-x} \frac{i}{x+i}\right) + x} \\ &= 1 + \theta\left(\frac{k-1-x}{\frac{k-1-x}{k-1} + \sum_{i=1}^{k-1-x} \frac{i}{x+i} + \theta x}\right). \end{aligned}$$

Now, we focus on the term $\sum_{i=1}^{k-1-x} \frac{i}{x+i}$. We find that:

$$\sum_{i=1}^{k-1-x} \frac{i}{x+i} = \sum_{i=1}^{k-1-x} \frac{x+i-x}{x+i} = \sum_{i=1}^{k-1-x} \left(1 - \frac{x}{x+i}\right) = k-1-x+x\sum_{i=x+1}^{k-1} \frac{1}{i},$$

where in the last stap we split the sum and relabelled the index i to $x+i$. Continuing from this, we obtain:

$$\begin{aligned} \text{POA} &\geq 1+\theta\left(\frac{k-1-x}{\frac{k-1-x}{k-1}+k-1+(\theta-1)x-x\sum_{i=x+1}^{k-1}\frac{1}{i}}\right) \\ &\geq 1+\theta\left(\frac{k-1-x}{\frac{k-1-x}{k-1}+k-1+(\theta-1)x-x\int_{x+1}^{k}\frac{1}{y}dy}\right) \\ &= 1+\theta\left(\frac{k-1-x}{\frac{k-1-x}{k-1}+k-1+x\left(\theta-1-\ln\left(\frac{k}{x+1}\right)\right)}\right). \end{aligned}$$

Now, we choose to set $x = \lfloor -W(-e^{-1-\theta})(k-1)\rfloor$. To make sure this is well defined, we require that $x \geq 1$. This holds whenever $-W(-e^{-1-\theta})(k-1)-1>0$ for any $\theta > 1+\frac{W(-2e^{-2})}{2}$. However, $W(-e^{-1-\theta})$ is a decreasing function in θ, so this is minimal when $\theta = 1$. Thus, it suffices to check when $-W(-e^{-2})(k-1)-1>0$, which we know from [3], holds for $k \geq 8$. With this in mind, we continue by plugging in for x:

$$\begin{aligned} \text{POA} &\geq 1+\frac{\theta\left(k-1-\lfloor -W(-e^{-1-\theta})(k-1)\rfloor\right)}{\frac{k-1-\lfloor -W(-e^{-1-\theta})(k-1)\rfloor}{k-1}+k-1+\lfloor -W(-e^{-1-\theta})(k-1)\rfloor\left(\theta-1-\ln\left(\frac{k}{\lfloor -W(-e^{-1-\theta})(k-1)\rfloor+1}\right)\right)} \\ &\geq 1+\frac{\theta\left(k-1-\left(-W(-e^{-1-\theta})(k-1)\right)\right)}{\frac{k-(-W(-e^{-1-\theta})(k-1))}{k-1}+k-1+\left(-W(-e^{-1-\theta})(k-1)-1\right)\left(\theta-1-\ln\left(\frac{k}{(-W(-e^{-1-\theta})(k-1))+1}\right)\right)} \\ &\geq 1+\frac{\theta\left(\left(1-\frac{1}{k}\right)+W(-e^{-1-\theta})\left(1-\frac{1}{k}\right)\right)}{\frac{k+W(-e^{-1-\theta})(k-1)}{k(k-1)}+\left(1-\frac{1}{k}\right)+\left(-W(-e^{-1-\theta})\left(1-\frac{1}{k}\right)-\frac{1}{k}\right)\left(\theta-1-\ln\left(\frac{k}{(-W(-e^{-1-\theta})(k-1))+1}\right)\right)}. \end{aligned}$$

Taking the limit $k \to \infty$ greatly simplifies our expression and yields:

$$\text{POA} \geq 1+\theta\left(\frac{1+W(-e^{-1-\theta})}{1-W(-e^{-1-\theta})\left(\theta-1+\ln\left(-W(-e^{-1-\theta})\right)\right)}\right).$$

Using that for any $z>0$ that $\ln(-W(-e^{-1-z})) = -1-z-W(-e^{-1-z})$ (see Appendix A), we obtain:

$$\begin{aligned} \text{POA} &\geq 1+\theta\left(\frac{1+W(-e^{-1-\theta})}{1-W(-e^{-1-\theta})\left(-2-W(-e^{-1-\theta})\right)}\right) \\ &= 1+\theta\left(\frac{1+W(-e^{-1-\theta})}{\left(1+W(-e^{-1-\theta})\right)^2}\right) \\ &= 1+\theta\left(\frac{1}{1+W(-e^{-1-\theta})}\right), \end{aligned}$$

as desired. □

For $\theta = 1$, the lower bound yields $1+\frac{1}{1+W(-e^{-2})} \approx 2.18$, the same result found in [3] for uniform price multi-unit auctions with only utility maximizers. Furthermore, for $\theta = 1$, the equilibrium above is the exact same as the equilibrium used in [3] to yield the same lower bound but for utility maximizing uniform price multi-unit auctions.

Chapter 4

Smoothness

4.1 Smoothness

In this section, we dive into an important technique for determining upper bounds on the price of anarchy for coarse correlated equilibria. With the *smoothness* technique introduced in [22], combined with the extension theorem we are able to achieve upper bounds on the CCE-POA for budgetless auctions, i.e., auctions where $\boldsymbol{B} = \boldsymbol{\infty}$. In the sections after, we develop a framework that allows us to analyze the efficiency of equilibria in auctions with budgets.

> **Definition 4.1** (rf. De Keijzer et al. [9]). Consider a $(\boldsymbol{v}, \boldsymbol{\theta})$ discriminatory price multi-unit auction and let x_i^* be the number of units allocated to player i under the optimal allocation. More precisely, $x_i^* = x_i(\text{argmax}_{\boldsymbol{b} \in \mathcal{B}} \text{POA}(\boldsymbol{b}))$. Furthermore, let $\beta_j(\boldsymbol{b})$ refer to the j-th lowest winning bid under the profile $\boldsymbol{b}$. We call this auction (λ, μ)-smooth for $\lambda > 0$ and $\mu \geq 0$ if for any player $i \in [n]$ and any strategy profile $\boldsymbol{b}_{-i} \in \mathcal{B}_{-i}$ there exists a mixed deviating bid χ_i', not dependent on $\boldsymbol{b}_{-i}$, such that:
>
> $$\mathop{\mathbb{E}}_{b_i' \sim \chi_i'} \left[o_i(b_i', \boldsymbol{b}_{-i}) \right] \geq \lambda v_i(x_i^*) - \mu \sum_{j=1}^{x_i^*} \beta_j(\boldsymbol{b}_{-i}),$$
>
> where we require $\chi_i' \in \mathcal{S}_i(\boldsymbol{b}_{-i})$.

The POA for smooth budgetless auctions can be upper bounded through the following extension theorem.

> **Theorem 4.2** (rf. De Keijzer et al. [9]). Consider a discriminatory price multi-unit auction $(\boldsymbol{v}, \boldsymbol{\theta})$ that is (λ, μ) smooth. Then the coarse correlated price of anarchy is at most $\max\{1, 1 + \mu - \theta_{\min}\}/\lambda$.

Proof. Suppose that χ is a CCE of our discriminatory price multi-unit auction. Let χ_i' be the deviating bid from the smoothness definition. Then for any player and any such deviating mixed bid χ_i', we have that

$$\mathop{\mathbb{E}}_{\boldsymbol{b} \sim \chi} [o_i(\boldsymbol{b})] \geq \mathop{\mathbb{E}}_{\substack{\boldsymbol{b} \sim \chi \\ b_i' \sim \chi_i'}} [o_i(b_i', \boldsymbol{b}_{-i})] = \mathop{\mathbb{E}}_{\boldsymbol{b} \sim \chi} \left[\mathop{\mathbb{E}}_{b_i' \sim \chi_i'} \left[o_i(b_i', \boldsymbol{b}_{-i}) \right] \right] \geq \lambda \mathop{\mathbb{E}}_{\boldsymbol{b} \sim \chi} [v_i(x_i^*)] - \mu \sum_{j=1}^{x_i^*} \mathop{\mathbb{E}}_{\boldsymbol{b} \sim \chi} [\beta_j(\boldsymbol{b}_{-i})].$$

Using that $v_i(x_i^*)$ is constant and summing over every player gives:

$$\sum_{i=1}^{n} \mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}[o_i(\boldsymbol{b})] \geq \lambda \,\mathrm{OPT} - \mu \sum_{i=1}^{n}\sum_{j=1}^{x_i^*} \mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}[\beta_j(\boldsymbol{b}_{-i})].$$

Next, since $\sum_{i=1}^{n} x_i^* = k$, we see that in the sum $\sum_{i=1}^{n}\sum_{j=1}^{x_i^*}$ appear k terms. Using that $\beta_j(\boldsymbol{b})$ increasing in j and that the expectation is linear, we obtain:

$$\sum_{i=1}^{n} \mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}[v_i(\boldsymbol{b})] - \sum_{i=1}^{n} \theta_i \mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}[p_i(\boldsymbol{b})] \geq \lambda \,\mathrm{OPT} - \mu \mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}\left[\sum_{j=1}^{k} \beta_j(\boldsymbol{b}_{-i})\right].$$

Here, note that $\sum_{j=1}^{k} \beta_j(\boldsymbol{b})$ is the total payment made of all players combined. Using this, and rearranging some terms now yields:

$$\mathrm{SW}(\chi) - (\theta_{\min} - \mu)\sum_{i=1}^{n} \mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}[p_i(\boldsymbol{b})] \geq \sum_{i=1}^{n} \mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}[v_i(\boldsymbol{b})] - \sum_{i=1}^{n}(\theta_i - \mu) \mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}[p_i(\boldsymbol{b})] \geq \lambda \,\mathrm{OPT}.$$

Now, there are two cases to consider. If $\mu \leq \theta_{\min}$, then we have that $\mathrm{SW}(\chi) \geq \lambda \,\mathrm{OPT}$ and the result follows. When $\mu > \theta_{\min}$ we get that $\mu - \theta_{\min} > 0$. By the RoS constraint, have that $\mathbb{E}_{\boldsymbol{b}\sim\chi}[p_i(\boldsymbol{b})] \leq \mathbb{E}_{\boldsymbol{b}\sim\chi}[v_i(\boldsymbol{b})]$. Using this then gives:

$$\mathrm{SW}(\chi) + (\mu - \theta_{\min})\,\mathrm{SW}(\chi) \geq \lambda \,\mathrm{OPT},$$

from which we obtain the result as well. □

To illustrate the strength of smoothness, we shall make a quick detour to value maximizing auctions. In other words, we limit ourselves to the class $\mathcal{M}_{\mathbf{0}}^{\mathrm{disc}}$. The following lemma finds gives a lower bound.

Lemma 4.3. The price of anarchy of budgetless discriminatory multi-unit auctions with only value maximizers is lower bounded by 2.

Proof. Consider the following auction instance with 2 players and $k+1$ units for $k \in \mathbb{N}$. $v_1 = (k+1, 0, 0, \ldots, 0)$ and $v_2 = (1, \ldots, 1)$, and let ties be broken in favor of player 1. Now, consider the bids $b_1 = (1, \ldots, 1)$ and $b_2(1, \ldots, 1)$ and note that this is a PNE. Under this equilibrium, we see that the social welfare is $\mathrm{SW}(\boldsymbol{b}) = k+1$, however, the optimal social welfare is given by $k+1+k$. This corresponds to the allocation where player 1 recieves 1 unit and player 2 receives the rest. Thus, we conclude that

$$\mathrm{POA}(\mathcal{M}_{\mathbf{0}}^{\mathrm{disc}}) \geq \frac{2k+1}{k+1},$$

for any $k \in \mathbb{N}$. Taking the limit then yields the result. □

Now, we shall prove that value maximizers are $(1,1)$ smooth.

Lemma 4.4. A budgetless discriminatory price multi-unit auction with only value maximizers is $(1,1)$-smooth

Proof. Note, if $x_i^* = 0$, then the smoothness inequality holds trivially. So, suppose that $x_i^* \geq 1$. Let $\boldsymbol{b}$ be any bidding profile and consider the deviating bid $b_i' = \frac{1}{x_i^*} v_i(x_i^*)$ for the first x_i^*-units and 0 everywhere else. Note that this deviating bid satisfies the RoS constraint by construction. Let $x = x_i(b_i', \boldsymbol{b}) \leq x_i^*$, then we find that $\frac{1}{x_i^*} v_i(x_i^*) \geq \beta_j(\boldsymbol{b}_{-i})$ for $1 \leq j \leq x$. Similarly, since player i is not allocated more items, we must also have that $\frac{1}{x_i^*}v_i(x_i^*) \leq \beta_j(\boldsymbol{b}_{-i})$ for $x+1 \leq j \leq x_i^*$. Thus, we find that:

$$\begin{aligned}0 &\geq \sum_{j=x+1}^{x_i^*} \frac{1}{x_i^*} v_i(x_i^*) - \sum_{j=x+1}^{x_i^*} \beta_j(\boldsymbol{b}_{-i}) \\ &= \sum_{j=1}^{x_i^*} \frac{1}{x_i^*} v_i(x_i^*) - \sum_{j=1}^{x} \frac{1}{x_i^*} v_i(x_i^*) - \sum_{j=x+1}^{x_i^*} \beta_j(\boldsymbol{b}_{-i}).\end{aligned}$$

Note, that the sums in the first inequality might be empty. However, this does not affect the argument. So, rearranging some terms now gives:

$$\frac{x}{x_i^*} v_i(x_i^*) = \sum_{j=1}^{x} \frac{1}{x_i^*} v_i(x_i^*) \geq v_i(x_i^*) - \sum_{j=x+1}^{x_i^*} \beta_j(\boldsymbol{b}_{-i}) \geq v_i(x_i^*) - \sum_{j=1}^{x_i^*} \beta_j(\boldsymbol{b}_{-i}).$$

Now, as we work with submodular valuations we have that $v_i(x) \geq \frac{x}{x_i^*} v_i(x_i^*)$. Hence, we find that:

$$o_i(b_i', \boldsymbol{b}_{-i}) = v_i(x) \geq v_i(x_i^*) - \sum_{j=1}^{x_i^*} \beta_j(\boldsymbol{b}_{-i}),$$

showing that the auction is $(1,1)$ smooth. □

Combining the results with the extension theorem, Theorem 4.2, we find the following theorem.

Theorem 4.5. The coarse correlated price of anarchy of budgetless discriminatory price multi-unit auctions with only value maximizers auctions is 2.

Proof. By the previous lemma we know that value maximizing auctions are $(1,1)$ smooth. By the extension theorem we now see CCE-POA ≤ 2. Combining this with Lemma 4.3 yields tightness of this bound. □

Note that in Lemmas 4.3 and 4.4 we only used pure bidding profiles. Yet, through the extension theorem, we were able to prove exact results for the price of anarchy not just w.r.t. pure equilibria, but even for coarse correlated equilibria.

For utility maximizing auctions, a similar upper bound can be found of $\frac{e}{e-1}$ through smoothness, see [9]. A complementary tightness result for coarse correlated equilibria also exists [23].

4.2 Welfare objective auctions

Before we move on to smoothness that optimizes for different types, we introduce a new type of auction, *welfare objective multi-unit auctions*. Essentially, the auction will be the same as the multi-unit auction before, but with a different objective. These auctions are just a formal construct which will be used to analyze the POA for conventional multi-unit auctions. This will be done by transforming a conventional multi-unit auction into a welfare objective auction where the price of anarchy of the former is upper bounded by the latter.

An instance of this new auction is determined by the tuple $A^w = (\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{B})$, where the superscript w denotes that in this auction, the bidders have an objective that takes into account their contribution to the effective welfare. In this auction, instead of players submitting pure bids and considering mixed extensions, the strategy space is distributions over bids. The objective of each player is then to maximize the objective:

$$o_i^w(\chi) = \min\left\{\mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}[v_i(x_i(\boldsymbol{b}))], B_i\right\} - \theta_i \mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}[p_i(\boldsymbol{b})],$$

where p is a pricing rule that is either discriminatory or uniform. The maximization is with respect to the same pointwise ROI and budget constraint as before. It is important to note that $o_i^w(\chi) \neq \mathbb{E}_{\boldsymbol{b}\sim\chi}\, o_i^w(\boldsymbol{b})$, as the expectation is within the minimum. Thus, o_i^w is a function of distributions rather than pure bid profiles. This means that this auction structure is not a mixed extension.

We further define given v_i the valuation $v_i' : [k] \to \mathbb{R}_{\geq 0}$ via $v_i'(\ell) = \min\{v_i(\ell), B_i\}$ and remark that v_i' is still a submodular function. Given a distribution χ over pure strategy profiles we often write $v_i'(\chi) := \mathbb{E}_{b\sim\chi}[\min\{v_i(x_i(b)), B_i\}]$. Lastly, we define for a distribution χ over pure strategy profiles $v_i^w(\chi) = \min\left\{\mathbb{E}_{\boldsymbol{b}\sim\chi}[v_i(x_i(\boldsymbol{b}))], B_i\right\}$. So, unless the function has a w as superscript, if it takes a distribution as input, e.g. v_i, v_i', o_i, then it is taken as an expectation over that function w.r.t. to the pure bids sampled from the distribution. Only functions with a w as a superscript take distributions literally as input. Next, we turn our attention to equilibrium notions in welfare objective auctions. The following example shows that we must tread carefully here as well with regards to pure equilibria.

Example 4.6. Consider a discriminatory price welfare objective auction with two players whose valuations are $v_1 = (8, 4)$ and $v_2 = (3, 3)$. Suppose furthermore that $\theta_1 = 1$, $\theta_2 = 0$ and $B_1 = 10$, $B_2 = \infty$ and that ties are broken in favor of player 1. Then we note that for the bidding profile $\boldsymbol{b} = (b_1, b_2)$ with $b_1 = (3, 0)$ and $b_2 = (3, 3)$ that no pure deviation can improve the objective o_i^w for any $i \in [2]$. To see this, we note that player 2 is a "value" maximizer that cannot possibly bid more to obtain more items. Player 1 could deviate to $b_1' = (3, 3)$ and obtain both items, however, in that case:

$$o_1^w(b_1, b_2) = \min\{8, 10\} - 3 = 5 > 4 = \min\{12, 10\} - 6 = o_1^w(b_1', b_2).$$

However, consider the mixed deviation χ_1' which is b_1 with probability 0.5 and b_1' with probability 0.5. In that case, $v_1(\chi_1', b_2) = 0.5 \cdot 8 + 0.5 \cdot 12 = 10$ and $p_1(\chi_1', b_2) = 0.5 \cdot 3 + 0.5 \cdot 6 = 4.5$. Thus, we find that:

$$o_1^w(\chi_1', b_2) = \min\{v_1(\chi_1', b_2), 10\} - p_1(\chi_1', b_2) = 10 - 4.5 = 5.5 > 5 = o_1^w(\boldsymbol{b}).$$

The above example shows that $\boldsymbol{b}$ would be a pure equilibrium if we considered only pure deviations, it would not be a mixed equilibrium. Thus, the usual equilibrium hierarchy is broken if we define pure equilibria in terms of pure deviations. So, instead, we define all the notions of equilibria, defined in chapter 2.1, in terms of mixed deviation to ensure that the conventional hierarchy is conserved, meaning that $\boldsymbol{b}$ in the above example is not a pure equilibrium.

Next, we define the POA in welfare objective multi-unit auctions. Given a distribution over pure bids χ, we define social welfare as $\mathrm{SW}^w(\chi) = \sum_{i\in[n]} v_i^w(\chi)$ and $\mathrm{OPT}^w := \sup_{\boldsymbol{b}} \mathrm{SW}^w(\boldsymbol{b})$ where the supremum is taken over pure bid profiles. The corresponding allocation is denoted by $\boldsymbol{x}^*$. The price of anarchy w.r.t to an equilibrium notion EQ $\in$ {PNE, MNE, CE, CCE} is then:

$$\text{EQ-POA}^w(A^w) := \sup_{\chi\in\mathrm{EQ}(A^w)} \frac{\mathrm{OPT}^w}{\mathrm{SW}^w(\chi)}.$$

For uniform pricing we limit ourselves only to bids that do not violate the no-overbidding assumption: for all $\ell \in [k]$, $\sum_{j=1}^{\ell} b_{ij} \leq v_i^w(\ell)$ and for distributions we require it to hold pointwise.

Lastly, it is important to remark that albeit this mechanism seems artificial, the budgetless welfare objective multi-unit auctions coincide with the mixed extension of the conventional budgetless multi-unit auctions in terms of equilibrium notions and price of anarchy. So, while we are unable to carry out smoothness for the standard multi-unit auctions with budget, we will show that smoothness framework does work for welfare objective multi-unit auctions. Hence, any upper bound that we find for the POA of budgetless welfare objective multi-unit auctions will automatically lift to upper bounds for the POA of budgetless multi-unit auctions. Thus, in the next section we cover typed smoothness for welfare objective multi-unit auctions generalizing upon the typed smoothness covered for budgetless simutaneous auctions in [6].

4.3 Typed smoothness

In the previous sections, the smoothness concept is a property of an auction. During the proof of Theorem 4.2 we used the upper bound $\theta_{\min} \leq \theta_i$ for any player $i \in [n]$. So, in an auction with a uniform objective parameter profile, $\boldsymbol{\theta}$, in other words, an auction where $\theta_i = \theta$ for all $i \in [n]$, we expect the bound in Theorem 4.2 to not be bad. However, in auctions with players with distinct objective parameters, this bound might no longer be good.

For example, if we consider an auction with only value and utility maximizers, the previous analysis would yield an upper bound of $\frac{2e}{e-1}$. This is in contrast with the result in [10], where they found for mixed equilibria a tight price of anarchy of 2.18 for simultaneous first-price auctions with only value and utility maximizers. Note that they do make use of the RoS constraint in expectation, instead of pointwise. Using the RoS constraint in expectation, we can find a similar auction in our discriminatory price multi-unit auction setting to give a lower bound of 2.18. However, no lower bound could come close to $\frac{2e}{e-1}$ suggesting that the upper bound is not tight.

So, in this section, we turn our analysis to type-dependent smoothness. Instead of defining smoothness as a property of the whole auction, we let the inequality be type-dependent. So, each type may have different (λ, μ)-parameters. Specifically, these parameters will depend on θ, the objective parameter. So, we let $T = \{\theta \in [0,1] \mid \exists i \in [n] \text{ s.t. } \theta_i = \theta\}$ be the set of types, and $N_\theta = \{i \in [n] : \theta_i = \theta\}$. Note that the collection $(N_\theta)_{\theta \in T}$ partitions the set of players.

Through this novel technique, we can reduce the upper bound of $\frac{2e}{e-1}$ to 2.18, a significant improvement. This technique, developed in [6], was applied to simultaneous first-price auctions and showed that the 2.18 in [10] was tight, not just for mixed Nash equilibria, but also correlated and coarse correlated equilibria.

> **Definition 4.7.** Consider a $A^w = (\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{B})$ discriminatory price welfare objective multi-unit auction. A player $i \in [n]$ with objective parameter $\theta_i = \theta$ is $(\lambda_\theta, \mu_\theta)$ smooth if and only if for any pure $\boldsymbol{b}_{-i}$ there exists a χ_i' where $(\chi_i', \boldsymbol{b}_{-i}) \in \mathcal{S}$, and χ_i' may not depend on $\boldsymbol{b}_{-i}$, such that:
>
> $$o_i^w(\chi', \boldsymbol{b}_{-i}) \geq \lambda_\theta v_i'(x_i^*) - \mu_\theta \sum_{j=1}^{x_i^*} \beta_j(\boldsymbol{b}_{-i}).$$

To combine the different λ and μ smoothness parameters into a more sophisticated bound, we use *calibration vectors* as introduced in [6].

> **Definition 4.8.** The set of calibration vectors of a discriminatory price welfare objective multi-unit auction instance $A^w = (\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{b})$ such that the players of type θ are $(\lambda_\theta, \mu_\theta)$ smooth is given

by:

$$\mathcal{C}(A^w) = \left\{ \boldsymbol{\delta} \in (0,1]^{|T|} \mid \max_{\theta \in T}(\delta_\theta(1-\theta)) + \max_{\theta \in T}(\delta_\theta \mu_\theta) \leq 1 \right\}.$$

Now, we are ready to prove the following extension result that incorporates multiple types.

Theorem 4.9. In an auction instance $A^w = (\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{B})$ where for any $\theta \in T$ and any $i \in N_\theta$ is $(\lambda_\theta, \mu_\theta)$ smooth, we have the following bound for the coarse correlated price of anarchy:

$$\text{CCE-POA}^w(A^w) \leq \min \left\{ \min_{\theta \in T} \lambda_\theta, \left(\max_{\theta \in T} \left(\frac{\mu_\theta}{\lambda_\theta} \right) + \max_{\theta \in T} \left(\frac{1-\theta}{\lambda_\theta} \right) \right)^{-1} \right\}^{-1}.$$

Before we prove this theorem, note that for a single type auction, Theorem 4.9 reduces to Theorem 4.2. The proof we present is adapted from [6].

Proof of Theorem 4.9. Consider a discriminatory price welfare objective multi-unit auction $A^w = (\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{B})$ and let $\chi \in \text{CCE}(A)$. Let $\boldsymbol{\delta} \in \mathcal{C}(A^w)$ and χ'_i be the deviation for player i in the smoothness definition, we compute:

$$\begin{aligned}
\text{SW}^w(\chi) &= \sum_{\theta \in T} \sum_{i \in N_\theta} v_i^w(\chi) \\
&= \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta v_i^w(\chi) + (1-\delta_\theta) v_i^w(\chi) \\
&\geq \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta v_i^w(\chi) + (1-\delta_\theta) p_i(\chi) && \text{RoS and budget constraints} \\
&= \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta o_i^w(\chi) + (1-\delta_\theta + \delta_\theta \theta) p_i(\chi) \\
&\geq \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta o_i^w(\chi) + (1 - \max_{\theta \in T}(\delta_\theta(1-\theta))) p_i(\chi)
\end{aligned}$$

Now, focusing on the first part, we obtain:

$$\begin{aligned}
\sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta o_i^w(\chi) &\geq \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta o_i^w(\chi'_i, \chi_{-i}) \\
&= \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta \left(\min \left\{ \mathop{\mathbb{E}}_{\boldsymbol{b}_{-i} \sim \chi_{-i}} \left[v_i(\chi'_i, \boldsymbol{b}_{-i}) \right], B_i \right\} - p_i(\chi'_i, \chi_{-i}) \right) \\
&\geq \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta \left(\mathop{\mathbb{E}}_{\boldsymbol{b}_{-i} \sim \chi_{-i}} \left[\min \{ v_i(\chi'_i, \boldsymbol{b}_{-i}), B_i \} \right] - \mathop{\mathbb{E}}_{\boldsymbol{b}_{-i} \sim \chi_{-i}} \left[p_i(\chi'_i, \boldsymbol{b}_{-i}) \right] \right) \\
&= \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta \mathop{\mathbb{E}}_{\boldsymbol{b}_{-i} \sim \chi_{-i}} \left[o_i^w(\chi'_i, \boldsymbol{b}_{-i}) \right] \\
&\geq \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta \lambda_\theta v'_i(x_i^*) - \delta_\theta \mu_\theta \mathop{\mathbb{E}}_{\boldsymbol{b}_{-i} \sim \chi_{-i}} \left[\sum_{j=1}^{x_i^*} \beta_j(\boldsymbol{b}_{-i}) \right] \\
&= \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta \lambda_\theta v'_i(x_i^*) - \delta_\theta \mu_\theta \sum_{j=1}^{x_i^*} \mathop{\mathbb{E}}_{\boldsymbol{b} \sim \chi} \beta_j(\boldsymbol{b}_{-i}) \\
&\geq \min_{\theta \in T}(\delta_\theta \lambda_\theta) \, \text{OPT}^w - \max_{\theta \in T}(\delta_\theta \mu_\theta) \sum_{i \in [n]} \sum_{j=1}^{x_i^*} \mathop{\mathbb{E}}_{\boldsymbol{b} \sim \chi} \beta_j(\boldsymbol{b}_{-i})
\end{aligned}$$

$$\geq \min_{\theta \in T}(\delta_\theta \lambda_\theta) \,\mathrm{OPT}^w - \max_{\theta \in T}(\delta_\theta \mu_\theta) \sum_{i \in [n]} p_i(\chi).$$

In this chain of inequalities, the first follows from the fact that $\chi \in \mathrm{CCE}(A^w)$. The second inequality follows from Jensen's inequality. For the third we use the smoothness definition, Definition 4.7. The fourth follows readily and the last inequality follows from a similar argument used in the proof of Theorem 4.2 to bound the β_j term.

Combining this with the above, we find that:

$$\begin{aligned} \mathrm{SW}^w(\chi) &\geq \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta o_i^w(\chi) + (1 - \max_{\theta \in T}(\delta_\theta(1-\theta))) p_i(\chi) \\ &\geq \min_{\theta \in T}(\delta_\theta \lambda_\theta) \,\mathrm{OPT}^w + (1 - \max_{\theta \in T}(\delta_\theta(1-\theta)) - \max_{\theta \in T}(\delta_\theta \mu_\theta)) \sum_{i \in [n]} p_i(\chi). \end{aligned}$$

Since $\chi \in \mathrm{CCE}(A^w)$ and $\boldsymbol{\delta} \in \mathcal{C}(A^w)$ were arbitrary, we get $\mathrm{CCE\text{-}POA}^w \leq (\min_{\theta \in T} \delta_\theta \lambda_\theta)^{-1}$. Optimizing this over all $\boldsymbol{\delta} \in \mathcal{C}(A^w)$ yields the result $\mathrm{CCE\text{-}POA}^w \leq (\max_{\boldsymbol{\delta} \in \mathcal{C}(A^w)} \min_{\theta \in T} \delta_\theta \lambda_\theta)^{-1}$. Now, let

$$O^* = \min \left\{ \min_{\theta \in T} \lambda_\theta, \left(\max_{\theta \in T} \left(\frac{\mu_\theta}{\lambda_\theta} \right) + \max_{\theta \in T} \left(\frac{1-\theta}{\lambda_\theta} \right) \right)^{-1} \right\}.$$

The optimization problem $\max_{\theta \in T} \min_{\theta \in T} \delta_\theta \lambda_\theta$ is solved if we find $\boldsymbol{\delta} \in \mathcal{C}(A^w)$ such that $\delta_{\theta_i} \lambda_{\theta_i} = \delta_{\theta_j} \lambda_{\theta_j}$ for any pair $i, j \in [n]$. To this end, define $\boldsymbol{\delta}^*$ such that $\delta^*_\theta = O^*/\lambda_\theta$. If we show that $\boldsymbol{\delta}^* \in \mathcal{C}(A^w)$, then we are done. Note for any $\theta' \in T$ that:

$$\delta^*_{\theta'} = \frac{O^*}{\lambda_{\theta'}} \leq \frac{\min_{\theta \in T} \lambda_\theta}{\lambda_{\theta'}} \leq 1,$$

showing that $\boldsymbol{\delta}^* \in (0, 1]^{|T|}$. Lastly, we have:

$$\max_{\theta \in T}(\delta^*_\theta(1-\theta)) + \max_{\theta \in T}(\delta^*_\theta \mu_\theta) = O^* \max_{\theta \in T} \left(\frac{1-\theta}{\lambda_\theta} \right) + O^* \max_{\theta \in T} \left(\frac{\mu_\theta}{\lambda_\theta} \right) \leq 1,$$

as desired. □

Note in the above theorem that we used that the pointwise RoS constraint implies the RoS constraint in expectation. Similarly for the budget constraint. Thus, when considering discriminatory price welfare objective multi-unit auctions with RoS constraint in expectation, the above theorem holds as well. Hence, any upper bound we derive through Theorem 4.9 will also be an upper bound for multi-unit auctions where the RoS constraint is required to hold in expectation.

4.4 Smoothness parameters

In this section we prove that players of type θ are $(\lambda_\theta, \mu_\theta)$-smooth for certain λ_θ and μ_θ parameters. For value maximizers we have already seen that they are $(1, 1)$-smooth for conventional budgetless discriminatory price multi-unit auctions. The proof easily extends to discriminatory price welfare objective multi-unit auctions yielding the following lemma.

Lemma 4.10. Value maximizers are $(1, 1)$-smooth.

For players with objective parameter $\theta > 0$, the following lemma holds.

Lemma 4.11 (rf. De Keijzer et al. [9])**.** Consider a discriminatory price welfare objective multi-unit auction instance $A = (\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{B})$ under discriminatory pricing. Then players of type $\theta \in T$ are

$(\lambda_\theta, \mu_\theta)$ smooth with $\lambda_\theta = \alpha G_\theta$ and $\mu_\theta = \alpha$, where $\alpha \geq -\frac{\theta}{\ln(1-\theta)}$ and

$$G_\theta = \frac{1}{\theta}\left(1 - e^{-\frac{\theta}{\alpha}}\right).$$

Proof. Let $\boldsymbol{b}_{-i}$ be arbitrary an arbitrary pure strategy profile. Fix $i \in [n]$ and let c_i be the bidding vector that is $v_i'(x_i^*)/x_i^*$ (note the prime) in the first x_i^* components and zero everywhere else. Note, we can assume w.l.o.g. that $x_i^* > 0$ as otherwise the smoothness inequality would hold trivially. Now, we consider the deviating bid $\chi_i' = t \cdot c_i$, where t is a random variable drawn from the distribution with pdf $f(t) = \alpha/(1-\theta_i t)$ with support $[0, G_\theta]$, with G_θ defined in the lemma. It can be verified that this indeed defines a probability distribution. However, for $\chi_i \in \mathcal{S}_i(\boldsymbol{b}_{-i})$ to hold, we must have $G_\theta \leq 1$, which holds whenever $\alpha \geq -\theta/\ln(1-\theta)$. For $\theta = 1$, we see that $G_\theta = 1 - e^{-\frac{1}{\alpha}}$ and so, $f(t)$ will define a distribution for any $\alpha \geq 0$.

Next, let k^* be the number of items player i would receive under $(G_\theta c_i, \boldsymbol{b}_{-i})$. Furthermore, for $j = 0, \ldots, k^*$, let γ_i denote the infimum value in $[0, G_\theta]$ such that such that player i would obtain exactly j items if they deviated to $\gamma_j c_i$. Note then that $\gamma_j v_i(x_i^*)/x_i^* = \beta_j(\boldsymbol{b}_{-i})$. For convenience, we also write $\gamma_{k^*+1} = G_\theta$.

Lastly, let $x_i(\chi_i', \boldsymbol{b}_{-i})$ denote the random variable for the number of items player i is allocated under the randomized bid χ_i'. Note that $x_i(\chi_i', \boldsymbol{b}_{-i}) \leq k^* \leq x_i^*$, by construction of χ_i'. Also, by definition of γ_j, we see that $x_i(\chi_i', \boldsymbol{b}_{-i}) = j$ if and only if $t \in (\gamma_j, \gamma_{j+1}]$. Now, as v_i' is submodular (because v_i is), we note that $v_i'(j)/j \geq v_i'(x_i^*)/x_i^*$ for any $1 \leq j \leq k^*$. Using these properties, we compute:

$$\begin{aligned}
o_i^w(\chi_i', \boldsymbol{b}_{-i}) &\geq \mathop{\mathbb{E}}_{b_i' \sim \chi_i'}\left[o_i^w(b_i', \boldsymbol{b}_{-i})\right] \\
&= \mathop{\mathbb{E}}_{b_i' \sim \chi_i'}\left[v_i'(x_i(b_i', \boldsymbol{b}_{-i})) - \theta_i p_i(b_i', \boldsymbol{b}_{-i})\right] \\
&\geq \sum_{j=1}^{k^*} \int_{\gamma_j}^{\gamma_{j+1}} \left(v_i'(j) - \theta_i \cdot j \frac{v_i'(x_i^*)}{x_i^*}\right) f(t)dt \\
&\geq \sum_{j=1}^{k^*} \frac{j}{x_i^*} v_i'(x_i^*) \int_{\gamma_j}^{\gamma_{j+1}} (1 - \theta_i t) f(t)dt \\
&= \frac{\alpha}{x_i^*} v_i'(x_i^*) \sum_{j=1}^{k^*} j(\gamma_{j+1} - \gamma_j),
\end{aligned}$$

where in the second inequality we used that if $x_i(b_i', \boldsymbol{b}_{-i}) = j$, that $p_i(b_i', \boldsymbol{b}) = \frac{j}{x_i^*} v_i(x_i^*)$ and in the third we used that $v_i(j)/j \geq v_i(x_i^*)/x_i^*$ for any $1 \leq j \leq k^*$. Continuing, we obtain:

$$\begin{aligned}
o_i^w(\chi_i', \boldsymbol{b}_{-i}) &\geq \frac{\alpha}{x_i^*} v_i'(x_i^*) \sum_{j=1}^{k^*} j(\gamma_{j+1} - \gamma_j) \\
&= \frac{\alpha}{x_i^*} v_i'(x_i^*) \left(k^* G_\theta - \sum_{j=1}^{k^*} \gamma_j\right) \\
&\geq \alpha G_\theta v_i'(x_i^*) - \alpha \sum_{j=1}^{x_i^*} \beta_i(\boldsymbol{b}_{-i}),
\end{aligned}$$

where in the last inequality we have used that $G_\theta \frac{v_i'(x_i^*)}{x_i^*} \leq \beta_j(\boldsymbol{b}_{-i})$ for $k^* + 1 \leq j \leq x_i^*$. Thus, we conclude that player i is $(\alpha G_\theta, \alpha)$-smooth for $\alpha \geq -\theta/\ln(1-\theta)$. $\square$

In the next chapter, we prove various upper bounds through the smoothness framework for the budgetless case. Note, that O^* is now a function of $|T|$ number of variables, one for each type. Thus, to find the best upper bound, i.e., the lowest possible upper bound, we will need to maximize O^* over these $|T|$ variables as CCE-POA $\leq {O^*}^{-1}$.

Chapter 5

POA in DPAs

In this chapter we investigate the price of anarchy of discriminatory price multi-unit auctions for more general solution concepts. First we cover two special cases of budgetless auctions and then move on to budgeted discriminatory price multi-unit auctions. The case with budgets we shall analyze through a transformation to discriminatory price welfare objective multi-unit auctions.

5.1 Uniform objective parameters

In this section we shall assume for every player $i \in [n]$ that $\theta_i = \theta$ for some $\theta \in [0, 1]$. The case where $\theta = 0$ we had already solved and found a tight upper bound for the EQ-POA of 2 for any equilibrium concept as seen in Theorem 4.5. Using Lemma 4.11 and Theorem 4.2 we obtain the following theorem.

Theorem 5.1. Let $A = (v, \boldsymbol{\theta})$ be an instance of a discriminatory price multi-unit auction where $\theta_i = \theta$ for all players $i \in [n]$. Then coarse correlated price of anarchy satisfies:

$$\text{CCE-POA}(A) \leq f(\theta) := \begin{cases} \frac{e}{e-1}, & 1 - \frac{1}{e} \leq \theta \leq 1, \\ 1 + \ln(1-\theta) - \frac{\ln(1-\theta)}{\theta}, & 0 < \theta < 1 - \frac{1}{e}. \end{cases}$$

Proof. Using Lemma 4.11 and Theorem 4.2, we find that the price of anarchy satisfies:

$$\text{CCE-POA}(A) \leq \frac{\max\{1, 1 + \alpha - \theta\}}{\frac{\alpha}{\theta}\left(1 - e^{-\frac{\theta}{\alpha}}\right)},$$

for $\alpha \geq -\theta / \ln(1-\theta)$. We note that the function above is minimized whenever $1 = 1 + \alpha - \theta$, concluding that this occurs at $\alpha = \theta$. Comparing this with the constraint $\alpha \geq -\theta / \ln(1-\theta)$, we see that this choice of α is only valid whenever $1 - \frac{1}{e} < \theta \leq 1$. Plugging $\alpha = \theta$ into the above upper bound, would then yield the $\frac{e}{e-1}$ part of the theorem.

When $0 < \theta < 1 - \frac{1}{e}$, we find that $\theta \leq -\theta / \ln(1-\theta)$. Thus, since α is constrainted to $\alpha \geq -\theta / \ln(1-\theta)$, we find that $\alpha \geq \theta$. In other words, $\max\{1, 1 + \alpha - \theta\} = 1 + \alpha - \theta$. Let us consider the function:

$$g(\alpha) = \frac{1 + \alpha - \theta}{\frac{\alpha}{\theta}\left(1 - e^{-\frac{\theta}{\alpha}}\right)}.$$

We find that for $\alpha \geq -\theta / \ln(1-\theta)$ that the function $g(\alpha)$ is increasing in α. Hence, we may plug in $\alpha = -\theta / \ln(1-\theta)$ to obtain the best possible upper bound and obtain the second case in the theorem. $\square$

An interesting point here is, that as θ approaches 0, we find that $1 + \ln(1-\theta) - \ln(1-\theta)/\theta$

tends to 2, showing that the bound in the previous theorem is continuous. We also see that the upper bound of $\frac{e}{e-1}$ for utility maximizing also follows.

Putting the lower bound of Lemma 3.7 and the upper bound of the previous theorem on a graph, we find the following diagram.

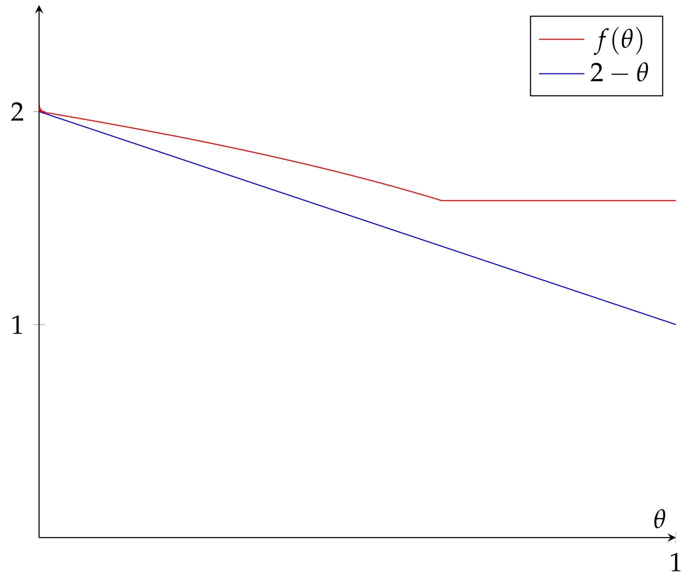

Figure 5.1: Upper bound against lower bound.

Adapting the lower bound of $\frac{e}{e-1}$ for the coarse correlated price of anarchy for utility maximizing auctions in [23], we see that the upper bound is tight for $1-\frac{1}{e} \leq \theta \leq 1$.

Example 5.2. Consider a single unit auction where both players have an objective parameter of $1-\frac{1}{e} \leq \theta \leq 1$. Let $v_1 = 1$, $v_2 = 0$ and $\boldsymbol{b} = (t,t)$ where t is sampled from a distribution with cdf $F(t) = \frac{\frac{1}{e}}{1-\theta t}$ where $t \in [0, \frac{1}{\theta}(1-e^{-1})]$. Note that the RoS constraint is satisfied pointwise as $1-\frac{1}{e} \leq \theta \leq 1$, meaning that the support of the distribution is a subset of $[0,1]$. Let ties be broken in favor of player 1 except when the tie occurs at 0. From the perspective of player 2, $\boldsymbol{b}$ is an equilibrium, as they will never have to pay, leaving their objective 0 in all outcomes and this cannot possibly be improved.

For player 1, their expected utility is:

$$\mathbb{E}\left[u_1(\boldsymbol{b})\right] = \int_0^{\frac{1}{\theta}(1-e^{-1})} (1-\theta t)F'(t)dt = \frac{\theta}{e}\int_0^{\frac{1}{\theta}(1-e^{-1})} \frac{1}{1-\theta t}dt = \frac{1}{e}.$$

Suppose now they deviate to a pure bid $b \in [0, \frac{1}{\theta}(1-e^{-1})]$. Their expected utility is then:

$$(1-\theta b)F(b) = \frac{1}{e}.$$

In case they bid larger than $\frac{1}{\theta}(1-e^{-1})$, their objective will be strictly lower than $\frac{1}{e}$. Thus, we see that $\boldsymbol{b}$ is an equilibrium.

Note that the probability of t being zero is $\frac{1}{e}$. The expected social welfare is then $1-\frac{1}{e}$, while the optimal welfare is 1. Hence, we conclude that the POA of this instance is $\frac{e}{e-1}$.

For uniform objective parameters $0 < \theta < 1-\frac{1}{e}$, we are still uncertain what the exact price of anarchy is. However, for $1-\frac{1}{e} \le \theta \le 1$ we have the following theorem.

Theorem 5.3. The CCE-POA of the class of budgetless discriminatory price auctions with uniform objective parameter $1-\frac{1}{e} \le \theta \le 1$ is $\frac{e}{e-1}$.

5.2 Bipartite DPAs

In this section we analyse discriminatory price multi-unit auctions with two types of players. One type being value maximizer and the other type being any $\theta \in (0,1]$. For an upper bound, we use Theorem 4.9 and obtain a similar result as in [6]. In the proof, we follow an analogous derivation as the one presented in [6].

Lemma 5.4. Consider a discriminatory price multi-unit auction $A = (\boldsymbol{v}, \boldsymbol{\theta})$ where $T = \{0, \theta\}$ for some $\theta \in (0,1]$. The coarse correlated price of anarchy satisfies the following bound:

$$\text{CCE-POA} \le \begin{cases} 1+\theta\left(1+W(-e^{-1-\theta})\right)^{-1}, & \text{if } \theta > 1+\frac{W(-2e^{-2})}{2} \\ 2, & \text{else.} \end{cases}$$

Proof. Firstly, note by Lemmas 4.10 and 4.11 that value maximizers are $(1,1)$ smooth and players in N_θ are $(\alpha G_\theta, \alpha)$ smooth for $\alpha \in D$ where $D = \{x \in \mathbb{R} : x \ge -\theta/\ln(1-\theta)\}$. Using the bound $1-e^{-\frac{\theta}{\alpha}} \le \frac{\theta}{\alpha}$, for any $\alpha \in \mathbb{R}^+$, we find $\alpha G_\theta \le 1$ for all $\alpha \in D$. Thus, we see $\min_{\theta \in T} \lambda_\theta = \alpha G_\theta$. Furthermore, taking the first derivative w.r.t. to α, we see that the following function is decreasing in α:

$$\frac{1-\theta}{\alpha G_\theta} = \frac{\theta(1-\theta)}{\alpha\left(1-e^{-\frac{\theta}{\alpha}}\right)}.$$

Hence, we have that this function is maximized for $\alpha = -\theta/\ln(1-\theta)$. Plugging this in, would yield the following quantity:

$$-(1-\theta)\frac{\ln(1-\theta)}{\theta}.$$

Now, using the inequality $\ln(x) < x-1$ for any $x \in \mathbb{R}^+$, we find by the transformation $x \mapsto \frac{1}{y}$ that $-\ln(y) < \frac{1-y}{y}$ for $y \in \mathbb{R}^+$. Now, making the substitution $y \mapsto 1-z$, we find that $-\ln(1-z) < \frac{z}{1-z}$ for $z \in (0,1)$. Using this last bound applied to $z = \theta \in (0,1)$, we find that:

$$\frac{1-\theta}{\alpha G_\theta} \le -(1-\theta)\frac{\ln(1-\theta)}{\theta} \le 1.$$

Note, if $\theta = 1$, then $\frac{1-\theta}{\alpha G_\theta} = 0$ and the inequality holds as well. From this we see that $\max_{\theta \in T} \frac{1-\theta}{\lambda_\theta} = 1$. Lastly, for $\alpha \in D$ we see that $\frac{1}{G_\theta} \ge 1$, showing that $\max_{\theta \in T} \frac{\mu_\theta}{\lambda_\theta} = \frac{1}{G_\theta}$.

The above considerations show that $O^*(\alpha)$ from Theorem 4.9 equals:

$$O^*(\alpha) = \min\left(\alpha G_\theta, \left(\frac{1}{G_\theta}+1\right)^{-1}\right).$$

Our goal is now to minimize this function for $\alpha \in D$. We will make a case distinction between two cases.

Suppose that $\theta \le 1+\frac{W(-2e^{-2})}{2}$. Plugging in $\alpha = -\theta/\ln(1-\theta)$ we find:

$$O^*(-\theta/\ln(1-\theta)) = \min\left(-\frac{\theta}{\ln(1-\theta)}, \frac{1}{2}\right).$$

However, note, if we solve for x in the equation $-x/\ln(1-x) = \frac{1}{2}$, then we get:

$$-2x = \ln(1-x)$$

$$e^{-2x} = 1 - x$$

$$2 = (2x-2)e^{2x}$$

$$2e^{-2} = (2x-2)e^{2x-2}$$

$$W(2e^{-2}) = 2x - 2.$$

Hence, from this, we find that $x = 1 + \frac{1}{2}W(2e^{-2})$. Taking the first derivative of $-\theta/\ln(1-\theta)$ w.r.t. θ shows that this function is decreasing in θ. So, since $\theta \leq 1 + \frac{W(-2e^{-2})}{2}$, we find:

$$-\frac{\theta}{\ln(1-\theta)} \geq -\frac{x}{\ln(1-x)} = \frac{1}{2}.$$

Thus, for $\theta \leq 1 + \frac{W(-2e^{-2})}{2}$, we find that $O^* = \frac{1}{2}$, yielding the second case of the theorem.

Now, suppose that $1 \geq \theta > 1 + \frac{W(-2e^{-2})}{2}$. In that case, the maximum of $O^*(\alpha)$ occurs when $\alpha G_\theta = \left(\frac{1}{G_\theta} + 1\right)^{-1} = \frac{G_\theta}{G_\theta + 1}$. This is equivalent to $\alpha(G_\theta + 1) = 1$. Now, we expand:

$$\frac{1}{\theta}\left(1 - e^{-\frac{\theta}{\alpha}}\right) + 1 = \frac{1}{\alpha}$$

$$1 - e^{-\frac{\theta}{\alpha}} + \theta = \frac{\theta}{\alpha}$$

$$1 + \theta - \frac{\theta}{\alpha} = e^{-\frac{\theta}{\alpha}}$$

$$e^{\frac{\theta}{\alpha}}\left(\frac{\theta}{\alpha} - 1 - \theta\right) = -1$$

$$e^{\frac{\theta}{\alpha} - 1 - \theta}\left(\frac{\theta}{\alpha} - 1 - \theta\right) = -e^{-1-\theta}$$

$$\frac{\theta}{\alpha} - 1 - \theta = W\left(-e^{-1-\theta}\right).$$

From the last equality we easily find that:

$$\alpha = \frac{\theta}{1 + \theta + W\left(-e^{-1-\theta}\right)}.$$

Plugging this value into G_θ, we find:

$$G_\theta(\alpha) = \frac{1}{\theta}\left(1 - e^{-\frac{\theta}{\alpha}}\right) = \frac{1}{\theta}\left(1 - e^{-\left(1+\theta+W\left(-e^{-1-\theta}\right)\right)}\right) = \frac{1}{\theta}\left(1 + W\left(-e^{-1-\theta}\right)\right),$$

where in the last equality we used that $W\left(-e^{-x}\right) = -e^{-x-W\left(-e^{-x}\right)}$, for $x > 1$, which we can use as $1 + \theta > 1$. So, in this case, we find that

$$O^* = \frac{1 + W\left(-e^{-1-\theta}\right)}{1 + \theta + W\left(-e^{-1-\theta}\right)}.$$

Hence, for O^{*-1} and the CCE-POA we get:

$$\text{CCE-POA} \le \frac{1+\theta+W\left(-e^{-1-\theta}\right)}{1+W\left(-e^{-1-\theta}\right)} = 1+\theta\left(1+W(-e^{-1-\theta})\right)^{-1},$$

as desired. $\square$

Combining this upper bound with the lower bound from Lemma 3.10 we obtain the following theorem.

Theorem 5.5. Let EQ $\in \{\text{PNE}, \text{MNE}, \text{CE}, \text{CCE}\}$, then for the class of budgetless bipartite discriminatory price multi-unit auctions the price of anarchy satisfies:

$$\text{EQ-POA} = \begin{cases} 1+\theta\left(1+W(-e^{-1-\theta})\right)^{-1}, & \text{if } \theta > 1+\frac{W(-2e^{-2})}{2} \\ 2, & \text{else.} \end{cases}$$

The above theorem closes the case for pointwise RoS constraint. Since the upper bound of Theorem 5.4 extends to bipartite discriminatory price multi-unit auctions with RoS constraint in expectation, we still need to find a lower bound for tightness. Note that the lower bound in Lemma 3.10 might not be a mixed equilibrium if we consider the RoS constraint in expectation. In [10], they construct a lower bound for simultaneous auctions with RoS constraint in expectation. The example below modifies this lower bound to fit into the discriminatory price multi-unit auction setting, yielding tightness.

Example 5.6. Consider a discriminatory price multi-unit auction with $k+1$ items and 2 players. Player 1 is a value maximizer and player 2 has an objective parameter $\theta \in \left(1+\frac{W(-2e^{-2})}{2}, 1\right]$. Let $v_1 = (v, 0, \dots, 0)$ and $v_2 = (1, \dots, 1, 0)$ where v will be determined later. Consider now the bids $b_1 = (1, t, t, \dots, t)$ and $b_2 = (0, \dots, 0)$, where t is sampled from a distribution T with CDF $F(t) = \frac{a}{1-\theta t}$ where $t \in [0, \frac{1-a}{\theta}]$ and $a \in [1-\theta, 1]$. A simple calculation shows that $\mathbb{E}(T) = \frac{1}{\theta}(1-a+a\ln(a))$. Hence, the expected payment of player 1 will be $1+\frac{k}{\theta}(1-a+a\ln(a))$. Choosing then v as this value means that the RoS constraint will be satisfied in expectation. Note that the RoS constraint is not satisfied pointwise.
Now, we will argue that this is indeed an equilibrium. From the perspective of player 1 this is an equilibrium as they receive will certainly receive one unit and about the other units they are indifferent. For player 2, suppose that they deviate to $b_2' = (c_1', \dots, c_k', 0)$, where we assume that the c_i' are in non-increasing order. Note that we may assume that $c_i' \in [0, \frac{1-a}{\theta}]$. To see this, note that in the bidding profile $\boldsymbol{b} = (b_1, b_2)$ each unit for player 2 contributes a to their expected objective. However, if they bid more than $\frac{1-a}{\theta}$ for a unit, they will certainly get that unit, however, the contribution to the utility would be strictly less than a. So, assuming that $c_i' \in [0, \frac{1-a}{\theta}]$, the expected objective of player 2 is:

$$\begin{aligned}\mathbb{E}(o_2(b_1, b_2')) &= \sum_{j=1}^{k}\left(j-\theta\sum_{\ell=1}^{j} c_\ell\right)\mathbb{P}(c_{j+1} < t < c_j) \\ &= \sum_{j=1}^{k}\left(\sum_{\ell=1}^{j} 1-\theta c_\ell\right)(F(c_j)-F(c_{j+1}))\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{k}\left(\sum_{\ell=1}^{j} 1-\theta c_\ell\right) F(c_j) - \sum_{j=1}^{k}\left(\sum_{\ell=1}^{j} 1-\theta c_\ell\right) F(c_{j+1}) \\
&= \sum_{j=1}^{k}\left(\sum_{\ell=1}^{j} 1-\theta c_\ell\right) F(c_j) - \sum_{j=2}^{k+1}\left(\sum_{\ell=1}^{j-1} 1-\theta c_\ell\right) F(c_j) \\
&= (1-\theta c_1)F(c_1) + \sum_{j=2}^{k}\left(\sum_{\ell=1}^{j} 1-\theta c_\ell\right) F(c_j) - \sum_{j=2}^{k}\left(\sum_{\ell=1}^{j-1} 1-\theta c_\ell\right) F(c_j) \\
&= (1-\theta c_1)F(c_1) + \sum_{j=2}^{k}(1-\theta c_j)F(c_j) \\
&= \sum_{j=1}^{k}(1-\theta c_j)F(c_j) \\
&= \sum_{j=1}^{k} a \\
&= ak.
\end{aligned}
$$

Note, that $\mathbb{E}(o_2(\boldsymbol{b})) = ak$ as well and thus we find that $\boldsymbol{b}$ is a equilibrium from player 2's perspective as well.
Now, we shall compute the POA of this instance. We have

$$\mathrm{OPT} = v + k = 1 + \frac{k}{\theta}(1 - a + a\ln(a)) + k.$$

However, the expected social welfare is:

$$\mathrm{SW}(\boldsymbol{b}) = v + ak = 1 + \frac{k}{\theta}(1 - a + a\ln(a)) + ak.$$

Taking the ratio, we find that:

$$\text{CCE-POA} \geq \frac{1 + \frac{k}{\theta}(1 - a + a\ln(a)) + k}{1 + \frac{k}{\theta}(1 - a + a\ln(a)) + ak}.$$

Now, taking the limit as $k \to \infty$, we find that:

$$\text{CCE-POA} \geq \frac{\frac{1}{\theta}(1 - a + a\ln(a)) + 1}{\frac{1}{\theta}(1 - a + a\ln(a)) + a}.$$

To find the best possible lower bound, we shall maximize this function for $a \in [1-\theta, 1]$ and find the CCE-POA $\geq 1 + \theta\left(1 + W(-e^{-1-\theta})\right)^{-1}$ yielding tightness of the upper bounds in Theorem 5.4. Let $g(a)$ be the function above. Differentiating this function w.r.t. a yields:

$$g'(a) = -\frac{\theta\left(\ln(a) - a + \theta + 1\right)}{\left(a\ln(a) + (\theta - 1)a + 1\right)^2}.$$

So, we wish to solve $\ln(a) - a + \theta + 1 = 0$. Note that this is equivalent to $a = e^{a-1-\theta}$. Rearringing yields $-ae^{-a} = -e^{-1-\theta}$ and using the Lambert-W function then gives $a = -W\left(-e^{-1-\theta}\right)$. Note for $x > 1$ that $W\left(-e^{-x}\right) = -e^{-x-W\left(-e^{-x}\right)}$. From this, we see that

$\ln\left(-W\left(-e^{-x}\right)\right) = -x - W\left(-e^{-x}\right)$. Hence, plugging this in yields:

$$g\left(-W\left(-e^{-1-\theta}\right)\right) = \frac{1 + W\left(-e^{-1-\theta}\right) - W\left(-e^{-1-\theta}\right)\ln\left(-W\left(-e^{-1-\theta}\right)\right) + \theta}{1 + W\left(-e^{-1-\theta}\right) - W\left(-e^{-1-\theta}\right)\ln\left(-W\left(-e^{-1-\theta}\right)\right) - \theta W\left(-e^{-1-\theta}\right)}$$
$$= \frac{1 + W\left(-e^{-1-\theta}\right) - W\left(-e^{-1-\theta}\right)\left(-1 - \theta - W\left(-e^{-1-\theta}\right)\right) + \theta}{1 + W\left(-e^{-1-\theta}\right) - W\left(-e^{-1-\theta}\right)\left(-1 - \theta - W\left(-e^{-1-\theta}\right)\right) - \theta W\left(-e^{-1-\theta}\right)}$$
$$= \frac{1 + 2W\left(-e^{-1-\theta}\right) + \theta W\left(-e^{-1-\theta}\right) + W\left(-e^{-1-\theta}\right)^2 + \theta}{1 + 2W\left(-e^{-1-\theta}\right) + W\left(-e^{-1-\theta}\right)^2}$$
$$= \frac{\left(1 + \theta W\left(-e^{-1-\theta}\right)\right)^2 + \theta\left(1 + W\left(-e^{-1-\theta}\right)\right)}{\left(1 + \theta W\left(-e^{-1-\theta}\right)\right)^2} = 1 + \theta\left(1 + W\left(-e^{-1-\theta}\right)\right)^{-1}.$$

Combining this example with upper bound obtain from smoothness, we obtain:

Theorem 5.7. For EQ $\in \{\text{MNE}, \text{CE}, \text{CCE}\}$ we have that the EQ-POA of the class of budgetless bipartite discriminatory price multi-unit auctions where the RoS constraint is taken in expectation is equal to that of Theorem 5.5.

5.3 Budgeted auctions

In this section we analyze the efficiency of discriminatory price multi-unit auctions with budgets. This gives us the following theorem.

Theorem 5.8. Let $\mathcal{C}$ be a class of multi-unit auctions (uniform or discriminatory price), not necessarily budgetless. Let $\theta_{\max}$ be the largest price sensitivity of any player in any auction in $\mathcal{C}$. Then, letting $\mathcal{C}'$ be the class of welfare objective multi-unit auctions such that all players in any auction have a price sensitivity satisfying $0 \leq \theta_i \leq \theta_{\max}$, under the same pricing rule as $\mathcal{C}$, we have that:

$$\text{CCE-POA}(\mathcal{C}) \leq \text{CCE-POA}^w(\mathcal{C}').$$

Proof. Consider a $A = (\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{B})$ a multi-unit auction with budgets and let $\chi \in \text{CCE}(A)$. We partition $[n]$ through the sets $N_1 := \{i \in [n] \mid v_i(\chi) > B_i\}$ and $N_2 = \{i \in [n] \mid v_i(\chi) \leq B_i\}$. Now, consider the welfare objective multi-unit auctions $A^w_\chi = (\boldsymbol{v}, \boldsymbol{\theta}', \boldsymbol{B})$ (under the same pricing rule as A) where for $i \in N_1$ we set $\theta'_i = 0$ and for $i \in N_2$ we set $\theta'_i = \theta_i$. We shall now reason that $\chi \in \text{CCE}(A^w_\chi)$.

Note that the RoS and budget constraints in both auctions are equal and thus the space of possible deviations remains conserved under this transformation. Next, we note that for $i \in N_1$ that in A^w_χ, $o^w_i(X) = v^w_i(X) = \min\{v_i(X), B_i\} \leq B_i$ for any distribution X over pure strategy profiles. By construction, $o^w_i(\chi) = B_i$ showing that the welfare objective is maximized for these players. Hence, these players have no incentive to deviate in A^w_χ. Suppose for $i \in N_2$ that there exists a deviation χ'_i such that $o^w_i(\chi'_i, \chi_{-i}) > o^w_i(\chi)$. By definition of N_2 we then see that:

$$o_i(\chi'_i, \chi_{-i}) \geq \min\{v_i(\chi'_i, \chi_{-i}), B_i\} - \theta_i p_i(\chi'_i, \chi_{-i}) = o^w_i(\chi'_i, \chi_{-i}) > o^w_i(\chi) = o_i(\chi),$$

contradicting the fact that $\chi \in \text{CCE}(A)$. Hence, $\chi \in \text{CCE}(A^w_\chi)$. Now, we have that:

$$\text{EW}_A(\chi) = \sum_{i\in[n]} \min\{v_i(\chi), B_i\} = \sum_{i\in[n]} v_i^w(\chi) = \text{SW}^w_{A^w_\chi}(\chi),$$

where the subscripts denote w.r.t. which auction the function is taken, and

$$\text{OPT}_A = \sup_{\boldsymbol{b}} \sum_{i\in[n]} \min\{v_i(x_i(\boldsymbol{b})), B_i\} = \sup_{\boldsymbol{b}} \sum_{i\in[n]} v_i^w(\boldsymbol{b}) = \sup_{\boldsymbol{b}} \text{SW}^w(\boldsymbol{b}) = \text{OPT}^w_{A^w_\chi}.$$

Thus, putting this together, we see that:

$$\frac{\text{OPT}_A}{\text{EW}_A(\chi)} = \frac{\text{OPT}^w_{A^w_\chi}}{\text{SW}^w_{A^w_\chi}(\chi)} \leq \text{CCE-POA}^w(A^w_\chi).$$

More specifically, for the auction A, we now obtain:

$$\text{CCE-POA}(A) \leq \sup_{\chi\in\text{CCE}(A)} \text{CCE-POA}^w(A^w_\chi).$$

For a general class of multi-unit auctions $\mathcal{C}$, we can obtain class $\mathcal{X}$ of welfare objective auctions such that $A^w \in \mathcal{X}$ iff there exists $A \in \mathcal{A}$ and $\chi \in \text{CCE}(A)$ such that $A^w_\chi = A^w$, where A^w_χ is the determined from A as above. Now, for $\mathcal{C}'$ in the theorem, we see that $\mathcal{X} \subseteq \mathcal{C}'$. Hence, by the above considerations, we obtain:

$$\text{CCE-POA}(\mathcal{C}) \leq \text{CCE-POA}^w(\mathcal{X}) \leq \text{CCE-POA}^w(\mathcal{C}')$$

□

Let us compute a POA as an example. Let $\mathcal{A}$ be the class of bipartite discriminatory multi-unit auctions with budgets where each player has a price sensitivity of 0 or θ and there is at least one player of each type. By considering the transformation above, we find that any such auction under any CCE equilibrium will be transformed into a bipartite auction with the same two types, or it will become a value maximizing auction. The latter has a POA^w of 2, while the former may be higher. Hence, by the upper bounds we found through smoothness in the previous section on budgetless bipartite auctions (the optimization problem is the exact same for welfare objective auctions with budgets) we have for $\text{EQ} \in \{\text{PNE}, \text{MNE}, \text{CE}, \text{CCE}\}$:

$$\text{EQ-POA}(\mathcal{A}) = \begin{cases} 1 + \theta\left(1 + W(-e^{-1-\theta})\right)^{-1}, & \text{if } \theta > 1 + \frac{W(-2e^{-2})}{2} \\ 2, & \text{else,} \end{cases}$$

where the lower bound follows from the pure equilibrium in Lemma 3.10 and the upper bound from the analysis above.

Now, we combine this result with our previous results on the POA of pure equilibria to give a complete description. Consider the class of discriminatory price multi-unit auctions $\mathcal{M}^{\text{disc}}_{\theta_1,\theta_2}$, where $A \in \mathcal{M}^{\text{disc}}$ iff $\theta_{\min} \geq \theta_1$ and $\theta_2 \geq \theta_{\max}$. We find that:

$$\text{POA}(\mathcal{M}^{\text{disc}}_{\theta_1,\theta_2}) = \begin{cases} 1 + \theta_2\left(1 + W(-e^{-1-\theta_2})\right)^{-1}, & \text{if } \theta_2 > 1 + \frac{W(-2e^{-2})}{2} \text{ and } \theta_1 = 0 \\ 2, & \text{else.} \end{cases}$$

It is important to remark that the upper bounds found through the above method might not always be tight. For example, let $\mathcal{A}$ be the class of discriminatory price multi-unit auctions under discriminatory pricing where no player is a value maximizer and consider $\mathcal{X}$ obtained through the above algorithm. In that case, we see that $\mathcal{X}$ contains the class of bipartite discriminatory price multi-unit auctions where one player is a value maximizer and one player is a utility maximizer. Thus, for CCE, $\text{POA}(\mathcal{A}) \leq 2.18$, where we used 2.18 as the approximation of the real quantity computed in the previous section. However, the results from Chapter 3 tell us that for pure equilibria $\text{POA}(\mathcal{A}) = 2$. Whether 2.18 is tight when considering CCEs remains unknown.

Some comments on the generalizability of the above method. The essential ingredients are that capped submodular valuations remain submodular, the extension theorem, and the smoothness per type. The smoothness proofs easily generalize to any pricing rule dominated by discriminatory pricing. Although not all pricing rules would admit an extension theorem like Theorem 4.9, we see that the uniform pricing rule has its own extension theorem for budgetless uniform price multi-unit auctions in Theorem 6.3. So, with this in mind, we conclude that the above method of finding the POA for CCE extends to uniform price multi-unit auctions.

Furthermore, it will also extend to simultaneous first-price auctions if we restrict player's valuations to submodular valuations, as they remain closed under capping and from [6] we see that it admits the exact same typed smoothness framework as discriminatory price multi-unit auctions. It should be noted that the method also works when taking the budget and RoS constraint in expectation, allowing for more flexibility in finding lower bounds.

Using the fact that this method extends to simultaneous first-price auctions with valuations restricted to submodular valuations, we can easily find that for utility maximizers the coarse correlated price of anarchy is at most 2.18. Thus, since additive valuations are submodular, the upper bound will also hold for additive valuations. The efficiency of simultaneous first-price auctions with budget constraints has been investigated in [2], where they adjusted the model to make the goods divisible. Below we shall give an example showing that the 2.18 for utility maximizing simultaneous first-price auctions (with indivisible goods) with budget constraint is tight. However, in accordance with conventions in the literature, for this example we shall take the budget constraints in expectation as in [6] and [2].

Example 5.9. Consider a simultaneous first-price auction with two bidders, both utility maximizers whose valuations are $v_1 = (1, 1)$ for $a \in (0, 1)$ and $v_2 = (0, 1)$. Ties in the first auction are broken in favor of player 1, ties in the second auction are broken in favor of player 2 if the tie break is at 0 and otherwise in favor of player 1.

Consider the distribution over bids χ where player 1 and player 2 both bid $(0, t)$, where t is sampled from a distribution with CDF $F(t) = \frac{a}{1-t}$ where $t \in [0, 1-a]$. Player 1 has a budget of $1 - a + a\ln(a)$ and player 2 has a budget of ∞. Using the analysis from example 5.6, we find that the budget constraint is satisfied in expectation for player 1.

Currently $u_2(\chi) = a$, if player 2 deviates and bids $(0, b)$ for $b \in [0, 1-a]$ instead, their utility becomes $u_2(\chi_{-2}, (0, b)) = (1 - b)F(b) = a$. Bidding $(0, b)$ for $b > 1 - a$ is never beneficial for them. Hence, player 2 has no incentive to deviate.

For player 1, the expected utility is:

$$u_1(\chi) = 1 + \int_0^{1-a} (1 - t) f(t) dt = 1 - a\ln(a).$$

If instead, they deviate and bid $(0, b)$ for $b \in (0, 1 - a)$, their utility would become:

$$u_1((0, b), \chi_{-1}) = 1 + (1 - b)F(b) = 1 + a.$$

Hence, if $1 - a\ln(a) \geq 1 + a$, then for any choice of b player 1 would also not have an incentive to deviate. For $a \leq \frac{1}{e}$ we see that the inequality holds. Hence, for the χ to be a CCE, we only require that $a \leq \frac{1}{e}$.
The optimal allocation considering effective welfare is: allocate the first item to player 1 and the second item to player 2. Thus, $\mathrm{OPT} = 2 - a + a\ln(a)$. However, the expected effective welfare is $1 + a\ln(a)$ yielding that for this auction,

$$\mathrm{POA} \geq \frac{2 - a + a\ln(a)}{1 + a\ln(a)},$$

which can be optimized to the known bound $\mathrm{POA} \geq 1 + \frac{1}{1+W(-e^{-2})} \approx 2.18$.

Since valuations in the above example are additive, we obtain the following theorem.

Theorem 5.10. The coarse correlated POA for simultaneous first-price auctions with utility maximizing players with submodular or additive valuations and budget constraints in expectation is 2.18.

Chapter 6

Uniform Price Multi-Unit Auctions

In this chapter we study uniform price multi-unit auctions. We first prove the existence of pure equilibria and then move on to the price of anarchy. Note that for the uniform price auction, we only consider *feasible* bids, in other words, bids that satisfy the no-overbidding assumption: for all $\ell \in [k]$, $\sum_{j=1}^{\ell} b_{ij} \leq v_i(\ell)$. We shall only study the budgetless case in this chapter as the case with budgets follows similarly to the analysis in the previous chapter.

6.1 Existence of pure equilibria

Unlike discriminatory auctions, the existence of a pure Nash equilibrium in uniform price is independent of the tie-breaking rule.

Theorem 6.1. Pure Nash equilibria in uniform price multi-unit auctions exist.

Proof. Let $(\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{B})$ be an instance of a uniform price multi-unit auction under uniform pricing with an arbitrary tie-breaking rule. Let $i \in [n]$, we define the *average budget* $A_i : [k] \to \mathbb{R}^+$ as $A_i(\ell) = \min\left(\frac{B_i}{\ell}, \frac{m_i(\ell)}{\theta_i}, \frac{v_i(\ell)}{\ell}\right)$, where if $\theta_i = 0$ we set $\frac{m_i(\ell)}{\theta_i} = \infty$. Viewing A_i as a k-tuple, we let $\boldsymbol{A}$ be the vector obtained by appending all tuples A_i, $i \in [n]$. Next, we order $\boldsymbol{A}$ from highest to lowest to obtain $\tilde{\boldsymbol{A}}$. For ease of notation, let $p = \tilde{\boldsymbol{A}}_k$ and define for each $i \in [n]$, $k_i := |\{\ell \in [k] : A_i(\ell) \geq p\}|$. Then, for $i \in [n]$ consider the bid defined by:

$$b_i = \begin{cases} (\underbrace{p, \ldots, p}_{k_i\text{-times}}, 0, \ldots, 0), & \text{if } k_i \neq 0 \\ (0, \ldots, 0), & \text{else.} \end{cases}$$

Now, consider the bidding profile $\boldsymbol{b} = (b_1, \ldots, b_n)$. Note that the payment for each unit for any player is 0 and thus no player will deviate to a bid that would result in them being allocated fewer items. Hence, to improve the objective, a player would need to deviate to a bid that would result in them being allocated more units. Thus, we only have to consider bids b'_i where $b'_{ij} \geq b_{ij}$ for $1 \leq j \leq k_i$. However, to be allocated more items, they would have to bid b'_i with $b'_i(k_i + 1) \geq p$, but, by construction, such bids cannot satisfy the no-overbidding assumption. Thus, we conclude that $\boldsymbol{b}$ is a PNE, as desired. $\square$

6.2 Smoothness

In this section, we cover the uniform price smoothness results corresponding to what we found for discriminatory price auctions. Note that the smoothness definition is independent of the pricing rule so that remains unchanged. We repeat the definition for convenience's sake.

Definition 6.2. Consider a $(\boldsymbol{v}, \boldsymbol{\theta})$ uniform price multi-unit auction and let x_i^* be the number of units allocated to player i under the optimal allocation. More precisely, $x_i^* = x_i(\text{argmax}_{\boldsymbol{b} \in \mathcal{B}} \text{POA}(\boldsymbol{b}))$. Furthermore, let $\beta_j(\boldsymbol{b})$ refer to the j-th lowest winning bid under the profile $\boldsymbol{b}$. A player of type θ is $(\lambda_\theta, \mu_\theta)$-smooth for $\lambda_\theta > 0$ and $\mu_\theta \geq 0$ if for any strategy profile $\boldsymbol{b}_{-i} \in \mathcal{B}_{-i}$ there exists a mixed deviating bid χ_i', not dependent on $\boldsymbol{b}_{-i}$, such that:

$$o_i^w(\chi_i', \boldsymbol{b}_{-i}) \geq \lambda_\theta v_i'(x_i^*) - \mu_\theta \sum_{j=1}^{x_i^*} \beta_j(\boldsymbol{b}_{-i}),$$

where we require $\chi_i' \in \mathcal{S}_i(\boldsymbol{b}_{-i})$. Lastly, we require that the deviation, χ_i' and the profile $\boldsymbol{b}_{-i}$ all satisfy the no-overbidding assumption.

Now, we cover the extension theorem for uniform price multi-unit auctions.

Theorem 6.3 (rf. De Keijzer et al. [9]). In a uniform price welfare objective auction instance $A^w = (\boldsymbol{v}, \boldsymbol{\theta}, \boldsymbol{B})$ where for any $\theta \in T$ any $i \in N_\theta$ is $(\lambda_\theta, \mu_\theta)$ smooth, the CCE-POA^w is at most $\frac{1+\max_{\theta \in T} \mu_\theta}{\min_{\theta \in T} \lambda_\theta}$.

Proof. Let χ be any CCE and χ_i' be the deviation from the smoothness definition. Then, we find that:

$$\text{SW}^w(\chi) = \sum_{i \in [n]} v_i^w(\chi) \geq \sum_{i \in [n]} o_i^w(\chi) \geq \sum_{i \in [n]} o_i^w(\chi_i', \chi_{-i}) \geq \sum_{i \in [n]} \mathop{\mathbb{E}}_{\boldsymbol{b}_{-i} \sim \chi_{-i}} \left[o_i^w(\chi_i', \boldsymbol{b}_{-i})\right]$$

where in the first inequality we use that the value upper bounds the objective for each player. We continue by rearranging the terms in the sum and applying the smoothness definition to obtain:

$$\begin{aligned}
\text{SW}^w(\chi) &\geq \sum_{\theta \in T} \sum_{i \in N_\theta} \mathop{\mathbb{E}}_{\boldsymbol{b}_{-i} \sim \chi_{-i}} \left[o_i^w(\chi_i', \boldsymbol{b}_{-i})\right] \\
&\geq (\min_{\theta \in T} \lambda_\theta) \, \text{OPT}^w - (\max_\theta \mu_\theta) \sum_{i \in [n]} \sum_{j=1}^{x_i^*} \mathop{\mathbb{E}}_{\boldsymbol{b}_{-i} \sim \chi_{-i}} \left[\beta_j(\boldsymbol{b}_{-i})\right] \\
&\geq (\min_{\theta \in T} \lambda_\theta) \, \text{OPT}^w - (\max_\theta \mu_\theta) \sum_{i \in [n]} \sum_{j=1}^{x_i^*} \mathop{\mathbb{E}}_{\boldsymbol{b} \sim \chi} \left[\beta_j(\boldsymbol{b})\right] \\
&\geq (\min_{\theta \in T} \lambda_\theta) \, \text{OPT}^w - (\max_\theta \mu_\theta) \, \text{SW}^w(\chi),
\end{aligned}$$

where in the last inequality, we used the no-overbidding assumption. From this, we find the desired result. ☐

So far, we have been unable to derive a more sophisticated upper bound using different types for the uniform price auction, as in Theorem 4.9. Most likely we need a more sophisticated type distinction, one that depends on more than just θ. Note, that the proofs in the typed smoothness results, Lemma 4.4 and Lemma 4.11, would hold for any pricing rule dominated by the discriminatory pricing rule. Hence, we immediately achieve the following results.

Lemma 6.4. In a uniform price welfare objective multi-unit auction, value maximizers are $(1,1)$-smooth and players of type $\theta > 0$, are $(\alpha G_\theta, \alpha)$-smooth for $\alpha \geq -\theta / \ln(1-\theta)$ where

$$G_\theta = \frac{1}{\theta}\left(1 - e^{-\frac{\theta}{\alpha}}\right).$$

In the next section, we shall use the extension theorem along with the smoothness results above to find upper bounds for the price of anarchy of uniform price multi-unit auctions with budgets.

6.3 Price of anarchy

In this section we analyze lower and upper bounds for the uniform price auction. We start with a universal lower bound, move on to uniform θ auctions and lastly cover budgets.

Proposition 6.5 (rf. De Keijzer et al. [9])**.** The price of anarchy of budgetless uniform price multi-unit auctions is at least 2.

Proof. Consider a two-player budgetless uniform auction with k units. Let player 1's valuation be $m_1 = (k, 0, \dots, 0)$, player 2's be $m_1 = (1, 1, \dots, 1)$ and suppose that ties are broken in favor of player 1. Lastly, let $\theta_1, \theta_2 \in [0, 1]$ be arbitrary. Consider $b_1 = (1, \dots, 1)$ and $b_2 = (0, \dots, 0)$. Note that both players satisfy the no-overbidding assumption and the RoS constraint. Furthermore, since ties are broken in favor of player 1, player 2 cannot win a single unit without violating the no-overbidding constraint. Since player 1 receives at least one unit, they also have no incentive to deviate. Hence, $\boldsymbol{b} = (b_1, b_2)$ is a PNE. The social welfare of this PNE is k, while the optimal social welfare is $k + (k - 1)$. Hence, we obtain that the price of anarchy is bounded below by:

$$\text{POA} \geq 2 - \frac{1}{k}.$$

Taking the limit yields the desired result. $\square$

From Lemma 4.10 we know that value maximizers are $(1, 1)$-smooth. Hence, by Theorem 6.3, we find that the price of anarchy of value maximizing uniform price multi-unit auctions is bounded above by 2. Combining this with the lower bound above, yields the following result.

Proposition 6.6. For EQ $\in \{\text{PNE}, \text{MNE}, \text{CE}, \text{CCE}\}$ the EQ-LPOA of value maximizing uniform price multi-unit auction is 2.

Next, we analyze uniform price multi-unit auctions where the objective parameter profile is uniform. In other words, $[n] = N_\theta$ for some $\theta \in [0, 1]$. The price of anarchy for the case where $\theta = 1$ is covered in [3] and is found to be $1 + \frac{1}{1+W(-e^{-2})} \approx 2.18$. The lower bound in Lemma 3.10 was modified from the lower bound presented in that paper and thus was originally constructed in the setting of uniform price multi-unit auctions. We note thus that the lower bound in Lemma 3.10 will also be a lower bound for uniform price auctions with any θ-profile. This yields the following lower bound.

Lemma 6.7. For a class budgetless uniform price auctions where for any auction and any player in that auction $\theta_i \leq \theta_{\max}$ we have for any EQ $\in \{\text{PNE}, \text{MNE}, \text{CE}, \text{CCE}\}$ that:

$$\text{EQ-POA} \geq \begin{cases} 1 + \theta_{\max} \left(1 + W(-e^{-1-\theta})\right)^{-1}, & \text{if } \theta_{\max} > 1 + \frac{W(-2e^{-2})}{2} \\ 2, & \text{else.} \end{cases}$$

For the upper bound, we expect that the analysis presented in [3] generalizes to show that for pure equilibria the above bound is tight.

Lastly, we shall analyze budgets. Consider the class uniform price multi-unit auctions with budgets $\mathcal{A}_{\theta_{\max}}$ such that for any auction and any bidder in that auction $\theta_i < \theta_{\max}$ for some $\theta_{\max} \in [0, 1]$. For this class, the following theorem gives an upper bound for CCE.

Theorem 6.8. Let $\theta_{\max} \in (0,1]$, then for EQ $\in \{\text{PNE}, \text{MNE}, \text{CE}, \text{CCE}\}$ we have:

$$\text{EQ-LPOA}(\mathcal{A}_{\theta_{\max}}) \leq \frac{2\theta_{\max}}{1 - e^{-\theta_{\max}}}.$$

Proof. By the method discussed in Section 5.3 extends to uniform price auctions. Thus, we want to bound EQ-POA^w for the class of auctions $\mathcal{X}$ where $A^w \in \mathcal{X}$ iff A^w contains a value maximizer and $\theta_i \leq \theta_{\max}$ for any bidder i. We have already seen that value maximizers are $(1,1)$-smooth and that players with a price sensitivity of θ are $(\alpha G_\theta, \alpha)$ smooth for $\alpha \geq 1$. Thus, we have that players with objective parameter θ are $(G_\theta, 1)$-smooth and since G_θ is decreasing in θ we find $\min_{\theta \in [0,\theta_{\max}]} \lambda_\theta = G_{\theta_{\max}}$ and $\max_{\theta \in [0,\theta_{\max}]} \mu_\theta = 1$. Hence, by Theorem 6.3 we obtain the bound above. □

From [9] we already knew that EQ-POA $\leq \frac{2e}{e-1}$ for utility maximizing budgetless uniform price auctions. However, the above theorem shows that the same bound holds when budgets are included. In fact, in [9], they obtain a slightly better bound of 3.148 and the same optimization can be applied here in the presence of budget constraints to obtain marginally better bounds. Interestingly, value maximizers are $(1,1)$-smooth, we find that EQ-LPOA $= 2$ for value maximizing auctions. Since $\lim_{\theta \to 0} \frac{2\theta}{1-e^{-\theta}} = 2$, we find that the above theorem can be extended to $\theta_{\max} \in [0,1]$, where we know it is tight for $\theta_{\max} = 0$.

Chapter 7

Incomplete Information Setting

In this chapter we introduce Bayesian Nash equilibria for budgetless auctions and show that the extension theorems in this context are completely analogous to the extension theorems for coarse correlated equilibria in budgetless auctions. Unless otherwise specified, we assume the discriminatory pricing rule. In the interest of brevity, we only study budgetless auctions for this chapter; however, we expect the welfare objective auction analysis method to extend to Bayesian equilibria as well. The main result is that the typed smoothness framework introduced in [6] extends to Bayesian notions of equilibrium. This extends the smoothness results presented in [9] where the efficiency of Bayesian equilibria is analyzed for multi-unit auctions with utility maximizers.

7.1 Equilibria

In the incomplete information model of Harsanyi, the valuation of each player $i \in [n]$ is drawn from a finite set V_i according to a discrete probability distribution $\pi_i : V_i \to [0,1]$. The actual valuation function drawn, from the distribution V_i, is private information. The valuation profile $\boldsymbol{v}$ is drawn from a publicly known distribution, the product distribution, $\pi : \boldsymbol{V} \to [0,1]$, of $\pi_1, \dots, \pi_n$, where $\boldsymbol{V} = V_1 \times \cdots \times V_n$. In this model, bids are a function B_i, of v_i and map to a distribution over bid vectors $B_i(v_i) := B_i^{v_i}$. We write $b_i \sim B_i^{v_i}$ for the bid vector drawn from the distribution $B_i^{v_i}$ and $\boldsymbol{B}_{-i}^{\boldsymbol{v}_{-i}}$ for the distribution over bid vectors other than i.

The RoS and budget constraint we shall again require to hold pointwise. More precisely, given a bidding profile $\boldsymbol{B}_{-i}$, we say that B_i satisfies the RoS and budget constraint if and only if for all $\boldsymbol{v} \in \operatorname{supp} \pi$ and any $\boldsymbol{b} \in \operatorname{supp} \boldsymbol{B}^{\boldsymbol{v}}$ $p_i(\boldsymbol{b}) \leq v_i(\boldsymbol{b})$ and $p_i(\boldsymbol{b}) \leq C_i$. Here, we changed the letter of the budget constraint to C_i, as we already use B_i for bids. Similarly to before, we let $\mathcal{B}_i(\boldsymbol{B}_{-i})$ be the set of bids B_i that satisfy the RoS and budget constraint given the profile $\boldsymbol{B}_{-i}$.

A *Bayesian Nash equilibrium* (BNE) is a strategy profile $\boldsymbol{B} = (B_1, \dots, B_n)$ if $B_i \in \mathcal{B}_i(\boldsymbol{B}_{-i})$ for all $i \in [n]$ and for every pure strategy $b_i' \in \mathcal{B}_i(\boldsymbol{B}_{-i})$ we have:

$$\mathop{\mathbb{E}}_{\substack{\boldsymbol{w}_{-i}|v_i \\ \boldsymbol{b} \sim \boldsymbol{B}^{(v_i, \boldsymbol{w}_{-i})}}} \left[o_i^{\boldsymbol{v}}(\boldsymbol{b})\right] \geq \mathop{\mathbb{E}}_{\substack{\boldsymbol{w}_{-i}|v_i \\ \boldsymbol{b} \sim \boldsymbol{B}^{(v_i, \boldsymbol{w}_{-i})}}} \left[o_i^{\boldsymbol{v}}(b_i', \boldsymbol{b}_{-i})\right] = \mathop{\mathbb{E}}_{\substack{\boldsymbol{w}_{-i}|v_i \\ \boldsymbol{b}_{-i} \sim \boldsymbol{B}^{\boldsymbol{w}_{-i}}}} \left[o_i^{\boldsymbol{v}}(b_i', \boldsymbol{b}_{-i})\right],$$

where the expectation is taken over public distribution bidders other than i given their own valuation, that is, $\boldsymbol{w}_{-i}|v_i$, and $o_i^{\boldsymbol{v}}$ is the objective of player i such that the value is v_i.

An auction instance is now written as $A = (\pi, \boldsymbol{\theta})$, where π is the product distribution over the valuation functions. Note, then if π is a constant distribution that the Bayesian equilibria coincide with mixed or pure equilibria depending on whether B^{v_i} is a constant distribution or not. Thus, for a multi-unit auction A we remark that $\mathrm{PNE}(A) \subseteq \mathrm{MNE}(A) \subseteq \mathrm{BNE}(A)$.

We define the social welfare function $\mathrm{SW}(\boldsymbol{v}, \boldsymbol{B}^{\boldsymbol{v}}) = \mathbb{E}_{\boldsymbol{b} \sim \boldsymbol{B}^{\boldsymbol{v}}} \left[\sum_{i \in [n]} v_i(\boldsymbol{b})\right]$. Furthermore, for $\boldsymbol{v} \in \boldsymbol{V}$, we let $\boldsymbol{x}^{\boldsymbol{v}}$ denote the socially optimal allocation. The expected optimal social welfare is then $\mathbb{E}_{v\sim\pi}[\mathrm{SW}(\boldsymbol{v},\boldsymbol{x}^{\boldsymbol{v}})] := \mathbb{E}_{v\sim\pi}\left[\sum_{i\in[n]} v_i(x_i^{\boldsymbol{v}})\right]$. The *Bayesian price of anarchy* (BPOA) for an auction instance $A = (\pi, \boldsymbol{\theta})$ is then defined as:

$$\mathrm{BPOA}(A) = \sup_{\boldsymbol{B}\in\mathrm{BNE}(A)} \frac{\mathbb{E}_{\boldsymbol{v}\sim\pi}[\mathrm{SW}(\boldsymbol{v},\boldsymbol{B}^{\boldsymbol{v}})]}{\mathbb{E}_{\boldsymbol{v}\sim\pi}[\mathrm{SW}(\boldsymbol{v},\boldsymbol{x}^{\boldsymbol{v}})]}.$$

Thus, from the above definition of BPOA we find for EQ $\in$ {PNE, MNE} that EQ-POA $\leq$ BPOA. Hence, any lower bound we previously found through pure or mixed equilibria will extend to a lower bound for the BPOA.

7.2 Smoothness

In this section we will prove the extension theorems for the BPOA. We find that the upper bounds obtained for the CCE-POA will be exactly the same for the BPOA as each type will have the same smoothness. First, we define smoothness in the context of incomplete information. We first define smoothness in the context of incomplete information and calibration vectors.

Definition 7.1. Consider a $(\pi, \boldsymbol{\theta})$ multi-unit auction. A player $i \in [n]$ with objective parameter $\theta_i = \theta$ is $(\lambda_\theta, \mu_\theta)$ smooth if and only if for any $\boldsymbol{v}$ and any $\boldsymbol{b}_{-i}$ there exists a $B_i' := B_i'^{v_i}$, not dependent on $\boldsymbol{b}_{-i}$, such that for $\boldsymbol{B} = (B_i', \boldsymbol{b}_{-i})$ we have $B_j \in \mathcal{B}_j(\boldsymbol{B}_{-j})$ for all $j \in [n]$ and:

$$\mathop{\mathbb{E}}_{b_i'\sim B_i'}\left[o_i^{\boldsymbol{v}}(b_i', \boldsymbol{b}_{-i})\right] \geq \lambda_\theta v_i(x_i^{\boldsymbol{v}}) - \mu_\theta \sum_{j=1}^{x_i^{\boldsymbol{v}}} \beta_j(\boldsymbol{b}_{-i}).$$

Definition 7.2. The set of calibration vectors of a multi-unit auction instance $A = (\boldsymbol{v}, \boldsymbol{\theta})$ such that the players of type θ are $(\lambda_\theta, \mu_\theta)$ smooth is given by:

$$\mathcal{C}(A) = \left\{\boldsymbol{\delta} \in (0,1]^{|T|} \mid \max_{\theta\in T}(\delta_\theta(1-\theta)) + \max_{\theta\in T}(\delta_\theta\mu_\theta) \leq 1\right\}.$$

Now, we shall prove that the extension theorem of 4.9 works for Bayesian mixed equilibria as well.

Theorem 7.3. In an auction instance $A = (\pi, \boldsymbol{\theta})$ where for any $\theta \in T$ and any $i \in N_\theta$ is $(\lambda_\theta, \mu_\theta)$ smooth, we have the following bound for the coarse correlated price of anarchy:

$$\mathrm{BPOA}(A) \leq \min\left\{\min_{\theta\in T}\lambda_\theta, \left(\max_{\theta\in T}\left(\frac{\mu_\theta}{\lambda_\theta}\right) + \max_{\theta\in T}\left(\frac{1-\theta}{\lambda_\theta}\right)\right)^{-1}\right\}^{-1}.$$

Proof of Theorem 4.9. Consider a multi-unit auction $A = (\pi, \boldsymbol{\theta})$ and let $\boldsymbol{B} \in \mathrm{BNE}(A)$. Let $\boldsymbol{\delta} \in \mathcal{C}(A)$ and B_i' be the deviation for player i in the smoothness definition, we compute for any $\boldsymbol{v}$:

$$\begin{aligned}
\mathrm{SW}(\boldsymbol{v}, \boldsymbol{B}^{\boldsymbol{v}}) &= \sum_{\theta\in T}\sum_{i\in N_\theta} \mathop{\mathbb{E}}_{\boldsymbol{b}\sim\boldsymbol{B}^{\boldsymbol{v}}}[v_i(\boldsymbol{b})] \\
&= \sum_{\theta\in T}\sum_{i\in N_\theta} \delta_\theta \mathop{\mathbb{E}}_{\boldsymbol{b}\sim\boldsymbol{B}^{\boldsymbol{v}}}[v_i(\boldsymbol{b})] + (1-\delta_\theta)\mathop{\mathbb{E}}_{\boldsymbol{b}\sim\boldsymbol{B}^{\boldsymbol{v}}}[v_i(\boldsymbol{b})] \\
&\geq \sum_{\theta\in T}\sum_{i\in N_\theta} \delta_\theta \mathop{\mathbb{E}}_{\boldsymbol{b}\sim\boldsymbol{B}^{\boldsymbol{v}}}[v_i(\boldsymbol{b})] + (1-\delta_\theta)\mathop{\mathbb{E}}_{\boldsymbol{b}\sim\boldsymbol{B}^{\boldsymbol{v}}}[p_i(\boldsymbol{b})] && \text{RoS constraint}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta \mathop{\mathbb{E}}_{\boldsymbol{b} \sim \boldsymbol{B}^{\boldsymbol{v}}} [o_i^{\boldsymbol{v}}(\boldsymbol{b})] + (1 - \delta_\theta + \delta_\theta \theta) \mathop{\mathbb{E}}_{\boldsymbol{b} \sim \boldsymbol{B}^{\boldsymbol{v}}} [p_i(\boldsymbol{b})] \\
&\geq \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta \mathop{\mathbb{E}}_{\boldsymbol{b} \sim \boldsymbol{B}^{\boldsymbol{v}}} [o_i^{\boldsymbol{v}}(\boldsymbol{b})] + (1 - \max_{\theta \in T}(\delta_\theta(1-\theta))) \mathop{\mathbb{E}}_{\boldsymbol{b} \sim \boldsymbol{B}^{\boldsymbol{v}}} [p_i(\boldsymbol{b})]
\end{aligned}
$$

Now, using smoothness and the fact that $\boldsymbol{B} \in \mathrm{BNE}(A)$, we shall focus on the first part. We obtain:

$$
\begin{aligned}
\sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta \mathop{\mathbb{E}}_{\substack{\boldsymbol{v} \sim \pi \\ \boldsymbol{b} \sim \boldsymbol{B}^{\boldsymbol{v}}}} [o_i^{\boldsymbol{v}}(\boldsymbol{b})] &= \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta \mathop{\mathbb{E}}_{v_i \sim \pi_i} \left[\mathop{\mathbb{E}}_{\substack{\boldsymbol{v}_{-i} \sim \pi_{-i} | v_i \\ \boldsymbol{b} \sim \boldsymbol{B}^{(v_i, \boldsymbol{v}_{-i})}}} [o_i^{\boldsymbol{v}}(\boldsymbol{b})] \right] \\
&= \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta \mathop{\mathbb{E}}_{\substack{v_i \sim \pi_i \\ b_i' \sim B_i'^{v_i}}} \left[\mathop{\mathbb{E}}_{\substack{\boldsymbol{v}_{-i} \sim \pi_{-i} | v_i \\ \boldsymbol{b} \sim \boldsymbol{B}^{(v_i, \boldsymbol{v}_{-i})}}} [o_i^{\boldsymbol{v}}(\boldsymbol{b})] \right] \\
&\geq \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta \mathop{\mathbb{E}}_{\substack{v_i \sim \pi_i \\ b_i' \sim B_i'}} \left[\mathop{\mathbb{E}}_{\substack{\boldsymbol{v}_{-i} \sim \pi_{-i} | v_i \\ \boldsymbol{b} \sim \boldsymbol{B}^{(v_i, \boldsymbol{v}_{-i})}}} \left[o_i^{\boldsymbol{v}}(b_i', \boldsymbol{b}_{-i})\right] \right] && \text{Def. BNE} \\
&= \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta \mathop{\mathbb{E}}_{v_i \sim \pi_i} \left[\mathop{\mathbb{E}}_{\substack{\boldsymbol{v}_{-i} \sim \pi_{-i} | v_i \\ \boldsymbol{b}_{-i} \sim \boldsymbol{B}_{-i}^{\boldsymbol{v}_{-i}}}} \left[\mathop{\mathbb{E}}_{b_i' \sim B_i'} \left[o_i^{\boldsymbol{v}}(b_i', \boldsymbol{b}_{-i})\right] \right] \right] \\
&\geq \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta \mathop{\mathbb{E}}_{v_i \sim \pi_i} \left[\mathop{\mathbb{E}}_{\substack{\boldsymbol{v}_{-i} \sim \pi_{-i} | v_i \\ \boldsymbol{b}_{-i} \sim \boldsymbol{B}_{-i}^{\boldsymbol{v}_{-i}}}} \left[\lambda_\theta v_i(x_i^{\boldsymbol{v}}) - \mu_\theta \sum_{j=1}^{x_i^{\boldsymbol{v}}} \beta_j(\boldsymbol{b}_{-i}) \right] \right] \\
&\geq \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta \lambda_\theta \mathop{\mathbb{E}}_{\boldsymbol{v} \sim \pi} [v_i(x_i^{\boldsymbol{v}})] - \sum_{\theta \in T} \sum_{i \in N_\theta} \delta_\theta \mu_\theta \mathop{\mathbb{E}}_{\substack{\boldsymbol{v} \sim \pi \\ \boldsymbol{b} \sim \boldsymbol{B}^{\boldsymbol{v}}}} \left[\sum_{j=1}^{x_i^{\boldsymbol{v}}} \beta_j(\boldsymbol{b}) \right] \\
&\geq \min_{\theta \in T}(\delta_\theta \lambda_\theta) \mathop{\mathbb{E}}_{\boldsymbol{v} \sim \pi} [\mathrm{SW}(\boldsymbol{v}, \boldsymbol{x}^{\boldsymbol{v}})] - \max_{\theta \in T}(\delta_\theta \mu_\theta) \sum_{i \in [n]} \mathop{\mathbb{E}}_{\substack{\boldsymbol{v} \sim \pi \\ \boldsymbol{b} \sim \boldsymbol{B}^{\boldsymbol{v}}}} p_i(\boldsymbol{b})
\end{aligned}
$$

where the last inequality follows from a similar argument used in the proof of Theorem 4.2 to bound the β_j term.

Combining this with the above, we find that:

$$
\begin{aligned}
\mathop{\mathbb{E}}_{\boldsymbol{v} \sim \pi} [\mathrm{SW}(\boldsymbol{v}, \boldsymbol{B}^{\boldsymbol{v}})] \geq{}& \min_{\theta \in T}(\delta_\theta \lambda_\theta) \mathop{\mathbb{E}}_{\boldsymbol{v} \sim \pi} [\mathrm{SW}(\boldsymbol{v}, \boldsymbol{x}^{\boldsymbol{v}})] \\
&+ (1 - \max_{\theta \in T}(\delta_\theta(1-\theta)) - \max_{\theta \in T}(\delta_\theta \mu_\theta)) \sum_{i \in [n]} \mathop{\mathbb{E}}_{\substack{\boldsymbol{v} \sim \pi \\ \boldsymbol{b} \sim \boldsymbol{B}^{\boldsymbol{v}}}} [p_i(\boldsymbol{b})]
\end{aligned}
$$

Since $\boldsymbol{B} \in \mathrm{BNE}(A)$ and $\boldsymbol{\delta} \in \mathcal{C}(A)$ were arbitrary, we get BPOA $\leq (\min_{\theta \in T} \delta_\theta \lambda_\theta)^{-1}$. Optimizing this over all $\boldsymbol{\delta} \in \mathcal{C}(A)$ yields the exact same result as in Theorem 4.9, as desired. □

Now, we shall determine the smoothness parameters for each type in the incomplete information setting. Note that in the smoothness lemma for value maximizers, Lemma 4.4, we only use pure deviations. Thus, this easily lifts to the Bayesian setting. Similarly, for non-value maximizers, Lemma 4.11, we only used a mixed deviation showing that this also lifts to the Bayesian setting presented here. Thus, we obtain the following lemma.

Lemma 7.4. *Consider a multi-unit auction $A = (\pi, \boldsymbol{\theta})$. In that case, the value maximizers are*

$(1,1)$-smooth and the players of type $\theta > 0$, are $(\alpha G_\theta, \alpha)$-smooth for $\alpha \geq -\theta/\ln(1-\theta)$ where

$$G_\theta = \frac{1}{\theta}\left(1 - e^{-\frac{\theta}{\alpha}}\right).$$

Combining this lemma with the previous theorem shows that all the upper bounds obtained for CCE-POA, will also be upper bounds for BPOA. We remark that our notion of BNE is not the most general. It is the Bayesian counter part of MNE. Similarly, one may define Bayesian correlated and Bayesian coarse correlated equilibria, and the smoothness framework presented above will then extend for Bayesian coarse correlated equilibria.

Thus, any lower bound where we used a pure equilibrium gives a valid lower bound for the BPOA. Hence, for bipartite auctions, we find matching lower and upper bounds even for BPOA. For uniform θ auctions, the presented lower bound uses coarse correlated equilibria, meaning that this lower bound is not valid for the BPOA. For first-price single item auctions with utility maximizers, the Bayesian price of anarchy is not $\frac{e}{e-1}$, but $\frac{e^2}{e^2-1}$ [17]. Contrasting this with value maximizers, single-item first-price auctions are always efficient, in any equilibrium concept, since bidding truthfully dominates any other bid below the player's value. Another way to see this, is to note that for single item auctions, value maximizers are $(1,0)$-smooth, thus yielding a POA of exactly 1.

For auctions with budgets we expect that the method in section 5.3 extends to Bayesian equilibria. Hence, all the upper bounds previously discussed for CCE can still be lifted to the setting of incomplete information when budgets are included.

Chapter 8

Conclusion and Open Problems

The main results of this thesis are the following. We showed the existence of pure equilibria in multi-unit auctions (both discriminatory and uniform price) with hybrid objective parameters in the presence of RoS and budget constraint. We proved a complete description of the efficiency of pure equilibria in the presence of budgets and in budgetless case. More specifically, for the class $\mathcal{M}^{\text{disc}}_{\theta_1,\theta_2}$, which is the class of all discriminatory price multi-unit auctions where all players have a price sensitivity $\theta_1 \leq \theta \leq \theta_2$ we found:

$$\text{POA}(\mathcal{M}^{\text{disc}}_{\theta_1,\theta_2}) = \begin{cases} 1 + \theta_2 \left(1 + W(-e^{-1-\theta_2})\right)^{-1}, & \text{if } \theta_2 > 1 + \frac{W(-2e^{-2})}{2} \text{ and } \theta_1 = 0 \\ 2, & \text{else.} \end{cases}$$

For the budgetless subclass of $\mathcal{M}^{\text{disc}}_{\theta_1,\theta_2}$, call it $\mathcal{A}^{\text{disc}}_{\theta_1,\theta_2}$ we found:

$$\text{POA}(\mathcal{A}^{\text{disc}}_{\theta_1,\theta_2}) = \begin{cases} 1 + \theta_2 \left(1 + W(-e^{-1-\theta_2})\right)^{-1}, & \text{if } \theta_2 > 1 + \frac{W(-2e^{-2})}{2} \text{ and } \theta_1 = 0 \\ 2, & \text{if } \theta_2 \leq 1 + \frac{W(-2e^{-2})}{2} \text{ and } \theta_1 = 0 \\ 2 - \theta_1, & \text{if } \theta_1 > 0. \end{cases}$$

Most importantly, we described a general framework in which the efficiency of coarse correlated equilibria can be analyzed even in the presence of budgets. This was done by transforming the auction at an equilibrium to a welfare objective auction where the transformation is equilibrium dependent. Upper bounds on the efficiency of the latter would then lift to upper bound on the POA of the former. In this transformation, a player's price sensitivity, θ_i, was either preserved, or set to 0 so they would become a "value maximizer" in the welfare objective auction. This demonstrates the importance of bipartite auctions because for a class of auctions, the transformation essentially enlarges the class to add value maximizers parameterizing the POA in terms of $\theta_{\max}$, the largest price sensitivity in the class.

This framework is extended to uniform price auction and can even be extended to a different auction format alltogether, e.g., simultaneous first-price auctions with submodular valuations and possibly even more general valuation functions.

Other results include determining the price of anarchy of bipartite auctions. These auctions have been analyzed in [10] and [6]. We show that the results for simultaneous auctions also hold for multi-unit auctions. In fact, for value/utility maximizing bipartite multi-unit auctions, we have for EQ $\in \{\text{PNE}, \text{MNE}, \text{CE}, \text{CCE}\}$ that EQ-POA $= \frac{2+W(-e^{-2})}{1+W(-e^{-2})}$, since the upper bound is for CCE, while the lower bound is in terms of a pure equilibrium, Lemma 3.10. For simultaneous first-price auctions, the lower bound is in terms of a mixed equilibrium.

Furthermore, we saw that 2.18 is an upper bound for CCE for the POA for utility maximizing simultaneous first-price auctions with budgets where valuations can be submodular or even additive. Example 5.9 shows that this is tight.

Still, many open questions remain in this new model with hybrid objectives. We discuss a few.

For single item auctions, uniform objective parameter θ, what is the price of anarchy (budgetless case)? For value maximizers, we know for all equilibrium notions treated in this thesis that the auction is efficient. For utility maximizers, the auction is efficient up to correlated equilibria. For coarse correlated equilibria, the price of anarchy becomes $\frac{e}{e-1}$. What is the price of anarchy for intermediate values of $\theta \in (0, 1)$?

Similarly, the Bayesian price of anarchy of single item utility maximizing auctions is $1 - \frac{1}{e^2}$ [17]. For value maximizers, the price of anarchy is 1? As a first step, it would be interesting to see how the lower bound would generalize once a θ parameter is introduced.

What is the price of anarchy (budgetless case) of the class of discriminatory price multi-unit auctions with a uniform objective parameter θ? We found tightness results for $1 - \frac{1}{e} \leq \theta \leq 1$, but what about the domain $0 < \theta < 1 - \frac{1}{e}$? Currently, the best lower bound is in terms of pure equilibria that yield $2 - \theta$.

Do the welfare objective auctions have any meaning or are they just an artifact resulting from the formalism that allows us to compute the efficiency of equilibria in the presence of budgets? Or is there perhaps a direct analysis possible without needing to transform into such an auction?

Can the method for determining the POA in terms of effective welfare in some way be extended to the stronger version in terms of liquid welfare?

The class of uniform auctions where players have a sensitivity of at most $\theta_{\max}$ appears to be exactly the same as that of discriminatory bipartite auctions. Is there a correspondence transforming one into the other and vice versa?

Using the type dependent smoothness framework, introduced in [6], yielded success in finding tight upper bounds for discriminatory multi-unit auctions. For uniform price multi-unit auctions, we have not been able to find a type dependent extension theorem. What types would enable a type dependent extension theorem for the uniform price auction? Would it be able to push the upper bound of $\frac{2e}{e-1}$ to the best current lower bound, 2.18? Or are there perhaps more inefficient lower bounds if we consider mixed or coarse correlated equilibria? What if we consider budgets?

What is the relation between discriminatory multi-unit auctions and first-price simultaneous auctions? They admit lower bounds of similar structure and the price of anarchy for many classes, such as bipartite, is very similar. The only separation found so far is that discriminatory multi-unit auctions, the hierarchy collapse at pure equilibria regarding the price of anarchy. However, for first-price simultaneous auctions, the best pure lower bound is 2, while the upper bound is 2.18.

The coarse correlated POA of simultaneous first-price utility maximizing auctions is 2.18. However, the lower bound is in terms of a coarse correlated equilibrium. Like in the budgetless case, can we find a lower bound in terms of mixed equilibria or are mixed equilibria more efficient than coarse correlated equilibria in this context?

Simultaneous first-price auctions, discriminatory multi-unit auctions, and generalized first-price auctions, see Appendix B, all admit the exact same extension theorem and smoothness parameters. Is there an underlying mechanism that captures all these different auctions?

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Appendix A

Lambert W function

In this part of the appendix we cover some details and useful identities about the Lambert-W function. The Lambert-W function is defined as the inverse of the function $f(x) = xe^x$ for $x \geq -e^{-1}$. The constraint $x \geq -e^{-1}$ ensures that the function f is injective on that domain. This is the principal branch of the Lambert-W function. The other real branch is the inverse of the function $f(x) = xe^x$ for $x \leq -e^{-1}$. However, for all our purposes, we will only deal with the principal branch.

We note the inverse of $f(x) = xe^x$ for $x \geq -e^{-1}$ as $W : \mathbb{R}_{\geq -e^{-1}} \to \mathbb{R}_{\geq -1}$, hence the name. The domain is thus $[-e^{-1}, \infty)$ and the range is $[-1, \infty)$. A useful identity that we will use throughout this thesis is that for $z \geq 0$ we have $\ln(-W(-e^{-1-z})) = -1 - z - W(-e^{-1-z})$. To prove this, we note that by definition of W that:

$$e^{-1-z-W(-e^{-1-z})} = \frac{e^{-1-z}}{e^{W(-e^{-1-z})}} = \frac{W(e^{-1-z})e^{-1-z}}{W(e^{-1-z})e^{W(-e^{-1-z})}} = \frac{W(-e^{-1-z})e^{-1-z}}{e^{-1-z}} = W\left(-e^{-1-z}\right),$$

as desired. We refer to [8] for a more extensive treatment of this function.

Appendix B

Generalized First-Price Auction

In this section we introduce the generalized first price (GFP) auction and derive an upper bound for the price of anarchy in terms of smoothness. We follow the model presented in [24].

An instance of an *ad-auction* is given by a set $[n]$ of players, a set $[m]$ of advertisement slots, for each player a valuation per click, $\bar{v}_i \in \mathbb{R}_{\geq 0}$ and for each $i \in [n]$, $j \in [m]$ we let a_{ij} be the click-through rate of player i at position j. We assume that for each $i \in [n]$ that a_{ij} is non-increasing in j. We say that the click-through rates are *separable* if $a_{ij} = \alpha_i \gamma_j$ for all i, j.

On this ad-auction, we consider a *first price mechanism* in which each player reports a bid per click, b_i and we allocate the slots according to a greedy algorithm: allocate position 1 to player $i_1 = \operatorname{argmax}_{j \in n} a_{j1} b_j$ and allocate position 2 to player $i_2 = \operatorname{argmax}_{j \in [n] \setminus \{i_1\}} a_{j2} b_j$, etc. If player i is allocated location j, they are charged $p_i = a_{ij} b_i$.

We let $j_i(\boldsymbol{b})$ be the allocated position of player i under the bidding profile $\boldsymbol{b}$. Next, we let $a_{i0} = 0$ for all $i \in [n]$ so that if player i is not allocated a position, we let $j_i(\boldsymbol{b}) = 0$, meaning, they would not be charged. Then, the objective of player i under the bidding profile $\boldsymbol{b}$ is given by:

$$o_i(\boldsymbol{b}) = a_{ij_i(\boldsymbol{b})}(\bar{v}_i - \theta_i b_i),$$

where $\theta_i \in [0, 1]$ is the objective parameter of player i.

In this model, the RoS constraint is defined is: $p_i(\boldsymbol{b}) \leq v_i(\boldsymbol{b})$ which is equivalent to $a_{ij_i(\boldsymbol{b})} b_i \leq a_{ij_i(\boldsymbol{b})} \bar{v}_i$, or simply: $b_i \leq \bar{v}_i$, while the budget constraint is defined as $p_i(\boldsymbol{b}) \leq B_i$. For distributions over bids, we consider the pointwise extension.

Pure Nash equilibria (PNE), coarse correlated equilibria (CCE) and liquid price of anarchy (LPOA) for these solution concepts are all similarly defined as in Chapter 2.

Similarly, we define welfare objective GFP auctions, GFP^w auctions where the objective is:

$$o_i^w(\chi) := \min \left\{ \mathop{\mathbb{E}}_{\boldsymbol{b} \sim \chi} \left[a_{ij_i(\boldsymbol{b})} \bar{v}_i \right], B_i \right\} - \theta_i \mathop{\mathbb{E}}_{\boldsymbol{b} \sim \chi} [b_i],$$

$v_i^w(\chi) := \min \left\{ \mathbb{E}_{\boldsymbol{b} \sim \chi} \left[a_{ij_i(\boldsymbol{b})} \bar{v}_i \right], B_i \right\}$ and $v_i'(j) := \min\{a_{ij}\bar{v}_i, B_i\}$. Equilibria and price of anarchy for these auctions are defined similarly as in Chapter 4. Now, we define the smoothness.

> **Definition B.1.** Consider a GFP auction with $[n]$, the set of players, $[m]$ the set of locations, $\bar{\boldsymbol{v}}$ the valuation profile, a_{ij} the click-through rates and $\boldsymbol{\theta}$ the objective parameter profile. Furthermore, let $\pi(j, \boldsymbol{b})$ be the player that is allocated location j under bidding profile $\boldsymbol{b}$ and j_i^* the location of player i under the optimal allocation. A player i of type θ is $(\lambda_\theta, \mu_\theta)$-smooth for $\lambda_\theta > 0$ and $\mu_\theta \geq 0$ if for any strategy profile $\boldsymbol{b}_{-i}$ there exists a possibly mixed bid χ_i', not dependent on $\boldsymbol{b}_{-i}$, such that:
>
> $$o_i^w(\chi_i', \boldsymbol{b}_{-i}) \geq \lambda_\theta v_i'(j_i^*) - \mu_\theta a_{\pi(j_i^*, \boldsymbol{b}_{-i}) j_i^*} b_{\pi(j_i^*, \boldsymbol{b}_{-i})}.$$

In the case where $n > m$, some players are allocated no positions. For such players $i \in [n]$, we set $j_i^* = 0$ and let $\pi(j_i^*, \boldsymbol{b})$ be an arbitrary player that is not allocated a position. Furthermore, the set of calibration vectors is defined analogously to Definition 4.8. We now state and prove the extension theorem.

Theorem B.2. Consider a GFP^w auction, A^w where each player of type θ is $(\lambda_\theta, \mu_\theta)$ smooth and the set of calibration vectors is $\mathcal{C}(A^w)$. The coarse correlated price of anarchy satisfies:

$$\text{CCE-POA}^w(A^w) \le \min\left\{\min_{\theta\in T}\lambda_\theta, \left(\max_{\theta\in T}\left(\frac{\mu_\theta}{\lambda_\theta}\right) + \max_{\theta\in T}\left(\frac{1-\theta}{\lambda_\theta}\right)\right)^{-1}\right\}^{-1}.$$

Proof. Let χ be a CCE and χ_i' be the deviating bid from the smoothness definition and let $\boldsymbol{\delta} \in \mathcal{C}(A^w)$. We have that:

$$\begin{aligned}
\text{SW}^w(\chi) &= \sum_{\theta\in T}\sum_{i\in N_\theta} v_i^w(\chi) \\
&= \sum_{\theta\in T}\sum_{i\in N_\theta} \delta_\theta v_i^w(\chi) + (1-\delta_\theta)v_i^w(\chi) \\
&\ge \sum_{\theta\in T}\sum_{i\in N_\theta} \delta_\theta v_i^w(\chi) + (1-\delta_\theta)p_i(\chi) && \text{RoS and budget constraints} \\
&= \sum_{\theta\in T}\sum_{i\in N_\theta} \delta_\theta o_i^w(\chi) + (1-\delta_\theta+\delta_\theta\theta)p_i(\chi) \\
&\ge \sum_{\theta\in T}\sum_{i\in N_\theta} \delta_\theta o_i^w(\chi) + (1-\max_{\theta\in T}(\delta_\theta(1-\theta)))p_i(\chi)
\end{aligned}$$

Now, using smoothness and the fact that $\chi \in \text{CCE}(A^w)$, we shall focus on the first part. We obtain:

$$\begin{aligned}
\sum_{\theta\in T}\sum_{i\in N_\theta} \delta_\theta o_i^w(\chi) &\ge \sum_{\theta\in T}\sum_{i\in N_\theta} \delta_\theta o_i^w(\chi_i', \chi_{-i}) \\
&\ge \sum_{\theta\in T}\sum_{i\in N_\theta} \delta_\theta \mathop{\mathbb{E}}_{\boldsymbol{b}_{-i}\sim\chi_{-i}}\left[o_i^w(\chi_i', \boldsymbol{b}_{-i})\right] \\
&\ge \sum_{\theta\in T}\sum_{i\in N_\theta} \delta_\theta\lambda_\theta v_i'(j_i^*) - \delta_\theta\mu_\theta \mathop{\mathbb{E}}_{\boldsymbol{b}_{-i}\sim\chi_{-i}}\left[a_{\pi(j_i^*,\boldsymbol{b}_{-i})j_i^*} b_{\pi(j_i^*,\boldsymbol{b}_{-i})}\right] \\
&\ge \min_{\theta\in T}(\delta_\theta\lambda_\theta)\,\text{OPT}^w - \max_{\theta\in T}(\delta_\theta\mu_\theta)\sum_{i\in[n]} \mathop{\mathbb{E}}_{\boldsymbol{b}_{-i}\sim\chi_{-i}}\left[a_{\pi(j_i^*,\boldsymbol{b}_{-i})j_i^*} b_{\pi(j_i^*,\boldsymbol{b}_{-i})}\right] \\
&\ge \min_{\theta\in T}(\delta_\theta\lambda_\theta)\,\text{OPT}^w - \max_{\theta\in T}(\delta_\theta\mu_\theta)\sum_{i\in[n]} \mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}\left[a_{\pi(j_i^*,\boldsymbol{b})j_i^*} b_{\pi(j_i^*,\boldsymbol{b})}\right] \\
&= \min_{\theta\in T}(\delta_\theta\lambda_\theta)\,\text{OPT}^w - \max_{\theta\in T}(\delta_\theta\mu_\theta)\sum_{i\in[n]} \mathop{\mathbb{E}}_{\boldsymbol{b}\sim\chi}\left[a_{ij_i(\boldsymbol{b})} b_i\right] \\
&= \min_{\theta\in T}(\delta_\theta\lambda_\theta)\,\text{OPT}^w - \max_{\theta\in T}(\delta_\theta\mu_\theta)\sum_{i\in[n]} p_i(\chi).
\end{aligned}$$

Combining this with the above, we find that:

$$\begin{aligned}
\text{SW}^w(\chi) &\ge \sum_{\theta\in T}\sum_{i\in N_\theta} \delta_\theta o_i^w(\chi) + (1-\max_{\theta\in T}(\delta_\theta(1-\theta)))p_i(\chi) \\
&\ge \min_{\theta\in T}(\delta_\theta\lambda_\theta)\,\text{OPT}^w + (1-\max_{\theta\in T}(\delta_\theta(1-\theta)) - \max_{\theta\in T}(\delta_\theta\mu_\theta))\sum_{i\in[n]} p_i(\chi).
\end{aligned}$$

Since $\chi \in \text{CCE}(A^w)$ and $\delta \in \mathcal{C}(A^w)$ were arbitrary, we get $\text{CCE-POA}^w \leq (\min_{\theta \in T} \delta_\theta \lambda_\theta)^{-1}$. Optimizing this over all $\delta \in \mathcal{C}(A^w)$ yields the result $\text{CCE-POA}^w \leq (\max_{\delta \in \mathcal{C}(A^w)} \min_{\theta \in T} \delta_\theta \lambda_\theta)^{-1}$. Following the same optimization proof like in Theorem 4.9, we obtain the desired result. □

Now that we have established a smoothness framework for this type of auction, let us prove the smoothness result per type. However, we shall limit ourselves to the budgetless case and remark that budgetless welfare objective GFP auctions coincide with conventional budgetless GFP auctions. Thus, the above theorem will also work for the standard budgetless GFP auction.

Lemma B.3. Consider a budgetless GFP auction. As before, value maximizers are $(1, 1)$-smooth and let

$$G_\theta := \frac{1}{\theta}\left(1 - e^{-\frac{\theta}{\alpha}}\right).$$

Then a player i for type $\theta > 0$ is $(\alpha G_\theta, \alpha)$-smooth where $\alpha \geq -\theta / \ln(1 - \theta)$.

Proof. In the interest of brevity, we shall omit the proof of value maximizers.

If $j_i^* = 0$, then the inequality holds trivially by choosing a zero bid. So, suppose that $j_i^* > 0$, let $\boldsymbol{b}_{-i}$ be arbitrary. Following the same proof structure as in [24], for player $i \in [n]$ we consider the mixed bid χ_i' which bids $t\bar{v}_i$ where t is drawn from a distribution with pdf $f(t) = \frac{\alpha}{1-\theta t}$ supported on $[0, G_\theta]$ where:

$$G_\theta = \frac{1}{\theta}\left(1 - e^{-\frac{\theta}{\alpha}}\right).$$

Note that if t satisfies $a_{ij_i^*} t\bar{v}_i \geq a_{\pi(j_i^*, \boldsymbol{b}_{-i}) j_i^*} b_{\pi(j_i^*, \boldsymbol{b}_{-i})}$, that player i will be allocated position j_i^* or higher. In other words, for such t, we have that $a_{ij_i(t\bar{v}_i)} \geq a_{ij_i^*}$. Letting

$$\gamma_i = \frac{a_{\pi(j_i^*, \boldsymbol{b}_{-i}) j_i^*} b_{\pi(j_i^*, \boldsymbol{b}_{-i})}}{a_{ij_i^*} \bar{v}_i},$$

we obtain:

$$\begin{aligned}
o_i(\chi_1', \boldsymbol{b}_{-i}) &= \int_0^{G_\theta} a_{ij_i(t\bar{v}_i, \boldsymbol{b}_{-i})} (\bar{v}_i - \theta_i t \bar{v}_i) f(t) dt \\
&\geq \int_{\gamma_i}^{G_\theta} a_{ij_i(t\bar{v}_i, \boldsymbol{b}_{-i})} (\bar{v}_i - \theta_i t \bar{v}_i) f(t) dt \\
&\geq a_{ij_i^*} \bar{v}_i \int_{\gamma_i}^{G_\theta} (1 - \theta_i t) f(t) dt \\
&= a_{ij_i^*} \bar{v}_i \alpha (G_\theta - \gamma_i) \\
&= \alpha G_\theta a_{ij_i^*} \bar{v}_i - \alpha a_{\pi(j_i^*, \boldsymbol{b}_{-i}) j_i^*} b_{\pi(j_i^*, \boldsymbol{b}_{-i})},
\end{aligned}$$

as desired. □

The lemma above will not hold when we consider budgets because the class of valuations we consider for the GFP auction are not closed under capping. However, for the budgetless case, we see that the typed smoothness framework introduced in [6] extends further than just simultaneous and multi-unit auctions.