Complexity Reduction in Combinatorial Set Theory

MSc Thesis *(Afstudeerscriptie)*

written by

Yipu Li (李一朴)
(born July 21, 2002 in Shanghai, China)

under the supervision of **Dr Yurii Khomskii** and **Dr Vera Fischer**, and submitted to the Examinations Board in partial fulfillment of the requirements for the degree of

MSc in Logic

at the *Universiteit van Amsterdam.*

Date of the public defense:	**Members of the Thesis Committee:**
August 25, 2026	Dr. Benno van der Berg (Chair)
	Dr. Yurii Khomskii (supervisor)
	Dr. Vera Fischer (co-supervisor)
	Dr. Alexi Block Gorman
	Dr. Tristan van der Vlugt

Institute for Logic, Language and Computation

Abstract

In this thesis, we study complexity reduction problems that originate from the analysis of definable combinatorial objects. We make significant progress in lifting a family of complexity reduction results to higher point classes. Specifically, we isolate a property we call Functional Projection that makes the complexity reduction proofs lift, and we establish that Functional Projection holds for all projective levels in the model L and $L[U]$. As a consequence, we derive the existence of combinatorial objects in $L[U]$ at the optimal projective level $\mathbf{\Pi}^1_2$.

Acknowledgements

First, I would like to thank my supervisor Yurii. Since I arrived at Amsterdam, you have provided me great support in my pursuit of set theory. From the projects we did to the supervision of this thesis, I deeply appreciate the insightful discussions we had and the kind encouragements you gave to me. Finally, in working towards this thesis, you have always been a source of inspiration for me.

Besides, I wish to thank Vera. Thank you for hosting me in Vienna, and working with me on the project that eventually led to this thesis. I held worries when I arrived at the unfamiliar city of Vienna, but you welcomed me with warmth, helped me discover the excitement of research in set theory and made my stay in Vienna a cherish-able and fruitful experience.

I'm also thankful towards the committee for carefully reading my thesis, as well as the brilliant young logician community I met in both Amsterdam and Vienna. I would like to thank Julia, for working with me on the project; I thank Tenyo, Spyros and Orestis for studying descriptive set theory with me; and I thank Sebastian, Jonathan and William for the discussions we had and the support you provided me.

Moreover, I would like to thank all the friends I made in Amsterdam. May our friendships never fade. I hold special appreciation to 一辰 and 怡萌. Together with you, I discovered how joyful and fruitful life could be.

Finally, to my Mom and Dad: Thank you for your support and understanding all along. And to 荻, thank you for everything in these tough but wonderful years.

Contents

Chapter 1

Introduction

A guiding theme of set theory is how definability of mathematical structures rules out pathological behaviors. In the study of descriptive set theory, projective hierarchy is a powerful notion that allows us to reason about and classify the definability and complexity of subsets of the real numbers. In this hierarchy, the complexity of a set is measured by the complexity of the formula in the language of second-order arithmetic that defines it. By classical results of Suslin and others, being "low" on the projective hierarchy leads to desirable properties such as being Lebesgue Measurable. On the other hand, combinatorial set theory studies a variety of combinatorial families on the real line, whose origin often come from other branches of mathematics. A canonical example is the so-called maximal almost disjoint family (MAD family). A family $A \subseteq [\omega]^\omega$ is said to be "almost disjoint" if for any $x \neq y \in A$, $x \cap y$ is finite. An almost disjoint family is maximal (thus MAD) if it is not properly contained in any other almost disjoint family. Recent years, there has been a lot of works studying at which level of projective hierarchy these combinatorial sets lie. The study sheds light on the innate complexity and pathological behavior of the notion of maximal almost disjointness, and many other combinatorial properties.

As a consequence of the attempt to derive the optimal complexity of certain combinatorial families, a series of complexity reduction results have been discovered for various combinatorial families in the last decade. These results share the following structure: if there is a definable $A \subseteq \mathbb{R}$ satisfying certain combinatorial property, then there will be $B \subseteq \mathbb{R}$ satisfying the combinatorial property, yet simpler in complexity of definability. These results allow us to reduce the projective complexity of combinatorial families, thus we call them "complexity reduction results".

One notable feature of these complexity reduction results (with few exceptions) is that they are restricted to a low projective level: the $\mathbf{\Sigma}^1_2$, $\mathbf{\Pi}^1_1$ level. In this thesis, we aim to lift these complexity reduction results to higher pointclasses. We make significant progress by isolating a property called Functional Projection, and show that it is sufficient for the complexity reductions to go through. Our contribution circumvents a key difficulty in lifting these results. Proofs of complexity reduction results in the literature rely on the "uniformization property", which is known to fail for higher projective levels. On the

other hand, we show that Functional Projection holds for *all* projective levels in certain inner models of set theory, thus lifting the complexity reduction results to higher levels.

To motivate our study, let us start by introducing some background in combinatorial set theory.

1.1 Background on Combinatorial Set Theory

The study of combinatorial properties of the real line has always been a central theme of set theory. George Cantor, whose work gave birth to the study of set theory, considered the problem of whether every uncountable subset of the real line must contain a nonempty perfect set, i.e. a closed set without isolated points. This property was known as the perfect set property: A set $A \subseteq \mathbb{R}$ has the *perfect set property* if it is either countable or contains a nonempty perfect set. Cantor's interests in this property was part of his endeavor to establish the continuum hypothesis (CH). If a set has the perfect set property, then it has to be either countable or in bijection with the continuum. Cantor attempted to establish CH by showing that all subsets of the real have this property.

But of course nowadays we know Cantor's endeavor is bound to fail, for Paul Cohen showed in 1963 ([Coh63]) that the continuum hypothesis was independent of the axiomatic system of Zermelo-Fraenkel with choice (ZFC), the standard axiomatic system of mathematics. But even prior to that, Felix Bernstein was able to construct a combinatorial set that does not have the perfect set property, indeed he constructed what is now known as the Berstein set: an uncountable set that both its complement and itself intersect every perfect set on the real line.

However, there are many legacies to be collected from Cantor's study of the perfect set property, and they are closely related to the definability aspect of the real line. By the Cantor-Bendixson theorem, all closed sets have the perfect set property. Suslin ([Sus17]) greatly improved this result and showed that all $\mathbf{\Sigma}^1_1$ sets, also known as analytic sets, have the perfect set property. Along this line, we see that for sets of reals, how being definable (being of low complexity as measured by the projective hierarchy) leads to good behavior.

On the other hand, whether sets of reals have the perfect set property beyond the $\mathbf{\Sigma}^1_1$ level turns out to be independent of ZFC. On one side, starting from the assumption that there is an inaccessible cardinal, Solovay ([Sol70]) forced a model where all projective sets have the perfect set property. On the other side of the story, in Gödel's Constructible Universe L ([Göd40]) it is easy to build a $\mathbf{\Sigma}^1_2$ witness to the failure of the perfect set property by recursion along the $\mathbf{\Sigma}^1_2$-good well-order of the reals. Moreover, we are able to reduce the complexity of this $\mathbf{\Sigma}^1_2$ instance and obtain a $\mathbf{\Pi}^1_1$ witness to the failure of the perfect set property. Thus in conclusion, the problem of whether the very next projective level beyond $\mathbf{\Sigma}^1_1$ has the perfect set property is independent of ZFC.

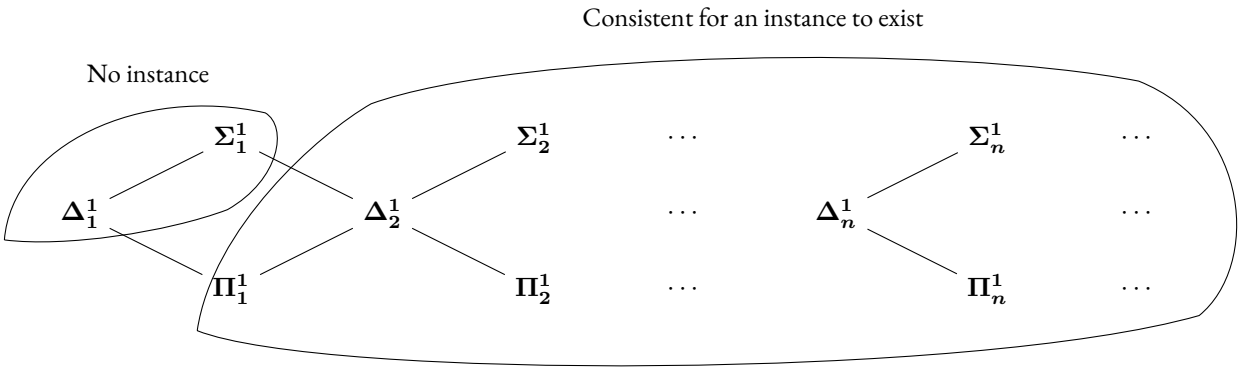
Recent decades have witnessed extensive studies of various other combinatorial sets on the real line, such as *maximal almost disjoint* (MAD) families, *maximal independent families* or *ultrafilters*.

Take MAD families as an example. By Mathias' well-known result [Mat77], there are no $\mathbf{\Sigma}^1_1$ in-

finite MAD families. On the other hand, in Gödel's Constructible Universe L it is easy to build a $\mathbf{\Sigma}^1_2$ witness to a infinite MAD family, by recursion along the $\mathbf{\Sigma}^1_2$-good well-order of the reals. In [Mil89], Miller developed a sophisticated coding technique to produce a $\mathbf{\Pi}^1_1$ infinite MAD family in L. This is consistently the optimal complexity of an infinite MAD family.

The same story is true for many other maximal combinatorial families: in [Mil89] Miller showed that there are no analytic maximal independent family by showing that a maximal independent family leads to a set without the property of Baire; while his coding method also produced a $\mathbf{\Pi}^1_1$ maximal independent family in L. In [FS22] Fischer and Schilhan showed that there are no analytic maximal towers while a coanalytic maximal tower exists in L.

We clearly see a pattern in these results: For a large class of combinatorial families (including the set without perfect set property, infinite MAD family, maximal independent family etc.), while they cannot exist on the $\mathbf{\Sigma}^1_1$ level, there is consistently an instance of them on the $\mathbf{\Pi}^1_1$ level.



A picture of the appearance of the combinatorial families we discussed on the projective hierarchy.

1.2 Complexity Reduction Results

Let us focusing our attention on obtaining a $\mathbf{\Pi}^1_1$ instance of certain combinatorial family (say MAD family) in L. By our discussion this is indeed the optimal level consistent with ZFC. One way is to appeal to Miller's coding method in [Mil89]. See also the paper [Vid12] by Vidnyánszky for an abstract discussion of Miller's method.

Yet it turned out that this sophisticated technique can be circumvented when Törnquist [Tör09] provided a direct combinatorial proof in ZFC[1] of the statement:

If there is a $\mathbf{\Sigma}^1_2$ MAD family then there is a $\mathbf{\Pi}^1_1$ MAD family.

[1] Indeed in his proof the use of choice is very minimal, so the theorem follows from certain weak forms of choice, say the principle Dependent Choice.

Thus starting from a $\mathbf{\Sigma}^1_2$ infinite MAD family, which can be easily constructed along the well-order of L, we apply the result and immediately get a $\mathbf{\Pi}^1_1$ instance of an infinite MAD family in L. But Törnquist's theorem is much stronger. As a ZFC result, the argument allows us to reduce the complexity of an optimal witness of infinite MAD family from $\mathbf{\Sigma}^1_2$ to $\mathbf{\Pi}^1_1$ in any model of set theory, thus it is very powerful in the study of optimal complexity of infinite MAD families.

Inspired by this result, various researchers established similar results for a variety of other combinatorial families, specifically for maximal independent families [BFK19], Maximal towers [FS22], Ultrafilter Bases [Sch22a], Maximal Ideal Independent Families [MS26], Maximal Orthogonal Families [FT10] and Hausdorff Gaps [Mil25] (the same statement for counterexamples of the Perfect Set Property was already well-known, see for instance the proof of Corollary 25.37 in Jech [Jec03]). They showed that for X ranging over the combinatorial families we discussed:

If there is a $\mathbf{\Sigma}^1_2$ combinatorial family X, then there is a $\mathbf{\Pi}^1_1$ combinatorial family X.

We call results of this type *complexity reduction* theorems. For a detailed survey that includes almost all currently known complexity reduction, the reader may refer to [Mil25, Chapter 2].

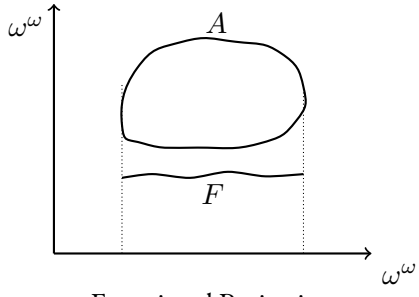
However, with the notable exception of Schilhan's result for Ultrafilter Bases ([Sch22a]), all of these results are restricted to the $\mathbf{\Sigma}^1_2$ and $\mathbf{\Pi}^1_1$ level. Thus a natural question might be asked whether these results can be lifted to higher pointclasses. Scholars including Törnquist, Schrittesser, Khomskii and Millhouse have raised the following question:

Question 1. *Is it the case that if there is a $\mathbf{\Sigma}^1_{n+1}$ combinatorial family X, then there is a $\mathbf{\Pi}^1_n$ instance, for $n > 1$?*

It turns out that a central barrier to lifting the complexity reduction results is that they rely on *Kôndo's $\mathbf{\Pi}^1_1$ Uniformization Theorem*, which informally states that $\mathbf{\Pi}^1_1$ relations admits a choice function that is $\mathbf{\Pi}^1_1$ definable, see Subsection 2.2.2 for a discussion. However, it is a ZFC result that $\mathbf{\Pi}^1_2$ uniformization fails. Thus, if we rely on this strong property of uniformization, we cannot lift the complexity reduction results to $\mathbf{\Sigma}^1_3$ and $\mathbf{\Pi}^1_2$ levels.

1.3 Main Contributions and Structure of the Thesis

The main contribution of this thesis is the isolation of a property we call *Functional Projection property*. It is a weakened version of the uniformization property. Let Γ be a point class in the projective hierarchy, a point class Γ satisfies Functional Projection (written as Γ-FP) if for every relation $A \subseteq \omega^\omega \times \omega^\omega$ in Γ, there is a partial function $F \subseteq \omega^\omega \times \omega^\omega$ that has the same projection as A, and the graph of F is in Γ.



Functional Projection

In the thesis, we show that (1) Functional Projection suffices for the complexity reduction results to lift to higher point classes, and (2) contrary to the strong property of uniformization, which is known not generalizable to certain higher point classes, we show that we can establish Function Projection for higher point classes in certain models.

It turns out that Functional Projection was studied by Kanovei and Lyubetsky [KL24] and [KL25] under a different name.[2] Among other things, in our terminology, they showed that $\mathbf{\Pi}^1_n$-FP holds in L for all levels $n \geq 1$, and that for $n \geq 2$, $\mathbf{\Pi}^1_n$-FP is independent of ZFC.

In Chapter 2, we introduce the background knowledge in descriptive set theory, combinatorial set theory and the theory of L and $L[U]$ on which our thesis is built on. In Chapter 3, we write out in detail a formal account of doing recursion along a projective good well-ordering. This is known to set theoriets, but for which it is not easy to find a written reference. This formal treatment will be important for our analysis of combinatorial objects in inner models like L and $L[U]$.

In Chapter 4, we define the Functional Projection property and then establish several elementary results about it. Moreover, we show that contrary to the uniformization property, Function Projection turns out to be genuinely weaker and can be established for higher point classes. Specifically, we establish Functional Projection for all projective levels in the inner model L and $L[U]$.[3]

Theorem 1 (Kanovei, Lyubetsky, Theorem 59 in Chapter 4). *In L, $\mathbf{\Pi}^1_n$-FP holds for all $n \geq 1$.*

Theorem 2 (Theorem 62 in Chapter 4). *In $L[U]$, $\mathbf{\Pi}^1_n$-FP holds for all $n \geq 1$.*

Moreover, we show that Functional Projection is a robust phenomenon for models with good definable well-orderings. This shows that Functional Projection is true in a wide range of models far beyond L and $L[U]$.

Theorem 3 (Theorem 58 in Chapter 4). *Let $<$ be a well-order of the reals of order type ω_1 s.t. $<$ is $\mathbf{\Delta}^1_n$ and it is $\mathbf{\Sigma}^1_n$ good ($n \geq 2$). Then $\mathbf{\Pi}^1_m$-FP holds for all $m \geq n+1$.*

[2] They called it 'uniform projection problem', but we chose to stick to our name as we think it better conveys the idea behind the concept.

[3] For the case for L, our proof is different and simpler from the proof if [KL25], and it was discovered by the author independently before he was aware of that paper.

Also, we determine the behavior of Functional Projection under the Axiom of Projective Determinacy (PD). This is an axiom that gives a detailed structural theory of projective sets.

Theorem 4 (Theorem 71 in Chapter 4)**.** *Under Projective Determinacy,* $\mathbf{\Pi}^1_{2n}$-FP *fails while* $\mathbf{\Pi}^1_{2n+1}$-FP *holds for all* $n \in \omega$.

The following picture summerizes the behavior of Functional Projection under different assumptions:

PD	$\boxed{\mathbf{\Pi}^1_1}$	$\mathbf{\Pi}^1_2$	...	$\boxed{\mathbf{\Pi}^1_{2n-1}}$	$\mathbf{\Pi}^1_{2n}$	$\cdots$
$V = L$ or $V = L[U]$	$\boxed{\mathbf{\Pi}^1_1}$	$\boxed{\mathbf{\Pi}^1_2}$	...	$\boxed{\mathbf{\Pi}^1_{2n-1}}$	$\boxed{\mathbf{\Pi}^1_{2n}}$	$\cdots$

The boxed point classes have the Functional Projection Property.

In Chapter 5, we show that Functional Projection suffices for the complexity reduction results to lift to higher point classes. Thus we make significant progress in addressing Question 1. We show that:

Theorem 5 (Theorem 72, 73, 85, 87, 93, 94 in Chapter 5)**.** *Assume* $\mathbf{\Pi}^1_n$-FP*, then the complexity reduction results lifts for all of the following objects:*

1. *Counterexample to perfect set property.*
2. *Infinite MAD family.*
3. *Maximal independent family.*
4. *Maximal (linear) towers.*
5. *Maximal Ideal independent family.*
6. *Van Douwen family.*

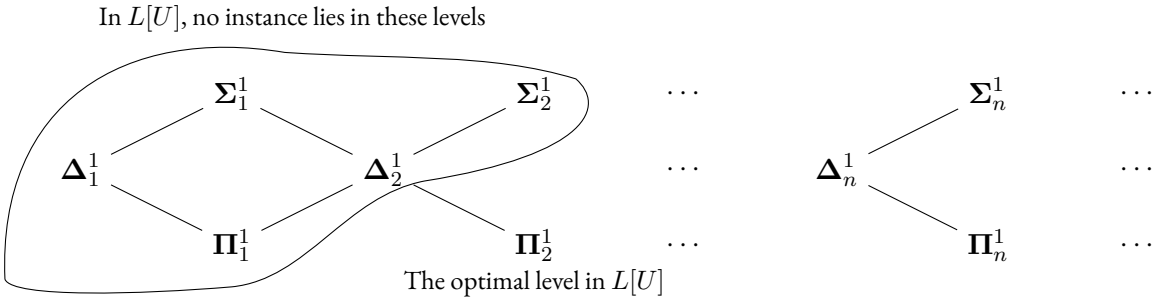
We are also interested in complexity reduction results of some strengthening of the combinatorial notions. These notions are often motivated by the study of forcing indestructibility of maximal combinatorial families. Among others, we are interested in tight MAD families (see Definition 76), we show that:

Theorem 6 (Corollary 82 in Chapter 5)**.** *Assume that* $\mathbf{\Pi}^1_n$-FP *holds, further assume that Cohen forcing is* $\mathbf{\Sigma}^1_{n+1}$ *absolute. Then if there is a* $\mathbf{\Sigma}^1_{n+1}$ *tight MAD family, then there is a* $\mathbf{\Pi}^1_n$ *tight MAD family.*

Having established Functional Projection for inner models, we are thus able to do complexity reduction proofs in these models and thus analyze optimal projective complexity of certain combinatorial families. In particular, we show that a wide range of combinatorial families admit a $\mathbf{\Pi}^1_2$ witness in $L[U]$:

Theorem 7 (Theorem 97 in Chapter 5)**.** *In $L[U]$, all the combinatorial families mentioned in Theorem 5 admit a $\mathbf{\Pi}^1_2$ witness.*

The fact that there is a $\mathbf{\Pi}^1_2$ counterexample to the perfect set property in $L[U]$ was already known to Silver and Solovay (see [Kan03]), but all the rest are new results. Moreover, the conclusion of Theorem 5 is indeed optimal in $L[U]$ for all the combinatorial families mentioned, except the maximal ideal independent family. That is because there can be no $\mathbf{\Sigma}^1_2$ instances of the combinatorial families mentioned above in $L[U]$. Whether there could be $\mathbf{\Sigma}^1_2$ instances of maximal ideal independent family remains open.



Finally, in Chapter 6, we collect other results the author obtained in his master period that do not perfectly fit into the main story of the thesis, but are nevertheless interesting.

In [BK13], Brendle and Khomskii observed that if the tower number $\mathfrak{t}$ (see Definition 23) is greater than ω_1, then there could not be $\mathbf{\Sigma}^1_2$ infinite MAD families, establishing an interesting link between cardinal invariants and definability of combinatorial families. Their proof leverages the fact that all $\mathbf{\Sigma}^1_2$ sets are $\aleph_1$ unions of Borel sets.

We investigate the relation between the tower number and $\mathbf{\Pi}^1_2$ combinatorial families. Using resources from Ramsey theory, we show that increasing the tower number $\mathfrak{t}$ kills $\mathbf{\Pi}^1_2$ Vitali sets.

Theorem 8 (Theorem 110 and Corollary 115 in Chapter 6)**.** *If $\mathfrak{t} > \omega_1$, then there is no $\mathbf{\Pi}^1_2$ E_0 transversal nor a $\mathbf{\Pi}^1_2$ Vitali set.*

Next, we investigate the combinatorial families under the general frame work of viewing them as independent sets of certain Borel graphs or hyper graphs, see [Sch20] for an overview. We develop a graph comparison notion called *p-relation* that allows us to transform a maximal independent set of one graph to that of another in a way that preserves definability.

Theorem 9 (Theorem 113 in Chapter 6)**.** *Let R be a Borel p-relation between G_0 and G_1. Then if I is a maximal independent set of G_1, $R^{-1}[I]$ is a maximal independent set of G_0. Moreover, if I is $\mathbf{\Sigma}^1_n$, then $R^{-1}[I]$ is $\mathbf{\Sigma}^1_n$.*

As an example of applying this notion, we reprove the folklore equivalence between the existence of a Vilati set and the existence of an E_0 transversal by revealing the p-relation between their underlying Borel graphs.

Chapter 2

Preliminaries

In this chapter, we develop the preliminary materials. Starting with an introduction to the coding functions and conventions we use, we review some basic facts about Descriptive Set Theory, Combinatorial Set Theory and Inner Models in Section 2.2, 2.3 and 2.4 respectively. All the material in this Chapter is standard and can be found in set theoretic textbooks such as [Mos09], [Jec03] and [Sch14].

2.1 Codings and Conventions

Before turning to the set-theoretic material, we introduce some coding conventions used throughout the thesis. Natural numbers are able to code pairs of natural numbers, let

$$\begin{aligned}\langle \cdot, \cdot \rangle : \omega \times \omega &\to \omega \\ (m, n) &\mapsto \frac{(m+n)(m+n+1)}{2} + n.\end{aligned}$$

This is the Cantor pairing function, it is bijective. By nesting this pairing function, we are able to develop a coding of ω^n for arbitrary n. Inductively, we define

$$\begin{aligned}\langle \cdot, \cdot, \cdot \rangle : \omega^3 &\to \omega \\ (n_0, n_1, n_2) &\mapsto \langle \langle n_0, n_1 \rangle, n_2 \rangle\end{aligned}$$

and with $\langle \cdot, \dots, \cdot \rangle : \omega^n \to \omega$ defined,

$$\begin{aligned}\langle \cdot, \cdot, \dots, \cdot \rangle : \omega^{n+1} &\to \omega \\ (n_0, \dots, n_n) &\mapsto \langle \langle n_0 \dots n_{n-1} \rangle, n_n \rangle\end{aligned}$$

These are coding functions of strings of natural numbers of a fixed size.

Moreover, we can code finite strings of natural number uniformly as natural numbers. Let $\{p_n \mid$

$n \in \omega\}$ enumerate the prime numbers increasingly, let $f : \omega^{<\omega} \to \mathbb{N}^+$ be the function

$$\begin{aligned} code_\omega : \omega^{<\omega} &\to \mathbb{N}^+ \\ s &\mapsto \prod_{i<|\operatorname{dom}(s)|} p_i^{s(i)+1} \end{aligned}$$

This is a bijective function by the fundamental Fundamental Theorem of Arithmetic.

Given any element $n \in \omega$, as the function $\langle \cdot, \cdot \rangle$ is bijective, there is unique $(m_0, m_1) \in \omega^2$ s.t. $\langle m_0, m_1 \rangle = n$, we write $(n)_0 := m_0$ and $(n)_1 := m_1$. These are decoder of a pair of natural numbers.

We can also code a pair of functions from natural number to natural number as a single such function. With an abuse of notation, we define,

$$\begin{aligned} \langle \cdot, \cdot \rangle : \omega^\omega \times \omega^\omega &\to \omega^\omega \\ (x, y) \mapsto \langle x, y \rangle : n &\mapsto \begin{cases} x(k) & \text{if } n = 2k \\ y(k) & \text{if } n = 2k+1 \end{cases} \end{aligned}$$

This is also a bijective coding. As long as we are clear with the type of the object we are coding, we would not confuse it with the coding with natural numbers. Similarly, this coding can be extended to code strings of elements in ω^ω of a fixed size $\langle \cdot, \dots, \cdot \rangle : (\omega^\omega)^n \to \omega^\omega$.

Finally, a single element in ω^ω is able to code countably many elements in ω^ω.

$$\begin{aligned} code_{\omega^\omega} : (\omega^\omega)^\omega &\to \omega^\omega \\ (x_0, x_1, \dots) &\mapsto code_{\omega^\omega}((x_n)_{n\in\omega}) : m \mapsto x_{(m)_0}((m)_1) \end{aligned}$$

This is also a bijective coder. For any $x \in \omega^\omega$, there is unique $(x_n, n \in \omega)$ s.t. $code_{\omega^\omega}(x_n, n \in \omega) = x$. We define the decoder function by $(x)_n = x_n$ for $n \in \omega$. The explicit form of this decoder is given by $(x)_n : m \mapsto x(\langle n, m \rangle)$.

Given an element x in 2^ω, we identify it with a subset of ω whose characteristic function is coded by x.

By $[x]^\omega$ we mean the set of countably infinite subsets of x; $[x]^{<\omega}$ is the set of finite subsets of x. Given $x \in [\omega]^\omega$, we write x^* for the function enumerating x increasingly.

2.2 Descriptive Set Theory

In this section we sketch the basics of Descriptive Set Theory. The reader may refer to [Mos09] for a detailed account. In descriptive set theory, we study Polish spaces, which is a generalization of the

concept of the real line.

Definition 1 (Polish Space)**.** *A* ***Polish Space*** *is a topological space that has a countable dense subset and admits a complete metric.*

We say a Polish space is ***perfect*** *if it contains no isolated points.*

Canonical examples of Polish spaces are as follows:

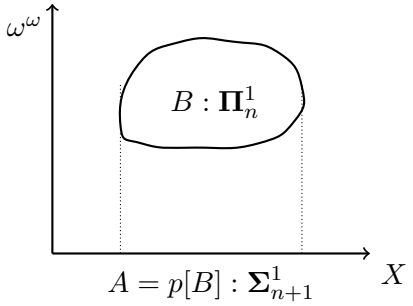
- ω equipped with the discrete topology. This is not a perfect Polish spaces.
- $\mathbb{R}$, $\mathbb{R}^n$ The real line and its products.
- The Baire space: ω^ω equipped with the product topology. The metric on the Baire space is defined by
$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 2^{-k} & \text{if } x \neq y, k = \min\{n \mid x(n) \neq y(n)\} \end{cases}$$
- The Cantor space: 2^ω equipped with the product topology. The Cantor space can be considered as a closed subspace of the Baire space.
- $[\omega]^\omega := \{x \subseteq \omega \mid x \text{ is infinite}\}$, considered as a G_δ subspace of the Cantor space. It turns out that $[\omega]^\omega$ is homeomorphic to the Baire space via the following map:
$$\begin{aligned} \pi : \omega^\omega &\to [\omega]^\omega \\ x &\mapsto \{n + \sum_{i=0}^{n} x(i) \mid n \in \omega\} \end{aligned}$$

In this thesis, In this thesis, we adopt the common set-theoretic convention of referring to elements of arbitrary Polish spaces, and in particular elements of Baire space, as "reals."

Upon a Polish space X, we define the following hierarchy, called the **Projective Hierarchy**. The idea behind this hierarchy is that it measures the complexity and definability of subsets of real number. For a set $A \subseteq X$:

A is $\mathbf{\Sigma}^1_1$, or **analytic**, if there is a closed $C \subseteq X \times \omega^\omega$ s.t. $A = p[C] := \{x \in X \mid \exists y \in \omega^\omega, (x,y) \in C\}$. Analytic sets are projections of closed sets. A is $\mathbf{\Pi}^1_1$, or **coanalytic**, if $X \setminus A$ is analytic.

Inductively, A is $\mathbf{\Sigma}^1_{n+1}$ if there is $B \subseteq X \times \omega^\omega$ that is $\mathbf{\Pi}^1_n$ s.t. $A = p[B] := \{x \in X \mid \exists y \in \omega^\omega, (x,y) \in B\}$. $\mathbf{\Sigma}^1_{n+1}$ sets are projections of $\mathbf{\Pi}^1_n$ sets. A is $\mathbf{\Pi}^1_{n+1}$ if $X \setminus A$ is $\mathbf{\Sigma}^1_{n+1}$. A is $\mathbf{\Delta}^1_n$ if it is both $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$.

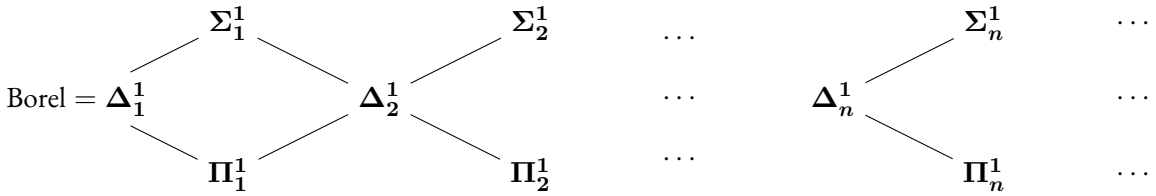


Projections of $\mathbf{\Pi}^1_n$ sets are $\mathbf{\Sigma}^1_{n+1}$ sets.

A fundamental fact is that $\mathbf{\Delta}^1_1$ sets are precisely the Borel sets.

Theorem 2 (Suslin, see 2E.2 in [Mos09]). *Upon a Polish space, a set is Borel iff it is in* $\mathbf{\Delta}^1_1$.

The inclusion relation of projective point classes is given by the following map, where point class on the right hand side of a link properly include point class on the let hand side of the link. Point classes that are not linked are incomparable.



On the other hand, we are also interested in the **Kleene point classes**, they are the effective version of the projective point class.

Fix an enumeration $(s_n, n \in \omega)$ of $\omega^{<\omega}$ according to $code_\omega$, i.e. $code_\omega(s_n) = n$, we write $[s] := \{x \in \omega^\omega \mid s \subseteq x\}$. For the sake of introducing of Kleene point classes, we identify $\omega^\omega \times \omega^\omega$ with ω^ω via the pairing map $\langle \cdot, \cdot \rangle : \omega^\omega \times \omega^\omega \to \omega^\omega$.

A set $A \subseteq \omega^\omega$ is **recursively open** if $A = \bigcup_{i \in \omega} [s_{\epsilon(i)}]$ for some recursive $\epsilon \in \omega^\omega$. A set is **recursively closed** if its complement is recursively open.

$A \subseteq \omega^\omega$ is Σ^1_1 if there is recursively closed $C \subseteq \omega^\omega \times \omega^\omega$ s.t. $A = p[C]$. Inductively, $A \subseteq \omega^\omega$ is Σ^1_{n+1} if there is Π^1_n set $C \subseteq \omega^\omega \times \omega^\omega$ s.t. $A = p[C]$. A is Π^1_{n+1} if its complement is Σ^1_{n+1}.

Finally, there is the parametrized version of the Kleene point class. For $a \in \omega^\omega$, $A \subseteq \omega^\omega$ is $\Sigma^1_n(a)$ (or $\Pi^1_n(a)$) if there is $C \subseteq \omega^\omega \times \omega^\omega$ that is Σ^1_n (or Π^1_n) s.t. $A = \{x \mid (x, a) \in C\}$. $\Delta^1_n(a) = \Sigma^1_n(a) \cap \Pi^1_n(a)$. It turns out that $\mathbf{\Sigma}^1_n = \bigcup_{a \in \omega^\omega} \Sigma^1_n(a)$ and $\mathbf{\Pi}^1_n = \bigcup_{a \in \omega^\omega} \Pi^1_n(a)$.

Similarly, the Kleene hierarchy can also be defined on subsets of ω.Let $* : \omega \times \omega^\omega \to \omega^\omega$ be the

following bijective coding function:

$$*(n,x)(i) = \begin{cases} n & \text{if } i = 0 \\ x(i-1) & \text{if } i > 0 \end{cases}$$

Then we say $a \subseteq \omega$ is Σ^1_1 if there is $C \subseteq \omega \times \omega^\omega$ s.t. $*[C]$ is recursively closed in ω^ω and s.t. $a = p[C]$. Inductively, $a \subseteq \omega$ is Σ^1_{n+1} if there is $C \subseteq \omega^\omega \times \omega^\omega$ s.t. $*[C]$ us Π^1_n and $a = p[C]$. A is Π^1_{n+1} if its complement is Σ^1_{n+1}. Paramatrized version is similarly defined.

The projective point classes are also called boldface point classes and the Kleene point classes are their lightface counterparts. For simplicity, our main focus in this thesis will be on boldface point classes, but we notice that most of the results we developed can easily be adapted to work for the more restrictive light face version, and we will explicitly discuss its failure if a result does not admit a light face version. We introduce the lightface point classes because they are necessary for introducing the absoluteness results in the later part of this chapter and they are necessary for Section 4.3.

By our way of defining the projective hierarchy, each projective set would correspond to a formula. For instance a $\Pi^1_1(a)$ set $A \subseteq \omega^\omega$ can be written as

$$A(x) \iff B(x,a) \iff \forall y \in \omega^\omega, C(x,y,a) \iff \forall y \in \omega^\omega, \exists i \langle x,y,a\rangle \in [s_{\epsilon(i)}]$$

Here $B \subseteq \omega^\omega \times \omega^\omega$ is Π^1_1, $C \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ is recursively open and ϵ is a recursive real. Let us write $\exists^{\mathbb{R}}, \forall^{\mathbb{R}}$ to indicate that they range over the Baire space (thus a real number quantifier), and $\exists^\omega, \forall^\omega$ to indicate that they range over ω. An inductive analysis immediately gives the following correspondence:

A set $A \subseteq X$ is $\Sigma^1_n(a)$ iff there is ϵ recursive s.t.
$x \in A \iff \exists^{\mathbb{R}} y_0 \dots Q^{\mathbb{R}}_{n-1} y_{n-1} Q^\omega_n i, \langle x, y_0 \dots y_{n-1}, a\rangle \in [s_{\epsilon(i)}]$ for a recursive ϵ
A set $A \subseteq X$ is $\Pi^1_n(a)$ iff there is ϵ recursive s.t.
$x \in A \iff \forall^{\mathbb{R}} y_0 \dots Q'^{\mathbb{R}}_n y_{n-1} Q'^\omega_{n+1} i, \langle x, y_0 \dots y_{n-1}, a\rangle \in [s_{\epsilon(i)}]$ for a recursive ϵ (*)

Here Q_i is $\forall$ if i is odd and $\exists$ if i is even. Q'_i is $\exists$ if i is odd and $\forall$ if i is even.

It is worth discussing how the interpretations of projective sets vary across models of set theory. For a projective $A \subseteq \omega^\omega$, it corresponds to a formula $\varphi(x)$ in the form of (*). The formula could have different interpretations in different models, depending on the interpretation of the basic open neighbourhood $[s]$ and the interpretation of the quantification over ω^ω.

Here we give an example to indicate how a projective predicate could have different interpretations across models:

Example 3. *Let X be a perfect Polish space and $U \subseteq X$ be some nonempty recursively open*

set, corresponding to $U(x) \iff \exists i \in \omega, x \in [s_{\epsilon(i)}]$ *for a recurive* ϵ*. Let* M *be a countable transitive model. Then the interpretation of* $\varphi(x) := \exists i \in \omega, x \in [s_{\epsilon(i)}]$ *is necessarily different in* M *from* V*. In* V*,* $\varphi(x)$ *being a nonempty open set, has to be uncountable. Yet* $\varphi^M(x)$*, as a subset of* M*, is a countable set evaluated in* V*, as* M *itself is countable. Thus the interpretation of* φ *is different in* M *from that in* V*.*

As open sets are Borel, this is a $\mathbf{\Delta}^1_1$ *set whose interpretation depends on models.*

In this thesis, we adopt a convention in the literature. For a projective A and a model M, by A^M we mean the evaluation of the formula $\varphi(x)$ in M, where $\varphi(x)$ is the formula defining A as in (*). i.e. $A^M := \{x \in \mathbb{R} \cap M \mid M \models \varphi(x)\}$. When A is evaluated in V, we usually drop the superscript and simply write A instead of A^V.

2.2.1 Some Classical Results in Descriptive Set Theory

For X, Y Polish spaces, a map $f : X \to Y$ is called **Borel** if for any open set $U \subseteq Y$, $f^{-1}[U]$ is Borel.

Lemma 4 (Lusin-Suslin, [Mos09, 1G.1 and 2E.4])**.** *$f : X \to Y$ is Borel iff its graph is Borel in $X \times Y$.*

It is immediate from this Lemma that if a Borel map is both injective and surjective, then its inverse would also be Borel. We call such map **Borel isomorphism**. It turns out that up to Borel isomorphism, there is only one perfect Polish space.

Theorem 5 ([Mos09, 1G.4])**.** *Perfect Polish spaces are Borel isomorphic to each other.*

The injective images of Borel sets by Borel functions are Borel:

Theorem 6 (Lusin-Suslin, [Mos09, 1G.5 and 2E.7])**.** *If $f : X \to Y$ is a Borel map and $B \subseteq X$ is a Borel set s.t. f is one-to-one on B, then $f[B]$ is Borel.*

As a consequence, $\mathbf{\Pi}^1_n$ sets are closed under Borel injections.

Lemma 7. *If $f : X \to Y$ is a Borel injection and $A \subseteq X$ is a $\mathbf{\Pi}^1_n$, then $f[A]$ is $\mathbf{\Pi}^1_n$.*

Proof. We may assume $A \neq X$ as otherwise $f[A]$ is Borel thus $\mathbf{\Pi}^1_n$ by Theorem 6. Fix $x_0 \notin A$ and extend f^{-1} to a function $g : Y \to X$, by

$$g(y) = \begin{cases} f^{-1}(y) & \text{if } y \in f[X] \\ x_0 & \text{if } y \notin f[X] \end{cases}$$

By Theorem 6, $f[X]$ is Borel and thus $Graph(g) = Graph(f^{-1}) \cup ((Y - f[X]) \times \{x_0\})$ is Borel. So g is a Borel function by Lemma 4. Thus $f[A] = g^{-1}[A]$ is $\mathbf{\Pi}^1_n$ as $\mathbf{\Pi}^1_n$ sets are closed under Borel pre-images. □

Notice there is a difference between the assumption on f in Theorem 6 and Lemma 7. In Theorem 6, f is assumed to be one-to-one on B while in Lemma 7 f has to be one-to-one on the whole space. The requirement in the later Lemma is stronger, and indeed we cannot weaken this assumption to f is one-to-one on A in that lemma.[1]

Theorem 8 (Sierpinski, [Jec03, Theorem 25.19]). *$\mathbf{\Sigma}^1_2$ sets are $\aleph_1$ unions of Borel sets.*

Finally, we collect the absoluteness results for low pointclasses:

Theorem 9 (Mostowski, [Jec03, Theorem 25.4]). *Let $P \subseteq \omega^\omega$ be a $\mathbf{\Sigma}^1_1$ predicate, say P is $\Sigma^1_1(a)$. Let M be any transitive model of ZFC without the powerset axiom containing a. Then*

$$P^M = P \cap M$$

On the other hand, $\mathbf{\Sigma}^1_2$ absoluteness requires more, in particular, the model has to contain ω_1.

Theorem 10 (Shoenfield, [Jec03, Theorem 25.20]). *Let $P \subseteq \omega^\omega$ be a $\mathbf{\Sigma}^1_2$ predicate, say P is $\Sigma^1_2(a)$. Let M be any transitive model of set theory containing a and ω_1. Then*

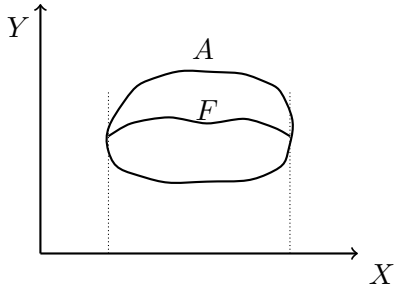
$$P^M = P \cap M$$

2.2.2 The Uniformization Problem

Definition 11 (Uniformization). *For X, Y Polish spaces and $A \subseteq X \times Y$, we say that $F \subseteq X \times Y$ **uniformizes** A if the following holds:*

- *$F \subseteq A$.*
- *For any $x \in X$, $\exists y \in Y, (x,y) \in A \iff \exists y \in Y, (x,y) \in F$.*
- *F is a partial function from X to Y, i.e, for any $x \in dom(F)$, there is unique y s.t. $(x,y) \in F$.*

[1] We can give an example where $A \subseteq X \times Y$ is $\mathbf{\Pi}^1_1$, $p : X \times Y \to X$ is continuous and one-to-one on A yet $p[A]$ is no longer $\mathbf{\Pi}^1_1$. We need resources from the next subsection: Let $C \subseteq X$ be a $\mathbf{\Sigma}^1_2$ set that is not $\mathbf{\Pi}^1_1$, let $C(x) \iff \exists y A(x,y)$ where $A \subseteq X \times Y$ is $\mathbf{\Pi}^1_1$. Let F be $\mathbf{\Pi}^1_1$ uniformize A, then the projection map $p : X \times Y \to X$ is one-to-one on F, yet $p[F] = p[A] = C$ which is not $\mathbf{\Pi}^1_1$.



Uniformization

It is clear that under Axiom of Choice, every relation can be uniformized. But it is a rule of thumb that whenever we apply AC, the object we get is hard to describe. In descriptive set theory we are interested in uniformizing a relation in a definable way.

A pointclass Γ is said to have the **uniformization property** if for any Polish spaces X, Y and $A \subseteq X \times Y$ that is Γ, there is F in Γ that uniformizes it. We write Γ-Unif if Γ has the uniformization property.

In a sense, Γ-Unif can be understood as Γ relations admit a 'definable choice function, of complexity Γ'.

By a celebrated result of Kondo, $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_2$ sets have the uniformization property.

Theorem 12 (Kondo, [Mos09, 4E.4])**.** $\mathbf{\Pi}^1_1$-Unif *and* $\mathbf{\Sigma}^1_2$-Unif *holds.*

Kondo's uniformization theorem is an essential resource on which most of the complexity reduction results we mentioned were built. (see [Tör09], [BFK19], [MS26], [FT10], [Mil25], [FS22])

On the other hand, the uniformization property fails for $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_2$ sets, This follows from the general observation that projective pointclass and its dual point classes cannot both have uniformization property. For Γ, write $\neg\Gamma$ as its dual pointclass, i.e. $A \in \Gamma$ iff $X \setminus A \in \neg\Gamma$ for $A \subseteq X$.

Theorem 13 (Folklore)**.** *Let* Γ *be one of* $\mathbf{\Sigma}^1_n$ *or* $\mathbf{\Pi}^1_n$*, then* Γ *and* $\neg\Gamma$ *cannot both have the uniformization property.*

Proof. Assume for contradiction that both Γ and $\neg\Gamma$ have the uniformization property.

Let $G \subseteq \omega^\omega \times \omega^\omega$ be a universal Γ set, meaning for any Γ set P in ω^ω, $P = G_y := \{x \in \omega^\omega \mid (y, x) \in G\}$ for some y. For existence of such set, see 1D.2 in [Mos09]. For $x, y \in \omega^\omega$ and $z = \langle x, y\rangle$, let's write $\langle z\rangle_0 = x$ and $\langle z\rangle_1 = y$.

Write

$$P(y,x) \iff G(\langle y\rangle_0, x) \qquad\qquad Q(y,x) \iff G(\langle y\rangle_1, x)$$

Since Γ has uniformization property, consider $M \subseteq \omega^\omega \times \omega^\omega \times \omega$ where

$$M(y,x,n) \iff (n = 0 \wedge P(y,x)) \vee (n = 1 \wedge Q(y,x))$$

Uniformize it by M^* that is Γ. Thus $P^*(y,x) \Leftrightarrow M(y,x,0)$, $Q^*(y,x) \Leftrightarrow M(y,x,1)$ are disjoint from each other, satisfies $P^* \subseteq P$, $Q^* \subseteq Q$ and $P^* \cup Q^* = P \cup Q$.

Consider $N \subseteq \omega^\omega \times \omega^\omega \times \omega$ where

$$N(y, x, n) \iff (n = 0 \wedge \neg P^*(y, x)) \vee (n = 1 \wedge \neg Q^*(y, x))$$

This is a $\neg\Gamma$ set and thus it can be uniformized by some N^* that is $\neg\Gamma$. Let $S(y, x) \Leftrightarrow N^*(y, x, 0)$, $T(y, x) \Leftrightarrow N^*(y, x, 1)$, they are disjoint from each other and satisfies $S \subseteq \neg P^*$ and $T \subseteq \neg Q^*$ and $S \cup T = \neg P^* \cup \neg Q^* = \omega^\omega \times \omega^\omega$. Thus $S = \neg T$ and both of them are Γ.

Now consider the set

$$D(x) \iff S(x, x) \qquad E(x) \iff \neg S(x, x)$$

These are Γ and therefore by universality of G, there is u, v s.t.

$$S(x, x) \iff D(x) \iff (u, x) \in G \iff (\langle v, u\rangle, x) \in Q \tag{2.1}$$

$$\neg S(x, x) \iff E(x) \iff (v, x) \in G \iff (\langle v, u\rangle, x) \in P \tag{2.2}$$

Now consider whether $(\langle v, u\rangle, \langle v, u\rangle) \in S$, we will show that both options lead to a contradiction. If $(\langle v, u\rangle, \langle v, u\rangle) \in S$, then $(\langle v, u\rangle, \langle v, u\rangle) \in Q$ by (2.1). If $(\langle v, u\rangle, \langle v, u\rangle) \in P$, then $(\langle v, u\rangle, \langle v, u\rangle) \in \neg S$ by (2.2), a contradiction. If $(\langle v, u\rangle, \langle v, u\rangle) \notin P$, by $Q \setminus P \subseteq Q^* \subseteq \neg S$, this is also a contradiction. The other case where $(\langle v, u\rangle, \langle v, u\rangle) \in \neg S$ is similar.

This concludes the proof. □

Corollary 14 (Folklore)**.** $\mathbf{\Sigma}^1_1$-Unif *and* $\mathbf{\Pi}^1_2$-Unif *fails.*

Proof. Immediate by Theorem 12 and 13. □

We remark that the above argument is quite general[2]. The result for $\mathbf{\Sigma}^1_1$ pointclass can indeed be greatly improved, indeed there is a closed set that is not uniformizable by any $\mathbf{\Sigma}^1_1$ set, see 4D.11 in [Mos09].

Thus, we see an innate difficulty in establishing uniformization property to higher point classes: $\mathbf{\Pi}^1_2$ uniformization is known to fail. Thus relying solely on the uniformization property, there is a difficulty lifting complexity reduction results to the $\mathbf{\Sigma}^1_3$, $\mathbf{\Pi}^1_2$ level.

Moreover, we remark that these results are the best possible under ZFC. Uniformization property for higher pointclasses would depend on stronger assumptions beyond ZFC. In particular, $\mathbf{\Pi}^1_n$ uniformization for even n seems to be a very unnatural assumption. In both the model L and models of projective determinacy, uniformization for all even $\mathbf{\Pi}^1_n$ levels fail, see 5A.2 and 6C.8 in [Mos09]. Hoffelner recently developed a technique of forcing uniformization for even $\mathbf{\Pi}$ levels, but his technique relies

[2]What we are really showing is that uniformization property of Γ leads to *reduction property* of Γ and *separation property* of the dual point class of Γ, and then we argue that no projective point class can satisfy reduction and separation at the same time. For what these property are and a reference of the proof presented above, see exercise 25.7 - 25.11 in [Jec03].

on large cardinal assumptions in the ground model, see [Hof22]. Thus, there is also tremendous difficulties in lifting complexity reduction results to the $\mathbf{\Sigma}^1_{n+1}$, $\mathbf{\Pi}^1_n$ level for even n, if we are only considering uniformization property.

2.3 Combinatorial Set Theory

In this section, we introduce several combinatorial families that we aim to analyze in the thesis.

Definition 15 (Maximal almost disjoint family)**.** *A family $A \subseteq [\omega]^\omega$ is said to be **almost disjoint** if for any $x \neq y \in A$, $x \cap y$ is finite.*

An almost disjoint family is maximal (thus MAD) if it is not properly contained in any other almost disjoint family.

Equivalently, an almost disjoint family A is maximal iff there is no $x \notin A$ s.t. for any $y \in A$, $x \cap y$ is finite.

Finite MAD family exists trivially, for instance $\{\omega\}$ is a MAD family. So we are interested in infinite MAD families. Henceforth unless otherwise specified, MAD families will refer to infinite MAD families.

Definition 16 (Maximal independent family)**.** *A family $A \subseteq [\omega]^\omega$ is **independent** if for all $k, l < \omega$ and distinct $x_0 \dots x_k, y_0 \dots y_l \in A$, the set $\bigcap_{i \leq k} x_i \setminus \bigcup_{j \leq l} y_j$ is infinite.*

*An independent family is **maximal (mif)** if it is not properly contained in any other independent family.*

Similarly, we have that an independent family is maximal iff there is no $x \in [\omega]^\omega$ s.t. for any distinct $x_0 \dots x_k, y_0 \dots y_l \in A$, both $(\bigcap_{i \leq k} x_i \setminus \bigcup_{j \leq l} y_j) \setminus x$ and $x \cap (\bigcap_{i \leq k} x_i \setminus \bigcup_{j \leq l} y_j)$ is infinite.

Definition 17 (Maximal ideal independent family)**.** *A family $I \subseteq [\omega]^\omega$ is **ideal independent** if for all $F \in [I]^{<\omega}$ and $a \in I \setminus F$, $a \not\subseteq^* \bigcup F$.*

An ideal independent family is maximal if it is not properly contained in any other ideal independent family.

Equivalently, an ideal independent family I is maximal iff for every $b \in [\omega]^\omega$ there is finite $F \subseteq I$ s.t. either

1. $b \subseteq^* \bigcup F$ or
2. there is $a \in I \setminus F$ s.t. $a \subseteq^* b \cup \bigcup F$.

Similar to MAD families, finite Maximal ideal independent family trivially exists, for instance $\{\omega\}$ is such a family. We are interested in infinite ones.

Definition 18 (Maximal eventually different family)**.** *A family $E \subseteq \omega^\omega$ is **eventually different** if for all $x \neq y \in E$, there is $n \in \omega$ s.t. for all $m \geq n$ $x(m) \neq y(m)$.*

An eventually different family is maximal if it is not properly contained in any other eventually different family.

Similarly, an eventually different family E is maximal iff for every $x \in \omega^\omega$, there is $y \in E$ s.t. there are infinitely many m s.t. $x(m) = y(m)$.

It turns out that all of the combinatorial families discussed above can be put under a very general framework, allowing many general theorems about them to be proved.

Let X be a Polish space, then $X^{<\omega} := \bigcup_{n<\omega} X^n$ admits a Polish space structure, considered as the disjoint union of $X^n, n \in \omega$. We say $E \subseteq X^{<\omega}$ is a set of **hyper-edges** if membership in E depends only on the finite set of entries, not on their order or repetition. Formally, E is a set of hypergraph if whenever two finite sequences $(x_i)_{i<n}$ and $(y_j)_{j<m}$ enumerate the same finite subset of X, we have $(x_i)_{i<n} \in E \iff (y_j)_{j<m} \in E$. If E is a set of hyper-edges, we abuse the notation and simply assume its argument is a finite set of elements from X.

In case $E \subseteq X^2$, then we say it is a set of binary edges. We say $G = (X, E)$ is a Borel hyper-graph if E is Borel. In the case that E is binary edges, we simply say $G = (X, E)$ is a binary Borel graph.[3]

We say $I \subseteq G$ is an **independent set** if for any distinct $x_0 \dots x_n \in G$, $\neg E\{x_0 \dots x_n\}$. i.e. none of the elements are connected. An independent set is maximal if it is not a proper subset of any other independent set.

Now, all of the maximal combinatorial families can be viewed as maximal independent sets of certain Borel hyper-graph. As examples, we give the graph behind MAD family and maximal independent families. Finding the graphs behind the other two families is routine.

Example 19. *MAD family corresponds to maximal independent set of the following binary graph $([\omega]^\omega, E_{MAD})$ where*

$$xE_{MAD}y \iff |x \cap y| = \infty$$

Notice that finite MAD families are also maximal independent sets in this graph, so we are interested in infinite maximal independent sets in this graph.

Example 20. *Maximal independent families corresponds to maximal independent set of the following hyper-graph $([\omega]^\omega, E_{MIF})$ where*

$$E_{MIF}(x_0 \dots x_n) \iff \exists s, t \leq n, \exists i_0 \dots i_s, j_0 \dots j_t \leq n,$$
$$(x_{i_0} \dots x_{i_s}, x_{j_0} \dots x_{j_t} \text{ are pairwise different} \wedge \bigcap_{m \leq s} x_{i_m} \setminus \bigcup_{m \leq t} x_{j_m} \text{ is finite.})$$

[3]Our definition of binary Borel graph is different from the standard definition of graph in the literature. In the literature, binary edges of graphs have to be irreflexive, but this properties turn out not to be essential for our purpose. Notice by our definition of hyper-edges, binary edges are still symmetric.

It is clear that E_{MIF} is a set of Borel hyper-edges.

The study of maximal combinatorial families under the perspective of hyper-graphs has been a quickly evolving field. See [Sch20] for an overview and [Sch22b], [Sch26] for pioneering results under this perspective. However, for our purpose, we are satisfied with some simple observations.

Say a set $I \subseteq G$ is **spanning** if for any $x \notin I$, there are distinct $x_0 \dots x_n \in I$ s.t. $E\{x_0 \dots x_n, x\}$.

Proposition 21. *For an independent set $I \subseteq G$, it is a maximal independent set iff it is spanning.*

Proof. $\Rightarrow$: If there is $x \notin I$, s.t. there is no distinct $x_0 \dots x_n \in I$ s.t. $E\{x_0 \dots x_n, x\}$, then $I \cup \{x\}$ is independent. Thus I is not maximal.

$\Leftarrow$: If I is not maximal, then there is I' independent s.t. $I' \supset I$, take $x \in I' \setminus I$, then by assumption of independence of I', there is no distinct $x_0 \dots x_n \in I$ s.t. $E\{x_0 \dots x_n, x\}$. □

This result explains the equivalent characterization of maximality we introduced before for almost disjoint families, independent families etc.

Proposition 22. *Let G be a Borel hyper-graph, if I is a $\mathbf{\Sigma}^1_n$ maximal independent set of G, then I is $\mathbf{\Delta}^1_n$.*

Proof. Consider the formula

$$x \notin I \iff \exists n \exists x_0 \dots x_n (x_0 \dots x_n \in I \wedge x, x_0 \dots x_n \text{ are distinct from each other} \wedge E\{x_0 \dots x_n, x\})$$

This gives a $\mathbf{\Sigma}^1_n$ definition of $X \setminus I$, thus making I $\mathbf{\Delta}^1_n$. To see this equivalence hold, $\Rightarrow$ follows from the fact that I is spanning, and $\Leftarrow$ follows from the fact that I is independent. □

There are other combinatorial families we are interested in, and they don't fall under the framework just introduced, so we introduce them separately. For $x, y \in 2^\omega$ we say x **is almost included in** y, written $x \subseteq^* y$, if $x \setminus y$ is finite.

Definition 23 (Maximal (Linear) Tower)**.** *A collection $A \subseteq [\omega]^\omega$ is said to be a **tower** if it is wellordered by the relation $\supseteq^*$. If A is only linearly ordered by $\supseteq^*$ we say it is a **linear tower**.*

A (linear) tower is maximal if it cannot be extended, i.e. there is no $z \in [\omega]^\omega \setminus A$ s.t. for all $x \in A$, $x \supseteq^* z$.

The tower is connected to the so called tower number.

Definition 24 (Tower Number)**.** *The **tower number** $\mathfrak{t}$ is the least cardinal κ s.t. there is a maximal tower of size κ.*

We are interested in combinatorial properties that are preserved under certain forcings. Let A be a maximal almost disjoint family, let $\mathbb{P}$ be arbitrary forcing notion and G be a $\mathbb{P}$ generic filter over V, then in $V[G]$ A would still be an almost disjoint family purely because the property of being almost disjoint is Borel, thus absolute by Theorem 9. The same can be said for independent families, ideal independent families and towers etc. However, whether A remains maximal in the forcing extension is a more interesting question.

Definition 25 (Forcing indestructibility)**.** *Let A be a maximal almost disjoint family (independent family, ideal independent family, eventually different family, tower), we say that A is $\mathbb{P}$ **indestructible** for a forcing notion $\mathbb{P}$ if for any G that is $\mathbb{P}$ generic over V, A remains maximal in $V[G]$.*

2.3.1 Ramsey Property and other Regularities

Regularity properties are desirable properties of sets of reals. One property that plays an important role in combinatorial set theory is the Ramsey property. For $x \subseteq \omega$, we write $[x]^\omega := \{y \subseteq x \mid y \text{ is infinite.}\}$. For $s \in \omega^{<\omega}$, we identify it with its range. Then for $x \in [\omega]^\omega$ s.t. $min(x) > max(s)$, we write $[s, x] := \{y \subseteq s \cup x \mid s \subseteq y \text{ and } y \text{ is infinite.}\}$

Definition 26 (Ramsey Property)**.** *On $[\omega]^\omega$, we say $A \subseteq [\omega]^\omega$ has the **Ramsey Property** if there is $x \in [\omega]^\omega$ s.t. either $A \cap [x]^\omega = \emptyset$ or $A \supseteq [x]^\omega$.*

We say $A \subseteq [\omega]^\omega$ is completely Ramsey if for any $[s, x]$ there is infinite $y \subseteq x$ s.t. $[s, y] \cap A = \emptyset$ or $A \supseteq [s, y]$.

We are also interested in the following properties:

Definition 27 (Regularity Properties)**.**

1. *Upon a Polish space X, $A \subseteq X$ has the **Perfect Set Property** if it is either countable or contains a homeomorphic copy of the Cantor space 2^ω.*
2. *A set $A \subseteq X$ has the **Baire Property** if there is a Borel set B s.t. $A \Delta B$ is meager.*
3. *A set $A \subseteq X$ is **Lebesgue Measurable** if there is a Borel set B s.t. $A \Delta B$ is null.*

We care about whether sets in a pointclass satisfy a certain property. In the case all sets in a point class Γ have the Ramsey property, perfect set property, the Baire property, Lebesgue Measurability, we write $\mathsf{RP}(\Gamma), \mathsf{PSP}(\Gamma), \mathsf{BP}(\Gamma), \mathsf{LM}(\Gamma)$ respectively.

ZFC proves that all analytic sets have the regularity properties listed above. The results for PSP, BP and LM were proved by Suslin. The result for Ramsey property was proved by Mathias.

Theorem 28 (Suslin, Mathias)**.** $\mathsf{RP}(\mathbf{\Sigma}^1_1), \mathsf{PSP}(\mathbf{\Sigma}^1_1), \mathsf{BP}(\mathbf{\Sigma}^1_1), \mathsf{LM}(\mathbf{\Sigma}^1_1)$.

Proof. For the result for PSP, see Theorem 11.18 in [Jec03]. For the rest, see Proposition 2.2.3 in [Kho12]. □

An important feature of the study of combinatorial properties of the real line is that regularity properties often forbid the existence of pathological combinatorial families. In the following, we collect several such results that we will use.

By a new result in [HLZ26], Ramsey regularity kills a series of pathological sets at the same $\mathbf{\Sigma}$ level.

Theorem 29 (He, Luo and Zhang, in [HLZ26])**.** *Assume* $\mathsf{RP}(\mathbf{\Sigma}^1_n)$*, then*

1. *There is no* $\mathbf{\Sigma}^1_n$ *MAD family.*[4]
2. *There is no* $\mathbf{\Sigma}^1_n$ *maximal independent family.*

On the other hand, in [FS22], Fischer and Schilhan observed that if a tower contains an uncountable perfect set, then $\subseteq^*$ well-orders that the perfect set as a subspace. This is impossible as there cannot be Borel well-order of Polish spaces. Thus perfect set property kills maximal towers:

Theorem 30 (Fischer and Schilhan, [FS22, Theorem 2.2])**.** *Let* T *be a tower, then* T *cannot contain a perfect set. In particular, maximal towers do not have perfect set property.*

As a corollary of these results and Theorem 28, infinite MAD families, maximal independent families and maximal towers cannot appear at the $\mathbf{\Sigma}^1_1$ level.

On the other hand, it seems that the status of maximal eventually different family is quite different from the combinatorial families discussed above. In [HS24], Horowitz and Shelah constructed a Borel maximal eventually different family without appealing to Axiom of Choice. Schrittesser further observed that in fact there is a closed one in [Sch17].

Given the existence of a simple instance of maximal eventually different family in the sense of projective complexity, this hints that the combinatorial notion of being maximally eventually different is not pathological at all, as it is able to coexist with every regularity notion we introduced. Thus, they stand in stark contrast to the other combinatorial families.

However, However, we are still interested in certain strengthenings of the notion of a maximal eventually different family$\cdots$, for instance Van Douwen families studied in [MS26]. These concepts will be introduced in subsection 5.1.5.

[4]We note that He, Luo and Zhang's result on MAD family does not yield the corresponding statement for lightface Σ^1_n point class, as they relied on extra real parameters. In the lightface language, their result can be stated as: If there is a Σ^1_n MAD family, then there is a real a and a $\Sigma^1_n(a)$ set that does not have Ramsey property.

2.4 The Theory of L and $L[U]$

In this section we sketch the basic theory of L and $L[U]$. Since the study of these two inner models is a complicated and deep subject, we will be satisfied with collecting the results we need instead of a systematic theory. The reader may refer to Chapter 13 and 19 in [Jec03] for a detailed theory.

Real numbers are able to code countable models. For our purpose, we care about models with a binary relation and a unary predicate. Thus for $x \in \omega^\omega$, we associate a model $M_x := (\omega, E^0_x, P^0_x)$ coded by it where

$$(n, m) \in E^0_x \iff x(2 \times \langle n, m\rangle) = 0$$

and

$$n \in P^0_x \iff x(2n + 1) = 0$$

i.e. the even coordinates of x code the binary relation, and the odd coordinates code the unary predicate. In the case that P^0_x is empty, we simply write $M_x = (\omega, E^0_x)$.

We desire to interpret E^0_x as the set membership relation, notice E^0_x is a well-founded relation can be expressed by the following formula, which is $\mathbf{\Pi}^1_1$:

$$\forall f \in \omega^\omega \neg\forall n \in \omega, \big(f(n + 1)E^0_x f(n)\big)$$

In this case we say that x codes a well-founded structure. We write $tr(M_x) = (\overline{tr(M_x)}, \in, P_x)$ as the transitive collapse of M_x, the collapsing map is denoted as π_x. From now on we stop distinguishing the domain of $tr(M_x)$ and the model itself.

A key feature of real codes for countable model is that any quantifiers over that model becomes natural number quantifiers. Thus satisfaction relation on that model is simple in complexity. This is a feature we will exploit in the following chapters.

Lemma 31 (See also [Jec03, Lemma 25.25])**.** *Let c code a well-founded structure. For any formula $\varphi(v, a_0 \dots a_n, P_x)$ where $a_o \dots a_n \in tr(M_c)$, for any $x \in \omega^\omega$, the statement $x \in tr(M_c) \wedge tr(M_c) \models \varphi(x, a_0 \dots a_n, P_c)$ is Δ^1_1 (Indeed arithmetical) in $x, a_0 \dots a_n, c$.*

Proof. The statement $x \in tr(M_c) \wedge tr(M_c) \models \varphi(x, a_0 \dots a_n, P_c)$ is equivalent to:

$$\exists i, m_0 \dots m_n \big(x = \pi_c(i) \wedge a_0 = \pi_c(m_0) \wedge \dots \wedge a_n = \pi_c(m_n) \wedge M_c \models \varphi(i, m_0 \dots m_n, P^0_c)\big) y$$

That $M_c \models \varphi(i, m_0 \dots m_n, P^0_c)$ is Δ^1_1 in c can be shown by an induction on the complexity of φ. It suffice to analyze the complexity of $x = \pi_c(i)$ where $x \in \omega^\omega, i \in \omega$. First we notice if $k \in \omega$, then

$$\pi_c(i) = k \iff \exists r_0 \dots r_k \in \omega(i = r_k \wedge M_c \models (r_0 = \emptyset \wedge r_1 = r_0 \cup \{r_0\} \wedge \dots \wedge r_k = r_{k-1} \cup \{r_{k-1}\}))$$

Thus if $x \subseteq \omega$,

$$\pi_c(i) = x \iff \forall n\big(nE_c^0 i \leftrightarrow \pi_c(n) \in x\big)$$

A similar formula would define $\pi_c(i) = x$ for $x \in \omega^\omega$.

This completes the proof. □

2.4.1 The Constructible Universe L

For the theory of L, the constructible universe, real numbers are able to code countable initial segments of L. In case $M_x = (\omega, E_x^0) \models V = L$, then $tr(M_x) \cong L_\alpha$ for some $\alpha < \omega_1$.

Upon L, we have the canonical well-order $<_L$. A remarkable fact about $<_L$ is that initial segments of L agrees with L on $<_L$ for elements already in L. Let $\varphi_<(x, y)$ be the formula s.t. $L \models \varphi_<(x, y) \iff x <_L y$.

Lemma 32 (see [Jec03, Lemma 13.19])**.** *For any real $x \in \omega^\omega$ and ordinal α s.t. $x \in L_\alpha$,*

1. *For all $y \in \omega^\omega \cap L$ s.t. $y <_L x$, $y \in L_\alpha$.*
2. *$L \models \varphi_<(x, y) \iff L_\alpha \models \varphi_<(x, y)$*

As a consequence, the canonical well-order restricted on the reals becomes $\mathbf{\Delta}^1_2$, and moreover becomes good in the following sense:

Definition 33 (Γ good well-ordering, see also [Mos09, 5A])**.** *Let $\leq$ be a well-ordering for a Polish space X, we say $\leq$ is $\mathbf{\Sigma}^1_n$ **good** if for Polish space Z and $P \subseteq Z \times X$ in $\mathbf{\Sigma}^1_m$ where $m \geq n$, the following relations are both in $\mathbf{\Sigma}^1_m$:*

$$Q(z, x) \iff \exists y \leq x, P(z, y)$$
$$R(z, x) \iff \forall y \leq x, P(z, y)$$

i.e. $\leq$ is good if doing bounded quantification under $\leq$ preserves $\mathbf{\Sigma}^1_m$ ness.

In the literature, there is another definition of goodness, see for instance [Jec03, Lemma 25.27]. Which states that $\leq$ is $\mathbf{\Sigma}^1_n$ good if the relation 'z codes the initial segments before x' is a $\mathbf{\Sigma}^1_n$ relation. The next proposition shows that they are equivalent:

Proposition 34. *Let $\leq$ be a $\mathbf{\Delta}^1_n$ well-order of the reals of order type ω_1, then $\leq$ is $\mathbf{\Sigma}^1_n$ good iff there is a relation R on ω^ω that is $\mathbf{\Sigma}^1_n$ s.t.*

$$R(z, x) \iff \{(z)_n \mid n \in \omega\} = \{y \mid y \leq x\}$$

Proof. $\Leftarrow$: Under the assumption, let $P \subseteq \omega^\omega \times \omega^\omega$ be a $\mathbf{\Sigma}^1_m$ relation for $m \geq n$, then

$$\exists y' \leq y, P(x, y') \iff \exists z(R(z, y) \wedge \exists n P(x, (z)_n))$$
$$\forall y' \leq y, P(x, y') \iff \exists z(R(z, y) \wedge \forall n P(x, (z)_n))$$

Showing that they are both $\mathbf{\Sigma}^1_m$. The case for arbitrary Polish space follows from Theorem 5.

$\Rightarrow$: Write $R(z, x) \iff \forall n (z)_n \leq x \wedge \forall x' \leq x, \exists n, (z)_n = x'$. □

Theorem 35 (Godel, see [Jec03, Lemma 25.27]). *In L, the canonical well-ordering $<_L$ restricted to the reals is a $\mathbf{\Delta}^1_2$ relation and is a $\mathbf{\Sigma}^1_2$ good well-ordering of order type ω_1^L.*

2.4.2 Measurable Cardinal

A cardinal κ is a **measurable cardinal** is a cardinal with a non-principal $< \kappa$-complete ultrafilter on it.

In the presence of a measurable cardinal, the assumption that $V = L$ fails badly.

Theorem 36 ([Kan03, 9.1(a)]). *If there is a measurable cardinal, then for any $a \in \omega^\omega$, $\omega_1^{L[a]} < \omega_1$. Indeed ω_1 is inaccessible in $L[a]$.*

Assuming there is a measurable cardinal, all Σ^1_2 sets have the regularity properties listed above.

Theorem 37 (Solovay, Silver). *Assume that there is a measurable cardinal, then all of the following holds:* $\mathsf{RP}(\mathbf{\Sigma}^1_2), \mathsf{PSP}(\mathbf{\Sigma}^1_2), \mathsf{BP}(\mathbf{\Sigma}^1_2), \mathsf{LM}(\mathbf{\Sigma}^1_2)$.

Proof. For the part for Ramsey, Baire property and Lebesgue measurability, by Corollary 2.2.7 in [Kho12], if for all $a \in \omega^\omega, \omega_1^{L[a]} < \omega_1$ then all $\mathbf{\Sigma}^1_2$ have the above properties. Apply Theorem 36 and we are done.

For the perfect set property part, use Theorem 14.10 in [Kan03] and Theorem 36. □

Moreover, it turns out that $\mathbf{\Sigma}^1_3$ sets are absolute between the ground model and extensions by small forcings. Compare this with Theorem 9 and 10.

Theorem 38 (Martin, Solovay). *Let κ be a measurable cardinal and $\mathbb{P}$ be a forcing poset with $|\mathbb{P}| < \kappa$. Let G be $\mathbb{P}$ generic over V, let P be a $\mathbf{\Sigma}^1_3$ predicate, then*

$$P^{V[G]} \cap V = P^V$$

More abstractly, we have the following definition:

Definition 39 (Generic Absoluteness). *We say that a forcing $\mathbb{P}$ is $\mathbf{\Sigma}^1_n$ **generic absolute** if for any G that is $\mathbb{P}$ generic over V, A that is $\mathbf{\Sigma}^1_n$,*

$$A^{V[G]} \cap V = A^V$$

Put in this language, what Theorem 38 showed is that if there is measurable cardinal, then every forcing of size less than that cardinal is $\mathbf{\Sigma}^1_3$ absolute.

2.4.3 The Canonical Model with One Measurable Cardinal $L[U]$

Let κ be the first measurable cardinal and U be a normal $< \kappa$ complete ultrafilter on κ, i.e. U is closed under diagonal intersection, see Chapter 10 in [Jec03]. Then the model $L[U]$ would satisfy κ is the unique measurable cardinal. This is the canonical model with one measurable cardinal.

A detailed theory of $L[U]$ is beyond the scope of the thesis and also unnecessary for our purpose. We merely recall some facts that will be used:

As U is a subset of $\mathcal{P}(\kappa)$, this means that when $\delta < \kappa$, the initial segment of $L[U]$, $L_\delta[U]$ receives no information from U. Thus in contrast to L, initial segments of $L[U]$ would not agree with it on the canonical order. This time, the models that plays the role of L_α in the case of L are called iterable premice.

Definition 40 (see also page 264 in [Kan03]). *A structure $(M, \in V)$ is a premouse if*

1. *$(M, \in V) \models V$ is a normal ultrafilter on κ.*
2. *$M \models ZFC^-$.*
3. *$M = L_\xi[V]$ for some ξ.*

$(M, \in, V)$ is called iterable if there is a system $(M_\alpha, V_\alpha, j_{\alpha\beta} \mid \alpha \leq \beta \in Ord)$ of transitive well-founded models s.t.

1. *$(M_0, V_0) = (M, V)$.*
2. *$j_{\alpha\beta} : M_\alpha \to M_\beta$ are elementary embeddings that commute.*
3. *If $\beta = \alpha + 1$ is a successor, $M_\beta = Ult(M_\alpha, V_\alpha)$ and $j_{\alpha\beta}$ is the ultrapower map.*
4. *If γ is a limit, then (M_γ, V_γ) is the transitive collapse of direct limit of $((M_\alpha, \in, V_\alpha), j_{\alpha,\beta} \mid \alpha \leq \beta < \gamma)$. $j_{\alpha\gamma}$ are given by the direct limit map composed with the collapsing map.*

The key fact of iterable premice is that they agree with $L[U]$ on initial segments of the constructible order. Let $\varphi_<(x, y, P)$ be the formula s.t. $L[P] \models \varphi_<(x, y, P) \iff x <_{L[P]} y$. Compare the following Theorem with Theorem 35.

Lemma 41 (see page 271 on [Kan03]). *For any iterable premouse $(M, \in, V)$, $x \in L[U] \cap \omega^\omega$, if $x \in M$,*

1. *For any* $y \in L[U] \cap \omega^\omega$ *s.t.* $y <_{L[U]} x$, $y \in M$ *and* $(M, \in, V) \models \varphi_<(y, x, V)$ *i.e.* $y <_{L[V]} x$.

2. $y <_{L[U]} x \iff (M, \in, V) \models \varphi_<(y, x, V) \iff y <_{L[V]} x$

A consequence is that $<_{L[U]}$ restricted to the reals is a $\mathbf{\Sigma}^1_3$ good well ordering.

Theorem 42 (Silver, see page 272 in [Kan03]). *Assume* $V = L[U]$,

1. *'*x *codes a model* (ω, E^0_x, P^0_x) *whose transitive collapse* $(tr(M_x), \in, P_x)$ *is an iterable premouse' is a* $\mathbf{\Pi}^1_2$ *relation for* x.

2. $<_{L[U]} |_{2^\omega}$ *is a* $\mathbf{\Delta}^1_3$ *well-order of the reals of order type* ω_1, *and it is* $\mathbf{\Sigma}^1_3$ *good.*

There is a natural well ordering on $(L_\alpha, \alpha < \omega_1)$, namely we compare them by their height α. The following notion gives a well-ordering of iterable premice, but of course it is considerably more complicated. Define the following ordering on iterable premice:

$(M, \in, U) <_{ip} (N, \in, W)$ iff there is F s.t. $(L_\xi[F], \in, F \cap L_\xi[F])$ is an iterate of $(M, \in, U)$, $(L_\eta[F], \in, F \cap L_\eta[F])$ is an iterate of $(N, \in, W)$ and $\xi < \eta$. By the comparison Lemma, this order is well-defined, see Lemma 20.8 in [Kan03].

Lemma 43 (see page 273 in [Kan03]). $<_{ip}$ *is a well-ordering on iterable set premice.*

Lemma 44 ([Kan03, Theorem 20.21]). *For* $y \in L[U] \cap \omega^\omega$, *the relation '*$tr(M_x)$ *is the* $<_{ip}$ *least countable iterable premouse containing* y*' is a* $\mathbf{\Pi}^1_2$ *relation for* x, y.

Chapter 3

Recursion Along Good Projective Well-orderings

In this chapter, we work out in detail the analysis of transfinite recursion on the reals along a good projective well-ordering. The analysis is known to set theorists, but for which it is not easy to find a written reference. The analysis would be important in our analysis of combinatorial objects in models with a good projective well-ordering, for instance L. The framework we are using here for transfinite recursion is from [Vid12].

First, we recall that real numbers are able to code countable ordinals. For $x \in \omega^\omega$, we associate a relation on ω by

$$n \preceq_x m \iff x(\langle n, m\rangle) = 0$$

Let

$$\begin{aligned} x \in LO \iff & \forall n, m \in \omega, (n \preceq_x m \vee m \preceq_x n) \\ & \wedge \forall n \in \omega, (n \preceq_x n) \\ & \wedge \forall n, m \in \omega, (n \preceq_x m \wedge m \preceq_x n \to n = m) \\ & \wedge \forall n, m, l \in \omega, (n \preceq_x m \wedge m \preceq_x l \to n \preceq_x l) \end{aligned}$$

Clearly $x \in LO$ iff $\preceq_x$ is a linear order.

The following predicate defines reals coding a well order:

$$x \in WO \iff x \in LO \wedge \neg\exists f \in \omega^\omega \forall n, \big(f(n+1) \prec_x f(n)\big)$$

This is a $\mathbf{\Pi}^1_1$ predicate. Indeed this is a $\mathbf{\Pi}^1_1$ complete predicate, in the sense that for any $\mathbf{\Pi}^1_1$ $A \subseteq X$ there is continuous $f : X \to \omega^\omega$ s.t. $x \in A \iff f(x) \in WO$. See for instance Theorem 25.12 in [Jec03].

Let X be a Polish space, then there is a natural Polish space structure on $X^{\leq\omega} := \bigcup_{\alpha\leq\omega} X^\alpha$, considered as the disjoint union of the spaces $X^n, n \in \omega$ and X^ω. Let B be an uncountable subset of any Polish space, this will be the set of codes of requirements each step of recursion has to respect. Recall that for a set $S \subseteq X \times Y \times Z$, the section of S along $(x, y) \in X \times Y$ is defined as $S_{x,y} := \{z \in Z \mid (x, y, z) \in S\}$. Fix a Borel operation $\oplus : X^{\leq\omega} \times X^{\leq\omega} \to X^{\leq\omega}$ such that $\text{ran}(A \oplus B) = \text{ran}(A) \cup \text{ran}(B)$.

Definition 45. *Let $F \subseteq X^{\leq\omega} \times B \times X$ and $BC \in X^{\leq\omega}$. Then we say $Y \subseteq X$ is **compatible with F from base case** BC if there is an enumeration of $B = \{p_\alpha \mid \alpha < \omega_1\}$, $Y = \{x_\alpha \mid \alpha < \omega_1\}$ and for every $\alpha < \omega_1$ a sequence $A_\alpha \in X^{\leq\omega}$ that enumerates $\{x_\beta \mid \beta < \alpha\}$ of order type $\leq \omega$ without repetition s.t. $\forall\alpha < \omega_1, (x_\alpha \in F_{A\oplus BC,p_\alpha})$ holds.*

The definition says that at each step α, the element in Y is picked from $F_{A\oplus BC,p_\alpha}$, depending on the set of previous choices A_α, the base case BC and a special requirement at this step p_α.

There are two notable features about this definition: Firstly, the definition is supposed to be used in models of CH, outside models of CH, B need not be enumerated by a set of order type ω_1. While this is sufficient for our purpose, this is still a significant restriction. Secondly, by this definition every set compatible with F is uncountable, thus we are not considering the recursion processes that halts after finite or countable steps. For our purpose, these restrictions does not matter, for in our study of recursion process producing a definable maximal combinatorial families, we are always interested in doing recursion in model of CH with a projective good well-ordering to produce an uncountable set.

Let's illustrate this definition with some examples, from now on until end of the section, we work in models of CH:

Example 46. *Consider arbitrary Borel hyper-graph $G = (X, E)$, we show that there is a $F_G \subseteq X^{\leq\omega} \times X \times X$ s.t. any I compatible with F_G from base case $\emptyset$ will be a maximal independent set for G.*

Proof. Define $F_G(A, p, x)$ holds iff EITHER:

all of the following are true:

1. $\text{ran}(A)$ is independent.
2. $\text{ran}(A) \cup \{p\}$ is independent.
3. There is $x_0 \dots x_n \in \text{ran}(A)$ s.t. $E\{p, x, x_0 \dots x_n\}$, and $\text{ran}(A) \cup \{x\}$ is independent.

OR 1 is true and 2 is false and $\text{ran}(A) \cup \{x\}$ is independent.

OR 1 is false.

We show that if I is compatible with F_G. Let $X = \{p_\alpha \mid \alpha < \omega_1\}$, $I = \{x_\alpha \mid \alpha < \omega_1\}$ and $\{A_\alpha \mid \alpha < \omega_1\}$ witness that I is compatible with F_G. Then by an induction, A_α is independent for

all $\alpha < \omega_1$. Thus $I = \bigcup_{\alpha<\omega_1} A_\alpha$ is independent. Moreover, I is spanning as for any $p_\alpha \in X$, either $\mathrm{ran}(A) \cup \{p\}$ is independent or 3 holds, either way, there is $x_0 \dots x_n \in F$ s.t. $E\{p, x, x_0 \dots x_n\}$. Thus by Proposition 21, I is a maximal independent set. □

Thus, we see that MAD families, maximal independent families, maximal ideal independent families and maximal eventually different families all fall under this framework: There are some recursion process F s.t. these combinatorial objects can be viewed as compatible sets of the recursion.

We have to be a bit cautious in the case of MAD family, finite MAD family are also maximal independent in the graph behind MAD family. To solve this we may start our recursion from some base case which code some fixed infinite partition of ω. The same is true for infinite maximal ideal independent family. We write it out explicitly:

Example 47 (MAD family)**.** *Let $G = (X, E_{MAD})$, where E_{MAD} is defined as example 19, let $\{a_n \mid n \in \omega\}$ be an infinite partition of ω, then for any I compatible with F_G from base case $A_0 : n \mapsto a_n$, $I \cup \{a_n \mid n \in \omega\}$ will be an infinite MAD family.*

Let's see another example that is not a maximal independent set, we construct a maximal tower along recursion.

Example 48. *Let $F_T \subseteq ([\omega]^\omega)^{\leq\omega} \times [\omega]^\omega \times [\omega]^\omega$ be defined by*

$$F_T(A, p, x) \iff \Big[\exists y \in WO\ \forall n, m \in \mathrm{dom}(A)\big(n \preceq_y m \leftrightarrow A(n) \supseteq^* A(m)\big)\Big] \\ \to \Big(|p \setminus x| = \infty \wedge \forall n \in \mathrm{dom}(A)(x \subseteq^* A(n))\Big).$$

Then compatible sets of F_T are maximal towers. Moreover, such compatible set exists. Finally, a simple complexity analysis gives that F_T is $\mathbf{\Sigma}^1_1$.

Proof. Let $X = \{p_\alpha \mid \alpha < \omega_1\}$, $T = \{x_\alpha \mid \alpha < \omega_1\}$ and $\{A_\alpha \mid \alpha < \omega_1\}$ witness that T is compatible with F_T. Then we can prove by induction that $x_\beta \supseteq^* x_\alpha$ iff $\beta \leq \alpha$ as x_α is a pseudo intersection of $A_\alpha = \{x_\beta \mid \beta < \alpha\}$. It is maximal as any $p \in [\omega]^\omega$ is enumerated as some p_α, and thus $p_\alpha \setminus x_\alpha$ is infinite, witnessing p_α is not a pseudo intersection of T.

To show existence of such compatible set, it suffice to show that for any countable tower $A = \{b_\beta \mid \beta \leq \alpha < \omega_1\}$ and infinite $p \subseteq \omega$, there is always a pseudo intersection x of A s.t. $p \setminus x$ is infinite. Let $A = \{a_n \mid n \in \omega\}$, recursively pick m_n s.t.

1. $m_n \in a_0 \cap \dots \cap a_n$.
2. $m_n > m_{n-1}$ and $p \cap (m_{n-1}, m_n) \neq \emptyset$.

Such m_n can be found as $a_0 \ldots a_n$ is linearly ordered by $\subseteq^*$, one of them is least by $\subseteq^*$, thus they have infinite intersection. And it is clear that $\{m_n \mid n \in \omega\} \subseteq^* a_n$ for all n and $p \setminus \{m_n \mid n \in \omega\}$ is infinite. □

Now we show that doing recursion along a good projective well-order does produce a projective set.

Theorem 49 (Folklore)**.** *Let $<$ be a $\mathbf{\Delta}^1_n$ well-ordering of the reals that is $\mathbf{\Sigma}^1_n$ good ($n \geq 2$) and is of order type ω_1. Let B be a $\mathbf{\Delta}^1_n$ subset of Polish space M and is uncountable, this is the set of requirements.*

Then for any $\mathbf{\Delta}^1_n$ $F \subseteq X^{\leq\omega} \times B \times X$, $BC \in X^{\leq\omega}$, s.t. for any $A \in X^{\leq\omega}, p \in X$, $F_{A\oplus BC,p} \setminus (ran(BC) \cup ran(A)) \neq \emptyset$, then there is a $\mathbf{\Sigma}^1_n$ set Y compatible with F from base case BC.

The requirement that $F_{A\oplus BC,p} \setminus (\text{ran}(BC) \cup \text{ran}(A)) \neq \emptyset$ makes sure that we always have an element to choose in each recursion step. The proof idea is to look for the $<$ least element that satisfies the requirement at each recursion step, and use real numbers to code recursion processes up to $\alpha < \omega_1$.

Proof of Theorem 49. Consider the following predicate $Rec \subseteq \omega^\omega \times X^{\leq\omega} \times X \times M^{\leq\omega} \times M$: $Rec(w, A, a, P, p)$ holds iff EITHER:

It is the base case, i.e. all of the following holds:

1.1 $dom(A) = dom(P) = \emptyset$.

1.2 p is the $<$ least element of B. $\forall p' \leq p, (p' \in B \to p = p')$[1].

1.3 a is the $<$ least element not in $\text{ran}(A \oplus BC)$ s.t. $F(A \oplus BC, p)$, i.e. $a \notin \text{ran}(A \oplus BC)$ and $F(A \oplus BC, p)$ and $\forall a' < a, (a' \notin \text{ran}(A \oplus BC) \to \neg F(A \oplus BC, p))$.

OR it is the recursion step, i.e. all of the following holds:

2.1 $w \in WO, dom(A) = dom(P) > 0$.

2.2 A, P, as ordered by the order coded by w, is a recursion process that picks the $<$ least element: For all $n \in dom(A)$ there is $A' \in X^{\leq\omega}, P' \in M^{\leq\omega}$ s.t.

- $\text{ran}(A') = \{A(m) \mid m \preceq_w n\} \wedge \text{ran}(P') = \{P(m) \mid m \preceq_w n\}$.
- $F(A' \oplus BC, P(n), A(n))$ and $\forall a' < A(n), \neg F(A' \oplus BC, P(n), a')$.

[1]This is a bizarre way to write $\forall p' \in B(p \leq p')$, but we have to write it this way for the complexity analysis to go through.

2.3 a is the $<$ least element not in $\text{ran}(A \oplus BC)$ that satisfies $F(A \oplus BC, p, a)$: $a \notin \text{ran}(A \oplus BC)$ and $F(A \oplus BC, p, a)$ and $\forall a' < a, a' \notin \text{ran}(A \oplus BC) \to \neg F(A \oplus BC, p, a')$

2.4 $\text{ran}(P) \cup \{p\}$ is an initial segment of B along the well order $<$. i.e. $\forall n(P(n) < p)$ and $\forall p' < p(p' \in B \to p' \in \text{ran}(P))$.

2.5 P as ordered by $\preceq_w$ agrees with $<$: $\forall n, m \in dom(P), (P(n) < P(m) \iff n \prec_w m)$.

We show that this is a $\mathbf{\Sigma}^1_n$ predicate.

It is easy to see that clause 1.1 and 2.1 are $\mathbf{\Sigma}^1_n$. Observe that $p' \in \text{ran}(P) \Leftrightarrow \exists n, P_n = p'$, thus 2.4 is $\mathbf{\Sigma}^1_n$ by the goodness of $<$. That 1.2, 1.3, 2.3 are $\mathbf{\Sigma}^1_n$ also follows from the goodness of $<$. Finally for 2.2,

$$\text{ran}(A') = \{A(m) \mid m \preceq_w n\} \iff \forall i \exists j \preceq_w n \big(A'(i) = A(j)\big) \wedge \forall j \preceq_w n \exists i (A'(i) = A(j)\big)$$

Thus it is also $\mathbf{\Sigma}^1_n$ by the goodness of $<$.

Finally let

$$Y(a) \iff \exists w \in \omega^\omega, A \in X^{\leq \omega}, P \in M^{\leq \omega}, p \in M, Rec(w, A, a, P, p)$$

Thus Y is $\mathbf{\Sigma}^1_n$. We show Y is compatible with F. For that purpose, we have to find enumeration of Y, B and a sequence $A_\alpha \in X^{\leq \omega}$ that enumerates $\{x_\beta \mid \beta < \alpha\}$ of order type $\leq \omega$ s.t. $\forall \alpha < \omega_1, (x_\alpha \in F_{A_\alpha \oplus BC}, p_\alpha)$.

The enumeration of B is easy to find, simply let $\{p_\alpha \mid \alpha < \omega_1\}$ enumerates B according to $<$. i.e. $p_\alpha < p_\beta$ iff $\alpha < \beta$. We recursively define an injection π from ω_1 to Y.

For the base case $\alpha = 0$, let $\pi(0)$ be the $<$ least element in $F_{\emptyset \oplus BC, p_0} \setminus \text{ran}(BC)$, such element exists by assumption, then $\pi(0) \in Y$ as for arbitrary w, 1.1, 1.2, 1.3 holds and thus $Rec(w, \emptyset, \pi(0), \emptyset, p_0)$, thus $\pi(0) \in Y$.

Given $\pi(\beta)$ defined for all $\beta < \alpha < \omega_1$, let A be a function that enumerates $\{\pi(\beta) \mid \beta < \alpha)\}$, P enumerates $\{p_\beta \mid \beta < \alpha\}$, then let $\pi(\alpha)$ be the least element in $F_{A \oplus BC, p_\alpha} \setminus (\text{ran}(BC) \cup \text{ran}(A))$, such element exists by assumption. Let $w \in WO$ code the order type α s.t. $\pi^{-1}(A(n)) < \pi^{-1}(A(m))$ iff $n \prec_w m$, then 2.1, 2.2, 2.3 and 2.4 holds and thus $Rec(w, A, \pi(\alpha), P, p_\alpha)$, therefore $\pi(\alpha) \in Y$.

Now we show that π is actually surjective. Suppose $x \in Y$, thus there is w, A, P, p s.t. $Rec(w, A, x, P, p)$. In the case that A is infinite, P is also infinite by clause 2.1. By clause 2.4, P is of the form $\{p_\beta \mid \beta < \alpha\}$ and $p = p_\alpha$, thus the order type of $\prec_w$ is α by clause 2.5. Let $A = \{x_\beta \mid \beta < \alpha\}$ be an enumeration of A by the order induced by $\prec_w$, i.e. $A(n) = x_\beta$ iff $\beta = ||\prec|_n||$. By an induction on $\beta < \alpha$ using 2.2, we show that $x_\beta = \pi(\beta)$. Thus $A = \{\pi(\beta) \mid \beta < \alpha\}$, thus by clause 2.3 we have $x = \pi(\alpha)$.

In the case that A is finite, say $dom(A) = n$, if $n = 0$, then we are in the base case and by 1.2 1.3, $x = \pi(0)$. If $n > 0$, then $dom(P) = n$ by 2.1. $P = \{p_i \mid i < n\}$ and $p = p_n$. By clause 2.5 $|| \prec_w || \geq n$, let $A = \{x_i \mid i < n\}$ be an enumeration of A by the order induced by $\prec_w$, i.e. if $x_i = A(j)$ and $x_{i'} = A(j')$, then $i < i'$ iff $j \prec_w j'$. Then by an induction $\pi(i) = x_i$, thus $x = \pi(n)$ by clause 2.3. This finishes the proof of surjectivity.

Thus the function π enumerates Y, let $Y = \{x_\alpha \mid \alpha < \omega_1\}$ be s.t. $x_\alpha = \pi(\alpha)$. Let A_α enumerate $\{x_\beta \mid \beta < \alpha\}$ of order type $\leq \omega$. This enumeration satisfies $\forall \alpha < \omega_1 x_\alpha \in F_{A_\alpha \oplus BC, p_\alpha}$ by our recursive definition of $\pi(\alpha)$. □

We can also start our recursion from any countable set that satisfies the recursion requirement, in the following sense:

Corollary 50. *In L, for any $\mathbf{\Delta}^1_2$ set of requirements B and any $\mathbf{\Delta}^1_2$ recursion $F \subseteq X^{\leq\omega} \times B \times X$, s.t. for any $A \in X^{\leq\omega}$, $p \in B$, $F_{A,p} \setminus ram(A)$ is non-empty. There is a $\mathbf{\Sigma}^1_2$ set Y compatible with F.*

Proof. Immediate from Theorem 49 and Theorem 35. □

Corollary 51. *In L, there is $\mathbf{\Sigma}^1_2$ instance of each of the following objects:*

Infinite MAD family, maximal independent family, infinite maximal ideal independent family, maximal tower.

Indeed for all but maximal tower, there is $\mathbf{\Delta}^1_2$ instance of them.

Proof. By the above corollary and the examples 46 and 48. For the second result, apply proposition 22. □

Under the framework for transfinite induction we just introduced, Vidnyánszky was able to abstract from Miller's construction of $\mathbf{\Pi}^1_1$ objects in L, and obtain a general theorem stating a sufficient condition for transfinite recursion to produce a coanalytic object. Informally, his theorem states that if at every recursion step our options are flexible enough (cofinal in Turing degree), then we are able to produce a coanalytic object. We state his theorem here:

Theorem 52 (Vidnyánszky, see [Vid12])**.** *In L, if B is an uncountable Borel set, $F \subseteq (\omega^\omega)^{\leq\omega} \times B \times (\omega^\omega)$ is a coanalytic set s.t. for any $p \in B$, $A \in (\omega^\omega)^{\leq\omega}$, the section $F_{A,p}$ is cofinal in Turing degrees, i.e. $\forall x \in \omega^\omega, \exists y \in F_{A,p}, x \leq_T y$, then there is a coanalytic X compatible with F.*

Chapter 4

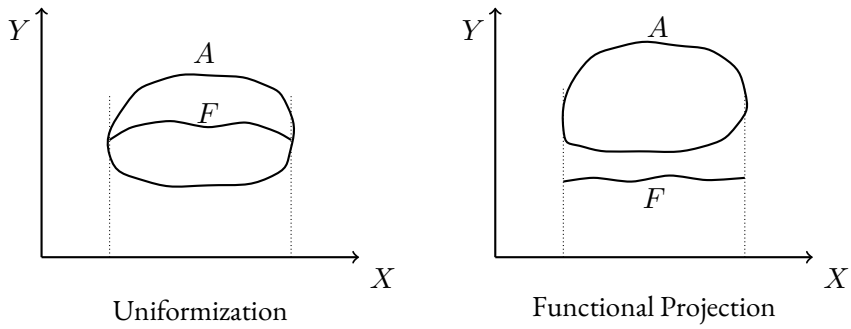
Functional Projection Property

In this Chapter, we introduce what we call the Functional Projection Property for pointclasses. We establish that it holds in L and $L[U]$ in Section 4.1; compare it with Uniformization in Section 4.2 and show it is a genuinely weaker property than Uniformization; and finally we determine the behavior of Functional Projection under determinacy hypotheses in Section 4.3. Most results in this chapter are original.

Definition 53 (Functional Projection)**.** *For a pointclass Γ, we say that Γ-**Functional Projection** (abbreviated by Γ-FP) holds if for any perfect Polish space X, Y and $A \subseteq X \times Y$ of complexity Γ, there exists $F \subseteq X \times Y$ in Γ, such that*

1. *For all $x \in X$, $\exists y\,(x,y) \in A \Leftrightarrow \exists y\,(x,y) \in F$.*
2. *F is a partial function from X to Y. i.e. for all $x \in dom(F)$ there is unique y s.t. $(x,y) \in F$.*

It is clear from definition that the difference between Γ-FP and Γ-Unif is that we no longer require F to be a subset of A. Thus, Γ-Unif is a stronger property and entails Γ-FP. The following picture shows the difference:



It is clear that for projective pointclasses closed under continuous images, Functional Projection property holds. That's because if Γ is a pointclass closed under continuous images, and $A \subseteq X \times Y$ is in Γ, then $p[A]$ is also in Γ. Then fix $y \in Y$ and $p[A] \times \{y\}$ would satisfy the requirements. In particular, Functional Projection holds for $\mathbf{\Sigma}^1_n$ for all $n \geq 1$.

Whether Functional Projection holds for $\mathbf{\Pi}^1_n$ pointclasses is the more interesting question, and will be the primary question we will investigate. But first, we notice that by Lusin-Suslin's theorem 6, Functional Projection property fails for closed sets:

Proposition 54. *There is a closed relation $C \subseteq \omega^\omega \times \omega^\omega$ s.t. it is not a projection of any Borel partial function B from ω^ω to ω^ω. As a consequence, Functional Projection property fails for closed sets and Borel sets.*

Proof. Let $A \subseteq \omega^\omega$ be any analytic set which is not Borel. Thus there is $C \subseteq \omega^\omega \times \omega^\omega$ s.t. $A = p[C]$.

By Theorem 6, the projection of any Borel partial function $B \subseteq \omega^\omega \times \omega^\omega$ is a Borel set. Thus there is no Borel partial function B s.t. $p[B] = p[C]$, as otherwise A would be Borel. □

On the other hand, Kondo's uniformization theorem 12 entails that Functional Projection holds for $\mathbf{\Pi}^1_1$ sets.

Proposition 55. $\mathbf{\Pi}^1_1$-FP *holds.*

Proof. By Theorem 12, and the fact that uniformization property for a pointclass entails Functional Projection property. □

Finally, we make the observation that for $\mathbf{\Pi}^1_n$ sets, as they are closed under Borel reflections, establishing Functional Projection for specific spaces suffices for establishing the Functional Projection property.

Lemma 56. *Let X, Y be fixed perfect Polish spaces. For any $n \geq 1$, the following are equivalent:*

1. $\mathbf{\Pi}^1_n$-FP.
2. *For any $A \subseteq X \times Y$ in Π^1_n, there exists a partial function $F \subseteq X \times Y$ with graph in $\mathbf{\Pi}^1_n$ such that:*
$$\exists y\, (x, y) \in A \;\Leftrightarrow\; \exists y\, (x, y) \in F$$

Proof. 1 to 2 is true by definition. 2 to 1: Let X', Y' be arbitrary perfect Polish spaces, fix Borel isomorphism $\pi_1 : X \to X'$ and $\pi_2 : Y \to Y'$.

For $A \subseteq X' \times Y'$, $A' = \{(\pi_1^{-1}(x), \pi_2^{-1}(y)) \mid (x,y) \in A\} \subseteq X \times Y$ and is $\mathbf{\Pi}^1_n$ as $\mathbf{\Pi}^1_n$ is closed under Borel reflections, thus there is $F \subseteq X \times Y$ in $\mathbf{\Pi}^1_n$ s.t. F is a partial function from X to Y and

$$\exists w \in Y, F(v,w) \iff \exists w \in Y, A'(v,w)$$

Thus $\{(\pi_1(v), \pi_2(w)) \mid (v,w) \in F\}$ is a partial function from X' to Y' in $\mathbf{\Pi}^1_n$ and has the same projection as A. □

As mentioned in the introduction, Kanovei and Lyubetsky in [KL24] and [KL25] proved that Functional Projection for $\mathbf{\Pi}^1_n$ where $n \geq 2$ is independent of ZFC.

Theorem 57 (Kanovei, Lyubetsky)**.** *For all $n \geq 2$, $\mathbf{\Pi}^1_n$-FP is independent of ZFC.*

4.1 Functional Projection in Inner Models

In this sections, we show that Functional Projection holds for all $\mathbf{\Pi}^1_n$ pointclasses in the inner model L and $L[U]$. First, we show that a projective good well-ordering of the reals necessarily leads to Functional Projection on high projective levels. Recall the definition of a Γ-good well-order in Definition 33.

Theorem 58. *Let $<_M$ be a well-ordering of the reals of order type ω_1 s.t. $<_M$ is $\mathbf{\Delta}^1_n$ and it is $\mathbf{\Sigma}^1_n$-good ($n \geq 2$). Then $M \models \mathbf{\Pi}^1_m$-FP for all $m \geq n+1$.*

Proof. By Lemma 56, it suffice to show for any $m \geq n+1$, any $Q \subseteq \omega^\omega \times \omega^\omega$ that is a $\mathbf{\Pi}^1_m$ relation, there is partial function $C \subseteq \omega^\omega \times \omega^\omega$ in $\mathbf{\Pi}^1_m$ that has the same projection as Q.

Let $P \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ be $\mathbf{\Sigma}^1_{m-1}$ such that

$$Q(x,w) \iff \forall u \in \omega^\omega, P(x,w,u)$$

We consider the following relation where $C(x, \langle w, u \rangle)$ iff all the following holds:

1. $Q(x,w)$
2. $\forall w' <_M w, \exists i \in \omega, \neg P(x, w', (u)_i)$
3. $\forall u' <_M u, \exists w' <_M w, \forall i \in \omega, P(x, w', (u')_i)$

First we show that C is $\mathbf{\Pi}^1_m$: clause 2 is $\mathbf{\Pi}^1_{m-1}$ in x, w, u as it is

$$\forall w' \in \omega^\omega \big(w' <_M w \to \exists i \in \omega, \neg P(x, w', (u)_i)\big)$$

Clause 3 is $\mathbf{\Sigma}^1_{m-1}$ in x, w, u by goodness of $<_M$.

Now we show that

$$\exists w \in \omega^\omega Q(x,w) \iff \exists w, u \in \omega^\omega C(x, \langle w, u\rangle)$$

$\Leftarrow$: trivial by clause 1.

$\Rightarrow$: Let w be the $<_M$ least w s.t. $Q(x,w)$, then for every $w' <_L w$, there is some $u_{w'}$ s.t. $\neg P(x, w', u_{w'})$. As $\{w' \mid w' <_M w\}$ is countable, there will be a u s.t. $\forall w' <_M w, \exists i \in \omega, \neg P(x, w', (u)_i)$. Let u be the $<_M$ least such element. i.e. $\forall u' <_M u, \exists w' <_M w, \forall i \in \omega, P(x, w', (u')_i)$. Then $C(x, \langle w, u\rangle)$ holds by construction.

Finally, we show if $C(x, \langle w, u\rangle)$, such w, u has to be unique.

Such w, u is unique as we are taking the $<_M$ least such element: If $C(x, \langle w, u\rangle)$ and $C(x, \langle w', u'\rangle)$ holds, assume for contradiction that $w <_M w'$, then by clause 1 of $C(x, \langle w, u\rangle)$, $Q(x,w)$ holds. Yet by clause 2 of $C(x, \langle w', u'\rangle)$, $\exists i \in \omega \neg P(x, w, (u')_i)$, entailing $\neg Q(x,w)$, a contradiction. Thus $w = w'$. To establish $u = u'$, assume for contradiction that $u <_M u'$, then clause 2 of $C(x, \langle w, u\rangle)$ contradicts clause 3 of $C(x, \langle w, u'\rangle)$.

This completes the proof. □

Next, Functional Projection holds for all levels in L:

Theorem 59 (Kanovei Lyubetsky, in [KL25]). *$L \models \mathbf{\Pi}^1_n$-FP for all $n \geq 1$.*

This result was proved in [KL25]. Our proof is different and it generalizes to other models beyond L.

It is clear that by Proposition 55 and applying Theorem 58 on the canonical well-ordering $<_L$, we have established $\mathbf{\Pi}^1_n$-FP for all levels except $\mathbf{\Pi}^1_2$. And indeed $\mathbf{\Pi}^1_2$-FP is the hard part. We sketch our proof idea first:

Let $R(x,w) \subseteq \omega^\omega \times \omega^\omega$ be a $\mathbf{\Pi}^1_2$ relation. We wish to find $\mathbf{\Pi}^1_2$ C s.t. $\exists w R(x,w) \iff \exists w Q(x,w) \iff \exists! w C(x,w)$.

The naive idea is just to look for the $<_L$ least element w s.t. $R(x,w)$. This is in itself not a $\mathbf{\Pi}^1_2$ process, but it is $\mathbf{\Pi}^1_2$ to say of some $c_0 \in \omega^\omega \cap L$ s.t. the model N_{c_0} coded by c_0 is some L_α and is the least such model containing w and moreover, N_{c_0} thinks w is the least element s.t. $R(x,w)$. Again in looking for the $<_L$ least element c_0 coding such a structure, we look at some N_{c_1} that think it to be so, and continue this process. Then the pair $w, \langle c_0, i \in \omega\rangle$ would be a unique code for x.

It turns out one more complication is needed. The statement that $\neg R(x,w')$ is a $\mathbf{\Sigma}^1_2$ formula and since we cannot use Shoenfield absoluteness for countable initial segments of L, the statement $\neg R(x,w')$

does not pass down to L_α for $w' <_L w$, here w is our desired $<_L$ least witness to $R(x,w)$. So w being the $<_L$ least element s.t. $R(x,w)$ does not necessarily mean that the premouse think this to be true. The trick to save this is the following: Assume $R(x,w) \iff \forall u P(x,w,u)$, P being $\mathbf{\Sigma}^1_1$. Since $\{w' \in 2^\omega \mid w' <_L w\}$ is countable, there is a single $u \in 2^\omega$ coding enough elements from $\{z \mid \neg P(x,w',z), w' <_L w\}$ for models with access to u to realize that $\neg R(x,w')$. Thus for countable model N_{c_0} containing both w and u, we could successfully pass the statement 'w is the $<_L$ least element s.t. $R(x,w)$' to countable models by Mostowski absoluteness.

Proof of Theorem 59. We work in L. By Theorem 35, the canonical well-order $<_L$ restricted to the reals is a $\mathbf{\Delta}^1_2$ relation that's $\mathbf{\Sigma}^1_2$ good and is of order type ω_1^L. Thus by Theorem 58, $L \models \mathbf{\Pi}^1_n$-FP for all $n \geq 3$. Thus, given Proposition 55, we are only left with the case $n = 2$.

For simplicity of notation we only prove the result for light face Π^1_2, parametrization is easy.

Let $R(x,w) \subseteq \omega^\omega \times \omega^\omega$ be a Π^1_2 relation. Assume $P \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ be Σ^1_1 such that

$$R(x,w) \iff \forall u P(x,w,u)$$

We consider the following relation: $C(x, \langle w,u,c\rangle)$ iff all of the following holds:

1. $R(x,w)$.
2. $(c)_0$ codes some L_α and α is the least ordinal s.t. L_α contains x, w, u.
3. $M_{(c)_0}$ satisfies the following:
   - 3.1 $\forall w' <_L w, \exists i \in \omega, \neg P(x,w',(u)_i)$.
   - 3.2 $\forall u' <_L u, \exists w' <_L w, \forall i \in \omega, P(x,w',(u')_i)$
4. For all n, $(c)_{n+1}$ codes some L_α and α is the least ordinal s.t. L_α contains $(c)_n$.
5. For all n, $M_{(c)_{n+1}}$ thinks $(c)_n$ is the $<_L$ least code of the structure it codes. i.e. $\forall c' <_L (c)_n$, $M_{c'} \ncong M_{(c)_n}$.

Claim 60. *C is Π^1_2*

Proof. Clause 1 is Π^1_2 by assumption, clause 2 is Π^1_2 in x, w, u, c as it can be expressed by the conjunction by the two sentences,

$$1. M_{(c)_0} \models V = L \wedge x, w, u \in M_{(c)_0}$$
$$2. \forall c' \big(M_{c'} \models V = L \wedge x, w, u \in M_{c'} \to M_{(c)_0} \text{ is an initial segment of } M_{c'}\big)$$
$$\iff \forall c' \big(M_{c'} \models V = L \wedge x, w, u \in M_{c'} \to \exists f \in \omega^\omega \forall n, m \in \omega (n E_{(c)_0} m \to f(n) E_{c'} f(m))\big)$$

This is Π^1_2 as $M_{c'} \models V = L \wedge x, w, u \in M_{c'}$ is Δ^1_1 in x, w, u, c'.

Clause 4 is similar.

Finally, clause 3 and clause 5 are arithmetical in x, w, u, c by Lemma 31. □

Claim 61.

$$\exists w R(x, w) \iff \exists! w, \exists! u, \exists! c, C(x, \langle w, u, c\rangle)$$

Proof. Right to left is easy as by conjunct 1 of the definition of C.

Left to Right: Assume $A(x)$ and let w be the $<_L$ least element such that $R(x, w)$. Hence for all $w' <_L w$, there is z s.t. $\neg P(x, w', z)$. Since $\{w' \in 2^\omega \mid w' <_L w\}$ is countable, there is u s.t. for all $w' <_L w$, $\exists n \neg P(x, w', (u)_n)$. Let u be the $<_L$ least such element. i.e. if $u' <_L u$ then there is $w' <_L w$ s.t. $\forall n P(x, w', (u')_n)$.

Let α_0 be least s.t. $x, w, u \in L_{\alpha_0}$. Let c_0 be the $<_L$ least element s.t. $tr(M_{c_0}) = (L_{\alpha_0}, \in)$. Then recursively pick α_{n+1} to be the least ordinal s.t. s.t. $c_n \in L_{\alpha_{n+1}}$ and c_{n+1} to be the $<_L$ least element coding a model whose collapse is $(L_{\alpha_{n+1}}, \in)$. Let c be such that $(c)_n = c_n$. We verify that $C(x, \langle w, u, c\rangle)$.

Clause 1, 2, 4 of C are true by construction. For clause 3 of C, let L_α be the collapse of the model coded by $(c)_0$.

For the conjunct 3.1, for arbitrary w', suppose $w' <_L w$, then by Theorem 32, $w' \in L_\alpha$ and hence by assumption $\exists n \neg P(x, w', (u)_n)$, this is Π^1_1. By Mostowski absoluteness Theorem 9, $L_\alpha \models \exists n \neg P(x, w', (u)_n)$. Hence 3.1 is verified.

For the conjunct 3.2, for any u' s.t. $u' < u$, again $u' \in L_\alpha$. We need to find w' s.t. $L_\alpha \models \forall n P(x, w', (u')_n)$ and $w' <_L w$. Again by Lemma 32, $u' <_L u$ and thus by minimality assumption of u, there is $w' <_L w$ and $\forall n P(x, w', (u')_n)$. But by Lemma 32, $w' \in L_\alpha$, moreover by Mostowski absoluteness Theorem 9, $\forall n P(x, w', (u')_n)$ passes down to L_α. This verifies conjunct 3.2.

Finally we are left with cluase 5 of C. This is a similar argument as clause 3 after observing that $M_{c'} \ncong M_{(c)_n}$ is the following Π^1_1 statement, thus absolute between $M_{(c)_{n+1}}$ and L:

$$\neg \exists f \in \omega^\omega \big[f \text{ is bijective } \wedge \forall n, m \in \omega \big(n E_{c'} m \leftrightarrow f(n) E_{(c)_n} f(m) \big) \big]$$

Now we argue that such w, u, c is unique. Assume $C(x, w, u, c)$ and $C(x, w', u', c')$. Suppose for contradiction that $w \neq w'$, without loss of generality, $w <_L w'$. Let $L_\alpha = tr(M_{(c')_0})$. Then $w \in L_\alpha$ and $L_\alpha \models w <_L w'$ by Lemma 32, but since $R(x, w)$ holds in L by point 1 of C and by downward absoluteness for Π^1_2 predicates, L_α also thinks $R(x, w)$, this is a contradiction to point 3 in C: as by point 3 in C and $w <_L w'$, $L_\alpha \models \exists n \neg P(x, w, (u')_n)$ and hence $\neg R(x, w)$. This concludes that $w = w'$. The argument that $u = u', c = c'$ are similar minimality arguments. □

This completes the proof. □

Using a similar proof idea, we show that Functional Projection also holds for all levels in $L[U]$.

Theorem 62. *$L[U] \models \mathbf{\Pi}^1_n$-FP for all $n \geq 1$.*

Proof. We work in $L[U]$. By Theorem 42, the canonical well-order $<_{L[U]}$ restricted to the reals is a $\mathbf{\Delta}^1_3$ relation that's $\mathbf{\Sigma}^1_3$ good and is of order type $\omega_1^{L[U]}$. Thus by Theorem 58, $L[U] \models \mathbf{\Pi}^1_n$-FP for all $n \geq 4$. Thus, given Proposition 55, we are only left with the case $n = 2$ and $n = 3$. Again we only deal with lightface pointclasses.

The case for $n = 2$ is entirely parallel to the case of $n = 2$ in the proof of Theorem 59. With the order $<_{L[U]}$ playing the role of $<_L$, iterable premice playing the role of L_α and $<_{ip}$ playing the role of the initial segment order on $L_\alpha, \alpha < \omega_1$. For sake of completeness, we include a sketch of the proof.

Let $R(x, w) \subseteq \omega^\omega \times \omega^\omega$ be a Π^1_2 relation. Assume $P \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ be Σ^1_1 such that

$$R(x, w) \iff \forall u P(x, w, u)$$

We consider the following relation: $C(x, \langle w, u, c\rangle)$ iff all of the following holds:

1. $R(x, w)$.
2. $tr(M_{(c)_0})$ is the $<_{ip}$ least countable iterable premouse containing x, w, u.
3. $tr(M_{(c)_0}) = (\overline{tr(M_{(c)_0})}, \in, V)$ satisfies:
   - $\forall w' <_{L[V]} w, \exists i \in \omega, \neg P(x, w', (u)_i)$.
   - $\forall u' <_{L[V]} u, \exists w' <_{L[V]} w, \forall i \in \omega, P(x, w', (u')_i)$.
4. For all n, $tr(M_{(c)_{n+1}})$ is the $<_{ip}$ least countable iterable premouse containing $(c)_n$.
5. For all n, $tr(M_{(c)_{n+1}}) = (\overline{tr(M_{(c)_{n+1}})}, \in, V_{n+1})$ satisfies $\forall c' <_{L[V_{n+1}]} (c)_n, M_{c'} \ncong M_{(c)_n}$.

Claim 63. *C is Π^1_2.*

Proof. Clause 1 is Π^1_2 by assumption. Clause 3 and 5 are Δ^1_1 by 31. 2 and 4 are Π^1_2 by Lemma 44. □

The argument that C is a partial function and has the same projection as R is entirely parallel to the proof of Theorem 59.

The case for $n = 3$. Let $R \subseteq \omega^\omega \times \omega^\omega$ be the Π^1_3 relation. Assume

$$R(x, w) \iff \forall u S(x, w, u) \iff \forall u \exists v T(x, w, u, v)$$

Where $S \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ is Σ^1_2 and $T \subseteq \omega^\omega \times \omega^\omega \times \omega^\omega$ is Π^1_1.

Consider the following relation: $C(x, \langle w, u, v, c\rangle)$ iff all of the following holds:

1. $R(x, w)$ and $\forall w' <_{L[U]} w, \exists i \in \omega, \neg S(x, w', (u)_i)$.
2. $tr(M_{(c)_0})$ is the $<_{ip}$ least countable iterable premouse containing x, w, u, v.
3. $tr(M_{(c)_0}) = (\overline{tr(M_{(c)_0})}, \in, V)$ satisfies:
   - 3.1 $\forall w' <_{L[V]} w, \exists i \in \omega, \neg S(x, w', (u)_i)$
   - 3.2 $\forall u' <_{L[V]} u, \exists w' <_{L[V]} w, \forall i, \exists j, T(x, w', (u')_i, (v)_j)$.
   - 3.3 $\forall v' <_{L[V]} v, \exists u' <_{L[V]} u, \forall w' <_{L[V]} w, \exists i, \forall j, \neg T(x, w', (u')_i, (v')_j)$
4. For all n, $tr(M_{(c)_{n+1}})$ is the $<_{ip}$ least countable iterable premouse containing $(c)_n$.
5. For all n, $tr(M_{(c)_{n+1}})$ thinks $(c)_n$ is the least code of the structure it codes, as ordered by its canonical order.

That C is a Π^1_3 predicate is the same argument as Claim 63, with the only exception of the sentence $\forall w' <_{L[U]} w, \exists i \in \omega, \neg S(x, w', (u)_i)$ in clause 1, which is easily seen to be Π^1_3.

Now we show that $\exists w R(x, w)$ iff there is unique w, u, v, c s.t. $C(x, \langle w, u, v, c\rangle)$. Right to Left is simply by clause 1. In case $\exists w R(x, w)$, let w be the $<_{L[U]}$ such w, let u be the $<_{L[U]}$ least element satisfying 3.1, let v be the $<_{L[U]}$ least element satisfying 3.2. Let c_0 be the $<_{L[U]}$ least element satisfying 2 for the chosen w, u, v and recursively find c_{n+1} satisfying 4 for c_n. Let c be such that $(c)_n = c_n$.

Thus 1,2,4 will be true by construction. That clause 5 is true is the exact same argument as before. We verify clause 3, notice all three sub-statements 3.1, 3.2, 3.3 are true in the ground model by construction, with $<_{L[V]}$ replaced by the ground model canonical order $<_{L[U]}$. And the ground model truth passes down successfully to the premouse $tr(M_{(c)_0})$ as by Lemma 41, $tr(M_{(c)_0})$ and $L[U]$ agrees on the reals we care about and their order, and all three predicates $\neg S, T, \neg T$, being $\Pi^1_2, \Pi^1_1, \Sigma^1_1$ are downward absolute by Theorem 9.[1]

Finally, the w, u, v, c we found are unique. If both $C(x, \langle w, u, v, c\rangle)$ and $C(x, \langle w', u', v', c'\rangle)$ holds, we argue that $\langle w, u, v, c\rangle = \langle w', u', v', c'\rangle$. Suppose without loss of generality that $w' \leq_{L[U]} w$, if $w' <_{L[U]} w, \exists i, \neg S(x, w', (u)_i)$ by clause 1, but we also have $R(x, w')$ by clause 1 of $C(x, \langle w', u', v', c'\rangle)$, thus $w' = w$. The minimality argument for u, v works as premice agrees on the reals we care about and their order and the predicate $T, \neg T$ are absolute between them. That c is unique is the same argument as before.

This completes the proof. □

[1] A detailed proof would go along the line of the proof of that Claim 61.

4.2 Functional Projection vs Uniformization

Having established Functional Projection for various models, we would like to discuss its relation with uniformization.

Given that $\mathbf{\Pi}^1_2$ Uniformization is known to fail by Theorem 13, the fact that Functional Projection holds for that level in L and $L[U]$ already shows the difference. We will see that the result leads to many consequences in the next chapter.

Moreover, what we have established is that Functional Projection is a genuinely weaker property than uniformization. First, it is well-known that L is a model where all uniformization on the $\mathbf{\Pi}^1_n$ side fails for $n \geq 2$.

Theorem 64 (Folklore)**.** *In L, $\mathbf{\Sigma}^1_n$-*Unif *holds for all $n \geq 2$, as a consequence, $\mathbf{\Pi}^1_n$-*Unif *fails for all $n \geq 2$.*

Proof. For $n = 2$, this is just Kondo's uniformization theorem 12. For $n > 2$, for arbitrary P that is $\mathbf{\Sigma}^1_n$, there is Q that is $\mathbf{\Pi}^1_{n-1}$ s.t.

$$P(x, y) \iff \exists z \in \omega^\omega, Q(x, y, z)$$

Consider the following predicate P^*, Q^* by:

$$\begin{aligned} Q^*(x, y, z) &\iff Q(x, y, z) \wedge \forall c = \langle y', z' \rangle <_L \langle y, z \rangle, \neg Q(x, y', z') \\ P^*(x, y) &\iff \exists z, Q^*(x, y, z) \end{aligned}$$

By goodness of $<_L$, Q^* is of the form $\mathbf{\Pi}^1_{n-1} \wedge \mathbf{\Sigma}^1_{n-1}$ and P^* is $\mathbf{\Sigma}^1_n$.

Moreover, $P^* \subseteq P$ is clear. We show that if $P^*(x, y)$ and $P^*(x, y')$, then $y = y'$. Assume without loss of generality that $y \neq y'$/ We have that there is z, z' s.t. $Q^*(x, y, z)$ and $Q^*(x, y', z')$, since $y \neq y'$, $\langle y, z \rangle \neq \langle y', z' \rangle$, assume without loss of generality that $\langle y, z \rangle < \langle y', z' \rangle$, this yields a contradiction as $Q^*(x, y, z)$ requires $Q(x, y, z)$ yet $Q^*(x, y', z')$ requires $\neg Q(x, y, z)$.

Finally, the second statement follows from Theorem 13. □

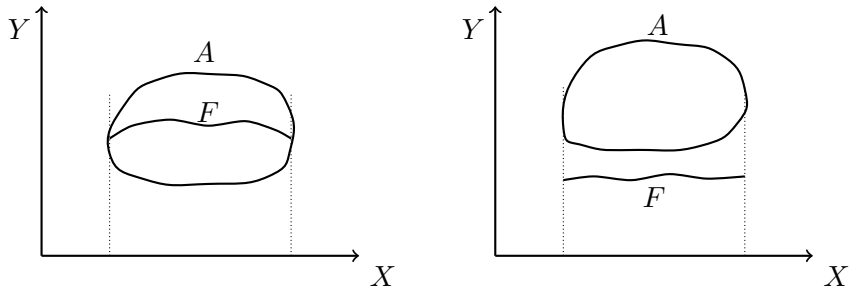
Therefore, we know that Functional Projection is genuinely weaker:

Corollary 65. *For all $n \geq 2$, it is consistent with ZFC that $\mathbf{\Pi}^1_n$-*FP *holds while $\mathbf{\Pi}^1_n$-*Unif *fails.*

Proof. Immediate by Theorem 59 and 64. □

In hindsight, it is not surprising that even though Functional Projection and uniformization property are seemingly similar, they turn out to be genuinely different. What Functional Projection lacks, i.e.

the requirement that the function has to be a subset of the original relation, carries significant importance.



While uniformization can be viewed as a definable choice principle, allowing one to choose specific elements of a relation in a definable way, Functional Projection loses this power. In descriptive set theory, Uniformization has a rich variety of consequences, for instance, it implies the reduction property, separation property for the dual point class and often leads to some version of a Basis Theorem (See [Mos09] for a reference). Yet Functional Projection, being different from choice principle, retains no such power. However, we will see that Functional Projection is powerful enough to imply many complexity reduction results in the next chapter.

4.3 Determinacy and Functional Projection

Besides the assumption that there is a projective well-ordering on the reals, the axiom of **Projective Determinacy** is another powerful assumption that leads to a clear structure of the projective hierarchy. In this section we investigate the behavior of the Functional Projection property under determinacy hypothesis. In particular, we show that under projective determinacy, Function Projection property fails for all even $\mathbf{\Pi}$ levels; while for odd $\mathbf{\Pi}$ levels, Function Projection property holds due to uniformization.

The key of our argument is the observation that $\mathbf{\Pi}^1_{n+1}$ pointclass being normed allows us to lift Lusin-Suslin theorem 6 to higher pointclasses, in particular, this would imply that $\mathbf{\Pi}^1_n$ does not satisfy the Functional Projection. Let us first recall the definition of Norm and some facts about it, We need to make essential use of Kleene pointclasses in this section.

Definition 66 (Norm). *For a Polish space X and a set $A \subseteq X$, a **norm** on A is a function $\varphi : A \to \lambda$ where λ is an ordinal.*

*Let Γ be a point class, we say φ is a Γ **norm** if there are relations $\leq^{\varphi}_{\Gamma}, \leq^{\varphi}_{\neg\Gamma}$ in Γ and $\neg\Gamma$ respectively s.t. for every y,*

$$P(y) \Rightarrow \forall x \big[\big(P(x) \wedge \varphi(x) \leq \varphi(y) \big) \iff x \leq^{\varphi}_{\Gamma} y \iff x \leq^{\varphi}_{\neg\Gamma} y \big]$$

*A pointclass Γ is **normed** if every set in Γ has a Γ-norm.*

For a Kleene point class Γ, if Γ is normed, then any parametrization $\Gamma(z)$ is also normed.

Let $\Delta = \Gamma \cap \neg\Gamma$, we say that a real $x \in \omega^\omega$ is **Δ-recursive** if $\{code_\omega(s) \mid s \in \omega^{<\omega}, s \subseteq x\}$ is in Δ, i.e. the set of codes of basic open sets x in is Δ as a subset of ω. If $x \in \omega^\omega$ is a Δ-recursive real, we also write $x \in \Delta$.

An important result is that if Γ is normed, then it is closed under restricted existential quantification. This generalizes the Spector-Gandy Theorem on the Δ^1_1 level.

Theorem 67 ([Mos09, page 4D.3])**.** *Let X be a Polish space, if $\Pi^1_n(a)$ is normed, then for any $Q \subseteq X \times \omega^\omega$, the set $P \subseteq X \times \omega^\omega$ defined by*

$$P(x,z) \iff \exists y \in \Delta^1_n(z,a) Q(x,y)$$

is also $\Pi^1_n(a)$.

As a result, we may lift one side of Lusin-Suslin theorem 6 under the assumption that higher point classes are normed.

Theorem 68. *If Π^1_{n+1} is normed, then $\mathbf{\Pi}^1_n$-FP fails.*

Proof. Let $A \subseteq \omega^\omega$ be a Σ^1_{n+1} set that is not $\mathbf{\Pi}^1_{n+1}$[2]. Then there is $B \subseteq \omega^\omega \times \omega^\omega$ s.t.

$$A(x) \iff \exists y B(x,y)$$

Assume for contradiction that there is a $F \subseteq \omega^\omega \times \omega^\omega$ that is a partial function with graph in $\mathbf{\Pi}^1_n$ s.t. it has the same projection as B. Assume that F has graph in $\Pi^1_n(a)$. Then if $(x,y) \in F$, we have for arbitrary $s \in \omega^{<\omega}$,

$$\begin{aligned} s \subseteq y &\iff \forall y'((x,y') \in F \to s \subseteq y') \\ &\iff \exists y'((x,y') \in F \wedge s \subseteq y') \end{aligned}$$

Showing that the $\{code_\omega(s) \mid s \in \omega^{<\omega} \wedge s \subseteq y\}$ is in $\Delta^1_{n+1}(x,a)$, thus y is $\Delta^1_{n+1}(x,a)$ recursive.

We thus obtain that

$$A(x) \iff \exists y, F(x,y) \iff \exists y \in \Delta^1_{n+1}(x,a), F(x,y)$$

and thus by Theorem 67 we have that A is $\Pi^1_{n+1}(a)$, contrary to our assumption. □

Now, we are ready to determine the picture of Functional Projection under determinacy hypotheses. Let us recall the relevant definitions and results.

[2] Existence of such set follows from Hierarchy Theorem of Kleene point class, see for instance [Mos09, 3E.8].

For $A \subseteq \omega^\omega$, we associate the **real number game** on A, denoted by $G(A)$ as follows: player I and Player II take turns playing natural numbers as their respective move, with I playing the even moves $n_0, n_2 \dots$ and II playing the odd moves $n_1, n_3 \dots$. I wins the game iff the resulting real $x = (n_0, n_1 \dots)$ is in A, and II wins otherwise.

I	n_0		n_2	n_4	$\cdots$
II		n_1		n_3	$\cdots$

$$x = (n_0, n_1, n_2, n_3, \dots) \in \omega^\omega, \qquad \text{I wins} \iff x \in A.$$

We say σ is a **strategy for player I** if it is a function with domain all finite sequences of ω of even length and values in ω. We say I follows σ in a run of game $G(A)$ if the resulting play $x = (n_0, n_1, n_2, n_3, \dots)$ satisfies $n_0 = \sigma(\emptyset)$ and $n_{2m} = \sigma(n_0 \dots n_{2m-1})$ for all $m \geq 1$.

Dually, we define the **strategy for player II** as a function with domain all finite sequences of ω of odd length and values in ω.

When player I follows σ and II follows τ, then the resulting play is uniquely determined where $n_0 = \sigma(\emptyset), n_1 = \tau(n_0)$ and recursively $n_{2m} = \sigma(n_0 \dots n_{2m-1}), n_{2m+1} = \tau(n_0 \dots n_{2m})$. We denote the resulting play as $\sigma * \tau$.

Definition 69 (Determinacy)**.** *σ is a **winning strategy for I** in $G(A)$ if for any strategy τ for II, $\sigma * \tau \in A$. i.e. no matter what II plays, I wins.*

*Dually, τ is a **winning strategy for II** in $G(A)$ if for any strategy σ for I, $\sigma * \tau \notin A$.*

*A set $A \subseteq \omega^\omega$ is said to be **determined** if either player I or player II has a winning strategy in the game $G(A)$. A point class Γ is said to be determined if every set in it is determined.*

The axiom **Projective Determinacy** states that all projective sets are determined. For a detailed theory of Projective Determinacy and the property of determinacy in general, the reader may refer to [Mos09, Chapter 6] and [Sch14, Chapter 12 and 13].

Martin and Moschovakis proved that under Projective Determinacy, normed point classes appear on the projective hierarchy periodically, oscillating between the Σ and Π side. And the same is true for uniformization property

Theorem 70 (The First and the Second Periodicity Theorem, Martin and Moschovakis [Mos09, 6B.1, 6B.2, 6C.3, 6C.5])**.** *Assume that $\mathbf{\Delta}^1_{2n}$ sets are determined. Then for all $k \leq n$, Σ^1_{2k+2} and Π^1_{2k+1} are normed, and Σ^1_{2k+2}-Unif, Π^1_{2k+1}-Unif hold.*

The following diagram dipicts the behavior of norm and Uniformization property under Projective Determinacy, with the underlined point classes exactly those that are normed and have the Uniformization property.

$$\begin{array}{ccccccc} \mathbf{\Sigma}^1_1 & \underline{\mathbf{\Sigma}^1_2} & \ldots & \mathbf{\Sigma}^1_{2n-1} & \underline{\mathbf{\Sigma}^1_{2n}} & \cdots \\ \underline{\mathbf{\Pi}^1_1} & \mathbf{\Pi}^1_2 & \ldots & \underline{\mathbf{\Pi}^1_{2n-1}} & \mathbf{\Pi}^1_{2n} & \cdots \end{array}$$

Underlined point classes are normed and have the Uniformization property.

As a consequence, we prove that PD determines the behavior of Functional Projection in the following way:

Theorem 71. *Assume that $\mathbf{\Delta}^1_{2n}$ sets are determined. Then for all $m \leq n$, $\mathbf{\Pi}^1_{2m}$-FP fails, while $\mathbf{\Pi}^1_{2m+1}$-FP hold.*

In particular, under Projective Determinacy, $\mathbf{\Pi}^1_{2n}$-FP fails while $\mathbf{\Pi}^1_{2n+1}$-FP hold for all $n \in \omega$.

Proof. For the even level $\mathbf{\Pi}^1_{2m}$, by the periodicity theorem, $\mathbf{\Pi}^1_{2m+1}$ is normed and thus by Theorem 68 $\mathbf{\Pi}^1_{2m}$-FP fails.

For odd level, this follows from the periodicity theorem and the fact that Uniformization implies Functional Projection. □

Thus, the following picture completely settles the behavior of Function Projection property under the assumption of Projective Determinacy (PD). The boxed point classes are exactly those that satisfy Function Projection. We compare this to the picture we got from the assumption $V = L$ or $V = L[U]$.

$$\begin{array}{c|ccccccc} PD & \boxed{\mathbf{\Pi}^1_1} & \mathbf{\Pi}^1_2 & \ldots & \boxed{\mathbf{\Pi}^1_{2n-1}} & \mathbf{\Pi}^1_{2n} & \cdots \\ V = L \text{ or } V = L[U] & \boxed{\mathbf{\Pi}^1_1} & \boxed{\mathbf{\Pi}^1_2} & \ldots & \boxed{\mathbf{\Pi}^1_{2n-1}} & \boxed{\mathbf{\Pi}^1_{2n}} & \cdots \end{array}$$

Boxed point classes have Functional Projection property.

4.4 Further Questions

We conclude with a few questions. All of the models we see in this chapter that separates Functional Projection from Uniformization has short continuum: they are of continuum ω_1. We are interested in models of long continuum where we can separate uniformization property and Functional Projection property.

Question 2. *Find a model of $\neg CH$ on which $\mathbf{\Pi}^1_n$-FP $+ \neg \mathbf{\Pi}^1_n$-Unif holds for $n \geq 2$.*

During a seminar discussion, Philipp Lücke raised the question whether $\mathbf{\Pi}^1_n$-FP $+ \neg \mathbf{\Pi}^1_n$-Unif should be viewed as a pathological behavior. Discussions in this chapter hints that it is truly pathological:

proof of separation of Functional Projection from Uniformization in L and $L[U]$ uses the well-ordering of the reals in a non-trivial way, while the strong regularity assumption of determinacy rules out such possibility.

But to truly settle whether $\mathbf{\Pi}^1_n$-FP $+ \neg \mathbf{\Pi}^1_n$-Unif is a consquence of the Axiom of Choice, it is worth looking at whether it holds in the Solovay model.

Question 3. *Determine the behavior of* $\mathbf{\Pi}^1_n$-FP *in the Solovay model for* $n \geq 2$. *More generally, is there an interesting point class* Γ *s.t. in the Solovay model,* Γ-FP $+ \neg \Gamma$-Unif *holds?*

Chapter 5

Functional Projection and Complexity Reduction

In this chapter, we show that under the assumption of Functional Projection, complexity reduction results in the literature lift to higher pointclasses in Section 5.1. And finally we apply the lifting and obtain $\mathbf{\Pi}^1_2$ instances of combinatorial families in $L[U]$ in Section 5.2. Most results in this chapter are original.

5.1 Lifting Complexity Reduction Results

5.1.1 Perfect Set Property

First we show that under the assumption of Functional Projection, if there is a $\mathbf{\Sigma}^1_{n+1}$ set without the perfect set property, then there is a $\mathbf{\Pi}^1_n$ set without the perfect set property.

Theorem 72. *Assume* $\mathbf{\Pi}^1_n$-FP. *Then all* $\mathbf{\Sigma}^1_{n+1}$ *sets have the* PSP *iff all* $\mathbf{\Pi}^1_n$ *sets have the* PSP.

Proof. For the non-trivial direction, suppose all $\mathbf{\Pi}^1_n$ sets have the Perfect Set Property. Let $A \subseteq X$ be an uncountable $\mathbf{\Sigma}^1_{n+1}$ set. By $\mathbf{\Pi}^1_n$-FP, there is a function $F : A \to \omega^\omega$ with a $\mathbf{\Pi}^1_n$ graph and with $\mathrm{dom}(F) = A$. Then F (identified with its graph) is also an uncountable set, so by assumption it contains a perfect subset $C \subseteq F$. Then the projection $p[C]$ is an analytic uncountable subset of A, so there is a perfect set $P \subseteq p[C] \subseteq A$ by the Perfect Set Theorem for analytic sets. □

5.1.2 Maximal Almost Disjoint Families

The following theorem lifts Törnquist's result in [Tör09], which is on the $\mathbf{\Sigma}^1_2$, $\mathbf{\Pi}^1_1$ level. The proof follows the same line as Törnquist's original result.

Theorem 73. *Assume* $\mathbf{\Pi}^1_n$-FP. *Then there is a* $\mathbf{\Sigma}^1_{n+1}$ *MAD family iff there is a* $\mathbf{\Pi}^1_n$ *MAD family.*

Proof. Let A be a $\mathbf{\Sigma}^1_{n+1}$ MAD family. By $\mathbf{\Pi}^1_n$-FP there is a partial function $Q \subseteq [\omega]^\omega \times 2^\omega$ with a $\mathbf{\Pi}^1_n$ graph such that

$$A(x) \iff \exists! w Q(x, w)$$

Define the following maps $g_0, g_1 : ([\omega]^\omega)^2 \to \mathcal{P}(3 \times \omega)$:

$$g_0 : x, w \mapsto (\{0\} \times x) \cup (\{2\} \times \{x^*(n) \mid n \in w\})$$
$$g_1 : x, w \mapsto (\{1\} \times x) \cup (\{2\} \times \{x^*(n) \mid n \notin w\})$$

Let $A' = g_0[Q] \cup g_1[Q]$. To see that A' is almost disjoint, pick $z \neq z' \in A'$, there are two cases:

1. $z = g_i(x, w), z' = g_i(x', w')$. Without loss of generality assume $i = 0$. We notice that it can't be s.t. $x = x'$, since if so then by the fact that Q is a function, $w = w'$ and thus $z = z'$. Thus $x \neq x'$. $z \cap z' \subseteq (\{0\} \times (x \cap x')) \cup (\{2\} \times (x \cap x'))$, which is finite since A is a MAD family.

2. $z = g_i(x, w), z' = g_{1-i}(x', w')$. Without loss of generality assume $i = 0$. If $x = x'$, then $w = w'$ and hence by construction of g_0, g_1, $z \cap z' = \emptyset$. If $x \neq x'$, $z \cap z' \subseteq \{2\} \times x \cap x'$ is finite.

   To see it is maximal, for $y \in [3 \times \omega]^\omega, i \in \{0, 1, 2\}$ write $y_i = \{n \in \omega \mid (i, n) \in y\}$. It suffice to show that any $y \in [3 \times \omega]^\omega$, there is $z \in A'$ s.t. $y \cap z$ is infinite. As $y \in [3 \times \omega]^\omega$, one of y_0, y_1, y_2 is infinite. If y_0 is infinite, then by the fact that A is MAD, there is $x \in A$ s.t. $|x \cap y_0| = \infty$. There is w s.t. $Q(x, w)$ and thus $g_0(x, w) \in A'$ and $y \cap g_0(x, w) \supseteq \{0\} \times (y_0 \cap x)$ is infinite. The case for y_1 is similar. For the case that y_2 is infinite, there is $x \in A$ s.t. $|x \cap y_2| = \infty$ and there is w s.t. $Q(x, w)$. Then y_2 intersects one of $\{x^*(n) \mid n \in w\}, \{x^*(n) \mid n \notin w\}$ infinitely, hence y intersects one of $g_0(x, w), g_1(x, w)$ infinitely.

   This finishes the proof that A' is an infinite MAD family.

Next we show that A' is in $\mathbf{\Pi}^1_n$. This is because g_1, g_2 are Borel injections from $([\omega]^\omega)^2 \to \mathcal{P}(3 \times \omega)$, thus by Lemma 7, $A' = g_1[Q] \cup g_2[Q]$ is $\mathbf{\Pi}^1_n$ as $\mathbf{\Pi}^1_n$ point class is closed under finite union. □

It is worth noticing that based on an observation by Fischer and Schrittesser in [FS21], with a sufficient absoluteness assumption, the above construction actually yields a forcing indistructible MAD family if we start from one. Recall the definition of $\mathbf{\Sigma}^1_n$ generic absoluteness from Definition 39.

Corollary 74. *Let* $\mathbb{P}$ *be a* $\mathbf{\Sigma}^1_{n+1}$ *generic absolute forcing notion, assume that* $\mathbf{\Pi}^1_n$-FP *holds. Then if there is a* $\mathbf{\Sigma}^1_{n+1}$ *MAD family that is* $\mathbb{P}$ *indestructible, then there is a* $\mathbf{\Pi}^1_n$ *MAD family that is* $\mathbb{P}$ *indestructible.*

Proof. We continue the proof in the above theorem. Let A be such a $\mathbf{\Sigma}^1_{n+1}$ MAD that is $\mathbb{P}$ indestructible, $Q \subseteq [\omega]^\omega \times 2^\omega$ a partial function with graph in $\mathbf{\Pi}^1_n$ s.t.

$$A(x) \iff \exists! w Q(x, w)$$

let $A' = g_1[Q] \cup g_2[Q]$. We've shown that it is a MAD family. It suffice to show that it is $\mathbb{P}$ indestructible.

Let $A^* = (p[Q])^{V[G]}$, the statement that '$p[Q]$ is an almost disjoint family' is

$$\varphi := \forall x \neq y, \big[\exists z_1, z_2 \big((x, z_1) \in Q \wedge (y, z_2) \in Q\big) \to |x \cap y| < +\infty\big]$$

This is a $\mathbf{\Pi}^1_{n+1}$ statement that is true in V, thus true in $V[G]$ by $\mathbf{\Sigma}^1_{n+1}$ absoluteness. (To see this, we may assume add a vacuous argument $\psi(z) := (\varphi \wedge z = z)$, then $\varphi \iff \exists z \psi(z) \iff \forall z \psi(z)$, apply absoluteness on this $\psi(z)$.) Hence A^* is an almost disjoint family in $V[G]$.

By $\mathbb{P}$ indestructibility of A, A is MAD in $V[G]$. Moreover, $A \subseteq A^*$ by absoluteness for Q. Thus $A = A^*$.

Now we show that $Q^V = Q^{V[G]}$. $\subseteq$ is by $\mathbf{\Pi}^1_n$ absoluteness. For $\supseteq$ side, first by absoluteness $Q^{V[G]}$ is a graph of a function. If $(x, w) \in Q^{V[G]}$, then $x \in A^*$ thus $x \in A$. Hence there is $w' \in V$ s.t. $(x, w') \in Q^V$, thus $(x, w') \in Q^{V[G]}$. By functionality of $Q^{V[G]}$, $w = w'$. Thus $(x, w) \in Q^V$.

Finally, $(g_1[Q] \cup g_2[Q])^{V[G]} = g_1[Q] \cup g_2[Q]$ holds since $Q = Q^{V[G]}$ and Borel functions are absolute on ground model reals.

Thus, the argument in the above theorem that A' is a MAD family can be run in $V[G]$. This concludes the proof. □

Notice that starting from a $\mathbf{\Pi}^1_n$ $\mathbb{P}$ indestructible MAD family A, the set A^V is a MAD family by forcing indestructibility. But it may not remain $\mathbf{\Pi}^1_n$ in the forcing extension, as the interpretation of A in $V[G]$ might change. But we make the following observation, which states that enough generic absoluteness guarantees that the definition persists. In particular, by Theorem 10, the following theorem entails that there is still a $\mathbf{\Pi}^1_1$ MAD family in the forcing extension if there is a $\mathbf{\Pi}^1_1$ $\mathbb{P}$ indestructible MAD family in the ground model.

Proposition 75. *Let $\mathbb{P}$ be a $\mathbf{\Sigma}^1_{n+1}$ generic absolute forcing notion, let A be a $\mathbf{\Sigma}^1_{n+1}$ MAD family that is $\mathbb{P}$ indestructible. Then $A^V = A^{V[G]}$.*

Proof. The following $\mathbf{\Pi}^1_{n+1}$ sentence states that A is an almost disjoint family:

$$\forall x \neq y (x, y \in A \to |x \cap y| < \infty)$$

Thus $A^{V[G]}$ is an almost disjoint family. Moreover by generic absoluteness, $A^V \subseteq A^{V[G]}$, by the fact that A^V remains MAD in $V[G]$, $A^V = A^{V[G]}$. □

We give some interesting examples of Corollary 74. For an almost disjoint family A, we associate an ideal $I(A) := \{x \in [\omega]^\omega \mid \exists n \exists a_0 \dots a_n \in A, x \subseteq^* a_0 \cup \dots \cup a_n\}$. It is clear that if A is infinite, then $I(A)$ has to be proper. The following notion was first considered by Malykhin [Mal90].

Definition 76. *We say an almost disjoint family is* ***tight*** *if for any $(x_n, n \in \omega) \subseteq [\omega]^\omega \setminus I(A)$, there is a $a \in A$ s.t. $|a \cap x_n| = \infty$ for all $n \in \omega$.*

Tight almost disjoint family is automatically MAD. It is a famous result that we can construct a tight MAD family along a well-order of the reals and tight MAD families are Cohen indestructible. For completeness, we collect these results here:

Lemma 77. *Let $\{a_n \mid n \in \omega\}$ be a countable almost disjoint family, let $\{x_n \mid n \in \omega\}$ be a set of elements not in $I(\{a_n \mid n \in \omega\})$, then there is $a \in [\omega]^\omega$ almost disjoint from each element of $\{a_n \mid n \in \omega\}$ and $a \cap x_n$ is infinite for all $n \in \omega$.*

Proof. Let $e : \omega \to \omega$ be a surjection s.t. for each $n \in \omega$, $e^{-1}(n)$ is infinite.

Recursively, let m_n be the least element in $x_{e(n)} \setminus (a_0 \cup \dots \cup a_n)$ that is greater than m_{n-1}. Such m_n exists as we assumed $x_{e(n)} \notin I(\{a_n \mid n \in \omega\})$. Let $x = \{m_n \mid n \in \omega\}$, then $x \cap a_n$ is finite as $m_{n'} \notin a_n$ for all $n' > n$ by construction. $x \cap x_n \supseteq \{m_s \mid e(s) = n\}$ and is thus infinite. □

Theorem 78 (Folklore)**.** *Let $<$ be a $\mathbf{\Delta}^1_n$ well ordering of the reals that's $\mathbf{\Sigma}^1_n$ good. Then there is a $\mathbf{\Sigma}^1_n$ infinite tight MAD family.*

Proof. Consider the following recursion scheme $F \subseteq ([\omega]^\omega)^{\leq\omega} \times ([\omega]^\omega)^\omega \times [\omega]^\omega$ as: $F(A, p, x)$ holds iff EITHER: all of the following are true:

1. $ran(A)$ is almost disjoint.
2. $\forall n, (p)_n \notin I(ran(A))$.
3. x is almost disjoint from every element in $ran(A)$ and $x \cap (p)_n$ is infinite for every $n \in \omega$.

OR 1 is true and 2 is false and x is almost disjoint from every element in $ran(A)$.

OR 1 is false.

Let B code an infinite partition of ω, then clearly A compatible with F from base case B is a tight MAD family as for every sequence $(x_n, n \in \omega)$ s.t. $x_n \notin I(A)$ for all n, suppose it is enumerated as p_α. By $x_\alpha \in F_{A_\alpha, p_\alpha}$, $ran(A_\alpha)$ is an almost disjoint family by an induction on α, $x_n \notin I(ran(A_\alpha))$ as $ran(A_\alpha) \subseteq A$, thus both 1 and 2 holds and thus $x_\alpha \in A$ intersects every element x_n infinitely.

Thus by applying Lemma 77 and Theorem 49 on F we obtain the conclusion. □

Indeed, it was shown by Hrušák and García-Ferreira in [HF03] that tight MAD families exists as long as $\mathfrak{b} = 2^{\aleph_0}$, where $\mathfrak{b}$ is the unbounded number. Moreover, they showed in the same paper that tight MAD families are Cohen indestructible.

Theorem 79 (Hrušák and García-Ferreira, [HF03]). *Tight MAD families are Cohen indestructible.*

Proof. Let A be a tight MAD family, let $\mathbb{C}$ be Cohen's forcing, $\dot{x}$ a name of a subset of ω, assume that $p \Vdash \dot{x}$ is infinite. We want to show that the set $D := \{q \in \mathbb{C} \mid q \Vdash \exists a \in A, |\dot{x} \cap a| = \infty\}$ is dense below p.

For arbitrary $q \leq p$, we want to find $r \leq q$ s.t. $r \in D$. May assume that $q \Vdash \dot{x} \notin I(A)$ as otherwise there is $r \leq q$ s,t, $r \Vdash \exists a_0 \dots a_n \in A, \dot{x} \subseteq^* a_0 \cup \dots \cup a_n$, which entails that $r \in D$ by an infinite pigeon hole principle.

Enumerate $\{r \in \mathbb{C} \mid r \leq q\}$ as $\{r_n \mid n \in \omega\}$, write $x_n = \{m \in \omega \mid \exists r \leq r_n, r \Vdash m \in \dot{x}\}$. First notice $x_n \notin I(A)$, as otherwise $r_n \Vdash \dot{x} \in I(A)$, a contradiction for assumption. Thus by tightness there is $a \in A$ s.t. $a \cap x_n$ is infinite for all $n \in \omega$. We show that $q \Vdash \dot{x} \cap a$ is infinite. Suppose not, then there is $k \in \omega$ and $r_n \Vdash (a \cap \dot{x}) \setminus k = \emptyset$, but as $x_n \cap a$ is infinite, there is $k' \in x_n \cap a$, $k < k'$ and $r \leq r_n$ s.t. $r \Vdash k' \in \dot{x} \cap a$. This is a contradiction.

This completes the proof. □

Thus we obtain the following results:

Corollary 80 (Folklore). *In L there is a $\mathbf{\Pi}^1_1$ Cohen indestructible MAD family.*

Proof. We apply Theorem 78 on the canonical well ordering of L, see Theorem 35, thus there is a $\mathbf{\Sigma}^1_2$ tight MAD family in L. By Theorem 79, it is Cohen indestructible and thus by Corollary 74, there is a $\mathbf{\Pi}^1_1$ Cohen indestructible MAD family in L. □

On the otherhand, we have a partial converse to Theorem 79, proved by Kurilić in [Kur01]. For an almost disjoint family $A \subseteq [\omega]^\omega$, we let the restriction of A on $x \in [\omega]^\omega$, denoted $A|x$ be the set $\{a \cap x \mid a \in A, |a \cap x| = \infty\}$. We say $A|x$ is a tight MAD family on $[x]^\omega$ if for any $\{y_n \mid n \in \omega\} \in [x]^\omega \setminus I(A|x)$, there is $z \in A|x$ s.t. $y_n \cap z$ is infinite for all n.

Theorem 81 (Kurilić). *If A is a Cohen indestructible MAD family then there is $x \in [\omega]^\omega$ s.t. $A|x$ is a tight MAD family on $[x]^\omega$.*

Proof. Assume for contradiction that there is no such $x \in [\omega]^\omega$. Then starting from $x_\emptyset = \omega$, we recursively build $(x_s \mid s \in \omega^{<\omega})$ satisfying the following:

1. $x_s \in [\omega]^\omega \setminus I(A)$.
2. $\{x_{s^\frown n} \mid n \in \omega\}$ witnesses x_s is not tight, i.e. there is no $a \in A|x_s$ s.t. $a \cap x_{s^\frown n}$ is infinite for all n.

Let c be a Cohen real, let y be a pseudo-intersection of $(x_{c|_n}, n \in \omega)$, we show that y witnesses that A is not MAD in the forcing extension, a contradiction to the assumption.

To see $V[c] \models \forall a \in A, |a \cap y| < \infty$, we notice for any $a \in A$, the following set is dense by construction:

$$D_a := \{s \in \omega^{<\omega} \mid |a \cap x_s| < \infty\}$$

Thus for any $a \in A$, there is n s.t. $c|_n \in D_a$, and y being the pseudo-intersection of $(x_{c|_n}, n \in \omega)$, we have $|a \cap y| < \infty$. □

Thus, we obtain the following fact, allowing us to reduce the complexity of a tight MAD family.

Corollary 82. *Assume that $\mathbf{\Pi}^1_n$-FP holds, further assume that Cohen forcing is $\mathbf{\Sigma}^1_{n+1}$ absolute. Then if there is a $\mathbf{\Sigma}^1_{n+1}$ tight MAD family, then there is a $\mathbf{\Pi}^1_n$ tight MAD family.*

Proof. If there is a $\mathbf{\Sigma}^1_{n+1}$ tight MAD family, then by Theorem 79, it is Cohen indestructible. By Corollary 74 there is a $\mathbf{\Pi}^1_n$ MAD family A that is Cohen indestructible, thus there is x s.t. $A|x$ is a tight MAD family on $[x]^\omega$. Let $f : x \to \omega$ increasingly enumerate x, then $\{f[y] \mid y \in A|x\}$ is a $\mathbf{\Pi}^1_n$ tight MAD family. □

This result is curious as Törnquist's construction as presented in Theorem 73 does not automatically preserve tightness. To see this, let A be the original $\mathbf{\Sigma}^1_{n+1}$ MAD family and A' be defined as in the proof. Let $\{x_n \mid n \in \omega\} \subseteq [\omega]^\omega \setminus I(A)$, then there is no element in A' hitting all elements in $\{\{0\} \times x_n \mid n \in \omega\} \cup \{\{1\} \times x_n \mid n \in \omega\}$ infinitely.

We also remark that all the above complexity results except Corollary 82 can be established for lightface point classes. For instance, the light face version of Theorem 73 holds: If Π^1_n-FP holds and there is a Σ^1_{n+1} MAD family, then there is a Π^1_n MAD family. But Corollary 82 is different as we relied on the extra parameter $x \in [\omega]^\omega$ on which the restriction $A|x$ is tight. Thus the best lightface version of Corollary 82 we can get is the following statement:

Corollary 83. *Assume that Π^1_n-FP holds, further assume that Cohen forcing is $\mathbf{\Sigma}^1_{n+1}$ absolute. Then if there is a Σ^1_{n+1} tight MAD family, then there is a real $a \in \omega^\omega$ and a $\Pi^1_n(a)$ tight MAD family.*

Finally, we make the observation that even though the construction in Theorem 73 does not preserve tightness, it does preserve a weaker property called weakly tightness. We say an almost disjoint family A is **weakly tight** if for any $(x_n, n \in \omega) \subseteq [\omega]^\omega \setminus I(A)$, there is a $a \in A$ s.t. $|a \cap x_n| = \infty$ for infinitely many $n \in \omega$. Similar to tight almost disjoint families, weakly tight almost disjoint families are automatically MAD. This concept was proposed by Hrušák and García-Ferreira in [HF03].

Proposition 84. *Assume $\mathbf{\Pi}^1_n$-FP. Then there is a $\mathbf{\Sigma}^1_{n+1}$ weakly tight MAD family iff there is a $\mathbf{\Pi}^1_n$ weakly tight MAD family.*

Proof. Let A be a $\mathbf{\Sigma}^1_{n+1}$ weakly tight MAD family. By $\mathbf{\Pi}^1_n$-FP there is a partial function $Q \subseteq [\omega]^\omega \times 2^\omega$ with a $\mathbf{\Pi}^1_n$ graph such that

$$A(a) \iff \exists! w Q(a, w)$$

For sake of convenience we recall the following maps $g_0, g_1 : ([\omega]^\omega)^2 \to \mathcal{P}(3 \times \omega)$:

$$g_0 : a, w \mapsto (\{0\} \times a) \cup (\{2\} \times \{a^*(n) \mid n \in w\})$$
$$g_1 : a, w \mapsto (\{1\} \times a) \cup (\{2\} \times \{a^*(n) \mid n \notin w\})$$

Let $A' = g_0[Q] \cup g_1[Q]$. In the proof of Theorem 73, we have that A' is a $\mathbf{\Pi}^1_n$ MAD family. We show that it is weakly tight, assuming A is.

For $y \in [3 \times \omega]^\omega, i \in \{0, 1, 2\}$ write $y_i = \{n \in \omega \mid (i, n) \in y\}$.

Let $\{x^n \mid n \in \omega\} \subseteq [3 \times \omega]^\omega \setminus I(A')$. Notice for each n, there is $i \in \{0, 1, 2\}$ s.t. $x^n_i \notin I(A)$, as otherwise say $x^n_0 \subseteq^* a^0_0 \cup \ldots a^{m_0}_0$, $x^n_1 \subseteq^* a^0_1 \cup \ldots a^{m_1}_1$ and $x^n_2 \subseteq^* a^0_2 \cup \ldots a^{m_2}_2$ for some $a^j_i \in A$, then

$$x^n \subseteq^* \bigcup_{j \leq m_0} g_0(a^j_0, w^j_0) \cup \bigcup_{j \leq m_1} g_1(a^j_1, w^j_1) \cup \bigcup_{j \leq m_2} g_0(a^j_2, w^j_2) \cup g_1(a^j_2, w^j_2)$$

contradicting $x^n \notin I(A')$. Here w^j_i is the witness s.t. $Q(a^j_i, w^j_i)$.

Thus there is $i \in \{0, 1, 2\}$ s.t. for infinitely many n, $x^n_i \notin I(A)$. Enumerate them as $\{z^m \mid m \in \omega\}$.

Case 1: $i = 0$ or 1, these two are symmetrical, we may assume $i = 0$. Since $z^m_0 \notin I(A)$, by weak tightness, there is $a \in A$ s.t. $a \cap z^m_0$ is infinite for infinitely many m. Let w be s.t. $Q(a, w)$. Thus $g_0(a, w) \cap z^m$ is infinite for infinitely many m, witnesses the weak tightness of A'.

Case 2: $i = 2$. Since $z^m_2 \notin I(A)$, by weak tightness, there is $a \in A$ s.t. $a \cap z^m_2$ is infinite for infinitely many m. Let w be s.t. $Q(a, w)$. Thus for one of $g_0(a, w), g_1(a, w)$, it intersects infinitely many z^m infinitely, witnesses the weakly tightness of A'. □

Comparing Corollary 82 and Proposition 84, we observe that reducing the complexity of tight MAD family need the assumption of generic absoluteness of Cohen forcing. We may ask whether this assumption is necessary, whether we can find a construction that preserves tightness directly:

Question 4. *Is it the case that under $\mathbf{\Pi}^1_n$-FP, there is a $\mathbf{\Sigma}^1_{n+1}$ tight MAD family iff there is a $\mathbf{\Pi}^1_n$ tight MAD family?*

5.1.3 Maximal independent family

The following theorem lifts Brendle, Fischer and Khomskii's result in [BFK19]. The proof follows the same line as their original proof.

Theorem 85. *Assume* $\mathbf{\Pi}^1_n$-FP *holds. Then there is a* $\mathbf{\Sigma}^1_{n+1}$ *maximal independent family iff there is a* $\mathbf{\Pi}^1_n$ *maximal independent family.*

Proof. Let A be a $\mathbf{\Sigma}^1_{n+1}$ MIF and write $A(x) \iff \exists! w \in 2^\omega F(x,w)$ where F is a $\mathbf{\Pi}^1_n$ function. We consider $\omega \cup 2^{<\omega}$ as the disjoint union of ω and $2^{<\omega}$.

Define

$$\begin{aligned} g : [\omega]^\omega \times 2^\omega &\to \mathcal{P}(\omega \cup 2^{<\omega}) \\ (x,w) &\mapsto x \cup \{w|_n \mid n \in \omega\} \end{aligned}$$

Let $A' = g[F]$, then by the argument as in [BFK19] it follows that A' is a MIF.

Finally, g is clearly Borel and injective. Thus $g[F]$ is a $\mathbf{\Pi}^1_n$ MIF. □

By an analogous argument as in Corollary 74, we obtain that:

Corollary 86. *Let* $\mathbb{P}$ *be a* $\mathbf{\Sigma}^1_{n+1}$ *generic absolute forcing notion, assume that* $\mathbf{\Pi}^1_n$-FP *holds. Then if there is a* $\mathbf{\Sigma}^1_{n+1}$ *maximal independent family that is* $\mathbb{P}$ *indestructible, then there is a* $\mathbf{\Pi}^1_n$ *maximal independent family that is* $\mathbb{P}$ *indestructible.*

A famous example of forcing indestructible maximal independent family is the Sacks indestructible maximal independent family. See [Fis18] for a construction of a coanalytic such family in L.

5.1.4 Maximal Ideal Independent Family

The following theorem lifts Millhouse and Schembecker's result in [MS26].

Theorem 87. *Assume* $\mathbf{\Pi}^1_n$-FP*. Then there is a* $\mathbf{\Pi}^1_n$ *maximal ideal independent family iff there is a* $\mathbf{\Sigma}^1_{n+1}$ *maximal ideal independent family.*

Proof. Let I be a $\mathbf{\Sigma}^1_{n+1}$ ideal independent family and assume

$$I(x) \iff \exists! w \in [\omega]^\omega G(x,w)$$

where G is a $\mathbf{\Pi}^1_n$ function.

We define the following function

$$\begin{aligned} \chi : [\omega]^\omega \times [\omega]^\omega &\to [\omega]^\omega \\ (x,w) &\mapsto \{3n \mid n \in x\} \cup \{3n+1 \mid n \in w\} \end{aligned}$$

Let $z = \{3n+1, 3n+2 \mid n \in \omega\}$. We let $I' = \chi[I] \cup \{z\}$.

The argument that it is a maximal ideal independent family is the same as [MS26].

Finally the function χ is clearly a Borel injection. Thus I' is a $\mathbf{\Pi}^1_n$ maximal ideal independent family. □

By an analogous argument as in Corollary 74, we obtain that:

Corollary 88. *Let $\mathbb{P}$ be a $\mathbf{\Sigma}^1_{n+1}$ generic absolute forcing notion, assume that $\mathbf{\Pi}^1_n$-FP holds. Then if there is a $\mathbf{\Sigma}^1_{n+1}$ maximal ideal independent family that is $\mathbb{P}$ indestructible, then there is a $\mathbf{\Pi}^1_n$ maximal ideal independent family that is $\mathbb{P}$ indestructible.*

5.1.5 Van Douwen Family

Van Douwen Family is a strengthened version of maximal eventually different family.

Definition 89 (Van Douwen Family)**.** *An eventually different family $A \subseteq \omega^\omega$ is called a Van Douwen family if for any infinite partial function $f : \omega \to \omega$, there is $h \in A$ s.t. there are infinitely many n s.t. $f(n)$ is defined and equals $h(n)$.*

Thus, a maximal eventually different family is Van Douwen if it also agrees with partial functions infinitely often.

Proposition 90. *There is a Borel recursion scheme F s.t. a Y compatible with F is a Van Douwen family. As a consequence, there are $\mathbf{\Delta}^1_2$ Van Douwen family in L.*

Proof. Let $f \in B$ iff f is a partial function from ω to ω.

Let $F_{VD} \subseteq (\omega^\omega)^{\leq\omega} \times B \times \omega^\omega$ be defined by $F_{VD}(A, f, g)$ iff EITHER all of the following holds:

1. $ran(A)$ is an eventually different family

2. There is no $h \in ran(A)$ s.t. $\forall m \exists n \geq m, f(n)$ is defined. $\wedge\, h(n) = f(n)$

3. $\forall m \exists n \geq m, f(n)$ is defined. $\wedge\, g(n) = f(n)$

OR 1 is true and 2 is false OR 1 is false.

Then Y compatible with F would be a Van Douwen family.

For the second statement, apply Corollary 51. □

In [Rag10], Raghavan showed that Van Douwen families cannot be analytic. This is a contrast to the weaker notion of maximal eventually different family, which admits a Borel instance by the result in [HS24]. At its core, what he showed is that the existence of a Van Douwen family leads to the existence of an infinite MAD family.

Theorem 91 (Raghavan, [Rag10, Corollary 2])**.** *If there is a $\mathbf{\Sigma}^1_n$ Van Douwen family, then there is a $\mathbf{\Sigma}^1_n$ infinite MAD family.*

Proof. Let A be such a Van Douwen family, by identifying each of its element with its graph, A is an almost disjoint family in $[\omega \times \omega]^\omega$. Notice A is necessarily infinite as no finite eventually different families can be maximal.

Let $a_0, a_1 \dots$ be a sequence of different elements from A, let $\{B_n \mid n \in \omega\}$ be an infinite partition of ω. Define a new function $f : \omega \to \omega$ by $f(n) = a_i(n)$ if $n \in B_i$. We show that $f \notin I(A)$, this is because for any $b_0 \dots b_n \in A$, there is a_i not equal to any of them, thus $a_i \cap b_j$ is finite for every $j \leq n$, thus $a_i|_{B_i} \not\subseteq^* b_0 \cup \dots \cup b_n$, thus $f \supseteq a_i|_{B_i}$ is not in $I(A)$.

Now consider the following family $B \subseteq [\omega \times \omega]^\omega$:

$$b \in B \iff \exists a \in A (b = f \cap a) \wedge |b| = \infty$$

We show that B is an infinite MAD family on f in the sense that for any $b_0, b_1 \in B$, they intersect finitely and for any $g \in [f]^\omega$, there is $b \in B$ s.t. $b \cap g$ is infinite.

B is almost disjoint on f as A is eventually different, and for any $g \in [f]^\omega$, it is an infinite partial function from ω to ω and thus we can find $a \in A$ s.t. $a \cap g$ is infinite, thus $a \cap f \in B$ and $a \cap f \cap g$ is infinite.

Finally, to show that B is indeed infinite, suppose for contradiction that B is finite, of size n. Suppose $B = \{a_0 \cap f, \dots, a_{n-1} \cap f\}$, where $a_0 \dots a_{n-1} \in A$, but as $f \notin I(A)$, $g := f \setminus a_0 \cup \dots \cup a_{n-1}$ is an infinite partial function from $\omega \to \omega$, thus there is $a \in A$ s.t. $a \cap g$ is infinite, thus $a \cap f \in B$ yet $a \cap f \neq a_i \cap f$ for all $i < n$, thus $|B| \geq n + 1$, a contradiction to our assumption.

Finally, it is easy to turn B in to a $\mathbf{\Sigma}^1_n$ infinite MAD family on ω via a bijection between f and ω. □

We remark that this theorem does not admit a lightface version, as we are relying on extra real parameters.

As a corollary, Ramsey property forbids the existence of Van Douwen families.

Corollary 92. *If $\mathbf{\Sigma}^1_n$ sets have the Ramsey property, then there is no $\mathbf{\Sigma}^1_n$ Van Douwen family.*

Proof. By Theorem 29 and Theorem 91. □

In [MS26], the complexity reduction result was established for Van Douwen families. We lift the result under Functional Projection property.

Theorem 93. *Assume $\mathbf{\Pi}^1_n$-FP, then if there is a $\mathbf{\Sigma}^1_{n+1}$ Van Douwen family, then there is a $\mathbf{\Pi}^1_n$ Van Douwen family.*

Proof. Define functions $\chi_0, \chi_1 : {}^\omega\omega \times 2^\omega \to {}^\omega\omega$ by

$$\chi_0(f, c)(n) := 2f(n) + c(n),$$

$$\chi_1(f, c)(n) := 2f(n) + 1 - c(n).$$

Now, assume that F is a $\mathbf{\Sigma}^1_{n+1}$ Van Douwen family, and let $H \subseteq {}^\omega\omega \times {}^\omega 2$ be a $\mathbf{\Pi}^1_n$ partial function such that F is the projection of H to the first component. Let

$$G := \chi_0[H] \cup \chi_1[H].$$

That G is a Van Douwen family is the same proof as in [MS26].

As for the definability of $\mathcal{G}$, clearly both χ_0 and χ_1 are injective Borel functions, thus by Lemma 7, G is $\mathbf{\Pi}^1_n$. □

5.1.6 Maximal Tower and Maximal Linear Tower

The following theorem lifts Fischer and Schilhan's result in [FS22].

Theorem 94. *Assume* $\mathsf{FP}(\mathbf{\Pi}^1_n)$*, then if there is a* $\mathbf{\Sigma}^1_{n+1}$ *maximal linear tower, then there exists a* $\mathbf{\Pi}^1_n$ *maximal linear tower. Moreover, if we start from a* $\mathbf{\Sigma}^1_{n+1}$ *maximal tower, then we get a* $\mathbf{\Pi}^1_n$ *maximal tower.*

Proof. Let A be a $\mathbf{\Sigma}^1_{n+1}$ maximal linear tower, and let $F \subseteq [\omega]^\omega \times 2^\omega$ be $\mathbf{\Pi}^1_n$ such that

$$x \in A \iff \exists! w \in 2^\omega \ (x, w) \in F.$$

For $x \in [\omega]^\omega$, let $x(n)$ be the n-th element of x. Define $g : [\omega]^\omega \times 2^\omega \to [\omega \times \omega]^\omega$ by letting

$$g(x, w) = (\{0\} \times x) \cup \big[\bigcup_{n\in\omega} \{n+1\} \times \big(x \setminus x(n + w(n))\big)\big].$$

We show that g is a Borel injection. That it is Borel is obvious. For $z \in [\omega \times \omega]^\omega$, let z_n be the nth section of z. For $(x, w) \neq (x', w')$, may assume $x = x'$ as otherwise $g(x, w)_0 \neq g(x', w')_0$. Let s be the largest common initial segment of w, w', may assume $w = s^\frown 0^\frown y$ and $w' = s^\frown 1^\frown y'$, then

$$g(x, w)_{|s|+1} = (x \setminus x(|s|)) \neq (x \setminus x(|s| + 1)) = g(x, w')_{|s|+1}$$

Thus g is injective.

The argument that $g[F]$ is a maximal linear Tower is the same as that in [FS22]. Thus $g[F]$ is a $\mathbf{\Pi}^1_n$ maximal linear Tower by Lemma 7.

The second part of the statement follows by observing that the injection preserves order: for $(x, w), (x', w') \in F$,

$$x \subseteq^* x' \iff g(x, w) \subseteq^* g(x', w')$$

Hence if A is well-ordered by $\supseteq^*$, so is $g[F]$. □

However, we notice that in this case, the complexity reduction proofs for forcing indestructible

instances (the parallel statement to Corollary 74) does not seem to easily generalize to the maximal tower case. The reason is that in the proof of Corollary 74, we used that if A, A' are both MAD families and $A \subseteq A'$, then $A = A'$. But this is not the case for maximal towers.

5.2 Combinatorial families in $L[U]$

As a consequence of what we have established, we are able to show that in $L[U]$ the combinatorial families discussed in this chapter all admit a $\mathbf{\Pi}^1_2$ instance.

First, by the fact that the canonical well-ordering on $L[U]$ is $\mathbf{\Sigma}^1_3$ good when restrcted to the reals, it is immediate that recursion in $L[U]$ can obtain a $\mathbf{\Sigma}^1_3$ object.

Proposition 95. *In $L[U]$, for any $\mathbf{\Delta}^1_3$ set of requirements B, base case $BC \in X^{\leq\omega}$ and any $\mathbf{\Delta}^1_3$ recursion condition $F \subseteq X^{\leq\omega} \times B \times X$, s.t. for any $A \in X^{\leq\omega}$, $p \in B$, $F_{\langle BC,A\rangle,p} \setminus (ran(BC) \cup ran(A))$ is non-empty, there is a $\mathbf{\Sigma}^1_3$ set Y compatible with F.*

Proof. Immediate from Theorem 49 and Theorem 42. □

Corollary 96. *In $L[U]$, there is $\mathbf{\Sigma}^1_3$ instance of each of the following objects: Infinite MAD family, maximal independent family, infinite maximal ideal independent family, maximal tower, Van Douwen family. Indeed for all but maximal tower, there is $\mathbf{\Delta}^1_3$ instance of them.*

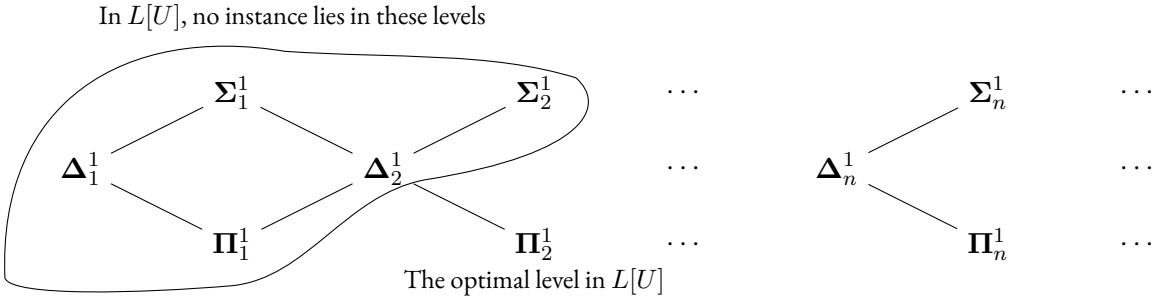
Proof. Same as Corollary 51. For Van Douwen family, apply Proposition 90. □

Theorem 97. *In $L[U]$, all of the following combinatorial families admit a $\mathbf{\Pi}^1_2$ instance.*

1. *Counterexample to perfect set property.*
2. *Infinite MAD family.*
3. *Maximal independent family.*
4. *Maximal (linear) towers.*
5. *Maximal Ideal independent family.*
6. *Van Douwen family.*

Proof. We work in $L[U]$. By Corollary , we get $\mathbf{\Sigma}^1_3$ instances of the objects. Moreover by Theorem 62, $\mathbf{\Pi}^1_2$-FP holds. Thus we may apply Theorems 72, 73, 85, 87, 93, 94 respectively to arrive at $\mathbf{\Pi}^1_2$ instances of the desired objects. □

We remark that for most of the combinatorial families listed above, we know that $\mathbf{\Pi}^1_2$ is the optimal complexity level at where they may exist in $L[U]$. Hence the instance we obtained in Theorem 97 is optimal in projective complexity:



Proposition 98. *For all but the maximal ideal independent family mentioned in Theorem 97, there is no $\mathbf{\Sigma}^1_2$ instance of it in $L[U]$.*

Proof. As there is a measurable cardinal in $L[U]$, Theorem 37 entails that all $\mathbf{\Sigma}^1_2$ sets are regular (have the perfect set property, Ramsey property, Lebesgue Measurability and Baire Property), thus by Theorem 29, there can't be $\mathbf{\Sigma}^1_2$ instance of infinite MAD family and maximal independent family; by Theorem 30 there can't be $\mathbf{\Sigma}^1_2$ instance of maximal tower; by Theorem 91 there can't be $\mathbf{\Sigma}^1_2$ instance of Van Douwen Family.

By Theorem 7.1 in [FS22], there is a maximal linear tower of complexity $\mathbf{\Sigma}^1_2$ iff there is a maximal tower of complexity $\mathbf{\Sigma}^1_2$, thus there is no maximal linear tower in $L[U]$ also. □

However, it remains open whether there can be a regularity property that forbids the existence of a maximal ideal independent family.

Question 5. *Is there a regularity property that forbids the existence of maximal ideal independent family? In particular, are there analytic maximal ideal independent families?*

Finally, we mention that Miller was able to show that there is a $\mathbf{\Pi}^1_1$ Hamel basis in L in [Mil89], thus it is natural to ask the status of Hamel basis in $L[U]$. On one hand, it is a folklore result that existence of a $\mathbf{\Sigma}^1_2$ Hamel basis leads to the existence of a $\mathbf{\Sigma}^1_2$ Vitali set (see for instance example **??** in Chapter 6), which cannot exist in $L[U]$, thus there is no $\mathbf{\Sigma}^1_2$ Hamel basis in $L[U]$. On the other hand, an easy application of transfinite recursion in $L[U]$ produces a $\mathbf{\Sigma}^1_3$ instance, appealing to Theorem 49 and Theorem 42. Thus, it remains to decide whether there is a $\mathbf{\Pi}^1_2$ Hamel basis in $L[U]$.

Question 6. *Is there a $\mathbf{\Pi}^1_2$ Hamel basis in $L[U]$?*

One approach to answer this question affirmatively is to find a complexity reduction result similar to those discussed in this chapter, however, this seems hard due to the combinatorics of Hamel bases:

Question 7. *Is it the case that under* $\mathbf{\Pi}^1_n$-FP*, the existence of a* $\mathbf{\Sigma}^1_{n+1}$ *Hamel basis entails the existence of a* $\mathbf{\Pi}^1_n$ *one?*

Chapter 6

Other Results

In this chapter, we collect some other results the author obtained during his Master period. In Section 6.1, we develop the theory of continuous relations and prove an interesting connection between the cardinal characteristics and definable combinatorics. In Section 6.2, we develop a graph comparision notion between Borel graphs and hypergraphs that is relevant for the study of maximal independent sets.

6.1 Tower Number and $\mathbf{\Pi}^1_2$ Combinatorics

Recall the definition of the tower number in Definition 24, we show that if the tower number $\mathfrak{t}$ is increased beyond ω_1, then certain combinatorial families cannot have $\mathbf{\Pi}^1_2$ instances.

6.1.1 Continuous Relation

The idea of continuous relation developed in this subsection comes from an unpublished exposition by William Chan on an early version of He, Luo and Zhang's paper [HLZ26].

Continuous relation is a generalizaiton of the concept of continuous function.

Definition 99 (Continuous Relation)**.** *Let X, Y be two topological spaces, let R be a relation on $X \times Y$ from X to Y, we say R is **continuous** if for each open $U \subseteq Y$, $R^{-1}[U] := \{x \mid \exists y \in U, R(x, y)\}$ is open in X.*

Clearly if R happen to be a function, then the notion of a continuous relation coincides with the concept of continuous functions.

We are mainly interested in relations from one Polish space to another Polish space. So without explicit mention, in the following we discuss continuous relations on Polish spaces.

First let's notice that when considering the notion of continuity of relations, the following fact breaks down. For functions $f : X \to Y$, if f is continuous, then f considered as a subset of $X \times Y$ is closed. For the $\Rightarrow$ side, for $\{(x_n, f(x_n)) \mid n \in \omega\} \subseteq f$ that converges to (x, y), by continuity of f, $f(x_n) \to f(x)$ and thus $(x, y) = (x, f(x)) \in f$.

But for relations, the nice fact breaks down:

Proposition 100. *There is $R \subseteq X \times Y$ s.t. R is continuous but not projective.*

And there is R that is closed but not continuous.

Proof. For the first part, first notice by a cardinality argument, a non-projective set exists. It is clear that since $\mathbf{\Sigma}^1_1$ sets are projections of closed sets and every closed set is $\mathbf{\Sigma}^1_1$, $\mathbf{\Sigma}^1_1$ sets on a fixed Polish space X has cardinality 2^ω, which is the number of closed sets on a Polish space. By an induction, for every n, $\mathbf{\Sigma}^1_n$ sets on a fixed Polish space X has cardinality 2^ω. Thus there are only 2^ω projective sets on a fixed Polish space, but there are 2^{ω^ω} many subsets of X.

Thus let $S \subseteq \mathbb{R}$ be some set that is not projective, consider $R = [\mathbb{R} \times (\mathbb{R} \setminus \{0\})] \cup [S \times \{0\}]$, which is continuous. It is continuous as for any nonempty $U \subseteq \mathbb{R}$ open, there is $0 \neq x \in U$ and thus $R^{-1}[U]$ is $\mathbb{R}$, thus open. Yet R cannot be projective as otherwise $S := \{x \in \mathbb{R} \mid (x, 0) \in \mathbb{R}\}$ would be projective.

For the second part, let $A \subseteq X$ be a $\mathbf{\Sigma}^1_1$ set that is not open, then there is $R \subseteq X \times Y$ closed and $R^{-1}[Y] = p[R] = A$ which is not open. □

We remark that by the first counter example, another good characterization of continuity is lost: Remember functions f is continuous iff f reflects closed sets: $f^{-1}[C]$ is closed for C closed. But in the case S is not closed, R clearly does not reflect closed set.

Also, a corollary of the first example is that there are 2^{ω^ω} many different continuous relations:

Corollary 101. *There are 2^{ω^ω} many different continuous relations on perfect Polish spaces X, Y.*

Proof. On one hand, there are 2^{ω^ω} different subsets of $X \times Y$ in total. On the other hand, each different subset of X yields a unique continuous relation by the construction in the first example of 100. □

For the second counter example, we are working with a relation that is not fully defined on its domain. But even requiring R to have full domain X won't save us here. For that, we need some more theory on Polish space.

Lemma 102 (See [Gao08, Proposition 1.1.2])**.** *Let X be a Polish space, if $U \subseteq X$ is open, then U with the subspace topology is Polish.*

Proof. Let d be the complete metric on X, let $F = X \setminus U$, define $d(x, F) = inf\{d(x, y) \mid y \in F\}$. Consider the following metric e on U, for $x, y \in U$:

$$e(x, y) = d(x, y) + \frac{1}{d(x, F)} + \frac{1}{d(y, F)}$$

It is easy to check that e and d generates the same topology on U. Moreover, if $x_0, x_1 \dots$ cauchies converges under e, then $x_0, x_1 \dots$ cauchies converges to some $x \notin F$ under d. Thus e is a complete metric on U. Thus U is Polish. □

Proposition 103. *There is $R \subseteq X \times Y$ s.t. $dom(R) := \{x \in X \mid \exists y \in Y, (x, y) \in R\} = X$, and is closed but not continuous.*

Proof. Let $U \subseteq Y$ be an open set that is not Y. Thus $X \times U$ with the subspace topology is Polish, and we can find closed $C \subseteq X \times U$ s.t. $p[C]$ is analytic but not open. C being closed in the subspace topology, there is C' closed in $X \times Y$ s.t. $C = C' \cap (X \times U)$ Consider the following relation

$$\begin{aligned} R &= C \cup [X \times (Y \setminus U)] \\ &= [(X \times U) \cap C'] \cup [X \times (Y \setminus U)] \\ &= [(X \times U) \cup X \times (Y \setminus U)] \cap [C' \cup X \times (Y \setminus U)] \\ &= (X \times Y) \cap [C' \cup X \times (Y \setminus U)] \end{aligned}$$

which is closed. But $R^{-1}[U] = p[R \cap (X \times U)] = p[C]$, is not open. □

Perhaps the analysis thus far shows that continuous relation in general can be weird. But we will see that for given projective (Borel) relation, considering its continuity will be a powerful tool of analysis.

First let's start by an important merit of relations. With relations, in contrast to functions, it makes sense to ask whether the union and intersection of continuous relations is still continuous.

Proposition 104. *Let $(R_i, i \in I)$ be a family of continuous relations on $X \times Y$, then $R = \bigcup_{i \in I} R_i$ is continuous.*

Proof. For $U \subseteq Y$ open, $R^{-1}[U] = \bigcup_{i \in I} R_i^{-1}[U]$, which is open. □

Theorem 105. *For X, Y Polish spaces, $(R_\alpha, \alpha < \omega_1)$ a decreasing sequence of continuous relations, decreasing in the sense that if $\alpha < \beta$ then $R_\alpha \supseteq R_\beta$. Then $R = \bigcap_{\alpha < \omega_1} R_\alpha$ is continuous.*

Proof. It suffice to show that for any basic open $V \subseteq Y$, for any $x \in X$ and $y \in V$ s.t. $R(x, y)$, there is basic open $U \subseteq X$ s.t. $x \in U$ $R^{-1}[V] \supseteq U$.

for any $x' \in U$, $R[x'] \cap V \neq \emptyset$. Here $R[x'] := \{y \in Y \mid R(x, y)\}$.

Let $\{N_m \mid m \in \omega\}$ enumerate the basic open set of X.

Define

$$m_\alpha = min\{m \mid \forall x \in N_m \wedge x' \in N_m, \exists y' \in V, (R_\alpha(x', y'))\} = min\{m \mid R_\alpha^{-1}[V] \supseteq N_m, x \in N_m\}$$

We observe that for $\alpha < \beta < \aleph_1$, as $R_\alpha \supseteq R_\beta$, $m_\alpha \leq m_\beta$.

Thus there is some m s.t. $m_\alpha \leq m$ for all $\alpha < \omega_1$, as otherwise

$$\bigcup_{m\in\omega} \{\alpha \mid m_\alpha \leq m\} = \omega_1$$

Which is absurd as under the assumption $\{m_\alpha \mid \alpha < \omega_1\}$ is unbounded in ω, each $\{\alpha \mid m_\alpha \leq m\}$ is countable.

Thus this N_m satisfies the requirement. □

The argument clearly produces a more general statement for spaces that are not Polish. Recall that $\mathcal{B}$ is a **local basis** of X at x if for any open U with $x \in U$, there is $B \in \mathcal{B}$ s.t. $x \in B \subseteq U$. The **local weight** of a space X at x, denoted $w(X, x)$, is the least size of a local basis of x. The **character** of X, denoted $\chi(X)$ is $sup_{x\in X} w(X, x)$.

Theorem 106. *For X, Y arbitrary topological spaces, if $\chi(X) = \lambda$ and $cf(\kappa) > \lambda$, $(R_\alpha, \alpha < \kappa)$ is a decreasing sequence of continuous relations, decreasing in the sense that if $\alpha < \beta < \kappa$ then $R_\alpha \supseteq R_\beta$. Then $R = \bigcap_{\alpha<\kappa} R_\alpha$ is continuous.*

Proof. By the same argument as above. □

We remark that in the above theorem, both $cf(\kappa) > \lambda$ and the fact that the relations are decreasing are important. Consider the following example: $\{0\} \times \mathbb{R} = \bigcap_{n\in\omega}(-\frac{1}{n}, \frac{1}{n}) \times \mathbb{R}$, this example shows that countable intersections of continuous relation may cease to be continuous.

On the other hand, if we drop the assumption of decreasing in the statement, then clearly the statement would imply that countable intersections of continuous relation is still continuous by making the tail constant to total relation. This is absurd by the counterexample we just provided.

For application to combinatorial set theory, we are interested in continuous relations on $[\omega]^\omega \times 2^\omega$.

For $R \subseteq X \times Y$, and $X' \subseteq X$, we say that R is **continuous on** X' if $R \cap (X' \times Y)$ is continuous as a relation from X' to Y, where X' inherits the subspace topology.

Mathias proved that if a function is Ramsey measurable, in the sense that the reflection of any open set has the Ramsey property, then the function is continuous on a Ramsey large set. He, Luo, Zhang observed that this also holds for relations. We state this fact for analytic relations.

For an element $y \in [x]^\omega$, let y^* be the function that increasingly enumerates y.

Theorem 107 (He, Luo, Zhang)**.** *Assume $R \subseteq [\omega]^\omega \times 2^\omega$ is an analytic relation, then for any infinite x, there is $y \in [x]^\omega$ s.t. R is continuous on $[y]^\omega$.*

Proof. Let $B_0 = \emptyset$ and $C_0 = x, t_0^0 = \emptyset$ and $\ell_0^0 = \emptyset, p_0^0 = (t_0^0, \ell_0^0) = (\emptyset, \emptyset)$. Define $P^{p_0^0} \subseteq [C_0]^\omega$ by:

$$g \in P^{p_0^0} \iff \exists h \in 2^\omega \big(h \restriction 0 = \ell_0^0 \wedge R(t_0^0{}^\frown g, h)\big)$$

This is an analytic relation, thus is completely Ramsey by Theorem 28. Pick an infinite $C^{p_0^0} \subseteq C_0$ which is homogeneous for $P^{p_0^0}$, meaning that $[C^{p_0^0}]^\omega \subseteq P^{p_0^0}$ or $[C^{p_0^0}]^\omega \cap P^{p_0^0} = \emptyset$.

Let $B_1 = B_0 \cup \{\min C^{p_0^0}\}$ and $C_1 = C^{p_0^0} \setminus B_1$.

Suppose B_n and C_n have been defined. Let $p_0^n, \ldots, p_{K_n}^n$ enumerate $[B_n]^{<\omega} \times 2^n$. For each $k \leq K_n$, let $t_k^n \in [B_n]^{<\omega}$ and $l_k^n \in {}^n2$ be such that $p_k^n = (t_k^n, l_k^n)$. Define $P^{p_0^n} \subseteq [C_n]^\omega$ by

$$g \in P^{p_0^n} \iff \exists h \in 2^\omega \big(h \restriction n = \ell_0^n \wedge R(t_0^n{}^\frown g, h)\big)$$

This is an analytic relation, thus is completely Ramsey by Theorem 28. Pick an infinite $C^{p_0^n} \subseteq C_n$ which is homogeneous for $P^{p_0^n}$.

Suppose that for some $k < K_n$, $C^{p_k^n}$ has been defined. Define $P^{p_{k+1}^n} \subseteq [C^{p_k^n}]^\omega$ by

$$g \in P^{p_{k+1}^n} \iff \exists h \in 2^\omega \big(h \restriction n = l_{k+1}^n \wedge R(t_{k+1}^n{}^\frown g, h)\big)$$

This is an analytic relation, thus is completely Ramsey by Theorem 28. Pick an infinite $C^{p_{k+1}^n} \subseteq C^{p_k^n}$ which is homogeneous for $P^{p_{k+1}^n}$.

After completing this finite construction, let $B_{n+1} = B_n \cup \{\min C^{p_{K_n}^n}\}$ and $C_{n+1} = C^{p_{K_n}^n} \setminus B_{n+1}$.

Let $y = \bigcup_{n\in\omega} B_n$. We claim that R is a continuous relation on $[y]^\omega$.

Let $f \in [y]^\omega$, $n \in \omega$, and $h \in 2^\omega$ be such that $R(f, h)$. Choose $k \leq K_n$ such that $h \restriction n = \ell_k^n$ and t_k^n is such that $f = t_k^n{}^\frown g$ for some $g \in [C_{n+1}]^\omega$. Since $l_k^n = h \restriction n$ and $g \in [C_{n+1}]^\omega$, it follows that $C_{n+1} \subseteq C^{p_k^n}$ must satisfy $[C_{n+1}]^\omega \subseteq P^{p_k^n}$, as witnessed by h.

Let $m' = |t_k^n|$ and let $m = m' + 1$. Suppose $w \in [y]^\omega$ is such that $w \restriction m = f \restriction m$. Write $w = (w \restriction m')^\frown u$ for some $u \in [y]^\omega$. Since $u(0) = w(m') = f(m') \in C_{n+1}$, we have $u \in [C_{n+1}]^\omega$ by the construction of y. Thus $u \in P^{p_k^n}$. Therefore, there is some $\bar{h} \in {}^\omega 2$ such that $R(t_k^n{}^\frown u, \bar{h})$ and $\bar{h} \restriction n = \ell_k^n = h \restriction n$.

But since $w \restriction m = f \restriction m, m = m' + 1, m' = |t_k^n|$, and $f = t_k^n{}^\frown g$, we have $(t_k^n{}^\frown u) = ((f \restriction m')^\frown u) = ((w \restriction m')^\frown u) = w$. Hence $R(w, \bar{h})$ and $\bar{h} \restriction n = h \restriction n$. This establishes that R is a continuous relation on $[y]^\omega$. □

Lemma 108. *Let $R \subseteq [\omega]^\omega \times 2^\omega$ be a relation that's E_0 invariant on the first argument. i.e. For all $x, y \in [\omega]^\omega$ and $g \in 2^\omega$, if $R(x, g)$ and $|x \Delta y| < \infty$ then $R(y, g)$.*

If R is continuous on $[x]^\omega$ and $y \subseteq^* x$, then R is continuous on $[y]^\omega$.

Proof. Assume $z \in [y]^\omega$ and $R(z, g)$. For arbitrary $n \in \omega$, we want to find $m \in \omega$ s.t. whenever $z'^* \restriction m = z^* \restriction m$ for $z' \in [y]^\omega$, there is g' s.t. $g' \restriction n = g \restriction n$ and $R(z', g')$.

Let $s = y \setminus x$, thus $z \setminus s \subseteq y \setminus s \in [x]^\omega$. By E_0 invariance $R(z \setminus s, g)$. By the assumption there is $m \in \omega$ s.t. whenever $z'^* \restriction m = (z \setminus s)^* \restriction m$ for $z' \in [x]^\omega$, there is g' s.t. $g' \restriction n = g \restriction n$ and

$R(z', g')$.

We show that $m+|s|$ satisfies the assumption. For $z'' \in [y]^\omega$ s.t. $z''^* \restriction (m+|s|) = z^* \restriction (m+|s|)$, $z'' \setminus s \in [x]^\omega$ and $(z'' \setminus s)^* \restriction m = (z \setminus s)^* \restriction m$. Thus there is g' s.t. $g' \restriction n = g \restriction n$ and $R(z'' \setminus s, g')$. Finally by E_0 invariance $R(z'', g')$. □

6.1.2 Tower Number and E_0 transversals

Definition 109. *$V \subseteq 2^\omega$ is called a E_0 **transversal** if:*

1. *For all $x \neq y \in V$, $|x \Delta y| = \infty$.*
2. *For all $x \in 2^\omega$ there is $y \in V$, $|x \Delta y| < \infty$.*

Again, the perspective of Borel graph kicks in, we can view E_0 transversal as a maximal independent set of the following Borel graph: $(2^\omega, E)$ where $xEy \Leftrightarrow x \Delta y$ is finite..

Theorem 110. *If $\mathfrak{t} > \omega_1$ then there is no $\mathbf{\Pi}^1_2$ E_0 transversal.*[1] *(And no $\mathbf{\Sigma}^1_2$ E_0 transversal, as every $\mathbf{\Sigma}^1_2$ E_0 transversal is automatically $\mathbf{\Pi}^1_2$.)*

Proof. We prove the converse. Assume V is a $\mathbf{\Pi}^1_2$ E_0 transversal, then there is Borel $B_\alpha, \alpha < \aleph_1$ s.t. $V = \bigcap_{\alpha<\aleph_1} B_\alpha$ by Theorem 8. We may assume B_α is decreasing as countable intersections of Borel sets are again Borel.

For $\alpha < \omega_1$, define $R_\alpha \subseteq [\omega]^\omega \times 2^\omega$ by

$$R_\alpha(x, a) \iff a \in B_\alpha \wedge |x \Delta a| < \infty$$

This is a Borel relation.

We build a tower of elements x_α, $x_\alpha \subseteq^* x_\beta$ iff $\alpha < \beta$ s.t. R_α is cont. on $x_{\alpha+1}$.

On successor step we apply Theorem 107 to $[x_\alpha]^\omega$ and R_α.

On limit step λ we take the pseudo-intersection of $x_\alpha, \alpha < \lambda$.

Now we argue that this tower $(x_\alpha, \alpha < \aleph_1)$ cannot have a pseudo-intersection, witnessing $\mathfrak{t} = \omega_1$. Assume for contradiction that it does, x is a pseudo-intersection of it. We observe that for each $\alpha < \omega_1$, the relation R_α is E_0 invariant on the first coordinate and thus by Lemma 108, we obtain that R_α is continuous on $[x]^\omega$.

Let $R_V \subseteq [\omega]^\omega \times 2^\omega$ be s.t.

$$R_V(x, a) \iff a \in V \wedge |x \Delta a| < \infty$$

[1] In a personal communication, Jialiang He was able to improve this result to if $\mathfrak{h} > \omega_1$, then there is no $\mathbf{\Pi}^1_2$ E_0 transversal. His method also uses the idea of continuous relation.

Thus $R_V = \bigcap_{\alpha<\aleph_1} R_\alpha$ is continuous on $[x]^\omega$ by Theorem 105.

We obtain a contradiction from there: choose $y, y' \in [x]^\omega$ s.t. $|y\Delta y'| = \infty$, thus $R_V(y, a), R_V(y', a')$ for some $a \neq a' \in V$. Let n be s.t. $a(n-1) \neq a'(n-1)$. By continuity there is m s.t. whenever $y''^* \restriction m = y \restriction m$ for $y'' \in [x]^\omega$, there is a'' s.t. $a'' \restriction n = a \restriction n$ and $(y'', a'') \in R_V$. Consider y'' given by

$$y''^*(i) = \begin{cases} y^*(i) & \text{if } i < m \\ y'^*(i+k) & \text{if } i \geq m, \text{ where } k \text{ is the least number satisfying } y'^*(m+k) > y^*(m-1). \end{cases}$$

Thus there is a'' s.t. $a'' \restriction n = a \restriction n$ and $(y'', a'') \in R_V$. Specifically, $a'' \neq a'$ as $a''(n-1) = a(n-1) \neq a'(n-1)$. Also notice $y''\Delta y'$ is finite as they agree on the tail. We thus have $|y''\Delta a''| < \infty$, $|y'\Delta a'| < \infty$ and thus $|a''\Delta a'| < \infty$ which is absurd as they are distinct elements from V. □

6.2 Graph Comparison and Maximal Independent Set

Recall the definition of Borel graphs in the section 2.3. Let's first focus our attention on binary graphs. For a detailed theory of Borel graph, the reader may refer to [KM20].

Definition 111 (Borel Graph)**.** *Let X be a Polish space and $E \subseteq X^2$ be symmetric. We say $G = (X, E)$ is a **Borel graph** if E is Borel.*

*$I \subseteq G$ is called an **independent set** if for any $x_0 \neq x_1 \in I$, $\neg x_0 E x_1$. An independent set is maximal if it is not a proper subset of any other independent set.*

*$I \subseteq G$ is called a **spanning set** if for any $x \in G \setminus I$, there is $y \in I$ s.t. xEy.*

Recall by Proposition 21, an independent set is maximal iff it is spanning.

This section is motivated by the following question, we try to look for a graph comparison notion that is useful for the study of maximal independent set:

Question: Is there a comparison notion for Borel graphs that allows one to transform a maximal independent set of one Borel graph to a maximal independent set of the another, preferably in a way that preserves definability?

Of course under AC, all Borel graphs have a maximal independent set simply by an application of Zorn's Lemma, thus the definability side of the question is the more interesting one.

The following is a satisfying notion:

Definition 112 (p-Relation)**.** *Let $G_0 = (X_0, E_0)$ and $G_1 = (X_1, E_1)$ be two Borel graphs. We say that $R \subseteq X_0 \times X_1$ is a **p-relation** if*

1. *For each $y \in X_1$, $R^{-1}[y]$ is independent.*

2. *xE_0x' and $xRy, x'Ry'$ entails yE_1y'.*

3. *For each $x \in X_0$ there is $y \in X_1$ s.t. xRy and y satisfies whenever $y' \in X_1$ satisfies yE_1y' then there is $x' \in X_0$ s.t. $x'Ry'$ and xE_0x'.*

A p-relation is called Borel if its graph is Borel.

The notion is a generalization of the famous notion of p-morphism (or bounded morphism) in modal logic. See for instance [BRV01].

Theorem 113. *Let R be a Borel p-relation between G_0 and G_1, then if I is a maximal independent set of G_1, then $R^{-1}[I]$ is a maximal independent set of G_0. Moreover, if I is $\mathbf{\Sigma}^1_n$, then $R^{-1}[I]$ is $\mathbf{\Sigma}^1_n$.*

Proof. Let $J = R^{-1}[I]$, for $x \neq x' \in J$, if xE_0x', then in particular there is no single y s.t. xRy and $x'Ry$, say $y \neq y'$ s.t. $xRy, x'Ry'$, thus yE_1y', a contradiction to the independence of I. Thus J is independent.

To show J is spanning, pick $x \in X_0 \setminus J$, there is $y \in X_1$ satisfying 3 of the definition. As I is spanning, there is $y' \in I$ s.t. yE_1y', thus there is $x'E_0x$ and $x' \in R^{-1}[y'] \subseteq R^{-1}[I]$, witnessing that J is spanning. □

To see that p-relation is indeed a powerful notion, consider the following example:

Let $G = (2^\omega, E_0)$, where xE_0y iff $x\Delta y$ is finite, i.e. they are eventually the same. A maximal independent set of this graph is a E_0 transversal.

Let $G_v = (\mathbb{R}, E_v)$, where xE_vy iff $x - y \in \mathbb{Q}$. A maximal independent set of this graph is known as the **Vitali set**.

Example 114. *There is a Borel p-relation R from G to G_v, thus if there is a $\mathbf{\Sigma}^1_n$ Vitali set, then there is a $\mathbf{\Sigma}^1_n$ E_0 transversal.*

Proof. Consider the following relation xRy iff:

$$\exists i \in \omega, \exists n_{m_0} \dots n_{m_i}, \Big(\forall j \leq i, (x(m_j) = 0) \\ \wedge y = \sum_{m=0}^{\infty} \frac{k_m}{(m+1)!}\Big), \text{ where } k_m = \begin{cases} x(m) & \text{if } m \notin \{m_0 \dots m_i\} \\ n_{m_j} & \text{if } m = m_j \end{cases}$$

In case xE_0x' and $xRy, x'Ry', y - y' \in \mathbb{Q}$ as they are different on finite bits.

For each $y \in \mathbb{R}$, first notice it admits a unique expansion into the form

$$y = a_0 + \sum_{m=1}^{\infty} \frac{a_m}{(m+1)!} \tag{*}$$

where for $m > 0$, $a_m \leq m$. This expansion can be done by taking $a_0 = \lfloor y \rfloor$ and recursively taking

$$a_m = \lfloor (y - a_0 - \frac{a_1}{2!} \cdots - \frac{a_{m-1}}{m!})(m+1)! \rfloor$$

Thus $R^{-1}[y]$ is either empty or a singleton, as xRy iff there are only finite bits in $(a_m, m \in \omega)$ that is not 0 and 1, and x is obtained by turning these bits into 0.

Finally, for any $x \in 2^\omega$, consider the real

$$y = \sum_{m=0}^{\infty} \frac{x(m)}{(m+1)!}$$

This is clearly a converging sequence, thus well-defined. For any $y' \in \mathbb{R}$ s.t. $y - y' = r \in \mathbb{Q}$, take least m s.t. there is some k s.t. $r = \frac{k}{(m+1)!}$, thus taking $(a_n, n \in \omega)$ as the expansion of y' into the form of (*), it is clear that $a : n \mapsto a_n$ and x agrees on elements beyond m. And thus x' given by

$$x'(n) := \begin{cases} a(n) & \text{if } a(n) \leq 1 \\ 0 & \text{if } a(n) > 1 \end{cases}$$

satisfies $x' R y'$ and $x' E_0 x$. □

We further make the observation that this relation R is the example is a union of countably many Borel functions: Let $A = \bigcup\{\omega^s \mid s \text{ is a finite set of natural numbers}\}$, thus A is contable, for each $p \in A$, define $f_p : 2^\omega \to \mathbb{R}$ as

$$f_p(x) = \sum_{m=0}^{\infty} \frac{k_m}{(m+1)!}, \text{ where } k_m = \begin{cases} x(m) & \text{if } m \notin dom(p) \\ p(m) & \text{if } m \in dom(p) \wedge x(m) = 0 \end{cases}$$

Then $R = \bigcup_{p \in A} f_p$. Given that observation, we conclude that given a Vitali set $V \subseteq G_v$, the reflection $R^{-1}[V]$ preserves the place in the projective hierarchy: If V is $\mathbf{\Pi}^1_n$, then $R^{-1}[V]$ is also $\mathbf{\Pi}^1_n$.

Thus we obtain that Theorem 110 leads to the following result for Vitali set:

Corollary 115. *If* $\mathfrak{t} > \omega_1$ *then there is no* $\mathbf{\Pi}^1_2$ *Vitali set.*

The reverse is also true, for that we need some theory from invariant descriptiove set theory.

Lemma 116 (See also [Gao08, Proposition 6.1.2 and 6.1.4]). *There is a Borel injection $f : \mathbb{R} \to 2^\omega$ s.t. $x_0 E_v x_1 \Leftrightarrow f(x_0) E_0 f(x_1)$.*[2]

Proof. Define $f_0 : \mathbb{R} \to \omega^\omega$ be determined by:

$$r = f_0(r)(0) + \sum_{m=1}^{\infty} \frac{f_0(r)(m)}{(m+1)!}$$

where $f_0(r)(m) \leq m$ for $m > 0$. This is an injective map s.t. $x_0 E_v x_1 \Leftrightarrow \{n \mid f_0(x_0)(n) \neq f_0(x_1)(n)\}$ is finite.

Then let $f_1 : \omega^\omega \to 2^\omega$ be given by

$$f_1(x)(\langle n, k\rangle) = \text{the } k+1\text{-th least significant digit of the binary expansion of } x(n)$$

This is an injective map s.t. $\{n \mid y_0(n) \neq y_1(n)\}$ is finite $\Leftrightarrow f_1(y_0) E_0 f_1(y_1)$.

Composing $f = f_1 \cdot f_0$ gives the desired map. □

Example 117. *There is a Borel p-relation R from G_v to G, thus if there is a $\mathbf{\Sigma}^1_n$ E_0 transversal, then there is a $\mathbf{\Sigma}^1_n$ Vitali set.*

Proof. Fix an ordering $<_{fin}$ on $[\omega]^{<\omega}$ in type ω. Let f be the function given by Lemma 116.

Let R be given by

$$xRy \iff |f(x)\Delta y| < \infty \wedge \forall q \in \mathbb{Q}, ((f(x)\Delta y) \leq_{fin} (f(x+q)\Delta y))$$

i.e. xRy if $f(x)$ and y are different by finite changes, and within the coset $x + \mathbb{Q}$, the finite changes $f(x)$ makes to reach y is the least. It is easy to verify that R is a Borel p-relation. □

Corollary 118 (ZF+DC). *There is a Vitali set iff there is an E_0-transversal.*

Thus we see that p-relation is indeed a powerful notion, being behind many implication results of combinatorial families. It allows us to transform one maximal independent set into another via comparing the graphs behind them. We are interested in finding more examples of the kind:

Question 8. *Are there other natural examples of the following kind: If there is a Γ maximal independent set of G, then there is a Γ maximal independent set of G'? And are they induced by p-relation between G and G'?*

Finally, we have seen many examples of hypergraph whose maximal independent set leads to failure of certain regularities. But what about the converse? We are interested in finding graphs whose maximal independent sets detect failure of regularity properties:

[2] This is known as a faithful Borel reduction from E_v to E_0.

Question 9. *Is there a Borel (hyper) graph G s.t. it has a maximal independent set iff a certain regularity property (Baire property, Ramsey property etc.) fails?*

Chapter 7

Conclusions and Questions

In this thesis, we studied the problem of lifting complexity reduction results for combinatorial families. We isolated the Functional Projection property which allows us to lift the complexity reduction results to higher point classes. Moreover, we established $\mathbf{\Pi}^1_n$-FP for both the model L and the model $L[U]$. This is in sharp contrast with the behavior of the uniformization property the behavior of the uniformization property, which fails at certain higher projective levels, such as $\mathbf{\Pi}^1_2$. Our result allowed us to derive the optimal projective complexity of the combinatorial objects in the model $L[U]$.

We believe that the property of Functional Projection we isolated in this thesis is of great interest in itself. As we have discussed in chapter 4, it weakens uniformization in a significant way, losing its power as a definable choice principle. On the other hand, it retains enough strength to support many complexity reduction arguments.

Coming back to the question of lifting complexity reduction results to higher pointclasses, even though we have isolated a sufficient condition for the lifting, namely the Functional Projection property, it remains open whether we can do complexity reductions without this assumption.

Question 10. *Can we lift these complexity reduction results to higher pointclasses without assuming Functional Projection? More precisely, is it the case that ZFC alone proves that if there is a $\mathbf{\Sigma}^1_{n+1}$ combinatorial family X, then there is a $\mathbf{\Pi}^1_n$ instance, for $n \geq 2$?*

At this point, the author expects the answer to be negative. I believe that for combinatorial family such as infinite MAD family, there are models where there are $\mathbf{\Sigma}^1_{n+1}$ instances of it but no $\mathbf{\Pi}^1_n$ of it ($n \geq 2$). In particular, we conjecture that:

Conjecture 1. *There is a model of ZFC where there is a $\mathbf{\Sigma}^1_3$ infinite MAD family but no $\mathbf{\Pi}^1_2$ infinite MAD family.*

Bibliography

[BFK19] Jörg Brendle, Vera Fischer, and Yurii Khomskii. "Definable maximal independent families". In: *The Journal of Symbolic Logic* 147 (2019), pages 3547–3557. https://doi.org/10.2178/jsl.7804080 (cited on pages 6, 18, 56, 57).

[BK13] Jörg Brendle and Yurii Khomskii. "Mad families constructed from perfect almost disjoint families". In: *The Journal of Symbolic Logic* 78.4 (2013), pages 1164–1180. ISSN: 00224812, 19435886. http://www.jstor.org/stable/43303702 (visited on 06/02/2026) (cited on page 9).

[BRV01] Patrick Blackburn, Maarten de Rijke, and Yde Venema. "Modal Logic". In: *Studia Logica* 76.1 (2001), pages 142–148 (cited on page 71).

[Coh63] Paul J. Cohen. "The Independence of the Continuum Hypothesis". In: *Proceedings of the National Academy of Sciences of the United States of America* 50.6 (1963), pages 1143–1148. https://doi.org/10.1073/pnas.50.6.1143 (cited on page 4).

[Fis18] Vera Fischer. *A co-analytic Sacks indestructible maximal independent family*. Preprint. 2018. https://www.logic.univie.ac.at/~vfischer/co_analytic_sacks_independent_2603.pdf (cited on page 57).

[FS21] Vera Fischer and David Schrittesser. "A Sacks indestructible co-analytic maximal eventually different family". English. In: *Fundamenta Mathematicae* 252.2 (2021). Publisher Copyright: © 2021 Instytut Matematyczny PAN., pages 179–201. ISSN: 0016-2736. https://doi.org/10.4064/fm842-1-2020 (cited on page 51).

[FS22] Vera Fischer and Jonathan Schilhan. "Definable Towers". English. In: *Fundamenta Mathematicae* 256.3 (2022). Publisher Copyright: © Instytut Matematyczny PAN, 2022., pages 221–241. ISSN: 0016-2736. https://doi.org/10.4064/fm917-9-2021 (cited on pages 5, 6, 18, 24, 60, 62).

[FT10] Vera Fischer and Asger Törnquist. "A co-analytic maximal set of orthogonal measures". In: *The Journal of Symbolic Logic* 75.4 (2010), pages 1403–1414. https://doi.org/10.2178/jsl/1286198154 (cited on pages 6, 18).

[Gao08] Su Gao. *Invariant Descriptive Set Theory*. Volume 293. Pure and Applied Mathematics. Boca Raton, FL: Chapman & Hall/CRC, 2008. ISBN: 9781584887935. https://doi.org/10.1201/9781584887942 (cited on pages 65, 73).

[Göd40] Kurt Gödel. *The Consistency of the Axiom of Choice and of the Generalized Continuum-Hypothesis with the Axioms of Set Theory*. Princeton, NJ: Princeton University Press, 1940 (cited on page 4).

[HF03] Michael Hruš'ak and Salvador García Ferreira. "Ordering Mad Families a la Katětov". In: *Journal of Symbolic Logic* 68.4 (2003), pages 1337–1353. https://doi.org/10.2178/jsl/1067620190 (cited on pages 53–55).

[HLZ26] Jialiang He, Jintao Luo, and Shuguo Zhang. *Ramsey Property and Pathological Sets: Almost Disjointness, Independence and Other Maximal Objects*. 2026. https://arxiv.org/abs/2604.26570 (cited on pages 24, 64).

[Hof22] Stefan Hoffelner. *Forcing the Π^1_n-Uniformization Property*. 2022. https://arxiv.org/abs/2103.11748 (cited on page 20).

[HS24] Haim Horowitz and Saharon Shelah. "A Borel maximal eventually different family". In: *Annals of Pure and Applied Logic* 175.1, Part B (2024). Kenneth Kunen (1943-2020), page 103334. ISSN: 0168-0072. https://doi.org/https://doi.org/10.1016/j.apal.2023.103334. https://www.sciencedirect.com/science/article/pii/S016800722300091X (cited on pages 24, 58).

[Jec03] Thomas Jech. *Set Theory: The Third Millennium Edition, revised and expanded*. Berlin, Heidelberg: Springer Science & Business Media, 2003. ISBN: 978-3-540-44085-7 (cited on pages 6, 11, 17, 19, 24–28, 30).

[Kan03] Akihiro Kanamori. *The Higher Infinite. Large Cardinals in Set Theory from Their Beginnings*. Springer Berlin, Heidelberg, 2003. https://doi.org/https://doi.org/10.1007/978-3-540-88867-3 (cited on pages 9, 27–29).

[Kho12] Yurii Daniilovich Khomskii. "Regularity Properties and Definability in the Real Number Continuum: Idealized forcing, polarized partitions, Hausdorff gaps and mad families in the projective hierarchy". PhD Thesis. Amsterdam: Universiteit van Amsterdam, 2012. https://uva.nl (cited on pages 24, 27).

[KL24] Vladimir Kanovei and Vassily Lyubetsky. "On the Uniform Projection Problem in Descriptive Set Theory". In: *Axioms* 14 (Dec. 2024), page 13. https://doi.org/10.3390/axioms14010013 (cited on pages 7, 38).

[KL25] Vladimir Kanovei and Vassily Lyubetsky. “On the Uniform Projection and Covering Problems in Descriptive Set Theory Under the Axiom of Constructibility”. In: *Mathematics* 13 (Jan. 2025), page 409. https://doi.org/10.3390/math13030409 (cited on pages 7, 38, 39).

[KM20] Alexander S. Kechris and Andrew S. Marks. *Descriptive Graph Combinatorics*. 2020. http://www.math.caltech.edu/~kechris/papers/combinatorics20book.pdf (cited on page 70).

[Kun80] Kenneth Kunen. *Set Theory: An Introduction to Independence Proofs*. Volume 102. Studies in Logic and the Foundations of Mathematics. Amsterdam: North-Holland, 1980. ISBN: 978-0-444-86839-8.

[Kur01] Miloš S. Kurilić. “Cohen-stable families of subsets of integers”. In: *Journal of Symbolic Logic* 66.1 (2001), pages 257–270. https://doi.org/10.2307/2694920 (cited on page 54).

[Mal90] V. I. Malykhin. “Topological Properties of Cohen Generic Extensions”. In: *Transactions of the Moscow Mathematical Society* 52 (1990), pages 1–32 (cited on page 53).

[Mat77] A.R.D. Mathias. “Happy families”. In: *Annals of Mathematical Logic* 12.1 (1977), pages 59–111. ISSN: 0003-4843. https://doi.org/https://doi.org/10.1016/0003-4843(77)90006-7. https://www.sciencedirect.com/science/article/pii/0003484377900067 (cited on page 4).

[Mil25] Julia Millhouse. “Definable Witnesses”. PhD thesis. Universität Wien, 2025 (cited on pages 6, 18).

[Mil89] Arnold W. Miller. “Infinite combinatorics and definability”. In: *Annals of Pure and Applied Logic* 41.2 (1989), pages 179–203. ISSN: 0168-0072. https://doi.org/https://doi.org/10.1016/0168-0072(89)90013-4. https://www.sciencedirect.com/science/article/pii/0168007289900134 (cited on pages 5, 62).

[Mos09] Yiannis N Moschovakis. *Descriptive Set Theory*. Volume 155. Providence, RI: American Mathematical Society, 2009. ISBN: 978-0-8218-4813-5 (cited on pages 11, 12, 14, 16, 18, 19, 26, 45–47).

[MS26] Julia Millhouse and Lukas Schembecker. *Coanalytic families of functions*. 2026. https://arxiv.org/abs/2510.07222 (cited on pages 6, 18, 24, 57, 59, 60).

[Rag10] Dilip Raghavan. “THERE IS A VAN DOUWEN MAD FAMILY”. In: *Transactions of the American Mathematical Society* 362.11 (2010), pages 5879–5891. ISSN: 00029947. http://www.jstor.org/stable/25747148 (visited on 05/25/2026) (cited on page 58).

[Sch14] Ralf Schindler. *Set Theory: Exploring Independence and Truth*. Jan. 2014. ISBN: 978-3-319-06724-7. https://doi.org/10.1007/978-3-319-06725-4 (cited on pages 11, 47).

[Sch17] David Schrittesser. *On Horowitz and Shelah's Borel maximal eventually different family*. 2017. https://arxiv.org/abs/1703.01806 (cited on page 24).

[Sch20] David Schrittesser. *Maximal discrete sets*. 2020. https://arxiv.org/abs/2012.14638 (cited on pages 9, 22).

[Sch22a] Jonathan Schilhan. "Coanalytic ultrafilter bases". In: *Archive for Mathematical Logic* 61.3 (2022), pages 567–581. https://doi.org/10.1007/s00153-021-00801-7. https://doi.org/10.1007/s00153-021-00801-7 (cited on page 6).

[Sch22b] Jonathan Schilhan. "Tree Forcing and Definable Maximal Independent Sets in Hypergraphs". In: *The Journal of Symbolic Logic* 87.4 (2022), pages 1419–1458. https://doi.org/10.1017/jsl.2022.36 (cited on page 22).

[Sch26] Jonathan Schilhan. "Maximal sets without choice". In: *Annals of Pure and Applied Logic* 177.4 (2026), page 103694. ISSN: 0168-0072. https://doi.org/https://doi.org/10.1016/j.apal.2025.103694. https://www.sciencedirect.com/science/article/pii/S0168007225001435 (cited on page 22).

[Sol70] Robert M. Solovay. "A Model of Set-Theory in Which Every Set of Reals Is Lebesgue Measurable". In: *Annals of Mathematics*. 2nd series 92.1 (1970), pages 1–56 (cited on page 4).

[Sus17] Mikhail Y. Suslin. "Sur une définition des ensembles mesurables B sans nombres transfinis". In: *Comptes Rendus de l'Académie des Sciences de Paris* 164 (1917), pages 88–91 (cited on page 4).

[Tör09] Asger Törnquist. "Σ^1_2 and Π^1_1 Mad Families". In: *The Journal of Symbolic Logic* 78.4 (2009), pages 1181–1182. https://doi.org/10.2178/jsl.7804080 (cited on pages 5, 18, 50).

[Tör18] Asger Törnquist. "Definability and almost disjoint families". In: *Advances in Mathematics* 330 (2018), pages 61–73. ISSN: 0001-8708. https://doi.org/https://doi.org/10.1016/j.aim.2018.03.005. https://www.sciencedirect.com/science/article/pii/S0001870818300938.

[Vid12] Zoltán Vidnyánszky. "Transfinite inductions producing coanalytic sets". In: *Fundamenta Mathematicae* 224 (Sept. 2012). https://doi.org/10.4064/fm224-2-2 (cited on pages 5, 30, 35).