A 2-Sketchy Approach to Type Theory

MSc Thesis *(Afstudeerscriptie)*

written by

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Abstract

In this thesis we show how several of the common categorical structures for modeling type theory, comprehension categories and categories with families, can be given in terms of 2-dimensional limit theories and sketches. In order to have precise control over the homomorphisms, so that they align with those found in the literature, we take use finite limit $\mathscr{F}$-theories and finite limit $\mathscr{F}$-sketches as the notion of 2-dimensional theory. Using these tools, we show in detail how comprehension categories fit into the framework, with the weak maps matching the literature, and sketch how categories with families fit as well. We then show how to extend the theory for categories with families with several type constructors: dependent product types, dependent sum types, and extensional identity types. We show how the language of $\mathscr{F}$-theories can be used to compare different models of type theory by showing the theory of categories with families equivalent to categories with attributes as $\mathscr{F}$-categories, adapting the traditional proof. Finally, we show how this framework differs from earlier ones, Uemura's categories with representable maps [29] and Coraglia's and Di Liberti's judgemental theories [8], by modeling Gratzer's multimode type theory [14], something which neither of the aforementioned frameworks can easily cover.

Contents

Chapter 1

Introduction

Since the introduction of dependent type theories, a plethora of categorical semantics have been considered. A good overview of the landscape of these different semantics and their relations to each other can be found in [1], but to paint a picture of their multitude, we include small sample of them here:

- Cartmell [7] introduced the more algebraic notions of categories with attributes (unpublished, but elaborated on by Moggi [23] and Pitts [26]).
- Jacobs [18] introduced comprehension categories as a minimal structure for studying context extension and comprehension, encompassing both categories with attributes and other notions of comprehension in logic, such as subset comprehension as found in toposes.
- Dybjer [11] gave an alternative formulation of categories with attributes, dubbed categories with families, making terms a core feature rather than a derived notion. These were later again reformulated by Awodey [3] and Fiore [13] in terms of representable maps of presheaves [27, Tag 0023]. This reformulation was named natural models by Awodey [3].

Furthermore, all of these semantics present only the most barebones of type theories, with only a notion of contexts, types, sometimes terms, and context extension. We can of course extend these semantics to account for e.g. dependent product types, but this is generally adhoc, with no framework to combine and compare different extensions.

In order to rectify the situation, Uemura [29] proposed a functorial semantics based on the natural models of [3], wherein a type theory is taken to be a finitely complete 1-category equipped with a class of special maps, dubbed representable maps, following the tradition of the functorial semantics introduced by [21]. The canonical example of a representable maps of is a category of presheaves equipped with its class of representable maps, and a model of a type theory as in [29] is a base category $\mathcal{C}$ of contexts equipped with a functor from the type theory to $\mathcal{C}$ preserving the structure of categories with representable maps. Considering the representable map category generated by a single non-trivial representable map, we then recover as its models exactly the natural models of [3]. However, while this semantics allows to capture many type theories that have been considered, it does not capture all the ones of interest. In particular, it does not capture multimode type theory (MTT) as introduced by Gratzer et al. [14], which garnered a lot of interest in recent times, in part due to its uniform treatment of multiple modalities. Since MTT essentially consists of several type theories with maps between them corresponding to the modalities, and so semantically to several models of type theory, each with its own category of contexts, it does not easily fit into the framework of [29], wherein a model has only one category of contexts.

Another approach to a more general understanding of type theories, and in fact deductive systems in general, can be found in [8]. Here Coraglia and Di Liberti [8] introduce the notion of a *judgemental theory*, consisting of a base category of contexts **ctx**, a family of *judgements* $\mathcal{J}$ consisting of functors $\mathcal{F} : F \to \mathbf{ctx}$, a family of *rules* $\mathcal{R}$ consisting of arbitrary functors, and a family *policies*, 2-dimensional filllers of triangles

$$\begin{tikzcd} A \arrow[rr, "\lambda\in\mathcal{R}"] \arrow[dr, "\mathcal{F}\in\mathcal{J}\cup\mathcal{R}"'] & & B \arrow[dl, "\mathcal{G}\in\mathcal{J}\cup\mathcal{R}"] \\ & C & \end{tikzcd}$$

Of course these classes have to satisfy some closure conditions and other properties. Now a natural model is a judgemental theory containing judgements, rules, and policies such that we have a diagram in **Cat** of the form

$$\begin{tikzcd} \dot{\mathcal{U}} \arrow[rr, "\mathrm{T}"'] \arrow[dr, "\dot{u}"'] \arrow[rr, phantom, "\perp"] & & \mathcal{U} \arrow[ll, bend right, "\mathrm{V}"'] \arrow[dl, "u"] \\ & \mathbf{ctx} & \end{tikzcd}$$

where the inner triangle commutes strictly and $u, \dot{u}$ are discrete fibrations. Here we use that a map of discrete fibrations corresponds to a representable map of presheaves if and only if it has a right adjoint, which was known already in [3]. In [8] they further show how these judgemental theories can give rise to deductive systems, and how the generated deductive syntax corresponding to these natural models is essentially the standard syntax of type theory. Finally they show how to model certain type constructors in this framework, which differs slightly from, and is simpler than, the approach via polynomial functors found in [3, 13, 29]. However, similarly to categories with representable maps, judgemental theories are defined with a specified base category of contexts, and so does not clearly capture multimode type theories. It must be noted, though, that the authors of [8] remark that there is no inherent reason, other than tradition, that a special category of contexts is singled out, and that it in fact makes their theory less clean than one might have wished.

With both of these previous approaches in mind, it seems natural that we should consider a model of type theory precisely as some sort of 2-dimensional diagram in **Cat**, since both Uemura [29] and Coraglia and Di Liberti [8] used the same formulation of natural models in terms of discrete fibrations. Such a digram in **Cat** is the same data as a 2-functor $D : \mathcal{T} \to \mathbf{Cat}$, and since we want to consider not arbitrary diagrams, but diagrams satisfying certain properties, we require some additional conditions. In particular, we see that $\mathcal{T}$, the shape of the diagram, needs to be of the form

$$\begin{tikzcd} \dot{\mathcal{U}} \arrow[rr, "\mathrm{T}"'] \arrow[rr, phantom, "\perp"] \arrow[dr, "\dot{u}"'] & & \mathcal{U} \arrow[ll, bend right, "\mathrm{V}"'] \arrow[dl, "u"] \\ & \mathcal{C} & \end{tikzcd}$$

Since adjunctions in 2-categories are preserved by any 2-functor, we need no restriction on D to make $D(\mathrm{T})$ a representable map of discrete fibrations. However, we do need to require that u and $\dot{u}$ are discrete fibrations. It is shown in [28] that we can define (Grothendieck) fibrations inside general 2-categories, and furthermore that it can be done in terms of finite 2-limits. The same extends to discrete fibrations. So if we ask $\mathcal{T}$ to have enough finite 2-limits to specify that u and $\dot{u}$ be discrete fibrations internal to $\mathcal{T}$, and then ask D to preserve those finite 2-limits, then we have exactly that the image of D consists of a natural model in the sense of Awodey [3], up to the Grothendieck construction. This idea, that a type theory is merely a certain a 2-dimensional theory in the same vein of Lawvere theories, is the heart of this thesis.

Of course, to talk about type theories as 2-dimensional theories, we first need a good notion of such theories. Many different, but related, notions of 2-dimensional have been previously suggested:

- Gray [15] takes a straightforward generalization of Lawvere theories to the 2-dimensional setting, defining a 2-algebraic theory to be a 2-category $\mathcal{A}$ equipped with a finite product preserving 2-functor from the (locally discrete) 2-category of finite sets to $\mathcal{A}$, and a model of the theory to be a finite product preserving 2-functor $\mathcal{A} \to \mathbf{Cat}$. These 2-algebraic theories allow for encoding 2-categorical structures such as monoidal and cartesian categories, but since we only ask for preservation of products, we lack more complex limits, like comma objects, powers by categories, and pullbacks. Furthermore, we note that this notion is inherently single-sorted, though this is easily fixed by only asking that the theory be a 2-category with finite products, without requiring the identity-on-objects and finite product preserving 2-functor.

- Nishizawa and Power [25] generalize Lawvere theories to be over finitely presentable enriched categories, rather than just set. Using their framework, one would define a 2-Lawvere theory to be a 2-category $\mathcal{A}$ equipped with an identity on objects and finite 2-limit preserving functor $\mathbf{Cat}_f \to \mathcal{A}$, where $\mathbf{Cat}_f$ is the full sub-2-category of $\mathbf{Cat}$ consisting of the finitely presentable categories. These theories allow for more complex constructions than the previous, for example supporting a theory of categories with pullbacks and terminal objects. Moreover, while the theories specialized to $\mathbf{Cat}$ are single-sorted, another choice of finitely presentable category to take the theories over, such as $\mathbf{Cat}^n$ for some finite cardinal n, would allow for n distinct sorts.

- In a similar vein to the previous example, Kelly [19] presents enriched finite limit theories, for which we again consider the special case of enrichment in $\mathbf{Cat}$. Here a theory is a 2-category with all finite 2-limits, and a model is a finite 2-limit preserving 2-functor into $\mathbf{Cat}$. Note, however, that the homomorphisms for such theories are inherently strict, and so for example the theory of monoidal categories has as homomorphisms those preserving the monoidal product strictly. Of course, this can be amended by asking for the homomorphisms to preserve the structure only up to isomorphism, but that must be treated with some care.

- Finally, we mention the $\mathscr{F}$-theories, which are finite limit theories in the sense of [19] in which the category $\mathscr{F}$ of full embeddings of categories (see definition 3.1.1) is taken as the base of enrichment. These were introduced by Arkor, Bourke, and Ko [2] in terms of $\mathscr{F}$-sketches, and any $\mathscr{F}$-sketch generates a $\mathscr{F}$-theory in this sense with the same (strict) models. The motivation for their introduction was to amend the notion of finite 2-limit sketches and theories in the sense of [19] to allow for fine control over which operations of the models are preserved strictly, and which are preserved only up to isomorphism. The prime example for a theory in which you want certain parts to preserved only up to isomorphism.

For our purposes, since we would like to not only recover the models of type theory, but also their morphisms, it turns out that the last of these options is the most easily applicable. For example, we wish that the pseudomorphisms of our natural models be pairs of maps of discrete fibrations preserving the structure up to isomorphism, as described in [24, Chapter 2]. Since maps of discrete fibrations are strictly commuting squares, this requires our theories to provide fine control over which parts of the structure is strictly preserved by homomorphisms, and which part is only weakly preserved, which is exactly what $\mathscr{F}$-theories provide.

The structure of the thesis is as follows:

- In chapter 2 we give some background definitions and theorems. We begin by recalling the notion of finite limit theories and sketches, as these serve as the ideological basis for the $\mathscr{F}$-theories and sketches considered in chapter 3. Then, for similar reasons, we cover some of the basic parts of the theory of categories enriched over cartesian closed categories, as well as their corresponding notion of their limits. We give special attention to the case of **Cat**-enriched categories, better known as 2-categories, as well as providing an explicit description of 2-limits. Finally, we cover the notion of cartesian morphisms and fibrations internal to 2-categories, and in particular their characterisation in terms of finite 2-limits.

- Chapter 3 first covers the basic features of $\mathscr{F}$-enriched categories and their limits. Then we introduce $\mathscr{F}$-theories and $\mathscr{F}$-sketches as follows from the general theory of [19], as well as the notions of models of $\mathscr{F}$-theories and $\mathscr{F}$-sketches and their properties as given [2]. We end the chapter with some examples of $\mathscr{F}$-theories, given in terms of $\mathscr{F}$-sketches, to give a feel for how we will construct and deal with $\mathscr{F}$-theories in the subsequent chapter.

- The main part of the thesis is chapter 4, wherein we show how comprehension categories [18], categories with attributes [6] (phrased in terms of discrete comprehension categories, see [18]), and categories with families [11] (phrased in terms of natural models [3]) are models of their respective $\mathscr{F}$-theories. We furthermore show that the theories for categories with attributes and categories with families are equivalent as $\mathscr{F}$-categories, thereby recovering the standard result that categories with attributes are indeed equivalent to categories with families. Furthermore, this provides an equivalnce of 2-categories of their respective models, whether one considers strict or pseudo homomorphisms. Next we show how to extend the theory of categories with families to model type theories with type formers such as dependent product types, dependent sum types, and extensional identity types, whose description turns out to be almost verbatim the same as is found in [8], with the only difference that we work in a more formal setting, replacing the setting of **Cat** with that of an abstract finite complete $\mathscr{F}$-category. Finally we show how we can capture more complex type theories using this framework, focusing on the multimode type theory of Gratzer et al. [14], which neither the categories with representable maps of [29] nor the judgemental theories of [8] are easily shown to model.

Decide on whether to include inductive types or not

Chapter 2

Preliminaries

This chapter is to lay out the groundworks of the concepts we will make use of in the rest of the thesis.

2.1 Finite limit theories and sketches

The basic notion of theory we will generalize is that of finite limit theories, or essentially algebraic theories, which allow for specifying not just total algebraic operations, but also partial ones whose domains are determined by algebraic equations. For example, any algebraic theory is a finite limit theory, but also the theory of categories.

Definition 2.1.1 (Finite limit theory)**.** A *finite limit theory* is a small, finitely complete category $\mathcal{T}$. A *model* of a finite limit theory in a finitely complete category $\mathcal{C}$ is a finite limit-preserving functor $\mathcal{T} \to \mathcal{C}$. A *homomorphism* of models $M, N : \mathcal{T} \to \mathcal{C}$ is a natural transformation $M \to N$. We denote by $\mathrm{Mod}_{\mathcal{T}}(\mathcal{C})$ the full subcategory of the functor category $\mathcal{C}^{\mathcal{T}}$ on models.

Literature 2.1.2.

To ease the construction of finite limit theories, we will make use of the notion of finite limit sketches. Furthermore, all of the theories we will construct in chapters 3 and 4 are given in terms of their sketch. The reason we consider theories still is that they provide a good way to say when two theories are equivalent, namely by categorical equivalence, which is slightly more involved for sketches.

Definition 2.1.3 (Finite limit sketch)**.** A *finite limit sketch* $(\mathcal{C}, \mathcal{S})$ consists of a small category $\mathcal{C}$ together with a collection $\mathcal{S}$ of finite cones over digrams in $\mathcal{C}$. A *model* of the finite limit sketch $(\mathcal{C}, \mathcal{S})$ in a category $\mathcal{D}$ is a functor $\mathcal{C} \to \mathcal{D}$ which maps every cone of $\mathcal{S}$ to a limit cone in $\mathcal{D}$. We denote the category of models in a category $\mathcal{D}$ by $\mathrm{Mod}_{(\mathcal{C},\mathcal{S})}(\mathcal{D})$.

Remark 2.1.4. Any finite limit theory $\mathcal{T}$ induces a sketch $(\mathcal{T}, \mathcal{S}_{\mathcal{T}})$ in which the cones are exactly the finite limit cones. Then a model of $(\mathcal{T}, \mathcal{S}_{\mathcal{T}})$ is exactly a finite limit-preserving functor, and so a model of $\mathcal{T}$. Hence the categories of models are the same.

Theorem 2.1.5. *Any finite limit sketch $(\mathcal{C}, \mathcal{S})$ may be extended to a finite limit theory $\mathcal{T}_{(\mathcal{C},\mathcal{S})}$, such that there is a fully faithful embedding $i : \mathcal{C} \to \mathcal{T}_{(\mathcal{C},\mathcal{S})}$ that provides a model of $(\mathcal{C}, \mathcal{S})$ inside* *the sketch induced by $\mathcal{T}_{(\mathcal{C},\mathcal{S})}$, and the restriction $i^* : \mathrm{Mod}_{\mathcal{T}}(\mathcal{D}) \to \mathrm{Mod}_{(\mathcal{C},\mathcal{S})}(\mathcal{D})$ is an equivalence for any finitely complete category $\mathcal{D}$.*

Literature 2.1.6. Sketches were first introduced by Ehresmann in [12] as a formalisation of the notion of theory. A further generalization in the context of enriched categories is given in provided by Kelly in [19], which we later make use of. The above theorem is also given in the generalized enriched context in [19].

2.2 Enriched categories

Enriched categories are generally defined in terms of symmetric monoidal closed categories, but for our purposes we restrict ourselves to cartesian closed categories, as these suffice for our purposes and lets us elide the definition of monoidal categories. The main reference for enriched categories used in this thesis is [19].

Definition 2.2.1 (Enriched categories)**.** Let $\mathscr{K}$ be a cartesian category. An *$\mathscr{K}$-enriched category* (or $\mathscr{K}$-category for short) $\mathbb{A}$ consists of

1. A class of objects $\mathbb{A}_0$,
2. For each pair of objects $A, B \in \mathbb{A}_0$, an object $\mathbb{A}(A, B)$ of $\mathscr{K}$,
3. For each object $A \in \mathbb{A}_0$ a morphism $\mathrm{id}_A : \mathbf{1} \to \mathbb{A}(A, A)$,
4. For each triple of objects $A, B, C \in \mathbb{A}_0$, a morphism $\mathrm{comp}_{A,B,C} : \mathbb{A}(B, C) \times \mathbb{A}(A, B) \to \mathbb{A}(A, C)$,

such that for each $A, B, C, D \in \mathbb{A}_0$, the following diagrams, corresponding to the unit and associativity laws, commute:

$$\begin{tikzcd} \mathbb{A}(A,B) \arrow[r, "1_{\mathbb{A}(A,B)} \times \mathrm{id}_A"] \arrow[dr, "1_{\mathbb{A}(A,B)}"'] & \mathbb{A}(A,B) \times \mathbb{A}(A,A) \arrow[d, "\mathrm{comp}_{A,A,B}"] \\ & \mathbb{A}(A,B) \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A}(A,B) \arrow[r, "\mathrm{id}_B \times 1_{\mathbb{A}(A,B)}"] \arrow[dr, "1_{\mathbb{A}(A,B)}"'] & \mathbb{A}(B,B) \times \mathbb{A}(A,B) \arrow[d, "\mathrm{comp}_{A,B,B}"] \\ & \mathbb{A}(A,B) \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{A}(C,D) \times \mathbb{A}(B,D) \times \mathbb{A}(A,B) \arrow[r, "\mathrm{comp}_{B,C,D} \times 1_{\mathbb{A}(A,B)}"] \arrow[d, "1_{\mathbb{A}(C,D)} \times \mathrm{comp}_{A,B,C}"'] & \mathbb{A}(B,D) \times \mathbb{A}(A,B) \arrow[d, "\mathrm{comp}_{A,B,D}"] \\ \mathbb{A}(C,D) \times \mathbb{A}(A,C) \arrow[r, "\mathrm{comp}_{A,C,D}"'] & \mathbb{A}(A,D) \end{tikzcd}$$

Example 2.2.2 (Locall small categories)**.** Locally small 1-categories are **Set**-enriched categories: they are equipped with a class of objects and a *set* of morphisms.

Example 2.2.3 (2-categories)**.** (Strict) 2-categories are exactly the **Cat**-enriched categories: the 1-cells are the objects of the hom-categories and the 2-cells the morphisms. Vertical composition is the composition of the hom-categories and horizontal composition is the action on morphisms of the composition bifunctor of the **Cat**-enriched category.

Example 2.2.4. Let $\mathbb{A}$ be a $\mathscr{K}$-enriched category. Then $\mathbb{A}^{\mathrm{op}}$, defined by taking the same objects and flipping the direction of objects (so $\mathbb{A}^{\mathrm{op}}(A, B) = \mathbb{A}(B, A)$), is a $\mathscr{K}$-enriched category.

Definition 2.2.5 (Enriched functors)**.** Let $\mathbb{A}, \mathbb{B}$ be $\mathscr{K}$-enriched categories for some cartesian category $\mathscr{K}$. An *$\mathscr{K}$-enriched functor* (or $\mathscr{K}$-functor) $F : \mathbb{A} \to \mathbb{B}$ consists of

1. A function $F_0 : \mathbb{A}_0 \to \mathbb{B}_0$, and
2. for each pair of objects $A, B \in \mathbb{A}_0$, a morphism $F_{A,B} : \mathbb{A}(A, B) \to \mathbb{A}(F_0A, F_0B)$

such that for each $A, B, C \in \mathbb{A}$, we have

$$F_{A,A}\mathrm{id}_A = \mathrm{id}_{F_0A} \qquad \text{and} \qquad F_{A,C}\,\mathrm{comp}_{A,B,C} = \mathrm{comp}_{F_0A,F_0B,F_0C}(F_{B,C} \times F_{A,B}).$$

The identity $\mathscr{K}$-functors and composition of $\mathscr{K}$-functors are defined in the obvious way.

Example 2.2.6 (Self-enrichment)**.** Suppose $\mathscr{K}$ is a cartesian closed category. Then $\mathscr{K}$ can canonically be made into a $\mathscr{K}$-category $\mathbb{K}$ as follows:

1. The objects are those of $\mathscr{K}$,
2. for objects $A, B \in \mathscr{K}$, the hom-object $\mathbb{K}(A, B)$ is exactly the internal hom B^A,
3. for each object $A \in \mathscr{K}$, the identity is exactly the transpose of the projection $\mathbf{1} \times A \to A$,
4. for each triple of objects $A, B, C \in \mathscr{K}$, the composition is the internal composition map $C^B \times A^B \to C^A$.

Example 2.2.7. Let $\mathbb{K}$ be as before, $\mathbb{A}$ be some $\mathscr{K}$-category, and X an object of $\mathbb{A}$. Then there is a $\mathscr{K}$-functor $\mathbb{A}(X, -) : \mathbb{A} \to \mathbb{K}$ given on objects by $\mathbb{A}(X, -)(A) = \mathbb{A}(X, A)$. The morphism component $\mathbb{A}(X, -)_{A,B} : \mathbb{A}(A, B) \to \mathbb{A}(X, B)^{\mathbb{A}}(X, A)$ is given by the transpose of the composition map $\mathrm{comp}_{X,A,B} : \mathbb{A}(A, B) \times \mathbb{A}(X, A) \to \mathbb{A}(X, B)$, for all objects $A, B \in \mathbb{A}$.

Definition 2.2.8 (Enriched natural transformation)**.** Let $\mathbb{A}, \mathbb{B}$ be $\mathscr{K}$-enriched categories for some cartesian category $\mathscr{K}$ and $F, G : \mathbb{A} \to \mathbb{B}$ enriched functors between them. An *$\mathscr{K}$-enriched natural transformation* (or $\mathscr{K}$-natural transformation) $\alpha : F \to G$ consists of a family $(\alpha_A : \mathbf{1} \to \mathbb{B}(F_0A, G_0A))_{A \in \mathbb{A}_0}$ such that for each $A, B \in \mathbb{A}$ the following square, corresponding to naturality, commutes:

$$\begin{tikzcd}
\mathbb{A}(A,B) \arrow[r, "\alpha_B \times F_{A,B}"] \arrow[d, "G_{A,B} \times \alpha_A"'] & \mathbb{B}(F_0B, G_0B) \times \mathbb{B}(F_0A, F_0B) \arrow[d, "\mathrm{comp}_{F_0A,F_0B,G_0B}"] \\
\mathbb{B}(G_0A, G_0B) \times \mathbb{B}(F_0A, G_0A) \arrow[r, "\mathrm{comp}_{F_0A,G_0A,G_0B}"'] & \mathbb{B}(F_0A, G_0B)
\end{tikzcd}$$

Example 2.2.9 (Enriched functor category)**.** Let $\mathscr{K}$ be a complete cartesian closed category. Then for any pair of $\mathscr{K}$-categories $\mathbb{A}, \mathbb{B}$ such that the objects of $\mathbb{A}$ form a set, we can form the $\mathscr{K}$-functor category $[\mathbb{A}, \mathbb{B}]$, whose objects are $\mathscr{K}$-functors $\mathbb{A} \to \mathbb{B}$ and hom-objects are an internalization of the $\mathscr{K}$-natural transformations. Specifically, the hom-object $[\mathbb{A}, \mathbb{B}](F, G)$ is given as the largest subobject of the product $\prod_{A\in\mathbb{A}} \mathbb{B}(F_0A, G_0A)$ such that for every $A, B \in \mathbb{A}$ the square

$$\begin{tikzcd}
{[\mathbb{A},\mathbb{B}](F,G) \times \mathbb{A}(A,B)} \arrow[r, "\pi_B \times F_{A,B}"] \arrow[d, "\sigma"'] & {\mathbb{B}(F_0B, G_0B) \times \mathbb{B}(F_0A, F_0B)} \arrow[dd, "\mathrm{comp}_{F_0A,F_0B,G_0B}"] \\
{\mathbb{A}(A,B) \times [\mathbb{A},\mathbb{B}](F,G)} \arrow[d, "G_{A,B}\times\pi_A"'] & \\
{\mathbb{B}(G_0A, G_0B) \times \mathbb{B}(F_0A, G_0A)} \arrow[r, "\mathrm{comp}_{F_0A,G_0A,G_0B}"'] & {\mathbb{B}(F_0A, G_0B)}
\end{tikzcd}$$

where $\sigma : [\mathbb{A},\mathbb{B}](F,G) \times \mathbb{A}(A,B) \to \mathbb{A}(A,B) \times [\mathbb{A},\mathbb{B}](F,G)$ is the symmetry isomorphism of the product. This can be described as an equalizer, and so is well-defined as a limit. Note that the global elements of the hom-object $[\mathbb{A},\mathbb{B}](F,G)$, i.e. the maps $\mathbf{1} \to [\mathbb{A},\mathbb{B}](F,G)$ are exactly the $\mathscr{K}$-natural transformations.

2.2.1 Limits in enriched categories

Given that the basis for this thesis is enriched finite limit theories, we will of course need to have and understanding of enriched limits. This is especially important as the conical approach of standard category theory doesn't work out quite as nicely for enriched categories. For example, unlike ordinary category theory, we find that for general $\mathscr{K}$, the enriched presheaf category $[\mathbb{A}^{\mathrm{op}}, \mathbb{K}]$ is not the free cocompletion. Furthermore, there are limit-like in for, example 2-categories, that we would like to capture, but which aren't conical in the classical sense. Therefore we introduce, and will make heavy use of, weighted limits.

For the remainder of this section, we fix a cartesian closed category $\mathscr{K}$, which will serve as the base of enrichment. We will use $\mathbb{K}$ to denote the canonical $\mathscr{K}$-category induced by $\mathscr{K}$ itself, as in example 2.2.6.

Definition 2.2.10 (Weights, Diagrams, Weighted cones)**.** Let $\mathbb{J}$ be a small $\mathscr{K}$-category. A *weight* is a $\mathscr{K}$-functor $W : \mathbb{J} \to \mathbb{K}$. For $\mathbb{A}$ another $\mathscr{K}$-category, a $\mathbb{J}$*-shaped diagram* in $\mathbb{A}$ is a $\mathscr{K}$-functor $D : \mathbb{J} \to \mathbb{A}$. For a weighted W and diagram D with common domain $\mathbb{J}$, a W*-weighted cone over* D based at $A \in \mathbb{A}$ is a $\mathscr{K}$-natural transformation $[\mathbb{J},\mathbb{K}](W, \mathbb{A}(A, D))$, where $\mathbb{A}(A,D) = \mathbb{A}(A,-) \circ D$.

Definition 2.2.11 (Weighted limit)**.** Let $W : \mathbb{J} \to \mathbb{K}$ and $D : \mathbb{J} \to \mathbb{A}$ be a weight and diagram respectively. The W*-weighted limit of* D is an object $L \in \mathbb{A}$ and a $\mathscr{K}$-natural isomorphism

$$\mathbb{A}(-, L) \cong [\mathbb{J},\mathbb{K}](W, \mathbb{A}(-, D)).$$

Example 2.2.12 (Conical limits)**.** We recover the standard conical limits by taking the weight to be constantly the terminal object.

Example 2.2.13 (Cotensors)**.** Let $X \in \mathscr{K}$ be some object, $\mathbb{A}$ some $\mathscr{K}$-category, and $A \in \mathbb{A}$ some object of $\mathbb{A}$. The *power* of A by X, written A^X, is the limit with of the diagram $D(*) = A$, using the terminal $\mathscr{K}$-category as shape, weighted by $W(*) = X$. Thus the power A^X is determined by a natural isomorphism

$$\mathbb{A}(-, A^X) \cong \mathbb{K}(X, \mathbb{A}(-, A)).$$

2.3 2-categories

We cover some basic properties of 2-categories and the limits therein. For a more thorough introduction to the subject of 2-categories, we refer the reader to chapter 7 of [4], which has also served as the reference for most of the definitions used here.

Definition 2.3.1 (2-categories)**.** A 2-category is a category enriched in **Cat**, as shown in example 2.2.3. A 2-functor between 2-categories $\mathcal{A} \to \mathcal{B}$ is a **Cat**-enriched functor, and a 2-natural transformations is a **Cat**-enriched natural transformation.

Notation 2.3.2. We will sometimes denote the objects of a 2-category by 0-cells, the objects of the hom-categories by morphisms, maps or 1-cells, and the morphisms of the hom-categories by 2-cells.

We use $\circ$ for the object component of the composition functor and the composition of 2-cells, and $\star$ for the morphism part of the composition functor, the horizontal composition of 2-cells. We may also write composition of 1-cells by juxtaposition where convenient.

Example 2.3.3. Since **Cat** is cartesian closed, it cannonically induces a 2-category, whose objects are categories, morphisms are functors, and 2-cells are natural transformations.

Similarly, many categories of structured categories are often naturall viewed as 2-categories. For example, consider the 2-category of finitely complete categories, whose objects are finitely complete categories, morphisms are lex functors, and 2-cells are natural transformations.

Example 2.3.4. Any category $\mathscr{A}$ can be viewed as a 2-category $\mathcal{A}$ by taking the hom-categories $\mathcal{A}(A, B)$ to be the discrete category associated with the homset $\mathscr{A}(A, B)$.

Definition 2.3.5 (Whiskering)**.** Let $\mathcal{A}$ be a 2-category, $f : A \to B$ a morphism of $\mathcal{A}$, and $\alpha : g \to h : B \to C$ a 2-cell of $\mathcal{A}$. We define the left whiskering of f and α, written $f\alpha : fg \to fh : A \to C$, to be the horizontal composite $1_f \star \alpha$.

Notation 2.3.6. We will generally write whiskering by juxtaposition, with left (resp. right) whiskering inferred from which is the 2-cell and which is the 1-cell. Note that, in a 2-category, we have that $f(g\alpha) = (fg)\alpha$ for composable 1-cells f, g and a 2-cell α, which means that parentheses can generally be elided.

Definition 2.3.7 (Pseudonatural transformations)**.** Let $F, G : \mathcal{A} \to \mathcal{B}$ be 2-functors. A pseudonatural transformation $\alpha : F \to G$ consists of the following data:

1. For each object $A \in \mathcal{A}$, a 1-cell $\alpha_A : FA \to GA$,
2. For each morphism $f : A \to B$ in $\mathcal{A}$, an isomorphism α_f as shown in the following square

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "\alpha_A"'] & FB \arrow[d, "\alpha_B"] \\ GA \arrow[r, "Gf"'] \arrow[ur, phantom, "\Nearrow \alpha_f"] & GB \end{tikzcd}$$

such that for every $A \in \mathcal{A}$ we have $\alpha_{1_A} = 1_{\alpha_A}$ and for any pair of composable 1-cells f, g in $\mathcal{A}$ we have that $\alpha_{f \circ g} = \alpha_f \circ \alpha_g$.

Definition 2.3.8 (Modifications)**.** Let $\alpha, \beta : F \to G : \mathcal{A} \to \mathcal{B}$ be two parallel pseudonatural transformations. A modification $m : \alpha \to \beta$ consists of a family $(m_A : \alpha_A \to \beta_A)_{A \in \mathcal{A}}$ such that for any $\varphi : f \to g : A \to B$ in $\mathcal{A}$ the equality $m_B \star F\varphi = G\varphi \star m_A$ holds.

Definition 2.3.9 (Sub-2-category)**.** A *sub-2-category* of a 2-category $\mathcal{A}$ is a 2-category $\mathcal{B}$ such that the objects of $\mathcal{B}$ form a subclass of the objects of $\mathcal{A}$, the hom-categories $\mathcal{B}(A, B)$ are subcategories of $\mathcal{A}(A, B)$, and identities and composition in $\mathcal{A}$ restrict to $\mathcal{B}$.

Definition 2.3.10 (Locally fully faithful)**.** A 2-functor $F : \mathcal{A} \to \mathcal{B}$ is *locally fully faithful* if for every $A, B \in \mathcal{A}$, the functor $F_{A,B} : \mathcal{A}(A, B) \to \mathcal{B}(FA, FB)$ is fully faithful.

Definition 2.3.11 (Locally full sub-2-category)**.** A sub-2-category $\mathcal{B} \subseteq \mathcal{A}$ is *locally full* if $\mathcal{B}(A, B) = \mathcal{A}(A, B)$ for each $A, B \in \mathcal{B}$.

2.3.1 2-limits

Given the definition of weighted limits in section 2.2.1, we may take the 2-limits to be exactly weighted limits in **Cat**-enriched categories. This subsection will cover some useful examples and properties of 2-limits, as they form the bases of the rest of the thesis. We begin by providing an explicit description of the weighted limits and cones, so as to provide some intuition for how to construct the correct weights for the limits we wish to describe, and then follow up with plenty of examples of 2-limits and their universal properties.

Proposition 2.3.12. *Let $\mathcal{J}$, $\mathcal{A}$ be 2-categories, $W : \mathcal{J} \to$ **Cat** be a weight, and $D : \mathcal{J} \to \mathcal{A}$ be a $\mathcal{J}$-shaped diagram in $\mathcal{A}$. Then a W-weighted cone (a, α) over D, i.e. an object $a \in \mathcal{A}$ and a 2-natural transformation $W \to \mathcal{A}(a, D)$ consists of the following data:*

- *for each object $x \in \mathcal{J}$ and $r \in W(x)$, a morphism $\alpha_{x,r} : a \to Dx$, and*
- *for each $p : r \to s \in W(x)$, a 2-cell $\alpha_{x,p} : \alpha_{x,r} \to \alpha_{x,s}$, such that*
- *for each morphism $f : x \to y$ in $\mathcal{J}$ and $r \in W(x)$, we have $Df \circ \alpha_{x,r} = \alpha_{y,W(f)(r)}$, and*
- *for each $p : r \to s$ in $W(x)$ we have $Df \star \alpha_{x,p} = \alpha_{y,W(f)(p)}$.*

Furthermore, a morphism $m : \alpha \to \beta$ of parallel cones $W \to \mathcal{A}(a, D)$, i.e. a modification, consists of a family $(m_{x,r} : \alpha_{x,r} \to \beta_{x,r})_{x \in \mathcal{J}, r \in W(x)}$ such that for every $f : x \to y$ in $\mathcal{J}$ and $r \in W(x)$ we have that

$$m_{y,Wf(r)} = Dfm_{x,r} : Df \circ \alpha_{x,r} = \alpha_{y,Wf(r)} \to \beta_{y,Wf(r)} = Df \circ \beta_{x,r}.$$

Proof. This holds immediately after unfolding all the layers of definition. $\square$

Definition 2.3.13 (Pullback of cones)**.** Let $\mathcal{J}, \mathcal{A}$ be 2-categories, $W : \mathcal{J} \to \mathcal{F}$ a weight, and $D : \mathcal{J} \to \mathcal{A}$ a diagram. Given a W-weighted cone (a, α) over D and a morphism $f : b \to a$ in $\mathcal{A}$, we define the pullback of (a, α) along f as the cone $(b, f^*\alpha)$ defined by

$$(f^*\alpha)_{x,r} = \alpha_{x,r} \circ f$$

on objects $x \in \mathcal{J}$ and $r \in W(x)$, and

$$(f^*\alpha)_{x,p} = \alpha_{x,p} \star f$$

on objects $x \in \mathcal{J}$ and morphisms $p : r \to s \in W(x)$.

Given a modification $m : \alpha \to \alpha' : W \to \mathcal{A}(a, D)$, we define the pullback $f^*m : f^*\alpha \to f^*\alpha'$ by

$$(f^*m)_{x,r} = m_{x,r} \star f$$

Furthermore, for $\varphi : f \to g : b \to a$ a 2-cell in $\mathcal{A}$, we define a modification $\varphi^*\alpha : f^*\alpha \to g^*\alpha$ by $(\varphi^*\alpha)_{x,r} = \alpha_{x,r} \star \varphi$.

These together form the data of the 2-functor $[\mathcal{J}, \mathbf{Cat}](W, \mathcal{A}(-, D)) : \mathcal{A} \to \mathbf{Cat}$ taking objects of $\mathcal{A}$ to their 2-categories of cones based at that object, 1-cells $f : b \to a$ of $\mathcal{A}$ to the pullback of cones as defined above, and similarly for 2-cells.

Proposition 2.3.14. *Given a weight $W : \mathcal{J} \to \mathbf{Cat}$, a diagram $D : \mathcal{J} \to \mathcal{A}$ has a limit in $\mathcal{A}$ if and only if there exists a W-weighted cone (ℓ, L) over D in $\mathcal{A}$ such that*

1. *for every cone $\alpha : W \to \mathcal{A}(a, D)$, there exists a unique morphism $h_\alpha : a \to \ell$ such that $\alpha = h_\alpha^* L$,*
2. *for every modification $m : \alpha \to \alpha' : W \to \mathcal{A}(a, D)$ there exists a unique 2-cell $h_m : h_\alpha \to h_{\alpha'}$ such that $h_m^* \ell = m$.*

We will call a cone satisfying these properties a limiting cone.

Proof. We start by unfolding the definition of the naturality squares of the isomorphism

$$h : [\mathcal{J}, \mathbf{Cat}](W, \mathcal{A}(-, D)) \cong \mathcal{A}(-, \ell)$$

which is supposed to exist for ℓ to be the W-weighted limit of D. For each $a \in \mathcal{A}$, we find that $h_a : [\mathcal{J}, \mathbf{Cat}](W, \mathcal{A}(a, D)) \cong \mathcal{A}(a, \ell)$ is an isomorphism in $\mathbf{Cat}$, i.e. an invertible functor. Naturality tells us that given a morphism $\varphi : b \to a$, we have that $h_b \circ \varphi^* = \mathcal{A}(\varphi, \ell) \circ h_a$, where φ^* denotes the operation of pulling back a cone along a morphism of $\mathcal{A}$. On cones $\alpha : W \to \mathcal{A}(b, \ell)$, this gives us that $h_b(\varphi^*\alpha) = h_a(\alpha) \circ \varphi$, and on modifactions $m : \alpha \to \alpha' : W \to \mathcal{A}(b, D)$, we have $h_b(\varphi^* m) = h_a(m) \star \varphi$.

Now, given a natural isomorphism h as above, we can consider the cone $L = h^{-1}(\mathrm{id}_\ell)$. Given any other cone $\alpha : W \to \mathcal{A}(a, D)$, naturality tells us that

$$h_a(h_a(\alpha)^* L) = h_\ell(L) \circ h_a(\alpha) = \mathrm{id}_\ell \circ h_a(\alpha) = h_a(\alpha),$$

so $h_a(\alpha)^* L = \alpha$ since h is an isomorphism. Similarly, for a modification $m : \alpha \to \alpha'$ we find that

$$h_a(h_a(m)^* L) = h_a(m) = h_\ell(L) \star h_a(m) = \mathrm{id}_\ell \star h_a(m) = h_a(m),$$

so again $h_a(m)^* L = m$.

In the other direction, suppose we are given an object $\ell \in \mathcal{A}$ and cone $L : W \to \mathcal{A}(\ell, D)$ such that (1) and (2) hold. Then we construct the natural isomorphism $h : [\mathcal{J}, \mathbf{Cat}](W, \mathcal{A}(-, D)) \cong \mathcal{A}(-, \ell)$ by $h_a(\alpha) = h_\alpha$ and $h_a(m : \alpha \to \alpha') = h_m$. Verification that h_a is indeed an isomorphism for each $a \in \mathcal{A}$ is simple to verify, so we focus instead only on naturality: given $\varphi : b \to a \in \mathcal{A}$, we must show that $h_b \circ \varphi^* = \mathcal{A}(\varphi, \ell) \circ h_a$. On cones, this amounts to showing that for any cone $\alpha : W \to \mathcal{A}(a, D)$, we have that $h_b(\varphi^* \alpha) = h_a(\alpha) \circ \varphi$. But

$$\varphi^* \alpha = \varphi^* h_a(\alpha)^* L = (h_a(\alpha) \circ \varphi)^* L,$$

so uniqueness of $h_b(h_a(\alpha) \circ \varphi)$ gives us that $h_b(\varphi^*\alpha) = h_a(\alpha) \circ \varphi)$. For the naturality condition on 2-cells, let $m : \alpha \to \alpha'$ be a modification. We must show that $h_b(\varphi^* m) = h_a(\alpha) \star m$, which much like for cones follows from uniqueness:

$$\varphi^* m = \varphi^*(h_a(m)^* L) = (h_a(m) \star \varphi)^* L,$$

so $h_b(\varphi^* m) = h_a(m) \star \varphi$. □

Example 2.3.15 (Arrow object)**.** Let $\mathcal{A}$ a 2-category and $A \in \mathcal{A}$ be given. The arrow-object of A, written $A^{\to}$, is the power by the category $2 = \{0 \to 1\}$. By the above explicit description, $A^{\to}$ is equipped with two arrows $\mathrm{dom}_A, \mathrm{cod}_A : A^{\to} \to A$ and a 2-cell $\lambda_A : \mathrm{dom}_A \to \mathrm{cod}_A$, and this classifies 2-cells into A, in the sense that for any 2-cell $\alpha : f \to g : X \to A$, there exists a unique 1-cell $h : X \to A^{\to}$ such that $f = \mathrm{dom}_A h$, $g = \mathrm{cod}_A h$, and $\alpha = \lambda_A h$, and a corresponding universal property for 2-cells.

Equivalently, there is for every $X \in \mathcal{A}$ an isomorphism of categories $\mathcal{A}(X, A^{\to}) \cong \mathcal{A}(X, A)^2$, where the latter denotes the arrow category, and this isomorphism is natural in X in a suitable sense.

Example 2.3.16 (Comma object)**.** Let $\mathcal{A}$ be a 2-category and $A \xrightarrow{f} C \xleftarrow{g} B$ a cospan in $\mathcal{A}$. The comma object for f, g is an object $f \downarrow g$ equipped with a limiting cone $(\mathrm{dom}_{f,g}, \mathrm{cod}_{f,g}, \lambda_{f,g})$ for the diagram given by the aforementioned cospan and weighted by $1 \xrightarrow{0_2} 2 \xleftarrow{1_0} 1$, where $2 = \{0_2 \to 1_2\}$ is the walking arrow and 1 is the terminal category. The universal property tells us that for any span $A \xleftarrow{p} X \xrightarrow{q} B$ with a 2-cell $\alpha : fp \to gq$, there exists a unique 1-cell $h : X \to f \downarrow g$ such that $\mathrm{dom}_{f,g} h = p$, $\mathrm{cod}_{f,g} h = q$, and $\lambda_{f,g} h = \alpha$, as well as a corresponding property for 2-cells.

Example 2.3.17 (1-limits)**.** All 1-limits, such as products, pullbacks, equalizers, etc. lift to 2-limits by taking the weight to be constantly the terminal object. As such, we will speak of these 1-limits also in the context of 2-categories.

Example 2.3.18 (Inserters)**.** Given two parallel 1-cells $f, g : a \to b$ in a 2-category $\mathcal{A}$, their *inserter* is the limit of these arrows weighted by

$$\begin{tikzcd} 1 \arrow[r, bend left, "0_2"] \arrow[r, bend right, "1_2"'] & 2 \end{tikzcd}$$

with 1 and 2 defined as above. Thus the inserter consists of an object v, a 1-cell $i : v \to a$, and a 2-cell $\lambda : fi \to gi$, with a corresponding universal property.

It can be also be constructed as the following pullback:

$$\begin{tikzcd} v \arrow[r, "\bar{\lambda}"] \arrow[d, "i"'] & b^{\to} \arrow[d, "(\mathrm{dom}_b, \mathrm{cod}_b)"] \\ a \arrow[r, "(f,g)"'] & b \times b \end{tikzcd}$$

where $\bar{\lambda}$ is the unique 1-cell such that $\lambda_b \bar{\lambda} = \lambda : fi \to gi$.

2.4 Fibrations

Originally introduced by Grothendieck in [16] in the context of descent theory, fibrations have turned out to be an immensely useful tool in the categorical modelling of substitution in type theory. In essence, a fibration encodes the idea of a family of categories $(\mathcal{E}_b)_{b\in\mathcal{B}}$ indexed over a base category $\mathcal{B}$, equipped with actions of morphisms $f^* : \mathcal{E}_b \to \mathcal{E}_{b'}$ for each morphism $f : b' \to b$. In essence, a fibration is a pseudofunctor $\mathcal{E}_{(-)} : \mathcal{B} \to \mathbf{Cat}$. However, rather than the complex algebraic structure that is a pseudofunctor, it turns out that we can instead describe this structure by a *functor* $p : \mathcal{E} \to \mathcal{B}$, where we call $\mathcal{E}$ the total category of the fibration p, with the idea that we can define $\mathcal{E}_b$ to be the fiber of p over b: the objects e are those of $\mathcal{E}$ which map to b, i.e. $p^{-1}(b) = \{e \mid p(e) = b\}$, and morphisms $k : e \to e'$ are those of $\mathcal{E}$ which lie over the identity, meaning that $p(k) = 1_b$. For a good textbook account of fibrations in **Cat** and their applications to logic, I recommend [17].

Since we will be working in general 2-categories, rather than only **Cat**, we will require a generalization of fibrations to this setting. This was initially done by Street in [28], and was further developed in [30]. A good overview of these generalizations, both to 2-categories and bicategories, can be found in [22]. Here we will simply provide the definitions that will be required for the later chapters, without much general motivation on why they are defined as they are.

The main concept underlying fibrations is that of *cartesian morphisms*. These are certain 2-cells satisfying a universal property with respect to some fixed 1-cells $p : A \to B$ in a 2-category $\mathcal{A}$, with the name coming from the universal property in a sence mimicing that of pullbacks.

Definition 2.4.1 (Cartesian morphisms)**.** Let $p : A \to B$ be a morphism of an 2-category $\mathcal{A}$. A 2-cell $\varphi : a_1 \to a_2 : X \to A$ is called *p-cartesian* if for every $g : Y \to X$, $a_0 : Y \to A$, $\alpha : a_0 \to a_2 g$, and $\beta : pa_0 \to pa_1 g$ with $\beta \circ p\varphi = p\alpha$, there exists a unique $\gamma : a_0 \to a_1 g$ such that $\beta = p\gamma$ and $\alpha = \varphi g \circ \gamma$. We will call the data (g, a_0, α, β) satisfying the equality $\beta \circ p\varphi = p\alpha$ a cartesian cone.

Example 2.4.2. Let $\mathrm{cod} : A^{\to} \to A$ denote the codomain projection for some object A in a finitely complete 2-category $\mathcal{A}$. A 2-cell $\alpha : a_1 \to a_2 : X \to A^{\to}$ is cartesian precisely if the following square is a pullback:

$$\begin{tikzcd}
\mathrm{dom} a_1 \arrow[r, "\mathrm{dom}\alpha"] \arrow[d, "\lambda a_1"'] & \mathrm{dom} a_2 \arrow[d, "\lambda a_2"] \\
\mathrm{cod} a_1 \arrow[r, "\mathrm{cod}\alpha"'] & \mathrm{cod} a_2
\end{tikzcd}$$

Definition 2.4.3 (Fibrations)**.** A map $p : A \to B$ in a 2-category $\mathcal{A}$ is a *fibration* if every $\varphi : b \to pa : X \to B$ has *p-cartesian lift*, that is, a p-carteisan 2-cell $\bar{\varphi} : \bar{b} \to a$ with $p\bar{b} = b$ and $p\bar{\varphi} = \varphi$.

Proposition 2.4.4. *Let $p : A \to B$ be a morphism in an 2-category $\mathcal{A}$ with finite weighted limits and $\varphi : a_1 \to a_2 : X \to A$ a 2-cell. Consider the diagram $D : \mathcal{J} \to \mathcal{A}$ given by*

$$\begin{tikzcd}
X \arrow[r, bend left, "a_1"{name=U}] \arrow[r, bend right, "a_2"'{name=D}] \arrow[from=U, to=D, phantom, "\Downarrow\varphi"] & A \arrow[r, "p"] & B
\end{tikzcd}$$

and weight $W : \mathcal{J} \to \mathbf{Cat}$ generated by

$$\begin{tikzcd} 1 \arrow[r, bend left, "1"] \arrow[r, bend right, "2"'] \arrow[r, phantom, "\Downarrow\varphi"] & H \arrow[r, "c"] & 3 \end{tikzcd}$$

where $H = \{0 \to 2 \leftarrow 1\}$ is the walking cospan, and $c : H \to 3$ maps the morphism $1 \to 2$ to the corresponding morphism in 3 and $0 \to 2$ to the composite $0 \to 1 \to 2$ in 3.

A W-weighted cone over D is exactly a cartesian cone over φ.

Proposition 2.4.5. *Let $\mathcal{A}$ be an 2-category with finite weighted limits, $p : A \to B$ a 1-cell in $\mathcal{A}$, and $\varphi : a_1 \to a_2 : X \to A$ a 2-cell in $\mathcal{A}$. Then φ is cartesian if and only if $\ell = (\mathrm{cod}, \mathrm{dom}, \varphi\mathrm{cod} \circ \lambda, p\lambda)$ is a universal cartesian cone with apex $A \downarrow a_1$.*

Proof. In the left-to-right direction, suppose φ is cartesian. We must show that for every object Y, the following hold:

(1) for every cartesian cone (g, a, α, β) with apex Y, there exists a unique morphism $h : Y \to A \downarrow a$ such that $g = \mathrm{cod}h$, $a = \mathrm{dom}h$, $\alpha = (\varphi\mathrm{cod} \circ \lambda)h$ and $\beta = p\lambda h$.

(2) for every morphism of cones $m : (g, a, \alpha, \beta) \to (g', a', \alpha', \beta')$ there exists a unique morphism $h_m : h \to h'$ such that $m = h_m^* \ell$.

For (1), note that since (g, a, α, β) is a cartesian cone and φ is cartesian, there exists a unique 2-cell $\gamma : a \to a_1 g$ such that $\alpha = \varphi \circ \gamma$ and $\beta = p\gamma$. By the universal property of $A \downarrow a_1$, there then exists a unique morphism $\bar{\gamma} : Y \to A \downarrow a_1$ with $\lambda\bar{\gamma} = \gamma$. In particular, we have that $\mathrm{cod}\bar{\gamma} = g$, $\mathrm{dom}\bar{\gamma} = a$, $\alpha = \varphi \circ \lambda\bar{\gamma}$, and $\beta = p\lambda\bar{\gamma}$, so taking $h = \bar{\gamma}$ we are done. Furthermore, $h = \bar{\gamma}$ is the unique such map by uniqueness of $\bar{\gamma}$ and γ.

For (2), suppose we are given a modification $m : (g, a, \alpha, \beta) \to (g', a', \alpha', \beta')$. Concretely, m consists of two morphisms $m_1 : g \to g'$ and $m_2 : a \to a'$ such that the following squares commute:

$$\begin{tikzcd} a \arrow[r, "m_2"] \arrow[d, "\alpha"'] & a' \arrow[d, "\alpha'"] \\ a_2 g \arrow[r, "a_2 m_1"'] & a_2 g' \end{tikzcd} \qquad \begin{tikzcd} pa \arrow[r, "pm_2"] \arrow[d, "\beta"'] & pa' \arrow[d, "\beta'"] \\ pa_1 g \arrow[r, "pa_1 m_1"'] & pa_1 g' \end{tikzcd}$$

We want to show that the square

$$\begin{tikzcd} a \arrow[r, "m_2"] \arrow[d, "\gamma"'] & a' \arrow[d, "\gamma'"] \\ a_1 g \arrow[r, "a_1 m_1"'] & a_1 g' \end{tikzcd}$$

commutes as well, as that induces the desired unique morphism $\bar{m} : \bar{\gamma} \to \bar{\gamma}'$ with the desired properties. To that end, consider the cartesian cone over Y given by $(g', a, \alpha'' = \alpha' \circ m_2, \beta'' =$

$\beta' \circ pm_2$). To see that this is indeed a cone, we must show that $p\alpha'' = p\varphi \circ \beta''$, but this follows from simple computation:

$$p\alpha'' = p(\alpha' \circ m_2) = p\alpha' \circ pm_2 = p\varphi \circ \beta' \circ pm_2 = p\varphi \circ \beta''$$

Since $(g', a, \alpha'', \beta'')$ is a cartesian cone and r is cartesian, there exists a unique lift γ'' with $p\gamma'' = \beta''$ and $\alpha'' = \varphi g' \circ \gamma''$. But we have already that

$$p(\gamma' \circ m_2) = p\gamma' \circ pm_2 = \beta' \circ pm_2 = \beta'' \quad \text{and} \quad \varphi g' \circ \gamma' \circ m_2 = \alpha' \circ m_2 = \alpha'',$$

and similarly

$$p(a_1 m_1 \circ \gamma) = pa_1 m_1 \circ p\gamma = pa_1 m_1 \circ \beta = \beta''$$

and

$$\varphi g' \circ a_1 m_1 \circ \gamma = a_2 m_1 \circ \varphi g \circ \gamma = a_2 m_1 \circ \alpha = \alpha'',$$

where we use the interhcange law to commute φ and m_1. In particular, we see that both $\gamma' \circ m_2$ and $a_1 m_1 \circ \gamma$ provide universal morphisms for the cartesian cone $(g', a, \alpha'', \beta'')$, from which we conclude that they must be identical by the uniqueness of such morphisms. Hence the diagram

$$\begin{tikzcd} a \arrow[r, "m_2"] \arrow[d, "\gamma"'] & a' \arrow[d, "\gamma'"] \\ a_1 g \arrow[r, "a_1 m_1"'] & a_1 g' \end{tikzcd}$$

commutes.

For the right-to-left direction, suppose that $(\mathrm{cod}, \mathrm{dom}, \varphi\mathrm{cod} \circ \lambda, p\lambda)$ is a limiting cone with apex $A \downarrow a_1$, and let (g, a, α, β) be any cartesian cone with apex Y. By the universal property of limiting cones, there exists a unique morphism $\bar{\gamma} : Y \to A \downarrow a_1$ such that $\mathrm{cod}\bar{\gamma} = g$, $\mathrm{dom}\bar{\gamma} = a$, $(\varphi\mathrm{cod} \circ \lambda)\bar{\gamma} = \alpha$, and $p\varphi\bar{\gamma} = \beta$. In particular, taking $\gamma = \lambda\bar{\gamma} : a \to a_1 g$, we have the desired morphism, and it is unique by the universal property of $A \downarrow a_1$. $\square$

Corollary 2.4.6. *Suppose $\mathcal{A}, \mathcal{B}$ are finitely complete 2-categories and $F : \mathcal{A} \to \mathcal{B}$ is a 2-functor preserving finite 2-limits. Further let $p : A \to B \in \mathcal{A}$ and $\alpha : a_1 \to a_2 : X \to A$ be given. If α is p-cartesian, then $F\alpha$ is Fp-cartesian.*

Proposition 2.4.7. *Let $p : A \to B$ be a morphism in an 2-category $\mathcal{A}$ with finite 2-limits. The following are equivalent:*

(1) p is a fibration.

(2) There exists a map $\ell : B \downarrow p \to A$ with $p\ell = \mathrm{dom}$ and p-cartesian 2-cell $\varepsilon : \ell \to \mathrm{cod}$ with $p\varepsilon = \lambda$, where $\lambda : \mathrm{dom} \to p\mathrm{cod}$ is the universal 2-cell for $B \downarrow p$.

(3) The map $t : A^{\to} \to B \downarrow p$ has a section $r : B \downarrow p \to A^{\to}$ which is p-cartesian.

(4) The canonical map $t : A^{\to} \to B \downarrow p$ has a right adjoint $r : B \downarrow p \to A^{\to}$ with identity counit.

(5) *The map $i : A \to B \downarrow p$ induced by the identity 1_p has a right adjoint $\ell : B \downarrow p \to A$ in the slice $\mathcal{A}/B$ with invertible unit.*

Proof. We show that (1) $\iff$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (2) and (2) $\iff$ (5).

For (1) $\implies$ (2), note that since p is a fibration, there exists a p-cartesian lift $\varepsilon : \ell \to \mathrm{cod} : B \downarrow p \to A$ for the 2-cell $\lambda : \mathrm{dom} \to p\mathrm{cod} : B \downarrow p \to B$, and this lift satisfies $p\varepsilon = \lambda$. This is exactly the map we require.

For (2) $\implies$ (1), note that since $\varepsilon : \ell \to \mathrm{cod}$ is p-cartesian, so is εx for all $f : X \to B \downarrow p$. In particular, for any 2-cell $\varphi : b \to pa$ with $b : X \to B$ and $a : X \to A$, we have an induced 1-cell $f : X \to B \downarrow p$ with $\lambda f = \varphi$. Then εf is p-cartesian, and by the properties of ε, we have that $p\varepsilon f = \lambda f = \varphi$, so εf is a cartesian lift for φ. Since φ was an arbitrary such 2-cell, we conclude that every such 2-cell has a caresian lift, which is exactly what it means for p to be a fibration.

For (2) $\implies$ (3), note that ε induces a map $r : B \downarrow p \to A^{\to}$ with $\lambda' r = \varepsilon$, where $\lambda' : \mathrm{dom}' \to \mathrm{cod}' : A^{\to} \to A$ is the universal 2-cell associated with $A^{\to}$. This map then satisfies that $\lambda tr = p\lambda' r = p\varepsilon = \lambda$, so that $tr = 1_{B\downarrow p}$, and r is p-cartesian by definition.

suppose that p is a fibration. We require first of all a map $r : B \downarrow p \to A^{\to}$, which we then show is a p-cartesian section to $t : A^{\to} \to B \downarrow p$. Such a map r is equivalently a 2-cell $\lambda r : \mathrm{dom} r \to \mathrm{cod} r : B \downarrow p \to A$. Since p is a fibration, there exists a p-cartesian lift $\bar{r} : x \to \mathrm{cod}' : B \downarrow p \to A$ for the map $\lambda' : \mathrm{dom}' \to p\mathrm{cod}' : B \downarrow p \to B$, where $x : B \downarrow p \to A$ is such that $px = \mathrm{dom}'$. This map $\bar{r}$ induces a 1-cell $r : B \downarrow p \to A^{\to}$ with $\lambda \bar{r} = r$. In particular, we see that $\lambda' tr = p\lambda r = p\bar{r} = \lambda'$, so that $tr = 1_{B\downarrow p}$, as desired. Hence $r : B \downarrow p \to A^{\to}$ is a p-cartesian section of t.

For (3) $\implies$ (4), suppose r is a p-cartesian 1-cell with $tr = 1_{B\downarrow p}$. We show that r is also a right-adjoint to t, for which we must find a unit $\eta : 1_{A^{\to}} \to rt$. By the universal property of $A^{\to}$, η is equivalent to a commuting square

$$\begin{tikzcd}
\mathrm{dom}' \arrow[r, "\eta_1"] \arrow[d, "\lambda'"'] & \mathrm{dom}' rt \arrow[d, "\lambda' rt"] \\
\mathrm{cod}' \arrow[r, "\eta_2"'] & \mathrm{cod}' rt
\end{tikzcd}$$

Since $\mathrm{cod}' rt = \mathrm{cod} trt = \mathrm{cod} t = \mathrm{cod}'$, it would suffices to find a map η_1 such that $\lambda' = \lambda' rt \circ \eta_1$, since $\eta_2 = 1_{\mathrm{cod}'}$ then makes the above square commute. Since r is a p-cartesian 1-cell, $\lambda' rt$ is a p-cartesian 2-cell, whereby there exists a unique universal map $\gamma : \mathrm{dom}' \to \mathrm{dom}' rt$ induced by the cone $(\mathrm{cod}', \mathrm{dom}', \lambda', 1_{p\mathrm{dom}'})$ with $p\gamma = 1_{p\mathrm{dom}'}$ and $\lambda' = \lambda' rt \circ \gamma$, so we may take $\eta_1 = \gamma$, and then η is the universal map induced by the square

$$\begin{tikzcd}
\mathrm{dom}' \arrow[r, "\gamma"] \arrow[d, "\lambda'"'] & \mathrm{dom}' rt \arrow[d, "\lambda' rt"] \\
\mathrm{cod}' \arrow[r, "1'_{\mathrm{cod}}"'] & \mathrm{cod}' rt
\end{tikzcd}$$

To verify that $\eta : 1_{A^{\to}} \to rt$ is indeed the unit of an adjunction $t \dashv r$, we must verify that $t\eta = 1_t$ and $\eta r = 1_r$. For the former, it suffices to show that $\mathrm{dom} t\eta = 1_{\mathrm{dom} t}$ and $\mathrm{cod} t\eta = 1_{\mathrm{cod} t}$. But these hold since

$$\mathrm{dom} t\eta = p\mathrm{dom}'\eta = p\gamma = 1_{p\mathrm{dom}'} = 1_{\mathrm{dom} t} \qquad \text{and} \qquad \mathrm{cod} t\eta = \mathrm{cod}'\eta = 1_{\mathrm{cod}'} = 1_{\mathrm{cod} t}.$$

Hence $t\eta = 1_t$. For the latter equation $\eta r = 1_r$, we must show that $\gamma r = 1_{\mathrm{dom}' r}$, for which it suffices to see that $1_{\mathrm{dom}' r}$ and γr both are universal morphisms for the same p-cartesian cone over $\lambda' r : \mathrm{dom}' r \to \mathrm{cod}' r : B \downarrow p \to A$. The cone we consider is specifically $(1_{B\downarrow p}, \mathrm{dom}' r, \lambda' r, 1_{p\mathrm{dom}' r})$ with apex $B \downarrow p$, which clearly satisfies $p\lambda' r = p\lambda' r \circ 1_{p\mathrm{dom}'}$. Furthermore, we clearly see that $1_{\mathrm{dom}' r}$ is a universal map over the cone, and that $p\gamma r = 1_{p\mathrm{dom}' r}$ and $\lambda' r \circ \gamma r = \lambda' rtr \circ \gamma r = (\lambda' rt \circ \gamma) r = \lambda' r$, so γr is another such universal map. Hence $\gamma r = 1_{\mathrm{dom}' r}$, from which we conclude that $\eta r = 1_r$. Hence η is the unit of an adjunction $t \dashv r$ with identity counit.

For (4) $\implies$ (2), we must show that the right adjoint $r : B \downarrow p \to A^{\to}$ is p-cartesian, so let (g, a, α, β) be a p-cartesian cone over $\lambda' r : \mathrm{dom}' r \to \mathrm{cod}' r$ with apex Y. Thus $g : Y \to B \downarrow p$, $a : Y \to A$, $\alpha : a \to \mathrm{cod}' rg$, and $\beta : pa \to p\mathrm{dom}' rg = \mathrm{dom} trg = \mathrm{dom} g$. $\square$

Notation 2.4.8. For any fibration $p : A \to B$ with a chosen lifting $\varepsilon : \ell \to \mathrm{cod}$ with $\ell : B \downarrow p \to A$ as in (2), and any object X and morphisms $a : X \to A$, $b : X \to B$, and $\sigma : b \to pa$, we write $A[\sigma] := \ell\bar{\sigma}$, where $\bar{\sigma} : X \to B \downarrow p$ is the map induced by σ.

Chapter 3

Finite limit $\mathscr{F}$-theories

In this chapter we introduce the notion of theories and models that underlie our treatment of the semantics of type theory in the subsequent chapter. Since the models of our theories ought to be categories equipped with some structure, it is sensible take take a 2-dimensional approach. The main notion underlying our treatment will be that of $\mathscr{F}$-categories (definition 3.1.5), as these will allow fine-grained control of the strictness of homomorphisms.

As was discussed in the introduction (chapter 1), many different notions of 2-dimensional theories have been introduced and used throughout the literature. The reason for using $\mathscr{F}$-theories, as opposed to some other notion, is precisely that they provide precise control over which operations are preserved strictly by homomorphisms and which are preserved only up to isomorphism. Since the weak morphisms of models of type theory, whether one considers comprehension categories or categories with families or something else, are usually defined with some squares commuting strictly and some only up to isomorphism, this fine control provided by $\mathscr{F}$-theories is what allows us to recover the correct morphisms as well. Furthermore, since $\mathscr{F}$-theories are based on [19] with a different notion of models, we can also generate these theories using $\mathscr{F}$-enriched sketches, as is done in [2]. It is also from [2] that we take the notion of model we use, so that our $\mathscr{F}$-theories align well with the $\mathscr{F}$-sketches of *op. cit.*.

We begin this chapter (section 3.1) by defining $\mathscr{F}$ (definition 3.1.1) and its enriched categories (definition 3.1.5), where we also relate them to standard 2-categories. Then we go on to provide an explcit description of its weighted limits (section 3.2), which provides the basis for the subsequent $\mathscr{F}$-theories and sketches (section 3.3). We end the section with a short informal relation between the syntax of deductive systems and $\mathscr{F}$-theoris, in a similar style to the descrption of syntax from the judgemental theories of [8].

3.1 $\mathscr{F}$ and its enriched categories

Definition 3.1.1 ([20, §3.1])**.** We define $\mathscr{F}$ to be the full subcategory of $\mathbf{Cat}^{\to}$, the arrow category of $\mathbf{Cat}$, on fully faithful and injective-on-objects functors. Given an object $A_\tau \hookrightarrow A_\lambda$ of $\mathscr{F}$, we call A_τ the tight part (with the object of A_τ being tight object), and we call A_λ the loose part (with the objects of A_λ being loose objects). We will generally identify A_τ with its image in A_λ, thereby viewing A_τ as a class of objects of A_λ.

Remark 3.1.2. $\mathscr{F}$-categories were first introduced in [20], where they were used to study what 2-limits lifted to different 2-categories of algebras for 2-monads.

To talk about enrichment in $\mathscr{F}$, as defined in section 2.2, we require that $\mathscr{F}$ is cartesian closed, which is establshed in [20, §3.1]. Unfolding the definition of $\mathscr{F}$-enriched categories, we see that, since an object of $\mathscr{F}$ is essentilly a category equipped with a class of objects, an $\mathscr{F}$-enriched category is essentially the same as a 2-category equipped with a class of morphisms closed under identites and composition, and these morphisms we will denote as tight. It is precisely this additional data that allows us to specify that homomorphisms should preserve only some maps strictly, and the rest up to isomorphism: we take homomorphisms to be pseudonatural transformations whose naturality 2-cells are identities on tight morphisms. This approach therefore allows the specific theory precise control over the strictness of morphisms. For example, in the theory of categories with finite products, we take the maps giving finite products to be loose, in other words to not be tight, so that homomorphisms preserve these maps only up to isomorphism, whereas in the theory of functors we take the one generating morphism to be tight, forcing homomorphisms to preserve it strictly.

Lemma 3.1.3. *The functors $\mathbb{1}_\lambda : \mathbb{0} \hookrightarrow \mathbb{1}$ and $\mathbb{1}_\tau : \mathbb{1} \hookrightarrow \mathbb{1}$ are the representables of $\mathscr{F}$, in the sense that for any $A \in \mathscr{F}$, we have $A_\tau \cong \mathscr{F}(\mathbb{1}_\tau, A)$ and $A_\lambda \cong \mathscr{F}(\mathbb{1}_\lambda, A)$, naturally in A.*

Corollary 3.1.4. *For any $A, B \in \mathscr{F}$, the exponential $[A, B] \in \mathscr{F}$ satisfies that $[A, B]_\tau \cong \mathscr{F}(A, B)$ and $[A, B]_\lambda \cong \mathbf{Cat}(A_\lambda, B_\lambda)$, naturally in A, B.*

Proof. We have natural isomorphisms

$$[A, B]_\tau \cong \mathscr{F}(\mathbb{1}_\tau, [A, B]) \cong \mathscr{F}(\mathbb{1}_\tau \times A, B) \cong \mathscr{F}(A, B)$$

and

$$[A, B]_\lambda \cong \mathscr{F}(\mathbb{1}_\lambda, [A, B]) \cong \mathscr{F}(\mathbb{1}_\lambda \times A, B) \cong \mathbf{Cat}(A_\lambda, B_\lambda),$$

where we have $(\mathbb{1}_\lambda \times A) \cong (\mathbb{0} \hookrightarrow A_\lambda)$ since limits are computed pointwise, so that $\mathscr{F}(\mathbb{1}_\lambda \times A, B)$ is uniquely determined on the tight part, irrespective of the loose part, and so $\mathscr{F}(\mathbb{1}_\lambda \times A, B) \cong \mathbf{Cat}(A_\lambda, B_\lambda)$. □

Definition 3.1.5 ([20, §3.1])**.** An *enhanced 2-category*, or $\mathscr{F}$-category, is a category enriched in $\mathscr{F}$.

Notation 3.1.6. We will use blackboard bold letters $\mathbb{A}, \mathbb{B}, \mathbb{C}, ...$ to denote $\mathscr{F}$-categories. Furthermore, we use $A \rightsquigarrow B$ to denote loose morphisms of $\mathbb{A}$, i.e. morphisms in $\mathbb{A}_\lambda$, and $A \to B$ to denote tight morphisms, i.e morphisms in $\mathbb{A}_\tau$.

Remark 3.1.7. Since an object of $\mathscr{F}$ is equivalently a category equipped with a class of tight objects, we can view an $\mathscr{F}$-category as a 2-category equipped with a class of tight morphisms closed under identities (as the identity assigning functors preserve tightness and in the terminal object of $\mathscr{F}$ all morphisms are tight) and composition (for similar reasons regarding the composition functor).

Any $\mathscr{F}$-category $\mathbb{A}$ gives rise two 2-categories: the 2-category of tight morphisms $\mathbb{A}_\lambda$ and its locally full sub-2-category of tight morphisms $\mathbb{A}_\tau$. We use $J_\mathbb{A}$ to denote the inclusion $\mathbb{A}_\tau \hookrightarrow \mathbb{A}_\lambda$ and

We let $\mathbb{F}$ denote the $\mathscr{F}$-category of $\mathscr{F}$-categories, as shown exists in example 2.2.6. Its tight morphisms are $\mathscr{F}$-functors and its loose morphisms are ordinary functors on the loose part. The 2-cells are $\mathscr{F}$-natural transformations, i.e. natural transformations on whose components are all tight. Furthermore, given two $\mathscr{F}$-categories $\mathbb{A}, \mathbb{B}$, we let $[\mathbb{A}, \mathbb{B}]$ denote the functor $\mathscr{F}$-category as described in example 2.2.9. Thus its objects are $\mathscr{F}$-functors $\mathbb{A} \to \mathbb{B}$, its tight morphisms are $\mathscr{F}$-natural transformations, its loose morphisms are natural transformations whose componenets need not be tight, and its 2-cells are modifications.

Definition 3.1.8 (Chordate and inchordate $\mathscr{F}$-categories)**.** Any 2-category K can be regarded as an $\mathscr{F}$-category in two natural ways: taking only identities to be tight yields what we call the *inchordate* $\mathscr{F}$-category K^-, and dually taking all morphisms to be tight yields the *chordate* $\mathscr{F}$-category K^+. The operations of taking a 2-category to its inchordate or chordate $\mathscr{F}$-category extends to functors $2\mathbf{Cat} \to \mathscr{F}\text{-}\mathbf{Cat}$, and these are respectively left and right adjoint to the functor $(-)_\lambda : \mathscr{F}\text{-}\mathbf{Cat} \to 2\mathbf{Cat}$ taking a $\mathscr{F}$-category to its underlying 2-category of loose morphisms.

3.2 Limits in $\mathscr{F}$-categories

Since we will be considering $\mathscr{F}$-enriched finite limit theories, it will be helpful to have an explicit description of $\mathscr{F}$-enriched limits in a similar style to that of proposition 2.3.14. In fact, much of that description works out the same for $\mathscr{F}$-limits, with tightness conditions threaded throughout. As with the case of 2-limits, the usefulness of having such a characterisation of $\mathscr{F}$-limits will be instrumental in understanding what weights are correct for the notion we want.

Proposition 3.2.1. *Let $\mathbb{J}$ be an $\mathscr{F}$-category. A $\mathbb{J}$-shaped weight is an $\mathscr{F}$-functor $W : \mathbb{J} \to \mathbb{F}$, where $\mathbb{F}$ is the $\mathscr{F}$-category induced by the cartesian closed structure of $\mathscr{F}$. Given a $\mathbb{J}$-shaped diagram $D : \mathbb{J} \to \mathbb{A}$ in an $\mathscr{F}$-category $\mathbb{A}$, a loose W-weighted cone (a, α) from $a \in \mathbb{A}$ to D is a loose natural transformation $W \rightsquigarrow \mathbb{A}(a, D)$, i.e. a W_λ-weighted cone for D_λ in terms of 2-categories.*

If furthermore $\alpha_{x,r}$ (as given by proposition 2.3.12) is a tight morphism of $\mathbb{A}$ for each tight objet $r \in W(x)$ and $x \in \mathbb{J}$, we say that the cone is tight*.*

As with proposition 2.3.14, we see that a morphism of two parallel cones $\alpha, \beta \in [\mathbb{J}, \mathbb{F}](W, \mathbb{A}(a, D))$, i.e. a modification, works out to be a family $(m_{x,r} : \alpha_{x,r} \to \beta_{x,r})_{x \in \mathbb{J}, r \in W(x)}$ such that for every $p : r \to s \in W(x)$ we have

$$m_{x,s} \circ \alpha_{x,p} = \beta_{x,p} \circ m_{x,r},$$

and for every $f : x \rightsquigarrow y \in \mathbb{J}$ and for every $r \in W(x)$ we have

$$m_{y,Wf(r)} = Dfm_{x,r} : Df \circ \alpha_{x,r} = \alpha_{y,Wf(r)} \to \beta_{y,Wf(r)} = Df \circ \beta_{x,r}.$$

Proof. As with proposition 2.3.12, this is simply given by unfolding the many layer of definitions. □

As with proposition 2.3.14, we will require a notion of pullback with cones for $\mathscr{F}$-weighted digrams as well.

Definition 3.2.2 (Pullback of cones)**.** Let $\mathbb{J}$, $\mathbb{A}$ be $\mathscr{F}$-categories, $W : \mathbb{J} \to \mathbb{F}$ a weight, and $D : \mathbb{J} \to \mathbb{A}$ a diagram. Given a W-weighted cone (a, α) over D and a morphism $f : b \to a$ in $\mathbb{A}$, we define the pullback of (a, α) along f as the cone $(b, f^*\alpha)$ defined by

$$(f^*\alpha)_{x,r} = \alpha_{x,r} \circ f$$

on objects $x \in \mathbb{J}$ and $r \in W(x)$, and

$$(f^*\alpha)_{x,p} = \alpha_{x,p} \star f$$

on objects $x \in \mathbb{J}$ and morphisms $p : r \to s \in W(x)$.

Given a modification $m : \alpha \to \alpha' : W \rightsquigarrow \mathbb{A}(a, D)$, we define the pullback $f^*m : f^*\alpha \to f^*\alpha'$ by

$$(f^*m)_{x,r} = m_{x,r} \star f$$

Furthermore, for $\varphi : f \to g : b \to a$ a 2-cell in $\mathbb{A}$, we define the modification $\varphi^*\alpha : f^*\alpha \to g^*\alpha$ by $(\varphi^*\alpha)_{x,r} = \alpha_{x,r} \star \varphi$.

We briefly recount the notion of enriched limit in the special case of $\mathscr{F}$-categories:

Definition 3.2.3 (Weighted limit (see definition 2.2.11))**.** Given a shape $\mathscr{F}$-category $\mathbb{J}$ and weight $W : \mathbb{J} \to \mathscr{F}$, the W-weighted limit of a diagram $D : \mathbb{J} \to \mathbb{A}$ in an $\mathscr{F}$-category $\mathbb{A}$ is an object $L \in \mathbb{A}$ such that there exists an isomorphism in $\mathscr{F}$

$$\mathbb{A}(-, L) \cong [\mathbb{J}, \mathbb{F}](W, \mathbb{A}(-, D)).$$

Proposition 3.2.4. *Given a weight $W : \mathbb{J} \to \mathscr{F}$, a diagram $D : \mathbb{J} \to \mathbb{A}$ has a limit in $\mathbb{A}$ if and only if there exists an object $\ell \in \mathbb{A}$ and a tight cone $L : W \to \mathbb{A}(\ell, D)$ with the property that*

1. *for every cone $\alpha : W \rightsquigarrow \mathbb{A}(a, D)$, there exists a unique morphism $h_\alpha : a \rightsquigarrow \ell$ such that $\alpha = h_\alpha^* L$, and h_α is tight if and only if α is tight,*
2. *for every modification $m : \alpha \to \alpha' : W \rightsquigarrow \mathbb{A}(a, D)$ there exists a unique 2-cell $h_m : h_\alpha \to h_{\alpha'}$ such that $h_m^* \ell = m$.*

We will call a cone satisfying these properties a limiting cone.

Proof. The proof for the isomorphisms is exactly the same as for proposition 2.3.14, and so we only consider the tightness preservation and reflection of the isomorphism h_a constructed as there. In particular, note that h_a is an isomorphism in $\mathscr{F}$, and so $h_a(\alpha)$ is tight if and only if α is. $\square$

Furthermore, as aluded to throughout this section, any $\mathscr{F}$-limit in an $\mathscr{F}$-category $\mathbb{A}$ produces a 2-limit in the 2-category $\mathbb{A}_\lambda$ of loose morphisms, and any 2-limit in $\mathbb{A}_\lambda$ for which the universal cone is tight and for which the universal morphisms induced by cones α are tight if and only if the cones α are tight is an $\mathscr{F}$-limit. In particular, we see that an $\mathscr{F}$-limit is essentially the same as a 2-limit that respects and detects tightness of morphisms in a specific way.

Since in a limit theory the models are limit preserving functors, and we're aiming to generalize those, we clearly need to generalize the preservation of limits:

Definition 3.2.5 (Preservation of limits [19, §3.2])**.** An $\mathscr{F}$-functor $F : \mathbb{A} \to \mathbb{B}$ preserves W-weighted limits for some weight $W : \mathbb{J} \to \mathbb{F}$ if, for every diagram $D : \mathbb{J} \to \mathbb{A}$ and limiting cone $L : W \to \mathbb{A}(\ell, D)$, the image FL of L under F, defined as the composite

$$W \xrightarrow{\lim^W D} \mathbb{A}(\ell, D) \xrightarrow{F} \mathbb{B}(F\ell, F \circ D),$$

is a limiting cone in $\mathbb{B}$ of the diagram $F \circ D$.

For a class Φ of weights, we say that F preserves Φ-weighted limits if it preserves W-weighted limits for every weight $W \in \Phi$.

Similarly, our limit theories are finitely complete categories, which means we also need completeness:

Definition 3.2.6 (Completeness)**.** For Φ a class of weights and $\mathbb{A}$ an $\mathscr{F}$-category, we say that $\mathbb{A}$ is Φ-complete if for every weight $W \in \Phi$ all diagrams $\mathbb{J}_W \to \mathbb{A}$ have a limit.

A key result for us is that, much like in the 1-categorical case, we can freely complete an $\mathscr{F}$-category under a chose class of weights. This free completion is what will allow us to turn finite limit $\mathscr{F}$-sketches into finite limit $\mathscr{F}$-theories later.

Theorem 3.2.7. *Let Φ be a set of weights with small domain. We can freely complete any $\mathscr{F}$-category $\mathbb{A}$ under $\mathscr{F}$-limits, in the sense that there exists an $\mathscr{F}$-category $\hat{\mathbb{A}}$ and fully faithful $\mathscr{F}$-functor $\iota : \mathbb{A} \hookrightarrow \hat{\mathbb{A}}$, with the universal property that for any Φ-complete $\mathscr{F}$-category $\mathbb{B}$ and $\mathscr{F}$-functor $F : \mathbb{A} \to \mathbb{B}$, there exists a unique (up to isomrphism) $\mathscr{F}$-functor $\hat{F} : \hat{\mathbb{A}} \to \mathbb{B}$ such that $\hat{F} \circ \iota \cong F$.*

Proof. See theorem 5.35 of [19]. □

3.3 Finite limit $\mathscr{F}$-theories and sketches

Generalizing the notion of finite limit theories, we would define an enhanced finite limit 2-theory, as a small enhanced 2-category with finite weighted limits. To aid with constructing these categories, and thus providing examples of such theories, we make use of enhanced finite limit 2-sketches, introduced in [2] as a specialization of enriched sketches defined by [19].

However, taking as $\mathscr{F}$-category of models the $\mathscr{F}$-category of $\mathscr{F}$-functors and strict $\mathscr{F}$-natural transformations is too strict for our purposes, as this would require models to preserve all structure of the theory strictly. Thus we would be unable to capture well structured categories with weak morphisms, such as the 2-category of cartesian categories. Therefore we shall introduce the following functor $\mathscr{F}$-categories, which allow for a controlled level of weakness in the categories of models, specified by the theory under consideration.

Definition 3.3.1 ([20, §4.1])**.** Let w be a flavor of weakness $w \in \{s, p\}$ (for *strict* and *pseudo*) and $F, G : \mathbb{A} \to \mathbb{B}$ be $\mathscr{F}$-functors. A *loose w-natural transformation* $F \to G$ is a w-natural transformation of 2-functors $\alpha : F_\lambda \to G_\lambda$ such α is strictly 2-natural on tight morphisms, in the sense that $\alpha_d \circ Ff = Gf \circ \alpha_c$ for tight morphisms $f : c \to d \in \mathbb{A}$. A *tight* w-natural transformation $F \to G$ is a loose w-natural transformation whose components are all tight.

These determine an $\mathscr{F}$-category $\mathbb{F}\mathrm{un}_w(\mathbb{A}, \mathbb{B})$ whose objects are $\mathscr{F}$-functors $\mathbb{A} \to \mathbb{B}$, loose (resp. tight) morphisms are loose (resp. tight) w-natural transformations, and 2-cells are modifications.

These special functor $\mathscr{F}$-categories are what provide our $\mathscr{F}$-categories of models, and so we are now prepared to give the notions of theories and sketches we will use throughout.

Definition 3.3.2 (Weighted Theory)**.** A Φ-*weighted $\mathscr{F}$-theory* is a small $\mathscr{F}$-category with all Φ-weighted limits. A model for a theory $\mathbb{L}$ is a $\mathscr{F}$-limit preserving functor $\mathscr{F}$-functor $M : \mathbb{L} \to \mathbb{F}$. We have for each flavor of weakness $w \in \{\text{strict}, \text{pseudo}\}$ an $\mathscr{F}$-category of w models, namely the full sub-$\mathscr{F}$-category $\text{Mod}_w(\mathbb{L}, \mathbb{F})$ of the functor $\mathscr{F}$-category $\mathbb{F}\text{un}_w(\mathbb{L}, \mathbb{F})$ on Φ-limit preserving $\mathscr{F}$-functors.

Definition 3.3.3 ([19, §6.3])**.** Let Φ be a class of weights in $\mathscr{F}$. An Φ-*weighted limit sketch* $(\mathbb{S}, \mathcal{C})$ is an $\mathscr{F}$-category $\mathbb{S}$ equipped with a set of Φ-weighted cones $C = \{\alpha : W_\alpha \to \mathbb{S}(a_\alpha, D_\alpha)\}$. In other words, for each $\alpha \in C$, we have $W_\alpha \in \Phi$.

A morphism of sketches $(\mathbb{A}, \mathcal{C}) \to (\mathbb{B}, \mathcal{D})$ is an $\mathscr{F}$-functor $F : \mathbb{A} \to \mathbb{B}$ such that every cone in $\mathcal{C}$ is mapped to a cone in $\mathcal{D}$.

Every $\mathscr{F}$-category $\mathbb{A}$ gives rise to a Φ-limit sketch by taking $\mathcal{C}$ to be the class of Φ-weighted limit cones in $\mathbb{A}$. We shall denote this sketch by $\mathcal{S}_\Phi(\mathbb{A})$.

Theorem 3.3.4. *Let Φ be a class of weights and $(\mathbb{A}, \mathcal{C})$ be a small sketch. Then there exists a Φ-complete $\mathscr{F}$-category $\text{Th}(\mathbb{A}, \mathcal{C})$ and faithful $\mathscr{F}$-functor $\iota : \mathbb{A} \to \text{Th}(\mathbb{A}, \mathcal{C})$ taking the cones in $\mathcal{C}$ to limit cones in $\text{Th}(\mathbb{A}, \mathcal{C})$, and for every Φ-complete $\mathscr{F}$-category $\mathbb{B}$, there exists an $\mathscr{F}$-equivalence*

$$\text{Mod}_w(\text{Th}(\mathbb{A}, \mathcal{C}), \mathbb{B}) \simeq [(\mathbb{A}, \mathcal{C}), \mathcal{S}_\Phi(\mathbb{B})]_w,$$

where $[(\mathbb{A}, \mathcal{C}), \mathcal{S}_\Phi(\mathbb{B})]_w$ denotes the $\mathscr{F}$-category of sketch-homomorphisms, w-natural transformations, and modifications. Furthermore, this equivalence natural in $\mathbb{B}$.

Remark 3.3.5. In the strict case, this is proven as theorem 6.23 in [19] for general enriched categories. A proof for the weak case generalizes relatively straightforwardly from this, though is quite invloved and so is omitted here.

3.3.1 Examples of $\mathscr{F}$-theories

To get a better understanding of how these theories work, and what their 2-categories of models look like, we present a few examples of $\mathscr{F}$-sketches and $\mathscr{F}$-theories.

Example 3.3.6. The simplest theory is the theory of a single category: the sketch consists of the terminal $\mathscr{F}$-category, with no cones. Models of this theory in **Cat** are exactly categories, homomorphisms are functors, and the transformations are exactly natural transformations.

Example 3.3.7. There are two theories of morphisms, both of which are generated by a sketch with only a single morphism. The difference is whether this morphism is tight or loose. In the theory of tight morphisms, models are categories A, B with a functor $f : A \to B$ between them, and homomorphisms $f \to f'$ are strictly commuting squares

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r] & B' \end{tikzcd}$$

Meanwhile, in the theory of loose morphisms, the models are the same, but the homomorphisms are squares

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\cong"] & A' \arrow[d, "f'"] \\ B \arrow[r] & B' \end{tikzcd}$$

commuting up to isomorphism. This illustrates what the additional data of the class of tight morphisms provides: they allow us to control the strictness of the *homomorphisms* of models.

Example 3.3.8. The theory Cart of cartesian categories is generated by a single object A, morphisms $e : 1 \rightsquigarrow A$ and $p : A \times A \rightsquigarrow A$, cones witnessing 1 as the terminal object and $A \times A$ as the product of A with itself, and 2-cells witnessing e (resp. p) right adjoint to the universal map $A \to 1$ (resp. $\langle \mathrm{id}_A, \mathrm{id}_A \rangle : A \to A \times A$).

A model of Cart consists of a category $\mathcal{C}$ together with adjunctions $\Delta_{\mathcal{C}} \dashv \times_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and $!_{\mathcal{C}} \dashv 1_{\mathcal{C}} : 1 \to \mathcal{C}$, where $\Delta_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is the diagonal and $!_{\mathcal{C}} : \mathcal{C} \to 1$ is the universal map. Thus we see that a model is exactly a category with (chosen) finite products, a cartesian category. Furthermore, a homomorphism of models $(\mathcal{C}, \times_{\mathcal{C}}, 1_{\mathcal{C}}) \to (\mathcal{D}, \times_{\mathcal{D}}, 1_{\mathcal{D}})$ consists of a functor $F : \mathcal{C} \to \mathcal{D}$ preserving these adjunctions *up to isomorphism*. Thus these are the expected, non-strict, maps of cartesian categories.

Example 3.3.9. Dually to the previous example, taking Cocart be the theory generated by the same data as Cart, except taking e, p to be *left* adjoints rather than right adjoints, we find that the models are exactly cocartesian categories, and the homomorphisms are functors preserving finite coproducts only up to isomorphism.

Example 3.3.10. The theory of fibrations Fib is generated by a tight morphism $p : \mathbb{E} \to \mathbb{B}$, a loose adjunction $t \dashv r : \mathbb{B} \downarrow p \rightsquigarrow \mathbb{E}^{\to}$ with $tr = \mathrm{id}_{\mathbb{P}\downarrow p}$ and t the universal morphism generated by the cone $\langle pd, \alpha, c \rangle$ with $\alpha : d \to c : E^{\to} \to E$. If the unit of the adjunction $\eta : 1_{p\downarrow\mathbb{B}} \to rt$ is further specified to be the identity, so that $rt = \mathrm{id}_{\mathbb{P}^{\to}}$, we have the theory of discrete fibrations.

3.4 From syntax to $\mathscr{F}$-theories

In this section we show a correspondence between the syntax of deductive theories, presented by judgements and inference rules, and finite limit $\mathscr{F}$-theories, much in the style of [8]. The connection is generally as follows: a (dependent) judgement $d\ \mathcal{D} \vdash j\ \mathcal{J}$ is interpreted is a tight morphism $j : \mathcal{J} \to \mathcal{D}$, with the idea that the fibers of j over any instance $d\ \mathcal{D}$ corresponds to $d\ \mathcal{D} \vdash j\ \mathcal{J}$. A non-dependent judgement can then be viewed as a judgement over the terminal object 1, so that the dependency is trival. Equivalently a non-dependent judgement is just an object. An additional feature of this scheme is that, for any non-dependent judgement $\mathcal{J}$, taking the tight arrow object $\mathcal{J}^{\to}$ gives a judgement of morphisms dependent on $\mathcal{J} \times \mathcal{J}$. In modelling type theory, we would for example have a judgement of contexts $\vdash$ **ctx**, and then a judgement of substitions $\Gamma\,\mathbf{ctx}, \Delta\,\mathbf{ctx} \vdash \sigma\,\mathbf{ctx}^{\to}$. Given a (dependent) judgement of types over contexts ty : **ty** $\to$ **ctx**, we would then want to formulate a rule for stability under such substitutions:

$$\frac{\Gamma\,\mathbf{ctx}, \Delta\,\mathbf{ctx} \vdash \sigma\,\mathbf{ctx}^{\to} \qquad \Delta\,\mathbf{ctx} \vdash A\,\mathbf{ty}}{\Gamma\,\mathbf{ctx} \vdash \mathrm{sub}(A, \sigma)\,\mathbf{ty}}$$

We capture this rule as a loose morphism $\mathrm{sub} : \mathbf{ctx}^{\to}{}_{\mathrm{cod}}\times_{\mathrm{ty}} \mathbf{ty} \to \mathbf{ty}$ for which the following triangle commutes.

$$\begin{tikzcd}
\mathbf{ctx}^{\to}{}_{\mathrm{cod}}\times_{\mathrm{ty}} \mathbf{ty} \arrow[rr, "\mathrm{sub}"] \arrow[dr, "\mathrm{dom}\pi_1"'] & & \mathbf{ty} \arrow[dl, "\mathrm{ty}"] \\
& \mathbf{ctx} &
\end{tikzcd}$$

Now, note that $\mathbf{ctx}^{\to}{}_{\mathrm{cod}}\times_{\mathrm{ty}} \mathbf{ty} \cong \mathbf{ctx} \downarrow \mathrm{ty}$, so we see that sub is exactly as the morphism $\ell : \mathbf{ctx} \downarrow \mathrm{ty} \to \mathrm{ty}$ specifying the domain of the lifting in a fibration $\mathbf{ty} \to \mathbf{ctx}$.

In short, we model non-dependent judgements as objects, dependent judgements as tight morphisms, and rules as loose morphisms.

Chapter 4

Type Theories

As was explained in the introduction, there have been a whole host of different categorical structure that have been introduced to model type dependency and type theories throughout the literature, ranging from display map categories to comprehension categories to categories with families and natural models. The goal of this chapter is to show how such structures in can in fact be described as finite 2-limit theories. Since there are so many different models, and covering them all would be both time consuming and reduntant, we will focus only on two sorts of models:

1. Comprehension categories and categories with attributes. The former was introduced by Jacobs in [18] as a minimal categorical model for type dependency, and was there also shown to generalize the latter. The latter was introduced by Cartmell in [7].

2. Categories with Families and Natural Models, which are really two different presentations of the same data. The former were introduced by Dybjer as an alternative to categories with attributes which was closer to the syntax of type theory, and the latter introduced by Awodey as a reformulation of categories with families to make formulating the rules of type theory easier in terms of universal properties.

In essence, what we aim to show here is that models of type theories can be identified with 2-functors from a 2-category describing the theory into **Cat**, and that the homomorphisms can be identified certain pseudo-natural transformations between these functors. This is similar in spirit to [29], where Uemura introduced categories with representable maps for a similar aim. The idea there was that, since representable maps of discrete fibrations are natural models, we can describe our type theory abstractly as a finitely complete category equipped with an additional class of maps closed under certain constructions mimicing that of the representable maps. Then a model is category of contexts $\mathcal{C}$ and a finite limit preserving functor into the discrete fibrations over $\mathcal{C}$ mapping the representable maps of the theory to representable maps of fibrations.

In contrast, our here theories are finitely $\mathcal{F}$-complete $\mathcal{F}$-categories, the models are $\mathcal{F}$-functors into $\mathbf{Cat}^+$ preserving finite $\mathcal{F}$-limits, and the homomorphisms are certain transformatons. The additional data $\mathcal{F}$-categories over 2-categories allows us finer control over the strictness of homomorphisms, which is what allows us to recover the standard notions of homomorphisms of, for example, comprehension categories. Note that an $\mathscr{F}$-limit theory is a 2-theory, but the additional data of tight morphisms in $\mathscr{F}$-categories allows us fine control of the strictness of the homomorphisms. We will use this to show how we can recover the standard notions of homomorphism of models, such as the pseudo morphisms of comprehension categories, just from the $\mathscr{F}$-limit theory. To further show the utility of this approach, we further show how to model more complex features of type theory, such as inductive types and the modes and modal types of multimode type theory. Note that the latter is not readily modeled in categories with representable maps, due to the requirement of multiple categories of contexts, one for each mode.

4.1 Comprehension Categories and Categories with Attributes

The notion of comprehension categories was introduced in [18] to provide a minimal categorical description of type dependency. They concist of a commutative triangle in **Cat** of the following form:

$$\begin{tikzcd} \mathcal{U} \arrow[rr, "\chi"] \arrow[dr, "u"'] & & \mathcal{C}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{C} & \end{tikzcd}$$

where $u : \mathcal{U} \to \mathcal{C}$ is a fibration, $\mathcal{C}^{\to}$ is the arrow category of $\mathcal{C}$, and χ preserves cartesian morphisms. As a model of type theory, we want to think of the fiber u_Γ of u at a context $\Gamma \in \mathcal{C}$ as the category of types in the context Γ, so the objects are types $\Gamma \vdash A\ \mathcal{U}$. The reindexing provided by u being a fibration then models the substitution of types, and the comprehension map χ models context extension, with the preservation of cartesian morphisms giving the expected universal property thereof, in the sense that for any type $\Gamma \vdash A\ \mathcal{U}$ and substitution $\sigma : \Delta \to \Gamma$, the following is a pullback square in $\mathcal{C}$:

$$\begin{tikzcd} \Delta.A[\sigma] \arrow[r, "\bar{\sigma}.A"] \arrow[d, "\pi A[\sigma]"'] & \Gamma.A \arrow[d, "\pi A"] \\ \Delta \arrow[r, "\sigma"'] & \Gamma \end{tikzcd}$$

where we write π for the 2-cell $\lambda\chi : \mathrm{dom}\chi \to u$ induced by χ.

To axiomatise this as a finite $\mathscr{F}$-limit theory, we first consider the 2-categorical structure and limits that are relevant. We already have a characterisation of fibrations in terms of finite 2-limits, and since χ is supposed to preserve cartesian morphisms, it suffices to only ask for preservation of the cartesian lifts; any cartesian morphism is, after all, isomorphic to a cartesian lift. This can be captured by asking that $\chi\ell$ is cartesian, where ℓ is the generic cartesian lift, or cleavage, specified by u being a fibration. We can thus see that all data in the structure of a comprehension category can be described in terms of finite 2-limits. Next we consider the homomorphisms. In general, maps of fibrations are strict commuting squares, which tells us that in our theory, u should be a tight map. By simlar reasoning, we find that cod also should be tight, which is further supported by taking $\mathcal{C}^{\to}$ to be the corresponding tight limit, for which the codomain projection is necessarily tight. However, there is no reason to ask that homomorphisms preserve context extension strictly, and in fact, the weak morphisms that appear in the literature do not require this (see [1]). Instead, we ask for preservation only up to isomorphism, and so take the comprehension map χ to be a loose morphism.

Taking all of these properties into account leads us to the following definition of the $\mathscr{F}$-theory of comprehension categories:

Definition 4.1.1. We define Comp to be the finite $\mathscr{F}$-limit theory generated by the sketch consisting of diagrams

$$
\begin{tikzcd}
\mathcal{U} \arrow[rr, squiggly, "\chi"] \arrow[dr, "u"'] & & \mathcal{C}^{\to} \arrow[dl, "\mathrm{cod}_{\mathcal{C}}"] \\
& \mathcal{C} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}^{\to} \arrow[d, bend right=50, "\mathrm{dom}_{\mathcal{C}}"'] \arrow[d, bend left=50, "\mathrm{cod}_{\mathcal{C}}"] \arrow[d, phantom, "\overset{\lambda_{\mathcal{C}}}{\Longrightarrow}"] \\
\mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \downarrow u \arrow[r, "\mathrm{cod}_{\mathcal{C}}^{u}"] \arrow[d, "\mathrm{dom}_{\mathcal{C}}^{u}"'] & \mathcal{U} \arrow[d, "u"] \\
\mathcal{C} \arrow[r, "1_{\mathcal{C}}"'] \arrow[ur, Rightarrow, "\lambda_{\mathcal{C}}^{u}"] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C} \downarrow u \arrow[d, squiggly, bend right=50, "r"'] \arrow[d, bend left=50, "\mathrm{cod}_{\mathcal{C}}^{u}"] \arrow[d, phantom, "\overset{\ell}{\Longrightarrow}"] \\
\mathcal{U}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{U} \downarrow r \arrow[r, "\mathrm{cod}_{\mathcal{U}}^{r}"] \arrow[d, "\mathrm{dom}_{\mathcal{U}}^{r}"'] & \mathcal{C} \downarrow u \arrow[d, squiggly, "r"] \\
\mathcal{U} \arrow[r, "1_{\mathcal{U}}"'] \arrow[ur, Rightarrow, "\lambda_{\mathcal{U}}^{r}"] & \mathcal{U}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}^{\to} \downarrow \chi r \arrow[r, "\mathrm{cod}_{\mathcal{C}^{\to}}^{\chi r}"] \arrow[d, "\mathrm{dom}_{\mathcal{C}^{\to}}^{\chi r}"'] & \mathcal{C} \downarrow u \arrow[d, squiggly, "\chi r"] \\
\mathcal{C}^{\to} \arrow[r, "1_{\mathcal{U}}"'] \arrow[ur, Rightarrow, "\lambda_{\mathcal{C}^{\to}}^{\chi r}"] & \mathcal{C}^{\to}
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{U} \downarrow r \arrow[r, "\mathrm{cod}_{\mathcal{U}}^{r}"] \arrow[rr, bend left=60, "\mathrm{dom}_{\mathcal{U}}^{r}"] \arrow[drr, bend right=30, "u\mathrm{dom}_{\mathcal{U}}^{r}"'] & \mathcal{C} \downarrow u \arrow[r, bend left, "\mathrm{cod}_{\mathcal{C}}^{u}"] \arrow[r, squiggly, bend right, "r"'] \arrow[r, phantom, "\ell\Uparrow"] \arrow[u, phantom, "\Downarrow \ell\mathrm{cod}_{\mathcal{U}}^{r}\circ\lambda_{\mathcal{U}}^{r}"] \arrow[d, phantom, "\Uparrow u\lambda_{\mathcal{U}}^{r}"] & \mathcal{U} \arrow[d, "u"] \\
& & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}^{\to} \downarrow \chi r \arrow[r, "\mathrm{cod}_{\mathcal{C}^{\to}}^{\chi r}"] \arrow[rr, bend left=60, "\mathrm{dom}_{\mathcal{C}^{\to}}^{\chi r}"] \arrow[drr, bend right=30, "\mathrm{cod}_{\mathcal{C}}\mathrm{dom}_{\mathcal{C}^{\to}}^{\chi r}"'] & \mathcal{C} \downarrow u \arrow[r, squiggly, bend left, "\chi\mathrm{cod}_{\mathcal{U}}^{r}"] \arrow[r, squiggly, bend right, "\chi r"'] \arrow[r, phantom, "\chi\ell\Uparrow"] \arrow[u, phantom, "\Downarrow \chi\ell\mathrm{cod}_{\mathcal{C}^{\to}}^{\chi r}\circ\lambda_{\mathcal{C}^{\to}}^{\chi r}"] \arrow[d, phantom, "\Uparrow \mathrm{cod}_{\mathcal{C}}\lambda_{\mathcal{C}^{\to}}^{\chi r}"] & \mathcal{C}^{\to} \arrow[d, "\mathrm{cod}_{\mathcal{C}}"] \\
& & \mathcal{C}
\end{tikzcd}
$$

subject to the equations

$$u = \mathrm{cod}_{\mathcal{C}}\chi \qquad ur = \mathrm{dom}_{\mathcal{C}}^{u} \qquad u\ell = \lambda_{\mathcal{C}}^{u}$$

and with limit cones those making $\mathcal{C}^{\to}$ a power with the tight arrow category 2, $\mathcal{C} \downarrow u$ the comma object of $1_{\mathcal{C}}$ and u, $\mathcal{U} \downarrow r$ the comma object of $1_{\mathcal{U}}$ and r, $\mathcal{C}^{\to} \downarrow \chi r$ the comma object of $\mathcal{C}^{\to} \downarrow \chi r$, and the cones making ℓ and $\chi\ell$ cartesian morphisms, i.e. the last two diagrams above.

A model for Comp in **Cat** is exactly a (cloven) comprehension category, and given two comprehension categories $(\mathcal{C}, \mathcal{U}, u, \chi)$ and $(\mathcal{D}, \mathcal{V}, v, \psi)$, a homomorphism $f : (\mathcal{C}, \mathcal{U}, u, \chi) \to$

$(\mathcal{D}, \mathcal{V}, v, \psi)$ is essentially a prism

$$\begin{tikzcd}
\mathcal{U} \arrow[rr, "\chi"] \arrow[drrr, "f_e"] \arrow[ddr, "u"'] & & \mathcal{C}^{\to} \arrow[drrr, "f_b^{\to}"] \arrow[ddl, "\mathrm{cod}_{\mathcal{C}}"] \arrow[d, phantom, "\cong"] & & & \\
& & & \mathcal{V} \arrow[rr, "\psi"] \arrow[ddr, "v"] & & \mathcal{D}^{\to} \arrow[ddl, "\mathrm{cod}_{\mathcal{D}}"] \\
& \mathcal{C} \arrow[drrr, "f_b"'] & & & & \\
& & & & \mathcal{D} &
\end{tikzcd}$$

where all sides except the top one are commuting squares, the top one is a coherent isomorphism, and the pair (f_e, f_b) constitutes a map of fibrations. Finally, we see that the 2-cells $\alpha : f \to g$ in the 2-category of pseudomorphisms of comprehension categories is a pair of natural transformations $\alpha_b : f_b \to g_b$ and $\alpha_e : f_e \to g_e$ compatible with the top isomorphism in a suitable manner.

Remark 4.1.2. In the definition of Comp, we do not require homomorphisms to preserve cartesian lifts strictly, since the lifting map $r : \mathcal{C} \downarrow u \to \mathcal{U}$ is loose. This aligns with the traditional definition of maps of fibrations, where we only ask for preservation of cartesian maps, and so only that the image of a carteisan lift under the homomorphisms is isomorphic to a cartesian lift in the codomain. However, if we choose to make r tight isntead, the homomorphisms to strictly preserve the choice of cartesian lifts, and so in the language of models of type theory, to strictly preserve substitution.

This shows the flexibility that $\mathscr{F}$-limit theories provide in the 2-categories of models: by carefully choosing what maps are tight and what maps are loose, we have precise control over what parts of the theory are preserved strictly by homomorphisms, and what parts are only up to isomorphism, without ad hoc definitions for every possible notion of homomorphisms we could come up with. Furthermore, we can naturally recover the different notions of homomorphisms present in literature, such as the weak maps of comprehension categories, the strict maps, the lax maps, etc. With a bit of care, we could give a theory of split comprehension categories, and then ask for maps which preserve the splitting strictly or loosely.

Using $t : \mathcal{C} \downarrow u \to \mathcal{U}^{\to}$ denote the morphism induced by $u\lambda_{\mathcal{U}} : u\mathrm{dom}_{\mathcal{U}} \to u\mathrm{cod}_{\mathcal{U}}$, we see that u is a discrete fibration if $rt = \mathrm{dom}^u_{\mathcal{C}}$ and $\ell t = 1_{1_{\mathcal{U}}}$, capturing that the lift of the image of a generic morphism in $\mathcal{U}$ under u is that same generic morphism. Adding this morphism t and the comma object and cone for $\mathcal{U}^{\to}$ as well as these equations to the sketch for comprehension categories, we then specify the comprehension categories whose fibration of types are discrete, also known as categories with attributes. We denote the theory generated by this augmeneted sketch by Attr, the theory of categories with attributes.

Definition 4.1.3. We let Attr be the finite limit $\mathscr{F}$-theory generated as Comp, but with the diagrams and cones expressing that ℓ is cartesian with diagrams and cones expressing that the lifting (r, ℓ) is the cleavage of a discrete fibration.

4.1.1 Terms in comprehension categories

In section 4.2.1, we will show that natural models are the same as categories with attributes. However, unlike natural models, the notion of term is not inherently part of the definition of comprehension categories, and so they need to be derived in some way to show this equivalence. A common interpretation is that a term $\Gamma \vdash a : A$ of type $A : X \to \mathcal{U}$ in a context $\Gamma = uA : X \to \mathcal{C}$ is a section to the projection map $\pi A : \{A\} \to A$, as is defined in [18]. Using finite 2-limits, we can in fact construct an object $\dot{\mathcal{U}}$ with the property that any map $a : X \to \dot{\mathcal{U}}$ induces a unique section $\tau a : u\mathrm{T}a \to \{\mathrm{T}a\}$ of $\pi\mathrm{T}a$, for some adequately chosen $\mathrm{T} : \dot{\mathcal{U}} \to \mathcal{U}$.

Definition 4.1.4. In Comp, let $\dot{\mathcal{U}}$ denote the $\mathscr{F}$-limit of the diagram

$$\begin{tikzcd} \mathcal{U} \arrow[r, bend left, "u"] \arrow[r, bend right, "\{-\}"'] \arrow[r, phantom, "\pi\Uparrow"] & \mathcal{C} \end{tikzcd}$$

with weight

$$\begin{tikzcd} 1 \arrow[r, bend left, "a"] \arrow[r, bend right, "b"'] \arrow[r, phantom, "r\Uparrow"] & S \end{tikzcd}$$

where $S = \{s : a \rightleftarrows b : r\}$ is the walking section/retraction pair, so that $rs = 1_a$.

We denote the universal map $\dot{\mathcal{U}} \to \mathcal{U}$ by T, and the universal 2-cell $u\mathrm{T} \to \{\mathrm{T}\}$ by τ. Note that by definition, we thus have that $\pi\mathrm{T} \circ \tau = 1_{\mathrm{T}}$, so that τ is a generic term, and any such section induces a universal map into $\dot{\mathcal{U}}$.

With this definition of terms comes also a notion of 2-cells of terms. From the definition of 2-limits, we see that a 2-cell $\alpha : a \to b : X \to \dot{\mathcal{U}}$ is equivalent to a 2-cell $\mathrm{T}\alpha : \mathrm{T}a \to \mathrm{T}b$ such that the diagram below commutes:

$$\begin{tikzcd} u\mathrm{T}a \arrow[r, "\tau a"] \arrow[d, "u\mathrm{T}\alpha"'] & \{\mathrm{T}a\} \arrow[r, "\pi a"] \arrow[d, "\{\mathrm{T}\alpha\}"] & u\mathrm{T}a \arrow[d, "u\mathrm{T}\alpha"] \\ u\mathrm{T}b \arrow[r, "\tau b"'] & \{\mathrm{T}b\} \arrow[r, "\pi\mathrm{T}b"'] & u\mathrm{T}b \end{tikzcd}$$

Note that the right square commutes trivially by the exchange law applied to $\pi : \{-\} \to u$ and $\mathrm{T}\alpha : \mathrm{T}a \to \mathrm{T}b$, so that really a map of terms $\alpha : a \to b : X \to \dot{\mathcal{U}}$ is the same as a map of types $\mathrm{T}\alpha : \mathrm{T}a \to \mathrm{T}b$ such that the above left square commutes.

Furthermore, we have a nice relation between terms and substitution in the form of the following lemma:

Lemma 4.1.5. *Let $A : X \to \mathcal{U}$ and $\sigma : \Gamma \to uA$ be given. Then there is a bijective correspondence between maps $\alpha : \Gamma \to \{A\}$ such that $\pi A \circ \alpha = \sigma$ and terms of type $\Gamma \vdash a : A[\sigma]$ (where we interpret a as a section of $\pi(A[\sigma]) : \{A[\sigma]\} \to \Gamma$)*

Proof. Consider the diagram

$$
\begin{tikzcd}
\Gamma \arrow[dr, dashed, "a"] \arrow[drr, dotted, bend left, "\alpha"] \arrow[ddr, bend right, "1_{\{A\}}"'] & & \\
& \{A[\sigma]\} \arrow[r, "\{\overline{\sigma}\}"] \arrow[d, "\pi(A[\sigma])"'] & \{A\} \arrow[d, "\pi A"] \\
& \Gamma \arrow[r, "\sigma"'] & uA
\end{tikzcd}
$$

Recall that the square to the bottom right is a pullback square, so that given α, such a term a is uniquely determined by the universal property of pullbacks. In the other direction, any term a uniquely determines a α by composition with $\{\overline{\sigma}\}$. □

This object of terms, and the maps T and $\dot{u} := u\mathrm{T}$, have some very nice properties. For example, we can show that T preserves and reflects cartesian morphisms, and that $\dot{u}$ is a fibration, so that we have a notion of substitution on terms. We begin by showing that T reflects cartesian morphisms, as that will be helpful in showing the other properties.

Lemma 4.1.6. *The map* T *reflects u-cartesian morphisms, in the sense that if* $\alpha : a \to b : X \to \dot{\mathcal{U}}$ *is such that* $\mathrm{T}\alpha$ *is u-cartesian, then* α *is* $\dot{u}$*-cartesian.*

Proof. Suppose we are given a $\dot{u}$-cartesian cone (f, c, φ, σ) over α, so that $\dot{u}(\alpha f) \circ \sigma = \dot{u}\varphi$. Then we also have that $u\mathrm{T}\alpha f \circ \sigma = u\mathrm{T}\varphi$, since $\dot{u} = u\mathrm{T}$ by definition. In particular, since $\mathrm{T}\alpha$ is cartesian, there then exists a map $\hat{\sigma} : \mathrm{T}c \to \mathrm{T}a$ with $\mathrm{T}\alpha f \circ \hat{\sigma} = \mathrm{T}\varphi$ and $u\hat{\sigma} = \sigma$. If the square

$$
\begin{tikzcd}
u\mathrm{T}c \arrow[r, "\tau c"] \arrow[d, "u\hat{\sigma}"'] & \{\mathrm{T}c\} \arrow[d, "\{\hat{\sigma}\}"] \\
u\mathrm{T}af \arrow[r, "\tau af"'] & \{\mathrm{T}af\}
\end{tikzcd}
$$

commutes, then $\hat{\sigma}$ is a morphism of terms, as established earlier. Thus $\hat{\sigma}$ would be of the form $\mathrm{T}\bar{\sigma}$ for some unique $\bar{\sigma} : c \to a$, and so by uniqueness of $\hat{\sigma}$, it would be the unique map such that $u\mathrm{T}\bar{\sigma} = \sigma$ and $\alpha f \circ \bar{\sigma} = \varphi$. To see that this holds, consider the following diagram:

$$
\begin{tikzcd}
u\mathrm{T}c \arrow[r, "\tau c"] \arrow[d, "u\hat{\sigma}"'] & \{\mathrm{T}c\} \arrow[d, "\{\hat{\sigma}\}"] & \\
u\mathrm{T}af \arrow[r, "\tau af"'] & \{\mathrm{T}af\} \arrow[r, "\pi\mathrm{T}af"] \arrow[d, "\{\mathrm{T}\alpha\}"'] & u\mathrm{T}af \arrow[d, "u\mathrm{T}\alpha"] \\
& \{\mathrm{T}bf\} \arrow[r, "\pi\mathrm{T}bf"'] & u\mathrm{T}bf
\end{tikzcd}
$$

Since α and φ are maps of terms, we have that

$$\{\mathrm{T}\alpha\} \circ \tau af \circ u\hat{\sigma} = \tau bf \circ u\mathrm{T}\alpha \circ \sigma = \tau bf \circ u\mathrm{T}\varphi$$

and

$$\{\mathrm{T}\alpha\} \circ \{\hat{\sigma}\} \circ \tau c = \{\mathrm{T}\alpha \circ \hat{\sigma}\} \circ \tau c = \{\mathrm{T}\varphi\} \circ \tau c = \tau bf \circ u\mathrm{T}\varphi.$$

Similarly we have that

$$\pi\mathrm{T}af \circ \{\hat{\sigma}\} \circ \mathrm{T}\tau c = u\hat{\sigma} \circ \pi\mathrm{T}c \circ \mathrm{T}\tau c = \sigma = \pi\mathrm{T}af \circ \tau af \circ u\hat{\sigma}.$$

Since $\mathrm{T}\alpha$ is cartesian, the bottom right square is a pullback, and so by

$$u\mathrm{T}\alpha \circ \sigma = u\mathrm{T}\varphi = \pi\mathrm{T}bf \circ \tau bf \circ u\mathrm{T}\varphi$$

there exists a unique map $h : u\mathrm{T}c \to \{\mathrm{T}af\}$ such that $\pi\mathrm{T}af \circ h = \sigma$ and $\{\mathrm{T}\alpha\} \circ h = \tau bf \circ u\mathrm{T}\varphi$. But as we saw, both $\{\hat{\sigma}\} \circ \tau c$ and $\tau af \circ u\hat{\sigma}$ satisfy this property, so they must be equal. Therefore the top-left square commutes. $\square$

Lemma 4.1.7. *The map $\dot{u} \coloneqq u\mathrm{T} : \dot{\mathcal{U}} \to \mathcal{C}$ is a fibration.*

Proof. We show that for any u-cartesian morphism $\alpha : A \to \mathrm{T}b$ lifts to a $\dot{u}$-cartesian morphism $\mathrm{T}\bar{\alpha} : a \to b$ with $\mathrm{T}\bar{\alpha} = \alpha$. Note that this requires that $\mathrm{T}a = A$, and so we must find a section $\tau a : uA \to \{A\}$ to $\pi A : \{A\} \to A$, and further this choice of section must make α into a map of terms $a \to b$. To find this section, consider the following diagram:

$$\begin{tikzcd}
uA \arrow[r, dashed, "\tau a"] \arrow[rr, bend left, "1_{uA}"] \arrow[d, "u\alpha"'] & \{A\} \arrow[r, "\pi A"] \arrow[d, "\{\alpha\}"] & uA \arrow[d, "u\alpha"] \\
u\mathrm{T}b \arrow[r, "\tau b"'] & \{\mathrm{T}b\} \arrow[r, "\pi\mathrm{T}b"'] & u\mathrm{T}b
\end{tikzcd}$$

Since α is u-cartesian, the right square is a pullback square, so by $\pi\mathrm{T}b \circ \tau b \circ u\alpha = u\alpha$ we have a unique map $\tau a : uA \to \{A\}$ with $\{\alpha\} \circ \tau a = \tau b \circ u\alpha$ and $\pi A \circ \tau a = 1_{uA}$. Thus τa induces a term $a : X \to \dot{\mathcal{U}}$ with $\mathrm{T}a = A$, and furthermore $u\alpha$ induces a map of terms $\bar{\alpha} : a \to b$ such that $\mathrm{T}\bar{\alpha} = \alpha$. Finally, since T reflects cartesian morphisms and α is cartesian, it follows that $\bar{\alpha}$ is cartesian as well.

Finally, to find a cartesian lift for $\sigma : \Gamma \to \dot{u}b$, we first lift it to a u-cartesian map $\hat{\sigma} : \mathrm{T}b[\sigma] \to \mathrm{T}b$, and then to a $\dot{u}$-cartesian map $\bar{\sigma} : b[\sigma] \to b$. Now note that $\dot{u}\bar{\sigma} = u\mathrm{T}\bar{\sigma} = u\hat{\sigma} = \sigma$ is a $\dot{u}$-cartesian lift of σ, as desired. $\square$

In preparation for section 4.2.1, we will prove a few useful theorems about T:

Lemma 4.1.8. T *preserves cartesian morphisms.*

Proof. Suppose $\alpha : a \to ; X \to \dot{\mathcal{U}}$ is $\dot{u}$-cartesian, and let $\alpha' : a' \to b$ denote the $\dot{u}$-cartesian lift of $\dot{u}\alpha$. Then there exists an isomorphism $h : a \to a'$ such that $\alpha' \circ h = \alpha$ and $\dot{u}h = 1_{\dot{u}a}$ by the essential uniqueness of cartesian lifts, and by construction $\mathrm{T}\alpha' : \mathrm{T}a' \to \mathrm{T}b$ is u-cartesian. But then since $\mathrm{T}h$ is a vertical isomorphism, we see that $\mathrm{T}\alpha = \mathrm{T}\alpha' \circ \mathrm{T}h$ is u-cartesian as well; to find the unique map for a u-cartesian cone over $\mathrm{T}\alpha$, we simply find it for $\mathrm{T}\alpha'$ and transport the solution accross $\mathrm{T}h$. □

Lemma 4.1.9. T *is faithful, in the sense that for any 2-cells* $\alpha, \beta : a \to b : X \to \dot{\mathcal{U}}$, *if* $\mathrm{T}\alpha = \mathrm{T}\beta$, *then* $\alpha = \beta$.

Proof. This follows from the 2-dimensional universal property of $\mathbb{T}$. □

4.2 Categories with Families

The second semantics we will fit into our framework is categories with families, which were introduced by Dybjer in [11] as a variant of Cartmell's categories with attributes with closer ties to the syntax of type theory, in particular modeling terms as a presheaf by themselves, rather than as a certain class of context maps. A further reformulation of CwFs, found independently by Fiore [13] and Awodey [3], gives context extension as a representable map of presheaves between the presheaves of terms the presheaf of types, rather than as a direct universal property as in [11]. This reformulation, dubbed natural models in [3], allows for one final change to fit into our framework: we replace the presheaves by discrete fibrations. One advantage of this step is that representability of a map of presheaves is equivalent to the existence of a right adjoint to the typing map between the categories of elements of the type and term fibrations (as already noted in [3]), which is easily axiomatized in our framework.

Definition 4.2.1 (Natural models of type theory)**.** A *natural model of type theory*, in the sense of [3], is a diagram in **Cat** of the form

$$\begin{tikzcd} \dot{\mathcal{U}} \arrow[rr, "\mathrm{T}"'] \arrow[rr, phantom, shift left=1.5ex, "\perp"] \arrow[dr, "\dot{u}"'] & & \mathcal{U} \arrow[ll, bend right, "\mathrm{V}"'] \arrow[dl, "u"] \\ & \mathcal{C} & \end{tikzcd}$$

where $u, \dot{u}$ are discrete fibrations and T is a map of discrete fibrations, so that $\mathrm{T}u = \dot{u}$. Note that V need not be, and generally is not, a map of fibrations.

Remark 4.2.2. In [8], the natural models are further generalized by weakening the discrete fibrations to arbitrary fibrations, and thus also asking that T preserves cartesian morphisms and that the unit and counit of the adjunction $\mathrm{T} \dashv \mathrm{V}$ are cartesian. In [9], these generalized natural models, there dubbed generalized categories with families, are shown to be biequivalent with comprehension categories. I conjecture that one can show the $\mathscr{F}$-theory of these generalized categories with families biequivalent to the one for comprehension categories introduced earlier, with a proof similar to the one found in *op. cit.* For now we treat only the case with discrete fibrations, as the theory of those is both simpler and more developed.

As with comprehension categories, we already know how to specify the data for a natural model in an $\mathscr{F}$-limit sketch. For example, the 2-cell $u : \mathcal{U} \to \mathcal{C}$ is a discrete fibration precisly if the map $t : \mathcal{U}^{\to} \to \mathcal{C} \downarrow u$ induced by $u\lambda_{\mathcal{U}} : u\mathrm{dom}_{\mathcal{U}} \to u\mathrm{cod}_{\mathcal{U}}$ has an inverse $r : \mathcal{C} \downarrow u \to \mathcal{U}^{\to}$ satisfying $\mathrm{cod}_{\mathcal{U}} r = \mathrm{cod}^{u}_{\mathcal{C}}$, which is evidently describable by an $\mathscr{F}$-sketch. We denote the $\mathscr{F}$-theory induced by this sketch by NM.

Remark 4.2.3. Note that there are multiple equivalent ways of describing discrete fibrations in terms of finite 2-limits, and so multiple $\mathscr{F}$-sketches for natural models. However, these $\mathscr{F}$-sketches all induce equivalent $\mathscr{F}$-theories, and so we speak of *the* theory of natural models NM.

Remark 4.2.4. Since we took much care in the definition of comprehension categories to show the ability of $\mathscr{F}$-theories to precisely control which operations are preserved strictly and which are preserved loosely, and this often turns out to be quite verbose, we will in the following sections elide such considerations. However, it should be clear from the previous section that it is possible to recover whatever notion of homomorphisms of e.g. natural models one wants by selecting the right tight morphisms. Note that we by default we will take the $\mathscr{F}$-limits to be as tight as possible.

With the basic theory of natural models at hand, we will spend the remainder of this section showing how further structure and properties, in particular dependent product types, dependent sum types, and extensional identity types, can be similarly described in terms of finite 2-limits. This allows us to construct finite $\mathscr{F}$-limit theories for any type theory with these features, whose models are the natural models with these features. The specific constructions we use are generally based on [8], rather than [3], as we have free access to 2-pullbacks and similar 2-limits, but not the representable discrete fibrations and exponentiabilty of representable maps of discrete fibrations.

Before we can show how to model the rules for these different type constructors, we will need a few useful definitions. First, we will need a the map corresponding to context extension. In analogy to comprehension categories, we will denote this morphism by $\{-\} : \mathcal{U} \to \mathcal{C}$, and we define it by $\{-\} := \dot{u}\mathrm{V}$. The counit of the adjunction $\mathrm{T} \dashv \mathrm{V}$ then gives us the expected projection map $\pi := u\varepsilon : \{-\} = u\mathrm{TV} \to u$. We also want to relate the terms $\dot{\mathcal{U}}$ to substitutions. Specifically, we have a section $\tau : u\mathrm{T} \to \{\mathrm{T}\}$ of $\pi\mathrm{T} : \{\mathrm{T}\} \to u\mathrm{T}$ given by the unit, as in $\tau = \dot{u}\eta$. Thus every term $a : X \to \dot{\mathcal{U}}$ with $A := \mathrm{T}a$ induces a substitution $\tau a : uA \to \{A\}$ for which $\pi A \circ \tau a = 1_{uA}$. In other words, every term $a : X \to \dot{\mathcal{U}}$ induces a term in the sense of comprehension categories.

Dependent products

Recall the formation and introduction rules for dependent product types:

$$\frac{\Gamma \vdash A\ \mathcal{U} \qquad \Gamma.A \vdash B\ \mathcal{U}}{\Gamma \vdash \Pi(A,B)\ \mathcal{U}} \qquad\qquad \frac{\Gamma \vdash A\ \mathcal{U} \qquad \Gamma.A \vdash b : B}{\Gamma \vdash \lambda(A,b) : \Pi(A,B)}$$

The formation rule tells us that given any type A and type B dependent on A, we can form the product type $\Pi(A,B)$. We model this as a morphism from some premise judgement P, corresponding to the pair A and $A \vdash B$, to the judgement of types $\mathcal{U}$. More formally, the premise is a judgement $p : P \to \mathcal{C}$ over contexts, since A and B are types lying over some context, and the formation rule is a map $\Pi : P \to \mathcal{U}$ such that $u\Pi = p$. To see what the premise

P should be, note that a type B is dependent over A if $uB = \{A\}$, so we should take P to be the pullback

$$\begin{tikzcd} \mathcal{U} {}_{\{-\}}\times_{u} \mathcal{U} \arrow[r, "B_F"] \arrow[d, "A_F"'] & \mathcal{U} \arrow[d, "u"] \\ \mathcal{U} \arrow[r, "\{-\}"'] & \mathcal{C} \end{tikzcd}$$

with $p = uA_F$ picking the underlying context Γ of A. In this way, a generalized element $H : X \to P$ is the same as a pair $H_1, H_2 : X \to \mathcal{U}$ with $uH_2 = \{H_1\}$, as desired, vice versa. In particular, the formation rule then says that given any pair of types $\alpha, \beta : X \to \mathcal{U}$ with $u\beta = \{\alpha\}$, we have $\Pi(\alpha, \beta) : X \to \mathcal{U}$, as expected.

Similarly, the introduction rule tells us that, given a type A and a term $A \vdash b : B$, we have a term $\lambda(A, b) : \Pi(A, B)$. This rule is also captured by a morphism, this time of the form $\lambda : P' \to \dot{\mathcal{U}}$ for an appropriate P. This time P consists of a pair of a type A and a term b, such that $\dot{u}b = \{A\}$, and so is captured by the pullback

$$\begin{tikzcd} \mathcal{U} {}_{\{-\}}\times_{\dot{u}} \dot{\mathcal{U}} \arrow[r, "b"] \arrow[d, "A'"'] & \dot{\mathcal{U}} \arrow[d, "\dot{u}"] \\ \mathcal{U} \arrow[r, "\{-\}"'] & \mathcal{C} \end{tikzcd}$$

To capture the typing of the λ-term, we need to ask that the following square commutes:

$$\begin{tikzcd} \mathcal{U} {}_{\{-\}}\times_{\dot{u}} \dot{\mathcal{U}} \arrow[r, "\lambda"] \arrow[d, "{(A', \mathrm{T}b)}"'] & \dot{\mathcal{U}} \arrow[d, "\mathrm{T}"] \\ \mathcal{U} {}_{\{-\}}\times_{u} \mathcal{U} \arrow[r, "\Pi"'] & \mathcal{U} \end{tikzcd}$$

where $(A', \mathrm{T}B)$ is the unique map into the pullback $\mathcal{U} {}_{\{-\}}\times_{u} \mathcal{U}$ given by the equation $\{A'\} = \dot{u}b = u\mathrm{T}b$ holding.

Finally, using the fact that terms of a dependent product $\Gamma \vdash t : \Pi(A, B)$ ought to be in 1-1 correspondence with terms $\Gamma.A \vdash t' : B$, we see that asking that the above square be a pullback gives us the expected inverse of the lambda rule; A term $t : X \to \dot{\mathcal{U}}$ with $\mathrm{T}t = \Pi(\alpha, \beta)$ for some adequate $\alpha, \beta : X \to \mathcal{U}$, we have a unique map $\bar{t} : X \to \mathcal{U} {}_{\{-\}}\times_{\dot{u}} \dot{\mathcal{U}}$ with $\lambda\bar{t} = t$, and vice versa. To capture the elimantion rule

$$\frac{\Gamma \vdash t : \Pi(A, B) \qquad \Gamma \vdash u : A}{\Gamma \vdash t \cdot u : B[\mathrm{id}_\Gamma, u]},$$

we see that the definition $t \cdot u := \bar{t}[\mathrm{id}_\Gamma, u]$ has the expected type, and by this definition for any term $\bar{t} : X \to \mathcal{U} {}_{\{-\}}\times_{\dot{u}}$, we find that $\lambda(\bar{t}) \cdot u = \bar{t}[\mathrm{id}_\Gamma, u]$, so that we validate the computation rule for dependent product types as well.

Putting all of this together, we have the following definition:

Definition 4.2.5. A natural model $(\mathcal{C}, \mathcal{U}, \dot{\mathcal{U}})$ has dependent products if there exists a pullback square of the form

$$\begin{tikzcd}
\mathcal{U}\,{}_{\{-\}}\!\times_{\dot{u}} \dot{\mathcal{U}} \arrow[r, "\lambda"] \arrow[d, "{(A', \mathrm{T}b)}"'] & \dot{\mathcal{U}} \arrow[d, "\mathrm{T}"] \\
\mathcal{U}\,{}_{\{-\}}\!\times_{u} \mathcal{U} \arrow[r, "\Pi"'] & \mathcal{U}
\end{tikzcd}$$

Since all data here is describable by finite 2-limits, we can simply add this to the sketch for natural models to obtain the theory of natural models with dependent product types.

Dependent sums

As with dependent product types, the type former for dependent sum types is modeled by a morphism $\Sigma : \mathcal{U}\,{}_{\{-\}}\!\times_u \mathcal{U} \to \mathcal{U}$. However, the introduction rule is more complicated:

$$\frac{\Gamma \vdash a : A \qquad \Gamma \vdash b : B[\mathrm{id}_\Gamma, t]}{\Gamma \vdash \langle a, b\rangle : \Sigma(A, B)}$$

In particular, the premise requires a term u depending on the *substitution* of a term t. An instance of the premise is thus a pair of maps $a, b : X \to \dot{\mathcal{U}}$ and a map $B : X \to \mathcal{U}$ such that $\dot{u}a = \dot{u}b$, $uB = \{\mathrm{T}a\}$, and $\mathrm{T}b = r(\tau a, B)$, where $(\tau a, B) : X \to \mathcal{C} \downarrow u$ is the map induced by the 2-cell $\tau a : \dot{u}a \to \{\mathrm{T}a\} = uB$ and $\tau = \dot{u}\eta : \dot{u} \to \dot{u}\mathrm{VT}$ is as defined above. In essence, the preimse consists of two terms a,b, and a type B, such that B depends on the type of a, and the type of b is $B[a]$. To construct an object corresponding to this, first consider the pullback

$$\begin{tikzcd}
\dot{\mathcal{U}}\,{}_{\{\mathrm{T}\}}\!\times_{u} \mathcal{U} \arrow[r, "B"] \arrow[d, "a"'] & \mathcal{U} \arrow[d, "u"] \\
\dot{\mathcal{U}} \arrow[r, "\{\mathrm{T}\}"'] & \mathcal{C}
\end{tikzcd}$$

This gives an object of terms a with types B depending on $\mathrm{T}a$. For ease of notation, let $P \coloneqq \dot{\mathcal{U}}\,{}_{\{T\}}\!\times_u \mathcal{U}$. To construct the desired premise, we now take the following pullback:

$$\begin{tikzcd}
P_{r(\tau a, B)}\!\times_{\mathrm{T}} \dot{\mathcal{U}} \arrow[r, "b"] \arrow[d, "p"'] & \dot{\mathcal{U}} \arrow[d, "\dot{u}"] \\
P \arrow[r, "{r(\tau a, B)}"'] & \mathcal{C}
\end{tikzcd}$$

where $(\tau a, B) : P \to \mathcal{C} \downarrow u$ is the map induced by the universal property of $\mathcal{C} \downarrow u$ with $\tau a : \dot{u}a \to \{\mathrm{T}a\} = uB$. With this, we say that the dependent pairing operation is a map $\langle -, - \rangle : P_{r(\tau a, B)}\!\times_{\mathrm{T}} \dot{\mathcal{U}} \to \dot{\mathcal{U}}$, and we can define dependent sums analogously to dependent products:

Definition 4.2.6. A natural model $(\mathcal{C}, \mathcal{U}, \dot{\mathcal{U}})$ has depedent sums if there exists a pullback square of the form

$$\begin{tikzcd}
P_{r(\tau a,B)} \times_{\mathrm{T}} \dot{\mathcal{U}} \arrow[r, "\langle -,- \rangle"] \arrow[d, "(\mathrm{T}a,B)p"'] & \dot{\mathcal{U}} \arrow[d, "\mathrm{T}"] \\
\mathcal{U}\ {}_{\{-\}}\!\times_u \mathcal{U} \arrow[r, "\Sigma"'] & \mathcal{U}
\end{tikzcd}$$

Again all the data is given in terms of finite 2-limits, and so can be added to the sketch of natural models to obtain the theory of natural models iwth dependent sum types, and of course it can be combined with the sketch of natural models with dependent product types just as well.

Extensional identity types

Finally we cover extensional identity types, our treatment of which are directly adapted from [3, §2.3]. The formation rule for the identity types is given by

$$\frac{\Gamma \vdash A\, \mathcal{U} \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash \mathrm{Id}_A(a,b)\, \mathcal{U}}$$

which we model as a 1-cell $\mathrm{Id} : \dot{\mathcal{U}}\ {}_{\mathrm{T}}\!\times_{\mathrm{T}} \dot{\mathcal{U}} \to \mathcal{U}$ with $u\mathrm{Id} = u\mathrm{T}p = u\mathrm{T}q$. Similarly, the introduction rule is given by

$$\frac{\Gamma \vdash a : A}{\Gamma \vdash \mathsf{refl}(a) : \mathrm{Id}_A(a,a)\, \mathcal{U}}$$

and so is modeled by a 1-cell $\mathsf{refl} : \dot{\mathcal{U}} \to \dot{\mathcal{U}}$ with $\mathrm{T}\mathsf{refl} = \mathrm{Id}\delta$, where $\delta : \dot{\mathcal{U}} \to \dot{\mathcal{U}} \times_{\mathcal{U}} \dot{\mathcal{U}}$ is induced map by the pullback square in the following diagram:

$$\begin{tikzcd}
\dot{\mathcal{U}} \arrow[dr, dashed, "\delta"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \dot{\mathcal{U}}\ {}_{\mathrm{T}}\!\times_{\mathrm{T}} \dot{\mathcal{U}} \arrow[r, "p"] \arrow[d, "q"'] & \dot{\mathcal{U}} \arrow[d, "\mathrm{T}"] \\
& \dot{\mathcal{U}} \arrow[r, "\mathrm{T}"'] & \mathcal{U}
\end{tikzcd}$$

If the square

$$\begin{tikzcd}
\dot{\mathcal{U}} \arrow[r, "\mathsf{refl}"] \arrow[d, "\delta"'] & \dot{\mathcal{U}} \arrow[d, "\mathrm{T}"] \\
\dot{\mathcal{U}}\ {}_{\mathrm{T}}\!\times_{\mathrm{T}} \dot{\mathcal{U}} \arrow[r, "\mathrm{Id}"'] & \mathcal{U}
\end{tikzcd}$$

is a pullback, then for any terms $a, b : X \to \dot{\mathcal{U}}$ with $\mathrm{T}a = \mathrm{T}b$, if there is term $e : X \to \dot{\mathcal{U}}$ with $\mathrm{T}e = \mathrm{Id}(a, b)$, then $a = b$. In other words, given such a square, the natural model has extensional identity types.

Definition 4.2.7. A natural model $(\mathcal{C}, \mathcal{U}, \dot{\mathcal{U}})$ has extensional identity types if there exists a pullback square of the form

$$\begin{tikzcd}
\dot{\mathcal{U}} \arrow[r, "\mathsf{refl}"] \arrow[d, "\delta"'] & \dot{\mathcal{U}} \arrow[d, "\mathrm{T}"] \\
\dot{\mathcal{U}}\ {}_{\mathrm{T}}\!\times_{\mathrm{T}} \dot{\mathcal{U}} \arrow[r, "\mathrm{Id}"'] & \mathcal{U}
\end{tikzcd}$$

As with both dependent product and dependent sum types, all constructions used were in terms of finite 2-limits, and so we can add these to our sketch for natural models, possibly with dependent sum and product types, to obtain a theory for natural models with identity types.

4.2.1 Equivalence to categories with attributes

Already when they were introduced in [11], Dybjer showed how categories with families are in fact equivalent to categories with attributes. When passing from a category with families to a category with attributes, you simply need to recover the comprehension map, which we've already had to do in describing the common type constructors, and show that this comprehension map has the properties required of a comprehension category. In the other direction, you take the object of types to be the object of sections to the comprehension of a type, constructed in an appropriate sense and shown to be a fibration. In fact, we can in this way show that the theories Attr and NM are biequivalent as $\mathscr{F}$-categories, and so that their categories of models are always equivalent, whether you consider strict morphisms, pseudomorphisms, or lax morphisms.

To construct such a biequivalence requires constructing $\mathscr{F}$-functors $F : \mathrm{Attr} \rightleftarrows \mathrm{NM} : G$ together with natural $\mathscr{F}$-equivalences $FG \equiv 1_{\mathrm{NM}}$ and $GF \equiv 1_{\mathrm{Attr}}$. To find such functors, it suffices to construct a model F of Attr inside the $\mathscr{F}$-category NM, and a model G of NM inside Attr, and then to show that the model FG inside NM (resp. GF inside Attr) are equivalent to the ambient model of the theory. We will therefore split the proof up into four parts: the constructions of the models are the first two parts, and then the equivalences the second.

The first part, the construction of a model of Attr inside NM, we've already essentially done, since the obvious comprehension map is the one induced by the map $\pi : \{-\} \to u$ we defined earlier. Thus it only remains to show that this is indeed a comprehension category.

Lemma 4.2.8. *Working inside* NM*, Let $\chi : \mathcal{U} \to \mathcal{C}^{\to}$ be the map induced by the 2-cell $\pi : \{-\} \to u$ described above. Then*

$$\begin{tikzcd}
\mathcal{U} \arrow[rr, "\chi"] \arrow[dr, "u"'] & & \mathcal{C}^{\to} \arrow[dl, "\mathrm{cod}"] \\
& \mathcal{C} &
\end{tikzcd}$$

is a category with attributes, in the sense that it is the image of a finite $\mathscr{F}$-limit preserving functor $F : \mathrm{Attr} \to \mathrm{NM}$.

Proof. The only thing left to show is that χ preserves cartesian 2-cells, and since u is a discrete fibration, every 2-cell in $\mathcal{U}$ is cartesian. Therefore we must show that χ itself is cartesian, which is equivalent to showing that for every $f : A \to B : X \to \mathcal{U}$, the following square is a pulback:

$$\begin{tikzcd}
\{A\} \arrow[r, "\{f\}"] \arrow[d, "\pi A"'] & \{B\} \arrow[d, "\pi B"] \\
uA \arrow[r, "uf"'] & uB
\end{tikzcd}$$

To see that this is indeed the case, consider an arbitrary span $uA \xleftarrow{\sigma} \Gamma \xrightarrow{\tau} \{B\} : X \to \mathcal{C}$ with $uf \circ \sigma = \pi B \circ \tau$. We must show that there exists a unique map $\varphi : \Gamma \to \{A\}$ with $\pi A \circ \varphi = \sigma$ and $\{f\} \circ \varphi = \tau$, for which it suffices to find a map $\bar{\varphi} : \bar{\Gamma} \to \mathrm{V}A$ with $\dot{u}(\mathrm{V}f \circ \bar{\varphi}) = \tau$ and $u(\varepsilon A \circ \mathrm{T}\bar{\varphi}) = \sigma$, for some $\bar{\Gamma}$ with $\dot{u}\bar{\Gamma} = \Gamma$. To this end, let $\bar{\tau} : \bar{\Gamma} \to VB$ denote the lift of $\tau : \Gamma \to \dot{u}VB$ in $\dot{u}$. Similarly let $\bar{\sigma}$ denote the u-lift of $\sigma : \Gamma \to uA$. Now we have

$$u(\varepsilon B \circ \mathrm{T}\bar{\tau}) = \pi B \circ \tau = u(f \circ \bar{\sigma}),$$

so since u-lifts are unique by u being a discrete fibration, we have that $\varepsilon B \circ \mathrm{T}\bar{\tau} = f \circ \bar{\sigma}$. In particular, we see that $\bar{\sigma}$ is a map $\mathrm{T}\bar{\Gamma} \to A$, so if we take

$$\bar{\varphi} \coloneqq \mathrm{V}\bar{\sigma} \circ \eta\bar{\Gamma} : \bar{\Gamma} \to \mathrm{VT}\bar{\Gamma} \to \mathrm{V}A,$$

we can compute

$$\begin{aligned}
\dot{u}(\mathrm{V}f \circ \bar{\varphi}) &= \dot{u}(\mathrm{V}(f \circ \bar{\sigma}) \circ \eta\bar{\Gamma}) \\
&= \dot{u}(\mathrm{V}(\varepsilon B \circ \mathrm{T}\bar{\tau}) \circ \eta\bar{\Gamma}) \\
&= \dot{u}(\mathrm{V}\varepsilon B \circ \mathrm{VT}\bar{\tau} \circ \eta\bar{\Gamma})) \\
&= \dot{u}(\mathrm{V}\varepsilon B \circ \eta\mathrm{V}B \circ \bar{\tau}) \\
6 = \dot{u}\bar{\tau} = \tau &
\end{aligned}$$

and

$$u(\varepsilon A \circ \mathrm{T}\bar{\varphi}) = u(\varepsilon A \circ \mathrm{TV}\bar{\sigma} \circ \mathrm{T}\eta\bar{\Gamma}) = u(\bar{\sigma} \circ \varepsilon\mathrm{T}\bar{\Gamma} \circ \mathrm{T}\eta\bar{\Gamma}) = u\bar{\sigma} = \sigma,$$

as desired. Therefore we have that $\varphi \coloneqq \dot{u}\bar{\varphi}$ yields $\{f\} \circ \varphi = \dot{u}(\mathrm{V}f \circ \bar{\varphi}) = \tau$ and $\pi A \circ \varphi = u(\varepsilon A \circ \mathrm{T}\bar{\varphi}) = \sigma$, showing that φ has the desired properties. Uniqueness of φ follows from the uniqueness of adjuncts and $\dot{u}, u$-lifts.

Since χ is cartesian, it follows that

$$\begin{tikzcd}
\mathcal{U} \arrow[rr, "\chi"] \arrow[dr, "u"'] & & \mathcal{C}^{\to} \arrow[dl, "\mathrm{cod}"] \\
& \mathcal{C} &
\end{tikzcd}$$

is a model of Attr, as we wanted to show. ☐

Lemma 4.2.9. *Working in* Attr*, the map* $\dot{u}$ *as defined in section 4.1.1 is a discerete fibration, the map* T *has a right adjoint* V*, and the diagram*

$$\begin{tikzcd} \dot{\mathcal{U}} \arrow[rr, "\mathrm{T}"'] \arrow[rr, phantom, bend left=20, "\perp"] \arrow[dr, "\dot{u}"'] & & \mathcal{U} \arrow[ll, bend right, "\mathrm{V}"'] \arrow[dl, "u"] \\ & \mathcal{C} & \end{tikzcd}$$

is a natural model in Attr*, in the sense that it is the image of a finite* $\mathscr{F}$*-limit preserving functor* $G : \mathrm{NM} \to \mathrm{Attr}$.

Proof. We first show that $\dot{u}$ is a discerete fibration, for which it suffices to show that all $\dot{u}$-vertical maps (those maps $\alpha : a \to b : X \to \dot{\mathcal{U}}$ for which $\dot{u}\alpha$ is identity) are identity, since $\dot{u}$ is a fibration by lemma 4.1.7. To see that this holds, suppose $\alpha : a \to b : X \to \dot{\mathcal{U}}$ is a vertical map. Then $u\mathrm{T}\alpha = \dot{u}\alpha$ is the identity, so since u is a discrete fibration, $\mathrm{T}\alpha$ is identity as well. Thus we have $\mathrm{T}a = \mathrm{T}b$ and $\mathrm{T}\alpha = 1_{\mathrm{T}a}$. Now we see by the construction of T that α must be identity, as desired.

For the right adjoint V of T, we follow the intuition of categories with families. Let $\varepsilon : \delta \to 1_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}$ denote the cartesian lift of $\pi : \{-\} \to u$. Then we have the following diagram:

$$\begin{tikzcd}
\{-\} \arrow[drr, bend left, "1_{\{-\}}"] \arrow[dr, dashed, "\tau\mathrm{V}"] \arrow[ddr, bend right, "1_{\{-\}}"'] & & \\
& \{\delta\} \arrow[r, "\{\varepsilon\}"] \arrow[d, "\pi\delta"'] & \{-\} \arrow[d, "\pi"] \\
& \{-\} \arrow[r, "\pi"'] & u
\end{tikzcd}$$

Since the bottom-right square is a pullback, there exists a section $\tau\mathrm{V}$ of $\pi\delta$, which then induces a term $\mathrm{V} : \mathcal{U} \to \dot{\mathcal{U}}$. Furthermore, since $\mathrm{TV} = \delta$, we see that the 2-cell $\varepsilon : \delta \to 1_{\mathcal{U}}$ is also a 2-cell $\mathrm{TV} \to 1_{\mathcal{U}}$, which will be the counit of the adjunction. Furthermore, we have that $\dot{u}\mathrm{V} = u\mathrm{TV} = u\delta = \{-\}$. Since for natural models the unit $\eta : 1_{\dot{\mathcal{U}}} \to \mathrm{VT}$ plays the role of the substitution induced by extending by the variable, we want η to be the lift of $\tau : u\mathrm{T} \to \{\mathrm{T}\} = \dot{u}\mathrm{VT}$. Let $\bar{\tau} : a \to \mathrm{VT}$ denote this lift. To allow taking $\eta := \bar{\tau}$ to be the unit, we must that $a = 1_{\dot{\mathcal{U}}}$. Note that $\mathrm{T}\bar{\tau}$ is the u-lift of $\tau : u\mathrm{T} \to u\mathrm{TVT} = u\delta\mathrm{T}$, so since $u(\varepsilon\mathrm{T} \circ \mathrm{T}\bar{\tau}) = \pi\mathrm{T} \circ \tau = 1_{\dot{u}}$ and u-lifts are unique, we have that $\varepsilon\mathrm{T} \circ \mathrm{T}\bar{\tau} = 1_{\mathrm{T}}$. In particular, it follows that $a = 1_{\dot{\mathcal{U}}}$, and so we can take $\eta = \bar{\tau}$. This also immediately gives us the one of the triangle equations: $\varepsilon\mathrm{T} \circ \mathrm{T}\eta = 1_{\mathrm{T}}$. For the other equation, $\mathrm{V}\varepsilon \circ \eta\mathrm{V} = 1_{\mathrm{V}}$, it suffices to show that $\dot{u}\mathrm{V}\varepsilon \circ \dot{u}\eta\mathrm{V} = 1_{\dot{u}\mathrm{V}}$, since $\dot{u}$ is a discerete fibration. But we have

$$\dot{u}\mathrm{V}\varepsilon \circ \dot{u}\eta\mathrm{V} = \{\varepsilon\} \circ \tau\mathrm{V} = 1_{\{-\}} = 1_{\dot{u}\mathrm{V}}$$

by the construction of V and η. Therefore ε and η are the counit and unit respectively of an adjunction $\mathrm{T} \dashv \mathrm{V}$, as desired.

We conclude that the diagram

$$\begin{tikzcd}
\dot{\mathcal{U}} \arrow[rr, "\mathrm{T}"'] \arrow[rr, phantom, shift left=2, "\perp"] \arrow[dr, "\dot{u}"'] & & \mathcal{U} \arrow[ll, bend right, "\mathrm{V}"'] \arrow[dl, "u"] \\
& \mathcal{C} &
\end{tikzcd}$$

forms a natural model in Attr, with $G : \mathrm{NM} \to \mathrm{Attr}$ the corresponding $\mathscr{F}$-functor. ☐

Remark 4.2.10. Most of this proof does not rely on discreteness of u, except in showing that $\dot{u}$ is also a discrete fibration. Therefore it could perhaps be adapted into an alternative proof of equivalence between the generalized categories of families of [8] and comprehension categories to the one presented in [9], though this would need more thorough checking.

Remark 4.2.11. Though not used explicitly here, note that ε as defined in this proof is the counit of a comonad. In particular, δ is a so-called weakening-contraction comonad, which are introduced and shown equivalent to comprehension categories in [17].

We are now equipped to show that $FG \cong 1_{\mathrm{NM}}$ and $GF \cong 1_{\mathrm{Attr}}$.

Theorem 4.2.12. *F and G as defined above are weak inverses.*

Proof. We first show that $GF \cong 1_{\mathrm{Attr}}$. In fact, $GF = 1_{\mathrm{Attr}}$, since GF fixes the type fibration $u : \mathcal{U} \to \mathcal{C}$ and, using χ' to denote the comprehension map of GF and $\pi' : \{-\}' \to u$ the corresponding 2-cell, we have that $\pi' = u\varepsilon = \pi$ by the definition of ε, so $\chi' = \chi$.

For $FG \cong 1_{\mathrm{NM}}$, we will show $\mathrm{T} : \dot{\mathcal{U}} \to \mathcal{U}$ is a limiting cone to the diagram defining $\dot{\mathcal{U}}'$, i.e. that $\dot{\mathcal{U}}$ is an object of sections to typing projections $\pi A : \{A\} \to uA$. Then we have from the uniqueness of limits that $\dot{\mathcal{U}} \cong \dot{\mathcal{U}}'$ in a way coherent with all the structure of a natural model, which shows that $FG \cong 1_{\mathrm{NM}}$. We first show that T indeed provides such a cone: that is, we must show that there is a map $\tau : u\mathrm{T} \to \{\mathrm{T}\}$ which is a section to $\pi\mathrm{T}$. But we already constructed this map τ earlier. To show that this cone is limiting, we appeal to proposition 3.2.4. Thus we must show that for every type $A : X \to \mathcal{U}$ and section $s : uA \to \{A\}$ of $\pi A : \{A\} \to uA$, there exists a unique map $\varphi_s : X \to \dot{\mathcal{U}}$ with $\mathrm{T}a = A$ and $\tau a = s$, and for every additional type $B : X \to \mathcal{U}$, section $t : uB \to \{B\}$ of $\pi B : \{B\} \to uB$, and 2-cell $h : A \to B$ with $\{h\} \circ s = t \circ uh$, there exists a unique 2-cell $\varphi_h : \varphi_s \to \varphi_b$ with $h = \mathrm{T}\varphi_h$.

We start by constructing $\varphi_s : X \to \dot{\mathcal{U}}$. Let $\bar{s} : a \to \mathrm{V}A$ denote the $\dot{u}$-lift of s. Since

$$u(\varepsilon A \circ \mathrm{T}\bar{s}) = \pi A \circ u\mathrm{T}\bar{s} = \pi A \circ s = 1_{uA}$$

has the unique u-lift 1_A, we must have that the domain of $\mathrm{T}\bar{s}$ is A, i.e. that $\mathrm{T}a = A$. It remains to show that $\tau a = s$, for which we first show that $\eta a = \bar{s}$:

$$\begin{aligned}
\bar{s} &= 1_{\mathrm{VT}a} \circ s \\
&= (\mathrm{V}\varepsilon \circ \eta\mathrm{V})\mathrm{T}a \circ \bar{s} \\
&= \mathrm{V}\varepsilon\mathrm{T}a \circ \eta\mathrm{VT}a \circ \bar{s} \\
&= \mathrm{V}\varepsilon\mathrm{T}a \circ \mathrm{VT}\bar{s} \circ \eta a \\
&= \mathrm{V}(\varepsilon\mathrm{T}a \circ \mathrm{T}\bar{s}) \circ \eta a \\
&= 1_{\mathrm{VT}a} \circ \eta a \\
&= \eta a
\end{aligned}$$

With this, we see that $\tau a = \dot{u}\eta a = \dot{u}\bar{s} = s$, so taking $\varphi_s = a$ gives the desired map with $\tau\varphi_s = a$. Furthermore, it is unique, since any other map $a' : X \to \dot{\mathcal{U}}$ satisfying $\tau a' = s$ has that $\dot{u}\eta a' = \dot{u}\eta\varphi_s$ and $\mathrm{T}a' = A$, so by the uniqueness of u-lifts, $\eta a' = \eta\varphi_s$.

Now, suppose we are given sections $s : uA \to \{A\}$ and $t : uB \to \{B\}$ and a map $h : A \to B$ with $\{h\} \circ s = t \circ uh$. We must construct a unique map $\varphi_h : \varphi_s \to \varphi_t$. Using the same notation as above, let $\bar{s} : \varphi_s \to A$ and $\bar{t} : \varphi_t \to B$ denote the $\dot{u}$-lifts of s and t respectively. Now let $\bar{h} : a \to \varphi_t$ denote the $\dot{u}$-lift of $uh : \dot{u}\varphi_s \to \dot{u}\varphi_t$, so that $\dot{u}\bar{h} = uh$. Then we have that

$$u\mathrm{T}\bar{h} = uh, \circ\eta\varphi_s) = \{h\} \circ s = t \circ uh = u(\mathrm{T}\eta\varphi_t \circ h),$$

and since u is a discrete fibration, this means that $\mathrm{T}\bar{h} = h$. Moreover, we have that

$$\dot{u}(\eta\varphi_t \circ \bar{h}) = t \circ uh = \{h\} \circ s = \dot{u}(\mathrm{V}h \circ \eta\varphi_s),$$

so since $\dot{u}$ is a discrete fibration and $\mathrm{V}h : \mathrm{V}A \to \mathrm{VT}\varphi_t$ has the same codomain as $\eta\varphi_t$, we see that $\eta\varphi_t \circ \bar{h} = \mathrm{V}h \circ \eta\varphi_s$. In particular, we have that $\bar{h} : \varphi_s \to \varphi_t$ and $\mathrm{T}\bar{h} = h$. Thus we can take $\varphi_h = \bar{h}$. To see that this is unique, suppose we were given another 2-cell $h' : \varphi_s \to \varphi_t$ with $\mathrm{T}h' = h$. Then

$$\dot{u}h' = u\mathrm{T}h' = uh = u\mathrm{T}\varphi_h = \dot{u}\varphi_h : \dot{u}\varphi_s \to \dot{u}\varphi_t,$$

so from the uniqueness of $\dot{u}$-lifts, we see that $h' = \varphi_h$.

Thus we have shown that (T, τ) forms a limiting cone for the object of sections of type projections $\pi : \{-\} \to u$, which gives us the coherent isomorphism $\dot{\mathcal{U}} \cong \dot{\mathcal{U}}'$ we required. From this, we conclude that $FG \cong 1_{\mathrm{NM}}$.

Since we have shown that $FG \cong 1_{\mathrm{NM}}$ and $GF \cong 1_{\mathrm{Attr}}$, we have exactly that F and G constitute an equivalence of NM and Attr. □

Corollary 4.2.13. *The 2-categories of natural models and categories with attributes are equivalent.*

4.3 Multimode and Modal type theories

Multimode dependent type theory (MTT) was introduced in [14] to provide a general family of type theories with different modalities. MTT is parameterized by a 2-category $\mathcal{M}$, a so-called mode theory, the objects of which are called *modes*, the 1-cells are called *modalities*, and the 2-cells are called *operations*. The goal of this section is to show how the semantics presented in [14] can be seen as the models of a finite $\mathscr{F}$-limit sketch. A interesting feature of the semantics of MTT which diffrentiates it from most other type theories, is that it more or less consists of a family of type theories, one at each mode, together with rules connecting the different type theories with the modalities and operations. as dictated by the mode theory. This is reflected in the semantics presented in [14] by having a family of natural models indexed by the modes, together with functors and natural transformations between the models corresponding to the modalities and operations. Because the semantics requires multiple categories of contexts, it is difficult to fit neatly into the functorial semantics of [29] or the framework of [8], which single out the context judgement, and so the category of contexts, as a primitive notion.

To show that the semantics of [14, §5] can be recovered as an finite $\mathscr{F}$-limit theory, we will presen the important features of *op. cit.* and show that these can be constructed by finite 2-limits. We begin with the defintions of a *modal context structure* and *modal natural models*.

Definition 4.3.1. Let $\mathcal{M}$ be a mode theory. A *modal context structure* is a 2-functor $\mathcal{C}[-] : \mathcal{M}^{coop} \to \mathbf{Cat}$.

A *modal natural model* on a modal context structure $\mathcal{C}[-]$ is an family of maps of discrete fibrations $\mathrm{T}_m : \dot{\mathcal{U}}[m] \to \mathcal{U}[m]$ over $\mathcal{C}[m]$, indexed over the modes $m \in \mathcal{M}$, such that for each modality $\mu : m \to n$, the map $\mathcal{C}[\mu]^*\mathrm{T}_m : \mathcal{C}[\mu]^*\dot{\mathcal{U}}[m] \to \mathcal{C}[\mu]^*\mathcal{U}[m]$ is a representable map of discrete fibrations.

Remark 4.3.2. In [14], they require the category of contexts to have a terminal object, corresponding to the empty context. Since we have elided such a requirement from the start, we will continue to do so now. Note, however, that it is trivial to add the requirement by simply adding a terminal object 1 in the sketch and right adjoints to the unique maps $\mathcal{C}[m] \to 1$.

We will describe a sketch for modal natural models directly, since the $\mathscr{F}$-sketch for modal context structures is just $((\mathcal{M}^{coop})^-, \emptyset)$, the sketch with underlying $\mathscr{F}$-category $\mathcal{M}^{coop}$ with only identities tight and with no cones.

Definition 4.3.3. The sketch for modal natural models consists of:

- For each mode m, a diagram

$$\begin{tikzcd} \dot{\mathcal{U}}_m \arrow[rr, squiggly, "\mathrm{T}_m"'] \arrow[dr, "\dot{u}_m"'] & & \mathcal{U}_m \arrow[dl, "u_m"] \\ & \mathcal{C}_m & \end{tikzcd}$$

  with cones and additional maps making $u_m, \dot{u}_m$ discrete fibrations.

- For each modality $\mu : m \to n$, squares

$$\begin{tikzcd} \text{🔒}_\mu^* \mathcal{U}_m \arrow[r] \arrow[d, "\text{🔒}_\mu^* u_m"'] & \mathcal{U}_m \arrow[d, "u_m"] \\ \mathcal{C}_n \arrow[r, "\text{🔒}_\mu"'] & \mathcal{C}_m \end{tikzcd} \qquad \begin{tikzcd} \text{🔒}_\mu^* \dot{\mathcal{U}}_m \arrow[r] \arrow[d, "\text{🔒}_\mu^* \dot{u}_m"'] & \dot{\mathcal{U}}_m \arrow[d, "\dot{u}_m"] \\ \mathcal{C}_n \arrow[r, "\text{🔒}_\mu"'] & \mathcal{C}_m \end{tikzcd}$$

  and an adjunction $\text{🔒}_\mu^* \mathrm{T}_m \dashv \mathrm{V}_\mu : \text{🔒}_\mu^* \mathcal{U}_m \to \text{🔒}_\mu^* \dot{\mathcal{U}}_m$, as well as cones and diagrams making the two squares pullbacks and $\text{🔒}_\mu^* \mathrm{T}_m : \text{🔒}_\mu^* \dot{\mathcal{U}}_m \to \text{🔒}_\mu^* \mathcal{U}_m$ the map induced by $\mathrm{T}_m : \dot{\mathcal{U}}_m \to \mathcal{U}_m$.

- For each operation $\alpha : \mu \to \nu$, a 2-cell $\text{🔑}_\alpha : \text{🔒}_\nu \to \text{🔒}_\mu$.

Additionally we add equalites so that the assignments $m \mapsto \mathcal{C}_m$, $(\mu : m \to n) \mapsto \text{🔒}_\mu$, and $(\alpha : \mu \to \nu) \mapsto \text{🔑}_\alpha$ together constitute a strict 2-functor from $\mathcal{M}^{coop}$.

Following the same structure as in section 4.2, we see that this produces a finite limit $\mathscr{F}$-sketch, and so also an induced finite limit $\mathscr{F}$-theory, which we will denote by MTT.

Using the same techinques for adding type formers as in section 4.2, we can now add type formers to the different mode theories.

Remark 4.3.4. With this definition, we see that, for every modality $\mu : m \to n \in \mathcal{M}$, we have a model of NM inside MTT with $\mathcal{U} \coloneqq \text{🔒}_\mu^* \mathcal{U}_n$, $\dot{\mathcal{U}} \coloneqq \text{🔒}_\mu^* \dot{\mathcal{U}}_n$, $\mathrm{T} \coloneqq \text{🔒}_\mu^* \mathrm{T}_n$, and $\mathrm{V} \coloneqq \mathrm{V}_\mu$.

Remark 4.3.5. Since we often want to consider type theories with additional type formers, and not just the bare bones type theories given by natural models and comprehension categories, it would be interesting in the future to see if we can specify multimode type theories in terms of some tensor product of theories with the mode theory $\mathcal{M}$, allowing you to specify a base type theory at each mode $m \in \mathcal{M}$. Thus could also allow for a more modular approach to the semantics of multimode type theories, easily allowing a different base semantics, such as for example comprehension categories, hopefully without having to reprove all the theorems we would have for the semantics presented here in terms natural models.

As with the other type formers, we can define the modal types by adding, for each modality

$\mu : n \to m \in \mathcal{M}$, a pullback square as follows:

$$\begin{tikzcd}
\text{🔒}_\mu^* \dot{\mathcal{U}}_n \arrow[r, "\mathrm{mod}_\mu"] \arrow[d, "\text{🔒}_\mu^* u_m"'] & \dot{\mathcal{U}}_m \arrow[d, "u_m"] \\
\text{🔒}_\mu^* \mathcal{U}_n \arrow[r, "\mathrm{Mod}_\mu"'] & \mathcal{U}_m
\end{tikzcd}$$

Remark 4.3.6. This definition is stronger than the one presented in [14], where it is shown that these modal types correspond to *dependent right adjoints*. Gratzer et al. [14] also provides a version of multimode type theory with a slgithyl weaker notion of modal types, modeled similarly to how [3] models intentional identity types. Much like with intentional identity types, I have not had the time to figure out how those approaches fit into this framework. Thus we will take the slightly stronger notion, which however still captures many examples of modal types.

Chapter 5

Conclusion

The goal of this thesis was to show how several of the categorical models of dependent type theory can be captured in terms of some sort of 2-dimensional algebraic theory. Given that many of these models can be phrased in terms of fibred categories, and fibrations are specific instances of finite 2-limits, it seemed like a good notion of 2-dimensional theory in this case was that of finite 2-limit theories, generalizing the finite limit theories of categorical universal algebra. To further make the homomorphisms of models to coincide with those already in literature, we took the step from **Cat**-enriched finite limit theories to $\mathscr{F}$-enriched ones, following [2], as these provide fine control over which operations of the theory are preserved strictly, and which are preserved only up to isomorphism.

After recounting the basics of $\mathscr{F}$-enriched limit theories we required, we showed how both comprehension categories and natural models were describable in the framework. We furthermore showed how to model the basic type constructors, following the general methods of [8], though rather than working in **Cat** directly, we were constructing the limits in abstract finite limit $\mathscr{F}$-sketches. We ended by showing how this framework can capture the multimode type theory of [14], which distinguishes the framework from the categories with representable maps of [29] and judgemental theories of [8].

- While we somewhat compared finite finite limit $\mathscr{F}$-theories with similar frameworks, such as categories with representable maps [29] and judgemental theories [8], the comparisons have been superficial, and mostly focused on the case of multimode type theory. However, we have not explored how the different frameworks are related.

  For example, given that the intended interpretation for categories with representable maps is given in terms of discrete fibrations, one can intuitively construct, given any such category, a finite limit $\mathscr{F}$-theory, such that their models coincide: simply consider the sketch embeds the given category with representable maps, such every map into the terminal object is made into a discrete fibration and any representable map is made into a left-adjoint. Of course, that the models of the induced $\mathscr{F}$-theory coincide with the models of the category with representable maps needs to be checked more carefully, and one might naturally ask whether this assignment is functorial, and if so, whether this functor from representable map categories to $\mathscr{F}$-theories commutes with the models.

  Similarly, one might expect that the models of any $\mathscr{F}$-theory describing a type theory yields a judgemental theory. This question can be further extended to asking whether the models of *any* $\mathscr{F}$-theory yields a judgemental theory. Following this comparison with judgemental theories, one discovers a deficiency with $\mathscr{F}$-theories: they do not allow for modeling contraviant operations. Similarly, it is impossible (to my knowledge) to model fibred Heyting algebras, which means that we cannot easily construct e.g. a theory corresponding to Lawvere's hyperdoctrines when compared to judgemental theories, something that is used by [8, §4] to show that first-order logic falls into the framework of judgemental theories.

- We have in this thesis only really made use of the basic features of $\mathscr{F}$-theories and 2-dimensional theories in general. In particular, while we have shown that type theories induce $\mathscr{F}$-theories, we have not made use of this fact to study their categories of models. This is closely connected with the 2-dimensional generalizations of locally finitely presentable and accessible categories, which are studied in [5, 10].

- In section 4.2.1 we showed how the classical equivalence between natural models and categories with attributes adapts to the setting of $\mathscr{F}$-theories. A more general version of the theorem is proven in [9], wherein a generalized notion of categories with families, allowing for general Grothendieck fibrations of types and terms, rather than only discrete ones, is proven equivalent to general comprehension categories. It would be interesting to see how their proof adapts to the setting of $\mathscr{F}$-theories, and whether one thus can show that theory of generalized categories with families is equivalent to the one for comprehension categories, in a way compatible with the semantics. This would partially simplify the proof, in the sense that you get equivalence of the 2-categories of models "for free", whether the morphisms are strict, pseudo, lax, or even colax.

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