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NEW FOUNDATIONS

A SURVEY OF QUINE'S SET THEORY

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Foreword

With this paper, I hope to provide a thorough introduction to **NF**, a set theory contending with **ZF** as a basis for mathematics.

The amount of literature available on **NF** is not so extensive, but working it through will undoubtedly cost more time (and perhaps headaches too) than reading my condensed survey of that material.

I'd like to express my thanks to Dr. Doets for drawing my attention to this interesting set theory when I got stuck in a subject concerning model-theoretic algebra, and for helping me to struggle through the literature.

I'd also like to thank Prof. Boffa for sending me some material I otherwise would have lacked.

Finally, I hereby express my deep regards to Prof. Quine for inventing the theory, so that we all could take notice of its beauty.

Contents

Introduction

The comprehension principle which asserts that the set $\{ x \mid \varphi \}$ exists for all formulae φ of the first order predicate calculus with binary relation $\in$ expressing membership, leads to three well-known antinomies:

- Cantor himself recognized the fact that the universal set $V=\{ x \mid x=x \}$ couldn't exist as it would be equal to its power set, which contradicts the fact that there can be no surjection of any set x onto Px;
- Burali-Forti derived a similar contradiction by considering the set OR of all ordinal numbers, well-ordered by the canonical $\leq$. The ordinal number of OR would be one exceeding all ordinal numbers;
- Russell derived a less intricate, purely logical, contradiction by considering the set $A=\{ x \mid x \notin x \}$. It follows that $A \in A$ if and only if $A \notin A$. Actually this contradiction is distilled from the proof involving Cantor's antinomy where the set $\{ x \mid x \notin f x \}$ is considered; in the supposition about the existence of V, f is taken to be the identity map.

To rid set theory of these apparent inconsistencies numerous proposals have been made, of which the system **ZF** (or Zermelo-Fraenkel set theory) obviously has been accepted by mathematicians all over the world as the basis for their work (sometimes they supplement it with the axiom of choice -or a weakened form of it- to be able to derive their results).

In **ZF** the antinomies are avoided by modification of the comprehension principle into the separation axiom: given a set b the set $\{ x \in b \mid \varphi \}$ exists. As we lose a lot of standard set-theoretical operations, these have to be postulated separately: the axiom of pairing, and the axioms of sum and power set. Of course, **ZF** also comprises the axiom of extensionality stating that sets are determined by their elements. To generate enough of mathematics, the existence of at least one infinite set is provided for by the axiom of infinity. Fraenkel spotted a shortcoming and added the axiom of substitution; and lastly one has the option of considering only regular sets by adding the axiom of regularity.

The three above-mentioned antinomies are excluded in **ZF** because the sets involved cannot be formed by means of the axioms.

Another way to keep the antinomies out is contained in (a modern variant of) Russell's type theory **TT**. In contradistinction with most other set theories, the underlying logic is many-sorted: it contains a denumerable hierarchy of typed variables x_i, y_i, z_i, ... (for each $i \in \omega$). The formulae of **TT** are built by logical combinations of modified forms of atomic formulae: only atoms of the form $x_i \in y_{i+1}$ or $x_i = y_i$ are admitted in the language. Formulae which are thus formed are called stratified.

The axioms of **TT** are the following axioms of extensionality and comprehension (for all $i \in \omega$):

EXT_{i+1}: $\forall x_{i+1}\, y_{i+1} [\forall z_i (z_i \in x_{i+1} \leftrightarrow z_i \in y_{i+1}) \to x_{i+1} = y_{i+1}]$

COMP_{i+1}: $()\ \exists y_{i+1} \forall x_i (x_i \in y_{i+1} \leftrightarrow \varphi)$, for stratified φ not containing y_{i+1} free.

(the initial pair of parentheses stands for universal quantification over the variables different from x which occur free in φ)

Models of type theory take the form $(M_0, M_1, M_2, ..., \in_M)$ where the M_i's are sets, and $\in_M \subseteq \bigcup\{ M_i \times M_{i+1} \mid i \in \omega \}$. M_0 has to be non-empty because we want to be able to use all the theorems of general many-sorted first order logic, in particular the scheme $\forall x_0\, \varphi \rightarrow \exists x_0\, \varphi$. The elements of M_0 are <u>individuals</u>: they contain no elements.

The effect of all this is that sets (except individuals) contain only members of next lower type and thus the antinomies are excluded by the fact that $x \notin x$ is a meaningless formula, V cannot be formed and ordinals are reproduced from type to type: the set of ordinals of a specific type has an ordinal number of a type one higher so it cannot be equated with that of its elements by means of <.

As it is apparent that $(B, PB, P^2B, ..., \in)$ is a model of **TT** for every non-empty set B, **TT** is consistent, and for adequacy purposes can be consistently enlarged by adding axioms of choice for all levels and an axiom of infinity stating in effect that there exist infinitely many objects of type 0 (by defining the set of finite sets with elements of type 0 and requiring that $\{ x_0 \mid x_0 = x_0 \}$ is not contained in it).

Though being consistent, **TT** has some serious drawbacks which some people may even find repugnant: notions reappear at each level from some point on upwards; examples of this phenomenon are the quasi-universal sets $V_{i+1} = \{ x_i \mid x_i = x_i \}$ and the quasi-empty sets $\Lambda_{i+1} = \{ x_i \mid x_i \neq x_i \}$, as are the natural numbers reappearing in each type from level 1 upwards. Even arithmetic comes to us in various types.

Moreover, **TT** doesn't admit the Boolean algebra of sets: unions and intersections are only defined for sets of equal type; the complement -x ceases to comprise all non-members of x, but comes to comprise only those non-members of x of next lower type.

Mathematicians are not accustomed to keeping track of the levels sets are in, and when one drops the subscripts on variables for better readability, assuming the subscripts to be big enough to allow existence of the sets under consideration, one has to be very careful to avoid unstratified formulae: whereas for instance $x \in y$ and $y \in x$ are perfectly stratified, their conjunction is not.

This reduplication of notions in various levels led Quine in [37] to the definition of his "New Foundations" or **NF** for short. In **NF**, the type differences in **TT** are dropped so that the language is the same as that of **ZF**; it is only in the comprehension axiom-scheme that we retain the stratification proviso. For this purpose, a formula φ is said to be stratified if it is *possible* to assign types to all variables in φ so that it becomes a formula of **TT**. Of course, bound variables may be renamed if necessary. So $x \in y$ and $\forall x (y \in x) \wedge x \in y$ are stratified, but $x \in x$ and $x \in y \wedge y \in x$ are not.

The axioms of **NF** are thus:

EXT: $\forall xy [\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y]$

COMP: $() \exists y \forall x (x \in y \leftrightarrow \varphi)$, for stratified φ not containing y free

Note that EXT and all instances of COMP are stratified .

Like **ZF**, **NF** is not known to be (in)consistent (not even relative to some other set theory), although it may appear that Cantor's antinomy can be derived in it as **NF** incorporates a universal set. That this is not the case is one of the peculiar facts in **NF**, at least for someone with **ZF**-glasses on.

The purpose of this paper is to shed some light on the partial and relative consistency results obtained in the last fifty years regarding **NF**.

The first chapter will make the reader familiar with the basic notions and operations in **NF**, which are used in chapter 2 to derive the remarkable fact -proved by Specker- that the (full) axiom of choice does not hold in **NF**, whereas the axiom of infinity (thus) does. Chapter 3 comprises the partial consistency results due to Jensen and Grishin (consistency of **NFU** and $\mathbf{NF}_3$), as well as the axiomatic reduction of **NF** to $\mathbf{NF}_4$ also due to Grishin.

Lastly, chapter 4 is about relative consistency results derived by means of the 'permutation method', in particular some relatively consistent facts about the comparison of certain finite cardinal numbers.

To keep things readable, I will not always be too formal in the definitions and derivations; the reader who wishes to check that everything has a strictly formal equivalent will easily be able to do so.

All the definitions and derivations will apply to **NF** when not explicitly indicated otherwise.

One final remark here: while it is perfectly possible to *define* equality in **NF** (as in fact Quine originally did), this brings with it its own complexities, see for instance [Cocchiarella 76], and we will assume equality to be a basic notion of the underlying logic, for which the identity axioms hold.

1. Preliminaries

§1. basic notions

Comprehension assures the existence of a set whose elements are just the ones that fulfil a stratified φ, and extensionality assures that there is exactly one such set. It is customary to denote this set by a new term $\{ x \mid \varphi \}$ which is called an abstraction. Note that in **NF** meaningfulness of $\{ x \mid \psi \}$ is only postulated for stratified ψ, but we will see shortly that some unstratified formulae also determine sets.

It is convenient to use these new terms in our language as a kind of abbreviations, but then of course we have to prescribe how to obtain abstractionless transforms of the formulae we write down. This is done in the following way (when writing down a formula with an abstraction, the abstraction will always occur in one of the following three ways, never alone):

$$y \in \{ x \mid \varphi \} \quad \equiv \quad \varphi[x/y] \ \ (\text{or}: \exists z [y \in z \wedge \forall x (x \in z \leftrightarrow \varphi)])$$
$$\{ x \mid \varphi \} \in y \quad \equiv \quad \exists z [z \in y \wedge \forall x (x \in z \leftrightarrow \varphi)]$$
$$y = \{ x \mid \varphi \} \quad \equiv \quad \forall x (x \in y \leftrightarrow \varphi)$$

Furthermore, when φ and ψ are (possibly unstratified) equivalent formulae, extensionality assures that $\{ x \mid \varphi \} = \{ x \mid \psi \}$ (when these sets exist). So we may freely switch between equivalents in our derivations.

From the above-mentioned abstractionless transforms it is clear that when $\{ x \mid \varphi \}$ is assigned a type one higher than that of x (so when φ has no free variables besides x, an arbitrary type), these transforms are stratified whenever φ is. This will help us checking stratification without having to transform formulae to perhaps monstrously long abstractionless ones.

When φ has at most x as a free variable, $\{ x \mid \varphi \}$ has no free variables and so is a constant. In type theory these constants 'reappear' in all levels from some point on upwards; in **NF** we say they are typeless and may be assigned arbitrary type. Two or more occurrences of the same constant in a formula may even be assigned different types to achieve stratification.

When φ has free variables besides x, $\{ x \mid \varphi \}$ obviously is an operation on these extra variables. In connection with these operations, it is convenient to speak about the relative type of terms. This is in fact a terminology borrowed from **TT**, but as the type subscripts have disappeared in **NF**, we can no longer talk about the exact type of sets.

Thus $\{x\}$ has a type one higher than that of x; $x \cup y = \{ z \mid z \in x \vee z \in y \}$ will have the same type as that of x and y, which have to be of equal type for stratification purposes, although subsequent substitution may circumvent this (see below).

We write Λ for the empty set $\{ x \mid x \neq x \}$ (or even $\{ x \mid \bot \}$ where $\bot$ is taken to be a false stratified sentence, for instance $\bot \equiv \forall x\,(x \neq x)$).
The universe is $V = \{ x \mid x = x \}$ (or even $V = \{ x \mid \neg\bot \}$).
One easily sees that **NF** admits the complete Boolean algebra of sets, while **ZF** does so too except for the complement $-x = \{ z \mid z \notin x \}$. Likewise, other set-theoretical operations like $\bigcap$ (intersection), $\bigcup$ (sum), $\{.\}$ (singleton), $\{.,.\}$(unordered pair) and P (power set) are seen to be given by stratified formulae.
Note that by instantiation we are able to form sets determined by unstratified formulae. For instance, $x \cup y$ exists for all x and y; by taking $\{x\}$ for y we get the set $x \cup \{x\} = \{ z \mid z \in x \vee z = x \}$.
One might fear to be able to derive Russell's antinomy by keen substitution, but apparently this is impossible: $x \in x$ does not seem to be equivalent to the conjunction or disjunction of two stratified formulae.
When we have available an operation F on variables $\bar{x}$, an obvious generalization of abstraction is

$$\{ F\bar{x} \mid \varphi \} = \{ z \mid \exists \bar{x}\,(z = F\bar{x} \wedge \varphi) \}$$

This notation will often be used for brevity. Of course, $z = F\bar{x} \wedge \varphi$ will have to be stratified in order to form the set by comprehension.
An important operation formed by this principle is $P_1 x = \{ \{y\} \mid y \in x \}$. It participates for instance in the definition of the set of natural numbers Nn (the finite cardinals of Frege-Russell type): 0 and 1 are given by $\{\Lambda\}$ and $P_1 V$ respectively (the sets of sets with 0 elements and 1 element), while addition (needed for the definition of Nn) is defined by

$$x+y = \{ z \cup w \mid z \in x \wedge w \in y \wedge z \cap w = \Lambda \}$$

Given an operation F on variables $\bar{x}$, we may be able to form the set which consists of the closure of some set y with respect to F:

$$\mathrm{Clos}\,(y, F) = \bigcap \{ w \mid y \subseteq w \wedge \forall \bar{x} z\,(\bar{x} \in w \wedge z = F\bar{x} \rightarrow z \in w) \}$$

Obviously, for stratification purposes, $\bar{x}$ and z must have the same type . So F must be a 'type-preserving' operation. Thus, for any type-preserving F, Clos (y, F) will exist. The type of Clos (y, F) is clearly the same as that of y.
We are now able to define the set of natural numbers which will become to be the set of finite cardinals:

$$\mathrm{Nn} = \mathrm{Clos}\,(\{0\}, .+1)$$

One might wonder why we haven't defined Nn like ω in **ZF,** which is formed by closure of $\{\Lambda\}$ under the operation $x^+=x\cup\{x\}$. The reason is that, though there is no difficulty in forming all separate elements of ω (see above), problems arise when we try to form ω itself by closure: $z=x^+$ is not stratified, so we have no easy way of showing that Clos $(\{\Lambda\},.^+)$ exists.

It is very easy to derive the following facts about Nn and + :

(1) $0\in \mathrm{Nn}$

(2) $\forall n\in \mathrm{Nn}\,(n+1\in \mathrm{Nn})$

(3) $\forall n\in \mathrm{Nn}\,(n+1\neq 0)$

(4) $\varphi 0 \wedge \forall n\in \mathrm{Nn}\,[\varphi n \rightarrow \varphi(n+1)] \rightarrow \forall n\in \mathrm{Nn}\ \varphi n$ for <u>stratified</u> φ

(5) + is commutative, associative and 0 acts as a neutral element.

(1) and (2) are direct consequences of the definition of Nn; (3) follows because, for all y and z, $y\cup\{z\}=\Lambda$ is impossible, so $\forall x\,(\Lambda\notin x+1)$; (4) is proved by considering the set $\{x \mid \varphi x\}$ which contains Nn by definition of Nn and the premiss. This is of course the reason that φ must be stratified. In fact we could have stated a less restrictive requirement: <u>it suffices that</u> $\{x \mid \varphi x\}$ <u>exists</u>. (5) again is a simple consequence of the definitions.

(1) to (4) constitute the Peano-axioms except for the facts that induction is (primarily) only possible for stratified formulae, and that injectiveness of $.+1$ on Nn is missing, but the latter will turn out to be equivalent to the axiom of infinity (see next section), which is provable in **NF**.

There may be some doubt whether Nn contains all 'natural numbers', that is, whether the apparently intended process of forming Nn by starting with 0, taking the successor, the successor thereof, and so on, may arrive at Λ at a certain point (and, *hence*, stop from there on) because the universe doesn't have any more elements than already contained in some $x\in \bigcup \mathrm{Nn}$. Intuitively, this amounts to saying that $\Lambda\notin \mathrm{Nn}$ if and only if V is infinite, which is in fact the case as is shown in the next section.

To define general cardinal numbers as sets of equinumerous sets, we have to define functions and thus relations, so we are in need of an ordered pair. Now it is possible in **NF** to define an ordered pair which doesn't raise the type, but this again is (technically spoken) an equivalent of the axiom of infinity which is not available until after the next chapter. For this reason, we temporarily assume to have available an ordered pair (x,y) for all x,y of the same type which lifts the type by k. Kuratowski's ordered pair $\{\{x\},\{x,y\}\}$ which is obviously present in **NF** has k=2, Quine's ordered pair as described in section 2 has k=0. For the moment, it suffices to use the former, but when we wish to show the equinumerosity of x and the Cartesian product of x with some singleton (for arbitrary x) we will have to use the latter.

We write $f: x \rightarrow y$ for "f is a function from x into y" and $f: x \sim y$ for "f is a bijection from x onto y". Defining the binary relation $\sim$ by $x \sim y \Leftrightarrow \exists f\,(f: x \sim y)$, $\sim$ is easily seen to be an equivalence relation so we are at the point where we can define our cardinal numbers. All this can be done in **NF**, as equivalence relations and equivalence classes are **NF**-notions. Cardinal numbers and the set of cardinals are defined by

$|x| = \{\, y \mid y \sim x \,\}$ (=the equivalence class of x with respect to $\sim$)
$NC = \{\, |x| \mid x \in V \,\}$

We define natural ordering relations $\leq$ and $<$ on NC by

$\leq = \{\, (m,n) \in NC^2 \mid \exists ab\,(a \in m \wedge b \in n \wedge a \subseteq b)\,\}$
$I = \{\, (x,y) \mid x = y \,\}$
$< = \leq - I$

$\leq$ is a partial ordering on NC; $|x| \leq |y|$ is equivalent to the statement that there is an $f: x \rightarrow y$ which is 1-1.

As V=PV, we have $|V| = |PV|$. Doesn't this contradict Cantor's theorem? No, it doesn't, because in its usual form Cantor's theorem is unstratified: $|x|$ has a type one lower than that of $|Px|$, so we have no obvious proof of $|x| < |Px|$ for arbitrary x. We would like to consider the set $\{\, y \in x \mid y \notin f y \,\}$ when $f: x \sim Px$, but we cannot form this set by comprehension as $y \in x \wedge y \notin f y$ is not stratified: $y \notin f y$ is an abbreviation for $\neg \exists z\,(y \in z \wedge (y,z) \in f)$. Actually, it is even impossible to prove $|x| \leq |Px|$ for arbitrary x, as the 'function' relating y to $\{y\}$ for $y \in x$ doesn't have to exist. Nevertheless, we <u>do</u> have an alternative form of the theorem:

<u>Theorem 1.1.1.</u> (Cantor) $\forall x\ |P_1 x| < |Px|$

The proof is just as in **ZF**: obviously $|P_1 x| \leq |Px|$ as $P_1 x \subseteq Px$; and when we suppose that $f: P_1 x \sim Px$, we derive a contradiction by considering the set $A = \{\, y \in x \mid y \notin f\{y\} \,\}$ which exists because $y \in x \wedge y \notin f\{y\}$ is stratified: $y \notin f\{y\}$ is an abbreviation for $\neg \exists z\,(y \in z \wedge (\{y\},z) \in f)$. $A \in Px$; suppose $A = f\{w\}$. One easily sees $w \in A \leftrightarrow w \notin A$.

For a lot of sets x, though, $|x| < |Px|$ holds because $x \sim P_1 x$. Sets x for which $x \sim P_1 x$ are called <u>Cantorian</u>, denoted by Can (x). Obviously $\neg$Can (V), as $|P_1 V| < |PV| = |V|$. In **ZF** all sets are Cantorian.

So maybe we should have tried to define an ordered pair for sets with a type difference of one (for instance $(x,y)=\{\{\{x\},y\},\{y\}\}$), but of course, when we do so we lose the identity map I, and can no longer define $<$ in an easy manner.

§2. the axiom of infinity

Proposition 1.2.1. $Nn \subseteq NC \cup \{\Lambda\}$

Proof : by induction we prove $\forall n \in Nn\,(n \in NC \vee n=\Lambda)$. $0=|\Lambda| \in NC$; when $n=\Lambda$ then also $n+1=\Lambda$, while if $n=|x|$ and $n+1 \neq \Lambda$, say $y \in n+1$, we have $y = z \cup \{w\}$ where $z \sim x$ and $w \notin z$. For arbitrary a,b with $a \sim x$, $b \notin a$, we then have $a \cup \{b\} \sim z \cup \{w\}$ because an $f: a \sim z$ can be easily extended by $f: b \mapsto w$. So $n+1 = |y|$.

As we would like to view Nn as the set of finite cardinal numbers (which will turn out to be appropriate after the proof of $\Lambda \notin Nn$ in section 2.2.), we take the finite sets to be the elements of $\bigcup Nn$, which we denote by Fin.
$V \notin Fin$ is the obvious sentence to take as the axiom of infinity AI.

In [Quine 45], the author defines a type-preserving ordered pair as follows:
first define a type-preserving operation F, next define (x,y) in terms of F:

$$\begin{cases} Fz = (z-Nn) \cup \{ n+1 \mid n \in z \cap Nn \} \\ (x,y) = \{ Fz \mid z \in x \} \cup \{ \{0\} \cup Fz \mid z \in y \} \end{cases}$$

Let us refer to this definition as Quine's pair.

Lemma 1.2.2. $\forall nm \in Nn\,(n+1=m+1 \rightarrow n=m) \rightarrow F$ is an injection

Proof: assume the premiss and suppose Fx=Fy, i.e.
$(x-Nn) \cup \{ n+1 \mid n \in x \cap Nn \} = (y-Nn) \cup \{ m+1 \mid m \in y \cap Nn \}$.
Let $z \in x$ be arbitrary. Then either $z \in x-Nn$ or $z \in x \cap Nn$. When $z \in x-Nn$, then also $z \in y-Nn$ ($z \notin Nn$ so $\neg \exists m \in Nn\,(z=m+1)$); when $z \in x \cap Nn$, we know that $z+1=m+1$ for some $m \in y \cap Nn$ (as $z+1 \in Nn$). So, by the premiss, $z=m \in y \cap Nn$. This proves $x \subseteq y$; the reverse inclusion $y \subseteq x$ is proved in the same way. Now apply the axiom of extensionality to derive x=y.

Lemma 1.2.3. $\forall nm \in NC\,(n+1=m+1 \in NC \rightarrow n=m)$

Proof: let $n=|x|$ and $m=|y|$, $n+1=m+1=|z|$. Then $z = a \cup \{u\} = b \cup \{w\}$ for certain a, b, u, w with $a \sim x$, $u \notin a$, $b \sim y$, $w \notin b$. Now n=m because $a \sim b$: when u=w this is immediate because then even a=b; when $u \in b$, $w \in a$, define $f: a \rightarrow b$ by fc=c when $c \neq w$, fw=u. Clearly $f: a \sim b$.

Proposition 1.2.4. The following are equivalent:

(1) $V \notin Fin$ (i.e. AI)

(2) $\Lambda \notin Nn$

(3) $Nn \subseteq NC$

(4) $\forall nm \in Nn\,(n+1=m+1 \rightarrow n=m)$

(5) Quine's pair acts as an ordered pair

Proof :

(1)→(2): we prove by induction $\forall n \in Nn\,(n \neq \Lambda)$. $0=\{\Lambda\}\neq\Lambda$; suppose $n \in Nn$ and $n+1=\Lambda$. Then $\forall x \in n\, \neg\exists y\,(y \notin x)$ so $\forall x \in n\,(V \subseteq x)$. As $\forall x\,(x \subseteq V)$, we get by extensionality: $\forall x \in n\,(x=V)$, so $n=\Lambda$ or $n=\{V\}$. The latter would imply $V \in Fin$, contrary to the assumption of (1). So $n=\Lambda$, contrary to the induction hypothesis.

(2)→(3): this follows from proposition 1.2.1.

(3)→(4): this follows from lemma 1.2.3.

(4)→(5): suppose $(x,y)=(u,v)$, i.e.

$$\{Fz \mid z \in x\} \cup \{\{0\} \cup Fz \mid z \in y\} = \{Fw \mid w \in u\} \cup \{\{0\} \cup Fw \mid w \in v\}.$$

Let $z \in x$ be arbitrary. As Fz does not contain 0, we have Fz=Fw for some $w \in u$. By lemma 1.2.2 and (4) we infer $z=w \in u$. So $x \subseteq u$; $u \subseteq x$ is proved analogously. Let $z \in y$ be arbitrary. Again, because Fa does not contain 0 for any a, we have $\{0\} \cup Fz = \{0\} \cup Fw$ for some $w \in v$. So Fz=Fw, and $z=w \in v$ by 1.2.2.. So $y \subseteq v$; analogously $v \subseteq y$. Applying EXT, we see that $x=u \wedge y=v$.

(5)→(1): this is postponed until after the definition of 2^m for cardinal numbers m in section 2.1.

Remark 1: since AI is provable in **NF**, all the statements in the proposition above are, but the proposition is interesting in its own right as, for the time being, five apparently completely different statements can all act as an axiom of infinity. Moreover,

Remark 2: for future reference, it must be noted that we made use of EXT a couple of times in the proof of Proposition 1.2.4.: we used $V \subseteq x \wedge x \subseteq V \rightarrow x=V$, and we needed EXT when proving that Quine's pair is an ordered pair (once in 1.2.2.; twice in the proof of (4)→(5)). When considering the (consistent) fragment **NFU** of **NF**, which we get when restricting EXT to non-empty sets, i.e. replacing it by

$$EXT^*: \quad \forall xy\,(\exists z\,(z \in x) \rightarrow [\forall z\,(z \in x \leftrightarrow z \in y) \rightarrow x=y])\,,$$

we accordingly have to restate the propositions 1.2.1. and 1.2.4.. When we define 0 not to be $\{\Lambda\}$ but as $\{x \mid \neg\exists y\,(y \in x)\}$, the set of empty sets, it is very easy to see that 1.2.1. becomes $Nn \subseteq NC \cup 0$: $n=\Lambda$ implies $n+1 \in 0$, but we don't know whether $n+1=\Lambda$ (where Λ is an arbitrarily chosen element of 0). Yet, as $n \in 0$ implies $n+1 \in 0$, we can prove 1.2.1. in the mentioned modified form. Furthermore, in 1.2.4., (2) has to be replaced by $Nn \subseteq V\text{-}0$ and we have to add in (5) that Quine's pair only acts as an ordered pair for non-empty constituents (note that, as $\exists x\,(x\in V)$, the use of EXT in the proof of (1)$\rightarrow$modified(2) can be replaced by an application of EXT*).

The next proposition seems too trivial to state, but in fact it is very important in the disproof of AC in the next chapter (together with strong induction below).

Proposition 1.2.5. $\forall n\in Nn\,(n+1\neq\Lambda \rightarrow n<n+1)$

Proof: by induction on n. $0=|\Lambda|<|\{\Lambda\}|=1$; assuming $n+2\neq\Lambda$ (where 2 is defined as 1+1), it follows that $n+1\neq\Lambda$ and $n\neq\Lambda$, so by 1.2.1. n, n+1, $n+2\in NC$. Obviously $n+1\leq n+2$ (as $n+2=|x\cup\{y\}|$ for some $x\in n+1$, $y\notin x$). Supposing n+1= n+2, we infer n = n+1 by 1.2.3., which is contradictory to the induction hypothesis. So $n+1<n+2$, which concludes the proof.

Lemma 1.2.6. $\forall nm\in NC\,(m<n+1\rightarrow m\leq n)$

Proof: supposing $m<n+1$, for some x, y, z we have $n=|x|$, $m=|y|$, $x\subset y\cup\{z\}$, $z\notin y$ (since $x=y\cup\{z\}$ implies m=n+1, it must be so that $x\neq y\cup\{z\}$). When $z\notin x$, we infer $x\subseteq y$ so $m\leq n$. When $z\in x$, then $w\notin x$ for some $w\in y$; $y\cup\{z\}-\{w\}\sim y$ and $x\subseteq y\cup\{z\}-\{w\}$, so in this case $m\leq n$ also holds.

Theorem 1.2.7. (strong induction) for stratified φ:

$$\varphi 0\wedge\forall n\in Nn\,[\forall m\in Nn\,(m\leq n\rightarrow\varphi m)\rightarrow\varphi(n+1)]\rightarrow\forall n\in Nn\cap NC\ \varphi n$$

Proof: let ψn be the formula $\forall m\in Nn\,(m\leq n\rightarrow\varphi m)$. Assume φ0 and $\forall n\in Nn\,(\psi n\rightarrow\varphi(n+1))$. By (weak) induction we prove $\forall n\in Nn\ \psi n$ (note that ψ is stratified as φ is stratified):

* ψ0 is the sentence $\forall m\in Nn\,(m\leq 0\rightarrow\varphi m)$ which is equivalent to φ0 as $m\leq 0\leftrightarrow m=0$.
* let $n\in Nn$, assume ψn. Let $m\in Nn$, $m\leq n+1$. $m\leq n+1$ means $m<n+1\vee m=n+1$.
  $m<n+1$: according to lemma 1.2.6., we have $m\leq n$, so φm follows from ψn.
  $m=n+1$: because ψn, we have φ(n+1), i.e. φm.

Let $n\in Nn\cap NC$. As $n\leq n$ (this is the reason that we require $n\in NC$!), we deduce φn from $\forall n\in Nn\ \psi n$.

Remark: when we try to prove strong induction for all φ for which $\{x\mid\varphi x\}$ exists, we encounter a problem: for the induction on ψ to succeed, we would have to show the existence of the set

$$\{x\mid\psi x\}=\{x\mid\forall m\in Nn\,(m\leq x\rightarrow\varphi m)\}$$
$$=\bigcap_{m\in Nn}[\{x\mid\neg m\leq x\}\cup\{x\mid\varphi m\}]$$
$$=\bigcap\{z\mid\exists m\in Nn\,(z=\{x\mid\neg m\leq x\}\cup\{x\mid\varphi m\})\}$$

At first sight this set exists -whether $\{ x \mid \varphi x \}$ exists or not- as unions always exist. But this is a superficial analysis: the set behind $\bigcap$ has to exist so its defining formula has to be stratified to form it by COMP. And we have no way of inferring this from the mere existence of $\{ x \mid \varphi x \}$. We would in fact like to apply the substitution axiom SUB from **ZF**, but SUB doesn't hold in **NF** (when **NF** is consistent): it implies the separation axiom, which implies $\{ x \mid x \notin x \} = \{ x \in V \mid x \notin x \} \in V$ (as $V \in V$), but then we can establish Russel's antinomy.

Let us note finally that, while the concept of stratification involves only atoms and the connectives $\wedge$, $\vee$ and $\rightarrow$, the stratification proviso in COMP affects only atoms and quantifiers: when trying to prove the existence of $\{ x \mid \varphi \}$ for arbitrary φ with induction on φ, there is no problem with $\neg$ and $\wedge$ (as complements and intersections always exist), but problems arise at (atoms and) the induction step for $\exists$: we would like to rewrite $\{ x \mid \exists y \, \varphi \}$ to $\bigcup \{ z \mid \psi \}$ where ψ is shorter than φ, but $x \in \bigcup \{ z \mid \psi \}$ if and only if $\exists z \, (x \in z \wedge \psi)$, so this transcription only works when φ is of the form $x \in y \wedge \psi$.

§3. the axiom of choice

One form of the axiom of choice is the sentence "every set of non-empty disjoint sets has a choice set", or in symbolic logic:

(AC) $\forall x\, [\, \forall y \in x\, (y \neq \Lambda) \wedge \forall yz \in x\, (y \neq z \rightarrow y \cap z = \Lambda) \rightarrow \exists w \forall y \in x \exists! z\, (z \in y \cap w)\,]$

(where $\exists! x\, \varphi x$ is short for $\exists x\, \varphi x \wedge \forall xy\, (\varphi x \wedge \varphi y \rightarrow x=y)$).

Thus written, it is almost unreadable, but it demonstrates the fact that AC is stratified (this fact is being used in section 3.6., and has been used in the introduction when speaking about AC in **TT**).

Rosser proved in [53] that some well-known equivalents of AC in **ZF** remain so in **NF**; for better legibility I state them in English (the notation Z_i has been taken from the mentioned book):

Z_2 : every set can be well-ordered;

Z_3 : every set of non-empty sets has a choice function;

Z_7 : every partially ordered set x in which every simply ordered subset has an upper bound in x, has a maximal element (Zorn's lemma).

One remark has to be made, though: a choice function usually states $fx \in x$ somewhere, but in order to achieve stratification (essential in the proof of the equivalence of the Z's to AC) we have to modify this to $\exists z \in x\, (fx = \{z\})$.

Actually, Rosser proved the equivalence of AC, Z_2, Z_3 and Z_7 restricted to sets of cardinality less than or equal to a certain cardinal number -varying according to which form of AC is considered- but these imply the equivalence of the 'full' variants of AC by taking $|V|$ as upper bounds .

An interesting point is that instead of Z_2 we can take the sentence " V can be well-ordered " as V is a set of which every set is a subset.

Moreover, other sentences proven to be equivalent to AC in **ZF** remain so in **NF** too, of which the most important one is the trichotomy theorem, used in the next chapter when disproving AC:

$\forall mn \in NC\, (\, m < n \vee m = n \vee m > n\,)$

(see lemma 2.2.1. for a proof of AC $\rightarrow$ trichotomy)

2. The axioms of choice and infinity in NF

§1. the cardinal operators T, $2^{\cdot}$ and Φ

In [53], Specker disproved the full axiom of choice by considering certain sets of cardinals 'close' to $\Omega = |V|$, thus indirectly proving the redundancy of 'axiom scheme 13' in [Rosser 53] (i.e. 1.2.4.(4)).

Before defining the three operators needed for the proof, we state

Proposition 2.1.1.
(a) $x \sim y \leftrightarrow P_1x \sim P_1y$
(b) $x \sim y \rightarrow Px \sim Py$
(c) $P_1Px \sim PP_1x$

Proof:
(a) when $f: x \sim y$, then $g: P_1x \sim P_1y$ by $g: \{z\} \mapsto \{fz\}$ for $z \in x$;
when $f: P_1x \sim P_1y$, then $g: x \sim y$ by $g: z \mapsto \bigcup f\{z\}$ for $z \in x$
(b) when $f: x \sim y$, then $g: Px \sim Py$ by $g: z \mapsto f[z]$ for $z \in Px$
(c) $f: P_1Px \sim PP_1x$ by $f: z \mapsto \{ \{w\} \mid w \in \bigcup z \}$ for $z \in P_1Px$.
[all these functions exist by the fact that originals and images are equal in type]

Definition 2.1.2.
$Tz = \{ y \mid \exists x (z = |x| \wedge y \sim P_1x) \}$
$2^z = \{ y \mid \exists x (z = |P_1x| \wedge y \sim Px) \}$
$\Phi z = \mathrm{Clos}(\{z\}, 2^{\cdot}) - \{\Lambda\}$

These formal definitions are merely given to show that the operations exist, but we will apply them only to cardinal numbers. Thus, by proposition 2.1.1.(a) & (b):

(2.1.3) $T|x| = |P_1x|$ and $2^{|P_1x|} = |Px|$, while

(2.1.4) $2^z \neq \Lambda \leftrightarrow \exists x (z = |P_1x|)$

For the remainder of this paper, m, n and p are assumed to denote cardinal numbers (exception: m and n may denote elements of Nn -of which we do not yet know whether it contains Λ or not- when explicitly indicated).

T is type-raising while $2^{\cdot}$ is type-preserving, which is why Φ could be defined as above. From the definition of Φ we deduce easily

(2.1.5.) $n \in A \wedge \forall m (m \in A \wedge 2^m \neq \Lambda \rightarrow 2^m \in A) \rightarrow \Phi n \subseteq A$, and

(2.1.6.) $m \in \Phi n \leftrightarrow m = n \vee \exists p \in \Phi n (m = 2^p \neq \Lambda)$.

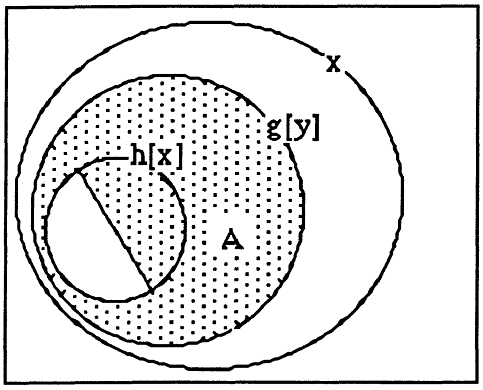
As we are going to need it in the sequel, this is a good place to state the theorem that $\leq$ is antisymmetric. In the proof we make use of the fact that any function can be viewed as a type-preserving unary operator (and vice versa):

* when f is a function, define F by $Fz=\{ u \mid \exists w [(z,w)\in f \wedge u\in w] \}$ (=f z when $z\in \mathrm{Dom}(f)$; $=\Lambda$ when $z\notin \mathrm{Dom}(f)$). F is clearly type-preserving.

*when F is a type-preserving unary operator and x an arbitrary set, define $f=\{ (z,Fz) \mid z\in x \}$. From $\forall z \exists! y (y=Fz)$ it follows that f is a function from x onto $\{ Fz \mid z\in x \}$.

Theorem 2.1.7. (Schröder-Bernstein) $n\leq m \wedge m\leq n \rightarrow n=m$

Proof: Let $n=|x|$, $m=|y|$; let $f: x\rightarrow y$ and $g: y\rightarrow x$ be 1-1. Then $h=gf: x\rightarrow x$ is 1-1. $h[x]\subseteq g[y]\subseteq x$; consider the sets $A=\mathrm{Clos}(g[y]-h[x], h)$ and $B=h[x]\cap A$



$h[A] \subseteq A$ by definition of A, so $h[A] \subseteq A\cap h[x]=B$; also
$B=h[x]\cap A=h[x]\cap ((g[y]-h[x])\cup h[A])=h[x]\cap h[A] \subseteq h[A]$, so $B=h[A]$ and $h: A\sim B$.
$g[y]-A=h[x]-A=h[x]-B$ so by $A\sim B$ it follows that $g[y]\sim h[x]$ (because $(g[y]-A)\cap A=(h[x]-B)\cap B=\Lambda$). As $h[x]\sim x$ and $g[y]\sim y$, we infer $x\sim y$.

Corollary 2.1.8. $n<m\rightarrow \neg m\geq n$

Before proceeding with an exploration of Φ, $2^{\cdot}$ and T, we first give the promised remaining proof of 1.2.4.(5)$\rightarrow$(1):

Lemma 2.1.9. (a) $1.2.4.(5) \rightarrow \forall mn \in NC\,(m+n \in NC)$

(b) $1.2.4.(5) \rightarrow (2^m \neq \Lambda \wedge 2^n \neq \Lambda \rightarrow 2^{m+n} \neq \Lambda)$

Proof:

(a) When $m=|x|$, $n=|y|$ it follows by the existence of the sets $\{ (z,(z,\Lambda)) \mid z \in x \}$ and $\{ (z,(z,V)) \mid z \in y \}$ (due to the *type-preserving* nature of Quine's pair) that $x \sim x \times \{\Lambda\}$ and $y \sim y \times \{V\}$ so $m+n = |(x \times \{\Lambda\}) \cup (y \times \{V\})|$ (as obviously $a \cap b = \Lambda \rightarrow |a| + |b| = |a \cup b|$).

(b) Assume $2^m \neq \Lambda \neq 2^n$; let $P_1 x \in m$, $P_1 y \in n$ (apply 2.1.4.). Then, by Quine's ordered pair and 2.1.1.(a), $P_1 x \sim P_1(x \times \{\Lambda\})$ and $P_1 y \sim P_1(y \times \{V\})$, so $P_1\big((x \times \{\Lambda\}) \cup (y \times \{V\})\big) = P_1(x \times \{\Lambda\}) \cup P_1(y \times \{V\}) \in |P_1 x| + |P_1 y| = m+n$.

Proof of 1.2.4.(5)→(1): $|P_1 V| < |V|$, so by 2.1.8. above $\neg |V| \leq |P_1 V|$, which implies that there is no set x for which $P_1 x \in |V|$, so $2^{|V|} = \Lambda$. On the other hand, a simple proof shows $\forall n \in Nn\,(2^n \neq \Lambda)$: $0 = |\Lambda| = |P_1 \Lambda|$ so $2^0 \neq \Lambda$. $1 = P_1 V = |\{\Lambda\}| = |P_1\{\Lambda\}|$, so $2^1 \neq \Lambda$. It follows with induction from 2.1.9.(a) that $Nn \subseteq NC$. Apply this fact, 2.1.9.(b) and induction to derive $\forall n \in Nn\,(2^n \neq \Lambda)$. The inference is that $\forall n \in Nn\,(n \neq |V|)$, whence $V \notin Fin$.

Some simple facts about the three operations under consideration are listed in the next proposition. Remember that $\Omega = |V|$.

Proposition 2.1.10.

(a) $2^m \neq \Lambda \rightarrow m < 2^m$

(b) $m \leq Tn \rightarrow \exists p \leq n\,(m = Tp)$

(c) $m \leq n \leq T\Omega \rightarrow 2^m \leq 2^n$

(d) $n \in \Phi m \rightarrow m \leq n$

(e) $m \leq T\Omega \rightarrow m,\ 2^m \in \Phi m$

(f) $m \leq n \leftrightarrow Tm \leq Tn$

(g) $\neg m \leq T\Omega \leftrightarrow 2^m = \Lambda \leftrightarrow \Phi m = \{m\}$

(h) $m \leq T\Omega \rightarrow 2^{Tm} = T2^m$

(i) $m \leq T\Omega \rightarrow \Phi m = \{m\} \cup \Phi 2^m$

(j) $m \leq T\Omega \rightarrow |\Phi m| = |\Phi 2^m| + 1$

Proof:

(a) this is a reformulation of Cantor's theorem 1.1.1.

(b) Let $m=|x|$, $n=|y|$. $|x| \leq T|y| = |P_1 y|$ implies the existence of an injection $f: x \rightarrow P_1 y$. Obviously $f[x] = P_1 z$ for some $z \subseteq y$ (viz. $z = \{ w \in y \mid \{w\} \in f[x] \}$), so $|x| = |P_1 z| = T|z|$. Clearly $|z| \leq |y|$.

(c) Assume $m \leq n$ and $n \leq T\Omega$. By (b), $n = |P_1 x|$ for some x, whence $m \leq T|x|$, so $m = |P_1 y|$ for some $y \subseteq x$ by the proof of (b). $Py \subseteq Px$ implies $2^m = |Py| \leq |Px| = 2^n$.

(d) Consider the set $A = \{ n \mid m \leq n \}$.

$m \in A$; when $p \in A$ and $2^p \neq \Lambda$, then $m \leq p < 2^p$ by (a) so $2^p \in A$. Hence $\Phi m \subseteq A$ by 2.1.5..

(e) $m \in \Phi m$ is obvious; $m \leq T\Omega$ implies $m = Tp$ for some $p = |x|$, so $2^m = |Px| \neq \Lambda$, whence $2^m \in \Phi m$ by 2.1.6.

(f) $\rightarrow$: $m = |x|$, $n = |y|$, $x \subseteq y$ so $P_1x \subseteq P_1y$, whence $Tm = |P_1x| \leq |P_1y| = Tn$.
$\leftarrow$: When $Tm \leq Tn$, then $Tm = Tp$ for some $p \leq n$ by (b), so $m = p \leq n$ by 2.1.1.(a)

(g) $\neg m \leq T\Omega$ is equivalent to $\neg\exists x\,(m = |P_1x|)$, which is equivalent to $2^m = \Lambda$ by 2.1.4.. From $2^m = \Lambda$ it follows that $\Phi m = \{m\}$: $\supseteq$ is obvious; $\subseteq$: $m \in \{m\}$, and when $p = 2^n \neq \Lambda$ for some $n \in \{m\}$, then $2^m \neq \Lambda$, contrary to the assumption. Conversely, $\Phi m = \{m\}$ implies $\neg m \leq T\Omega$ because of (a) and (e).

(h) Assuming $m \leq T\Omega$, we deduce $m = |P_1x|$ for some x; applying proposition 2.1.1.(c) we find $2^{Tm} = 2^{|P_1P_1x|} = |PP_1x| = |P_1Px| = T\,|Px| = T\,2^m$.

(i) Assume $m \leq T\Omega$, i.e. $2^m \neq \Lambda$ (so $\Phi 2^m \neq \Lambda$).
*$\Phi m \subseteq \{m\} \cup \Phi 2^m$: $m \in \{m\} \cup \Phi 2^m$; when $n \in \{m\} \cup \Phi 2^m$ and $2^n \neq \Lambda$, then clearly $2^n \in \Phi 2^m$ (by 2.1.6.). Now apply 2.1.5..
*$\Phi 2^m \subseteq \Phi m - \{m\}$: $2^m \in \Phi m$ by (e), while $2^m \neq m$ by (a). If $n \in \Phi m - \{m\}$ and $2^n \neq \Lambda$, then $2^n \in \Phi m$ by 2.1.6., and $m \leq n$ by (d), so $m < 2^m \leq 2^n$ by (a) and (c), which implies $2^n \neq m$.

(j) Assume $m \leq T\Omega$. $m \in \Phi 2^m$ would imply $2^m \leq m$ by (d), but $m < 2^m$ by (a). So $m \notin \Phi 2^m$, that is, $\{m\} \cap \Phi 2^m = \Lambda$, so $|\Phi m| = |\{m\} \cup \Phi 2^m| = |\Phi 2^m| + 1$.

From 2.1.10.(a) & (h) we see that the following situation exists:

$$\Omega \;\overset{T}{\underset{2^{\cdot}}{\overset{\rightarrow}{\underset{\leftarrow}{>}}}}\; T\Omega \;\overset{T}{\underset{2^{\cdot}}{\overset{\rightarrow}{\underset{\leftarrow}{>}}}}\; T^2\Omega \;\overset{T}{\underset{2^{\cdot}}{\overset{\rightarrow}{\underset{\leftarrow}{>}}}}\; T^3\Omega \;\overset{T}{\underset{2^{\cdot}}{\overset{\rightarrow}{\underset{\leftarrow}{>}}}}\; \ldots\ldots$$

[In fact this requires a metamathematical proof by induction: we know $2^{T\Omega} = \Omega$; assuming $2^{T^{n+1}\Omega} = T^n\Omega$, we deduce $2^{T^{n+2}\Omega} = T2^{T^{n+1}\Omega} = TT^n\Omega = T^{n+1}\Omega$].

Having this infinite descending chain of cardinals doesn't contradict AC, however, because it is seemingly impossible to form the *set* $\{\Omega, T\Omega, TT\Omega, \ldots\ldots\}$: an attempt to define this set with the aid of Clos will result in failure as T is not type-preserving. Our disproof of AC nevertheless depends on the fact that AC implies the well-ordering of NC by $\leq$ (see next section).

The following three propositions are going to be needed in the next section; as they don't depend upon AC they are stated and proved here.

Proposition 2.1.11. ΦTm finite $\rightarrow \Phi m$ finite; that is:
$\forall n \in Nn \forall m (|\Phi Tm| = n \rightarrow \Phi m \in Fin)$

Proof: If $\neg m \leq T\Omega$, then $\Phi m = \{m\} \in 1$.

If $m \leq T\Omega$, then $|\Phi m| = |\Phi 2^{m}|+1$. Also $Tm \leq T\Omega$ so $|\Phi Tm| = |\Phi 2^{Tm}|+1 = |\Phi T2^{m}|+1$. According to proposition 1.2.5., we have $|\Phi T2^{m}| < |\Phi Tm|$ when $|\Phi T2^{m}| \in Nn$.

Applying strong induction, the induction hypothesis gives us $\Phi 2^{m} \in Fin$, so $\Phi m \in Fin$. The actual conclusion reached at by the application of 1.2.7 reads $\forall n \in Nn \cap NC$...., but we can skip NC because $|\Phi Tm| \in NC$ for all m.

Remark: Note that we had to use strong induction because we do not know (yet) whether $n+1 = |\Phi T2^{m}|+1$ implies $n = |\Phi T2^{m}|$. Also note that nothing is said (nor can be said) about which natural number contains Φm.

In the following, 3n is an abbreviation for n+n+n (remember that + is associative).

Proposition 2.1.12. $\forall n \in Nn \cap NC [\exists m \in Nn \cap NC (n = 3m \vee n = 3m+1 \vee n = 3m+2) \wedge$
$\forall mp \neg (n = 3m = 3p+1 \vee n = 3m = 3p+2 \vee n = 3m+1 = 3p+2)]$

Proof: by induction on n.

$0=3 \cdot 0$; $\forall p (0 \neq 3p+1 \wedge 0 \neq 3p+2)$ (as $x+1 \neq 0$ for all x).

Let $n+1 \in Nn \cap NC$. Then $n \in Nn \cap NC$, so by the induction hypothesis there is an $m \in Nn \cap NC$ such that $n = 3m \vee n = 3m+1 \vee n = 3m+2$. Hence
$n+1 = 3m+1 \vee n+1 = 3m+2 \vee n+1 = 3m+3 = 3(m+1)$.

Note that, in the last case, $m+1 \in NC$ because $m+1 \in Nn$ and $m+1 \neq \Lambda$ (as $m+1=\Lambda$ implies $n+1=3.\Lambda=\Lambda$).

As $n+1 \in NC$, it is impossible that n+1 modulo 3 is two of 0,1,2 at the same time because of lemma 1.2.3. and the induction hypothesis ($n+1 = 3p$ implies $p \neq 0$, so $p \geq 1$, and from this it follows that $p = q+1$ for some $q \in NC$: see the proof of lemma 2.3.2. on page 22).

Lemma 2.1.13. $\forall mn (m+n \in NC \rightarrow T(m+n) = Tm+Tn)$

Proof: let $m+n = |x|$, $x = y \cup z$ where $y \in m$, $z \in n$, $y \cap z = \Lambda$.

$T(m+n) = |P_1 x| = |P_1(y \cup z)| = |P_1 y \cup P_1 z| = |P_1 y| + |P_1 z| = Tm+Tn$ (as $P_1 y \cap P_1 z = \Lambda$)

Lemma 2.1.14. $\forall n \in Nn \cap NC (Tn \in Nn \cap NC)$

Proof: induction on n: $T0=0$; when $n+1 \in Nn \cap NC$, then $T(n+1) = Tn+T1 = Tn+1$.

$T(n+1) \in Nn$ by the induction hypothesis; $T(n+1) \in NC$ because $n+1 \in NC$.

<u>Proposition 2.1.15.</u> $\forall m \in Nn \cap NC\,(m \neq Tm+1 \wedge m \neq Tm+2)$

<u>Proof</u>: assume $m \in Nn \cap NC$ and suppose $m=Tm+1$. We derive a contradiction with the aid of 2.1.12. to 2.1.14.:

* $m=3p$: $m = 3p = T(3p)+1 = 3Tp+1$
* $m=3p+1$: $m = 3p+1 = T(3p+1)+1 = 3Tp+2$
* $m=3p+2$: $m = 3p+2 = T(3p+2)+1 = 3(Tp+1)$

These three statements all contradict 2.1.12. because of lemma 2.1.14.; $m = Tm+2$ leads analogously to a contradiction.

§2. disproof of AC; proof of AI

Assuming AC, NC is simply ordered by ≤. But there is more: NC is even well-ordered by ≤. In **ZF**, this is immediate when cardinals are defined as finite or initial von Neumann-ordinals, because ≤ is the restriction of the well-ordering of the class of ordinals to the class of cardinals. Here we have defined cardinals in the Frege-Russell way, so we have to prove something. Ordinals are defined in **NF** as equivalence-classes of ordinally similar well-orderings. NO, the set of ordinals, is well-ordered by $\leq_0$ defined by

$\alpha \leq_0 \beta \quad \equiv \quad \exists R\in\alpha\, \exists S\in\beta$ "R is ordinally similar to an initial segment of S"

In particular, $\alpha \leq_0 \beta$ implies $|Dom(R)| \leq |Dom(S)|$ for all $R\in\alpha$, $S\in\beta$.

Lemma 2.2.1.(AC) ≤ well-orders NC

Proof: First we prove that NC is simply ordered by ≤: let x, y be arbitrary sets. Well-order x and y with respectively R and S by means of AC. As NO is well-ordered by $\leq_0$, we have $No(R) \leq_0 No(S) \vee No(R) >_0 No(S)$ (where No(R') denotes the equivalence class of the well-ordering R' under ordinal similarity). This implies $|x| \leq |y| \vee |x| > |y|$ (as x=Dom(R) and y=Dom(S)).

Let A be a non-empty set of cardinals; $B=\{\, x \mid \exists n\in A\,(n=|x|)\,\}$ ($=\bigcup A$). For all x, $Fx = \{\, R \mid R$ well-orders $x\,\} \neq \Lambda$ by AC. Let $C=\{\, Fx \mid x\in B\,\}$. When $x, y\in B$ and $R\in Fx\cap Fy$, it follows that x=y (as R well-orders x and y) so Fx=Fy. So $x, y\in B$ and $Fx\neq Fy$ imply $Fx\cap Fy = \Lambda$.

Applying AC again, let D be a choice-set for C: $\forall x\in B\, \exists! R\in D\,(R$ well-orders $x)$. Let No(S) be the $\leq_0$-least element of $\{\, No(R) \mid R\in D \wedge Dom(R)\in B\,\}$; $|Dom(S)|\in A$ and $\forall R\in D\,(Dom(R)\in B \rightarrow |Dom(R)| \geq |Dom(S)|)$. When $|x|\in A$, there is an $R\in D$ which well-orders x, so $|x| = |Dom(R)| \geq |Dom(S)|$. So |Dom(S)| is the ≤-least element of A.

Lemma 2.2.2.(AC) (a) $2^m=\Lambda \rightarrow |\Phi Tm| = 2$ or 3

(b) Φm finite $\rightarrow (|\Phi Tm| = T|\Phi m|+1 \vee |\Phi Tm| = T|\Phi m|+2)$, that is:

$\forall n\in Nn \forall m\,(|\Phi m| = n \rightarrow |\Phi Tm| = T|\Phi m|+1$ or $2)$

Proof:

(a) $2^m = \Lambda$ implies $\neg m \leq T\Omega$. Trichotomy gives us $m > T\Omega$, which implies $Tm \geq TT\Omega$ and $2^{Tm} \geq 2^{TT\Omega} = T\Omega$. So we consider two cases:

* $2^{Tm} = T\Omega$: $\Phi Tm = \{Tm, T\Omega, \Omega\}\in 3$
* $2^{Tm} > T\Omega$: $\Phi Tm = \{Tm, 2^{Tm}\}\in 2$

(b) By strong induction on n; note that the statement is stratified so induction is possible.

If $|\Phi m| = 1$, then $2^m = \Lambda$ so $|\Phi Tm| = 2$ or 3 by (a); as $T|\Phi m| = T1 = 1$, we infer $|\Phi Tm| = T|\Phi m|+1$ or 2.

If $|\Phi m| > 1$, then $2^m \neq \Lambda$ so $|\Phi m| = |\Phi 2^m|+1$ by 2.1.10.(j). $|\Phi 2^m| < |\Phi m|$ because of proposition 1.2.5.; the induction hypothesis says

$$|\Phi T2^m| = T|\Phi 2^m|+1 \text{ or } 2, \text{ so}$$

$$|\Phi Tm| = |\Phi 2^{Tm}|+1 = |\Phi T2^m|+1 = T|\Phi 2^m|+2 \text{ or } 3 = T(|\Phi 2^m|+1)+1 \text{ or } 2 = T|\Phi m|+1 \text{ or } 2.$$

Applying strong induction, we infer

$$\forall n \in Nn \cap NC\, \forall m\, (|\Phi m| = n \rightarrow |\Phi Tm| = T|\Phi m|+1 \text{ or } 2).$$

The clause $n \in NC$ is clearly superfluous.

Remark: note that we only used the following consequence of AC: $\forall m\, (m \leq T\Omega \vee m > T\Omega)$, and that we only used it to start the induction.

Theorem 2.2.3. (Specker) **NF** $\vdash \neg$AC

Proof: Consider the set $A = \{ m \mid \Phi m \in Fin \}$. $A \neq \Lambda$ as $\Phi\Omega = \{\Omega\}$. Assuming AC, let n be the $\leq$-least element of A (apply 2.2.1.).

As Φn is finite, ΦTn is finite according to 2.2.2.(b) and lemma 2.1.14., so, by minimality of n, $n \leq Tn$. So $n = Tp$ for some $p \leq n$. ΦTp is finite, so Φp is finite by 2.1.11., which implies $n \leq p$. So $n=p$ and $Tn = Tp = n$. But then 2.2.2.(b) states that $|\Phi n| = T|\Phi n|+1$ or 2, contradictory to proposition 2.1.15..

From 2.2.3. we infer that V cannot be well-ordered. When we prove that finite sets *can* be well-ordered, we obtain as a corollary $V \notin Fin$.

Proposition 2.2.4. $\forall n \in Nn\, \forall x \in n$ (x can be well-ordered)

Proof: very easy by (weak) induction. Λ can be well-ordered; when $x \in n+1$, we have $x = y \cup \{z\}$ for some $y \in n$, $z \notin y$; the well-ordering R on y given by the induction hypothesis can clearly be extended to x by taking z as smallest element: x is well-ordered by $R \cup \{ (z,w) \mid w \in x \}$.

So **NF** $\vdash$ AI, and by 1.2.4. we infer $Nn \subseteq NC$, the injectiveness of .+1 on Nn, and the availability of Quine's ordered pair. These facts will be tacitly assumed in subsequent chapters.

§3. Rosser's counting axiom

From 2.1.13. (and the fact that $Nn \subseteq NC$) we see that

(1) $T0 = 0$
(2) $\forall n \in Nn\ (Tn = n \rightarrow T(n+1) = n+1)$

One would be apt to say that by induction it follows from (1) and (2) that

(CA) $\forall n \in Nn\ Tn = n$,

but as T is type-raising we cannot apply COMP to show existence of the necessary set, neither does it seem that any other method would supply us with it. In [Rosser 53], this statement is proven to be equivalent to what Rosser calls the "Counting Axiom":

$\forall n \in Nn\ (\{ m \in Nn \mid 0 < m \leq n \} \in n)$,

which implies that when a finite set x is 'counted' by relating its elements to consecutive natural numbers, starting at 1, arrival at n when x is exhausted means that x has exactly n elements (when we let '$x \in n$' stand for 'x has n elements' for $n \in NC$). This principle is extremely important in everyday mathematics as Rosser points out, for instance in complex function theory where it is stated that the number of zeros minus the number of poles of a meromorphic function f inside a contour C equals

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{f'(z)} dz .$$

All theorems of mathematics that have to do with counting depend in one way or another on CA. So it would be nice indeed to be able to prove CA, but, as Orey showed in [64], CA cannot be proved in **NF** (unless **NF** is inconsistent). So, in particular, $\neg\exists x \forall y\ (y \in x \leftrightarrow y = Ty)$. More on this in section 4.2.

Just for the record, we here give the proof that CA is equivalent to the counting axiom as Rosser formulated it.

Lemma 2.3.1. $\forall n \in Nn\, \forall x \in n\, (P_1^{\,2}x \sim \{\, m \in Nn \mid 0 < m \leq n \,\})$

Proof: by induction on n; note that by introducing $P_1^{\,2}$ we have made this a stratified statement.
When n=0, then $x=\Lambda$ whence the statement is obvious; when $x \in n+1$, there exist y, z for which $x = y \cup \{z\}$, $y \in n$, $z \notin y$. By the induction hypothesis $P_1^{\,2}y \sim \{\, m \in Nn \mid 0 < m \leq n \,\}$. So $P_1^{\,2}x = P_1^{\,2}y \cup P_1^{\,2}\{z\} \sim \{\, m \in Nn \mid 0 < m \leq n \,\} \cup \{n+1\} = \{\, m \in Nn \mid 0 < m \leq n+1 \,\}$
(apply 1.2.5. and 1.2.6. for the last equality)

Lemma 2.3.2. $\forall nm \in Nn\, (n < m \rightarrow \exists p\, (m = n+p+1))$

Proof: by induction on n. If $m > 0$, then $m \geq 1$ so $m = p+1$ for some p: $x \in m$, $y \in V$, $x \supseteq \{y\}$ implies $m = 1 + |x - \{y\}|$. Assume $n+1 < m$. Then also $n < m$ so $\exists p\, (m = n+p+1)$ by the induction hypothesis. As $m \neq n+1$, we infer $p \neq 0$. So $p > 0$, whence $p = 1+q$ for some $q \in NC$. Hence $m = (n+1)+q+1$.

Lemma 2.3.3. $\forall n \in Nn\, \forall m\, (m < n \vee m=n \vee m > n)$

Proof: by induction on n. $\forall m\, (m \geq 0)$ is obvious; let $n \in Nn$, $m \in NC$. The induction hypothesis yields $n < m \vee n=m \vee n > m$. When $m \leq n$, we infer $m < n+1$ (as $n+1 \in NC$: $Nn \subseteq NC$); when $m > n$, we infer $m = n+p+1$ for some p by 2.3.2. So $n+1 \leq m$.

Theorem 2.3.4. $\forall n \in Nn\, (Tn=n) \leftrightarrow \forall n \in Nn\, (\{\, m \in Nn \mid 0 < m \leq n \,\} \in n)$

Proof:

$\rightarrow$: when $x \in n \in Nn$, by the premiss Can(x), so $x \sim P_1x \sim P_1^{\,2}x$. By 2.3.1. then $x \sim \{\, m \in Nn \mid 0 < m \leq n \,\}$. As $Nn \subseteq NC$, we get $\{\, m \in Nn \mid 0 < m \leq n \,\} \in n$.

$\leftarrow$: when $x \in n \in Nn$, by the premiss and 2.3.1.(and the fact that $Nn \subseteq NC$): $x \sim P_1^{\,2}x$. $P_1x \in Fin$ because of 2.1.14., say $P_1x \in m \in Nn$. By 2.3.3., $m < n \vee m=n \vee m > n$. Let's consider these three possibilities.

(*) $m < n$: by 2.3.2., $n = m+p+1$, so $x = y \cup z$ for some $y \in m$, $z \in p+1$, $y \cap z = \Lambda$. From $y \sim P_1x$ it follows that $P_1y \sim P_1^{\,2}x \sim x$. From $y \subseteq x$ we deduce $P_1y \subseteq P_1x$, so $n = |x| = |P_1y| \leq |P_1x| = m$. This is contradictory to $m < n$.

(*) $m > n$ analogously leads to a contradiction.

So m=n, whence $x \sim P_1x$, i.e. Can(x), and $Tn = n$.

3. The consistency problem for NF

§1. model-theoretic considerations

For a stratified formula φ of **TT**, define φ^+ as the formula resulting from φ when all type-subscripts are raised by unity.

Obviously, $\mathbf{TT} \vdash \varphi \Rightarrow \mathbf{TT} \vdash \varphi^+$ for all stratified φ, as a proof of φ can be "plussed" in totality to yield a proof of φ^+(and when σ is an axiom of **TT**, σ^+ also is an axiom of **TT**).

The reverse implication $\mathbf{TT} \vdash \varphi^+ \Rightarrow \mathbf{TT} \vdash \varphi$ does not hold: taking the sentence $\exists x_0 y_0 (x_0 \neq y_0)$ for φ, we see that $\mathbf{TT} \models \varphi^+$ (hence $\mathbf{TT} \vdash \varphi^+$) because $\mathbf{TT} \models \Lambda_1 \neq V_1$; but $\mathbf{TT} \not\models \varphi$ because $M \not\models \varphi$ when $|M_0| = 1$.

To come to a set theory, originating from **TT**, where the reverse does hold, several ways are open to us:

(1) $\overline{\mathbf{TT}}$ = **TT** with the deduction rule $\dfrac{\varphi^+}{\varphi}$ (rule of ambiguity) added;

(2) $\mathbf{TT}^*$ = **TT** with all axioms of ambiguity added, i.e. all sentences of the form $\sigma \leftrightarrow \sigma^+$ where σ is stratified;

(3) **TNT** = theory of negative types (Wang): the analogue of **TT** for types ranging over the set of integers **Z** instead of ω. The axioms of **TNT** are thus:

EXT_k: $\forall x_k y_k [\forall z_{k-1} (z_{k-1} \in x_k \leftrightarrow z_{k-1} \in y_k) \rightarrow x_k = y_k]$ $(k \in \mathbf{Z})$

COMP_k: $() \exists y_k \forall x_{k-1} (x_{k-1} \in y_k \leftrightarrow \varphi)$ ($k \in \mathbf{Z}$; φ a stratified formula of **TNT**; y_k not free in φ)

Models of **TNT** are of the form $M = (\ldots, M_{-2}, M_{-1}, M_0, M_1, M_2, \ldots, \in_M)$ where $\in_M \subseteq \bigcup \{ M_k \times M_{k+1} \mid k \in \mathbf{Z} \}$;

(4) **NF**, where φ and φ^+ become the same formula after the type-subscripts have been dropped.

In this section we are going to consider some connections between $\mathbf{TT}^*$ and **NF**, but it is interesting to remark that the following are equivalent:

(a) $\overline{\mathbf{TT}} \vdash \varphi$

(b) $\mathbf{TNT} \vdash \varphi$

(c) $\exists i \in \omega\ \mathbf{TT} \vdash \varphi^{++\ldots+}$ (i times +)

(a)⇒(b): the rule of ambiguity is provable in **TNT**.

(b)⇒(c): as a deduction of φ in **TNT** contains only finitely many formulae, we can add a certain amount $i \in \omega$ to all occurring indices so that the deduction takes place in **TT**.

(c)⇒(a): simply apply the rule of ambiguity i times.

When $M = (M_0, M_1, M_2, ..., \in)$ is a model of **TT**, so is $M^+ = (M_1, M_2, M_3, ..., \in)$ (where for brevity $\in$ is used both times; in fact the second $\in$ is the first $\in$ minus $M_0 \times M_1$).

Proposition 3.1.1 (a) $M \models \sigma^+ \Leftrightarrow M^+ \models \sigma$, for all stratified sentences σ;
(b) $M \models \mathbf{TT}^* \Leftrightarrow M \models \mathbf{TT} \,\&\, M \equiv M^+$

Proof:

(a) trivial

(b) $\Rightarrow$: assuming that M is a model of $\mathbf{TT}^*$, it clearly is a model of **TT**; let σ be a stratified sentence. From $M \models \sigma \leftrightarrow \sigma^+$ it follows that $M \models \sigma \Leftrightarrow M \models \sigma^+ \Leftrightarrow M^+ \models \sigma$, the last equivalence being given by (a). So $M \equiv M^+$.

$\Leftarrow$: assuming $M \models \mathbf{TT}$ and $M \equiv M^+$, it remains to show that M is a model of all axioms of ambiguity. Let σ be a stratified sentence. From (a) and the elementary equivalence of M and M^+, we see that $M \models \sigma \Leftrightarrow M^+ \models \sigma \Leftrightarrow M \models \sigma^+$, which proves $M \models \sigma \leftrightarrow \sigma^+$.

Models of $\mathbf{TT}^*$ are called ambiguous models of **TT**, models M of **TT** for which $M \cong M^+$ are called shifting models of **TT**.

Of course, an isomorphism f: M$\rightarrow$N from one **TT**-model onto the other is a sequence $(f_i)_{i \in \omega}$ of bijections f_i from M_i onto N_i for which the following holds:

for all $i \in \omega$, and all $x \in M_i$, $y \in M_{i+1}$: $x \in_M y \Leftrightarrow f_i x \in_N f_{i+1} y$

An isomorphism f from M onto M^+ then is a sequence of bijections $M_0 \xrightarrow{f_0} M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \ldots$ (preserving $\in$) which is why f is called a shift and M a shifting model of **TT**.

In [Specker 62], an interesting connection between **NF** and $\mathbf{TT}^*$ is virtually stated and proved:

Theorem 3.1.2. **NF** and $\mathbf{TT}^*$ have the same stratified theorems

The proof is model-theoretic and the one presented here, taken from [Boffa 77a] makes use of the following isomorphism theorem (theorem 6.1.15 in [Chang-Keisler 73]).

Lemma 3.1.3 Two models are elementary equivalent if and only if they have isomorphic ultrapowers

To apply this lemma here, we first redefine the concept ultrapower for **TT**-models $M = (M_0, M_1, M_2, ..., \in_M)$. For convenience, the concept is explained for models of a one-sorted theory first.

Let $I \neq \emptyset$. An ultrafilter D over I is a set of subsets $D \subseteq PI$ for which the following holds:

(1) $I \in D$

(2) $X, Y \in D \Rightarrow X \cap Y \in D$

(3) $X \in D$ & $X \subseteq Z \subseteq I \Rightarrow Z \in D$

(4) $X \in D \Leftrightarrow I\text{-}X \notin D$

When $\mathcal{A}_i = (A_i, ...)$ is a model of some language for each $i \in I$, and D is an ultrafilter over I, we define an equivalence relation $\sim_D$ on the product $\prod_{i \in I} A_i = \{ f: I \rightarrow \bigcup_{i \in I} A_i \mid \forall i \in I\, (f i \in A_i) \}$ as follows:

$$f \sim_D g \Leftrightarrow \{ i \in I \mid f i = g i \} \in D$$

The reduced product of the A_i's modulo D is the set $\prod_D A_i = \{ |f| : f \in \prod_{i \in I} A_i \}$ where $|f|$ is the equivalence class of f under $\sim_D$. The ultraproduct $\prod_D \mathcal{A}_i$ then is the model with as universe $\prod_D A_i$ and as interpretations for the symbols in the language the following (a symbol with the subscript i denotes the interpretation of that symbol in $\mathcal{A}_i$):

(1) when R is an n-placed relation symbol, R^* is defined by

$$R^*(|f_1|, |f_2|, ..., |f_n|) \Leftrightarrow \{ i \in I \mid R_i(f_1 i, f_2 i, ..., f_n i) \} \in D$$

(2) when F is an n-placed function symbol, F^* is defined by

$$F^*(|f_1|, |f_2|, ..., |f_n|) = |g| \Leftrightarrow \{ i \in I \mid F_i(f_1 i, f_2 i, ..., f_n i) = g i \} \in D$$

(3) when c is a constant,

$$c^* = |(c_i)_{i \in I}|$$

When all the $\mathcal{A}_i$'s are the same model $\mathcal{A} = (A, ...)$, the ultraproduct is called an ultrapower of $\mathcal{A}$, denoted by $\mathcal{A}^I/D$ (its universe is denoted by A^I/D). A fundamental theorem states that $\mathcal{A}^I/D$ is elementary equivalent to $\mathcal{A}$ for any I and D.

For **TT**-models $M = (M_0, M_1, M_2, ..., \in_M)$ we define the ultrapower of M for appropriate I and D as follows:

$M^I/D = (M_0{}^I/D, M_1{}^I/D, ..., \in_M^*)$ with the $\in$-relation defined as usual:

for $|f| \in M_j{}^I/D, |g| \in M_{j+1}{}^I/D$: $\quad |f| \in_M^* |g| \Leftrightarrow \{ i \in I \mid f i \in_M g i \} \in D \quad (j \in \omega)$

Keisler's isomorphism theorem holds in this context: if $M \equiv N$ (i.e. they satisfy the same stratified sentences), then $M^I/D \cong N^I/D$ for some I,D.

When $M \models \mathbf{TT}^*$, by 3.1.1.(b) $M \equiv M^+$ so for some I and D: $M^I/D \cong (M^+)^I/D$. Because $(M^+)^I/D = (M^I/D)^+$ and any model is elementary equivalent to all its ultrapowers, every ambiguous model of **TT** is elementary equivalent to a shifting **TT**-model.

When N is a shifting model of **TT**, where f is the shift, a model of **NF** is given by $(N_0, \in)$ where $x \in y$ is defined as $x \in_N f_0 y$: EXT and COMP follow directly from the corresponding axioms of **TT** and the fact that f_0 is bijective.

Consider the following diagram:

$$\begin{array}{ccccccc}
N_0 & \xrightarrow{f_0} & N_1 & \xrightarrow{f_1} & N_2 & \xrightarrow{f_2} & \cdots \\
\downarrow id & & \downarrow f_0^{-1} & & \downarrow f_0^{-1} f_1^{-1} & & \cdots \\
N_0 & \xrightarrow[id]{} & N_0 & \xrightarrow[id]{} & N_0 & \xrightarrow[id]{} & \cdots
\end{array}$$

When $i \in \omega$, $x \in N_i$ and $y \in N_{i+1}$, then $x \in_N y$ is equivalent to $f_0^{-1} f_1^{-1} \ldots f_{i-1}^{-1} x \in_N f_1^{-1} f_2^{-1} \ldots f_i^{-1} y$ (as f is a shift). This in turn is equivalent to $f_0^{-1} f_1^{-1} \ldots f_{i-1}^{-1} x \in f_0^{-1} f_1^{-1} \ldots f_i^{-1} y$ because of the definition of $\in$. So the sequence $(id, f_0^{-1}, f_0^{-1} f_1^{-1}, \ldots)$ is an isomorphism from N onto $N_0^{\#} = (N_0, N_0, N_0, \ldots, \in)$.

Combined, the two facts above give us that every ambiguous model of **TT** is elementary equivalent to a **TT**-model of the form $N^{\#}$ where $N \models \mathbf{NF}$.

Conversely, for every model N of **NF**, $N^{\#}$ clearly is a an ambiguous (even a shifting) model of **TT**.

So we have established the following

Lemma 3.1.4. $\quad M \models \mathbf{TT}^* \Leftrightarrow \exists N \models \mathbf{NF}\,(M \equiv N^{\#})$

proof of theorem 3.1.2.: a theorem of $\mathbf{TT}^*$ is a theorem of **NF** too because **NF** strictly contains $\mathbf{TT}^*$. Conversely, when σ is a stratified theorem of **NF**, it holds in all models of the form $N^{\#}$ where $N \models \mathbf{NF}$ (because N and $N^{\#}$ satisfy the same stratified sentences). By lemma 3.1.4., we infer that $M \models \sigma$ for all ambiguous **TT**-models M. So $\mathbf{TT}^* \models \sigma$. Because of the completeness of (many-sorted) first order predicate calculus, it follows that $\mathbf{TT}^* \vdash \sigma$.

Corollary 3.1.5. every stratified theorem of **NF** has a stratified proof (i.e. a proof composed of stratified theorems)

Moreover, as $\bot$ is stratified, we have

Corollary 3.1.6. **TT*** and **NF** are equiconsistent.

As $\neg$AC is stratified, AC is refutable in **TT*** by theorems 3.1.2. and 2.2.3.. Crabbé proved in [84] that adding two (specific) axioms of ambiguity to **TT** suffices to disprove AC, whereas **TT**+AC+ ($\sigma \leftrightarrow \sigma^+$) is consistent for any stratified σ (so AC cannot be disproved in **TT**+ one axiom of ambiguity).

§2. Boolean algebras

Definition 3.2.1. A Boolean algebra is a structure $(X, \cap, \cup, ^c)$ where $X \subseteq PY$ for some set Y, $\cap$, $\cup$ and ^{c} are the usual set-theoretic operations intersection, union and complement, and X is closed under these three operations (and thus contains $\emptyset$ and Y).

For a Boolean algebra $(X, \cap, \cup, ^c)$, the binary relation $\leq$ on X is defined by $x \leq y \Leftrightarrow x \cap y = x$.

An atom of a Boolean algebra is an element $x \neq \emptyset$ for which $\emptyset \leq y \leq x$ implies $y = \emptyset \vee y = x$.

An atomic Boolean algebra is one in which every element $\neq \emptyset$ contains an atom.

For brevity, we use the notation $(X, \leq)$ for a Boolean algebra, assuming $\cap$, $\cup$ and ^{c} to be basic operations on X. Obviously, when $X \subseteq PY$ and X contains $P_1 Y$, then X is atomic.

The result needed in the sequel is the following.

Theorem 3.2.2. all infinite atomic Boolean algebras are elementary equivalent

Proof: (sketch) Let $(X_1, \leq)$ and $(X_2, \leq)$ where $X_1 \subseteq PY_1$ and $X_2 \subseteq PY_2$ be two infinite atomic Boolean algebras. The elementary equivalence of $(X_1, \leq)$ and $(X_2, \leq)$ is proved by means of an Ehrenfeucht-game: a winning strategy for the second player in a game of length n is given by the following.

By choosing an element of X_1 (X_2) at step i, the first player subdivides Y_1 (Y_2) in 2^i parts. The second player takes care that he chooses an element of X_2 (X_1) so that the "atomic" sizes of the resulting parts resemble those in X_1 (X_2): when such a part contains only finitely many atoms (possibly only one), his choice will be one giving the corresponding part an equal number of atoms; when such a part contains infinitely many atoms, his choice will be so that in the rest of the game he will never be outwitted to reveal the possible finitude of the number of atoms contained in the corresponding part: a number greater than or equal to 2^{n-i} is effectively infinite, as the remaining n-i steps are insufficient to reveal its finitude.

Note that the infinity of X_1 and X_2 makes it possible to follow this procedure: it is easy to show that an infinite atomic Boolean algebra contains infinitely many atoms.

A statement whose proof very much resembles the one given above is lemma 3.4.5.: two countably infinite Boolean algebras which satisfy two further conditions, are isomorphic.

§3. k-stratified fragments of **TT** and **NF**

For $k>0$, $\mathbf{TT}_k$ is defined as the fragment of **TT** generated by the first k types 0, 1,...., k-1. So the language of $\mathbf{TT}_k$ only has variables of type 0 to k-1. Formulae of $\mathbf{TT}_k$ are called k-stratified. The axioms of $\mathbf{TT}_k$ (for $k \geq 2$) are: extensionality for sets of type 1 to k, and all k-stratified comprehension axioms. $\mathbf{NF}_k$ results from $\mathbf{TT}_k$ when dropping the types; so the axioms of $\mathbf{NF}_k$ (for $k \geq 2$) are extensionality, and comprehension for formulae which are (or rather: can be) k-stratified. The language of $\mathbf{NF}_k$ is the same as that of **NF**.

Note that EXT is 2-stratified and that instances of COMP are at least 2-stratified, so $\mathbf{NF}_1$ and $\mathbf{TT}_1$ are both the elementary theory of equality.

$\mathbf{TT}^*_k$ is defined as $\mathbf{TT}_k$ with all k-stratified axioms of ambiguity added, that is, all sentences of the form $\sigma \leftrightarrow \sigma^+$ where σ is a (k-1)-stratified sentence.

By a modified Specker-argument, we see that $\mathbf{NF}_k$ and $\mathbf{TT}^*_k$ have the same k-stratified theorems:

For $M = (M_0, M_1, M_2, ..., M_{k-1}, \in) \models \mathbf{TT}_k$ $(k \geq 2)$, define $M^- = (M_0, M_1, ..., M_{k-2}, \in)$ and $M^+ = (M_1, M_2, ..., M_{k-1}, \in)$. M^- and M^+ are models of $\mathbf{TT}_{k-1}$.

Lemma 3.3.1. $\quad M \models \mathbf{TT}^*_k \Leftrightarrow M \models \mathbf{TT}_k \;\&\; M^- \equiv M^+$

Proof:

$\Rightarrow$: assume $M \models \mathbf{TT}^*_k$, and let σ be a (k-1)-stratified sentence. $M \models \sigma \leftrightarrow \sigma^+$, so

$$M^- \models \sigma \Leftrightarrow M \models \sigma \Leftrightarrow M \models \sigma^+ \Leftrightarrow M^+ \models \sigma$$

$\Leftarrow$: assume $M \models \mathbf{TT}_k \;\&\; M^- \equiv M^+$, and let σ be a (k-1)-stratified sentence. Clearly

$$M \models \sigma \Leftrightarrow M^- \models \sigma \Leftrightarrow M^+ \models \sigma \Leftrightarrow M \models \sigma^+.$$

Lemma 3.3.2. $\quad M \models \mathbf{TT}^*_k \Leftrightarrow \exists N' \models \mathbf{NF} \qquad M \equiv (N', \in)_{i<k}$

Proof:

$\Leftarrow$: obvious from lemma 3.3.1.

$\Rightarrow$: when M is a model of $\mathbf{TT}^*_k$, we have $M^- \equiv M^+$ by 3.3.1., so M^- and M^+ have isomorphic ultrapowers by 3.1.3.: for some I and D we have $(M^-)^I/D \cong (M^+)^I/D$. So for $N = M^I/D \models \mathbf{TT}_k$ we have $N^- \cong N^+$, which implies that there is a shift $N_0 \xrightarrow{f_0} N_1 \xrightarrow{f_1} .... \xrightarrow{f_{k-2}} N_{k-1}$, while $N \equiv M$. As in the proof of 3.1.4., this leads to the fact that M is elementary equivalent to a model of the form $(N', \in)_{i<k}$ where $N' \models \mathbf{NF}$.

Theorem 3.3.3. $\mathbf{NF}_k$ and $\mathbf{TT}_k^*$ have the same k-stratified theorems

Proof: let σ be a k-stratified sentence. When $\mathbf{TT}_k^* \vdash \sigma$, also $\mathbf{NF}_k \vdash \sigma$, as $\mathbf{NF}_k$ includes $\mathbf{TT}_k^*$.

Conversely, when $\mathbf{NF}_k \vdash \sigma$, σ holds in all models of $\mathbf{NF}_k$, so by lemma 3.3.2., $\mathbf{TT}_k^* \models \sigma$ (as N and $(N, \in)_{i<k}$ satisfy the same k-stratified sentences for models $(N, \in) \models \mathbf{NF}$). Hence $\mathbf{TT}_k^* \vdash \sigma$.

So $\mathbf{NF}_k$ and $\mathbf{TT}_k^*$ are equiconsistent.

As **NF** is in a certain sense the limit of $\mathbf{NF}_k$ for $k \to \infty$, it seems worthwhile to investigate the consistency of $\mathbf{NF}_k$ for k=1,2,3,.... in order to get some grip on the consistency problem for **NF**. We will see in the next two sections that $\mathbf{NF}_3$ is consistent, and that $\mathbf{NF}=\mathbf{NF}_4$. But let us first show that $\mathbf{TT}_k$-models may be assumed to be of a certain standard form.

Lemma 3.3.4. every $\mathbf{TT}_k$-model is, up to isomorphism, of the form
$M = (M_0, M_1, M_2, ..., M_{k-1}, \in_M)$
where $M_{i+1} \subseteq PM_i$ (i=0, 1,...., k-2) and $\in_M = \bigcup_{i<k-1} \{ (a,b) \in M_i \times M_{i+1} \mid a \in b \}$

Proof: Let $N = (N_0, N_1, N_2, ..., N_{k-1}, \in_N)$ be a model of $\mathbf{TT}_k$. Define M inductively by
$M_i = h_i[N_i]$, where $h_0 = \mathrm{id}$ and $h_i: N_i \to PM_{i-1}$ is the function defined by
$h_i: a \mapsto h_{i-1}[\{ c \in N_{i-1} \mid c \in_N a \}]$ ($i=1,....,k-1$; $a \in N_i$). Obviously $M_{i+1} \subseteq PM_i$ for all $i<k-2$; defining $\in_M$ as above, M becomes a model of $\mathbf{TT}_k$ which is isomorphic to N: for all $i<k-1$, $a \in N_i$, $b \in N_{i+1}$, we have $a \in_N b \Leftrightarrow h_i a \in h_{i+1} b$:

$$h_0 a \in h_1 b \quad \Leftrightarrow a \in \{ c \in N_0 \mid c \in_N b \}$$
$$\Leftrightarrow a \in_N b;$$

for $i>0$:

$$h_i a \in h_{i+1} b \quad \Leftrightarrow h_{i-1}[\{ c \in N_{i-1} \mid c \in_N a \}] \in h_i[\{ d \in N_i \mid d \in_N b \}]$$
$$\Leftrightarrow \exists d \in N_i (h_i d = h_{i-1}[\{ c \in N_{i-1} \mid c \in_N a \}] \,\&\, d \in_N b)$$
$$\Leftrightarrow a \in_N b \quad :$$

($\Rightarrow$): let $e \in N_{i-1}$ be arbitrary. Then

$$e \in_N a \Leftrightarrow e \in \{ c \in N_{i-1} \mid c \in_N a \} \Leftrightarrow h_{i-1} e \in h_i d \Leftrightarrow e \in_N d$$

(the last equivalence being given by the induction hypothesis; the next to last by the premiss).
As $N \models \mathrm{EXT}_i$, it follows that $a = d$, so $a \in_N b$ follows from $d \in_N b$.
($\Leftarrow$): when $a \in_N b$, we have $h_i a = h_{i-1}[\{ c \in N_{i-1} \mid c \in_N a \}]$ by definition.

Remark: the only thing needed for the proof is $N \models \mathrm{EXT}_i$ for $0<i<k-1$.

Let (for $k \geq 2$) $\mathbf{TT}_k^{\infty}$ ($\mathbf{TT}^{\infty}$) be the theory resulting from $\mathbf{TT}_k$ ($\mathbf{TT}$) by adding the axioms σ_n ($n \in \omega$), where $\sigma_n = \exists \bar{x}_0 \prod_{1 \leq i < j \leq n} (x_{i0} \neq x_{j0})$ asserts the existence of at least n individuals.

Proposition 3.3.5. All models of $\mathbf{TT}_k^*$ are models of $\mathbf{TT}_k^{\infty}$

Proof: Let M be a model of $\mathbf{TT}_k^*$. Assume that M is a standard model as described in 3.3.4. above. Suppose that M_0 is finite, say $|M_0| = m$. Because $M_1 = PM_0$ ($M_1 \supseteq P_1M_0$ and M_1 is closed under unions), it follows from $m < m+1 \leq 2^m$ that $M \not\models \sigma_{m+1}$ but $M \models \sigma^+_{m+1}$.

So M_0 must be infinite, whence M is a model of $\mathbf{TT}_k^{\infty}$.

§4. consistency of $\mathbf{NF}_3$

The following proposition, as well as much of the next section, comes from [Boffa 77b]. From now on, we will always assume our $\mathbf{TT}_k$-models to be standard.

Proposition 3.4.1. $\mathbf{TT}_2$ is equivalent to the elementary theory of atomic Boolean algebras (that is, they are interpretable in each other)

Proof: when $(M_0, M_1, \in) \models \mathbf{TT}_2$, we have $M_1 \supseteq P_1 M_0$, so $(M_1, \subseteq)$ is an atomic Boolean algebra. When $(B, \leq)$ is an atomic Boolean algebra, we define $M_B = (M_0, M_1, \in)$ by $M_0 =$ the set of atoms of B; M_1=B; $\in\, = \leq$. $(B, \leq)$ is elementary equivalent to the Boolean algebra $B^* = (PX, \subseteq)$ for a certain X (trivial for finite B; all infinite atomic Boolean algebras are elementary equivalent by theorem 3.2.2.). As $M_{B^*} = (P_1X, PX, \subseteq) \models \mathbf{TT}_2$, and $(B, \leq) \equiv B^*$ implies $M_B \equiv M_{B^*}$, we have $M_B \models \mathbf{TT}_2$.

Corollary 3.4.2. (a) $\mathbf{TT}_2 = \mathrm{EXT}_1 +$ existence of singleton, union and complement;

(b) $\mathbf{NF}_2 = \mathrm{EXT}+$ existence of singleton, union and complement;

(c) all models of $\mathbf{TT}_2^\infty$ are elementary equivalent;

(d) the models of $\mathbf{TT}_3^*$ are exactly the models of $\mathbf{TT}_3^\infty$.

Proof: (a) to (c) are obvious. For (d), 3.3.5. states that any model of $\mathbf{TT}_3^*$ is a model of $\mathbf{TT}_3^\infty$; when M is a model of $\mathbf{TT}_3^\infty$, (c) tells us that M^- and M^+ are elementary equivalent $\mathbf{TT}_2$-models, whence $M \models \mathbf{TT}_3^*$ by 3.3.1. (also see [Boffa-Crabbé 75]).

Corollary 3.4.3. (Grishin) $\mathbf{NF}_3$ is consistent

The original proof of 3.4.3. given by Grishin in [69] is more intricate and makes use of the compactness theorem in a subtle way. For this reason, we here give a sketch of his proof.

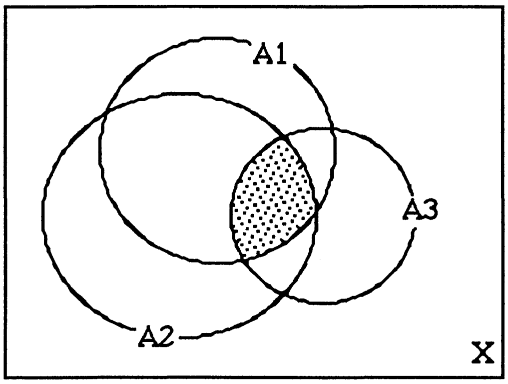
Definition 3.4.4. a Grishin-algebra on X is a countably infinite Boolean algebra $\mathbf{A} \subseteq PX$ which includes P_1X, and for which any infinite $A \in \mathbf{A}$ can be decomposed into two equinumerous parts, that is, there is an $A^* \in \mathbf{A}$ so that $A^* \subseteq A$ and $|A| = |A - A^*|$.

Lemma 3.4.5. any two Grishin-algebras are isomorphic with respect to inclusion.

Proof: by a back-and-forth argument: let $\mathbf{A} \subseteq PX$ and $\mathbf{B} \subseteq PY$ be two Grishin-algebras. Fix enumerations of **A** and **B**. We construct new enumerations $\{A_i\}_{i\in\omega}$, $\{B_i\}_{i\in\omega}$ of **A** and **B** respectively so that for any $n\in\omega$ the 2^n parts in which X is divided by $A_1,....,A_n$ have the same powers as their counterparts in Y.

We start the enumeration by setting $A_1=X$, $B_1=Y$.

For even n, we take A_{n+1} to be the first element of our fixed enumeration not yet selected and construct a B_{n+1} which behaves with respect to $B_1,....,B_n$ in the same way as A_{n+1} does with respect to $A_1,....,A_n$. Example for n=2:



A_1 to A_n divide X in 2^n parts; consider one of these parts, for instance $A=A_1\cap A_2$. When $A_3\cap A$ is finite, we can form a corresponding equinumerous part $B\subseteq B_1\cap B_2$ by the fact that **B** contains all singletons and unions; when it is infinite, $B_1\cap B_2$ is infinite too, so it can be decomposed in two infinite parts. One of these parts is taken to be B. B_{n+1} then is defined as the union of all so constructed equinumerous parts when we iterate this procedure 2^n times.

For odd n, the roles of **A** and **B** are interchanged.

When $i, j\in\omega$, we have

$$A_i\subseteq A_j \Leftrightarrow |A_i\cap A_j^c|=0 \Leftrightarrow |B_i\cap B_j^c|=0 \Leftrightarrow B_i\subseteq B_j$$

so f: $\mathbf{A}\to\mathbf{B}$, $A_i\mapsto B_i$ $(i\in\omega)$, is an isomorphism with respect to $\subseteq$.

Grishin's proof of 3.4.3.: Suppose $M=(M_0, M_1, M_2, \in)$ is a model of $\mathbf{TT}_3$, where M_1 and M_2 are Grishin-algebras. Let ϕ: $M_1\to M_2$ be the $\subseteq$-isomorphism given by lemma 3.4.5.. Then the functions f_0: $M_0\to M_1$ and f_1: $M_1\to M_2$ defined by

$$\begin{cases} f_0\, a=b \Leftrightarrow \phi\{a\}=\{b\} & (a\in M_0,\ b\in M_1) \\ f_1=\phi \end{cases}$$

yield a shift of M: f_0 and f_1 are obviously bijective (as ϕ is), and for all $a\in M_0$, $b\in M_1$:

$$a\in b \Leftrightarrow \{a\}\subseteq b \Leftrightarrow \phi\{a\}\subseteq \phi b \Leftrightarrow \{f_0\, a\}\subseteq f_1 b \Leftrightarrow f_0\, a\in f_1 b$$

Consequently, $\mathbf{NF}_3$ is consistent. So it remains to construct a model of the form mentioned above.

Consider the theory $T=\mathbf{TT}_7^{\infty}+\{\forall x_{i+1}\, Fin\,(x_{i+1}) \mid i=0,1,2\}$ where $Fin\,(x_{i+1})$ expresses the fact that x_{i+1} is finite (for instance: $Fin\,(x_{i+1}) \equiv x_{i+1} \in \bigcup Nn_{i+3}$; the definition $Nn_5 = \bigcap \{ x_5 \mid 0_4 \in x_5 \wedge \forall y_4 \in x_5\,(y_4+1_4 \in x_5) \}$ requires seven types).

By compactness, T is consistent, so by the Löwenheim-Skolem-Tarski theorem it has a countable model. The part of this model formed by the first three levels M_0, M_1, M_2 is a model of $\mathbf{TT}_3$. Obviously M_1 and M_2 are countably infinite Boolean algebras including all singleton subsets; it remains to show that every infinite element of M_1 (M_2) can be decomposed into two equinumerous parts. Denote by $Zr\,(x_{i+1})$ (i=0,1) the formula of $\mathbf{TT}_7$ expressing "either x_{i+1} or x_{i+1} without one element decomposes into two equinumerous disjoint sets".

In T, the sentence $\forall x_{i+1}\, Zr\,(x_{i+1})$ (i=0,1) is provable as $Fin\,(x_{i+1})$ implies $Zr\,(x_{i+1})$; the intricacy in the proof is that x_{i+1}, though externally possibly infinite, can be considered as internally finite.

The provability of the sentence above implies that for every $A \in M_i$ (i=1,2), either A or A without one element decomposes into two equinumerous sets; thus when A is infinite it decomposes into two infinite sets, which completes the proof.

§5. axiomatic reduction of **NF** to $\mathbf{NF}_4$

Define $E_3=\{\ \{\{x_0\},y_1\}\mid x_0\in y_1\ \}$. The existence of E_3 is assured in $\mathbf{TT}_k$ for $k\geq 4$: it is stated by a 4-stratified comprehension axiom: "E_3 exists" (or "E_3" for short) is the sentence

$$\exists z_3\forall w_2\,(\,w_2\in z_3\leftrightarrow\exists x_0 y_1\,[\,\forall u_1\,(\,u_1\in w_2\leftrightarrow[\,u_1=y_1\vee\forall v_0\,(\,v_0\in u_1\leftrightarrow v_0=x_0\,)\,]\,)\wedge x_0\in y_1\,]\,)$$

$\mathbf{TT}_k^+$ is defined as the theory that results when all axioms of $\mathbf{TT}_k$ are 'plussed'.

Proposition 3.5.1. $\quad \mathbf{TT}_4=\mathbf{TT}_3+\mathbf{TT}_3^+ +(\,E_3\text{ exists}\,)$

Proof: obviously $\mathbf{TT}_4$ implies the theory on the right. So it suffices to prove the axioms of $\mathbf{TT}_4$ in the right hand theory. All EXT-axioms of $\mathbf{TT}_4$ and all 3-stratified COMP-axioms are axioms of $\mathbf{TT}_3+\mathbf{TT}_3^+$. Accordingly it is left to prove every 4-stratified COMP-axiom in the right hand theory. The problem is that when φ is 4-stratified, it possibly contains variables of type 0 to 3, so we have to do something about this before being able to apply COMP-axioms in $\mathbf{TT}_3+\mathbf{TT}_3^+$.

Our plan is to rewrite φ to an equivalent formula φ^* which does not contain variables of type 0. So we have to express atoms of the form $v_0\in v_1$ or $v_0=w_0$ in a form that does not contain variables of type 0 any more. Luckily, E_3 codes $v_0\in v_1$, while $v_0=w_0$ can be expressed by $\{v_0\}=\{w_0\}$. As it is possible that some occurrence of a variable of type 0 was bound by $\exists$ (or $\forall$), one further adjustment has to be made for these cases: defining $S_2=P_1V_1=P_1\{\,x_0\mid x_0=x_0\,\}$, the complete substitution plan to get φ^* from φ looks as follows:

$$\left\{\begin{array}{ccc} v_0\in v_1 & \mapsto & \{\{v_0\},v_1\}\in E_3 \\ v_0=w_0 & \mapsto & \{v_0\}=\{w_0\} \\ \exists v_0\,\psi\,(\ldots.\{v_0\}\ldots.) & \mapsto & \exists v_1\in S_2\,\psi\,(\ldots.v_1\ldots.) \end{array}\right.$$

Clearly φ^* and φ are equivalent in $\mathbf{TT}_3+\mathbf{TT}_3^+ +E_3$ (we need $\mathbf{TT}_3$ for the existence of $\{x_0\},\{x_1,y_1\}$ and S_2).

When φ is a 4-stratified formula, $\{\,x_0\mid\varphi(x_0)\,\}=\bigcup\{\,v_1\in S_2\mid\varphi^*(v_1)\,\}$, and $\{\,x_i\mid\varphi\,\}=\{\,x_i\mid\varphi^*\,\}$ $(i=1,2)$ exist by means of $\mathbf{TT}_3$ ($\bigcup x_2$) and appropriate COMP-axioms of $\mathbf{TT}_3^+$.

Proposition 3.5.2. $\mathbf{TT}_5 = \mathbf{TT}_4 + \mathbf{TT}_4^+$;

$\mathbf{TT}_6 = \mathbf{TT}_5 + \mathbf{TT}_5^+$;

etcetera;

$\mathbf{TT} = \mathbf{TT}_4 + \mathbf{TT}_4^+ + \mathbf{TT}_4^{++} + \ldots.$

Proof: except for the last equality, the proof is analogous to that of the proposition above; existence of E_3 doesn't have to be assumed here because it is already supplied for in $\mathbf{TT}_k$ for $k \geq 4$.

Lastly,

$$\mathbf{TT} = \mathbf{TT}_4 + \mathbf{TT}_5 + \ldots = \mathbf{TT}_4 + (\mathbf{TT}_4 + \mathbf{TT}_4^+) + (\mathbf{TT}_4 + \mathbf{TT}_4^+ + \mathbf{TT}_4^{++}) + \ldots =$$

$$= \mathbf{TT}_4 + \mathbf{TT}_4^+ + \mathbf{TT}_4^{++} + \ldots.$$

Let, in **NF**, E be defined by $E = \{ \{\{x\}, y\} \mid x \in y \}$

Theorem 3.5.3. [Grishin 72] $\mathbf{NF} = \mathbf{NF}_4 = \mathbf{NF}_3 + \text{existence of } E$

Proof: Clearly **NF** contains the other two theories. To prove the axioms of **NF** in $\mathbf{NF}_4$ and $\mathbf{NF}_3 + E$ we note that, by 3.1.2., for stratified sentences τ, $\mathbf{NF} \vdash \tau$ is equivalent to $\mathbf{TT}^* \vdash \tau$. From 3.5.1. and 3.5.2. we see that

$$\begin{aligned}
\mathbf{TT}^* &= \mathbf{TT} + \{ \sigma \leftrightarrow \sigma^+ \mid \sigma \text{ a stratified sentence} \} \\
&= \mathbf{TT}_4 + \mathbf{TT}_4^+ + \ldots + \{ \sigma \leftrightarrow \sigma^+ \} \\
&= \mathbf{TT}_4 + \{ \sigma \leftrightarrow \sigma^+ \} \\
&= \mathbf{TT}_3 + \mathbf{TT}_3^+ + E_3 + \{ \sigma \leftrightarrow \sigma^+ \} \\
&= \mathbf{TT}_3 + E_3 + \{ \sigma \leftrightarrow \sigma^+ \}
\end{aligned}$$

So, when $\mathbf{NF} \vdash \tau$, we have $\mathbf{TT}_4 + \{ \sigma \leftrightarrow \sigma^+ \} \vdash \tau$ and $\mathbf{TT}_3 + E_3 + \{ \sigma \leftrightarrow \sigma^+ \} \vdash \tau$. Consequently, $\mathbf{NF}_4 \vdash \tau$ and $\mathbf{NF}_3 + E \vdash \tau$. As all axioms of **NF** are stratified, this completes the proof.

Boffa proves in [75b] that the reduction of **NF** to $\mathbf{NF}_3$+E is optimal (corollary 3.5.5. below).

Theorem 3.5.4. For each 3-stratified sentence σ:

$$\mathbf{NF} \vdash (\sigma \rightarrow \mathbf{NF}_3 + \sigma \text{ is consistent})$$

Proof: let us work in **NF**+σ. Define $M=(P_1{}^2V, P_1V, V, \in_M)$ where

$\{\{x\}\} \in_M \{y\} \Leftrightarrow \{x\} \in_M y \Leftrightarrow x \in y.$

One easily sees that M is a model of $\mathbf{TT}_3^\infty$+σ. So $\mathbf{TT}_3^\infty$+σ is consistent, which entails the consistency of $\mathbf{NF}_3$+σ (as $\mathbf{TT}_3^\infty$ and $\mathbf{NF}_3$ have the same 3-stratified theorems by 3.3.3. and 3.4.2.(d))

Corollary 3.5.5. There is no consistent set of 3-stratified axioms which entails **NF**

Proof: let AI^* be a 3-stratified version of the axiom of infinity , for instance

$$AI^* \equiv \exists x\, [\, x \neq \Lambda \wedge \forall y \in x\, \exists z \in x\, (\, y \subseteq z \wedge y \neq z\,)\,]$$

(note that $V \notin Fin$ is not 3-stratified: $V_1 \notin Fin_2 = \bigcup Nn_3$ requires 5 types)

Now suppose that Σ is a consistent set of 3-stratified axioms entailing **NF**. The theorem above tells us that (as $\mathbf{NF} \vdash AI^*$) we can prove the consistency of $\mathbf{NF}_3$+AI^* in Σ. This is in contradiction with Gödel's second incompleteness theorem, as we can develop Peano-arithmetic in $\mathbf{NF}_3$+AI^*, hence in Σ. (see for instance [Boffa 81])

Finally, let us remark that in [44], Hailperin reduced **NF** to EXT+finitely many 6-stratified COMP-axioms. As **NF**=$\mathbf{NF}_3$+E, we can improve on this result by stating that **NF** can be axiomatized by EXT + one 4-stratified COMP-axiom + finitely many 3-stratified COMP-axioms.

§6. consistency of NFU; NF versus NFU

Replacing EXT by

EXT^*: $\forall xy\,(\exists z\,(z\in x)\rightarrow[\forall z\,(z\in x\leftrightarrow z\in y)\rightarrow x=y\,])$

we come to the system **NFU** (U for Urelements). Jensen proved in [68] that this modification of **NF** is consistent. The proof of the consistency of **NFU** given here has been taken from [Boffa 77a].

TTU is the analogue of **NFU** in **TT**: we replace EXT_{i+1} by

EXT^*_{i+1}: $\forall x_{i+1}y_{i+1}\,(\exists z_i\,(z_i\in x_{i+1})\rightarrow[\forall z_i\,(z_i\in x_{i+1}\leftrightarrow z_i\in y_{i+1})\rightarrow x_{i+1}=y_{i+1}\,])$

Let $M=(M_i,\in)_{i\in\omega}$ be a model of **TTU**. For every increasing sequence of natural numbers $i_0<i_1<i_2<\ldots$. we define a new model $N=(M_{i_n},\in)_{n\in\omega}$ of **TTU** as follows: for $a\in M_{i_n}$, $b\in M_{i_{n+1}}$:

$$N\vDash a\in b \quad\Leftrightarrow\quad M\vDash\exists z\,(a\in z\wedge b=\{z\}^{i_{n+1}-i_n-1})$$

where $\{z\}^m=\{\ldots.\{z\}\ldots.\}$ (m times $\{.\}$); $\{z\}^0=z$

Note that in M, z has type i_n+1, so b has type i_{n+1}, as required.

Though a tedious thing to do, we check that N is a model of **TTU**.

EXT^*_{n+1}: assume $a,b\in M_{i_{n+1}}$ and $N\vDash\exists z\,(z\in a)\wedge\forall z\,(z\in a\leftrightarrow z\in b)$, that is,

(1) $M\vDash\exists z\exists w\,(z\in w\wedge a=\{w\}^{i_{n+1}-i_n-1})$, and

(2) $M\vDash\forall z\,[\exists w\,(z\in w\wedge a=\{w\}^{i_{n+1}-i_n-1})\leftrightarrow\exists w\,(z\in w\wedge b=\{w\}^{i_{n+1}-i_n-1})\,]$.

We have to prove $N\vDash a=b$, that is, $M\vDash a=b$. As $M\vDash\text{EXT}^*_{i_{n+1}}$ it is sufficient to prove that

(3) $M\vDash\exists z\,(z\in a)$,and

(4) $M\vDash\forall z\,(z\in a\leftrightarrow z\in b)$.

(a) when $i_{n+1}=1+i_n$, (1) reads $M\vDash\exists z\exists w\,(z\in w\wedge a=w)$, so (3) holds; while (2) reads $M\vDash\forall z\,[\exists w\,(z\in w\wedge a=w)\leftrightarrow\exists w\,(z\in b\wedge b=w)\,]$, so (4) holds.

(b) when $i_{n+1}\geq 2+i_n$, we deduce from (1):

$M\vDash\exists w\,(\{w\}^{i_{n+1}-i_n-2}\in a)$ so (3) holds for $z=\{w\}^{i_{n+1}-i_n-2}$.

Assume $c\in M_{i_{n+1}-1}$ and $M\vDash c\in a$. From (1) it follows that

$M\vDash\exists z\exists w\,(z\in w\wedge a=\{w\}^{i_{n+1}-i_n-1}\wedge c=\{w\}^{i_{n+1}-i_n-2})$.

By (2) then $M\models\exists z\exists w\,(z\in w\wedge b=\{w\}^{i_{n+1}-i_n-1})$ so $M\models c\in b$.

$M\models c\in b\rightarrow c\in a$ is proved analogously.

$COMP_{n+1}$: let φ be a stratified formula not containing y_{n+1} free; let φ^* be the (stratified) formula resulting from φ when all types n are replaced by i_n.

We have to prove

$N\models\exists y_{n+1}\forall x_n\,(x_n\in y_{n+1}\leftrightarrow\varphi)$, that is,

(5) $\quad M\models\exists y_{i_{n+1}}\forall x_{i_n}\,[\exists z_{1+i_n}(x_{i_n}\in z_{1+i_n}\wedge y_{i_{n+1}}=\{z_{1+i_n}\}^{i_{n+1}-i_n-1})\leftrightarrow\varphi^*]$

(a) when $i_{n+1}=i_n+1$, $COMP_{i_{n+1}}$ gives us $M\models\exists y_{i_{n+1}}\forall x_{i_n}(x_{i_n}\in y_{i_{n+1}}\leftrightarrow\varphi^*)$, so (5) holds.

(b) when $i_{n+1}\geq i_n+2$, $COMP_{1+i_n}$ gives $M\models\exists z_{1+i_n}\forall x_{i_n}(x_{i_n}\in z_{1+i_n}\leftrightarrow\varphi^*)$, supposing φ^* does not contain z_{1+i_n} free, so by taking $y_{i_{n+1}}=\{z_{1+i_n}\}^{i_{n+1}-i_n-1}$ we see that (5) holds.

Note that we have used $M\models\forall x_i$ "$\{x_i\}^m$ exists" for all i and m.

We say that N has been <u>extracted</u> from M; notation: $N\leq M$. $\leq$ is transitive.

We say that M <u>forces</u> a stratified sentence σ (notation: $M\Vdash\sigma$) when $N\models\sigma$ for all $N\leq M$;

M <u>decides</u> σ when $M\Vdash\sigma$ or $M\Vdash\neg\sigma$.

When M decides σ, clearly $M\models\sigma\leftrightarrow\sigma^+$ (as $M^+\leq M$).

<u>Lemma 3.6.1.</u> (Ramsey) If X is an infinite set and $[X]^n\subseteq G_1\cup G_2$, where $[X]^n$ denotes the set of all n-element subsets of X, then there is an infinite subset Y of X such that $[Y]^n\subseteq G_1$ or $[Y]^n\subseteq G_2$.

(no proof)

<u>Lemma 3.6.2.</u> (extraction lemma) for all stratified sentences σ,

$$M\models\mathbf{TTU}\Rightarrow\exists N\leq M\,(N\text{ decides }\sigma)$$

<u>Proof</u>: let k be greater than all types appearing in σ. Define a partition G_1, G_2 of $[\omega]^{k+1}$ as follows:

$G_1=\{\{i_0,i_1,\ldots,i_k\}\mid i_0<i_1<\ldots<i_k\;\&\;(M_{i_0},M_{i_1},\ldots,M_{i_k},\ldots)\models\sigma\}$;

$G_2=\{\{i_0,i_1,\ldots,i_k\}\mid i_0<i_1<\ldots<i_k\;\&\;(M_{i_0},M_{i_1},\ldots,M_{i_k},\ldots)\models\neg\sigma\}$.

By lemma 3.6.1., let X be an infinite set of natural numbers $i_0<i_1<\ldots<i_n<\ldots$ such that $[X]^{k+1}\subseteq G_1$ or $[X]^{k+1}\subseteq G_2$, and set $N=(M_{i_0},M_{i_1},\ldots,M_{i_n},\ldots)$. When $[X]^{k+1}\subseteq G_1$, then N forces σ; when $[X]^{k+1}\subseteq G_2$, then N forces $\neg\sigma$.

Theorem 3.6.3. for all models M of **TTU**, **NFU**+$\{ \sigma \mid M \Vdash \sigma \}$ is consistent

Proof: define $\Sigma = \{ \sigma \mid M \Vdash \sigma \}$. By a variant of Speckers method, it is sufficient to show the consistency of $\mathbf{TTU}^*+\Sigma$. A finite part Σ' of $\mathbf{TTU}^*+\Sigma$ can be divided in three finite parts:

(1) axioms of **TTU**: $\sigma_1, \ldots, \sigma_n$

(2) axioms of ambiguity: $\tau_1 \leftrightarrow \tau_1^+, \ldots, \tau_m \leftrightarrow \tau_m^+$

(3) sentences forced by M: $\upsilon_1, \ldots, \upsilon_p$

Define $\Sigma'' = \{ \sigma_1, \ldots, \sigma_n, \tau_1, \ldots, \tau_m, \upsilon_1, \ldots, \upsilon_p \}$. When a model $N \leq M$ decides all sentences in Σ'', then $N \models \mathbf{TTU}$ so $N \models \sigma_i$ (i=1,....n); $N \models \tau_i \leftrightarrow \tau_i^+$ (i=1,....m) by the remark made just above lemma 3.6.1.; and $N \models \upsilon_i$ (i=1,....p) because M forces υ_i. So $N \models \Sigma'$. By compactness, it is thus sufficient to show that there exists an $N \leq M$ deciding all sentences in Σ''; this is easily established by iterating lemma 3.6.2. n+m+p times ($\leq$ is transitive).

Corollary 3.6.4. **NFU** + AI + AC is consistent

Proof: it suffices to show that AI and AC are being forced by some model of **TTU**. Obviously, $(B, PB, P^2B, \ldots, \in)$ where B is an infinite set does the trick.

In **NFU**, 0 is the set of empty sets $\{ x \mid \neg\exists y (y \in x) \}$. Let Λ be an arbitrarily chosen element of 0. In **NFU** + AI, we have available Quine's ordered pair (x,y) for $x,y \notin 0$, as has been remarked in section 1.2. (remark 2 after 1.2.4.).

Let H be the statement that there are no more urelements then non-empty sets:

(H) $|0| \leq |V-0|$

Theorem 3.6.4. **NF** is equiconsistent with **NFU** + AI + H

Proof: We form a model of **NF** in **NFU** + AI + H. Let $f: 0 \to V-0$ be injective. Define $g: V \to V$ by

$$\begin{cases} gx = (fx, 0) & \text{for } x \in 0-\{\Lambda\} \\ g(fx, n) = (fx, n+1) & \text{for } x \in 0-\{\Lambda\},\ n \in Nn \\ gx = x & \text{otherwise} \end{cases}$$

By AI, g is injective. Its range clearly is $V-(0-\{\Lambda\})$.

Define $\in^* \subseteq V^2$ by $x \in^* y \Leftrightarrow x \in g\,y$. Then $(V, \in^*)$ is a model of **NF**:

EXT: $\forall z (z \in^* x \leftrightarrow z \in^* y)$ is equivalent to $\forall z (z \in g\,x \leftrightarrow z \in g\,y)$. Because of the fact that the only empty set in the range of g is Λ, we infer (by EXT^*) $gx = gy \ \vee\ gx = \Lambda = gy$. So x=y because g is injective.

COMP: $\exists y \forall x\,(x \in^* y \leftrightarrow \varphi)$ is equivalent to $\exists y \forall x\,(x \in g\,y \leftrightarrow \varphi)$. Obviously, $y = g^{-1}(\{x \mid \varphi\})$ is what we are looking for (more precisely, we infer $\exists y \forall x\,(x \in g\,y \leftrightarrow \varphi)$ from $\exists z \forall x\,(x \in z \leftrightarrow \varphi)$ and $\forall z \exists y\,(z = g\,y)$).

So we have established that consistency of **NFU** + AI + H implies consistency of **NF**. Conversely, **NF** includes **NFU** + AI + H (as **NF** $\vdash$ AI $\wedge$ $0 = \{\Lambda\}$).

As stated in [Boffa 73], we obtain

Corollary 3.6.5. **NF** is equiconsistent with **NFU** + Can(0)

Proof: As $P_1 0 \subseteq V\text{-}0$, H follows from $0 \sim P_1 0$. Boffa states that Specker's proof of **NF** $\vdash \neg$AC can be reproduced in **NFU** except for the point that PP_1x is equinumerous to P_1Px for arbitrary x. Here we must read Py as in the usual definition: $Py = \{ z \mid z \subseteq y \}$. This means that $0 \subseteq Py$ for all y. Defining $P'y = Py - (0\text{-}\{\Lambda\})$, for which it follows by the same argument as in 2.1.1.(c) that $P'P_1x \sim P_1P'x$ for all x (when we define $P_1x = x$ for $x \in 0$), let us reconsider the proof of $PP_1x \sim P_1Px$. Obviously, $Py = P'y \cup (0\text{-}\{\Lambda\})$, so $PP_1x = P'P_1x \cup (0\text{-}\{\Lambda\})$ and $P_1Px = P_1(P'x \cup (0\text{-}\{\Lambda\})) = P_1P'x \cup P_1(0\text{-}\{\Lambda\})$. From $0 \sim P_1 0$ it follows directly that $0\text{-}\{\Lambda\} \sim P_1(0\text{-}\{\Lambda\})$; the equinumerosity of $P'P_1x$ and $P_1P'x$ together with the disjointness of the unions involved then yield $PP_1x \sim P_1Px$. So (according to Boffa) Specker's disproof of AC can be re-established, so AI can be proved. Theorem 3.6.4. now tells us that consistency of **NFU**+Can(0) entails consistency of **NF**. Conversely, in **NF** $0 = \{\Lambda\} = P_1 0$.

4. Relative consistency results

§1. Quine's individuals; the permutation method

In [37], Quine wrote:
" "($x\in y$)" states that x is a member of y. *Prima facie*, this makes sense only when y is a class. However, we may agree on an arbitrary supplementary meaning for the case where y is an *individual* or non-class: we may interpret "($x\in y$)" in this case as stating that x *is* the individual y. "
In a footnote, he remarks: "this interpretation -along with extensionality- results in the fusion of every individual with its unit class, but this is harmless."
Scott proved in [62] that it is independent of **NF** whether there exists an individual in Quine's sense; this is, consistency of **NF** implies consistency of the two theories **NF**+$\exists x\,(x=\{x\})$ and **NF**+$\neg\exists x\,(x=\{x\})$.
The method used by Scott is known as the permutation method. We will now proceed to describe it.

Definition 4.1.1. a permutation is a bijective type-preserving unary operator

Remember that functions and unary operators are interchangeable notions as remarked above theorem 2.1.7..

So when π is a type-preserving unary operator in **NF**, π is a permutation when
NF$\vdash \forall y\exists z\forall x\,(y=\pi x\leftrightarrow x=z)$.

For a permutation π and a formula φ, the π-transform φ^{π} of φ is defined as the formula resulting from φ when all occurrences of $x\in y$ in φ are replaced by $x\in\pi y$ (where x and y are arbitrary variables).

Theorem 4.1.2. for all permutations π and all formulae φ: **NF**$\vdash\varphi\Rightarrow$ **NF**$\vdash\varphi^{\pi}$

Proof: as the π-transform of a proof of φ is the tree consisting of all π-transforms of the formulae in the proof, and the rules of inference are not influenced by taking π-transforms, it suffices to prove the statement in the case where φ is an axiom of **NF**.

EXT: when φ is $\forall xy\,[\forall z\,(z\in x\leftrightarrow z\in y)\rightarrow x=y]$, then φ^{π} is
$\forall xy\,[\forall z\,(z\in\pi x\leftrightarrow z\in\pi y)\rightarrow x=y]$. Supposing $\forall z\,(z\in\pi x\leftrightarrow z\in\pi y)$, we deduce $\pi x=\pi y$ by EXT. So x=y because π is injective.

COMP: let φ be $\exists y \forall x\,(x \in y \leftrightarrow \psi)$ where ψ is a stratified formula not containing y free. Then φ^{π} is $\exists y \forall x\,(x \in \pi y \leftrightarrow \psi^{\pi})$.y does not occur free in ψ^{π} and ψ^{π} is stratified as π is type-preserving, so by COMP we infer $\exists z \forall x\,(x \in z \leftrightarrow \psi^{\pi})$. As π is surjective, there is a y so that $z = \pi y$, which concludes the proof.

Proposition 4.1.3. when π is a permutation such that $\mathbf{NF} \vdash \varphi^{\pi}$, and **NF** is consistent, then **NF**+φ is consistent

Proof: suppose $\mathbf{NF}+\varphi \vdash \bot$. Then $\mathbf{NF} \vdash \neg\varphi$, so by theorem 4.1.2.: $\mathbf{NF} \vdash (\neg\varphi)^{\pi}$. As $(\neg\varphi)^{\pi}$ is $\neg\varphi^{\pi}$, we derive a contradiction from $\mathbf{NF} \vdash \varphi^{\pi}$ and $\mathbf{NF} \nvdash \bot$.

Theorem 4.1.4. $\exists x\,(x=\{x\})$ and $\neg\exists x\,(x=\{x\})$ are relatively consistent to **NF**

Proof: it all comes to finding two permutations π and ρ so that $\mathbf{NF} \vdash \varphi^{\pi} \wedge \neg\varphi^{\rho}$, where φ is the sentence $\exists x\,(x=\{x\})$. φ is an abbreviation for $\exists x \forall y\,(y \in x \leftrightarrow y=x)$, so for permutations π, φ^{π} is $\exists x \forall y\,(y \in \pi x \leftrightarrow y=x)$, which can be abbreviated to $\exists x\,(\pi x = \{x\})$.

When describing permutations, we will only indicate where they differ from the identity.

Let π be the permutation given by $\Lambda \leftrightarrow \{\Lambda\}$ (so Λ and $\{\Lambda\}$ are interchanged; $\pi x = x$ for $x \neq \Lambda, \{\Lambda\}$). As $\pi\Lambda = \{\Lambda\}$ we infer $\mathbf{NF} \vdash \varphi^{\pi}$.

We have to work harder to get ρ:

define two type-preserving operations F and G by

$$F: \begin{cases} \Lambda \mapsto \{\Lambda\} & \\ x \mapsto \Lambda & \text{when } x \neq \Lambda \text{ and } \Lambda \notin x \\ x \mapsto \{\{\Lambda\}\} & \text{when } \Lambda \in x \text{ and } \{\{\Lambda\}\} \notin x \\ x \mapsto \{\Lambda\} & \text{when } \Lambda \in x \text{ and } \{\{\Lambda\}\} \in x \end{cases}$$

$$G: \{x\} \mapsto \{x, Fx\}$$

We infer:

(1) $\forall x\,(x \neq Fx)$: clear from the definition of F

(2) $\forall xy\,(G\{x\} \neq \{y\})$: clear from (i) and the definition of G

(3) $\forall x\,(G\{x\} \neq x)$: $G\{x\}=x$ means $x=\{x,Fx\}$, so $x \neq \Lambda$; we derive $\Lambda \in x \leftrightarrow \Lambda \notin x$: suppose $\Lambda \notin x$, then $Fx=\Lambda$ so $\Lambda \in \{x,Fx\}=x$. Suppose $\Lambda \in x = \{x,Fx\}$, then it must be so that $Fx = \Lambda$ (as $x \neq \Lambda$). But then, by definition of F, $\Lambda \notin x$.

(4) $\forall xy\,(G\{x\}=G\{y\}\rightarrow x=y)$: suppose $G\{x\}=G\{y\}$ and $x\neq y$. Then we must have $Fx=y\wedge Fy=x$, so x is Λ, $\{\Lambda\}$ or $\{\{\Lambda\}\}$, whence Fx is (respectively) $\{\Lambda\}$, $\{\{\Lambda\}\}$ or Λ; and FFx is (respectively) $\{\{\Lambda\}\}$, Λ or $\{\Lambda\}$. But then $Fy=FFx\neq x$.

From (2) it follows that $P_1V\cap G[P_1V]=\Lambda$. Define the type-preserving operation ρ by $\{x\}\leftrightarrow G\{x\}$. Then $\forall x\,(\rho\rho x=x)$ and ρ is a permutation: for arbitrary y and $z=\rho y$:

$$y=\rho x\Leftrightarrow\rho\rho y=\rho x\Leftrightarrow\rho z=\rho x\Leftrightarrow z=x$$

where the last implication to the right follows from $P_1V\cap G[P_1V]=\Lambda$, (4) and $\{x\}=\{y\}\rightarrow x=y$. When $\rho x=\{x\}$ for some x, we infer $x=\rho\rho x=\rho\{x\}=G\{x\}$, in contradiction with (3). So we have $\mathbf{NF}\vdash\forall x\,(\rho x\neq\{x\})$, that is, $\mathbf{NF}\vdash\neg\varphi^{\rho}$.

The following extension of the method has been described in [Henson 73b].
When π is a permutation, define the sequence of type-preserving operators π_0, π_1, ... inductively by:

$$\begin{cases}\pi_0 x=x\\ \pi_1 x=\pi x\\ \pi_{n+1}x=\{\pi_n y\mid\pi_{n-1}y\in\pi_n x\} & \text{when } n\geq 1\end{cases}$$

The following lemma is easily proved.

<u>Lemma 4.1.5.</u> (a) for all $n\geq 1$: $\pi_{n-1}y\in\pi_n x\leftrightarrow\pi_n y\in\pi_{n+1}x$
(b) for all $n\geq 1$: π_n is a permutation

Note that some elimination of abstractions is necessary to render meaningfulness to φ^{π_n} for $n\geq 2$.

<u>Theorem 4.1.6.</u> Let φ be a formula with $\bar{x}=x_1....x_n$ as free variables. When φ can be stratified by assigning the types $\bar{k}=k_1....k_n$ to $x_1....x_n$ (respectively), then

$$\mathbf{NF}\vdash\varphi^{\pi}\leftrightarrow\varphi\,(\pi_{k_1}(x_1),....,\pi_{k_n}(x_n))$$

<u>Proof</u>: by induction on φ.When φ is the atom $x_1=x_2$, the statement follows from the fact that π_k is injective for all k; when φ is the atom $x_1\in x_2$, then apply lemma 4.1.5.(a) k_1 times ($k_2=k_1+1$). The induction steps for $\neg$ and $\vee$ are trivial, so consider the case where φ is $\exists x_0\,\psi$. By the induction hypothesis we have

$$\mathbf{NF} \vdash \psi^{\pi} \leftrightarrow \psi(\pi_{k_0}(x_0), \pi_{k_1}(x_1), \dots \pi_{k_n}(x_n))$$

(where $k_0\bar{k}$ is a ψ-stratifying assignment to $x_0\bar{x}$). As π_{k_0} is a surjective operation, the desired result follows.

Definition 4.1.7. A sentence σ is invariant if, for any permutation π, $\mathbf{NF} \vdash \varphi \leftrightarrow \varphi^{\pi}$.
An extension S of **NF** is invariant if all its axioms are invariant.

Note that every stratified sentence is invariant; Henson proves CA to be invariant in [73b], which by the theorem below (the analogue of 4.1.3.) shows that permutation methods can not be applied to show relative consistency of CA to **NF**.

Theorem 4.1.8. Let S be an invariant extension of **NF**. When there is a permutation π such that $S+\varphi^{\pi}$ is consistent, then $S+\varphi$ is consistent.

Proof: When $S \vdash \neg\varphi$, then there are axioms $\psi_1, \dots, \psi_n$ of S such that $\vdash (\psi_1 \wedge \dots \wedge \psi_n) \to \neg\varphi$. By theorem 4.1.2., we infer $\mathbf{NF} \vdash (\psi_1^{\pi} \wedge \dots \wedge \psi_n^{\pi}) \to \neg\varphi^{\pi}$. Invariance of $\psi_1, \dots, \psi_n$ implies $\mathbf{NF} \vdash (\psi_1 \wedge \dots \wedge \psi_n) \to \neg\varphi^{\pi}$, so $S \vdash \neg\varphi^{\pi}$.

For the rest of this section, let S denote an invariant extension of **NF**.
The proof of theorem 4.1.4. shows that $\exists x\,(x=\{x\})$ and $\neg\exists x\,(x=\{x\})$ are consistent relative to S.
Let us prove that the class of Quine-individuals doesn't have to be a set. To do this, we need some more information about ordinal numbers.
There is a type-raising operation U defined on ordinal numbers analogous to the operation T on cardinal numbers:

$$U(No(R)) = No(RP_1(R))$$

where R is a well-ordering, No(R) its ordinal number and $RP_1(R) = \{ (\{x\},\{y\}) \mid (x,y) \in R \}$.
The following lemma is proved analogous to the corresponding one for T; its proof can be found in [Rosser 53] and [Henson 73a]. α, β and γ denote arbitrary ordinal numbers.

Lemma 4.1.9.
(a) $\alpha=\beta \leftrightarrow U\alpha = U\beta$
(b) $\alpha \leq_0 \beta \leftrightarrow U\alpha \leq_0 U\beta$
(c) $\alpha \leq_0 U\beta \to \exists\gamma(\gamma \leq_0 \beta \wedge \alpha = U\gamma)$

Theorem 4.1.10. $\neg\exists y\forall x\,(x\in y\leftrightarrow x=\{x\})$ is consistent relative to S.

Proof: let σ be the mentioned sentence. Let π be the permutation given by $\{\alpha\}\leftrightarrow U\alpha$ ($\alpha\in NO$). The π-transform of σ is $\neg\exists y\forall x\,(x\in \pi y\leftrightarrow \pi x=\{x\})$. Suppose that there is a set y which satisfies $x\in \pi y\leftrightarrow \pi x=\{x\}$. Then we have $\alpha\in \pi y\leftrightarrow \alpha=U\alpha$, so $\{\,\alpha\mid\alpha\neq U\alpha\,\}=-\pi y\cap NO$ exists. Therefore there is a $\leq_0$-least $\beta\in NO$ for which $\beta\neq U\beta$. But this contradicts the lemma above: when $\beta<_0 U\beta$ we have $\beta=U\gamma$ for some $\gamma<_0\beta$, but then $\gamma\neq U\gamma$ which contradicts the minimality of β; when $U\beta<_0\beta$, we have $UU\beta<_0 U\beta$ so $U\beta\neq UU\beta$, which also contradicts the fact that β is minimal.

As an application of the extended permutation method, we prove that there may be a set equal to its power set, distinct from V.

Theorem 4.1.11. $\exists x\,(x\neq V\wedge x=Px)$ is relatively consistent to S.

Proof: the function f defined on P_1V by $f\{x\}=P_1x$ is a bijection from P_1V onto PP_1V; P_1x is a singleton only when x is a singleton, say $x=\{z\}$, and in that case $f\{x\}=\{x\}$. So there is a permutation π for which $\mathbf{NF}\vdash\pi\{x\}=P_1x\wedge\pi\pi x=x$. In particular,

$$\pi_2\{V\}=\{\pi x\mid x\in\pi\{V\}\}=\{\pi x\mid x\in P_1V\}=\{\pi\{z\}\mid z\in V\}=\{P_1z\mid z\in V\}$$
$$=\{P_1z\mid z\subseteq V\}=PP_1V=P\pi\{V\}=P\pi_1\{V\}$$

Now x=Py can be stratified by assigning the types 2 to x and 1 to y, so its π-transform is equivalent in **NF** to $\pi_2x=P\pi_1y$. The π-transform of $x\neq V$ is equivalent in **NF** to $\pi_1x\neq V$, so letting x=y and noting that $\pi_1\{V\}\neq V$, this proves the π-transform of $\exists x\,(x\neq V\wedge x=Px)$. So that sentence is consistent relative to S.

§2. comparing n, Tn and 2^{Tn}

For a set x, and $n=|x|$, we have $|P_1x|=Tn$ and $|Px|=2^{|P_1x|}=2^{Tn}$. So comparing $|x|$, $|P_1x|$ and $|Px|$ comes to the same thing as comparing n, Tn and 2^{Tn}. Cantor's theorem states $Tn<2^{Tn}$ for all n, and this is almost the only restriction on the possible relationships between the three cardinals. The only other restriction is

Theorem 4.2.1. $\forall n\in Nn\,(n\neq 2^{Tn})$

Proof: (sketch) when $\Delta\in NC$ and $\Delta=2^{T\Delta}$, the situation resembles the one for Ω: $\Delta>T\Delta>T^2\Delta>$.....
By a straightforward modification of Specker's disproof of AC, we can prove $\Delta\notin Nn$. We will sketch what this modification looks like.
The main ingredients of the proof of $\mathbf{NF}\vdash\neg AC$ were:
(1) $AC\rightarrow\leq$ well-orders NC;
(2) ΦTm finite $\rightarrow\Phi m$ finite;
(3) $\forall m\,(m\leq T\Omega\vee m>T\Omega)\rightarrow[\,\Phi m$ finite $\rightarrow|\Phi Tm|=T|\Phi m|+1$ or $2\,]$
(see the remark below 2.2.2.)
Then we considered the set $\{m\mid\Phi m\in Fin\}$, took n to be the minimal element of this set, existing by means of (1), and derived n =Tn by means of (2) and (3); hence $|\Phi n|=T|\Phi n|+1$ or 2, contradicting 2.1.15.
First of all, we have to avoid the use of AC here. This can be achieved by considering $WC=\{\,|x|\mid x$ can be well-ordered $\}$ instead of NC.
We want to treat Δ as the 'greatest' cardinal, so we have to replace Ω everywhere in chapter 2 by Δ (and $2^m\neq\Lambda$ by $m\leq T\Delta$), and we have to replace Φ by the operation Ψ defined by $\Psi m=\Phi m\cap\{p\mid p\leq\Delta\}$. Then we can prove
(1') $\leq$ well-orders WC;
(2') ΨTm finite $\rightarrow\Psi m$ finite;
(3') $\forall m\,(m\leq T\Delta\vee m>T\Delta)\rightarrow[\,\Psi m$ finite $\rightarrow|\Psi Tm|=T|\Psi m|+1$ or $2\,]$
Next we consider the set $\{m\in WC\mid\Psi m\in Fin\}$, which is non-empty when we further assume $\Delta\in WC$. The rest of the proof is analogous, and we conclude from the contradiction reached at that
$\Delta\notin WC\vee\neg\forall m\,(m\leq T\Delta\vee m>T\Delta)$
So we infer $\Delta\notin Nn$ from the lemmata 2.3.3., 2.2.4. and 2.1.14..

A model-theoretic method for proving the relative consistency of certain sentences, intimately related to the permutation method, has been described in [Henson 69]. We will use it to show that for finite cardinals n, the remaining inequalities $n<Tn<2^{Tn}$, $Tn<n<2^{Tn}$ and $Tn<2^{Tn}<n$ are all possible.

Let $\mathcal{A}$=(A,R) be an {∈}-model. For automorphisms θ of $\mathcal{A}$ we define a new {∈}-model $\mathcal{A}^\theta=(A,R^\theta)$ by $xR^\theta y \Leftrightarrow xR(\theta y)$.

Lemma 4.2.2. (a) θ is an automorphism of $\mathcal{A}^\theta$

(b) let φ be a formula with $\bar{x}$ as free variables, and let $\bar{k}$ be a φ-stratifying assignment to $\bar{x}$. Then for all $\bar{a}\in A$:

$$\mathcal{A}^\theta \models \varphi[\bar{a}] \Leftrightarrow \mathcal{A} \models \varphi[\theta^{k_1}(a_1),\ldots,\theta^{k_n}(a_n)]$$

Proof:

(a) let a,b∈ A. Then $(\theta a,\theta b)\in R^\theta \Leftrightarrow (\theta a,\theta^2 b)\in R$

$\Leftrightarrow (a,\theta b)\in R$ (as θ is an automorphism of $\mathcal{A}$)

$\Leftrightarrow (a,b)\in R^\theta$

(b) analogous to the proof of theorem 4.1.6.

In particular we have $\mathcal{A}^\theta\models\sigma\Leftrightarrow\mathcal{A}\models\sigma$ for stratified sentences σ.

Definition 4.2.3. $Nn^*(n) \equiv n\in Nn \wedge n=Tn$

$n<^*m \equiv Nn^*(n)\wedge Nn^*(m)\wedge n<m$

Note that neither of the formulae $Nn^*(n)$ and $n<^*m$ is stratified.
Let $\mathcal{A}$=(A,R) be an {∈}-model. Define the model $\mathcal{A}^*=(A^*,R^*)$ by

$$\begin{cases} A^*=\{a\in A \mid \mathcal{A}\models Nn^*(a)\} \\ aR^*b \Leftrightarrow a,b\in A^* \;\&\; \mathcal{A}\models a<^*b \end{cases}$$

When $\mathcal{A}$ is a model of **NF**, then $\mathcal{A}^*$ is an infinite linear ordering (as **NF** is a model of n =Tn for all n∈ ω).

Theorem 4.2.4. Let S be a consistent stratified extension of **NF**; let (B, <') be an arbitrary linear ordering; let B_1, B_2, B_3 be an arbitrary partition of B. Then there is a model **B** of S such that

(a) $\forall b_1b_2\in B\ [\ \mathbf{B}\models b_1\in Nn \;\&\; (b_1<' b_2 \Leftrightarrow \mathbf{B}\models b_1<b_2)\ ]$

$$\text{(b)}\begin{cases} \forall b\in B_1 & \mathbf{B}\models b<Tb \\ \forall b\in B_2 & \mathbf{B}\models b=Tb \\ \forall b\in B_3 & \mathbf{B}\models Tb<b \end{cases}$$

Proof: let $(\mathbf{Z}, <'')$ be the natural ordering of the integers. Define $<_1$ on $B\times\mathbf{Z}$ by

$$(b_1, z_1) <_1 (b_2, z_2) \Leftrightarrow b_1 <' b_2 \text{ or } (b_1 = b_2 \;\&\; z_1 <'' z_2).$$

Identifying (b,0) with b for convenience, $(B, <')$ becomes a subordering of $(B\times\mathbf{Z}, <_1)$.

As noted above, when $\mathcal{A}$ is an S-model, then $\mathcal{A}^*$ is an infinite linear ordering. So we may apply a theorem of Ehrenfeucht-Mostowski ([56]; [Vaught 66]) to obtain a model $\mathcal{A}$=(A,R) of S with the following properties:

(i) $(B\times\mathbf{Z}, <_1)$ is a subordering of $\mathcal{A}^*$;
(ii) every automorphism of $(B\times\mathbf{Z}, <_1)$ can be extended to an automorphism of $\mathcal{A}$.

Let the automorphism θ of $(B\times\mathbf{Z}, <_1)$ be given by

$$\theta(b,z) = \begin{cases} (b, z-1) & \text{if } b\in B_1 \\ (b, z) & \text{if } b\in B_2 \\ (b, z+1) & \text{if } b\in B_3 \end{cases}$$

Let θ also denote the extension of this automorphism to $\mathcal{A}$. $\mathcal{A}^\theta$ now is the model **B** whose existence is stated in the theorem.

$\mathcal{A}^\theta \models S$ because all axioms of S are stratified and $\mathcal{A} \models S$.

If $(b_1, z_1) <_1 (b_2, z_2)$, then $\mathcal{A} \models (b_1, z_1) < (b_2, z_2)$ because of (i). So (by lemma 4.2.2.(b); $x<y$ can be stratified by assigning the same type to x and y): $\mathcal{A}^\theta \models (b_1, z_1) < (b_2, z_2)$. In particular, when $b\in B$ then

(1) $\mathcal{A}^\theta \models (b,-1) < b < (b,1)$

Moreover, by the same argument: when $b_1, b_2 \in B$ and $b_1 <' b_2$ then $\mathcal{A}^\theta \models b_1 < b_2$. For $b\in B$ we have by (i):

(2) $\mathcal{A} \models Nn^*(b)$,

so $\mathcal{A} \models b\in Nn$ and $\mathcal{A}^\theta \models b\in Nn$ for all $b\in B$ by lemma 4.2.2.(b). This proves (a).

From (2) it follows that for all $b\in B$: $\mathcal{A} \models b = Tb$, hence $\mathcal{A} \models \theta b = T(\theta b)$. $x=Ty$ can be stratified by an assignment giving x a type one higher than that of y, so, again applying lemma 4.2.2.(b) we get

(3) $\mathcal{A}^\theta \models b = T(\theta b)$.

* when $b\in B_1$, then $\theta b=(b,-1)$ so by (1) we have $\mathcal{A}^\theta \models \theta b < b$, whence by (3): $\mathcal{A}^\theta \models \theta b < T(\theta b)$. 4.2.2.(b) now yields $\mathcal{A}^\theta \models b < Tb$.
* when $b\in B_2$, then $\theta b=b$ so (3) implies $\mathcal{A}^\theta \models b = Tb$.
* when $b\in B_3$, then $\theta b=(b,1)$ so by (1) we have $\mathcal{A}^\theta \models b < \theta b$, whence by (3) $\mathcal{A}^\theta \models T(\theta b) < \theta b$ and thus $\mathcal{A}^\theta \models Tb < b$.

Corollary 4.2.5. Let S be a consistent stratified extension of **NF**. Then S remains consistent after adding the following two sentences:

$$\exists n \in Nn\,(n < Tn)$$
$$\exists n \in Nn\,(Tn < n)$$

Corollary 4.2.6. CA is not a theorem of any consistent stratified extension of **NF**

So, as CA is not provable in **NF** but $n = Tn$ *is* provable for all $n \in \omega$, **NF** and all its stratified extensions are ω-incomplete.

Theorem 4.2.7. $\mathbf{NF} \vdash \exists n \in Nn\,(Tn < n) \rightarrow \exists n \in Nn\,(2^{Tn} < n)$

Hence consistent stratified extensions of **NF** remain consistent after adding

$$\exists n \in Nn\,(2^{Tn} < n)$$

Proof: (sketch) consider the set $\Phi 1$. As $n < 2^{n}$ for all $n \in Nn$, $\Phi 1$ is an unbounded subset of Nn. Let f enumerate the elements of $\Phi 1$ in increasing order: $f0 = 1$; $f(n+1) = 2^{fn}$. In fact there is a formula $\varphi(m,n)$ which is stratified when m and n are given the same type and which satisfies $fm = n \leftrightarrow \varphi(m,n)$: take $\varphi(m,n)$ to be the the formula

$\exists g: \{ p \in Nn \mid p < m+1 \} \rightarrow \Phi 1\ [g0 = 1 \wedge \forall p < m\,(g(p+1) = 2^{gp}) \wedge gm = n]$.

Strong induction on Nn then gives that the relation f defined by $f = \{(m,n) \mid \varphi(m,n)\}$ is the function which enumerates $\Phi 1$ in the mentioned way.

We prove by induction on $m \in Nn$ that $f(Tm) = T(fm)$: (the induction is possible because the mentioned sentence is equivalent to $\forall m \in Nn \forall n \in Nn\,[\varphi(m,n) \leftrightarrow \varphi(Tm,Tn)]$ which is stratified)

* $fT0 = f0 = 1 = T1 = T(f0)$
* $fT(m+1) = f(Tm+1) = 2^{fTm} = 2^{Tfm} = T2^{fm} = Tf(m+1)$

Let $n \in Nn$ satisfy $Tn < n$. Then $T(n+1) = Tn+1 < n$ (as $Tn+1 \neq n$ by 2.1.15.). So $fT(n+1) < fn$, whence $2^{Tfn} = T2^{fn} = Tf(n+1) = fT(n+1) < fn$, so fn is the desired element of Nn.

Theorem 4.2.8. $\mathbf{NF} \vdash \forall mn \in Nn\, [\, Tm < m < n = Tn \rightarrow T(m+n) < m+n < 2^{T(m+n)}\,]$

Hence consistent stratified extensions of **NF** remain consistent after adding

$\exists n \in Nn\, (\, Tn < n < 2^{Tn}\,)$

Proof: by an informal argument using cardinal arithmetic as developed in [Rosser 53]. Let $m, n \in Nn$ be such that $Tm < m < n = Tn$.

$T(m+n) = Tm+Tn = Tm+n < m+n$ (the last inequality can be proved by induction on n);

$2^{T(m+n)} = 2^{Tm+Tn} = 2^{Tm+n} = 2^{Tm} \cdot 2^{n}$.

As $1 = T1$, we have $Tm > 1$, so $2^{Tm} > 2$. Thus $2 \cdot 2^{n} < 2^{Tm} \cdot 2^{n}$. As $m < n$, we have $m+n < 2 \cdot n$, so

$m+n < 2 \cdot n < 2 \cdot 2^{n} < 2^{Tm} \cdot 2^{n} = 2^{T(m+n)}$.

Pétry completed the fan of all possible relationships between n, Tn and 2^{Tn} by showing in [75 & 77] that any consistent stratified extension of **NF** remains consistent after adding

$\exists n\, (\, n \not\leq Tn \not\leq n\,)$ and

$\exists n\, (\, n \not\leq 2^{Tn} \not\leq n\,)$

(n is necessarily infinite when one of these sentences holds).

Epilogue

It has been an interesting journey through a mind-boggling set theory. Now it is time to reflect. What of **NF** as New Foundations, as a new set-theoretical basis for mathematics? (as Rosser intended when writing his extensive book [53]). Those mathematicians who need AC as an axiom will not be pleased to use **NF** instead of **ZF**, but as **ZF**+¬AC is as consistent as **ZF**+AC is (namely, as consistent as **ZF**) their choice of assuming AC seems rather arbitrary; **NF** is simply more outspoken in this instance: AC does not hold. Nevertheless, one of the basic theorems of topology is Tychonoff's theorem stating product-invariance of compactness, which undoubtedly is equivalent to AC in **NF** as well as in **ZF**. So topologists had better stick to **ZFC**.

Then we have the problems surrounding the counting axiom CA, which is a theorem of **ZF** but not of **NF**. It is not impossible that CA is independent of **NF** although theorem 4.2.4. does not furnish a proof of the relative consistency of CA to **NF**. When someone provides us with a proof that CA is indeed relatively consistent to **NF**, then one can equally well assume CA as refute it. For everyday mathematics, it seems a better course to assume it, just as in some parts of mathematics it is better to assume the axiom of choice. The fact that CA intuitively is more correct than AC is no reason to require of a set theory that CA can be derived, making no problem about the independence of AC; so adding CA to **NF** for the purpose of developing mathematics easier is no odder than adding AC to **ZF**. **NF** just seems to be more general than set theories in which every set is Cantorian.

Another problem is that induction is not possible for all formulae, but only for stratified ones (or, more generally, for formulae φ for which $\{ x \mid \varphi x \}$ exists). It is remarked in [Fraenkel/Bar-Hillel/Levy 73, p163] that mathematical induction for *all* formulae can be added as an additional axiom scheme to **NF** (although I don't see why).
Moreover, large sets (among which V) can not be well-ordered, but according to Quine [69, p296] we can still reserve the right to assume that Cantorian sets can be well-ordered.
Here I have to give some credit to Fraenkel c.s. for remarking that the aesthetically fine basis of **NF** (EXT+just one axiom scheme) becomes rather disturbed when we add to it all kinds of axioms to provide for a trouble-free development of mathematics, but then again, I don't see why this would bother us much: mathematicians were accustomed to using AC long before its relative consistency to **ZF** was proved, so we can equally well assume additional axioms in **NF** for which it has been proved, or perhaps will be proved in the future, that they are relatively consistent to **NF**. It is just the basis of **NF**, which is insufficient to *prove* these additional axioms, which forced us to make a choice between assuming them and refuting them.

Finally, there is the problem whether **NF** is at all consistent. Some partial consistency results have been proved in this paper, and the reader might be convinced that **NF** is consistent, just as probably everyone has the same happy feelings about **ZF**.

Gödel's second incompleteness theorem tells us that we will not be able to prove the consistency of **NF** in **NF** itself (when **NF** is indeed consistent), so we have to look for a proof in some other set theory -like **ZF**- or possibly a proof that **NF** is consistent when **ZF** is (and vice versa?). The search for such a proof is one of the major tasks of the metamathematics of set theory. An interesting approach of this goal has been made in [Boffa 88], where **NF** is related to the theory **ZFJ**, which results from **ZF** when adding a unary function symbol J to the language and adding axioms stating that J is an automorphism (which also leads to a quick proof of the consistency of **NFU**).

In view of this as yet unsolved problem, I'd like to quote the title of chapter 15 of [Rosser 53]: "We rest our case".

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