An Intuitionistic Temporal Logic for Brouwer's Creating Subject Arguments

MSc Thesis *(Afstudeerscriptie)*

written by

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"Każdy przecież początek
to tylko ciąg dalszy,
a księga zdarzeń
zawsze otwarta w połowie."

Wisława Szymborska,
Miłość od pierwszego wejrzenia

Abstract

We develop an intuitionistic temporal logic for Niekus' formalization of Brouwer's Creating Subject arguments. This formalization differs from standard ones, such as the Theory of the Creating Subject (**TCS**), which results in a different treatment of the sequences used in these arguments. Niekus notes that these sequences have incomplete descriptions and are therefore instances of individual choice sequences, an observation arising in connection with his proposal.

To formalize the arguments in accordance with Niekus' approach, we introduce an intuitionistic temporal logic with temporal modalities $\Box$ and $\bigcirc$, where $\bigcirc$ admits an interpretation as a least fixed point operator. The resulting system, **TICS-2**, builds on earlier formalizations of Niekus' proposal. We establish soundness and completeness of **TICS-2** with respect to a class of bi-relational Kripke frames. For its $\Box$-fragment $\mathbf{TICS\text{-}2}_{\Box}$, we develop an alternative semantics based called *modal cover semantics*, as recently introduced by Valliappan, and obtain a constructive completeness proof via a canonical model construction. This fragment corresponds to Intuitionistic Epistemic Logic **IEL**, as developed by Artemov and Protopopescu; we discuss results for this logic and its connection to our system. We moreover introduce $\mathbf{QE^{+}\text{-}TICS\text{-}2}_{\Box}$, a first-order extension of $\mathbf{TICS\text{-}2}_{\Box}$ with an explicit existence predicate as in E^{+}-logic, and establish soundness and completeness for this system.

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Contents

Chapter 1

Introduction

L. E. J. Brouwer introduced the notion of a *choice sequence* as part of his intuitionism. A choice sequence is an infinite sequence with an incomplete description for the determination of its terms. Its values are not completely fixed in advance; rather, its description typically only determines the range of admissible choices for each term. A general requirement is that each term must be chosen from mathematical objects that have previously been constructed.

Brouwer used choice sequences in his analysis of the intuitionistic continuum. In most of his work, as well as in standard theories of choice sequences, the focus is on the totality of choice sequences rather than individual instances.

Joop Niekus studies Brouwer's use of *individual* choice sequences [Nie05; Nie10; Nie25]. He finds occurrences of such sequences in Brouwer's arguments involving the *Creating Subject*, the mathematician who carries out mathematical constructions, including proofs.

The notion of the Creating Subject was formalized by Kreisel and Troelstra, leading to the development of the *Theory of the Creating Subject* (**TCS**). These formalizations were prompted by Brouwer's use of the notion in several arguments against classical properties of the continuum. However, as noted by Niekus, due to the way they treated the notion, they overlooked the presence of individual choice sequences in Brouwer's Creating Subject arguments. According to Niekus, the overly strong assumptions built into these formalizations caused this aspect of Brouwer's arguments to remain unnoticed, as they made the sequences arising in the arguments appear to have complete descriptions.

Niekus proposes an alternative formalization of Brouwer's Creating Subject arguments, in which the connection to choice sequences becomes apparent. Using this formalization, he observes that the sequences arising in the arguments should be characterized as choice sequences, since their descriptions are incomplete. The value of a term is not fully determined when the sequence is defined, as it depends on the temporal order in which the Creating Subject carries out future mathematical constructions. Niekus argues that this observation matches the close connection between choice sequences and the Creating Subject in Brouwer's conception, as suggested by other aspects of Brouwer's work, in which every choice sequence may be understood as a Creating Subject sequence [Nie87]. Consequently, Niekus maintains that the theory of the Creating Subject is not separate from the theory of choice sequences: the method of the Creating Subject consists precisely in the use of individual choice sequences, rather than the general properties of choice sequences. He reconstructs Brouwer's arguments within his framework and argues that it avoids problems in existing formalizations, such as Troelstra's paradox [Nie17].

Niekus' formalization, developed in [Nie87], uses the operator $\vdash_n$, as does the Theory of the Creating Subject (**TCS**). In both formalizations, this is later replaced by $\Box_n$. In **TCS**, $\Box_n A$ expresses that "at stage n, the Creating Subject has a proof of A". In Niekus' system, by contrast, $\Box_n A$ expresses that "after n *further* stages, *we* have a proof of A". This difference in interpretation gives rise to the different (logical) properties. Niekus provided a Kripke semantics for his system, obtained by extending **IPC** with the operators $\Box_n$ and $\exists n \Box_n$, which he later called the *Theory of Individual Choice Sequences* (**TICS**) [Nie25].

To obtain a more expressive formalization of the temporal aspects of Brouwer's Creating Subject arguments, Niekus and Lex Hendriks later extended **TICS** with an additional temporal operator. The resulting intuitionistic propositional modal logic is equipped with two temporal operators: $\Box$, read as *from the next stage onwards*, and $\bigcirc$, read as *eventually.* Niekus and Dick de Jongh provided an axiomatization of the system together with a sound Kripke semantics and established its completeness. The resulting system retained the name **TICS**. To date, these results remain unpublished.

In this thesis, we build on this work by considering a variant of **TICS** obtained by removing one axiom. This leads to the intuitionistic temporal logic **TICS-2**. The language of **TICS-2** still uses the temporal modalities $\Box$ and $\bigcirc$, where we note that $\bigcirc$ can be interpreted as a least fixed point operator. We define a class of bi-relational Kripke frames for this logic, in which we work with an intuitionistic preorder $\leq$ and a temporal accessibility relation R. We prove that **TICS-2** is sound and complete with respect to this class of frames. For **TICS-2**$_\Box$, the $\Box$-fragment of **TICS-2**, we furthermore develop an alternative semantics called *modal cover semantics*, as recently introduced by Valliappan [Val26]. We use this to obtain a constructive completeness proof for this fragment via a canonical model construction, which gives an alternative to Goldblatt's order-theoretic approach [Gol11]. This fragment turns out to correspond to Intuitionistic Epistemic Logic **IEL**, as developed by Artemov and Protopopescu [AP16]. We discuss this correspondence and mention existing results for **IEL**, such as a sequent calculus and a constructive algebraic completeness proof, which may be useful for further work on **TICS-2**. We furthermore introduce **QE**$^+$**-TICS-2**$_\Box$, a first-order extension of **TICS-2**$_\Box$ with an explicit existence predicate as in E^+-logic, for which we establish completeness. In the conclusion, we consider how the results found for the $\Box$-fragment of the logic could be extended to the full logic **TICS-2**. Throughout the thesis, as well as in the detailed philosophical motivation, we discuss how all of these logics are useful to analyze Brouwer's Creating Subject arguments, and thereby his use of individual choice sequences.

The thesis is structured as follows. Chapter 2 provides a detailed philosophical motivation for the logics developed in this thesis. We examine Brouwer's intuitionism and the central role of time in his philosophy of mathematics. We then consider the notion of a choice sequence and the Creating Subject, and explain how Niekus notes that these notions come together in Brouwer's Creating Subject arguments. In particular, we present Niekus' critique of the standard treatment of sequences based on the Creating Subject's activity, and discuss his proposed alternative perspective. Building on this, we argue for the usefulness of the logical systems developed in this thesis for formalizing the Creating Subject sequences, and thereby the arguments in which they arise, following Niekus' approach.

Chapter 3 provides an overview of the logic **TICS**, as developed by Joop Niekus, Lex Hendriks, and Dick de Jongh. Although this work has not yet been published elsewhere, it forms the starting point of the present thesis, and it is therefore important to make explicit the framework on which the current work is based. The syntax and semantics are presented, and the soundness and completeness proofs are given.

Chapter 4 introduces the intuitionistic temporal logic developed in this thesis, **TICS-2**. Its syntax and Kripke semantics are described, and we establish soundness and completeness with respect to a class of bi-relational Kripke frames for **TICS-2**. We show how the system can be applied to Creating Subject arguments and briefly discuss the considerations involved in developing Beth semantics for this logic in order to obtain a constructive completeness proof.

Chapter 5 focuses on the $\Box$-fragment of **TICS-2**. We develop an alternative semantics based on Valliappan's generalization of Goldblatt's modal cover systems and give a constructive completeness proof via a canonical model construction. We also discuss the correspondence between this fragment and Intuitionistic Epistemic Logic (**IEL**).

Chapter 6 introduces $\mathbf{QE^{+}\text{-}TICS\text{-}2_{\Box}}$, a first-order extension of $\mathbf{TICS\text{-}2_{\Box}}$ with an explicit existence predicate as in E^{+}-logic. Building on results for the first-order extension of **IEL** with equality and on the work on E^{+}-logic described in [TD88], we develop a sound and complete semantics for this logic.

Chapter 7 concludes the thesis and discusses directions for further work. In particular, as Chapter 5 and 6 only develop results for the $\Box$-fragment of the logic, we outline possible ways to extend these to full **TICS-2**.

Chapter 2

Philosophical motivation

> "If our labours are not to be like those of Sisyphus, then these logics must have some use:
> – the labour itself will not be its own reward."
>
> W.B. Ewald (in [Ewa78])

In this chapter, we provide the philosophical motivation and background for the development of our intuitionistic temporal logic. We begin with Brouwer's intuitionism and consider the central role of time in this philosophy. We then turn to the notions of choice sequences and the Creating Subject. Following Niekus, we relate these concepts through the sequences arising in Brouwer's Creating Subject arguments (CS-sequences). We discuss Niekus' critique of standard formalizations of these arguments, which, he argues, overlook the use of individual choice sequences by making CS-sequences appear to have complete descriptions. We then present his alternative formalization, in which CS-sequences are understood as choice sequences. Finally, we explain how the temporal logics developed in this thesis provide a formal framework for Brouwer's Creating Subject arguments in accordance with Niekus' proposal.

2.1 Brouwer's intuitionism and the role of time

L.E.J. Brouwer is the founder of what is today known as intuitionism. Intuitionism holds that mathematics is grounded in the intuition of the human mind. Brouwer himself situates his view in relation to Kant's philosophy, according to which space and time function as the a priori forms of human intuition [Bro12]. However, the emergence of non-Euclidean geometry challenged Kant's account of the apriority of spatial intuition. Brouwer abandons the apriority of spatial intuition while retaining temporality as the fundamental form of mathematical intuition, and calls his position "neo-intuitionism" to mark this contrast. The prefix "neo" was later dropped and the position is now simply referred to as *intuitionism* [Kui04].

Brouwer's formulation of the *first act of intuitionism* provides a clear presentation of his conception of mathematics:

> "Completely separating mathematics from mathematical language and hence from the phenomena of language described by theoretical logic, recognizing that intuitionistic mathe-

matics is an essentially languageless activity of the mind having its origin in the perception of a move of time. This perception of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory." [Bro81]

This characterizes intuitionistic mathematics as grounded in temporal intuition. Mathematical objects are understood as mental constructions, and logic does not determine this process but instead reflects upon its already-constructed results; it is thus posterior to mathematical activity. Brouwer's account of the natural numbers likewise rests on this temporal intuition [Iem08]. Their origin lies in the *two-ity*: a moment falling apart into a first part, which passes away, and a second, which succeeds it while the first is retained by memory. Iterating this temporal separation generates the natural numbers, forming a (discrete) basis of mathematics for Brouwer.

As a consequence of his conception of mathematics, Brouwer rejects the idea that truth is independent of human activity. Instead, a mathematical statement is true only in virtue of a construction that establishes it, namely a proof. Since mathematical constructions are carried out at particular moments in time, mathematical truth is inherently temporal: new truths may become established as further constructions are performed.

Under Brouwer's interpretation of the logical connectives, later formalized in the Brouwer-Heyting-Kolmogorov (BHK) interpretation (see, for example, [TD88]), the meaning of a formula is determined by what constitutes a construction providing a proof of that formula. A proof of a conjunction $A \wedge B$ consists of a proof of A together with a proof of B. A proof of a disjunction $A \vee B$ consists of a proof of one of the disjuncts together with an indication of which disjunct has been proved. A proof of an implication $A \to B$ is a construction that transforms any proof of A into a proof of B. Finally, a proof of a negation $\neg A$ is a construction showing that any proof of A would lead to a contradiction.

This constructive interpretation explains Brouwer's rejection of the law of excluded middle, $A \vee \neg A$. Since the justification of a formula depends on the availability of suitable constructions, the truth of a statement need not be determined at a given moment but may emerge only through future mathematical activity. In particular, for an arbitrary proposition A, we may have, here and now, neither a proof of A nor a construction establishing that a proof of A would lead to a contradiction. Therefore, $A \vee \neg A$ is not generally justified from an intuitionistic perspective. The establishment of a statement as true or false may depend on mathematical constructions that have not yet been carried out and may never be carried out.

As we will see, the open-ended character of mathematical activity is also central to Brouwer's Creating Subject arguments. In these arguments, the values of the relevant sequences depend on future mathematical constructions carried out by the Creating Subject. Although the details will be discussed later, this already indicates why an intuitionistic logic with explicit temporal operators is useful for describing Brouwer's arguments.

2.2 Choice sequences

The *second act of intuitionism* introduces the intuitionistic continuum, in which Brouwer recovers the continuous from the discrete. He characterises the continuum as consisting of "more or less freely proceeding infinite sequences of mathematical entities previously acquired" [Bro81]. This is formalized by the intuitionistic notion of a *spread* [Nie17]. A spread is a law that regulates the construction of infinitely proceeding sequences, which are called the *elements* of the spread. It is determined by a founding sequence of already constructed objects together with a building rule that specifies which choices are permitted at each stage of the construction. An element of a spread may remain open-ended: at any given stage, only a finite initial segment is determined, while its continuation is not necessarily fixed in advance.

In the literature, as well as in Brouwer's own work, a distinction is made between two kinds of sequences occurring in the continuum. Following Troelstra, we use the term *lawlike* for sequences whose values are determined completely by a law of choice fixed from the outset, reserving *choice sequence* for sequences whose development is not fixed in this way.[1] Brouwer himself once used the terms *fertig* and *unfertig* for these two cases, and also referred to lawlike sequences as *sharp* or *determined*.

In Niekus' terminology, these two types of sequences can therefore be distinguished in terms of their descriptions: lawlike sequences are *complete* objects, since their values are fully determined by a finite description, whereas choice sequences are *incomplete*, since their values are not completely determined [Nie10]. Both types of sequences are needed to obtain the full intuitionistic continuum. The part of the continuum consisting only of lawlike sequences forms the so-called *reduced* continuum, which contains only countably many elements because each sequence is determined by a finite description. The inclusion of choice sequences is therefore essential for recovering the full continuum. For a choice sequence, as an open-ended element of a spread, only a finite initial segment is available at any given moment. Brouwer takes this open-ended character to be characteristic of points of the intuitionistic continuum in general.

The traditional theories on choice sequences originate in research initiated by Kreisel and later developed together with Troelstra. They treat the totality of choice sequences and therefore concern global properties. A central example is the Continuity Principle, which states that any function defined on all choice sequences depends only on a finite initial segment of its argument, from which it follows that all total functions on the intuitionistic continuum are continuous (see, for example, [Vel21]). The standard reference on choice sequences is [Tro77]. This work contains formal systems of choice sequences, for which Troelstra proves elimination theorems, showing that anything provable using choice sequences can already be proved without them. Niekus notes that this conclusion was consistent with the observation that, apart from one disputed example, no instances of individual choice sequences had been identified in Brouwer's work [Nie17].

Brouwer's own treatment is likewise largely global in character; the task of giving a precise account of what a choice sequence is occupied him throughout his career. Against this background, Niekus turns his attention to individual sequences, arguing that no adequate theory of them has been developed.

According to Niekus, individual choice sequences appear in Brouwer's work in connection with the method of the *Creating Subject*. For Niekus, the so-called Theory of the Creating Subject (**TCS**) is not a separate theory from the theory of choice sequences, but simply an application of individual choice sequences rather than of the theory of choice sequences as a totality. However, as he notes:

> "No connection has been made between the standard theories of choice sequences and the TCS; the two theories have been developed completely separately." [Nie10]

Niekus argues that this connection was overlooked because existing formalizations of the Creating

[1] Terminology varies in the literature. Some authors use *choice sequence* as a general term for all infinite sequences in the spread, distinguishing lawlike sequences as a special class and using terms such as *lawless* for sequences whose future choices remain open.

Subject method imposed assumptions that made the sequences arising in Brouwer's arguments appear to have complete descriptions. As a result, their character as choice sequences remained unnoticed.

According to Niekus, the only other potential instance of an individual choice sequence occurs in Brouwer's proof of the negative continuity theorem. However, he argues that this proof admits several reconstructions, including ones based entirely on lawlike sequences [Nie05]. In [Nie25], Niekus concludes that this proof is Brouwer's first unsuccessful attempt to use an individual choice sequence. He therefore concludes that there is no unambiguous evidence for a distinctive role of individual choice sequences outside the Creating Subject arguments. We examine these arguments in the next section.

2.3 Brouwer's Creating Subject arguments

In his Creating Subject arguments, Brouwer constructs sequences whose values depend directly on the activity of the *Creating Subject* (CS). The Creating Subject is typically conceived as an idealized mathematician who carries out mathematical constructions and proves mathematical statements through discrete stages of time. This idealization abstracts away from limitations of time, space, and human error. There are different interpretations of the Creating Subject. For example, Van Atten gives a phenomenological interpretation of this notion (see [Att06]), while Niekus argues that the Creating Subject should be understood as representing the constructor itself, suggesting that we should read "the Creating Subject" as "we" or "I" (see [Nie87]).

To see how Brouwer uses this notion, we look at an example that appears in Brouwer's 1948 paper *Essentially Negative Properties* [Bro48], where he gives a weak counterexample[2] to the claim that the order relation on the intuitionistic continuum is linear. He takes α to be a mathematical assertion that cannot currently be tested, meaning that no method is known either to prove $\neg\alpha$ or to prove $\neg\neg\alpha$. He describes a sequence (a_n) that is being constructed as follows:

- if at stage n the Creating Subject has no proof of α or $\neg\alpha$, set $a_n = 0$;
- if between stage $k-1$ and k the Creating Subject obtains a proof of α, set $a_n = 2^{-k}$ for all $n \geq k$;
- if between stage $k-1$ and k the Creating Subject obtains a proof of $\neg\alpha$, set $a_n = -2^{-k}$ for all $n \geq k$.

Let ρ denote the real number determined by this sequence. Brouwer then argues that neither $\rho > 0$ nor $\rho < 0$ holds, but that $\rho = 0$ is absurd. He argues in the following way:

> "If for this real number ρ the relation $\rho > 0$ were to hold, then $\rho < 0$ would be impossible, so it would be certain α could never be proved to be absurd, so the absurdity of the absurdity of α would be known, so α would be tested, which it is not. Thus the relation $\rho > 0$ does not hold. Further, if for the real number ρ the relation $\rho < 0$ were to hold, then $\rho > 0$ would be impossible, so it would be certain that α could never be proved to be true, so the absurdity of α would be known, so again α would be tested, which it is not. Thus the relation $\rho < 0$ does not hold. Finally let us suppose that the relation $\rho = 0$ holds. In this case neither $\rho < 0$ nor $\rho > 0$ could ever be proved, so neither the absurdity nor the truth

[2] A *weak counterexample* is an argument showing that a proposition cannot be established by a constructive proof, without proving its negation. In Brouwer's terminology, such an argument establishes only a *weak negation*: it shows that no proof of the proposition is currently available, rather than showing that assuming the proposition leads to a contradiction, as would be required for the intuitionistic negation $\neg A$ [Nie17].

of α could ever be proved, so the absurdity as well as the absurdity of the absurdity of α would be known. This is a contradiction, so the relation $\rho = 0$ is absurd, in other words the real numbers ρ and 0 are different." [Bro48]

It is clear in which sense the terms of this sequence depend on the activity of the Creating Subject. This, for instance, motivates the idea of the Creating Subject as an *idealized* mathematician: since the sequence being defined is infinite, the Creating Subject must be able to continue the construction indefinitely. The idealization is that the Creating Subject is not limited by the finite duration of a human life, but is assumed to have unlimited time in which to carry out mathematical constructions.

In this sense, the Creating Subject functions as the idealized agent through which mathematical constructions are performed. The notion was given a formal treatment after Brouwer's introduction of the expression. Kreisel gave the first formalization [Kre68], which was later elaborated by Troelstra [Tro69]. This approach is referred to as the Kreisel-Troelstra *Theory of the Creating Subject*[3] (**TCS**). In their system, $\vdash_n A$ expresses that the Creating Subject has evidence for A at stage n; in the notation of Troelstra and van Dalen, this becomes $\Box_n A$, read as "by stage n the Creating Subject has made A evident" [Att18]. In the strong formalization[4], which is the one Troelstra follows, the axioms are:

CS1 $\Box_n A \vee \neg \Box_n A$ for all n

CS2 $\Box_n A \to \Box_{n+m} A$ for all n, m

CS3 $A \to \exists n \, \Box_n A$

CS4 $\exists n \, \Box_n A \to A$

Recently, this formalization has been defended by Van Atten [Att18]. He appeals to Brouwer's view that having a finite algorithm for determining a proof can itself count as having a proof. According to him, the axioms can be justified as follows. For CS1, note that at each stage n, either the Creating Subject has a proof of the proposition or not. To establish the status of $\Box_n A$, if stage n lies in the past, the subject inspects its perfect memory; if stage n lies in the future, it can proceed through the finitely many intermediate stages and determine whether a proof of A has been obtained by stage n. CS2 expresses perfect memory: once a proposition is established at a stage, it remains established at all later stages. CS3 states that whenever a proposition holds, i.e. is established by a proof, the Creating Subject constructs that proof at some particular stage, which can therefore be made explicit. CS4 expresses the correctness of the Creating Subject: if it establishes a proposition at some stage, then that proposition holds, i.e. the Creating Subject's proof is valid. If the stage n at which the proof is obtained lies in the future, there is still a finite procedure for obtaining it: one can progress through the finitely many intermediate stages and retrieve the proof at stage n.

Let us return to the example. We see how our lack of information about the future activity of the Creating Subject prevents us from proving either $\rho > 0$ or $\rho < 0$. Since the development of the sequence depends on mathematical activity that has not yet taken place, the value of the corresponding real number cannot be settled. However, we can conclude that it is impossible to already establish that neither α nor $\neg\alpha$ will ever be proved. Hence, we may conclude that $\rho \neq 0$.

[3] Troelstra originally used the term "Theory of the *Creative* Subject". Since Brouwer himself used the expression "*Creating* Subject", and Niekus [Nie87] and later Van Atten [Att18] argue that this terminology better reflects Brouwer's own usage, we adopt the term *Creating* Subject here.

[4] In the weak formalization, CS3 is replaced by $A \to \neg\neg\exists n \, \Box_n A$.

This kind of reasoning can be formalized in the language of **TCS**. For example, Brouwer's claim that "it would be certain that α could never be proved to be true, and hence the absurdity of α would be known" can be represented by the formula

$$\neg\exists n\, \Box_n \alpha \rightarrow \neg\alpha. \tag{2.1}$$

The sequences defined in Brouwer's Creating Subject arguments are commonly referred to as *CS-sequences* or, in some contexts, as *empirical* sequences [Att18]. The latter terminology reflects concerns about whether these arguments are genuinely mathematical, since they may appear too empirical in nature. See, for example, Heyting's critical discussion [Hey66].

This kind of sequence can be compared with other sequences used by Brouwer. In a similar way, Brouwer namely constructs a sequence intended to provide a weak counterexample to the linear ordering of the *reduced* continuum, i.e. the continuum restricted to lawlike sequences. He does this in the following way:

> "Let d_v denote the v-th digit after the decimal point in the decimal expansion of π, and let $m = k_n$, where k_n is defined as the position at which, in the ongoing decimal expansion of π, the block $d_m d_{m+1} \ldots d_{m+9}$ forms the sequence 0123456789 for the n-th time. Let furthermore $c_v = (-1/2)^{k_1}$ if $v \geq k_1$, otherwise $c_v = (-1/2)^v$. Then the infinite sequence $(c_1, c_2, c_3, \ldots)$ defines a real number r for which neither $r = 0$, nor $r > 0$, nor $r < 0$ holds." [Bro25]

As Van Atten notes [Att18], the choices in the sequence c_v are not defined in terms of the moment at which the Creating Subject comes to know the truth of a proposition, as in the previous example. The decimal expansion of π is lawlike, and it is therefore decidable for every v whether $v \geq k_1$. Hence, each value of the infinite sequence can in principle be calculated, and the resulting real number r is itself lawlike. Nevertheless, this does not suffice to determine which of the disjuncts holds, since we do not currently know whether, and if so at what point in the decimal expansion, k_1 will be found. It is precisely this lack of present epistemic access that leads to the failure of linearity. By now, k_1 has been found, but as Brouwer remarks in a footnote, r can be defined using any decidable property of natural numbers for which one knows neither that it has an instance nor that it cannot have one. Since this construction uses a decidable property concerning the decimal expansion of π, we will refer to sequences of this kind as *π-sequences*, in order to distinguish them from CS-sequences.

Under **TCS**, the CS-sequences are treated much like π-sequences: although their values may not currently be known, they are completely determined by a fixed recipe. CS1 states that, at every stage n, either the Creating Subject has a proof of α or it does not (and likewise for $\neg\alpha$). This completely determines the value of the sequence at each term, so the sequence is fixed by a decidable law.

Troelstra, for example, refers to such sequences as lawlike. He nevertheless distinguishes *absolutely lawlike* (or *mathematical*) sequences, whose values are determined by a complete description independent of the Creating Subject, from empirical sequences [Tro69]. At times, he groups both kinds of sequences under the broader category of *fixed by a recipe.*

Another formalization of the Creating Subject arguments is given by the Kripke Schema (KS), proposed by Kripke in 1965. It states that for any proposition A, there is a sequence a such that

$$(\forall x\,(a(x) = 0 \vee a(x) = 1)) \wedge ((\exists x\,(a(x) = 1)) \rightarrow A) \wedge ((\forall x\,(a(x) = 0)) \rightarrow \neg A)$$

The schema is intended to capture the use of CS-sequences. The idea is that the sequence is assigned the value 0 as long as no proof of A has been found, and switches to the value 1 from the moment that a proof of A is found. Van Atten shows that KS is derivable in the Theory of the Creating Subject, with the derivation relying crucially on CS1 and CS4; see [Att18]. As in **TCS**, the sequence in KS is represented as a completely described object: although its values may not yet be known, the rule determining them is fixed in advance, and one can find the values by using this rule. According to some authors, including Van Atten, an instance of KS can be identified in Brouwer's work, whereas others, such as Niekus, reject this interpretation.

Already in 1987, Niekus investigated Brouwer's method of the Creating Subject and concluded that the Kreisel-Troelstra formalization was too strong for Brouwer's purposes [Nie87]. He argued that, due to these overly strong assumptions, the CS-sequences were incorrectly interpreted as lawlike sequences. In his view, the objects arising in Brouwer's Creating Subject arguments are instead instances of choice sequences: their descriptions are incomplete because their future values depend on mathematical activity that has not yet taken place. He gives a different formalization of the Creating Subject arguments, in which the principles CS1 and CS4 are no longer valid, and argues that the CS-sequences should consequently be characterized as choice sequences rather than lawlike sequences.

2.4 Niekus' view

Niekus' proposal starts from a different reading of the notion of the Creating Subject. He argues that existing formalizations treated the Creating Subject as a special notion newly introduced by Brouwer, thereby overlooking its connection with Brouwer's earlier arguments using sequences based on mathematical activity [Nie87]. Before adopting the expression *Creating Subject*, Brouwer had already used formulations such as "we" and "I" to refer to the mathematician carrying out mathematical constructions. On Niekus' reading, the Creating Subject therefore does not denote a new mathematical entity, but is simply Brouwer's new way of referring to the mathematician who is constructing the sequence. Reasoning about the activity of the Creating Subject therefore amounts to reasoning about what may happen in *our* mathematical activity.

When formalized in the language of **TCS**, the statement $\Box_n A$ should therefore be read as "at the n-th stage *from now* we shall have a proof of A". Although $\Box_n A$ could also be interpreted as saying that a proof of A is already available now and will still be available at stage n, this reading is not relevant for Brouwer's Creating Subject arguments, where the point is precisely to reason about proofs that may only be obtained in the future. Niekus therefore takes $\Box_n A$ to express only the former interpretation.

Under this interpretation the axioms CS1 and CS4 of **TCS** are not valid. Niekus gives the following counterargument, where G_n (which expresses "in n steps from now, it is going to be the case that") replaces $\Box_n$:

> "That
>
> $$\exists n\, G_n\varphi \to \varphi \tag{7}$$
>
> is not valid can be seen as follows. Let B be an undecided proposition. We define a sequence $b_1, b_2, b_3, \ldots$ as follows. We choose $b_n = 0$ until we have found a proof of $B \vee \neg B$, after which we keep $b_n = 1$.
>
> Take φ to be $(b_{n_0} = 0 \vee b_{n_0} = 1)$. We do not have $(b_{n_0} = 0 \vee b_{n_0} = 1)$ for any n_0, because

this would mean that we would have one of the disjuncts, and so we would already know now whether we shall have a proof of $B \vee \neg B$ at the time of the choice of b_{n_0}. On the other hand, we do have $G_{n_0+1}(b_{n_0} = 0 \vee b_{n_0} = 1)$, i.e. we do have $G_{n_0+1}\varphi$, so (7) does not hold. In a similar way, it follows that

$$\forall n(G_n\varphi \vee \neg G_n\varphi) \tag{8}$$

is not valid. Let the sequence $b_1, b_2, b_3, \ldots$ be defined as above and take φ to be $b_{n_0} = 1$. If (8) were to hold, we would know already now whether we shall have a proof of $B \vee \neg B$ at the time of the choice of b_{n_0}, which is of course not generally true. Hence (8) does not hold either." [Nie10]

The validity of these principles would require the Creating Subject to decide, already now, what will be known at a future stage of its own mathematical activity, which is precisely what Niekus rejects. On his interpretation, the axioms CS2 and CS3 remain valid, and he argues that these are the only principles required to formulate Brouwer's Creating Subject arguments in the language of **TCS**. He develops a Kripke semantics for his alternative reading of $\Box_n$ and $\exists n \Box n$, under which, of the **TCS** axioms, precisely CS2 and CS3 are validated [Nie87]. In several works, he shows that Brouwer's original arguments can indeed be reconstructed from these assumptions alone [Nie05; Nie10; Nie17]. We refer to these papers for the technical details.

An important consequence of Niekus' proposal is that Creating Subject sequences are no longer regarded as lawlike. For a lawlike sequence (a_n), one would expect that, for each term, we have

$$a_n = 0 \vee a_n \neq 0,$$

since the law determining the specific value a_n of the sequence provides a finite method for deciding this disjunction. According to Niekus, such reasoning is not justified for CS-sequences, and this is reflected in the failure of CS1 in his formalization. The values of the CS-sequences depends on future mathematical activity that has not yet taken place, and therefore we cannot determine which of the two disjuncts holds, since one cannot simply calculate the value a_n.

In recent work on Creating Subject arguments in favour of **TCS** and KS, Van Atten responds to this objection. He acknowledges that CS-sequences are not completely determined by their description, so they are not lawlike in the same sense of what Troelstra called *absolutely lawlike* sequences. According to Van Atten, an essential degree of freedom remains, since the Creating Subject's freedom in carrying out mathematical activity continues to influence the values that the sequence will take. Nevertheless, Van Atten argues that each individual value is still determined by a finite procedure: for any given stage n, the Creating Subject can carry out the finitely many steps needed to reach that stage and then check whether evidence for A has appeared at one of the preceding stages [Att18]. In this sense, we have a finite method for finding the value a_n, any thereby for deciding whether any term a_n of a CS-sequence is equal to zero or not.

This interpretation treats a CS-sequence as an ongoing process whose values become available as the activity of the Creating Subject unfolds. Niekus rejects this way of understanding the sequence. As he points out, the idea that one can move into the future to inspect later values would make a CS-sequence depend on the individual carrying out the description, since different individuals may carry out the relevant future mathematical activity differently and thus arrive at different values [Nie10]. This is not the case for lawlike sequences: by following the defining rule, any individual would obtain the same values. Brouwer himself highlights this distinction between a CS-sequence s and a π-sequence r in his 1934 Geneva lecture:

> "When one hundred different persons are constructing the number r, one is always certain that any interval chosen by one of these persons is always covered, at least partly, by every interval chosen by one of the others. That is different for s. If I would give the definition of s to one hundred different persons, who are all going to work in a different room, it is possible that one of these one hundred persons at one time will choose an interval not covered by an interval chosen by one of the others."

Niekus thus says, while the values of a CS-sequence depend on future mathematical activity, the sequence should not be regarded as a process whose future values can later be inspected. Rather, a sequence is given by its description, and mathematical claims about it must be justified solely on the basis of that description. In this way, he argues, one avoids the strongly solipsistic and intensional character often attributed to these mathematical objects, which has led some to dismiss them as serious mathematical entities.

The description of a CS-sequence is incomplete, since it does not give us a way to determine its values. This marks a clear difference between CS-sequences and sequences such as the π-sequence discussed above, whose descriptions are complete. For a sequence like π, any digit can, in principle, be computed by carrying out a sufficiently long calculation using information available at the present time. In contrast, the values of a Creating Subject sequence depend on mathematical activity that has not yet taken place and cannot simply be presumed to occur. Therefore, while we can reason about all terms of a lawlike sequence, since the information determining each individual term is in principle available, this is not the case for CS-sequences.

The contrast between the two main defenses of formalizations of Brouwer's Creating Subject arguments and their corresponding sequences in the current literature can therefore be summarised as follows. For Van Atten, who supports the Kreisel-Troelstra formalization (**TCS**), passing to a fixed future stage n provides a legitimate finite method for obtaining a proof if one is available by that stage, which results in the validity of CS1 and CS4. For Niekus, by contrast, all mathematical reasoning must be justified from the present standpoint: we may reason about possible future stages, but we cannot appeal to information that only becomes available through future mathematical activity. In his formalization, the principles CS1 and CS4 are therefore not included. According to their respective interpretations, they treat the individual values of CS-sequences differently: under Van Atten's interpretation, the individual values are in principle obtainable, whereas on Niekus' interpretation they are not.

TCS has become the established formal framework for studying the Creating Subject arguments. The goal of this thesis is to develop Niekus' proposal into a full formal framework, providing a way to investigate his interpretation as an alternative to existing formalizations. Moreover, since Niekus explicitly connects the formalization of the Creating Subject arguments with Brouwer's use of individual choice sequences, such a framework can be helpful in further developing the theory of choice sequences.

In the next section, we discuss why the logics developed in this thesis provide an appropriate framework for formalizing Creating Subject arguments in accordance with Niekus' proposal.

2.5 A formal framework for Niekus' interpretation

The starting point for our formalization is Niekus' proposal, which he formulated within the language of **TCS** together with a semantics for his interpretation of $\Box_n$ and $\exists n, \Box_n$ [Nie87]. Building on this work, as well as unpublished work by Niekus together with Lex Hendriks and Dick de Jongh, we formulate the system in a framework with an operator $\Box$ expressing "from the next stage onwards", from which $\Box_n$ and $\exists n \,\Box_n$ can be derived. This framework additionally includes the temporal operator $\bigcirc$, expressing "eventually". We further extend it to a first-order setting with an explicit existence predicate E. In this section, we motivate these changes and extensions and explain their role in the formalization of Brouwer's Creating Subject arguments.

Our framework replaces the indexed modality $\Box_n$ of **TCS** with a single temporal operator $\Box$. For Niekus, $\Box_n A$ expresses that, after n further steps, A will hold, regardless of how the construction proceeds. Consequently, $\exists n \,\Box_n A$ states that there is some stage n such that, after n further steps, A will hold, regardless of how the construction proceeds. This behaviour is captured by the modality $\Box$, expressing that, regardless of how the construction proceeds, A will hold from the next stage onwards. The operators $\Box_n$ and $\exists n \,\Box_n$ can then be defined from this modality.

We also introduce the temporal operator $\bigcirc$, which is closely related to, but weaker than, the operator $\exists n \,\Box_n$. A formula of the form $\bigcirc A$ expresses that, regardless of how the construction proceeds, there will eventually be a stage from which on A holds. Unlike $\exists n \,\Box_n A$, however, it does not talk about a particular finite number of steps after which A is guaranteed to hold. This makes $\bigcirc$ better suited to expressing Brouwer's Creating Subject arguments, since Brouwer himself does not refer to a specific number of stages in these arguments. For example, Brouwer's claim that "it would be certain that α could never be proved to be true, and hence the absurdity of α would be known" can in our extended language be represented by the formula

$$\neg \bigcirc \alpha \to \neg\alpha. \tag{2.2}$$

As we will show, our system proves $\bigcirc A \to \neg\neg A$, and intuitionistically we have $\neg\neg\neg A \leftrightarrow \neg A$, which together validate this expression. The operator $\bigcirc$ cannot, in general, be defined as $\exists n \,\Box_n$. The latter requires a uniform stage n after which A holds in all possible continuations of the construction. However, no such bound exists in general: different continuations of the Creating Subject's activity may establish A at different stages. For this reason, $\bigcirc$ is taken as a primitive modality. We later show, using our semantics, that it is independent of $\exists n \,\Box_n$. When the temporal structure is a finitely branching tree, however, the two operators coincide.

In our axiomatization of $\Box$ and $\bigcirc$, as well as in the corresponding semantics, we ensure that the interpretation of these operators matches Niekus' proposal. It should reflect the asymmetry between present and future information: anything that is available at the current stage remains available at all later stages, whereas information that becomes available only at a later stage cannot, in general, be accessed from the present stage.

We further extend the language with an explicit existence predicate E, as suggested by Niekus himself [Nie87]. The intended reading is that $E(t)$ expresses that the term t exists, i.e. that it refers to an existing object. For a choice sequence (a_n), this means that $E(a_n)$ holds once one can calculate the value of a_n. For a lawlike sequence, every term exists from the outset, since its value is determined by the law defining the sequence, and we can therefore identify the mathematical object to which the term corresponds. For a CS-sequence, however, no such determination is available in advance: if the value of a term depends on information that only becomes available at a later stage, then that term does not yet exist. Once the required information becomes available, the corresponding mathematical object can be identified, and we may conclude $E(a_n)$ and reason about the term.

The addition of the existence predicate to the language is thus useful for expressing the difference between Van Atten's and Niekus' interpretations of CS-sequences. Since Van Atten argues that there is a finite procedure for determining the value of a term a_n by proceeding to the relevant future stage, he accepts that it is decidable for every term whether it is equal to zero, which can be expressed as

$$\forall n\, (a_n = 0 \vee a_n \neq 0).$$

Niekus, by contrast, holds that such a disjunction is not justified in general. If the value a_n only becomes available at a future stage, which is the case for CS-sequences, then neither $a_n = 0$ nor $a_n \neq 0$ can be established before that point. For Niekus, decidability therefore applies only to existing objects; in our extended language, this can be expressed as

$$\forall n\, (E(a_n) \to (a_n = 0 \vee a_n \neq 0)).$$

In the case of CS-sequences, the value of a_n becomes available exactly after n further stages, which can be expressed using the existence predicate as follows:

$$\text{for every } n \text{ we have } \Box_n E(a_n).$$

The advantage of using the existence predicate in combination with the operator $\Box$ is that it allows us to derive statements such as

$$\text{for every } n \text{ we have } \Box_n (a_n = 0 \vee a_n \neq 0)$$

for CS-sequences. The combination of the existence predicate and the temporal operator thus enables us to represent what is justified about the future development of the sequences based on the information available at each stage, without requiring us to actually proceed into the future.

The same idea can also be applied to arbitrary choice sequences: the existence predicate could be used to express that every term of a choice sequence (a_n) will eventually come into existence by writing

$$\forall n \bigcirc E(a_n).$$

The use of $\bigcirc$ is essential here: for a general individual choice sequence there need not be any fixed stage at which the value of a term is guaranteed to be defined. Since, for Niekus, a formalization of the Creating Subject is intended as a formalization of the use of individual choice sequences in general, the resulting theory should be applicable to any kind of choice sequence brought into existence through the activity of the Creating Subject.

In the remainder of this thesis, we develop the formal systems as described above. Where relevant, we discuss the philosophical implications of the results in light of the different interpretations of Creating Subject arguments. The main focus, however, will be on the technical development and analysis of the logics themselves.

Chapter 3

TICS

This chapter gives an overview of **TICS**, a system with two modal operators: $\Box$, read as "from the next stage onwards", and $\bigcirc$, read as "eventually". Both are meant to be read in the universal sense: $\Box A$ means that A holds in any successor stage and $\bigcirc A$ means that any possible continuation eventually reaches a stage at which the formula holds. The system was developed by Joop Niekus, Lex Hendriks, and Dick de Jongh. Although these results have not previously been published, an overview of the system is included here because it provides the necessary background for the developments in the remainder of this thesis. All results in this chapter are due to these three authors.

3.1 Syntax

The language of **TICS** extends the propositional language of **IPC** by adding two modalities $\Box$ and $\bigcirc$. Thus, its formulas are of the form

$$A ::= p \mid \bot \mid (A \wedge B) \mid (A \vee B) \mid (A \to B) \mid \Box A \mid \bigcirc A,$$

where p ranges over propositional variables. We define $A \leftrightarrow B := (A \to B) \wedge (B \to A)$ and $\neg A := A \to \bot$. Niekus and de Jongh axiomatize **TICS** by extending **IPC** with the following axioms:

B1 $A \to \bigcirc A$

B2 $\Box \bigcirc A \to \bigcirc A$

B3 $(A \to B) \to ((\Box B \to B) \to (\bigcirc A \to B))$

4 $(A \to B) \to (\bigcirc A \to \bigcirc B)$

5 $\bigcirc \bigcirc A \to \bigcirc A$

6 $\Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B)$

7 $(A \to B) \to (\Box A \to \Box B)$

8 $\Box(A \to B) \to (\bigcirc A \to \bigcirc B)$

9 $\Box(A \to B) \to (A \vee (A \to B))$

10 $\bigcirc A \to \Box \bigcirc A$

11 $\bigcirc A \to \neg\neg A$

TICS consists of all substitution instances of these axioms together with all substitution instances of intuitionistic propositional tautologies and is closed under Modus Ponens (MP). Since this system extends **IPC** by adding only modal axioms and no new inference rules, the deduction theorem remains valid. We write $\vdash A$ if A is a theorem of **TICS**, and $\Gamma \vdash A$ if A is derivable from Γ together with the axioms of **TICS**.

3.2 Semantics

The semantics of **TICS**, also due to Niekus, Hendriks, and De Jongh, are given by a special class of intuitionistic Kripke frames.

Definition 1. *A structure $F = \langle W, < \rangle$ is a* ***CS-frame*** *if $\langle W, < \rangle$ is a rooted tree (i.e., a strict preorder with a unique minimal element such that the predecessors of every point are linearly ordered by $<$), every point has a successor (i.e., for every $x \in W$ there exists $y \in W$ with $x < y$), and every point has finite depth (i.e., for every $x \in W$, there are only finitely many $y \in W$ with $y < x$). We write $\leq$ for the reflexive closure of $<$.*

CS-frames correspond to the rooted trees with infinite branches used in [**Nie87**].

Definition 2. *A structure $M = \langle W, \leq, [\![\cdot]\!] \rangle$ is a* ***CS-Kripke model*** *if $\langle W, \leq \rangle$ is (the reflexive closure of) a CS-frame and*

$$[\![\cdot]\!] : \mathrm{At} \to \mathcal{P}(W)$$

is a valuation assigning to each atomic formula a subset of W, monotone along $\leq$:

$$w \in [\![p]\!] \text{ and } w \leq v \quad \Longrightarrow \quad v \in [\![p]\!].$$

This valuation induces a forcing relation $\Vdash \subseteq W \times \mathrm{Form}$ between worlds and formulas of ***TICS****, defined inductively by:*

$$\begin{aligned}
w \Vdash p &\iff w \in [\![p]\!], \\
w \Vdash \bot &\iff \text{never}, \\
w \Vdash A \wedge B &\iff w \Vdash A \text{ and } w \Vdash B, \\
w \Vdash A \vee B &\iff w \Vdash A \text{ or } w \Vdash B, \\
w \Vdash A \to B &\iff \forall v \geq w\, (v \Vdash A \Rightarrow v \Vdash B), \\
w \Vdash \Box A &\iff \forall v > w\, (v \Vdash A), \\
w \Vdash \bigcirc A &\iff \text{for every infinite chain } w = w_0 < w_1 < w_2 < \cdots, \\
&\qquad \exists n \in \mathbb{N}\, (w_n \Vdash A).
\end{aligned}$$

We say that A is **valid** on a CS-Kripke model M, notation $M \models A$, if $w \Vdash A$ for every $w \in W$. We say that A is **valid** on a CS-frame F, notation $F \models A$, if it is valid on every CS-Kripke model based on F. Finally, A is **valid on the class of CS-frames**, notation $\models A$, if it is valid on every CS-frame.

3.2.1 Soundness

We first show the persistence of **TICS**-formulas in CS-Kripke frames in the following lemma.

Lemma 3 (Persistence)**.** *For every formula A and every CS-Kripke model $M = \langle W, \leq, [\![\cdot]\!]\rangle$: if $w \Vdash A$ and $w \leq v$, then $v \Vdash A$.*

Proof. By induction on the structure of A.

$\mathbf{A = p}$. Immediate from the persistence condition on the valuation $[\![\cdot]\!]$.

$\mathbf{A} = \bot$. Immediate.

$\mathbf{A = A_1 \wedge A_2}$. If $w \Vdash A_1 \wedge A_2$, then $w \Vdash A_1$ and $w \Vdash A_2$, so by the induction hypothesis $v \Vdash A_1$ and $v \Vdash A_2$, i.e. $v \Vdash A_1 \wedge A_2$.

$\mathbf{A = A_1 \vee A_2}$. If $w \Vdash A_1 \vee A_2$, then $w \Vdash A_1$ or $w \Vdash A_2$. By the induction hypothesis, $v \Vdash A_1$ or $v \Vdash A_2$, i.e. $v \Vdash A_1 \vee A_2$.

$\mathbf{A = A_1 \to A_2}$. Suppose $w \Vdash A_1 \to A_2$ and $w \leq v$. Let $u \geq v$ with $u \Vdash A_1$; since $v \leq u$ gives $w \leq u$, and $w \Vdash A_1 \to A_2$, we get $u \Vdash A_2$. As u was arbitrary, $v \Vdash A_1 \to A_2$.

$\mathbf{A} = \Box\mathbf{A_0}$. Suppose $w \Vdash \Box A_0$ and $w \leq v$. Let $u > v$; since $w \leq v < u$ gives $w < u$, and $w \Vdash \Box A_0$, we get $u \Vdash A_0$. As u was arbitrary, $v \Vdash \Box A_0$.

$\mathbf{A} = \bigcirc\mathbf{A_0}$. Suppose that $w \Vdash \bigcirc A_0$ and that $w \leq v$. Let

$$v = v_0 < v_1 < v_2 < \cdots$$

be an arbitrary infinite chain starting at v. If $w = v$, then this is also an infinite chain starting at w, so by hypothesis there exists some $n \in \mathbb{N}$ such that $v_n \Vdash A_0$. Hence $v \Vdash \bigcirc A_0$. Now suppose that $w < v$. Then

$$w < v = v_0 < v_1 < v_2 < \cdots$$

is an infinite chain starting at w. Since $w \Vdash \bigcirc A_0$, some point on this chain satisfies A_0. If this point is w, then by the induction hypothesis, applied to $w < v$, we obtain $v \Vdash A_0$. Otherwise, it is some v_n, in which case $v_n \Vdash A_0$. In either case, the original chain starting at v contains a point satisfying A_0. Since the choice of the chain starting at v was arbitrary, it follows that $v \Vdash \bigcirc A_0$. □

Theorem 4 (Soundness)**.** *If $\vdash A$ then $\models A$.*

Proof. Every CS-frame validates all substitution instances of intuitionistic tautologies and is closed under Modus Ponens. This follows because a CS-Kripke model is a Kripke model for intuitionistic logic: the relation $\leq$ is a preorder, the semantic clauses for $\wedge$, $\vee$, $\to$, and $\bot$ are the standard intuitionistic ones, and the valuation is monotone along $\leq$. It remains to check axioms B1–B3 and 4–11.

Fix a CS-Kripke model $M = \langle W, \leq, [\![\cdot]\!]\rangle$. We use Lemma 3 repeatedly. By the semantics of intuitionistic implication, to prove the validity of an implication $A \to B$ it suffices to take an arbitrary world w satisfying the antecedent A and show that the consequent B holds at w.

B1: $A \to \bigcirc A$**.** Let w be arbitrary and suppose $w \Vdash A$; we show $w \Vdash \bigcirc A$. For any infinite chain $w = w_0 < w_1 < \cdots$, the first point $w_0 = w$ already satisfies A. Hence $w \Vdash \bigcirc A$.

B2: $\Box \bigcirc A \to \bigcirc A$**.** Let w satisfy $w \Vdash \Box \bigcirc A$. We show $w \Vdash \bigcirc A$. Let $w = w_0 < w_1 < w_2 < \cdots$ be an arbitrary infinite chain. Since $w_1 > w$, we have $w_1 \Vdash \bigcirc A$. Applying this to the tail $w_1 < w_2 < \cdots$, there exists some $n \geq 1$ such that $w_n \Vdash A$. Hence $w \Vdash \bigcirc A$.

B3: $(A \to B) \to ((\Box B \to B) \to (\bigcirc A \to B))$**.** Let $w \Vdash A \to B$. Let $w' \geq w$ satisfy $w' \Vdash \Box B \to B$. We show that $w' \Vdash \bigcirc A \to B$. Let $w'' \geq w'$ satisfy $w'' \Vdash \bigcirc A$. We show $w'' \Vdash B$.

By persistence, $w'' \Vdash A \to B$ and $w'' \Vdash \Box B \to B$. Suppose $w'' \nVdash B$. Then $w'' \nVdash \Box B$, since otherwise the implication clause would give $w'' \Vdash B$. Hence there exists $v_1 > w''$ with $v_1 \nVdash B$. Repeating this argument, using persistence of $\Box B \to B$, we obtain an infinite chain

$$w'' = v_0 < v_1 < v_2 < \cdots$$

such that $v_n \nVdash B$ for every n. Since $w'' \Vdash \bigcirc A$, some $v_n \Vdash A$. By persistence, $v_n \Vdash A \to B$, and by the implication clause applied at v_n, we obtain $v_n \Vdash B$, contradicting $v_n \nVdash B$. Hence $w'' \Vdash B$.

4: $(A \to B) \to (\bigcirc A \to \bigcirc B)$. Let $w \Vdash A \to B$. Let $w' \geq w$ satisfy $w' \Vdash \bigcirc A$. We show $w' \Vdash \bigcirc B$. By persistence, $w' \Vdash A \to B$. Let $w' = v_0 < v_1 < \cdots$ be an arbitrary infinite chain. Since $w' \Vdash \bigcirc A$, some $v_n \Vdash A$. Since $v_n \geq w'$ and $w' \Vdash A \to B$, the implication clause gives $v_n \Vdash B$. Hence $w' \Vdash \bigcirc B$.

5: $\bigcirc \bigcirc A \to \bigcirc A$. Let $w \Vdash \bigcirc \bigcirc A$. Let $w = w_0 < w_1 < w_2 < \cdots$ be an arbitrary infinite chain. Since $w \Vdash \bigcirc \bigcirc A$, some $w_n \Vdash \bigcirc A$. Applying this to the tail starting at w_n, there exists some $m \geq n$ such that $w_m \Vdash A$. Hence $w \Vdash \bigcirc A$.

6: $\Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B)$. For every $v > w$,

$$v \Vdash A \wedge B \iff (v \Vdash A \text{ and } v \Vdash B).$$

Therefore, $w \Vdash \Box(A \wedge B)$ iff $w \Vdash \Box A$ and $w \Vdash \Box B$.

7: $(A \to B) \to (\Box A \to \Box B)$. Let $w \Vdash A \to B$. Let $w' \geq w$ satisfy $w' \Vdash \Box A$. For any $v > w'$, we have $v \Vdash A$. Since $v \geq w$ and $w \Vdash A \to B$, the implication clause gives $v \Vdash B$. Hence $w' \Vdash \Box B$.

8: $\Box(A \to B) \to (\bigcirc A \to \bigcirc B)$. Let $w \Vdash \Box(A \to B)$. Let $w' \geq w$ satisfy $w' \Vdash \bigcirc A$. By persistence, $w' \Vdash \Box(A \to B)$. Let $w' = v_0 < v_1 < \cdots$ be an arbitrary infinite chain. Since $w' \Vdash \bigcirc A$, some $v_n \Vdash A$. Choose $k = \max(n, 1)$. By persistence, $v_k \Vdash A$, and since $v_k > w'$, $w' \Vdash \Box(A \to B)$ gives $v_k \Vdash A \to B$. Hence $v_k \Vdash B$, so $w' \Vdash \bigcirc B$.

9: $\Box(A \to B) \to (A \vee (A \to B))$. Let $w \Vdash \Box(A \to B)$. If $w \Vdash A$, then we are done. Suppose instead that $w \nVdash A$. We show that $w \Vdash A \to B$. Let $v \geq w$ with $v \Vdash A$. Since $w \nVdash A$, we have $v > w$. Therefore, by $w \Vdash \Box(A \to B)$, we obtain $v \Vdash A \to B$. Hence, by the implication clause, $v \Vdash B$. Since v was arbitrary, $w \Vdash A \to B$. Therefore $w \Vdash A \vee (A \to B)$.

10: $\bigcirc A \to \Box \bigcirc A$. Let $w \Vdash \bigcirc A$ and let $v > w$. We show $v \Vdash \bigcirc A$. Let $v = v_0 < v_1 < \cdots$ be an arbitrary infinite chain. Then $w < v_0 < v_1 < \cdots$ is an infinite chain from w. Hence some point on this chain satisfies A. If this point is w, persistence gives $v_0 \Vdash A$; otherwise it is already some v_n. Therefore $v \Vdash \bigcirc A$.

11: $\bigcirc A \to \neg\neg A$. Let $w \Vdash \bigcirc A$. We show $w \Vdash \neg\neg A$. Let $w' \geq w$ with $w' \Vdash \neg A$, i.e. $w' \Vdash A \to \bot$. By persistence, $w' \Vdash \bigcirc A$. Consider any infinite chain

$$w' = u_0 < u_1 < u_2 < \cdots .$$

Then some $u_n \Vdash A$. Since $u_n \geq w'$ and $w' \Vdash A \to \bot$, the implication clause gives $u_n \Vdash \bot$, impossible. Hence no such w' exists, and therefore $w \Vdash \neg\neg A$. $\Box$

3.2.2 Completeness

In this section we prove completeness of **TICS** with respect to the class of CS-frames. Fix a formula E with $\nvdash E$; we construct a CS-Kripke model refuting E. First, we establish the relevant definitions and the Lindenbaum lemma. Note that this version of it does not require the Axiom of Choice since we work with a countable language.

Definition 5. *A* ***DP-theory*** *is a set of formulas Δ closed under derivability ($\Delta \vdash C \Rightarrow C \in \Delta$), consistent ($\Delta \nvdash \bot$), and possessing the* ***disjunction property****: if $C \vee D \in \Delta$ then $C \in \Delta$ or $D \in \Delta$.*

Lemma 6 (Lindenbaum)**.** *If $\Gamma \nvdash B$, then there is a DP-theory Γ' with $\Gamma \subseteq \Gamma'$ and $\Gamma' \nvdash B$.*

Proof. Let $C_0, C_1, C_2, \ldots$ enumerate all formulas. Define $\Delta_0 = \Gamma$ and

$$\Delta_{n+1} = \begin{cases} \Delta_n \cup \{C_n\} & \text{if } \Delta_n, C_n \nvdash B, \\ \Delta_n & \text{otherwise,} \end{cases}$$

and let $\Gamma' = \bigcup_n \Delta_n$. Clearly $\Gamma \subseteq \Gamma'$.

We first show that $\Gamma' \nvdash B$. By induction, $\Delta_n \nvdash B$ for every n: this holds for $n = 0$ by hypothesis, and the recursion preserves it by construction. If $\Gamma' \vdash B$, then, since derivations are finite, some finite subset of Γ' derives B, hence $\Delta_n \vdash B$ for some n, which gives a contradiction. So $\Gamma' \nvdash B$. In particular, this gives that Γ' is consistent, otherwise by ex falso we would have $\Gamma' \vdash B$. Deductive closure of Γ' is immediate from $\Gamma' \nvdash B$ together with the construction: if $\Gamma' \vdash C_n$ then $\Delta_n, C_n \nvdash B$ must have held at stage n (otherwise $\Gamma' \vdash B$, contradicting the above), so C_n was added, i.e. $C_n \in \Gamma'$.

Now we show the disjunction property. Suppose $C_i \vee C_j \in \Gamma'$ but $C_i, C_j \notin \Gamma'$. By the construction we must have had $\Delta_i, C_i \vdash B$ for C_i to have been rejected; so $\Gamma', C_i \vdash B$, and similarly $\Gamma', C_j \vdash B$. By disjunction elimination, $\Gamma', C_i \vee C_j \vdash B$, and since $C_i \vee C_j \in \Gamma'$, $\Gamma' \vdash B$, which contradicts $\Gamma' \nvdash B$. Hence $C_i \in \Gamma'$ or $C_j \in \Gamma'$. $\square$

Definition 7. *Define*

$$\Phi_E = \mathrm{Sub}(E) \cup \{\Box(B_1 \wedge \cdots \wedge B_k) : \Box B_i \in \mathrm{Sub}(E)\},$$

where $\mathrm{Sub}(E)$ *is the set of subformulas of E. Note that since E has finitely many subformulas, Φ_E is finite.*

We build the countermodel M as an infinite tree; each node δ carries a DP-theory Δ. The root γ carries a DP-theory Γ obtained by noting $\emptyset \nvdash E$, so Lemma 6 gives a DP-theory Γ such that $\Gamma \nvdash E$. Every node δ with theory Δ receives finitely many direct $<$-successors constructed by the following procedures, applied to each formula in Φ_E falling under one of four cases. We fix, for the whole construction, an enumeration of the $\bigcirc$-formulas of Φ_E. The construction sometimes refers to *pending formulas / (pending) obligations.* This denotes a formula for which some action still has to be performed. Once the relevant action has been taken, such an obligation is said to be *discharged.* If no procedure applies to Δ, then δ is repeated as its own successor.

Procedure (1) (treats $\bigcirc C \in \Phi_E \setminus \Delta$, for every such C). Let $\bigcirc D$ be the first pending obligation on the list with $\bigcirc D \in \Delta$, $D \notin \Delta$ (if none exists, omit D from what follows). Let $\Box B_1, \ldots, \Box B_k \in \Phi_E$ be the formulas such that $\Box B_i \in \Delta$ and $B_i \notin \Delta$. Using axiom 6 and the definition of Φ_E, there can all be combined into one $\Box B \in \Phi_E$ such that $\Box B \in \Delta$ and $B \notin \Delta$. If no such formula exists, omit B in what follows.

Claim: for each $\bigcirc C \in \Phi_E \setminus \Delta$ we have $\Delta, D, B \nvdash \bigcirc C$. Suppose otherwise. By the deduction theorem,

$$\Delta \vdash B \to (D \to \bigcirc C).$$

By axiom 7, $\Delta \vdash \Box B \to \Box(D \to \bigcirc C)$. Since $\Box B \in \Delta$, we have $\Delta \vdash \Box(D \to \bigcirc C)$. By axiom 8, $\Delta \vdash \bigcirc D \to \bigcirc\bigcirc C$, and by axiom 5, $\Delta \vdash \bigcirc D \to \bigcirc C$. Since $\bigcirc D \in \Delta$, we have $\Delta \vdash \bigcirc C$, so $\bigcirc C \in \Delta$ by deductive closure, which contradicts $\bigcirc C \notin \Delta$.

By Lemma 6 applied to $\Delta \cup \{D, B\}$ against $\bigcirc C$, for each $\bigcirc C \in \Phi_E \setminus \Delta$, we obtain a DP-theory Δ'_C with

$$\Delta \subseteq \Delta'_C, \qquad D, B \in \Delta'_C, \qquad \bigcirc C \notin \Delta'_C.$$

Each Δ'_C is the DP-theory for a separate created successor of δ, produced for the formula $\bigcirc C \in \Phi_E \setminus \Delta$. Since D is placed into every Δ'_C, its obligation is discharged on each of these successors (as $D \in \Delta'_C$).

Procedure (2) (treats $F \to G \in \Phi_E \setminus \Delta$). As in Procedure (1), combine all $\Box B_i \in \Delta \cap \Phi_E$ with $B_i \notin \Delta$ into a single witness B via axiom 6 (omit B if there is none). Assume $\Delta \nvdash F$ (since, if $\Delta \vdash F$ and $\Delta \nvdash G$, then the point itself witnesses the failure of the implication).

Claim: $\Delta, B, F \nvdash G$. Suppose otherwise; by the deduction theorem, $\Delta \vdash B \to (F \to G)$. By axiom 7, we have $\Delta \vdash \Box B \to \Box(F \to G)$, and since $\Box B \in \Delta$, we conclude $\Delta \vdash \Box(F \to G)$. By axiom 9, $\Delta \vdash F \vee (F \to G)$, so by the disjunction property of Δ, $\Delta \vdash F$ or $\Delta \vdash F \to G$, which are both excluded by hypothesis. Contradiction.

By Lemma 6, obtain Δ' with $\Delta \cup \{B, F\} \subseteq \Delta'$ and $G \notin \Delta'$. The obligation is permanently discharged, since Δ' provides a witness with $F \in \Delta'$ and $G \notin \Delta'$. For any later node $\Delta'' \supseteq \Delta'$, if $F \in \Delta''$ and $G \notin \Delta''$, then Δ'' itself witnesses the failure of $F \to G$. Otherwise, $G \in \Delta''$, so $\Delta'' \vdash G$, hence since also $F \in \Delta''$, by the deduction theorem we have $\Delta'' \vdash F \to G$. Thus no further treatment of this particular obligation is required.

Procedure (3) (treats $\Box A \in \Phi_E \setminus \Delta$). Combine boxed formulas as before into a single $\Box B \in \Delta$, $B \notin \Delta$. Consider $\bigcirc D \in \Delta$ such that $D \notin \Delta$, the first one on the list (omitting either if absent). We seek Δ' with $B, D \in \Delta'$, $A \notin \Delta$. If this extension is consistent, Lemma 6 immediately gives the required successor. However, if adding both B and D forces A, we need to construct a different successor, which is the purpose of Case B.

Case A: $\Delta \nvdash B \to (D \to A)$. Then $\Delta, B, D \nvdash A$, and Lemma 6 directly gives the required Δ'.

Case B: $\Delta \vdash B \to (D \to A)$. Here $\Delta, B, D \vdash A$, so Δ' cannot be produced by extending Δ with B and D together. In this case, we place $\Box A$ (rather than D) into a successor. By axiom B3, instantiated with D for A and A for B,

$$\vdash (D \to A) \to \big((\Box A \to A) \to (\bigcirc D \to A)\big).$$

Since $\Delta \vdash B \to (D \to A)$, prefixing $B \to (-)$ throughout gives

$$\Delta \vdash B \to \big((\Box A \to A) \to (\bigcirc D \to A)\big),$$

and hence, by a standard propositional rearrangement,

$$\Delta \vdash \big(B \to (\Box A \to A)\big) \to \big(B \to (\bigcirc D \to A)\big).$$

Claim: $\Delta \nvdash B \to (\bigcirc D \to A)$. Suppose otherwise. By axiom 7,

$$\Delta \vdash \Box B \to \Box(\bigcirc D \to A),$$

and applying axiom 7 again inside the consequent,

$$\Delta \vdash \Box B \to (\Box \bigcirc D \to \Box A).$$

By axiom 10,

$$\Delta \vdash \Box B \to (\bigcirc D \to \Box A).$$

Since $\Box B, \bigcirc D \in \Delta$, this gives $\Delta \vdash \Box A$, so $\Box A \in \Delta$ by deductive closure, which contradicts the standing hypothesis $\Delta \nvdash \Box A$ of Procedure (3). This proves the claim. By the claim together with the implication derived above,

$$\Delta \nvdash B \to (\Box A \to A).$$

By Lemma 6, we obtain Δ' with

$$\Delta \subseteq \Delta', \qquad B \in \Delta', \qquad \Box A \in \Delta', \qquad A \notin \Delta',$$

and, since $\bigcirc D \in \Delta \subseteq \Delta'$, also $\bigcirc D \in \Delta'$. This Δ' is the successor for Case B.

Procedure (4) (treats $\bigcirc D \in \Delta \cap \Phi_E$ with $D \notin \Delta$, and $\Box B \in \Delta \cap \Phi_E$ with $B \notin \Delta$, when no other procedure applies. Here again, we see $\Box B$ as the formula combining all boxed formulas $\Box B_i$ with $\Box B_i \in \Delta \cap \Phi_E$ and $B_i \notin \Delta$).

Suppose first that $\bigcirc D \in \Delta$ and $D \notin \Delta$. We show that D can be consistently added to Δ. Suppose otherwise, i.e., suppose that $\Delta \vdash \neg D$. By axiom 11, we have $\vdash \bigcirc D \to \neg\neg D$, and since $\bigcirc D \in \Delta$, it follows that $\Delta \vdash \neg\neg D$. Together with $\Delta \vdash \neg D$, this gives $\Delta \vdash \bot$, contradicting the consistency of Δ. Hence $\Delta \cup \{D\}$ is consistent, and Lemma 6 provides a DP-theory $\Delta' \supseteq \Delta \cup \{D\}$.

The argument for $\Box B \in \Delta$ and $B \notin \Delta$ is analogous. The only additional ingredient is the derivability of $\vdash \Box B \to \neg\neg B$ in **TICS**. For a proof of this result, one may combine axiom 11 with the implication $\Box A \to \bigcirc A$, established in Proposition 74 of Appendix A. The latter proposition is proved using the equivalent axiomatization $1'$–$7'$ of **TICS**, as shown in the same appendix. Therefore, if $\Delta \vdash \neg B$, then $\Delta \vdash \neg\neg B$ from $\Box B \in \Delta$, giving a contradiction. Thus $\Delta \cup \{B\}$ is consistent, and Lemma 6 applies.

Finally, suppose that both $\bigcirc D \in \Delta$ and $\Box B \in \Delta$, with $D, B \notin \Delta$. We show that $\Delta \cup \{D, B\}$ is consistent. Assume otherwise. Then $\Delta \vdash D \to (B \to \bot)$, and hence $\Delta \vdash D \to \neg B$. Since $\Box B \in \Delta$, the previous argument gives $\Delta \vdash \neg\neg B$. Therefore $\Delta \vdash \neg D$. But $\bigcirc D \in \Delta$ and $\vdash \bigcirc D \to \neg\neg D$, so $\Delta \vdash \neg\neg D$, again contradicting consistency. Hence $\Delta \cup \{D, B\}$ is consistent, and Lemma 6 gives the required Δ'.

This defines the tree via the direct $<$-successor construction. We define the valuation $[\![\cdot]\!]$ by $\delta \in [\![p]\!] \iff p \in \Delta$. Now we can show the Truth Lemma.

Lemma 8 (Truth Lemma)**.** *For every node δ of M with theory Δ, and every $C \in \Phi_E$:*

$$C \in \Delta \iff \delta \Vdash C.$$

Proof. By induction on the structure of C.

$\mathbf{C = p}$. By definition of the valuation, $\delta \in [\![p]\!] \iff p \in \Delta$.

$\mathbf{C = \bot}$. We have never have $\delta \Vdash \bot$ by definition and never have $\bot \in \Delta$ since all Δ consistent.

$\mathbf{C = C_1 \wedge C_2}$. Since Φ_E is closed under subformulas, we have $C_1, C_2 \in \Phi_E$. We show both directions. If $C_1 \wedge C_2 \in \Delta$, then by conjunction elimination, $C_1 \in \Delta$ and $C_2 \in \Delta$. By the induction hypothesis, $\delta \Vdash C_1$ and $\delta \Vdash C_2$, and hence $\delta \Vdash C_1 \wedge C_2$. Conversely, suppose that $\delta \Vdash C_1 \wedge C_2$. By definition, $\delta \Vdash C_1$ and $\delta \Vdash C_2$. By the induction hypothesis, $C_1 \in \Delta$ and $C_2 \in \Delta$. By conjunction introduction, $C_1 \wedge C_2 \in \Delta$. Therefore,

$$C_1 \wedge C_2 \in \Delta \iff \delta \Vdash C_1 \wedge C_2.$$

$\mathbf{C = C_1 \vee C_2}$. Again, $C_1, C_2 \in \Phi_E$. Suppose first that $C_1 \vee C_2 \in \Delta$. Since Δ is a DP-theory, it has the disjunction property, and therefore $C_1 \in \Delta$ or $C_2 \in \Delta$. By the induction hypothesis, $\delta \Vdash C_1$ or $\delta \Vdash C_2$, and hence $\delta \Vdash C_1 \vee C_2$. Conversely, suppose that $\delta \Vdash C_1 \vee C_2$. By the semantic clause for disjunction, $\delta \Vdash C_1$ or $\delta \Vdash C_2$. By the induction hypothesis, $C_1 \in \Delta$ or $C_2 \in \Delta$. By the disjunction introduction rules, $C_1 \vee C_2 \in \Delta$. Hence,

$$C_1 \vee C_2 \in \Delta \iff \delta \Vdash C_1 \vee C_2.$$

$\mathbf{C = F \to G}$. Suppose $F \to G \in \Delta$. For any $\delta' \geq \delta$ with theory $\Delta' \supseteq \Delta$: if $\delta' \Vdash F$, then by the induction hypothesis $F \in \Delta'$, so by modus ponens $G \in \Delta'$ (using $F \to G \in \Delta \subseteq \Delta'$), so $\delta' \Vdash G$ by the induction hypothesis. Hence $\delta \Vdash F \to G$.

Conversely, suppose $F \to G \notin \Delta$. By Procedure (2), δ has a successor δ' (either one constructed at δ itself to witness this failure or inherited from an earlier ancestor on the branch where the obligation was first treated) with $F \in \Delta'$ and $G \notin \Delta'$. By the induction hypothesis, $\delta' \Vdash F$ and $\delta' \nVdash G$. Hence $\delta \nVdash F \to G$.

$\mathbf{C = \Box A}$. Suppose $\Box A \in \Delta$. Each of Procedures (1)–(4), when applied to a theory containing $\Box A$ with A not yet present, places A into the immediate successor by construction; once A is present, it persists to all further points in the tree. So of any successor $\delta' > \delta$, the DP-theory Δ contains A. By the induction hypothesis, for any successor δ' of δ we have $\delta' \Vdash A$, so by the semantics, $\delta \Vdash \Box A$.

Conversely, suppose $\Box A \notin \Delta$. By Procedure (3), δ has an immediate successor δ' with $A \notin \Delta'$. By the induction hypothesis, $\delta' \nVdash A$, so $\delta \nVdash \Box A$.

$\mathbf{C = \bigcirc C_0}$. Suppose $\bigcirc C_0 \in \Delta$. Let $\delta = \delta_0 < \delta_1 < \cdots$ be an arbitrary infinite path, with theories $\Delta = \Delta_0 \subseteq \Delta_1 \subseteq \cdots$. We find n with $C_0 \in \Delta_n$; then $\delta_n \Vdash C_0$ by the induction hypothesis, and since the path was arbitrary, $\delta \Vdash \bigcirc C_0$.

If $C_0 \in \Delta_0$ we are done. Otherwise $\bigcirc C_0$ is pending at δ_0. Since the infinite path lies on an infinite branch that is constructed step by step according to the procedures, we distinguish three cases depending on the nature of these procedures along the branch.

(1) Infinitely many transitions of Procedure (1) occur along the branch. Recall that at each Procedure (1) transition the formula at the front of the pending list of $\bigcirc$-formulas is discharged. The pending list contains only formulas $\bigcirc D$ with $\bigcirc D \in \Phi_E$. Since Φ_E is finite, only finitely

many distinct formulas can ever occur on the list. At δ_0, $\bigcirc C_0$ is pending and hence occupies a finite position in the list. Although later transitions may add new pending formulas before $\bigcirc C_0$, these formulas are also drawn from the finite set of $\bigcirc$-formulas in Φ_E. Moreover, once a formula is discharged by a Procedure (1) transition, it is never added to the pending list again. Since infinitely many Procedure (1) transitions occur, the front of the list eventually reaches $\bigcirc C_0$, which is then discharged. By the definition of a Procedure (1) transition, the successor theory contains C_0. Hence $C_0 \in \Delta_n$ for some n.

(2) Infinitely many transitions of Procedure (3) occur along the branch. Each Procedure-(3) transition of Case B uses some $\Box A \in \Phi_E$ which is not a member of the current DP-theory and places $\Box A$ into the successor theory; by persistence, $\Box A$ remains in every later theory, so the same $\Box A$ can never again satisfy $\Box A \notin \Delta$, further along the branch. Hence each $\Box A \in \Phi_E$ serves as the principal formula according to Procedure (3) Case B at most once over the whole branch. So, after finitely many steps, the transitions happen according to Case A. Then everything is exactly as in Case 1, so each time the first $\bigcirc D$ of the list is taken care of and after a while the whole list is.

(3) Neither Case (1) nor Case (2) applies. Note that only finitely many transitions of Procedure (2) can occur along any branch. Indeed, there are only finitely many formulas of the form $F \to G$ in Φ_E, and each such implication is treated at most once on a branch: once a successor witnessing its failure has been created, the corresponding obligation is permanently discharged. From some point on, only Procedure (4) is applied. Every pending obligation from that point on, in particular $\bigcirc C_0 \in \Delta_n$, $C_0 \notin \Delta_n$, is a Procedure (4) obligation and is discharged at the very next node by Procedure (4), giving $C_0 \in \Delta_{n+1}$.

Conversely, suppose $\bigcirc C_0 \notin \Delta$. By Procedure (1), some immediate successor δ' of δ satisfies $\bigcirc C_0 \notin \Delta'$. Repeating this at δ', and so on, produces an infinite branch $\delta = \delta_0 < \delta_1 < \cdots$ with $\bigcirc C_0 \notin \Delta_n$ for every n. We claim $C_0 \notin \Delta_n$ for every such n. If $C_0 \in \Delta_n$ for some n, then since $\vdash C_0 \to \bigcirc C_0$ (axiom B1), this gives $\Delta_n \vdash \bigcirc C_0$, so $\bigcirc C_0 \in \Delta_n$ by deductive closure, contradicting $\bigcirc C_0 \notin \Delta_n$. So $C_0 \notin \Delta_n$ for all n, and by the induction hypothesis $\delta_n \not\Vdash C_0$ for all n. This branch witnesses $\delta \not\Vdash \bigcirc C_0$. $\Box$

Now we may conclude the completeness theorem.

Theorem 9 (Completeness). *If $\nvdash E$ then $\not\models E$.*

Proof. Since $\nvdash E$, Lemma 6 gives a DP-theory Γ with $\Gamma \nvdash E$; let Γ be the theory of the root γ of the model M constructed relative to Φ_E. As $E \in \Phi_E$ and $E \notin \Gamma$ (else $\Gamma \vdash E$ by deductive closure), Lemma 8 (Truth Lemma) gives $\gamma \not\Vdash E$. The underlying frame of M is a CS-frame: it is constructed as a rooted tree, every node has an immediate successor (by construction, a genuine successor which is obtained by one of the procedures, or itself when no procedure applies), and every node has only finitely many predecessors, since it is constructed step-by-step from the root upwards. The valuation clearly satisfies monotonicity. Hence E is refuted at γ in a CS-Kripke model. $\Box$

Corollary 10. ***TICS** is sound and complete with respect to the class of CS-frames: for every **TICS**-formula A, we have $\vdash A$ if and only if $\models A$.*

Chapter 4

TICS-2

In the previous chapter we presented **TICS** and saw that it is sound and complete with respect to a class of Kripke frames in which the intuitionistic preorder $\leq$ is taken to be the reflexive closure of the temporal accessibility relation $<$. In this chapter we consider a different semantic setting, in which the intuitionistic accessibility relation $\leq$ and the temporal accessibility relation, now denoted by R, are allowed to behave more independently. We take $R \subseteq \leq$, but this time we do not have that the reflexive closure of R equals $\leq$: we allow non-reflexive $\leq$-steps which are not R-steps.

This change has an important consequence for the resulting logic. The completeness proof for **TICS** relies on an axiom equivalent to $\Box B \to (A \vee (A \to B))$, which is validated by the semantics since the worlds quantified over in the semantic clause for implication are exactly the current world and its temporal successors. Once we allow non-reflexive $\leq$-steps which are not R-steps, the axiom is no longer valid. The axiom precisely captures the requirement that such steps do not exist.

One can see this clearly by considering what the axiom expresses. Reading $\Box$ as "from the next stage onwards", it says that if B will be established at the next stage, then for every statement A we either currently have a proof of A or a proof of $A \to B$, i.e. a procedure to obtain a proof of B from any proof of A. This is a strong principle, since A can be any unrelated statement. It can be understood as valid precisely when obtaining a proof of A, if we did not already have one, necessarily moves us to a stage at which B is proved regardless. In this case, if we do not have a proof of A, we have a method of obtaining a proof of B from a proof of A: if we obtain a proof of A, we only have to note that this means we have reached a future stage, at which we know that we have a proof of B. Thus, the axiom expresses exactly the idea that a genuine increase of information means that a temporal step has been made.

The alternative setup used in this chapter separates the notions of temporal progression and information growth more explicitly. The intuitionistic order $\leq$ represents general information growth: moving upwards in the order corresponds to passing to a state containing at least as much information as the current state. Temporal development is one specific mechanism through which information can become available, but it need not coincide with every form of information growth. The temporal relation R can therefore be understood as describing a distinguished temporal progression rather than every moment at which new information is acquired. This perspective allows us to consider less fine-grained timesteps. For example, if $\Box$ is understood as referring to what becomes available "tomorrow", we can still allow information to become available "today". The phrase "next stage" can thus refer to particular kinds of stages, such as minutes, days, or years. Such an interpretation can also be useful when modelling choice sequences: we may want the n-th term of a sequence to be determined at the start of the n-th day, rather than at the n-th moment. In this setting, $\Box$ describes a particular temporal step relevant for the interpretation, while $\leq$ captures the more general notion of informational increase. Consequently, the implication connective can again refer to non-reflexive $\leq$-steps that are not temporal steps, which is precisely why the axiom $\Box B \to (A \vee (A \to B))$ is no longer valid.

We show that the resulting system, **TICS-2**, omits the above axiom while remaining complete with respect to the corresponding bi-relational semantics. Apart from this principle, the resulting systems coincide. Since the axiom is not used in any of our applications, working without it provides a more minimal system for our intended purposes.

The chapter is structured as follows. We first introduce the syntax of **TICS-2**. We then describe its Kripke semantics and establish soundness and completeness. We use this to show how the system can be used for our philosophical project. The soundness and completeness proofs for our Kripke semantics rely on classical reasoning in the metatheory. To obtain a constructive alternative to the completeness proof, we turn to Beth semantics. In this chapter, we do not aim to give a complete account of Beth semantics; instead, we outline what a suitable approach should look like. The full construction is carried out in the next chapter for the $\Box$-fragment of the logic.

4.1 Syntax

TICS-2 is an intuitionistic propositional modal logic in the same language as **TICS**. It is obtained by adding the following modal axioms to **IPC**:

T1 $A \to \Box A$

T2 $\Box(A \to B) \to (\Box A \to \Box B)$

T3 $\neg\Box\bot$

T4 $A \to \bigcirc A$

T5 $\Box \bigcirc A \to \bigcirc A$

T6 $(A \to B) \to ((\Box B \to B) \to (\bigcirc A \to B))$

In this chapter, we write $\vdash$ for $\vdash_{\textbf{TICS-2}}$. Since **TICS-2** is obtained by extending **IPC** with additional modal axioms and no rules, the deduction theorem clearly remains valid.

To compare this axiomatization with that of **TICS**, we first note that **TICS** admits the following equivalent axiomatization:

1' $A \to \Box A$

2' $\Box(A \to B) \to (\Box A \to \Box B)$

3' $\neg\Box\bot$

4' $A \to \bigcirc A$

5' $\Box \bigcirc A \to \bigcirc A$

6' $(A \to B) \to ((\Box B \to B) \to (\bigcirc A \to B))$

7' $\Box B \to (A \vee (A \to B))$

For the proof of this equivalence, we refer to Appendix A. Through this equivalent presentation, we see that the only difference between the **TICS** and **TICS-2** is the presence of axiom 7'. Thus, **TICS-2** can essentially be obtained syntactically from **TICS** by removing axiom 7' from the axiomatization.

An important observation is that the unary modality $\bigcirc$ can be seen as a *least fixed point operator*. In recent years, there has been growing interest in fixed point extensions of intuitionistic modal logics, collectively known as *Intuitionistic Fixed Point Modal Logics* (IFPMLs) [Afs+24]. Examples include intuitionistic temporal logics, such as those studied in [Afs+23; BDF22], and the intuitionistic modal μ-calculus, for example as developed in [Pac23].

In our setting, $\bigcirc$ is not introduced as an explicit least fixed point operator, as in the modal μ-calculus, but rather axiomatically. Nevertheless, the axioms characterize $\bigcirc$ precisely as the least fixed point of a suitable monotone operator. More specifically, $\bigcirc A$ is the least pre-fixed point of the monotone propositional function

$$p \mapsto A \vee \Box p.$$

The monotonicity of this function follows from the monotonicity of the $\Box$ operator, which is derivable from axioms T1 and T2. For a propositional function F, a formula B is called a *pre-fixed point* of F if $F(B) \to B$. A *least pre-fixed point* is a pre-fixed point that implies every other pre-fixed point. By the Knaster-Tarski theorem, the least fixed point of a monotone operator coincides with its least pre-fixed point (see, e.g., [Tar55]), which is the characterization that we use.

To establish that $\bigcirc A$ is a least fixed point operator, we should thus show that it is the least pre-fixed point of the function $p \mapsto A \vee \Box p$. We see this by noting that axioms T4 and T5 give us that

$$A \vee \Box \bigcirc A \to \bigcirc A,$$

so $\bigcirc A$ is a pre-fixed point. Moreover, the induction principle T6 expresses that it is the least such pre-fixed point: whenever

$$A \vee \Box B \to B,$$

we have

$$\bigcirc A \to B.$$

Thus $\bigcirc A$ implies every pre-fixed point of the operator. Hence, in the notation of fixed point logics, where $\mu p F(p)$ denotes the least fixed point of the operator F, we may write

$$\bigcirc A = \mu p(A \vee \Box p).$$

We therefore see that both **TICS** and **TICS-2** are intuitionistic modal logics equipped with a monotone modal operator $\Box$ and a least fixed point operator $\bigcirc$.

4.2 Kripke semantics

Bi-relational Kripke semantics provide the standard framework for interpreting intuitionistic modal logics (IMLs) by combining two accessibility relations: an intuitionistic preorder $\leq$, representing the growth of information, and a modal accessibility relation R, used to interpret the modal operators.

Semantics of this style originate with Fischer Servi [Fis77; Fis80]; we refer to Simpson [Sim94] for a general development. Different choices of the semantic clauses for the intuitionistic and modal connectives, together with different interaction conditions imposed on the relations $\leq$ and R, give rise to different semantic variants of intuitionistic modal systems.

In Kripke semantics for intuitionistic modal logic, as in Kripke semantics for intuitionistic logic more generally, forcing is required to be persistent with respect to the intuitionistic ordering $\leq$. That is, if a formula is forced at a world, then it remains forced at every extension of that world. This reflects the informal interpretation of forcing in intuitionistic semantics as having a proof of A: if we have a proof of A at a given stage, then we continue to have it at any later stage where additional information has become available.

In our setting, the modal relation R is interpreted as a temporal step relation. Besides the usual persistence requirement with respect to the intuitionistic order, we also require persistence along the temporal relation R: a proof available at one point in time should remain available at any later point in time. In the semantics developed here, this is enforced by requiring $R \subseteq \leq$, which ensures that every temporal transition is also an intuitionistic extension. If the two relations were treated as genuinely independent, this persistence condition would instead need to be imposed explicitly in a different way.

One could also consider alternative formalizations of Brouwer's arguments in which persistence along the temporal dimension is not required for all formulas. For example, Brouwer's weak negation reflects a notion of *present* unavailability: a proposition may not have a proof at the current stage, while a proof may become available at a later stage. In such a setting, the corresponding formula need not be persistent with respect to the temporal relation R. In this thesis, we do not develop such a system; instead, we work with the stronger persistence assumptions described above. Nevertheless, similar techniques could be adapted to study frameworks in which temporal persistence is not required for all formulas.

4.2.1 Frames and models

We begin with the definition of a frame for our logic.

Definition 11. *A **TICS-2 frame** is a tuple $(W, \leq, R)$, where W is a non-empty set of worlds, $\leq$ is a partial order on W, and R is a binary relation on W satisfying:*

- *R is serial;*
- *$R \subseteq \leq$;*
- *triangularity: for all $x, y, z \in W$, if $x \leq y$ and yRz, then xRz.*

The triangularity condition is illustrated by the following diagram, explaining the name:

$$\begin{tikzcd}
y \arrow[r, "R"] & z \\
x \arrow[u, "\leq"] \arrow[ur, dashed, "R"'] &
\end{tikzcd}$$

Note that the condition can also be expressed by the equation $\leq \circ R \subseteq R$.

Definition 12. *A **TICS-2 model** is a tuple $(W, \leq, R, V)$, where $(W, \leq, R)$ is a TICS-2 frame and V is a valuation function satisfying, for every atomic proposition p and all $x, y \in W$,*

$$x \leq y \text{ and } x \in V(p) \implies y \in V(p).$$

That is, V is monotone with respect to $\leq$. The forcing relation $\Vdash$ is defined inductively as follows:

- $k \nVdash \bot$;
- $k \Vdash p \iff k \in V(p)$;
- $k \Vdash A \wedge B \iff k \Vdash A$ *and* $k \Vdash B$;
- $k \Vdash A \vee B \iff k \Vdash A$ *or* $k \Vdash B$;
- $k \Vdash A \to B \iff \forall k' \geq k,\ (k' \Vdash A \implies k' \Vdash B)$;
- $k \Vdash \Box A \iff \forall k' \,(kRk' \implies k' \Vdash A)$;
- $k \Vdash \bigcirc A \iff$ *for every infinite R-path $k = k_0 R k_1 R \ldots$, there exists $n \geq 0$ such that $k_n \Vdash A$.*

We say that A is **valid** on a TICS-2 model M, notation $M \models A$, if $k \Vdash A$ for every $k \in W$. We say that A is **valid** on a TICS-2 frame F, notation $F \models A$, if it is valid on every model based on F. Finally, A is **valid in TICS-2**, notation $\models A$, if it is valid on every **TICS-2** frame.

4.2.2 Soundness

We first note that the forcing relation is persistent.

Proposition 13. *All formulas are persistent with respect to $\leq$. As a result, all formulas are also persistent with respect to R.*

Proof. We prove persistence with respect to $\leq$ by induction on the complexity of formulas. The cases for the intuitionistic connectives are standard and follow as in **IPC**. For the modal operators, suppose that $k \leq k'$. If $\Box A$ holds at k, then every R-successor of k satisfies A. By triangularity, every R-path starting from k' is also an R-path starting from k, and therefore every R-successor of k' satisfies A. Hence, $\Box A$ is persistent with respect to $\leq$. The argument for $\bigcirc A$ is analogous: any R-path from k' is also an R-path from k, so forcing of $\bigcirc A$ is preserved under $\leq$-extensions. Finally, since $R \subseteq \leq$, persistence with respect to $\leq$ immediately implies persistence with respect to R. $\Box$

We use this to establish the validity of the modal axioms on our frames.

Proposition 14. *All modal axioms of **TICS-2** are valid on any **TICS-2** frame.*

Proof. We show all the modal axioms separately. Note that only the proof of soundness of T6 relies on classical reasoning in the metatheory.

T1: $A \to \Box A$. Suppose $k \Vdash A$ and take k' such that kRk'. By persistence of all **TICS-2** formulas with respect to R, we have $k' \Vdash A$. Therefore, $k \Vdash \Box A$.

T2: $\Box(A \to B) \to (\Box A \to \Box B)$. Suppose $k \Vdash \Box(A \to B)$. We need to show $k \Vdash \Box A \to \Box B$. Take any $k' \geq k$ such that $k' \Vdash \Box A$. Take any k'' with $k'Rk''$. Since $k' \geq k$, by persistence with respect to $\leq$ we have $k' \Vdash \Box(A \to B)$. Therefore, $k'' \Vdash A \to B$ and $k'' \Vdash A$, so $k'' \Vdash B$. Therefore, $k' \Vdash \Box B$.

T3: $\neg\Box\bot$**.** Holds by seriality of R: every world has an R-successor, where $\bot$ is not forced, so $\Box\bot$ is never forced.

T4: $A \to \bigcirc A$**.** Suppose $k \Vdash A$. Take any infinite R-path $k_0 R k_1 R \ldots$ with $k_0 = k$. Take $n = 0$, then $k_0 = k \Vdash A$.

T5: $\Box \bigcirc A \to \bigcirc A$**.** Suppose $k \Vdash \Box \bigcirc A$. Take any infinite R-path $k_0 R k_1 R \ldots$ with $k_0 = k$. We have $k_0 R k_1$, so $k_1 \Vdash \bigcirc A$. The tail $k_1 R k_2 R \ldots$ is an infinite R-path from k_1, so some k_n with $n \geq 1$ satisfies $k_n \Vdash A$.

T6: $(A \to B) \to ((\Box B \to B) \to (\bigcirc A \to B))$**.** Let $w \Vdash A \to B$. Let $w' \geq w$ satisfy $w' \Vdash \Box B \to B$. We show that $w' \Vdash \bigcirc A \to B$. Let $w'' \geq w'$ satisfy $w'' \Vdash \bigcirc A$. We show $w'' \Vdash B$. By persistence, $w'' \Vdash A \to B$ and $w'' \Vdash \Box B \to B$. Suppose $w'' \nVdash B$. Then $w'' \nVdash \Box B$, since otherwise the implication clause would give $w'' \Vdash B$. Hence there exists v_1 with $w'' R v_1$ and $v_1 \nVdash B$. Repeating this argument, using persistence of $\Box B \to B$, we obtain an infinite R-chain

$$w'' = v_0 R v_1 R v_2 R \cdots$$

such that $v_n \nVdash B$ for every n. Since $w'' \Vdash \bigcirc A$, some $v_n \Vdash A$. By persistence, $v_n \Vdash A \to B$, and by the implication clause applied at v_n, we obtain $v_n \Vdash B$, contradicting $v_n \nVdash B$. Hence $w'' \Vdash B$. $\Box$

As in the case of **TICS**, we obtain soundness with respect to our class of Kripke frames. The intuitionistic part of the argument follows from the standard soundness of intuitionistic Kripke semantics: if the modal relation R is ignored, the underlying structure of a **TICS-2** model is precisely an intuitionistic Kripke model, with the usual intuitionistic forcing clauses. The remaining modal axioms are verified in the previous proposition. We therefore have the following theorem.

Theorem 15 (Soundness)**.** *For every formula A,*

$$\vdash A \implies \models A.$$

4.2.3 Completeness

In order to prove completeness, we note the following.

Proposition 16.

$$\vdash \Box(A \wedge B) \leftrightarrow \Box A \wedge \Box B.$$

Proof. See Appendix A, Proposition 73. $\Box$

Proposition 17. *The following are derivable in **TICS-2**:*

1. $\Box(A \to B) \to (\bigcirc A \to \bigcirc B)$;
2. $\bigcirc \bigcirc A \to \bigcirc A$;
3. $\bigcirc A \to \neg\neg A$;
4. $\Box A \to \bigcirc A$.

Proof. See Appendix A, Corollary 75. $\Box$

We prove completeness for **TICS-2** with respect to the semantics introduced above, working in a classical metatheory. The proof strategy is the one of Niekus, Hendriks and De Jongh; the countermodel construction is mainly taken from the completeness proof for **TICS**, as given in the previous chapter. We reuse the notion of DP-theory and the Lindenbaum Lemma (Lemma 6) without change, so we only need to redefine the model construction to accommodate the different semantics for **TICS-2**.

Unlike the model for **TICS**, whose tree relation $<$ is used in the interpretation of both the intuitionistic connectives and modalities, the model for **TICS-2** carries two separate relations: an inclusion order $\leq$, used exactly as in ordinary intuitionistic Kripke models to model the intuitionistic fragment, and a relation R, constructed below, used to interpret $\Box$ and $\bigcirc$. Consequently, implications are falsified directly along $\leq$-successors, and no analogue of Procedure (2) from the **TICS** construction (which falsifies an implication along an immediate $<$-successor) is required here.

Fix a formula E with $\nvdash E$; we construct a model M refuting E. The worlds of the model M are DP-theories:

$$W := \{\Delta : \Delta \text{ is a DP-theory}\}.$$

We set

$$\Delta \leq \Delta' \quad\iff\quad \Delta \subseteq \Delta',$$

and define the valuation V canonically by $\Delta \in V(p) \iff p \in \Delta$. It remains to define R. As before, we work with the finite set

$$\Phi_E = \mathrm{Sub}(E) \cup \{\Box(B_1 \wedge \cdots \wedge B_k) : \Box B_i \in \mathrm{Sub}(E) \text{ for all } i\}.$$

By Proposition 16,

$$\Box B_1 \wedge \cdots \wedge \Box B_k \leftrightarrow \Box(B_1 \wedge \cdots \wedge B_k),$$

so whenever several formulas $\Box B_1, \ldots, \Box B_k \in \Phi_E \cap \Delta$ occur together, we may replace them by the single formula $\Box B := \Box(B_1 \wedge \cdots \wedge B_k) \in \Phi_E \cap \Delta$. We use this throughout the construction below and speak simply of *the* boxed formula $\Box B \in \Delta$. We also fix an enumeration of the $\bigcirc$-formulas in Φ_E.

We first define a relation R^-, and then use it to obtain R. The root of the construction is the DP-theory Γ obtained by noting $\emptyset \nvdash E$, so Lemma 6 gives a DP-theory Γ such that $\Gamma \nvdash E$. For a DP-theory Δ, an R^--successor Δ' is produced by one of the following procedures, applied to a formula of Φ_E falling under one of the cases below. Again, the construction sometimes refers to *pending formulas / (pending) obligations*. This denotes a formula for which some action still has to be performed. Once the relevant action has been taken, such an obligation is said to be *discharged*. If none of the procedures applies to Δ, we let Δ be its own R^--successor.

Procedure (1) (treats $\bigcirc C \in \Phi_E \setminus \Delta$). Let $\bigcirc D \in \Phi_E$ be the first formula, on the fixed enumeration, with $\bigcirc D \in \Delta$, $D \notin \Delta$ (omitted below if no such D exists), and let $\Box B \in \Phi_E$ be the combined boxed formula with $\Box B \in \Delta$, $B \notin \Delta$ (again omitted if absent). We construct Δ' with $D, B \in \Delta'$ and $\bigcirc C \notin \Delta'$.

Claim: $\Delta, D, B \nvdash \bigcirc C$. Suppose otherwise. By the deduction theorem,

$$\Delta \vdash D \wedge B \to \bigcirc C, \qquad \text{so} \qquad \Delta \vdash B \to (D \to \bigcirc C).$$

By T1, $\Delta \vdash \Box(B \to (D \to \bigcirc C))$, and by T2, $\Delta \vdash \Box B \to \Box(D \to \bigcirc C)$. Since $\Box B \in \Delta$, we have $\Delta \vdash \Box(D \to \bigcirc C)$. By Proposition 17(1), $\Delta \vdash \bigcirc D \to \bigcirc \bigcirc C$, and by Proposition 17(2), $\Delta \vdash \bigcirc D \to \bigcirc C$. Since $\bigcirc D \in \Delta$, $\Delta \vdash \bigcirc C$, so $\bigcirc C \in \Delta$ by deductive closure, which contradicts $\bigcirc C \notin \Delta$.

By Lemma 6, we obtain Δ' with $\Delta \cup \{D, B\} \subseteq \Delta'$ and $\bigcirc C \notin \Delta'$. Repeating this for every $\bigcirc C \in \Phi_E \setminus \Delta$ against the same D and B gives, for each such C, a separate R^--successor of Δ that does not contain $\bigcirc C$ while it contains D and B. Since D is a member of any R^--successor built in this way, the obligation for witnessing D is discharged in every one of these successors.

Procedure (2) (treats $\Box A \in \Phi_E \setminus \Delta$). Let $\Box B \in \Delta$, $B \notin \Delta$, and let $\bigcirc D \in \Delta \cap \Phi_E$, $D \notin \Delta$, be the first such formula on the enumeration (again, either omitted if absent). We seek Δ' with $B, D \in \Delta'$ and $A \notin \Delta'$. If this extension is consistent, Lemma 6 immediately gives the required successor. However, if adding both B and D forces A, we need to construct a different successor, which is the purpose of Case B.

Case A: $\Delta \nvdash B \to (D \to A)$. Then $\Delta, B, D \nvdash A$, and Lemma 6 directly gives the required Δ'.

Case B: $\Delta \vdash B \to (D \to A)$. By T6,

$$\vdash (D \to A) \to \big((\Box A \to A) \to (\bigcirc D \to A)\big),$$

so, prefixing $B \to (-)$ throughout and rearranging,

$$\Delta \vdash \big(B \to (\Box A \to A)\big) \to \big(B \to (\bigcirc D \to A)\big).$$

Claim: $\Delta \nvdash B \to (\bigcirc D \to A)$. Suppose otherwise. By T1 and T2,

$$\Delta \vdash \Box(B \to (\bigcirc D \to A)), \text{ so } \Delta \vdash \Box B \to \Box(\bigcirc D \to A), \text{ and thus } \Delta \vdash \Box B \to (\Box \bigcirc D \to \Box A).$$

By T1, instantiated at $\bigcirc D$, we have $\bigcirc D \to \Box \bigcirc D$, so

$$\Delta \vdash \Box B \to (\bigcirc D \to \Box A).$$

Since $\Box B, \bigcirc D \in \Delta$, this gives $\Delta \vdash \Box A$, so $\Box A \in \Delta$ by deductive closure, which contradicts the standing hypothesis $\Box A \notin \Delta$. This proves the claim. By the claim together with the implication derived above, $\Delta \nvdash B \to (\Box A \to A)$. By Lemma 6, we obtain Δ' with $\Delta \subseteq \Delta'$, $B, \Box A \in \Delta'$, $A \notin \Delta'$, and, since $\bigcirc D \in \Delta \subseteq \Delta'$, also $\bigcirc D \in \Delta'$. This Δ' is the successor for Case B.

Procedure (3) (treats $\bigcirc D \in \Delta \cap \Phi_E$ with $D \notin \Delta$, or $\Box B \in \Delta \cap \Phi_E$ with $B \notin \Delta$, when neither Procedure (1) nor Procedure (2) applies).

Suppose first $\bigcirc D \in \Delta$, $D \notin \Delta$. We show D can be consistently added to Δ. Suppose not, i.e. $\Delta \vdash \neg D$. This is impossible: by Proposition 17(3), $\vdash \bigcirc D \to \neg\neg D$, and since $\bigcirc D \in \Delta$, $\Delta \vdash \neg\neg D$; together with $\Delta \vdash \neg D$ this gives $\Delta \vdash \bot$, contradicting consistency of Δ. Hence $\Delta \cup \{D\}$ is consistent.

The argument for $\Box B \in \Delta$, $B \notin \Delta$, is analogous, using Proposition 17(4) ($\Box B \to \bigcirc B$) together with Proposition 17(3) ($\bigcirc B \to \neg\neg B$) to infer $\Box B \to \neg\neg B$. Hence $\Delta \cup \{B\}$ is consistent as well.

If both $\bigcirc D, \Box B \in \Delta$, we show $\Delta \cup \{D, B\}$ is consistent. Suppose not; then $\Delta \vdash D \to (B \to \bot)$, i.e. $\Delta \vdash D \to \neg B$. Since $\Box B \in \Delta$, the argument above gives $\Delta \vdash \neg\neg B$, and hence $\Delta \vdash \neg D$. But $\bigcirc D \in \Delta$ and $\vdash \bigcirc D \to \neg\neg D$, so also $\Delta \vdash \neg\neg D$, which gives a contradiction. Therefore $\Delta \cup \{D, B\}$ is consistent, and Lemma 6 gives the required Δ'.

This completes the construction of R^- from the root upwards. We define R^- on the worlds generated in this construction (which are elements of W) by setting $\Delta R^-\Delta'$ if and only if either Δ' is obtained from Δ by one of Procedures (1)-(3), or none of the procedures is applicable to Δ and $\Delta' = \Delta$. By construction, R^- is serial and satisfies $R^- \subseteq \leq$, that is, $\Delta R^-\Delta'$ implies $\Delta \subseteq \Delta'$. We define R in the following way:

$$\Delta R \Delta'' \iff \exists \Delta'(\Delta \subseteq \Delta' \text{ and } \Delta' R^- \Delta'').$$

It follows directly from the corresponding properties of R^- that R is serial and satisfies $R \subseteq \leq$. Importantly, R also satisfies triangularity:

$$\text{if } \Delta \subseteq \Delta' \text{ and } \Delta' R \Delta'', \text{ then } \Delta R \Delta''.$$

To see this, assume $\Delta \subseteq \Delta'$ and $\Delta' R \Delta''$. Then by definition of R, there exists Γ such that $\Delta' \subseteq \Gamma$ and $\Gamma R^- \Delta''$. By transitivity of $\subseteq$, we have $\Delta \subseteq \Gamma$. Therefore, by definition of R, we may conclude $\Delta R \Delta''$. This shows triangularity. We now check that R has the properties required for the Truth Lemma.

Lemma 18. *Let $\Box B \in \Phi_E$. If $\Box B \in \Delta$ and $\Delta R \Delta'$, then $B \in \Delta'$.*

Proof. Take Δ with $\Box B \in \Delta \cap \Phi_E$. First, we show that this property holds for R^-, and then we show that it holds for R.

Fix an arbitrary Δ' with $\Delta R^- \Delta'$. We show $B \in \Delta'$. If $B \in \Delta$ already, then $B \in \Delta'$ since $R^- \subseteq \leq$. Otherwise $B \notin \Delta$, so $\Box B \in \Delta$ with $B \notin \Delta$ is a pending obligation at Δ: together with any other boxed formulas $\Box B_i \in \Delta \cap \Phi_E$ with $B_i \notin \Delta$, it is combined via Proposition 16 into the single formula treated by whichever procedure constructs Δ'. Each of Procedures (1)–(3) places this combined formula, and hence B in particular, into the successor theory it builds. Repetition (the default successor) occurs only when no procedure applies, i.e. when no boxed obligation is pending, which is excluded here since $B \notin \Delta$ is exactly such an obligation. Hence $B \in \Delta'$ in every case that can occur.

Now suppose we have some Γ such that $\Delta \subseteq \Gamma$ and $\Gamma R^- \Delta'$. If $\Box B \in \Delta$, then $\Box B \in \Gamma$ since $\Delta \subseteq \Gamma$, so since the property holds for R^- we may conclude $B \in \Delta'$. $\Box$

Lemma 19. *If $\bigcirc D \in \Delta$, then every infinite R-path from Δ contains a node Δ' with $D \in \Delta'$.*

Proof. Let

$$\Delta_0 R \Delta_1 R \Delta_2 R \cdots$$

be an arbitrary infinite R-path with $\Delta_0 = \Delta$. By definition of R, we can rewrite this into

$$\Delta_0 \subseteq \Delta_0' R^- \Delta_1 \subseteq \Delta_1' R^- \Delta_2 \subseteq \Delta_2' R^- \cdots$$

Since $\bigcirc D \in \Delta$ and every step along this path is either a $\subseteq$-step or an R^--step, $\bigcirc D$ persists through all intermediate nodes on the path. Consequently, every occurrence of $\bigcirc D$ at a node Δ_0 from which an infinite R-path starts is carried through infinitely many R^--transitions. We show that this eventually forces D to be added.

If $D \in \Delta_0$, there is nothing to prove. Otherwise, $\bigcirc D$ is pending at Δ_0. We distinguish three cases.

(1) Infinitely many transitions of Procedure (1) occur along the path. Recall that at each Procedure (1) transition the formula at the front of the pending list of $\bigcirc$-formulas is discharged. The pending list contains only formulas $\bigcirc C$ with $\bigcirc C \in \Phi_E$. Since Φ_E is finite, only finitely many distinct

formulas can ever occur on the list. At Δ_0, $\bigcirc D$ is pending and hence occupies a finite position in the list. Although later transitions may add new pending formulas before $\bigcirc D$, these formulas are also drawn from the finite set of $\bigcirc$-formulas in Φ_E. Moreover, once a formula is added by a Procedure (1) transition, it is never added to the pending list again. Since infinitely many Procedure (1) transitions occur, the front of the list eventually reaches $\bigcirc D$, which is then discharged. By the definition of a Procedure (1) transition, the successor theory, say Δ_k for some k, contains D.

(2) Infinitely many transitions of Procedure (2) occur. Each Procedure (2) transition of Case B uses some $\Box A \in \Phi_E$ and places $\Box A$ into the successor theory. By persistence, $\Box A$ remains in every later theory, and therefore the same $\Box A$ can never again satisfy $\Box A \notin \Delta'_i$ further along the path. Hence, each $\Box A \in \Phi_E$ can serve as the principal formula in Procedure (2), Case B, at most once along the entire path. Therefore, after finitely many steps, the transitions are governed by Case A. The situation is then exactly as in Case (1): each time the first pending $\bigcirc$-formula is added, so eventually our D is added to some Δ_k.

(3) Neither Case (1) nor Case (2) applies. Hence, from some point onwards, say at Δ'_{k-1}, only Procedure (3) is applied. Every pending obligation from this point onwards, in particular $\bigcirc D \in \Delta'_{k-1}$ with $D \notin \Delta'_{k-1}$, is therefore a Procedure (3) obligation and is added to the next node by Procedure (3). Consequently, $D \in \Delta_k$.

In any of the cases, D is eventually added to some theory on the original path, which shows the statement. □

Now we can prove the Truth Lemma.

Lemma 20 (Truth Lemma). *For every node Δ of M and every $A \in \Phi_E$:*

$$\Delta \Vdash A \iff A \in \Delta.$$

Proof. By induction on the complexity of A.

$\mathbf{A = p}$, $\mathbf{A = \bot}$, $\mathbf{A = B \wedge C}$, $\mathbf{A = B \vee C}$. These cases are verified exactly as in the proof of the Truth Lemma for **TICS** (Lemma 8): the argument uses only the definition of the valuation, the consistency of Δ, and the disjunction property of a DP-theory, none of which depends on R, so it carries over unchanged.

$\mathbf{A = B \to C}$. Suppose $B \to C \in \Delta$. Let $\Delta' \geq \Delta$ with $\Delta' \Vdash B$. By the induction hypothesis, $B \in \Delta'$. Since $\Delta \subseteq \Delta'$ and Δ is deductively closed, $B \to C \in \Delta'$, so $C \in \Delta'$ by modus ponens, and hence $\Delta' \Vdash C$ by the induction hypothesis. As Δ' was an arbitrary $\leq$-successor, $\Delta \Vdash B \to C$.

Conversely, suppose $B \to C \notin \Delta$, i.e. $\Delta \nvdash B \to C$. By Lemma 6 there is a DP-theory $\Delta' \supseteq \Delta$ with $B \in \Delta'$ and $C \notin \Delta'$; let Δ' be the corresponding node, so $\Delta \leq \Delta'$. By the induction hypothesis, $\Delta' \Vdash B$ and $\Delta' \nVdash C$. Hence $\Delta \nVdash B \to C$.

$\mathbf{A = \Box B}$. Suppose $\Box B \in \Delta$ and let $\Delta R \Delta'$. By Lemma 18, $B \in \Delta'$, so $\Delta' \Vdash B$ by the induction hypothesis. As Δ' was an arbitrary R-successor, $\Delta \Vdash \Box B$.

Conversely, suppose $\Box B \notin \Delta$. By Procedure (2), Δ has an R^--successor Δ' with $B \notin \Delta'$, so $\Delta' \nVdash B$ by the induction hypothesis. Since we have $\Delta \subseteq \Delta R^- \Delta'$ and thus $\Delta R \Delta'$, this Δ' is also an R-successor. Hence $\Delta \nVdash \Box B$.

$\mathbf{A} = \bigcirc\mathbf{B}$. Suppose $\bigcirc B \in \Delta$, and let $\Delta = \Delta_0 R \Delta_1 R \cdots$ be an arbitrary infinite R-path. By Lemma 19, some node Δ_n on this path has $B \in \Delta_n$, so $\Delta_n \Vdash B$ by the induction hypothesis. As the path was arbitrary, $\Delta \Vdash \bigcirc B$.

Conversely, suppose $\bigcirc B \notin \Delta$. By Procedure (1), Δ has an R^--successor Δ' with $\bigcirc B \notin \Delta'$; repeating this at Δ', and so on, produces an infinite R^--path $\Delta = \Delta_0 R^- \Delta_1 R^- \cdots$ with $\bigcirc B \notin \Delta_n$ for every n. Since we can rewrite this path into $\Delta_0 \subseteq \Delta_0 R^- \Delta_1 \subseteq \Delta_1 R^- \cdots$, this is also an infinite R-path $\Delta = \Delta_0 R \Delta_1 R \cdots$. Now, as $B \to \bigcirc B$ is an instance of T4, $B \in \Delta_n$ would give $\bigcirc B \in \Delta_n$ by deductive closure, contradicting $\bigcirc B \notin \Delta_n$; hence $B \notin \Delta_n$ for every n. By the induction hypothesis, $\Delta_n \nVdash B$ for every n, so this path witnesses $\Delta \nVdash \bigcirc B$. $\square$

Now we may conclude the completeness theorem.

Theorem 21 (Completeness)**.** *If* $\nvdash E$ *then* $\not\models E$.

Proof. Since $\nvdash E$, Lemma 6 gives a DP-theory Γ with $\Gamma \nvdash E$; let Γ be the theory of the root Γ of the model M constructed relative to Φ_E. As $E \in \Phi_E$ and $E \notin \Gamma$ (else $\Gamma \vdash E$), Lemma 20 (Truth Lemma) gives $\Gamma \nVdash E$. Hence E is refuted at Γ in M. $\square$

Corollary 22. ***TICS-2*** *is sound and complete with respect to its semantics: for every* ***TICS-2****-formula* A*, we have* $\vdash A$ *if and only if* $\models A$*.*

4.3 Application of the results

We can apply the soundness and completeness with respect to our class of Kripke frames to make some observations about **TICS-2**.

Example 23. *The formula*

$$\Box B \to (A \vee (A \to B))$$

is not valid on ***TICS-2*** *frames. Consider the following* ***TICS-2*** *model:*

$$\begin{tikzcd} w \arrow[r, "\leq"] \arrow[rr, bend left=60, "R"] & v \arrow[r, "\leq"] \arrow[r, bend left, "R"] & z \arrow[loop above, "R"] \end{tikzcd}$$

$$V(w) = \emptyset \qquad V(v) = \{p\} \quad V(z) = \{p, q\}$$

First, we note that R *is serial and* $R \subseteq \leq$*. On top of that, triangularity holds: whenever* $x \leq y$ *and* yRz*, we also have* xRz*. The valuation* V *is clearly monotone with respect to* $\leq$*, so it is a* ***TICS-2*** *model. We now show that the formula fails at* w*. Since the only* R*-successor of* w *is* z *and* $z \Vdash q$*, we have*

$$w \Vdash \Box q.$$

Furthermore, $w \nVdash p$ *because* $p \notin V(w)$*. Also,*

$$v \geq w, \qquad v \Vdash p, \qquad v \nVdash q.$$

Therefore, by the forcing clause for implication,

$$w \nVdash p \to q.$$

Hence,

$$w \nVdash p \vee (p \to q).$$

Consequently,

$$w \nVdash \Box q \to (p \vee (p \to q)),$$

and the formula is not valid on TICS-2 frames.

Using the same semantics, we can extend the language with iterated box operators, which express that a formula holds after a fixed number of temporal steps.

Definition 24. *For every $n \geq 0$, the iterated box operators $\{\Box_n\}_n$ are defined inductively by*

$$\Box_0 A := A, \qquad \Box_{n+1} A := \Box(\Box_n A).$$

An easy induction on n shows that $k \Vdash \Box_n A$ if and only if every world that is reachable from k by an R-path of length at least n forces A. We further extend the language with the operator $\exists n\, \Box_n$, which expresses that a formula holds after some finite number of iterations of the box operator. Its forcing relation is defined as follows.

Definition 25. *The forcing relation for $\exists n\, \Box_n$ is given by*

$$k \Vdash \exists n\, \Box_n A$$

if and only if there exists a natural number n such that

$$k \Vdash \Box_n A.$$

We establish the following persistence property of the iterated box operators.

Proposition 26. *For all $n, m \geq 0$,*

$$\vdash \Box_n A \to \Box_{n+m} A.$$

Proof. We prove the statement by induction on m. For $m = 0$, the result is immediate. For the induction step, suppose

$$\vdash \Box_n A \to \Box_{n+m} A.$$

By T1 and T2, we obtain

$$\vdash \Box(\Box_n A) \to \Box(\Box_{n+m} A).$$

By axiom T1, we have

$$\vdash \Box_n A \to \Box(\Box_n A).$$

Combining these two implications gives

$$\vdash \Box_n A \to \Box(\Box_{n+m} A).$$

By the definition of the iterated box operators,

$$\Box(\Box_{n+m}A) = \Box_{n+m+1}A,$$

and therefore

$$\vdash \Box_n A \to \Box_{n+m+1}A.$$

Hence the result follows by induction. $\square$

As an immediate consequence, we obtain that every formula implies its iterated box versions.

Proposition 27. *For every $n \geq 0$,*

$$\vdash A \to \Box_n A.$$

Consequently,

$$\vdash A \to \exists n\, \Box_n A.$$

Proof. Taking $n = 0$ and $m = n$ in the previous proposition gives

$$\vdash \Box_0 A \to \Box_n A.$$

Since $\Box_0 A = A$, we obtain

$$\vdash A \to \Box_n A.$$

The second claim follows immediately. $\square$

However, the converse implication does not hold.

Example 28. *The formula*

$$\exists n\, \Box_n A \to A$$

is not valid. Consider the following ***TICS-2*** *model:*

$$\begin{tikzcd} w \arrow[r, "R"] & v \arrow[loop above, "R"] \end{tikzcd}$$

$$V(w) = \emptyset \qquad V(v) = \{p\}$$

Here, $\leq$ is taken to be the reflexive closure of R. This is clearly a model for ***TICS-2****. At w we have*

$$w \nVdash p,$$

while

$$w \Vdash \Box p,$$

because the only R-successor of w is v, where p holds. Hence

$$w \Vdash \exists n\, \Box_n p,$$

(namely $n = 1$) but

$$w \nVdash p.$$

Therefore,

$$w \nVdash \exists n\, \Box_n p \to p.$$

Example 29. *The formula*

$$\Box_n A \vee \neg \Box_n A$$

*is not valid for any n. Consider the following **TICS-2** model:*

$$\begin{tikzcd}
V(w)=\emptyset & V(v)=\{p\} \\
w \arrow[r, "R"] \arrow[dr, "R"] & v \arrow[loop right, "R"] \\
& z \arrow[loop right, "R"] \\
& V(z)=\emptyset
\end{tikzcd}$$

*Here, $\leq$ is taken to be the reflexive closure of R. The frame satisfies all **TICS-2** conditions. In particular, triangularity holds: from $w \leq v$ and vRv we obtain wRv, which is included in the frame. The relation R is serial and clearly $R \subseteq \leq$ by definition. Moreover, V maps propositional variables to up-sets, so this is a model. We now evaluate the formula at the root w. Since wRz and $z \nVdash p$, we have*

$$w \nVdash \Box p.$$

However, we have wRv implies $w \leq v$ and

$$v \Vdash \Box p,$$

because the only R-successor of v is itself and

$$v \Vdash p.$$

Therefore,

$$w \nVdash \neg \Box p,$$

since there exists an extension $v \geq w$ where $\Box p$ holds. Hence,

$$w \nVdash \Box p \qquad \textit{and} \qquad w \nVdash \neg \Box p.$$

Consequently,

$$w \nVdash \Box_1 p \vee \neg \Box_1 p.$$

*This type of counterexample generalizes immediately to show the failure of decidability for $\Box_n A$ for every $n \geq 0$. Thus decidability does not hold for any iterated box operator in **TICS-2**.*

Based on the preceding propositions and counterexamples, we conclude that CS1 and CS4 are not valid in our system, whereas CS2 and CS3 are valid. This coincides exactly with Niekus' proposal.

The same conclusions apply to **TICS**. The proofs of CS2 and CS3 rely only on axioms T1 and T2, which are also valid in **TICS**. Likewise, the counterexamples to CS1 and CS4 can be transformed into

TICS-models by replacing each reflexive endpoint with an infinite R-branch consisting of repetitions of the same node. Hence CS1 and CS4 are not valid in **TICS** either, while CS2 and CS3 are.

The eventuality operator $\bigcirc$ cannot be expressed in terms of the iterated box operators. The reason is that $\bigcirc$ expresses eventual truth along every temporal path, whereas $\exists n\,\Box_n A$ requires the existence of a uniform finite bound after which A holds on all paths. This distinction is illustrated by the following example.

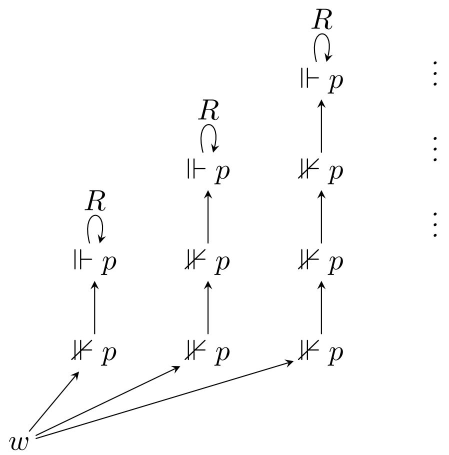
Example 30. *There exists a* ***TICS-2*** *model in which*

$$w \Vdash \bigcirc p$$

but

$$w \nVdash \exists n\,\Box_n p.$$

Consider the following model, where the arrows indicate the generating relation, whose transitive closure is the temporal accessibility relation R, and where $\leq$ is the reflexive closure of R.



Every infinite R-path eventually reaches a p-world, and therefore

$$w \Vdash \bigcirc p.$$

However, for every n, there is an R-path of length n ending in a world where p does not hold. Hence

$$w \nVdash \Box_n p$$

for every n, and therefore

$$w \nVdash \exists n\,\Box_n p.$$

Thus $\bigcirc$ is independent from the iterated box operators.

The countermodel above relies essentially on the fact that w has infinitely many successors: such a phenomenon cannot occur in finitely branching tree models. In finitely branching trees, if A holds eventually along every path, then there is a uniform finite bound after which A holds on all paths. This follows from an application of the Fan Theorem. Consequently, in such models, $\bigcirc A$ and $\exists n\,\Box_n A$ are equivalent.

Note again that there are no non-reflexive $\leq$-steps used here which are different from R-steps.

Therefore, by unravelling the reflexive endpoint into an infinite branch, we obtain a **TICS**- countermodel as well. This illustrates that the operator $\bigcirc$ allows both systems to express temporal properties that cannot be captured using the operator $\Box$ alone.

The only argument above that was specific to **TICS-2** was the first one. This is to be expected, since it establishes the failure of axiom 7', which is valid in **TICS**. The remaining arguments hold for both systems and demonstrate their relation to **TCS**, as well as the added expressive power of $\bigcirc$ over $\Box$ in both systems. Thus, both **TICS** and **TICS-2** serve as more expressive alternatives to **TCS** in accordance with Niekus' proposal: in both systems, CS2 and CS3 are valid, whereas CS1 and CS4 are not. The only difference between the two systems is the inclusion of axiom 7'.

4.4 Towards a constructive completeness proof

In this chapter, we established the soundness and completeness of **TICS-2** with respect to its Kripke semantics. It is worth noting, however, that these metatheoretic results involved rely on classical principles. For example, the countermodel construction requires extending an arbitrary consistent theory to a DP-theory via the Lindenbaum lemma, a step that is not available constructively, because it relies on assuming the law of the excluded middle.

In this section, we explore whether this non-constructive step can be avoided. We look at *Beth semantics*, which replaces the demand that every node already decides a disjunction with the weaker demand that disjunctions are settled *eventually*, along every branch. This removes the need for Lindenbaum's lemma, but, as we will see, designing the semantics appropriately gets complicated once the temporal structure of **TICS-2** is taken into account. Working through these difficulties will motivate the approach we adopt in the next chapter, where we prove constructive completeness for the $\Box$-fragment of **TICS-2** using *modal cover semantics* instead.

4.4.1 Beth semantics

Beth semantics is an alternative relational semantics for intuitionistic logic, originally introduced by Beth as a constructive account of intuitionistic validity [Bet56]. For a detailed presentation of Beth semantics, we refer to [TD14]. Here, we only outline the basic setup.

Originally, Beth semantics was formulated in terms of Beth trees. However, it can also be presented more generally using a preorder equipped with a cover relation, a framework known as *cover semantics* or *Beth-Kripke-Joyal semantics*. The original tree semantics is a special case of this more general framework. In the following, we use the tree presentation to explain the underlying intuition, while our formal development will be based on the more general framework.

Whereas Kripke semantics, based on Kripke frames, interprets formulas at individual states of information related by an ordering that represents the growth of information, Beth semantics is based on *Beth trees*, whose nodes represent states of partial information. The branching structure of a Beth tree captures the possible developments of information, and the truth of formulas is evaluated relative to this structure of possible developments.

Forcing in a Beth tree is designed to satisfy two conditions. First, it should be persistent with respect to the underlying preorder: if a formula is forced at a node, then it remains forced at every refinement of that node. This is the same requirement as in Kripke semantics. Second, it should satisfy a *covering* condition: if, along every possible continuation, there is some later point at which a formula is forced, then the formula is already forced at the present node.

Consequently, truth at a node is determined not only by the information available at that node, but also by the possible ways in which this information can be developed along the branches of the tree, to a greater extent than in Kripke semantics. For example, in Beth semantics, a disjunction $A \vee B$ may hold at a node even though neither disjunct holds there individually, provided that every possible continuation eventually establishes one of them. Although Beth semantics uses a different structure from Kripke semantics at the level of frames and valuations, the two semantics agree at the level of validity: both are sound and complete with respect to intuitionistic logic. Moreover, every Kripke model can be transformed into an equivalent Beth model [Kri65].

We now turn to the question of what a Beth-style semantics for **TICS-2** should look like. In this setting, the semantic structure must incorporate both a preorder $\leq$ and a cover structure underlying Beth semantics, together with a temporal relation R interpreting the modal operators. The first important observation is that R cannot simply be identified with a subrelation of the preorder $\leq$, as it was in the Kripke semantics developed above.

In Beth semantics, as mentioned, the preorder $\leq$ is equipped with a covering condition: if every possible $\leq$-continuation of a node contains a later point at which a formula A is forced, then A is already forced at the current node. This condition is appropriate for the intuitionistic refinement structure, but it should not be imposed on temporal progression. To see why, suppose that we assumed $R \subseteq \leq$. Then every infinite R-path would also be a $\leq$-continuation. Now consider $\bigcirc A$: by its intended meaning, $\bigcirc A$ holds at a node if along every infinite R-path there is some later point at which A holds. Under the assumption $R \subseteq \leq$, this would satisfy the antecedent of the covering condition for $\leq$, forcing A to already hold at the current node. Hence we would validate

$$\bigcirc A \to A,$$

which is not a principle intended to hold in **TICS-2**. The same problem arises for $\Box A$.

We should therefore work with two genuinely separate relations: a preorder $\leq$, representing the growth of mathematical information, and R, representing temporal progression. We incorporate the covering property through an explicit cover relation $\triangleleft$ associated with $\leq$. Intuitively, $k \triangleleft U$ means that U is *reachable* from k: every possible continuation of the information state k eventually reaches a node in U.

As in Kripke semantics, forcing should be persistent with respect to both $\leq$ and R, in accordance with the intended interpretation of these relations. However, since R is no longer assumed to be contained in $\leq$, this persistence is no longer obtained automatically; instead, it must be guaranteed by suitable interaction conditions between R, $\leq$ and $\triangleleft$. In addition, the cover relation $\triangleleft$ must satisfy a localization condition: if $s \triangleleft U$ for some node s and upward closed set of nodes U, and a formula A is forced at every node in U, then A is already forced at s. This reformulates the covering condition in terms of the explicit cover relation. The main challenge is then to identify conditions on these relations that ensure the required persistence and localization properties, and to construct a canonical model satisfying them.

One existing approach to Beth semantics for intuitionistic modal logic is given by [Nik22]. In that framework, however, forcing is required to be persistent only with respect to the intuitionistic preorder $\leq$. Since our setting also requires persistence along the temporal relation R, we must adopt a different approach. It is worth noting as well that, although [Nik22] formulate their semantics in terms of Beth models, their completeness proof still relies on classical means. In particular, they first construct a classical Kripke countermodel and then transform it into a Beth model. This strategy is not available in our setting, where the aim is to obtain a constructive completeness proof, and such a proof cannot be obtained by first passing through a classical Kripke countermodel.

Satisfying all of the conditions on R, $\leq$, and $\lhd$ simultaneously turns out to be difficult to achieve in the canonical model. This issue is not specific to our setting: Goldblatt encounters the same problem in the completeness proof for an intuitionistic modal logic with respect to Beth-style cover semantics equipped with an accessibility relation R [Gol11]. Valliappan [Val26] identifies the underlying source of this difficulty: a relation R and a cover relation $\lhd$ are, in general, difficult to integrate smoothly within a canonical model construction. Rather than resolving this tension directly within relational Beth semantics for intuitionistic modal logic, which requires simultaneously handling two relations, $\leq$ and R, and a cover relation $\lhd$, we instead adopt the approach proposed by [Val26] in the next chapter. Specifically, we move to *modal cover semantics*, which replaces the accessibility relation R with a modal covering relation. The interaction between the two covering relations is more natural than that between a cover relation and an accessibility relation, since both are based on the same underlying structure: relations from worlds to sets of worlds. This makes the construction of a canonical model more manageable.

In the next chapter, we introduce modal cover semantics and use it to establish constructive (soundness and) completeness for the $\Box$-fragment of **TICS-2**. The restriction to the $\Box$-fragment is motivated by the fact that it turns out that Valliappan's strategy of constructivizing the canonical model construction is available in this setting. This enables us to establish a constructive completeness theorem for $\mathbf{TICS\text{-}2}_{\Box}$ with respect to Valliappan's modal cover semantics [Val26]. In the last chapter of this thesis, we discuss possible extensions of this approach to the full logic, including the $\bigcirc$-operator.

Chapter 5

TICS-2$_\Box$

In this chapter, we consider the $\Box$-fragment of **TICS-2**. This fragment also provides a natural setting for developing an alternative to the Theory of the Creating Subject, since the original theory uses only the operators $\Box_n$ and $\exists n \Box_n$ in its language. Restricting attention to the $\Box$-fragment is therefore already useful for discussing how these arguments can be formalized, while avoiding the additional complexity introduced by $\bigcirc$.

The restriction to the $\Box$-fragment is motivated by the fact that Valliappan's strategy turns out to be applicable in this setting. More precisely, by constructivizing the (Scott-style) canonical model construction, we obtain a constructive completeness theorem for **TICS-2**$_\Box$ with respect to Valliappan's modal cover semantics [Val26].

It turns out that the $\Box$-fragment coincides with Intuitionistic Epistemic Logic (**IEL**), developed by Artemov and Protopopescu [AP16]. We discuss this correspondence and review some of the existing literature on **IEL**.

Constructive completeness for **IEL** has already been established by Rogozin [Rog21] with respect to Goldblatt's modal cover systems using order-theoretic methods, and syntactically by Hagemeier and Kirst [HK21]. By contrast, Valliappan's framework admits a constructive version of the canonical model construction. We apply this construction to the $\Box$-fragment of **TICS-2**, thereby also obtaining a new constructive completeness theorem for **IEL** with respect to Valliappan's modal cover semantics.

5.1 Syntax

The $\Box$-fragment of **TICS-2**, denoted by **TICS-2**$_\Box$, is obtained by removing the formulas and axioms involving $\bigcirc$. It is therefore axiomatized by adding the following modal axioms to **IPC**:

T1 $A \to \Box A$

T2 $\Box(A \to B) \to (\Box A \to \Box B)$

T3 $\neg\Box\bot$

In this chapter, we write $\Gamma \vdash A$ for $\Gamma \vdash_{\mathbf{TICS\text{-}2}_\Box} A$.

5.2 Modal cover semantics

In the previous chapter, we saw that a Beth-style semantics for **TICS-2** requires several ingredients: a preorder $\leq$ and a temporal relation R, together with a covering relation $\triangleleft$ with respect to $\leq$ that must interact correctly with both. This combination turns out difficult to realize in a single canonical model construction. *Modal cover semantics*, recently introduced by Valliappan [Val26] as a generalisation of Goldblatt's modal cover systems [Gol11], offers a way around this difficulty.

Goldblatt [Gol11] introduced cover semantics as an alternative semantics for intuitionistic modal logics, generalising Beth trees for intuitionistic logic. He starts with a cover system, which consists of a preorder together with a covering relation. This corresponds exactly to the more general presentation of Beth semantics discussed in the last chapter. Modalities are then interpreted using additional accessibility relations that satisfy suitable compatibility conditions with the preorder and the cover relation. The general setup is thus the same as the one we proposed in the previous section for **TICS-2**.

As explained by Valliappan [Val26], a distinctive feature of Goldblatt's approach is that modalities are not interpreted by directly adopting the usual Kripke semantics for modal operators. Instead, both $\Box$- and $\Diamond$-like modalities are interpreted through existential accessibility clauses:

$$M, w \Vdash \Box A \iff \exists v \, (wR_{\Box}v \text{ and } M, v \Vdash A),$$

$$M, w \Vdash \Diamond A \iff \exists v \, (wR_{\Diamond}v \text{ and } M, v \Vdash A).$$

The difference in their logical behaviour is thus not determined by the form of the forcing clauses, but by additional conditions imposed on the accessibility relations $R_{\Box}$ and $R_{\Diamond}$.

Relational cover semantics of this kind, however, introduces exactly the difficulty we ran into with Beth semantics for **TICS-2**. To ensure that the forcing relation for modal formulas satisfies the covering condition, one needs a *modal localization* condition connecting the modal accessibility relation with the covering relation, and preserving this interaction throughout a canonical model construction turns out to be difficult. Goldblatt therefore proves completeness using order-theoretic methods rather than by constructing a canonical model.

Valliappan [Val26] proposes an alternative that avoids this obstacle. Instead of interpreting a modality through an accessibility relation between individual worlds, a *modal covering relation* assigns to each world a collection of possible successor sets, mirroring the way the covering relation $\triangleleft$ associates each world with a collection of covering sets. In this way, the intuitionistic cover relation and the modal relation have the same structure: both relate worlds to sets of worlds. This generalises Goldblatt's framework while allowing the required modal localization property to be built directly into the model. Consequently, a canonical model construction à la Scott becomes available, in contrast to Goldblatt's order-theoretic approach. To clarify what is meant by such a canonical model construction, consider the following.

The completeness proof for **TICS-2** with respect to Kripke semantics presented was not based on the standard canonical model construction from modal logic, where the modal accessibility relation is defined globally from the modal formulas contained in each theory, but rather on a step-by-step tree construction. This construction was essential for ensuring the correct interaction between the $\Box$- and $\bigcirc$-formulas. If we had considered only the $\Box$-fragment, we could instead have defined

$$\Gamma R \Gamma' \iff \Box A \in \Gamma \to A \in \Gamma'.$$

In that case, the requirement of restricting attention to a finite set Φ_E would also disappear.[1] The relevant properties of this canonical relation arise from the axioms of the logic for which the completeness proof is being established. Completeness proofs based on canonical model constructions with relations of this kind belong to a tradition originating with Scott and others (see [LS77]). We will refer to the resulting models and constructions as *Scott-style* or *à la Scott*. In the Kripke semantics setting, the Lindenbaum lemma is required in these canonical model constructions, since the worlds are DP-theories. Valliappan's strategy for constructive completeness can be understood as developing a constructive version of the Scott-style canonical model construction, thereby avoiding the need for DP-theories.

For a diamond modality, the canonical accessibility relation can be defined by

$$\Gamma R \Gamma' \iff A \in \Gamma' \to \Diamond A \in \Gamma.$$

Since the modalities in Goldblatt-style semantics are of this semantic shape, this type of construction is the type of construction that is relevant to our setting. We found that such a Scott-style relation construction works for the $\Box$-fragment, but we did not get it to work for the fragment including $\bigcirc$, as will be discussed in the concluding chapter of this thesis. We therefore restrict our attention to the $\Box$-fragment of **TICS-2**.

A subtlety in the canonical model construction used in the constructive completeness proof below is that we cannot assume decidability of whether $\bot$ is provable from a given finite set of formulas. Consequently, the construction proceeds without deciding the consistency of the finite sets that occur, and some nodes of the canonical model may therefore be labelled by inconsistent sets of formulas. As a result, the canonical model, and hence the semantics, allows worlds at which $\bot$ is forced, since ruling out such nodes would itself require deciding the consistency of every finite set of formulas in the logic.

In this section, we thus apply Valliappan's approach to the $\Box$-fragment of our logic. We introduce modal cover systems and use this to establish constructive completeness for **TICS-2**$_\Box$ through a canonical model construction.

5.2.1 Modal cover systems

In this subsection, we define the modal cover semantics for **TICS-2**$_\Box$, following the framework developed in [Val26]. We begin with the notion of a cover system, which provides an alternative presentation of Beth semantics for intuitionistic logic.

Definition 31 (Cover system)**.** *A **cover system** $\mathcal{C} = (W, \leq, \lhd)$ consists of a set W of worlds, a reflexive-transitive **refinement** relation $\leq$ on W, and a **covering** relation $\lhd \subseteq W \times \mathcal{P}(W)$. We write $w \lhd \alpha$ (or $\alpha \rhd w$) and say α **covers** w. We define a refinement relation $\preceq$ on subsets of W by $\alpha \preceq \alpha'$ iff for every $v' \in \alpha'$ there exists $v \in \alpha$ with $v \leq v'$. The covering relation is subject to the following conditions:*

- ***Refinement.*** *If $w \leq w'$ and $w \lhd \alpha$, then there exists α' such that $w' \lhd \alpha'$ and $\alpha' \succeq \alpha$.*
- ***Inclusion.*** *If $w \lhd \alpha$, then $\{w\} \preceq \alpha$.*
- ***Identity.*** *$w \lhd \{w\}$.*

[1] A completeness proof for the $\Box$-fragment of our logic based on this canonical accessibility relation is given in the next chapter. There, it can be seen that no finite set Φ_E is required.

- ***Transitivity.*** *If $w \lhd \alpha$ and for every $v \in \alpha$ there exists α_v with $v \lhd \alpha_v$, then $w \lhd \bigcup_{v \in \alpha} \alpha_v$.*

We define the operator $\langle \lhd \rangle : \mathcal{P}(W) \to \mathcal{P}(W)$ by

$$\langle \lhd \rangle X \;=\; \{w \in W \mid \exists\, \alpha.\, w \lhd \alpha \subseteq X\}.$$

A subset $X \subseteq W$ is an *up-set* if $w \leq w'$ and $w \in X$ imply $w' \in X$, and is *localized* if $\langle \lhd \rangle X \subseteq X$. A *localized up-set* is a subset that is both an up-set and localized.

Definition 32 (Cover model)**.** *A **cover model** $\mathcal{M} = (\mathcal{C}, V)$ for Intuitionistic Propositional Calculus (**IPC**) consists of a cover system*

$$\mathcal{C} = (W, \leq, \lhd)$$

together with a valuation function

$$V : \mathrm{Atom} \to \mathcal{P}(W)$$

assigning to each propositional atom a localized up-set of W.

Given a cover model $\mathcal{M}$, the forcing relation $\Vdash$ is defined inductively by:

$$\begin{aligned}
\mathcal{M}, w \Vdash p &\iff w \in V(p),\\
\mathcal{M}, w \Vdash \bot &\iff w \lhd \emptyset,\\
\mathcal{M}, w \Vdash A \wedge B &\iff \mathcal{M}, w \Vdash A \text{ and } \mathcal{M}, w \Vdash B,\\
\mathcal{M}, w \Vdash A \vee B &\iff \exists \alpha \left(w \lhd \alpha \wedge \forall v \in \alpha\, (\mathcal{M}, v \Vdash A \text{ or } \mathcal{M}, v \Vdash B)\right),\\
\mathcal{M}, w \Vdash A \to B &\iff \forall w' \geq w\, (\mathcal{M}, w' \Vdash A \Rightarrow \mathcal{M}, w' \Vdash B).
\end{aligned}$$

We extend the satisfaction relation to contexts and write $\mathcal{M}, w \Vdash \Gamma$ to denote that $\mathcal{M}, w \Vdash A_i$ for all formulas $A_i \in \Gamma$. We define the truth set $|A|_{\mathcal{M}}$ of a formula A in a model $\mathcal{M}$ as the set of worlds where A holds, and extend this to contexts by

$$|A|_{\mathcal{M}} = \{\, w \in W \mid \mathcal{M}, w \Vdash A \,\}, \qquad |A_1. \ldots . A_n|_{\mathcal{M}} = |A_1|_{\mathcal{M}} \cap \cdots \cap |A_n|_{\mathcal{M}}.$$

The paper shows the following:

Lemma 33. *For any formula A and cover model $\mathcal{M}$ of **IPC**, the truth set $|A|_{\mathcal{M}}$ is a localized up-set. Consequently, $|\Gamma|_{\mathcal{M}}$ is a localized up-set for any context Γ.*

We consider an extension of cover systems, namely *modal* cover systems. The definition is taken from [Val26], with additional conditions tailored to our setting. The conditions Modal Identity, Modal Confluence, and Modal Emptiness ensure precisely that the axioms T1, T2, and T3 are valid in our modal cover systems.

Definition 34 (**TICS-2$_\Box$**-Modal cover system)**.** *A **TICS-2$_\Box$-modal cover system** $(\mathcal{C}, \blacktriangleleft)$ extends a cover system $\mathcal{C} = (W, \leq, \lhd)$ with a modal covering relation $\blacktriangleleft \subseteq W \times \mathcal{P}(W)$. We write $w \blacktriangleleft \beta$ and say that β **modally covers** w. We define the operator $\langle \blacktriangleleft \rangle : \mathcal{P}(W) \to \mathcal{P}(W)$ by*

$$\langle \blacktriangleleft \rangle X = \{\, w \in W \mid \exists \beta.\, w \blacktriangleleft \beta \subseteq X \,\}.$$

The relation $\blacktriangleleft$ is required to satisfy the following conditions:

- ***Modal Refinement.*** *If $w \le w'$ and $w \blacktriangleleft \beta$, then there exists β' such that $w' \blacktriangleleft \beta'$ and $\beta' \succeq \beta$.*
- ***Modal Localization.*** *If $w \lhd \alpha \subseteq \langle \blacktriangleleft \rangle X$, then there exists β such that*

$$w \blacktriangleleft \beta \subseteq \langle \lhd \rangle X.$$

- ***Modal Identity.*** *$w \blacktriangleleft \{w\}$.*
- ***Modal Confluence.*** *If $w \blacktriangleleft \beta_1$ and $w \blacktriangleleft \beta_2$, then there exists γ such that*

$$w \blacktriangleleft \gamma, \qquad \beta_1 \preceq \gamma, \qquad \beta_2 \preceq \gamma.$$

- ***Modal Emptiness.*** *If $w \blacktriangleleft \beta \subseteq \langle \lhd \rangle \emptyset$, then $w \lhd \emptyset$.*

Definition 35 (**TICS-2**$_\Box$-modal cover model)**.** *A* **TICS-2**$_\Box$***-modal cover model*** *is a structure*

$$\mathcal{M} = (\mathcal{C}, \blacktriangleleft, V)$$

where $(\mathcal{C}, \blacktriangleleft)$ is a modal cover system and V assigns to each propositional atom a localized up-set of W.

For each modal cover model, we can consider the truth set of a formula in the model.

Definition 36. *Let $\mathcal{M}$ be a* **TICS-2**$_\Box$*-modal cover model. The* ***truth set*** *$|A|^{\mathcal{M}} \subseteq W$ of a formula A is defined inductively by:*

$$\begin{aligned}
|p|^{\mathcal{M}} &= V(p), \\
|\bot|^{\mathcal{M}} &= \langle \lhd \rangle \emptyset, \\
|A \wedge B|^{\mathcal{M}} &= |A|^{\mathcal{M}} \cap |B|^{\mathcal{M}}, \\
|A \vee B|^{\mathcal{M}} &= \langle \lhd \rangle \big(|A|^{\mathcal{M}} \cup |B|^{\mathcal{M}} \big), \\
|A \to B|^{\mathcal{M}} &= \{\, w \in W \mid \forall w' \ge w.\ w' \in |A|^{\mathcal{M}} \Rightarrow w' \in |B|^{\mathcal{M}} \,\}, \\
|\Box A|^{\mathcal{M}} &= \langle \blacktriangleleft \rangle |A|^{\mathcal{M}}.
\end{aligned}$$

We verify that truth sets of the form $|A|^{\mathcal{M}}$ are localized up-sets.

Lemma 37. *For any formula A and* **TICS-2**$_\Box$*-modal cover model $\mathcal{M}$, the truth set $|A|^{\mathcal{M}}$ is a localized up-set. Consequently, $|\Gamma|^{\mathcal{M}}$ is a localized up-set for any context Γ.*

Proof. By induction on the formula A. The cases for propositional connectives follow from [Val26]. The case $\Box A$ follows from the fact that $\blacktriangleleft$ satisfies Modal Refinement and Modal Localization, exactly the two properties used in Valliappan's proof for $|\heartsuit A|$ being a localized up-set. $\Box$

The satisfaction relation $\mathcal{M}, w \Vdash A$ is defined by $\mathcal{M}, w \Vdash A$ iff $w \in |A|^{\mathcal{M}}$, and expands to the following clauses:

$$
\begin{aligned}
\mathcal{M}, w \Vdash p &\iff w \in V(p)\\
\mathcal{M}, w \Vdash \bot &\iff w \lhd \emptyset\\
\mathcal{M}, w \Vdash A \wedge B &\iff \mathcal{M}, w \Vdash A \text{ and } \mathcal{M}, w \Vdash B\\
\mathcal{M}, w \Vdash A \vee B &\iff \exists\, \alpha.\ w \lhd \alpha \text{ and } \forall v \in \alpha.\ \mathcal{M}, v \Vdash A \text{ or } \mathcal{M}, v \Vdash B\\
\mathcal{M}, w \Vdash A \to B &\iff \forall w' \geq w.\ \mathcal{M}, w' \Vdash A \Rightarrow \mathcal{M}, w' \Vdash B\\
\mathcal{M}, w \Vdash \Box A &\iff \exists\, \beta.\ w \blacktriangleleft \beta \text{ and } \forall v \in \beta.\ \mathcal{M}, v \Vdash A
\end{aligned}
$$

A formula A is *valid* in a **TICS-2**$_\Box$-modal cover model $\mathcal{M}$, written $\mathcal{M} \models A$, if $\mathcal{M}, w \Vdash A$ for all $w \in W$. A formula A is **TICS-2**$_\Box$-valid, written $\models A$, if it is valid in every **TICS-2**$_\Box$-modal cover model.

5.2.2 Soundness

Theorem 38 (Soundness)**.** *If* $\vdash A$*, then* $\models A$*.*

Proof. The soundness of the intuitionistic propositional fragment follows from Valliappan's local cover semantics. It therefore remains to verify the additional modal axioms. Note that all axioms are proved constructively.

T1 $A \to \Box A$**.** For any proposition A, we show that $|A| \subseteq \langle\blacktriangleleft\rangle|A|$. Let $w \in |A|$. By Modal Identity, $w \blacktriangleleft \{w\}$, and since $\{w\} \subseteq |A|$, we have $w \in \langle\blacktriangleleft\rangle|A|$. Hence $X \subseteq \langle\blacktriangleleft\rangle X$, so $A \to \Box A$ is valid.

T2 $\Box(A \to B) \to (\Box A \to \Box B)$**.** We show that $\langle\blacktriangleleft\rangle|A \to B| \cap \langle\blacktriangleleft\rangle|A| \subseteq \langle\blacktriangleleft\rangle|B|$. Suppose that $w \blacktriangleleft \beta \subseteq |A \to B|$ and $w \blacktriangleleft \gamma \subseteq |A|$. By Modal Confluence, there exists a cover δ such that $w \blacktriangleleft \delta$, $\beta \preceq \delta$, and $\gamma \preceq \delta$. Let $v \in \delta$. Since $\beta \preceq \delta$, there exists $b \in \beta$ with $b \leq v$, and since $\gamma \preceq \delta$, there exists $a \in \gamma$ with $a \leq v$. As truth sets are up-sets, $b \in |A \to B|$ implies $v \in |A \to B|$, and $a \in |A|$ implies $v \in |A|$. Hence $v \in |B|$. Therefore $\delta \subseteq |B|$, so $w \in \langle\blacktriangleleft\rangle|B|$. Thus $\Box(A \to B) \to (\Box A \to \Box B)$ is valid.

T3 $\neg\Box\bot$**.** We show that $\langle\blacktriangleleft\rangle\langle\lhd\rangle\emptyset \subseteq \langle\lhd\rangle\emptyset$. Suppose $w \in \langle\blacktriangleleft\rangle\langle\lhd\rangle\emptyset$. Then there exists a modal cover $w \blacktriangleleft \beta$ such that $\beta \subseteq \langle\lhd\rangle\emptyset$. By Modal Emptiness, $w \lhd \emptyset$, that is, $w \in \langle\lhd\rangle\emptyset$. Hence $|\Box\bot| = \langle\blacktriangleleft\rangle|\bot| = \langle\blacktriangleleft\rangle\langle\lhd\rangle\emptyset \subseteq \langle\lhd\rangle\emptyset = |\bot|$, which is precisely the semantic content of $\neg\Box\bot$. Therefore $\neg\Box\bot$ is valid. ∎

5.2.3 Completeness

Now we show completeness. The canonical model is defined as follows.

Definition 39 (Canonical modal cover model)**.** *Worlds are finite contexts of formulas, which we regard as finite sets, ordered by entailment. We write* Ctx *for the set of all such contexts. The **antecedents** of A, denoted by* Ant(A)*, are the contexts under which A is derivable:*

$$\mathrm{Ant}(A) = \{\Gamma \in \mathrm{Ctx} \mid \Gamma \vdash A\}.$$

The local covering relation $\lhd \subseteq \mathrm{Ctx} \times \mathcal{P}(\mathrm{Ctx})$ is defined inductively by:

$$\frac{}{\Gamma \lhd \{\Gamma\}} \qquad \frac{\Gamma \vdash \bot}{\Gamma \lhd \emptyset} \qquad \frac{\Gamma \vdash A \vee B \quad \Gamma.A \lhd \alpha_1 \quad \Gamma.B \lhd \alpha_2}{\Gamma \lhd \alpha_1 \cup \alpha_2}.$$

The modal covering relation $\blacktriangleleft \subseteq \mathrm{Ctx} \times \mathcal{P}(\mathrm{Ctx})$ *is defined inductively by:*

$$\frac{}{\Gamma \blacktriangleleft \{\Gamma\}} \qquad \frac{\Gamma \vdash \Box C}{\Gamma \blacktriangleleft \{\Gamma.C\}} \qquad \frac{\Gamma \vdash \bot}{\Gamma \blacktriangleleft \emptyset} \qquad \frac{\Gamma \vdash A \vee B \quad \Gamma.A \blacktriangleleft \alpha_1 \quad \Gamma.B \blacktriangleleft \alpha_2}{\Gamma \blacktriangleleft \alpha_1 \cup \alpha_2}.$$

For a propositional atom p*, we define the valuation*

$$V(p) = \{\Gamma \in \mathrm{Ctx} \mid \Gamma \vdash p\} = \mathrm{Ant}(p).$$

As we will see in the proof of Modal Localization, the overlap between the definitions of local and modal cover is precisely what ensures that this property holds.

With respect to the canonical cover construction in Valliappan's paper [Val26], the only change we make in the non-modal fragment is to replace the order $\subseteq$ on contexts in the canonical model by the entailment preorder

$$\Gamma \leq \Delta :\iff \forall A.\, \Gamma \vdash A \Rightarrow \Delta \vdash A.$$

The reason for this change is that it is needed in the proof of Modal Confluence. In the case where both derivations end in $\Box$, the proof uses the fact that $\Gamma.A, \Gamma.B \leq \Gamma.(A \wedge B)$. This does not hold if $\leq$ is defined by context inclusion. Therefore, we use the entailment preorder, which compares contexts according to the information they derive rather than their syntactic shape. Since this preorder extends the inclusion ordering, as $\Gamma \subseteq \Delta$ implies $\Gamma \leq \Delta$, it abstracts away from the particular presentation of contexts while preserving the essential property of the inclusion ordering: moving upwards corresponds to having at least as much derivable information. Since the entailment ordering is standard in canonical model constructions for completeness proofs of intuitionistic logic with respect to cover semantics, the non-modal arguments carry over unchanged, and we do not repeat them here.

Lemma 40 (Canonical modal cover model)**.** *The relations* $\triangleleft$ *and* $\blacktriangleleft$ *defined on* Ctx *satisfy the cover system conditions (Refinement, Inclusion, Identity, Transitivity) and the modal cover system conditions (Modal Refinement, Modal Localization, Modal Identity, Modal Confluence, Modal Emptiness), respectively. Moreover,* $V(p)$ *is a localized up-set for any propositional atom* p*. Therefore, the canonical cover model is a **TICS-2**$_\Box$ modal cover model.*

Proof. The cover conditions for $\triangleleft$ are established in [Val26] and carry over; the proof is standard and also works with the entailment preorder. Moreover, since $V(p)$ and $\triangleleft$ are defined exactly as in [Val26], the localization of $V(p)$ carries over directly. Finally, the fact that $V(p) = \mathrm{Ant}(p)$ is an up-set follows immediately from the definition of the preorder. We therefore only need to verify the modal cover conditions for $\blacktriangleleft$.

- **Modal Refinement.** If $\Gamma \leq \Gamma'$ and $\Gamma \blacktriangleleft \alpha$, then there exists α' such that $\Gamma' \blacktriangleleft \alpha'$ and $\alpha' \succeq \alpha$. The proof proceeds by induction on the derivation of $\Gamma \blacktriangleleft \alpha$.
  - Γ-rule: $\Gamma \blacktriangleleft \{\Gamma\}$. Take $\alpha' = \{\Gamma'\}$.
  - $\Box$-rule: if $\alpha = \{\Gamma.A\}$ and $\Gamma \in \mathrm{Ant}(\Box A)$, then $\Gamma' \in \mathrm{Ant}(\Box A)$ and $\Gamma' \blacktriangleleft \{\Gamma'.A\}$. Take $\alpha' = \{\Gamma'.A\}$.
  - $\bot$-case: if $\Gamma \in \mathrm{Ant}(\bot)$ then $\Gamma' \in \mathrm{Ant}(\bot)$ and both are modally covered by $\emptyset$.
  - $\vee$-case: if $\Gamma \in \mathrm{Ant}(A \vee B)$ with premises $\Gamma.A \blacktriangleleft \alpha_1$ and $\Gamma.B \blacktriangleleft \alpha_2$, apply the induction hypothesis to obtain modal covers $\alpha_1' \succeq \alpha_1, \alpha_2' \succeq \alpha_2$ for $\Gamma'.A$ and $\Gamma'.B$, and conclude $\Gamma' \blacktriangleleft \alpha_1' \cup \alpha_2'$.

Moreover, the construction preserves the length of the derivation: if $\Gamma \blacktriangleleft \alpha$ has a derivation of length $|D|$, then the corresponding derivation of $\Gamma' \blacktriangleleft \alpha'$ also has length $|D|$. This follows immediately in the first three cases, where the new derivation consists of the same single rule application. In the $\vee$-case, by the induction hypothesis the two premises are transformed without increasing the lengths of their derivations, and the final $\vee$-rule application adds exactly the same additional step as in the original derivation. Hence the resulting derivation has the same length.

- **Modal Localization.** If $\Gamma \triangleleft \alpha \subseteq \langle\blacktriangleleft\rangle X$, then there exists β such that $\Gamma \blacktriangleleft \beta \subseteq \langle\triangleleft\rangle X$. The proof proceeds by induction on the derivation of $\Gamma \triangleleft \alpha$.
  - Singleton case: $\alpha = \{\Gamma\}$, so $\Gamma \in \langle\blacktriangleleft\rangle X$. By definition of $\langle\blacktriangleleft\rangle X$ there exists β with $\Gamma \blacktriangleleft \beta \subseteq X \subseteq \langle\triangleleft\rangle X$.
  - $\bot$-case: take $\beta = \emptyset$; then $\Gamma \blacktriangleleft \emptyset \subseteq \langle\triangleleft\rangle X$.
  - $\vee$-case: if $\Gamma \in \mathrm{Ant}(A \vee B)$ with $\Gamma.A \triangleleft \alpha_1 \subseteq \langle\blacktriangleleft\rangle X$ and $\Gamma.B \triangleleft \alpha_2 \subseteq \langle\blacktriangleleft\rangle X$, apply the induction hypothesis to both branches to obtain β_1, β_2 with $\Gamma.A \blacktriangleleft \beta_1 \subseteq \langle\triangleleft\rangle X$ and $\Gamma.B \blacktriangleleft \beta_2 \subseteq \langle\triangleleft\rangle X$, and conclude with $\beta = \beta_1 \cup \beta_2$.
- **Modal Identity.** Immediate from the rule $\Gamma \blacktriangleleft \{\Gamma\}$.
- **Modal Emptiness.** If $\Gamma \blacktriangleleft \beta \subseteq \mathrm{Ant}(\bot)$, then $\Gamma \in \mathrm{Ant}(\bot)$. The proof proceeds by induction on the derivation of $\Gamma \blacktriangleleft \beta$.
  - Γ-rule: $\Gamma \blacktriangleleft \{\Gamma\} \subseteq \mathrm{Ant}(\bot)$ gives $\Gamma \in \mathrm{Ant}(\bot)$ immediately.
  - $\Box$-rule ($\beta = \{\Gamma.A\} \subseteq \mathrm{Ant}(\bot)$, $\Gamma \in \mathrm{Ant}(\Box A)$): from $\Gamma.A \vdash \bot$ obtain $\Gamma \vdash \neg A$. By T1, $\Gamma \vdash \neg A \to \Box\neg A$, so $\Gamma \vdash \Box\neg A$. By T2, $\Gamma \vdash \Box\neg A \to (\Box A \to \Box\bot)$, so $\Gamma \vdash \Box\bot$. By T3, $\Gamma \vdash \neg\Box\bot$, hence $\Gamma \vdash \bot$.
  - $\bot$-case: $\Gamma \in \mathrm{Ant}(\bot)$ by hypothesis.
  - $\vee$-case ($\Gamma \in \mathrm{Ant}(A \vee B)$, $\Gamma.A \blacktriangleleft \alpha_1 \subseteq \mathrm{Ant}(\bot)$, $\Gamma.B \blacktriangleleft \alpha_2 \subseteq \mathrm{Ant}(\bot)$): by the induction hypothesis, $\Gamma.A \in \mathrm{Ant}(\bot)$ and $\Gamma.B \in \mathrm{Ant}(\bot)$. By $\vee$-elimination with $\Gamma \vdash A \vee B$, $\Gamma \vdash \bot$.
- **Modal Confluence.** If $\Gamma \blacktriangleleft \alpha$ and $\Gamma \blacktriangleleft \beta$, then there exists γ such that $\Gamma \blacktriangleleft \gamma$, $\alpha \preceq \gamma$, $\beta \preceq \gamma$.

  We prove this by strong induction on $n = |D_1| + |D_2|$, where D_1 and D_2 are the given derivations of $\Gamma \blacktriangleleft \alpha$ and $\Gamma \blacktriangleleft \beta$ respectively. We case on the last rule used in D_1 and in D_2.
  - **One of the derivations ends in $\bot$.** Say D_1 ends in $\bot$, so $\Gamma \vdash \bot$ and $\alpha = \emptyset$. Take $\gamma = \emptyset$; then $\Gamma \blacktriangleleft \emptyset$ by the $\bot$ rule, $\alpha = \emptyset \preceq \emptyset$ trivially, and $\beta \preceq \emptyset$ holds vacuously (there is nothing in $\emptyset$ to find a witness for), regardless of what β or D_2 are. The case where D_2 ends in $\bot$ is symmetric.
  - **One of the derivations ends in identity** (and neither ends in $\bot$). Say D_1 ends in identity, so $\alpha = \{\Gamma\}$. Take $\gamma = \beta$. We need to show that $\alpha = \{\Gamma\} \preceq \gamma = \beta$, i.e. that for all $\Gamma' \in \beta$ we have $\Gamma \leq \Gamma'$. We show this by induction on the rules defining $\blacktriangleleft$:
    - Γ-rule: $\beta = \{\Gamma\}$. Clearly $\Gamma \leq \Gamma$ by reflexivity of the entailment preorder.
    - $\Box$-rule: $\beta = \{\Gamma.C\}$ with $\Gamma \vdash \Box C$. Since $\Gamma \subseteq \Gamma.C$, we have $\Gamma \leq \Gamma.C$.
    - $\bot$-case: $\beta = \emptyset$. The claim holds vacuously, since there is no $\Gamma' \in \beta$.

* $\vee$-case: $\beta = \alpha_1 \cup \alpha_2$, where $\Gamma.A \blacktriangleleft \alpha_1$ and $\Gamma.B \blacktriangleleft \alpha_2$. By the induction hypothesis, every $\Gamma' \in \alpha_1$ satisfies $\Gamma.A \leq \Gamma'$, and every $\Gamma' \in \alpha_2$ satisfies $\Gamma.B \leq \Gamma'$. Since $\Gamma \leq \Gamma.A$ and $\Gamma \leq \Gamma.B$ by inclusion of contexts, transitivity gives $\Gamma \leq \Gamma'$ for every $\Gamma' \in \alpha_1 \cup \alpha_2$.

Note that $\beta \preceq \gamma = \beta$ holds trivially. The symmetric case is identical.

- **Both derivations end in $\Box$.** Say D_1 gives $\Gamma \vdash \Box A$, $\alpha = \{\Gamma.A\}$, and D_2 gives $\Gamma \vdash \Box B$, $\beta = \{\Gamma.B\}$. Then $\Gamma \vdash \Box A \wedge \Box B$, so by distribution of $\Box$ over $\wedge$, $\Gamma \vdash \Box(A \wedge B)$, giving $\Gamma \blacktriangleleft \{\Gamma.(A \wedge B)\}$ by the Box rule. Since $A \wedge B \vdash A$ and $A \wedge B \vdash B$, we have $\Gamma.A \leq \Gamma.(A \wedge B)$ and $\Gamma.B \leq \Gamma.(A \wedge B)$. Take $\gamma = \{\Gamma.(A \wedge B)\}$; then $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

- **One derivation ends in $\Box$, the other in $\vee$.** Say D_1 gives $\Gamma \vdash \Box A$, $\alpha = \{\Gamma.A\}$ (so $|D_1| = 1$), and D_2 gives $\Gamma \vdash C \vee D$, $\beta = \beta_1 \cup \beta_2$, with sub-derivations $D_2^\ell : \Gamma.C \blacktriangleleft \beta_1$ and $D_2^r : \Gamma.D \blacktriangleleft \beta_2$, so $|D_2| = 1 + |D_2^\ell| + |D_2^r|$.

  By weakening, $\Gamma.C \vdash \Box A$ and $\Gamma.D \vdash \Box A$, so by the $\Box$ rule we get fresh derivations $E^\ell : \Gamma.C \blacktriangleleft \{\Gamma.C.A\}$ and $E^r : \Gamma.D \blacktriangleleft \{\Gamma.D.A\}$, each of size 1.

  Apply the induction hypothesis to the pair (E^ℓ, D_2^ℓ): this is legitimate since $|E^\ell| + |D_2^\ell| = 1 + |D_2^\ell| < 1 + |D_2| \leq n$ (using $|D_2| = 1 + |D_2^\ell| + |D_2^r| \geq 1 + |D_2^\ell|$ and $|D_1| = 1$). This gives γ_1 with $\Gamma.C \blacktriangleleft \gamma_1$, $\{\Gamma.C.A\} \preceq \gamma_1$, $\beta_1 \preceq \gamma_1$. Symmetrically, applying the induction hypothesis to (E^r, D_2^r) (again strictly smaller than n) gives γ_2 with $\Gamma.D \blacktriangleleft \gamma_2$, $\{\Gamma.D.A\} \preceq \gamma_2$, $\beta_2 \preceq \gamma_2$.

  By the $\vee$ rule (using $\Gamma \vdash C \vee D$), $\Gamma \blacktriangleleft \gamma_1 \cup \gamma_2$. Since $\Gamma.A \leq \Gamma.C.A$ and $\Gamma.A \leq \Gamma.D.A$, transitivity gives $\{\Gamma.A\} \preceq \gamma_1 \cup \gamma_2$, i.e. $\alpha \preceq \gamma_1 \cup \gamma_2$; and $\beta = \beta_1 \cup \beta_2 \preceq \gamma_1 \cup \gamma_2$ follows componentwise. Take $\gamma = \gamma_1 \cup \gamma_2$. The symmetric case is identical with the roles of D_1, D_2 swapped.

- **Both derivations end in $\vee$.** Say D_1 gives $\Gamma \vdash A \vee B$, $\alpha = \alpha_1 \cup \alpha_2$, with sub-derivations $D_1^\ell : \Gamma.A \blacktriangleleft \alpha_1$, $D_1^r : \Gamma.B \blacktriangleleft \alpha_2$; and D_2 gives $\Gamma \vdash C \vee D$, $\beta = \beta_1 \cup \beta_2$, with sub-derivations $D_2^\ell : \Gamma.C \blacktriangleleft \beta_1$, $D_2^r : \Gamma.D \blacktriangleleft \beta_2$.

  By weakening, $\Gamma.C \vdash A \vee B$ and $\Gamma.D \vdash A \vee B$. Applying Modal Refinement to D_1^ℓ along $\Gamma.A \leq \Gamma.A.C = \Gamma.C.A$, and to D_1^r along $\Gamma.B \leq \Gamma.B.C = \Gamma.C.B$, we obtain derivations $F_C^\ell : \Gamma.C.A \blacktriangleleft \alpha_1' \succeq \alpha_1$ and $F_C^r : \Gamma.C.B \blacktriangleleft \alpha_2' \succeq \alpha_2$, with $|F_C^\ell| = |D_1^\ell|$ and $|F_C^r| = |D_1^r|$ (as Modal Refinement preserves size). By the $\vee$ rule, this gives a fresh derivation $E_C : \Gamma.C \blacktriangleleft \alpha_1' \cup \alpha_2'$ with $|E_C| = 1 + |D_1^\ell| + |D_1^r| = |D_1|$.

  Symmetrically, we obtain $E_D : \Gamma.D \blacktriangleleft \alpha_1'' \cup \alpha_2''$ with $\alpha \preceq \alpha_1'' \cup \alpha_2''$ and $|E_D| = |D_1|$.

  Now apply the induction hypothesis to (E_C, D_2^ℓ): this is legitimate since $|E_C| + |D_2^\ell| = |D_1| + |D_2^\ell| < |D_1| + |D_2| = n$ (as $|D_2^\ell| < |D_2|$). This gives γ_1 with $\Gamma.C \blacktriangleleft \gamma_1$, $(\alpha_1' \cup \alpha_2') \preceq \gamma_1$, $\beta_1 \preceq \gamma_1$. Symmetrically, applying the induction hypothesis to (E_D, D_2^r) (again strictly smaller than n) gives γ_2 with $\Gamma.D \blacktriangleleft \gamma_2$, $(\alpha_1'' \cup \alpha_2'') \preceq \gamma_2$, $\beta_2 \preceq \gamma_2$.

  By the $\vee$ rule, $\Gamma \blacktriangleleft \gamma_1 \cup \gamma_2$. Since $\alpha = \alpha_1 \cup \alpha_2 \preceq (\alpha_1' \cup \alpha_2') \cup (\alpha_1'' \cup \alpha_2'') \preceq \gamma_1 \cup \gamma_2$ and $\beta = \beta_1 \cup \beta_2 \preceq \gamma_1 \cup \gamma_2$, take $\gamma = \gamma_1 \cup \gamma_2$. □

Now we show the Truth Lemma.

Lemma 41 (Truth Lemma). *In the canonical model, for every formula A and every world $\Gamma \in \mathrm{Ctx}$,*

$$\Gamma \in |A|^{\mathcal{N}} \iff \Gamma \vdash A.$$

Proof. The cases for propositional letters, $\bot$, and the propositional connectives follow immediately from [Val26]. We treat only the modal case. We show $|\Box A|^{\mathcal{N}} = \langle \blacktriangleleft \rangle |A|^{\mathcal{N}}$ by showing both inclusions.

Left to right. Suppose $\Gamma \blacktriangleleft \beta \subseteq |A|^{\mathcal{N}}$. We show $\Gamma \vdash \Box A$ by induction on the derivation of $\Gamma \blacktriangleleft \beta$.

- *Identity* ($\beta = \{\Gamma\}$, $\Gamma \in |A|^{\mathcal{N}}$): by the induction hypothesis $\Gamma \vdash A$, so $\Gamma \vdash \Box A$ by T1.
- $\Box$-*rule* ($\Gamma \vdash \Box C$, $\beta = \{\Gamma.C\}$, $\Gamma.C \in |A|^{\mathcal{N}}$): by the induction hypothesis $\Gamma.C \vdash A$, i.e. $\Gamma \vdash C \to A$, so by T1: $\Gamma \vdash \Box(C \to A)$, and by T2: $\Gamma \vdash \Box C \to \Box A$. Since $\Gamma \vdash \Box C$, we get $\Gamma \vdash \Box A$.
- $\bot$-*case* ($\Gamma \vdash \bot$): $\Gamma \vdash \Box A$ by ex falso.
- $\vee$-*case* ($\Gamma \vdash B_1 \vee B_2$, $\Gamma.B_1 \blacktriangleleft \alpha_1 \subseteq |A|^{\mathcal{N}}$, $\Gamma.B_2 \blacktriangleleft \alpha_2 \subseteq |A|^{\mathcal{N}}$): by the induction hypothesis on both sub-derivations, $\Gamma.B_1 \vdash \Box A$ and $\Gamma.B_2 \vdash \Box A$. By $\vee$-elimination, $\Gamma \vdash \Box A$.

Right to left. If $\Gamma \vdash \Box A$, then since $\Box A \in \Phi_E$, the $\Box$-rule gives $\Gamma \blacktriangleleft \{\Gamma.A\}$. We have $\Gamma.A \vdash A$, hence $\Gamma.A \in |A|^{\mathcal{N}}$. By the induction hypothesis applied to the context $\Gamma.A$, we may conclude $\{\Gamma.A\} \subseteq |A|^{\mathcal{N}}$. Hence $\Gamma \in \langle \blacktriangleleft \rangle |A|^{\mathcal{N}} = |\Box A|^{\mathcal{N}}$. $\Box$

Theorem 42 (Completeness)**.** *If $\models E$, then $\vdash E$.*

Proof. Assume $\models E$. Then E is true at every world of every **TICS-2**$_\Box$-modal cover model, in particular at the empty context $\emptyset$ of the canonical model. Hence

$$\emptyset \Vdash E.$$

By the Truth Lemma,

$$\emptyset \Vdash E \iff \emptyset \vdash E.$$

Therefore $\emptyset \vdash E$, i.e. $\vdash E$. $\Box$

We have therefore obtained the following constructive result.

Corollary 43. ***TICS-2***$_\Box$ *is sound and complete with respect to modal cover semantics: for every* ***TICS-2***$_\Box$*-formula A, we have $\vdash A$ if and only if $\models A$.*

5.3 Intuitionistic Epistemic Logic (IEL)

Our system **TICS-2**$_\Box$ turns out to correspond to another logic, called *Intuitionistic Epistemic Logic*. In order to see this, we first note that, in the presence of axioms T1 and T2, axiom T3 can equivalently be replaced by the axiom

$$\Box A \to \neg\neg A.$$

The following proposition establishes the equivalence of these two axioms over **IPC** extended with T1 and T2.

Proposition 44. *Over **IPC** together with T1 and T2, the following are equivalent:*

T3 $\neg\Box\bot$*;*

T3′ $\Box A \to \neg\neg A$.

Proof. We show both directions separately. When we write "IPC", we mean that a suitable theorem of **IPC** is applied to the formulas occurring in that step.

T3 $\Rightarrow$ T3′. We derive

$$\vdash_{\mathbf{TICS\text{-}2}_\Box} \Box A \to \neg\neg A.$$

1	$\Box(A \to \bot) \to (\Box A \to \Box\bot)$	(T2)
2	$A \to \bot$	(assumption)
3	$\Box(A \to \bot)$	(T1, 2)
4	$\Box A \to \Box\bot$	(MP 1, 3)
5	$\neg\Box\bot$	(T3)
6	$\Box A \to \neg(A \to \bot)$	(IPC 4,5; discharge 2)
7	$\Box A \to \neg\neg A$	(definition)

T3′ $\Rightarrow$ T3. We derive

$$\vdash_{\mathbf{IPC}+\mathrm{T1}+\mathrm{T2}+\mathrm{T3}'} \neg\Box\bot.$$

1	$\Box\bot \to \neg\neg\bot$	(T3′)
2	$\neg\neg\bot \to \bot$	(IPC)
3	$\Box\bot \to \bot$	(IPC 1, 2)
4	$\neg\Box\bot$	(definition)

□

This gives an alternative axiomatization of $\mathbf{TICS\text{-}2}_\Box$, namely as $\mathbf{IPC} + \mathrm{T1} + \mathrm{T2} + \mathrm{T3}'$. This corresponds precisely to the axiomatization of Intuitionistic Epistemic Logic (**IEL**), introduced by Artemov and Protopopescu [AP16].

Definition 45. ***Intuitionistic Epistemic Logic (IEL)*** *extends intuitionistic propositional logic with a unary epistemic operator K, intended to represent intuitionistic knowledge. Its axiomatization consists of:*

- *all axioms of Intuitionistic Propositional Calculus;*
- $K(F \to G) \to (KF \to KG)$ *(distribution);*
- $F \to KF$ *(co-reflection);*
- $KF \to \neg\neg F$ *(intuitionistic reflection).*

The only inference rule is modus ponens. By removing the intuitionistic reflection axiom, one obtains ***Intuitionistic Doxastic Logic (IEL$^-$)****.*

Under the identification $K = \Box$, the distribution, co-reflection, and intuitionistic reflection axioms correspond exactly to T2, T1, and T3', respectively. Therefore, the $\Box$-fragment of **TICS-2** coincides with **IEL**.

The intended interpretation of the knowledge modality in **IEL**, as explained in [KY15], is that of *verified truth*, following Williamson [Wil92]. A proof of KA represents that A has been verified: it provides conclusive verification that a proof of A exists, without necessarily providing such a proof itself. The co-reflection principle

$$A \to KA$$

is taken to be valid, since it expresses that any proof of A can itself be effectively verified. In contrast, Artemov and Protopopescu note that the converse reflection principle

$$KA \to A,$$

which is valid for knowledge in many classical epistemic settings, is not intuitionistically valid. A verification that A holds does not need to contain the construction required for a proof of A itself. They illustrate this with a zero-knowledge protocol, in which a prover convinces a verifier that a proposition has been proved without revealing any information from which a proof could be reconstructed. At the end of such a protocol, the verifier may therefore be entitled to *believe*, or even to *know*, that the proposition holds, while possessing no proof of it themselves.

A second example concerns mathematical knowledge based on authority. Mathematical practice often relies on results established by other experts: one can understand and use a theorem without possessing its proof. They illustrate this with Fermat's Last Theorem, which a mathematician may reasonably regard as known without being able to produce the proof personally. This shows again that verification of A need not amount to possession of a proof of A, explaining why the reflection principle $KA \to A$ is not intuitionistically valid.

The intuitionistic reflection axiom $KA \to \neg\neg A$ is what distinguishes **IEL** from the weaker doxastic logic $\mathbf{IEL}^-$, which is obtained by dropping this axiom. To see why this particular principle marks the boundary between belief and knowledge, it is worth recalling how both notions arise in the first place. Artemov and Protopopescu's starting point is a BHK interpretation of epistemic statements on which knowledge and belief are both cast as results of a process of *verification*. Belief, as modelled by $\mathbf{IEL}^-$, requires nothing beyond this: to believe A is simply to possess some verification of it, of whatever strength, with no further guarantee that A actually holds. Knowledge is a strictly stronger notion: to know A is to possess a verification strong enough to guarantee that A cannot be false, i.e. that $\neg\neg A$ holds. Intuitionistic reflection is thus what turns mere *belief* into *knowledge*: it is the extra condition, imposed on top of co-reflection and distribution, that upgrades a verification into one that certifies the impossibility of falsity. Because of this, intuitionistic knowledge under **IEL** lies strictly between intuitionistic and classical truth:

$$A \to KA \to \neg\neg A.$$

Every proof gives rise to knowledge, and every piece of knowledge implies the double negation of truth, while neither implication is reversible in general [HK21].

5.3.1 The $\Box$-operator in IEL

In **IEL**, a distinction is made between possessing a proof of A and merely having a verification that a proof exists. This distinction is analogous to our interpretation of $\Box$ in terms of the Creating Subject: in both cases, there is a guarantee of a proof without the proof itself being presently available. Reading K as verified truth also provides useful conceptual background for our interpretation of the $\Box$-operator. The formula KA expresses that the existence of a proof of A is verified, whereas $\Box A$ expresses that a proof of A will be available at the next stage. From the perspective of the present stage, $\Box A$ can therefore be understood as expressing a verification of future proof availability.

This shared structure, namely the absence of a currently available constructive proof, also appears in another illustration of why the reflection principle $KA \to A$ fails intuitionistically. This case involves existential statements, which motivates the first-order extension of **IEL** in [SS23]. Artemov and Protopopescu describe the following scenario [AP16]: suppose your wallet has been stolen on the subway. The available evidence (the missing wallet and a cut pocket) justifies knowing that *someone* stole it, that is, asserting $K(\exists x\, S(x))$, where $S(x)$ says that x stole your wallet. If $K(\exists x\, S(x)) \rightarrow \exists x\, S(x)$ were intuitionistically valid, this would give a constructive proof of $\exists x\, S(x)$, requiring an actual witness a together with a proof that $S(a)$ holds. The evidence, however, only provides verification that a thief exists, not a constructive witness identifying the thief.

Similar considerations motivate Niekus' rejection of the reflection-like principles of the Theory of the Creating Subject, in particular the axiom CS4: $\exists n\, \Box_n A \rightarrow A$. A guarantee that A will be proved at some later stage does not itself provide a constructive proof of A at the present stage, since the constructive information required to establish A may not yet be available. The counterexample he gives, as discussed earlier in the thesis, concerns the decidability of the values of a sequence. Suppose that the value a_{n_0} is determined only at stage n_0. Then, at stage $n_0 + 1$, we can certainly prove that

$$a_{n_0} = 0 \vee a_{n_0} \neq 0,$$

since the value has become available and one of the two disjuncts can be shown to hold. Hence

$$\Box_{n_0+1}(a_{n_0} = 0 \vee a_{n_0} \neq 0)$$

holds. At the current stage, however, the value a_{n_0} is not yet available, and we therefore do not possess a constructive proof of $a_{n_0} = 0 \vee a_{n_0} \neq 0$. Such a proof would require already knowing which disjunct holds. The statement is guaranteed to become decidable at a later stage, but it is not yet constructively settled at the present stage, exactly because a witness, namely the value of a_{n_0}, which picks out one of the disjuncts, is not available.

There are thus some parallels between the motivation underlying **IEL** and our formalization of the Creating Subject. Another connection of this kind has already been established by Su and Sano, who employ their first-order Kripke semantics for **IEL** to interpret Brouwer's Creating Subject [SS23]. In doing so, they refer to the following description of an idealized mathematician from Van Dalen:

> "Think of an idealized mathematician (in this context traditionally called the *creative subject*), who extends both his knowledge and his universe of objects in the course of time. At each moment k he has a stock Σ_k of sentences, which he, by some means, has recognized as true and a stock A_k of objects which he has constructed (or created). Since at every moment k the idealized mathematician has various choices for his future activities (he may even stop altogether), the stages of his activity must be thought of as being *partially ordered*, and not necessarily linearly ordered." [Dal04]

This picture of the idealized mathematician moving through a partially ordered succession of temporal stages is exactly what is represented by our Kripke models for $\Box$. In accordance with the design of our models, Van Dalen emphasizes that the accessibility relation between stages need not be linear: a given stage may have several possible successors, each corresponding to a different way in which the mathematician's stock of proofs and constructed objects can be extended.

Su and Sano represent the knowledge of the idealized mathematician using first-order Kripke models for their logic. The formulas true at a state w are identified with the formulas in the mathematician's stock of proved statements, Proved_w, which matches the intuitionistic reading of truth. They further distinguish verified statements by defining

$$A \in \text{Verified}_w \quad \Longleftrightarrow \quad KA \in \text{Proved}_w.$$

Since their system validates $A \to KA$, every proved statement is also verified, so

$$\text{Proved}_w \subseteq \text{Verified}_w.$$

The modal relation is then defined by

$$wRv \quad \Longleftrightarrow \quad \text{Verified}_w \subseteq \text{Proved}_v,$$

so that an R-successor of w represents a later state in which everything that is currently verified has become proved.

This interpretation is closely related to our reading of $\Box$. In our setting, $\Box A$ expresses that A will hold after the next R-step: every possible temporal successor must contain a proof of A. Thus, as in the semantics above, the distinction is between what is currently proved and what is guaranteed to hold at some future extension. A formula A may fail to hold at the present stage while $\Box A$ holds, because the information required to establish A constructively may only become available at the R-successors.

We thus see that the systems match not only on a technical level, but also in their applications. Although **IEL** was introduced relatively recently, it has since become a well-established framework for studying intuitionistic knowledge. In the next subsection, we discuss some of the existing work on **IEL**, which is also relevant for our project.

5.3.2 Existing work on IEL

For **IEL**, a wide range of results has already been developed in its semantics and proof theory, including results established in an explicitly constructive metatheory. We briefly review the most relevant ones below, together with a shorter account of some of the additional literature.

Completeness results Kripke semantics. In their paper introducing **IEL** [AP16], Artemov and Protopopescu prove soundness and completeness with respect to Kripke semantics for exactly the same class of frames as the one considered earlier in this thesis for **TICS-2**. Su and Sano extend these results to a first-order setting: they prove soundness and completeness of first-order intuitionistic epistemic logic (**QIEL**) with respect to Kripke models with growing domains [SS19], and subsequently do the same for the language with equality (**QIEL**$^{=}$) [SS23], where the underlying Kripke frames are exactly the frames considered for **TICS-2**.

Proof theory. Proof-theoretic results for **IEL** have also been developed in some detail. For example, Krupski and Yatmanov introduce sequent calculi for both **IEL**$^{-}$ and **IEL** [KY15]. However, as observed in [SS19], their calculus for **IEL** does not satisfy the subformula property. This issue is addressed in [SS19] by developing a sequent calculus for **QIEL** that does satisfy the subformula property, for which the authors moreover establish results such as cut-elimination and Craig interpolation. In [SS23], the same authors develop a sequent calculus for **QIEL**$^{=}$. On the natural deduction side, Brogi develops a natural deduction calculus for $\mathbf{IEL}^-$ and proves normalization for its deductions [Per21].

Constructive meta-theory. In addition to Kripke semantics, **IEL** and $\mathbf{IEL}^-$ have also been studied using cover semantics in the order-theoretic framework introduced by Goldblatt [Gol11]. Rogozin uses this framework to obtain constructive completeness results for $\mathbf{IEL}^-$, **IEL**, and their predicate extensions [Rog21]. Independently, Hagemeier and Kirst study the meta-theory of **IEL** in a fully constructive setting [HK21]. Based on a syntactic decidability proof for a cut-free sequent calculus, they provide a mechanised constructive proof of completeness for **IEL** with respect to finite contexts, together with constructive versions of the finite model property and Su and Sano's semantic cut-elimination result. Consequently, the constructive completeness result obtained in this thesis for the $\square$-fragment of **TICS-2**, and thus for **IEL**, is not the first constructive completeness result for **IEL**. Rather, the contribution of our approach is to provide a constructive completeness proof for **IEL** through a constructive version of a Scott-style canonical model construction using modal cover techniques, rather than the order-theoretic methods of Goldblatt and Rogozin or the syntactic methods of Hagemeier and Kirst.

Other work. **IEL** has also been studied in several other directions, which we only briefly mention here. On the type-theoretic and categorical side, Rogozin studies $\mathbf{IEL}^-$ through a Curry-Howard correspondence with a modal lambda calculus and a corresponding categorical semantics [Rog21]. On the complexity side, Krupski and Yatmanov show that the derivability problem for their sequent calculus for **IEL** is decidable in polynomial space, so that **IEL** is PSPACE-complete [KY15]. Fiorino extends this line of work by giving sequent calculi for **IEL** and $\mathbf{IEL}^-$ that satisfy the subformula property and whose deductions are bounded in linear depth, which gives a proof-search procedure that additionally returns a Kripke countermodel of minimal depth for invalid formulas [Fio23]. Protopopescu shows that the verification notion underlying **IEL** admits an arithmetical semantics [Pro20]. On the philosophical side, **IEL** has been related to the justified-true-belief analysis of knowledge from a Brouwerian perspective [Art18]. On top of that, **IEL**-style modalities have also been extended to multi-agent epistemic settings, including distributed knowledge [SMS21]. Finally, on the modal side, Balbiani extends the box-based language of **IEL** and $\mathbf{IEL}^-$ with an independent diamond modality and establishes completeness for the resulting two-operator logics [Bal25]. We do not discuss these lines of work further here, but flag them to indicate the range of directions in which **IEL** has been taken up.

In the further work section of the conclusion, we discuss how the results presented in this chapter, both our own contributions and those concerning **IEL**, may be extended to the full system. One result concerning **IEL**, however, can already be applied directly. In the next chapter, we consider how the soundness and completeness results for $\mathbf{QIEL}^=$ developed in [SS23] can be extended to a first-order version of $\mathbf{TICS\text{-}2}_\square$ with equality and an explicit existence predicate.

Chapter 6

$\mathbf{QE^+}$-TICS-$2_\Box$

In this chapter, we consider a first-order extension of **TICS-2**$_\Box$ with equality, which makes use of an explicit existence predicate as in E^+-logic. We first introduce E^+-logic. We then consider the system **QE$^+$-TICS-2**$_\Box$ by defining its language, semantics and Hilbert-style proof system, and establish soundness and completeness. Finally, we use these results to make some observations about the logics involved and their interpretations.

6.1 E^+-logic

In many mathematical settings, terms do not always need to denote an existing object. For example, partial functions such as $x \mapsto x^{-1}$ on the real numbers are undefined at $x = 0$. To account for this, one can extend the logic with an *existence predicate* E. The formula Et is read as "the term t exists" or "t is defined". Quantifiers are then interpreted as ranging only over existing objects, and instantiation of quantified formulas is permitted only for terms whose existence has been established.

There are two closely related versions of logic with an existence predicate, described in [TD88]. The main difference lies in the interpretation of free variables. In E*-logic*, introduced by Scott in [Sco06], free variables are treated purely schematically, so any term may be substituted for a free variable. In E^+*-logic*, as described by Beeson in [Bee12], free variables are instead understood to range only over existing objects.

For example, if x is a free variable and f is a partial function, then in E^+-logic the expression $f(x)$ is immediately understood as the value of f at an existing object (although $f(x)$ itself may still be undefined). This matches the usual mathematical convention that variables range over the objects in the domain. In contrast, in E-logic it is not automatic that the free variable itself denotes an existing object.

The behaviour of the existence predicate is reflected in the proof rules. In E-logic, the quantifier rules are modified so that quantifiers range only over existing objects in the following way:

$$\forall I^E \ \frac{\begin{array}{c}[Ex]\\ \vdots \\ A\end{array}}{\forall y\, A[x/y]} \qquad\qquad \forall E^E \ \frac{\forall x\, A \qquad Et}{A[x/t]}$$

$$\exists I^E \frac{A[x/t] \qquad Et}{\exists x\, A} \qquad\qquad \exists E^E \frac{\exists y\, A[x/y] \qquad \begin{matrix}[A]\ [Ex] \\ \vdots \\ C\end{matrix}}{C}$$

The schematic role of free variables is expressed by the substitution rule

$$\text{SUB}\ \frac{A}{A[x/t]},$$

where the derivation of A does not rely on assumptions containing x free. In E^+-logic, every variable is assumed to denote an existing object by including the axiom Ex for each variable x. The quantifier rules are then $\forall I$, $\exists E$, $\exists I^E$, and $\forall E^E$. The first two are the ordinary quantifier rules. Since free variables are no longer purely schematic, the substitution rule should instead take the form

$$\frac{A \qquad Et}{A[x/t]},$$

but this rule is derivable from $\forall I$ followed by $\forall E^E$.

In logics with an existence predicate and equality, two terms are said to be equal if and only if they both denote existing objects and those objects are equal. Accordingly, we have the axiom schema

$$t = t \leftrightarrow Et.$$

This already provides a basis for first-order logic with E and equality. As observed in [TD88], however, it is convenient to additionally assume *strictness* for all primitive function and relation symbols:

$$\begin{aligned} Ef(t_0, \ldots, t_{n-1}) &\to Et_i \qquad (i < n), \\ R(t_0, \ldots, t_{n-1}) &\to Et_i \qquad (i < n). \end{aligned}$$

Under these strictness axioms, the meaning of Et and $t = s$ for compound terms can be reduced to equality between simpler terms. In particular, the following formulas are derivable:

$$Et \leftrightarrow t = t \leftrightarrow \exists x\, (t = x),$$

$$t = s \leftrightarrow \exists x\, (t = x \wedge s = x),$$

and

$$f(t) = x \leftrightarrow \exists y\, (t = y \wedge f(y) = x).$$

Similarly,

$$R(t) \leftrightarrow \exists y\, (t = y \wedge R(y)).$$

We use this framework as a foundation for extending $\mathbf{TICS\text{-}2}_{\Box}$ to a first-order system with equality and an existence predicate. The resulting system is based on E^+-logic together with the strictness axioms. In the final section of this chapter, we show how the existence predicate can be used to formalize the stage-by-stage existence of the values of a choice sequence. For example, a CS-sequence may be viewed as a partial function whose nth value becomes defined only at stage n. Before that stage, the value does not yet exist and no conclusions can be drawn about it; once it has come into existence, it can be used in further reasoning. We return to the details of this after setting up the system.

6.2 $\mathbf{QE^+}$-TICS-2$_\Box$

We introduce a system based on the $\Box$-fragment of **TICS-2**, rather than on the full **TICS-2**. The motivation for this choice is that a sound and complete semantics for the first-order version of **TICS-2**$_\Box$ with equality has already been established in [SS23] under the name $\mathbf{QIEL^=}$, which is Artemov-Protopopescu's first-order intuitionistic epistemic logic with equality. We use this result to obtain a sound and complete system for the first-order extension of the $\Box$-fragment of our modal logic with equality and an existence predicate, namely $\mathbf{QE^+}$**-TICS-2**$_\Box$. Equivalently, $\mathbf{QE^+}$**-TICS-2**$_\Box$ may be viewed as an extension of $\mathbf{QIEL^=}$ by an existence predicate interpreted according to E^+-logic. We denote this extension by $\mathbf{QIELE^+}$.

In [TD88], Troelstra and van Dalen describe a method for obtaining a sound and complete semantics for $\mathbf{IQCE^+}$, intuitionistic first-order logic with equality and an existence predicate interpreted according to E^+-logic, from a sound and complete semantics for $\mathbf{IQC^=}$, intuitionistic first-order logic with equality. Our setting involves precisely the same type of extension, namely enriching an intuitionistic first-order logic with equality by adding an existence predicate interpreted according to E^+-logic. We therefore follow the same strategy.

Our starting point is therefore the system $\mathbf{QIEL^=}$, as described in [SS23]. We extend its language with an existence predicate E. The underlying Kripke models remain the same as those of $\mathbf{QIEL^=}$. The essential semantic change is that function symbols are interpreted as partial functions on the domain. Following the strategy of Troelstra and van Dalen, the semantic clauses are then reformulated via

$$Et := \exists x\,(t = x),$$

$$(t = s) := \exists y\,(t = y \wedge s = y),$$

and

$$(f(\vec{t}) = x) := \exists \vec{y}\,\big(\vec{t} = \vec{y} \wedge f(\vec{y}) = x\big).$$

Likewise, the truth of atomic formulas involving compound terms is determined by strictness:

$$R(\vec{t}) := \exists \vec{y}\,\big(\vec{t} = \vec{y} \wedge R(\vec{y})\big).$$

To obtain a proof system, we modify the Hilbert axiom system in the same way that Troelstra and Van Dalen extend the system for $\mathbf{IQC^=}$ to obtain one for $\mathbf{IQCE^+}$. We work out the details of this throughout this section.

6.2.1 Language

Fix a first-order signature consisting of a countably infinite set of function symbols $\mathcal{F} = \{f_j\}_j$, where each f_j has a fixed finite arity m_j, and a countably infinite set of predicate symbols $\mathcal{P} = \{P_i\}_i$, where each P_i has a fixed finite arity n_i. Constants are identified with 0-ary function symbols. Let $\mathcal{V}$ be a countably infinite set of variables. The terms over this signature are defined by

$$t ::= v \mid f_j(t_1, \ldots, t_{m_j}).$$

The formulas of **QE$^+$-TICS-2$_\Box$** over this signature are defined by

$$A ::= P_i(t_1, \ldots, t_{n_i}) \mid t = s \mid Et \mid \bot \mid A \wedge B \mid A \vee B \mid A \to B \mid \forall x A \mid \exists x A \mid \Box A.$$

6.2.2 Frames and models

We now define frames and models for **QE$^+$-TICS-2$_\Box$** over the fixed first-order signature.

Definition 46 (Frame)**.** *A **frame** is a tuple $(W, \leq, R, D, \sim)$ where:*

- *$(W, \leq)$ is a preorder;*
- *$R \subseteq W \times W$ satisfies $R \subseteq \leq$, $\leq \circ R \subseteq R$, and R is serial;*
- *D assigns each $k \in W$ a non-empty set $D(k)$, with $k \leq l \Rightarrow D(k) \subseteq D(l)$;*
- *each $\sim_k$ is an equivalence relation on $D(k)$, with $k \leq l \Rightarrow \sim_k \subseteq \sim_l$.*

These are precisely the underlying frames of the models for **QIEL$^=$** described in [SS23]. In turn, the skeletons (so the structure without D and $\sim$) correspond to our **TICS-2** Kripke frames.

Definition 47 (Model)**.** *A **model** is $M = (W, \leq, R, D, \sim, I)$, a frame together with:*

- *For each function symbol $f_j \in \mathcal{F}$ and each world $k \in W$, a partial function*

$$f_j^k : D(k)^{m_j} \rightharpoonup D(k)$$

  satisfying the following two conditions:

  - ***Rigidity:** if $k \leq l$ and $f_j^k(\bar{d})$ is defined, then $f_j^l(\bar{d})$ is defined and*

$$f_j^k(\bar{d}) \sim_l f_j^l(\bar{d}).$$

  - ***Congruence:** if $\bar{d} \sim_k \bar{d}'$ componentwise and $f_j^k(\bar{d})$ is defined, then $f_j^k(\bar{d}')$ is defined and*

$$f_j^k(\bar{d}) \sim_k f_j^k(\bar{d}').$$

- *For each predicate symbol $P_i \in \mathcal{P}$ and each world $k \in W$, a relation*

$$P_i^k \subseteq D(k)^n$$

  satisfying the following two conditions:

  - ***$\sim_{\mathbf{k}}$-invariance:** if $\bar{d} \sim_k \bar{d}'$ componentwise, then*

$$\bar{d} \in P_i^k \iff \bar{d}' \in P_i^k.$$

- ◦ ***Persistence:*** *if $k \leq l$, then*
$$P_i^k \subseteq P_i^l,$$
where we identify $D(k)^n$ with its inclusion into $D(l)^n$.

Except for the partiality of the functions f_j^k and the corresponding adjustments, this definition coincides with that of a **QIEL**$^=$ model. Interpreting all function symbols as total functions recovers the usual notion of a **QIEL**$^=$ model, so the present definition is a proper generalization.

Definition 48 (Satisfaction)**.** *Let $\mathcal{M} = (W, \leq, R, D, \sim, I)$ be a model and $k \in W$. Denote by $\mathcal{L}(\mathcal{C} \cup D(k))$ the language $\mathcal{L}$ expanded with a name $\underline{d}$ for each $d \in D(k)$. For closed formulas $A \in \mathcal{L}(\mathcal{C} \cup D(k))$, define $k \Vdash A$ by simultaneous recursion on the complexity of A, with atomic formulas treated by a further recursion on the total size of their terms.*
For $d, d' \in D(k)$, tuples $\bar{d} \in D(k)^n$ and $\bar{a} \in D(k)^{m_j}$:

$$\begin{aligned}
k \Vdash \underline{d} = \underline{d'} &\iff d \sim_k d', \\
k \Vdash P_i(\underline{\bar{d}}) &\iff \bar{d} \in P_i^k, \\
k \Vdash f_j(\underline{\bar{a}}) = \underline{d'} &\iff f_j^k(\bar{a}) \text{ is defined and } f_j^k(\bar{a}) \sim_k d'.
\end{aligned}$$

For closed terms $t, s, \bar{t} = (t_1, \ldots, t_n)$, let $\bar{y} = (y_1, \ldots, y_n)$ be fresh variables and let $\bar{t} = \bar{y}$ abbreviate $\bigwedge_i t_i = y_i$. Then:

$$\begin{aligned}
k \Vdash f_j(\bar{t}) = x &\iff k \Vdash \exists \bar{y}(\bar{t} = \bar{y} \wedge f_j(\bar{y}) = x), \\
k \Vdash t = s &\iff k \Vdash \exists y(t = y \wedge s = y), \\
k \Vdash Et &\iff k \Vdash \exists y(t = y), \\
k \Vdash P_i(\bar{t}) &\iff k \Vdash \exists \bar{y}(\bar{t} = \bar{y} \wedge P_i(\bar{y})).
\end{aligned}$$

The remaining clauses are:

$$\begin{aligned}
k \Vdash \bot &\iff \text{never}, \\
k \Vdash A \wedge B &\iff k \Vdash A \text{ and } k \Vdash B, \\
k \Vdash A \vee B &\iff k \Vdash A \text{ or } k \Vdash B, \\
k \Vdash A \to B &\iff \forall l \geq k\, (l \Vdash A \Rightarrow l \Vdash B), \\
k \Vdash \forall x A &\iff \forall l \geq k\, \forall d \in D(l)\; l \Vdash A(\underline{d}/x), \\
k \Vdash \exists x A &\iff \exists d \in D(k)\; k \Vdash A(\underline{d}/x), \\
k \Vdash \Box A &\iff \forall l\, (kRl \Rightarrow l \Vdash A).
\end{aligned}$$

For an assignment $\sigma : \mathcal{V} \to D(k)$, define

$$k \Vdash A[\sigma] \iff k \Vdash A(\underline{\sigma(x_1)}/x_1, \ldots, \underline{\sigma(x_n)}/x_n),$$

where $x_1, \ldots, x_n$ are the free variables of A. We write $\mathcal{M} \models A$ if $k \Vdash A[\sigma]$ for every $k \in W$ and every assignment σ, and $\models A$ if $\mathcal{M} \models A$ for every model $\mathcal{M}$. For a set of formulas Γ, write $\Gamma \models A$ if every model satisfying all formulas in Γ also satisfies A.

Term interpretation is incorporated into the satisfaction definition. The names $\underline{d}$ denote domain elements directly and are therefore always defined, while ordinary constants are treated as 0-ary function symbols and thus follow the function symbol clause. Thus,

$$\underline{d}^k = d,$$

and, whenever defined,

$$(f_j(t_1, \ldots, t_n))^k = f_j^k(t_1^k, \ldots, t_n^k).$$

Lemma 49 (Persistence)**.** *For every closed $A \in \mathcal{L}(\mathcal{C} \cup D(k))$ and $k \leq l$,*

$$k \Vdash A \Rightarrow l \Vdash A.$$

Proof. All forcing clauses except the atomic clause for function symbols applied to non-compound terms correspond to forcing in Kripke models for **QIEL**$^=$, where persistence is already established (see [SS23]). It remains to consider the case

$$k \Vdash f_j(\underline{\bar{a}}) = \underline{d'}.$$

Then $f_j^k(\bar{a})$ is defined and

$$f_j^k(\bar{a}) \sim_k d'.$$

By rigidity,

$$f_j^l(\bar{a}) \text{ is defined and } f_j^l(\bar{a}) \sim_l f_j^k(\bar{a}).$$

By transitivity and monotonicity of $\sim$, we obtain

$$f_j^l(\bar{a}) \sim_l d',$$

and hence

$$l \Vdash f_j(\underline{\bar{a}}) = \underline{d'}.$$

□

6.2.3 The Hilbert system $\mathbf{H_2}$-$\mathbf{QTICS2}_{\Box}^{+}$

We now introduce a Hilbert-style axiom system $\mathbf{H_2}$-$\mathbf{QTICS2}_{\Box}^{+}$ for our logic. It also gives a proof system for **QIELE**$^+$, by replacing T3 by T3' and $\Box$ by K. We follow the same strategy as in [TD88]: the propositional axiom schemas and inference rules are retained, the quantifier axioms schemas are modified to incorporate the existence predicate as in E^+-logic, suitable axioms for equality and existence are added, and strictness and replacement principles are included. On top of that, we add the modal axioms.

We have the following axioms schemas (where A, B, C are formulas; t, s, t' terms; x a variable; t free for x where relevant):

($\wedge$-Ax)	$A \wedge B \to A,\ A \wedge B \to B,\ A \to (B \to (A \wedge B))$
($\vee$-Ax)	$A \to A \vee B,\ B \to A \vee B,\ (A \to C) \to ((B \to C) \to (A \vee B \to C))$
($\to$-Ax)	$A \to (B \to A),\ (A \to (B \to C)) \to ((A \to B) \to (A \to C))$
($\bot$-Ax)	$\bot \to A$

$$(\exists\text{-Ax}_2^E)\ A(t/x) \wedge Et \to \exists x A, \qquad (\forall\text{-Ax}_2^E)\ \forall x A \wedge Et \to A(t/x),$$

$$\text{(EAX)}\quad Ex, \qquad Et \leftrightarrow \exists x(x = t), \qquad \forall x(x = x),$$

$$t = s \wedge t' = s \to t = t',$$

$$\text{(STR)}\quad P_i(t_1, \ldots, t_n) \to Et_i, \qquad Ef_j(t_1, \ldots, t_n) \to Et_i \quad (1 \le i \le n),$$

$$\text{(REPL)}\quad P_i(\bar{s}) \wedge \bar{s} = \bar{t} \to P_i(\bar{t}), \qquad E(f_j\bar{s}) \wedge \bar{s} = \bar{t} \to f_j\bar{s} = f_j\bar{t},$$

$$\text{(T1) } A \to \Box A, \qquad \text{(T2) } \Box(A \to B) \to (\Box A \to \Box B), \qquad \text{(T3) } \neg\Box\bot.$$

and rules:

$$(\to E)\ \frac{A \quad A \to B}{B}, \qquad (\forall I_2)\ \frac{A \to B}{A \to \forall y\, B(y/x)}, \qquad (\exists E_2)\ \frac{B \to A}{\exists y\, B(y/x) \to A},$$

where $x \notin FV(A)$ and $y = x$ or $y \notin FV(B)$. We write $\Gamma \vdash A$ and $\vdash A$ as usual for our proof system.

We first note that the Deduction Theorem holds for our proof system. We can see this by noting that it holds for the Hilbert-style proof system for $\mathbf{IQCE}^+$ by its equivalence with the corresponding natural deduction system, as shown in [TD88]. Since our system is obtained by extending $\mathbf{IQCE}^+$ with three modal axiom schemas, the Deduction Theorem continues to hold.

6.2.4 Soundness

Lemma 50 (Term evaluation and substitution)**.** *Let M be a model, $k \in W$, σ an assignment into $D(k)$, t a term free for x in A.*

1. *If $k \Vdash Et[\sigma]$, there is a unique $d \in D(k)$, up to $\sim_k$, with $k \Vdash (t = \underline{d})[\sigma]$ (call such d the value of t at k, σ).*

2. *If d is a value of t at k, σ, then for every formula A, $k \Vdash A(t/x)[\sigma] \iff k \Vdash A[\sigma, x{:=}d]$.*

Proof. **(1)** By induction on t. If $t = v$ is a variable, then $\sigma(v)$ is the unique value (up to $\sim_k$) by the atomic equality clause for names. If $t = f_j(t_1, \ldots, t_{m_j})$, then by the existence clause $k \Vdash \exists x(f_j(t_1, \ldots, t_{m_j}) = x)$ and by the compound function clause $k \Vdash \exists x(\exists \bar{y}(\bar{y} = \bar{t} \wedge f_j(\bar{y}) = x))$. Therefore, each t_i has a value d_i by the induction hypothesis, so we get $k \Vdash \exists x(f_j^k(\bar{d}) = x)$. By the forcing clause for the existential quantifier, there is $d \in D(k)$ such that $k \Vdash f_j^k(\bar{d}) = \underline{d}$. This d is the value of t. This value is clearly unique up to $\sim_k$ by congruence of f_j^k, using the uniqueness of each d_i.

(2) By induction on A. For atomic formulas, we unfold the corresponding clauses for compound terms. The resulting conditions are identical once every occurrence of t is replaced by its value d at k, σ. This follows from part (1), which guarantees that such values are unique up to $\sim_k$. The Boolean, quantifier, and $\Box$ cases are routine by induction hypothesis, since substitution commutes with these connectives in the usual way. $\Box$

Theorem 51 (Strong Soundness)**.**

$$\Gamma \vdash A \quad \Rightarrow \quad \Gamma \models A.$$

Proof. We proceed by induction on derivations from Γ. Let M be a model, k a world, and σ an assignment, and assume that

$$k \Vdash B[\sigma]$$

for every $B \in \Gamma$. We show that $k \Vdash A[\sigma]$.

- **Assumption.** If $A \in \Gamma$, then $k \Vdash A[\sigma]$ by assumption.

- **Intuitionistic and modal axiom schemas and inference rules.** These follow from the strong soundness of $\mathbf{QIEL}^{=}$ in [SS23], since the corresponding forcing clauses are unchanged and the same axiom schemas and inference rules are present in $\mathbf{QIEL}^{=}$. The only difference is that $\mathbf{QE^{+}\text{-}TICS\text{-}2}_{\Box}$ uses T3′ instead of T3; however, since T3 and T3′ are provably equivalent, the soundness argument carries over unchanged.

- **($\exists$-$\mathbf{Ax}_2^E$).** If
$$k \Vdash (A(t/x) \wedge Et)[\sigma],$$
then t has a value d at k, σ by Lemma 50(1), and Lemma 50(2) gives $k \Vdash A[\sigma, x := d]$. Hence $k \Vdash \exists x A[\sigma]$.

- **($\forall$-$\mathbf{Ax}_2^E$).** If
$$k \Vdash (\forall x A \wedge Et)[\sigma],$$
let d be the value of t at k, σ. From $k \Vdash \forall x A[\sigma]$ we obtain $k \Vdash A[\sigma, x := d]$, and Lemma 50(2) gives $k \Vdash A(t/x)[\sigma]$.

- **EAX.** The axiom Ex is immediate since $\sigma(x) \in D(k)$. The equivalence $Et \leftrightarrow \exists x(x = t)$ follows directly from the existence clause. The validity of $\forall x(x = x)$ follows from reflexivity of $\sim_k$. For transitivity of equality,
$$t = s \wedge t' = s \to t = t',$$
values d, e, e' of t, s, t' satisfy $d \sim_k e$ and $e' \sim_k e$, hence $d \sim_k e'$ by symmetry and transitivity.

- **STR.** The clauses for $P_i(\vec{t})$ and $f_j(\vec{t})$ require values for all arguments, giving respectively
$$P_i(\vec{t}) \to Et_i \qquad \text{and} \qquad Ef_j(\vec{t}) \to Et_i.$$

- **REPL.** For
$$P_i(\bar{s}) \wedge \bar{s} = \bar{t} \to P_i(\bar{t}),$$
the values $\bar{d}, \bar{e}$ of $\bar{s}, \bar{t}$ satisfy $\bar{d} \sim_k \bar{e}$. By $\sim_k$-invariance of P_i^k, $\bar{d} \in P_i^k$ implies $\bar{e} \in P_i^k$. The function replacement axiom
$$E(f_j\bar{s}) \wedge \bar{s} = \bar{t} \to f_j\bar{s} = f_j\bar{t}$$
is analogous, using congruence of the partial functions f_j^k. □

6.2.5 Completeness

In this subsection, we define the canonical model and prove completeness. To accommodate the existence predicate, we modify the notion of saturatedness and extend the saturation lemma to ensure that the resulting theories satisfy this strengthened notion of saturation.

Definition 52 (Existentially saturated)**.** *Let C be a set of constants. A set of sentences T in the language $\mathcal{L}(C)$ is C-**existentially saturated** iff*

1. *T is consistent;*

2. $T \vdash A$ implies $A \in T$;

3. $T \vdash A \vee B$ implies $A \in T$ or $B \in T$;

4. $T \vdash \exists x\, A(x)$ implies that for some $c \in C$,

$$A(c), E(c) \in T.$$

Lemma 53 (Existential Saturation Lemma). *Suppose $T \nvdash A$, where T and A are in a language $\mathcal{L}_0$. Let*

$$C = \{c_0, c_1, c_2, \ldots\}$$

be a countable set of constants not occurring in $\mathcal{L}_0$, and let $\mathcal{L}_0(C)$ be the extension of $\mathcal{L}_0$ by C. Then there exists a C-existentially saturated theory

$$T_\omega \supseteq T$$

such that

$$T_\omega \nvdash A.$$

Proof. The proof of saturation is standard, and we refer the reader to [TD88, Lemma 6.3, p. 88] for the details. The only modification required to obtain an existentially saturated theory instead of a saturated theory concerns condition 4 of Definition 52. Specifically, when $T \vdash \exists x\, A(x)$, we require not only that $A(c) \in T$, but also that $E(c) \in T$. Consequently, only the existential case of the construction needs to be modified. We do this as follows. Whenever

$$\Gamma_{2n} \vdash \exists x A_n(x),$$

instead of defining

$$\Gamma_{2n+1} := \Gamma_{2n} \cup \{A_n(c_{g(k)})\},$$

we define

$$\Gamma_{2n+1} := \Gamma_{2n} \cup \{A_n(c_{g(k)}), E(c_{g(k)})\}.$$

It remains to show that this extension is consistent. Suppose, towards a contradiction, that

$$\Gamma_{2n} \cup \{A_n(c_{g(k)}), E(c_{g(k)})\}$$

is inconsistent. Then

$$\Gamma_{2n} \cup \{A_n(c_{g(k)})\} \vdash \neg E(c_{g(k)}),$$

so we get

$$\Gamma_{2n} \vdash A_n(c_{g(k)}) \to \neg E(c_{g(k)}).$$

Since $c_{g(k)}$ is fresh, we get

$$\Gamma_{2n} \vdash \forall x (A_n(x) \to \neg Ex).$$

Because Ex is an axiom, we have

$$\Gamma_{2n} \vdash \forall x (A_n(x) \to \bot).$$

We now show that this contradicts

$$\Gamma_{2n} \vdash \exists x A_n(x).$$

Indeed, by ($\forall$-Ax$_2^E$) and the axiom Ey, we obtain, for any y,

$$\forall x(A_n(x) \to \bot) \vdash A_n(y) \to \bot.$$

Applying ($\exists E_2$), it follows that

$$\Gamma_{2n} \vdash \exists x A_n(x) \to \bot,$$

that is,

$$\Gamma_{2n} \vdash \neg\exists x A_n(x),$$

contradicting the assumption that

$$\Gamma_{2n} \vdash \exists x A_n(x).$$

Therefore,

$$\Gamma_{2n} \cup \{A_n(c_{g(k)}), E(c_{g(k)})\}$$

is consistent. The remainder of the construction is identical to that in [TD88] and carries over without further modification. This gives the existentially saturated set of sentences T^ω. $\square$

Definition 54 (Canonical model)**.** *Let Γ_0 be any theory in $\mathcal{L}_0$. Let*

$$C_0, C_1, C_2, C_3, \ldots$$

be a countable sequence of pairwise disjoint countable sets of constants not occurring in $\mathcal{L}_0$. Define

$$C_n^* := C_0 \cup C_1 \cup \cdots \cup C_n.$$

Define $M = (W, \leq, R, D, \sim, I)$ as follows:

- $W := \{\Gamma : \Gamma \supseteq \Gamma_0,\ \mathcal{L}(\Gamma) = \mathcal{L}_0 \cup C_n^*,\ \Gamma$ *is C_n^*-existentially saturated for some* $n\}$.
- $\Gamma \leq \Omega \iff \Gamma \subseteq \Omega, \qquad \Gamma R \Omega \iff \Box^{-1}(\Gamma) \cup \Gamma \subseteq \Omega,$ *where* $\Box^{-1}(\Gamma) := \{A : \Box A \in \Gamma\}$.
- $D(\Gamma) := \{\, t \in \mathrm{Term}(\mathcal{L}(\Gamma)) : t$ *is closed and* $\Gamma \vdash Et \,\}$.
- $t \sim_\Gamma s :\iff \Gamma \vdash t = s$.
- $f_j^\Gamma(\bar{t}) := f_j(\bar{t})$.
- $P_i^\Gamma := \{\bar{t} \in D(\Gamma)^n : \Gamma \vdash P_i(\bar{t})\}$.

By construction of the existentially saturated theories Γ, for every $c \in C_n^*$ we have $Ec \in \Gamma$. However, constants already present in $\mathcal{L}_0$ need not satisfy $\Gamma \vdash Ec$, and the same holds for compound terms.

Lemma 55. *The structure M of Definition 54 is a model in the sense of Definition 47.*

Proof. We verify the required conditions.

- **Frame conditions.** Clearly $(W, \subseteq)$ is a preorder. Moreover,

$$\Gamma R \Omega \implies \Gamma \subseteq \Omega,$$

so $R \subseteq \leq$. If $\Gamma \subseteq \Omega$ and $\Omega R \Theta$, then

$$\Box^{-1}(\Gamma) \cup \Gamma \subseteq \Box^{-1}(\Omega) \cup \Omega \subseteq \Theta,$$

hence $\Gamma R \Theta$.

For seriality, suppose

$$\Gamma \cup \Box^{-1}(\Gamma) \vdash \bot.$$

This means that for some $B_1, \ldots, B_n$ with $\Box B_i \in \Gamma$,

$$\Gamma \vdash B_1 \to \cdots \to (B_n \to \bot).$$

By T1 and repeated applications of T2,

$$\Gamma \vdash \Box B_1 \to \cdots \to (\Box B_n \to \Box\bot).$$

Since each $\Box B_i \in \Gamma$, deductive closure gives $\Gamma \vdash \Box\bot$, and hence by T3, $\Gamma \vdash \bot$, contradicting consistency. Therefore

$$\Gamma \cup \Box^{-1}(\Gamma) \nvdash \bot.$$

By the existential saturation lemma, it extends to some $\Omega \in W$, and thus $\Gamma R \Omega$.

- **Domains.** If $\Gamma \subseteq \Omega$ and $t \in D(\Gamma)$, then $t \in \mathrm{Term}(\mathcal{L}(\Gamma)) \subseteq \mathrm{Term}(\mathcal{L}(\Omega))$ and $\Gamma \vdash Et$. Then $\Omega \vdash Et$ so $t \in D(\Omega)$. Hence domains are monotone. Also,

$$\Gamma \vdash t = s \implies \Omega \vdash t = s,$$

so $\sim_\Gamma$ is monotone.

- **Equivalence relations.** For $t \in D(\Gamma)$ we have $Et \in \Gamma$. From $\forall x(x = x)$ and ($\forall$-Ax_2^E) we obtain

$$\Gamma \vdash t = t,$$

so $\sim_\Gamma$ is reflexive.

If $\Gamma \vdash t = s$, then using the equality axiom

$$t = s \wedge s = s \to s = t$$

together with reflexivity gives

$$\Gamma \vdash s = t.$$

Hence $\sim_\Gamma$ is symmetric.

If $\Gamma \vdash t = s$ and $\Gamma \vdash s = u$, symmetry gives $\Gamma \vdash u = s$, and the equality axiom

$$t = s \wedge u = s \to t = u$$

gives

$$\Gamma \vdash t = u.$$

Thus $\sim_\Gamma$ is transitive.

- **Predicates.** Suppose $\bar{t} \sim_\Gamma \bar{t}'$ and $\Gamma \vdash P_i(\bar{t})$. By replacement,

$$\Gamma \vdash P_i(\bar{t}) \wedge \bar{t} = \bar{t}' \to P_i(\bar{t}'),$$

and therefore

$$\Gamma \vdash P_i(\bar{t}').$$

Hence P_i^Γ is $\sim_\Gamma$-invariant. Persistence follows immediately from $\Gamma \subseteq \Omega$.

- **Functions.** For congruence, suppose $\bar{t} \sim_\Gamma \bar{t}'$ and $f_j^\Gamma(\bar{t})$ is defined. Then

$$\Gamma \vdash Ef_j(\bar{t}).$$

Since $\bar{t} \sim_\Gamma \bar{t}'$, we have

$$\Gamma \vdash \bar{t} = \bar{t}'.$$

By the replacement axiom

$$E(f_j\bar{t}) \wedge \bar{t} = \bar{t}' \to f_j\bar{t} = f_j\bar{t}',$$

we obtain

$$\Gamma \vdash f_j(\bar{t}) = f_j(\bar{t}').$$

Thus

$$\Gamma \vdash \exists x(x = f_j(\bar{t}')),$$

so $f_j^\Gamma(\bar{t}')$ is defined. Since $\Gamma \vdash f_j(\bar{t}) = f_j(\bar{t}')$ we have

$$f_j^\Gamma(\bar{t}) \sim_\Gamma f_j^\Gamma(\bar{t}').$$

Therefore f_j^Γ is congruent.

For rigidity, suppose that $f_j^\Gamma(\bar{t})$ is defined. Then

$$\Gamma \vdash Ef_j(\bar{t}).$$

By monotonicity,

$$\Omega \vdash Ef_j(\bar{t}),$$

so $f_j^\Omega(\bar{t})$ is defined. By the definition of the canonical interpretation,

$$f_j^\Gamma(\bar{t}) := f_j(\bar{t}) =: f_j^\Omega(\bar{t}).$$

By reflexivity,

$$f_j(\bar{t}) \sim_\Omega f_j(\bar{t}),$$

and therefore

$$f_j^\Gamma(\bar{t}) \sim_\Omega f_j^\Omega(\bar{t}). \qquad \square$$

Lemma 56 (Truth Lemma)**.** *For every $\Gamma \in W$ and every formula A in the language of Γ,*

$$A \in \Gamma \iff \Gamma \Vdash A.$$

Proof. By induction on A.

- **Atomic formulas.** For $A := t = s$, immediate from definition and the fact that Γ is deductively closed.

  For $A := P_i(\bar{t})$, also immediate from definition and the fact that Γ is deductively closed.

  For $A := Et$, we prove both directions. If $\Gamma \vdash Et$, then $t \in D(\Gamma)$ by definition of the domain. Taking $y = t$, we have $\Gamma \Vdash t = t$, since $t \sim_\Gamma t$ by reflexivity. Hence $\Gamma \Vdash \exists y(y = t)$, so $\Gamma \Vdash Et$.

  Conversely, if $\Gamma \Vdash Et$, then $\Gamma \Vdash \exists y(y = t)$. Thus there is some $s \in D(\Gamma)$ with $\Gamma \Vdash s = t$. By the atomic equality case just established, $\Gamma \vdash s = t$. Since $s \in D(\Gamma)$, by $\exists$-$\mathbf{Ax}_2^E$ we get $\Gamma \vdash \exists x(x = t)$, so by the axiom $Et \leftrightarrow \exists x(x = t)$ we get $\Gamma \vdash Et$. Therefore

$$\Gamma \Vdash Et \iff \Gamma \vdash Et \iff Et \in \Gamma.$$

- **Propositional connectives.** The cases for $\wedge, \vee, \rightarrow, \bot$ are standard, using the induction hypothesis and the saturation properties of Γ.

- **Universal quantifier.** Suppose first that $\forall x A \in \Gamma$. Let $\Omega \geq \Gamma$ and $c \in D(\Omega)$. By persistence, $\forall x A \in \Omega$, and since $c \in D(\Omega)$ we also have $Ec \in \Omega$. Therefore the quantifier elimination axiom gives $A(c/x) \in \Omega$. By induction, $\Omega \Vdash A(c/x)$, and hence $\Gamma \Vdash \forall x A$.

  Conversely, suppose $\forall x A \notin \Gamma$ and Γ C_n^*-existentially saturated. Take any $c \in C_{n+1}$. If $A(c/x) \in \Gamma$ then since c is fresh, by $(\forall I_2)$ we would obtain $\forall x A \in \Gamma$. Hence we have $A(c/x) \notin \Gamma$. Extend Γ to a C_{n+1}^*-existentially saturated theory Ω such that $\Omega \nvdash A(c/x)$. Then $c \in D(\Omega)$ and, by induction hypothesis, $\Omega \nVdash A(c/x)$. Therefore $\Gamma \nVdash \forall x A$.

- **Existential quantifier.** Suppose $\exists x A \in \Gamma$. By existential saturation there is a constant $c \in C_n^*$ such that $A(c/x), Ec \in \Gamma$. The induction hypothesis gives $\Gamma \Vdash A(c/x)$, and since $c \in D(\Gamma)$ we get $\Gamma \Vdash \exists x A$.

  Conversely, suppose $\exists x A \notin \Gamma$. For every $c \in D(\Gamma)$ we have $A(c/x) \notin \Gamma$, since otherwise $Ec \in \Gamma$ together with $(\exists I^E)$ would imply $\exists x A \in \Gamma$. Hence by induction hypothesis, $\Gamma \nVdash A(c/x)$ for every $c \in D(\Gamma)$, and thus $\Gamma \nVdash \exists x A$.

- **Modal operator.** Suppose $\Box A \in \Gamma$ and $\Gamma R \Omega$. By the definition of R, $\Box A \in \Gamma$ implies $A \in \Omega$. Hence by induction hypothesis, $\Omega \Vdash A$, and therefore $\Gamma \Vdash \Box A$.

  Conversely, suppose $\Box A \notin \Gamma$. We claim that $\Gamma \cup \Box^{-1}(\Gamma) \nvdash A$. Otherwise, for some finitely many formulas from $\Box^{-1}(\Gamma)$, say $B_1, \dots, B_n$ with $\Box B_i \in \Gamma$, we would have

$$\Gamma \vdash B_1 \to \cdots \to (B_n \to A).$$

Using T1 and repeated applications of T2 gives

$$\Gamma \vdash \Box B_1 \to \cdots \to (\Box B_n \to \Box A),$$

and since each $\Box B_i \in \Gamma$, deductive closure gives $\Gamma \vdash \Box A$, contradiction. Hence $\Gamma \cup \Box^{-1}(\Gamma)$ is consistent and can be extended to an existentially saturated theory Ω such that $\Omega \nvdash A$. Then $\Gamma R \Omega$ and $A \notin \Omega$, so by induction $\Omega \nVdash A$. Therefore $\Gamma \nVdash \Box A$. $\Box$

Theorem 57 (Strong Completeness)**.**

$$\Gamma_0 \models A \quad \Longrightarrow \quad \Gamma_0 \vdash A.$$

Proof. Suppose that $\Gamma_0 \nvdash A$. Construct the canonical model M based on Γ_0 as above. By the construction, every world Δ of M extends Γ_0, and hence

$$\Gamma_0 \subseteq \Delta.$$

Therefore, by the Truth Lemma, every formula in Γ_0 is forced at every world of M, so

$$M \models \Gamma_0.$$

Now consider an existentially saturated extension Γ of Γ_0 such that $A \notin \Gamma$. Again by the Truth Lemma, since Γ is a world of M, we have

$$\Gamma \nVdash A.$$

It follows that

$$M \not\models A.$$

Therefore,

$$\Gamma_0 \not\models A.$$

Taking the contrapositive gives

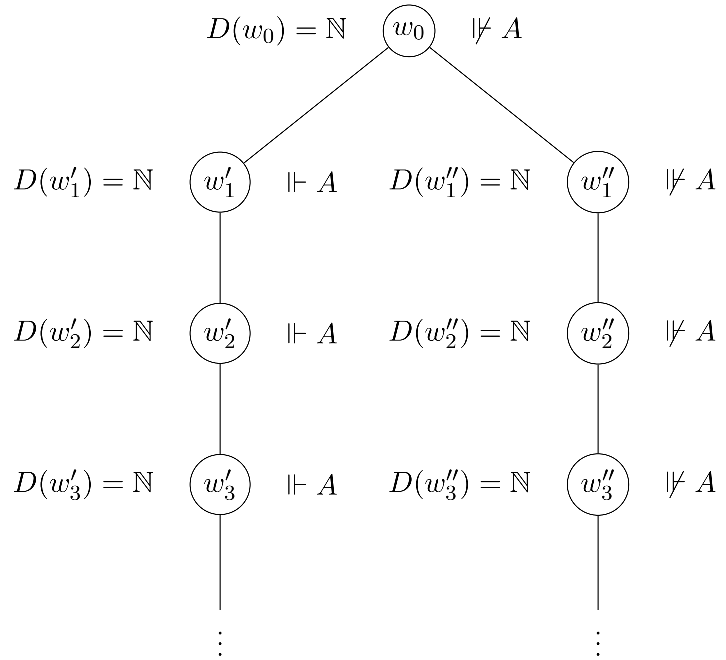
$$\Gamma_0 \models A \Longrightarrow \Gamma_0 \vdash A.$$ $\Box$

Corollary 58. *For every formula A of **QE$^+$-TICS-2**, we have* $\Gamma \vdash A \iff \Gamma \models A$.

6.3 Application of the results

We now apply this system to the original aim of formalizing Brouwer's Creating Subject arguments and the corresponding sequences arising from these arguments. The system may also have applications related to the philosophical motivations underlying **IEL**, as it provides a version of this logic enriched with an existence predicate. We discuss how our system could be useful in relation to Intuitionistic Epistemic Logic in the further work section of the next chapter. Here we focus on the CS-sequences.

The system allows us to formalize arguments showing that, for a sequence (a_n), whenever a_n exists, one can determine whether $a_n = 0$ or $a_n \neq 0$. We can also show that this does not hold without the existence assumption: the issue is that, if a_n is not yet available in the current domain, neither disjunct is necessarily justified. Equality requires the term a_n to exist and to be equal to 0, whereas inequality requires that, whenever the term comes into existence, it is not equal to 0. Thus, before a_n becomes available, we cannot always conclude that either $a_n = 0$ or $a_n \neq 0$ holds. This can be illustrated by the following counterexample:



The lines indicate the generating relation, whose transitive closure is the temporal accessibility relation R. The direction of the relation is taken downwards along the lines, and $\leq$ is the reflexive closure of R. The domain is constant and given by $\mathbb{N}$, with the identity equivalence relation. A world is said to be at stage n if it is of the form w_n' or w_n'', with w_0 being the unique world at stage 0. We consider a language with a constant symbol $\bar{n}$ for each natural number n and one function symbol a. Let a represent the CS-sequence (a_n), and write $a(\bar{i})$ for the term corresponding to the i-th element a_i of this sequence.

For a world k, write $a(\bar{i})^k$ for the interpretation of the term $a(\bar{i})$ at k. At any world at stage n, the values $a(\bar{0})^k, \ldots, a(\bar{n})^k$ are defined. These values are determined inductively: if k is a world at stage n and k' is its predecessor on the same branch, then

$$a(\bar{i})^k = a(\bar{i})^{k'} \qquad \text{for all } i < n.$$

Thus, the values assigned at earlier stages are preserved, and only the new value $a(\bar{n})^k$ is determined at stage n. It is defined according to whether A holds at k:

$$a(\bar{n})^k = \begin{cases} 0, & \text{if } k \nVdash A, \\ 1, & \text{if } k \Vdash A. \end{cases}$$

Now consider $a(\bar{1})$ at the root w_0. Since $a(\bar{1})$ is not yet defined at stage 0, we have

$$w_0 \nVdash a(\bar{1}) = 0.$$

However, we also have

$$w_0 \nVdash a(\bar{1}) \neq 0,$$

because there is an extension $w_1'' \geq w_0$ where A is not forced, and hence

$$w_1'' \Vdash a(\bar{1}) = 0.$$

Therefore,

$$w_0 \nVdash a(\bar{1}) = 0 \vee a(\bar{1}) \neq 0.$$

The same construction gives a counterexample for every $n > 0$. The case $n = 0$ can be treated by shifting the construction by one stage: we let $a(\bar{0})$ remain undefined at the root w_0, and define it only at the successor worlds w_1' and w_1'', according to whether A holds there. Hence, for every $n \in \mathbb{N}$, there is a tree model in which

$$a(\bar{n}) = 0 \vee a(\bar{n}) \neq 0$$

is not forced at the root.

We now show how to express that, after a sufficient number of temporal steps, a term $a(\bar{n})$ is defined and that we can therefore reason about its value. We work in the system **QE$^+$-TICS-2$_\Box$** and consider a language containing constant symbols $\bar{n}$ for each $n \in \mathbb{N}$, one function symbol a, and a unary predicate N. The theory T contains the following sentences. First, we include the sentences

$$N(\bar{n})$$

for every $n \in \mathbb{N}$. We also include the sentence expressing decidability of equality between objects satisfying N:

$$\forall x \forall y \big((N(x) \wedge N(y)) \to (x = y \vee x \neq y) \big).$$

Furthermore, we include the sentence

$$\forall x \big(N(x) \to (E(a(x)) \to N(a(x))) \big),$$

ensuring that whenever $a(x)$ is defined for a natural number x, its value is again a natural number. Since equality is decidable for natural numbers, we obtain

$$\forall x \big(N(x) \to (E(a(x)) \to (a(x) = \bar{0} \vee a(x) \neq \bar{0})) \big).$$

To obtain the temporal extension $T_\Box$, we add, for every $n \in \mathbb{N}$, the sentence

$$\Box_n E(a(\bar{n})).$$

Therefore, for every $n \in \mathbb{N}$, we may conclude

$$\Box_n (a(\bar{n}) = \bar{0} \vee a(\bar{n}) \neq \bar{0}).$$

The conclusion follows from Proposition 27 and axiom T2.

Likewise, we can extend the system **QE$^+$-TICS-2**$_\Box$ to **QE$^+$-TICS-2** by adding formulas of the form $\bigcirc A$ and the axioms T4, T5 and T6. To obtain the temporal extension $T_\bigcirc$, we add the sentence

$$\forall x\big(N(x) \to \bigcirc E(a(x))\big)$$

to T. This theory proves

$$\bigcirc(a(\bar{n}) = \bar{0} \vee a(\bar{n}) \neq \bar{0}).$$

This follows from

$$\vdash_{\mathbf{TICS\text{-}2}} (A \to B) \to (\bigcirc A \to \bigcirc B),$$

as shown in Proposition 72 in Appendix A.

We note that we have not developed the semantics of **QE$^+$-TICS-2** in this chapter; the discussion above concerns only the syntactic system. Extending the semantics to this system is relatively straightforward, as we discuss in the next chapter. The main difficulty lies not in defining the semantics, but in establishing completeness.

These examples provide a first indication of how the system can be used in our philosophical project. We do not develop the formalization of these arguments in full detail here; rather, they illustrate how the combination of temporal operators and the existence predicate can capture the distinctions relevant to Niekus' interpretation of Creating Subject sequences. Further development of these applications and their formal details remains a topic for future work, as will be discussed in the next chapter.

Chapter 7

Conclusion

In this concluding chapter, we summarize the principal results of the thesis, briefly discuss the relation between our choice of **TICS-2** and the original logic **TICS**, and outline several directions for future work on our intuitionistic modal logic, including possible extensions of the results for the $\Box$-fragment to the full language.

7.1 Results

In this thesis, we introduced **TICS-2**, an intuitionistic modal logic that combines the $\Box$-operator with the least fixed point operator $\bigcirc$, which can be defined as $\bigcirc A = \mu p(A \vee \Box p)$. A formula $\Box A$ expresses that A holds from the next stage onward, regardless of which successor stage is reached. A formula $\bigcirc A$ expresses eventuality along all possible continuations: regardless of the path taken, a stage will eventually be reached from which onward A holds. These temporal operators are intended to represent the future activity of the Creating Subject, the (idealized) mathematician who proves mathematical statements through time.

The system is closely related to **TICS**, the system from which we started, which was developed with the same goal. **TICS** is strictly stronger than **TICS-2**, differing only by the inclusion of axiom 7': $\Box B \to (A \vee (A \to B))$. In Section 4.3, we showed that both systems provide an alternative to the standard formalization known as the Theory of the Creating Subject (**TCS**), which consists of the principles CS1, CS2, CS3, and CS4 and uses the operator $\Box_n$. In both **TICS** and **TICS-2**, CS1 and CS4 are not valid, whereas CS2 and CS3 are. Moreover, both systems include the additional temporal operator $\bigcirc$, which is genuinely more expressive than any iteration of the $\Box$-operator alone. Consequently, both systems provide suitable frameworks for capturing a richer temporal structure for reasoning about the Creating Subject's activity, in accordance with Niekus' proposal. The only difference between the two systems is the presence of axiom 7' in **TICS**. Since none of the Creating Subject applications considered in this thesis relies on axiom 7', **TICS-2** provides a weaker alternative to **TICS** that is sufficient for the purposes considered here.

We established soundness and completeness of **TICS-2** with respect to its Kripke semantics. Furthermore, we gave a constructive soundness and completeness proof for the $\Box$-fragment of **TICS-2** with respect to modal cover semantics. We extended this fragment to first-order logic with equality and an existence predicate, calling the resulting system $\mathbf{QE^{+}\text{-}TICS\text{-}2_{\Box}}$, and again established soundness and completeness for this first-order extension with respect to Kripke semantics. We briefly showed how this system can be used in formalizing Brouwer's Creating Subject arguments. For all logics considered, we worked with a Hilbert-style proof system.

We observed that the $\Box$-fragment of **TICS-2** corresponds exactly to Intuitionistic Epistemic Logic (**IEL**). As a consequence, the results obtained for the $\Box$-fragment can be viewed as results for **IEL**, and vice versa. In particular, by developing the theory for **TICS-2**$_\Box$, we constructively obtained sound and complete modal cover semantics for **IEL**. Furthermore, **QE**$^+$**-TICS-2**$_\Box$ provides a first-order extension of **IEL** with equality and an explicit existence predicate (**QIELE**$^+$), based on an existing first-order version of **IEL**. We identified, in addition to the technical connections, a philosophical link between the two frameworks: the $\Box$-operator in **TICS-2**, which describes statements that will be proved by the Creating Subject at future stages, plays a role analogous to the knowledge operator K in **IEL**, understood as describing verified statements for which a constructive proof is not necessarily currently available. This connection is already reflected in the literature: one of the applications of **QIEL** is precisely related to an interpretation of the Creating Subject. This parallel can be made even more precise in the setting of **QIELE**$^+$, as we discuss in the last section of this chapter. We thus see that our work also contributes to the study of **IEL**.

7.2 TICS instead of TICS-2

In our development of **TICS-2**, we made the choice to separate the temporal accessibility relation R from the information growth relation $\leq$. From a Brouwerian perspective, however, one may argue that every genuine increase in information should correspond to a temporal transition, since time itself is understood as the "falling apart of a life moment into two distinct things", as described in Brouwer's first act of intuitionism. On this interpretation, R should be identified with the strict part of $\leq$, so that any strict increase in information, viewed as an instance of *two-ity*, corresponds to a temporal step. This is precisely the choice made in **TICS**.

For those who favour such a setup, the original **TICS** remains available. However, as we have seen, identifying R with the strict part of $\leq$ results in the additional axiom $\Box B \to (A \vee (A \to B))$. It would therefore be necessary to investigate whether the results for the $\Box$-fragment remain valid once this axiom is added, resulting in **TICS**$_\Box$. For the soundness and completeness results with respect to Kripke semantics, the corresponding adaptation should be straightforward. For the results based on modal cover semantics, however, it remains to be investigated whether the existing arguments continue to apply and whether the overall framework remains suitable in this setting. Moreover, the correspondence with **IEL** would no longer hold, since **IEL** does not validate the principle $KB \to (A \vee (A \to B))$. Consequently, the results obtained via this connection would no longer apply directly.

One could investigate whether there are other interesting connections to draw, or applications to find, for the $\Box$-fragment of **TICS**. Directly adding the axiom $KB \to (A \vee (A \to B))$ to **IEL** would not match the intended interpretation of that system, since the principle would express that having a verification of B already provides, for every statement A, either a proof of A or a proof of $A \to B$. Other systems may, however, provide a more natural point of comparison. For example, the modalized Heyting calculus (**mHC**), as described in [Esa06], already closely resembles the $\Box$-fragment of **TICS**: it extends **IPC** by precisely the axioms 1', 2', and 7' of **TICS**. This system is also used as the basis for the temporal Heyting calculus (**tHC**), where a backward-looking (existential) diamond is added alongside the (universal) box. The $\Box$-fragment of **tHC** thus closely resembles the $\Box$-fragment of our **TICS**, and the corresponding semantics are also very similar, in the sense that the temporal relation R is again taken to be the strict version of the intuitionistic relation $\leq$. The only difference is that **mHC** lacks the axiom $\neg\Box\bot$. It would be worth investigating what adding $\neg\Box\bot$ would amount to with respect to the existing results on **mHC**; given the temporal interpretation the system already supports, this addition would presumably not cause any difficulties. There is, moreover, existing work on modal algebras and duality theory for **mHC**, in terms of frontal Heyting algebras and a corresponding adjusted Esakia duality, as described in [CSS10], which could provide further interesting directions for future research on **TICS**.

In this thesis, however, we have focused on **TICS-2**, and therefore in the next section of this chapter we discuss directions for further work on that logic instead.

7.3 Further work

The results of this thesis leave open several directions for further work, which we group into four themes: extending the constructive and first-order results for the $\Box$-fragment to the full logic **TICS-2**; developing a proof theory for the systems considered here; using this framework to develop the theory of choice sequences, as suggested by Niekus; and broadening the scope of the framework beyond its original motivation of formalizing Brouwer's Creating Subject arguments.

7.3.1 Extending the $\Box$-fragment results to TICS-2

The results obtained in the previous two chapters concern only the $\Box$-fragment of **TICS-2**. A natural next step is therefore to extend these results to the full logic by incorporating the temporal operator $\bigcirc$. This involves two separate extensions: obtaining a constructive soundness and completeness proof for the full propositional system, and extending the soundness and completeness results for full **TICS-2** to the first-order setting with equality and the existence predicate. These extensions are non-trivial and have not been pursued in this thesis. In this section, we indicate the main difficulties that arise, as well as the aspects that can nevertheless be established.

For the constructive completeness of full **TICS-2**, one possible approach is to extend Valliappan's modal cover framework. Valliappan himself observes that his framework naturally extends to multiple modalities by generalizing the approach used for multi-modal systems in Goldblatt's framework. In the present setting, this would amount to associating a modal cover relation with each modality: $\blacktriangleleft_{\Box}$ for the $\Box$-operator and $\blacktriangleleft_{\bigcirc}$ for the $\bigcirc$-operator. Imposing suitable conditions on these modal covering relations would then ensure that the two modalities behave in the intended way. For $\blacktriangleleft_{\Box}$, we have already seen that, in addition to modal refinement and modal localization, modal identity, modal confluence and modal emptiness manage to capture the $\Box$-behaviour. For $\blacktriangleleft_{\bigcirc}$, in addition to modal refinement and modal localization, we would at least need modal identity, corresponding to T4. Moreover, we would need further conditions relating $\blacktriangleleft_{\Box}$ and $\blacktriangleleft_{\bigcirc}$ in order to validate the interaction axioms T5 and T6 of **TICS-2**. While these conditions appear feasible in principle, the axiom T6 presents an additional challenge. It is not immediate how to formulate a modal condition that both ensures soundness of the axiom and is guaranteed to hold in the canonical model by the axiom itself. This may be related to the fact that T6 reflects the least fixed point, and hence inductive, character of the operator.

A more natural approach may therefore be to exploit the least fixed point characterization of $\bigcirc$.

As shown earlier, the axioms of **TICS-2** characterize $\bigcirc A$ as the least fixed point of the operator

$$F(p) = A \vee \Box p.$$

Rather than introducing a separate modal cover relation for $\bigcirc$, one could therefore define $\blacktriangleleft_{\bigcirc}$ inductively using $\blacktriangleleft$, the modal covering relation for $\Box$. The validity of axioms T4, T5, and T6 would then follow constructively from the inductive definition of $\blacktriangleleft_{\bigcirc}$ and the properties of a modal cover model. More precisely, one could define the relation $\blacktriangleleft_{\bigcirc}$ inductively from $\blacktriangleleft$ and $\triangleleft$ by the following rules:

$$\frac{}{w \blacktriangleleft_{\bigcirc} \{w\}} \qquad \frac{w \blacktriangleleft \beta \quad \forall q \in \beta.\, q \blacktriangleleft_{\bigcirc} \alpha_q}{w \blacktriangleleft_{\bigcirc} \bigcup_{q \in \beta} \alpha_q} \qquad \frac{w \triangleleft \alpha \quad \forall v \in \alpha.\, v \blacktriangleleft_{\bigcirc} \gamma_v}{w \blacktriangleleft_{\bigcirc} \bigcup_{v \in \alpha} \gamma_v}$$

The final clause in this definition ensures that the truth set of $\bigcirc A$ is a localized up-set in any modal cover model. The canonical model construction would then only need to produce the modal covering relation for $\Box$, with the interpretation of $\bigcirc$ obtained through the inductive construction above. The main remaining challenge is to establish the Truth Lemma in this setting.

This approach also connects naturally with the algebraic semantics associated with cover semantics. In particular, the correspondence between (modal) cover systems and algebraic semantics developed in [Gol11; Val26; Rog21] suggests that the $\bigcirc$-operator could be treated as an additional least fixed point operator on the algebra of localized up-sets. This would align with the order-theoretic techniques underlying constructive completeness proofs for cover semantics and may provide a route towards constructive completeness for the full **TICS-2** system. Moreover, it may also provide a basis for extending these results to the first-order setting. In particular, [Rog21] gives a constructive algebraic completeness proof for predicate extensions of **IEL**. It would therefore be interesting to investigate whether similar techniques apply to the present setting once the least fixed point behaviour of $\bigcirc$ is incorporated into the algebraic semantics, potentially leading to completeness results for first-order extensions of full **TICS-2**.

For the first-order extension, not only the constructive completeness result but also a non-constructive completeness result for the full system remains to be found. Again, for the $\Box$-fragment, the situation is already well understood. As mentioned before, and as used extensively in the previous chapter, sound and complete Kripke semantics have already been developed for the first-order extensions of **TICS-2**$_{\Box}$, namely in the form of **QIEL** and **QIEL**$^{=}$. A natural next step would therefore be to extend these results to the full system by incorporating the $\bigcirc$-operator. Syntactically, this amounts to adding formulas of the form $\bigcirc A$ together with the axioms T4, T5 and T6. Semantically, one would use the same Kripke models with growing domains, extending the forcing relation by

$$\begin{aligned} k \Vdash \bigcirc A \iff\ & \text{for every infinite chain } k = k_0 R k_1 R k_2 R \cdots, \\ & \text{there exists } n \in \mathbb{N} \text{ such that } k_n \Vdash A. \end{aligned}$$

Soundness of this extension is immediate, since it follows directly from the soundness of T4, T5 and T6 over the underlying propositional Kripke frames. Moreover, in the propositional setting we already saw that the addition of the $\bigcirc$-operator when passing from **TICS-2**$_{\Box}$ to **TICS-2** only adds a clause to the forcing relation and does not require any change to the underlying Kripke semantics: both systems, **IEL** and **TICS-2**, are sound and complete with respect to the same class of frames. It is therefore natural to expect that the same class of first-order frames should suffice when extending from **QIEL** to the first-order version of **TICS-2**.

Proving this completeness, however, presents a challenge. The difficulty lies in the combination of the $\bigcirc$-operator and the quantifiers. In the propositional setting, the countermodel construction relied crucially on the existence of a finite set of formulas, namely Φ_E, over which the step-by-step construction was performed. This construction ensured that the modal requirements imposed by the $\Box$- and $\bigcirc$-formulas were satisfied. By restricting the construction to this finite set, we could ensure that all formulas required for the Truth Lemma were eventually accounted for. This finite adequate set is no longer available in the first-order setting, since Φ_E would need to be closed under infinitely many substitution instances for the Truth Lemma to go through in the quantifier cases. Consequently, it is no longer possible to choose Φ_E to be finite.

For the $\Box$-fragment, this difficulty does not arise, since the first-order completeness proof uses a Scott-style canonical model construction, as discussed in the previous chapter, for which a finite Φ_E is not required. At present, it is not clear how this construction could be adapted to the setting with the $\bigcirc$-operator, and we leave this as a direction for future research.

Once soundness and completeness with respect to Kripke semantics with growing domains have been established for first-order **TICS-2**, equality can be added in the standard way. The resulting first-order logic with equality can then be extended with an existence predicate by applying the same strategy used in the previous chapter for $\mathbf{QE^+}$**-TICS-2**$_\Box$, following the method of Troelstra and Van Dalen. This results in a first-order version of full **TICS-2** with equality and an existence predicate, which we call $\mathbf{QE^+}$**-TICS-2**.

7.3.2 Proof theory

The present thesis develops Hilbert-style axiomatizations for the logics considered in this thesis, but does not investigate their proof theory in further detail. This leaves open several proof-theoretic questions. A natural starting point is provided by the existing proof-theoretic results for **IEL**, as discussed above. In particular, sequent calculi with desirable properties are already available for the $\Box$-fragment of **TICS-2**, as well as for its first-order extensions, both with and without equality. These results could serve as a basis for investigating corresponding proof systems for the full language of **TICS-2**. As noted throughout this thesis, the $\bigcirc$-operator can be characterised as a least fixed point operator. Techniques from proof theory for fixed point logics may therefore be useful for obtaining a suitable sequent calculus for full **TICS-2**. One possible starting point is the work of [Afs+24], who develop a cyclic proof system for an intuitionistic modal logic with a fixed point operator, called the *master modality*. This modality arises as the *greatest* fixed point of the propositional operator

$$p \mapsto A \wedge \Box p.$$

This is closely related to the present setting, with a dual situation arising here from the fact that we work with the *least* fixed point

$$p \mapsto A \vee \Box p.$$

Investigating whether similar techniques can be adapted to the $\bigcirc$-operator provides a natural direction for developing a proof system for **TICS-2**, and, if successful, such techniques might extend to the first-order setting as well.

A further direction would be to develop a sequent calculus for the first-order extension of **TICS-2**$_\Box$

with equality and an existence predicate. The sequent calculus for $\mathbf{QIEL}^{=}$ developed in [SS23] provides a basis for this development, into which an existence predicate could be incorporated. One could build here on existing work on sequent calculi with an existence predicate, such as the approach of Baaz and Iemhoff [BI06], who give a Gentzen calculus for intuitionistic logic with an existence predicate (without equality), or the more recent work of [Ind21]. If suitable techniques for handling the $\bigcirc$-operator as a least fixed point operator are developed along the lines discussed above, these may eventually also provide a route towards obtaining a sequent calculus for $\mathbf{QE}^{+}$**-TICS-2**.

Another direction for future work is to investigate the decidability and computational complexity of the logics developed in this thesis. For **TICS**, we can note that the countermodel construction for an unprovable formula E already essentially produces a finite model: the constructed tree model is formally infinite, but along any branch there is a stage at which all requirements arising from formulas in the finite set Φ_E have been satisfied, after which the current node is used repeatedly to construct the remainder of the infinite branch, so that the tree contains only finitely many genuinely distinct worlds. As pointed out to the author by Dick de Jongh, this can be made precise by taking Φ_E to serve as a *finite adequate set* for the construction.

Finite adequate sets are a standard technique for establishing the finite model property (FMP), see [Sol76]. The basic idea is that, given a formula E, one identifies a finite set Σ_E of formulas containing all the information relevant to the countermodel construction, and then reruns the construction using, instead of full DP-theories, only "Σ_E-DP-theories": sets $\Delta \subseteq \Sigma_E$ that are consistent, deductively closed within Σ_E and have the disjunction property with respect to disjunctions using formulas in Σ_E. Since Σ_E is finite, there are only finitely many such Σ_E-DP-theories, so the model built from them is finite. For **TICS**, it should be easy to rerun the countermodel construction in this way, making the observation above precise and giving the FMP. Together with the finite axiomatization of the logic over the decidable **IPC**, this establishes that provability in **TICS** is decidable. The computational complexity of the resulting decision procedure could then be investigated. For **TICS-2**, a similar strategy suggests itself. For the $\Box$-fragment of **TICS-2**, existing decidability and complexity results for **IEL** transfer directly.

7.3.3 Connection to the theory of choice sequences

The framework developed in this thesis can be helpful in developing the theory of choice sequences, following Niekus' proposal. This perspective is motivated by Niekus' observation that the sequences appearing in Brouwer's Creating Subject arguments can be understood as choice sequences, as reflected in his use of the term *Theory of Individual Choice Sequences*. Since the Creating Subject arguments provide examples of such sequences in Brouwer's work, they naturally served as the starting point for the present development. However, the resulting framework is not restricted to these particular examples; it can be used more generally, within the theory of choice sequences, to reason about individual sequences.

An indication of this broader applicability is already provided by the interaction between the temporal operator $\bigcirc$ and the existence predicate E. Together, they make it possible to reason about objects or information that become available only at later stages, without requiring a predetermined stage at which this occurs. This goes beyond the Creating Subject arguments considered in this thesis, where the value of the nth term is determined at stage n. In general, however, an individual choice sequence is simply an open-ended element of a spread, and there is no reason to assume that the stage at which a particular value becomes available is fixed in advance, only that it will *eventually* happen.

A natural direction for future research is therefore to see how one can incorporate the present framework into formalizations of choice sequences to account for *individual* choice sequences. Such an addition could investigate which principles are characteristic of working with individual choice sequences, independently of the specifics of the Creating Subject arguments.

7.3.4 The existence predicate in IEL and further applications

Except for expanding the original project, new applications of the developed logics could also be explored. One such direction concerns combining the existence predicate with **IEL**. In extending **TICS-2**$_\Box$ to **QE**$^+$**-TICS-2**$_\Box$, we also obtained, as a byproduct, a first-order extension of **IEL** with equality and an existence predicate, which we called **QIELE**$^+$.

A connection between **IEL** and the Creating Subject was made in [SS23], where Kripke models with growing domains are used to describe the stock of statements proved and verified by the Creating Subject at each stage. In Van Dalen's original description of this idealized mathematician, however, the stock of defined mathematical objects also plays a role. This aspect can be formalized using our Kripke models for **QIELE**$^+$ by taking the terms satisfying Et to represent the mathematical objects that have been defined at a given stage.

It may also be that the existence predicate is useful in connection with the general motivation of first-order **IEL**. As discussed earlier, the principle $K(\exists x\, S(x)) \to \exists x\, S(x)$ fails, since knowledge that an object satisfying S exists does not necessarily provide a nameable witness. An explicit existence predicate allows one to distinguish between merely knowing that such an object exists and having a witness in the form of a term that denotes it. This may provide a more refined analysis of knowledge of existential claims and the conditions under which the reflection principle holds for existential statements. It would be worthwhile to explore whether this extension with an existence predicate supports further applications analogous to those developed for **IEL**, such as its connection to distributed knowledge in multi-agent settings, and to broader questions concerning epistemic concepts in intuitionistic settings.

Beyond these connections to intuitionistic epistemic logic, it would also be of interest to investigate further applications of **QE**$^+$**-TICS-2**$_\Box$ in other contexts. Since this system combines an explicit existence predicate with a $\Box$-operator whose intended reading is temporal rather than epistemic, it may be of independent interest for reasoning about the coming into existence of objects over time. This perspective may be relevant for settings in which the domain of discourse develops over time and new objects or states become available, including computational and constructive settings. Such applications would make the system useful beyond its original motivation via the Creating Subject and its correspondence with **IEL**.

The same question arises for the full logic **TICS-2**. For instance, it would be interesting to investigate whether the modal operator $\bigcirc$ has natural applications in intuitionistic epistemic logic, in addition to its temporal reading. More generally, the correspondence between **TICS-2**$_\Box$ and **IEL** shows that the operators of **TICS-2** can admit interpretations beyond the temporal perspective considered in this thesis. It would therefore be worthwhile to explore further applications of **TICS-2** and the range of interpretations supported by its operators.

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Appendix A

Some syntactic results for TICS and TICS-2

In this appendix, we present some syntactic results concerning **TICS** and **TICS-2**. First, we show that the two axiomatizations of **TICS** used in the thesis are equivalent. We then derive several theorems of **TICS-2** using this equivalence.

A.1 Equivalence TICS axiomatizations

Let $\mathcal{B} = \{B1, \ldots, 11\}$ be the original axiomatization of **TICS**, given by

B1 $A \to \bigcirc A$

B2 $\Box \bigcirc A \to \bigcirc A$

B3 $(A \to B) \to ((\Box B \to B) \to (\bigcirc A \to B))$

4 $(A \to B) \to (\bigcirc A \to \bigcirc B)$

5 $\bigcirc \bigcirc A \to \bigcirc A$

6 $\Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B)$

7 $(A \to B) \to (\Box A \to \Box B)$

8 $\Box(A \to B) \to (\bigcirc A \to \bigcirc B)$

9 $\Box(A \to B) \to (A \vee (A \to B))$

10 $\bigcirc A \to \Box \bigcirc A$

11 $\bigcirc A \to \neg\neg A$

and let $\mathcal{P} = \{1', \ldots, 7'\}$ the alternative one, given by

1' $A \to \Box A$

2' $\Box(A \to B) \to (\Box A \to \Box B)$

3' $\neg\Box\bot$

4' $A \to \bigcirc A$

5' $\Box \bigcirc A \to \bigcirc A$

6' $(A \to B) \to ((\Box B \to B) \to (\bigcirc A \to B))$

7' $\Box B \to (A \vee (A \to B))$

We show that the two axiomatizations are equivalent over **IPC**.

A.1.1 $\mathcal{P} \vdash \mathcal{B}$

Observe first that B1=$4'$, B2=$5'$, and B3=$6'$, so these axioms are common to both systems. In the rest of this subsection, we show that the remaining axioms in $\mathcal{B}$ are derivable from $\mathcal{P}$ over **IPC**. Whenever "IPC" appears in a derivation, it indicates that an appropriate theorem of **IPC** is applied to the relevant formulas.

Lemma 59 (Necessitation)**.** *For any **TICS**-formula A, if $\mathcal{P} \vdash A$, then $\mathcal{P} \vdash \Box A$.*

Proof. Since $A \to \Box A$ is an instance of $1'$, if $\mathcal{P} \vdash A$, modus ponens gives $\mathcal{P} \vdash \Box A$. $\Box$

Proposition 60 (Axiom **7**)**.**

$$\mathcal{P} \vdash (A \to B) \to (\Box A \to \Box B).$$

Proof.

1	$(A \to B) \to \Box(A \to B)$	(1')
2	$\Box(A \to B) \to (\Box A \to \Box B)$	(2')
3	$(A \to B) \to (\Box A \to \Box B)$	(IPC 1,2)

$\Box$

Proposition 61 (Axiom **6**)**.**

$$\mathcal{P} \vdash \Box(A \wedge B) \leftrightarrow (\Box A \wedge \Box B).$$

Proof. We show $\mathcal{P} \vdash \Box(A \wedge B) \to (\Box A \wedge \Box B)$ and $(\Box A \wedge \Box B) \to \Box(A \wedge B)$ separately.
($\Rightarrow$)

1	$A \wedge B \to A$	(IPC)
2	$(A \wedge B \to A) \to (\Box(A \wedge B) \to \Box A)$	(Proposition 60)
3	$\Box(A \wedge B) \to \Box A$	(MP 1,2)

Similarly,

$$\mathcal{P} \vdash \Box(A \wedge B) \to \Box B.$$

Hence, by **IPC**,

$$\mathcal{P} \vdash \Box(A \wedge B) \to (\Box A \wedge \Box B).$$

(⇐)

1	$A \to (B \to A \wedge B)$	(IPC)
2	$\Box(A \to (B \to A \wedge B))$	(Necessitation)
3	$\Box(A \to (B \to A \wedge B)) \to (\Box A \to \Box(B \to A \wedge B))$	(2')
4	$\Box A \to \Box(B \to A \wedge B)$	(MP 2,3)
5	$\Box(B \to A \wedge B) \to (\Box B \to \Box(A \wedge B))$	(2')
6	$\Box A \to (\Box B \to \Box(A \wedge B))$	(IPC 4,5)
7	$(\Box A \wedge \Box B) \to \Box(A \wedge B)$	(IPC 7)

$\square$

Proposition 62 (Axiom **4**)**.**

$$\mathcal{P} \vdash (A \to B) \to (\bigcirc A \to \bigcirc B).$$

Proof.

1	$B \to \bigcirc B$	(4')
2	$\Box(B \to \bigcirc B)$	(Necessitation)
3	$\Box(B \to \bigcirc B) \to (\Box B \to \Box \bigcirc B)$	(2')
4	$\Box B \to \Box \bigcirc B$	(MP 2,3)
5	$\Box \bigcirc B \to \bigcirc B$	(5')
6	$\Box B \to \bigcirc B$	(IPC 4,5)
7	$A \to \Box A$	(1')
8	$\Box(A \to B) \to (\Box A \to \Box B)$	(2')
9	$(A \to B) \to (A \to \Box B)$	(IPC 7,8)
10	$(A \to B) \to (A \to \bigcirc B)$	(IPC 6,9)
11	$(A \to \bigcirc B) \to ((\Box \bigcirc B \to \bigcirc B) \to (\bigcirc A \to \bigcirc B))$	(6')
12	$(A \to \bigcirc B) \to (\bigcirc A \to \bigcirc B)$	(IPC 5,11)
13	$(A \to B) \to (\bigcirc A \to \bigcirc B)$	(IPC 10,12)

$\square$

Proposition 63 (Axiom **5**)**.**

$$\mathcal{P} \vdash \bigcirc \bigcirc A \to \bigcirc A.$$

Proof.

1	$\bigcirc A \to \bigcirc A$	(IPC)
2	$(\bigcirc A \to \bigcirc A) \to ((\Box \bigcirc A \to \bigcirc A) \to (\bigcirc \bigcirc A \to \bigcirc A))$	(6')
3	$(\Box \bigcirc A \to \bigcirc A) \to (\bigcirc \bigcirc A \to \bigcirc A)$	(MP 1,2)
4	$\Box \bigcirc A \to \bigcirc A$	(5')
5	$\bigcirc \bigcirc A \to \bigcirc A$	(MP 3,4)

$\square$

Proposition 64 (Axiom **8**)**.**

$$\mathcal{P} \vdash \Box(A \to B) \to (\bigcirc A \to \bigcirc B).$$

Proof.

1	$\Box(A \to B)$	(assumption)
2	$\Box A \to \Box B$	(2', 1)
3	$A \to \Box A$	(1')
4	$A \to \Box B$	(IPC 2,3)
5	$B \to \bigcirc B$	(4')
6	$\Box B \to \Box \bigcirc B$	(Axiom 7,5)
7	$\Box \bigcirc B \to \bigcirc B$	(5')
8	$\Box B \to \bigcirc B$	(IPC 6,7)
9	$A \to \bigcirc B$	(IPC 4,8)
10	$(A \to \bigcirc B) \to (\bigcirc A \to \bigcirc B)$	(6')
11	$\bigcirc A \to \bigcirc B$	(MP 9,10)
12	$\Box(A \to B) \to (\bigcirc A \to \bigcirc B)$	(IPC 1,11; discharge 1)

□

Proposition 65 (Axiom **9**).

$$\mathcal{P} \vdash \Box(A \to B) \to (A \vee (A \to B)).$$

Proof.

1	$\Box(A \to B) \to (A \vee (A \to (A \to B)))$	(7')
2	$(A \to (A \to B)) \to (A \to B)$	(IPC)
3	$(A \vee (A \to (A \to B))) \to (A \vee (A \to B))$	(IPC 2)
4	$\Box(A \to B) \to (A \vee (A \to B))$	(IPC 1,3)

□

Proposition 66 (Axiom **10**).

$$\mathcal{P} \vdash \bigcirc A \to \Box \bigcirc A.$$

Proof. This is the instance of $1'$ obtained by substituting $\bigcirc A$ for A. □

Proposition 67 (Axiom **11**).

$$\mathcal{P} \vdash \bigcirc A \to \neg\neg A.$$

Proof.

1	$(A \to \bot) \to ((\Box\bot \to \bot) \to (\bigcirc A \to \bot))$	(6')
2	$\Box\bot \to \bot$	(3')
3	$(A \to \bot) \to (\bigcirc A \to \bot)$	(IPC 1,2)
4	$\bigcirc A \to \neg\neg A$	(IPC 3)

□

Theorem 68. *Every axiom of $\mathcal{B}$ is derivable from $\mathcal{P}$.*

Proof. The axioms B1, B2, and B3 coincide with $4'$, $5'$, and $6'$, respectively. The remaining axioms are established in the preceding Propositions 60-67. □

A.1.2 $\mathcal{B} \vdash \mathcal{P}$

We show that every axiom of $\mathcal{P}$ is valid on the class of CS-frames. Since $\mathcal{B}$ is sound and complete with respect to this class, validity of the axioms $1'$-$7'$ implies their derivability from $\mathcal{B}$.

Proposition 69. *For every CS-frame,*

$$1', \ldots, 7'$$

are valid.

Proof. Validity of $4', 5'$ and $6'$ follows from the fact that they correspond to B1, B2 and B3 respectively, which are axioms in $\mathcal{B}$, so we know they are valid on any CS-frame. For the other axioms, fix a CS-Kripke model $M = \langle W, \leq, [\![\cdot]\!] \rangle$ based on a CS-frame. We use persistence from Lemma 3 throughout.

$1'$: $A \to \Box A$. Let $w \models A$ and let $v > w$. By persistence, since $w \leq v$, we have $v \models A$. Hence every strict successor of w satisfies A, so $w \models \Box A$. Thus $w \models A \to \Box A$.

$2'$: $\Box(A \to B) \to (\Box A \to \Box B)$. Let $w \models \Box(A \to B)$ and let $v \geq w$ satisfy $v \models \Box A$. We show that $v \models \Box B$. Let $u > v$. Since $w \leq v < u$, we have $u > w$, and hence $u \models A \to B$ by $w \models \Box(A \to B)$. Moreover, $u \models A$ because $v \models \Box A$. Therefore, $u \models B$. Since $u > v$ was arbitrary, $v \models \Box B$, and hence $w \models \Box(A \to B) \to (\Box A \to \Box B)$.

$3'$: $\neg\Box\bot$. Let w be arbitrary. Since every point of a CS-frame has a successor, choose $v > w$. By definition, $v \not\models \bot$. Therefore, $w \not\models \Box\bot$, and hence $w \models \neg\Box\bot$.

$7'$: $\Box B \to (A \vee (A \to B))$. Let $w \models \Box B$. If $w \models A$, then immediately $w \models A \vee (A \to B)$. Suppose instead that $w \not\models A$. We show $w \models A \to B$. Let $v \geq w$ and assume $v \models A$. Since $w \not\models A$, we cannot have $v = w$, because then $w \models A$. Hence $v > w$. By $w \models \Box B$, it follows that $v \models B$. Therefore every extension of w satisfying A also satisfies B, so $w \models A \to B$. Hence in either case, $w \models A \vee (A \to B)$. Therefore all axioms $1'$–$7'$ are valid on every CS-frame. □

Theorem 70. *Every axiom of $\mathcal{P}$ is derivable from $\mathcal{B}$.*

Proof. By the proposition above, every axiom of $\mathcal{P}$ is valid on all CS-frames. Since $\mathcal{B}$ is complete with respect to CS-frames, each of these axioms is derivable in $\mathcal{B}$. □

Corollary 71. *The axiomatizations $\mathcal{P}$ and $\mathcal{B}$ are equivalent.*

A.2 Some derivations in TICS-2

Some of the following facts are used in the completeness proof for **TICS-2**. Recall that **TICS-2** is axiomatized by

T1 $A \to \Box A$

T2 $\Box(A \to B) \to (\Box A \to \Box B)$

T3 $\neg\Box\bot$

T4 $A \to \bigcirc A$

T5 $\Box \bigcirc A \to \bigcirc A$

T6 $(A \to B) \to ((\Box B \to B) \to (\bigcirc A \to B))$

This corresponds to $\mathcal{P}^- = \{1', \ldots, 6'\}$, i.e. $\mathcal{P}$ without axiom $7'$. Inspection of the derivations of Lemma 59 and Propositions 60-64, 66, 67 shows that none of them rely on $7'$: the only place where $7'$ is used is in the derivation of axiom 9 in Proposition 65, which is, in turn, used in no other place. Therefore, we see that:

Proposition 72. *Lemma 59 and Propositions 60-64, 66, 67 all hold with $\mathcal{P}$ replaced by $\mathcal{P}^-$, which in turn corresponds to the axiomatization T1-T6. Therefore, all these results are also valid in **TICS-2**.*

We use this to establish some of the following facts about **TICS-2**.

Proposition 73.

$$\vdash_{\textbf{\textit{TICS-2}}} \Box(A \wedge B) \leftrightarrow \Box A \wedge \Box B.$$

Proof. This is precisely axiom 6, so it follows from Proposition 61. □

Proposition 74.

$$\vdash_{\textbf{\textit{TICS-2}}} \Box A \to \bigcirc A.$$

Proof.

1	$A \to \bigcirc A$	(T4)
2	$(A \to \bigcirc A) \to \Box(A \to \bigcirc A)$	(T1)
3	$\Box(A \to \bigcirc A)$	(MP 1,2)
4	$\Box A \to \Box \bigcirc A$	(T2)
5	$\Box \bigcirc A \to \bigcirc A$	(T5)
6	$\Box A \to \bigcirc A$	(IPC 4,5)

□

Since the derivation above essentially uses the axioms $1'$, $2'$, $4'$, and $5'$, it also goes through in **TICS**; hence, the result holds for this system as well.

Corollary 75. *The following are derivable in* ***TICS-2****:*

1. $\Box(A \to B) \to (\bigcirc A \to \bigcirc B)$ *(Proposition 64);*
2. $\bigcirc \bigcirc A \to \bigcirc A$ *(Proposition 63);*
3. $\bigcirc A \to \neg\neg A$ *(Proposition 67);*
4. $\Box A \to \bigcirc A$ *(Proposition 74).*