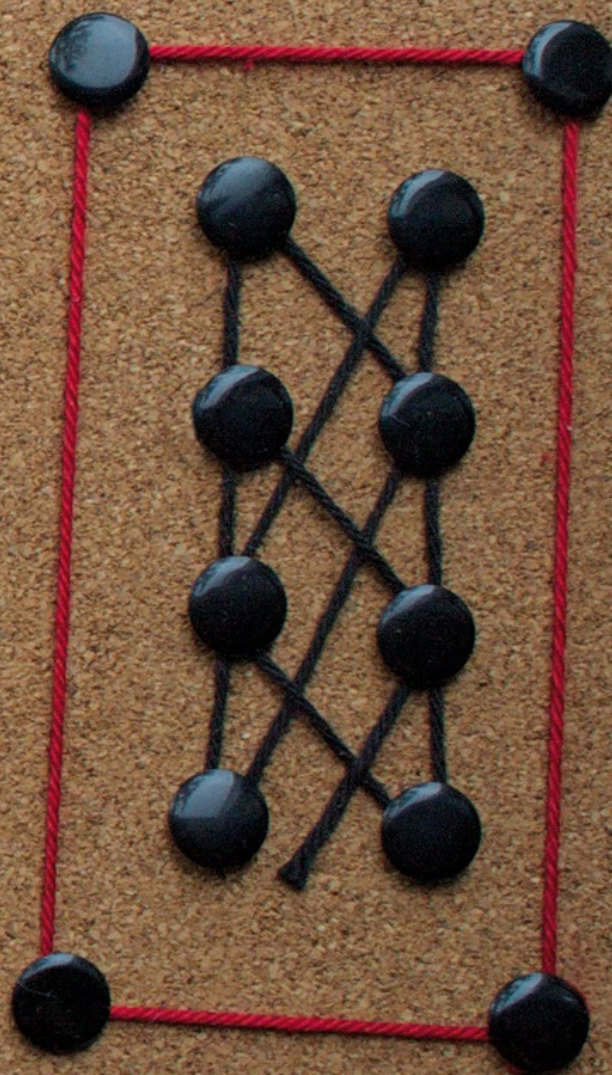
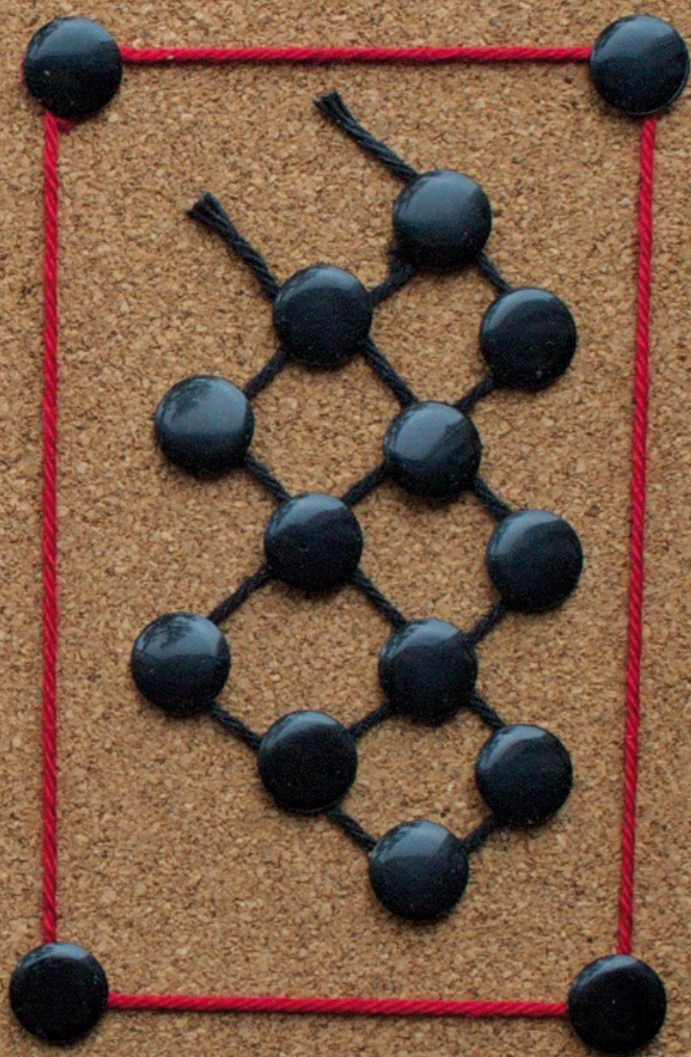


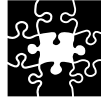
Representations and model theory of free Heyting algebras



Rodrigo Nicolau Almeida

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Chapter 1

Introduction

Prologue – Mathematical Conspiracies

We often picture contemporary research in mathematics as a series of basic steps:

1. Pick a domain of analysis and a question one has about it.
2. Devise a formal tool to analyse it. Uncover the axiomatics of the tool, leading the way to new phenomena and questions.
3. Construct a representation of the tool to make the questions easier. Turn the representation into a tool, and iterate.

However, this process – guided as it is by analogy, simile, metaphor, and mathematical communities’ intuitions, dreams and folklore – has a way of carving unspoken pathways between mathematical ideas: to draw a philosophical term these form something like *penumbral connections*¹ between different fields of mathematics. After enough time passes, these connections gain the aspect of a *conspiracy*: the mathematical ideas themselves appear to conspire to hide deep connections, and keep certain problems outside of the reach of our conceptual technology.

The purpose of this thesis is to study a web of problems lying at the heart of intuitionistic logic and non-classical logic more broadly, and to bring forth a number of plausible suspects for further interrogation. For this reason, like every detective novel, it is bound to bring forth innocents and red herrings – though at this point in time, we can only apologize for such narrative inconsistencies to the future reader.

Introduction

For us, the domain of analysis of this thesis is *Intuitionistic Propositional Logic* (IPC), and our foremost tool are *Heyting algebras*. The particular focus of the

¹Specifically, this is borrowed from [113].

this thesis is on *Free Heyting algebras*, a special class of Heyting algebras which capture faithfully most information about intuitionistic logic. The order on the algebras coincides exactly with derivability in IPC, their elements being equivalence classes of formula, giving us an avenue into a deeper understanding of logical properties of the system. By employing a Stone-like-representation, namely *Esakia duality*, such algebraic phenomena gain a geometric meaning, with the derivability of the system becoming itself a geometric object.

In keeping with the above movement, in this thesis we therefore study representations of free Heyting algebras, through algebraic and order-topological methods. These representations allow us to ask questions about the *model theory* of free Heyting algebras. Model theory means for us two things: the study of classes of algebras as models of a formal theory, in the universal algebraic sense; but also quite prevalently, the theory of definability, of subsets and higher-order structures, in free Heyting algebras and more broadly in all Heyting algebras.

In this introduction, we contextualize the themes of the thesis, and give some brief background history on intuitionistic logic, Heyting algebras, and duality theory, from the point of view of algebraic logic. We then discuss some of the questions that had been open in the literature, and explain how free algebras figure at the center of all these problems. We then explain how the work of this thesis contributes to either solving some of these problems, or advancing their study – hopefully unraveling some of the threads of the mathematical conspiracy that it crafts. The exposition will be informal, assuming familiarity with the basic concepts of the field, which will be formally discussed in the subsequent chapters.

1.1 Intuitionistic logic and its semantics

The logic IPC traces its origins to Arend Heyting’s formalization [144] of L.J. Brouwer’s philosophical understanding of mathematical truth, and of the meaning of “constructing” mathematical objects [73]. The proposal to reject classical logic, and embrace a more limited set of valid inferences, came from a simple switch in priorities: whether a mathematical statement φ is *metaphysically true or not*, should not be the main issue; instead we should ask, whether we can *prove* that statement [99, p. 2]. Simple examples of this are abundant: for the intuitionist, from a mathematical point of view, the Goldbach conjecture is neither true nor false, seeing that we have neither a proof of its truth nor a proof of its falsity; and so-called “proofs” which provide a disjunction – such as the classic argument proving “either $\sqrt{2}^{\sqrt{2}}$ is irrational or $\sqrt{2}^{\sqrt{2}^{\sqrt{2}}}$ is irrational” – are taken to be illegitimate, since they are asserted without determining which of the two disjuncts holds.

In an attempt to provide a solid foundation to Brouwer’s ideas, Heyting introduced the *intuitionistic propositional calculus*, IPC, which outlined the logical principles considered valid by an intuitionist. Symbolically, the formulas in

this system are written over propositional letters p, q, r , as well as the symbol $\perp$ (falsum or contradiction), by using the connectives $\wedge, \vee, \rightarrow$. It is common to introduce the abbreviations $\neg p$ for $p \rightarrow \perp$, and $p \leftrightarrow q$ for $(p \rightarrow q) \wedge (q \rightarrow p)$. The system IPC is then axiomatized in this language by imposing a few basic axioms on the connectives (see Definition 2.3.3), which essentially restrict classical logic CPC by abandoning the law of excluded middle, and its many equivalent forms, and giving the implication connective a dedicated role. As a consequence of having such weaker principles, one *gains* several desirable logical properties: for example, the *disjunction property* holds in IPC, meaning that a disjunction $\varphi \vee \psi$ cannot be proven without proving either φ or ψ .

Being one of the most heavily studied propositional logics of the 20th century, we cannot provide in this introduction a comprehensive history of the mathematical analysis of IPC (see for e.g. [54, Introduction] or [82]). However, we can identify a few important historical threads.

1.1.1 A brief history of the study of IPC

Algebraic semantics

Originally discussed in the work of Kurt Gödel [133] and Alfred Tarski [198], and popularized by Helena Rasiowa [226], amongst others, the introduction of an **algebraic semantics** allowed the tools of universal algebra to be employed in studying superintuitionistic logics. *Heyting algebras* or as they are still referred in some places in the literature “pseudo-Boolean algebras” [226, 82], were introduced in [198, 197] as algebraic structures to study intuitionistic logic. For the reader familiar with basic lattice theory, they can be characterized as those bounded lattices $(L, \leq)$ where for each pair of elements $a, b \in L$ there is an element $a \rightarrow b$ satisfying for each $c \in L$,

$$c \wedge a \leq b \iff c \leq a \rightarrow b,$$

where $c \wedge a$ is the infimum of the two elements. It is well known that they form a variety [198], and include several subclasses of independent interest: Boolean algebras, Gödel algebras and De Morgan Heyting algebras, for example, appear frequently elsewhere in mathematical logic. Moreover, though Heyting algebras arose in the context of logic, they do not appear exclusively here, but in many other places in mathematics:

1. In *topology*, where lattices of open subsets of topological spaces are always complete Heyting algebras;
2. More generally in *locale theory* or *point-free topology*, where locales are precisely complete Heyting algebras;

3. In *category theory* where posetal cartesian closed categories are precisely Heyting algebras;
4. In *domain theory*, where continuous distributive lattices are Heyting algebras;
5. In *topos theory*, where all lattices of subobjects in a topos are Heyting algebras;
6. In *provability logic* where Lindenbaum-Tarski algebras of intuitionistic arithmetical theories are naturally seen as Heyting algebras;

Relational semantics

Despite their theoretical flexibility, algebraic semantics were seen as a rather difficult tool to handle. The popularization of **relational methods** in the 1950s and 1960s, especially through Kripke semantics [172, 171], allowed for a geometrically intuitive semantics, closer to the usual truth-table semantics of classical logic. This is done by considering structures $\mathfrak{M} = (P, \leq, c, r)$ where $(P, \leq)$ is a partially ordered set, its elements being referred to as *worlds*, c is a *valuation*, assigning for each world $x \in P$, a subset $c(x) \subseteq \text{Prop}$, a collection of propositional variables which is upwards closed; and r is the *root*. One writes $\mathfrak{M}_{r'}$ for $r' \in P$ to mean the upwards closed subset $\uparrow r'$, and the restriction of the structure of the model to this subset. One then specifies the truth-condition of a formula at a world (true at the root), denoted $\mathfrak{M} \Vdash \varphi$, by induction on the construction of formulas:

1. If $\varphi = p$, then $\mathfrak{M} \Vdash \varphi$ holds if $p \in c(r)$.
2. $\mathfrak{M} \Vdash \varphi \wedge \psi$ if and only if $\mathfrak{M} \Vdash \varphi$ and $\mathfrak{M} \Vdash \psi$;
3. $\mathfrak{M} \Vdash \varphi \vee \psi$ if and only if $\mathfrak{M} \Vdash \varphi$ or $\mathfrak{M} \Vdash \psi$;
4. $\mathfrak{M} \Vdash \varphi \rightarrow \psi$ if and only if for all $r' \geq r$, $\mathfrak{M}_{r'} \Vdash \varphi$ implies that $\mathfrak{M}_{r'} \Vdash \psi$.

Given a poset $(P, \leq, r)$ with a root, we say that a formula φ is *valid* if for any valuation c , $\mathfrak{M} = (P, \leq, c, r) \Vdash \varphi$. It was shown by Kripke that validity in IPC can be captured by this semantics, which became known as *Kripke completeness*. It was also shown that in fact it suffices to consider *finite* posets – a property known as the *finite model property* (FMP). Other properties which are easily seen from this semantics include the aforementioned disjunction property.

Such a semantics marked the work of authors like Vadim Yankov² [264] and Dick De Jongh [160], who gave methods of extracting axiomatizations of intuitionistic logics through geometric means (see [37] for an extended discussion of

²Following [87], we transliterate the Russian to ‘Yankov’, though ‘Jankov’ is likewise common.

the consequences of this). It led to much of the development of the field, and marked many of the core questions in the study of extensions of IPC: which ones are Kripke complete, or which have the FMP, what are the adequate semantics for all superintuitionistic extensions, and how can one classify the complexity of such extensions in terms of their semantics.

Duality theory

Starting with the foundational work of Marshall Stone [246], it has become a standard feature of algebraic analysis to correspond algebraic objects – such as Boolean algebras – to topological structures – like compact Hausdorff zero-dimensional spaces, so-called *Stone spaces*. In the context of non-classical logics, the work of Bjarni Jónsson and Tarski [161] on Boolean algebras with operators, and the work of Hilary Priestley [223] on distributive lattices, provided some of the foundational work in this area. In terms of the study of intuitionistic logic, the key contribution is due to Leo Esakia [106] (see also [102]).

To each Heyting algebra H , it associates a specific order-topological space $\text{Spec}(H)$ called its “dual”, which captures all the information about H . Such spaces have become known as *Esakia spaces*. They can be characterised as order-topological spaces $(X, \leq, \tau)$ with a compact topology, where the relation $\leq$ is closed in $X \times X$, and is continuous (meaning that $\downarrow U = \leq^{-1}[U]$ is open whenever U is open). The theory of such spaces, and the related Priestley spaces in the duality of distributive lattices, has been extensively developed over the last 40 years, and productively exploited in analysing intuitionistic logic and the theory of Heyting algebras. Moreover, much like Heyting algebras, it should be pointed out that Esakia spaces also arise curiously in other fields of mathematics: in the theory of real algebraic geometry, the real spectrum of so-called “excellent rings” is always definitionally equivalent to an Esakia space (see e.g. [18, Prop. 6.1]).

Having a duality means that all questions about Heyting algebras can be asked in an equivalent form for Esakia spaces. Sometimes such questions become easier, on a purely conceptual or intuitive level, by looking at them through the lens of ordered spaces; other times, the opposite happens. Though it is definitional, we stress the following: *a duality is an equivalence*. It is always possible to translate any proof about Esakia spaces into a proof about Heyting algebras making no mention of the dual spaces. In turn, most suitable statements about Heyting algebras can be made to talk solely about logic. However, the main ingredient that might be lost is **intuition**.

Structuralization

Whilst the first three trends correspond to semantic frameworks, the last two we wish to emphasize frame the methodological shift in the field. First, a general trend in the study of non-classical logics, as well as in mathematics at large, is that *properties become structure*. What were once theorems about a specific system – such as the deduction theorem, interpolation, or as above, Kripke completeness, or

the disjunction property – become research questions on their own, and one goes from asking *whether* such a property P is true, to asking *how one can recognize when* the property P is true in general.

Early moves in this direction were made by Tsutomu Hosoi [148], A. Kuznetsov [174], Kit Fine [111] and Wim Blok [68], with key contributions made by Larisa Maksimova [184, 187] and Leo Esakia [110], Alex Citkin [86] amongst many others (see the historical notes on [82] for an extended overview). Some examples of these kinds of structural characterizations for superintuitionistic logics include:

1. The characterization of *tabular superintuitionistic logics* – those logics which are determined by a single finite poset – in the form of identifying so-called *pre-tabular logics* (due to Esakia and Grigolia [110]).
2. The classification of extensions of IPC with the Craig interpolation property [184];
3. The result by A. Kuznetsov [175] showing that all superintuitionistic logics are conservative over their modalized expansion by a so-called KM-modality;
4. The proof, independently due to Wim Blok [68] and Leo Esakia³ of an isomorphism between the lattice of subvarieties of **HA** and the lattice of subvarieties of **Grz**.

The search for general structural methods lead to the invention of methods like Zacharyashev’s “canonical formulas” (see [82, Ch. 9]) and the related methods of “stable canonical formulas” (see [39]), as well as the trends we will see in the next point.

Categorification

A **categorification** of the field, where the language of category theory has become increasingly common to describe the tools and properties at play, and progressively used not just to find universal solutions, but as inspiration for further research questions. Elsewhere in algebraic logic, the study of *coalgebraic logic* (see e.g. [253] for a good introduction) bridged the role of categories in explaining many of the constructions which researchers had been employing for several decades.

In the study of superintuitionistic logics, the role of category theory can be seen from early on in the work of Andrew Pitts [217, 218], as well as the work of Mamuka Jibladze and Dito Patariaia [108, 213], where there is a clear interplay between questions purely about intuitionistic logic and its extensions, and the categorical structure of the algebras corresponding to such extensions.

³It should be remarked that the contents of the original proof by Esakia of the theorem now known as the Blok-Esakia theorem has been seemingly lost. The only traces of it are in [102, A. 4], where the author makes reference to the use of “almost direct products” and Feferman-Vaught theory.

One of the most clear bodies of work exemplifying this trend lies in the work of Silvio Ghilardi, especially in joint work with Marek Zawadowski [130, 128]. Their work on model completions constituted a clear move in the field, towards questions of logical definability as they arise out of category theoretic – and implicitly, geometrically inspired – investigations. As an illustration of the content of such research, in the cited work the authors showed the equivalence between uniform interpolation of a logical system, the existence of model completions for varieties of Heyting algebras, and closure under sheaf-theoretic images. This then allowed to contextualize the existing combinatorial properties of uniform interpolation in terms of a definability result.

1.1.2 Open problems

After these historical threads, we now highlight some outstanding questions, which remain open about IPC and its extensions in the year of 2026. The following are a small selection of the problems of the field – all chosen due to their bearing on the research in this thesis:

- *Kuznetsov hierarchy*: It was originally noted by Gödel [134] that for intuitionistic logic, there can be no single truth-table semantics – IPC is not *tabular*. As noted above, the work of Kripke showed that taking the class of posets, and in fact finite posets, is indeed enough. The connections between intuitionistic logic and topology were also noted as early as 1930 [133], where it was shown that topological spaces provide a complete semantics for intuitionistic logic.

In the 1970s A. Kuznetsov, and his Chişinău group, became interested in the complexity of semantics of superintuitionistic logics, establishing what is now known as the *Kuznetsov hierarchy* (see [50]): from *tabular* logics, to *locally tabular*, to logics with the FMP, Kripke complete, topologically complete, and so on. It was open for a long time whether every superintuitionistic logic was Kripke complete, a question which was resolved, in the negative, by Shehtman [241]; the problem for topological completeness, posed by A. Kuznetsov, however, remains open, and has resisted most current techniques (see e.g. [45, 193] for recent advances).

- *Local tabularity*: at the topmost part of Kuznetsov’s hierarchy, lie the *tabular logics* which are fully understood (see [82, Ch. 12]), with a full classification and a very thorough understanding of the logical properties of such systems. The next class there considered are *locally tabular logics*. Recall that an algebra A is locally finite when, given a finite set $B \subseteq A$, the subalgebra $\langle B \rangle \leq A$ generated by B is finite. A logic L is *locally tabular* if all the algebras validating that logic are locally finite.

Such a property is remarkably useful in semantic analysis, reducing most universal algebraic questions to questions of finite combinatorics. It is well understood in various settings such as modal logics [185] (see also [239] for recent developments in this direction). Nevertheless, over IPC the problem appears to be rather involved: it was shown by Mardaev [189] that the cardinality of pre-locally tabular logics is that of the continuum, and several conjectured characterizations of local tabularity have been proven to not exhaust the whole class of locally tabular logics: Maksimova's ω -closure criterion was shown to have counterexamples by Mardaev in [191]; Bezhanishvili and Grigolia's [48] 2-finiteness criterion, saying that $\mathcal{F}_L(2)$ being finite is enough for L to be locally tabular, was shown to admit counterexamples by several authors [150]. Indeed, only in very limited contexts has local tabularity been shown to admit tame behaviour, such as when there is global control over the structure of frames: finite depth (by the same arguments as in the **S4** case) or having width ≤ 2 (see [247]). The question of whether it is algorithmically possible to decide whether a logic is locally tabular (as it is for example with tabularity [110]) remains open.

- *Higher-order definability and the Heyting-to-topos problem:* a surprising result of Pitts [218], showed that the full second-order propositional logic admits a faithful translation to IPC, implying the related property of uniform interpolation: whenever $\vdash \varphi(\bar{p}, q) \rightarrow \psi(q, \bar{r})$, there are 'largest' and 'least' (with respect to the derivability order) interpolants, respectively $A_q(\varphi)$ and $E_q(\varphi)$. This and related properties (see [135] for a recent survey) parametrize the relationship between IPC and its higher-order variants [104, 108], being intimately related to a question popularized by Pitts: whether every Heyting algebra arise as the algebra of truth values of a topos. Despite a complex history (see [216]), such a problem has remained almost entirely open.
- *Modal Extensions and Translations:* already in the work of Gödel [133], it was noted that intuitionistic logic carried a natural modal meaning, which made it translate into **S4**; later work showed that IPC could more appropriately be seen to be related to **Grz** [141]. The work of Blok [68] and Esakia [103] proved an isomorphism between lattices of extensions. Later work by Kuznetsov and Muravitsky found a particular modal extension of intuitionistic logic – nowadays known as **KM**-logic [206], and a similar translations into **GL**, and similarly [260] found such an isomorphism with respect to tense variants of **Grz**. The logic **KM** was moreover shown to be intimately connected to IPC, through a general conservativity result [206, 155]. Similar results for monadic intuitionistic logic have proven difficult [44, 43]. The question of what extensions of IPC remain conservative over it, and which ones allow for translations to modal systems provides thus a large open area

of research.

The above problems, though of very different natures, havin in common that they can be boiled down to problems about *free Heyting algebras*: whether free Heyting algebras can be suitably obtained from a well-understood class of algebras (Kuznetsov’s problem), whether they are themselves reasonably simple (local tabularity), whether they carry higher-order structure (higher-order definability), or whether they admit suitable homomorphisms to other free algebras (modal extensions and translations). In the next section we explain this centrality further, and discuss how they can be employed in mounting lines of attack on the above problems.

1.2 Free Heyting algebras

Free algebras are natural objects from many points of view. In logical terms they are often called *Lindenbaum-Tarski algebras* [82, 66, 122]. The following is a general recipe for their construction: suppose one works with a given logical system L , with a given class $\mathbf{A}_L$ providing its algebraic semantics, and suppose we are given a set $\mathbf{Prop}$ of proposition letters. The algebra $\mathcal{F}_L(\mathbf{Prop})$ can be equivalently described as the algebra of equivalence classes of formulas $[\varphi]_{\approx_L}$, where the equivalence is given by equiderivability

$$\varphi \approx_L \psi \iff \vdash_L \varphi \leftrightarrow \psi.$$

Birkhoff’s theorem, one of the building blocks of universal algebra, states that the equational theory of $\mathcal{F}_L(\omega)$, where ω is an infinite set of generators, is the same as the equational theory of $\mathbf{A}_L$. This provides a deep connection between properties shared by all algebras of L , properties of the logical system L , and concrete special properties of $\mathcal{F}_L(\omega)$: for example, the fact that admissible rules over $\vdash_L$ coincide with the quasi-equations valid in $\mathcal{F}_L(\omega)$ (see e.g. [235]).

On the other hand, free algebras satisfy special *universal mapping properties*, which amount to saying that the class $\mathbf{A}_L$ has a forgetful functor to $\mathbf{Set}$ which has a left adjoint, constructing precisely for each set X , the algebra $\mathcal{F}_L(X)$. This point of view allows us not only to understand how free algebras relate to all others, but gives us an array of interesting constructions: coproducts, pushouts, and other colimits in classes of algebras, all of which can be generically described as quotients of specific free algebras.

1.2.1 Representations of free Heyting algebras

Because of this fundamental role, it has often been desirable to study special *representations* of free algebras; these tend to be connected to the existence of well-behaved normal forms of the associated logics [125], and to the existence of

well-behaved locally finite reducts yielding “filtrations” [126, 36]. For some classes of interest in algebraic logic – like Boolean algebras and distributive lattices – such descriptions exist to a very satisfactory degree: for example, one can describe for any set X the free Boolean algebra on X many generators, by looking at the Stone space that is dual to it, which coincides with $\mathbf{2}^X$, the X -fold product of copies of $\mathbf{2}$, the two point discrete space. Such constructions have proved instrumental, for instance, in describing many properties of theories of Boolean algebras (e.g., [167, Ch. 18]).

Free Heyting algebras are much less well-behaved. Already the 1-generated free Heyting algebra – the well-known Rieger-Nishimura lattice [231, 209] (see Section 3.1) – is infinite. To approach the n -generated free Heyting algebras, different techniques have been introduced.

An early way of visualizing such an algebra came from the technique of *colorings*, or *universal models*: one constructs, recursively, from a finite set of points, an infinite poset $\mathcal{U}(n)$, which captures all the irreducible n -generated models. This is due independently to Bellissima [30], Grigolia [138] and Shehtman [242], though some of its key features can also be seen already in the work of De Jongh [160] (see [82, Ch. 8.8] or [54, Sec. 3.2] for modern presentations). The resulting representation of $\mathcal{F}(n)$ is through the definable upsets of a given poset $\mathcal{U}(n)$. Whilst $\mathcal{U}(n)$ is highly impractical to write down – even for $n = 2$ – the simple combinatorial conditions generating each new layer have lead to it finding use in many questions about intuitionistic logic (e.g., in [65, 234, 40, 151]). A key peculiarity, though, is that $\mathcal{U}(n)$ is not the *dual* of $\mathcal{F}(n)$; the dual, $\text{Spec}(\mathcal{F}(n))$, is a much larger space, where $\mathcal{U}(n)$ sits as a dense open upset. The precise mechanism of such a representation bears a resemblance to the relationship to other dual constructions – like representations of duals of products using non-compact topological sums. Despite the success of universal models, not much has been studied about general representations of this kind (with a few notable exceptions, see e.g [123]).

A different representation of $\mathcal{F}(n)$ was given by Ghilardi [127], generalizing previous work by Urquhart [251], and in turn being generalized by Butz [75]. This representation makes use of the relationship between Heyting algebras and distributive lattices: the dual $\text{Spec}(\mathcal{F}(n))$ is presented as an inverse limit of a sequence

$$P_0^n \xleftarrow{r_0} P_1^n \xleftarrow{r_1} P_2^n \xleftarrow{r_2} \dots,$$

where P_0^n is the dual of the free distributive lattice on n -many generators, and each step is obtained by a power-set-like operation which adds freely collections of rooted subsets. Such a description allows a different stratification of $\mathcal{F}(n)$: whilst with $\mathcal{U}(n)$ the *completely join-irreducible* elements of $\mathcal{F}(n)$ gain a transparent shape, here the plain *join-irreducible* elements are more easily visible. This makes it easy to show, for instance, that free finitely generated Heyting algebras are bi-

Heyting algebras⁴.

Though the step-by-step representation presents an attractive way of capturing all formulas of IPC dually, the restriction of such a representation to the *finitely generated case*, meant that several constructions of interest which could be developed for Boolean algebras – like coproducts and pushouts of general Heyting algebras – remained beyond the reach of existing methods.

1.2.2 Model theory of free algebras

As we have emphasized above, specific properties of free Heyting algebras are often equivalent to logical properties of IPC and its extensions. General examples of this are abundant:

1. Questions about *local tabularity* or *tabularity* boil down to problems of size, and existence of bounds, on free algebras;
2. Questions about *interpolation* or *Beth definability*, depend on the structure of free algebras, their lattice theory, and their category theoretic properties;
3. Questions about *uniform interpolation* often involve problems of the existence of adjoints to inclusion maps between free algebras;
4. Questions about the existence of *translations* or *uniform conservativity* results can be phrased as the existence of specific maps between free algebras.

Such problems can all be seen as problems about the *model theory* of free Heyting algebras. Here we take model theory to mean the *study of definability* in the free Heyting algebra, and consequently, definability in the logical system itself. For instance, a natural conjecture about local tabularity is that it corresponds to a syntactic property: the existence of a uniform bound on how complex a formula is, in terms of having n -many nested implications and not being equivalent to any formula with fewer nested implications. This can be seen to be equivalent to asking whether free algebras for these systems admitting a simple representation which is likewise parametrized by such uniform bounds.

Similarly, questions of higher-order extensions and ways of relating propositional theories to higher-order ones have been long the subject of debate and questioning [216]. Through usual methods, they can be seen as problems of definability of higher-order quantifiers in free algebras. To address such problems it is necessary to find how one can “complete” Heyting algebras into toposes.

⁴We briefly remark that a complement to these two representations was likewise given by Ghilardi and Zawadowski [130], representing $\mathcal{F}(n)$ as a special sublattice of the lattice of subsheaves of a sheaf topos. Such a representation will not play as central a role as the universal model and step-by-step representations in the developments of this thesis (but see Appendix A.3.2 for a comparison).

1.3 Structure and contributions of this thesis

The problems outlined in the previous section form the bulk of our study in this thesis. These are split in two parts.

Part [One](#) discusses representations of free Heyting algebras. In this part we present some new and generalized representations, of free algebras and of several free constructions, employing duality theory systematically. Our key contributions can be summarised as follows:

1. (Locally Esakia spaces and universal models) We contextualize the theory of universal models through the concept of *locally Esakia spaces*. These provide a more flexible method of representing a Heyting algebra H via different ordered spaces, which are not necessarily compact. These different duals are connected to the Esakia dual by *order-compactifications*. With this level of generality, we are able to study locally Esakia representations of products of Heyting algebras, and of so-called “Bellissima algebras”, in relation to a question by Darnière and Juncker [\[91\]](#).
2. (Step-by-step constructions of Heyting algebras) We discuss a *general theory of step-by-step constructions*, with the key contribution being the *generalization of the Ghilardi construction* – named here a *Ghilardi complex* – to the case of arbitrary Heyting algebras. This is used to present colimits of Heyting algebras and polynomial Heyting algebras dually. In particular, such a step by step construction is given by specifying the left adjoint of the inclusion of HA into DL, and manipulating such functors. As a key application of general Ghilardi complexes, we study two interpretations of IPC in positive logic PL (its implication-free fragment): one which acts homomorphically on connectives, but is non-conservative, and another which is conservative, yielding *uniform positive approximations*.
3. (Variations of the step-by-step construction) We also study several key variations of the Ghilardi construction. We present the step-by-step procedure for subvarieties of Heyting algebras, showing how it simplifies in such cases. We also show how a discrete Ghilardi-complex can be developed, and use it to solve an open problem in coalgebraic intuitionistic modal logic (see [\[179, Rem. 8\]](#) and [\[140\]](#)), concerning a coalgebraic semantics for intuitionistic modal logics. Such work is presented here only in brief, and has been continued in [\[15\]](#), where it is shown that such methods can be applied to well-known logics, such as Fischer-Servi’s IK (see [\[114\]](#)).

In Part [Two](#), we look at some questions concerning model theory and definability in free Heyting algebras. Here we make two studies on issues of definability, both addressing concrete open problems in the field, and both being contextualized in a broader set of research questions. Our main contributions are the following:

1. (Uniform local tabularity) To understand Kuznetsov’s hierarchy, and especially local tabularity, in a deeper way, we give a detailed study of *uniform local tabularity*, first analyzed by Shehtman [240] (under the name of “finite formula depth property”). We axiomatize the smallest extensions of intuitionistic logic which are uniformly locally tabular for $n \leq 3$. This can be seen as an explanation of the tractability of the Gödel logic LC, and provides a new hierarchy for a more fine-grained analysis of logical complexity. We also answer a question of Shehtman concerning the existence of locally tabular but non-uniformly locally tabular logics, showing that the logic of “weak Peirce’s law” wPL is one such example.
2. (Topos-admissibility) To study *higher-order extensions*, one needs a notion of “power-objects”, which internalise the process of forming higher-order structures like sets of sets. For this it turns out to be useful to impose stronger restrictions on a homomorphism, working with so called *étale maps*.

We provide here a systematic study of étale Heyting algebras, as introduced in [177, 155], and employ a combination of step-by-step representations and duals of products of algebras to describe the free 1-generated algebra for varieties of étale-Heyting algebras. We also show how such a representation becomes vastly easier when simplifying assumptions – such as local linearity and boundedness – are brought in.

An especially strong assumption is *étale-finiteness*. As a consequence of it, all free constructions collapse to distributive lattice free constructions. In this class of Heyting algebras, we make progress on the Heyting-to-topos question. In particular, we provide a construction which shows that all étale-finite Heyting algebras arise as lattices of truth values of elementary toposes.

We have strived to adhere to two principles in writing this thesis: (1) to keep the thesis self-contained and streamlined wherever possible; (2) to make the thesis serve the research community. In pursuit of (1), we have included a long technical introduction, Chapter 2, which details many well-known, and some less well-known results, with precise references and proofs wherever appropriate. In pursuit of (2), we have included a dedicated Appendix A containing proofs of several folklore results, some simplified proofs and small extensions of known results, and some questions and connections which seem to us of interest to record for further analysis. Attributions have been made to the best of the knowledge of the author, sometimes to oral communications, due to the lack of written records.

To aid the reader, we include a flowchart in Figure 1.1 with the conceptual dependencies between chapters.

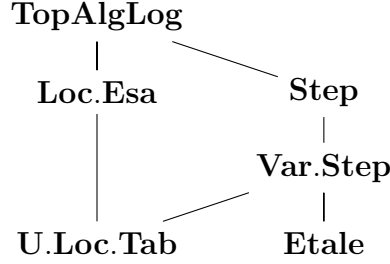


Figure 1.1: Diagram of dependencies of chapters

1.3.1 Sources

The contents of this thesis are collected from several papers, preprints and notes, many of which obtained in collaboration. In some cases, substantial parts of the texts were edited to better fit in the narrative and style of the thesis; for that reason, *all mistakes found should be attributed solely to the author, and not to any the co-authors mentioned below*:

1. The contents of Chapter 2 were based and expanded from several sources: (1) the lecture notes for Mathematical Structures in Logic, written together with Nick Bezhanishvili [14]; (2) the preliminaries of [17], written together with Silvio Ghilardi; (3) and the preliminaries for [11], written together with Guram Bezhanishvili.
2. The results from Chapter 3 concerning locally Esakia spaces and their theory were obtained in collaboration with N. Bezhanishvili and G. Bezhanishvili. The results about locally Esakia spaces were collated in [12], whilst the results concerning the combinatorics of the universal model, dual representation of products, ultraproducts, and Bellissima algebras, are collected here for the first time.
3. The results from Chapter 4 and from Chapter 5 were originally made public in the preprint [6]. The text found in this thesis has been entirely rewritten to focus on a more uniform and more transparent presentation of the results. Several new results concerning normal forms, relative anarchy, and the interpretations of IPC in its positive fragment PL are also included here for the first time.

The mentioned results concerning coalgebraic representations of intuitionistic modal logics appeared in joint work with N. Bezhanishvili [13] as well as N. Bezhanishvili and Sarah Dukic [15]. The results mentioned concerning regular Heyting algebras are also recorded in [6], and were presented in [8]. The results mentioned concerning interpolation of bi-Gödel algebras were obtained in collaboration with Qian Chen, and are recorded in the forthcoming [16].

4. The results from Chapter 6 were obtained by the author, and were published at AIML 2026 [9].
5. The results from Chapter 7 concerning the existence of a topos for étale-finite Heyting algebras were obtained together with Marco Abbadini and Igor Arrieta, and compiled in [3]. The results concerning other classes of étale Heyting algebras are due to the author and collected here for the first time.
6. The discussion from Chapter 8 that concerns questions and further directions in the interplay of definability and geometry, stems from ongoing research jointly with Matteo De Berardinis and Lingyuan Ye.
7. The Appendix A contains several results which are the product of collaboration: the proof of Lemma A.1.1 stems from joint work with G. Bezhanishvili; the results from Section A.2.2 arose from joint work with N. Bezhanishvili in the context of Esakia order-compactification; Sections A.2.3, A.2.4 and A.5 come from our joint work with M. Abbadini and I. Arrieta; Sections A.3.1 and A.3.2 were remarks obtained in discussion with Luca Carai; and Section A.4 is joint work with M. De Berardinis.

1.3.2 Attribution and AI

For matters of ethics and transparency, we declare our usage of artificial intelligence in the process of writing the present thesis:

1. At numerous points during the writing process, Anthropic’s “Claude Opus 4.6–4.8” was used to assist in searching for references;
2. In Chapter 4, the same model was asked to clean up notation and optimize readability on the proof of Proposition 4.5.9, before the author rewrote the proof;
3. In Chapter 7, some simplifications of the main arguments were worked out after interaction with OpenAI’s ChatGPT5. Namely the idea of using a simplifying lemma like Proposition 7.3.12 for simplifying power object arguments, was made by the model.
4. The above notwithstanding, none of the text contained in the present thesis was generated using artificial intelligence, and all authorship is retained by the author and his co-authors as detailed above.

Chapter 2

Topology, Algebra, and Logic

In this chapter we give an overview of all necessary background material for this thesis, and take the opportunity to fix notation and terminology for the sequel. We assume the reader is generically familiar with a wide variety of topics: the basics of lattice theory [137], namely the basic theory of distributive lattices and their morphisms; the basics of category theory [182], namely the notions of categories, functors and categorical equivalences; the basics of topology [101], in particular the theory of continuous functions, and familiarity with some of the more common classes of topological spaces; and the basics of universal algebra [34]. Despite these broad assumptions, many of the core definitions are recalled here, and basic results are stated in full, for ease of reference throughout the text.

Outline of the Chapter

1. In Section 2.1 we discuss the basics of orders, lattices and categories, with a special focus on Heyting algebras and their classes therein, as well as on Priestley and Esakia duality.
2. In Section 2.2 we give the basic properties of classes of algebras and logics which we will refer to throughout the thesis.
3. In Section 2.3 we present intuitionistic logic, with its order-theoretic semantics and some of the basic properties of IPC and its extensions.

2.1 Orders, Lattices and Categories

2.1.1 Posets and Categories

Partially ordered sets

By a *poset* we will mean a *partially ordered set*, often denoted by $(P, \leq)$, that is, a set equipped with a reflexive, transitive and antisymmetric relation $\leq$. We will

write $<$ for the irreflexive version of this order, We will identify a poset with its carrier set whenever this does not cause confusion.

Definition 2.1.1 (Maximal and minimal sets). Given a poset P an element $x \in P$ is *maximal* if whenever $x \leq y$ then $x = y$. We denote by $\max(P)$ the set of maximal elements in P . Minimal elements are defined similarly and we write $\min(P)$ for the set of minimal elements. For a subset $C \subseteq P$, $\max(C)$ ($\min(C)$) will mean the set of maximal (minimal) elements of C .

A *linear order* is a poset $(L, \leq)$ such that for each $x, y \in L$, $x \leq y$ or $y \leq x$. Note that the order duals of linear orders, $(L, \geq)$ are again linear orders, which we denote throughout by L^{op} . A subset $C \subseteq P$ of a poset which is a linear order as a subposet is a *chain*. A collection $W \subseteq P$ where no two elements are comparable, i.e., for each $a, b \in W$, $a \not\leq b$ and $b \not\leq a$, is an *antichain*.

Definition 2.1.2 (Upset and downset). Given a poset $(P, \leq)$, a subset $U \subseteq P$ is said to be *upwards-closed* (downwards-closed) or simply an *upset* (downset) if whenever $x \in U$ and $x \leq y$ ($y \leq x$) then $y \in U$. We denote by $\text{Up}(P)$ ($\text{Down}(P)$) the collection of upsets (downsets) of P .

Given a subset C , the least upset containing it always exists, and is denoted by $\uparrow C$. We shorten $\uparrow x$ by $\uparrow\{x\}$, and refer to this as the *rooted upset*, with x being the *root*. Given a subset $C \subseteq P$, we denote by:

$$\Box_{\leq} C = \{x \in P : \forall y \geq x, y \in C\},$$

often simply $\Box C$; given two upsets U, V we employ the following notation:

$$U \Rightarrow V := -U \cup V \text{ and } U \rightarrow V := \Box(U \Rightarrow V).$$

Category theory

Definition 2.1.3 (Adjoint functors). Given two functors $F : \mathbf{C} \rightarrow \mathbf{D}$ and $G : \mathbf{D} \rightarrow \mathbf{C}$ we say that F is the *left adjoint of G* (resp. G is *right adjoint of F*), written $F \dashv G$, if and only if: for each $A \in \mathbf{D}$, there is a morphism $\eta_A : A \rightarrow GF(A)$ such that for each object $B \in \mathbf{C}$, and every morphism $g : A \rightarrow G(B)$, there is a unique morphism $f : F(A) \rightarrow B$ with $G(f)\eta_A = g$ (see Figure 2.1)

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GF(A) \\ & \searrow g & \downarrow G(f) \\ & & G(B) \end{array}$$

Figure 2.1

The map η , the *unit*, is associated to a natural transformation; a dual natural transformation $\varepsilon : FG \rightarrow 1_{\mathbf{D}}$ always exists, and is referred to as the *co-unit*.

Example 2.1.4 (Examples of adjoint functors).

1. (Reflective subcategory) If $\mathbf{C} \leq \mathbf{D}$ is a full subcategory, and I is the inclusion functor, a left adjoint $R \dashv I$ is called a *reflector*; in this case $\mathbf{C}$ is a *reflective subcategory* of $\mathbf{D}$.
2. (Poset adjoint maps) If P, Q are posets, we can see them as categories. A functor is then an order preserving map $f : P \rightarrow Q$; a map $g : Q \rightarrow P$ is then the (right) adjoint if it satisfies:

$$f(x) \leq y \iff x \leq g(y).$$

One particular structure naturally induced by adjunctions is that of a *monad*:

Definition 2.1.5 (Monad). A *monad* is a triple (T, η, μ) , consisting of an endofunctor $T : \mathbf{C} \rightarrow \mathbf{C}$, and two natural transformations $\eta : 1_{\mathbf{C}} \rightarrow T$ and $\mu : T^2 \rightarrow T$ subject to the following compatibility laws (Figure 2.2) for each $B \in \mathbf{C}$:

$$\begin{array}{ccc} T(B) & \xrightarrow{\eta_{TB}} T^2(B) & \xleftarrow{T(\eta_B)} T(B) \\ & \searrow 1 & \downarrow \mu \\ & & T(B) \end{array} \qquad \begin{array}{ccc} T^3(B) & \xrightarrow{T(\mu)} T^2(B) & \\ \mu_{T(B)} \downarrow & & \downarrow \mu \\ T^2(B) & \xrightarrow{\mu} T(B) & \end{array}$$

Figure 2.2: Monad laws

Example 2.1.6 (Examples of monads).

1. (Posetal monads=Closure operators) Given $c : P \rightarrow P$, an order-preserving map on a poset P , we have that c is a monad precisely when it is a *closure operator*: for each $p \in P$, $p \leq c(p)$, and $cc(p) = c(p)$.
2. (Monad of an adjunction) Given an adjunction $F \dashv G$ where $F : \mathbf{C} \rightarrow \mathbf{D}$, the composition $GF : \mathbf{C} \rightarrow \mathbf{C}$ forms a monad, when equipped with the unit η of the adjunction, and the multiplication $\mu := G(\varepsilon_{F(X)})$.

The example 2.1.6.(2) is the key example where we will see monads in this thesis. In particular, we will consider *algebras* for such monads.

Definition 2.1.7 (Algebra for a monad). An *algebra for the monad* T , where (T, η, μ) is a monad on the category $\mathbf{C}$, is a pair (B, α) , where α is a map $\alpha : TB \rightarrow B$, satisfying the following compatibility laws:

$$\begin{array}{ccc}
T^2(B) & \xrightarrow{T(\alpha)} & T(B) \\
\mu_B \downarrow & & \downarrow \alpha \\
T(B) & \xrightarrow{\alpha} & B
\end{array}
\qquad
\begin{array}{ccc}
X & \xrightarrow{\eta_B} & T(B) \\
& \searrow 1_B & \downarrow \alpha \\
& & B
\end{array}$$

Figure 2.3: Algebra over a monad; on the left, the multiplication, and on the right, the unit conditions.

We assume the reader is familiar with basic *limits and colimits* of categories, and associated mentions of *categorical products*, *pullbacks*, amongst other concepts.

Notation 2.1.1 (Notation in categories with finite limits). Given a category $\mathbf{C}$ with all finite limits, we will denote the following objects:

- The *terminal object* by $1_{\mathbf{C}}$ (often without the subscript);
- The *categorical product* by $A \otimes A$, whenever $A \in \mathbf{C}$;
- The *categorical pullback* by $B \otimes_A C$ whenever there are $f: B \rightarrow A$ and $g: C \rightarrow A$ a specified pair of morphisms.

2.1.2 Lattices and Heyting algebras

Definition 2.1.8 (Lattice). A poset $(L, \leq)$ is a *lattice* if it has all finite infima and suprema. It is a *complete lattice* if L has all infima and suprema.

Remark 2.1.9 (All lattices are bounded). Note that in the light of the above definition, we assume the convention throughout this thesis that all lattices are bounded (since empty suprema and infima give us the bounds).

Given a lattice L , we denote by $a \wedge b$ the infimum of a, b , by $a \vee b$ the supremum of a, b , by $0 = \bigvee \emptyset$ and by $1 = \bigwedge \emptyset$. Given $(L, \leq)$ a lattice, note that $(L, \geq)$ is likewise a lattice.

Definition 2.1.10 (Distributive lattice). A lattice L is *distributive* if for all $a, b, c \in L$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

We view lattices implicitly in the signature containing $(\wedge, \vee, 0, 1)$, so a lattice homomorphism is a map preserving meets, joins and bounds. We denote by **DL** the category of distributive lattices with lattice homomorphisms.

Definition 2.1.11 (Complemented element). An element $a \in L$ in a lattice is *complemented* if there exists some element $b \in L$ such that $a \wedge b = 0$ and $a \vee b = 1$.

Remark 2.1.12 (Unique complementation). It is well-known [137, Lem. 1, pp. 57] that in a distributive lattice, every element which is complemented is *uniquely* complemented. Moreover, a distributive lattice homomorphism preserves (though it need not reflect) complements of elements.

Definition 2.1.13 (Boolean algebra). A distributive lattice B is a *Boolean algebra* if every element is complemented.

We write **BA** for the category of Boolean algebras and Boolean homomorphisms – lattice homomorphisms preserving and reflecting complements. As it is well-known, this coincides with the full subcategory of **DL** consisting of distributive lattices where every element is complemented [137, pp. 100].

Definition 2.1.14 (Heyting algebra). A lattice $(H, \leq)$ is a *Heyting algebra* if for all $a, b \in H$ there is an element $a \rightarrow b \in H$ such that for all $c \in H$:

$$c \wedge a \leq b \iff c \leq a \rightarrow b.$$

Such an element is called the *Heyting implication* or the *relative pseudocomplement* of a relative to b .

Heyting algebras are always considered in the signature $(\wedge, \vee, \rightarrow, 0, 1)$, with Heyting homomorphisms preserving the relative pseudocomplement. We write **HA** for the category of Heyting algebras with Heyting homomorphisms.

There is a natural connection between Heyting algebras and specific distributivity assumptions on lattices:

Theorem 2.1.15. *Let L be a lattice.*

1. *If L is a Heyting algebra, then L is distributive.*
2. *If L is a complete lattice, then L is a Heyting algebra if and only if it satisfies for each $a \in H$ and $(a_i)_{i \in I}$ a family of elements in H ,*

$$a \wedge \bigvee_{i \in I} a_i = \bigvee_{i \in I} a \wedge a_i$$

Consequently, a lattice admits at most one Heyting algebra structure.

Proof. For (1) see [158, Lem. 1.10], and for (2) see [158, Thm. 4.2, pp.26]. ■

Remark 2.1.16 (Complete Heyting algebras are locales). Complete Heyting algebras are sometimes called *locales* [215], and feature prevalently in the study of constructive topology.

The following examples can easily be verified to be Heyting algebras (using Theorem 2.1.15, for the latter two examples):

Example 2.1.17. [Examples of Heyting algebras]

1. (Linear Heyting algebras) If L is a linear order, L is a Heyting algebra; the meet and join are given by the binary supremum and maximum, and for $a, b \in L$, $a \rightarrow b = 1$ when $a \leq b$ and $a \rightarrow b = b$ when $a \not\leq b$. Such algebras are called *linear Heyting algebras*.
2. (Boolean algebras) A Boolean algebra B is a Heyting algebra, where we define $a \rightarrow b = \neg a \vee b$.
3. (Lattice of upsets) Given a poset P , the lattice $(\mathbf{Up}(P), \subseteq)$ forms a Heyting algebra; the implication operation is given by $U \rightarrow V$.
4. (Topological Heyting algebra) If (X, τ) is a topological space, the lattice $(\mathbf{Op}(X), \subseteq)$ of open subsets is a Heyting algebra, with the implication being given by $\text{int}(U \Rightarrow V)$; note that the case of $\mathbf{Up}(P)$ is a special instance, where one considers the Alexandroff topology on P .

Definition 2.1.18 (Irreducible and prime). Let L be a lattice. An element $a \in L$ is said to be:

1. *join-irreducible* (join-prime; meet-irreducible; meet-prime) if whenever $a = b \vee c$ ($a \leq b \vee c$; $a = b \wedge c$; $a \geq b \wedge c$) then $a = b$ or $a = c$ ($a \leq b$ or $a \leq c$; $a = b$ or $a = c$; $a \geq b$ or $a \geq c$);
2. *completely join-prime* (completely join-irreducible) if whenever $a \leq \bigvee_{i \in I} b_i$ ($a = \bigvee_{i \in I} b_i$) then for some i , $a \leq b_i$ ($a = b_i$);
3. an *atom* if $a \neq 0$ and whenever $0 < b \leq a$ then $a = b$.

Remark 2.1.19 (Prime=Irreducible). For distributive lattices, the classes of join-irreducible and join-prime elements coincide. Similarly, the classes of meet-irreducible and meet-prime elements likewise coincide. This fact will be used throughout, often implicitly.

Definition 2.1.20 (Filters, Ideals). Let L be a lattice. A subset $F \subseteq L$ is a *filter* if it is an upset, and whenever $a, b \in F$ then $a \wedge b \in F$; it is called *proper* if $0 \notin F$; it is called *prime* if whenever $a \vee b \in F$ then $a \in F$ or $b \in F$; it is called *principal* if there is an element $a \in F$ such that for each $b \in F$, $a \leq b$.

A subset $I \subseteq L$ is called an *ideal* if it is a downset and whenever $a, b \in I$ then $a \vee b \in I$; it is called *proper* if $1 \notin I$; it is called *prime* if whenever $a \wedge b \in I$ then $a \in I$ or $b \in I$; it is called *principal* if there is an element $a \in I$ such that for each $b \in I$, $b \leq a$.

In some instances we will need special filters which exist most naturally on Boolean algebras:

Definition 2.1.21 (Ultrafilter). A filter $F \subseteq B$ on a Boolean algebra is called an *ultrafilter* if for each $a \in B$ either $a \in F$ or $\neg a \in F$.

Definition 2.1.22 (Filter generated, lattice of filters). Given a lattice L , and a subset $S \subseteq L$, we denote by $\text{Fil}(S)$ (respectively $\text{Id}(S)$) the *filter generated by S* (respectively the *ideal generated by S*). Given a lattice L , we denote by $\text{Fil}(L)$ the (complete) lattice of its filters ordered by inclusion.

In Heyting algebras, filters capture the role of algebraic congruences: given a congruence Θ , we define a filter $F_\Theta := \{a : (a, 1) \in \Theta\}$, and given a filter F , we define $\Theta(F) := \{(a, b) : a \leftrightarrow b \in F\}$. The following is then well-known:

Proposition 2.1.23 (Filter-congruence correspondence). Given a Heyting algebra H , the map $\Theta : \text{Fil}(H) \rightarrow \mathbf{Con}(H)$ is a lattice isomorphism, with its inverse being given by $F_{(-)}$.

2.1.3 Classes of Heyting algebras

In this section we collect some basic classes of Heyting algebras which will appear throughout in the thesis.

Definition 2.1.24 (Completely distributive lattice). A complete lattice is called *completely distributive* if it validates the law:

$$\bigwedge_{i \in I} \bigvee_{j \in J} a_{i,j} \leq \bigvee_{f \in J^I} \bigwedge_{i \in I} a_{i,f(i)}.$$

Definition 2.1.25 (Compact element). An element $a \in L$ in a lattice is called *compact* if for each $a \in L$, whenever $a \leq \bigvee S$ for $S \subseteq L$ there is some $s \in S$ such that $a \leq s$.

Definition 2.1.26 (Algebraic lattice). A lattice is called *algebraic* if each element is a join of compact elements.

The following was shown by Raney [225].

Proposition 2.1.27 (Raney duality). The following are equivalent for a Heyting algebra H :

1. H is complete, completely distributive and algebraic;
2. H is complete and completely generated by its completely join-irreducible elements.
3. There is a poset P such that $H \cong \text{Up}(P)$.

Since as noted in Example 2.1.17, the lattices of the form $\text{Up}(P)$ are always Heyting algebras, this gives us a lattice theoretic characterization of Heyting algebras arising as lattices of upsets. One is sometimes also interested in special subclasses of this:

Definition 2.1.28 (Locally finite). A poset P is called *locally finite*¹ if for each $x \in P$, $\uparrow x$ is finite.

Definition 2.1.29 (Profinite algebra). Let $\mathbf{A}$ be a category of algebras. An algebra $A \in \mathbf{A}$ is called *profinite* if it is an inverse limit of finite algebras.

Definition 2.1.30 (Finite index property). An algebra A is said to have the *finite index property* if whenever $a, b \in A$ and $a \neq b$ then there is a finite algebra B and a surjective homomorphism $f : A \rightarrow B$ such that $f(a) \neq f(b)$.

Theorem 2.1.31 (Profinite Heyting algebras). *The following are equivalent for a Heyting algebra H :*

1. H is a profinite Heyting algebra.
2. H is complete, completely distributive and has the finite index property.
3. $H \cong \text{Up}(P)$ where P a locally finite poset.

Proof. This was proven in [38, Thm. 3.6]. ■

Subdirectly irreducible Heyting algebras appear very frequently in our analysis; as a consequence of Proposition 2.1.23, and usual facts about subdirectly irreducible algebras (see [74, Thm. 8.4]) we have the following characterization:

Proposition 2.1.32 (Subdirectly irreducible Heyting algebras). A Heyting algebra is subdirectly irreducible if and only if it has a second largest element².

Proof. See [102, A.1.1]. ■

Definition 2.1.33 (Co-Heyting algebra, bi-Heyting algebra). A lattice $(L, \leq)$ is called a *co-Heyting algebra* if $(L, \geq)$ is a Heyting algebra. Equivalently, there is for each $a, b \in H$ an element $a - b$ such that:

$$a - b \leq c \iff a \leq b \vee c.$$

It is called a *bi-Heyting algebra* if it is both a Heyting algebra and a co-Heyting algebra.

¹Such posets were named *image-finite* in [38]; the term locally finite was used in [33], and is used here for consistency with analogous “local properties”: local linearity, local directedness, etc.

²In some literature this element is called *monolith* or an *opremum*.

Bi-Heyting algebras were introduced by Rauszer [227], and were studied by Esakia [105], Wolter [258], and others. We consider them as a category **biHA** with Heyting algebra homomorphisms preserving the co-implication.

Example 2.1.34 (Examples and counterexamples of bi-Heyting algebras).

1. (Linear bi-Heyting algebra) Every linear order, or lattice of upsets, is a bi-Heyting algebra.
2. (Cofinite are not bi-Heyting) Consider the lattice $\mathcal{P}_{\text{cof}, \emptyset}(\omega)$, containing the cofinite subsets of the natural numbers and 0. This is a complete Heyting algebra, but not a bi-Heyting algebra. We have that $\emptyset = \bigwedge_{n \in \omega} \omega \setminus [2n] = \bigwedge_{n \in \omega} (\omega \setminus [2n] \cup \{0\})$, and so:

$$\{0\} = \{0\} \cup \left(\bigwedge_{n \in \omega} \omega \setminus [2n] \right) \neq \bigwedge_{n \in \omega} (\omega \setminus [2n] \cup \{0\}) = \emptyset;$$

therefore, by the dual statement Theorem 2.1.15, the algebra is not bi-Heyting.

Definition 2.1.35 (Regular and complemented). An element $a \in H$ in a Heyting algebra is called *regular* if $a = \neg\neg a$. It is called *complemented* if $a \vee \neg a = 1$. We denote by $\text{Reg}(H)$ the set of regular elements of H , and $\text{Cen}(H)$ the set of complemented elements.

The set $\text{Cen}(H)$ when equipped with the restriction of the operations from H is a Boolean algebra. Likewise $\text{Reg}(H)$ forms a Boolean algebra when equipped with $\wedge, \neg$ and the operation $\vee_{\text{reg}}$ defined by:

$$a \vee_{\text{reg}} b := \neg\neg(a \vee b).$$

Moreover, if $f : H \rightarrow H'$ is a Heyting algebra homomorphism, it is not difficult to see that both $f \upharpoonright_{\text{Cen}}$ and $f \upharpoonright_{\text{Reg}}$ will be Boolean algebra homomorphisms. Indeed, we have the following:

Theorem 2.1.36 (Adjunctions to forgetful $\mathbf{HA} \rightarrow \mathbf{BA}$). *The Booleanisation functor $\text{Reg} : \mathbf{HA} \rightarrow \mathbf{BA}$ is left adjoint to the inclusion $I : \mathbf{BA} \rightarrow \mathbf{HA}$, and the functor $\text{Cen} : \mathbf{HA} \rightarrow \mathbf{BA}$ is the right adjoint to I .*

Proof. See the main results from [109]. ■

Definition 2.1.37 (Dense elements). Given two elements $a, b \in H$, we say that a is *dense over* b if either of the following equivalent conditions hold³:

1. $b = c \vee (c \rightarrow a)$ for some $a \leq c$;

³See [104, Prop. 4, 5] for a proof of these equivalences.

2. $a \leq b$ and $b \rightarrow a = a$.
3. $b \rightarrow a \leq a$.

We denote by $\mathcal{D}_a$ the *filter* of elements dense over a ⁴. In general this filter may not be principal – for instance, take the algebra $1 + \omega^{\text{op}}$, where every element is dense over 0, but there is no least dense element. This motivates the following:

Definition 2.1.38 (KM Heyting algebras). A Heyting algebra H is called a *KM Heyting algebra* if for each $a \in H$, $\mathcal{D}_a$ is principal.

We can see KM-algebras as a *modal Heyting algebra*, by enriching them with an operation $\square$, satisfying the following axioms (see [104, 206]):

Definition 2.1.39 (KM-algebras). A Heyting algebra $(H, \leq, \tau)$ equipped with a unary operation $\tau: H \rightarrow H$ is called a *KM-algebra* if it satisfies⁵:

- (Normality) $\tau(a \wedge b) = \tau a \wedge \tau b$;
- (Inflationarity) $a \leq \tau a$;
- (Cantor-Bendixson) $\tau a \leq b \vee (b \rightarrow a)$;
- (Strong Löb) $\tau a \rightarrow a \leq a$.

We denote by **KM** the class of KM-algebras with Heyting algebra homomorphisms preserving the operation τ , sometimes also called the *successor* operation [77, Exa. 5.1].

Example 2.1.40 (Examples of KM Heyting algebras). We give some examples of KM Heyting algebras:

1. (Finite) Every finite Heyting algebra is **KM**, since every filter is principal in a finite algebra.
2. (Boolean) Every Boolean algebra is a **KM** Heyting algebra, with $\tau a := 1$.
3. (Duals of well-founded posets) Let $H \cong \mathbf{Up}(P)$ where P is a well-founded poset, for instance if H is a profinite Heyting algebra (see Theorem 2.1.31). Then H is a **KM** Heyting algebra. In fact if P is a poset, then $\mathbf{Up}(P)$ admits the structure of a **KM**-algebra if and only if P is well-founded [206, Prop. 13].

⁴See e.g. [155] for a proof that this forms a filter

⁵The terminology of “Cantor-Bendixson” comes from [179], whilst the name “Strong Löb” comes from [132].

4. (Scattered chain) Recall that a linear order $(L, \leq)$ is *scattered* if it contains no subset $C \subseteq L$ which is a dense linear order (see [233, Sec. 2.3])⁶. Then L is a KM-Heyting algebra.

As a result of carrying a uniquely definable structure, KM Heyting algebras have some interesting closure properties inside of the class of Heyting algebras. The following is implicit in [77]:

Proposition 2.1.41 (Closure under products and homomorphic images). The class of KM Heyting algebras is closed under products and homomorphic images.

Proof. It was shown in [155, Prop. 8] that for each Heyting algebra H , and $a \in H$, an arbitrary element $s_a \in H$ is equivalent to the successor τa if and only if for each $b \in H$

$$(s_a \rightarrow b) = (a \rightarrow b) \wedge ((b \rightarrow a) \rightarrow a).$$

We can thus consider the first-order sentence:

$$\forall a \exists s_a \forall b ((s_a \rightarrow b) = (a \rightarrow b) \wedge ((b \rightarrow a) \rightarrow a)).$$

This sentence is a Horn formula, and it is positive. Thus by the classical preservation theorems of Horn formulas (see e.g. [146, Sec. 9]), whereby Horn formulas are preserved by direct products, and positive formulas (see e.g. [146, Thm. 10.3.3]), whereby positive formulas are preserved by surjective homomorphisms, it is preserved under homomorphic images and products. This shows the result. ■

Remark 2.1.42 (Boolean are only uniformly KM variety). We can deduce from Proposition 2.1.41, and some results mentioned later in this chapter, that Boolean algebras are the only variety of Heyting algebras where all algebras are KM Heyting algebras. Indeed, note that as a consequence of Proposition 2.3.39, every non-Boolean variety $\mathbf{V}$ is contained in $\mathbf{Sme}$. On the other hand, $\mathbf{Sme}$ contains algebras which are not KM – consider for instance the algebra $\mathcal{P}_{\emptyset, \text{cof}}(\mathbb{N})$ (see Chapter 7.5 and Appendix A.2.4 for further discussion).

2.1.4 Ordered Topological Spaces and Duality Theory

Having seen the main algebraic structures of interest, we now turn to their dual representations. For that purpose we begin by recalling Hilary Priestley’s duality [223] for distributive lattices, before turning to Esakia duality [106] for Heyting algebras.

⁶By a classic result due to Hausdorff, such linear orders can be constructed from ordinals by concatenation, order duality, and exponentiation; see [233, Ch. 4] for references. Also note that the term *scattered* has topological bearing: a linear order is scattered if and only if it is a scattered topological space when equipped with the order-topology.

Stone and Priestley duality

Definition 2.1.43 (Ordered topological space). An *ordered topological space*, often just an *ordered space*, is a tuple $(X, \leq, \tau)$ of a poset $(X, \leq)$ with a topology.

The study of ordered topological spaces was pioneered by Nachbin. His main findings were collected in [207], to which we refer the reader throughout this section.

Notation 2.1.2. We often write only $(X, \leq)$, or simply X , to denote an ordered space, whenever this causes no confusion.

We denote by $\text{Up}(X)$ the set of upsets of an ordered space X , and write $\text{ClopUp}(X)$ for the *clopen upsets* of X , $\text{ClopDown}(X)$ for the *clopen downsets* of X , and $\text{Clop}(X)$ for the clopens of X .

Definition 2.1.44 (Order-homeomorphism, order-embedding). A map $f: X \rightarrow Y$ between ordered spaces is an *order-homeomorphism* if it is a homeomorphism between the underlying topologies, and a poset isomorphism. An injective map $f: X \rightarrow Y$ is called an *order-embedding* if $f: X \rightarrow f[X]$ is an order-homeomorphism.

Definition 2.1.45 (Priestley space). An ordered space X is called a *Priestley space* if it is compact and satisfies the *Priestley separation axiom*: whenever $x, y \in X$ and $x \not\leq y$, then there is a clopen upset U such that $x \in U$ and $y \notin U$.

Remark 2.1.46 (Priestley implies Stone). It is clear that Priestley separation implies being totally separated⁷ and hence, that X is a Stone space. Moreover, if X is a Stone space, then $(X, =)$, the discretely ordered space, will be a Priestley space, since then the Priestley separation axiom reduces to the usual clopen separation axiom which holds of all Stone spaces.

Notation 2.1.3. Whenever $C \subseteq X$ is seen as an ordered subspace, we take it to have the induced order and topology, unless otherwise specified.

Below we collect some fundamental facts about Priestley spaces which will be used recurringly.

Proposition 2.1.47 (Basic properties of Priestley). Let X be a Priestley space.

1. The collection $\text{ClopUp}(X) \cup \text{ClopDown}(X)$ forms a subbasis for the topology on X ;
2. For each $x \in X$, $\uparrow x$ and $\downarrow x$ are closed. More generally for any C a closed subset, $\uparrow C$ and $\downarrow C$ are closed.

⁷Recall that a space X is said to be *totally separated* if whenever $x \neq y$ are points in X , then there is a clopen $W \subseteq X$ such that $x \in W$ and $y \notin W$ (sometimes also unfortunately called *zero-dimensional*, see [101, Thm. 6.2.1]).

3. If A is a closed upset and B is a closed downset, such that $A \cap B = \emptyset$, then there is a clopen upset U such that $A \subseteq U$ and $B \subseteq X \setminus U$.
4. If C is a closed set, then it is a Priestley space (with the induced order and topology).
5. For any closed set $C \subseteq X$, the set $\max(C)$ is non-empty.

Proof. See e.g. [124, Sec. 3] and the references therein contained. ■

Remark 2.1.48 (Complementation closed subbasis). We will make recurring use of the fact that a Priestley space has a subbasis closed under complementation. As a result of this, a map $f: X \rightarrow Y$ between Priestley spaces is continuous if and only if whenever $U \subseteq Y$ is a clopen upset, $f^{-1}[U]$ is a clopen subset.

We denote by **Pries** the category of Priestley spaces with continuous and order-preserving maps. The following is the celebrated result by H. Priestley:

Theorem 2.1.49 (Priestley duality). *The category **Pries** is dually equivalent to the category **DL**.*

Proof. See [223]. ■

Priestley duality closely parallels Stone duality for Boolean algebras. We briefly recall the two transformations involved in it.

Spaces to Algebras

From Priestley spaces to distributive lattices, we have a functor:

$$\begin{aligned} \text{ClopUp}: \mathbf{Pries} &\rightarrow \mathbf{DL} \\ X &\mapsto \text{ClopUp}(X) \\ f: X \rightarrow Y &\mapsto f^{-1}: \text{ClopUp}(Y) \rightarrow \text{ClopUp}(X), \end{aligned}$$

sending each Priestley space to its lattice of clopen upsets:

$$(\text{ClopUp}(X), \cap, \cup, \emptyset, X),$$

easily seen to be a distributive lattice, and each continuous and order-preserving map to the distributive lattice homomorphism $f^{-1}[-]$.

Algebras to Spaces

From distributive lattices to Priestley spaces, we have a functor:

$$\begin{aligned} \text{Spec}: \mathbf{DL} &\rightarrow \mathbf{Pries} \\ D &\mapsto \text{Spec}(D) \\ f: D \rightarrow D' &\mapsto f^{-1}: \text{Spec}(D') \rightarrow \text{Spec}(D), \end{aligned}$$

where $\mathbf{Spec}(D) = \{F \subseteq D : F \text{ is a prime filter}\}$ (Definition 2.1.20), ordered by inclusion, called its *prime spectrum*. To specify the topology, we make use of specific subsets:

$$\varphi(a) = \{F \in \mathbf{Spec}(D) : a \in F\},$$

Having such subsets, to define a topology, either a subbasis by the following collection:

$$\{\varphi(a) : a \in D\} \cup \{\mathbf{Spec}(D) \setminus \varphi(b) : b \in D\},$$

or, equivalently, a basis:

$$\{\varphi(a) \setminus \varphi(b) : a, b \in D\};$$

the resulting topology is sometimes called the *patch topology* on $\mathbf{Spec}(D)$. With such a topology, the sets of the form $\varphi(a)$ turn out to correspond precisely to the clopen upsets of X . For this reason we sometimes speak of an element $a \in D$ *being represented* by a clopen upset. Moreover, there is a natural relationship between the lattice connectives and their set-theoretic representation, namely, for each $a, b \in D$:

$$\begin{aligned} \varphi(0) &= \emptyset \\ \varphi(1) &= \mathbf{Spec}(D) \\ \varphi(a \wedge b) &= \varphi(a) \cap \varphi(b) \\ \varphi(a \vee b) &= \varphi(a) \cup \varphi(b). \end{aligned}$$

Given a lattice homomorphism $f : D \rightarrow D'$ the preimage $f^{-1}[-]$ becomes a continuous order-preserving map $f^{-1} : \mathbf{Spec}(D') \rightarrow \mathbf{Spec}(D)$.

Duality ensures, for each distributive lattice D , the existence of a natural transformation providing an isomorphism between $D \cong \mathbf{ClopUp}(\mathbf{Spec}(D))$, and for each homomorphism $f : D \rightarrow D'$, an isomorphism $f \cong (f^{-1})^{-1}$ and similarly. Equivalent statements hold for each Priestley space and morphism between Priestley spaces.

It is clear by Remark 2.1.46 that the category **Stone** of Stone spaces and continuous order-preserving maps is, up to an equivalence of definitions, a subcategory of **Pries**: each Stone space corresponds to the Priestley space $(X, =)$, where the order is the identity. This inclusion is dual to the inclusion functor $\mathbf{l} : \mathbf{BA} \hookrightarrow \mathbf{DL}$. This functor $\mathbf{l}$ has a left adjoint denoted by $\mathbf{B} : \mathbf{DL} \rightarrow \mathbf{BA}$ the Boolean envelope functor. This is easily described dually:

Proposition 2.1.50 (Boolean envelope dually). The functor $\mathbf{F} : \mathbf{Pries} \rightarrow \mathbf{Stone}$ which assigns to a Priestley space $(X, \leq)$ the underlying Stone space, and to each order-preserving continuous function the underlying continuous function, is dual to the Boolean envelope functor.

Proof. See e.g. [124, Prop. 3.4]. ■

Esakia duality

Esakia spaces were originally introduced as ordered Stone spaces $(X, \leq)$ where $\leq$ is closed as a relation in $X \times X$ and continuous: whenever $U \subseteq X$ is open, then $\leq^{-1}[U]$ is open; note that $\leq^{-1}[A] = \downarrow A$ by definition. Nowadays it is standard to give their definition by specializing Priestley duality⁸:

Definition 2.1.51 (Esakia space). A Priestley space X is called an *Esakia space* if whenever W is a clopen subset, $\downarrow W$ is clopen.

Some unique properties of Esakia spaces are collected in the next proposition:

Proposition 2.1.52 (Basic properties of Esakia). Let $(X, \leq)$ be an Esakia space:

1. If C is a closed upset, then C is an Esakia space (with the induced order and topology).
2. If W is a clopen subset, then W is an Esakia space (with the induced order and topology).
3. If C is a closed subset, then $\max(C)$ is closed and non-empty. Moreover, if $x \in C$, then $x \in \downarrow \max(C)$.
4. For each subset A , $\overline{\uparrow A} = \uparrow \overline{A}$.

In addition to the objects being different, the morphisms between Esakia spaces are more specialized than general continuous and order-preserving maps:

Definition 2.1.53 (p-morphism). Given two posets P, Q , an order-preserving map $f : P \rightarrow Q$ is called a *p-morphism* if whenever $f(x) \leq y$ where $x \in P$ and $y \in Q$, there is some $x' \geq x$ such that $f(x') = y$.

Remark 2.1.54 (Different presentations of p-morphisms).

1. It is well known (see, e.g., [102, Prop. 1.4.12]) that for an order-preserving map $f : X \rightarrow Y$, the following are equivalent:
 - f is a p-morphism.
 - If U is an upset of X , then $f[U]$ is an upset of Y .
 - For each $x \in X$, $\uparrow f(x) \subseteq f[\uparrow x]$ (the equality holds since it is order-preserving);
 - For each $y \in Y$, $f^{-1}[\downarrow y] \subseteq \downarrow f^{-1}[y]$.
2. Note that while order-homeomorphisms are necessarily p-morphisms, not every order-embedding is a p-morphism.

⁸The equivalence with the above original definition was shown by Esakia, see [102, Chapter 3.1].

We denote by **Esa** the category of Esakia spaces and continuous p-morphisms.

Theorem 2.1.55 (Esakia duality). *The category **Esa** is dually equivalent to the category **HA**.*

Proof. See [102, Cor. 3.4.8.(2)]. ■

The duality is obtained by restricting the functors **ClopUp** and **Spec** appropriately. There are two crucial aspects in Esakia duality *viz.* Priestley duality: the modelling of the implication connective, and the structure of the maps. Concerning the first, if D is any distributive lattice, and $a, b \in D$ are such that $a \rightarrow b$ exists, then:

$$\varphi(a \rightarrow b) = \mathbf{Spec}(D) \setminus \downarrow(\varphi(a) \setminus \varphi(b)) = \Box(\varphi(a) \Rightarrow \varphi(b)) = \varphi(a) \rightarrow \varphi(b),$$

that is, we model the Heyting implication in a modal way, using the order. Concerning the maps, given $f: H \rightarrow H'$ a distributive lattice homomorphism, it will be a Heyting algebra homomorphism if and only if $f^{-1}: \mathbf{Spec}(H') \rightarrow \mathbf{Spec}(H)$ is a continuous p-morphism.

bi-Esakia duality

Occasionally we will also talk about duals of bi-Heyting algebras. For a detailed account of this the reader is referred to e.g., [64].

Definition 2.1.56 (Co-Esakia, bi-Esakia). A Priestley space $(X, \leq)$ is called a *co-Esakia space* if $(X, \geq)$ is an Esakia space: whenever W is a clopen subset, $\uparrow W$ is clopen. X is called a *bi-Esakia space* if it is both an Esakia space and a co-Esakia space.

Definition 2.1.57 (Co-p-morphism). An order-preserving map between posets $f: (P, \leq) \rightarrow (Q, \leq)$ is called a *co-p-morphism* if $f: (P, \geq) \rightarrow (Q, \geq)$ is a p-morphism. It is called a *bi-p-morphism* if it is both a p-morphism and a co-p-morphism.

We denote by **biEsa** the category of bi-Esakia spaces and continuous bi-p-morphisms. The following is the appropriate restriction of Esakia duality to this setting:

Theorem 2.1.58 (Duality for bi-Esakia). *The category **biEsa** is dually equivalent to the category **biHA**.*

Proof. See e.g. [102, Prop A.11.1]. ■

The modelling of the coimplication/difference is given, in specular fashion, for any $a, b \in D$ in a distributive lattice where $a - b$ exists,

$$\varphi(a - b) = \uparrow(\varphi(a) \setminus \varphi(b)).$$

Moreover, bi-Heyting homomorphisms correspond to bi-p-morphisms in the expected way.

Remark 2.1.59 (Order duality). Due to the symmetry in the definitions, when working with bi-Esakia spaces we often take for granted the order-dual properties of those that hold for Esakia spaces: for example a clopen subset of a bi-Esakia space is again bi-Esakia, in light of Proposition 2.1.52.(2).

KM spaces

In this section we recall a characterization of the duals of KM Heyting algebras; these were mentioned without proof, for example, in [104].

Definition 2.1.60 (KM spaces). We say that an Esakia space X is a *KM Esakia space* if $\text{ClopUp}(X)$ is a KM Heyting algebra.

Proposition 2.1.61 (Duality for KM algebras). An Esakia space X is a KM space if and only if one of the following equivalent conditions holds:

1. For each $U \subseteq X$ a clopen upset, $\tau(U) = U \cup \max(X \setminus U)$ is a clopen (upset);
2. For each $U \subseteq X$ a clopen upset, $\max(X \setminus U)$ is clopen;
3. For each $W \subseteq X$ a clopen subset, $\max(W)$ is clopen.

Proof. First note that the conditions ((1)–(3)) are indeed equivalent: given (1) holds, then $\tau(U) \setminus U = \max(X \setminus U)$ will always be clopen, so we obtain (2); (2) $\Rightarrow$ (1), and (3) $\Rightarrow$ (2) are obvious. Finally we show that (2) $\Rightarrow$ (3). Given any clopen subset W , we have:

$$\max(W) = \max(\downarrow W),$$

and since $\downarrow W$ is clopen, since X is Esakia, and it will be a downset, we have that $X \setminus \downarrow W$ will be a clopen upset. Thus by hypothesis, $\max(X \setminus (X \setminus \downarrow W)) = \max(\downarrow W) = \max(W)$ will be clopen.

We now show that the conditions are equivalent to being the dual of a KM Heyting algebra. So first assume that X is dual to a KM Heyting algebra H . Then we will show that

$$\varphi(\tau a) = \tau(\varphi(a)) = \varphi(a) \cup \max(X \setminus \varphi(a));$$

if $x \in \varphi(\tau a)$, and $x \notin \varphi(a)$, we will show that whenever $x < y$, then $y \in \varphi(a)$. So assume that $x < y$, which by hypothesis means that there is some $b \in H$ where $y \in \varphi(b)$ and $x \notin \varphi(b)$. Since $x \in \varphi(\tau a)$, we have by the Cantor-Bendixsson axiom that $x \in \varphi(b) \cup \varphi(b \rightarrow a)$; since $x \not\leq y$, then also $y \in \varphi(b \rightarrow a)$, which means that $y \in \varphi(a)$. For the converse inclusion, by inflationarity, we have that $\varphi(a) \subseteq \varphi(\tau a)$. So assume that we have $x \in \max(X \setminus \varphi(a))$; since $x \notin \varphi(a)$, and by the Strong Löb axiom we have $\varphi(\tau a \rightarrow a) \subseteq \varphi(a)$, we have that there must be some $y \geq x$ such that $y \in \varphi(\tau a) \setminus \varphi(a)$; however by maximality, we conclude that $x = y$, meaning that $x \in \varphi(\tau(a))$. We conclude the equality of the sets. Since the

sets of the form $\varphi(a)$ are precisely the clopen upsets of X , by condition (1), we conclude that X is a KM space.

Now assume that (1) holds. Then we can verify that this definition yields a successor operation on the algebra $\text{ClopUp}(X)$, by verifying the axioms of KM-algebra:

1. (Normality) First assume that $x \in \tau(U \cap V)$. If $x \in \max(X \setminus (U \cap V))$, which implies that $x \in \max(X \setminus U) \cap \max(X \setminus V)$, and so $x \in \tau(U) \cap \tau(V)$. If $x \in U \cap V$, certainly also $x \in \tau(U) \cap \tau(V)$. So we obtain that $\tau(U \cap V) \subseteq \tau(U) \cap \tau(V)$.

Conversely, if $x \in \tau(U) \cap \tau(V)$, note that if $x \in U$ and $x \in V$, then $x \in U \cap V$, so $x \in \tau(U \cap V)$. If $x \in U$ and $x \in \max(X \setminus V)$, then also $x \in \max(X \setminus (U \cap V))$, since whenever $x < y$, then $y \in U \cap V$ because U is a clopen upset and $x \in U$. Similarly, if $x \in V \cap \max(X \setminus U)$. Finally if $x \in \max(X \setminus U) \cap \max(X \setminus V)$, then whenever $x < y$, $y \in U \cap V$, so $x \in \max(X \setminus (U \cap V))$. In all cases we conclude that $x \in \tau(U \cap V)$, showing that $\tau(U) \cap \tau(V) \subseteq \tau(U \cap V)$.

2. (Inflationarity) This is obvious, since $U \subseteq \tau(U)$.
3. (Cantor Bendixson) Let V be any other clopen upset. Suppose that $x \in \tau(U)$, and $x \notin V$, and let us show that $x \in V \rightarrow U$. If $x \in U$, certainly this holds since $U \subseteq V \rightarrow U$. Otherwise, $x \in \max(X \setminus U)$. Let $x \leq y$, and $y \in V$. Since $x \neq y$, by assumption of maximality, $y \in U$. Thus we conclude that $\tau(U) \subseteq V \cup V \rightarrow U$.
4. (Strong Löb) Suppose that $x \notin U$. Then $x \in \downarrow(X \setminus U)$, and so $x \in \downarrow \max(X \setminus U)$ by Proposition 2.1.52. Let $x \leq y$ where $y \in \max(X \setminus U)$. Then $y \in \tau(U)$, but $y \notin U$. Thus $x \notin \tau(U) \rightarrow U$. This shows by contraposition that $\tau(U) \rightarrow U \subseteq U$.

We thus conclude that $\text{ClopUp}(X)$ is a KM Heyting algebra, establishing the result. ■

2.1.5 Duality and constructions

Having introduced the basic duality, we now spell out the correspondence between several topological and algebraic constructions. We summarise all these results with a duality dictionary in Table 2.1, to which we will refer throughout the text.

Products, quotients and factors of algebras

Notation 2.1.4 (Sums of spaces). Given two (ordered) spaces X, Y , we denote by $X \sqcup Y$ the sum of spaces: the carrier set is the disjoint union, the topology is

the sum topology, and the order is the one on each disjoint component. Given a family $(X_i)_{i \in I}$, we denote by $\bigsqcup_{i \in I} X_i$ the sum of the topologies and the order. Whenever relevant, this is distinguished from $\bigoplus_{i \in I} X_i$, the coproduct of these spaces.

Proposition 2.1.62 (Products are dually coproducts). Let D, D' be two distributive lattices. Then:

$$\text{Spec}(D \times D') \cong \text{Spec}(D) \sqcup \text{Spec}(D').$$

Proof. See e.g. [94, Ex. 11.8]. ■

Proposition 2.1.63 (Duality for maps). Let $f: D \rightarrow D'$ be a lattice homomorphism, and let $f^{-1}: \text{Spec}(D') \rightarrow \text{Spec}(D)$ the dual continuous and order-preserving map. Then:

1. f is injective if and only if f^{-1} is surjective.
2. f is surjective if and only if f^{-1} is an embedding.

Proof. See e.g. [124, Prop. 3.28]. ■

Recall that given a distributive lattice D , and $S \subseteq D \times D$ we can consider the smallest congruence generated by S , $\Theta(S)$ (see Section 2.2). We can thus construct $D/\Theta(S)$ in the usual manner in universal algebra [74, Sec. 2.6]. A dual representation of this process is given as follows:

Proposition 2.1.64 (Congruences to closed subsets). If D is a distributive lattice and $S \subseteq D \times D$, then, we have that

$$\text{Spec}(D)_S := \{x \in \text{Spec}(D) : \forall (a, b) \in S, x \in \varphi(a) \iff x \in \varphi(b)\},$$

is the Priestley dual of $D/\Theta(S)$.

Proof. See e.g. [124, Def. 4.14, Prop. 4.15]. ■

In the case of Heyting algebras, the correspondence is more precisely with filters:

Proposition 2.1.65 (Duality between filters and closed upsets). Let H be a Heyting algebra, and F a filter. Then

$$\mathcal{C}(F) = \{x \in \text{Spec}(H) : \forall a \in F, a \in x\},$$

is a closed upset of $X = \text{Spec}(H)$. Conversely, given a closed upset C , the set $F(C) = \{a \in H : C \subseteq \varphi(a)\}$ is a filter. This assignment is one-to-one and onto and an order-isomorphism, and we have moreover that:

$$\mathcal{C}(\text{Fil}(S)) = \bigcap_{a \in S} \varphi(a).$$

Proof. This was proved in [102, Thm. 3.4.16]. ■

Proposition 2.1.66 ((Finitely) Subdirectly irreducible algebras). Let H be a Heyting algebra and $X = \mathbf{Spec}(H)$. An element $a \in H$ is join irreducible if and only if $\varphi(a)$ is rooted. It is completely join-irreducible if and only if $\varphi(a)$ is rooted at r and $\{r\}$ is clopen.

Proof. See [102, Prop. A.1.1, Prop. A.1.2]. ■

Remark 2.1.67 (Strongly rooted Esakia space). An Esakia space X which is in the stronger conditions of Proposition 2.1.66 – it is rooted, and the rooted is an isolated point – is called *strongly rooted*.

As a consequence of this duality and correspondence of congruences with spatial structures, we can prove some easy facts which are rather specific to categories of lattices:

Proposition 2.1.68 (Binary commutativity of homomorphic images and products). Let A, B be distributive lattice based algebras. Then we have:

$$\text{Con}(A \times B) \cong \text{Con}(A) \times \text{Con}(B).$$

Consequently if $f: A \times B \rightarrow C$ is a surjective homomorphism, then $C \cong f \upharpoonright_{\pi_A} [A] \times f \upharpoonright_{\pi_B} [B]$.

Proof. An algebraic proof can be found in [195, Thm. 2.70]. The dual argument follows easily from Propositions 2.1.62 and 2.1.63, by observing that congruences, seen equivalently as homomorphic images, will be closed subsets of each of the two disjoint components, and hence, can be factored as congruences of each of the two factors. ■

Products and pullbacks of spaces

Notation 2.1.5 (Cartesian products and pullbacks). Throughout, given (X_i) a family we denote by $\prod_{i \in I} X_i$ the cartesian product of this family. We denote by $\pi_i: \prod_{i \in I} X_i \rightarrow X_i$ the i -th projection. When we take a binary product $X \times Y$, we denote by π_X, π_Y the projections onto each factor. We denote by X^I the I -fold product of X .

Given $f: X \rightarrow Y$ and $g: Z \rightarrow Y$ a pair of continuous and order-preserving maps, we denote by $X \times_Y Z$ the set-theoretic pullback, namely

$$X \times_Y Z = \{(x, z) \in X \times Z : f(x) = g(z)\}.$$

Proposition 2.1.69 (Product of Priestley). Given $(X_i)_{i \in I}$ a family of Priestley spaces, $\prod_{i \in I} X_i$ with the product topology and order is a Priestley space. Moreover, if each X_i is Esakia, $\prod_{i \in I} X_i$ is likewise Esakia⁹

⁹We warn the reader that this is *not* the categorical product of Esakia spaces, in general: while the cone $X \leftarrow X \times Y \rightarrow Y$ lives inside the category, the unique comparison function from another cone is not in general a p-morphism. This will be the theme of Chapter 4, Section 4.4.2.

Proof. For (1-2) about Priestley see [94, Chapter 11]. For the facts about Esakia spaces, see [102, A.8.1] for products. ■

For pullbacks, we have that these are also computed as in **Set**, and satisfy some desirable properties:

Proposition 2.1.70 (Pullback is Priestley). $f: X \rightarrow Y$ and $g: X \rightarrow Z$ continuous and order-preserving maps, $X \times_Y Z$, is a Priestley space. Moreover, if f is a surjective map, then π_Z will likewise be surjective.

Proof. The fact that $X \times_Y Z$ is a Priestley space follows straightforwardly from Proposition 2.1.69, using the fact that $X \times_Y Z$ is a closed subset of $X \times Z$. The fact that surjective maps are stable under pullback is a well-known fact, which we prove for completeness: if $z \in Z$ is arbitrary, look at $g(z)$; by surjectivity of f , there is some $x \in X$, such that $f(x) = g(z)$; then $(x, z) \in X \times_Y Z$, and so $\pi_Z(x, z) = z$, as desired. ■

Directed and inverse systems

Definition 2.1.71 (Directed system, inverse system). A family $(A_n, \iota_{i,j})_{n,i,j \in \omega}$ of objects in a category **C** is called an $(\omega$ -indexed) *directed system* if for each $i, j \in \omega$, $\iota_{i,j}: A_i \rightarrow A_j$ is a morphism, subject to the conditions (1) $\iota_{i,i}: A_i \rightarrow A_i$ is the identity, and (2) for $i \leq j \leq k$, we have $\iota_{i,k} = \iota_{j,k} \iota_{i,j}$.

Dually, a family $(X_n, p_{i,j})_{n,i,j \in \omega}$ of objects in a category **C** is called an *inverse system* if for each $i, j \in \omega$, $p_{i,j}: X_j \rightarrow X_i$ is a morphism subject to the dual conditions: (1) $p_{i,i}: X_i \rightarrow X_i$ is the identity, and (2) for $i \leq j \leq k$ we have $p_{i,k} = p_{i,j} p_{j,k}$.

Definition 2.1.72 (Direct colimit, inverse limit). Given a directed system (A_n) in a category **C**, we say that an object A_ω is the *directed colimit* if there exist morphisms $\iota_n: A_n \rightarrow A_\omega$ witnessing that A_ω is the *colimit*¹⁰ in **C** of the directed system.

Dually, given an inverse system $(X_n, p_{i,j})$, we say that an object X_ω is the *inverse limit* if there exist projections $\pi_n: X_\omega \rightarrow X_n$ witnessing that X_ω is the limit of this diagram.

Remark 2.1.73 (Caveat on inverse systems). Note that in general, if one does not assume that the maps $\iota_{i,j}$ are embeddings (respectively the maps $p_{i,j}$ are surjective), it well-possible that the limits are empty. This will not be a problem for us, since in all cases at hand, the limits will be well-defined.

The following proposition is a well-known fact, holding in much greater generality:

¹⁰The reader should notice that in some literature this is called the directed limit. We use the term directed colimit to emphasize the duality with inverse limits.

Proposition 2.1.74 (Directed colimit is a Heyting algebra). Let $(D_n, \iota_{i,j})$ be a chain of distributive lattices, connected by embeddings $\iota_{i,j} : D_i \rightarrow D_j$. If for each n , D_n is a Heyting algebra (bi-Heyting algebra), and for each i, j , $\iota_{i,j}$ is a Heyting algebra homomorphism, then D_ω is a Heyting algebra.

Proof. See e.g. [146, Thm. 2.4.4] for a proof of a more general fact concerning universal formulas. ■

Proposition 2.1.75 (Inverse limits are (bi-)Esakia). If $(X_n, p_{i,j})$ be an inverse system of Priestley spaces, connected by surjective homomorphisms $p_{i,j} : X_j \rightarrow X_i$. If for each n , X_n is an Esakia (bi-Esakia) space, and each $p_{i,j}$ is a p-morphism (bi-p-morphism) then X_ω is an Esakia space (bi-Esakia space).

Proof. This follows immediately by Proposition 2.1.74, by tracing the concepts along duality (consulting Table 2.1). ■

Remark 2.1.76 (Explicit description of inverse limits). Since we will need to work often with the topology of inverse limits, we recall the explicit description of the inverse limit X_ω of an inverse system $(X_n, p_{i,j})$ of Priestley spaces. This consists of the set

$$X_\omega = \left\{ x \in \prod_{n \in \omega} X_n : \forall n \leq m, p_{m,n} \pi_m(x) = \pi_n(x) \right\}.$$

We equip this with the pointwise order induced by $\prod_{n \in \omega} X_n$, and the subspace topology induced on the product topology. Explicitly, this means that the clopen upsets of X_ω are of the form $\pi_n^{-1}[U]$ for $U \subseteq X_n$ a clopen upset.

Notation 2.1.6. Given an inverse system $(X_n, p_{i,j})$ of Priestley spaces, and X_ω its inverse limit, given $x \in X_\omega$ we will write simply $x(n)$ to mean $\pi_n(x)$, leaving the projection maps implicit.

Adjoint maps and dual adjoints

Since we will frequently discuss adjoint maps to Heyting homomorphisms, especially in the context of dual spaces, we record the following fact:

Proposition 2.1.77 (Adjoint maps of Heyting algebra homomorphisms). Let $f : X \rightarrow Y$ be a p-morphism between Esakia spaces. Suppose that the dual map $f^{-1}[-] : \text{ClopUp}(Y) \rightarrow \text{ClopUp}(X)$ has a left adjoint $\exists_{f^{-1}[-]}$ and a right adjoint $\forall_{f^{-1}[-]}$. Then:

1. $\exists_{f^{-1}[-]}$ is precisely the map $f[-]$; so f maps, through the direct image, clopen upsets to clopen upsets;
2. $\forall_{f^{-1}[-]}$ is precisely the map $Y \setminus \downarrow f[X \setminus (-)]$; so the downset of a direct image of a clopen downset is a clopen.

Proof. See [136, Sec. 2], and [229, Lem. 6.3]. ■

Vietoris spaces

An especially useful construction on spaces that will serve us extensively throughout the thesis, the so-called *Vietoris space*, due to Leopold Vietoris [256] (see also [49]).

Definition 2.1.78 (Vietoris space). Let X be a Stone space. We denote by $\mathcal{V}(X)$, the *Vietoris space over X* , whose carrier is the set of non-empty closed subsets of X . We give it the *hit-or-miss topology* by considering a subbasis consisting of sets,

$$\llbracket U \rrbracket = \{C \in \mathcal{V}(X) : C \subseteq U\} \text{ and } \langle V \rangle = \{C \in \mathcal{V}(X) : C \cap V \neq \emptyset\}.$$

Given $(X, \leq)$ a Priestley space, we denote by $(\mathcal{V}(X), \supseteq)$ the *Vietoris space over $(X, \leq)$* , equipped with the *reverse inclusion*.

Lemma 2.1.79 (Vietoris space is Priestley). Whenever X is a Stone space, the space $(\mathcal{V}(X), \supseteq)$ is a Priestley space.

Proof. The fact that $\mathcal{V}(X)$ is compact is a standard fact (see e.g. [202, Thm. 4.2]). The Priestley separation axiom was shown in [49, Prop. 3.4]. ■

Vietoris spaces have been long used from a coalgebraic point of view: notably, one of Esakia's original definitions of an Esakia space consisted of a Stone space with an order, such that the map $\uparrow: X \rightarrow \mathcal{V}(X)$ is order-preserving and continuous:

Lemma 2.1.80 (Esakia spaces from up-arrow map). For each Priestley space X , X is an Esakia space if and only if $\uparrow: X \rightarrow \mathcal{V}(X)$ is continuous.

Proof. See [106]. ■

We will see various variations of this construction in Chapters 4 and 5. For reference, we finish with a collection of basic properties about Vietoris spaces:

Lemma 2.1.81 (Basic properties of Vietoris spaces).

1. If X is a Stone space, then the space $\mathcal{V}(X)$ is a Stone space.
2. If $f: X \rightarrow Y$ is a continuous map between Stone spaces, then the function

$$\begin{aligned} f[-]: \mathcal{V}(X) &\longrightarrow \mathcal{V}(Y) \\ C &\longmapsto f[C] \end{aligned}$$

is continuous.

3. If W is a clopen subset of a Stone space X , then the function

$$\begin{aligned} r_W: \mathcal{V}(X) &\longrightarrow \mathcal{V}(X) \\ C &\longmapsto C \cap W \end{aligned}$$

is continuous.

4. If X is an Esakia space, then $\mathcal{V}^\uparrow(X)$ is a closed subset of $\mathcal{V}(X)$.

Proof. (2) This is well known (for a proof see e.g. [173, Lem. 3.9]).

(3) Let W be a clopen subset of a Stone space X . The map $r_W: \mathcal{V}(X) \rightarrow \mathcal{V}(X)$ is well-defined since W is closed. It is moreover continuous, since, for any subbasic clopen in $\mathcal{V}(X)$ of the form $\llbracket W_0 \rrbracket$ for a clopen $W_0 \subseteq X$, we have:

$$r_W^{-1}[\llbracket W_0 \rrbracket] = \{C \in \mathcal{V}(X) : C \cap W \subseteq W_0\} = \llbracket W \Rightarrow W_0 \rrbracket,$$

which is clopen; moreover, it follows that the set $r_W^{-1}[\langle W_0 \rangle] = \mathcal{V}(X) \setminus r_W^{-1}[\llbracket X \setminus W_0 \rrbracket]$ is clopen.

(4) Let X be an Esakia space. Let $K \in \mathcal{V}(X) \setminus \mathcal{V}^\uparrow(X)$; we prove that there is a clopen subset of $\mathcal{V}(X)$ to which K belongs and that is disjoint from $\mathcal{V}^\uparrow(X)$. Since K is not an upset, there are $x \in K$ and $y \in X$ such that $x \leq y$ and $y \notin K$. By zero-dimensionality, there is a clopen subset W such that $K \subseteq W$ and $y \notin W$. Then, we consider

$$\llbracket W \rrbracket \cap \langle \downarrow(X \setminus W) \rangle.$$

Note that $\downarrow(X \setminus W)$ is clopen because X is Esakia. Moreover, note that K belongs to this set, since $K \subseteq W$ and $K \ni x \leq y \notin W$. Furthermore, for every $K' \in \llbracket W \rrbracket \cap \langle \downarrow(X \setminus W) \rangle$, by definition we have $K' \subseteq W$ and there are $x' \in K'$ and $y' \in X$ such that $x' \leq y'$ and $y' \notin W$, which implies that K' is not upwards closed. ■

2.2 Algebras of Logic

In this section we summarise the basics of universal algebra and algebraic logic which will be necessary throughout.

Notation 2.2.1 (Logical and algebraic notations). Whenever working with logics or algebras we will assume that we are working in a (often finitary) functional signature $\mathcal{L}$ comprising at least a constant symbol. The set of *terms* (also *propositional formulas* or just *formulas*) over a set Prop will be denoted by $\text{Fm}_{\mathcal{L}}(\text{Prop})$. We use the letters $p, q, r, \dots$ for variables, and compactly represent a tuple of distinct variables as $\bar{p}$; we use the letters $\varphi, \psi, \dots$ for $\mathcal{L}$ -formulas, and reserve the symbols $\Gamma, \Delta, \dots$ for sets of formulas. When $\mathcal{L}$ is understood, we drop the subscripts. The notation $\varphi(\bar{p})$ means that the formula φ has free variables

included in the tuple $\bar{p}$. Since our *tuples of variables* are assumed to be formed by *distinct* elements, we emphasize that when we write e.g. $\varphi(\bar{p}, \bar{q})$, we mean that the tuples $\bar{p}, \bar{q}$ are made of distinct variables and are also disjoint from each other.

Notation 2.2.2. We fix some algebraic notation. Given an algebra A and $S \subseteq A$, we denote by $\langle S \rangle$ the subalgebra of A generated by S . We often write $\Theta, \Xi, \dots$ for congruences of algebras. Given pairs $(a_1, b_1), \dots, (a_n, b_n)$ of elements, we denote by $\Theta((a_1, b_1), \dots, (a_n, b_n))$ the congruence generated by these pairs. We denote the quotient algebra A/Θ , and denote by $p_\Theta : A \rightarrow A/\Theta$ the canonical projection.

2.2.1 Algebraizable logics and equivalent algebraic semantics

The uniqueness of construction of formulas makes $\text{Fm}_{\mathcal{L}}(\text{Prop})$ into an algebra, namely the $\mathcal{L}$ -algebra of formulas. This is also sometimes referred to as the *absolutely free* $\mathcal{L}$ -algebra.

Definition 2.2.1 (Substitution). A map $\sigma : \text{Prop} \rightarrow \text{Fm}_{\mathcal{L}}(\text{Prop})$ is called a *substitution*; equivalently this is given by an endomorphism $\bar{\sigma} : \text{Fm}_{\mathcal{L}}(\text{Prop}) \rightarrow \text{Fm}_{\mathcal{L}}(\text{Prop})$.

Notation 2.2.3. Notations like $\varphi(\bar{\psi}/\bar{p})$ (or just $\varphi(\bar{\psi})$) denote the result of substituting $\bar{p}$ by $\bar{\psi}$ inside of φ , though we will occasionally use $\sigma(\varphi)$ to emphasize the specific substitution being used. For a set of formulas Γ , we write $\sigma\Gamma$ for $\{\sigma(\psi) : \psi \in \Gamma\}$.

Definition 2.2.2 (Finitary logic). A logic ¹¹ in the language $\mathcal{L}$, over a set of variables P , is a relation $\vdash \subseteq \wp(\text{Fm}_{\mathcal{L}}(\text{Prop})) \times \text{Fm}_{\mathcal{L}}(\text{Prop})$, subject to the following rules for $\Gamma \cup \Delta \cup \{\varphi\} \subseteq \text{Fm}_{\mathcal{L}}(\text{Prop})$:

- (Identity): For each $\Gamma \subseteq \text{Fm}_{\mathcal{L}}(\text{Prop})$, and $\varphi \in \Gamma$, $\Gamma \vdash \varphi$;
- (Monotonicity): If $\Gamma \vdash \varphi$ and $\Gamma \subseteq \Delta$, then $\Delta \vdash \varphi$.
- (Transitivity) If $\Gamma \vdash \varphi$ and $\Delta \vdash \psi$ for each $\psi \in \Gamma$, then $\Delta \vdash \varphi$.
- (Structurality): If $\Gamma \vdash \varphi$ then $\sigma\Gamma \vdash \sigma(\varphi)$.
- (Finitarity): If $\Gamma \vdash \varphi$ there is a finite set $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash \varphi$.

We write $\Gamma \vdash \Delta$ to mean that $\Gamma \vdash \varphi$ for each $\varphi \in \Delta$. Elements $(\Gamma, \varphi) \in \wp(\text{Fm}_{\mathcal{L}}(\text{Prop})) \times \text{Fm}_{\mathcal{L}}(\text{Prop})$ are called *rules*. We often denote them as Γ/φ . We denote by $\text{Ext}(\vdash)$ the lattice of (axiomatic) extensions of $\vdash$: the set of all logics $\vdash' = \vdash \cup \{(\emptyset, \varphi_i) : i \in I\}$ for some formulas φ_i .

¹¹All logics considered in this thesis will be finitary, so this property is taken as part of the definition; the reader interested in more general definitions can consult [116].

When working with specific systems, it is sometimes useful to discuss rules in the context of Hilbert-style derivation systems.

Definition 2.2.3 (Hilbert system for logics). Let Σ be a set of rules in the language $\mathcal{L}$. A sequence $(\varphi_0, \dots, \varphi_n)$ of $\mathcal{L}$ -formulas is a Σ -*derivation* of φ from Γ , if and only if:

1. For each φ_i one of the following holds:
 - (a) $\varphi_i \in \Gamma$;
 - (b) There are a substitution σ , $(\Delta, \psi) \in \Sigma$, and such that $1 \leq j_n < i$, $\varphi_i = \sigma(\psi)$, and $\sigma[\Delta] = \{\varphi_{j_1}, \dots, \varphi_{j_k}\}$.
2. $\varphi_n = \varphi$.

We write $\Gamma \vdash_{\Sigma} \varphi$ to mean that there exists a derivation of φ from Γ using the rules from Σ .

Given a language $\mathcal{L}$, the $\mathcal{L}$ -formulas can likewise be seen as terms in a first-order language. $\mathcal{L}$ -equations are denoted as $\varphi \approx \psi$. $\mathcal{L}$ -algebras are universal algebras with the functional type $\mathcal{L}$, and we often denote them by $A, B, C, \dots$. Throughout the thesis, logics will always refer to *algebraizable logics*, meaning that there is a class $\mathbf{A}$ of $\mathcal{L}$ -algebras, and essentially inverse structural transformers between $\mathcal{L}$ -formulas and $\mathcal{L}$ -equations, mapping $\vdash$ -rules to $\mathbf{A}$ -valid quasi-equations and vice-versa (see [116, Definition 3.11] for the precise definitions). Such a $\mathbf{A}$ is called the *equivalent algebraic semantics* of $\vdash$, and denoted as $\text{Alg}(\vdash)$, whilst $\vdash$ is referred to as the *logic associated to $\mathbf{A}$* , denoted as $\text{Log}(\mathbf{A})$.

Definition 2.2.4 (Variety, lattice of subvarieties). A class $\mathbf{A}$ of algebras is a *variety* if it is closed under homomorphic images, subalgebras and products. We denote by $\text{Var}(\mathbf{A})$, the lattice of subvarieties of a variety $\mathbf{A}$.

We denote class operators by $\mathbb{I}$ (isomorphisms), $\mathbb{H}$ (homomorphic images), $\mathbb{S}$ (subalgebras), $\mathbb{P}$ (products), $\mathbb{P}_S$ (subdirect products) and $\mathbb{P}_U$ (ultraproducts). It is well-known that varieties correspond to equational classes (Birkhoff's theorem) and to classes of the form $\mathbb{HSP}(\mathbf{A})$ (Tarski's theorem). We denote by $\mathbb{V}$ the variety closure operator. It is moreover known that if a logic $\vdash$ is algebraizable, the class $\mathbf{A}$ given above is unique; we will throughout assume that it is actually a variety.

Most of the logics we will be interested in this thesis carry additional structure beyond being algebraizable:

Definition 2.2.5 (Implicative logic). A logic $\vdash$ over a language $\mathcal{L}$ is *implicative* if it contains a connective $\rightarrow$ satisfying the following additional properties:

1. $\vdash p \rightarrow p$;

2. $p \rightarrow q, q \rightarrow r \vdash p \rightarrow r$;
3. For each n -ary formula λ for $n \geq 1$, $\{p_i \rightarrow q_i, q_i \rightarrow p_i : 1 \leq i \leq n\} \vdash \lambda(p_1, \dots, p_n) \rightarrow \lambda(q_1, \dots, q_n)$;
4. $p, p \rightarrow q \vdash q$;
5. $p \vdash q \rightarrow p$;

Remark 2.2.6 (Conventions on implicative logics). Implicative logics naturally carry a connective 1, given by $p \rightarrow p$, and in all the cases of interest we will likewise consider it to be a part of the language $\mathcal{L}$. In all cases of interest the language $\mathcal{L}$ will likewise have a connective $\wedge$, satisfying $p, q \dashv\vdash p \wedge q$. We abbreviate $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ by $\varphi \leftrightarrow \psi$.

The structural transformers for implicative logics are very simple: one maps a formula φ to the equation $\varphi \approx 1$, and an equation $\varphi \approx \psi$ to the formula $\varphi \leftrightarrow \psi$. As such, we will often simply write $A \models \varphi$ for φ an $\mathcal{L}$ -formula to mean $A \models \varphi \approx 1$. Likewise $\mathbf{A} \models \varphi$ means that a class of algebras validates φ .

Remark 2.2.7 (Centrality of implicative logics). Because of the focus of the present thesis first and foremost on intuitionistic logic and its close relatives, **all results of general nature will be stated for implicative logics with a conjunction connective $\wedge$** , and all logics treated in this thesis will be assumed to contain such an implication.

Remark 2.2.8 (Positive logic). A singular exception to the previous remark as regards to logics containing an implication connective, will be so-called *positive logic* (PL): this corresponds to the logic of distributive lattices when enriched with an internal sequent operation. Explicitly, it is given by the pairs $\varphi \leq \psi$ where $\varphi, \psi \in \mathcal{L}_{\mathbf{DL}}(\text{Prop})$. The symbol $\leq$ is interpreted as the order on a distributive lattice, in such a way that all results concerning algebraic completeness go through; the reader is referred to [56] for details on this procedure. For our purposes it is sufficient to know the precise statement of algebraic completeness, namely, for $\varphi(\bar{p}), \psi(\bar{p}) \in \mathcal{L}_{\mathbf{DL}}$, we have:

$$\begin{aligned} \vdash_{\text{PL}} \varphi(\bar{p}) \leq \psi(\bar{p}) &\iff \forall D \in \mathbf{DL} (D \models \forall \bar{x} (\varphi(\bar{x}) \leq \psi(\bar{x}))) \\ &\iff \mathcal{F}_{\mathbf{DL}}(\bar{p}) \models \varphi(\bar{p}) \leq \psi(\bar{p}). \end{aligned}$$

The following expresses the connection between the extensions of a logic and the quasivarieties and varieties of its equivalent algebraic semantics:

Proposition 2.2.9 (Isomorphism of lattices of extensions and varieties). Given a logic $\vdash$ with equivalent algebraic semantics $\mathbf{A}$, the lattices $\text{Var}(\mathbf{A})$ and $\text{Ext}(\vdash)$ are complete lattices which are dually isomorphic.

Proof. See [116, pp. 82–85]. ■

In light of this we will often move between logics on one hand, and classes of algebras on the other.

2.2.2 Basic classes of algebras

Notation 2.2.4. Given an algebra A , we denote by $\text{Con}(A)$ the complete lattice of its congruences.

Definition 2.2.10 (Subdirectly irreducible algebra). Let A be an algebra. We say that A is *subdirectly irreducible* if $\text{Con}(A)$ has a second least element.

The following is a central result, due to Birkhoff:

Proposition 2.2.11 (Birkhoff's subdirect decomposition theorem). Every algebra A is isomorphic to a subdirect product of its subdirectly irreducible factors. Consequently, for each variety $\mathbf{A}$, we have $\mathbf{A} = \mathbb{V}(\mathbf{A}_{\text{si}})$.

Proof. See e.g. [74, Thm. 8.6, Corollary 9.7] ■

A crucial class of objects which will be the subject of many of our results are the *free algebras*:

Definition 2.2.12 (Free algebras). Let $\mathbf{A}$ be a class of algebras closed under subdirect products, and X a set of generators. We denote by $\mathcal{F}_{\mathbf{A}}(X)$ the free $\mathbf{A}$ -algebra on X many generators. We denote the unique (up to isomorphism) κ -generated $\mathbf{A}$ -free algebra by $\mathcal{F}_{\mathbf{A}}(\kappa)$.

This crucial role is justified by the following:

Proposition 2.2.13. Given $\mathbf{A}$ a variety, we have:

$$\mathbf{A} = \mathbb{V}(\mathcal{F}_{\mathbf{A}}(\omega)) = \mathbb{V}(\{\mathcal{F}_{\mathbf{A}}(n) : n \in \omega\}).$$

Proof. See e.g. [74, Exercise 11.2]. ■

In this thesis we will frequently use two essential descriptions of free algebras in varieties algebraizing a given logic $\vdash$:

- $\mathcal{F}_{\mathbf{A}}(X)$ is the algebra of formulas over X up to $\vdash$ -provability;
- $\mathcal{F}_{\mathbf{A}}(X)$ satisfies the universal mapping property with respect to $\mathbf{A}$ -algebras, over the generators X , or in other words, it is the image of the left adjoint to the forgetful functor $G : \mathbf{A} \rightarrow \mathbf{Set}$.

As a consequence of this, arbitrary free algebras can be constructed by using the 1-generated free algebras; recall that we denote coproducts in arbitrary categories as $\bigsqcup_{i \in I}$.

Proposition 2.2.14 (Free algebras determined by coproducts and 1-free algebras). Let $\mathbf{A}$ be a variety of algebras. Then for each X , we have:

$$\mathcal{F}_{\mathbf{A}}(X) \cong \bigoplus_{x \in X} \mathcal{F}_{\mathbf{A}}(\{x\}).$$

Proof. Since the functor $\mathcal{F}(-) : \mathbf{Set} \rightarrow \mathbf{A}$ is a left adjoint, and left adjoints preserve existing colimits, we have:

$$\mathcal{F}_{\mathbf{A}}(X) \cong \mathcal{F}_{\mathbf{A}}\left(\bigsqcup_{x \in X} \{x\}\right) = \bigoplus_{x \in X} \mathcal{F}_{\mathbf{A}}(\{x\}).$$

■

Remark 2.2.15 (Free algebras do not become easy). The previous proposition does not mean that free algebras become easy when one knows the 1-generated algebras though, as it is well-known (see Chapter 4 for a longer discussion) that colimits of algebras might be quite involved, indeed, often involving the same complications as those of free algebras themselves.

Two classes intimately related to free algebras are those of *finitely presented* and *finitely generated* algebras:

Definition 2.2.16 (Finitely generated, finitely presented). An algebra A is *finitely generated* if there exist finitely many elements $a_1, \dots, a_n$ such that $\langle a_1, \dots, a_n \rangle = A$. It is *finitely presented* if $A \cong \mathcal{F}(X)/\Theta((\lambda_1, \gamma_1), \dots, (\lambda_n, \gamma_n))$ for X a finite set.

Notation 2.2.5 (Subclasses of algebras). Given $\mathbf{A}$ a class of algebras we write:

1. $\mathbf{A}_{\text{fg}}$ the class of finitely generated $\mathbf{A}$ -algebras;
2. $\mathbf{A}_{\text{fp}}$ the class of finitely presented $\mathbf{A}$ -algebras;
3. $\mathbf{A}_{\text{si}}$ the class of subdirectly irreducible $\mathbf{A}$ -algebras.

2.2.3 Properties of logics and algebras

In this section we will dive in to several properties of algebraizable logics, and relate them to their equivalent algebraic semantics. We provide a summary of these properties in Table 2.2. Note that throughout, *logic* will mean an implicative logic with equivalent algebraic semantics given by a variety $\mathbf{A}$ of $\mathcal{L}$ -algebras.

General universal algebra

Definition 2.2.17 (Congruence-distributivity). We say that an algebra A is *congruence-distributive* if $\text{Con}(A)$ is a distributive lattice. We say that a class $\mathbf{A}$ of algebras is congruence-distributive if every algebra in it is congruence-distributive.

The following lemma by Jónsson is frequently useful:

Lemma 2.2.18 (Jónsson's Lemma). Let $\mathbf{A}$ be a class of algebras which is congruence-distributive. Then $A \in \mathbb{V}(\mathbf{A})_{\text{si}}$ if and only if $A \in \text{HSP}_U(\mathbf{A})$. In particular, if $\mathbf{A}$ is a finite class of finite algebras, then $A \in \mathbb{V}(\mathbf{A})_{\text{si}}$ if and only if $A \in \text{HS}(\mathbf{A})$.

Proof. See e.g. [74, Cor. 6.9]. ■

We recall also the following well-known fact:

Proposition 2.2.19 (Lattice expansions are congruence-distributive). Let $\mathcal{L}$ be a language extending the signature of lattices $\mathcal{L}_{\text{Lat}}$. Assume that $\mathbf{A}$ is a variety of $\mathcal{L}$ -algebras such that each $\mathcal{L}_{\text{Lat}}$ -subreduct is a lattice. Then $\mathbf{A}$ is a congruence-distributive variety.

Proof. See e.g. [74, Thm. 12.3, Example pp.87]. ■

Definition 2.2.20 (Congruence Extension Property). A class $\mathbf{A}$ of algebras has the *congruence extension property* (CEP) if whenever $f : A \rightarrow B$ is an injective homomorphism, and $g : A \rightarrow C$ is a surjective homomorphism, there is a surjective homomorphism $h_1 : B \rightarrow E$ and an injective homomorphism $h_2 : C \rightarrow E$ such that $h_1 f = h_2 g$.

The following is an interesting consequence of the congruence extension property:

Proposition 2.2.21 (CEP implies $\mathbb{H}$ and $\mathbb{S}$ commute). Let $\mathbf{A}$ be a class of algebras with the congruence extension property. Then $\mathbb{H}\mathbb{S}(\mathbf{A}) = \mathbb{S}\mathbb{H}(\mathbf{A})$.

Proof. See e.g. [74, pp. 46]. ■

Tabularity and Local Tabularity

Definition 2.2.22 (Finitely generated variety). A variety $\mathbf{A}$ is *finitely generated* if $\mathbf{A} = \mathbb{V}(A)$ for some finite $\mathbf{A}$ -algebra.

Definition 2.2.23 (Tabular logic). A logic $\vdash$ with $\mathbf{A} = \text{Alg}(L)$ is said to be *tabular* if $\mathbf{A}$ is a finitely generated variety.

Proposition 2.2.24 (Finite generation = Finitely many subdirectly irreducibles). Let $\mathbf{A}$ be a congruence-distributive variety. Then $\mathbf{A}$ is finitely generated if and only if $\mathbf{A}_{\text{si}}$ is finite and contains only finite algebras.

Proof. This follows immediately from Lemma 2.2.18. ■

Definition 2.2.25 (Locally finite algebras and classes). Let A be an $\mathcal{L}$ -algebra. We say that A is *locally finite* if whenever $B \leq A$ is a finitely generated algebra, then B is finite. We say that a class of algebras $\mathbf{A}$ is *locally finite* if every algebra in $\mathbf{A}$ is locally finite.

Definition 2.2.26 (Local tabularity). A logic $\vdash$ with $\mathbf{A} = \text{Alg}(\vdash)$ is *locally tabular* if for each $\bar{p} = \{p_1, \dots, p_n\}$, there are only finitely many formulas over $\bar{p}$ up to equivalence under $\vdash$.

Proposition 2.2.27 (Equivalents of local finiteness). Given a logic $\vdash$ with $\mathbf{A} = \text{Alg}(\vdash)$, the following are equivalent:

1. $\mathbf{A}$ is locally tabular;
2. For each n , there is a k such that there are at most k many n -generated $\mathbf{A}$ -algebras;
3. For each n , $\mathcal{F}_{\mathbf{A}}(n)$ is finite.

Proof. See e.g. [74, Thm. 10.15]. ■

Finite Model Property

Definition 2.2.28 (Finite model property). A logic $\vdash$ has the *finite model property* (FMP) if $\mathbf{A} = \text{Alg}(\vdash)$, is generated by its finite algebras

Definition 2.2.29 (Finitely approximable). We say that an algebra A is *finitely approximable*¹² if there is a class $\mathbf{A}$ of finite algebras in the same type such that $A \in \mathbb{P}_S(\mathbf{A})$.

Remark 2.2.30 (Finitely approximable are closed under subalgebras). Note that if A is a finitely approximable, and $B \leq A$, then B is also finitely approximable: given $A \hookrightarrow \prod_{i \in I} A_i$ where A_i are finite, we can take the restriction of such an embedding to B , and by factoring, obtain a subdirect embedding of B into a product of finite algebras.

The following is a folklore consequence of the definitions; recall the finite-index property (Definition 2.1.30):

Lemma 2.2.31 (Finite approximable = finite-index property). An algebra A is finitely approximable if and only if it has the finite-index property.

Proof. If $f : A \rightarrow \prod_{i \in I} B_i$ is a subdirect embedding, then given any $a \neq b \in A$, $f(a) \neq f(b)$, so there is some i such that $f(a)(i) \neq f(b)(i)$; thus the composition of the projection with f witnesses the finite-index property.

For the converse, let A have the finite index property. Note that for each $a \neq b$, if $f : A \rightarrow B$ exists where B is finite, and $f(a) \neq f(b)$, then a subdirectly irreducible B exists (since B is a subdirect product of its subdirectly irreducible factors all of which are finite). Hence let $\{A_i : i \in I\}$ be (representatives of) all subdirectly irreducible homomorphic images of A , with $\pi_i : A \rightarrow A_i$ the homomorphism. Then defining $p : A \rightarrow \prod_{i \in I} A_i$ by $p(a)(i) = \pi_i(a)$, we have a subdirect embedding, since whenever $a \neq b$, for some i , $\pi_i(a) \neq \pi_i(b)$. ■

¹²In the universal algebra literature, this property is sometimes called *residual finiteness* (see e.g. [196]). The name given here reflects the logical nature of the same property (see [82, Ch. 5]).

The following is likewise folklore, and gives us a general criterion for the FMP in terms of free algebras:

Proposition 2.2.32 (FMP and finite approximability). Given a logic $\vdash$ with $\mathbf{A} = \text{Alg}(\vdash)$, the following are equivalent:

1. $\vdash$ has the FMP;
2. The algebra $\mathcal{F}_{\mathbf{A}}(\omega)$ is finitely approximable.
3. For each n , $\mathcal{F}_{\mathbf{A}}(n)$ is finitely approximable.

Proof. (1) $\Rightarrow$ (2) Since $\mathbf{A}$ is generated by finite algebras, by [74, Thm. 11.11], $|X| \geq \aleph_0$, then $\mathcal{F}_{\mathbf{A}}(X) \in \mathbb{P}_S(\mathbf{A})$. By definition of finite approximability, this means that $\mathcal{F}_{\mathbf{A}}(\omega)$ is finitely approximable.

(2) $\Rightarrow$ (3) As noted in Remark 2.2.30, finitely approximable algebras are closed under subalgebras. Since $\mathcal{F}_{\mathbf{A}}(n)$ always embeds into $\mathcal{F}_{\mathbf{A}}(\omega)$, if the latter is finitely approximable, so is the former.

(3) $\Rightarrow$ (1) Assume that $\mathcal{F}_{\mathbf{A}}(n)$ is finitely approximable for each n . Let $\varphi \approx \psi$ be an equation, and assume that $\mathbf{A} \not\models \varphi \approx \psi$; then $\mathcal{F}_{\mathbf{A}}(n) \not\models \varphi \approx \psi$ for n large enough to contain the variables in φ and ψ , and so by assumption of finite approximability, and Lemma 2.2.31, there is a finite quotient A of $\mathcal{F}_{\mathbf{A}}(n)$ such that $A \not\models \varphi \approx \psi$. This means that $\text{Eq}(\mathbf{A}_{\text{fin}}) = \text{Eq}(\mathbf{A})$. By algebraic completeness, $\mathbf{A} = \mathbb{V}(\mathbf{A}_{\text{fin}})$, showing that $\mathbf{A}$ is generated by its finite algebras. ■

Interpolation Properties

Often throughout this thesis we will refer to *interpolation properties*, since these play a key role in issues of definability and in the study of free algebras:

Definition 2.2.33 (Deductive interpolation, Craig interpolation). We say that a logic $\vdash$ has the:

- *deductive interpolation property* (DIP) if whenever $\varphi(\bar{p}, \bar{q}) \vdash \psi(\bar{q}, \bar{r})$ there is a formula $\chi(\bar{q})$ such that $\varphi(\bar{p}, \bar{q}) \vdash \chi(\bar{q})$ and $\chi(\bar{q}) \vdash \psi(\bar{q}, \bar{r})$;
- *Craig interpolation property* (CIP) if whenever $\vdash \varphi(\bar{p}, \bar{q}) \rightarrow \psi(\bar{q}, \bar{r})$ there is a formula $\chi(\bar{q})$ such that $\vdash \varphi(\bar{p}, \bar{q}) \rightarrow \chi(\bar{q})$ and $\vdash \chi(\bar{q}) \rightarrow \psi(\bar{q}, \bar{r})$.

The key properties corresponding to interpolation properties are forms of the *amalgamation property*:

Definition 2.2.34 (Amalgams and superamalgams). Given $f : A \rightarrow B$ and $g : A \rightarrow C$ two injective homomorphisms, we call (A, B, C) an *amalgamation triple*. An algebra D is called an *amalgam*, if there exist maps $h : C \rightarrow D$ and $h' : B \rightarrow D$ such that $h'f = hg$.

We say that D is a *superamalgam* if:

1. $h[C] \cap h'[B] = hg[A] = h'f[A]$, and
2. whenever $h(b) \leq h'(c)$ for $b \in B$ and $c \in C$, there is some $a \in A$ such that $b \leq f(a)$ and $g(a) \leq c$.

Definition 2.2.35 ((Super)Amalgamation property). A class of algebras $\mathbf{A}$ has the *amalgamation property* (AP) if whenever (A, B, C) is an amalgamation triple of algebras in $\mathbf{A}$, there is an amalgam $D \in \mathbf{A}$. We say that $\mathbf{A}$ has the *superamalgamation property* if there is a superamalgam $D \in \mathbf{A}$ for any such amalgamation triple.

A general correspondence between interpolation and amalgamation can be easily formulated for varieties with the congruence extension property:

Proposition 2.2.36 (Deductive interpolation = Amalgamation). Let $\vdash$ be a logic such that $\mathbf{A} = \text{Alg}(\vdash)$ has the CEP. Then $\vdash$ has the deductive interpolation property if and only if $\mathbf{A}$ has the amalgamation property.

Proof. See e.g. [200, Thm. 27] ■

For most cases of interest, we likewise have the following equivalence:

1. The logic $\vdash$ has the CIP;
2. The variety $\mathbf{A}$ has the superamalgamation property.

For instance, this equivalence holds for superintuitionistic logics [184] and modal logics [187], which will be enough for our purposes (see [200, Sec. 7] for a broader discussion).

Uniform interpolation properties

We will likewise be interested in properties stronger than usual interpolation.

Definition 2.2.37 (Right-uniform deductive interpolation). We say that a logic $\vdash$ has *right uniform deductive interpolation* if for any finite sets $\bar{p}, \bar{q}$ of propositional variables, and formula $\varphi(\bar{p}, \bar{q})$, there exists a formula $\chi(\bar{q})$ such that for any finite set of propositional variables $\bar{r}$ and for any formula $\psi(\bar{q}, \bar{r})$

$$\varphi(\bar{p}, \bar{q}) \vdash \psi(\bar{q}, \bar{r}) \iff \chi(\bar{q}) \vdash \psi(\bar{q}, \bar{r}).$$

Definition 2.2.38 (Coherence). A variety $\mathbf{A}$ is said to be *coherent* if whenever $A, B \in \mathbf{A}$ and $A \leq B$ where B is finitely presented in $\mathbf{A}$, and A is finitely generated, then A is finitely presented in $\mathbf{A}$.

The following is shown in [169, Thm. 2.3]:

Theorem 2.2.39. $\vdash$ has right uniform deductive interpolation if and only if it has the deductive interpolation property and $\mathbf{A}$ is coherent.

There is a specular *left uniform deductive interpolation property* saying that for any finite sets $\bar{q}, \bar{r}$ of propositional variables, and formula $\varphi(\bar{q}, \bar{r})$ there exists a formula $\mu(\bar{q})$ such that for any finite set of propositional variables $\bar{p}$ and for any formula $\psi(\bar{p}, \bar{q})$

$$\psi(\bar{p}, \bar{q}) \vdash \varphi(\bar{q}, \bar{r}) \iff \psi(\bar{p}, \bar{q}) \vdash \mu(\bar{q}).$$

The two properties are not equivalent¹³. These concepts are closely related to model-theoretic concepts, namely the existence of *model companions* (see [130, 135, 252] for the basic definitions), and some effectiveness assumptions on such model companions:

Corollary 2.2.40 (Model completions from uniform interpolation). *If $\vdash$ is a logic, with $\mathbf{A} = \text{Alg}(\vdash)$, and $\vdash$ has both the left and right uniform deductive interpolation properties, then the first order theory of $\mathbf{A}$ has a model completion.*

In addition to these *global* properties, a special role is played by *local properties of interpolation*, that is, properties involving the implication.

Definition 2.2.41 (Uniform CIP). We say that a logic $\vdash$ has the *right uniform Craig interpolation property* if for any finite sets $\bar{p}, \bar{q}$ of propositional variables, and formula $\varphi(\bar{p}, \bar{q})$, there exists a formula $\chi(\bar{q})$ such that for any finite set of propositional variables $\bar{r}$ and for any formula $\psi(\bar{q}, \bar{r})$,

$$\vdash \varphi(\bar{p}, \bar{q}) \rightarrow \psi(\bar{q}, \bar{r}) \iff \vdash \chi(\bar{q}) \rightarrow \psi(\bar{q}, \bar{r}).$$

In this case we denote $\chi(\bar{q})$ by $\tilde{\exists}\bar{p}\varphi(\bar{p}, \bar{q})$. We say that it has the *left uniform Craig interpolation property* if for any finite sets $\bar{q}, \bar{r}$ of propositional variables, and formula $\varphi(\bar{q}, \bar{r})$ there exists a formula $\mu(\bar{q})$ such that for any finite set of propositional variables $\bar{p}$ and for any formula $\psi(\bar{p}, \bar{q})$,

$$\vdash \psi(\bar{p}, \bar{q}) \rightarrow \varphi(\bar{q}, \bar{r}) \iff \vdash \psi(\bar{p}, \bar{q}) \rightarrow \mu(\bar{q});$$

in this case we denote $\mu(\bar{q})$ by $\tilde{\forall}\bar{r}\varphi(\bar{q}, \bar{r})$.

The right and left uniform Craig interpolants, in implicative logics, typically act like genuine. To give them a purely algebraic meaning we use the following condition (from [125, p. 229]):

Definition 2.2.42 (Beck-Chevalley for free inclusions). Let $\mathbf{A}$ be a class of algebras such that inclusions between finitely generated free algebras have left adjoints, i.e., For each $\bar{p}$, the inclusion $i_p : \mathcal{F}_{\mathbf{A}}(\bar{p}) \rightarrow \mathcal{F}_{\mathbf{A}}(\bar{p}, q)$ has a left adjoint $\exists_{i_p} : \mathcal{F}_{\mathbf{A}}(\bar{p}, q) \rightarrow \mathcal{F}_{\mathbf{A}}(\bar{p})$. We say that $\mathbf{A}$ satisfies the *Beck-Chevalley* condition for each substitution $\sigma : \mathcal{F}_{\mathbf{A}}(\bar{p}) \rightarrow \mathcal{F}_{\mathbf{A}}(\bar{q})$, and each formula $\psi(\bar{p}, p)$ we have:

$$\exists_{i_q}\sigma(\varphi(\bar{p}, p)) = \sigma(\exists_{i_p}\varphi(\bar{p}, p)).$$

¹³Equivalence can hold in very special contexts, typically when compact congruences are Boolean (this is the case of [55, Sec. 4] for the presence of a universal modality).

It is known [130, pp. 57] that if $\mathbf{A}$ is a class of algebras having additionally *right* adjoints between finitely generated algebras $\forall_{i_p}$, then in fact the Beck-Chevalley condition entails the corresponding fact for these operators, namely

$$\forall_{i_q} \sigma(\varphi(\bar{p}, p)) = \sigma(\forall_{i_p} \varphi(\bar{p}, p)).$$

The connection to interpolation lies in the following:

Proposition 2.2.43 (Beck-Chevalley and interpolation). Let $\vdash$ be a logic with $\mathbf{A} = \text{Alg}(\vdash)$. Suppose that inclusions $i_p: \mathcal{F}_{\mathbf{A}}(\bar{p}) \rightarrow \mathcal{F}_{\mathbf{A}}(\bar{p}, q)$ have left (right) adjoints. Then $\vdash$ has the Craig interpolation property if and only if $\mathbf{A}$ satisfies the Beck-Chevalley condition.

Proof. See [218, Thm. 13], see also [125]. ■

The following thus captures the essence of uniform Craig interpolation; the proof is essentially a reworking of the categorical formulation given in [130, Ch. 4]:

Theorem 2.2.44 (Uniform Craig interpolation and adjoints). *Let $\vdash$ be a logic and $\mathbf{A} = \text{Alg}(\vdash)$. Then the following are equivalent:*

1. $\vdash$ has the right (left) uniform Craig interpolation;
2. In $\mathbf{A}$ inclusions of finitely generated free algebras have left (right) adjoints satisfying the Beck-Chevalley condition;
3. $\vdash$ has the CIP and in $\mathbf{A}$, inclusions of finitely generated algebras have left (right) adjoints.

Proof. We prove this for right interpolation, since left interpolation is similar. The implication from (1) to (2) is clear, since the formula witnessing the right uniform Craig interpolation property is easily seen to satisfy the adjunction property, and moreover, seen to satisfy the Beck-Chevalley condition on account of $\psi(\bar{q}, \bar{r})$ having arbitrary extra variables. The equivalence of (2) and (3) follows from Proposition 2.2.43. We thus show that (2) implies (1). Assume that inclusions between finitely generated free algebras have left adjoints satisfying the Beck-Chevalley condition. Consequently, given $\varphi(\bar{p}, \bar{q})$, by composing the adjoints, we obtain a formula $\tilde{\exists} \bar{p} \varphi(\bar{p}, \bar{q}) \in \mathcal{F}_{\mathbf{A}}(\bar{p})$, in the equivalence class of $\exists_{i_{\bar{p}}} \varphi(\bar{p}, \bar{q})$. Now let $\psi(\bar{q}, \bar{r})$. Assume that $\vdash \varphi(\bar{p}, \bar{q}) \rightarrow \psi(\bar{q}, \bar{r})$; this holds if and only if:

$$i_{\bar{r}}(\varphi(\bar{p}, \bar{q})) \leq i_{\bar{r}}(\psi(\bar{q}, \bar{r}));$$

then by applying the adjoint forgetting $\bar{p}$, we get

$$\exists_{i_{\bar{p}}} i_{\bar{r}}(\varphi(\bar{p}, \bar{q})) \leq \exists_{i_{\bar{p}}} i_{\bar{r}}(\psi(\bar{q}, \bar{r})) \leq \psi(\bar{q}, \bar{r});$$

by the Beck-Chevalley condition, we have that $\exists_{i_{\bar{p}}} i_{\bar{r}}(\varphi(\bar{p}, \bar{q})) = i_{\bar{r}}(\exists_{i_{\bar{p}}} \varphi(\bar{p}, \bar{q}))$, which means precisely that $\vdash \tilde{\exists} \bar{p} \varphi(\bar{p}, \bar{q}) \rightarrow \psi(\bar{q}, \bar{r})$. The other direction follows by similar arguments. ■

The reader may wonder what connection there is between the right/left uniform deductive interpolation properties and their Craig-versions. As it turns out, without additional assumptions, the notions are completely independent: the basic modal logic $\mathbf{K}$ has both left and right uniform Craig interpolation [125, Thm. 6.1], whilst it is not coherent [169, Cor. 5.5], and so by Theorem 2.2.39, also shown by the same authors, it does not enjoy right uniform deductive interpolation. The following example in the converse direction appears to be known (see e.g. [135, Exa. 14]), though we could not find an argument in the literature. The following argument is due to M. Abbadini ¹⁴:

Example 2.2.45 (MV-algebras do not have uniform Craig interpolation). The logic $\mathbf{L}_{\infty f}$ is the *finitary companion* of Łukasiewicz logic; its equivalent algebraic semantics is given by the class of *MV-algebras*. This logic is known to be implicative, though it fails to have a deduction theorem [116, pp. 478]. As noted in [252, Exa. 3.7], this variety has right-uniform deductive interpolation. On the other hand the following argument shows that $\mathbf{L}_{\infty f}$ does not have right uniform Craig interpolation: consider the term $g(x, y) = y \vee \neg y$. The interpretation on the algebra $[0, 1]$ is a function $p: [0, 1]^2 \rightarrow [0, 1]$ such that, fixing x_0 , the minimum of $y \mapsto g(x_0, y)$ is $\frac{1}{2}$. However, there is no term $f(x)$ constantly equal to $\frac{1}{2}$. Hence if we look at the inclusion $i: \mathcal{F}_{\mathbf{MV}}(x) \rightarrow \mathcal{F}_{\mathbf{MV}}(x, y)$, this cannot have a right adjoint, since $\exists_i(g(x, y))$ would precisely correspond to such a term.

Deduction theorems

Definition 2.2.46 (Deduction detachment). A logic $\vdash$ has a *deduction-detachment theorem* if there is a finite set of formulas $I(x, y) = \{\lambda_i(x, y) : i \leq n\}$, such that for every set $\Gamma \cup \{\varphi, \psi\}$,

$$\Gamma, \varphi \vdash \psi \iff \Gamma \vdash I(\varphi, \psi).$$

Given an algebra A , and elements $a, b \in A$ we denote by $\Theta(a, b)$ the congruence generated by (a, b) . Recall that we call such a congruence *principal*.

Definition 2.2.47 (EDPC). A variety $\mathbf{A}$ is said to have *equationally definable principal congruences* (EDPC), or be an *EDPC variety*, if there is a finite set $\Phi(x, y, z, v)$ such that for every $A \in \mathbf{A}$, and $a, b, c, d \in A$:

$$(a, b) \in \Theta(c, d) \iff A \models \Phi(a, b, c, d).$$

The following was proved by Blok and Pigozzi:

Theorem 2.2.48 (Deduction bridge theorem). *A logic $\vdash$ has a deduction-detachment theorem if and only if $\mathbf{A}$ is an EDPC variety.*

¹⁴Private communication with Marco Abbadini (2026), reproduced here with permission.

Proof. See [69, Thm. 5.5]. ■

Let us say that a logic $\vdash$ has a *strong deduction-detachment theorem* if there exists a term $\delta(x)$ such that for each pair of formulas φ, ψ we have that:

$$\varphi \vdash \psi \iff \vdash \delta(\varphi) \rightarrow \psi.$$

The interest of this notion lies in the following connection:

Proposition 2.2.49 (Strong deduction and uniform interpolation). If $\vdash$ has a strong deduction detachment theorem and right (left) uniform Craig interpolation, then it has right (left) uniform deductive interpolation.

Proof. Assume that $\varphi(\bar{p}, \bar{q}) \vdash \psi(\bar{q}, \bar{r})$. Then $\vdash \delta(\varphi(\bar{p}, \bar{q})) \rightarrow \psi(\bar{q}, \bar{r})$, which by uniform Craig interpolation means that $\vdash \exists \bar{p} \delta(\varphi(\bar{p}, \bar{q})) \rightarrow \psi(\bar{q}, \bar{r})$, which entails by the definition of implication that $\exists \bar{p} \delta(\varphi(\bar{p}, \bar{q})) \vdash \psi(\bar{q}, \bar{r})$. Conversely assume that $\exists \bar{p} \delta(\varphi(\bar{p}, \bar{q})) \vdash \psi(\bar{q}, \bar{r})$; since $\vdash \delta(\varphi(\bar{p}, \bar{q})) \rightarrow \exists \bar{p} \delta(\varphi(\bar{p}, \bar{q}))$, we have that $\varphi(\bar{p}, \bar{q}) \vdash \exists \bar{p} \delta(\varphi(\bar{p}, \bar{q}))$, which means precisely that $\varphi(\bar{p}, \bar{q}) \vdash \psi(\bar{q}, \bar{r})$. The case for left uniform interpolation is similar but easy, since then ψ sits uniformly on the left of the turnstile. ■

Relative anarchy and interpretations

We finish this section with an ensemble of relational properties – properties relating two distinct logical systems – which, despite having been frequently studied, has seemingly so far gone without a name:

Definition 2.2.50 (Relative anarchy). Let $\mathbf{A}$ be a variety of $\mathcal{L}$ -algebras, and $\mathbf{A}_0$ a variety of $\mathcal{L}_0$ -algebras where $\mathcal{L} \subseteq \mathcal{L}_0$. We say that $\mathbf{A}$ is $\mathbf{A}_0$ -*anarchic*¹⁵ if:

1. (Inclusion) We have that $\mathbf{A} \upharpoonright_{\mathcal{L}} \subseteq \mathbf{A}$.
2. (Unique) For each $A \in \mathbf{A}$ there is at most one $B \in \mathbf{A}_0$ such that $B \upharpoonright_{\mathcal{L}} \cong A$.
3. (Anarchy) For each X where $|X| \leq \aleph_0$, the algebra $\mathcal{F}_{\mathbf{A}}(X)$ is the reduct of some algebra in $\mathbf{A}_0$.

¹⁵We stress that the usage of “anarchic” refers not to the colloquial use, i.e., without rules, but rather to the collectivist, mostly Russian, tradition of anarchism, prefigured by M. Bakunin. To somewhat loosely interpret the ideas, we can say that a system is *lawless* when it has the minimum amount of laws assumed to be necessary; but it is only considered *social anarchism* if these laws are capable of, organically, giving rise to important structures like collective organization or redistribution. The analogy with free algebras is thus clear: in general these are lawless, and they will be anarchic (relative to some algebraic expansion) if the uniquely definable structures are actually fully defined there.

The conditions (1-2) above are connected with the context of *algebraically expandable classes* studied extensively by Campercholli and Vaggione (see e.g. [78]), as well as in non-classical logics in [77], and is likewise connected to the theory of implicit operations [80], and the study of connectives in non-classical logic (see e.g. [90] for a discussion in the context of orthologic). Examples of this phenomenon are abundant:

1. Free modal algebras are tense algebras [125, Thm. 6.3].
2. Free distributive lattices are bi-Heyting algebras (a folklore fact, see Section A.1 for a proof);
3. Free Heyting algebras are bi-Heyting algebras [127] (see also Section 4.3).
4. Every free Gödel algebra is a bi-Heyting algebra by the main results of [79].

To provide a logical interpretation of relative anarchy, we begin by recalling the meaning of inclusion and uniqueness:

Proposition 2.2.51. Let $\mathbf{A}$ be a variety of $\mathcal{L}$ -algebras and $\mathbf{A}_0$ a variety of $\mathcal{L}_0$ -algebras where $\mathcal{L} \subseteq \mathcal{L}_0$. Then:

1. (Inclusion) holds if and only if whenever $\lambda \approx \gamma$ is an $\mathcal{L}$ -equation valid in $\mathbf{A}$, then $\mathbf{A}_0 \models \lambda \approx \gamma$.
2. (Unique) holds if and only if an isomorphism of $\mathbf{A}$ -algebras is an isomorphism of $\mathbf{A}_0$ -algebras.

Equivalently, inclusion and uniqueness hold if and only if the reduct forgetful functor $\downarrow_{\mathcal{L}}: \mathbf{A}_0 \rightarrow \mathbf{A}$ is faithful and injective on objects.

Proof. Both conditions are easy to verify by unfolding the definitions. ■

A categorical reading of this situation makes it quite transparent: our assumptions taken together, imply that there exists a free functor $\mathbf{F}: \mathbf{A} \rightarrow \mathbf{A}_0$, which is left adjoint to the forgetful functor $G: \mathbf{A}_0 \rightarrow \mathbf{A}$, and the algebras for the induced monad $G\mathbf{F}$ are precisely the $\mathbf{A}_0$ -algebras – there exists for each such algebra one monad structure, since they lie in the essential image of the forgetful functor, and at most one by our assumptions. This puts us in the context of *property-like structures* (see e.g. [162, Thm. 7.1]).

To understand the meaning of relative anarchy, we briefly introduce the idea of a “structural interpretation”; the reader should compare such definitions with the notion of translations discussed in [204]:

Definition 2.2.52 (Structural interpretation). Let $\mathcal{L} \subseteq \mathcal{L}_0$ be logical languages. Let $\mathbf{A}$ be a variety of $\mathcal{L}$ -algebras and $\mathbf{A}_0$ a variety of $\mathcal{L}_0$ -algebras. We say that an assignment $T: \text{Form}_{\mathcal{L}_0}(\text{Prop}) \rightarrow \text{Form}_{\mathcal{L}}(\text{Prop})$ is a *structural interpretation* if:

1. (Identity on formulas) For each formula $\varphi \in \text{Form}_{\mathcal{L}}(\text{Prop})$ we have $\mathbf{A}_0 \models T(\varphi) \approx \varphi$;
2. (Pseudo-Homomorphism) For each symbol $t(x_1, \dots, x_n) \in \mathcal{L}_0$, we have:

$$\mathbf{A}_0 \models T(t(p_1, \dots, p_n)) \approx T(t(T(p_1), \dots, T(p_n))).$$

Note that as a consequence of these conditions, T is a homomorphism on the connectives from $\mathcal{L}$, and “as close as possible” to being a homomorphism on $\mathcal{L}_0$. We can then prove the following:

Proposition 2.2.53 (Logical characterization of relative anarchy). Let $\mathbf{A}$ be a variety of $\mathcal{L}$ -algebras and $\mathbf{A}_0$ a variety of $\mathcal{L}_0$ -algebras, where $\mathcal{L} \subseteq \mathcal{L}_0$, satisfying (Inclusion) and (Uniqueness). Then there exists a (unique) structural interpretation of $\mathbf{A}_0$ in $\mathbf{A}$ if and only if $\mathbf{A}$ is $\mathbf{A}_0$ -anarchic.

Proof. As mentioned above, the categorical assumptions at play mean that there exists a left adjoint, free functor $\mathbf{F}: \mathbf{A} \rightarrow \mathbf{A}_0$, left adjoint to the forgetful functor $G: \mathbf{A}_0 \rightarrow \mathbf{A}$, which is injective on objects. The requirement of relative anarchy is equivalent to saying that the algebras $\mathcal{F}_{\mathbf{A}}(X)$ all have a (necessarily unique) $\mathbf{A}_0$ -structure.

Let us unfold what such a structure must look like, in terms of the laws for algebras over a monad (See Definition 2.1.7): the existence of an algebra structure, corresponds to the existence of a homomorphism $f: G\mathbf{F}(\mathcal{F}_{\mathbf{A}}) \rightarrow \mathcal{F}_{\mathbf{A}}$. Given an object $A \in \mathbf{A}$, the unit $\eta_A: A \rightarrow G\mathbf{F}(A)$ simply sends each a element to its image in the free $\mathbf{A}_0$ -generated algebra.

Let us then unfold what it means for $\mathcal{F}_{\mathbf{A}}(\omega)$ to carry a $\mathbf{A}_0$ -structure. Recall that:

1. (Unit) For each object $A \in \mathbf{A}$, the unit $\eta: A \rightarrow G\mathbf{F}(A)$ takes an element $a \in A$ and sends it to itself in the free $\mathbf{A}_0$ -algebra generated by A ;
2. (Counit) For each object $A \in \mathbf{A}_0$, the counit $\varepsilon_A: \mathbf{F}G(A) \rightarrow A$ works by “evaluation”: it takes a formula $\varphi(a_1, \dots, a_n)$, built over the elements of A , and sends it to the interpretation in A .
3. (Multiplication) By tracing the meaning of the counit, and recalling that the multiplication is $G(\varepsilon_{\mathbf{F}(A)})$, the multiplication $\mu_A: G\mathbf{F}G\mathbf{F}(A) \rightarrow G\mathbf{F}(A)$ works by reducing: given a formula $\varphi(x_1, \dots, x_n)$ taken in the language $\text{Form}_{\mathcal{L}_0}(\text{Prop})$, and $\psi_1, \dots, \psi_n$ formulas over $\mathcal{L}_0$ with parameters from A , note that we can see the formula $\varphi(\psi_1, \dots, \psi_n)$ in two different ways: as a formula $\varphi^*(\psi_1, \dots, \psi_n)$ with ψ_i parameters, and as a formula $\varphi(\psi_1, \dots, \psi_n)$ over a , obtained by substitution of $x_i \mapsto \psi_i$. The multiplication acts by:

$$\mu(\varphi^*(\psi_1, \dots, \psi_n)) = \varphi(\psi_1, \dots, \psi_n),$$

essentially evaluating the terms.

For simplicity of notation, letting $F = \mathcal{F}_{\mathbf{A}}(X)$, and writing $T := G\mathbf{F}$, we can now work out what the meaning of the unit and multiplication conditions are when taken for this particular adjunction, assuming that F has a $\mathbf{A}_0$ -structure:

1. The (Unit) condition. We require that for each formula $\varphi \in \text{Form}_{\mathcal{L}}(\text{Prop})$, seen as an element of F^{16} , we require that $f\eta_F = 1_F$. Given the conservativity of $\mathbf{A}_0$ over $\mathbf{A}$, this equivalently means that:

$$\mathbf{A}_0 \models f(\varphi) \approx \varphi,$$

for each formula φ in the language $\mathcal{L}$.

2. The (Multiplication condition): For each $t(x_1, \dots, x_n) \in \mathcal{L}_0$, unfolding the multiplication condition, we obtain that $\mathbf{A}_0$ should prove the following equality:

$$\mathbf{A}_0 \models f(t(p_1, \dots, p_n)) \approx f(t(f(p_1), \dots, f(p_n))).$$

By recalling that the algebra $\mathbf{F}(\mathcal{F}_{\mathbf{A}}(X))$ will be $\mathcal{F}_{\mathbf{A}_0}(X)$, it is clear from the above discussion that the existence of an $\mathcal{L}_0$ -algebra structure on $\mathcal{F}_{\mathbf{A}}(X)$ is equivalent to the existence of a structural interpretation of $\mathbf{A}_0$ in $\mathbf{A}$. $\blacksquare$

2.3 Intuitionistic Logic

2.3.1 Basic Definitions and Properties

Definition 2.3.1 (Intuitionistic language). The *intuitionistic language* over a set of propositions Prop , denoted by $\mathcal{L}_{\text{IPC}}(\text{Prop})$, consists of the formulas φ obtained by the following rules:

$$\varphi ::= p \mid \perp \mid \top \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \varphi \rightarrow \psi,$$

where $p \in \text{Prop}$.

We make the usual abbreviations $\varphi \leftrightarrow \psi$ and $\neg\varphi := \varphi \rightarrow \perp$. Note that such a language is precisely the same as $\mathcal{L}_{\mathbf{HA}}(\text{Prop})$.

Definition 2.3.2 (Implication depth). Given $\varphi \in \mathcal{L}_{\text{IPC}}(\text{Prop})$, the *depth* of φ , specifically its *implication depth*, is defined recursively by letting $d(p) = d(\perp) = d(\top) = 0$, $d(\varphi \wedge \psi) = d(\varphi \vee \psi) = \max(d(\varphi), d(\psi))$ and for implication, $d(\varphi \rightarrow \psi) = \max(d(\varphi), d(\psi)) + 1$.

Definition 2.3.3 (Superintuitionistic logic). We say that a logic $\vdash$, in the language $\mathcal{L}_{\text{IPC}}(\text{Prop})$, is a *superintuitionistic logic* (s.i.) if it contains the sequents,

¹⁶To avoid the reader needing to pattern match concepts, we are identifying ω and Prop , and treating them as the same.

1. $\vdash q \rightarrow (p \rightarrow q)$;
2. $\vdash (r \rightarrow (p \rightarrow q)) \rightarrow ((r \rightarrow p) \rightarrow (r \rightarrow q))$;
3. $\vdash p \wedge q \rightarrow p$ and $\vdash p \wedge q \rightarrow q$;
4. $\vdash p \rightarrow (q \rightarrow (p \wedge q))$;
5. $\vdash p \rightarrow p \vee q$ and $\vdash q \rightarrow p \vee q$;
6. $\vdash (p \rightarrow q) \rightarrow ((r \rightarrow q) \rightarrow ((p \vee r) \rightarrow q))$;
7. $\vdash \perp \rightarrow p$ and $\vdash p \rightarrow \top$,

and is closed under the following rules:

1. Uniform substitution, i.e., $\varphi(p_1, \dots, p_n) \in L$ then for each set of formulas $\psi_1, \dots, \psi_n$, $\varphi(\psi_1, \dots, \psi_n) \in L$;
2. Modus Ponens, i.e., $\vdash \varphi \rightarrow \psi$ and $\vdash \varphi$ then $\vdash \psi$.

We denote by $\vdash_{\text{IPC}}$ the smallest superintuitionistic logic.

Intuitionistic logic enjoys the deduction theorem in its full strength (see [82, Thm. 2.42]):

Theorem 2.3.4 (Deduction theorem for s.i. logics). *For each $\varphi, \psi \in \mathcal{L}_{\text{IPC}}(\text{Prop})$ we have that $\varphi \vdash_{\text{IPC}} \psi$ if and only if $\vdash_{\text{IPC}} \varphi \rightarrow \psi$.*

In light of the deduction theorem, it is common to identify $\vdash_{\text{IPC}}$ with its set of theorems, IPC. Consequently, we will also refer to *superintuitionistic logics* (s.i.) simply as axiomatic extensions of IPC.

Notation 2.3.1 (Smallest logic generated). Given an s.i. logic L , and a set S of formulas, we will denote by $L \oplus S$ the smallest s.i. logic containing $L \cup S$.

Throughout this thesis we will encounter several superintuitionistic logics. The following will be recurring:

Definition 2.3.5 (Basic superintuitionistic logics).

1. $\text{CPC} = \text{IPC} \oplus (p \vee \neg p) = \text{IPC} \oplus \neg\neg p \rightarrow p$ – Classical propositional logic.
2. $\text{Sme} = \text{IPC} \oplus \neg p \vee q \vee (q \rightarrow p)$ – called *Smetanich logic* the logic of *Stone-Heyting algebras*, or LC_2 ¹⁷.

¹⁷The term Smetanich logic appears in [82]. The axiomatization given here is due to Hosoi [149]. This logic is likewise known as the logic of “Here-and-there” in the logic programming community (see e.g. [214]).

3. $\text{LC} = \text{IPC} \oplus (p \rightarrow q) \vee (q \rightarrow p)$ – the *Gödel-Dummet logic*, *Gödel logic*, or *linear calculus*.
4. $\text{KC} = \text{IPC} \oplus (\neg p \vee \neg\neg p) = \text{IPC} \oplus \neg(a \wedge b) \leftrightarrow (\neg a \vee \neg b)$ – sometimes called *DeMorgan logic*, the *logic of weak excluded middle* or *Yankov logic*.
5. $\text{BD}_n = \text{IPC} \oplus \text{bd}_n$ where $\text{bd}_1 = p_1 \vee \neg p_1$ and $\text{bd}_{n+1} = p_{n+1} \vee (p_{n+1} \rightarrow \text{bd}_n)$, the *logics of dounded depth n* .
6. $\text{BW}_n = \text{IPC} \oplus \bigvee_{i=0}^n (p_i \rightarrow \bigvee_{j \neq i} p_j)$, the *logics of bounded width n* .

We say that a logic L is *consistent* if $\perp \notin L$; otherwise we say that it is *inconsistent*. Note that due to Modus Ponens, there is a unique inconsistent logic – **Form**. Then we have (see e.g. [82, Thm. 4.1]):

Proposition 2.3.6 (Classical logic is largest consistent). For each superintuitionistic logic L , if L is consistent, $L \subseteq \text{CPC}$.

The logic IPC , and all of its extensions, are algebraizable, and indeed implicative. The algebras that are its equivalent algebraic semantics are the Heyting algebras we met in Definition 2.1.14. The following is the precise, and fundamental, connection between Heyting algebras and intuitionistic logic:

Theorem 2.3.7 (Algebraic completeness). *The logic IPC is algebraizable (implicative) and **HA** is its equivalent algebraic semantics.*

Proof. This goes back to at least [226]. ■

Notation 2.3.2. Given a superintuitionistic logic L , we write $\mathbf{V}_L$ for the variety of Heyting algebras validating L . We reserve the symbol $\mathbf{V}$ to denote varieties of Heyting algebras, and write $\text{Log}(\mathbf{V})$ for the logic associated to a variety of Heyting algebras.

Some properties of superintuitionistic logic are not shared with its classical counterpart, and are peculiar to the intuitionistic framework:

Definition 2.3.8 (Disjunction property). A superintuitionistic logic L has the *disjunction property* if whenever $\varphi \vee \psi \in L$ then either $\varphi \in L$ or $\psi \in L$.

Proposition 2.3.9 (IPC has the disjunction property). The logic IPC has the disjunction property. None of the other logics in the above list have this property.

Proof. This goes back at least to [181]. See [82, Chapter 15] for a modern proof. ■

2.3.2 Order-theoretic Semantics

In this section we recall the basics of Kripke semantics of IPC and its extensions, as well as the general semantics using Esakia spaces. This will allow us to work throughout with correspondence between axioms valid in algebras, and dual properties valid in spaces and posets.

Definition 2.3.10 (Kripke model). We say that a triple $(\mathfrak{M}, r) = (P, \leq, c, r)$ where $(P, \leq)$ is a rooted poset, with root r , and $c : P \rightarrow \mathcal{P}(\bar{p})$ is an order-preserving function, is a (Kripke) *model over the propositional letters $\bar{p}$ over P* .

Definition 2.3.11 (Kripke semantics). If $(\mathfrak{M}, r) = (P, \leq, c, r)$ is a model and φ is a formula, we define the forcing relation $(\mathfrak{M}, r) \Vdash \varphi$ by induction on formulas:

- $\mathfrak{M}, r \Vdash \perp$ never;
- $\mathfrak{M}, r \Vdash p$ if and only if $p \in c(x)$.
- $\mathfrak{M}, r \Vdash \varphi \wedge \psi$ if and only if $\mathfrak{M}, r \Vdash \varphi$ and $\mathfrak{M}, r \Vdash \psi$;
- $\mathfrak{M}, r \Vdash \varphi \vee \psi$ if and only if $\mathfrak{M}, r \Vdash \varphi$ or $\mathfrak{M}, r \Vdash \psi$;
- $\mathfrak{M}, r \Vdash \varphi \rightarrow \psi$ if and only if whenever $r \leq q$ and $\mathfrak{M}, q \Vdash \varphi$ then $\mathfrak{M}, q \Vdash \psi$.

Definition 2.3.12 (Kripke and Esakia validities). For a poset P , we write $P \Vdash \varphi$ and say that φ is *valid over P* , if every model over P forces φ . P is said to validate a logic L if it validates all axioms in L . For an Esakia space $(X, \leq, \tau)$, we similarly write $X \Vdash \varphi$ and say that X *validates φ* if for each model over X such that for each p , $c^{-1}[\{p\}] \in \mathbf{ClopUp}(X)$, and each $x \in X$, $(\uparrow x, \leq, c, x) \Vdash \varphi$.

Notation 2.3.3 (Logic of posets and Esakia spaces). Given P a poset (resp. Esakia space) we write,

$$\text{Log}(P) = \{\varphi : P \Vdash \varphi\},$$

for the *logic of P* , which is always an element of $\text{Ext}(\text{IPC})$. Indeed, it should be clear that for an Esakia space X , $\text{Log}(X) = \text{Log}(\mathbf{ClopUp}(X))$, by definition.

Definition 2.3.13 (Kripke completeness). A logic L is said to be *Kripke complete* if there is a class $\mathbf{P}$ of posets such that $L = \text{Log}(\mathbf{P})$.

Remark 2.3.14 (Kripke semantics of positive logic). Recall from Remark 2.2.8 the positive logic PL. Since such a logic is a fragment of intuitionistic logic, it admits a natural Kripke semantics obtained by restriction of the intuitionistic Kripke semantics. This is peculiar in that the semantics, restricted to the connectives $\top, \perp, \wedge, \vee$, is nevertheless entirely local; the partial order, in such a case, can be seen as representing the possibility of worlds “gaining knowledge”, despite no actual movement occurring as far as the logical connectives allow.

Remark 2.3.15 (Frames=Posets+Esakia spaces). When we wish to make no distinction between posets and Esakia spaces we will refer simply to a *frame*.

We will now take a look at some basic properties which will recurringly appear in our work.

Definition 2.3.16 (Properties of frames). A frame $(F, \leq)$ is said to be:

1. *locally discrete* if for all $x, y \in F$, $x \leq y$ if and only if $x = y$.
2. *locally linear* if for all $x, y, z \in F$, whenever $x \leq y, z$ then either $y \leq z$ or $z \leq y$.
3. *locally directed* if for all $x, y, z \in F$, if $x \leq y, z$ there is some $w \in F$ such that $y \leq w$ and $z \leq w$.
4. a *tree* if it is rooted, and for each $x \in F$, $\downarrow x$ is linear.

Definition 2.3.17 (Depth in a frame). Let $(F, \leq)$ be a frame. Given $x \in F$, the *depth of x* , denoted $d_p(x)$, is the largest n such that there exists a chain $x = x_1 \leq \dots \leq x_n$, and there is no such chain of size $n + 1$, or ∞ if such an n does not exist. We denote by $d_p(F) = \sup\{d_p(x) : x \in F\}$.

Definition 2.3.18 (Width in a frame). Let $(F, \leq)$ be a frame. Given $x \in F$ *width of x* , denoted $w_p(x)$ is the largest n such that there is an antichain of size n and no antichain of size $n + 1$ in $\uparrow x$, or ∞ if such an n does not exist. We denote by $w_p(F) = \sup\{w_p(x) : x \in F\}$.

The following correspondences are well-known [82, Ch. 2, Ch. 8]:

Proposition 2.3.19 (Correspondence for superintuitionistic logics). For F a frame, we have:

1. $F \Vdash \text{CPC}$ if and only if F is discrete. Moreover, $\text{Log}(\text{CPC}) = \{F : F \text{ a discrete frame with a single point}\}$.
2. $F \Vdash \text{Sme}$ if and only if F is locally linear and has depth at most 2, if and only if $F \Vdash \text{BD}_2\text{W}_1$. Moreover, $\text{Log}(\text{Sme}) = \{F : F \text{ a 2-element chain}\}$.
3. $F \Vdash \text{LC}$ if and only if F is locally linear. Moreover, $\text{Log}(\text{LC})$ equals $\text{Log}(\{F : F \text{ a finite chain}\})$.
4. $F \Vdash \text{KC}$ if and only if F is locally directed. Moreover, if F is an Esakia space, this is likewise equivalent to $\max(\uparrow x)$ being a singleton for each $x \in F$. Moreover, $\text{Log}(\text{KC}) = \{F : F \text{ a finite poset with a maximal element}\}$.
5. $F \Vdash \text{BD}_n$ if and only if each point in F has depth less than or equal to n .
6. $F \Vdash \text{BW}_n$ if and only if each point in F has width at most n .

2.3.3 Bisimulations and k-bisimulations

A key technique for studying syntactic properties of superintuitionistic logics is the notion of *bisimulation* and the more specialized *bounded bisimulation*. In this section we recall some of their basic properties.

Definition 2.3.20 (Bisimulation). Given two models $\mathfrak{M} = (P, \leq, c, x)$ and $\mathfrak{N} = (P', \leq, c', x)$ over $\bar{p}$, a relation $S \subseteq P \times P'$ is a *bisimulation* based on (x, x') if:

1. Whenever $(w, w') \in S$ then $p \in c(w)$ if and only if $p \in c'(w')$;
2. Whenever $(w, w') \in S$ and $w \leq t$ there is some $t' \geq w'$ such that $(t, t') \in S$;
3. Whenever $(w, w') \in S$ and $w' \leq t'$, there is some $w \leq t$ such that $(t, t') \in S$.

Definition 2.3.21 (k-bisimulation). Given two models $\mathfrak{M} = (P, \leq, c, x)$ and $\mathfrak{N} = (P', \leq, c', x)$ over $\bar{p}$, we say that a relation $S_k \subseteq P \times P'$ is a *k-bisimulation* based on (x, x') if there are relations $S_k \subseteq \dots \subseteq S_j \subseteq \dots \subseteq S_0$ for each $0 \leq j \leq k$, such that xS_kx' and:

1. Whenever wS_0w' then $p \in c(w) \iff p \in c'(w')$ for each $p \in \bar{p}$;
2. For each $j < k$, whenever $wS_{j+1}w'$ and $w \leq v$ there is some $w' \leq v'$ such that vS_jv' .
3. For each $j < k$, whenever $wS_{j+1}w'$ and $w' \leq v'$ there is some $w \leq v$ such that vS_jv' .

Definition 2.3.22 (bisimilarity, k-bisimilarity). We say that $(\mathfrak{M}, x)$ is *(k-)bisimilar* with $(\mathfrak{N}, x')$ if there exists a *(k-)bisimulation* between them. We write $(\mathfrak{M}, x) \sim_{\bar{p}} (\mathfrak{N}, x')$ to mean that there is a bisimulation in this language, and $(\mathfrak{M}, x) \sim_{\bar{p}, k} (\mathfrak{N}, x')$ to mean that there is a *k-bisimulation* in this language.

As described in [130, Chapter 4.1], *k-bisimulations* can be characterized in game-theoretic terms: given a pair $(\mathfrak{M}, x)$ and $(\mathfrak{N}, x')$, Player I is said to win a 0-bisimulation game if $c(x) \neq c'(x')$ and otherwise Player II wins; and Player I wins a $k + 1$ -bisimulation game if there is $z \in \mathfrak{M}$, such that Player II cannot choose any $w \in \mathfrak{N}$ and win a *k-bisimulation* game with $(\mathfrak{M}_z, z)$ and $(\mathfrak{N}_w, w)$. Full bisimulations correspond to the same when the game is allowed to run for an arbitrary number of steps, up to ω . Given this point of view, it is sometimes convenient to consider an *asymmetric* variant of the game:

Definition 2.3.23 (Order-bisimulations, order-k-bisimulations). Given two Kripke models $\mathfrak{M}$ and $\mathfrak{N}$, we write $(\mathfrak{M}, x) \leq_{\bar{p}} (\mathfrak{N}, x')$, (respectively $(\mathfrak{M}, x) \leq_{\bar{p}, k} (\mathfrak{N}, x')$), to mean that there is some $x \leq z$ such that $(\mathfrak{M}_z, z) \sim_{\bar{p}} (\mathfrak{N}, x')$ (respectively $(\mathfrak{M}_z, z) \sim_{\bar{p}, k} (\mathfrak{N}, x')$).

With these game-theoretic intuitions, the following basic properties of (k -)bisimulations are straightforward:

Proposition 2.3.24 (Basic properties of k -bisimulations). For all models $(\mathfrak{M}, x)$, $(\mathfrak{N}, x')$ and $(\mathfrak{L}, x'')$, over $\bar{p}$ we have:

1. If $(\mathfrak{M}, x) \sim_{\bar{p}} (\mathfrak{N}, x') \sim_{\bar{p}} (\mathfrak{L}, x'')$ then $(\mathfrak{M}, x) \sim_{\bar{p}} (\mathfrak{L}, x'')$, and if $(\mathfrak{M}, x) \sim_{\bar{p},k} (\mathfrak{N}, x') \sim_{\bar{p},k} (\mathfrak{L}, x'')$ then $(\mathfrak{M}, x) \sim_{\bar{p},k} (\mathfrak{L}, x'')$.
2. $(\mathfrak{M}, x) \leq_{\bar{p}} (\mathfrak{N}, x')$ if and only if we have the following: for each formula $\varphi(\bar{p})$, $(\mathfrak{M}, x) \Vdash \varphi$ implies $(\mathfrak{N}, x') \Vdash \varphi$.
3. $(\mathfrak{M}, x) \leq_{\bar{p},k} (\mathfrak{N}, x')$ if and only if we have the following: for each formula $\varphi(\bar{p})$ such that $d(\varphi(\bar{p})) \leq n$, $(\mathfrak{M}, x) \Vdash \varphi$ implies $(\mathfrak{N}, x') \Vdash \varphi$.

Proof. See [130, Prop. 4.1]. ■

2.3.4 Properties of IPC and its extensions

In this final section we briefly correspond the logical and algebraic properties mentioned throughout with the main logical systems of interest just introduced. We begin with the general universal algebraic ones:

Proposition 2.3.25 (Universal algebraic properties). All extensions of IPC by theorems have a deduction theorem; all varieties $\mathbf{V}$ of Heyting algebras are congruence-distributive and have the CEP.

Proof. See [82, Thm. 2.42] for the deduction theorem, which is obviously inherited by extensions by axioms. By Proposition 2.2.19, since Heyting algebras are lattice expansions, they are congruence-distributive. The congruence extension property follows from the deduction theorem, which is equivalent to EDPC (Theorem 2.2.48), under the particular assumptions on **HA** we have established (see [224, Thm. 8.2.(iii)] and references therein contained). ■

As far as completeness properties goes, we have the following picture:

Proposition 2.3.26 (Completeness properties). The following properties hold of the systems outlined:

1. The logics **CPC**, **Sme** and **BD₂W₂** are tabular.
2. The logics **LC** and **BD_n** for each $n > 1$ are locally tabular, but not tabular.
3. The logics **IPC**, **KC**, **BW_n** for $n \geq 2$ are not locally tabular, but have the finite model property.

Proof. See [82, Ch. 4, Ch. 8]. ■

As far as interpolation properties, the original interpolation theorem was shown by Schütte [237], whilst the following characterization results are due to Maksimova[184]:

Theorem 2.3.27 (Maksimova’s theorem on CIP). *For L a superintuitionistic logic, L has the CIP if and only if L has deductive interpolation, if and only if L is one of the following logics: IPC, KC, LC, Sme, BD_2 , BD_2W_2 or CPC.*

We note that the negative part of this result – showing that all other logics fail to have interpolation – goes through the equivalence of the CIP and the superamalgamation property discussed in Section 2.2.3, namely, by refuting the amalgamation property for all other logics.

Maksimova’s theorem was strengthened by Pitts [218] and Ghilardi and Zawadowski [130]:

Theorem 2.3.28 (Characterization of UIP). *For L a superintuitionistic logic, all the logics from Theorem 2.3.27 have uniform interpolation.*

Proof. See [130, Ch. 4.5–4.7] ■

It will be nice at this point to remark on the semantics of such a property, which in intuitionistic logic become rather concrete, and are connected to a simple definability property:

Definition 2.3.29 (Adjoint definability property). We say that a logic $L \supseteq \text{IPC}$ has the *adjoint definability property* if for each n , the inclusions $i: \mathcal{F}_L(n) \rightarrow \mathcal{F}_L(n+1)$ have left and right adjoints.

Remark 2.3.30 (Adjoint definability $\neq$ uniform interpolation). Note that adjoint definability is true for any locally tabular logic, and hence, has no a priori connection to interpolation. As explained in [125], it is rather the Beck-Chevalley condition from Theorem 2.2.44 which guarantees interpolation – which in the presence of adjoint definability, becomes *uniform* interpolation.

Given p, q propositional letters, let $\mathcal{L}_2(\bar{p})$ be the language of intuitionistic logic containing propositional quantifiers $\forall p$ and $\exists p$.

Definition 2.3.31 (Bisimulation quantifiers). Given a formula $\varphi \in \mathcal{L}_2(\bar{p})$ we give semantics to φ over Kripke models $(\mathfrak{M}, x)$ by:

1. If φ is a propositional letter, $\perp$ or $\top$, or the primary connective of φ from $\mathcal{L}$, then the semantics is the usual one.
2. $\mathfrak{M}, x \Vdash \exists p_i \varphi$ if and only if there is a model $\mathfrak{N}, y$ over $\bar{p}$, such that $\mathfrak{M}, x \sim_{\bar{p} \setminus p_i} \mathfrak{N}, y$, and $\mathfrak{N}, y \Vdash \varphi$.
3. $\mathfrak{M}, x \Vdash \forall p_i \varphi$ if and only if for each $x \leq x'$, and any model $\mathfrak{N}, y$ over $\bar{p}$ such that $\mathfrak{M}, x \sim_{\bar{p} \setminus p_i} \mathfrak{N}, y$ and $\mathfrak{N}, y \Vdash \varphi$.

Such quantifiers are often known as *bisimulation quantifiers*. The following is their connection to adjoint definability (for a proof, see [135, Prop. 5]):

Proposition 2.3.32 (Adjoint definability and bisimulation quantifiers). Assume that L is a logic which has the FMP. Suppose that for each $\varphi(\bar{p}, q)$, there is a formula $\psi(\bar{q})$ such that for each finite model $\mathfrak{M}, x$:

$$\mathfrak{M}, x \Vdash \psi(\bar{p}) \iff \mathfrak{M}, x \Vdash \exists q \varphi(\bar{p}, q),$$

and a formula $\psi'(\bar{p})$ such that for each finite model $\mathfrak{M}, x$:

$$\mathfrak{M}, x \Vdash \psi'(\bar{p}) \iff \mathfrak{M}, x \Vdash \forall q \varphi(\bar{p}, q);$$

Then L has the adjoint definability property.

Remark 2.3.33 (Adjoint definability beyond intuitionistic logic). We note that Proposition 2.3.32 holds in much greater generality, for modal logics, bi-intuitionistic logics, and several other related systems. Under reasonable assumptions, it can be turned into an equivalence – see [135, Prop. 10] for a recent discussion of this, following the main results from [130, Ch. 4, Ch. 5].

From the bisimulation quantifiers several interesting properties of the uniform interpolants can be extracted, the following being perhaps the most peculiar:

Proposition 2.3.34 (Join-distributivity). For each pair of formulas φ, ψ in intuitionistic logic, we have:

$$\forall q(\varphi(\bar{p}, q) \vee \psi(\bar{p}, q)) \equiv \forall q\varphi(\bar{p}, q) \vee \forall q\psi(\bar{p}, q).$$

Proof. This was observed by Pitts [218, Exa. 10]; for a proof using bisimulation quantifiers see [257, Thm. 5.4]. ■

This property can be seen as a form of *internal disjunction property*. In fact, we have the following:

Proposition 2.3.35 (Join-distributivity implies disjunction property). Let L be a consistent superintuitionistic logic with the adjoint definability property. If L has join-distributivity of right adjoints, then L has the disjunction property.

Proof. Assume that $\varphi(\bar{p}) \vee \psi(\bar{p}) \in L$. By definition, this means that in the free algebra for this logic, $\top \leq_L \varphi(\bar{p}) \vee \psi(\bar{p})$. By the definition of the right adjoints, then $\forall_{\bar{p}}(\varphi(\bar{p}) \vee \psi(\bar{p})) \in L$ as well. By assumption of distributing over disjunctions, $\forall_{\bar{p}}\varphi(\bar{p}) \vee \forall_{\bar{p}}\psi(\bar{p}) \in L$. However, note that $\forall_{\bar{p}}\varphi(\bar{p}) \equiv_L \top$ or $\forall_{\bar{p}}\varphi(\bar{p}) \equiv_L \perp$ and similarly for ψ . If both were equivalent to $\perp$, then $\perp \in L$, which would make the logic inconsistent. Thus either $\forall_{\bar{p}}\varphi(\bar{p}) \equiv_L \top$ or $\forall_{\bar{p}}\psi(\bar{p}) \equiv_L \top$. This means, applying the adjunction, that $\top \leq \varphi(\bar{p})$ or $\top \leq \psi(\bar{p})$, meaning that $\varphi \in L$ or $\psi \in L$. ■

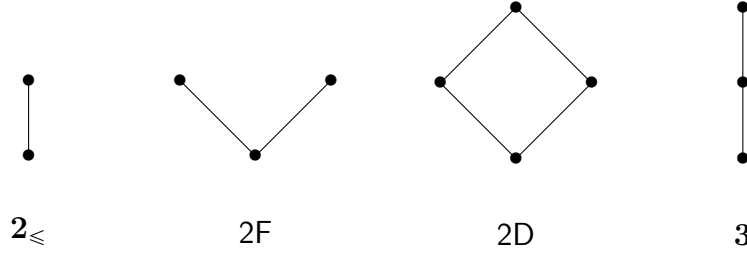


Figure 2.4: Forbidden frames in lattice of extensions

2.3.5 Lattice of extensions of IPC

As we have remarked before, Kuznetsov's hierarchy discussed in the introduction stratifies certain logics as more complex than others. This intuition is reflected in the structure of the *lattice of superintuitionistic logics* $\text{Ext}(\text{IPC})$. Since this intuition will serve us throughout, we briefly recall some basic properties of this lattice, and some of the basic machinery used to study it. Central to it is the following theorem, due to Yankov (see [37] for an extensive discussion):

Theorem 2.3.36 (Yankov's theorem). *For a finite poset P and a rooted poset Q , $P \not\models \mathcal{J}(Q)$ if and only if Q is a rooted subframe of a p -morphic image of P .*

Proof. See e.g. [37, Thm. 3.3]. ■

A key interest of Yankov formulas lies in their possibility to axiomatize logics purely in terms of their associated frames. This is employed systematically in several lines of research (see e.g. [40, 192, 65]). The key insight, originating from the work of Jónsson, and heavily exploited by Blok, is the use of *splittings*:

Definition 2.3.37 (Splitting logic). A pair $L_1, L_2 \in \text{Ext}(\text{IPC})$ is called a *splitting* of $\text{Ext}(\text{IPC})$ if $L_1 \not\subseteq L_2$ and we have

$$\text{Ext}(\text{IPC}) = \uparrow L_1 \cup \downarrow L_2.$$

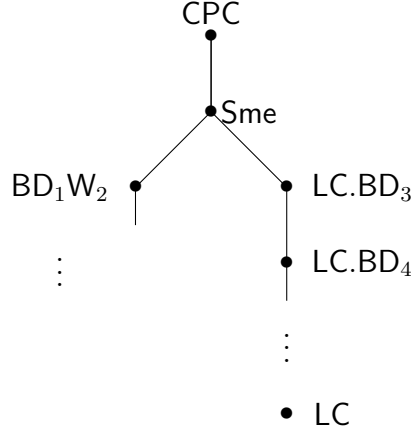
The logic L_1 is called a *splitting logic*. A logic L is called a *union-splitting* if there are a family of splitting logics L_i such that $L = \bigoplus_{i \in I} L_i$.

The connection between these concepts comes from the following theorem:

Theorem 2.3.38 (Splitting theorem). *A logic $L \in \text{Ext}(\text{IPC})$ is a splitting logic if and only if there is a finite rooted poset Q such that $L = \text{IPC} \oplus \mathcal{J}(Q)$.*

Proof. See e.g. [37, Thm. 3.7]. ■

We now recall the structure of the lattice of extensions of IPC, namely its topmost part:

Figure 2.5: Top part of lattice $\text{Ext}(\text{IPC})$.

Proposition 2.3.39 (Structure of lattice of extensions of IPC). The top part of the lattice $\text{Ext}(\text{IPC})$ has the following structure:

1. CPC is the largest consistent extension of IPC;
2. The pair (CPC, Sme) is a splitting. The logic $\text{Sme} = \text{Log}(\mathbf{2}_{\leq})$ (Figure 2.4).
3. The logic LC is a union splitting, and it is $\mathcal{J}(2F) \oplus \mathcal{J}(2D)$.

Proof. (1) this was noted in Proposition 2.3.6.

(2) and (3) To show that CPC is split by Sme, and LC is split by the frames 2F and 2D one can argue algebraically and dually. We do this for Sme as an illustration, with the case of LC being discussed in [82, Ch. 8], and following very similar arguments.

We have to show that any Heyting algebra which is not Boolean is contained in the variety **Sme**; by algebraic completeness, this yields the result. So assume that H is not Boolean. By Proposition 2.3.19, then $X = \text{Spec}(H)$ must contain some point x such that $|\uparrow x| > 1$. Let $X_0 := \uparrow x$.

Since $\max(X_0)$ is closed (Proposition 3.3.23), and $x \notin \max(X_0)$, by basic properties of Priestley spaces, there is a clopen upset $U \supseteq \max(X_0)$ such that $x \notin U$. Then define $f: X_0 \rightarrow \mathbf{2}_{\leq}$ by sending each $y \in U$ to 1, and each $y \notin U$ to 0. This is order-preserving, since U is an upset, and is a clopen upset since each $y \in X_0$ sees a maximal element. Thus $\mathbf{2}_{\leq}$ is a p-morphic image of X_0 , showing by duality that $H \in \mathbf{Sme}$. ■

Table 2.1: Duality dictionary

Dist. Lattices Heyting algebras	Priestley duality	Esakia duality
Distributive lattice	Priestley space	-
Heyting algebra	-	Esakia space
Meets	Intersection	
Joins	Union	
Relative pseudocomplements	$\Box(U \Rightarrow V)$	
Homomorphism	Continuous Order Preserving map	Continuous p-morphism
Homomorphic Image	Closed subset	Closed upset
Surjective homomorphism	Embedding	Injective p-morphism Generated subframe
Injective homomorphism	Surjective cont. ord. preserving	Surjective cont. p-morphism
Filter	Closed upset	
Finite Product	Finite Sum of Spaces	Finite Sum of spaces
Coproduct	Cartesian product	-
Pushout	Cartesian pullback	-
Directed system Directed limit	Inverse system Inverse limit	
Element of algebra	Clopen upset	
Join irreducible element	Rooted clopen upset	
Completely join irreducible element	Rooted clopen upset with isolated root	

Table 2.2: Logic and algebraic properties, and equivalents.

Logic	Algebra	Equivalents
Tabularity	Finitely Generated Variety	$ \mathbf{A}_{s,i} < \omega$
Local Tabularity	Locally Finite Variety	$\forall n, \mathcal{F}_{\mathbf{A}}(n)$ finite
Finite Model Property	Fin. Approximable Variety	$\forall n, \mathcal{F}_{\mathbf{A}}(n)$ fin. approximable
Deduction theorem	EDPC variety	-
Craig interpolation		Superamalgamation
Maehara Interpolation		Amalgamation + CEP
Right Uniform D. Interpolation		Amalgamation + Coherence
Left+Right Uniform D. Interpolation		Model Completion of $\mathbf{A}$ exists.
Left/Right Uniform Craig Interpolation	-	$\mathcal{F}_{\mathbf{A}}(n) \rightarrow \mathcal{F}_{\mathbf{A}}(n+1)$ has left/right adjoint satisfying Beck-Chevalley
Structural interpretation	Relative anarchy	-

Part One

Representations of free Heyting algebras

In the first part of this thesis, we study two main constructions of free Heyting algebras: through universal models, in the style of Bellissima [30], Esakia and Grigolia [107] and Shehtman [242]; and through step-by-step constructions, in the vein of Urquhart [251] and Ghilardi [127]. We present such results in a contextualized way, in as much generality as we make use of them in this thesis.

1. In Chapter 3 we consider the universal model construction, and present it in the general case of superintuitionistic logics with the FMP. The representation of the free algebra is here given by “definable upsets”. Due to the difficulties of providing explicit characterizations of these definable upsets, we look to characterizing such kind of representations, where one uses, instead of a duality, a more general class of structures. This is done by developing a basic theory of *locally Esakia spaces*. As applications of such a theory, we study Bellissima algebras, which arise naturally out of studying the universal model, as well as locally Esakia representations of duals of products and ultraproducts.
2. In Chapter 4, we present the step by step construction for general Heyting algebras generated by a distributive lattice, by introducing the theory of *Ghilardi complexes*. This allows us to describe many constructions of interest for the theory of Heyting algebras, like coproducts, pushouts and polynomial Heyting algebras, in a dual manner, and to obtain a finer grasp on it. This allows us to explicitly study a an interpretation of IPC to its implication-free fragment, shown to be structural but not conservative. A study of a special property of Ghilardi complexes, called here *G-conservativity* allows us in turn to describe a *conservative* translation from IPC to its implication-free fragment.
3. In Chapter 5, we provide a method for generalizing the step by step construction to subvarieties, and apply it in four cases of interest – CPC, Sme, LC and KC. We use this method to provide normal forms for these logics, and prove analogues of known results by Ghilardi [127] for them. We also provide an analogue of the step by step construction for posets and locally-finite posets. We finish by discussing several other applications with reach beyond the scope of the present thesis.

Chapter 3

Universal Models and locally Esakia spaces

In this chapter we discuss our first representation of free Heyting algebras, via so-called “universal models”. Typically credited to Bellissima [30], as well as Esakia and Grigolia [107], this construction was studied independently, and in various forms found by several different people, including De Jongh [160] and Shehtman [242]. In all these instances, such a study arose out of an attempt to generalize the representation of the Rieger-Nishimura lattice. Despite being rather more involved than the Rieger-Nishimura, the universal model construction and close variations have been employed to great use in the study of admissibility [234], structural completeness [65] and positive preservation results [62], amongst many other settings. The construction has also been studied in its own right by several people, most notably, in recent years, in [123, 91].

As a representation, the universal model on n -generators, $\mathcal{U}(n)$ is connected to the dual of $\mathcal{F}(n)$, denoted by $\mathcal{X}(n)$, via “definable subsets”. The precise nature of this connection has been fleshed out, most notably in [123] as well as in [136]. The perspective we adopt here is that of seeing such spaces as *generalized representations* of the algebra, namely, by non-compact spaces. From this point of view, a similar situation appears in the the dual of the product of Heyting algebras $(H_i)_{i \in I}$, with duals $X_i = \mathbf{Spec}(H_i)$, through the topological sum of Esakia spaces $\bigoplus_{i \in I} X_i$, which will fail to be compact whenever I is infinite. In both of these cases, we connect such generalized representations back to the Esakia dual through *order-compactifications*. This presents $\mathcal{U}(n)$ as a special kind of *locally Esakia space*, a concept introduced here. Through this we obtain a topological description of duals of what we call *Bellissima algebras*, which are generated out of the universal model, and correspond to the least Esakia order-compactification. In the special case of logics of bounded width 2 we obtain a full description of the duals of such algebras.

Outline of the chapter

1. In Section 3.1 we present the general idea of how free algebras can be represented using geometric refutation patterns.
2. In Section 3.2 we briefly recap the construction of universal models for superintuitionistic logics with the FMP;
3. In Section 3.3 we introduce locally Esakia spaces. These are sought with the goal of providing a flexible framework for duality for non-compact structures, systematically relating them to Esakia duality. We prove in particular that locally Esakia spaces are a reflective subcategory of Esakia spaces, explicitly describing the reflector.
4. In Section 3.4, we provide two applications of locally Esakia spaces: to the description of Bellissima algebras as least Esakia order-compactifications, and to the dual description of products of Heyting algebras.

3.1 Free algebras: geometrically encoded refutation patterns

One of the natural features of CPC from a semantic point of view is the existence of truth-tables – single point frames over which we can vary valuations to obtain all possible models of a given formula. This simplicity means that one can use these truth tables to obtain a representation of all formulas, and with that, of free Boolean algebras: by looking at the models one obtains, and knowing how to dually represent Boolean algebras, we obtain the usual representation of free finitely generated Boolean algebras by truth-assignments (see Section 4.1.2).

This simplicity is shared with very few other systems: it was originally shown by Gödel [134] that intuitionistic logic IPC cannot be tabular, and neither can it be captured by finitely many models over finitely many propositional variables. Nevertheless as is well-known (Section 2.3.4) finite models are enough to capture the semantics of IPC.

One may thus attempt to describe the structure of free algebras by using these finite models as *refutation patterns*: starting from a given formula $\varphi(\bar{p})$, consider a finite collection of model which, in an appropriate sense, are minimal in being able to refute $\varphi(\bar{p})$. This is the underlying principle of the construction of the *Rieger-Nishimura ladder* denoted by RN, which, when dualized, produces the free Heyting algebra on one generator, the *Rieger-Nishimura lattice* [231, 209] (see Figure 3.1).

The frame RN can be constructed recursively (see e.g. [47]) in the following way:

Definition 3.1.1 (Rieger-Nishimura ladder). We inductively define a collection of points. We start with a point 1, a point labelled with 0 which is incomparable

with it, and a point labelled with 0 which has 1 as its unique immediate successor; call this set X_0 . We then construct X_{n+1} by picking for each antichain of points in X_n $x_0, \dots, x_n$ such that there is no point y immediately below each x_i ; as it turns out, one can show that X_{n+1} always adds only two points. We let $\mathcal{U}(1) = \bigcup_{n \in \omega} X_n \cup \{\infty\}$, placing ∞ below every other point.

The poset $\mathcal{U}(1)$ is equipped with a topology by letting the opens be the finite sets, and the cofinite sets containing ∞ . The resulting space is denoted by $\mathcal{X}(1)$, and is called the *Rieger-Nishimura ladder*.

From this we can recursively extract a description of the formulas in one variable, by reading their semantics in the Esakia dual:

Definition 3.1.2 (Rieger-Nishimura polynomials). The *Rieger-Nishimura polynomials*, are constructed by recursion, on the basis of the sets used in constructing $\mathcal{U}(1)$:

1. The formula $g_{-1}(p) = \perp$, corresponding to the empty set;
2. The formula $g_0(p) = p$ corresponding to the point 1, and the formula $g_1(p) = \neg p$ corresponding to the point 0.
3. Given any $n > 0$, the formula $f_n(p) = g_{n-1}(p) \vee g_n(p)$, representing the union of the points associated with $g_n(p)$ and $g_{n+1}(p)$; we make the purely formal definition $f_{-1}(p) = \perp$.
4. Given any $n > 1$, the formula $g_n(p) = g_{n-1}(p) \rightarrow f_{n-3}(p)$, representing the point having as its unique successors the elements in the set given by $f_{n-3}(p)$.

Each of the points represented by $g_n(p)$ can be seen to be a minimal model refuting the formula it captures, in the sense that no two nodes of it satisfy the same formulas. The generalization of this to several variables leads to the concept of a *De Jongh formula* studied by De Jongh in [160], which capture, in a fixed number of variables, the same property as Yankov formulas (see Theorem 2.3.36).

3.2 Universal models for logics with FMP

Recall from Section 2.3.2 that for us models (over n -many variables) consist of tuples $\mathfrak{M} = (P, \leq, c, r)$, where $(P, \leq)$ is a poset, with a root r , and valuations on n many variables. It can be convenient to see such variables as functions from n to the poset $\mathbf{2}_{\leq}$, with the pointwise order, normally referred to as *colors*.

Definition 3.2.1 (Total covering relation). Let P be a poset, $A \subseteq P$ a subset and $x \in A$. We say that A *totally covers* x and write $x \preceq A$ if A is the set of all immediate successors of x .

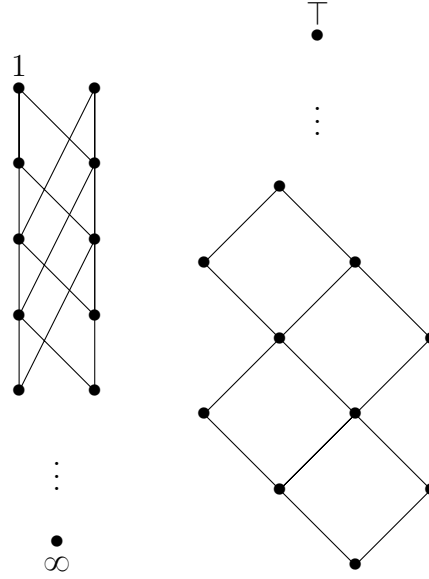


Figure 3.1: Rieger-Nishimura Ladder RN and its dual, the Rieger-Nishimura lattice

We write $x \leq y$ to mean $x \leq \{y\}$; note that in this notation, this means that x is the *only* immediate successor of y (as opposed to the usual $x < y$, which means that there are no points strictly between the two).

Remark 3.2.2 (Specifying relations on locally finite posets). Note that if $(P, \leq)$ is a locally finite poset, then $\leq$ is the reflexive and transitive closure of the relation $\leq$. Consequently, to define a locally finite poset it will suffice to specify such a relation on P , a fact we will use below.

Definition 3.2.3 (Irreducible model). Let $\mathfrak{M} = (P, \leq, c, r)$ be a Kripke model. We say that $\mathfrak{M}$ is *irreducible* if whenever $x, y \in P$, and $\mathfrak{M}_x$ and $\mathfrak{M}_y$ are bisimilar, then $x = y$.

Note that on any model, the relation of being bisimilar is an equivalence. Hence, it is always possible to go from a model to an irreducible one by quotienting:

Definition 3.2.4 (Bisimulation quotient). Given a model $\mathfrak{M} = (P, \leq, c, r)$, the *bisimulation quotient* $\mathfrak{M}^\sim = (P^\sim, \leq^\sim, c^\sim, [r])$ is defined by letting $P^\sim = \{[x]^\sim : x \in P\}$, $[x]^\sim \leq [y]^\sim$ if and only if $\exists x' \in [x]$ and $y' \in [y]$ such that $x' \leq y'$ and $c^\sim([x]^\sim) = c(x)$.

Proposition 3.2.5 (Canonical quotient defines bisimulation). Given $\mathfrak{M} = (P, \leq, c, r)$, the canonical quotient map $\text{pr}: P \rightarrow P^\sim$ is a p-morphism, its graph defines a bisimulation, and $\mathfrak{M}^\sim$ is an irreducible model.

Proof. This is a routine check. ■

From Proposition 3.2.5, we obtain a way to reduce any model to an irreducible one; we now show how to enumerate them.

Notation 3.2.1 (Adjoining new root). Given P a finite poset, we denote by ${}^rP$ the poset obtained by adjoining r as a new minimal element to the disjoint union $P \cup \{r\}$.

Proposition 3.2.6 (Recursive enumeration of irreducible models). Given a superintuitionistic logic L , and n a natural number, the class of finite irreducible models for L over n many variables, is enumerated by the following recursive procedure:

1. We define $I_L(n)_0$ by setting it to contain, for each $f \in 2^m$, the singleton $\{f\}$, with $c(f) = f$;
2. Assuming that $I_L(n)_k$ has been defined, we specify:
 - (a) For each $\mathfrak{M} \in I_L(n)_k$ where $\mathfrak{M} = (P, \leq, c, x)$ is a model, and each $f < c(x)$, such that ${}^rP$ is a frame for L , the model $\mathfrak{M}_{f,P} = \{y_{f,P}\} \cup P$, where $c(y_{f,P}) = f$;
 - (b) For every finite set $S = \{\mathfrak{N}_1 = (T_1, \leq, c_1, r_{T_1}), \dots, \mathfrak{N}_j = (T_j, \leq, c_j, r_{T_j})\}$ of elements from $I_L(n)_k$, where $|j| > 1$, and every color f such that $f \leq c_j(r_{T_j})$, and ${}^r(\bigcup_{i=1}^j T_i)$ is a frame for L , the model $\mathfrak{M}_{f,S} = \{y_{f,S}\} \cup \bigcup_{i=1}^j T_i$ where $c(y_{f,S}) = f$;

and $I_L(n)_{k+1}$ is the union of $I_L(n)_k$ with the set of all $\mathfrak{M}_{f,w}$ and $\mathfrak{M}_{f,S}$.

Proof. We prove by induction on height of a poset P , that for any model $\mathfrak{M} = (P, \leq, c, r)$, $\mathfrak{M}$ over n many variables, $\mathfrak{M}$ is irreducible if and only if it is generated by the above process. If the height of P is 1, then certainly P is irreducible, and appears in the above process. So assume that this holds for posets of height $< k$, and we show it for posets of height k . First let $\mathfrak{M} = (P, \leq, c, r)$ be a model, where P has height k , generated by the above procedure, and suppose towards a contradiction that $\mathfrak{M}$ is not irreducible. Then there exist two points $x, y \in P$ such that $\mathfrak{M}_x$ and $\mathfrak{M}_y$ are bisimilar. Note that we cannot have that both x and y are proper successors of r , since then by hypothesis we would have that $\mathfrak{M}_x$ and $\mathfrak{M}_y$ would both be irreducible, and thus $x = y$. So we assume, without loss of generality that $x = r$. We have two cases:

1. If x is the unique immediate successor of r , the only possibility is that $\mathfrak{M}_r$ is of the form $\mathfrak{M}_{f,P'}$ with $P' = \uparrow p$. Since $\mathfrak{M}_r$ and $\mathfrak{M}_y$ are bisimilar, we would have that $c(r) = c(x)$, which is a contradiction;

2. Otherwise, by construction, there is some immediate successor y of r such that $x \neq y$. Since $\mathfrak{M}$ is bisimilar to $\mathfrak{M}_x$, there must be some $x' \geq x$ such that $\mathfrak{M}_{x'}$ is bisimilar with $\mathfrak{M}_y$ – which is a contradiction to the induction hypothesis.

In all cases we get that $\mathfrak{M}$ and $\mathfrak{M}_x$ cannot be bisimilar, which means that $\mathfrak{M}$ is irreducible.

Conversely, assume that $\mathfrak{M}$ is irreducible. Let $x_1, \dots, x_n$ be all of the immediate successors of r . By assumption, $\mathfrak{M}_{p_i}$ are likewise irreducible, so by induction hypothesis, they were all generated by the above procedure (up to an isomorphism). Then by looking at the color of r , and the number of x_i , and using irreducibility, one obtains that $\mathfrak{M}$ is generated by the above procedure: if there is a single x_i , by irreducibility the color of r must be strictly below that of x_i ; and otherwise there are several x_i , and by persistence of valuations, the color must be below that of the immediate successors. The result follows by induction. ■

We are now ready to define universal models¹ for logics with the FMP². This construction is independently due to Bellissima [30], Esakia and Grigolia [107] and with inspiration of the latter authors, Shehtman [242].

Definition 3.2.7 (n-universal model). Let L be a superintuitionistic logic with the FMP. We define the *n-universal model over L* , $\mathcal{U}_L(n) = (\mathcal{U}_L(n), \leq, c)$, recursively, by setting $\mathcal{U}_L(n) = \bigcup_{k \in \omega} I_L(n)_k$; note that as above, this union is not disjoint.

To obtain the correctness of this construction, one verifies the following:

Proposition 3.2.8. Let L be a superintuitionistic logic with the FMP. Given any formula $\varphi(\bar{p})$ over n many variables, we have that $\varphi(\bar{p}) \in L$ if and only if $\mathcal{U}_L(n) \Vdash \varphi(\bar{p})$.

Proof. If $\varphi(\bar{p}) \in L$, by construction, each point of $\mathcal{U}_L(n)$ is a frame for L , so $\mathcal{U}_L(n) \Vdash \varphi(\bar{p})$. Conversely if $\varphi(\bar{p}) \notin L$, by the FMP, there is a model $\mathfrak{M} \not\Vdash \varphi(\bar{p})$; by Proposition 3.2.5, we can take $\mathfrak{M}$ to be a finite and irreducible model, and by Proposition 3.2.6, such a model is up to isomorphism equal to a model arising in the construction of $\mathcal{U}_L(n)$. ■

Having Proposition 3.2.8 we know that universal models completely capture the validity of formulas. But that begs the question of what their relationship is to free algebras $\mathcal{F}_L(n)$.

¹Note that such structures are not models in the sense of Definition 2.3.10; instead each of its rooted upsets will be one such model.

²Note that this definition makes sense for any logic L , though for logics without the FMP, one does not have that it precisely captures derivability in the logic.

Definition 3.2.9 (Definable upsets). Let L be a superintuitionistic logic with the FMP. For each n , and each $1 \leq i \leq n$, we denote by $c_i(n) \subseteq \mathcal{U}(n)$ the subset of all points in $\mathcal{U}(n)$ validating the proposition labelled with i :

$$c_i(n) = \{x \in \mathcal{U}_L(n) : c(x)(i) = 1\}.$$

We say that a subset $A \subseteq \mathcal{U}(n)$ is *definable* if it belongs to the Heyting algebra generated by the set $\{c_i(n) : 1 \leq i \leq n\}$. We denote by $\text{Def}_L(n)$ the algebra of definable upsets of $\mathcal{U}_L(n)$.

Proposition 3.2.10 (Free algebra is the algebra of definable upsets). The algebra $\mathcal{F}_L(n)$ is isomorphic to $\text{Def}_L(n)$.

Proof. Given $\{p_1, \dots, p_n\}$ we consider the map ι assigning p_i to $c_i(n)$, and uniformly lifted by the universal mapping property. Since the algebra is generated by such subsets, this map is surjective. Assume that $\iota(\varphi) = \iota(\psi)$; then $\iota(\varphi \leftrightarrow \psi) = 1$. Thus we see that $\mathcal{U}_L(n) \Vdash \varphi \leftrightarrow \psi$, which by Proposition 3.2.8 means that $\varphi \leftrightarrow \psi \in L$, and so $\varphi = \psi$ in $\mathcal{F}_L(n)$. This shows that ι is an isomorphism as desired. ■

Notation 3.2.2 (n -Henkin/ n -canonical model). Throughout this chapter we will use $\mathcal{X}_L(n) = \text{Spec}(\mathcal{F}_L(n)) \cong \text{Spec}(\text{Def}_L(n))$ to denote the dual space of the free algebra on n many generators for L .

One might hope that Proposition 3.2.10 might be improved into showing that $\text{Up}(\mathcal{U}_L(n))$ – the *n-canonical model* – is isomorphic to $\mathcal{F}_L(n)$ – but this was disproved by Bellissima [30, Lem. 4.1] (see Appendix A.2.2 for more on this). As noted in [91, Prob. 2], all descriptions of the definable upsets of $\mathcal{U}(n)$ for $n > 1$ seem to rely on additional combinatorial structures than those given by the universal model – namely, by relying on n -bisimulations (See Section 2.3.3), which brings us closer to the representation discussed in Chapter 4. Such a complexity might be unavoidable in the case of full IPC.

3.3 Locally Esakia spaces

Since $\mathcal{U}(n)$ is not compact, one could never expect that $\mathcal{U}(n) \cong \mathcal{X}(n)$, even for $n = 1$. Nevertheless, the appeal of the above representation is clear, in that it gives us a clear access to the geometric properties of free Heyting algebras, via a simple recursive construction. An analogous situation happens with the representation of duals of products of Heyting algebras: whilst the dual of $H_1 \times H_2$ is simply $\text{Spec}(H_1) \sqcup \text{Spec}(H_2)$, given $(H_i)_{i \in I}$ a family of Heyting algebras we have that

$$\bigsqcup_{i \in I} \text{Spec}(H_i),$$

the topological and order union of all these spaces, cannot be compact, and therefore not the dual of $\prod_{i \in I} H_i$. Nevertheless, it seems possible to encode all the relevant information to recover the dual space from this. To help with dealing uniformly with such situations, we describe in the following sections a theory of Esakia order-compactifications, and of the locally Esakia spaces which give a precise meaning to being a space for which one can provide an adequate non-compact representation.

3.3.1 Order-compactifications

It is a classic result that a topological space X admits a compactification iff it is completely regular, and that the Stone-Čech compactification βX is the largest compactification of X (see, e.g., [101, Ch. 3.5]). In fact, the correspondence $X \mapsto \beta X$ extends to a functor which is left adjoint to the inclusion of the category of compact Hausdorff spaces into the category of completely regular spaces (see, e.g., [158, pp.130-131]).

In the context of duality theory, as we know, a special role is played by those compact Hausdorff spaces that are zero-dimensional as they serve as Stone duals of Boolean algebras, and are widely known as *Stone spaces*. This motivates the study of zero-dimensional compactifications, the theory of which resembles that of all compactifications: a topological space X admits a zero-dimensional compactification iff it is a zero-dimensional (Hausdorff) space, and there exists the largest zero-dimensional compactification $\beta_0 X$, which is the Stone dual of the Boolean algebra $\mathbf{Clop}(X)$ of all clopen subsets of X [26]. The compactification $\beta_0 X$ is known as the *Banaschewski compactification* and the correspondence $X \mapsto \beta_0 X$ extends to a functor which is left adjoint to the inclusion of the category of Stone spaces into the category of zero-dimensional spaces [26] (see also [25]).

Nachbin generalized the theory of compactifications to what is known as order-compactifications (see, e.g., [207]), which was further developed by numerous authors. In this theory, the role of zero-dimensional compactifications is played by Priestley order-compactifications [168, 52]. Priestley spaces are of special importance in algebraic logic as they serve as Priestley duals of distributive lattices, thus generalizing Stone duality for Boolean algebras. For ordered spaces, the analog of zero-dimensionality is that of order-zero-dimensionality, and an ordered space X admits a Priestley order-compactification iff it is order-zero-dimensional [52, Cor. 3.8]. Moreover, there exists the largest Priestley order-compactification $\eta_0 X$, which is the Priestley dual of the distributive lattice $\mathbf{ClopUp}(X)$ of all clopen up-sets of X , and the correspondence $X \mapsto \eta_0 X$ extends to a functor which is left adjoint to the inclusion of the category of Priestley spaces into the category of order-zero-dimensional spaces [52, Rem. 3.10].

Throughout we refer the reader to [207], where the main concepts of this section were introduced.

Definition 3.3.1 (Nachbin space). An ordered space X is a *Nachbin space* if X is compact and the order $\leq$ is closed (in the product space $X \times X$).

It is well known (see [207, p. 26]) that the order $\leq$ being closed is equivalent to the following order-separation axiom (see e.g. [194]):

Definition 3.3.2 (Order-Hausdorff). An ordered space X is *order-Hausdorff* if for each $x, y \in X$, if $x \not\leq y$ then there exist an upset neighbourhood U of x and a downset neighbourhood V of y such that $U \cap V = \emptyset$.

In particular, an order-Hausdorff space is a Hausdorff space.

Remark 3.3.3 (Convention on ordered properties). As a general rule, we will refer to order-topological properties as “order-P,” e.g., order-zero-dimensionality. Some authors call such spaces zero-dimensional ordered spaces (see, e.g., [208]). Since the latter could alternatively be interpreted as an ordered space that happens to be zero-dimensional, we prefer the usage of “order-P” to avoid any unnecessary confusion.

Definition 3.3.4. Let X be an ordered space.

1. (Priestley separation) We say that X satisfies the *Priestley separation axiom* provided for each $x, y \in X$, if $x \not\leq y$ then there is a clopen upset U such that $x \in U$ and $y \notin U$.
2. (Order-zero-dimensionality) We call X *order-zero-dimensional* if
 - (a) X satisfies the Priestley separation axiom, and
 - (b) the set $\{U \setminus V : U, V \in \text{ClopUp}(X)\}$ is a basis for the topology.

It is clear that Priestley separation implies order-Hausdorffness. Moreover, it is easy to see that Priestley spaces are exactly those order-zero-dimensional spaces which are compact. We now study the natural ordered analogue of a compactification³:

Definition 3.3.5 (Order-embedding). A map $f : X \rightarrow Y$ between ordered spaces is an *order-embedding* if

- f is a topological embedding;
- $x \leq y$ iff $f(x) \leq f(y)$.

³We note that this concept has two natural generalizations to the setting of ordered spaces. The more restrictive of the two was already studied by Nachbin [207], which we called in [12] *N-order-compactification*; the more general one was later introduced by Blatter [67], and is the one presented below

Definition 3.3.6 (Order-compactification). Let X be an ordered space. An *order-compactification* of X is a pair (Y, e) , where Y is a Nachbin space and $e : X \rightarrow Y$ is an order-embedding such that $e[X]$ is dense in Y .

Remark 3.3.7. When it does not cause any confusion, we identify X with its image $e[X]$ in Y , view X as a dense subspace of Y , and Y as an order-compactification of X . Then, Y is an N-order-compactification of X iff the order on X is dense in the order on Y .

It is a classic result that a topological space X has a compactification iff it is completely regular, in which case the collection of (equivalence classes of) all compactifications has a natural partial order and the Stone-Čech compactification βX is the largest compactification in this partial order [101, pp.166-168]. These results generalize to the setting of order-compactifications [67]; for this we need the notion of complete order-regularity, originally from [207, Sec. II.1]:

Definition 3.3.8. An ordered space X is *completely-order-regular* if

1. For each $x, y \in X$, if $x \not\leq y$ then there is a continuous order-preserving $f : X \rightarrow [0, 1]$ such that $f(x) > f(y)$.
2. For each $x \in X$ and closed set F with $x \notin F$, there exist a continuous order-preserving $f : X \rightarrow [0, 1]$ and a continuous order-reversing $g : X \rightarrow [0, 1]$ such that $f(x) = 1 = g(x)$ and $F \subseteq f^{-1}(0) \cup g^{-1}(0)$.

Theorem 3.3.9 (Order-compactification = Complete-order-regularity). *An ordered space has an order-compactification iff it is completely-order-regular.*

Proof. See [67, Thm. 1.5(i)]. ■

We also have an obvious order on order-compactifications:

Definition 3.3.10 (Order on compactifications). Let (Y_1, e_1) and (Y_2, e_2) be two order-compactifications of a completely-order-regular space X . We write $(Y_1, e_1) \leq (Y_2, e_2)$ if there exists a continuous order-preserving $f : Y_2 \rightarrow Y_1$ such that $f e_2 = e_1$.

Since $e_1[X]$ is dense in Y_1 , such an f is always onto. Moreover, as for compactifications, we have that $(Y_1, e_1) \leq (Y_2, e_2)$ and $(Y_2, e_2) \leq (Y_1, e_1)$ iff there is an order-homeomorphism $f : Y_1 \rightarrow Y_2$ such that $f e_1 = e_2$, and this defines a partial order on the collection of (equivalence classes of) order-compactifications [67, p. 57], which we denote by $\mathcal{K}(X)$.

The largest element of this poset was constructed by Nachbin [207, App. 2] by generalizing the construction of the Stone-Čech compactification to the setting of ordered spaces. We refer to it as the *Nachbin order-compactification* and denote it by ηX .⁴ Similar to the Stone-Čech compactification, the Nachbin order-compactification has the following universal mapping property:

⁴The Nachbin order-compactification is sometimes denoted by $\beta_0 X$, $\beta_1 X$ (see, e.g., [84, 230]), or nX (see, e.g., [47, 52]).

Theorem 3.3.11 (Universal property of Nachbin order-compactification). *If Z is a Nachbin space, then each continuous order-preserving $f : X \rightarrow Z$ has a unique extension to a continuous order-preserving $g : \eta X \rightarrow Z$.*

Proof. See e.g. (see, e.g., [84, p. 56]). ■

For various constructions of ηX we refer to [84, pp. 56–57]. As with the Stone-Čech compactification, the construction of ηX is involved, but things simplify if X is order-zero-dimensional. The following was studied in [168, 52]:

Definition 3.3.12 (Priestley order-compactification). We call an order-compactification Y of X a *Priestley order-compactification* if Y is a Priestley space (equivalently, Y is order-zero-dimensional).

We point out that X has a Priestley order-compactification iff X is order-zero-dimensional [52, Thm. 3.5]. For such an X , let $\mathcal{K}_0(X)$ be the subposet of $\mathcal{K}(X)$ consisting of Priestley order-compactifications. Whenever X is order-zero-dimensional, we denote by $\eta_0 X$ the largest Priestley order-compactification of X ; this is obtained precisely by taking the Priestley dual of $\mathbf{ClopUp}(X)$ [52, Theorem 3.5]. The order-compactification map given by

$$\begin{aligned} \varepsilon_X : X &\rightarrow \eta_0 X \\ x &\mapsto \{U \in \mathbf{ClopUp}(X) : x \in U\}; \end{aligned}$$

Note also that by Priestley duality, the clopen upsets of $\eta_0 X$ are up to isomorphism exactly those in $\mathbf{ClopUp}(X)$.

3.3.2 Heyting and Esakia order-compactifications

Having summarised the existing theory on Priestley order-compactifications, we turn to study those Priestley order-compactifications that result in an Esakia space, which have as far as we are aware, not been the subject of analysis. For this we will need a few basic concepts:

Definition 3.3.13 (Continuous ordering). Let X be an ordered space. We say that

1. the order $\leq$ on X is *continuous* provided
 - (i) $\uparrow x$ is closed for each $x \in X$ and (ii) U open $\implies \downarrow U$ is open;
2. X is *continuously ordered* provided $\leq$ is a continuous order.

Note that an order-zero-dimensional space which is continuously ordered will therefore be compact if and only if it is an Esakia space [102, Thm. 3.1.2] (see Section 2.1.4).

We now consider two conditions that naturally arise when considering how to extend Priestley order-compactifications to the setting of Esakia spaces:

Definition 3.3.14 (Heyting, Esakia ord.-comp.). Let X be order-zero-dimensional. We say that a Priestley order-compactification Y of X is

1. a *Heyting order-compactification* if Y is an Esakia space;
2. an *Esakia order-compactification* if it is a Heyting order-compactification and $\uparrow_X x$ is dense in $\uparrow_Y x$ for each $x \in X$.

Given two Heyting order-compactifications (Y_1, e_1) and (Y_2, e_2) , we write,

$$(Y_1, e_1) \leq_H (Y_2, e_2),$$

whenever there is a continuous order-preserving map $f : Y_2 \rightarrow Y_1$ such that $fe_2 = e_1$. As with order-compactifications, we have that $(Y_1, e_1) \leq_H (Y_2, e_2)$ and $(Y_2, e_2) \leq_H (Y_1, e_1)$ if and only if there is an order-homeomorphism $f : Y_1 \rightarrow Y_2$ such that $fe_1 = e_2$, and this defines a partial order on the collection of equivalence classes of Heyting order-compactifications. We denote this poset by $(\mathcal{K}_H(X), \leq_H)$, and note that it is a subposet of the poset $(\mathcal{K}_0(X), \leq)$ of Priestley order-compactifications of X .

We also let $(\mathcal{K}_E(X), \leq_E)$ be the subposet of $(\mathcal{K}_H(X), \leq_H)$ consisting of Esakia order-compactifications. In other words, $\leq_E$ is the restriction of $\leq_H$ to $\mathcal{K}_E(X)$. Interestingly, the order $\leq_E$ is determined by continuous p-morphisms between Esakia order-compactifications, as was shown in [12, Proposition 4.13]⁵ :

Proposition 3.3.15. Let X be an order-zero-dimensional space, and let (Y_1, e_1) and (Y_2, e_2) be two Esakia order-compactifications of X . Then $(Y_1, e_1) \leq_E (Y_2, e_2)$ iff there is a continuous p-morphism $f : Y_2 \rightarrow Y_1$ such that $fe_2 = e_1$.

Observe that not every order-zero-dimensional space admits a Heyting or Esakia order-compactification. Indeed, if X is a Priestley space which is not Esakia, then X does not admit any Heyting order-compactification, and hence it also does not admit any Esakia order-compactification. This leads us to the following definitions:

Definition 3.3.16 (H/E-order-zero-dimensional). We call an order-zero-dimensional space X

1. *H-order-zero-dimensional* provided we have that X admits a Heyting order-compactification (meaning that $\mathcal{K}_H(X) \neq \emptyset$);
2. *E-order-zero-dimensional* provided we have that X admits an Esakia order-compactification (meaning that $\mathcal{K}_E(X) \neq \emptyset$).

⁵The proof provided there exploits the theory of *Esakia bases*. These are most interesting in the intrinsic theory of order-compactifications. Our focus here being representations and locally Esakia spaces, we have opted to not discuss this concept here.

We can then show that admitting an Esakia order-compactification is equivalent to the order being continuous.

Theorem 3.3.17 (Esakia order-compactifications and continuous order). *For an order-zero-dimensional space X , the following are equivalent.*

1. X is continuously ordered.
2. $\eta_0 X$ is an Esakia order-compactification.
3. X is E-order-zero-dimensional.

Proof. (2) $\Rightarrow$ (3) is obvious. To see why (1) $\Rightarrow$ (2) holds, assume that X is continuously ordered; then $\mathbf{CloUp}(X)$ is a Heyting algebra, and indeed a Heyting subalgebra of $\mathbf{Up}(X)$, since if W is clopen, then $\downarrow W$ is open (by $(X, \leq)$ being continuously ordered), and closed (by it being order-Hausdorff). Moreover, assume that $x \in X$, we show that $\uparrow_X x$ is dense in $\uparrow_Y x$; let A be a clopen subset of $\eta_0 X$ and $A \cap \uparrow_Y x \neq \emptyset$. Note that by construction A can be taken to be of the form $\varphi(U) \setminus \varphi(V)$ for $U, V \subseteq X$ clopen upsets. Then note that by our remark on $\mathbf{CloUp}(X)$ being a Heyting subalgebra of $\mathbf{Up}(X)$, we have $\downarrow(\varphi(U) \setminus \varphi(V)) = \eta_0 \setminus \varphi(U \rightarrow V)$, so if $x \leq y$ and $y \in \varphi(U) \setminus \varphi(V)$, $x \notin \varphi(U \rightarrow V)$, and so by definition, $x \notin U \rightarrow V$, which means simply that $x \leq x'$ for $x' \in U \setminus V$. Thus $x' \in \uparrow_X x \cap A$, showing density.

(3) $\Rightarrow$ (1) Let Y be an Esakia order-compactification of X . It suffices to show that U open in X implies that $\downarrow_X U$ is open in X . We have $U = W \cap X$ for some open $W \subseteq Y$. Since Y is an Esakia space, $\downarrow_Y W$ is open in Y . We show that

$$\downarrow_X U = \downarrow_Y W \cap X.$$

One inclusion is obvious. For the other, suppose that $x \in X$ and $x \in \downarrow_Y W$. Then $\uparrow_Y x \cap W \neq \emptyset$. Since Y is an Esakia order-compactification, $\uparrow_X x$ is dense in $\uparrow_Y x$, so $\uparrow_X x \cap W \neq \emptyset$. Therefore, there is $y' \in U$ such that $x \leq y'$, yielding that $x \in \downarrow_X U$. Thus, $\downarrow_X U$ is open in X . $\blacksquare$

We note that H-order-zero-dimensionality is not equivalent to E-order-zero-dimensionality:

Example 3.3.18 (H-order-zero-dimensional not E-order-zero-dimensional). Let C be the Cantor space with the trivial order. We fix a dense subset D of C , let $D' = \{x' : x \in D\}$ be a copy of D , and define X to be the topological sum $C \cup D'$ with the order given by the reflexive closure of $\{(x, x') : x \in D\}$ inside $C \cup D'$. In Figure 3.2, the white circle denotes a point $y' \notin D'$ which is omitted in our space $C \cup D'$.

It is straightforward to check that X is order-zero-dimensional. Moreover, since D' is clopen in X , if X were continuously ordered, $\downarrow D' = D' \cup D$ would be clopen in X , which would imply that D is clopen in C , a contradiction. Thus,

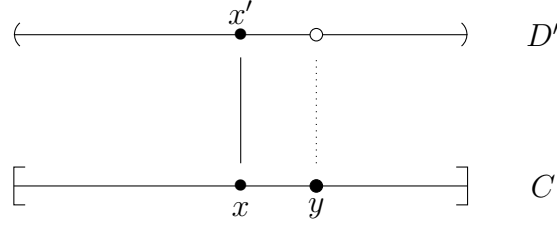


Figure 3.2: H-order-zero-dimensional space that is not E-order-zero-dimensional

X is not continuously ordered, and hence it is not E-order-zero-dimensional by Theorem 3.3.17.

We show that X is H-order-zero-dimensional. Let C' be a copy of C and let Y be the topological sum $C \cup C'$ with the order given by the reflexive closure of $\{(x, x') : x \in C\}$ in $C \cup C'$. To see that Y is an Esakia space, we clearly have that $\uparrow y$ is closed for each $y \in Y$. Moreover, each clopen of Y is of the form $U \cup V'$, with U, V clopen of C , so $\downarrow(U \cup V') = (U \cup V') \cup V$, which is clopen of Y . In addition, X is (homeomorphic to) a dense subspace of Y . Thus, Y is a Heyting order-compactification of X , and hence X is H-order-zero-dimensional.

3.3.3 Locally Esakia spaces and functoriality

As noted before, the classic construction of taking the Stone-Čech compactification is left adjoint to the inclusion of the category of compact Hausdorff spaces into that of completely regular spaces (see, e.g., [158, pp.130-131]); likewise, taking the Banaschewski compactification is left adjoint to the inclusion of the category of Stone spaces into that of zero-dimensional (Hausdorff) spaces [26]. The latter result extends to yield that η_0 is left adjoint to the inclusion of the category of Priestley spaces into that of order-zero-dimensional spaces (see, e.g., [52, Rem. 3.10]).

In this section we will prove analogues of these results for the category **Esa** of Esakia spaces. As we saw in Theorem 3.3.17, when X is E-order-zero-dimensional, $\eta_0 X$ is the largest Esakia order-compactification of X . Therefore, a natural category to work with appears to be the category of E-order-zero-dimensional spaces. However, the embedding $X \hookrightarrow \eta_0 X$ may in general not be a p-morphism. We thus need to restrict our attention to those E-order-zero-dimensional spaces in which the embedding $X \hookrightarrow \eta_0 X$ is indeed a p-morphism. The following gives a characterization of this situation:

Lemma 3.3.19 (Upset is Esakia). Let X be an order-zero-dimensional space and Y a Heyting order-compactification of X . If X is an upset of Y , then $\uparrow_X x$ is an Esakia space for each $x \in X$.

Proof. Since Y is an Esakia space, $\uparrow_Y x$ is an Esakia space (Proposition 2.1.52). But since X is an upset of Y , we have $\uparrow_X x = \uparrow_Y x$, yielding the result. ■

Lemma 3.3.19 suggests that the following definition captures the desired class of spaces:

Definition 3.3.20 (Locally Esakia space).

1. We call an ordered space X *image-compact* provided $\uparrow x$ is compact for each $x \in X$.
2. We call an ordered space X *locally Esakia* provided it is E-order-zero-dimensional and image-compact.
3. We denote by **LocEsa** the category of locally Esakia spaces and continuous p-morphisms.

Remark 3.3.21 (Locally Esakia spaces generalize Esakia and locally finite posets). The above notion provides a joint generalization of locally finite posets (Definition 2.1.28) and Esakia spaces: a locally finite poset is a locally Esakia space where the topology is discrete, whilst an Esakia space is a locally Esakia space where the topology is compact.

The next result characterizes when X is an upset in its order-compactification that is an Esakia space.

Proposition 3.3.22 (Esakia order-compactifications and image-compactness). Let X be order-zero-dimensional and Y a Heyting order-compactification of X . Then X is an upset of Y iff X is image-compact and Y is an Esakia order-compactification of X .

Proof. First suppose that X is an upset of Y . Then X is image-compact by Lemma 3.3.19. Moreover, since X is an upset in Y , we have $\uparrow_X x = \uparrow_Y x$ for each $x \in X$, so $\uparrow_X x$ is dense in $\uparrow_Y x$, and hence Y is an Esakia order-compactification of X .

Conversely, suppose that X is image-compact and Y is an Esakia order-compactification of X . Let $x \in X$ and $x \leq_Y y$. Consider the family

$$\mathcal{F} = \{\uparrow_Y x \cap U \cap X : y \in U\} \cup \{\uparrow_Y x \setminus (V \cap X) : y \notin V\},$$

where $U, V \in \mathbf{ClopUp}(Y)$. If $\mathcal{F}$ does not have the finite intersection property, then

$$\uparrow_Y x \cap U_1 \cap \dots \cap U_n \cap X \subseteq (V_1 \cup \dots \cup V_m) \cap X.$$

for some $U_1, \dots, U_n, V_1, \dots, V_m \in \mathbf{ClopUp}(Y)$. Let $U = U_1 \cap \dots \cap U_n$ and $V = V_1 \cup \dots \cup V_m$. Then $U, V \in \mathbf{ClopUp}(Y)$ and $\uparrow_Y x \cap U \cap X \subseteq V \cap X$. Therefore, $\uparrow_Y x \cap X \subseteq (Y \setminus U) \cup V$. Since Y is an Esakia order-compactification of X , $\uparrow_X x$ is dense in $\uparrow_Y x$, so

$$\uparrow_Y x = \overline{\uparrow_Y x \cap X} \subseteq (Y \setminus U) \cup V.$$

Thus, $x \in U \rightarrow_{\text{ClopUp}(Y)} V$, and hence $y \in U \rightarrow_{\text{ClopUp}(Y)} V$, which is a contradiction since $y \in U$ and $y \notin V$. Consequently, $\mathcal{F}$ has the finite intersection property, and because $\uparrow_Y x$ is compact, there is

$$y' \in \bigcap \{\uparrow_Y x \cap U \cap X : y \in U\} \cap \bigcap \{\uparrow_Y x \setminus (V \cap X) : y \notin V\}.$$

Since y and y' are contained in the same clopen upsets of Y , it follows from the Priestley separation axiom that $y = y'$, so $y \in X$, and hence X is an upset of Y . $\blacksquare$

From this we obtain the following version of Theorem 3.3.17 for image-compact spaces.

Theorem 3.3.23 (Locally Esakia spaces via order-compactifications). *For an order-zero-dimensional space X , the following are equivalent:*

1. X is a locally Esakia space.
2. X has an Esakia order-compactification Y in which X is an upset.
3. $\eta_0 X$ is an Esakia order-compactification in which X is an upset.

Proof. (3) $\Rightarrow$ (2) is obvious, and (1) $\Rightarrow$ (3) follows from Theorem 3.3.17 and Proposition 3.3.22. We prove (2) $\Rightarrow$ (1). That X is image-compact follows from Proposition 3.3.22. We show that X is continuously ordered. Let $U \subseteq X$ be open. Then $U = V \cap X$ for some open $V \subseteq Y$. Since X is an upset of Y ,

$$\downarrow_X U = \downarrow_X (V \cap X) = \downarrow_Y V \cap X$$

(because $x \in X$, $x \leq_Y y$, and $y \in V$ imply that $y \in V \cap X$). Since Y is an Esakia space, $\downarrow_Y V$ is open. Therefore, $\downarrow_X U$ is open, and hence X is continuously ordered. Thus, it is left to apply Theorem 3.3.17. $\blacksquare$

Let **OZD** be the category of order-zero-dimensional spaces and continuous order-preserving maps. As we pointed out at the beginning of this section, η_0 is left adjoint to the embedding **Pries** $\hookrightarrow$ **OZD**. Thus, each **OZD**-morphism $f : X \rightarrow Z$, where $X \in \mathbf{OZD}$ and $Z \in \mathbf{Pries}$, has a unique lift $\eta_0 f : \eta_0 X \rightarrow Z$. We give a convenient description of the lift, generalizing [47, Lem. 3.1].

Proposition 3.3.24 (Calculation of lifting of maps). Let $X \in \mathbf{OZD}$, $Z \in \mathbf{Pries}$, and $f : X \rightarrow Z$ be continuous and order-preserving.

1. The unique lift $\eta_0 f : \eta_0 X \rightarrow Z$ is given by letting $(\eta_0 f)(x)$ be the unique element in $\bigcap \mathfrak{F}_x^f \cap \bigcap \mathcal{I}_x^f$ for each $x \in \eta_0 X$, where

$$\begin{aligned} \mathfrak{F}_x^f &= \{\overline{\uparrow f[U]} : U \in \text{ClopUp}(X), U \in x\} \text{ and} \\ \mathcal{I}_x^f &= \{\overline{\downarrow f[X \setminus V]} : V \in \text{ClopUp}(X), V \notin x\}. \end{aligned}$$

2. For each $x \in \eta_0 X$ and $U, V \in \mathbf{ClopUp}(Z)$,

$$x \in (\eta_0 f)^{-1}[U \setminus V] \iff f^{-1}[U] \in x \text{ and } f^{-1}[V] \notin x.$$

Proof. (1) Let $x \in \eta_0 X$. First we show that $\bigcap \mathfrak{F}_x^f \cap \bigcap \mathcal{I}_x^f$ is a singleton. If the intersection was empty, we would have

$$\uparrow \overline{f[U_1]} \cap \dots \cap \uparrow \overline{f[U_n]} \cap \downarrow \overline{f[X \setminus V_1]} \cap \dots \cap \downarrow \overline{f[X \setminus V_m]} = \emptyset$$

for some $U_1, \dots, U_n, V_1, \dots, V_m \in \mathbf{ClopUp}(X)$. This implies that

$$f[U_1 \cap \dots \cap U_n \cap (X \setminus V_1) \cap \dots \cap (X \setminus V_m)] = \emptyset,$$

so $U_1 \cap \dots \cap U_n \cap (X \setminus V_1) \cap \dots \cap (X \setminus V_m) = \emptyset$, and hence $U_1 \cap \dots \cap U_n \subseteq V_1 \cup \dots \cup V_m$, a contradiction since $U_1 \cap \dots \cap U_n \in x$ but $V_1 \cup \dots \cup V_m \notin x$.

For the uniqueness, if the intersection contained $y_1 \not\leq y_2$, then there would exist $W \in \mathbf{ClopUp}(Z)$ such that $y_1 \in W$ and $y_2 \notin W$. If $f^{-1}[W] \in x$, then $y_2 \in \uparrow f f^{-1}[W] \subseteq W$; and if $f^{-1}[W] \notin x$, then $y_1 \in \downarrow f f^{-1}[Z \setminus W] \subseteq Z \setminus W$. The obtained contradiction completes the proof of uniqueness.

Next we prove that $\eta_0 f$ is continuous. Let $W \in \mathbf{ClopUp}(Z)$. We show that

$$(\eta_0 f)^{-1}[W] = \bigcup \{ \varphi(U) : \uparrow \overline{f[U]} \subseteq W \}.$$

If $\uparrow \overline{f[U]} \subseteq W$ and $x \in \varphi(U)$, then $(\eta_0 f)(x) \in W$ by construction. Conversely, if $(\eta_0 f)(x) \in W$, then $\bigcap_{U \in x} \uparrow \overline{f[U]} \subseteq W$. By compactness, there are $U_1, \dots, U_n$ such that $U_i \in x$ and $\bigcap_{i=1}^n \uparrow \overline{f[U_i]} \subseteq W$. Thus, letting $U = U_1 \cap \dots \cap U_n$, we obtain $\uparrow \overline{f[U]} \subseteq W$ and $U \in x$, so $x \in \varphi(U)$. A similar argument shows that if W is a clopen downset of Z , then

$$(\eta_0 f)^{-1}[W] = \bigcup \{ \eta_0 X \setminus \varphi(V) : \downarrow \overline{f[X \setminus V]} \subseteq W \}.$$

Since clopens form a basis, we conclude that $\eta_0 f$ is continuous.

It is left to show that $\eta_0 f$ is order-preserving. Let $x, y \in \eta_0 X$ with $x \leq y$ and let $W \in \mathbf{ClopUp}(Z)$. If $\eta_0 f(x) \in W$, then $x \in (\eta_0 f)^{-1}[W]$, which is a union of clopen upsets, and hence is an upset. Therefore, $y \in (\eta_0 f)^{-1}[W]$, and so $(\eta_0 f)(y) \in W$. Thus, by the Priestley separation axiom, $\eta_0 f(x) \leq \eta_0 f(y)$.

(2) First, suppose that $(\eta_0 f)(x) \in U \setminus V$. If $f^{-1}[U] \notin x$, then $(\eta_0 f)(x) \in \downarrow f f^{-1}[Z \setminus U]$ by (1). But $f f^{-1}[Z \setminus U] \subseteq Z \setminus U$, and since the latter is a closed downset, we obtain that $\downarrow f f^{-1}[Z \setminus U] \subseteq Z \setminus U$, implying that $(\eta_0 f)(x) \notin U$, a contradiction. Therefore, $f^{-1}[U] \in x$, and a similar argument yields that $f^{-1}[V] \notin x$. Conversely, suppose that $f^{-1}[U] \in x$ and $f^{-1}[V] \notin x$. By (1), $(\eta_0 f)(x) \in \uparrow f f^{-1}[U] \subseteq U$ and $(\eta_0 f)(x) \in \downarrow f f^{-1}[Z \setminus V] \subseteq Z \setminus V$, so $(\eta_0 f)(x) \in U \setminus V$. ■

We use Proposition 3.3.24 to prove that η_0 is left adjoint to the full subcategory inclusion $\mathbf{Esa} \hookrightarrow \mathbf{LocEsa}$. For this, we will make use of the following classic result:

Proposition 3.3.25 (Esakia's lemma). Let X be an Esakia space and $\mathfrak{F}$ a downwards directed family (with respect to inclusion) of nonempty closed subsets of X . Then

$$\downarrow \bigcap \{F : F \in \mathfrak{F}\} = \bigcap \{\downarrow F : F \in \mathfrak{F}\}.$$

Proof. See [106]. ■

The next result generalizes [47, Lem. 4.3].

Proposition 3.3.26 (Lifting to p-morphisms). Let X be E-order-zero-dimensional. If Z is an Esakia space and $f : X \rightarrow Z$ is a continuous p-morphism, then the unique lift $\eta_0 f : \eta_0 X \rightarrow Z$ is a p-morphism.

Proof. By Remark 2.1.54.1, it suffices to show that

$$(\eta_0 f)^{-1}[\downarrow y] = \downarrow(\eta_0 f)^{-1}[y]$$

for each $y \in Z$. Since each singleton of Z is the intersection of the clopen sets containing it (given this is a Stone space), and clopens are unions of sets of the form $U \setminus V$, by Esakia's lemma,

$$\downarrow(\eta_0 f)^{-1}[y] = \bigcap \{\downarrow(\eta_0 f)^{-1}[U \setminus V] : y \in U \text{ and } y \notin V\},$$

where U, V range over $\mathbf{ClopUp}(Z)$. We show that

$$\downarrow(\eta_0 f)^{-1}[U \setminus V] = (\eta_0 f)^{-1}[\downarrow(U \setminus V)].$$

Since $\eta_0 f$ is order-preserving, $\downarrow(\eta_0 f)^{-1}[U \setminus V] \subseteq (\eta_0 f)^{-1}[\downarrow(U \setminus V)]$. For the reverse inclusion, if $x \in (\eta_0 f)^{-1}[\downarrow(U \setminus V)]$, then $x \in (\eta_0 f)^{-1}[Z \setminus (U \rightarrow_{\mathbf{ClopUp}(Z)} V)]$ (see Fact 2.1.54). Therefore, by Proposition 3.3.24.2, $f^{-1}[U \rightarrow_{\mathbf{ClopUp}(Z)} V] \notin x$. Since f is a continuous p-morphism, f^{-1} is a Heyting homomorphism (see Fact 2.1.54). Thus, $f^{-1}[U] \rightarrow_{\mathbf{ClopUp}(X)} f^{-1}[V] \notin x$, and hence

$$x \notin \varphi(f^{-1}[U] \rightarrow_{\mathbf{ClopUp}(X)} f^{-1}[V]) = \varphi(f^{-1}[U]) \rightarrow_{\mathbf{ClopUp}(\eta_0 X)} \varphi(f^{-1}[V]),$$

where the equality holds by $\mathbf{ClopUp}(X)$ forming a Heyting subalgebra of $\mathbf{Up}(X)$. Therefore, there is $y \geq x$ such that $y \in \varphi(f^{-1}[U])$ and $y \notin \varphi(f^{-1}[V])$. Thus, there is $y \geq x$ such that $f^{-1}[U] \in y$ and $f^{-1}[V] \notin y$. Using Proposition 3.3.24.2 again, $y \in (\eta_0 f)^{-1}[U \setminus V]$, and so $x \in \downarrow(\eta_0 f)^{-1}[U \setminus V]$.

Consequently, using Esakia's lemma again,

$$\begin{aligned}
\downarrow(\eta_0 f)^{-1}[y] &= \bigcap \{(\eta_0 f)^{-1}[\downarrow(U \setminus V)] : y \in U \text{ and } y \notin V\} \\
&= (\eta_0 f)^{-1} \bigcap \{\downarrow(U \setminus V) : y \in U \text{ and } y \notin V\} \\
&= (\eta_0 f)^{-1} \downarrow \bigcap \{U \setminus V : y \in U \text{ and } y \notin V\} \\
&= (\eta_0 f)^{-1}[\downarrow y],
\end{aligned}$$

and hence $\eta_0 f$ is a p-morphism. ■

The following allows us to conclude that **Esa** is a reflective subcategory of **LocEsa**.

Theorem 3.3.27 (Left adjoint to locally Esakia spaces). $\eta_0 : \mathbf{LocEsa} \rightarrow \mathbf{Esa}$ is left adjoint to the inclusion functor $\mathbf{Esa} \hookrightarrow \mathbf{LocEsa}$.

Proof. Let X be a locally Esakia space. Since X is image-compact, the embedding $X \hookrightarrow \eta_0 X$ is a continuous p-morphism. Also, since X is E-order-zero-dimensional, a continuous p-morphism $f : X \rightarrow Y$ to an Esakia space Y has a unique lift $\eta_0 f : \eta_0 X \rightarrow Y$ in **Esa**. Thus, $\eta_0 : \mathbf{LocEsa} \rightarrow \mathbf{Esa}$ is left adjoint to the inclusion functor $\mathbf{Esa} \hookrightarrow \mathbf{LocEsa}$ (see, e.g., [182, p. 91]). ■

3.4 Locally Esakia spaces in dual constructions

In this final section we provide some examples of how one can see constructions on Heyting algebras using locally Esakia spaces, and how they can be connected back to Esakia duality in a systematic way through order-compactifications.

3.4.1 Products through Esakia order-compactifications

As a first application of locally Esakia spaces, we turn to the duals of products of Heyting algebras. Such characterizations are known in the context of distributive lattices [23, 24], and distributive-lattice based algebras [76] [168, Proposition 3.5], generalizing some classic results about Boolean algebras [100].

Definition 3.4.1 (Topological sum of ordered spaces). Given $(X_i)_{i \in I}$ a family of ordered spaces we denote by $\bigsqcup_{i \in I} X_i$ the ordered space obtained by taking the topological sum, and the disjoint union of the orders.

Given a subset $U \subseteq \bigsqcup_{i \in I} X_i$ we write $U_i := U \cap X_i$. Note that since the topology on $\bigsqcup_{i \in I} X_i$ is the sum topology, a set U is a clopen (upset) if and only if U_i is clopen (upset) for each i .

Lemma 3.4.2 (Properties preserved under sum). Let $(X_i)_{i \in I}$ be an ordered spaces, and $X = \bigsqcup_{i \in I} X_i$. Then:

1. If for each $i \in I$ X_i is order-zero-dimensional, then so is X .
2. If for each $i \in I$, X_i is locally Esakia, then so is X .

Proof. (1) Note that $\{U \setminus V : U, V \in \mathbf{ClopUp}(X)\}$ will certainly be a basis, since $U(i) \setminus V(i)$ is a basis for each i . Moreover if $x \not\leq y$, then $x, y \in X_i$, so there is some $W_i \subseteq X_i$ a clopen upset containing x and not y . Thus W_i will likewise be a clopen upset in X , obtaining the result.

(2) Note that if $\mathbf{ClopUp}(X_i)$ is a Heyting algebra for each i , then certainly we have that $\mathbf{ClopUp}(X)$ will likewise be a Heyting algebra. Moreover, image-compactness depends only on the order inside X_i , hence X will be image-compact as well. ■

Notation 3.4.1 (Coproduct of Priestley spaces). Given $(X_i)_{i \in I}$ a collection of Priestley spaces or Esakia spaces, we denote by $\bigoplus_{i \in I} X_i$ the coproduct of these spaces in the category **Pries** (respectively **Esa**).

Theorem 3.4.3 (Dual representation of products). *For any collection $(D_i)_{i \in I}$ of distributive lattices, and $(X_i)_{i \in I}$ the collection of its dual Priestley spaces, we have*

$$\prod_{i \in I} D_i \cong \mathbf{ClopUp}\left(\eta_0\left(\bigsqcup_{i \in I} X_i\right)\right) \cong \bigoplus_{i \in I} X_i.$$

Additionally, this is also an isomorphism of Heyting algebras if each H_i is a Heyting algebra and the product is taken with respect to that signature.

Proof. Since by Lemma 3.4.2 $X = \bigsqcup_{i \in I} X_i$ is order-zero-dimensional, we know by the theory of Priestley order-compactifications that $\eta_0(X) = \mathbf{Spec}(\mathbf{ClopUp}(X))$. Hence $\mathbf{ClopUp}(\eta_0(X)) \cong \mathbf{ClopUp}(X)$, with the isomorphism being naturally induced by Priestley duality. To obtain an isomorphism between $\prod_{i \in I} D_i$ and this set, we consider the assignment:

$$a \mapsto \bigsqcup_{i \in I} \varphi(a(i));$$

it is not difficult to show that this is an isomorphism of distributive lattices. This proves the result. ■

As usual, the role of compactification points makes the understanding of such products a highly non-trivial task, for instance, see the examples discussed in [24]. However, the fact that it corresponds to the Nachbin-like order-compactification η_0 , which by Theorem 3.3.27 satisfies a universal lifting property, can be useful when using products to parametrize specific constructions. We will see an example of this in Section 7.4. As a simple application, in this and the next section we present some dual proofs of well-known algebraic results. We start with a dual proof of Birkhoff's subdirect decomposition theorem in the context of Heyting algebras.

Recall that given $f : A \rightarrow \prod_{i \in I} B_i$ an embedding of an algebra into a product, f is called a *subdirect embedding* if for each $i \in I$, $\pi_i f$ is surjective [74, Chapter 2.8]. The following dual representation is a direct generalization of [38, Thm. 3.1].

Proposition 3.4.4 (Dual subdirect representations). Let H be a Heyting algebra, and $\{H_i : i \in I\}$ a family of Heyting algebras. Let $X = \mathbf{Spec}(H)$ and $X_i = \mathbf{Spec}(H_i)$. Then H is a subdirect product of $\{H_i : i \in I\}$ if and only if for each $i \in I$ there is an injective continuous p-morphism $p_i : X_i \rightarrow X$, and $\bigcup_{i \in I} p[X_i]$ is dense in X .

Proof. First assume that $e : H \rightarrow \prod_{i \in I} H_i$ is a subdirect embedding. Then by duality it is clear that $p_i := \pi_i^{-1}$ are injective maps. Moreover, if U, V are clopen upsets of X , assume that $U = \varphi(a)$ and $V = \varphi(b)$. Then if $U - V \neq \emptyset$ this means that $a \not\leq b$. Because e is an embedding, $e(a) \not\leq e(b)$, so for some $i \in I$, $e(a)(i) \not\leq e(b)(i)$. Hence by duality, $X_i \cap U - V \neq \emptyset$, meaning that $\bigcup_{i \in I} p_i[X_i]$ is dense, as desired.

Conversely, if $p_i : X_i \rightarrow X$ is an injective continuous p-morphism, and $\bigcup_{i \in I} p[X_i]$ is dense in X , note that the map

$$p : \bigsqcup_{i \in I} X_i \rightarrow X$$

given by $p(x) = p_i(x)$ whenever $x \in X_i$ is a continuous p-morphism, and factors through the injections. By Proposition 3.3.26, then $\eta_0 p : \eta_0(\bigsqcup_{i \in I} X_i) \rightarrow X$ is a continuous p-morphism as well; we check that it is, additionally, surjective. Indeed, if not, note that $(\eta_0 p)[\eta_0(\bigsqcup_{i \in I} X_i)]$ will be a closed subset of X , so $X \setminus (\eta p)[\eta(\bigsqcup_{i \in I} X_i)]$ will be a non-empty open subset. Hence

$$\bigcup_{i \in I} p[X_i] \setminus (\eta_0 p)\left[\eta_0\left(\bigsqcup_{i \in I} X_i\right)\right] \neq \emptyset,$$

which is clearly a contradiction. By reductio we have that $\eta_0 p$ is surjective, and hence, p^{-1} is a subdirect embedding. $\blacksquare$

Recall from Proposition 2.1.66 that an Esakia space is *strongly rooted* – it has a minimal element x , and $\{x\}$ is clopen – if and only if its dual Heyting algebra is subdirectly irreducible. We also recall from 2.1.52.(3) the *Esakia-Fine maximality principle*⁶, stating that in an Esakia space X , whenever $C \subseteq X$ is a closed subset, $\max(C)$ is closed and non-empty. Then we obtain the following theorem:

Proposition 3.4.5 (Birkhoff’s Subdirect Decomposition for Heyting Algebras). Each Heyting algebra is a subdirect product of subdirectly irreducible Heyting algebras.

⁶This terminology was introduced in [11], where this principle was studied and shown to be equivalent, under ZF, to the Axiom of Choice.

Proof. Let $\{X_i : i \in I\}$ be the collection of all strongly rooted Esakia closed subspaces of $X = \mathbf{Spec}(H)$. By Proposition 3.4.4 it suffices to show that $\bigcup_{i \in I} X_i$ is dense in X . Let $x \in U \setminus V$ where U, V are clopen upsets. By the Esakia-Fine Maximality Principle, there is some point $z \in \max(U \setminus V)$. Note that then $\uparrow z \cap (U \setminus V) = \{z\}$ so $\{z\}$ is isolated in $\uparrow z$ by $U \setminus V$. So $\uparrow z = X_i$ for some X_i , and so

$$(U \setminus V) \cap X_i \neq \emptyset;$$

this shows the result. ■

3.4.2 Ultraproduct decompositions

As an elaboration on our discussion on products, we provide here a proof of a little discussed, yet quite powerful decomposition of infinite products. The authors in [168] discussed distributive lattices, though in light of Propositions 3.3.24, 3.3.27, we can develop a similar representation for Heyting algebras which applies with minor modifications to Heyting algebras.

Let X_i be a family of Esakia spaces. For each i , let $t_i : X_i \rightarrow \bullet_i$ be the terminal map to the one-element space. This family of maps lifts to a map between ordered spaces $t : \bigsqcup_{i \in I} X_i \rightarrow I$, where we identify I with the discrete space. By the description of left adjoints, this lifts to a map:

$$\eta_0 t : \bigoplus_{i \in I} X_i \rightarrow \beta_0(I).$$

By the formula given in Proposition 3.3.24, we have that for each $x \in \bigoplus_{i \in I} X_i$, $x \in \eta_0 t^{-1}[W]$ for W a clopen subset of $\beta_0(I)$ if and only if $t^{-1}[W] \in x$. A clopen subset of $\beta_0(I)$ can of course be identified with a subset of I , and we have that:

$$t^{-1}[W] = \chi_W = \bigcup_{i \in W} X_i,$$

that is, it picks the whole component on X_i , if $i \in W$, and otherwise is empty.

Now, given $U \in \beta_0(I)$ an ultrafilter, denote by $X_U = t^{-1}[\{U\}]$. Note that this presents a decomposition of $\bigoplus_{i \in I} X_i$ into disjoint and order-disjoint components (since the order on $\beta_0(I)$ is trivial. That, is we have shown:

Proposition 3.4.6 (Koubek-Sichler decomposition). For each $(X_i)_{i \in I}$ a family of Esakia spaces, we have an order-isomorphism:

$$\bigoplus_{i \in I} X_i \cong \bigsqcup_{U \in \beta_0(I)} X_U,$$

where $X_U = t^{-1}[\{U\}]$.

Proposition 3.4.7 (Representation of ultraproducts). Let U be an ultrafilter on $\prod_{i \in I} H_i$. Then we have:

$$\prod_{i \in I} H_i / U \cong \text{ClopUp}(X_U).$$

Proof. Recall that by classical model theory (see e.g. [74, Sec. 5.2]), an ultraproduct over an ultrafilter U is defined by specifying a congruence on $\prod_{i \in I} H_i$, θ_U as follows: for $a, b \in \prod_{i \in I} H_i$:

$$(a, b) \in \theta_U \iff \llbracket a = b \rrbracket \in U,$$

where $\llbracket a = b \rrbracket = \{i \in I : a(i) = b(i)\}$. Since we are dealing with Heyting algebras, we can equivalently specify a filter F_U :

$$a \in F_U \iff \llbracket a \rrbracket \in U,$$

where $\llbracket a \rrbracket = \{i \in I : a(i) = 1\}$. Let us write $\mathcal{U}(F_U)$ for the closed subset of $\bigoplus_{i \in I} X_i$ specified in this way. Note that then for each $x \in \bigoplus_{i \in I} X_i$, we have that $x \in \mathcal{U}(F_U)$ if and only if $x \in \varphi(a)$ whenever $a \in F_U$.

On the other hand, let X_U be the order-connected component which corresponds to this ultrafilter under the Koubek-Sichler decomposition of Proposition 3.4.6. Then we know that $x \in X_U$ if and only if the following holds: for each $S \subseteq I$, letting $\chi_S = \bigsqcup_{i \in S} X_i$, $\chi_S \in x$ if and only if $S \in U$.

Note that in light of the representation of the product in terms of clopen upsets of the locally Esakia space $\bigsqcup_{i \in I} X_i$, given in Theorem 3.4.3, we have that for each $x \in \bigoplus_{i \in I} X_i$, for $a \in \prod_{i \in I} H_i$, $x \in \varphi(a)$ if and only if $a \in x$, and $a = \bigcup_{i \in I} a(i)$. We use this identification tacitly below.

So first assume that $x \in X_U$, and let $a \in F_U$. Since $\llbracket a \rrbracket \in U$, letting $S = \llbracket a \rrbracket$, we have by hypothesis that $\chi_S \in x$. But note that then, necessarily, $\chi_S \subseteq \bigsqcup_{i \in I} \varphi(a(i))$ (since on each order-component distinct from $i \in S$, χ_S is empty, whilst $\varphi(a(i))$ might contain some non-empty clopen). Thus $x \in \varphi(a)$, given our representation of the product.

Conversely, assume that $x \in \mathcal{U}(F_U)$. Suppose that $S \in U$; then note that the clopen $\chi_S \in F_U$ by definition, and so $x \in \varphi(\chi_S)$. Conversely, assume that $S \notin U$; then $I \setminus S \in U$, and so by the same argument $\chi_{I \setminus S} \in F_U$, and $x \in \varphi(\chi_{I \setminus S})$; if also $\chi_S \in x$, then $\chi_S \cap \chi_{I \setminus S} \in x$, which is a contradiction, since this intersection is empty and x is a prime filter, hence, a proper filter. $\blacksquare$

We can use this representation to obtain some simple yet useful corollaries on the structure of products of Heyting algebras:

Definition 3.4.8 (Order-connected space). An ordered space X is called *order-connected* if for each $x, y \in X$ there is a finite sequence $z_1, \dots, z_n$ such that $x = z_1$, $y = z_n$, and: (1) for each i , either $z_i \leq z_{i+1}$ or $z_{i+1} \leq z_i$ and (2) whenever $z_i \leq z_{i+1}$ then $z_{i+1} \geq z_{i+2}$.

Corollary 3.4.9 (Products to ultraproducts). *Let $(H_i)_{i \in I}$ be a family of Heyting algebras, and let H be a Heyting algebra with an order-connected dual space. If H is a homomorphic image of $\prod_{i \in I} H_i$, then H is a homomorphic image of an algebra $H^+ \in \mathbb{P}_U(\{H_i : i \in I\})$.*

Proof. If H is a homomorphic image of $\prod_{i \in I} H_i$, by duality, $\text{Spec}(H)$ embeds via a p-morphism into $\bigoplus_{i \in I} \text{Spec}(H_i)$. By Proposition 3.4.6, we know that all the X_U are independent in terms of their order. Since H is order-connected, and disjoint unions can never be order-connected, $\text{Spec}(H)$ must embed into a single X_U . The result thus follows by duality and Proposition 3.4.7. ■

As a simple corollary of this, note that if H is a finitely subdirectly irreducible, then each order-connected component of $\text{Spec}(H^I)$ is rooted. A different application of such a Corollary can be found in Appendix A.5. We finish by giving a dual proof of Jónsson's lemma (Lemma 2.2.18):

Theorem 3.4.10 (Jónsson's Lemma for Heyting algebras). *Let $\mathbf{K}$ be a class of Heyting algebras. If $H \in \mathbb{V}(\mathbf{K})$ is subdirectly irreducible, then $H \in \mathbb{HSP}_U(\mathbf{K})$.*

Proof. First, notice that by the congruence extension property, we know that $\mathbb{HSP} = \mathbb{SHIP}$ (Proposition 2.2.21). Thus, assume that $H \in \mathbb{SHIP}(\mathbf{K})$ and that H is subdirectly irreducible. Dually, we have that $X = \text{Spec}(H)$ is a p-morphic image of $g: Y \rightarrow X$, where $\text{ClopUp}(Y) \in \mathbb{HIP}(\mathbf{K})$. Let $r \in X$ be its root. Because X is strongly rooted, $g^{-1}[\{r\}]$ is a clopen subset. Because we are in a Priestley space (Proposition 2.1.69), let $r' \in \max(g^{-1}[\{r\}])$ be some element. Then note that $g|_{\uparrow r'}: \uparrow r' \rightarrow X$ will still be a surjective p-morphism, and $\uparrow r'$ is an Esakia space, where $\text{ClopUp}(\uparrow r') \in \mathbb{H}(\text{ClopUp}(Y)) \subseteq \mathbb{HIP}(\mathbf{K})$. Moreover, since $g^{-1}[\{r\}]$ was clopen, and r' is maximal, the root will be isolated. Thus, we have that H is a homomorphic image of H_0 , where H_0 is subdirectly irreducible and $H_0 \in \mathbb{HIP}(\mathbf{K})$. By Corollary 3.4.9, we then know that $H_0 \in \mathbb{HIP}_U(\mathbf{K})$, so we have that $H \in \mathbb{SHIP}_U(\mathbf{K})$, which by the congruence-extension property again means that $H \in \mathbb{HSP}_U(\mathbf{K})$. ■

3.4.3 Least Esakia order-compactifications and Bellissima algebras

Theorem 3.3.27, can be understood as showing that every locally Esakia space admits the greatest order-compactification. It therefore makes sense to ask when *least order-compactifications* exist – which are analogues in our setting of the Alexandroff compactification.

Lemma 3.4.11 (Image of map in locally finite poset). *Let P be a locally finite poset (i.e., a discrete locally Esakia space). Let $e: P \rightarrow Y$ be an Esakia order-compactification. Then for each $x \in P$, $e[\uparrow x]$ is a clopen upset.*

Proof. We need to show that the (clopen) upset $e[\uparrow x]$ is always a clopen in any Esakia order-compactification (it is always an upset by Proposition 3.3.22). It is finite in Y since e is a p-morphism, and P is locally finite; from this we have that it is closed, since Y is Hausdorff. Since e is a homeomorphism to the image, it follows for each $x \in P$, $e[\{x\}]$ is clopen in $e[P]$, since $\{x\}$ is clopen in P ; but then it will be clopen in Y : if there is a clopen $W \subseteq Y$ such that $W \cap e[P] = \{x\}$, since $e[P]$ is open, it is of the form

$$e[P] = \bigcup_{V \subseteq e[P]} V,$$

for V clopen upset. Thus $\{x\} = \bigcup_{V \subseteq e[P]} (W \cap V)$, so for some V , $\{x\} = W \cap V$. Since $e[\uparrow x]$ is finite, it is a finite union of clopens, and it will be clopen, as desired. ■

Seeing that the algebra of definable upsets $\text{Def}(\mathcal{U}(n))$ is not isomorphic to the algebra of all upsets, it is reasonable to ask whether there is a manageable collection of upsets which captures it. A reasonable option would be to take exactly the algebra generated by finite upsets, as studied by Bellissima:

Definition 3.4.12 (Bellissima algebra). Given a superintuitionistic logic L , for every n , we denote by $\text{Bel}_L(n)$, the *Bellissima n -algebra* the smallest Heyting algebra generated in $\text{Up}(\mathcal{U}_L(n))$ by the finite upsets $\{\uparrow x : x \in \mathcal{U}_L(n)\}$.

In [30, Theorem 4.4], the following was shown:

Proposition 3.4.13 (Bellissima algebra infinitely generated). For each $n > 1$, $\text{Bel}(n)$ is an infinitely generated algebra, with the property that for each formula φ over n many variables, $\text{Bel}(n) \models \varphi$ if and only if $\varphi \in \text{IPC}$.

This results extends to the case of L a logic where $\text{Bel}_L(n)$ is infinite. It follows from it being infinitely generated that $\text{Bel}_L(n)$ is not isomorphic to $\mathcal{F}_L(n)$ either, despite capturing exactly the valid formulas in $\bar{p}$ many variables.

By general considerations, we can show that Bellissima algebras are dual to least Esakia order-compactifications. We denote by $\mathbf{B}_n = \text{Spec}(\text{Bel}(n))$ (see Section 3.4.4).

Theorem 3.4.14 (Bellissima algebra via order-compactifications). *Let n be arbitrary. Then $\mathbf{B}_n$ is dual to the least Esakia order-compactification of $\mathcal{U}(n)$.*

Proof. To see that $e: \mathcal{U}(n) \rightarrow \mathbf{B}_n$ is an Esakia order-compactificatiwe verify that the map $e: \mathcal{U}(n) \rightarrow \mathbf{B}_n$ sending x to $\{U : \uparrow x \subseteq U\}$ is an order-embedding and that $e[\mathcal{U}(n)]$ is dense in this space.

Assuming that $x \leq y$, we certainly have that $e(x) \leq e(y)$; conversely if $e(x) \leq e(y)$, note that $\uparrow x \in e(x)$, so $\uparrow x \in e(y)$ implies that $\uparrow y \subseteq \uparrow x$, which means that $x \leq y$. This also shows that e is injective.

Note that for each $x \in \mathcal{U}(n)$, $e(x)$ is clopen, corresponding to $\varphi(\uparrow x)$. Thus e is an open map to its image, and it is certainly continuous since the topology on $\mathcal{U}(n)$ is discrete. Thus the map $e: \mathcal{U}(n) \rightarrow \mathbf{B}_n$ is an order-embedding.

Let $U, V \in \mathbf{Bel}(n)$, and look at $\varphi(U) \setminus \varphi(V)$, a subset of $\mathbf{B}_n$. If $\varphi(U) \setminus \varphi(V) \neq \emptyset$, then $\varphi(U) \Rightarrow \varphi(V) \neq \mathbf{B}_n$, and so $\varphi(U \rightarrow V) \neq \mathbf{B}_n$, and so $U \rightarrow V \neq \mathcal{U}(n)$. We can then pick a point $x \notin U \rightarrow V$, and find that $e(x) \in \varphi(U) \setminus \varphi(V)$. Thus $e[\mathcal{U}(n)]$ is dense in $\mathbf{B}_n$.

Finally we show that $\mathbf{B}_n$ is the least Esakia order-compactification. For that purpose, note that if Y is another compactification, we can compare $\mathbf{ClopUp}(\mathbf{B}_n)$ and $\mathbf{ClopUp}(Y)$; by Lemma 3.4.11, we have that each upset of the form $\uparrow x$ for $x \in \mathcal{U}(n)$ is clopen in Y . Since $\mathbf{ClopUp}(Y)$ is a Heyting subalgebra of $\mathbf{Up}(\mathcal{U}(n))$ (see the proof of Theorem 3.3.17). Thus, $\mathbf{Bel}(n) \subseteq \mathbf{ClopUp}(Y)$; by Esakia duality, we get a map $p: Y \rightarrow \mathbf{B}_n$, which commutes with the compactification maps. ■

The reader may convince themselves that such algebras and their dual spaces will be typically quite complicated; see Appendix A.2.2 for more on this.

3.4.4 Bellissima algebras of width 2

As a final positive result, we analyze a special instance of the Bellissima algebra, in logics of bounded width 2, i.e. for $L = \mathbf{BW}_2$ (see Proposition 2.3.19). Such a case includes several logics of interest (see [54] for all the extensions of $\mathbf{KG}$, as well as [65, 41]). They also have rather peculiar logical properties: whilst it was shown in [150] that there exist varieties for which all n -generated algebras are finite but there exists an infinite $n + 1$ -generated algebra, the same does not happen for logics of width 2 (see [247] for an extended discussion⁷). In this context, it is interesting to understand whether any combinatorial properties of the universal model can account for this behaviour. In this line of reasoning, we provide below a full description of the dual of the algebra $\mathbf{Bel}_{\mathbf{BW}_2}(n)$ for any n . This begins with the following observation:

Lemma 3.4.15 (Finite maximum lemma). If $x, y \in \mathcal{U}_L(n)$, then $\downarrow x \cap \downarrow y$ has a finite maximum.

Proof. We may assume that $\{x, y\}$ is an antichain otherwise this is obvious. Let m be a point in the maximum of $\downarrow x \cap \downarrow y$. Then m must have necessarily two immediate successors, m_1, m_2 . We must have by maximality that $m_1 \in \downarrow x - \downarrow y$ and $m_2 \in \downarrow y - \downarrow x$. Moreover, we have that $[m_1, x]$ must be a chain (otherwise, if there were two incomparable points there, we would have together with m_2 an antichain of three elements). Since the poset is n -generated, this chain has finitely many elements. Similarly for m_2 . Hence the choices for such maximal points can

⁷In fact, we have recently learned that T. Moraschini has proved, in currently unpublished work, that for logics of width 2, local finiteness is equivalent with omitting the Rieger-Nishimura ladder – that is, the logic $\mathbf{RN}$ is the unique pre-locally tabular logic.

be given by taking all chains going down from x , i.e., points such that $[a, x]$ is linear, and all chains down from y , and picking pairs of such points. This is easily seen to then be finite, since inside of these chains the color must consistently go down. ■

Note that if the width is greater than 2, such a property may fail (see A.2.5.(2)). We will likewise need some properties of the universal model, which we will prove in slightly greater generality. We recall now the notion of depth (Definition 2.3.16):

Definition 3.4.16 (Depth of points, levels). Let P be a locally finite poset. For each $x \in P$, the *depth* of x , denoted by $d_p(x)$, is the length of the largest chain $x = x_1 \leq x_2 \leq \dots \leq x_k$ in P . Given $k \in \omega$, we denote by:

$$\text{Lev}_k(P) = \{x \in P : d_p(x) = k\}.$$

Definition 3.4.17 (Level-finite poset). A locally finite poset P is called *level-finite* if for each $n \in \omega$ the set $\text{Lev}_n(P)$ is finite.

Lemma 3.4.18 (Key properties of level-finite posets). Let P be a level-finite poset. Then:

1. Whenever $x \in P$, then,

$$P - \downarrow x := \uparrow x \rightarrow \bigcup_{x < y} \uparrow y,$$

where $x < y$ denotes the immediate successor relation.

2. For each $x \in P$, we have that there exists T a finite subset, such that:

$$\uparrow x = P - \downarrow T.$$

Proof. (1): suppose that $y_0 \not\leq x$, and $y_0 \leq r$ and $x \leq r$. If for each $x < y$, $y \not\leq r$, we would conclude that $r = x$, which would imply, since $y_0 \leq r$ and $y_0 \not\leq x$ a contradiction. So for some such y , $y \leq r$. Conversely, assume that $y_0 \in \uparrow x \rightarrow \bigcup_{x < y} \uparrow y$; then if $y_0 \leq x$, we would obtain a contradiction, so certainly $y_0 \not\leq x$. This shows the equality.

(2) Let $x \in P$. Consider:

$$T = \{y \in P : d_p(y) \leq d_p(x), x \not\leq y\} \cup \{y \in P : d_p(y) = d_p(x) + 1\};$$

since P is level-finite, this set is finite. For any $r \in P$, if $x \leq r$, and $r \leq y$ where $y \in T$, we would have that $x \leq y$, a contradiction; conversely assume that $r \notin \downarrow T$. Then $d_p(r) \leq d_p(x)$, since otherwise, r would be below some point with depth at least $d_p(x) + 1$. Hence if $x \not\leq r$, then $r \in T$, a contradiction – i.e., $x \leq r$. ■

Corollary 3.4.19 (Characterization of Bellissima algebras for logics of width 2). *Let $L \supseteq \text{BW}_2$ be a logic with the FMP. Then the Bellissima algebra $\text{Bel}_L(n)$ is given exactly by the elements of the form:*

$$\{\mathcal{U}_L(n) - \downarrow S : S \text{ a finite subset}\}.$$

Proof. Throughout the proof, let $X := \mathcal{U}_L(n)$. First we verify that elements of the given form indeed form a Heyting subalgebra. Note that if $\downarrow S, \downarrow T$ are two such subsets then their union is again a subset of this form. Moreover:

$$(\downarrow S) \cap (\downarrow T) = \bigcup_{s \in S, t \in T} \downarrow s \cap \downarrow t,$$

and since by Lemma 3.4.15, this has a finite maximum, we have that it will again be of this kind. Finally, consider:

$$(X \setminus \downarrow S) \rightarrow (X \rightarrow \downarrow T);$$

its complement is $(X \setminus \downarrow S) \cap \downarrow T = \bigcup_{t \in T} \downarrow t \setminus (\downarrow S)$. It suffices to prove that $\downarrow t \setminus \downarrow S$ always has a finite maximum. Indeed, note that this is of the form:

$$\downarrow t \setminus \downarrow S = \downarrow t \cap \bigcap_{s \in S} X \setminus \downarrow s = \bigcap_{s \in S} \downarrow t \setminus \downarrow s.$$

Now for each s, t , if $t \leq s$, $\downarrow t \setminus \downarrow s = \emptyset$; if $t \not\leq s$ note that $t \in \downarrow t \setminus \downarrow s$, so the latter is simply $\downarrow t$.

Now by Lemma 3.4.18. 2, we have that every finite upset is in this algebra, meaning that it extends the Bellissima algebra. By 3.4.18.1, every element in it is generated from the finite upsets, entailing that it coincides with $\text{Bel}_L(n)$. ■

Let $\mathbf{B}_n^L = \text{Spec}(\text{Bel}_L(n))$ be the least Esakia order-compactification of $\mathcal{U}_L(n)$, as described in Theorem 3.4.14. Then we can obtain the following concrete description of this space:

Theorem 3.4.20 (Characterization of Bellissima duals for width 2). *For $L \supseteq \text{BW}_2$ a logic with the finite model property, the space $\mathbf{B}_n^L$ is isomorphic to the subspace of $\text{Spec}(\text{Def}_L(\mathcal{U}(n)))$ obtained by taking only the points of depth ω , i.e., points with infinitely many successors such that each of its proper successors has only finitely many.*

Proof. First, let $x \in \text{Spec}(\text{Def}_L(\mathcal{U}(n)))$ be a point of depth ω . Consider:

$$\mathcal{F} = \{U_L(n) \setminus \downarrow y : x \not\leq y, y \in \mathcal{U}(n)\} \text{ and } \mathcal{I} = \{U_L(n) \setminus \downarrow z : x \leq z, z \in \mathcal{U}(n)\}.$$

It is clear that this defines a prime filter and ideal pair in $\mathbf{B}_n^L$, which will be disjoint, and generate a point x' , seeing all the same finite points. Conversely if $x \in \mathbf{B}_n^L$ is a prime filter, note that we can realize it in $X_n = \text{Spec}(\text{Def}(\mathcal{U}(n)))$ by looking at $X_n \setminus U_L(n)$ – which is closed – and picking the maximum above a given prime filter in the intersection of the above sets. This map easily yields an isomorphism. ■

Chapter 4

Step-by-Step Constructions

In the previous chapter we saw that despite its strong geometric intuition, the universal model does not easily yield a full representation of the free Heyting algebra, since the definable upsets of $\mathcal{U}(n)$ remain difficult to describe. In this chapter we present a second concrete construction of free Heyting algebras, which gives a faithful representation of free Heyting algebras, at the cost of some additional complexity. This construction traces back to Ghilardi [127], inspired by the work of Urquhart [251], having been subsequently studied by several authors (e.g., in [75] and [57]). Its key idea relies on the duality between finite distributive lattices and finite posets, and the idea of filtration of Heyting algebras via distributive lattices (as explained in [126]).

We present such a construction, and its methodology, for arbitrary Priestley spaces, with no finiteness assumptions. This involves some non-trivial work in finding topological analogues of Ghilardi’s one-step extensions. In categorical terms, this amounts to providing a dual and concrete description of the functor which is left adjoint to the inclusion of **HA**, the category of Heyting algebras, in **DL**. As far as we are aware, such a generalization is new, and provides the main technical novelty of this chapter. This added level of generality leads us to the setting of *Ghilardi complexes*, which we use to study several constructions through a dual lens: coproducts of Heyting algebras, pushouts and polynomial algebras, to name a few.

As a corollary of this analysis, we prove the existence of two rather surprising interpretations of IPC into its positive fragment: a structural interpretation, which fails to be conservative, and a non-recursive, but conservative, translation. The same analysis yields a *uniform positive approximation* theorem.

Outline of the chapter

1. In Section 4.1 we outline the general motivation for step-by-step constructions, stemming from normal form representations, and outline the general methodology, emphasizing the functoriality of such a construction, and the

precise assumptions used to obtain the step-by-step representation of left adjoints.

2. In Section 4.2 we introduce the key algebraic and dual concepts allowing for the construction of Ghilardi complexes. With it, we generalize Ghilardi's construction of the free Heyting algebra generated by a finite distributive lattice to the general case.
3. In Section 4.3 we describe the structure of free Heyting algebras, recovering the results of Ghilardi [127] and [251]: all free finitely generated Heyting algebras are bi-Heyting, and join-irreducibly generated. Employing the uniform interpolation theorem, we extend this, along lines suggested by Ghilardi, to all free Heyting algebras. We also provide explicit calculations of the normal forms for intuitionistic logic.
4. In Section 4.5 we study the interpretation of IPC in positive logic using Ghilardi complexes. This is shown to be non-conservative, and its logic is characterized. We introduce the notion of G -conservative object, which allow a specific adjoint to be definable on the basis of the Ghilardi complex structure. We show that all countable locally finite Heyting algebras are G -conservative. As a corollary, we obtain two interrelated definability results: the existence of a different conservative translation from IPC to positive logic PL, and the existence of *uniform positive approximations* for all intuitionistic formulas.

4.1 Dual representations of free constructions

4.1.1 Free algebras: semantically organized syntax

When one considers classical propositional logic CPC from an algebraic point of view, one of its attractive features is the wide variety of *normal forms*¹. Disjunctive normal form (DNF), conjunctive normal form (CNF), negation normal form (NNF) are some well-known examples. The special structure of such normal forms is at the heart of the proof of some of the deepest theorems about Boolean algebras, and often allows extending finitary structure to infinitary settings: for example, the famous *Sikorski extension theorem*, concerning the injectivity of the class of complete Boolean algebras [243, Chapter 33], makes extensive use of the specific form of Boolean polynomials.

One particularly nice way of obtaining such normal forms is through duality theory: one *represents* a free algebra using finite sets, and from this obtains the

¹Throughout our mentions of normal forms, we implicitly assume an abstract term rewriting system over our syntax, which allows us to apply some rules called *reduction rules* or simply *reductions*, where a normal form is a formula which cannot be further reduced. This perspective will not be further emphasized throughout this thesis.

above normal forms as simple corollaries. In other words: a (dual) representation of a free algebra, amounts to *semantically organizing syntax*. This is the core idea which we will pursue throughout this chapter.

Normal forms have a long history in modal and superintuitionistic logics, going back at least to [112]. As discussed in [125], in such a setting, normal forms encode the models of a given logical system, and the reduction rules are given by geometric operations of “gluing” or “cutting” models. Applications of this have been abundant in the literature on non-classical logics: they provide ways of showing completeness [112], uniform conservativity phenomena [127], as well as proof-theoretic properties like cut-elimination [129] or bounded proof properties for proofs [58].

Showing that each formula is L -equivalent to one in a given normal form is thus equivalent to representing the free L -algebra – to do so via duality is nothing more than a convenient way to give spatial intuition to this process. This equivalence comes from the latter’s role as the Lindenbaum-Tarski algebra of L , with formulas being the elements of the algebra, and the representation providing the normal form. From this several consequences follow:

1. The description of a class of L -normal forms;
2. A dual description of colimits of algebras, following a similar structure to descriptions of free algebras;
3. A proof that the variety $\text{Alg}(L)$ is $\mathbf{A}$ -anarchic with respect to some suitable variety $\mathbf{A}$ in an enriched language. Moreover, with due care, an *explicit structural interpretation* can be extracted from such a proof.

4.1.2 Classical and positive logic

To illustrate what is meant by semantically organizing syntax, let us recall the representation of free algebras of $\mathbf{BA}$ and $\mathbf{DL}$.

Notation 4.1.1. We denote by $\mathbf{2}$ the two-element set with the discrete topology. We denote by $\mathbf{2}_{\leq}$ the two element poset $\{0 \leq 1\}$.

Let S be a set, which we think of as a set of propositions. A *classical model* consists simply of an assignment $v: S \rightarrow \mathbf{2}$, specifying which propositions from X are true or false. In $\mathbf{CPC}$ this immediately fixes the meaning of all classical formulas, and all models are maximal – no additional formulas can be forced at them without a contradiction. In $\mathbf{PL}$ (see Remark 2.2.8), a model is likewise specified solely by a valuation, but the lack of a negation connective means that truth may increase, which motivates the use of partially ordered sets, rather than sets, to represent the space of all models.

By thinking of such valuations as points, one can obtain a space, by forming the X -fold product, either of $\mathbf{2}$ or $\mathbf{2}_{\leq}$; these coincide as spaces, differing only on

the order: the former has discrete order, the latter, induces the pointwise order. We thus obtain:

Proposition 4.1.1 (Duals of free BA and free DL). For each set S :

1. The algebra $\mathbf{Clop}(\mathbf{2}^S)$ is the free Boolean algebra on S many generators.
2. The algebra $\mathbf{ClopUp}(\mathbf{2}_{\leq}^S)$ is dual to the free distributive lattice on S many generators.

Proof. We show (2), with (1) being analogous. First, note that we have a map $i: S \rightarrow \mathbf{ClopUp}(\mathbf{2}_{\leq}^S)$ sending s to the clopen upset $\pi_s^{-1}[\{1\}]$. Let D be a distributive lattice, and $f: S \rightarrow D$ a function. For each $s \in S$, $\varphi(f(s))$, is then a clopen upset of $\mathbf{Spec}(D)$, and induces a function $f_s: \mathbf{Spec}(D) \rightarrow \mathbf{2}_{\leq}$ defined by:

$$x \mapsto 1 \iff x \in \varphi(v(s)).$$

This is order-preserving and continuous since $\varphi(v(s))$ is a clopen upset. Thus, by the universal property of the product, this defines a unique map $f: \mathbf{Spec}(D) \rightarrow \mathbf{2}_{\leq}^S$. The dual $f^{-1}: \mathbf{ClopUp}(\mathbf{2}_{\leq}^S) \rightarrow D$ then gives a unique lifting of f , since $f^{-1}(i(s)) = f(s)$ by construction. ■

From this representation, we proceed to extract the DNF (and CNF) representation of all formulas:

Definition 4.1.2 (Atomic and join-irreducible formulas).

1. A formula $\varphi(p_1, \dots, p_n) \in \mathcal{L}_{\text{CPC}}$ is said to be *atomic* if it is of the form $\bigwedge_{i=1}^n j(p_i)$ where $j(p_i) \in \{p_i, \neg p_i\}$.
2. A formula $\varphi(p_1, \dots, p_n) \in \mathcal{L}_{\text{PL}}$ is said to be *join irreducible* if it is of the form $\bigwedge_{j=1}^m p_{i_j}$ for $\{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$; it is said to be *meet-irreducible* if it is of the form $\bigvee_{j=1}^m p_{i_j}$ for $\{i_1, \dots, i_m\} \subseteq \{1, \dots, n\}$.

The process for obtaining such normal is as follows: one takes $\bar{p}$ a set of propositional formulas. Consider $\mathcal{F}_{\text{BA}}(\bar{p})$. By Proposition 4.1.1, the dual space (definitionally equivalent to a set) is simply

$$\mathcal{P}(\bar{p}).$$

We can assign these elements to atoms: if $S \subseteq \bar{p}$ let us write $\chi_S(p)$ for the function:

$$\chi_*(S)(p_i) = \begin{cases} p_i, & \text{if } p_i \in S \\ \neg p_i & \text{if } p_i \notin S. \end{cases}$$

Then given a set S , we obtain an atom:

$$\text{At}(S) := \bigwedge_{i=1}^n \chi_S(p_i).$$

Since by definition, the elements of $\mathcal{F}_{\mathbf{BA}}(\bar{p})$ are the formulas, and by duality, they correspond to the subsets of $\mathcal{P}(\bar{p})$, they correspond to disjunctions of formulas. That is, we have essentially shown the CPC part of the following (the PL case is very similar):

Proposition 4.1.3 (Normal forms for CPC and PL).

1. For each formula $\varphi(\bar{p}) \in \text{Fm}_{\mathcal{L}_{\text{CPC}}}(\text{Prop})$ there is a unique set S of atomic formulas over $\bar{p}$ such that $\varphi(\bar{p}) \equiv \bigvee_{\psi \in S} \psi(\bar{p})$.
2. For each $\varphi(\bar{p}) \in \text{Fm}_{\mathcal{L}_{\text{PL}}}(\text{Prop})$ there is a unique set S of join-irreducible formulas over $\bar{p}$ such that $\varphi(\bar{p}) \equiv \bigvee_{\psi \in S} \psi(\bar{p})$, and a unique set T of meet-irreducible formulas such that $\varphi(\bar{p}) \equiv \bigwedge_{\psi \in T} \psi(\bar{p})$.

By inspecting our description of the free algebras for **BA** and **DL** we can likewise study the duals of coproducts and other colimits. Indeed, by recalling 2.2.14, and noting the general description of free Boolean algebras, we can expect the following:

Proposition 4.1.4 (Categorical products in **Pries** and **Stone**). Let $(X_i, \leq)_{i \in I}$ be a family of Priestley spaces.

1. The Priestley space $(\prod_{i \in I} X_i, \leq)$ is the categorical product in **Pries** of $(X_i, \leq)_{i \in I}$;
2. The space $\prod_{i \in I} X_i$ is the categorical product of $(X_i)_{i \in I}$ in **Stone**.

Proof. See e.g. [124, pp. 69]. ■

Finally, one can use these normal forms to extract additional structure. To understand how, take a formula $\varphi(\bar{p})$ and $\psi(\bar{p})$ in $\text{Fm}_{\mathcal{L}_{\text{PL}}}(\text{Prop})$. By Proposition 4.1.3, we can assume that:

$$\varphi(\bar{p}) = \bigvee_{i=1}^n \varphi_i(\bar{p}) \text{ and } \psi(\bar{p}) = \bigwedge_{j=1}^m \psi_j(\bar{p}),$$

where for each $1 \leq i \leq n$, φ_i is join-irreducible, and for each $1 \leq j \leq m$, ψ_j is meet-irreducible. Note that by some basic properties of relative pseudocomplements (see Definition 2.3.3), we have that in any algebra:

$$\varphi(\bar{p}) \rightarrow \psi(\bar{p}) = \left(\bigvee_{i=1}^n \varphi_i(\bar{p}) \right) \rightarrow \left(\bigwedge_{j=1}^m \psi_j(\bar{p}) \right) = \bigwedge_{i=1}^n \bigwedge_{j=1}^m \varphi_i(\bar{p}) \rightarrow \psi_j(\bar{p}).$$

Whenever one takes φ_i a join-irreducible formula, and ψ_j a meet-irreducible formula, *the relative pseudocomplement is easily seen to be expressible*. As above,

we can identify join-irreducible formulas with a unique subset $S \subseteq \bar{p}$, by the assignment $J(S) := \bigwedge \{p \in \bar{p} : p \in S\}$. Dually, one assigns subsets to formulas by complement, $M(S) := \bigvee \{p \in \bar{p} : p \notin S\}$.

With this in mind, suppose that $\varphi_i = J(S)$ and $\psi_j(\bar{p}) = M(T)$. Then one can make the following definition:

$$\varphi_i \rightarrow \psi_j := \begin{cases} \top & \text{if } S \setminus T \neq \emptyset \\ \psi_j & \text{otherwise.} \end{cases}$$

The reader can verify that this indeed satisfies the relative pseudocomplement laws (see Appendix A.1 for a proof). Thus, all relative pseudocomplements between positive formulas are expressible, meaning we can prove the following:

Proposition 4.1.5 (Distributive lattices (bi-)Heyting anarchic). For each set X , the free algebra $\mathcal{F}_{\mathbf{DL}}(X)$ is a (bi-)Heyting algebra. Thus, the variety **DL** is **HA**-(**biHA**-)anarchic.

Proof. See Appendix A.1.1 for a proof. ■

The cases of **CPC** and **DL** provide good advertising to why a good description of free algebras can be desirable – but they are certainly very special. Notably, the varieties **BA** and **DL** are finitely generated (see Section 2.2.3), implying for instance that over finitely many propositional variables there exist, up to equivalence, only finitely many formulas. By contrast, the $\Box$ -connective in modal logic (when contrasted with the classical connectives), or the $\rightarrow$ in intuitionistic logic (when contrasted with the connectives of positive logic), can be nested arbitrarily deep. We can broadly refer to these as *non-locally tabular connectives*². We turn to such general methodology in the next section.

4.1.3 Step-by-step building free algebras

The step-by-step method is a methodology for constructing free algebras, in situations where one may not be able to rely on local finiteness of the algebras. In this guise, it is typically used for finitely presented algebras, and coupled with a dual analysis, and is typically focused on describing normal forms and on analysing the shape of free algebras (see e.g. [89] and [59] for modal contexts, as well as [57] and [126] for modal and superintuitionistic settings). In this section, we stress the general categorical picture associated with such a method, implicit in much of the above literature.

To ease presentation, in this chapter we will make some simplifying assumptions (see Chapter 5 for some discussion). We assume throughout that:

²We advise caution when discussing this, as the concept of local tabularity for connectives only makes sense in the context of an agreed upon reduct: as is well-known the reduct $\text{IPC}_{\wedge, \rightarrow}$ of intuitionistic logic is also locally tabular, by a famous theorem of Diego [98]. However, throughout this thesis we will focus on the distributive-lattice reduct.

1. $\mathcal{L}_0$ is a given finite algebraic signature, containing a subsignature $\mathcal{L}$ called its *base-reduct*;
2. the equational axioms of the variety $\mathbf{A}_0$ which involve the operations in $\mathcal{L}_0 \setminus \mathcal{L}$ contain *no nested applications of operations* in $\mathcal{L}_0 \setminus \mathcal{L}$.

These assumptions can be removed, but they would make our presentation substantially more cumbersome³.

Notation 4.1.2 (Language reducts). Throughout, given two languages $\mathcal{L} \subseteq \mathcal{L}_0$ we will write $A \upharpoonright_{\mathcal{L}}$ for the $\mathcal{L}$ -reduct of an $\mathcal{L}_0$ -algebra. We will also write $\mathbf{A} \upharpoonright_{\mathcal{L}}$ for the class of $\mathcal{L}$ -reducts of a class of $\mathcal{L}_0$ -algebras.

For the rest of the section we fix:

- $\mathbf{A}_0$ a variety of $\mathcal{L}_0$ -algebras, and
- $\mathbf{A}$ a variety of $\mathcal{L}$ -algebras,

where for every $A \in \mathbf{A}_0$ $A \upharpoonright_{\mathcal{L}} \in \mathbf{A}$. We stress that for the purposes of this construction *no other requirements are needed* – all developments will be purely categorical. The two main assumptions that will be employed occasionally are the following:

1. (Local finiteness) The variety $\mathbf{A}$ is locally finite;
2. (One-step injectivity) For each $A \in \mathbf{A}$, the inclusion into the general one-step extension (see Definition 4.1.10) is monic.

The assumption of local finiteness corresponds to the idea that “Free algebras and colimits for $\mathbf{A}$ are well-understood.”. Indeed, most descriptions of the step-by-step assume local finiteness of the variety $\mathbf{A}$. The second assumption will figure crucially later in this chapter (see Appendix A.3.3 for discussion on this).

Suppose now that we are trying to describe free algebras for the variety $\mathbf{A}_0$. From the set-up we have, rather than begin from scratch, we can take the variety $\mathbf{A}$ as an intermediate step. This amounts to factoring the free functor $\mathcal{F}_{\mathbf{A}_0}(-): \mathbf{Set} \rightarrow \mathbf{A}_0$, by describing a functor $\mathbf{F}$, as illustrated in Figure 4.1.

The construction of $\mathbf{F}$ is the step-by-step part of the construction: what it does is to add freely, step-by-step, the elements which are required to make it a $\mathbf{A}_0$ -algebra. This one-step extension is given as follows:

³These limitations will be removed in other chapters of the thesis, where the simple modifications needed will be discussed: Chapter 5 we will have occasion to work with nested applications, and in Chapter 7 with infinite languages. We stress that this level of generality was already implicit in previous work [59, 63, 89] and we claim no essential originality of the present work beyond describing this for general varieties.

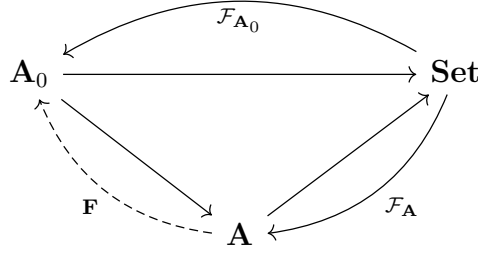


Figure 4.1

Definition 4.1.6 (One-step extension). Let $A \in \mathbf{A}$ be arbitrary. We define $E(A, \mathbf{A}_0)$, the *one-step extension* of $A \in \mathbf{A}$ into an $\mathbf{A}_0$ -algebra, to be the algebra:

$$E(A, \mathbf{A}_0) := \left(A \oplus_{\mathbf{A}} \mathcal{F}_{\mathbf{A}} \left(\bigcup_{t \in \mathcal{L}_0 \setminus \mathcal{L}} A^{\text{ar}(t)} \right) \right) / \Theta_1,$$

where Θ_1 is the smallest congruence containing, for each equational axiom $\lambda(\bar{x}) \approx \gamma(\bar{x})$ of $\mathbf{A}_0$, and each tuple $\bar{a}$ from A , the pair $(\lambda(\bar{a}), \gamma(\bar{a}))$. We define $i_0: A \rightarrow E(A, \mathbf{A}_0)$ to be the coproduct inclusion to the first summand.

With this definition, the coproduct inclusion $i_0: A \rightarrow E(A, \mathbf{A}_0)$ is an $\mathcal{L}$ -homomorphism, and the latter algebra is a partial $\mathbf{A}_0$ -algebra. It additionally satisfies a universal property with respect to partial $\mathbf{A}_0$ -algebras; an elegant way to express this, due to [127], is by taking $\mathbf{A}$ -homomorphisms to determine which operations must exist in the codomain:

Definition 4.1.7 (f -partial algebra). Given $f: A \rightarrow B$ a $\mathbf{A}$ -homomorphism of $\mathcal{L}$ -algebras, we say that B is f - $\mathbf{A}_0$ if for each $t(x_1, \dots, x_n) \in \mathcal{L}_0$ and $a_1, \dots, a_n \in A$, there exists an element $t(f(a_1), \dots, f(a_n)) \in B$ satisfying the axioms of the operation t imposed by $\mathbf{A}_0$.

We can then formulate the universal property for $(E(A, \mathbf{A}_0), i_0)$, which follows basically from the construction:

Proposition 4.1.8 (Univ₀). Let B be a $\mathbf{A}$ -algebra, and $f: A \rightarrow B$ a $\mathbf{A}$ -morphism such that B is f - $\mathbf{A}_0$. Then there exists a unique extension $\bar{f}: E(A, \mathbf{A}_0) \rightarrow B$ such that for each $t(x_1, \dots, x_n) \in \mathcal{L}_0$, and $\bar{a} \subseteq A$, we have $\bar{f}(t(i_0(a_1), \dots, i_0(a_n))) = t(f(a_1), \dots, f(a_n))$, and such that $\bar{f}i_0 = f$.

Proof. The existence and uniqueness of such a map depends solely on the existence and uniqueness of the coproduct map. The important fact is that the homomorphism f induces, for each $t \in \mathcal{L}_0 \setminus \mathcal{L}$, a map $f^{\text{ar}(t)}: A^{\text{ar}(t)} \rightarrow B$, obtained by sending $\bar{a}$ to $t(\bar{a})$, which since B is f - $\mathbf{A}_0$ is assumed to exist. The universal property of free algebras and the quotient by Θ_0 then ensure that this produces a unique map in the desired conditions. ■

$$\begin{array}{ccc}
A_0 & \xrightarrow{i_0} & A_1 \\
& \searrow f & \swarrow \bar{f} \\
& B &
\end{array}$$

We want to iterate this construction. However, on the second step an additional concern is necessary: for $t(x_1, \dots, x_n) \in \mathcal{L}_0 \setminus \mathcal{L}$ and elements $a_1, \dots, a_n$, we need to ensure that when we form the algebra $\mathbf{E}_2(A, \mathbf{A}_0)$, the “new” elements in the extension of t agree with the old ones. So the congruence Θ_2 must additionally contain pairs of the form:

$$(t(i_0(a_1), \dots, i_0(a_n)), i_0(t(a_1, \dots, a_n))),$$

for t such an operation and a_i elements from A . Similar to before, this can be easily ensured by taking homomorphisms as parameters:

Definition 4.1.9 (*f*-open maps). Given $f: A \rightarrow B$, and $g: B \rightarrow C$, where B is $f\text{-}\mathbf{A}_0$ and C is $g\text{-}\mathbf{A}_0$, we say that g is an *f*-open map, if for each $t(x_1, \dots, x_n) \in \mathcal{L}_0 \setminus \mathcal{L}$, and $a_1, \dots, a_n \in A$, $g(t(f(a_1), \dots, f(a_n))) = t(gf(a_1), \dots, gf(a_n))$.

Definition 4.1.10 (General one-step extension). Given $g: A_0 \rightarrow A$ a map, where A is $g\text{-}\mathbf{A}_0$. We define $\mathbf{E}(A, \mathbf{A}_0, g)$, the *g*-one-step extension of $A \in \mathbf{A}$ into an $\mathbf{A}_0$ -algebra, to be the algebra:

$$\mathbf{E}(A, \mathbf{A}_0, g) := \left(A \oplus_{\mathbf{A}} \mathcal{F}_{\mathbf{A}} \left(\bigcup_{t \in \mathcal{L}_0 \setminus \mathcal{L}} A^{\text{ar}(t)} \right) \right) / \Theta_2$$

where Θ_2 is the smallest congruence containing:

1. For each equational axiom $\lambda(\bar{x}) \approx \gamma(\bar{x})$ of $\mathbf{A}_0$, and each tuple $\bar{a}$ from A , the pair $(\lambda(\bar{a}), \gamma(\bar{a}))$;
2. For each $t \in \mathcal{L}_0 \setminus \mathcal{L}$, $a_1, \dots, a_n \in A_0$, the pair

$$(t(g(a_1), \dots, g(a_n)), g(t(a_1, \dots, a_n))).$$

We define $i_g: A \rightarrow \mathbf{E}(A, \mathbf{A}_0, g)$ to be the coproduct inclusion to the first summand.

With this definition, the coproduct inclusion $i_g: A \rightarrow \mathbf{E}(A, \mathbf{A}_0, g)$ becomes a *g*-open map in addition to an $\mathcal{L}$ -homomorphism. The universal property becomes the following general “one-step extension”⁴:

Proposition 4.1.11 (Univ_g). Let $g: A_0 \rightarrow A$ a homomorphism where A is $g\text{-}\mathbf{A}_0$, and and $f: A \rightarrow B$ a *g*-open map such that B is $f\text{-}\mathbf{A}_0$. Then there exists a unique extension $\bar{f}: \mathbf{E}(A, \mathbf{A}_0, g) \rightarrow B$, an *i_g*-open map, and such that $\bar{f}i_g = f$.

⁴If the variety in question has initial algebras – as will always be the case for us – this universal property includes the former, by formally defining A_{-1} to be the initial algebra.

Proof. The key observation is that by imposing the extra axioms from Θ_2 , then as soon as f is g -open, $\bar{f}$ becomes well-defined. The fact that this extension is i_g -open, that the diagram commutes, and that it is unique, follows exactly as in Proposition 4.1.8, and amounts to chasing the universal properties. ■

We emphasize that Proposition 4.1.11 is the heart of the step-by-step construction: it is from it that all general categorical facts follow, once one constructs the proper machinery.

Notation 4.1.3 (Inducing directed colimit). Throughout, when defining a directed system $(A_n, i_{i,j})_{n,i,j \in \omega}$, we will simply specify the objects A_n and the maps $i_n: A_n \rightarrow A_{n+1}$. This will follow the convention, occasionally reminded throughout, that the connecting map $i_{i,j}: A_i \rightarrow A_j$ is thus given by:

$$i_{i,j} = i_j \circ i_{j-1} \circ \cdots \circ i_i.$$

A similar conventions will often be employed with inverse systems.

Definition 4.1.12 (Chain of extensions). Given $A \in \mathbf{A}$, we define a directed chain $(A_n, i_{j,k})_{n \in \omega}$ of algebras as follows:

1. $A_0 = A$, and $A_1 = E(A, \mathbf{A}_0)$, with $i_0: A_0 \rightarrow A_1$ as in Definition 4.1.6.
2. For each $k > 1$, $A_{k+1} = E(A_k, \mathbf{A}_0, i_k)$ and $i_{k+1} := i_{i_k}$ as in Definition 4.1.10.

$$A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} \dots$$

We define $A_\omega := \lim_{n \in \omega} A_n$, with the inclusion maps $\iota_n: A_n \rightarrow A_\omega$, which are i_n -open maps.

It is a completely general fact that since $\mathbf{A}$ is a variety, the forgetful functor $\mathbf{A} \rightarrow \mathbf{Set}$ creates filtered colimits (see [70, p. 4.3.2], also [4, Remark 3.4.(4)]). In other words, the elements of A_ω are constructed as equivalence classes of the disjoint union $\bigsqcup_{n \in \omega} A_n$, under the equivalence relation induced by identification by i_n (i.e. $(a, j) \sim (b, k)$, if and only if $j \leq k$ and $i_{j,k}(a) = b$, or vice-versa). A_ω is a $\mathcal{L}$ -algebra as a directed colimit of algebras. This assignment is in fact functorial:

Definition 4.1.13 (Colimit maps). Suppose that $A, B \in \mathbf{A}$ and $f: A \rightarrow B$ is a $\mathcal{L}$ -homomorphism. We define a sequence of maps $f_k: A_k \rightarrow B_k$ commuting with the directed chains $(A_n, i_{j,k}^A)_{n \in \omega}$ and $(B_n, i_{j,k}^B)_{n \in \omega}$, as follows:

1. $f_0: A_0 \rightarrow B_0$ is simply f ;
2. $f_{k+1}: A_{k+1} \rightarrow B_{k+1}$ is defined to be the unique i_k^A -open map which is an extension of $i_k^B f_k$ such that $f_{k+1} i_k^A = i_k^B f_k$.

We define $f_\omega: A_\omega \rightarrow B_\omega$ the *colimit map* induced by this sequence. Explicitly, given $[a] \in A_\omega$ where $a \in A_n$, $f_\omega([a]) = [f_n(a)]$.

The given specifications ensure that colimit maps are well-defined. From all of this, we can obtain our desired left adjoint:

Theorem 4.1.14 (Left adjoint from $\mathbf{A}$ to $\mathbf{A}_0$). *Given $A \in \mathbf{A}$, we have:*

1. *The algebra A_ω is an $\mathbf{A}_0$ -algebra, when equipped with the interpretations canonically induced by the directed chain.*
2. *Whenever $f: A \rightarrow B$ is an $\mathcal{L}$ -homomorphism, the limit map $f_\omega: A_\omega \rightarrow B_\omega$ is an $\mathcal{L}_0$ -homomorphism. Moreover, $(fg)_\omega = f_\omega g_\omega$.*
3. *The algebra A_ω has the following universal property: for each homomorphism $\bar{f}: A \rightarrow B$ where B is a $\mathbf{A}_0$ algebra, there is a unique map $\bar{f}: A_\omega \rightarrow B$ where $\bar{f}$ is a $\mathcal{L}_0$ -homomorphism, and $\bar{f}\iota_0 = f$.*
4. *The functor $\mathbf{F}: \mathbf{A} \rightarrow \mathbf{A}_0$ is left adjoint to the forgetful functor $\mathbf{A}_0 \rightarrow \mathbf{A}$.*

Proof. (1) As noted, A_ω is $\mathbf{A}$ -algebra by general categorical (or universal algebraic) facts. For each $t(x_1, \dots, x_n) \in \mathcal{L}_0 \setminus \mathcal{L}$, we define $t: A_\omega^n \rightarrow A_\omega$ as follows: given $[a_1], \dots, [a_n]$, equivalence classes in A_ω , we find k large enough that there exist $b_i \in [a_i]$, and $b_i \in A_k$. Then we set $t([a_1], \dots, [a_n]) = t(i_k(b_1), \dots, i_k(b_n))$. This is well-defined, in the sense that whenever we have a collection $c_i \in [a_i]$ of elements in $A_{k'}$ such that $t(c_1, \dots, c_n)$ is well-defined, by the fact that each i_{k+1} is i_k -open, $t(c_1, \dots, c_n)$ will be the same element (up to identification by a map $i_{j,k}$) as $t(i_k(b_1), \dots, i_k(b_n))$.

The fact that A_ω satisfies all the axioms in $\mathbf{A}_0$ follows from the fact that, by construction, each of A_n validates such equations for all existing elements, and the fact that $\iota_{n+1}: A_{n+1} \rightarrow A_\omega$ is an i_n -open map: for each equational axiom $\lambda(\bar{x}) \approx \gamma(\bar{x})$, and $[a_1], \dots, [a_n]$, we take $b_1, \dots, b_n \in A_k$ large enough, and let $c_j = i_k(b_j)$ for each j . Then by construction in A_{k+1} :

$$\lambda(c_1, \dots, c_n) = \gamma(c_1, \dots, c_n),$$

and since ι_{k+1} is i_n -open, we have:

$$\begin{aligned} \lambda(\iota_{k+1}c_1, \dots, \iota_{k+1}c_n) &= \iota_{k+1}(\lambda(c_1, \dots, c_n)) \\ &= \iota_{k+1}(\gamma(c_1, \dots, c_n)) \\ &= \gamma(\iota_{k+1}c_1, \dots, \iota_{k+1}c_n); \end{aligned}$$

showing that $\lambda([a_1], \dots, [a_n]) = \gamma([a_1], \dots, [a_n])$ by the definition.

(2) As noted above, the fact that the assignment $f \mapsto f_\omega$ is functorial, and therefore, $(fg)_\omega = f_\omega g_\omega$ is a general categorical fact ([4, Remark 3.4]). the additional fact that f_ω is an $\mathcal{L}_0$ -homomorphism follows from the definitions given: for

each $[a_1], \dots, [a_n] \in A_\omega$, pick representatives $b_1, \dots, b_n \in A_k$ for k large enough; then:

$$\begin{aligned}
 f_\omega(t([a_1], \dots, [a_n])) &= [f_{k+1}(t(i_k^A(b_1), \dots, i_k^A(b_n)))] \\
 &= [(t(f_{k+1}i_k^A(b_1), \dots, f_{k+1}i_k^A(b_n)))] \\
 &= [(t(i_k^B f_k(b_1), \dots, i_k^B f_k(b_n)))] \\
 &= t([f_k(b_1)], \dots, [f_k(b_n)]) \\
 &= t(f_\omega[a_1], \dots, f_\omega[a_n])
 \end{aligned}$$

where the first equality follows from definition of f_ω ; the second from the fact that f_{k+1} is i_k^A -open; the third from the assumption of commutativity on f_{k+1} ; the fourth by the definition of t on B_ω ; and the final by definition of the limit map. This shows that f_ω is a $\mathcal{L}_0$ -homomorphism as desired.

(3) This follows essentially from applying iteratively Proposition 4.1.11: from the map $f: A \rightarrow B$, applying this Proposition, we obtain a sequence of maps $f_k: A_k \rightarrow B$ such that f_{k+1} is i_k -open (noting that B , being a $\mathbf{A}_0$ -algebra, is always f_k - $\mathbf{A}_0$). The map $\bar{f}: A_\omega \rightarrow B$ is the limit map, and will be a $\mathcal{L}_0$ -homomorphism by the exact same arguments as those given in (2). The uniqueness of $\bar{f}$ is immediate from the uniqueness of each extension f_k (granted by Proposition 4.1.11) and the uniqueness of directed colimit maps.

(4) By (1) and (2), the assignment:

$$\begin{aligned}
 \mathbf{F}: \mathbf{A} &\rightarrow \mathbf{A}_0 \\
 A &\mapsto A_\omega \\
 f &\mapsto f_\omega
 \end{aligned}$$

is a well-defined functor. For each $A \in \mathbf{A}$, we have an $\mathcal{L}$ -homomorphism $\iota_0: A \rightarrow A_\omega$. Therefore the statement of (3) proves precisely that $\mathbf{A}_0 \rightarrow \mathbf{A}$ is the right adjoint of $\mathbf{F}$ (see Definition 2.1.3). $\blacksquare$

Having such a left adjoint does not immediately entail a description of colimits of $\mathbf{A}_0$ in terms of colimits from $\mathbf{A}$ – despite left adjoints preserving colimits, this only works *in the image of the functor*. However, as we will see in Section 4.4, a similar methodology to the step-by-step can be employed which achieves this goal in great generality (see also Appendix A.3.3 for discussion on this).

4.2 Step-by-step Heyting algebras

In Section 4.1 we laid out the intended goals of representing our algebras using duality, and the general algebraic methodology for doing so in a step-by-step manner. Now we specialize all of this to the setting at hand.

The conditions placed on the arbitrary variety $\mathbf{A}$ are met by $\mathbf{DL}$, including the optional ones – every Heyting algebra is a distributive lattice (Theorem 2.1.15),

distributive lattices are locally finite; moreover, as we will see below (Proposition 4.2.8 and 4.2.12), the maps witnessing the general one-step are dual to injective homomorphisms. We check that the axioms do not involve any nesting of implications.

The axioms invoked in the definitions of one-step extensions can be listed in our case, as given in [57]: given a distributive lattice D , one axiomatizes the operation $\rightarrow: D \times D \rightarrow D$ by the following (see [57, Def. 5.1]):

Definition 4.2.1 (Heyting algebra axiomatization). Given D a distributive lattice, an operation $\rightarrow: D \times D \rightarrow D$ determines a relative pseudocomplement:

1. $a \rightarrow a = 1$;
2. $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$;
3. $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$;
4. $(a \rightarrow b) \wedge (b \rightarrow c) \leq a \rightarrow c$;
5. $a \leq a \rightarrow b$;
6. $a \wedge (a \rightarrow b) \leq b$.

Our task now boils down to finding the duals of the constructions to those of Section 4.1.3. Since the latter are specified by universal properties, once all definitions are properly in place, it will suffice to show that our desired constructions satisfy appropriate duals of such universal properties. The upshot will be that, as promised, we will obtain a very general method to describe all sorts of colimit constructions in Heyting algebras dually.

It should be noted that the general picture outlined in the following sections, involving the Vietoris functor $\mathcal{V}_r$ on Priestley spaces, was mentioned without proof in [57, Lemma 6.1].

4.2.1 g-openness

The dual to a g -partial algebra (Definition 4.1.7) – in our case, a g -Heyting algebra – is given in an obvious way:

Definition 4.2.2 (g -Esakia). Given continuous and order preserving map between Priestley spaces $g: X \rightarrow Y$, we say that X is g -Esakia if for all clopen upsets $U, V \subseteq Y$, $\downarrow(g^{-1}[U] \setminus g^{-1}[V])$ is open.

Note that in light of the description of relative pseudocomplements in Priestley spaces (see Table 2.1), we immediately get:

Lemma 4.2.3 (g -Heyting if and only if g -Esakia). A distributive lattice homomorphism $g: D_0 \rightarrow D$ is such that D is g -Heyting if and only if $\text{Spec}(D)$ is g^{-1} -Esakia.

The concept of g -openness has a more transparent dual representation, due to Ghilardi:

Definition 4.2.4 (g -open map, g -open set). Let $g: X \rightarrow Y$ be a continuous and order-preserving map between Priestley spaces where X is g -Esakia. A continuous and order-preserving map $f: Z \rightarrow X$, where Z is f -Esakia, is said to be g -open if:

$$\forall x \in Z, y \in X, (f(x) \leq y \rightarrow \exists x' \geq x, fg(x') = g(y)).$$

A subset $C \subseteq X$ is called g -open if the inclusion is a g -open map; explicitly,

$$\forall x \in C, y \in X, (x \leq y \rightarrow \exists x' \in C, g(x') = g(y)).$$

The following justifies this nomenclature:

Proposition 4.2.5 (Duality for g -open). Let $g: D_0 \rightarrow D$ and $f: D \rightarrow D'$ be distributive lattice homomorphisms. Then f is g -open if and only if f^{-1} is g^{-1} -open.

Proof. First first assume that f is g -open. For ease of notation, write $f_* = f^{-1}$ and $g_* = g^{-1}$. Throughout, we identify via duality $f = f_*^{-1}$ and $g = g_*^{-1}$. Assume that $x \in Z$ and $y \in X$ and $f_*(x) \leq y$. Assume towards a contradiction that

$$\uparrow x \cap f_*^{-1}g_*^{-1}[\uparrow g_*(y)] \cap f^{-1}g_*^{-1}[\downarrow g_*(y)] = \emptyset;$$

By Priestley duality, we have that $\uparrow g_*(y) = \bigcap_{g_*(y) \in U} U$, where U are clopen upsets, and similarly $\downarrow g_*(y) = \bigcap_{g_*(y) \in V} V$ where V are clopen downsets. So by compactness, we have for some U a clopen upset such that $g(y) \in U$, and V a clopen downset such that $g(y) \in V$, we have:

$$\uparrow x \cap f_*^{-1}g_*^{-1}[U] \cap f_*^{-1}g_*^{-1}[V] = \emptyset;$$

it follows that $\uparrow x \subseteq f_*^{-1}g_*^{-1}[-U \cup -V]$, and so $x \in \square(f_*^{-1}g_*^{-1}[-U \cup -V])$. Because $f = f_*^{-1}$ is g -open, we have that then $x \in f_*^{-1}[\square(g_*^{-1}[-U \cup -V])]$, so $\uparrow f_*(x) \subseteq g_*^{-1}[-U \cup -V]$ – a contradiction, since $f_*(x) \leq y$, and $g_*(y) \in U \cap V$. Hence we have that there exists a point $x' \geq x$ such that $g_*(y) \geq f_*g_*(x) \geq g_*(y)$.

Conversely, assume that f_* is g_* -open. Note that by order-preservation of the maps f_* and g_* , we always have that:

$$f_*^{-1}[g_*^{-1}[U] \rightarrow g_*^{-1}[V]] \subseteq f_*^{-1}[g_*^{-1}[U]] \rightarrow f_*^{-1}[g_*^{-1}[V]];$$

we show the other inclusion holds as well. Assume that $x \in f_*^{-1}[g_*^{-1}[U]] \rightarrow f_*^{-1}[g_*^{-1}[V]]$. Suppose that $f_*(x) \leq y$, and $y \in g_*^{-1}[U]$. Because f_* is g_* -open, let $x' \geq x$ such that $f_*g_*(x') = g_*(y)$; then $x' \in f_*^{-1}g_*^{-1}[U]$, so $x' \in f_*^{-1}g_*^{-1}[V]$, which by assumption means that $y \in g_*^{-1}[V]$. ■

4.2.2 Vietoris g-spaces

Let us consider the following objects:

- $g: D_0 \rightarrow D$ is a distributive lattice homomorphism, where D is g -Heyting;
- the one-step extension of a distributive lattice D has the shape $D \oplus_{\mathbf{DL}} \mathcal{F}_{\mathbf{DL}}(D \times D)/\Theta$, where
- Θ contains the axioms from Definition 4.2.1, together with the g -openness axioms identifying $g(a) \rightarrow g(b)$ with $g(a \rightarrow b)$ in $\mathcal{F}_{\mathbf{DL}}(D \times D)$.

Using Propositions 4.1.1 and 4.1.4 we can dualize this construction, to obtain a space C_X , for $X = \mathbf{Spec}(D)$. Explicitly, C_X is the closed subspace of $X \times \mathbf{2}_{\leq}^X$ which is determined by the congruence Θ (as explained in Proposition 2.1.64). It is from this object that the analysis from [57, Thm. 4.2] was carried out.

We will however take a more indirect route through a Vietoris construction, which we will show, by categorical arguments, to coincide with the above space (see Remark 4.2.14 for the connection between the two spaces).

Definition 4.2.6 (Vietoris functor). [173, Lem. 3.9]) The *Vietoris functor* is the functor $\mathcal{V}: \mathbf{Pries} \rightarrow \mathbf{Pries}$ which assigns to each Priestley space $(X, \leq)$ the Priestley space $(\mathcal{V}(X), \supseteq)$ (see Lemma 2.1.79), and sends each continuous and order-preserving map $f: X \rightarrow Y$ between Priestley spaces to:

$$\begin{aligned} \mathcal{V}(f): \mathcal{V}(X) &\rightarrow \mathcal{V}(Y) \\ C &\mapsto f[C]. \end{aligned}$$

Definition 4.2.7 (Rooted Vietoris). Let X be a Priestley space. We denote by $\mathcal{V}_r(X)$ the *rooted Vietoris*, the subspace:

$$\mathcal{V}_r(X) = \{C \in \mathcal{V}(X) : C \text{ is rooted}\}.$$

We see $\mathcal{V}_r$ as a functor by restriction of $\mathcal{V}$: whenever $f: X \rightarrow Y$ is a continuous and order-preserving map, $\mathcal{V}_r(f)[C] = f[C]$.

Proposition 4.2.8 (Basic properties of rooted Vietoris). Given a Priestley space X , we have:

1. The space $(\mathcal{V}_r(X), \supseteq)$, is a Priestley space.
2. The map $r: \mathcal{V}_r(X) \rightarrow X$ sending a rooted closed subset to its root is continuous, order-preserving, and surjective.

Proof. (1) To show that $\mathcal{V}_r(X)$ is a Priestley space, by Proposition 2.1.47.(4) it suffices to show that $\mathcal{V}_r(X)$ is closed. For that purpose, consider the following two subsets of $\mathcal{V}(X) \times X$:

$$\text{Subroot}(X) = \{(C, x) : C \subseteq \uparrow x\} \text{ and } \ni_X = \{(C, x) : x \in C\}.$$

We will show that both subsets are closed. We begin with $\ni_X$: if (C, x) is such that $r \notin C$, then $\{x\} \cap C = \emptyset$. Since X is a Stone space, let W be a clopen subset such that $x \in W$ and $C \subseteq X - W$. Then we can consider the subset $\llbracket X \setminus W \rrbracket \times W$, which by construction is a neighbourhood of (C, x) . If $(D, y) \in \llbracket X \setminus W \rrbracket \times W$, then certainly $y \notin D$, since $y \in W$ whilst $D \subseteq W$. So we conclude that $\ni_X$ is closed.

We now show $\text{Subroot}(X)$ is closed: suppose that $(C, x) \notin \text{Subroot}(X)$. By definition, then, $C \not\subseteq \uparrow x$, so there is a point $y \in C$ such that $x \not\leq y$. By the Priestley separation axiom, there is a clopen upset U such that $x \in U$ and $y \notin U$. So consider $\langle X \setminus U \rangle \times U$, which is a neighbourhood of (C, x) . If $(D, y) \in \langle X \setminus U \rangle \times U$, then there is some $z \in D$ such that $z \notin U$. Since U is an upset, we must have $y \not\leq z$, thus $D \not\subseteq \uparrow y$. This shows that $\text{Subroot}(X)$ is closed.

Now note that,

$$\mathcal{V}_r(X) = \pi_{\mathcal{V}(X)}[\text{Subroot}(X) \cap \ni_X].$$

Since π is a closed map (given it is a continuous surjection between Stone spaces), we then have that $\mathcal{V}_r(X)$ is a closed subspace of $\mathcal{V}(X)$, as desired.

(2) To show that r is continuous, it suffices to show that the preimage of a clopen upset is clopen (see Remark 2.1.48). But note that given $U \subseteq X$ a clopen upset, we have

$$r^{-1}[U] = \{C \in \mathcal{V}_r(X) : r(C) \in U\} = \llbracket U \rrbracket,$$

where the equality follows because U is an upset. Thus r is continuous. Moreover, it is surjective since for each $x \in X$, the set $\uparrow x$ is a closed set containing x as a root. ■

We could at this point show that $(\mathcal{V}_r(X), r)$ satisfies the dual property to (Univ_0) (Definition 4.1.8). We will instead obtain the relevant facts for $\mathcal{V}_r$ as an edge case of a more general definition.

Definition 4.2.9 (*g-open Vietoris*). Let $g: X \rightarrow Y$ be a continuous and order-preserving map between Priestley spaces, where X is *g*-Esakia. We denote by

$$\mathcal{V}_g(X) = \{C \in \mathcal{V}_r(X) : C \text{ is } g\text{-open}\}.$$

We moreover let $r_g: \mathcal{V}_g(X) \rightarrow X$ be the function assigning each rooted *g*-open subset to its root.

Remark 4.2.10 (Remarks on *g*-open sets).

1. Note that if $g: X \rightarrow \{\bullet\}$ is the terminal map to a single element Priestley space, then $\mathcal{V}_g(X) = \mathcal{V}_r(X)$.
2. Throughout, given a subset $C \subseteq P$ of a poset and $x \in C$, we write $C_x = \uparrow x \cap C$. Note that if C is *g*-open, then C_x will again be *g*-open.

We will now characterize the set of *g*-open subsets.

Lemma 4.2.11 (Closed characterization of g -open sets). Let $g: X \rightarrow Y$ be a continuous and order-preserving map, where X is g -Esakia. A closed subset K of X is g -open if and only if for every clopen upset A , and all clopen upsets B, C of Y , we have:

$$K \in \llbracket A \Rightarrow g^{-1}[B \Rightarrow C] \rrbracket \Rightarrow \llbracket A \Rightarrow (g^{-1}[B] \rightarrow g^{-1}[C]) \rrbracket.$$

Proof. Assume first that K is g -open, and $K \in \llbracket A \Rightarrow g^{-1}[B \Rightarrow C] \rrbracket$. Suppose that $x \in K \cap A$. We want to show that $x \in g^{-1}[B] \rightarrow g^{-1}[C]$. So let $x \leq x'$ such that $g(x') \in B$; by the g -openness condition let $y \in C$ be such that $x \leq y$, $g(x') = g(y)$. Since A is an upset, and $x \in A$, then $y \in A$, and so $y \in g^{-1}[B]$ implies $y \in g^{-1}[C]$, which implies that $x' \in g^{-1}[C]$.

For the converse, assume that $x \in K$. We need to show that:

$$\uparrow x \subseteq g^{-1}[g[K \cap \uparrow x]].$$

Since $g[K \cap \uparrow x]$ is closed, and the space is Stone, we can write it as the intersection of the clopen subsets of Y in which it is contained. Thus:

$$g^{-1}[g[K \cap \uparrow x]] = \bigcap \{g^{-1}[B \Rightarrow C] : g[K \cap \uparrow x] \subseteq B \Rightarrow C\},$$

where B, C are clopen upsets of Y . Thus it is enough to prove that whenever $x \leq y$, then for every such B, C , $y \in g^{-1}[B \Rightarrow C]$. Indeed, given such $B \Rightarrow C$, since $g[K \cap \uparrow x] \subseteq B \Rightarrow C$, then $K \cap \uparrow x \subseteq g^{-1}[B \Rightarrow C]$. Since the latter is clopen, and $\uparrow x = \bigcap_{x \in A} A$, where A is a clopen upset, we conclude that there is a clopen upset A such that $K \cap A \subseteq g^{-1}[B \Rightarrow C]$, i.e., $K \subseteq A \Rightarrow g^{-1}[B \Rightarrow C]$. By hypothesis, then $K \subseteq A \Rightarrow (g^{-1}[B] \rightarrow g^{-1}[C])$.

So now, if $x \leq y$, then note that $y \in A$ by construction, so if $g(y) \in B$, then because $x \in g^{-1}[B] \rightarrow g^{-1}[C]$, then $g(y) \in C$. This concludes the proof. ■

Proposition 4.2.12 (Basic properties of g -open Vietoris). Given $g: X \rightarrow Y$ a continuous and order-preserving map, and X a g -Esakia space, we have:

1. The space $\mathcal{V}_g(X) \subseteq \mathcal{V}_r(X)$ is closed, and hence, $(\mathcal{V}_g(X), \supseteq)$ is a Priestley space.
2. The restriction $r_g: \mathcal{V}_g(X) \rightarrow X$ is a surjective, continuous, order-preserving map.

Proof. (1) Let $V(g)$ denote the set of closed subsets of X (not necessarily rooted) which are g -open. By Lemma 4.2.11, we have that:

$$V(g) = \bigcap_{A \in \text{CloUp}(X), B, C \in \text{CloUp}(Y)} \llbracket A \Rightarrow g^{-1}[B \Rightarrow C] \rrbracket \Rightarrow \llbracket A \Rightarrow (g^{-1}[B] \rightarrow g^{-1}[C]) \rrbracket.$$

Since all these sets are closed, we have that $V(g)$ itself is closed.

Now notice that $\mathcal{V}_g(X) = \mathcal{V}_r(X) \cap V(g)$. Since by Lemma 4.2.8, $\mathcal{V}_r(X)$ is closed, it follows that $\mathcal{V}_g(X)$ is a closed subspace, i.e., $(\mathcal{V}_g(X), \supseteq)$ is a Priestley space.

(2) The fact that r_g is continuous is obvious as it is the restriction of a continuous function. The fact that it is surjective follows from Remark 4.2.10.(2), since $\uparrow x$ will be a g -open rooted subset. ■

4.2.3 Dual one-step extensions and Ghilardi complexes

With the definitions of the previous section, we are ready to prove the dual of (Univ_g) (Proposition 4.1.11):

Proposition 4.2.13 (Dual one-step extension). Let $g: X \rightarrow X_0$ be a continuous and order-preserving map where X is g -Esakia. Let $f: Y \rightarrow X$ be a continuous and order-preserving map where Y is f -Esakia, and f is g -open. Then there exists a unique continuous and order-preserving map $\bar{f}: Y \rightarrow \mathcal{V}_g(X)$ such that $\bar{f}$ is r_g -open, and $r_g \bar{f} = f$.

Proof. First we show *existence* of $\bar{f}$. For that purpose, we define

$$\begin{aligned} \bar{f}: Y &\rightarrow \mathcal{V}_g(X) \\ y &\mapsto f[\uparrow y]. \end{aligned}$$

Such a function is well-defined: because f is a continuous map between Stone spaces, the image of a closed set is closed. Moreover, $f[\uparrow y]$ is rooted at $f(y)$. Moreover, it is g -open since f is a g -open map: whenever $y \leq y'$ and $f(y') \leq x$, then by the definition of g -openness, there is some $y'' \geq y'$ such that $gf(y'') = g(x)$. That means that $f(y'')$ is the witness to g -openness as desired.

It is clear to see that $\bar{f}$ is order-preserving. Moreover, since $f(y)$ is the root of $\bar{f}(y)$, we see that $r_g \bar{f} = f$.

We now show that $\bar{f}$ is continuous. For that purpose, it suffices to show that given $W \subseteq X$ a clopen subset, $(\bar{f})^{-1}[\llbracket W \rrbracket]$ is clopen; the latter corresponds to

$$(\bar{f})^{-1}[\llbracket W \rrbracket] = \{y \in Y : f[\uparrow y] \subseteq W\} = \Box f^{-1}[W],$$

and the latter is clopen because Y is f -Esakia.

Next we show that $\bar{f}$ is r_g -open: if $f[\uparrow y] \supseteq C$, let $r_g(C) = x'$; by assumption, there is some $y' \geq y$ such that $f(y') = x'$. Then $\bar{f}(y) \supseteq \bar{f}(y')$ and $r_g \bar{f}(y') = y$; this establishes that $\bar{f}$ is r_g -open. From this we conclude that $\bar{f}$ in the given conditions exists.

We now prove *uniqueness*. Assume that $f^*: Y \rightarrow \mathcal{V}_g(X)$ is another map in the same conditions. We will show that then $f^*(y) = \bar{f}(y)$. For the right to left inclusion, assume that $y \leq y'$, and let us show that $f(y') \in f^*(y)$. Since $f(y') \in \bar{f}(y)$, we have that $\bar{f}(y)_{f(y')} = \bar{f}(y) \cap \uparrow f(y')$ is g -open by Remark 4.2.10.(2).

Because f^* is r_g -open, there is some $y'' \geq y'$ such that $r_g f^*(y'') = r_g \bar{f}(y)_{f(y')} = f(y')$; since $f^*(y') \supseteq f^*(y'')$, this shows that $f(y') \in f^*(y')$, and so $\bar{f}(y) \subseteq f^*(y)$.

In the opposite direction, assume that $x \in f^*(y)$; then since $f^*(y) \supseteq f^*(y)_x$ (since the latter is again a g -open and rooted subset), because f^* is r_g -open there is some $y' \geq y$ such that $r_g f^*(y') = r_g f^*(y)_x = x$. Thus because $r_g f^*(y') = f(y')$, we have $f(y') = x$. This shows that $x \in f[\uparrow x]$. We conclude that $f^*(y) = \bar{f}(y)$, proving uniqueness and concluding the proof. ■

Remark 4.2.14 (Explicit isomorphism). Whilst we know that an isomorphism between $\mathbf{ClopUp}(\mathcal{V}_g(X))$ and the algebra described in Definition 4.1.10 exists simply because both satisfy the same universal property, it might be interesting to see such an explicit isomorphism, dually: given $g: X \rightarrow X_0$ a continuous and order-preserving map, we have a map

$$f_{(-)}: \mathcal{V}_g(X) \rightarrow X \times \mathbf{2}_{\leq}^X,$$

which sends x to the function $f_C: D \times D \rightarrow \mathbf{2}_{\leq}$, defined by

$$f_x(a, b) = 1 \iff x \in \llbracket \varphi(a) \Rightarrow \varphi(b) \rrbracket.$$

The map $f_{(-)}$, can be shown to be continuous, and yield an isomorphism when restricted to the subspace C_X , corresponding to the axioms Θ described at the beginning of Section 4.2.2.

Having proven Proposition 4.2.13, we will now define the duals of Definition 4.1.12 and Definition 4.1.13.

Definition 4.2.15 (Ghilardi complex). Let X be a Priestley space. The *Ghilardi complex over X* , denoted $\mathcal{G}(X) = (X_n, r_n)_{n \in \omega}$ is the inverse system:

$$X_0 \xleftarrow{r_0} X_1 \xleftarrow{r_1} X_2 \dots,$$

where:

1. $X_0 = X$ and $X_1 = \mathcal{V}_r(X)$;
2. $r_0 = r$ is the root map from $\mathcal{V}_r(X) \rightarrow X$;
3. for $k \geq 1$, $X_{k+1} = \mathcal{V}_{r_k}(X_k)$, and $r_{k+1} = r_{r_k}$.

We denote by $\mathcal{V}_G(X)$ the inverse limit of this inverse system in the category of Priestley spaces.

Notation 4.2.1 (Ghilardi complex notation). When handling Ghilardi complexes, we fix the following notation: given X a Priestley space, and $\mathcal{G}(X) = (X_n, r_n)$ the Ghilardi complex over X , we will write $\pi_n: \mathcal{V}_G(X) \rightarrow X_n$ for the projection onto the n -coordinate. Given $n \leq m \in \omega$ we will let $r_{n,m}: X_m \rightarrow X_n$ be equal to the identity on X_n if $n = m$, and otherwise, be equal to $r_{m-1} \circ \dots \circ r_n$.

Explicitly, elements of $\mathcal{V}_G(X)$ will be infinite sequences $(x_n)_{n \in \omega}$ where $x_n \in X_n$, subject to the condition that whenever $n \leq m$, $x_n = r_{m,n}(x_m)$.

Example 4.2.16 (Example Ghilardi complex). Let us calculate the first four steps of $\mathcal{G}(\mathbf{2}_{\leq})$ in Figure 4.2. As proven below (Theorem 4.3.1), this computes some approximations of the dual of the Rieger-Nishimura ladder (see Figure 3.1). Note however that the individual steps of the step-by-step construction do not in general agree with the universal model construction provided in Chapter 3.

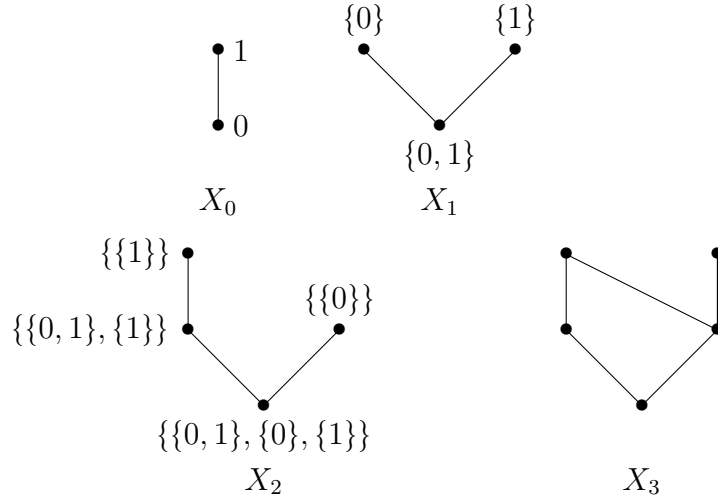


Figure 4.2: Four steps of the construction of the Rieger-Nishimura

Remark 4.2.17 (Intuitive meaning of Ghilardi complexes). Example 4.2.16 shows how one may think of Ghilardi complexes intuitively, in the same lines as the constructions described for CPC and PL from Section 4.1.2: in both cases, we associate to free algebras all possible (irreducible) models. When doing so over an arbitrary Priestley space X , the intended meaning is that X constitutes the poset of valuations. In the special case where $X = \mathbf{2}_{\leq}$ we are taking such values to be the classical valuations.

The r_k -maps can be seen in a *game-theoretic* way: they encode the equality of resolution under k -bisimulation games (see Section 2.3.3). Specifically, if we see two points $\mathfrak{M}, \mathfrak{N} \in X_{n+1}$ as models (as subsets of X_n), pointed at $x = r_n(\mathfrak{M})$ and $y = r_n(\mathfrak{N})$, then for each $k \leq n$, $(\mathfrak{M}, x) \sim_k (\mathfrak{N}, y)$ if and only if $r_{n,k}(\mathfrak{M}) = r_{n,k}(\mathfrak{N})$.

Definition 4.2.18 (Dual limit maps). Given X and Y two Priestley spaces, a *Ghilardi morphism* between $\mathcal{G}(X) = (X_n, r_n^X)$ and $\mathcal{G}(Y) = (Y_n, r_n^Y)$ is a sequence $(f_n)_{n \in \omega}$ of maps, such that:

1. For each n , $f_n: X_n \rightarrow Y_n$ is a continuous and order-preserving maps;
2. For each n , f_{n+1} is r_n^Y -open;

3. For each n , $r_n^Y f_{n+1} = f_n r_n^X$.

Given such a morphism, we let $f_\omega: \mathcal{V}_G(X) \rightarrow \mathcal{V}_G(Y)$ be the limit map.

In light of Definitions 4.2.15 and 4.2.18, we can conclude the following Theorem, by dualizing Theorem 4.1.14. Recall that since we are working in a dual category we will be describing the *right adjoint* to the inclusion functor $\mathbf{Esa} \rightarrow \mathbf{Pries}$:

Theorem 4.2.19 (Right adjoint from $\mathbf{Esa}$ to $\mathbf{Pries}$). *Given $X \in \mathbf{Pries}$, we have:*

1. *The space $\mathcal{V}_G(X)$ is an Esakia space.*
2. *Whenever $f: X \rightarrow Y$ is a continuous and order-preserving map where $Y \in \mathbf{Pries}$, then f_ω is a continuous p -morphism. Moreover, $(fg)_\omega = f_\omega g_\omega$.*
3. *The space $\mathcal{V}_G(X)$ has the following universal property: for each continuous and order-preserving map $f: Y \rightarrow X$ where Y is an Esakia space, there is a unique continuous p -morphism $\bar{f}: Y \rightarrow \mathcal{V}_G(X)$ such that $\pi_0 \bar{f} = f$.*
4. *The functor $\mathcal{V}_G: \mathbf{Pries} \rightarrow \mathbf{Esa}$ is left adjoint to the inclusion functor $\mathbf{Esa} \rightarrow \mathbf{Pries}$.*

Proof. Simply trace Theorem 4.1.14 along the duality dictionary in Table 2.1, using the dual one-step extension established in Proposition 4.2.13. ■

4.3 Free Heyting algebras, normal forms and relative anarchy

Having established the dual of the step-by-step construction in Theorem 4.2.19, we can now proceed to analyze intuitionistic logic in the same way we did CPC and PL in Section 4.1.2. In this section we will ask the same questions, namely:

1. What does the dual perspective on free algebras tell us?
2. What does the dual perspective tell us about normal forms for IPC?
3. What kind of relative anarchy results hold in this context?

We begin by describing free algebras dually:

Theorem 4.3.1 (Duals of free Heyting algebras). *For each set S , the algebra $\mathcal{F}_{\mathbf{HA}}(S)$ is dual to $\mathcal{V}_G(\mathbf{2}_{\leq}^S)$.*

Proof. In Proposition 4.1.1 we showed that $\mathbf{2}_{\leq}^S$ is dual to the free distributive lattice on S many generators, which gives the object part of the left adjoint to the forgetful functor $\mathbf{DL} \rightarrow \mathbf{Set}$; by Theorem 4.2.19 and 4.1.14, the dual of $\mathcal{V}_G$ gives the left adjoint to the forgetful functor $\mathbf{HA} \rightarrow \mathbf{DL}$. Thus by uniqueness of adjoints, the composite functor yields the dual of $\mathcal{F}_{\mathbf{HA}}(S)$. ■

We begin with a lemma of a more general nature, concerning adjoints between the posets; we stress that such adjoints *are not guaranteed to be continuous* (see Section 4.5):

Lemma 4.3.2 (Properties of adjoints in Ghilardi complexes). Let X be a Priestley space and $\mathcal{G}(X)$ the Ghilardi complex over X .

1. For each n , the map

$$\begin{aligned} \uparrow_n: X_n &\rightarrow X_{n+1} \\ x &\mapsto \uparrow x \end{aligned}$$

is left adjoint of the map $r_n: X_{n+1} \rightarrow X_n$.

2. For each n , and $x \in X_n$, define $l_n(x)(k) = r_{n,k}(x)$ for $k \leq n$, and for $k \geq n$, by induction, $l_n(x)(k+1) = \uparrow_k(l_n(x)(k))$. Then $l_n(x) \in \mathcal{V}_G(X)$, and the map $l_n: X_n \rightarrow \mathcal{V}_G(X)$ is the left adjoint to the projection $\pi_n: \mathcal{V}_G(X) \rightarrow X_n$.
3. The maps $r_n: X_{n+1} \rightarrow X_n$ are co-p-morphisms (Definition 2.1.57).

Proof. (1) We show that for each $C \in X_{n+1}$, and $x \in X_n$, we have

$$\uparrow_n x \leq C \iff x \leq r_n(C).$$

Recall that the order is reverse inclusion. Now by definition of $\mathcal{V}_r$ (see Definition 4.2.7), if C is rooted, $C \subseteq \uparrow r_n(C)$; so if $x \leq r_n(C)$, certainly $\uparrow_n x \supseteq \uparrow_n r_n(C) \supseteq C$. Conversely if $\uparrow_n x \supseteq C$, then by definition, we must have $x \leq r_n(C)$. This establishes the result.

(2) Note that the definition of l_n as a map ensures that for each $x \in X_n$, $l_n(x) \in \mathcal{V}_G(X)$. To see it is the left adjoint, we show that for each $\bar{y} \in \mathcal{V}_G(X)$, we have

$$l_n(x) \leq \bar{y} \iff x \leq \pi_n(\bar{y}).$$

From left to right this follows by definition of the order being pointwise and the definition of l_n . Right to left, notice that by applying (1) for each $k \geq n$, we obtain by induction that for each such k , $l_n(x)(k) \leq \bar{y}(k)$; thus $l_n(x) \leq \bar{y}$ as desired.

(3) To prove that $r_n(C)$ is a co-p-morphism, assume that $x \in X_{n+1}$ and $y \in X_n$, and $r_n(x) \geq y$. By (1), this means that $\uparrow_n y \leq x$, and by definition, $r_n(\uparrow_n y) = y$. ■

Definition 4.3.3 (Join-irreducibly generated distributive lattice). A distributive lattice D is said to be *join-irreducibly generated* if for each $a \in D$, there are join-irreducible elements $b_1, \dots, b_n \in D$ such that $a = b_1 \vee \dots \vee b_n$.

With this we obtain some structural results on finitely generated Heyting algebras, including an often cited lemma from [251] and the titular result from [127]:

Proposition 4.3.4 (Urquhart and Ghilardi’s theorems). For each n , the following hold:

1. The algebra $\mathcal{F}_{\mathbf{HA}}(n)$ is join-irreducibly generated.
2. The algebra $\mathcal{F}_{\mathbf{HA}}(n)$ is a bi-Heyting algebra.

Proof. (1) by Theorem 4.3.1, we have that $\mathcal{V}_G(\mathbf{2}_{\leq}^n)$ is dual to $\mathcal{F}_{\mathbf{HA}}(n)$. By definition of the topology on inverse systems (see Section 2.1.5), clopen upsets of this set are of the form $\pi_k^{-1}[V]$ for V a (clopen) upset of $(\mathbf{2}_{\leq}^n)_k$. Since the latter is a finite poset, we have that $V = \bigcup_{x \in V} \uparrow x$ for finitely many $x \in V$; note that each of the subsets $\uparrow x$ is join-irreducible. We show that

$$\pi_k^{-1}[\uparrow x] \text{ is join-irreducible;}$$

from this, we immediately obtain the result, since $\pi_k^{-1}[V]$ will be a finite join of join-irreducibles. To see this, we show that $\pi_k^{-1}[\uparrow x] = \uparrow l_k(x)$, where $l_k(x)$ is the image of the adjoint map from Lemma 4.3.2.(2). Indeed, for each $\bar{y} \in \mathcal{V}_G(\mathbf{2}_{\leq}^n)$, note that:

$$\begin{aligned} \bar{y} \in \pi_k^{-1}[\uparrow x] &\iff \pi_k(\bar{y}) \in \uparrow x \\ &\iff x \leq \pi_k(\bar{y}) \\ &\iff l_k(x) \leq \bar{y}; \end{aligned}$$

where the equivalences follow from l_k being left adjoint to the projection. This shows the equivalence, proving that $\mathcal{F}_{\mathbf{HA}}(n)$ is join-irreducibly generated.

(2) Since $\mathbf{2}_{\leq}^n$ is finite, and consequently $(\mathbf{2}_{\leq}^n)_k$ will always be finite, all of them are bi-Esakia. By Lemma 4.3.2.(3) all the maps r_n are moreover co-p-morphisms. Hence by Proposition 2.1.74, $\mathcal{V}_G(\mathbf{2}_{\leq}^n)$ will be bi-Esakia as well. ■

Remark 4.3.5 (Not meet-irreducibly generated). Given free distributive lattices are both join-irreducibly generated and also *meet-irreducibly generated* – every element is a meet of meet-irreducibles – it is reasonable to ask whether this is true of $\mathcal{F}_{\mathbf{HA}}(n)$. As it turns out, this follows from the fact that the Bellissima algebra $\text{Bel}(n)$ (Definition 3.4.12) is different from the free algebra (see Proposition A.2.8 in the Appendix).

Proposition 4.3.4 can be extended further, by using the uniform interpolation theorem: we recall from Theorem 2.3.28, and the characterization of uniform interpolation therein discussed, that each of the inclusions $i_n: \mathcal{F}(n) \rightarrow \mathcal{F}(n+1)$ has a right adjoint $\forall_i: \mathcal{F}(n+1) \rightarrow \mathcal{F}(n)$. Moreover, by Proposition 2.3.34, such an adjoint distributes over disjunctions. Moreover, we have the following fact⁵:

⁵This fact, using the slightly stronger assumption that $\forall_i$ distributes over disjunction, should be credited to Silvio Ghilardi, from whom we learned this argument, and which seems to not have appeared in print before. Also the fact that $\mathcal{F}_{\mathbf{HA}}(\omega)$ is a bi-Heyting algebra following is owed to Ghilardi. The minor sharpening given here will turn out useful in Chapter 5, Section 5.1.4. We also remark that for bi-Heyting algebras the converse to Proposition 4.3.6 likewise holds, giving us a characterization of those homomorphisms with adjoints.

Proposition 4.3.6 (Frobenius and co-Heyting). Let $i: H \rightarrow H'$ be an injective homomorphism of Heyting algebras, and assume that it has a right adjoint $\forall_i$ which satisfies the Frobenius Reciprocity/Constant Domain axiom, $\forall_i(a \vee f(b)) = \forall_i a \vee b$. Then i is a co-Heyting homomorphism, i.e., it preserves all existing coimplications.

Proof. Assuming that $\forall_i$ satisfies the Frobenius reciprocity axiom, taking $a, b \in H$, such that we have $a - b \in H$, we show that $i(a - b) = i(a) - i(b) \in H'$. Indeed, for each $c \in H'$, we need to show

$$i(a - b) \leq c \iff i(a) \leq i(b) \vee c.$$

We focus on the harder right to left implication. Assuming that $i(a) \leq i(b) \vee c$, then $a = \forall_i i(a) \leq \forall_i(i(b) \vee c) = b \vee \forall_i c$, where the first equality follows from i being an injective homomorphism, and the second from the condition. Then it follows that $a - b \leq f \forall_i c$, so $i(a - b) \leq i f \forall_i c \leq c$ by the unit condition for poset adjoints. ■

From this we obtain the following:

Theorem 4.3.7 (Free Heyting is bi-Heyting and join-irreducibly generated). *The free Heyting algebra $\mathcal{F}_{\mathbf{HA}}(\omega)$ is a bi-Heyting algebra, and it is join-irreducibly generated.*

Proof. We know that $\mathcal{F}_{\mathbf{HA}}(\omega)$ is the directed colimit of the maps $i_n: \mathcal{F}(n) \rightarrow \mathcal{F}(n+1)$. By Proposition 4.3.4.(2), $\mathcal{F}(n)$ is always a bi-Heyting algebra; by Proposition 4.3.6, and Proposition 2.3.34, we know that i_n is a bi-Heyting homomorphism. Thus we can define the co-implications on the directed colimit, and they will be well-defined.

The fact that $\mathcal{F}_{\mathbf{HA}}(\omega)$ is join-irreducibly generated follows similarly: we show that for each $\varphi \in \mathcal{F}(n)$, if $\iota_n: \mathcal{F}(n) \rightarrow \mathcal{F}(\omega)$ is the inclusion, then $\iota_n(\varphi)$ is still join-irreducible; by the definition of directed colimits, this will make every element of the directed colimit be a join of join-irreducibles.

To see that it is join-irreducible, equivalently we show that it is join-prime in the free Heyting algebra (Remark 2.1.19). For this, it suffices to show that $i_n(\varphi) \in \mathcal{F}(n+1)$ remains join-prime. So assume that $i_n(\varphi) = \psi \vee \chi$. Applying the right adjoint, we have:

$$\varphi = \forall_i i_n(\varphi) = \forall_i(\psi \vee \chi) = \forall_i \psi \vee \forall_i \chi.$$

Since φ is join-prime, then $\varphi = \forall_i \psi$ or $\varphi = \forall_i \chi$. Thus either $i_n(\varphi) \leq i_n(\forall_i \psi) \leq \psi$ or $i_n(\varphi) \leq i_n(\forall_i \chi) \leq \chi$. This shows the result. ■

As an immediate consequence we have:

Corollary 4.3.8 (bi-Heyting anarchy). *The variety $\mathbf{HA}$ is biHA-anarchic.*

We conclude by recovering the normal forms for intuitionistic logic. This will be done by fixing the number of propositional variables:

Definition 4.3.9 (Normal forms of IPC). Let $\bar{p}$ be a set of propositional variables. We define by induction on n the class of n -normal forms for $\bar{p}$, by recourse to the poset $(\mathbf{2}_{\leq}^{\bar{p}})_n$:

1. For $n = 0$, the normal forms will be the join-irreducible formulas over $\bar{p}$ (Definition 4.1.2). Each such formula corresponds to $\uparrow x$ for a unique $x \in \mathbf{2}_{\leq}^{\bar{p}}$, and we denote it by χ_x ;
2. For each n , and each $x \in (\mathbf{2}_{\leq}^{\bar{p}})_{n+1}$, we define

$$\chi_x := \bigwedge_{y \not\leq x} \left(\chi_y \rightarrow \bigvee_{w \not\leq y} \chi_w \right).$$

By the way these normal forms are defined, and Theorem 4.3.1, the following is immediate:

Proposition 4.3.10 (Intuitionistic normal form). For each formula $\varphi(\bar{p})$ in the language $\text{Form}(\bar{p})$, and each implication depth n such that $\varphi(\bar{p})$ has implication depth at most n , there is a unique finite set S of n -normal forms such that $\varphi(\bar{p}) \equiv \bigvee_{\chi_x \in S} \chi_x$.

Proof. Since the formula $\varphi(\bar{p})$ can be seen as an element of the free algebra, and thus, dually, as a clopen upset in $\mathcal{V}_G(\mathbf{2}_{\leq}^{\bar{p}})$, we can find some $V \subseteq (\mathbf{2}_{\leq}^{\bar{p}})_k$ such that $\pi_k^{-1}[V]$ is precisely the clopen upset corresponding to the formula. Thus, by taking the minimal elements $x \in V$, we obtain that $\pi_k^{-1}[V] = \bigcup_{x \in \min(V)} \pi_k^{-1}[\uparrow x]$. The latter are join-irreducible, and correspond to χ_x . This set is unique once we fix these parameters. Note moreover that the only difference between such elements and considering it in higher-implication depth is that rather than $\uparrow x$, one would take its iterated adjoints via the description given in Proposition 4.3.2. ■

4.4 Duality for free constructions of Heyting algebras via Ghilardi complexes

Having applied Ghilardi complexes to the study of free algebras, in this section we show how they can yield descriptions of a number of general constructions in universal algebra and category theory. As a first step, we need a very modest generalization of the notion of Ghilardi complex previously employed; since this generalization was not needed for free algebras, we opt to make a separate definition here. The added generality will be beneficial in allowing us to systematically “change base”.

Definition 4.4.1 (g-Ghilardi complex). Let $g: X \rightarrow Y$ be a continuous and order-preserving map between Priestley spaces, where X is g -Esakia. The g -Ghilardi complex over X , denoted $\mathcal{G}^g(X) = (X_n, r_n)_{n \in \omega}$ is the inverse system:

$$X_0^g \xleftarrow{r_0} X_1^g \xleftarrow{r_1} X_2^g \dots,$$

where:

1. $X_0 = X$ and $X_1 = \mathcal{V}_g(X)$;
2. $r_0 = r_g$ is the root map from $\mathcal{V}_g(X) \rightarrow X$;
3. for $k \geq 1$, we set $X_{k+1} = \mathcal{V}_{r_k}(X_k)$, and $r_{k+1} = r_{r_k}$.

We denote by $\mathcal{V}_G^g(X)$ the inverse limit of this system in the category of Priestley spaces.

Example 4.4.2 (Ghilardi complexes yield g-Ghilardi complexes). Let X be any Priestley space, and $\mathcal{G}(X) = (X_n, r_n)$ the Ghilardi complex over X . Then the sequence $(X_{n+1}, r_{n+1})_{n \in \omega}$ is the r_0 -Ghilardi complex over X_1 .

Example 4.4.3 (All Esakia spaces). Given that g can be any map, note that we can consider a g -Ghilardi complex containing copies of the same space. Let $g: X \rightarrow X$ be the identity map. If X is a general Priestley space, $\mathcal{V}_g(X)$ will add as many relative pseudocomplements as necessary; but if X is an Esakia space, then in fact

$$\mathcal{V}_g(X) \cong X;$$

this is easily seen by Proposition 4.2.13: given $f: Y \rightarrow X$ a map which is g -open, if g is the identity, then f will be a p -morphism, and consequently, will be id_X -open.

The introduction of the parameter g does not affect the main results from the previous sections:

Theorem 4.4.4 (Adjunction theorem for g-Ghilardi). *Let $g: X \rightarrow X'$ be a continuous and order-preserving map, where $X, X' \in \mathbf{Pries}$. Then:*

1. *The space $\mathcal{V}_G^g(X)$ is an Esakia space;*
2. *The space $\mathcal{V}_G^g(X)$ has the following universal property: for each continuous and order-preserving g -open map $f: Y \rightarrow X$, where Y is an Esakia space, there is a continuous p -morphism $\bar{f}: Y \rightarrow \mathcal{V}_G^g(X)$ such that $\pi_0 \bar{f} = f$.*

Proof. (1) in the proof of Theorem 4.2.19, we noted that the proof that $\mathcal{V}_G^g(X)$ is an Esakia space followed from Theorem 4.1.14 by duality. The proof of the latter fact – that $\mathbf{CloUp}(\mathcal{V}_G(X))$ is a Heyting algebra – proceeded, when specialized to Heyting algebras, by showing that the definition of the Heyting implication can be

made on equivalence classes, since all maps preserve such implications from some point onwards. It is clear to see that adding to this an additional homomorphism $g^{-1}: \mathbf{ClopUp}(X') \rightarrow \mathbf{ClopUp}(X)$ in the first step which is used to preserve some existing Heyting implications in $\mathbf{ClopUp}(X)$ does not change anything about the argument.

(2) the proof of this Proposition proceeded also by duality, showing that one can use the general one-step extension lemma to obtain a sequence of maps and then take a limit map. Also there, the addition of the extra parameter does not change how the extension is made. ■

The example from 4.4.2 when coupled with this Theorem gives us one of the main interests of the construction – it allows us to “shift the level”, in the following sense:

Lemma 4.4.5 (Shifting level in g -Ghilardi complexes). Let $g: X \rightarrow Y$ be a continuous and order-preserving map. There is a natural isomorphism:

$$\mathcal{V}_G^g(X) \cong \mathcal{V}_G^{r_g}(\mathcal{V}_g(X)).$$

Proof. This follows from the universal property in Theorem 4.4.4: we show that $\mathcal{V}_G^{r_g}(\mathcal{V}_g(X))$ likewise has the universal property of $\mathcal{V}_G^g(X)$. Indeed, let $f: Y \rightarrow X$ be a continuous and order-preserving g -open map. By the one-step extension (Proposition 4.2.13), there is unique continuous and order-preserving map $\bar{f}: Y \rightarrow \mathcal{V}_g(X)$ which is r_g -open. By the universal property of $\mathcal{V}_G^{r_g}(-)$, this lifts uniquely to a continuous p -morphism $\bar{f}^*: Y \rightarrow \mathcal{V}_G^{r_g}(\mathcal{V}_g(X))$. The uniqueness of $\bar{f}^*$ as a map to this structure now follows from the uniqueness of all constructions just mentioned. ■

Also generally useful is the fact that if the map we start with is already a p -morphism, then this property is preserved:

Lemma 4.4.6 (Unit maps are p -morphisms). Let $g: X \rightarrow Y$ be a continuous p -morphism, where X is g -Esakia, and Y is an Esakia space. Then the unit map $g\pi_0: \mathcal{V}_G^g(X) \rightarrow Y$ is a p -morphism.

Proof. This is easily seen algebraically: given $g^{-1}: H_0 \rightarrow H$ a map preserving all relative pseudocomplements, if the relative pseudocomplements from H_0 are preserved at every step of the way, then when mapping to the directed colimit, one obtains a Heyting algebra homomorphism. We sketch the dual argument, for the reader who may wish to avoid such dual verifications: if $\bar{x} \in \mathcal{V}_G^g(X)$, and $g(\bar{x}(0)) \leq y$, because g is a p -morphism, then there is some $x' \geq x(0)$ such that $g(x') = y$. Because $x(1)$ is g -open, there is some $x^* \in x(1)$ such that $x(0) \leq x^*$ and $g(x^*) = g(x') = y$. By setting $(x')(1) = x(1)_{x^*} = x(1) \cap \uparrow x^*$, we obtain an inductive argument which allows us to construct $\bar{x}^* \geq \bar{x}$ with the property that $g(\bar{x}^*) = y$. This proves the result. ■

4.4.1 Finitely presented algebras and finitary complexes

As noted in Chapter 2, for many logical purposes one is often interested in the category of finitely presented algebras (Definition 2.2.16), which serve as algebraic analogues of formulas. In this section we recover the results from [75] characterizing finitely presented Heyting algebras in terms of g -Ghilardi complexes.

On Heyting algebras finitely presented algebras can be seen as quotients of free algebras by single formulas:

Lemma 4.4.7 (Principal filters and finitely presented algebras). *A Heyting algebra H is finitely presented if and only if $H \cong \mathcal{F}_{\mathbf{HA}}(n)/F(\varphi)$, where $F(\varphi) = \uparrow\varphi$ is the principal filter generated by this formula.*

Proof. This follows immediately from the definition 2.2.16, as well as Proposition 2.1.23. ■

Corollary 4.4.8 (Finitely presented Heyting algebra and clopen upsets). *A Heyting algebra H is finitely presented if and only if $\mathbf{Spec}(H)$ is isomorphic to a clopen upset of $\mathbf{Spec}(\mathcal{F}_{\mathbf{HA}}(n))$ for some n .*

Proof. Immediate from Lemma 4.4.7 and Proposition 2.1.65. ■

Whilst this corollary is certainly a characterization, it does not fundamentally simplify things. To obtain a more manageable representation, we turn to the following:

Proposition 4.4.9 (Preservation of embeddings). *Let $f: X \rightarrow Y$ be a continuous embedding map between Priestley spaces, and suppose that $g_1: X \rightarrow Z$ and $g_2: Y \rightarrow Z$ is a map such that $g_2 f = g_1$. Then $\bar{f}: \mathcal{V}_G^{g_1}(X) \rightarrow \mathcal{V}_G^{g_2}(Y)$ is an injective p-morphism. Moreover, if f is an open map, then $\bar{f}[\mathcal{V}_G^{g_1}(X)]$ is a clopen upset in $\mathcal{V}_G^{g_2}(Y)$.*

Proof. The fact that $\bar{f}$ is injective is easy to observe inductively, since if $f: X \rightarrow Y$ is injective, then $f[-]$ will always be injective as well, which directly yields this fact in the limit.

To see that $\bar{f}[\mathcal{V}_G^{g_1}(X)]$ is clopen under the assumption that f is an open map, we first show that:

$$\bar{f}[\mathcal{V}_G^{g_1}(X)] = \pi_1^{-1}[f_1[\mathcal{V}_{g_1}(X)]],$$

where $f_1(-) = f[-]$ is the lifting of f . One inclusion is easy: if $\bar{x} \in \mathcal{V}_G(X)$, then $f_1(\bar{x}(1))$ by definition belongs to $\mathcal{V}_{g_2}(X)$, and hence to $f_1[\mathcal{V}_{g_1}(X)]$. Conversely, assume that $\bar{x}(1) = f_1(y_1)$ for some $y_1 \in \mathcal{V}_{g_1}(Y)$. Note that if $c \in \bar{x}(2)$, then $f_1(y_1) \supseteq c$, and $f_1(y) \subseteq f[X]$. If r is the root of c , this means that $r \in f[y]$, and so there is some $r' \in y$ such that $f(r') = r$; this is unique since f is injective. Thus $f^{-1}[c] \subseteq y_1$ is a closed subset (since f is continuous) and rooted. It is likewise g_1 -open: assume that $d \in f^{-1}[c]$ and $d \leq d'$. then $f(d) \leq f(d')$. Since c

is g_2 -open, there is some $d'' \geq f(d)$ such that $g_2(d'') = g_2 f(d')$, and $d'' \in c$. Since $c \subseteq f[y_1]$, there is some unique d^* such that $f(d^*) = d''$. Since f is an embedding, and $f(d) \leq f(d^*)$, we have $d \leq d^*$, and $d^* \in f^{-1}[c]$. Moreover, we have

$$g_1(d^*) = g_2 f(d^*) = g_2(d'') = g_2 f(d') = g_1(d'),$$

showing that $f^{-1}[c]$ is g_1 -open as desired. Injectivity and the assumption that $f[y_1] \supseteq c$ guarantee that $f f^{-1}[c] = c$. Thus, define

$$y_2 := \{f^{-1}[c] : c \in x(2)\} = f_1^{-1}[x(2)].$$

We then have that $f_2(y_2) = f_1[y_2] = x(2)$ by construction, and y_2 is a closed set (as the preimage of a continuous function) and rooted at y_1 . Moreover, it is r_2 -open: if $f^{-1}[c] \subseteq d$ for some $d \in \mathcal{V}_1(Y)$, then $c \subseteq f[d]$; since $c \in x(2)$, by assumption, there is some c' such that $c \subseteq c'$ such that $r_0(c) = r_0(c')$. Then $f^{-1}[c] \subseteq f^{-1}[c']$, and we observe that:

$$r_0(f^{-1}[c']) = f^{-1}[r_0(c')] = f^{-1}[r_0(f[d])] = f^{-1}f(r_0(d)) = r_0(d),$$

as desired. In general, assuming we have defined y_k such that $f_k(y_k) = x(k)$, we define $y_{k+1} = \{f_{k+1}^{-1}[c] : c \in x(k+1)\} = f_k^{-1}[x(k+1)]$ which follows the exact same arguments, noting that f_k will all be injective.

To conclude the proof, we check that $f_1[\mathcal{V}_1(X)]$ is a clopen subset. Let $C \in f_1[\mathcal{V}_1(X)]$; then there is some $D \in \mathcal{V}_1(X)$ such that $f[D] = C$. Consider the clopen subbasic set $\llbracket f[X] \rrbracket$, which is clopen because f is assumed to be open. Then $C \in \llbracket f[X] \rrbracket$. We will show that $\llbracket f[X] \rrbracket \subseteq f_1[\mathcal{V}_1(X)]$ which then concludes the proof. Taking $E \in \llbracket f[X] \rrbracket$, we have that $E \subseteq f[X]$; by considering $f^{-1}[E]$, we have that $f^{-1}[E] \subseteq X$ and it is rooted, because f is injective. This shows the proof. ■

Using this we obtain a more general description of duals of finitely presented Heyting algebras using g -Ghilardi complexes:

Theorem 4.4.10 (Characterization of finitely presented algebras). *For a Heyting algebra H , the following are equivalent:*

1. H is finitely presented.
2. H is isomorphic to $\text{CloUp}(\mathcal{V}_G^g(P))$ for some $g: P \rightarrow Q$ an order-preserving map between finite posets.

Proof. (1) $\Rightarrow$ (2) Assume that $U \subseteq \mathcal{V}_G(\mathcal{P}(n))$ is a clopen upset. Then U will be of the form $\pi_k^{-1}[V]$ for $V \subseteq \mathcal{V}_k(\mathcal{P}(n))$, so we can take the order-preserving map $r_k: V \rightarrow r_k[V]$, and it is easy to see that $V \cong \mathcal{V}_G^{r_k}(V)$.

(2) $\Rightarrow$ (1) First we show that if P is any poset, then $\mathcal{V}_G(P)$ is dual to a finitely presented algebra. For that purpose, let $\uparrow: P \rightarrow \mathcal{P}(P)$ be the upset map⁶. This is an order-embedding, so by Proposition 4.4.9, $\mathcal{V}_G(P)$ sits as a clopen upset in $\mathcal{V}_G(\mathcal{P}(P))$; by Theorem 4.3.1, noting that $\mathcal{P}(P) \cong \mathbf{2}_{\leq}^P$, we have that the latter is a free algebra, so by Corollary 4.4.8, $\mathcal{V}_G(P)$ is dual to a finitely presented algebra.

Now consider the map $r_g: \mathcal{V}_g(P) \rightarrow P$, the embedding $i: \mathcal{V}_g(P) \rightarrow \mathcal{V}_r(P)$ as well as the root map $r: \mathcal{V}_r(P)$. Then we obtain by Proposition 4.4.9 that $\mathcal{V}_G^{r_g}(\mathcal{V}_g(P))$ is a clopen upset in $\mathcal{V}_G^r(\mathcal{V}_r(P))$. But note that by Lemma 4.4.5 there are natural isomorphisms:

$$\mathcal{V}_G^{r_g}(\mathcal{V}_g(P)) \cong \mathcal{V}_G^g(P) \text{ and } \mathcal{V}_G^r(\mathcal{V}_r(P)) \cong \mathcal{V}_G(P);$$

by what we just showed, then, $\mathcal{V}_G^g(P)$ is dual to a finitely presented algebra. ■

Remark 4.4.11 (Comparison with sheaf-theoretic duality). The reader may wish to compare this presentation of finitely presented Heyting algebras with that given in [130]. For an extended discussion on this, see Section A.3.2.

4.4.2 Coproducts and colimits of Heyting algebras

As alluded to in Section 4.1.3, descriptions of free algebras are closely related to descriptions of colimits. However, the reader should not be tempted to simply use the adjoints to compute colimits. To understand why, note that dually, this amounts to trying to compute limits of Esakia spaces using Priestley duality. The fact that right adjoints preserve limits, and both categories are complete, means that given Esakia spaces X, Y , we have:

$$\mathcal{V}_G(X) \otimes \mathcal{V}_G(Y) \cong \mathcal{V}_G(X \times Y),$$

That is, for *spaces in the range of the functor*, products are calculated as desired. For spaces outside of the range, we can nevertheless adjust the situation. For that we need a slight modification of the one-step extension lemma:

Definition 4.4.12 (Multiple g-open Vietoris). Let $g_i: X \rightarrow X_i$ be a collection of continuous and order-preserving maps, such that for each $i \in I$, X is g_i -Esakia. We denote by $\mathcal{V}_I(X)$ the Priestley space:

$$\mathcal{V}_I(X) := \bigcap_{i \in I} \mathcal{V}_{g_i}(X),$$

and write $r_I: \mathcal{V}_I(X) \rightarrow X$ for the restriction of the root map.

⁶The very categorically minded reader will be happy to note that this is essentially the Yoneda embedding.

Note that we are justified in saying that $\mathcal{V}_I(X)$ is a Priestley space: by Proposition 4.2.12 we know that $\mathcal{V}_{g_i}(X)$ is a closed subspace of $\mathcal{V}_r(X)$, which is Priestley, and we are taking an arbitrary intersection of closed subspaces. Moreover, the restriction $r_I: \mathcal{V}_I(X) \rightarrow X$ is both still continuous and surjective, since there was a uniform witness to surjectivity from X to $\mathcal{V}_{g_i}(X)$.

Lemma 4.4.13 (Multiple g-step extension). Let $g: X \rightarrow X_i$ be a collection of continuous and order-preserving maps where X is g_i -Esakia for each i . Let $f: Y \rightarrow X$ be a continuous and order-preserving map where Y is f -Esakia and f is g_i -open for each i . Then there exists a unique continuous and order-preserving map $\bar{f}: Y \rightarrow \mathcal{V}_I(X)$ such that $\bar{f}$ is r_I -open and $r_I \bar{f} = f$.

Proof. We derive this from Proposition 4.2.13: by that result we obtain that for each i , the map $\bar{f}_i: Y \rightarrow \mathcal{V}_{g_i}(X)$ is given by $y \mapsto f[\uparrow y]$. It follows that by setting:

$$\begin{aligned} \bar{f}: Y &\rightarrow \mathcal{V}_I(X) \\ y &\mapsto f[\uparrow y], \end{aligned}$$

this becomes a well-defined map, sending y to a set which is g_i -open for each i , and will have all desired properties as these are given pointwise. ■

Definition 4.4.14 (Product Ghilardi complex). Let $(X_i)_{i \in I}$ be a family of Esakia spaces, and let $\pi_i: \prod_{i \in I} X_i \rightarrow X_i$ be the projections. The *product Ghilardi complex* is defined to be the r_I -Ghilardi complex over $\mathcal{V}_I(\prod_{i \in I} X_i)$, where I is the family of projections. We write $\tilde{\pi}_i$ for the composed map $\pi_i \pi_0: \mathcal{V}_G^{r_I}(\mathcal{V}_I(\prod_{i \in I} X_i)) \rightarrow X_i$.

We know that $\prod_{i \in I} X_i$ is π_i -Esakia, in fact, an Esakia space (Proposition 2.1.69). We then obtain:

Theorem 4.4.15 (Product of Esakia spaces). *Given a family $(X_i)_{i \in I}$ of Esakia spaces, the space $\mathcal{V}_G^{r_I}(\mathcal{V}_I(\prod_{i \in I} X_i))$ together with the projections $\tilde{\pi}_i$ is the categorical product of these Esakia spaces.*

Proof. We know that $\mathcal{V}_G^{r_I}(\mathcal{V}_I(\prod_{i \in I} X_i))$ is an Esakia space by Theorem 4.4.4. Moreover, by Lemma 4.4.6, since $\pi_i: \prod_{i \in I} X_i \rightarrow X_i$ are all p-morphisms, $\tilde{\pi}_i$ is likewise a p-morphism. So it suffices to check the universal property of the product.

Given Y an Esakia space, $p_i: Y \rightarrow X_i$ a family of p-morphisms, note that since $\prod_{i \in I} X_i$ is the product of Priestley spaces (Proposition 4.1.4), we have a unique continuous and order-preserving map $p: Y \rightarrow \prod_{i \in I} X_i$. Note that p is in fact π_i -open for each $i \in I$: whenever $p(y) \leq \bar{z}$, then by the pointwise order, for each i , $p(y)(i) \leq \bar{z}(i)$; since p_i is a p-morphism, and $p(y)(i) = p_i(y)$, we find $y' \geq y$ such that $p_i(y') = \bar{z}(i)$, and so $p(y')$ witnesses π_i -openness.

Thus by Proposition 4.4.13 there is a unique extension $\bar{p}: Y \rightarrow \mathcal{V}_I(X)$ to an r_I -open map such that $r_I \bar{p} = p$. In turn by Theorem 4.4.4, this induces a unique

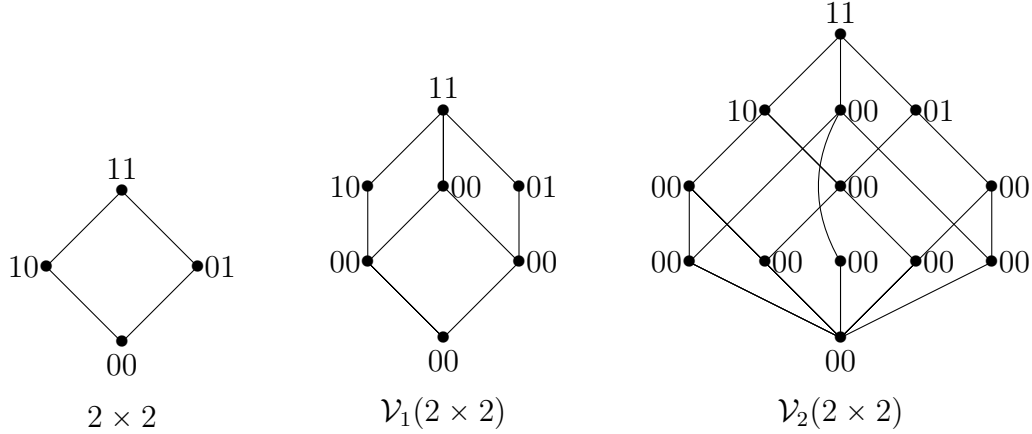


Figure 4.3

extension $\bar{p}^*: Y \rightarrow \mathcal{V}_G^I(\mathcal{V}_I(X))$, with the property that $\pi_0 \bar{p}^* = \bar{p}$. It follows that for each $i \in I$, we have

$$\tilde{\pi}_i \bar{p}^* = \pi_i r_I \pi_0 \bar{p}^* = \pi_i r_I \bar{p} = p_i;$$

and so the diagram commutes. The uniqueness is a direct consequence of all the constructions involved being universal, and can be easily verified. ■

We note that whilst the above construction can be used to construct products, very often such a product is infinite, regardless of whether the original spaces were infinite or not, as the next example shows:

Example 4.4.16 (Example product calculation). Consider the product $\mathbf{2}_{\leq} \otimes \mathbf{2}_{\leq}$. In Figure 4.3 we display the first three steps of the product Ghilardi complex $\mathcal{G}^{\pi_1, \pi_2}(\mathbf{2}_{\leq} \times \mathbf{2}_{\leq})$, where one preserves the condition of being an upset on each coordinate, on the first level.

An inductive argument then shows that each level $V_n(\mathbf{2}_{\leq} \times \mathbf{2}_{\leq})$ will have more elements than the previous one: for each $V_n(\mathbf{2}_{\leq} \times \mathbf{2}_{\leq})$, the subset $\{\uparrow x : x \in V_n(\mathbf{2}_{\leq} \times \mathbf{2}_{\leq})\}$ will root at the root of $V_n(\mathbf{2}_{\leq} \times \mathbf{2}_{\leq})$, and is not identical to $\uparrow r_n(V_n(\mathbf{2}_{\leq} \times \mathbf{2}_{\leq}))$ by an easy induction.

Remark 4.4.17. It is worthy to remark that elements of the product have a bisimulation-like flavour: given X an Esakia space, if $U \in V_1^{\pi_1, \pi_2}(X \times X)$, i.e., U is a closed, rooted and π_1, π_2 -open subset of $X \times X$, then U will likewise be a bisimulation between $(x, y) = r(U)$, and any closed bisimulation between (x, y) will give rise to such a pair. We expect that such a characterization can be used in the study of effective equivalence relations on Esakia spaces, as first studied by Ghilardi and Zawadowski [130, pp.101]. Such constructions are intimately related to *coequivalence relations*, capturing logically the concept of definable equivalence (see Appendix A.4).

A similar argument yields us a description of categorical pullbacks. For that we need a preliminary verification:

Lemma 4.4.18 (Cartesian pullback is π_i -Esakia). Let $f: X_1 \rightarrow X_0$ and $h: X_2 \rightarrow X_0$ be a pair of continuous p-morphisms between Esakia spaces. Then $(X_1 \times_{X_0} X_2)$ is π_i -Esakia for $i \in \{1, 2\}$.

Proof. Without loss of generality, let $U, V \subseteq X_1$ be clopen upsets; we will show that $\downarrow \pi_1^{-1}[U \setminus V]$ is a clopen, by showing that

$$\downarrow \pi_1^{-1}[U \setminus V] = \pi_1^{-1}[\downarrow(U \setminus V)].$$

One inclusion is clear: if $(x_1, x_2) \leq (z_1, z_2)$, and $z_1 \in U \setminus V$, then certainly $x_1 \in \downarrow(U \setminus V)$. Conversely, assume that $x_1 \leq z_1$ and $z_1 \in U \setminus V$. Then we have $f(x_1) \leq f(z_1)$; since $f(x_1) = g(x_2)$, and g is a p-morphism, there is some $z_2 \geq x_2$ such that $g(z_2) = f(z_1)$. Thus $(x_1, x_2) \leq (z_1, z_2)$ and $(z_1, z_2) \in \pi_1^{-1}[U \setminus V]$, showing that $(x_1, x_2) \in \downarrow \pi_1^{-1}[U \setminus V]$, and concluding the proof. ■

From this we can make the following definition:

Definition 4.4.19 (Pullback Ghilardi complex). Let $f: X_1 \rightarrow X_0$ and $h: X_2 \rightarrow X_0$ be a pair of continuous p-morphisms between Esakia spaces. The *pullback Ghilardi complex* for this diagram is the r_{π_1, π_2} -Ghilardi complex over $\mathcal{V}_{\pi_1, \pi_2}(X_1 \times_{X_0} X_2)$.

Theorem 4.4.20 (Pullback of Esakia spaces). *Given a diagram $f: X_1 \rightarrow X_0$ and $h: X_2 \rightarrow X_0$ with continuous p-morphisms between Esakia spaces, the Esakia space $\mathcal{V}_G^{r_{\pi_1, \pi_2}}(\mathcal{V}_{\pi_1, \pi_2}(X_1 \times_{X_0} X_2))$ together with the projections $\tilde{\pi}_1$ and $\tilde{\pi}_2$ is the categorical pullback.*

Proof. By Lemma 4.4.6 we know that $\tilde{\pi}_i$ are p-morphisms. They commute with the diagram by construction. So we show the universal property of the pullback. Let $p_1: Y \rightarrow X_1$ and $p_2: Y \rightarrow X_2$ be a pair of continuous p-morphisms such that $f p_1 = h p_2$. This induces a unique continuous and order-preserving map $(f \times h): Y \rightarrow X_1 \times_{X_0} X_2$ making the diagram commute, since the latter is the pullback in the category of Priestley spaces (Proposition 2.1.70). Note that this map is π_1, π_2 -open: if $y \in Y$, and $(f \times h)(y) \leq (x_1, x_2)$, then $f(y) \leq x_1$, so since f is a p-morphism, there is some $y' \geq y$ such that $f(y') = x_1$. Thus $(f \times h)(y')$ witnesses π_1 -openness. Thus by Lemma 4.4.13, we have a unique lifting to a map to $\mathcal{V}_{\pi_1, \pi_2}(X_1 \times_{X_0} X_2)$. The rest of the argument proceeds as in the case for products. ■

To finish this section, we show how one can use this description to “lift amalgamation” (see Definition 2.2.34) using duality. We note that here, the assumption (One-step injectivity) is crucial:

Theorem 4.4.21 (Amalgamation of Heyting algebras). *The variety $\mathbf{HA}$ of Heyting algebras has the amalgamation property.*

Proof. Let $f^*: H_0 \rightarrow H_1$ and $k^*: H_0 \rightarrow H_2$ be injective Heyting algebra homomorphisms. Let $X_i := \text{Spec}(H_i)$, which by duality (Table 2.1), yields continuous and surjective p-morphisms $f: X_1 \rightarrow X_0$ and $g: X_2 \rightarrow X_0$. By Proposition 2.1.70, we know that the projections $\pi_i: X_1 \times_{X_0} X_2 \rightarrow X_i$ are therefore both surjective as well. By Theorem 4.4.20, we obtain an Esakia space $Y := \mathcal{V}_G^{\pi_1, \pi_2}(\mathcal{V}_{\pi_1, \pi_2}(X_1 \times_{X_0} X_2))$, with projections $\tilde{\pi}_i: Y \rightarrow X_i$ for $i \in \{1, 2\}$ which are continuous surjective p-morphisms making the whole diagram commute. We prove that $\tilde{\pi}_i$ is surjective. By definition, this is equal to $\pi_0 r_{\pi_1, \pi_2} \pi_i$. The map π_i is surjective; the root map is surjective as discussed in Proposition 4.2.8 and 4.2.12. The fact that all the root maps are surjective then implies that π_0 is surjective: given $C \in \mathcal{V}_{\pi_1, \pi_2}(X)$, we define $C_0 = C$ and $C_{n+1} = \uparrow C_n \in (\mathcal{V}_{\pi_1, \pi_2}(X))_{n+1}$ and obtain that the sequence $(C_n)_{n \in \omega}$ belongs to the limit. This provides $\overline{C} \in \mathcal{V}_G^{\pi_1, \pi_2}(\mathcal{V}_{\pi_1, \pi_2}(X))$ such that $\overline{C}(0) = C$. Consequently, we obtain that $\tilde{\pi}_i$ is surjective, since it is a composition of surjective maps.

Now by duality this gives us a Heyting algebra $H' = \text{ClopUp}(Y)$, and a pair of injective maps $g_i: H_i \rightarrow H'$ making the diagram commute, proving amalgamation. $\blacksquare$

4.4.3 Polynomial algebras

The final construction described here will be a special case of a colimit:

Definition 4.4.22 (Polynomial Heyting algebra). Let H be a Heyting algebra. We denote by $H[x]$ the *Heyting algebra of polynomials over H* , defined as $H \oplus_{\mathbf{HA}} \mathcal{F}_{\mathbf{HA}}(1)$.

Polynomial Heyting algebras appear naturally in the study of uniform interpolation: a particular formulation similar to what was discussed in Proposition 2.2.43 is to say that the inclusion $H \rightarrow H[x]$ has a left and a right adjoint. We will also see polynomial algebras again in Chapter 7, where we will study a special variation of polynomial Heyting algebras, shown to be intimately related to subobject classifiers in specific subcategories.

From Theorem 4.4.15 we immediately obtain a description of polynomial Heyting algebras. However, one can make this more economic:

Definition 4.4.23. Given an Esakia space X , the *polynomial Ghilardi complex* over X , $\mathcal{G}^{\text{pol}}(X) = (X_n^p, r_n)$ is the π_X -Ghilardi complex over $X \times \mathbf{2}_{\leq}$.

We know by Theorem 4.4.4 that $\mathcal{V}_G^{\pi_X}(X \times \mathbf{2}_{\leq})$ is an Esakia space. In fact we show the following:

Proposition 4.4.24 (Dual polynomial Heyting algebras). Given X an Esakia space, and $\pi_X: X \times \mathbf{2}_{\leq} \rightarrow X$, the projection map from the cartesian product, we have an isomorphism $\mathcal{V}_G^{\pi_X}(X \times \mathbf{2}_{\leq}) \cong X \otimes \mathbf{RN}$. Thus for $H = \mathbf{ClopUp}(X)$, $\mathcal{V}_G^{\pi_X}(X \times \mathbf{2}_{\leq})$ is the dual of $H[x]$.

Proof. Note that $\mathcal{V}_G^{\pi_X}(X \times \mathbf{2}_{\leq})$ certainly has a projection to X , but it is not obvious it maps to $\mathbf{RN}$. Note that by Theorem 4.3.1, the latter is given by the inverse limit $\mathcal{V}_G(\mathbf{2}_{\leq})$.

By composing the projections $\pi_{\mathbf{2}_{\leq}} \pi_0$ we get a map $\mathcal{V}_G^{\pi_X}(X \times \mathbf{2}_{\leq}) \rightarrow \mathbf{2}_{\leq}$. By the adjunction, this lifts uniquely to a p-morphism $\mathcal{V}_G^{\pi_X}(X \times \mathbf{2}_{\leq}) \rightarrow \mathbf{RN}$. Then we have two projections which are continuous p-morphisms. Now using the same arguments as in Theorem 4.4.15, we can prove that this satisfies the universal property of the product. Thus the isomorphism holds. ■

4.5 G-conservative objects and translations

In this final section, we provide several logical applications of the step-by-step construction. These were designed to illustrate some of the potential of exploiting the theory of Ghilardi complexes for theoretical analysis of IPC.

First, we study the general interpretation of IPC in PL, guaranteed by relative anarchy (see 2.2.3), and show that it is not conservative.

By studying the theory of Ghilardi complexes, we find general conditions for the existence of a second kind of map (in addition to the canonical monad-induced algebra maps) which serve as Heyting algebra structures, shown to exist for what we call “G-conservative objects”. In Section 4.5.2 we prove that the free algebra $\mathcal{F}_{\mathbf{DL}}(\omega)$ is dual to a G -conservative Esakia space. From this, in Section 4.5.3 we give a conservative translation between IPC and PL. We finish in Section 4.5.4 by showing that this translation can be thought of as calculating *uniform positive approximations* of intuitionistic formulas, yielding a *uniform positive interpolation* theorem.

4.5.1 Structural interpretations and conservativity

Recall that in Proposition 4.1.5 we showed that $\mathbf{DL}$ is $\mathbf{HA}$ -anarchic. By the characterization of Proposition 2.2.53, this corresponds to the existence of a structural interpretation:

$$\mathfrak{S}: \mathbf{Form}_{\mathbf{PL}}(\mathbf{Prop}) \rightarrow \mathbf{Form}_{\mathbf{IPC}}(\mathbf{Prop}),$$

that is, a map which is the identity on positive formulas and which is a pseudo-homomorphism: it is a distributive lattice homomorphism, and satisfies the weak homomorphism law for implication:

$$\mathfrak{S}(\varphi \rightarrow \psi) = \mathfrak{S}(\mathfrak{S}(\varphi) \rightarrow \mathfrak{S}(\psi)).$$

A different notion of interpretation goes by requiring additionally that it *reflect* truth:

Definition 4.5.1 (Conservative translation). A map T from $\text{Form}_{\text{IPC}}(\text{Prop})$ to $\text{Form}_{\text{PL}}(\text{Prop})$ will be called a *conservative translation* if:

1. (Positive Homomorphism) T is a distributive lattice homomorphism;
2. (Identity on formulas) For each φ a formula in the language of PL , we have $\vdash_{\text{IPC}} T(\varphi) \leftrightarrow \varphi$.
3. (Conservativity) for each formula φ in the language of IPC , we have:

$$\vdash_{\text{IPC}} \varphi \iff \vdash_{\text{PL}} T(\varphi).$$

Having our step-by-step construction, we can obtain explicit calculations of this map in a straightforward way: we consider the free distributive lattice $\mathcal{F}_{\text{DL}}(\omega)$, and its dual, the space $\mathbf{2}_{\leq}^{\omega}$. The map $\mathfrak{S}$ is induced by the unique algebra map on the monad $\mathbf{F}_{\mathbf{HA}}I$, where $\mathbf{F}_{\mathbf{HA}}: \mathbf{DL} \rightarrow \mathbf{HA}$ is the free functor dual to $\mathcal{V}_G$, and I is the subcategory inclusion. Such a unique map is in fact given by the counit

$$\varepsilon_{I(\mathcal{F}_{\text{DL}}(\omega))}: \mathbf{FI}(\mathcal{F}_{\text{DL}}(\omega)) \rightarrow \mathcal{F}_{\text{DL}}(\omega);$$

by construction, $\mathbf{FI}(\mathcal{F}_{\text{DL}}(\omega)) \cong \mathcal{F}_{\mathbf{HA}}(\omega)$, and the counit is the “evaluation map”, sending each formula to its interpretation in the Heyting algebra structure of the distributive lattice. Dually, this counit becomes the unit map, the injective p-morphism:

$$\eta_{\mathbf{2}_{\leq}^{\omega}}: \mathbf{2}_{\leq}^{\omega} \rightarrow \mathcal{V}_G(\mathbf{2}_{\leq}^{\omega}),$$

Calculating such a map in the Ghilardi complex $\mathcal{G}(\mathbf{2}_{\leq}^{\omega})$, we have that this is given by the recursive family of maps:

1. $f_0: \mathbf{2}_{\leq}^{\omega} \rightarrow \mathbf{2}_{\leq}^{\omega}$, the identity;
2. For each k , $f_{k+1}(x) = f_k[\uparrow x]$.

Having this description, we can ask some questions about such an interpretation. A natural such question is whether this is in fact a *conservative* interpretation. Given how different the free algebras look, the negative answer to this appears quite natural:

Theorem 4.5.2 (Structural interpretation not conservative). *The structural interpretation $\mathfrak{S}$ is not conservative.*

Proof. We show that the formula $\neg p \vee \neg\neg p$ is valid under the interpretation, that is:

$$\mathcal{F}_{\text{DL}}(\omega) \models \mathfrak{S}(\neg p) \vee \mathfrak{S}(\neg\neg p) = 1.$$

To see why, it suffices to show that for any element $a \in \mathcal{F}_{\mathbf{HA}}(\omega)$ we have $\mathfrak{S}(\neg a) \vee \mathfrak{S}(\neg\neg a) = 1$. Using the representation from Theorem 4.3.1, we let a be a clopen upset $U \subseteq \mathcal{V}_G(\mathbf{2}_{\leq}^\omega)$. By construction these are of the form:

$$U = \pi_n^{-1}[V],$$

where V is a clopen upset in $(\mathbf{2}_{\leq}^\omega)_n$. Note that $f_n^{-1}[V]$ is likewise a clopen upset, then. By definition of the translation, we have:

$$\mathfrak{S}^{-1}[\pi_n^{-1}[V]] = f_n^{-1}[V];$$

We have that by construction, $\neg\pi_n^{-1}[V] = \pi_{n+1}^{-1}[\llbracket -V \rrbracket]$. Moreover, we can show that:

$$f_{n+1}^{-1}[\llbracket -V \rrbracket] = f_1^{-1}[\llbracket f_n^{-1}[V] \rrbracket] :$$

indeed, note that $f_{n+1}(x) \in \llbracket -V \rrbracket$ if and only if $f_n[\uparrow x] \in \llbracket -V \rrbracket$, if and only if $f_n[\uparrow x] \subseteq -V$, if and only if $\uparrow x \in \llbracket f_n^{-1}[-V] \rrbracket$; since $f_1(x) = \uparrow x$, this gets us the equality. Similarly, we show:

$$f_{n+2}^{-1}[\llbracket -\llbracket -V \rrbracket \rrbracket] = f_2^{-1}[\llbracket -\llbracket f_n^{-1}[V] \rrbracket \rrbracket].$$

We can thus simply let $V_0 := f_n^{-1}[V]$, and show that

$$\mathbf{2}_{\leq}^\omega = f_1^{-1}[\llbracket -V_0 \rrbracket] \cup f_2^{-1}[\llbracket -\llbracket -V_0 \rrbracket \rrbracket].$$

Now we calculate these elements. Notice that $f_1^{-1}[\llbracket -V_0 \rrbracket] = \{x \in \mathbf{2}_{\leq}^\omega : \uparrow x \subseteq -V_0\} = \neg V_0$, with the relative pseudocomplement calculated in $\mathbf{2}_{\leq}^\omega$. Slightly more elaborately, we see that:

$$f_2^{-1}[\llbracket -\llbracket -V_0 \rrbracket \rrbracket] = \{x \in \mathbf{2}_{\leq}^\omega : f_1(x) \subseteq \langle V_0 \rangle\}.$$

Unfolding this, such a set defines the set of x , such that whenever $x \leq y$, there is some $z \geq y$ such that $z \in V_0$. But this is simply the semantic meaning of the set $\neg\neg V_0$.

Now, since $\mathbf{2}_{\leq}^\omega$ has a maximal element, this implies that $\mathbf{2}_{\leq}^\omega = \neg V_0 \cup \neg\neg V_0$ (see Proposition 2.3.19).

This shows that under the translation $\mathfrak{S}$, the principle $\neg p \vee \neg\neg p$ is valid. ■

Remark 4.5.3 (Logic of the interpretation). The arguments of Example 4.5.2 are completely general, in that the set of logical principles which become valid under the interpretation $\mathfrak{S}$ will precisely coincide with the logic of $\mathcal{F}_{\mathbf{DL}}(\omega)$. For more on this, see Appendix A.3.4.

In light of this result, we can ask whether *any* conservative translation can exist between IPC and PL, on the level of theorems, which shares some of the same properties as $\mathfrak{S}$; naturally no other structural interpretation can exist, but alternative assumptions may prove equally natural. This is what we turn to next.

4.5.2 G-conservative Priestley spaces

The abstract situation witnessed by the structural interpretation is, as discussed in 2.2.3, the fact that each Heyting algebra carries a unique algebra structure for the monad induced by the adjunction. The unit map η_H is then split by the composite counit. Dually, the counit map $\varepsilon: \mathcal{V}_G(X) \rightarrow X$, which is simply the projection π_0 , is split by the unique lifting of the map $\uparrow$, as described in the previous section.

However, as we saw in Section 4.3, there is another possible way of extending $\uparrow: X \rightarrow \mathcal{V}_r(X)$ to a map going to the remaining elements of the Ghilardi complex: we can consider the sequence of maps

1. $\uparrow^0: X \rightarrow \mathcal{V}_r(X)$, and
2. For each k , $\uparrow^{k+1}: X \rightarrow X_{k+2}$ is defined by composing $\uparrow_k: X_k \rightarrow X_{k+1}$ with $\uparrow^k$.

Such maps always exist, as far as the order is concerned, in the Ghilardi complex. They cannot coincide with the liftings induced by the one-step extension except in trivial cases. If X is a finite poset – as we saw in Section 4.3 – then such maps are obviously continuous; but in general the situation is less clear. This motivates the following definition:

Definition 4.5.4 (G-conservative object). Let X be a Priestley space and $\mathcal{G}(X) = (X_n, r_n)$ the Ghilardi complex over X . X is called *G-conservative* if for each n , the map $\uparrow^n: X \rightarrow X_n$ is continuous.

Remark 4.5.5 (Remarks on G-conservativity).

1. (G-conservative are Esakia) Note that trivially, if X is *G-conservative*, then it is an Esakia space: this is because then the map $\uparrow: X \rightarrow \mathcal{V}_r(X)$ will be continuous, and so, by composing with the continuous inclusion of $\mathcal{V}_r(X) \subseteq \mathcal{V}(X)$ we will have one of the equivalent conditions for being an Esakia space from Lemma 2.1.80.
2. (Existence of infinite map) If X is *G-conservative*, note that there is a continuous and order-preserving map $\uparrow^\omega: X \rightarrow \mathcal{V}_G(X)$ which is left adjoint to the projection $\pi_0: \mathcal{V}_G(X) \rightarrow X$. Indeed, by Lemma 4.3.2, we have that the maps $\uparrow^k$ are compatible with the inverse system structure of $\mathcal{G}(X)$. If all of them are continuous, by the universal property of inverse limits, there is a unique continuous and order-preserving map $\uparrow^\omega: X \rightarrow \mathcal{V}_G(X)$ which sends x to the sequence $(\uparrow^k x)_{k \in \omega}$. Such a map is then the left adjoint of π_0 , since it coincides with the map l_0 described in Lemma 4.3.2.

As noted every finite poset is *G-conservative*. However, few more conditions seem to exist, and even fewer if we parametrize by considering *g*-Ghilardi complexes. We provide an extended discussion about this situation in Chapter 8.

Our goal in this section will be to extend the known class of G -conservative Priestley spaces, from the finite posets, to include the duals of every countable and *locally finite* Heyting algebra. This will allow us in the next section to construct a conservative translation.

Definition 4.5.6 (Locally finite Heyting algebra). Let H be a Heyting algebra. We say that H is *locally finite* if it is the directed union of finite Heyting algebras via a diagram consisting of Heyting algebra homomorphisms.

Remark 4.5.7 (Equivalence with usual local finiteness). The above is equivalent to the more usual definition of local finiteness saying that every finitely generated subalgebra is finite (see 6): indeed, given finitely many generators, they occur in finitely many of the algebras in the directed system, and by directedness, in a single finite Heyting algebra H_0 ; then the subalgebra generated will be a subalgebra of H_0 , and therefore, finite.

The following lemma is a standard fact on chain constructions :

Lemma 4.5.8 (Locally finite from chain). For H a Heyting algebra, H is countable and locally finite if and only if H is the limit of a countable chain of finite Heyting algebras via Heyting homomorphisms.

Proof. See e.g. [4, Lemma 1.6]. ■

We thus treat countable and locally finite Heyting algebras, as limits of chains of the form:

$$H_0 \hookrightarrow H_1 \hookrightarrow H_2 \hookrightarrow \cdots \hookrightarrow H_\omega$$

Dually, such a chain becomes the inverse limit:

$$P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots \leftarrow X_\omega,$$

where X_ω is the dual of H , and P_n the finite poset dual to H_n .

At this point it seems intuitive to apply the functors $\mathcal{V}_r(-)$ on each element of the chain. However, it is not obvious in general that such a functor preserves colimits (as it forms a part of a *right adjoint* functor). This and several related facts will follow from a rather intricate combinatorial property of the step-by-step construction purely working with finite posets, which we proceed to show.

Before the next, quite technical proof, we briefly recall for the benefit of the reader our notational conventions:

Notation 4.5.1. Below we will work with a p-morphism $f: P \rightarrow Q$ between finite posets. We denote by P_k and Q_k the k -th posets in the Ghilardi complexes $\mathcal{G}(P)$ and $\mathcal{G}(Q)$. The order on these posets, which we will remind the reader of throughout, is *reverse inclusion*. The map $f_0 = f$ and for each k , $f_{k+1} = f_k[-]$. We write $r_k: P_{k+1} \rightarrow P_k$ and $r_k: Q_{k+1} \rightarrow Q_k$. We believe this causes no confusion since the roots will always be taken in objects which will be easily identifiable as

being inside of P_{k+1} or Q_{k+1} . We write $r_{k,j}: P_k \rightarrow P_j$; note that this means that under this notation $r_k = r_{k+1,k}$.

For each k , we define $\uparrow^k: P \rightarrow P_k$ and also $\uparrow^k: Q \rightarrow Q_k$ (again we do not distinguish the posets, the location being clear from context), inductively: $\uparrow^0$ is the identity map, and for each k , $\uparrow^{k+1} = \uparrow_k \circ \uparrow^k$, that is, we compose the adjoint with the upset map $P_k \rightarrow P_{k+1}$.

The reader is encouraged to consult Figure 4.4 to understand the example at hand. We are now ready to prove the main combinatorial argument:

Proposition 4.5.9 (Beck-Chevalley for p-morphisms). Let $f: P \rightarrow Q$ be a p-morphism between finite posets. Then for every $n \geq 0$,

$$f_n \circ \uparrow^n = \uparrow^n \circ f.$$

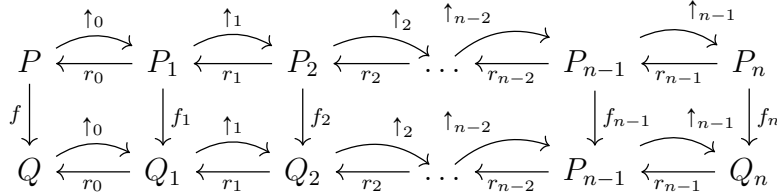


Figure 4.4: Finite extension property

Proof. We argue by induction on n . The case $n = 0$ is exactly the p-morphism property of f (see Remark 2.1.54). Assume $n \geq 1$ and that the equality holds at all smaller indices. Fix $p \in P$. Unwinding $f_n = f_{n-1}[-]$, we must show

$$f_n(\uparrow^n p) = f_{n-1}[\uparrow^n p] = \uparrow^n f(p) \quad (\text{as subsets of } Q_{n-1}).$$

($\subseteq$) This inclusion follows relatively easily. Given $C \in \uparrow^n p$, we want to show that $f_{n-1}(C) \in \uparrow^n f(p)$. Note that the hypothesis means that $C \subseteq \uparrow^{n-1} p$; thus $f_{n-1}(C) \subseteq f_{n-1}(\uparrow^{n-1} p)$. By induction hypothesis, we have that $f_{n-1}(\uparrow^{n-1} p) = \uparrow^{n-1} f(p)$, so we conclude $f_{n-1}(C) \subseteq \uparrow^{n-1} f(p)$. In other words, $f_{n-1}(C) \in \uparrow^n f(p)$ by definition of the iterated upset.

($\supseteq$) This inclusion is the non-trivial one, and will require some extra constructions. Let $C \in \uparrow^n f(p)$, so $C \in Q_{n-1}$ and $r_{n-1,0}(C) \geq f(p)$. We will construct $\tilde{C} \in \uparrow_n p$ with $f_{n-1}(\tilde{C}) = C$ via the following recursion.

Definition 4.5.10 (Auxiliary $M_k(p_0, D)$ construction). For each $k \geq 1$, each $p_0 \in P$, and each $D \in Q_k$, we say that the pair (p_0, D) satisfies the *type condition* if $f(p_0) = r_{k,0}(D)$. Whenever (p_0, D) satisfies the type condition, we define

$$\begin{aligned} M_1(p_0, D) &:= \uparrow p_0 \cap f^{-1}[D], \\ M_{k+1}(p_0, D) &:= \{ M_k(p_1, e) : e \in D, p_1 \in P, p_1 \geq p_0, f(p_1) = r_{k,0}(e) \}. \end{aligned}$$

The following claim contains the main properties of the correctness of the construction:

Claim 4.5.11 (Correctness of construction of $M_k(p_0, D)$). For each $n - 1 \geq k \geq 1$, $p_0 \in P$ and $D \in Q_k$ satisfying the type condition, we have:

1. $M_k(p_0, D)$ is a rooted subset of P_{k-1} . For $k = 1$, its root is p_0 , and for $k \geq 2$, its root is $M_{k-1}(p_0, r_{k-1}(D))$. In particular, $r_{k,0}(M_k(p_0, D)) = p_0$, and so $M_k(p_0, D) \supseteq \uparrow^{k-1} p_0$ as subsets of P_{k-1} .
2. The operation $M_k(-, -)$ is order preserving in both coordinates: if $p_0 \leq p'_0$ both elements in P , and $D \supseteq D'$ both elements in Q_k , such that (p_0, D) and (p'_0, D') satisfy the type condition, then $M_k(p_0, D) \supseteq M_k(p'_0, D')$.
3. $f_k(M_k(p_0, D)) = f_{k-1}[M_k(p_0, D)] = D$.
4. For $k \geq 2$, the set $M_k(p_0, D)$ is r_{k-2} -open,
5. For each $k \geq 1$, $M_k(p_0, D) \in P_k$.

Proof. We prove (1)–(4) by simultaneous induction on k .

(Base case for $k = 1$) Consider $p_0 \in P$ and $D \in Q_1$ such that (p_0, D) satisfy the type condition.

For (1), note that the set $M_1(p_0, D) = \uparrow p_0 \cap f^{-1}[D]$ contains p_0 by the type condition (since $f(p_0) = r_0(D) \in D$) and is rooted at p_0 . For (2), if $p_0 \leq p'_0$ then $\uparrow p'_0 \subseteq \uparrow p_0$; if also $D \supseteq D'$, then $f^{-1}[D'] \subseteq f^{-1}[D]$. So we get $\uparrow p'_0 \cap f^{-1}[D'] \subseteq \uparrow p_0 \cap f^{-1}[D]$, as desired. For (3), the inclusion $p[M_1(p_0, D)] \subseteq D$ is by definition, and conversely each $d \in D$ satisfies $d \geq r_0(D) = f(p_0)$, so the p-morphism property produces $p_1 \geq p_0$ with $f(p_1) = d \in D$. So we have equality. (4) is vacuous for this case. (5) holds since for $k = 1$, P_1 is isomorphic to the collection of rooted subsets of P by definition.

(Inductive case assuming k)

Fix $D \in Q_{k+1}$ with $f(p_0) = r_{k+1,0}(D)$.

(1) The element $r_k(D) \in D$ satisfies $r_{k+1,1}(r_0(D)) = r_{k+1,0}(D) = f(z)$. Thus, the pair $(p_0, r_k(D))$ satisfies the type condition. By induction hypothesis, we have that $M_k(p_0, r_k(D))$ is in P_k by induction hypothesis. For each $e \in D$, $r_k(D) \geq e$, and each $p_1 \geq p_0$, by induction hypothesis, we have that $M_k(p_0, r_k(D)) \subseteq M_k(p_1, e)$, so $M_k(p_0, r_k(D))$ is the root of $M_{k+1}(p_0, D)$. We then have $r_{k+1,0}(M_{k+1}(p_0, D)) = r_{k,0}(M_k(p_0, r_k(D))) = p_0$ by induction hypothesis.

(2) This is immediate by construction: assume that $p_0 \leq p_1$, and $D \supseteq D'$. Suppose that $p_1 \leq p_2$ and $e' \in D'$, and $f(p_2) = r_{k,0}(e')$, such that $M_k(p_2, e') \in M_{k+1}(p_1, D')$. Then by definition $p_0 \leq p_2$ and $e' \in D$,

so $M_k(p_2, e') \in M_{k+1}(p_0, D)$ as well. This shows that $M_{k+1}(p_0, D) \supseteq M_{k+1}(p_1, D')$.

(3) Note that:

$$\begin{aligned} f_{k+1}(M_{k+1}(p_0, D)) &= f_k[M_{k+1}(p_0, D)] \\ &= \{f_k(M_k(p_1, e)) : e \in D, p_1 \geq p_0, r_{k,0}(e) = f(p_1)\}; \end{aligned}$$

For each $e \in D$ and $p_1 \geq p_0$ satisfying the type condition, we have $f_k(M_k(p_1, e)) = e$ by induction hypothesis, so $f_{k+1}(M_{k+1}(p_0, D)) \subseteq D$. Note that since $k \leq n-1$, the type condition ensures that $D \in \uparrow^k f(p_0)$, and by induction hypothesis on n (the global induction), this is equal to $f_k[\uparrow^k p_0]$. Thus for each $e \in D$, there is some $p_1 \geq p_0$ and some F such that $f_{k-1}(F) = e$ such that $r_{k-1,0}(F) = p_1$. Since f_{k-1} commutes with the root maps, we have

$$r_{k,0}(e) = r_{k,0}f_{k-1}(F) = fr_{k,0}(F) = f(p_1).$$

Thus, (p_1, e) satisfies the type condition, which means by construction that $M_k(p_1, e) \in M_{k+1}(p_0, D)$. By induction hypothesis (this time on k), we have that $f_k(M_k(p_1, e)) = e$. This proves that $D \subseteq f_{k+1}(M_{k+1}(p_0, D))$. The equality is thus established, proving (3).

(4) We show that $M_{k+1}(p_0, D)$ is r_{k-1} -open. So assume that we have $M_k(p_1, e) \supseteq S$ where S is an element of P_k . Consider the root $r_{k-1}(S)$; since this is an element of $M_k(p_1, \epsilon)$, by induction hypothesis, this means that there is some $p_2 \geq p_1$ and $\epsilon \in e$ satisfying the type condition, such that $M_{k-1}(p_2, \epsilon) = r_{k-1}(S)$. Define

$$e' := e \cap \uparrow \epsilon.$$

Note that then we have, by definition, that (p_2, e') satisfy the type condition, since $r_{k,0}(e') = r_{k-1}r_{k-1,0}(\epsilon) = f(p_2)$. Since $D \in P_{k+1}$, it is r_{k-1} open. Since $e \supseteq e'$, by hypothesis there is some $e'' \in D$ such that $r_{k-1}(e'') = r_{k-1}(e') = \epsilon$. Thus the pair (p_2, e'') also satisfy the type condition.

We claim that then $M_k(p_2, e'')$ witnesses the r_{k-1} -openness above $M_k(p_1, e)$. Indeed, we have:

1. Since $p_1 \leq p_2$ and $e \supseteq e''$, by inductive hypothesis on monotonicity ((2)), we have that $M_k(p_1, e) \supseteq M_k(p_2, e'')$.
2. By induction hypothesis on (4), we have that $M_k(p_2, e'')$ is an element of P_k .
3. By induction hypothesis on (1), we have that $r_{k-1}(M_k(p_2, e'')) = M_{k-1}(p_2, \epsilon)$, by construction of e'' .

This shows (4). Finally, (5) follows from (1) and (4), since we showed that $M_{k+1}(p_0, D)$ is rooted and r_{k-1} -open. This concludes the induction and the proof. ■

Now to prove our result, instantiate $k = n - 1$, $p_0 = p$, and $D = C$. Then by Claim 4.5.11 we obtain an element of P_n , $\tilde{C} = M_{n-1}(p, C)$ which has the property that $\tilde{C} \supseteq \uparrow^{n-1}p$, i.e., $\tilde{C} \in \uparrow^n p$. Moreover, by construction, $p_{n-1}(\tilde{C}) = C$. Thus we have showed that $C \in f_{n-1}[\uparrow^n p] = f_n(\uparrow^n p)$. This concludes the induction on n , and with it, the proof. ■

One consequence of Proposition 4.5.9 is the following:

Proposition 4.5.12 (Surjectivity of induced maps). Let $f: P \rightarrow Q$ be a surjective p-morphism between finite posets. Then for each n , $f_n: P_n \rightarrow Q_n$ is surjective.

Proof. We show the following by induction: for each n , the minimal elements of Q_{n+1} are of the form $\uparrow x$ for $x \in Q_n$ a minimal element. Certainly if $x \in Q_n$ is minimal, then $\uparrow x \in Q_{n+1}$ will be minimal. On the other hand, if $C \in Q_{n+1}$ is minimal, consider $r_n(C)$; let y be a minimal element below $r_n(C)$. Then $\uparrow y \leq C$ because $\uparrow$ is the left adjoint, so because C is minimal, $C = \uparrow y$.

With this fact at hand, we now prove the statement: let n be arbitrary. Let $C \in P_n$ be arbitrary. By what we just showed, there is some $x \in Q$, such that $\uparrow^{n-1}x \supseteq C$, since the shape of minimal elements of Q_n is given by applying the adjoints iteratively. Since f is surjective, there is some $y \in P$ such that $f(y) = x$. By Proposition 4.5.9, we have:

$$f_n(\uparrow^{n-1}y) = \uparrow^{n-1}f(y) = \uparrow^{n-1}x.$$

By assumption of C being above $\uparrow^{n-1}x$, we have $C \in f_n(\uparrow^{n-1}y)$; thus there is some $D \in \uparrow^{n-1}y$ such that $f_{n-1}[D] = C$, i.e., $f_n(D) = C$. This shows that f_n is surjective, as desired. ■

From this we can extract a representation result for the duals of countable locally finite Heyting algebras (see Figure 4.5). This result can be seen as a form of continuity of the functors $\mathcal{V}_g$ which we have been using so far, with respect to such directed colimits.

Proposition 4.5.13 (Continuity of one-step extension). Let X_ω be the inverse limit of p-morphisms $f_n: P_{n+1} \rightarrow P_n$ between finite posets. Then:

1. For each k , $(\mathcal{V}_k(P_n), (f_n)_k)_{n \in \omega}$ is a directed system, and moreover, for each k , the root maps $r_k^n: \mathcal{V}_{k+1}(P_n) \rightarrow \mathcal{V}_k(P_n)$ are compatible with the directed systems.

$$\begin{array}{ccccc}
X_\omega & & \varprojlim^{n \in \omega} \mathcal{V}_r(P_n) & & \varprojlim^{n \in \omega} \mathcal{V}_{r_0}(P_n) \\
& & \vdots & & \\
& & \vdots & & \\
P_2 & \xleftarrow{r_0^2} & \mathcal{V}_r(P_2) & \xleftarrow{r_1^2} & \mathcal{V}_{r_0}(P_2) & \dots \\
f_1 \downarrow & & \downarrow (f_1)_1 & & \downarrow (f_1)_2 & \\
P_1 & \xleftarrow{r_0^1} & \mathcal{V}_r(P_1) & \xleftarrow{r_1^1} & \mathcal{V}_{r_0}(P_1) & \dots \\
f_0 \downarrow & & \downarrow (f_0)_1 & & \downarrow (f_0)_2 & \\
P_0 & \xleftarrow{r_0^0} & \mathcal{V}_r(P_0) & \xleftarrow{r_1^0} & \mathcal{V}_{r_0}(P_0) & \dots
\end{array}$$

Figure 4.5

2. For each k , $\mathcal{V}_k(X_\omega)$, the k -step in the Ghilardi complex is isomorphic to $\varprojlim^{n \in \omega} \mathcal{V}_k(P_n)$, and the root map $r_k: \mathcal{V}_{k+1}(X_\omega) \rightarrow \mathcal{V}_k(X_\omega)$ is the unique limit map lifting the maps $r_k^n: \mathcal{V}_{k+1}(P_n) \rightarrow \mathcal{V}_k(P_n)$.

Proof. (1) The fact that $(f_n)_k$ form a directed system is obvious by induction, since the maps are simply obtained by taking the direct image, a functorial operation. Moreover, the fact that the root maps is compatible with this system is also easy to see: for $k = 0$, and $n > 0$, and $C \in \mathcal{V}_r(P_n)$, by definition, $(f_{n-1})_1(C) = f_{n-1}[C]$, which is rooted at $f_{n-1}(r_0^n(C))$. The inductive case follows completely similarly.

Consequently, for each k , there exists a limit map $r_k^\omega: \varprojlim^{n \in \omega} \mathcal{V}_r(P_n) \rightarrow X_\omega$.

(2) We first prove that the pair $(\varprojlim^{n \in \omega} \mathcal{V}_r(P_n), r_0^\omega)$ has the universal property of the one-step extension from Proposition 4.2.13. Indeed, let Y be a Priestley space, and let $g: Y \rightarrow X_\omega$ be a continuous and order-preserving map, such that Y is g -Esakia. By composing with the projections we obtain continuous and order-preserving maps $g^n: Y \rightarrow P_n$, such that Y is g^n -Esakia. For each such map, we have a unique lifting $(g^n)_1: Y \rightarrow \mathcal{V}_r(P_n)$ to an r_0^n -open map. Observe that by definition, for each n , and $y \in Y$ we then have:

$$(f_{n+1})_1(g^{n+1})_1(y) = f_{n+1}[g^{n+1}[\uparrow y]] = g^n[\uparrow y],$$

which holds because of the definition of the maps g^n , and the fact that X_ω forms a directed system. By the universal property of inverse limits, this induces a map $g_1: Y \rightarrow \varprojlim^{n \in \omega} \mathcal{V}_r(P_n)$, explicitly, such that $g_1(y)(n) = (g^n)_1(y)$. By definition, we have that $(r_0)^\omega g_1(y) = g(y)$. If we show that such a map is $(r_0)^\omega$ -open, then it will be unique in such conditions making the diagram commute, by the universal property of inverse limit.

To prove this, assume that for $y \in Y$, $g_1(y) \leq \bar{x}$. We prove that for each n , we have:

$$\uparrow y \cap (g_n)_1^{-1}[(r_0^n)^{-1}[x(n)]] \neq \emptyset;$$

To see this, note that for each n , by hypothesis, $(g^n)_1(y) \leq x(n)$. Since $(g^n)_1$ is r_0^n -open by construction, there is some $y' \geq y$ such that $r_0^n(g^n)_1(y') = r_0^n(x(n))$. Thus, by compactness, noting that all of these are closed sets, we have that:

$$\uparrow y \cap \bigcap_{n \in \omega} (g_n)_1^{-1}[(r_0^n)^{-1}[x(n)]] \neq \emptyset.$$

Picking some $y' \geq y$ in such a set, we have that by construction, for each n , $r_0^n(g^n)_1(y') = r_0^n(x(n))$. But by definition of the lifting of the root map, then $r_0^\omega g_1(y') = r_0^\omega(\bar{x})$, which was to show.

The case for arbitrary $\mathcal{V}_{k+1}(P_n)$ follows along exactly the same lines, with the only addition being that one must show that the map r_k^ω is r_{k-1}^ω -open. But this is shown in exactly the same way as in the case for Y an arbitrary Priestley space. ■

We are now finally ready to show the main theorem of this section:

Theorem 4.5.14 (G-conservativity for duals of countable and locally finite). *For each X_ω , the dual of a countable locally finite Heyting algebra, we have that X_ω is G -conservative.*

Proof. We make use of Proposition 4.5.13. First, fix an arbitrary k . Consider for each n , the maps

$$\uparrow_n^k: P_n \rightarrow \mathcal{V}_k(P_n).$$

Note that as noted before, such a map will be the left adjoint to $r_{k,0}^n$. Moreover, by Proposition 4.5.9, such maps are compatible with the inverse system structure of the systems (P_n, f_n) and $(\mathcal{V}_k(P_n), (f_n)_k)$. As such, it lifts to a limit map $(\uparrow^k)^\omega: X_\omega \rightarrow \mathcal{V}_k(P_n)$ where we use the isomorphism guaranteed by Proposition 4.5.13. Then note that for $x \in X_\omega$ and $C \in \mathcal{V}_k(X_\omega)$ we have:

$$\begin{aligned} (\uparrow^k)^\omega(x) \leq C &\iff \forall n ((\uparrow^k)_n(x(n)) \leq C(n)) \\ &\iff \forall n (x(n) \leq r_{k,0}^n(C)) \\ &\iff x \leq r_{k,0}^\omega(C). \end{aligned}$$

This shows that as a map of posets, $(\uparrow^k)^\omega$ is the left adjoint of $r_{k,0}: \mathcal{V}_k(X_\omega) \rightarrow X_\omega$. Since adjoints are unique, this map then coincides with the iterated upset map, $\uparrow^k: X_\omega \rightarrow \mathcal{V}_k(X_\omega)$. But since the former was a limit map, it was continuous; therefore $\uparrow^k$ is continuous as well, completing the proof. ■

As a corollary, we obtain:

Corollary 4.5.15 (Existence of continuous left adjoint). *For each X_ω , dual to a countable and locally finite Heyting algebra, the projection map $\pi_0: \mathcal{V}_G(X) \rightarrow X$ has a continuous and order-preserving left adjoint $\uparrow^\omega: X \rightarrow \mathcal{V}_G(X)$ which is its retraction.*

Proof. This follows from Theorem 4.5.14, by applying the observation from Remark 4.5.5.(2). The fact that $\uparrow^\omega$ is a retraction follows from noticing that for each x , $\pi_0(\uparrow^\omega x) = x$ must hold by construction of the step-by-step iteration of the map. ■

4.5.3 Conservative translation of IPC in positive logic

We now apply the tools of Section 4.5.2 to provide a conservative translation from intuitionistic logic in positive logic (see Definition 4.5.1). To obtain such a translation, we will first show that $\mathcal{F}_{\mathbf{DL}}(\omega)$ is in fact a locally finite-Heyting algebra. This follows at once from the following lemma:

Lemma 4.5.16 (Free distributive lattices are locally finite Heyting). *For each n , the map $i_n: \mathcal{F}_{\mathbf{DL}}(n) \rightarrow \mathcal{F}_{\mathbf{DL}}(n+1)$ is a Heyting algebra homomorphism. Consequently, the algebra $\mathcal{F}_{\mathbf{DL}}(\omega)$ is a countable and locally finite Heyting algebra.*

Proof. Dually, we note that the map dual to i_n is the projection $p_n: \mathbf{2}_{\leq}^{n+1} \rightarrow \mathbf{2}_{\leq}^n$ which takes the restriction of a function to the first n many arguments. This is easily a p-morphism: whenever $p_n(f) \leq g$ then $f \leq g^+$, where $g^+(n+1) = 1$.

It is a well-known fact [4, pp.143] that $\mathcal{F}_{\mathbf{DL}}(\omega)$ is a directed colimit of the chain $i_n: \mathcal{F}_{\mathbf{DL}}(n) \rightarrow \mathcal{F}_{\mathbf{DL}}(n+1)$, which as shown, by duality, are injective Heyting algebra homomorphisms. Consequently $\mathcal{F}_{\mathbf{DL}}(\omega)$ is countable and locally finite as desired. ■

From this we obtain:

Corollary 4.5.17 (Existence of right adjoint to inclusion of free distributive). *The inclusion $\eta: \mathcal{F}_{\mathbf{DL}}(\omega) \rightarrow \mathcal{F}_{\mathbf{HA}}(\omega)$ has a right adjoint $\eta^\dagger: \mathcal{F}_{\mathbf{HA}}(\omega) \rightarrow \mathcal{F}_{\mathbf{DL}}(\omega)$ which is a distributive lattice homomorphism, and such that $\eta^\dagger \eta$ is the identity on $\mathcal{F}_{\mathbf{DL}}(\omega)$.*

Proof. By Lemma 4.5.16, the algebra is countable and locally finite, so by Corollary 4.5.15, there is a right adjoint to the unit map (recalling that the dual of the unit is the projection π_0). ■

We can now define our conservative translation:

Definition 4.5.18 (Adjoint translation). *For each formula φ , we define the translation $\mathfrak{A}(\varphi)$ on equivalence classes $[\varphi] \in \mathcal{F}_{\mathbf{DL}}(\omega)$ as follows: we let ψ be some positive formula in $\eta^\dagger([\varphi])$, and set for each $\chi \in [\varphi]$, $\mathfrak{A}(\chi) = \psi$.*

Note that in light of Corollary 4.5.17, we have:

Proposition 4.5.19 (Adjoint translation is conservative). The assignment $\mathfrak{A}$ is a well-defined conservative translation.

Proof. The map is well-defined by force. It is a positive homomorphism, since it is defined on the basis of the homomorphism $\eta^\dagger$; it is the identity on formulas, since for each φ in the language of positive logic, we have: $\eta^\dagger([\varphi]) = \eta^\dagger\eta(\varphi) = [\varphi]$. Finally, for conservativity, one direction is obvious: if $[\varphi] = 1$, then $\eta^\dagger([\varphi]) = 1$, which since these are free algebras, means that if $\vdash_{\text{IPC}} \varphi$ then $\vdash_{\text{PL}} \mathfrak{A}(\varphi)$. Conversely, assume that $\mathfrak{A}(\varphi) = 1$. Then $\eta^\dagger([\varphi]) = 1$, so $1 \leq \eta^\dagger([\varphi])$. Since this is the right adjoint, we obtain $1 = \eta(1) \leq [\varphi]$, so $[\varphi] = 1$. In other words, if $\vdash_{\text{PL}} \mathfrak{A}(\varphi)$ then $\vdash_{\text{IPC}} \varphi$. $\blacksquare$

Remark 4.5.20 (Non-contradictory). The reader familiar with the work of Jerabek [153] may wonder whether the existence of a conservative translation such as $\mathfrak{A}$ is contradictory. Indeed, in [153, Example 3.3, Theorem 3.6] it was shown that no conservative translation can exist between classical logic, and the logic of the monotone fragment $(\wedge, \vee, \top, \perp)$. If our translation was of the kind studied by Jerabek, then putting $P(-)$ together with the classical double negation translation (e.g. [166]) would yield a contradiction. But Jerabek studies a stronger notion of translation, namely, that from the assumption that $\varphi \vdash_{\text{CPC}} \psi$ we have that this holds if and only if $T(\varphi) \vdash_{\text{CPC}_{\wedge, \vee, \perp, \top}} T(\psi)$. The fact that the entailment relation there is *global* is thereby crucial.

Whilst the existence of such a translation is interesting, the above is certainly, syntactically, deeply unsatisfying. As such, we proceed to extract a proper syntactic description of this translation.

Remark 4.5.21 (Syntactic calculations). Let φ be a formula in intuitionistic logic. Let k be its implication depth. Then by Theorem 4.3.1 φ is represented in the algebra of clopen upsets of $\mathcal{V}_G(\mathbf{2}_{\leq}^\omega)$ by $\pi_k^{-1}[V]$ where $V \subseteq \mathcal{V}_k(\mathbf{2}_{\leq}^\omega)$ is a clopen upset. In turn, by Proposition 4.5.13, V is of the form $\pi_n^{-1}[W]$ where $W \subseteq (\mathbf{2}_{\leq}^n)_k$ is a clopen upset. By all the above commutations, we then have:

1. By definition of $\uparrow^\omega$, we have $(\uparrow^\omega)^{-1}\pi_k^{-1}[V] = (\uparrow^k)^{-1}[V]$.
2. By definition of $\uparrow^k$ as a limit map of the iterated adjoints, $\uparrow_n^k$, we get $(\uparrow^k)^{-1}[\pi_n^{-1}[W]] = \pi_n^{-1}[(\uparrow_n^k)^{-1}[W]]$.

Note that the particular n chosen does not matter by Proposition 4.5.9: given two distinct n, n' , and distinct W' such that $\pi_{n'}^{-1}[W'] = \pi_n^{-1}[W]$, since the connecting maps are always surjective, then $\pi_n^{-1}[(\uparrow_n^k)^{-1}[W]]$ and $\pi_{n'}^{-1}[(\uparrow_{n'}^k)^{-1}[W]]$ will be the same.

Using all of this, and the normal forms of IPC from Definition 4.3.9, we can make the following syntactic construction of $\mathfrak{A}$:

Definition 4.5.22 (Syntactic construction of $\mathfrak{A}$). Let $\bar{p}$ be a set of variables. Let $x \in (\mathbf{2}_{\leq}^{\bar{p}})_n$ be some element, and let χ_x be the associated normal form. We define $\mathfrak{A}(\chi_x)$ by recursion as follows:

1. If $n = 0$, we define $\mathfrak{A}(\chi_x) = \chi_x$.
2. If $d(\chi_x) = n + 1$, then χ_x is of the form:

$$\bigwedge_{y \not\leq x} \left(\chi_y \rightarrow \bigvee_{w \leq y} \chi_w \right),$$

so we define $\mathfrak{A}(\chi_x) = \bigvee_{w \leq y} \mathfrak{A}(\chi_w)$.

Given any formula $\psi(\bar{p})$, where $\psi \equiv \bigvee_{i=1}^n \chi_{x_i}$, we set $\mathfrak{A}(\psi) = \bigvee_{i=1}^n \mathfrak{A}(\chi_{x_i})$.

To check the correctness of this translation, in light of Remark 4.5.21, we thus need only the following lemma:

Lemma 4.5.23 (Correctness of $\mathfrak{A}$). Let $U \subseteq (\mathbf{2}_{\leq}^{\bar{p}})_n$ be an upset, and write $U = \bigcup_{x \in U} \uparrow x$. If $\psi(\bar{p})$ is the formula corresponding to U , that is $\varphi(\psi(\bar{p})) = U$, then we have $\varphi(\mathfrak{A}(\psi)) = (\uparrow^n)^{-1}[U]$.

Proof. We prove this by induction. The case for $n = 0$ is obvious. Assuming the result for n , we show it for $n + 1$. Since the preimage commutes with disjunction, we show this for $\varphi(\chi_x) = \uparrow x$. Assume that we have $\uparrow x = \varphi(\chi_x) = U$, of the form

$$U = \bigcap_{y \not\leq x} (\llbracket \varphi(\chi_y) \rrbracket \Rightarrow \bigcup_{w \leq y} \llbracket \varphi(\chi_w) \rrbracket).$$

As for U , we have $\varphi(\chi_y) = \uparrow y$ as a subset of $(\mathcal{P}(\bar{p}))_n$. Note that $\uparrow^{-1}[U]$ is explicitly given by:

$$\bigcap_{y \not\leq x} \square(\uparrow y \Rightarrow \bigcup_{w \leq y} \uparrow w),$$

where $\square$ is taken with respect to the order in $(\mathcal{P}(\bar{p}))_n$. Note that $\bigcup_{w \leq y} \uparrow w = (\mathcal{P}(\bar{p}))_n \setminus \downarrow y$ by an easy argument. But now observe⁷ that relative pseudocomplements of these pairs are easy to compute: for any poset P , $a, b \in P$, $\square(\uparrow a \Rightarrow X \setminus \downarrow b) = X \setminus \downarrow(\uparrow a \cap \downarrow b)$. But $\downarrow(\uparrow a \cap \downarrow b) = \downarrow b$ in case $a \leq b$, and otherwise is empty. So we obtain:

$$\square(\uparrow y \Rightarrow \bigcup_{w \leq y} \uparrow w) = \bigcup_{w \leq y} \uparrow w,$$

i.e., we drop the antecedent of the implication. ■

⁷This was likewise noted in [11, Lemma 2.7].

Remark 4.5.24 (Complexity considerations). The reader may wonder whether a simpler definition of the translation $\mathfrak{A}(-)$ is possible. Whilst this is not universally ruled out, we note that for complexity reasons, this seems unlikely. By a classical result of Statman [245], validity in IPC is **PSPACE**-complete. By contrast, it is well-known that the complexity of validity of CPC, and consequently of its positive fragment, is **coNP**-complete. So unless **PSPACE** = **coNP**, the translation $\mathfrak{A}(-)$ must impose some form of computational blow up in complexity.

4.5.4 Uniform positive interpolants

We finish this section by noting that besides providing a conservative translation, the map $\mathfrak{A}$ also proves an interesting property relating positive and intuitionistic logic: we show the existence of *uniform positive approximations*, as we proceed to explain.

Definition 4.5.25 (Positive approximation). Let φ be a formula in $\mathcal{L}_{\text{IPC}}$. A formula $\psi \in \mathcal{L}_{\text{PL}}$ is said to be a *positive approximation* of φ if $\vdash_{\text{IPC}} \psi \rightarrow \varphi$.

Every formula has a trivial positive approximation: $\perp \rightarrow \varphi$ is always a theorem. But some are better than others:

Definition 4.5.26 (Fineness order). Given a formula φ , and two positive approximations ψ_1, ψ_2 , ψ_1 is said to be *finer* than ψ_2 if $\vdash_{\text{IPC}} \psi_2 \rightarrow \psi_1$.

A consequence of our translation $\mathfrak{A}$ corresponding to the right adjoint $\mathcal{F}_{\text{DL}}(\omega)$ into $\mathcal{F}_{\text{HA}}(\omega)$, is the following:

Theorem 4.5.27 (Uniform positive approximation). *For each positive formula $\varphi \in \mathcal{L}_{\text{IPC}}$ there exists a uniform positive approximation, the finest positive approximation, $P(\varphi)$: for each positive formula ψ , $\vdash_{\text{IPC}} \psi \rightarrow \varphi$ if and only if $\vdash_{\text{IPC}} \psi \rightarrow P(\varphi)$.*

Proof. By setting $P(\varphi) = \mathfrak{A}(\varphi)$, this is simply a restatement of the fact that $\eta^\dagger$ is the right adjoint of η . ■

Remark 4.5.28 (Uniform positive approximations and uniform interpolants). The terminology chosen for uniform positive approximations reflects the conceptual closeness with uniform interpolation. In fact, after deriving the above results, we found a direct proof, invoking the uniform interpolation, of the existence of uniform positive approximations, from which one can in fact derive the translation $\mathfrak{A}$. The reader may consult this in Appendix A.3.5. We note that one advantage of the above construction is the fact that it provides a very clear and explicit description of this translation by working with normal forms, whilst the more direct uniform interpolation-based proof requires ranging over a collection of positive approximations for a formula.

Chapter 5

Variations and consequences of step-by-step constructions

Having seen in Chapter 4 how step-by-step constructions can be used to describe free objects and colimits, we turn in this chapter to several variations on this construction. Attending to the difference of computation of colimits and free objects in subvarieties, we expend some effort in explaining how the step-by-step construction can be adapted when dealing with subvarieties of **HA**. Following recent work [79], it is shown that in some good cases this general recipe can be substantially simplified, allowing us to obtain many of the same kinds of results as in the previous chapter, often in improved manner: normal forms, reduction principles and relative anarchy.

Whilst the construction of Ghilardi complexes on Priestley spaces presents one possible generalization of the original construction by Silvio Ghilardi, since such a construction was done on finite posets, an orthogonal *discrete* generalization can be done by considering a step-by-step construction connecting the category of posets and the category of locally finite posets. This construction is naturally related to problems of coalgebraic intuitionistic modal logic.

We conclude by briefly sketching some applications which these constructions have found in related settings. Further applications will be given in subsequent chapters of this thesis.

Outline

1. In Section 5.1, we adapt the step-by-step construction to deal with some notable subvarieties of **HA**. By specialising to **BA** we recover the classical Boolean envelope functor. By looking at **Sme** and **LC**, we obtain a construction which terminates in 1 or 2 steps, analogous to similar work done by Carai [79], and use the intuitions of this construction to extract normal forms for these logics. By using it to **KC**, we obtain normal forms, and recover a relative anarchy result of **KC** with respect to **biKC**.

2. In Section 5.2 we adapt the construction to deal with a topology-free reduct, obtaining an adjunction between the categories **Pos** and **LocFinPos**.
3. In Section 5.3, we briefly describe some of the applications of this construction in different, slightly tangential settings, employing both of the constructions from Chapter 4 as well as Sections 5.1 and 5.2.

5.1 Subvarieties of Heyting algebras

Unlike products, coproducts of algebras can vary drastically when one moves from a variety **V** to one of its subvarieties: whilst in the category **Grp**, coproducts are calculated as free groups under quotienting, a word construction, in the category **AbGrp** of abelian groups, they coincide with products. The general picture is somewhat analogous: in general, calculating coproducts, free algebras, and other colimits in subvarieties tends to make things easier, though certainly not always.

Let us return briefly to the general setting of step-by-step discussed in Section 4.1.3. We thus have a variety **K** in a language $\mathcal{L}$ and a variety **K**₀ in a language $\mathcal{L}_0$, where we wish to construct **K**₀-algebras out of **K**. To make the presentation slightly simpler, we made one key simplifying assumption in passing: *all axioms are assumed to contain no nested implications*. However, when dealing with subvarieties of Heyting algebras this is more or less inevitable: consider for instance the Yankov axiom, the law of weak excluded middle:

$$\neg p \vee \neg\neg p.$$

The reason this makes step-by-step representations harder is that once we start with our algebra A , we cannot then in the first step $\mathbf{E}(A, \mathbf{K}_0)$ we do not yet have enough implications to make the quotienting which ensures the operation $\rightarrow$ will satisfy the Yankov axiom.

As noted earlier in [126, 59, 89], this is not a problem: one can wait some steps to begin applying the quotienting. It is likewise possible to flatten some equations to quasi-equations, though this will not be discussed in what follows.

Having this in mind, for the purposes of dual representations, and with the same goals as in Chapter 4, we can outline our general methodology: (1) we extract the semantic meaning of the quotienting axioms applied step-by-step; (2) we then proceed to construct, for a given space X , the specified subspace of $\mathcal{V}_g(X)$ which corresponds to those quotients. Having the one-step extension, the methods of representing free algebras of Chapter 4 become available.

5.1.1 General Heyting subvarieties step-by-step

In this section we briefly outline the general methodology at hand for varieties of Heyting algebras. Note that for each distributive lattice D , and each variety **V** of

Heyting algebras, just as for all Heyting algebras, there is a forgetful functor $\mathbf{V} \rightarrow \mathbf{DL}$ which preserves all limits; it has a left adjoint $\mathbf{F}_{\mathbf{V}}: \mathbf{DL} \rightarrow \mathbf{V}$ computing the free $\mathbf{V}$ -algebra generated by a given distributive lattice. However, by uniqueness of adjoints, we can factor this adjunction as

$$\mathbf{D} \xrightarrow{\mathbf{F}} \mathbf{HA} \xrightarrow{\mathbf{V}_{\text{ref}}} \mathbf{V},$$

where $\mathbf{V}_{\text{ref}}$ is the $\mathbf{V}_{\text{ref}}$ -reflection (see [4, Sec. 3]). This can be calculated explicitly for a free Heyting algebra $\mathcal{F}_{\mathbf{HA}}(X)$, by taking a and a formula $\varphi(p_1, \dots, p_n)$, and considering the filter:

$$\{\varphi(\psi_1, \dots, \psi_n) : \psi_i \in \mathcal{F}_{\mathbf{HA}}(X)\}.$$

In other words, we force all elements to be equal to 1. Such a quotienting could also be done *along* the construction step-by-step. Then, in place of the one-step-extension $\mathbf{E}(A, \mathbf{HA})$, we would, for $\varphi(p_1, \dots, p_n)$ of implication depth n :

$$\mathbf{E}_n(A, \mathbf{HA} + \varphi) = \mathbf{E}_n(A, \mathbf{HA}) / \Theta(\varphi \approx 1).$$

This amounts to forcing each element originating from A to identify φ with 1. Note that this is simply a way of presenting the step-by-step construction of the left adjoint, from which we obtain, amongst other results, a variant of Theorem 4.1.14, which is proven in exactly the same way:

Theorem 5.1.1 (Left adjoint in subvarieties). *Let A be a distributive lattice, and $\mathbf{V}$ a variety of Heyting algebras. Then:*

1. *The algebras $\mathbf{E}_n(A, \mathbf{V})$ together with the inclusion maps, forms a directed system¹.*
2. *The algebra $A_{\omega}^{\mathbf{V}}$, the directed colimit of the above directed system, is a $\mathbf{V}$ -algebra, when equipped with the interpretations canonically induced by the directed chain.*
3. *The functor $\mathbf{F}_{\mathbf{V}}: \mathbf{DL} \rightarrow \mathbf{V}$, computing the colimit $A_{\omega}^{\mathbf{V}}$, is left adjoint to the forgetful functor $\mathbf{V} \rightarrow \mathbf{HA}$.*

Having this in mind, we now explain how this process of quotienting works on the dual spaces:

Definition 5.1.2 (Formula interpretations). Let X be a Priestley space and let $\mathcal{G}^g(X) = (X_n, r_n)$ be the Ghilardi complex over X . Given $U_1, \dots, U_n \subseteq X$, we define the n -th level *formula interpretation* of φ on this sequence on the Ghilardi complex, denoted $\mathcal{I}_n^{\varphi}(U_1, \dots, U_n) = \mathcal{I}_n^{\varphi}(\overline{U})$, induction on the complexity of φ :

¹We note that as an exception to our general rule, we do not force this directed system to be injective – in general it may fail to be, though it will still have a well-defined colimit.

1. If $\varphi = p_i$, then $\mathcal{I}_n^\varphi(\overline{U}) = r_{n,0}^{-1}[U_i]$, and if $\varphi = \top$ then $\mathcal{I}_n^\varphi(\overline{U}) = X_n$ and if $\varphi = \perp$ then $\mathcal{I}_n^\varphi(\overline{U}) = \emptyset$.
2. If $\varphi = \psi_0 \wedge \psi_1$ then $\mathcal{I}_n^\varphi(\overline{U}) = \mathcal{I}_n^{\psi_0}(\overline{U}) \cap \mathcal{I}_n^{\psi_1}(\overline{U})$;
3. If $\varphi = \psi_0 \vee \psi_1$ then $\mathcal{I}_n^\varphi(\overline{U}) = \mathcal{I}_n^{\psi_0}(\overline{U}) \cup \mathcal{I}_n^{\psi_1}(\overline{U})$;
4. If $\varphi = \psi_0 \rightarrow \psi_1$ and $n > 0$, then

$$\mathcal{I}_n^\varphi(\overline{U}) = \llbracket \mathcal{I}_{n-1}^{\psi_0} \Rightarrow \mathcal{I}_{n-1}^{\psi_1} \rrbracket.$$

Remark 5.1.3 (Formula interpretation is a semantics). The above formula interpretation provides a form of semantics for intuitionistic logic over Ghilardi complexes. This intuition is carried in the developments that come, as we do a form of “correspondence theory” for such a semantics.

Definition 5.1.4 (φ one-step extension). Let X be a Priestley space and $\mathcal{G}(X) = (X_n, r_n)$ be the Ghilardi complex over X . Given a formula $\varphi(p_1, \dots, p_n)$ of implication depth n , we consider the subspace

$$X_n^\varphi := \{C \in X_n : (\forall U_1, \dots, U_n \subseteq X), C \in \mathcal{I}_n^\varphi(\overline{U})\}.$$

This is called the φ -relative one-step extension at level n .

Relative one-step extensions serve precisely the same purpose as delayed quotients, as we discussed in the introduction to this chapter. This leads to the following definition:

Definition 5.1.5 (Relative Ghilardi complex). Let L be a superintuitionistic logic, and $\mathbf{V} = \text{Alg}(L)$. Given a Priestley space X , we denote by $\mathcal{G}_{\mathbf{V}}(X) = (X_n^{\mathbf{V}}, r_n)$ the relativised Ghilardi complex over X , obtained as follows:

1. $X_{0,-}^{\mathbf{V}} = X$.
2. For each $n > 0$, $X_{n,-}^{\mathbf{V}} = \bigcap_{\varphi \in L, d(\varphi) \leq n} X_n^\varphi$.

We denote the $\mathbf{V}$ Ghilardi complex to be the set of Priestley spaces which are in the essential image of the construction: for each n , $X_0^{\mathbf{V}} = \bigcap_{n \in \omega} r_{n,0}[X_{n,-}^{\mathbf{V}}]$. We denote by $\mathcal{V}_G^{\mathbf{V}}(X)$ the inverse limit of the $\mathbf{V}$ -Ghilardi complex, equipped with the usual inverse limit structure of root maps.

Theorem 5.1.6 (Relative duals of free algebras). *For each variety of Heyting algebras $\mathbf{V}$ and Priestley space X , the space $\mathcal{V}_G^{\mathbf{V}}(X)$ is dual to the free $\mathbf{V}$ -algebra generated by $\text{CloUp}(X)$.*

Proof. This follows immediately by duality and Theorem 5.1.1. ■

One interesting feature of this representation is that it allows us to find a different way of understanding completeness properties. In Appendix A.3.1, this is done to discuss the FMP. But for local tabularity and tabularity, a lot more is true (see Section 2.2.3 for the connection between local tabularity and the algebraic correspondents):

Theorem 5.1.7 (Local tabularity and tabularity in free algebras). *Let $\mathbf{V}$ be a variety of Heyting algebras. Then:*

1. *If $\mathbf{V}$ is a locally finite variety, and, then: given P be a finite poset, with $\mathcal{G}^{\mathbf{V}}(P) = (P_n, r_n)$ being the $\mathbf{V}$ -Ghilardi complex over P , there is some n such that $P_n \cong P_{n+1}$ via the root map.*
2. *If $\mathbf{V}$ is a finitely generated variety, then: there exists an n such that for each Priestley space X , if $\mathcal{G}^{\mathbf{V}}(X) = (X_n, r_n)$ is the $\mathbf{V}$ -Ghilardi complex over X , then $X_{n+1} \cong X_n$ via the root map.*

Proof. (1). Let P be a finite poset, and embed $P \subseteq \mathbf{2}_{\leq}^P$. Since $\mathcal{V}_G^{\mathbf{V}}$ is a right adjoint, as in Theorem 4.4.10 we can see that $\mathcal{V}_G^{\mathbf{V}}(P) \subseteq \mathcal{V}_G^{\mathbf{V}}(\mathbf{2}_{\leq}^P)$. Since by Theorem 5.1.6 and Theorem 4.3.1, the latter correspond to the duals of free $\mathbf{V}$ -algebras, and the variety is assumed to be locally finite, then $\mathcal{V}_G^{\mathbf{V}}(\mathbf{2}_{\leq}^P)$, yielding that $\mathcal{V}_G^{\mathbf{V}}(\mathbf{2}_{\leq}^P)$ is finite. Since inverse limits are finite if and only if on a cofinal tail the inverse maps are trivial (since we forced the elements appearing in the complex to be in the surjective image), this implies the result.

(2). This is easiest to see algebraically. Consider a distributive lattice D , and consider the chain:

$$D \hookrightarrow D_1^{\mathbf{V}} \hookrightarrow D_2^{\mathbf{V}} \hookrightarrow \dots,$$

where $D_i^{\mathbf{V}} = \text{CloUp}(X_i)$ with X_i the i -th space in the $\mathbf{V}$ -Ghilardi complex over $X = \text{Spec}(D)$. Note that because $\mathbf{V}$ is finitely generated, it is generated by a finite algebra H . Every distinct formula $\varphi(\bar{x})$ corresponds therefore to a distinct function $H \rightarrow H$ over $\bar{p}$ many variables, by tabularity. Since H is finite, there is a maximum implication depth amongst formulas in H : every formula $\varphi(p_1, \dots, p_n)$ corresponds to a function $f(x_1, \dots, x_n): H^n \rightarrow H$ by tabularity. By the pigeonhole principle², if we fix n , there is a maximum implication depth i_n over n many variables. By picking $f_{i_1}, f_{i_2}, \dots$ a sequence of functions over k many variables of maximum implication depth, we again by the pigeonhole principle obtain that there is a maximum implication depth, which will be some fixed n . Then it follows that $D_n^{\mathbf{V}} \cong D_{n+1}^{\mathbf{V}}$ since each formula over elements from D will be equivalent to one with fewer implications, and consequently, already introduced in a previous stage. This shows the desired result. ■

To illustrate this general methodology, we will take a few select simple cases. We will proceed along the Kuznetsov hierarchy: from the simplest logic, CPC, to the one closest to IPC, the Yankov logic KC:

²For more precise bounds see Chapter 6.

5.1.2 Boolean algebras

To characterize the one step-extension of Boolean algebras, take the characteristic axiom $p \vee \neg p$. We to define the one-step extension for this axiom by removing from $\mathcal{V}_r(X)$ all the closed rooted subsets which do not satisfy $\llbracket U \rrbracket \cup \llbracket -U \rrbracket$ for each clopen upset $U \subseteq X$.

Lemma 5.1.8 (Boolean one-step extension). Let X be a Priestley space. Given $C \in \mathcal{V}_r(X)$, we have that $C \in \llbracket U \rrbracket \cup \llbracket -U \rrbracket$ for each clopen upset $U \subseteq X$ if and only if C is a singleton.

Proof. If $C = \{x\}$ is a singleton then either $x \in U$ or $x \notin U$, and so either $C \in \llbracket U \rrbracket$ or $C \in \llbracket -U \rrbracket$. Now assume that C has two points x, y . Then either $x \not\leq y$ or $y \not\leq x$; without loss of generality assume the former. Then $x \in U$ and $y \notin U$ for some clopen upset. So $C \notin \llbracket U \rrbracket$ since $y \notin U$, and $C \notin \llbracket -U \rrbracket$ since $x \in U$, as desired. ■

We denote by $\text{Disc}(X) \subseteq \mathcal{V}_r(X)$ the set of singletons of X , which by Lemma 5.1.8 is a closed subspace, defining precisely the points satisfying the law of excluded middle. Then we have:

Lemma 5.1.9 (One-step termination). The space X is homeomorphic to $\text{Disc}(X)$ via the map $s: x \mapsto \{x\}$. Moreover the map $r: \mathcal{V}_r(\text{Disc}(X)) \rightarrow \text{Disc}(X)$ is an isomorphism.

Proof. If W is a clopen, then by construction of the map $s^{-1}[\llbracket W \rrbracket] = W$; since the sets $\llbracket W \rrbracket$ form a basis, and preimages commute with complement, this is enough to show that the map is continuous. It is clear that it is bijective, and bijective continuous functions between Stone spaces are homeomorphisms.

Note also that if Y is a discretely ordered Priestley space (definitionally equivalent to a Stone space), then the root map $r: \mathcal{V}_r(Y) \rightarrow Y$ is a homeomorphism, being the inverse of the singleton map just described. This implies that $\mathcal{V}_r(\text{Disc}(X)) \cong \text{Disc}(X)$, through the root map, as claimed. ■

From Theorem 5.1.6 it follows that $\text{Disc}(X)$ is precisely the dual to the free Boolean algebra generated by the dual distributive lattice $\mathbf{CloUp}(X)$. This is nothing more than the well-known dual representation of the Boolean envelope construction given in Proposition 2.1.50. From this point of view, one obtains that the results of Section 4.1.2 are subsumed under our framework: one can construct representations of free Boolean algebras, normal forms for classical logic, and descriptions of coproducts of Boolean algebras. We also note that in light of Theorem 5.1.7, the fact that the construction terminates after some number of steps is to be expected by the tabularity of CPC – though we will return later to the question of why precisely one step is enough.

5.1.3 Smetanich and LC-Algebras

Next we consider the case of Gödel algebras, denoted as **GA**, and the special case of **GA**-algebras of depth 2, which we call Smetanich algebras, or **Sme**-algebras³. Following the methodology outlined above, we consider the axioms $(p \rightarrow q) \vee (q \rightarrow p)$, and $\neg p \vee q \vee (q \rightarrow p)$ (see Definition 2.3.5), and characterize the meaning of them when looking at elements of $\mathcal{V}_g(X)$, for $g: X \rightarrow Y$ a continuous and order-preserving map:

Proposition 5.1.10 (Locally linear one-step extension). Let $g: X \rightarrow Y$ be a continuous and order-preserving function. Then for each $C \in \mathcal{V}_g(X)$, we have:

1. $C \in \llbracket U \Rightarrow V \rrbracket \cup \llbracket V \Rightarrow U \rrbracket$ for each clopen upset $U, V \subseteq X$ if and only if C is a chain.
2. $C \in \llbracket -U \rrbracket \cup \llbracket V \rrbracket \cup \llbracket V \Rightarrow U \rrbracket$ for each clopen upset $U, V \subseteq X$ if and only if C is a chain of size 2.

Proof. (1) Assume that $C \notin \llbracket U \Rightarrow V \rrbracket \cup \llbracket V \Rightarrow U \rrbracket$. This amounts to saying that $C \not\subseteq U \Rightarrow V$ and $C \not\subseteq V \Rightarrow U$. In other words, $x \in C$ and $x \in U \setminus V$ and $y \in C$ and $y \in V \setminus U$. Since these sets are clopen upsets, this means that $x \not\leq y$ and $y \not\leq x$. Hence C is not a chain.

Conversely, assume that C is not a chain. Since $x \not\leq y$, there is some clopen upset U such that $x \in U$ and $y \notin U$, and since $y \not\leq x$, there is some clopen upset V such that $y \in V$ and $x \notin V$. These two subsets then witness that $C \notin \llbracket U \Rightarrow V \rrbracket \cup \llbracket V \Rightarrow U \rrbracket$.

(2) If $C \notin \llbracket -U \rrbracket \cup \llbracket V \rrbracket \cup \llbracket V \Rightarrow U \rrbracket$, this unfolds to tell us that there is an element $x \in C$ such that $x \in U$; an element $y \notin V$ and an element $z \notin V \Rightarrow U$. If these elements are not comparable, then we are done, since C is then not a chain; otherwise, note that C will be a chain of size at least 3, since these elements cannot be equal.

For the converse, note that since the Smetanich axiom implies the Gödel-Dummett axiom, if C fails to be a chain, then by (1) we are done. So assume that C is a chain containing at least three elements: at least the elements $r(C) \leq x \leq y$, for x, y distinct. Using the Priestley separation axiom we find some V such that $x \in V$ and $r(C) \notin V$, and some U such that $y \in U$ and $x \notin U$. It follows that $C \notin \llbracket -U \rrbracket \cup \llbracket V \rrbracket \cup \llbracket V \Rightarrow U \rrbracket$, giving us the result. ■

Given $g: X \rightarrow Y$ a continuous and order-preserving map, let us denote by $\text{Nerve}_g(X)$ the subspace of X given by the φ -one step extension where φ is the Gödel-Dummett axiom; denote by $\text{Nerve}_g^2(X)$ the subspace of the Smetanich one step extension. Again, by Theorem 5.1.6, we obtain the duals of free Gödel algebras, respectively Smetanich algebras, over a distributive lattice, by forming

³In the literature these are also sometimes denoted as **GA**₂, see e.g. [79].

Ghilardi complexes and taking inverse limits. But just like in the Boolean case, by Theorem 5.1.7 we can expect some simplifications, since **Sme** is tabular, and **LC** is locally tabular (Proposition 2.3.26). In fact, we will show that in both cases the construction terminates after two steps.

Notation 5.1.1 (Iterated Gödel extensions). We denote by

$$\text{Nerve}_!(X) := \text{Nerve}_!(X) \text{ and } \text{Nerve}_{k+1}(X) := \text{Nerve}_{r_k}(\text{Nerve}_k(X)),$$

where r_k is the restriction of the root map, and $!$ is the terminal map to a one element poset.

Lemma 5.1.11 (Comparability in nerve space). Let $C, D \in \text{Nerve}_2(X)$. Suppose that $r_1(C) = r_1(D)$ and that C and D are comparable. Then $C = D$.

Proof. Assume that $C \in \text{Nerve}_2(X)$, and let R be its root. We want to show that for each $A \in C$, we have:

$$A = (\uparrow r_0(A)) \cap r_1(C).$$

One inclusion is clear. For the other, assume that $r_0(A) \leq x$ where $x \in r_1(C)$. Then $E = r_1(C) \cap \uparrow x$ will be an element in C . Since $A \in C$, and C is linearly ordered, we have that either $A \subseteq E$ or $E \subseteq A$. In the former case, since $r_0(A) \leq x$, $r_0(A) \in E$ implies that $r_0(A) = x$, and so $x \in A$. Conversely, we obtain directly that $x \in A$. We conclude the equality.

Now take $C, D \in \text{Nerve}_2(X)$ and $r_1(C) = r_1(D)$, and $D \subseteq C$. Suppose that $A \in C$; then $r_1(C) \supseteq A$, so $r_1(D) \supseteq A$; because D is r_0 -open, there is some $A' \in D$ such that $r_0(A') = r_0(A)$. Then $A' \in C$ as well, and by what was shown:

$$A' = (\uparrow r_0(A')) \cap r_1(C) = (\uparrow r_0(A)) \cap r_1(C) = A.$$

Hence $A \in D$ as desired. This implies that $C = D$. ■

Theorem 5.1.12 (2-step convergence). For each $n \geq 2$, $\text{Nerve}_n(X) \cong \text{Nerve}_2(X)$. Consequently $\text{Nerve}_2(X)$ is isomorphic to the dual of the free Gödel algebra generated by $\text{ClopUp}(X)$.

Proof. It suffices to show that the map $r_2: \text{Nerve}_3(X) \rightarrow \text{Nerve}_2(X)$ is an isomorphism; if this is the case, then r_2 -open subsets are forced to be in 1-1 correspondence with elements of $\text{Nerve}_3(X)$, and inductively, this implies that $\text{Nerve}_2(X)$ is isomorphic to the inverse limit.

First, note that if $C \in \text{Nerve}_2(X)$, then $\uparrow C$ is an r_2 -open chain. It is clear that it is r_2 -open (Remark 4.2.10). To see that it is a chain, we will show that in fact whenever $C \supseteq D$, then there exists a (unique) element $a \in C$ such that $D \cong \uparrow a$. Indeed, note that if $C \supseteq D$, then taking $C_{r(D)} = C \cap \uparrow r(D)$, we obtain two comparable chains with the same root; thus by Lemma 5.1.11, $D = C_{r(D)}$.

From this, if $C \supseteq B, B'$, it easily follows that either $B \supseteq B'$ or $B' \supseteq B$, and so $\uparrow C$ is a chain.

This map shows that $\uparrow: \text{Nerve}_2(X) \rightarrow \text{Nerve}_3(X)$ is a well-defined map. As usual, we can easily see that for each $C \in \text{Nerve}_2(X)$, $r_2(\uparrow C) = C$. Given $E \subseteq \text{Nerve}_2(X)$ a chain which is r_1 -open, we now wish to show that:

$$E = \uparrow r_2(E).$$

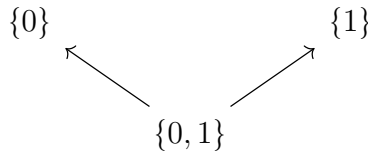
The left to right inclusion is obvious. For the right to left, assume that $r_2(E) \supseteq A$. For each $B \in A$, consider $A_B = \uparrow B \cap A$; then $r_2(E) \supseteq A_B$, and E is r_1 -open, so there exists some $B' \in E$ such that $r_1(B') = B$. This holds in particular for $r_1(A)$, yielding some A' . So consider:

$$E_{r_1(A)} = \uparrow r_1(A) \cap E.$$

Note that then $A \subseteq r_2(E_{r_1(A)})$: for each $B \in A$ such that $B \neq r_1(A)$, we have that $r_1(A) \supseteq B$; since by what we just noted, $B' \in E$, and E is a chain, if we have $B' \subseteq E_{r_1(A)}$ we would obtain that $B = r_1(B') = r_1(A)$, a contradiction. But then we have that A and $E_{r_1(A)}$ are both r_1 -open chains, and $r_1(A) = r_1(E_{r_1(A)})$, so by Lemma 5.1.11 we have $A = E_{r_1(A)}$; it follows that $A \in E$. This shows that $E = \uparrow r_2(E)$.

We have thus proved that $\uparrow$ is, order-theoretically, an inverse of r_2 , meaning that r_2 is an order-isomorphism. It is therefore a continuous bijection between Stone spaces, which makes it furthermore an order-homeomorphism, proving the result. ■

Remark 5.1.13 (One step is not enough). Consider the poset $P = \mathbf{2}_{\leq}$. Then $\text{Nerve}^2(P) = \text{Nerve}(P)$, and in both cases it contains exactly three elements: $\{0\}, \{1\}, \{0, 1\}$:



it is clear that such a construction is not locally linear.

Remark 5.1.14. Reading the above theorem 5.1.12 algebraically, or via the categorical adjunction, this implies that every formula over LC is equivalent to one with implication depth at most 2. This suggests a definition, which sits intermediate to those of Theorem 5.1.7:

Definition 5.1.15 (Uniform local tabularity). A superintuitionistic logic L is *n-uniform* if each formula $\varphi(\bar{p}) \in \mathcal{L}_{\text{IPC}}$ is equivalent to a formula $\psi(\bar{p})$ of implication depth n , modulo L . We say that L is *uniformly locally tabular* if L is n -uniform for some n .

We will study this property, in depth, using different methods, in Chapter 6.

The added simplicity of the above representation suggests that it should be possible to essentially “package” the whole step-by-step approach in one single construction. This approach was pursued by Carai [79], developed independently and in parallel with the present work⁴. Since we will later make use of it, we briefly recall this construction:

Definition 5.1.16 (Carai’s construction). Given a Priestley space X , we denote by $(\text{CC}(X), \leq)$ the following space:

1. $\text{CC}(X)$ carries the same space and topology as $\text{Nerve}(X)$;
2. For each C, D , we have that $C \leq D$ if and only if $C \supseteq D$ and $D = C \cap \uparrow r(D)$.

We denote by $\text{CC}_2(X)$ the subspace consisting of the elements from $\text{Nerve}^2(X)$.

In essence, the only change relative to $\text{Nerve}(X)$ is that one changes the order relation, and with it manages to work with a single iteration of the one-step construction. Then we have:

Theorem 5.1.17 (Duals of free Gödel algebras). *Let X be a Priestley space and $D = \text{CloUp}(X)$:*

1. $(\text{CC}(X), \leq)$ is an Esakia space, and is dual to the free Gödel algebra generated by D .
2. $(\text{CC}_2(X), \leq)$ is an Esakia space, and is dual to the free Smetanich algebra generated by D .

Proof. This was proven by Carai in [79, Theorem 3.19, Theorem 3.26]. ■

Normal forms and polynomial algebras for Smetanich logic

One of the interests of normal forms is in great part computational. Thus, it is valuable to obtain from free algebra representations, not just normal forms, but effective *reduction principles* – algebraic equivalences showing how to reduce formulas to a given normal form.

Example 5.1.18 (Semilinearity reduction principles). It is well-known that in **GA**-algebras (see e.g. [120]), the “semilinearity” reduction principles $p \rightarrow (q \vee r) = (p \rightarrow q) \vee (p \rightarrow r)$ and $(p \wedge q) \rightarrow r = (p \rightarrow r) \vee (q \rightarrow r)$ are valid.

⁴For a comparison with that approach one can check [79, Sec. 6], which discusses the relationship between Carai’s construction and ours. We also have some remarks in Appendix A.3.2, explaining how one can see the precise connection between the two approaches from the point of view of the sheaf theoretic duality.

Some more reduction principles, holding generally over **HA** and **GA**-algebras, can be summarised in the following:

Lemma 5.1.19 (Basic reductions). The following principles all hold for **HA**:

1. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$ and $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$;
2. $1 \rightarrow x = x$ and $x \rightarrow 1 = 1$ and $0 \rightarrow x = 1$.
3. $\neg(x \rightarrow y) = \neg\neg x \wedge \neg y$.

Moreover, the semilinearity principles $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$ and $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$ hold for **GA** algebras.

Proof. The principles (1-2) are standard equalities of **HA**, and semilinearity principles hold in **GA** as noted in example 5.1.18. We prove (3) for completeness. Certainly since $y \leq (x \rightarrow y)$, $\neg(x \rightarrow y) \leq \neg y$, and since $\neg x \leq x \rightarrow y$, $\neg(x \rightarrow y) \wedge \neg x \leq \neg(x \rightarrow y) \wedge (x \rightarrow y) = 0$, so $\neg(x \rightarrow y) \leq \neg\neg x \wedge \neg y$. For the converse, note that:

$$\neg\neg x \wedge \neg y \leq \neg(x \rightarrow y) \iff \neg\neg x \wedge \neg y \wedge (x \rightarrow y) \leq 0.$$

Since it holds in any Heyting algebra that $(x \rightarrow y) \wedge (y \rightarrow z) \leq x \rightarrow z$, it follows that $(x \rightarrow y) \wedge \neg y \leq \neg x$; so the condition on the right holds, since $\neg x \wedge \neg\neg x = 0$. ■

By Theorem 5.1.12, we have that each formula over **Sme** must be equivalent to one of implication depth 2. This can be partly sharpened: we can prove that only double negations are necessary. For this purpose, we need to show that formulas of implication depth 2 can have the last implication flattened to a double negation. If these are formulas of the form $p \rightarrow (q \rightarrow r)$, this is certainly equivalent to $(p \wedge q) \rightarrow r$; so the key problem arises when the implications are nested on the left. For that we use the following⁵:

Proposition 5.1.20 (Flattening). The following is valid in Smetanich algebras:

$$(x \rightarrow y) \rightarrow z = ((x \rightarrow y) \wedge z) \vee \neg(x \rightarrow y) \vee ((y \rightarrow z) \wedge x) \quad (1\text{-flat.})$$

Proof. First note that each of the disjuncts on the right implies $(x \rightarrow y) \rightarrow z$, in any Heyting algebra:

⁵For the mathematically-practically-minded reader, we stress the methodology: this principle was obtained by (1) analyzing Carai's construction over three variables, (2) calculating an equivalent formula without nested implications, and then (3) finding a simple equivalent of it. The algebraic proof given here is much shorter and elegant, though it obscures these underlying intuitions.

1. $(x \rightarrow y) \wedge z = ((x \rightarrow y) \rightarrow z) \wedge (x \rightarrow y) \leq ((x \rightarrow y) \rightarrow z)$;
2. Since $0 \leq z$, then $\neg a \leq a \rightarrow z$; hence $\neg(x \rightarrow y) \leq (x \rightarrow y) \rightarrow z$;
3. We have $(y \rightarrow z) \wedge x \leq (x \rightarrow y) \rightarrow z$ if and only if $(y \rightarrow z) \wedge x \wedge (x \rightarrow y) \leq z$; but since $(x \rightarrow y) \wedge (y \rightarrow z) \leq x \rightarrow z$, this certainly holds.

We now show the converse direction, by considering evaluations in the 3-element Heyting algebra. For that purpose, we consider the possible evaluations of x, y, z . By linearity, we have three cases: (1) $x \leq y$, or (2) $y < x$ and $y = 0$, or (3) $y = a$ and $x = 1$.

1. In the first case, $x \rightarrow y = 1$, so $(x \rightarrow y) \rightarrow z = z = (x \rightarrow y) \wedge z$, and we are done.
2. If $0 = y < x$, then $x \rightarrow y = y = 0$, and so $(x \rightarrow y) \rightarrow z = (0 \rightarrow z) = 1$; on the other hand also $\neg(x \rightarrow y) = 1$, so we are done.
3. Finally if $y = a$ and $x = 1$, then $x \rightarrow y = a$, so $(x \rightarrow y) \rightarrow z = a \rightarrow z$. In case $a \leq z$, then we obtain on the left 1; then $(y \rightarrow z) \wedge x = 1$ as well. In case $z = 0$, then $a \rightarrow z = 0$, which will be trivially smaller than anything. So we are done.

This concludes all cases, showing the validity of the principle. ■

Theorem 5.1.21 (Normal forms for Smetanich logic). *Every formula $\varphi(\bar{p})$ over **Sme** is equivalent to a formula of the form:*

$$\bigvee_{i=1}^n \left(\bigwedge_{j=1}^m (p_{ij} \rightarrow q_{ij}) \wedge \bigwedge_{j'=1}^{m'} \chi_{ij'}(p_{ij'}) \right),$$

where $\chi_{ij'}(p_{ij'}) \in \{p_{ij'}, \neg p_{ij'}, \neg \neg p_{ij'}\}$.

Proof. Considering an arbitrary formula $\varphi(\bar{p})$, we show that it can be placed in this form by applying the principles from Lemma 5.1.19 and the flattening formula from Proposition 5.1.20. For this purpose, assume that $\varphi(\bar{p})$ has implication depth 1. By disjunctive normal form, we know it can be written as:

$$\bigvee_{i=1}^n \bigwedge_{j=1}^m \psi_{ij} \rightarrow \chi_{ij},$$

where ψ_{ij}, χ_{ij} are disjunctions of conjunctions of literals, $\top$ and $\perp$. Applying the basic principles of **HA** and the semilinearity laws, we obtain that $\varphi(\bar{p})$ is equivalent to:

$$\varphi(\bar{p}) \equiv \bigvee_{i=1}^n \left(\bigwedge_{j=1}^m (\chi_{ij}(p_{ij}) \rightarrow \chi_{ij}(q_{ij})) \wedge \bigwedge_{j'=1}^{m'} \chi_{ij'}(p_{ij'}) \right);$$

Note that any implication of the form $p \rightarrow \neg q$ can be turned into one of the form $\neg(p \wedge q) = \neg p \vee \neg q$, so we can always assume that in the consequent we have a non-negative literal. If we have $\neg p \rightarrow q$, by applying (1-flattening) this is equivalent to:

$$(\neg p \wedge q) \vee \neg \neg p.$$

If we consider implications between formulas $\varphi(\bar{p})$ in this form, then 1-flattening guarantees that we again obtain an equivalent formula of implication depth 1. Thus we have that every formula must be equivalent to one of the desired shape. ■

This allows us to obtain a simple description of Smetanich polynomial algebras:

Definition 5.1.22 (Logic relative polynomial algebras). Let H be an L -algebra for L a superintuitionistic logic. We denote by $H[x]^L = H \oplus_{\mathbf{V}_L} \mathcal{F}_{\mathbf{V}_L}$, the Heyting algebra of polynomials over L .

We note the following:

Proposition 5.1.23 (Polynomial Smetanich algebra). The elements of $H[x]^{\text{Sme}}$ are all of the form,

$$\bigvee_{i=1}^n \chi_i \text{ where } , \exists T \subseteq W \ \chi_i := \bigwedge T.$$

for $a_i, b_i, c_i \in H$ and $W = \{(x \rightarrow a_i), (b_i \rightarrow x), \neg \neg x, c_i\}$.

Proof. By Theorem 5.1.21, we know that every element of this algebra can be represented in normal form. Note that if $a, b \in H$, then $a \rightarrow b \in H$, and same for negations and conjunctions. Moreover we can simply $(x \rightarrow a) \wedge (x \rightarrow b) = x \rightarrow (a \wedge b)$, and same with $(c \rightarrow x) \wedge (d \rightarrow x) = (c \vee d) \rightarrow x$. So by simplifying the expression, we get that the χ_i are precisely all the shapes which can be formed by conjunctions. ■

For **LC**, the normal forms are slightly more involved, and appeared before in [267], where they were studied using term rewriting systems. We highlight the most important such principle:

Proposition 5.1.24 (Unnesting principle). The following principle is valid in all Gödel algebras:

$$(p \rightarrow q) \rightarrow r = (q \rightarrow r) \wedge (((p \rightarrow q) \rightarrow q) \vee r) \quad (\text{Unn})$$

Proof. See [267, Table 1, C11]. ■

Note that from the unnesting principle, it follows that iterated implications of depth greater than 2 can be reduced: given $((p \rightarrow q) \rightarrow r) \rightarrow s$: we begin by applying unnesting to $(p \rightarrow q) \rightarrow r$ inside of this formula, and obtain the following as a valid sequence of equalities:

$$\begin{aligned}
& ((q \rightarrow r) \wedge (((p \rightarrow q) \rightarrow q) \vee r)) \rightarrow s \\
&= ((q \rightarrow r) \rightarrow s) \vee (((p \rightarrow q) \rightarrow q) \vee r) \rightarrow s \\
&= ((q \rightarrow r) \rightarrow s) \vee (((p \rightarrow q) \rightarrow q) \rightarrow s) \wedge (r \rightarrow s) \\
&= ((q \rightarrow r) \rightarrow s) \vee (((q \rightarrow s) \wedge (((p \rightarrow q) \rightarrow q) \rightarrow q) \vee s) \wedge (r \rightarrow s)) \\
&= ((q \rightarrow r) \rightarrow s) \vee (((q \rightarrow s) \wedge ((p \rightarrow q) \vee s) \wedge (r \rightarrow s));
\end{aligned}$$

The first equality follows by the semilinearity axiom $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$; the second equality by the usual axiom $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$; the third applies the unnesting principle again; the fourth, which no longer contains nested implications of depth greater than 2, follows from the fact, true in any Heyting algebra, that $((p \rightarrow q) \rightarrow q) \rightarrow q = p \rightarrow q$ ⁶. The upshot of this is the following:

Theorem 5.1.25 (Normal form theorem for LC). *Every formula $\varphi(\bar{p})$ over **LC** is equivalent to a formula of the form:*

$$\bigvee_{i=1}^n \left(\bigwedge_{j=1}^m \chi_{ij}(\alpha_{ij}, \beta_{ij}) \right),$$

where $\alpha_{ij}, \beta_{ij} \in \{p_i : i \leq n\} \cup \{\top, \perp\}$, and $\chi_{ij}(\alpha_{ij}, \beta_{ij}) \in \{\alpha_{ij} \rightarrow \beta_{ij}, (\alpha_{ij} \rightarrow \beta_{ij}) \rightarrow \beta_{ij}\}$.

Proof. This follows from Proposition 5.1.3, and our calculations above, by the same arguments as in Theorem 5.1.21. ■

As a Corollary we get:

Corollary 5.1.26 (Polynomial Gödel algebra). *The elements of $H[x]^{\text{LC}}$ are disjunctions of finite conjunctions of elements of the form:*

$$(x \rightarrow a_i), (b_i \rightarrow x), ((c_i \rightarrow x) \rightarrow x), d_i, ((x \rightarrow e_{ij}) \rightarrow e_{ij}),$$

where for each type of element, except for the last, we may assume that there is a single conjunct of that type.

Proof. Simply run the same simplifications as in Proposition 5.1.23 to obtain the elements in this form. Note that conjunctions of elements of the form $(x \rightarrow e) \rightarrow e$ cannot be further simplified. ■

⁶Note that this is simply an instance of the law of triple negation elimination when one relativises to the Heyting algebra $[a, 1]$.

5.1.4 DeMorgan Heyting algebras

We finally turn to **KC**. Unlike **CPC**, **Sme** and **LC**, and similarly to **IPC** **KC** is not locally tabular. As such, we should not expect a fundamental simplification of the step-by-step procedure. Nevertheless, we can follow our methodology to find a modified Ghilardi complex representation, which we will employ to study normal forms and relative anarchy for this variety. We begin with our correspondence Lemma:

Lemma 5.1.27 (Locally directed one-step extension). Let $g: X \rightarrow Y$ be a continuous and order-preserving map. Then for each $C \in V_g(X)$ we have that the following are equivalent:

1. For each clopen upset $U, V \subseteq X$ $C \in \llbracket -(U \cap V) \rrbracket$ if and only if $C \in \llbracket -U \rrbracket \cup \llbracket -V \rrbracket$;
2. C has a largest element.

Proof. (1) $\Rightarrow$ (2) Assume that C does not have a largest element. Since C is a closed subset, we know that $\max(C) \neq \emptyset$, and this is a closed set (see Proposition 2.1.52.(3)); let $x, y \in \max(C)$ be two distinct elements, so that $x \not\leq y$ and $y \not\leq x$. Let $x \in U$ where $y \notin U$. Since $\max(C) \setminus U$ is closed, and we have that $\downarrow(\max(C) \cap U) \cap \uparrow(\max(C) \setminus U) = \emptyset$ (since elements of $\max(C)$ are pairwise incomparable), we can find V a clopen upset such that $\max(C) \setminus U \subseteq V$ and $\max(C) \cap U \subseteq -V$. Then note that $C \subseteq -(U \cap V)$: for each $c \in C$, we can pick $c \leq c'$, where $c' \in \max(C)$; if $c' \in U$, then by construction, $c' \notin V$, so $c \notin V$ given this is an upset. On the other hand $C \not\subseteq -U$ and $C \not\subseteq -V$ since $x \in U$ and $y \in V$ by construction.

(2) $\Rightarrow$ (1) Assume that C has a largest element. Notice that $\llbracket -U \rrbracket \cup \llbracket -V \rrbracket \subseteq \llbracket -(U \cap V) \rrbracket$ because the order is reverse inclusion. So assume that $C \in \llbracket -(U \cap V) \rrbracket$. Let $c \in C$ the maximal element; then either $c \notin U$ or $c \notin V$ by hypothesis. Since U, V are upsets, and c is the largest element, $x \leq c$, which implies that $x \notin U$ or $x \notin V$, for whichever holds of C . Thus $C \subseteq -U$ or $C \subseteq -V$, as desired. ■

Denote by $\mathcal{V}_r^{\mathbf{KC}}(X)$ the subspace consisting of rooted subsets with a largest element. As noted, we can of course obtain $\mathcal{V}_G^{\mathbf{KC}}(X)$, the limit of the modified Ghilardi complex, to obtain the dual of free **KC**-algebras. However, this turns out to not be necessary from the second step onwards:

Lemma 5.1.28 (Largest elements preserved). For each Priestley space X if we denote by $\mathcal{V}_2^{\mathbf{KC}}(X) := \mathcal{V}_{r_0}^{\mathbf{KC}}(\mathcal{V}_r^{\mathbf{KC}}(X))$, then we have that $\mathcal{V}_{r_1}(\mathcal{V}_2^{\mathbf{KC}}(X)) = \mathcal{V}_{r_1}^{\mathbf{KC}}(\mathcal{V}_2^{\mathbf{KC}}(X))$.

Proof. First we will show that $\mathcal{V}_2^{\mathbf{KC}}(X)$ is locally directed, i.e., each $C \in \mathcal{V}_2^{\mathbf{KC}}(X)$ has a largest element above it. Let $C \supseteq D$ be some maximal element (these exist by Proposition 2.1.47). We claim that then $D = \{c\}$ for some $c \in \mathcal{V}_r^{\mathbf{KC}}(X)$.

Assume not, say $c < d$ are two elements in D . Since $d \in D$, look at $D_d = D \cap \uparrow d$; then $D \supseteq D_d$, but by our assumption, $c \notin D_d$; since D is maximal, this is a contradiction.

We also note that the element c must itself be of the form $\{x\}$ for some $x \in X$. Otherwise, if $x, y \in c$, say $x < y$, one could look at $c \supseteq c_y$, and obtain by r_0 -openness of D that there is some c' , such that $c \supseteq c'$ and $c' \in D$, and $r_0(c') = y$, a contradiction.

By construction, C contains a largest element, c ; we then have that $C \supseteq \{c\}$. Now assume that D is any maximal element above C , let us show that $D = \{c\}$. By what we just showed, we know that $D = \{\{x\}\}$; thus $\{x\} \in C$ and since c is the largest, $\{x\} \supseteq c$; but then certainly $c = \{x\}$. This shows that every maximal element above C coincides, as desired.

Now suppose that $R \in \mathcal{V}_{r_1}(\mathcal{V}_2^{\mathbf{KC}}(X))$; we want to show that R has a largest element, and thus, that $R \in \mathcal{V}_{r_1}^{\mathbf{KC}}(\mathcal{V}_2^{\mathbf{KC}}(X))$. Let $R_2 = r_2(R)$ be the root, and let D be the largest element such that $R_2 \supseteq D$, which exists by our argument above; note that we constructed it as $D = \{z\}$ for z the largest element in R_2 . Since R is r_1 -open, there is some $D' \in R$ such that $r_1(D') = z$. Since z is largest in R_2 , and so there can be no element above it in D' , we must have that $D' = D = \{z\}$. Then if $E \in R$, we can pick $E \supseteq E'$ where E' is some maximal element, say $E' = \{c\}$, and by the above argument, we will obtain that $c = z$, which means that R has a largest element, as desired. ■

We thus obtain:

Corollary 5.1.29 (Duals of free $\mathbf{KC}$ -algebras). *Given a Priestley space X , the space $V_G^{r_1}(\mathcal{V}_2^{\mathbf{KC}}(X))$ is dual to the free $\mathbf{KC}$ -algebra generated by $\text{CloUp}(X)$.*

Proof. By Theorem 5.1.6 and Lemma 5.1.27, we know that $V_G^{\mathbf{KC}}(X)$ is dual to the free $\mathbf{KC}$ -algebra generated by $\text{CloUp}(X)$, namely the inverse limit of the chain:

$$X \xleftarrow{r_0} \mathcal{V}_1^{\mathbf{KC}}(X) \xleftarrow{r_1} \mathcal{V}_2^{\mathbf{KC}}(X) \dots;$$

This is (definitionally) isomorphic to $\mathcal{V}_G^{\mathbf{KC}, r_1}(\mathcal{V}_2^{\mathbf{KC}}(X))$, i.e., the inverse limit of the Ghilardi complex obtained by iterating $\mathcal{V}_g^{\mathbf{KC}}(X)$, modifying the parameter g along the way. But by Lemma 5.1.28, we have that iterating such construction from that point on is the same as iterating simply $\mathcal{V}_g(X)$; so the latter inverse limit is isomorphic to $\mathcal{V}_G^{r_1}(\mathcal{V}_2^{\mathbf{KC}}(X))$. ■

We can then obtain the following analogue of Proposition 4.3.4:

Proposition 5.1.30 (Lattice-theoretic properties of free $\mathbf{KC}$ -algebras). *For each n , the following hold:*

1. The algebra $\mathcal{F}_{\mathbf{KC}}(n)$ is join-generated by join-irreducible elements.
2. The algebra $\mathcal{F}_{\mathbf{KC}}(n)$ is a bi-Heyting algebra.

Proof. (1). Let n be fixed; then by Corollary 5.1.29 we know that $\mathcal{F}_{\mathbf{KC}}(n)$ will be dual to the limit of the chain

$$\mathcal{P}(n) \xleftarrow{r_0} P_1 \xleftarrow{r_1} P_2 \dots,$$

where $P_1 = \mathcal{V}_r^{\mathbf{KC}}(\mathcal{P}(n))$, $P_2 = \mathcal{V}_{r_0}^{\mathbf{KC}}(P_1)$, and P_{n+1} for $n > 2$ is defined as in the g -Ghilardi complex (Definition 4.4.1). Note that by Lemma 4.3.2, for every $n > 1$, r_n will have an adjoint map $\uparrow: P_n \rightarrow P_{n+1}$. So similarly to Proposition 4.3.4 we can show that for each $U \subseteq P_n$ where $U = \uparrow x$, $\pi_n^{-1}[U]$ will be join-irreducible in $\mathcal{F}_{\mathbf{KC}}(n)$.

(2). Since r_n are bi-p-morphisms by Lemma 4.3.2, the result follows again by Proposition 2.1.74, since the connecting maps are co-p-morphisms. $\blacksquare$

Remark 5.1.31 (Alternative derivation of properties of $\mathbf{KC}$). The interested reader may notice that there is an alternative way of establishing Proposition 5.1.30: by Maksimova's arguments from [186], the $\mathbf{KC}$ logic is axiomatized by a “conservative axiom”. This implies that in fact $\mathcal{F}_{\mathbf{KC}}(n)$ can be obtained by a quotient $\mathcal{F}_{\mathbf{HA}}(n)/F$ where $F = \text{Fil}(\{\neg p_i \vee \neg\neg p_i : i \leq n\})$; such a quotient is implicitly what we take above. From this it follows by duality that the dual space of $\mathcal{F}_{\mathbf{KC}}(n)$ is a clopen upset in $\mathcal{X}(n)$. By Proposition 2.1.52 (when dualized), clopen subsets of a co-Esakia space will be co-Esakia. Since $\mathcal{X}(n)$ is bi-Esakia by duality and Proposition 4.3.4, the result follows.

Relative anarchy in $\mathbf{KC}$

In order to conclude that also $\mathcal{F}_{\mathbf{KC}}(\omega)$ is bi-Heyting, we will need to establish some properties of the right adjoints of maps $\mathcal{F}_{\mathbf{KC}}(n) \hookrightarrow \mathcal{F}_{\mathbf{KC}}(n+1)$. Since these do not seem to be available in the literature, we prove this in this short subsection, allowing us to conclude a relative anarchy result for the variety $\mathbf{KC}$. We will need the notions of bisimulation introduced in Section 2.3.3, to which we refer the reader.

Lemma 5.1.32 (Downwards propagation of bisimulation). Let $\mathfrak{M} = (P, \leq, c, r)$ be a model over $\bar{p}, q$, where P has a greatest element. Suppose that there is a point $x \in P$ and a model $\mathfrak{N}$, where $\mathfrak{N} = (Q, x', \leq, v')$ is a model over $\bar{p}$, where Q has a greatest element, and such that $(\mathfrak{M}, x) \sim_{\bar{p}} (\mathfrak{N}, x')$. Then there exists a model $\mathfrak{N}' = (Q', r', \leq, v')$, where Q' has a greatest element, over $\bar{p}, q$, such that $(\mathfrak{M}, r) \sim_{\bar{p}} (\mathfrak{N}', r')$ and $\mathfrak{N}' \leq_{\bar{p}} \mathfrak{N}$.

$$\begin{array}{ccc} \mathfrak{M}, x & \xrightarrow{\sim_{\bar{p}}} & \mathfrak{N}, x' \\ \uparrow & & \uparrow \\ \mathfrak{M}, r & \xrightarrow{\sim_{\bar{p}, q}} & \mathfrak{N}', r' \end{array}$$

Proof. Consider the set of pairs:

$$\text{Bis}(P, Q) = \{(w, w') : w \in P, w' \in Q, w \leq_{\bar{p}} w'\},$$

ordered pointwise, and for $p_i \in \bar{p}$, $(w, w') \in v(p_i)$ if and only if $w \in v(p_i)$, and $(w, w') \in v(q)$ if and only if $w' \in v(q)$. Note that this is rooted at (r, x') . Note that if t_P and t_Q are the greatest elements of $\mathfrak{M}$ and $\mathfrak{N}$, then the pair (t_P, t_Q) belongs here: since $x \leq t_P$, there must be some successor $x' \leq y$ such that $t_P \sim_{\bar{p}} y$; but then $y \leq t_Q$, and since t_P is maximal, $t_P \sim_{\bar{p}} t_Q$.

First we claim that $\pi_Q : \text{Bis}(P, Q) \rightarrow Q$ is a p-morphism. The fact that the order is preserved is clear because the order is pointwise. For the back condition, if $(w, w') \leq t$, then by hypothesis, $w \leq k$, $k \sim_{\bar{p}} w'$; if $w' \leq t$, then it follows that there is some $k \leq k'$ such that $k' \sim_{\bar{p}} t$. Hence by definition $(k, t) \in \text{Bis}(P, Q)$, and certainly $(w, w') \leq (k, t)$, which witnesses the back condition. From this we obtain that $(\text{Bis}(P, Q), (w, w')) \leq_{\bar{p}, q} (\mathfrak{N}, w')$: for each $p_i \in \bar{p}$, $w \in v(p_i)$ implies $w' \in v(p_i)$, and for q , this is in fact an equivalence.

We next claim that the projection $\pi_P : \text{Bis}(P, Q) \rightarrow P$ is a p-morphism. The pointwise ordering ensures that the map is order-preserving. Now assume that $\pi_P(w, w') \leq k$. By definition, $w \leq_{\sim} w'$ means that $w \leq k'$ where $k' \sim_{\bar{p}} w'$. Since P is directed, we have that there must be some t such that $k, k' \leq t$. Since $k' \sim_{\bar{p}} w'$, knowing $k' \leq t$ implies that there is some $w' \leq t'$ such that $t \sim_{\bar{p}} t'$. But then $k \leq_{\sim, \bar{p}} t'$, and by construction, $(w, w') \leq (k, t')$. From this we obtain, given the definition of the valuation that $(\text{Bis}(P, Q), (w, w')) \sim_{\bar{p}} (\mathfrak{M}, w)$. $\blacksquare$

Recall the *bisimulation semantics* of the formulas $\forall_q \varphi(\bar{p}, q)$ (see Proposition 2.3.32). This takes on the following form:

Lemma 5.1.33 (Bisimulation quantifiers). For each model $\mathfrak{M} = (P, r, \leq, v)$ over $(\bar{p}, q)$, and each formula $\varphi(\bar{p}, q)$, we have that $\mathfrak{M} \models \forall_q \varphi(\bar{p}, q)$ if and only if for each model $\mathfrak{N}$ over $\bar{p}, q$ $\mathfrak{M} \leq_{\bar{p}} \mathfrak{N}$, we have that $\mathfrak{N} \models \varphi(\bar{p}, q)$.

Note that the quantifier $\forall_q$ satisfies the usual normality rules, i.e., it distributes over conjunction and has $\top$ as a fixed point. It however satisfies more in general, even for **KC**, as form of the constant domain axiom, also known as dual Frobenius reciprocity:

Corollary 5.1.34 (Frobenius reciprocity for adjoints in KC). Assume that $\mathfrak{M} = (P, r, \leq, v)$ is a model over $\bar{p}, q$, where P has a largest element. Then for each formula $\varphi(\bar{p}, q)$ and $\psi(\bar{p})$ we have:

$$\mathfrak{M} \models \forall_q (\varphi(\bar{p}, q) \vee \psi(\bar{p})) \leftrightarrow (\forall_q \varphi(\bar{p}, q)) \vee \psi(\bar{p}).$$

Proof. Notice that since q does not occur in $\bar{p}$, $\forall_q \psi(\bar{p}) \equiv \psi(\bar{p})$. Then if $\mathfrak{M} \models \forall_q \varphi(\bar{p}, q) \vee \forall_q \psi(\bar{p})$, by monotonicity of the quantifier, $\mathfrak{M} \models \forall_q (\varphi(\bar{p}, q) \vee \psi(\bar{p}))$. Conversely, assume that $\mathfrak{M} \models \forall_q (\varphi(\bar{p}, q) \vee \psi(\bar{p}))$, and suppose that $\mathfrak{M} \not\models \psi(\bar{p})$.

Suppose that $\mathfrak{M} \leq_{\bar{p}} \mathfrak{N}$ is arbitrary. By Lemma 5.1.32, let $\mathfrak{M} \sim_{\bar{p},q} \mathfrak{M}'$ be a model over a frame with a greatest element, where $\mathfrak{M}' \leq_{\bar{p}} \mathfrak{N}$. By definition of the bisimulation quantifier, then $\mathfrak{M}' \models \varphi(\bar{p}, q) \vee \psi(\bar{p})$, but by the bisimulation invariance of formulas, and the fact that $\mathfrak{M} \not\models \psi(\bar{p})$, we have that $\mathfrak{M}' \not\models \psi(\bar{p})$. Thus $\mathfrak{M}' \models \varphi(\bar{p}, q)$. But then $\mathfrak{N} \models \varphi(\bar{p}, q)$ as well by preservation of truth. This shows the equivalence. ■

Remark 5.1.35 (Join-distributivity and the disjunction property). In Proposition 2.3.34, we noted that the right adjoints of inclusions of free Heyting algebras distribute over joins, which corresponds to $\forall_q$ as in Corollary 5.1.34 above distributing over disjunctions. By Proposition 2.3.35 this fails in **KC**: if **KC** algebras had such a property, then **KC** would have the disjunction property, whilst neither disjunct of the Yankov axiom $\neg p \vee \neg \neg p$ can be valid.

Denote by **biKC** the extension of **biIPC** by the Yankov axiom $\neg p \vee \neg \neg p$; denote the variety likewise by **biKC**. Then we have:

Corollary 5.1.36 (Relative anarchy for **KC**). *The variety **KC** is **biKC**-anarchic.*

Proof. We proceed analogously to the proof of Proposition 4.3.7. We know that right adjoints to the inclusion maps $i: \mathcal{F}_{\mathbf{KC}}(n) \rightarrow \mathcal{F}_{\mathbf{KC}}(n+1)$ exist [130, pp.101], and that $\mathcal{F}_{\mathbf{KC}}(n)$ are bi-Heyting algebras by Proposition 5.1.30. By Corollary 5.1.34, we know that these right adjoints satisfy the Frobenius reciprocity property (since **KC** is complete with respect to finite posets with greatest elements by Proposition 2.3.19). By Proposition 4.3.6 this implies that each map i is a bi-Heyting homomorphism as well. Then $\mathcal{F}_{\mathbf{KC}}(\omega)$ is a bi-Heyting algebra, as a limit of bi-Heyting algebras via bi-Heyting homomorphisms. ■

5.2 Categories of posets with p -morphisms

We now turn our attention to how the step-by-step construction can be understood in a “discrete” setting – i.e., by considering posets, rather than Esakia spaces. The expectation that analogous results should exist for such categories as for Priestley spaces and Esakia spaces can be motivated from a categorical point of view.

5.2.1 Ind and Pro completions

We recall the picture, outlined in [158, Chapter VI], relating categories of finite algebras and several of their completions; we refer the reader to [193] where these and other dualities are outlined. To start with, we have **FinBA**, the category of finite Boolean algebras, and its dual, **FinSet**. For such categories, there are two natural “completions”:

- The *ind-completion*, $\text{Ind}(-)$, which freely adds all directed colimits;

- The *pro-completion*, $\mathbf{Pro}(-)$, which freely adds all inverse (codirected) limits.

For categories of algebras, the pro-completion can equivalently be seen as consisting of the profinite algebras of the given type, whilst the ind-completion can be seen as obtaining all the *locally finite* algebras of that same type – which in the case of Boolean algebras or distributive lattices, will be all algebras. As such, we have that:

$$\mathbf{Ind}(\mathbf{FinBA}) = \mathbf{BA} \text{ and } \mathbf{Pro}(\mathbf{FinBA}) = \mathbf{CABA}$$

where $\mathbf{BA}$ is the category of all Boolean algebras with Boolean algebra homomorphisms, and $\mathbf{CABA}$ is the category of complete and atomic Boolean algebras with complete homomorphisms. Correspondingly, we have that

$$\mathbf{Ind}(\mathbf{FinSet}) = \mathbf{Set} \text{ and } \mathbf{Pro}(\mathbf{FinSet}) = \mathbf{Stone}.$$

The facts that $\mathbf{Set}^{op} \cong \mathbf{CABA}$ (Tarski duality) and $\mathbf{Stone}^{op} \cong \mathbf{BA}$ (Stone duality) then amount to the basic relationship between Ind and Pro-completions, namely, for any category $\mathbf{C}$:

$$(\mathbf{Ind}(\mathbf{C}^{op}))^{op} \cong \mathbf{Pro}(\mathbf{C}) \text{ and } (\mathbf{Pro}(\mathbf{C}^{op}))^{op} \cong \mathbf{Ind}(\mathbf{C}).$$

A similar picture can be drawn for distributive lattices: there the relevant category of finite objects is the category $\mathbf{FinPos}$ of finite posets with order-preserving maps, which is dual to $\mathbf{FinDL}$, finite distributive lattices with lattice homomorphisms, and we have that

$$\begin{aligned} \mathbf{Ind}(\mathbf{FinDL}) &= \mathbf{DL} \text{ and } \mathbf{Pro}(\mathbf{FinDL}) = \mathbf{C}_\infty\mathbf{DL} \\ \mathbf{Ind}(\mathbf{FinPos}) &= \mathbf{Pos} \text{ and } \mathbf{Pro}(\mathbf{FinPos}) = \mathbf{Pries}, \end{aligned}$$

where $\mathbf{DL}$ is the category of all distributive lattices with lattice homomorphisms; $\mathbf{C}_\infty\mathbf{DL}$ is the subcategory of complete and completely distributive lattices (see Definition 2.1.24) with complete lattice homomorphisms; $\mathbf{Pos}$ is the category of all posets with order-preserving maps.

If we are interested in Heyting algebras, it follows from Esakia duality that the category of finite Heyting algebras with Heyting algebra homomorphisms is dual to the category $\mathbf{FinPos}_p$ of finite posets with p-morphisms. The Ind-completion of $\mathbf{FinHA}$ is not necessarily $\mathbf{HA}$, since not every Heyting algebra is a directed colimit of finite Heyting algebras; however, one may use this intuition to “guess” a discrete category which allows the generalization of the above results. Indeed, a natural place to look is the Ind-completion of $\mathbf{FinPos}_p$. Such objects are those which are directed colimits of finite posets via p-morphisms – but this means that the finite posets must embed into the top of the object, i.e., such objects must be locally finite posets, in other words

$$\mathbf{Ind}(\mathbf{FinPos}_p) \cong \mathbf{LocFinPos}$$

where **LocFinPos** is the category of (the now appropriately named) locally finite posets, with p -morphisms; this is, indeed, the main result from [38], where it is shown that these are dual to the category of profinite Heyting algebras (see Theorem 2.1.31). A similar picture to this one has been discussed for finite modal algebras, and their duals, finite Kripke frames, in the recent work [33], where the above dualities are also sketched.

As a result of this relationship, it is natural to look for an adjunction holding between the inclusions of **ProHA** into **C_∞DL**, or equivalently, for a right adjoint to the inclusion of **LocFinPos** into **Pos**. This is what we now outline.

Our constructions in this section mirror very much the key constructions from Chapter 4:

Definition 5.2.1 (Discrete g -open rooted subsets). Let P and Q be posets and $g : P \rightarrow Q$ order-preserving maps. We denote by

$$\mathcal{P}_g(P) = \{C \subseteq P : C \text{ is rooted, } g\text{-open, and finite}\}.$$

We denote by $r_g : \mathcal{P}_g(P) \rightarrow P$ the root map.

We write $\mathcal{P}_r(P)$ for the collection of rooted subsets of P . Note that regardless of the structure of P , $(\mathcal{P}_g(P), \supseteq)$ will always be a locally finite poset.

Definition 5.2.2 (Discrete Ghilardi complex). Let P, Q be posets with $g : P \rightarrow Q$ an order-preserving map. The *discrete Ghilardi complex* over P , $\mathcal{G}_D(P) = (P_n, r_n)$ is the sequence:

$$(\mathcal{P}_0(P), \mathcal{P}_1(P), \dots)$$

connected by order-preserving maps $r_i : \mathcal{P}_{i+1}(P) \rightarrow \mathcal{P}_i(P)$ such that:

1. $\mathcal{P}_0(P) = P$;
2. $r_0 = g$;
3. For $i \geq 0$, $\mathcal{P}_{i+1}(P) := \mathcal{P}_{r_i}(\mathcal{P}_i(P))$;
4. $r_{i+1} := r_{r_i} : \mathcal{P}_{i+1}(P) \rightarrow \mathcal{P}_i(P)$ is the root map.

We write $\mathcal{P}_G^g(P)$ for the *locally finite part* of the inverse limit of the above sequence in the category of posets with order-preserving maps. In other words, we take

$$\{x \in \varprojlim \mathcal{P}_n(P) : \uparrow x \text{ is finite}\}.$$

The following gives an analogous one-step lemma:

Lemma 5.2.3 (Discrete one-step lemma). Let $g : P \rightarrow Q$ be an order-preserving map between posets. Let $f : R \rightarrow P$ be an order-preserving and g -open map, where R is locally finite. Then there exists a unique order-preserving and r_g -open map such that the triangle in Figure 5.1 commutes.

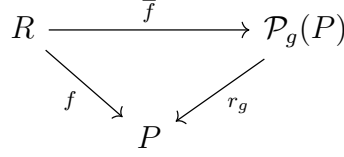


Figure 5.1: Commuting Triangle of Posets

Proof. The arguments are the same, albeit simpler, than in Proposition 4.2.13. The crucial difference is that one needs to show that given $q \in Q$, $\bar{f}(q) := f[\uparrow q]$ is a finite subset; this follows immediately from $\uparrow q$ being finite. ■

From Lemma 5.2.3, we can run the arguments showing that $\mathcal{P}_G$ is the left adjoint to the inclusion are entirely similar to the case of Priestley spaces and Esakia spaces, essentially dualizing those of Theorem 4.1.14:

1. We define the notion of limit map for discrete Ghilardi complexes, using the usual notion of limit map on inverse limits of posets, restricted to the locally finite part.
2. We show that any order-preserving map $f: Q \rightarrow P$ from a locally finite poset to P lifts to a p-morphism $\bar{f}: Q \rightarrow \mathcal{P}_G(P)$. This has as a consequence that limit maps are p-morphisms, turning $\mathcal{P}$ into a functor.
3. From this previous point, it straightforwardly follows that $\mathcal{P}_G(-)$ is the right adjoint to the inclusion of **LocFinPos** in **Pos**.

However, we stress that special care is needed when applying these arguments. Notice that before, we working with duals of categories of algebras, where compactness is implicit or explicitly present all the time. When working with discrete categories, limit constructions can fail to work as desired, namely, maps may fail to land in the right category when they are lifted (see 5.2.7 below). In particular, the key point where local finiteness becomes necessary is to show that given a map $f: Q \rightarrow P$, the lifting $\bar{f}: Q \rightarrow \mathcal{P}_G(P)$ is indeed a p-morphism. We spell this out in full detail:

Proposition 5.2.4 (Adjunction property for locally finite). Given a poset P , $\mathcal{P}_G^g(P)$ has the following universal property: whenever $f: Q \rightarrow P$ is an order-preserving map from a locally finite poset Q , there is a unique extension of f to a p-morphism $\bar{f}: Q \rightarrow \mathcal{P}_G(P)$.

Proof. Using Lemma 5.2.3 repeatedly we construct a map $\bar{f}: Q \rightarrow \varprojlim \mathcal{P}_n(P)$. Such a map is given explicitly by mapping the element q to the sequence $(f_n(q))_{n \in \omega}$, where $f_0 = f$, and for each k , $f_{k+1}(-) = f_k[\uparrow(-)]$.

We will show that such a map is in fact a p-morphism; if we do that, then since Q is locally finite, it will factor through the locally finite part of the inverse

limit: for each $q \in Q$, since $\bar{f}$ is a p-morphism, we have (see (Remark 2.1.54)) $\uparrow \bar{f}(q) = \bar{f}[\uparrow q]$. Since the latter is finite (as the direct image of a finite set), the former is finite as well.

We thus show that $\bar{f}$ is a p-morphism. Assume that $q \in Q$, and suppose that

$$\bar{f}(q) \leq \bar{p}$$

Suppose that $\bar{p} = (p_n)_{n \in \omega}$. Now let n be arbitrary. Then consider p_{n+1} . By assumption on the order being pointwise, we have that $f_{n+1}(q) \supseteq p_{n+1}$, and $f_{n+1}(q) = f_n[\uparrow q]$. Since f_{n+1} is r_n -open by construction, there is then some $q' \geq q$ such that $r_n f_{n+1}(q') = r_n(p_{n+1}) = p_n$. In other words, $\bar{f}(q')(k) = \bar{p}(k)$ for each $k \leq n$. Since this holds for each n , and q has only finitely many successors, by the pidgenohole principle, there must be a successor $q^* \geq q$ such that $\bar{f}(q^*) = \bar{p}$. This shows the result as desired. $\blacksquare$

We thus obtain the following:

Theorem 5.2.5 (Discrete adjunction). *The map $\mathcal{P}_G : \mathbf{Pos} \rightarrow \mathbf{LocFinPos}$ is a functor; indeed it is right adjoint to the inclusion $\mathbf{LocFinPos} \rightarrow \mathbf{Pos}$.*

Remark 5.2.6 (Adjunction for categories of profinite algebras). The dual picture of Theorem 5.2.5 shows that there exists a left adjoint to the inclusion $\mathbf{ProHA} \rightarrow \mathbf{C}_\infty \mathbf{DL}$. Following the main ideas of [33], the above describes an adjunction which splits the adjunction between profinite Heyting algebras and the category of sets. Moreover, as noted in [33], the construction $\mathcal{P}_G$ given above can be thought of as a generalization of the n -universal model from Chapter 3 – if one starts with the dual poset of the free distributive lattice on n -generators, $\mathcal{P}_G$ will produce precisely this same model (see also Appendix A.3.1).

Remark 5.2.7 (Discrete adjunctions and universal models). As mentioned in passing, Theorem 5.2.5 cannot be expected to be extended to yield an adjunction between $\mathbf{Pos}_p$ – the category of posets with p-morphisms – and $\mathbf{Pos}$ as above. This is because of a failure of the former category to exhibit enough nice structure: as noted in [51], $\mathbf{Pos}_p$ does not have binary products, and if $\mathbf{Pos}_p \rightarrow \mathbf{Pos}$ had a right adjoint, all limits in $\mathbf{Pos}$ would have to be preserved.

Remark 5.2.8 (Connection with toposes of posets). In recent work [266], Ye showed that $\mathbf{LocFinPos}$ is an infinitary extensive category failing to be Barr-exact (see [28]). Such a category admits a regular exact completion precisely in the sheaf topos described in [130]. The “nerve construction” [266, Section 4.1] thereby described, corresponds precisely to the adjunction $I \dashv \mathcal{P}_G$ which we have outlined here.

5.3 Other applications of the step-by-step construction

In this final section, we make a number of brief remarks concerning some other technical uses of the step-by-step constructions outlined in this chapter as well as chapter 4.

5.3.1 Coalgebraic semantics for intuitionistic modal logics

Coalgebraic semantics of classical modal and positive modal logics have been thoroughly investigated [255, 211, 253, 173]. In the case of classical modal logic this is done via the powerset coalgebras in the context of Kripke frames and by the Vietoris coalgebras for descriptive frames (see [173, 255]). For positive modal logic this is obtained via the convex set functor and its topological analogue [20, 63, 211, 49].

Intuitionistic modal logics, on the other hand, escaped this kind of analysis for quite a while. Such logics are quite varied, stemming from several distinct proposals over what is the appropriate notion of a “constructive” version of modal logic, see, e.g., [244, 114, 115, 104, 259]. The question of how to represent such frames as coalgebras for an appropriate functor on the category of posets with p-morphisms or respectively on the category of Esakia spaces and continuous p-morphisms (which is dual to the category of Heyting algebras) was open in the research community for a long time. It was raised explicitly by Litak [179] and by de Groot and Pattinson⁷ [140], where the existence of such a representation was left as an open question.

In [13], and more recently in [15], we tackled the question by exploiting the step-by-step representation, as we proceed to explain:

Definition 5.3.1 (Modal intuitionistic frame). Let $(P, \leq, R)$ be a triple where $(P, \leq)$ is a partial order, $R \subseteq P \times P$. We say this is a *modal intuitionistic frame* if it satisfies

$$R = \leq \circ R \circ \leq.$$

Given two modal intuitionistic frames $(P, \leq, R)$ and $(Q, \leq, R)$ we say that a map $f : P \rightarrow Q$ is a *modal p-morphism* if f is a p-morphism with respect to $\leq$ and R . Let **LocFinK** be the category of locally finite modal intuitionistic frames with modal p-morphisms.

Now, if $f : P \rightarrow \mathbf{Up}(P)$ is a coalgebra for this functor, then we can think of this map as picking for each $p \in P$ a set of modal successors; and dually, given a modal intuitionistic frame $(P, \leq, R)$, the map $R_P : P \rightarrow \mathbf{Up}(P)$ will provide a coalgebra. Hence we have:

⁷We point out that de Groot and Pattinson [140] also use the so-called dialgebraic representations in their work.

Proposition 5.3.2 (Coalgebraic semantics for modal frames). Modal intuitionistic frames $(P, \leq, R)$, are in 1-1 correspondence with order-preserving maps $R_P : P \rightarrow \mathbf{Up}(P)$ defined by

$$p \mapsto R[p].$$

This suggests that the category of coalgebras for the endofunctor $\mathbf{Up}(-)$ should be equivalent to the category of modal intuitionistic frames with modal p-morphisms. However, following the discussion in [179], note that we are working within the category $\mathbf{Pos}$, where maps are only required to be order-preserving, which means that there may be coalgebra morphisms which are not p-morphisms for the $\leq$ -relation. If one tentatively restricts to $\mathbf{Pos}_p$, then the problem of coalgebra morphisms failing to be p-morphisms can be avoided. Indeed, $\mathbf{Up}(-)$ restricts to an endofunctor on this (non-full) subcategory. However, this means that the above assignment $R_P : P \rightarrow \mathbf{Up}(P)$ might no longer be a morphism in this category, since there is no guarantee that it will be a p-morphism.

The solution we provided in is summarised in the following theorem:

Theorem 5.3.3 (Coalgebraic semantics for modal intuitionistic logic). *Let $(P, \leq)$ be a locally finite poset. Then the following are in one-to-one correspondence:*

1. *Modal intuitionistic frames over P ;*
2. *Order-preserving maps $f : P \rightarrow \mathbf{Up}(P)$;*
3. *P-morphisms $f' : P \rightarrow \mathcal{P}_G(\mathbf{Up}(P))$.*

Moreover, the categories $\mathbf{LocFinK}$ and the category $\mathbf{CoAlg}(\mathcal{P}_G(\mathbf{Up}))$ of coalgebras for the functor $\mathcal{P}_G \circ \mathbf{Up}$.

We note that the above restriction to $\mathbf{LocFinPos}$ is crucial in light of Remark 5.2.7, since for the larger category $\mathbf{Pos}_p$, no adjoint functor like the one given above can be obtained. The construction $\mathcal{P}_G(\mathbf{Up}(-))$ amounts to lifting the functor $\mathbf{Up}$, as seen in Figure 5.2, and amounts precisely to computing the left Kan extension of $\mathbf{Up} : \mathbf{Pos} \rightarrow \mathbf{Pos}$ along $\mathcal{P}_G$ ⁸.

$$\begin{array}{ccc} \mathbf{Pries} & \xrightarrow{F} & \mathbf{Pries} \\ \uparrow I & & \downarrow \mathcal{P}_G \\ \mathbf{Esa} & \xrightarrow{F^*} & \mathbf{Esa} \end{array}$$

Figure 5.2: Intuitionistic Lifting of functor F

This approach is quite general, and serves to provide, for any modal logic over a positive base – that is, one whose Kripke semantics is given over tuples $(P, \leq$

⁸We are indebted to Alexander Kurz for this observation.

, $R_1, \dots, R_n$) where R_i are relations satisfying some compatibilities with respect to $\leq$. This does not cover, however, several logics of interest, most notably, Fischer-Servi's logic $\mathbf{IK}$. In [15], this was remedied by applying the technique of quotienting along the step-by-step through axioms, as exposed in the present chapter. With it, a similar result to Theorem 5.3.3 for Fischer-Servi logic was obtained, though the details are quite technical.

5.3.2 Free Heyting extensions of Boolean algebras

As is well-known (see Theorem 2.1.36), there is a chain of adjunctions between the categories $\mathbf{HA}$ and $\mathbf{BA}$:

$$\mathbf{F} \dashv \mathbf{Reg} \dashv \mathbf{I} \dashv \mathbf{Center},$$

where $\mathbf{I}$ is the inclusion, $\mathbf{Reg}$ is the Booleanisation functor⁹, and $\mathbf{Center}$ is the center functor. The functor $\mathbf{F} : \mathbf{BA} \rightarrow \mathbf{HA}$, is guaranteed to exist by the Adjoint Functor theorem, as the left adjoint of $\mathbf{Reg}$; a description using a word construction can be derived from the work of Moraschini [203] (see also [7, Example 4.18]). It was likewise studied in [250], where it was shown to be fully faithful.

Using the step-by-step techniques, one can likewise give a description of such a functor, allowing a transparent proof of its categorical properties. This was done in [6, Section 6], and the reader is referred there for the proper definitions and all proofs of the following facts.

The first observation concerns the dual representation of the functor $\mathbf{Reg}$, as essentially shown in [35, Section 3]:

Proposition 5.3.4 (Dual of regularization functor). The functor $\max : \mathbf{Esa} \rightarrow \mathbf{Stone}$ assigning to each Esakia space its maximum, and the restriction of a p-morphism, is right adjoint to the inclusion $\mathbf{I} : \mathbf{Stone} \rightarrow \mathbf{Esa}$. Consequently, $\max$ is the dual functor to $\mathbf{Reg}$.

To obtain the dual of the functor $\mathbf{F}$, one employs the following modification of the first step in the step-by-step construction:

Lemma 5.3.5 (Max-space Vietoris). Given X be a Stone space, the space $\mathcal{V}_{\max}(X) = \{C \in \mathcal{V}_r(\mathcal{V}(X)) : \forall D \in C, \forall x \in D, \{x\} \in C\}$ is a Priestley space.

Remark 5.3.6 (Team semantics and flatness). It is noteworthy that the condition of $\mathcal{V}_{\max}(X)$ corresponds precisely to the *flatness property* often considered in team semantics (see e.g. [262, 263]). The connection with team semantics follows from other relationships noted between these functors and inquisitive logic, as studied by Ciardelli [85].

⁹This terminology arose in locale theoretic literature, see [27]; the reader should not confuse this with the Boolean envelope construction.

Using a suitable modification of $\mathcal{V}_G$, which amounts to preserving the first step one obtains a functor $\mathbf{M}: \mathbf{Stone} \rightarrow \mathbf{Esa}$ which is right adjoint to $\max: \mathbf{Esa} \rightarrow \mathbf{Stone}$.

A very concrete connection of this construction with algebraic work on team semantics is with the work [60], where an algebraic approach to such logics was presented which gives them a semantics in terms of special Heyting algebras called *regularly generated* Heyting algebras, or *regular* for short:

Definition 5.3.7 (Regular Heyting algebras). Let H be a Heyting algebra. We say that H is *regular* if $H = \langle H_{\neg\rightarrow} \rangle$, i.e., it is generated as a Heyting algebra by its regular elements.

In [61] and [139], these Heyting algebras and their dual Esakia spaces were studied in connection with inquisitive logic. Such algebras were then shown to arise out of the functor $\mathbf{F}$ in the following sense:

Theorem 5.3.8 (Regular Heyting algebras categorically). *A Heyting algebra H is regular if and only if the counit of the adjunction at H , $\varepsilon_H: \mathbf{F}(\mathbf{Reg}(H)) \rightarrow H$, is surjective.*

The above answers, in a categorical fashion, a question left open in [139] as to the structure of Esakia duals of regularly generated Heyting algebras. It also opens the door to several questions concerning the relationship between the above categories and the kind of “cohesiveness” present in certain categories of toposes [199] – indeed, the “discrete” version of the functor $\mathbf{F}$ was recently described by Ye [266], corresponding to a cohesive adjunction.

5.3.3 Interpolation for bi-intuitionistic logics

It is notable that of all the superintuitionistic logics, the one with the least direct proof is perhaps $\mathbf{LC}$: if one considers locally linear Esakia spaces¹⁰ that is, their cartesian product may fail to be locally linear. To fix this, one may employ the techniques of Section 4.4, combined with the description of free products of Section 5.1.3:

Theorem 5.3.9 (Pullbacks of locally linear Esakia). *Given X, Y, Z locally linear Esakia spaces, $f: X \rightarrow Y$ and $g: Z \rightarrow Y$ two p -morphisms, the space $\mathbf{CC}_{\pi_1, \pi_2}(X \times_Y Z)$ is the categorical pullback.*

The key difficulties of amalgamating frames persist when working with *bi-Gödel algebras*, $\mathbf{biGD}$; these consist of the extension of $\mathbf{biHA}$ by the Gödel-Dummett axiom $(p \rightarrow q) \vee (q \rightarrow p)$. Such logics have been studied for example in [64, 192]. The difficulties are compounded in that the maps at play are

¹⁰Note that these are called *Esakia root systems* in [79], which is where the notation $\mathbf{ERS}$, employed in this section for consistency, comes from.

bi-p-morphisms, and thus, involve more rigid manipulations. Nevertheless, the following can be shown:

Theorem 5.3.10. *The category **biERS** of locally linear bi-Esakia spaces with continuous bi-p-morphisms has co-amalgamation; in fact, **biGD** has the superamalgamation property.*

The key idea of the proof of Theorem 5.3.10 uses Theorem 5.3.9 in a crucial way: given a triple $f: X \rightarrow Y$ and $g: Z \rightarrow Y$, where f, g are bi-p-morphisms, and X, Y, Z are bi-Esakia root systems, we consider $\mathbf{CC}_{\pi_1, \pi_2}(X \times_Y Z)$, which will be an Esakia root system. It may fail to be a bi-Esakia space, but this is not necessary: knowing that it is an Esakia root system, we can consider $\mathbf{Up}(\mathbf{CC}_{\pi_1, \pi_2}(X \times_Y Z))$, and form the amalgam of the dual diagram with respect to this. For such an approach to work, one needs in particular to show that the composite projections $p_X: \mathbf{CC}(X \times_Y Z) \rightarrow X$ and $p_Z: \mathbf{CC}(X \times_Y Z) \rightarrow Z$ are both bi-p-morphisms, where having a representation via chains of the pullback becomes essential.

Using such a representation, in the forthcoming work [16], a full Maksimova-like [184] characterization of subvarieties of **biGD** with superamalgamation and with amalgamation is given; this can be stated in logical terms, in the following:

Theorem 5.3.11 (Interpolation in extensions of **biGD**). *There exist exactly six extensions of **biGD** with the deductive interpolation property, and exactly four of these have the Craig interpolation property.*

Part Two

Model theory of free Heyting algebras

In the previous chapters we have seen how to construct free Heyting algebras as well as several related universal constructions, like products and polynomial Heyting algebras, and we have seen some of the basic consequences of this for algebraic logic. In the second part of the thesis, we will two model-theoretic and structure-theoretic questions concerning Heyting algebras and certain special classes therein, which are in a crucial way connected to questions of representation of free Heyting algebras. In particular, we will look at the following questions:

1. In Chapter 6 we look at a question of Shehtman concerning whether uniform local tabularity is equivalent to local tabularity. Employing bounded bisimulations and a logical system introduced by Mordukhaiovich [189] and Esakia [102, A.5], we answer this question in the negative, and provide the basic structure theory for uniformly locally tabular logics with degree of uniformity lower than 3;
2. In Chapter 7, we will look at the problem, due to Andrew Pitts, concerning whether every Heyting algebra is *topos-admissible* – it appears as the lattice of truth-values of some elementary topos. We do so by exploring the theory of *étale Heyting algebras*, introduced by Jibladze and Kuznetsov [155], which is studied systematically there, in connection with their logical calculi, deemed *internal H-calculi*. As a main result, we prove that every Heyting algebra which is *étale-finite* is topos-admissible.

Despite their different nature, and independent interest, these problems employ the tools, intuitions and key insights we developed in the previous chapters, and therefore show the value of having such general tools. We conclude with some final considerations in Chapter 8.

Chapter 6

Uniform Local Tabularity

In Chapter 5 we saw that whilst in general the step by step construction of the free algebra is highly involved, in special cases like LC, this construction terminates in only two steps – a property called there *uniform local tabularity*. One can see uniform local tabular logics as those which *admit a simplified theory of free objects*. This phenomenon was previously studied by Shehtman [240] under the name of “finite formula depth property”. There Shehtman asked the following:

Problem 6.0.1 (Shehtman’s problem on uniform local tabularity). Is every locally tabular logic uniformly locally tabular?

In this chapter we will answer this problem in the negative, confirming Shehtman’s conjecture.

Outline of the chapter

1. In Section 6.1 we give some background context on problems of local tabularity in modal logics and extensions of IPC, and recall the basic definitions used for finite posets and their bisimulation games.
2. In Section 6.2 we introduce a hierarchy of locally tabular logics based on the concept of n -uniform logics, which are those admitting a uniform bound of n in the nesting of implications. We show that the least n -uniform logic always exists, and axiomatize the n -uniform logics for $n \leq 3$, as well as axiomatizing a natural infinite depth 4-uniform logic which is not 3-uniform.
3. In Section 6.2.3, by taking inspiration from the analysis of 3-uniformity and the logic of “weak Peirce’s law”, introduced by Esakia [102, A.5] and Mardaev [189], we provide a negative answer to Shehtman’s problem.
4. In Section 6.3 we further present an example of a pre-uniformly locally tabular logic, and use this to show that amongst extensions of the system KG, uniform local tabularity is decidable.

5. We summarise the resulting hierarchy of locally tabular logics in Section 6.4.

6.1 Background

It is a classical result due to Segerberg [238] and Maksimova [185] that an extension of **S4** is locally tabular if and only if it is of finite depth, if and only if it does not extend the logic **Grz.3**. This provides an easy criterion and algorithm to verify local tabularity, making it a decidable problem.

By contrast, in the system **IPC**, the situation is quite different. It was shown by Mardaev [190, 189] that there is a continuum of pre-locally tabular logics, all above **KC**. It is open whether every locally tabular logic is contained in a pre-locally tabular one and whether local tabularity is decidable. In terms of algebraic models – which correspond in algebraic logical terms to the logics – it is thus open whether the category of locally finite Heyting algebras admits any easy and logically insightful description (see [150] for recent discussion).

Given these difficulties it is reasonable to seek for strengthenings of local tabularity. In [242], Shehtman observed that one can use bisimulation games to characterize when certain logics are locally tabular, introducing a property called “finite formula depth”. We will likewise make use of bisimulation games; for this purpose the reader is reminded of some of the key definitions technical lemmas on bisimulations defined in Section 2.3.3.

6.1.1 Classes of finite posets

Recall that $d(\varphi)$ denotes the implication depth of a formula.

Definition 6.1.1 (Kripke models and classes of models). We say that a triple $(\mathfrak{M}, x) = (P, \leq, c, x)$ where $(P, \leq)$ is a rooted poset, with root x , and $c : P \rightarrow \mathcal{P}(\bar{p})$ is an order-preserving function, is a *model over the propositional letters $\bar{p}$* . We say that $(\mathfrak{M}, x)$ is a model of a logic L (over the propositional letters $\bar{p}$) if $(P, \leq)$ is a frame for L . Moreover, given $\mathbf{P}$ a class of posets, we say that $(\mathfrak{M}, x)$ a model over $\bar{p}$ is a *$\bar{p}$ -model of $\mathbf{P}$* if there is some poset $Q \in \mathbf{P}$ such that P is a p-morphic image of a rooted upset of Q . We denote by $\mathcal{M}_{\bar{p}}(\mathbf{P})$ the class of $\bar{p}$ -generated models over $\mathbf{P}$. We also denote by $M_{\mathbf{P}}(\varphi) = \{(\mathfrak{M}, x) \in \mathcal{M}_{\bar{p}}(\mathbf{P}) : (\mathfrak{M}, x) \Vdash \varphi\}$.

Remark 6.1.2.

1. (**Notation**) Whenever we do not wish to emphasize the specific propositional letters used in the language, we will identify the set $\{p_1, \dots, p_n\}$ with the natural number n , and consider models over $\bar{p}$ to be maps $c : P \rightarrow 2^n$; the image for a point $x \in P$ will be called its *color*, denoted by a tuple $c(x) = \langle i_1, \dots, i_n \rangle$. We will further shorten tuples of the form $\langle 1, \dots, 1, 0, \dots, 0 \rangle$,

consisting of a sequence of 1 followed by a sequence of 0, by the notation $1^{n-k}0^k$. When n is given, we will often omit the order and the valuation, and write simply (P, x) for a model. We occasionally write $\mathcal{M}_n(\mathbf{P})$ for the class of models over n from $\mathbf{P}$.

2. (Models are rooted) All statements given below for posets (P, x) or models thereof assume that P is *rooted at x* . This is implicit in our notion of model, but we repeat it for the sake of the reader familiar with frame theoretic arguments.
3. (Reductions and n -generation) Given an n , say that a model (P, x) over n is *n -generated* if P does not contain distinct bisimilar points. A well-known characterization of n -generation which we will use throughout is through α and β -reductions (see [54, Chapter 3], compare with Chapter 3). Namely, given a model P , we say that we can apply an *alpha-reduction* (α) whenever $x \in P$ has an immediate successor x' such that $\uparrow x = (\uparrow x' \cup \{x\})$ and $c(x) = c(x')$, obtained by identifying x with x' ; we say we can perform a *beta-reduction* (β) if x, x' are two points, such that $\uparrow x - \{x\} = \uparrow x' - \{x'\}$, and $c(x) = c(x')$, obtained by identifying x and x' . A model is n -generated precisely when there are no α or β -reductions which can be performed on the model. A convenient fact which we will exploit throughout is that for n -generated models, *bisimulation implies isomorphism*.

We will also need the notion of order-bisimulation introduced in Section 2.3.3. The interest of this property for us will be the following, originally in [130] and [257]:

Proposition 6.1.3 (n-Order-bisimulation and truth-preservation). For each pair of models (P, x) and (Q, y) over $\bar{p}$ the following are equivalent:

1. $(P, x) \leq_n (Q, y)$;
2. Whenever $d(\varphi(\bar{p})) \leq n$ and $(P, x) \models \varphi$ then $(Q, y) \models \varphi$.

It follows from this that for finite models, $(P, x) \leq_\infty (Q, y)$ if and only if for every formula φ , $(P, x) \models \varphi$ then $(Q, y) \models \varphi$. The following is completely analogous to [130, Chapter 4, Exercise 11], where we relativise to a specific subclass of posets, to capture different equivalence in different logics:

Proposition 6.1.4 (Equivalence criterion for order-bisimulation). Let $\mathbf{P}$ be a class of finite posets, $L = \text{Log}(\mathbf{P})$ and let $\bar{p}$ be a set of propositional variables. Then we have:

1. For each $\varphi(\bar{p}), \psi(\bar{p})$, we have:

$$L \vdash \varphi \leftrightarrow \psi \iff \mathcal{M}_{\mathbf{P}}(\varphi) = \mathcal{M}_{\mathbf{P}}(\psi).$$

2. The following are equivalent:

- (a) $S \subseteq \mathcal{M}_{\bar{p}}(\mathbf{P})$ is the class of models from $\mathcal{M}_{\bar{p}}(\mathbf{P})$ satisfying a formula ψ with $d(\psi) \leq n$;
- (b) S is closed under $\leq_n$ -bisimulation within $\mathcal{M}_{\bar{p}}(\mathbf{P})$: whenever $(P, x) \in S$ and $(Q, y) \in \mathcal{M}_{\bar{p}}(\mathbf{P})$ and $(P, x) \leq_n (Q, y)$ then $(Q, y) \in S$

Proof. (1) is an immediate consequence of L being the logic of a class of finite posets, and therefore, if $\varphi \leftrightarrow \psi \notin L$, there being a model of φ which is not a model of ψ .

(2) If S is the class of models satisfying a formula ψ with $d(\psi) \leq n$, then by Proposition 6.1.3, we have that S is closed under $\leq_n$ -bisimulation. Conversely, assume that S is closed under $\leq_n$ -bisimulation, and consider:

$$\text{Th}_n(S) = \{\psi(\bar{p}) : d(\psi) \leq n, \forall (P, x) \in S, (P, x) \Vdash \psi\}.$$

Since there are only finitely many formulas of implication depth at most n , $\text{Th}(S)$ is finite, and so we can consider $\chi(S) = \bigwedge \text{Th}(S)$. We claim that S is precisely the class of models from $\mathcal{M}_{\bar{p}}(\mathbf{P})$ satisfying $\chi(S)$. It is clear that if $(P, x) \in S$, $(P, x) \Vdash \chi(S)$. On the other hand, assume that $(Q, y) \notin S$. Then for each $(P, x) \in S$, $(P, x) \not\leq_n (Q, y)$, which by Proposition 6.1.3 means that there is a formula $\varphi(\bar{p})$ of implication depth at most n such that $(P, x) \Vdash \varphi_P$ and $(Q, y) \not\Vdash \varphi_P$. Let $\mu_P = \bigvee_{P \in S} \varphi_P$; note that up to equivalence, μ_P is finite, and $\mu_P \in \text{Th}_n(S)$. Since $(Q, y) \not\Vdash \mu_P$, then $(Q, y) \not\Vdash \chi(S)$, which concludes the proof. ■

6.1.2 Uniform local tabularity

Definition 6.1.5 (Uniform local tabularity). Let $L \supseteq \text{IPC}$ be a superintuitionistic logic. We say that L is *n-uniform* if each formula φ is equivalent to a formula of implication depth n over L . We say that L is *uniformly locally tabular* if L is n -uniform for some n . We denote by ULTab_n the class of n -uniform logics, and by ULTab their union, the class of all uniformly locally tabular logics.

We also denote by FinDep the class of logics of finite depth; by LTab the class of all locally tabular logics; Tab the class of tabular logics.

Notation 6.1.1 (Least Heyting algebra). Recall that given H a Heyting algebra, $A \subseteq H$ a subset, the least Heyting subalgebra of H containing A can be constructed as:

$$\langle A \rangle = \{t(a_1, \dots, a_k) : a_i \in A, t(x_1, \dots, x_n) \text{ is a formula}\}.$$

Local finiteness implies that for each H , and $A \subseteq H$ a finite subset, there is an n such that $\langle A \rangle$ is generated by formulas of implication depth at most n . This is where we can impose uniformity:

Definition 6.1.6 (n-uniform algebras). Let H be a Heyting algebra. We say that H is *n-uniformly finite*, or simply *n-uniform*, if whenever $A \subseteq H$ is finite, then:

$$\langle A \rangle = \{t(a_1, \dots, a_k) : a_i \in A, t(x_1, \dots, x_k) \text{ has implication depth } \leq n\}.$$

In this case we say that $\langle A \rangle$ is *n-uniformly generated*. We say that a class of algebras $\mathbf{A}$ is *n-uniformly finite* if every algebra in $\mathbf{A}$ is *n-uniformly finite*.

The motivation for the former definition lies in a certain modal understanding of Heyting algebras: one sees the implications as modalities and their nesting as a way to parametrize the complexity of a given logical system or algebra.

Proposition 6.1.7 (Algebraic characterization of n-uniformity). A superintuitionistic logic L is *n-uniform* if and only if $\text{Alg}(L)$ is *n-uniformly finite*. Moreover, the following inclusions hold between classes of Heyting algebras:

$$\text{Tab} \subseteq \text{FDep} \subseteq \text{ULTab} \subseteq \text{LTab}.$$

Proof. The equivalence follows essentially by definition. We concentrate on the inclusions. $\text{Tab} \subseteq \text{FDep}$, since each finite poset P satisfies bd_k where k is its height; the fact that $\text{FDep} \subseteq \text{ULTab}$ was shown by Shehtman [240], and amounts to the fact that for posets of finite depth, there is a uniform bound for how complex n -bisimulation can be; the fact that $\text{ULTab} \subseteq \text{LTab}$ is obvious. ■

Most converses of the inclusions of Proposition 6.1.7 are known not to hold. Namely (Definition 2.3.5), the logic BD_2 is known not to be tabular, even though it is of finite depth. The logic LC is known to be uniformly locally tabular (see [240], and the proof given in Theorem 5.1.12) but is not of finite depth. We will take a look at the last converse – $\text{LTab} \subseteq \text{ULTab}$ in Section 6.2.

6.1.3 Basic properties and criteria for uniform local tabularity

Proposition 6.1.8 (Basic criteria for uniform local tabularity). Let $\mathbf{P}$ be a class of finite posets. Then $L = \text{Log}(\mathbf{P})$ is $n + 1$ -uniformly locally tabular if and only if for each $\bar{p}$, whenever (P, x) and (Q, y) are two models in $\mathcal{M}_{\bar{p}}(\mathbf{P})$ and $(P, x) \leq_{n+1} (Q, y)$, then $(P, x) \leq_{\infty} (Q, y)$. Moreover, if n -bisimilarity implies full bisimilarity, the latter holds.

Proof. If L is $n + 1$ -uniformly locally tabular, then all formulas are equivalent to formulas of implication depth $\leq n + 1$. Let $(P, x), (Q, y) \in \mathcal{M}_{\bar{p}}(\mathbf{P})$ be two models, and assume that $(P, x) \leq_{n+1} (Q, y)$. Then whenever $(P, x) \Vdash \varphi$ for any φ , $(Q, y) \Vdash \varphi$. Thus $(P, x) \leq_{\infty} (Q, y)$ by our remark after Proposition 6.1.3.

Conversely, let $\varphi(\bar{p})$ be an arbitrary formula. Consider $S = \{(P, x) \in \mathcal{M}_{\bar{p}}(\mathbf{P}) : (P, x) \Vdash \varphi\}$. Now assume that $(P, x) \leq_{n+1} (Q, y)$. By assumption, $(P, x) \leq_{\infty} (Q, y)$, and consequently, $(Q, y) \Vdash \varphi$ as well. This means that S is closed under $\leq_{n+1}$ -bisimulation within $\mathcal{M}_{\bar{p}}(\mathbf{P})$, which by Proposition 6.1.4.(2) that S is the class of models from $\mathcal{M}_{\bar{p}}(\mathbf{P})$ satisfying a formula ψ with $d(\psi) \leq n+1$. By Proposition 6.1.4.(1), this means that $L \vdash \varphi \leftrightarrow \psi$, i.e., φ is equivalent to a formula of implication depth $\leq n+1$. Thus L is $n+1$ -uniformly locally tabular, as desired.

If we assume that n -bisimilarity implies full bisimilarity instead, then if we assume that $(P, x) \leq_{n+1} (Q, y)$, by considering $y \leq y$, we have that there is some $x \leq w$ such that $(P_w, w) \sim_n (Q, y)$. Since $(P, x) \Vdash \varphi$, then $(P_w, w) \Vdash \varphi$ as well; since n -bisimilarity implies full bisimilarity, by invariance of truth under bisimilarity, $(Q, y) \Vdash \varphi$. We thus obtain the same conclusion. ■

We close this section by noting an interesting property of n -uniform Heyting algebras. Let $\mathbf{HA}_{lf}$ be the category of locally finite Heyting algebras with Heyting algebra homomorphisms, and let $\mathbf{HA}_{lf}^n$ be the category of n -uniformly finite Heyting algebras with the same maps.

Proposition 6.1.9 (n -uniform Heyting algebras form a variety). The category $\mathbf{HA}_{lf}$ is closed under subalgebras, homomorphic images and finite products. For each $n \in \omega$, $\mathbf{HA}_{lf}^n$ is a variety.

Proof. The fact about subalgebras is trivial. To see the fact about homomorphic images, say H is locally finite (n -uniformly finite), and A is a homomorphic image. Let $\{b_1, \dots, b_n\} \subseteq A$ be finitely many elements. Then we can find finitely many elements $\{h_1, \dots, h_n\} \subseteq H$, mapping surjectively onto b_i ; by restricting the homomorphism from H to A , we find that $\langle \{h_1, \dots, h_n\} \rangle$ maps surjectively to $\langle \{b_1, \dots, b_n\} \rangle$. Thus the latter is finite (n -uniformly generated) by hypothesis.

To see the fact about products, note that if H_0, H_1 are locally finite, and $(a_0, b_0), \dots, (a_n, b_n)$ where $a_i \in H_0$ and $b_i \in H_1$, then

$$\begin{aligned} \langle (a_0, b_0), \dots, (a_n, b_n) \rangle &= \{t((a_0, b_0), \dots, (a_n, b_n)) : \\ &\quad d(t) \text{ is the maximum of that of } \langle A \rangle \text{ and } \langle B \rangle\}. \end{aligned}$$

The same argument works for infinite products if we assume the algebras are n -uniformly finite, since then, rather than taking the maximum of $\langle A \rangle$ and $\langle B \rangle$ one can take the bound n . ■

The above Proposition entails that there is the least n -uniform logic for each n . On the other hand, $\mathbf{HA}_{lf}$ is not a variety – it contains every finite Heyting algebra, and hence the smallest variety containing it is $\mathbf{HA}$ itself.

6.2 The hierarchy of locally tabular logics

In this section we will study the hierarchy of locally tabular logics – the *top* of the Kuznetsov hierarchy [50]. In 6.2.1 by characterizing the 0, 1, 2, 3-uniformly locally tabular logics. In Section 6.2.2 we make some remarks on 3-uniform logics of infinite depth, connecting them to recent studies of local tabularity in bi-intuitionistic logic [192]. We then give in Section 6.2.3 an example of a locally tabular but not uniformly locally tabular logic.

6.2.1 0-uniform, 1-uniform, 2-uniform, 3-uniform logics

The reader will notice that due to our definition, the only superintuitionistic logic which is 0-uniform is the trivial logic. 1-uniformity turns out to be quite a restrictive property: only CPC has it. 2-uniformity will be characteristic of extensions of LC, whilst 3-uniformity and beyond will turn out to have a rather complex structure.

For the next result, the reader should recall the structure of the lattice of superintuitionistic logics from Section 2.3.5, especially Proposition 2.3.39.

Theorem 6.2.1 (0,1,2-uniform logics). *For L a superintuitionistic logic, we have:*

1. L is 0-uniform if and only if L is inconsistent.
2. L is 1-uniform if and only if $L = \text{CPC}$;
3. L is 2-uniform if and only if $L \supseteq \text{LC}$.

Proof. As noted, the only 0-uniform logic is the inconsistent one. Note that CPC is 1-uniform: by definition of implication depth, every formula in CPC is equivalent to a conjunction of disjunctions of formulas $p, \neg p$. On the other hand, the Smetanich logic Sme is not 1-uniform: we see in Figure 6.1 models M_0 and M_1 , based on frames validating Sme , and $M_1 \leq_1 M_0$, certainly, but $M_1 \not\leq_\infty M_0$, since no point in M_1 labelled with 0 is 1-bisimilar with the singleton containing 0. By Proposition 6.1.8, this means that the logic is not 1-uniform, and neither is any logic extended by it.

As noted before, it was shown by Shehtman [240] that LC is 2-uniform; we give a short proof: if (P, x) and (Q, y) are two linearly ordered posets rooted at x, y , they are n -generated, and they are 1-bisimilar, then notice that by n -generation, there cannot be successive nodes with the same color (otherwise we could identify them). But then by an easy induction we see that the two chains $\uparrow x$ and $\uparrow y$ must be isomorphic. It follows that if $(P, x) \sim_1 (Q, y)$ for n -generated models, then (P, x) and (Q, y) are isomorphic, and consequently, fully bisimilar. We can then appeal to the same proposition as above.

Now assume that L is some logic and $\text{LC} \not\subseteq L$; then $L \not\models \mathcal{J}(2\text{F})$ or $L \not\models \mathcal{J}(\text{Dia})$, so either the 2-fork or the diamond are frames for L . If 2F is a frame for L , note

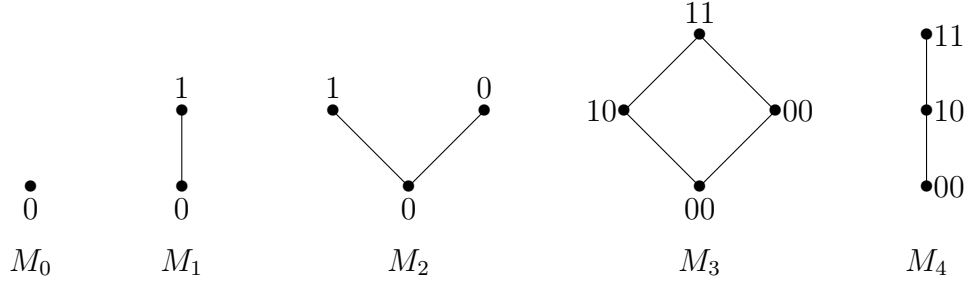


Figure 6.1

that in Figure 6.1, we have the models M_1 and M_2 , and $M_2 \leq_2 M_1$: if we pick in M_1 the top node labelled with 1, there is a node in M_2 with the same label, obtaining an isomorphism; and if we pick in M_1 the root, node that $M_1 \sim_1 M_2$. However, $M_2 \not\leq_3 M_1$, since by picking the root of M_1 , there is no 2-bisimilar node in M_2 . Similarly, the models M_3 and M_4 (the frame of which is a p-morphic image of M_3) have the property that $M_3 \leq_2 M_4$ (there is an isomorphic node in M_3 for each node but the root, and the roots are 1-bisimilar), but $M_2 \not\leq_3 M_4$. This shows that L cannot be 2-uniformly locally tabular again by Proposition 6.1.8. ■

The case of 3-uniform logics is a fair bit more complicated and deserves further consideration. For that purpose, recall the logic **wPL**, the logic of the “weak Peirce’s law”:

$$\mathbf{wPL} := \text{IPC} \oplus (q \rightarrow p) \vee ((p \rightarrow q) \rightarrow p) \rightarrow p.$$

It is known (see [102, A.9]) that a Heyting algebra H validates **wPL** if and only if it is a *cascade Heyting algebra*. Their variety was named by Esakia **CHL**, and was studied in [41] in the context of analyzing profiniteness and representability of Heyting algebras. The finite members of such a variety are the “sums” of Boolean algebras; we will make use of the dual posets of such algebras:

Definition 6.2.2 (Boolean sum). A finite (rooted) poset $(P, \leq)$ is called a *Boolean sum* if $d_p(P) = n$, and:

1. Whenever $x < y$, i.e., y is an immediate successor of x , then $d_p(x) = d_p(y) + 1$;
2. Each point x of depth $k + 1$ sees all points of depth k for each k .

When working with Boolean sums, note that the notion of depth of a point captures precisely its *level*: for P a Boolean sum, $\text{Lev}_k(P) = \{x \in P : d_p(x) = k\}$ consists of incomparable points which are all seen by the points in $\text{Lev}_{k-1}(P)$, and all see the points in $\text{Lev}_{k+1}(P)$. The following is a consequence of Esakia’s characterization:

Proposition 6.2.3 (Frames of **wPL** are Boolean sums). A finite rooted poset P is a frame for **wPL** if and only if it is a Boolean sum.

Proof. See [102, A.9.2]. ■

The following axiomatization was in turn shown by Mardaev [191, Theorem 5]:

Theorem 6.2.4 (Yankov axiomatization of weak Peirce's logic). *The logic **wPL** can be axiomatized as:*

$$\mathbf{wPL} := \mathbf{IPC} \oplus \mathcal{J}(Q_1) \oplus \mathcal{J}(Q_2) \oplus \mathcal{J}(Q_3),$$

where $\mathcal{J}(Q_i)$ is the Yankov formula for the posets in Figure 6.2. Moreover, this logic is locally tabular.

We will be concerned with special extensions of this logic. For that we will need some definitions:

Definition 6.2.5 (Stack-depth). Let P be a Boolean sum of depth n . For each n , let c_n be the cardinality of the set of elements of depth n . Given $\langle c_1, \dots, c_n \rangle$ we say that a consecutive subsequence $c_i, \dots, c_{i+k}$ where $c_{i+j} > 1$ for $0 \leq j \leq k$ is a k -stack. We say that P has *stack-depth* k if there exists a k -stack but not a $k+1$ -stack in P .

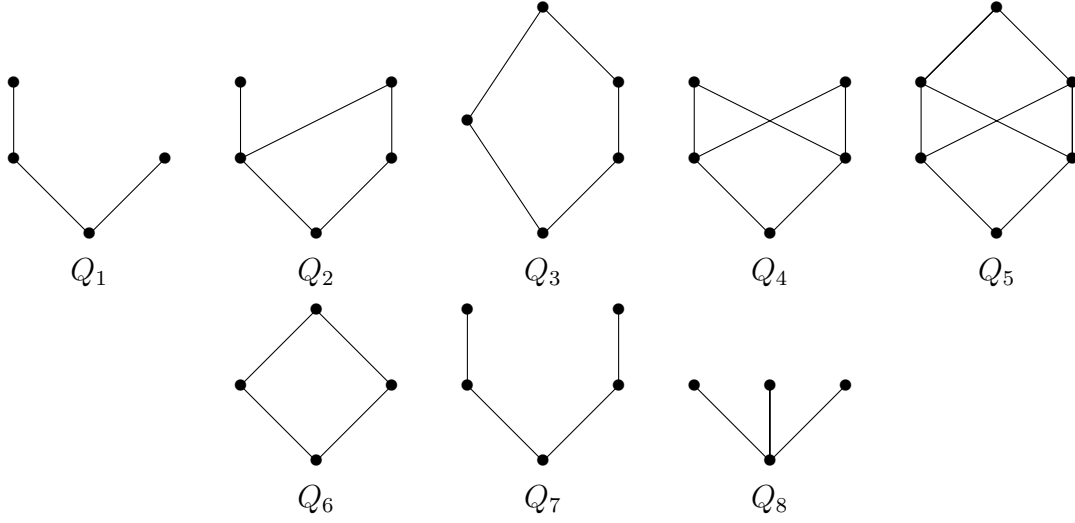
The stack depth of a Boolean sum thus measures how many consecutive levels there can be which contain more than one point.

Remark 6.2.6 (Diamond Heyting algebras). In [41] a subvariety of **CHL** was considered in connection to the problem of profinite completions, called the variety of *Diamond Heyting algebras* **DHL**. Such algebras are of width 2, have a maximal element, and they satisfy the Yankov formula $\mathcal{J}(Q_5)$. Thus, their finite rooted posets are precisely Boolean sums of stack depth 1, width 2 having a greatest element (see also Section 6.3, where the associated logic will be called **Box**).

Lemma 6.2.7 (Stack-depth 1 axiomatization). Let (P, x) be a finite poset which is a Boolean sum. Then $P \models \mathcal{J}(Q_i)$, for $i \in \{4, 5\}$ in Figure 6.2, if and only if P has stack-depth 1.

Proof. First note that both of the posets Q_4, Q_5 have stack depth 2. If Q_4, Q_5 is a rooted subframe of Q , which is a p-morphic image of P , by pulling along the p-morphism we will see that P has stack depth at least 2.

Conversely, assume that P has stack depth at least 2. Let $x \in P$ be a maximal point such that $\uparrow x$ has stack-depth 2. We may thus assume that x has at least two successors y_0, y_1 , and that each of them has two successors (otherwise we could have picked y_i as our maximal point) z_0, z_1 . Then there are two cases: either z_0 and z_1 have no common successor, or they do. In the first case, we define a

Figure 6.2: Q_i posets

map to Q_4 as follows: we send z_0 to the left top point in Q_4 , z_1 and all remaining points of that depth to the right top point; y_0 to the left middle point, y_1 and all remaining points of that depth to the middle right point; and x to itself. This is a surjective p-morphism onto Q_4 . The case of Q_5 is similar, except we send all points above z_0, z_1 to the maximal point. ■

Given a rooted frame (P, x) based on a Boolean sum with stack depth 1, let us say that a point $y \in P$ is *solitary* if y is the only point of depth k for some k . Otherwise we say that y is *accompanied* by a point y' . The importance of stack depth for us comes from the following result:

Lemma 6.2.8 (Boolean sums of stack-depth 1, 2-bisimulations). Let (P, x) be an n -generated model based on a finite Boolean sum of stack depth 1. If $(P, x) \sim_1 (P_y, y)$ for some $y \in \uparrow x$, then $x = y$.

Proof. Assume first that x is a solitary point. Then for each y , either $x \leq y$ or $y \leq x$ must hold. If $y \leq x$, note that by n -generation, any predecessor of x must have a color strictly smaller than x , and thus we cannot have $(P, y) \sim_1 (P, x)$. On the other hand, if $x \leq y$, let $S(x)$ be the set of immediate successors of x . If $S(x)$ is a singleton, we likewise have that the color of x must be strictly smaller, and so $(P, y) \sim_1 (P, x)$ cannot hold. If $S(x) = \{x_0, x'_0\}$, then since the stack depth is 1, these points must have distinct color, and either $c(x_0) \neq y$ or $c(x'_0) \neq y$; say $c(x_0) \neq y$ without loss of generality. If $(P, y) \sim_1 (P, x)$, then y would have to have a successor $y \leq y_0$ such that $c(y_0) = c(x_0)$. However, note that if z is the unique immediate successor of x_0, x'_0 , $c(x_0), c(x'_0) < c(z)$, so $y_0 = x_0$ or $y_0 = x'_0$. Thus if $x \neq y$, we would have a contradiction, since $x \leq y \leq y_0$ would force $y = y_0$ (since

y_0 must be an immediate successor of x), whilst we chose $c(y) \neq c(y_0)$. It follows that $(P, x) \sim_1 (P, y)$ can only hold if $x = y$.

Next assume that x is accompanied by x' . If $x < y$, then by n -generation, $c(x) \neq c(y)$. Moreover, $c(x) \neq c(x')$, so $(P, x) \sim_1 (P, x')$ cannot hold. Finally if $y < x$, then by the same arguments as in the previous case, if $(P, x) \sim_1 (P, y)$, we would obtain a contradiction. We thus obtain the result. ■

Proposition 6.2.9 (2-bisimulation implies full bisimulation for Boolean sums with stack depth 1). Let P, Q be two finite Boolean sum frames of stack depth 1. Assume that $(P, x) \sim_2 (Q, y)$ are n -generated models which are 2-bisimilar. Then they are isomorphic.

Proof. Let $S_2 \subseteq S_1 \subseteq S_0 \subseteq P \times Q$ be a 2-bisimulation. We claim that S_1 defines an isomorphism, mapping x to y . Indeed, assume that $y, y' \in P$ and $(y, z) \in S_1$ and $(y', z) \in S_1$. Then we have that $(P, y) \sim_1 (Q, z) \sim_1 (P, y')$, so $(P, y) \sim_1 (P, y')$. By Lemma 6.2.8, we have that $y = y'$. Thus S_1 is injective; a similar argument applied to Q shows that S_1 is functional, and the bisimulation clauses guarantee that it is a surjective relation. Thus S_1 is an isomorphism as desired. ■

Let **3Uni** be the following logic, which we refer to as the *logic of 3-uniformity*:

$$\mathbf{3Uni} := \text{IPC} \oplus \{\mathcal{J}(Q_i) : i \in \{1, \dots, 5\}\} = \mathbf{wPL} \oplus \mathcal{J}(Q_4) \oplus \mathcal{J}(Q_5).$$

Then we have:

Theorem 6.2.10 (**3Uni** is logic of 3-uniformity). *The logic **3Uni** is the least 3-uniform logic.*

Proof. First, note that in Figure 6.3, we see that all the posets Q_i for $1 \leq i \leq 5$ admit 2-bisimilar models (obtained by p-morphic images of Q_i , denoted Q_i^l and Q_i^r whether they occur on the right or left) which violate our criterion. Namely, we have that $Q_i^l \leq_3 Q_i^r$, since for each node except the root in Q_i^r there is a point isomorphic to it in Q_i^l , and the root of Q_i^r is 2-bisimilar to Q_i^l ; but $Q_i^l \not\leq_4 Q_i^r$, because the root in Q_i^r has no 3-bisimilar point on the left models.

Hence, no logic which contains them can be 2-uniform. By our axiomatization of the logic **2Uni**, any logic which is not an extension of it will therefore contain one of these posets by Yankov's theorem.

On the other hand, by Lemma 6.2.7, Proposition 6.2.9 and Corollary 6.1.8, we have that **2Uni** is 3-uniform. ■

To finish this section we note that there will only be countably many logics which are 3-uniform, we recall that we can consider an order on finite rooted posets: $Q \leq P$ if and only if Q is a p-morphic image of a rooted upset of P ; this is known to form a partial order, which is well-founded, since the posets are finite. Moreover the following is well-known (see [37, Section 3]):

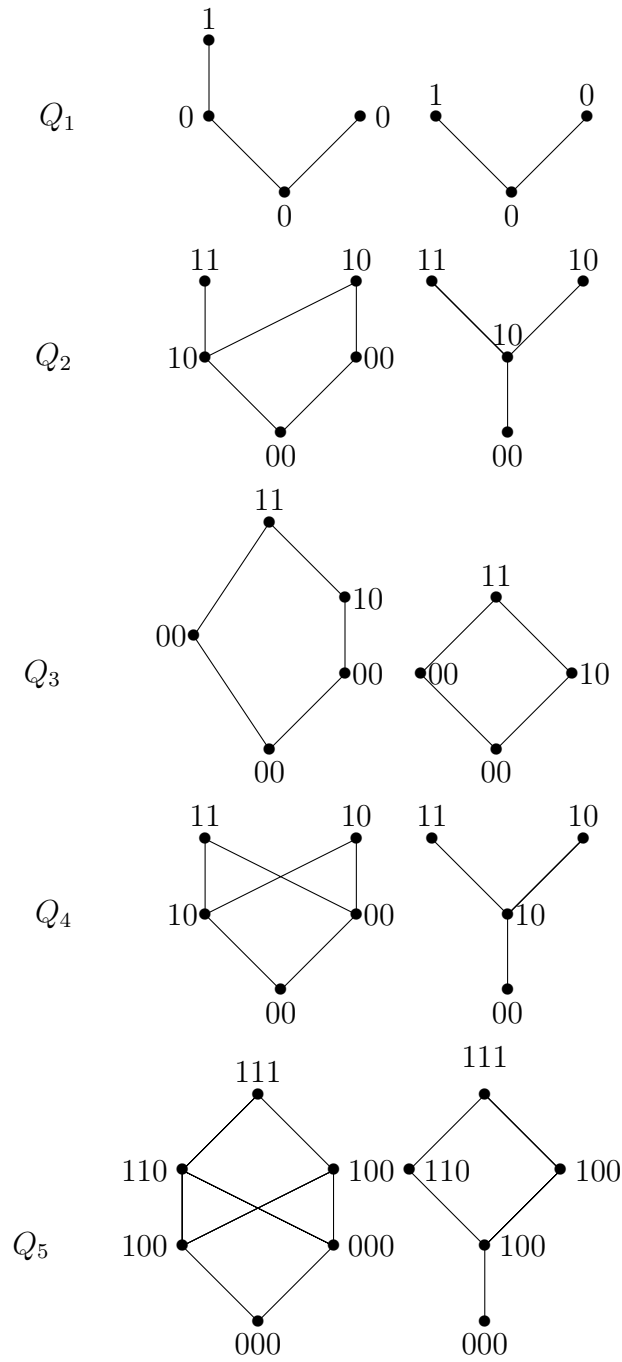


Figure 6.3: Models witnessing failure of 3-uniformity

Theorem 6.2.11 (Axiomatization of locally tabular logics through finite posets). *Let L be a locally tabular logic, and $(\mathbf{RP}(L), \leq)$ the poset of rooted posets of L ordered by $\leq$. If $(\mathbf{RP}(L), \leq)$ contains no infinite antichains, then L has only countably many extensions, and all of them are finitely axiomatizable.*

Corollary 6.2.12 (Countably many 3-uniform logics). *There exist only countably many 3-uniform logics.*

Proof. Note that by Lemma 6.2.7, we have that any rooted poset P corresponds to a pair, of a finite sequence and a natural number, $((k_0, \dots, k_{n-1}), k_n)$ where the sequence has k_0 always 1, and the numbers are alternatingly 1. We denote by $\text{seq}(P)$ this tuple associated to each poset. We order such pairs as:

$$((k_0, \dots, k_{n-1}), k_n) \leq ((j_0, \dots, j_{t-1}), j_t),$$

if and only if $(k_0, \dots, k_{n-1})$ is a subsequence of $(j_0, \dots, j_{t-1})$, and $k_n \leq j_t$. Then note that:

$$Q \leq P \iff \text{seq}(Q) \leq \text{seq}(P);$$

indeed, first note that if Q is a rooted upset of P' , it is in fact a p-morphic image as well, since one can collapse all lower levels to the root of Q . Hence we have that $Q \leq P$ if and only if there is a p-morphic image from P onto Q . Now if this happens, then what can happen to P is that the number of levels in P can get shorter, or the number of elements in the levels may get smaller – the p-morphism defines an inclusion of $\text{seq}(Q)$ in $\text{seq}(P)$, with the last element being preserved. The converse is likewise easy to see.

We thus obtain an isomorphism of $(\mathbf{RP}(L), \leq)$ with $\omega^{<\omega} \times \omega$ ordered by the product of the subsequence order and the natural number ordering. By Higman's lemma [145], the set $\omega^{<\omega}$ ordered by subsequence is a well-partial order, and the product of well-partial orders is again one by Dickson's lemma [97]. So $\mathbf{RP}(L)$ contains no infinite antichains, which by Theorem 6.2.11 means that there are only countably many extensions of 3Uni, all of which are finitely axiomatizable. ■

Remark 6.2.13 (5-uniform will be uncountable). Corollary 6.2.12 should be contrasted with the fact that there are uncountably many logics of depth 3 (see [37, Lemma 3.19, Antichain Ω_2], and the fact noted by Shehtman [240], that all logics of depth 3 are 5-uniform.

6.2.2 4-uniformity and The Logic of Finite Combs

It is clear from the aforementioned discussion that 3-uniformity is substantially more complicated than 2-uniformity, and that it is far from exhausting all uniformly locally tabular logics. In this section we study a basic example of a 4-uniform logic of infinite depth, unrelated to wPL, and which might be interesting

in its own right. The logic provided is the intuitionistic analogue of the bi-intuitionistic logic introduced by Martins and Moraschini [192], **biLFC** – the logic of finite combs.

Definition 6.2.14 (*n*-combs). A poset P is called an *n-comb* if $P = P_0 \sqcup P_1$, where $P_0 = \{x_1, \dots, x_n\}$ and $P_1 = \{y_1, \dots, y_n\}$, and we have that for each $x \in P$,

1. y_i is maximal;
2. $x_i \leq x$ if and only if $x = x_j$ and $i \leq j$, or $x = y_j$ and $i \leq j$.

A subposet $Q \subseteq P$ is called a *broken n-comb* if $P_0 \subseteq Q$. We call a poset a (broken) comb if it is a (broken) *n-comb* for some n .

The following is easy to verify:

Lemma 6.2.15 (p-morphic images of combs are broken). A poset is a p-morphic image of an *n-comb* if and only if it is a broken *m-comb*.

Proof. Simply note that for *n-combs*, the maximal points must be identified with each other. ■

Note that the posets in Figure 6.1 and in Figure 6.4 are all models over combs and broken combs. The poset Q_1 is likewise an example, and the depiction in Figure 6.3 shows that 2-bisimilarity is not enough to capture full bisimilarity for such models.

Proposition 6.2.16 (3-bisimilarity implies bisimilarity for broken *n-combs*). Let P and Q be broken *n-combs*. Let (P, x) and (Q, x') be *n-generated* models which are 3-bisimilar. Then they are isomorphic.

Proof. Assume that the two posets are *n-generated* for some n . Now, if (P, x) and (Q, x') are 3-bisimilar, notice that they must have the same maximal elements. Suppose that (P, x) has a largest non-maximal element which has a single immediate successor, x_k ; if (Q, y) did not end have a similar looking point, look at x'_k , its last point in the base of the comb, which will have two immediate successors. Then by the 3-bisimulation, there must be a point x_j which is 2-bisimilar with x'_k . Such a point must see two distinct maximal elements, and hence it must be some point $x_j < x_k$.

First assume that x_j has a different color from x_k ; then this forces that some point of color x_k will be above x'_k , and by 3-bisimulation, we get a contradiction. So if x_j has the same color, we can assume that it sees exactly two maximal elements, and hence, is covered by x_k . By 3-bisimulation, there must be a point $x''_k \in Q$ which is 2-bisimilar with x_k – but by the configuration, such an element cannot exist. So we have that x'_k must have a single maximal element.

Now let x_{k-1} be the immediate predecessor of the maximal point in P . We show that Q must have a point with the exact same structure. If x_{k-1} has a single

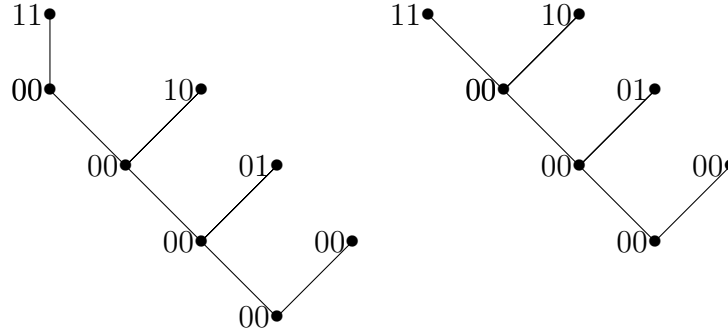


Figure 6.4: Comb and broken comb models

successor, x_k , the argument is similar to the one given in Proposition 6.2.9. If x_{k-1} has another immediate successor y_{k-1} , then we proceed as in the previous paragraph to ensure that Q has such a point in the immediate predecessor of x'_k . ■

Let LFC be the following logic:

$$\text{LFC} := \text{Log}(\{P : P \text{ is an } n\text{-comb}\}).$$

We have thus shown:

Theorem 6.2.17 (Logic of finite combs is uniformly locally tabular). *The logic LFC is uniformly locally tabular. Indeed it is 4-uniform, but not 3-uniform.*

Proof. By Corollary 6.1.8, and Proposition 6.2.16, we have the 4-uniformity of LFC. The fact that it is not 3-uniform follows from Theorem 6.2.10. ■

We conclude by noting that this logic admits a simple axiomatization with Yankov formulas. For that we need the following lemma:

Theorem 6.2.18 (Axiomatization of LFC). *The logic:*

$$\text{IPC} \oplus \{\mathcal{J}(Q_i) : i \in \{2, 4, 5, 6, 7, 8\}\},$$

is locally finite, and its finite models are the finite combs. Consequently, it coincides with LFC.

Proof. For a given n we will describe the n -universal model for this logic, as described in Chapter 3. Such a model will be shown to be finite, from which it will follow that the logic is locally tabular.

Indeed, note that we can start with 2^n many elements. Now if x has one of the maximal elements as its cover, it can only have at most two of them, otherwise we obtain a copy of Q_8 as a rooted upset. Now let x be a point such that it is

not covered by a maximal element; enumerate its successors, say $y_1, \dots, y_n$. Note that if y_i and y_j never see the same maximal elements, then $n = 1$; otherwise we would obtain a p-morphism onto Q_7 .

1. Assume that y_i and y_j both see the same maximal element z . Suppose that there is no other z' . Then we get a p-morphism onto Q_6 .
2. Assume that there is at least another maximal element z' . Identify all other maximal elements with z' . Then either y_i and y_j both now see z' , or one sees both z and z' and the other only sees z . Then we obtain a p-morphism either onto Q_2 or Q_4 .

This shows that $n = 1$ is the only possibility. Hence if x does not see any maximal element, its color must be strictly smaller than that of its unique immediate successor. This shows that P must look like a broken k -comb for some k . As such, we get that this coincides with LFC, since it is not hard to show that every poset of LFC validates the formulas of the above logic: otherwise, they would contain one of the given posets as a p-morphic image of a rooted upset, in contradiction to Lemma 6.2.15. ■

6.2.3 A Non-uniformly locally tabular logic

The analysis of 3-uniformity sheds some light on the possibilities for logics to be n -uniform but not $(n + 1)$ -uniform. Namely, note that $\mathbf{Log}(Q_4)$ will be an extension of $\mathbf{wPL}$, and it will be 4-uniform but not 3-uniform by what was just shown. A more general fact holds which will allow us to provide an example of a locally tabular logic which is not uniformly locally tabular.

Definition 6.2.19 (Box models). Let $n > 2$ and $k + 1 < n$. Let M_n^k, N_n^k for $k \leq n$ be the models depicted in Figure 6.6. Formally, we define:

1. M_n^k has $k + 1$ many layers of two elements; if $L_j(M_n^k)$ is the j -th layer for $0 \leq j \leq k$, then the left element in $L_j(M_n^k)$ is labelled with $1^{n-j}0^j$, and the right element with $1^{n-(j+1)}0^{j+1}$. There is also a root element labelled with $1^{n-k-1}0^{k+1}$.
2. N_n^k has likewise $k + 1$ many layers of two elements; if $L_j(N_n^k)$ is the j -th layer for $0 \leq j \leq k$, then the left element in $L_j(N_n^k)$ is labelled with $1^{n-j}0^j$, and the right element with $1^{n-(j+1)}0^{j+1}$. There is also a root element labelled with $1^{n-k-2}0^{k+2}$.

Thus the only difference between M_n^k and N_n^k is in the label of the root. The order in both cases is given by stacking the layers on top of each other.

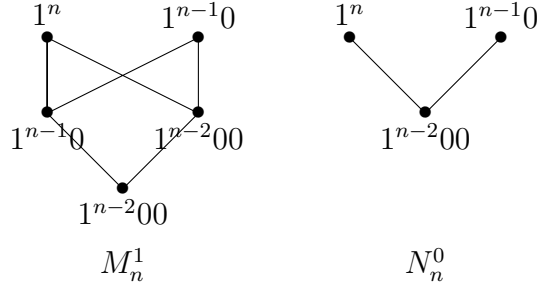
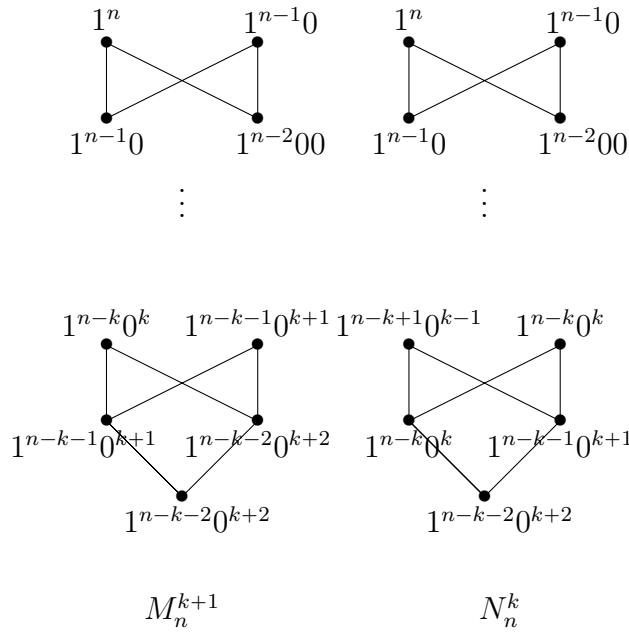


Figure 6.5

Figure 6.6: General models M_n^{k+1} and N_n^k

Note that these models are always n -generated. Indeed, the key difference between these models is that the root of N_n^k goes down in color, relative to its right immediate successor, whilst in M_n^k the color is the same. Figure 6.5 shows the cases for $k = 1$, for any $n \geq 2$

The following lemma provides the relationship between such models:

Lemma 6.2.20 (n-bisimulation separation for box models). Let $n > 2$ be arbitrary, and $k + 1 < n$. Then M_n^k is k -bisimilar, but not $(k + 1)$ -bisimilar with N_n^{k-1} .

Proof. This is shown by induction. Note that for $k = 1$, this is easy to verify, since N_n^1 will be of depth 2: looking at Figure 6.5, we have that the two roots have the same color, and they see points with the same colors, so they will be 1-bisimilar. However, assume that they were 2-bisimilar; let x be the point in M_n^1

with the label $1^{n-1}0$ which is the successor of the root. Then by 2-bisimilarity, there is some point in N_n^0 which is 1-bisimilar with it; but the only point which could be is the maximal point, which is a contradiction.

Now, assume that this holds for k . Consider M_n^{k+1} and N_n^k . Note that they have roots of the same color, labelled with $1^{n-k-2}0^{k+2}$, and indeed, if z is the immediate successor of M_n^{k+1} with the same color, $\uparrow z$ is isomorphic to N_n^k .

To understand the bisimulation, recall the game-theoretic description of n -bisimulations discussed in Section 2.3.3. Assume then that Player I picks the root of N_n^k or any of its successors; then Player II has an obvious response by choosing the element inside of $\uparrow z$. Thus the only way that Player I can hope to win the game is to pick either the root of M_n^k , or the left successor of the root. If Player I picks the left successor of the root, say y_0 , note that $\uparrow y_0 \cong M_n^k$; on the other hand, the right successor of the root of N_n^k is isomorphic to N_n^{k-1} , so Player II wins by induction hypothesis. Finally, if Player I picks the root of M_n^{k+1} , Player II is forced to show that this is k -bisimilar to the root of N_n^k ; by the arguments just shown Player I is in turn forced to play the root of M_n^{k+1} for all k many steps of the game, and loses when Player II finally picks the root, given the two roots have the same color. This shows that M_n^{k+1} is k -bisimilar to N_n^k as desired.

Now assume that M_n^{k+1} is $(k+1)$ -bisimilar to N_n^k . Then Player I can pick the left successor of the root of M_n^{k+1} , y_0 , labelled with $1^{n-k-1}0^{k+1}$, and Player II is forced to respond with the right successor of N_n^k , y_1 ; as noted $\uparrow y_0 \cong M_n^k$ and $\uparrow y_1 \cong N_n^{k-1}$. Since by induction hypothesis these models are not k -bisimilar, Player II then loses, meaning that M_n^{k+1} and N_n^k are not $(k+1)$ -bisimilar. ■

Hence we have the following:

Theorem 6.2.21 (Locally tabular $\Rightarrow$ Uniformly locally tabular). *The logic $\mathbf{wPL}$ is locally tabular but not uniformly locally tabular.*

Proof. All the models M_n^k for each n, k are based on frames which validate $\mathbf{wPL}$; hence by Corollary 6.1.8, $\mathbf{wPL}$ is not uniformly locally tabular, since we notice that $M_n^{k+1} \leq_{k+1} N_n^k$ but not $M_n^{k+1} \leq_{k+2} N_n^k$. On the other hand, it is locally tabular as shown by Esakia [102] and Mardaev [189]. ■

6.3 Pre-uniform local tabularity

As noted before the problem of working with locally tabular intermediate logics presents a challenge since, as shown by Mardaev [190], there are uncountably many such pre-locally tabular logics. By contrast, there are exactly three pre-tabular logics [110], and every non-tabular logic is contained in one of them, and all of them are decidable. Together, this makes the problem of deciding, given an axiom φ , whether $\mathbf{IPC} \oplus \varphi$ is tabular or not decidable.

Intrinsic in this is the idea of “pre-local tabularity”. The idea of “pre- P ”-logics, for P a property, has been extensively used by several authors (e.g., [110,

238, 185]) to characterize certain logical properties. We can make such a definition also in our setting:

Definition 6.3.1 (pre-uniformly locally tabular). Let L be a superintuitionistic logic. We say that L is *pre-uniformly locally tabular* if L is not uniformly locally tabular but all of its extensions are so.

We will now characterize pre-uniform local tabularity amongst extensions of the logic **KG**, of Kuznetsov-Gerchik (see next section). To work towards that, first recall the bounded width formula,

$$\mathbf{bw}_2 := (p_0 \rightarrow (p_1 \vee p_2)) \vee (p_1 \rightarrow (p_0 \vee p_2)) \vee (p_2 \rightarrow (p_0 \vee p_1)),$$

a finite frame validates this axiom if and only if they have width at most 2, i.e., for each point x in such a frame, $\uparrow x$ contains antichains of size at most 2. Recall also the formula:

$$\neg p \vee \neg \neg p,$$

the weak law of excluded middle; frames for such a formula are exactly those containing a greatest point above each point.

Definition 6.3.2 (Box logic as an extension of **wPL**). The logic **Box** is defined to be the extension of **wPL** by the $\mathbf{Box} = \mathbf{wPL} \oplus \mathbf{bw}_2 \oplus (\neg p \vee \neg \neg p)$.

We will see that rooted frames for **Box** look very much like those from Figure 6.6, with a point in the maximum – if one thinks of four adjacent points there, these look like stacks of boxes.

Remark 6.3.3. Note that **Box**, as an extension of **wPL** is locally tabular. For that reason, we will assume throughout that all logics L extending it are locally tabular, and consequently have the finite model property.

We will now show that **Box** is pre-uniformly locally tabular. For that purpose, let S_n be the result of adding a maximal point to the underlying frame of N_n^{n-2} (see Figure 6.6). We denote by $\mathbf{Lev}_j(S_k) = \{x \in S_k : d_p(x) = j\}$, the j -th level. Then first we show:

Proposition 6.3.4 (Characterization of **Box** models). We have that $\mathbf{Box} = \mathbf{wPL} \oplus \mathbf{bw}_2 \oplus (\neg p \vee \neg \neg p) = \mathbf{Log}(\{S_n : n \in \omega\})$.

Proof. It is clear that each S_n validates all the axioms of **Box**, since they have a maximal point, have width at most 2, and are frames for **wPL** as noted in Theorem 6.2.21. Conversely, by the semantic characterizations given of these formulas, any finite rooted poset P validating **Box** will be a Boolean sum (as it validates **wPL**), it will have width at most 2 by $\mathbf{bw}_2$, and a maximal element by the axiom $\neg p \vee \neg \neg p$. Let k be the depth of the root of P . Then for any $j \leq k$, either there is a unique point x such that $d_p(x) = j$, or there are two. This defines a p-morphism from S_k onto P . Thus, every finite rooted poset of **Box** can be obtained by p-morphisms from one of the form S_n , showing the result. ■

Recall the notion of stack depth (Definition 6.2.5); for frames with bounded width 2, having stack depth k amounts to having a bound of k on the number of consecutive levels with two points. Thus we obtain:

Proposition 6.3.5 (Control on stack depth). Let $\mathbf{Box} \subsetneq L'$ be a logic of infinite depth. Then there is some n such that posets in L' have stack depth n .

Proof. If we assume that there is no bound on the stack depth of posets for L' , then for each n , let P be a frame for L' containing a stack of at least $n + 2$ many elements. Let p be one of the minimal elements of the stack, and m be an element situated at n many levels above p ; then $\uparrow p$ is a frame for L' . Define a p -morphism onto S_n by sending m , the other element in its level, and all their successors to the top element in S_n , and all elements in the stack mapped through an isomorphism to S_n . Thus $S_n \models L'$. Since this holds for any n , by Proposition 6.3.4 we have that $L' = \mathbf{Box}$. ■

We thus obtain that the extensions of $\mathbf{Box}$ are controlled by the depth, which can be infinite or finite, and the stack-depth, which can be finite or infinite. We show the following:

Proposition 6.3.6 (Uniform local tabularity from bounded stack depth). Assume that $L \supseteq \mathbf{Box}$ is a logic of infinite depth such that all its elements have at most stack-depth $k \geq 1$. Then L is $(k + 2)$ -uniformly tabular. Consequently, $\mathbf{Box}$ is pre-uniformly locally tabular.

Proof. The fact that $\mathbf{Box}$ is not uniformly locally tabular follows by a simple modification of Theorem 6.2.21, adding a point on top to the models M_n^k with a distinguished color.

Assume that $L \supseteq \mathbf{Box}$ is a logic such that its models have stack depth at most k . Then we show the following modification of Lemma 6.2.8: if (P, x) is a Boolean sum of stack depth at most k and $(P, x) \sim_k (P_y, y)$ then $x = y$. This proceeds by a double induction on k and m , where m is the number of stacks above x of size at least k . For $k = 1$ and arbitrary m , this was provided by Lemma 6.2.8. Assume that it holds for k , and we first show it for $k + 1$ and $m = 1$; In this case, note that since $(P, x) \sim_{k+1} (P_y, y)$, we can pick an immediate successor of x , x' , and by k -bisimilarity, $(P_{x'}, x') \sim_k (P_{y'}, y')$ for some successor of y ; then the stack above $P_{x'}$ must be of size at most k , so by induction hypothesis, $x' = y'$ must hold. Since this holds for any immediate successor of x , we conclude that $x = y$. Assuming this for arbitrary m , we then show it for $m + 1$, by noting that when we move to an immediate successor, the immediate successor will see one fewer $k + 1$ -stack, and so the induction hypothesis can be carried through.

From this fact, we proceed as in Proposition 6.2.9: assuming that we have $(P, x), (Q, y)$, n -generated models based on frames for L , where L has a bound on the stack-depth of k , such that $(P, x) \sim_{k+1} (Q, y)$. Then we have an S_{k+1} -bisimulation, defined by a chain $S_{k+1} \subseteq S_k \subseteq \cdots \subseteq S_0$; then the relation S_k

defines an isomorphism, and so (P, x) and (Q, y) are bisimilar. From this it follows by the same arguments as in Proposition 6.2.9 that L is $k + 2$ -uniformly locally tabular. ■

6.3.1 Pre-uniform local tabularity in KG

In this section, we will characterize pre-uniform local tabularity in the case of extensions of the system **KG** of Kuznetsov-Gerciu [176] (for an overview of the system see [54, Chapter 4]). Such a system has a very broad range of extensions, and a rich semantics, which has made it useful for several purposes in the study of superintuitionistic logic (e.g. [44]). One of its key extensions is the logic $\mathbf{RN} = \mathbf{Log}(\mathbf{RN})$ – the logic of the frame of the Rieger-Nishimura ladder (see Figure 6.7), which is dual to the *Rieger-Nishimura lattice*, the free 1-generated Heyting algebra.

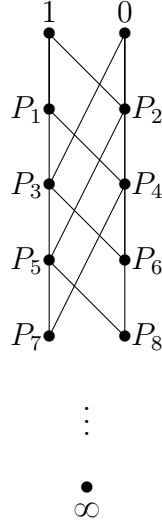


Figure 6.7: Rieger-Nishimura Ladder RN

Recall from [127, 75] (see also [57] and [6]) that $\mathbf{RN}$, the Rieger-Nishimura ladder, can be constructed as the inverse limit of a sequence:

$$P_1 \xleftarrow{r_1} P_2 \xleftarrow{r_2} P_3 \xleftarrow{r_3} \dots$$

where r_i is a map which is a p-morphism with respect to “sufficiently many” parts of P_3 . In fact these admit an explicit description:

1. The posets P_i are precisely the upsets as labelled in Figure 6.7.
2. The map $r_{2n+1} : P_{2n+2} \rightarrow P_{2n+1}$ sends the P_{2n+2} -root point, and the P_{2n} -root point, to P_{2n+1} , and leaves the rest untouched;

3. The map r_{2n+2} sends the P_{2n+3} and the P_{2n+1} -root points to P_{2n+2} , and leaves everything else untouched.

For our purposes we will need the following lemma, shown essentially in [54, Chapter 4]:

Lemma 6.3.7 (Box posets from Rieger-Nishimura). For each n , the poset S_n is an upset of a p-morphic image of $P_{2n+1} \oplus 1$.

Proof. One proceeds inductively as follows: first identify the point labelled 1 in Figure 6.7 with the top element. This produces one top level containing the maximum, followed by a level containing P_1 and 0, seen by P_3 . Then to obtain the next layer, identify P_2 with its now unique successor, and we obtain a new layer, consisting of P_3 and P_4 , seen by P_6 . We thus proceed in this way recursively, obtaining all boxes as desired. ■

To characterize uniform local tabularity, we will rely on the fact that already local tabularity above **KG** is characterized. In principle this is not necessary, as uniform local tabularity is potentially simpler; however in this simple case, it will be convenient to rely on the following, shown in [54, Theorem 4.6.5]:

Theorem 6.3.8 (Local tabularity in **KG**). A logic $\mathbf{KG} \subseteq L$ is not locally tabular if and only if $L \subseteq \mathbf{RN.KC} = \mathbf{RN} \oplus (\neg p \vee \neg \neg p)$.

We will also need one more technical notion:

Definition 6.3.9 (Degree of uniformity). We say that a poset P has *degree of uniformity* k if whenever (P, x) is an n -generated model, and (Q, y) is another n -generated model such that $S_k \subseteq P \times Q$, is a k -bisimulation between x and y , then S_k is an isomorphism.

Remark 6.3.10. The reader should notice that for a fixed poset P , a degree of uniformity need not be minimal – for the purposes where we will use it, the notion of a minimal degree of uniformity will not be needed.

It is well-known that any finite poset has a degree of uniformity (see e.g. [130, Proposition 4.5]).

Definition 6.3.11 (Vertical sum). Given rooted posets P, Q , we write $P \oplus Q$ for the *vertical sum* (see [54, Def. 4.1.13]) the poset based on the disjoint union of P, Q , where for each $x \in P$ and $y \in Q$, $x \leq y$. We denote the repeated vertical sum as $P_1 \oplus \cdots \oplus P_n$.

Lemma 6.3.12 (Degree of uniformity preserved under sums). If $P_1, \dots, P_n$ is a sequence of posets with degree of uniformity k_i , then $P_1 \oplus \cdots \oplus P_n$ has degree of uniformity $k' = \max(\{k_i : i \leq n\}) + 1$.

Proof. Write $P = P_1 \oplus \dots \oplus P_n$. Let (Q, y) be an arbitrary n -generated model, and assume that $(P, x) \sim_{k'} (Q, y)$. Let x_n be the root of P_n ; by k' -bisimilarity, we have that there is some $y_n \in Q$, such that $(P_{x_n}, x_n) \sim_{k'-1} (Q_{y_n}, y_n)$; since $k_n \leq k' - 1$, this relation defines an isomorphism. Note that any predecessor of x_n or y_n must have a properly smaller color. Let $Q_n = \uparrow y_n$. By restricting the relation $S_{k'}$ to $P \setminus (P_n \setminus \{x_n\})$ and $Q \setminus (Q_n \setminus \{y_n\})$, we claim that we still have a k' -bisimulation: for each $x' \geq x$, such that $x' \in P \setminus (P_n \setminus \{x_n\})$, there must be $y' \geq y$ which is $k' - 1$ -bisimilar, and this must lie also in $Q \setminus (Q_n \setminus \{y_n\})$. Thus we can proceed by induction, showing that each such restriction of S_k is an isomorphism. ■

We can thus conclude:

Theorem 6.3.13 (Characterization of pre-uniform local tabularity in $\mathbf{KG}$). *If $L \supseteq \mathbf{KG}$ is any logic, then L is not uniformly locally tabular if and only if $L \subseteq \mathbf{Box}$.*

Proof. If $\mathbf{KG} \subseteq L$, and L is not locally tabular, we thus have by Theorem 6.3.8 that $L \subseteq \mathbf{RN} \subseteq \mathbf{Box}$. Hence assume that L is locally tabular but not uniformly locally tabular. We can assume by Theorem 6.1.7 that L is not of finite depth. Since L is locally tabular, there must be an n which is maximal such that $\uparrow P'_n$ is a frame for the logic. Hence by [54, Corollary 4.3.9], the finitely generated frames for L must be finite vertical sums of p-morphic images of upsets of P_n for $k \leq n$ (not necessarily rooted).

Now as noted in Lemma 6.3.7, for large enough P_n we can obtain S_n , and for large enough vertical sums, we can obtain all the S_n . Suppose that there is some n such that we cannot obtain S_n as a frame for L . Then there must be a bound on the size of the posets Q with a root and a maximal element which appear in vertical sums in L (otherwise using the technique of Lemma 6.3.7 we could obtain arbitrarily large S_n). Using Lemma 6.3.12 we can conclude that posets for L will have a maximal degree of uniformity k for some k – which implies (and is stronger than the fact that) whenever $(P, x) \leq_k (Q, y)$ for (P, x) and (Q, y) two models based on frames for L , then $(P, x) \leq_\infty (Q, y)$. By Proposition 6.1.8 we obtain that L is uniformly locally tabular – a contradiction.

So we must assume that L contains arbitrarily large vertical sums of P_n as frames. Thus as noted, we can obtain S_n for each n . By Proposition 6.3.4 we have that $L \subseteq \mathbf{Box}$ as desired. ■

Corollary 6.3.14 (Decidability of pre-uniform local tabularity). *Above $\mathbf{KG}$, uniform local tabularity is decidable: if $L = \mathbf{KG} \oplus \varphi$, it is decidable whether L is uniformly locally tabular.*

Proof. Since the logic $\mathbf{Box}$ is finitely axiomatized and locally finite, it is decidable. Given L a decidable logic, one simply checks whether $L \subseteq \mathbf{Box}$, by checking whether $\mathbf{Box}$ contains the axiom φ , which is enough by Theorem 6.3.13. ■

6.4 Summary

A visual summary of these results in terms of the hierarchy of superintuitionistic logics is given in Figure 6.8.

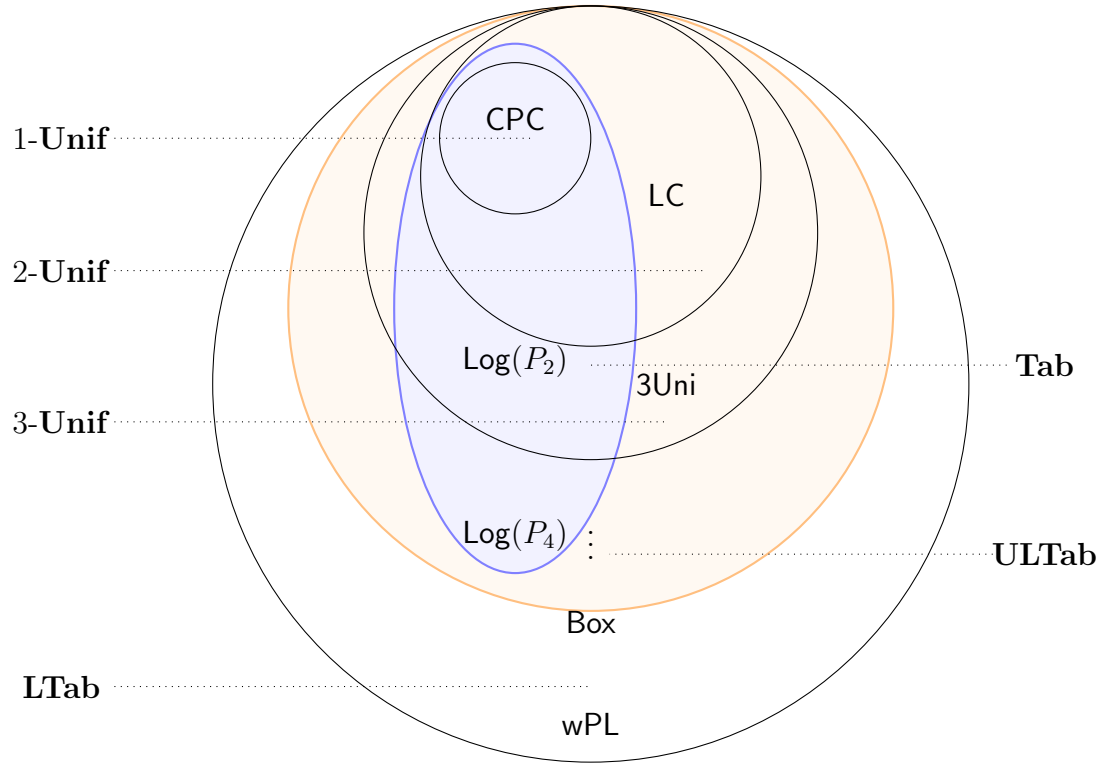


Figure 6.8: Hierarchy of Locally Tabular Superintuitionistic Logics; blue oval covers the tabular logics.

Chapter 7

Étale Heyting algebras and Toposes

In [218], Andrew Pitts asked whether every Heyting algebra arises as the lattice of truth-values of an elementary topos – what we call being a *topos-admissible Heyting algebra*. Despite interest in the community, and a complex history, the problem has remained largely open. In an attempt to answer this question in the positive, and with a view to addressing several other persistent open problems in the field, Jibladze and Evgeny Kuznetsov [177, 155] introduced the concept of *étale Heyting algebras*. In this chapter we provide a basic logical and algebraic analysis of the kind of logical calculi associated to these classes of algebras. We then zoom in on a special class of algebras – étale-finite Heyting algebras – and prove that all of them are topos-admissible.

Outline of the chapter

1. In Section 7.1 we provide some basic background on the Heyting-to-topos problem, and the motivations to introduce étale-Heyting algebras, as well as recalling some basic concepts needed moving forward.
2. In Section 7.2 we provide the basic theory of varieties of étale Heyting algebras, the dual étale Esakia spaces, and also introduce their logical counterpart in the form of *internal H -calculi*, denoted by Int_H . We study dual representations of free étale Heyting algebras in general, and their simplifications in the case of Smetanich algebras. We also present the basic theory of étale-finite Heyting algebras.
3. In Section 7.3 we provide the main theorem of this Chapter: every étale-finite Heyting algebra appears as the lattice of truth values of an elementary topos.
4. In Section 7.4 we identify some key properties of the toposes produced by our method, which we call *finitely propositional toposes*. We show that every finitely propositional topos carries an étale-finite Heyting algebra as

its lattice of truth-values, providing a key obstruction to extending the methods of this chapter beyond this case.

5. We conclude in Section 7.5, by pointing to future avenues of development, inspired by the study of étale Smetanich algebras and the locally Esakia spaces of Chapter 3.

7.1 Heyting algebras and toposes

In [218], Andrew Pitts proved that, for each Heyting algebra H , the inclusion $i: H \hookrightarrow H[x]$ has a left and a right adjoint, establishing consequently, that IPC has uniform interpolation (See Section 2.2.3, Theorem 2.2.44). In the introduction to that paper, the author explains how this result, which implies uniform interpolation (see Section 2.2.3), arose out of an attempt to solve, in the negative, the following question:

Problem 7.1.1 (Heyting-to-topos problem). Given a Heyting algebra H , is there an elementary topos $\mathcal{E}$ such that the lattice $\text{Sub}_{\mathcal{E}}(1)$ of subobjects of the terminal object 1 is isomorphic to H ?

Such a problem can be formulated in purely logical terms: does every propositional intuitionistic theory T have a higher-order extension T' such that

1. T' is conservative over T , i.e., T' proves no new propositional sentences,
2. and, for every higher-order sentence φ , T' proves that φ is equivalent to a propositional sentence?

The lattice $\text{Sub}_{\mathcal{E}}(1)$ is also known as *the lattice of truth values of $\mathcal{E}$* . A Heyting algebra isomorphic to $\text{Sub}_{\mathcal{E}}(1)$ for some elementary topos $\mathcal{E}$ will be called *topos-admissible*. Since Pitts' paper, two classes of Heyting algebras have been shown to be topos-admissible:

- *Complete Heyting algebras* (i.e., locales), where one can take the topos of sheaves on the given locale [183, Ch. IX.5];
- *Boolean algebras*, where a construction of Peter Freyd is known to produce a topos with the desired properties (see [159, Exercise 9.11], and also [219, Example 4.8] for a similar construction).

Some basic closure conditions are known on topos-admissible Heyting algebras: they are closed under products and homomorphic images. With a few isolated exceptions (see Appendix A.5.3), this constitutes at time of writing the boundary of knowledge on topos-admissible Heyting algebras.

This problem appeared multiple times in the literature during the 1990s and early 2000s (see e.g. [130, Introduction], [219]). In the early 2000s, Dimitri (Dito)

Patariaia began work on this problem; part of his analysis is outlined in a lecture given by Peter Johnstone [156] on the subject of “hochas” (see A.5.1) – higher-order cylindric Heyting algebras – and involved adapting methods related to the step-by-step construction of finitely generated free Heyting algebras [127] to adding higher-order quantifiers. Though there were claims in the community of a solution, Patariaia passed away without leaving a written proof. Over the years, the problem kept being asked in different contexts (see e.g., [2, 1]). In recent years there has been a renewed interest in the problem: in [178], a state of the art was given on the problem, outlining some of the difficulties of extending it beyond the Boolean case, and, in [216], the problem was asked once more, leading to more attention by the research community (e.g., see [135, Question 16]). Throughout, we will generically refer to this problem as the “Heyting-to-topos” problem¹.

Patariaia’s approach with hochas—similar to Pitts’ tripos-to-topos construction [219]—amounts to an algebraic encoding of the structure of a minimal topos (see Appendix A.5.1 for a discussion). By contrast, Freyd’s approach is strongly geometric, and heavily reminiscent of the methods of Grothendieck topos theory: given a Boolean algebra, one considers its Stone dual $\text{Spec}(B)$, and local homeomorphisms from Stone spaces X to $\text{Spec}(B)$. The fact that maps between Stone spaces are closed, and the properties of local homeomorphisms, makes it such that the subterminals are easily seen to form the lattice of clopens of $\text{Spec}(B)$, which by Stone duality is isomorphic to B itself.

Motivated by this sheaf-theoretic intuition, E. Kuznetsov and Jibladze [177, 155] introduced *étale Heyting algebras*: these can be seen as pairs (H, f) where $f: H_0 \rightarrow H$ is a Heyting homomorphism which additionally *preserve the validity of all equations with constants from H* . Jibladze and E. Kuznetsov employed étale Heyting algebras to solve a question related to KM algebras: they gave a transparent algebraic proof that every variety of Heyting algebras is generated by its KM algebras (a result originally due to A. M. Kuznetsov [175]). This is related to the question of whether every variety of Heyting algebras is generated by its complete Heyting algebras (see e.g. [46]).

Amongst étale Heyting algebras, An especially natural class of such algebras are those satisfying *Jibladze’s laws* [164, 154], which give a generalized law of excluded middle, and which we call *étale-finite Heyting algebras*. These will play a substantial role in this chapter: we will show that every étale-finite Heyting algebra is topos-admissible, and show that in a technical sense, these form the largest class for which a topos can be constructed using local homeomorphisms on categories of compact spaces.

¹In private communication with Andrew Pitts, we learned that such a problem was mentioned to Pitts by Peter Freyd, though it is seemingly not known who originally posed the question.

7.2 Étale structures and internal H-calculi

In this section, we study étale Heyting algebras from an algebraic and logical point of view. We then move on to analyzing free étale algebras in general and in the special case of Smetanich algebras; we conclude by outlining the basic theory of étale-finite Heyting algebras.

7.2.1 Internal H-calculi

Definition 7.2.1 (H-algebras). Given H_0 a Heyting algebra, a pair (H, f) where $f: H_0 \rightarrow H$ is a homomorphism is called an H_0 -algebra.

One may see H_0 -algebras alternatively as pairs (H, f) , or as Heyting algebras H with an interpretation of the constants of H_0 subject to the relationships between them:

Definition 7.2.2 (Extended calculus). Let H_0 be a Heyting algebra. We denote by $\mathcal{L}_{\text{IPC}}(H_0)$ the language expanded by a fresh constant c_a for each element of $a \in H_0$. We denote by $\text{IPC}(H_0)$ the extension of IPC in the language $\mathcal{L}_{\text{IPC}}(H_0)$ by the equational axioms of H_0 , i.e., for all $a, b \in H_0$,

1. $c_1 \leftrightarrow \top$ and $c_0 \leftrightarrow \perp$;
2. $c_{a \star b} \leftrightarrow c_a \star c_b$ for $\star \in \{\wedge, \vee, \rightarrow\}$.

An extension $L \supseteq \text{IPC}(H_0)$ is called an H_0 -superintuitionistic logic, if it is closed under Modus Ponens and uniform substitution. We say that L is H_0 -consistent if for each $a \neq b \in H_0$, $c_a \leftrightarrow c_b \notin L$.

Notation 7.2.1 (Identification of constants). Whenever it does not cause confusion, we identify c_a with the element a itself.

Formally, for any $L \supseteq \text{IPC}(H_0)$ we define $\vdash_L$ in the manner introduced in Section 2.2, and obtain that it is a proper algebraic logic, though in a possibly infinitary signature. Note that $\vdash_L$ will still be an implicative logic, and it will enjoy the same deduction theorem (see Section 2.2.3) as IPC:

Proposition 7.2.3 (Deduction theorem). For each Heyting algebra H_0 , and $L \supseteq \text{IPC}(H_0)$ an H_0 -s.i. logic, and for each $\varphi, \psi \in \mathcal{L}_{\text{IPC}}(H_0)$, we have:

$$\varphi \vdash_L \psi \iff \vdash_L \varphi \rightarrow \psi.$$

An H_0 -algebra is then simply an algebraic model of $\text{IPC}(H_0)$. The usual algebraic completeness theorem yields that $\text{IPC}(H_0)$ is sound and complete with respect to H_0 -algebras, which form a variety – homomorphisms between (H, f) and (H', f') are homomorphisms $g: H \rightarrow H'$ such that $f'g = f$. We will assume throughout all the basic algebraic completeness results as discussed in Chapter 2.

Given an H_0 -s.i. logic L , we denote by $\mathbf{Alg}(L)$ its H_0 -algebras. We denote by $\mathbf{HA}_{H_0}$ the variety of all H_0 -algebras. We will be mostly interested in a special subvariety of it which appears to have been first explicitly considered in [177]:

Definition 7.2.4 (Étale Heyting algebras). Given a Heyting algebra H_0 , an H_0 -algebra (H, f) is said to be H_0 -étale if for each finite tuple $\bar{a} = (a_1, \dots, a_n)$ for $a_i \in H_0$, and each formula $\varphi(a_1, \dots, a_n, \bar{y}) \in \mathcal{L}_{\mathbf{IPC}}(H_0)$:

$$H \models \forall \bar{y}(\varphi(\bar{a}, \bar{y}) = 1) \implies H' \models \forall \bar{y}(\varphi(f\bar{a}, \bar{y}) = 1).$$

Denote by $\mathbf{Et}_{H_0}$ the class of H_0 -étale algebras.

Definition 7.2.5 (H_0 -internal calculus). We denote by $\mathbf{Int}_{H_0}$, the H_0 -internal calculus, the extension of $\mathbf{IPC}(H_0)$ by the axioms $\varphi(\bar{c}_{\bar{a}}, \bar{p})$ such that $H_0 \models \forall \bar{y} \varphi(\bar{a}, \bar{y})$.

The following was noted in [177, pp.10]:

Proposition 7.2.6 (Completeness of Internal H_0 -calculi). For each Heyting algebra H_0 , $\mathbf{Et}_{H_0} = \mathbf{Var}(H_0, \text{id}_{H_0})$, $\mathbf{Int}_{H_0}$ is H_0 -consistent, and is sound and complete with respect to $\mathbf{Et}_{H_0}$.

We refer to $\mathbf{Et}_{H_0}$ as *the variety of H_0 -étale algebras*. A class $\mathbf{V}$ of the form $\mathbf{Et}_{H_0}$ for some H will be called generically a variety of étale algebras.

Example 7.2.7 (Classical logic is the $\mathbf{2}$ -internal calculus). Note that by construction, $\mathbf{IPC}(\mathbf{2})$ is term equivalent to $\mathbf{IPC}$, since the axioms force the additional constants to be interpreted as $\top$ and $\perp$. On the other hand, $\mathbf{Int}_{\mathbf{2}}$ adds as axioms the equational theory of $\mathbf{2}$, which is classical logic. Thus $\mathbf{Int}_{\mathbf{2}}$ is equivalent to $\mathbf{CPC}$.

Example 7.2.8 (KM tabular Gödel logics are chain-internal calculi). If one takes the n -element chain $[n]$, then $\mathbf{IPC}([n])$ is a form of n -valued intuitionistic logic, with no additional connectives. On the other hand, $\mathbf{Int}_{[n]}$ can be seen to be term equivalent to the logic in the language $\mathcal{L}_{\mathbf{IPC}, \Box}$ axiomatized by,

$$\mathbf{KM} \oplus (p \rightarrow q) \vee (q \rightarrow p) \oplus \mathbf{bd}_n.$$

To see this note that the variety corresponding to this logic is $\mathbf{Var}_{\mathbf{KM}}([n])$. Moreover, the presence of the $\mathbf{KM}$ -modality makes every element in $[n]$ a definable constant: for each $k \in [n]$, this is represented by $\Box^{k-1} \perp$. Conversely, any algebra in $\mathbf{Var}([n], \text{id}_n)$ can be seen to be a $\mathbf{KM}$ -algebra (see Example A.2.4).

The situation from Example 7.2.7 is completely general: the internal H_0 -calculi always correspond to the strongest superintuitionistic logic, in an extended language, containing a given Heyting algebra as a model:

Proposition 7.2.9 (Internal H_0 -calculi are largest H_0 -consistent). For each $L \supseteq \mathbf{IPC}(H_0)$ a superintuitionistic H_0 -logic which is H_0 -consistent, we have $L \subseteq \mathbf{Int}_{H_0}$.

Proof. Let $\mathbf{V}$ be the variety of L . It suffices to show that $(H_0, \text{id}_{H_0}) \in \mathbf{V}$. For that purpose, let $(\mathcal{F}_{\mathbf{V}}(\omega), f) \in \mathbf{V}$ be the free ω -generated algebra in this variety. Then consider $f: H_0 \rightarrow \mathcal{F}_{\mathbf{V}}(\omega)$ the witnessing map. Then f is injective: if we had:

$$\mathcal{F}_{\mathbf{V}}(\omega) \models f(a) = f(b),$$

then $c_a \leftrightarrow c_b = 1$, which would be a contradiction to L being H_0 -consistent. Since f is injective, then $f[H_0] \cong H_0$, and since this is a variety, hence closed under subalgebras, so we have $(f[H_0], \text{id}_{f[H_0]}) \in \mathbf{V}$; by closure under isomorphisms, $(H_0, \text{id}_{H_0}) \in \mathbf{V}$.

We have that $f[H_0]$ will be a subalgebra of H , and hence $(f[H_0], f) \in \mathbf{V}$. By algebraic completeness, we conclude that $L \subseteq \text{Int}_{H_0}$. $\blacksquare$

We thus obtain that Int_{H_0} serves as a boundary. In this context, one may wonder what is the connection between the H_0 -étale algebra (H, f) and some further H -étale algebra. This was noted in [155]:

Proposition 7.2.10. [155, Corollary 2.11] Let H_0 be a Heyting algebra, (H, f) be H_0 -étale, and (H', g) is H -étale. Then we have that (H', gf) is H_0 -étale.

Remark 7.2.11 (Inductive rules and étale algebras). The Heyting homomorphisms $f: H_0 \rightarrow H$ which witness the fact that (H, f) is H_0 -étale have a close connection with some morphisms appearing elsewhere in the study of Heyting algebras. This is connected with so-called Π_2 -rules, i.e., rules which have the first-order structure

$$\forall \bar{x} (\forall \bar{z} \varphi(\bar{x}, \bar{z}) \rightarrow \psi(\bar{x})),$$

where φ and ψ are (conjunctions of) equations. Such formulas are special Horn formulas, allowing for universal quantifiers in the antecedent, and therefore, generalizing quasi-equations. In [5], we discussed the model-theoretic characterization of classes of algebras axiomatized by such formulas: they are precisely those classes closed under the operators $\mathbb{I}$, $\mathbb{P}$, $\mathbb{P}_U$ and the special operator $\mathbb{S}_V$ corresponding to $\forall$ -subalgebras:

Definition 7.2.12 ($\forall$ -subalgebras). A subalgebra $A \leq B$ is a $\forall$ -subalgebra if whenever $A \models \lambda(\bar{a}, \bar{x}) \approx \gamma(\bar{a}, \bar{x})$, then $B \models \lambda(\bar{a}, \bar{x}) \approx \gamma(\bar{a}, \bar{x})$. $\forall$ -embeddings are defined in the obvious corresponding way, by composing with isomorphisms.

It is clear, then, that if (H, f) is a H_0 -algebra, where f is injective, then f is a $\forall$ -subalgebra if and only if (H, f) is H_0 -étale.

In general, the characterization of when a given f is an étale homomorphism, and relatedly, the axiomatization of Int_H , may be quite complex problems, given that the structure of H may itself be rather complex. We turn to this problem in the manner characteristic of this thesis: by attempting to describe the free objects of such calculi, both algebraically and dually.

7.2.2 Free étale Heyting algebras

In this section we describe some general facts about the structure of free 1-generated Heyting algebras.

Definition 7.2.13 (Free 1-generated algebra in internal calculus). Let H_0 be a Heyting algebra. We denote by $H_0\langle\iota\rangle$ the free algebra in one generator in the variety $\mathbf{Et}_{H_0}$.

The algebras $H_0\langle\iota\rangle$ algebras played a key role in the work of Jibladze and E. Kuznetsov in [155], where they were used to provide a one-step “enrichment” of Heyting algebras by KM-operations.

Notation 7.2.2 (Functional notation for elements of powers). Given H a Heyting algebra, we denote by $c_h: H \rightarrow H$ the constant map with value $h \in H$, and by $d: H \rightarrow H$ the identity map.

Proposition 7.2.14 (Explicit construction of free étale). [155, Proposition 4.1] For any Heyting algebra H_0 , we have that $H_0\langle\iota\rangle$ is isomorphic to the subalgebra of H^H generated by $\{c_h : h \in H\} \cup \{d\}$.

Note also the following:

Lemma 7.2.15 (Iterated description of free Heyting algebras). For each Heyting algebra H , we have that the map $\iota^2: H_0 \rightarrow (H_0\langle\iota_{H_0}\rangle)\langle\iota_{H_0\langle\iota_{H_0}\rangle}\rangle$, given by composition of canonical inclusions, is isomorphic to the free Heyting algebra on two generators in $\mathbf{Et}_{H_0}$. Moreover the inclusion map $\iota_{H_0\langle\iota_{H_0}\rangle}$ is the canonical inclusion between such algebras.

Proof. Note that by Proposition 7.2.10, this will be an object in $\mathbf{Et}_{H_0}$. Therefore, one simply has to check the universal property: for each (H, f) and each pair of elements (a, b) , we obtain from f and a a unique homomorphism $\bar{f}: H_0\langle\iota_{H_0}\rangle \rightarrow H$ sending ι_{H_0} to a . Given any formula $\varphi(\bar{x}, \iota_{H_0}, f\bar{a})$ for $\bar{a} \subseteq H_0$, note that by the construction of $H_0\langle\iota_{H_0}\rangle$ given in Proposition 7.2.14, we can find an equivalent formula $\psi(\bar{x}, x, f\bar{a})$, where ι_{H_0} is replaced by a free variable; then we have that $H \models \forall \bar{x} \varphi(\bar{x}, \bar{f}\iota_{H_0}, \bar{f}\bar{a})$, if and only if $H \models \forall \bar{x}, x \psi(\bar{x}, x, \bar{f}\bar{a})$. Thus, if it holds that $H_0\langle\iota_{H_0}\rangle \models \forall \bar{x}, x \varphi(\bar{x}, \iota_{H_0}, f\bar{a})$, this implies that $H_0 \models \forall \bar{x}, x \psi(\bar{x}, x, \bar{a})$. Since (H, f) is H_0 -étale, $H \models \forall \bar{x}, x \psi(\bar{x}, x, \bar{f}\bar{a})$, which implies that $H \models \forall \bar{x}, x \varphi(\bar{x}, x, \bar{f}\bar{a})$.

This shows that $\bar{f}: H_0\langle\iota_{H_0}\rangle \rightarrow H$ is such that $(H, \bar{f})$ is $H_0\langle\iota_{H_0}\rangle$ -étale. Thus using the universal property of $(H_0\langle\iota_{H_0}\rangle)\langle\iota_{H_0\langle\iota_{H_0}\rangle}\rangle$, we obtain a unique extension $\bar{f}^2: (H_0\langle\iota_{H_0}\rangle)\langle\iota_{H_0\langle\iota_{H_0}\rangle}\rangle \rightarrow H$, sending $\iota_{H_0\langle\iota_{H_0}\rangle}$ to b . It is straightforward to verify that this factors through the composition ι^2 , and that such a map is indeed unique as a map of H_0 -algebras. ■

To obtain a better grasp of these objects, we combine the methods of the previous chapters to describe $H_0\langle\iota\rangle$ for a generic H_0 . The reader may this point

wish to consult the description given of the coproduct of Esakia spaces given in Section 3.4.1, and the step-by-step construction of Heyting algebras discussed throughout Chapter 4, especially the polynomial complex from Section 4.4.3.

Definition 7.2.16 (Representable polynomial complex). Let H be a Heyting algebra, $X = \text{Spec}(H)$, and let $\mathcal{G}^{\pi x}(X \times \mathbf{2}_{\leq}) = ((X \times \mathbf{2}_{\leq})_n, r_n)$ the polynomial Ghilardi complex². Let $\bigsqcup_{h \in H} X_h$ be the topological and ordered sum of H many copies of X .

1. Let $f_0 : \bigsqcup_{h \in H} X_h \rightarrow X \times \mathbf{2}_{\leq}$ be the map sending $x \in X_h$ to $(x, 1)$ if and only if $h \in x$. We denote by $\text{Rep}_0 = f_0[\bigsqcup_{h \in H} X_h]$.
2. For each n , $f_{n+1} : \bigsqcup_{h \in H} X_h \rightarrow (X \times \mathbf{2}_{\leq})_n$ is defined by sending x to $f_n[\uparrow x]$; notice that this is well-defined, since for each $h \in H$, f_n is a closed map, as it restricts to a map $f_h : X_h \rightarrow (X \times \mathbf{2}_{\leq})_n$ summand X_h . We denote by $\text{Rep}_{n+1} = f_{n+1}[\bigsqcup_{h \in H} X_h]$.

Points from Rep_k are called *representable*. The clear advantage of them is that one has a very easy method of constructing them, and of checking their validity. However, they do not often form a closed subspace. For that reason we need to consider the following:

Definition 7.2.17 (Internal Ghilardi complex). The H_0 -internal Ghilardi complex, is the sequence:

$$\overline{\text{Rep}_0} \xleftarrow{r_0} \overline{\text{Rep}_1} \xleftarrow{r_1} \dots \xleftarrow{r_n} \overline{\text{Rep}_n} \xleftarrow{r_{n+1}} \dots$$

where $\overline{\text{Rep}_k}$ is the topological closure in $(X \times \mathbf{2}_{\leq})_n$ of Rep_n , and r_n are the restrictions of the root maps.

Note that this chain is well-defined:

Lemma 7.2.18 (Internal Ghilardi complex is well-defined). The maps r_n are surjective continuous maps when restricted to the representable Ghilardi complex. Moreover, for each n , the unique lifting $\overline{f_{n+1}} : \bigoplus_{h \in H} X_h \rightarrow (X \times \mathbf{2}_{\leq})_n$ has its image equal to $\overline{\text{Rep}_{n+1}}$, and commutes with the diagram.

Proof. Note that $r_n : \text{Rep}_{n+1} \rightarrow \text{Rep}_n$ is surjective, since if $x \in \text{Rep}_n$, there is some $h \in H$ and $y \in X_h$ such that $f_n(y) = x$; then by definition, $r_n f_{n+1}(y) = r_n f_n[\uparrow y] = x$. Since r_n is a continuous map between the Stone spaces $(X \times \mathbf{2}_{\leq})_{n+1}$ and $(X \times \mathbf{2}_{\leq})_n$, it is closed. Thus:

$$r_n[\overline{\text{Rep}_{n+1}}] = \overline{r_n[\text{Rep}_{n+1}]} = \overline{\text{Rep}_n},$$

²See Proposition 4.4.24.

which shows surjectivity. The fact that the lifting works as desired follows by similar considerations:

$$\begin{aligned}
 \overline{f_{n+1}}\left[\eta\left(\bigsqcup_{h \in H} X_h\right)\right] &= \overline{f_{n+1}}\left[\bigsqcup_{h \in H} X_h\right] \\
 &= \overline{f_{n+1}}\left[\bigsqcup_{h \in H} X_h\right] \\
 &= \overline{f_{n+1}}\left[\bigsqcup_{h \in H} X_h\right] \\
 &= \overline{\text{Rep}_{n+1}};
 \end{aligned}$$

the fact that $\overline{f_{n+1}}$ commutes with the diagram follows by entirely analogous arguments. $\blacksquare$

Notation 7.2.3 (Iterated top elements). Let X be an Esakia space. For each n , define $\top_n$ by: (1) $\top_0 = X \times \{1\}$, and (2) for each $n \geq 0$, $\top_{n+1} = r_n^{-1}[\top_n]$.

To proceed we will need a logical interpretation for the elements of the H_0 -representable Ghilardi complex. For the following, recall Definition 5.1.2, giving the formula interpretations over Ghilardi complexes:

Lemma 7.2.19 (Invariance of truth for internal Ghilardi complexes). Let $n \in \omega$, $x \in \bigsqcup_{h \in H} X_h$, and let K_h be the clopen upset representing h . For each formula φ of depth at most n , and for each $V_1, \dots, V_n \subseteq X$, if $x \in X_h$, then $x \in \varphi(V_1, \dots, V_n, K_h)$ if and only if $f_n(x) \in \mathcal{I}^\varphi(V_1, \dots, V_n, \top_n)$.

Proof. We show this by induction. The base case is obvious by definition of X_h , and the map f_0 . Now assume that this holds for $n = 0$. Now assume that φ is of depth $n + 1$; then we can assume that φ is a disjunction of conjunctions of implications, so without loss of generality, take it to be an implication $\chi_1 \rightarrow \chi_2$. We want to show that $f_{n+1}(x) \in \chi_1(\overline{V}, \top_n) \rightarrow \chi_2(\overline{V}, \top_n)$. Suppose that $x \leq y$, and $f_n(y) \in \chi_1(\overline{V}, \top_n)$; by induction, we have that $y \in \chi_1(\overline{V}, K_h)$. Since $x \in \chi_1 \rightarrow \chi_2$, $y \in \chi_2(\overline{V}, K_h)$, which by induction shows that $f_n(y) \in \chi_2(\overline{V}, \top_n)$. The other direction is similar. $\blacksquare$

Corollary 7.2.20 (Characterization of elements of the internal Ghilardi complex). Let X be an Esakia space, and let $x \in (X \times \mathbf{2}_{\leq})_n$ be an element of the polynomial Ghilardi complex. Then the following are equivalent:

1. $x \in \overline{\text{Rep}_n(X)}$;
2. Whenever φ, ψ are formulas of depth at most n , and $V_1, \dots, V_n \subseteq X$, if $\varphi(V_1, \dots, V_n, V) \subseteq \psi(V_1, \dots, V_n, V)$ for each $V \subseteq X$ a clopen upset, then $x \in \mathcal{I}^\varphi(V_1, \dots, V_n, \top_n)$ implies that $x \in \mathcal{I}^\psi(V_1, \dots, V_n, \top_n)$.

Proof. (1) $\Rightarrow$ (2) Assume that $x \in \overline{\text{Rep}_n(X)}$, and φ, ψ are formulas of depth at most n , such that for $V_1, \dots, V_n \subseteq X$, we have for each $V \subseteq X$:

$$\varphi(V_1, \dots, V_n, V) \subseteq \psi(V_1, \dots, V_n, V).$$

Assume that $x \in \mathcal{I}^\varphi(V_1, \dots, V_n, \top_n)$ but $x \notin \mathcal{I}^\psi(V_1, \dots, V_n, \top_n)$; then by closure, there is some representable point $x' \in \text{Rep}_n(X)$, such that $x' \in \mathcal{I}^\varphi(V_1, \dots, V_n, \top_n)$ and $x' \notin \mathcal{I}^\psi(V_1, \dots, V_n, \top_n)$, where $x' = f_n(y)$. By Lemma 7.2.19 we obtain that

$$y \in \varphi(V_1, \dots, V_n, V) \setminus \psi(V_1, \dots, V_n, V),$$

which is a contradiction to our assumption on such a formula.

(2) $\Rightarrow$ (1) Assume that x has the designated property. We must show that each neighbourhood of x intersects the representable points. Consider:

$$\mathcal{I}^\varphi(V_1, \dots, V_n, \top_n) \setminus \mathcal{I}^\psi(V_1, \dots, V_n, \top_n),$$

one such neighbourhood, consisting of interpretations of formulas of depth at most n , over H_0 , which is possible by construction of the clopen upsets of the polynomial Ghilardi complex. By the given property, there must exist some $V \subseteq X$ a clopen upset such that:

$$\varphi(V_1, \dots, V_n, V) \not\subseteq \psi(V_1, \dots, V_n, V).$$

Let y be a point in this neighbourhood. Then again by Lemma 7.2.19, $f_n(y)$ will be a point such that $f_n(y) \in \mathcal{I}^\varphi(V_1, \dots, V_n, \top_n) \setminus \mathcal{I}^\psi(V_1, \dots, V_n, \top_n)$. This concludes the proof. $\blacksquare$

Let $\mathcal{V}_{\text{Rep}}(X)$ be the projective limit of the H_0 -representable Ghilardi. We now show that this construction has the desired universal property. From this it will follow that it is isomorphic to $H_0\langle\iota\rangle$.

Lemma 7.2.21 (Internal Vietoris limit is Esakia). The space $\mathcal{V}_{\text{Rep}}(X)$ is an Esakia space.

Proof. By Lemma 7.2.18, we have that the liftings $\overline{f_n}: \bigoplus_{h \in H} X_h \rightarrow (X \times \mathbf{2}_{\leq})_n$ has its image equal to $\overline{\text{Rep}_n}$ and commutes with the diagram. Thus there is a unique p-morphism $\overline{f_\infty}: \bigoplus_{h \in H} X_h \rightarrow \mathcal{V}_G^{\pi_X}(X \times \mathbf{2}_{\leq})$. By construction of $\overline{\text{Rep}_n}$ at each stage, it is clear that whenever $x \in \bigoplus_{i \in I} X_i$, we have $\overline{f_\infty}(x) \in \mathcal{V}_{\text{Rep}}(X)$. Moreover, if $\overline{x} \in \mathcal{V}_{\text{Rep}}(X)$, then for each n , we have that $\overline{x}(n) \in \overline{\text{Rep}_n} = \overline{f_n}[\bigoplus_{h \in H} X_h]$. By compactness, and the universal property of limit maps, we conclude that $\overline{x} \in \overline{f_\infty}[\bigoplus_{h \in H} X_h]$.

Since $\bigoplus_{h \in H} X_h$ is an Esakia space, $\mathcal{V}_{\text{Rep}}(X)$ is then a p-morphic image of an Esakia space, and thus, is an Esakia space itself by duality (see Table 2.1). $\blacksquare$

Let $p: \mathcal{V}_{\text{Rep}}(X) \rightarrow X$ be the composition of $\pi_X \pi_0$. We will now show that this is dual to an object in $\mathbf{Et}_{H_0}$, and the free 1-generated algebra in this variety.

Theorem 7.2.22 (Dual description of free étale Heyting algebras). *For each H_0 a Heyting algebra, the Esakia space $(\mathcal{V}_{\text{Rep}}(\text{Spec}(H_0)), p)$ is dual to $H_0\langle\iota\rangle$.*

Proof. We will show the universal property of $H_0\langle\iota\rangle$. Let $X = \text{Spec}(H_0)$. Dually, the universal property takes the following form: let $g: Y \rightarrow X$ be a p-morphism between Esakia spaces which is dual to $(\text{ClopUp}(Y), g^{-1}) \in \mathbf{Et}_{H_0}$. Then for each clopen upset $U \subseteq Y$, there is a unique extension $\bar{g}: Y \rightarrow \mathcal{V}_{\text{Rep}}(X)$ such that $\bar{g}^{-1}[\pi_0^{-1}[X \times \{1\}]] = U$.

For this purpose, first note $\text{Rep}_0 = X \times \mathbf{2}_{\leq}$, because for each $x \in X$, we can find a coordinate h where $h \notin x$. Thus we begin by defining $g_0: Y \rightarrow X \times \mathbf{2}_{\leq}$ by sending y to (y, i) if and only if $y \in U$. Note that this map is π_X -open since U is a clopen upset. Thus by the universal property of one-step extensions of g -Ghilardi complexes (see Theorem 4.4.4), this induces in a canonical way a map along the polynomial Ghilardi complex in a unique way: $g_{k+1}: Y \rightarrow (X \times \mathbf{2}_{\leq})_{k+1}$ sends y to $g_k[\uparrow y]$. All that is necessary is to show that such maps factor through the internal Ghilardi complex.

Assume then that $g_k: Y \rightarrow (X \times \mathbf{2}_{\leq})_k$ has been defined, such that $g_k[Y] \subseteq \overline{\text{Rep}_n}$ and such that $g_n^{-1}[\top_n] = U$. We want to show that $g_{n+1}(y) \in \overline{\text{Rep}_{n+1}}$. Indeed, suppose not. Using Corollary 7.2.20, let φ, ψ be some formula such that $g_{n+1}(y) \in \varphi(V_1, \dots, V_n, U_n)$ but $g_{n+1}(y) \notin \psi(V_1, \dots, V_n, U_n)$. Then we have that $y \in \varphi(g^{-1}[V_1], \dots, g^{-1}[V_n], U)$ and $y \notin \psi(g^{-1}[V_1], \dots, g^{-1}[V_n], U)$, using the fact that g is a p-morphism, and thus its dual is a Heyting algebra homomorphism, commuting with all connectives. But note that by assumption, we have that $(\varphi \rightarrow \psi)(V_1, \dots, V_n, V)$ holds for each $V \subseteq X$ a clopen upset. Hence this is an equation that holds over H_0 , and by assumption, $(\text{ClopUp}(Y), g^{-1})$ is étale – a contradiction. Thus $g_{n+1}(y) \in \overline{\text{Rep}_{n+1}}$, as desired. The fact that $g_{n+1}^{-1}[\top_{n+1}] = U$ follows by the universality of the definition.

By taking the unique extension $\bar{g}: Y \rightarrow \mathcal{V}_G^{\pi_X}(X \times \mathbf{2}_{\leq})$, we thus obtain that this factors through $\mathcal{V}_{\text{Rep}}(X)$, which proves the result. $\blacksquare$

One can see Theorem 7.2.22 as presenting the general recipe for constructing étale Heyting algebras. However, just like in Chapter 5, we can hope to extract more information about these algebras, and develop further intuition about their fundamental properties, by studying some simpler cases. We turn to this in the next section.

7.2.3 Étale Smetanich algebras

We now move to a much simpler, yet non-trivial, case of étale Heyting algebras: Étale Smetanich algebras. For this we exploit the normal form theory developed in Section 5.1.3, and consider a variation of Carai's construction (see Definition 5.1.16).

Definition 7.2.23 (Polynomial Smetanich space). Let X be a **Sme**-space. We denote by:

$$\mathbf{CC}_2^{\text{pol}}(X) = \{C \in \mathbf{CC}_2(X \times \mathbf{2}_{\leq}) : \forall (x, i) \in C, x \leq y \Rightarrow \exists j \geq i, (y, j) \in C\},$$

where we have $C \leq D$ if and only if $C \supseteq D$ and $D = C \cap \uparrow r(D)$. We write simply r for the composite map $\pi_X r$ sending each C , with root (x, i) , to x .

Notation 7.2.4. Throughout when working with $\mathbf{CC}_2^{\text{pol}}(X)$, we write $\top := X \times \{1\}$, and write simply $A, B, \dots$ for $A \times \{0, 1\}$.

Proposition 7.2.24 (Dual representation of Smetanich polynomial algebras). Let H be a Smetanich algebra, and $X = \mathbf{Spec}(H)$. The polynomial Smetanich algebra $H[x]^{\text{Sme}}$ is dual to the polynomial Smetanich space $\mathbf{CC}_2^{\text{pol}}(X)$, and the inclusion map is dual to $r: \mathbf{CC}_2^{\text{pol}}(X) \rightarrow X$.

Proof. It is easy to see, by the way the order is defined, that $\mathbf{CC}_2^{\text{pol}}(X)$ is an upset in $\mathbf{CC}_2(X \times \mathbf{2}_{\leq})$. We show that it is closed in this space. Let $C \notin \mathbf{CC}_2^{\text{pol}}(X)$. By definition, we have that $(x, i) \in C$ and $x \leq y$ but $(y, j) \notin C$ for $j \geq i$. Thus either $i = 0$, and $(y, j) \notin C$ for any j , or $i = 1$ and $(y, 1) \notin C$. We may take $x \neq y$, and so $y \not\leq x$; let U be a clopen upset such that $y \in U$ and $x \notin U$. Note that then $x \in \neg\neg U$, since X is dual to a Smetanich algebra, and since $x < y$, y must be the unique point immediately above x . Thus consider the following neighbourhood:

$$(\langle \neg\neg U \times \{0\} \rangle \cap \llbracket -U \rrbracket) \cup ((\langle \neg\neg U \times \{1\} \rangle) \cap \llbracket \top \Rightarrow -U \rrbracket).$$

By the assumption on C , this is a neighbourhood of C . Now assume that D is any set in this neighbourhood. Let $z = r(D)$. Then either $(z, 0) \in \neg\neg U \times \{0\}$, and $D \subseteq \llbracket -U \rrbracket$, meaning that $z \leq z'$ where $(z', i) \notin D$ for any i ; or $(z, 1) \in D$ where $z \in \neg\neg U$, meaning that $z \leq z'$ for a unique immediate predecessor where $z' \in U$, but similarly $(z', 1) \notin D$. In all cases we see that then $D \notin \mathbf{CC}_2^{\text{pol}}(X)$, which proves that the above is a neighbourhood of C entirely outside of this subset.

We conclude that $\mathbf{CC}_2^{\text{pol}}(X)$ is a closed upset, and thus an Esakia space (Proposition 2.1.52). To show that it is dual to $H[x]^{\text{Sme}}$, we show the dual universal property, i.e., for any Y a **Sme**-space, and $f: Y \rightarrow X$ a p-morphism, there is a unique p-morphism $\bar{f}: Y \rightarrow \mathbf{CC}_2^{\text{pol}}(X)$ such that $r\bar{f} = f$, and such that $\bar{f}^{-1}[\llbracket \top \rrbracket] = U$.

Let Y be any **Sme**-space, $f: Y \rightarrow X$ is a p-morphism, and $U \subseteq Y$ is a clopen upset. Define a map $\bar{f}: Y \rightarrow \mathbf{CC}_2^{\text{pol}}(X)$ by sending y to $\chi_y = \{(f(z), i) : y \leq z, (i = 1 \iff z \in U)\}$. Note that this is a subset with at most two elements, since Y is a **Sme**-space, and such an assignment makes the diagram commute. If $y \in U$, $\bar{f}(y) \subseteq X \times \{1\}$, and conversely, if $\bar{f}(y) \subseteq X \times \{1\}$, then by definition, $y \in U$.

We show that $\bar{f}$ is a continuous map: a subbasis for the space $\mathbf{CC}_2^{\text{pol}}(X)$ is given by the sets $\llbracket W \times \{i\} \rrbracket$ for W clopen and $i \in \{0, 1\}$, and their complements. But then note:

$$\bar{f}^{-1}[\llbracket W \times \{i\} \rrbracket] = \{y \in Y : \chi_y \subseteq W \times \{i\}\}.$$

If $i = 0$, this is precisely the set $\Box W \cap \neg U$; if $i = 1$, this is the set $\Box W \cap U$, where $\Box W$ is calculated in the Esakia topology on Y . In both cases it is clopen, meaning that the preimage is clopen, which immediately entails continuity. The definition easily entails that $\bar{f}$ is order-preserving. Notice moreover that $\bar{f}$ is a p-morphism: if $\bar{f}(y) \leq C$, note that whenever $\bar{f}(y) \neq C$, then $C = \{(f(z), i)\}$ (recalling that $C \subseteq \bar{f}(y)$), where by construction, there is some $y \leq z$, and $i = 1$ if and only if $z \in U$; thus $\bar{f}(z) = C$ must happen.

We now prove that $\bar{f}$ is unique. Note that if $g: Y \rightarrow \mathbf{CC}_2^{\text{pol}}(X)$ is another map, then first note that if y is maximal making the diagrams commute, then if $y \in U$, $g(y) \in \llbracket \top \rrbracket$, so $g(y) = \{(y, 1)\} = \bar{f}(y)$. Similarly if $y \notin U$ and y is maximal, then $g(y) = \{(y, 0)\} = \bar{f}(y)$. Thus on maximal elements, g and $\bar{f}$ agree. This forces non-maximal elements to agree, since both of the maps are p-morphisms, yielding full equality of the maps. ■

It is easy to see that even for a finite Smetanich algebra, $H[x]^{\text{Sme}}$ need not be isomorphic to $H\langle\iota\rangle$, though the latter is, by general arguments, a quotient of the former:

Example 7.2.25 (Polynomial does not coincide with étale free). Consider $H_0 = [3]$; its dual is isomorphic to the poset $\mathbf{2}_{\leq} = \{a < b\}$ (letters added for clarity of notation). Running the construction from Proposition 7.2.24, we have that the dual space is isomorphic to the poset Y of Figure 7.1, with the root map $r: Y \rightarrow \mathbf{2}_{\leq}$, assigning each point to its left label. The preimages corresponding to the elements from H_0 are precisely Y , $\emptyset$, and $V_a = \{(a, 0)_l, (a, 0)_r, (a, 1)\}$. Now, consider the upset $U = \{(b, 0)_r, (a, 1)\}$. Note that over $[3] = (\perp < 1 < \top)$ we have:

$$[3] \models (x \leftrightarrow \top) \vee (x \leftrightarrow \perp) \vee (x \leftrightarrow 1).$$

Thus, in order for $\mathbf{Up}(Y)$ to be the dual of $H_0\langle\iota\rangle$, we must have that:

$$Y = (U \leftrightarrow Y) \cup (U \leftrightarrow \emptyset) \cup (U \leftrightarrow V_a);$$

this simplifies to $Y = U \cup \neg U \cup (V_a \leftrightarrow U_a)$; note that $\neg U = \{(a, 0)_l, (b, 0)_l\}$, and $V_a \leftrightarrow U = \{(a, 1), (b, 1)\}$. Notably, the point $(a, 0)_r$ does not appear in any of them.

On the other hand, it is not difficult to see that $Y \setminus \{(a, 0)_r\}$ does satisfy this principle – and as we will see below, is dual to $H_0\langle\iota\rangle$.

Example 7.2.25 suggests that the problem lies in including points with (01)-“color” which are not witnessed by the poset or Esakia space – in the terminology

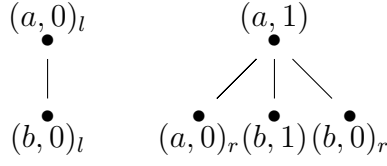


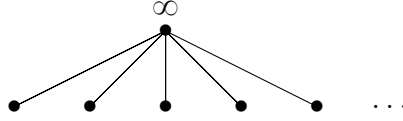
Figure 7.1

of Section 7.2.2, such points are not representable. Thus, this suggests the following definition:

Definition 7.2.26 (Separable upset). Given X a **Sme**-space, we denote by $\text{Sep}_2(X) \subseteq \text{CC}_2^{\text{pol}}(X)$ the space containing all C such that either C is a singleton, or $C = \{(x, 0), (y, 1)\}$ and there exists some $U \subseteq X$ a clopen upset such that $y \in U$ and $x \notin U$. Let $\text{CC}_2^*(X) := \overline{\text{Sep}_2(X)}$.

We give an illustrative example, which will moreover show that taking the closure is necessary.

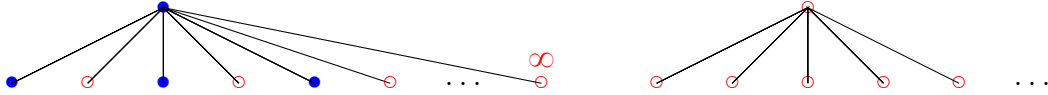
Example 7.2.27 (Separable upset for Smetanich). Consider the space $\alpha\mathbb{N}$ with ∞ as top, and its dual algebra $\mathcal{P}_{\emptyset, \text{cof}}(\mathbb{N})$. The Esakia space $X = \alpha\mathbb{N}$ has the accumulation point ∞ as its top element and $\mathbb{N}$ discretely ordered.



Let Ω be the space given by three copies of $\alpha\mathbb{N}$, $A_i = \{(n, i) : n \in \omega\} \cup \{(\infty, i)\}$, ordered as follows:

1. A_1 and A_2 are ordered by letting $(n, 1) \leq (\infty, 1)$;
2. For each $x \in A$, $(x, 3) \leq (\infty, 2)$.

A depiction of Ω is given in Figure 7.2; blue full points represent the space A_1 . The reader can check that by taking the blue points as labelled with 1, and the red

Figure 7.2: The space Ω .

points labelled with 0, one has that Ω is precisely $\overline{\text{Sep}_2(X)}$ for $(X, \leq)$ the Esakia space from Example 7.2.41.(3). The red point labelled with ∞ corresponds to the chain $\{(\infty, 0), (\infty, 1)\}$, which is not in $\text{Sep}_2(X)$, but which compactifies the set of chains $\{(n, 0), (\infty, 1) : n \in \omega\}$.

With all of this we can give our dual representation of free, one-generated, étale Smetanich algebras:

Proposition 7.2.28 (Closure of separable is Esakia). *If H is a **Sme**-algebra, and $X = \text{Spec}(H)$, the space $\text{CC}_2^*(X)$ is an Esakia space, and the restriction $r: \text{CC}_2^*(X) \rightarrow X$ is surjective.*

Proof. Note that $\text{Sep}_2(X)$ is an upset by definition. By definition, CC_2 is closed, and in Esakia spaces closure of an upset is an upset (see Proposition 2.1.52), and so $\text{CC}_2^*(X)$ is a closed upset. Since closed upsets of Esakia spaces are Esakia by the same Proposition, we obtain that it is an Esakia space. To see the surjectivity of r , note that for each $x \in X$, $\uparrow x \times \{0\}$ will be a chain rooting back at x . ■

Theorem 7.2.29 (Smetanich étale algebras). *Whenever H is a **Sme**-algebra, and $X = \text{Spec}(H)$ we have that $\text{ClopUp}(\text{CC}_2^*(X)) \cong H_0\langle\iota\rangle$.*

Proof. By Proposition 7.2.28 we know that $\text{CC}_2^*(X)$ is an Esakia space, and r is surjective. We verify that r^{-1} witnesses the étale condition. Consider the following three subsets of $\text{CC}_2^*(X)$:

1. $U_1 = \llbracket \top \rrbracket$;
2. $U_2 = \llbracket X \times \{0\} \rrbracket$;
3. $U_3 = \langle \top \rangle \setminus \llbracket \top \rrbracket$.

Note that all these sets are clopen in $\text{CC}_2^*(X)$. They are disjoint, by definition, and they are exhaustive: for each $C \in \text{CC}_2^{\text{pol}}(X)$, either all of its (at most two) elements are of the form $(y, 1)$, or all of the form $(y, 0)$, or C is of depth exactly 2, and its maximal element is of the form $(y, 1)$, and its minimal element of the form $(z, 0)$ for $z \leq y$. Such “types” identify C : if C, D are of the same type, and they have the same root, then they must be the same.

Now assume that $\varphi(\bar{x}, \bar{a})$ is a formula in the language $\mathcal{L}_{H_0}$. Note that up to a substitution, when evaluating this formula in $\text{ClopUp}(\text{CC}_2^*(X))$, we can assume that φ is of the form $\varphi(x, \bar{a})$ for $\bar{a} \in H_0$ ³. By Proposition 5.1.23, giving us the normal forms for Smetanich logic, we can take it to be a disjunction of χ_i , where $W_i = \{\llbracket \top \Rightarrow A_i \rrbracket, \llbracket B_i \Rightarrow \top \rrbracket, \langle \top \rangle, \llbracket D_i \rrbracket\}$ and $\exists T \subseteq W$ such that $\chi_i = \bigcap T$. Now assume that there exists some $x \notin \bigcup_{i=1}^n \chi_i$. Then for each i , there is some Z_i such that $C \notin Z_i$. Note that if $Z_i = \llbracket \top \Rightarrow A_i \rrbracket$ and $Z_j = \llbracket \top \Rightarrow A_j \rrbracket$, we have

$$Z_i \cup Z_j = \llbracket \top \Rightarrow (A_i \cup A_j) \rrbracket,$$

so since we assume that $C \notin Z_i \cup Z_j$, we may assume that there is a single element of this type, i.e., we may assume that there are A, B, D such that $W = \{\llbracket \top \Rightarrow$

³More precisely, if the original formula was valid in H , then the modified formula would be as well by uniform substitution, and the refutation of the latter allows us to conclude the desired result.

A], $\llbracket B \Rightarrow \top \rrbracket$, $\langle \top \rangle$, $\llbracket D \rrbracket$, and some $T \subseteq W$, such that $C \notin \bigcup T$. Note that $\bigcup T$ is clopen, so if C exists in these conditions, by the definition of closure, we may assume $C \in \mathbf{Sep}_2(X)$.

We divide the analysis in three cases:

1. $C \in \llbracket \top \rrbracket$: then T cannot contain $\langle \top \rangle$ or $\llbracket B \Rightarrow \top \rrbracket$. If $C \notin \llbracket \top \Rightarrow A \rrbracket$, so we pick U a clopen upset such that $C \cap (U \setminus A) \neq \emptyset$. Then we have that $H \not\models \varphi(x, \bar{a})[U/x]$, with the element from C serving as a witness.
2. $C \in \llbracket -\top \rrbracket$: then T cannot contain $\llbracket \top \Rightarrow A \rrbracket$. Similarly, let U be a clopen upset such that $C \cap (B \setminus U) \neq \emptyset$, and notice that $H \not\models \varphi(x, \bar{a})[U/x]$.
3. $C \in \langle \top \rangle \setminus \llbracket \top \rrbracket$: in this case, we can assume that $\langle \top \rangle$ does not appear, all other disjuncts are present, since all other cases are simplifications. We may assume that $C = \{(x, 0), (y, 1)\}$, and by assumption of separability, $x \neq y$. Since $C \notin \llbracket \top \Rightarrow A \rrbracket$, we have that $y \notin A$, and since $C \notin \llbracket B \Rightarrow \top \rrbracket$, $x \notin B$, $\{x, y\} \subseteq -D$. By assumption of separability, let U be a clopen upset such that $y \in U$ and $x \notin U$; then $H \not\models \varphi(x, \bar{a})[U/x]$.

This shows that if $H \models \forall \bar{x} \varphi(\bar{x}, \bar{a})$, then $\mathbf{CloUp}(\mathbf{CC}_2^*(X)) \models \forall \bar{x} \varphi(\bar{x}, \iota \bar{a})$, which shows the algebra belongs to the variety.

Let $g: Y \rightarrow X$ be such that $(\mathbf{CloUp}(Y), g^{-1}) \in \mathbf{Et}_{H_0}$, and let $U \subseteq Y$ a clopen upset. By Proposition 7.2.24, there is a unique continuous p-morphism $\bar{g}: Y \rightarrow \mathbf{CC}_2^{\text{pol}}(X)$, such that $\bar{g}^{-1}[\llbracket \top \rrbracket] = U$. The argument showing that $\bar{g}$ factors through $\mathbf{CC}_2^*(X)$ is then exactly the same as that given above to show that $\mathbf{CC}_2^*(X)$ is dual to an algebra in $\mathbf{Et}_{H_0}$. ■

Remark 7.2.30 (Root map properties). Note that in some cases, the root map $r: \mathbf{CC}_2^*(X) \rightarrow X$ can satisfy special properties: if one takes the Example 7.2.27, the root map $r: \Omega \rightarrow \alpha\mathbb{N}$ is in fact a local homeomorphism. However, in general this need not hold: see Section 7.5 for discussion on this.

7.2.4 Étale-finite Heyting algebras

We now move finally to the most special case of étale Heyting algebras, for which we have a vastly more developed theory: étale-finite Heyting algebras.

Definition 7.2.31 (Étale-finite Heyting algebra). Let H_0 be a finite Heyting algebra. A Heyting algebra homomorphism $f: H_0 \rightarrow H$ is said to satisfy *Jibladze's law*⁴ if, for every $x \in H$,

$$\bigvee_{x_0 \in H_0} (x \leftrightarrow f(x_0)) = 1.$$

⁴The terminology “Jibladze’s law” is introduced here. Frame-theoretic analogues of this law seem to have first appeared in print in [164], where they are attributed to Jibladze (March 1990); they have been studied more systematically by Jibladze in [154].

Given an arbitrary Heyting algebra H and a finite Heyting algebra H_0 , we say that H is H_0 -étale if there is a homomorphism $f: H_0 \rightarrow H$ satisfying Jibladze's law. We say that H is *étale-finite* if there is a finite Heyting algebra H_0 such that H is H_0 -étale.

Remark 7.2.32.

1. (Jibladze's law generalises the excluded middle) Jibladze's law can be seen as a generalisation of the law of excluded middle: taking $H_0 = 2$ and $f: 2 \rightarrow H$ to be the unique Heyting homomorphism, the equation defining Jibladze's law becomes

$$(x \leftrightarrow 1) \vee (x \leftrightarrow 0) = 1,$$

i.e.,

$$x \vee \neg x = 1,$$

which is the law of excluded middle. Similar generalisations of the law of excluded middle are already present in the frame-theoretic literature [164] (see also [154]). From this perspective, étale-finiteness may be regarded as a finitary analogue of such relative principles.

2. (Étale-finite $\leftrightarrow$ finite subalgebra) Replacing H_0 by the image $f[H_0]$, we observe that a Heyting algebra H is étale-finite if and only if there is a finite Heyting *subalgebra* H_0 of H such that, for every $x \in H$, in H we have

$$\bigvee_{x_0 \in H_0} (x \leftrightarrow x_0) = 1.$$

In Proposition 7.4.6 we will show that it suffices that H_0 is a finite *subset* of H , not necessarily a *subalgebra*.

The following was shown in [177, pp. 11–12]:

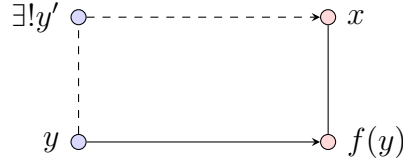
Proposition 7.2.33 (Étale-maps and reducts). For H_0 a finite Heyting algebra and H an arbitrary Heyting algebra, the following are equivalent:

1. H is an H_0 -étale Heyting algebra;
2. H is the Heyting reduct of some algebra in $\mathbf{Et}_{H_0}$.

Remark 7.2.34 (Étale-finite $\Rightarrow$ locally finite). Every étale-finite Heyting algebra is locally finite. Indeed, by Proposition 7.2.33, an étale-finite Heyting algebra H is the Heyting reduct of some algebra in $\mathbf{Et}_{H_0}$ for H_0 a finite Heyting algebra; thus $H \in \mathbf{Var}(H_0)$ as well. Since every finitely generated variety is locally finite, every algebra in $\mathbf{Et}_{H_0}$ is locally finite. Thus, the Heyting algebra H is locally finite.

We will be especially concerned with the duals of homomorphisms $f: H_0 \rightarrow H$ satisfying Jibladze's law. We recall from [177] the following notion of a strict p-morphism:

Definition 7.2.35 (Strict p-morphism). Let $f: Y \rightarrow X$ be an order-preserving map between posets. We say that f is a *strict p-morphism* if, for all $y \in Y$ and $x \in X$ with $f(y) \leq x$, there is a unique $y' \geq y$ such that $f(y') = x$.



The following is an equivalent formulation of the same concept:

Remark 7.2.36 (Strict p-morphism on principal upsets, [177, Lem. 2.10]). An order-preserving map $f: Y \rightarrow X$ is a strict p-morphism if and only if, for each $y \in Y$, f restricts to an order-isomorphism between $\uparrow y$ and $\uparrow f(y)$.

Definition 7.2.37 (Étale-finite Esakia space). We say that an Esakia space X is *étale-finite* if there is a continuous strict p-morphism from X to a finite Esakia space X_0 .

Theorem 7.2.38 (Strictness is dual to Jibladze's law, [177, Cor. 4.7]). Let $\acute{e}: X \rightarrow X_0$ be a continuous p-morphism from an Esakia space X to a finite Esakia space X_0 , and let $f: H_0 \rightarrow H$ be the dual Heyting algebra homomorphism. The p-morphism $\acute{e}$ is strict if and only if f satisfies Jibladze's law.

Corollary 7.2.39 (Étale-finite Esakia spaces $\leftrightarrow$ étale-finite Heyting algebras). An Esakia space is étale-finite if and only if its dual Heyting algebra is étale-finite.

Although we defined étale-finite Esakia spaces, we note that these are the same as “étale-finite Priestley spaces”:

Lemma 7.2.40 (Priestley étale-finite is Esakia). If $\acute{e}: X \rightarrow X_0$ is a continuous strict p-morphism with X a Priestley space and X_0 a finite poset, then X is an Esakia space.

Proof. Since $\acute{e}$ is a continuous strict p-morphism, for each $x_0 \in X_0$ the fibre $S_{x_0} = \acute{e}^{-1}[\{x_0\}]$ is clopen.

Let $x_0 \leq y_0$ in X_0 . By strictness of the p-morphism, for every $x \in S_{x_0}$ there is a unique point $y \in S_{y_0}$ such that $x \leq y$. We denote it by $p_{x_0, y_0}(x)$, thus obtaining a map $p_{x_0, y_0}: S_{x_0} \rightarrow S_{y_0}$. For every $A \subseteq S_{y_0}$, we have

$$p_{x_0, y_0}^{-1}[A] = S_{x_0} \cap \downarrow A. \quad (7.1)$$

Indeed, $x \in S_{x_0}$ is sent into A precisely when the unique point of S_{y_0} above x belongs to A ; equivalently, when $x \leq a$ for some $a \in A$. In particular, if $A \subseteq S_{y_0}$ is closed, then $\downarrow A$ is closed in X (since the downward closure of a closed set is closed in a Priestley space, see e.g. [124, Exercise 3.2.7.b]); hence, the set $p_{x_0, y_0}^{-1}[A] = S_{x_0} \cap \downarrow A$ is closed. Thus, p_{x_0, y_0} is continuous.

Now let $W \subseteq X$ be clopen. Since the fibres S_{y_0} partition X , we have

$$W = \bigcup_{y_0 \in X_0} (W \cap S_{y_0}). \quad (7.2)$$

Using (7.2) and (7.1), we obtain

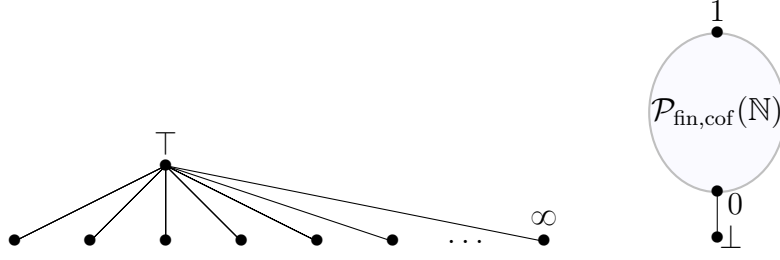
$$\begin{aligned} \downarrow W &= \bigcup_{y_0 \in X_0} \downarrow (W \cap S_{y_0}) = \bigcup_{y_0 \in X_0} \bigcup_{x_0 \leq y_0} (S_{x_0} \cap \downarrow (W \cap S_{y_0})) \\ &= \bigcup_{y_0 \in X_0} \bigcup_{x_0 \leq y_0} p_{x_0, y_0}^{-1}[W \cap S_{y_0}]. \end{aligned}$$

For each y_0 , the set $W \cap S_{y_0}$ is clopen in S_{y_0} . Since p_{x_0, y_0} is continuous, its inverse image is clopen in S_{x_0} , and hence clopen in X , because S_{x_0} is clopen in X . Since X_0 is finite, $\downarrow W$ is a finite union of clopen subsets of X . $\blacksquare$

Example 7.2.41 (Examples of étale-finite Heyting algebras and Esakia spaces).

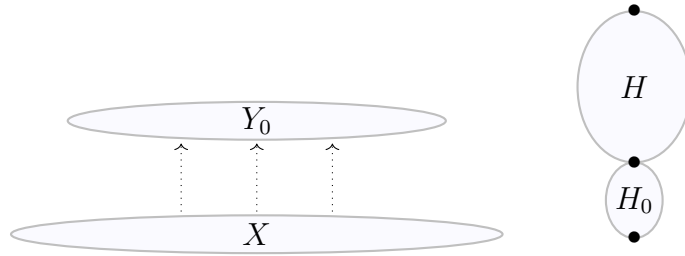
1. (Finite Heyting algebras, finite Esakia spaces) All finite Heyting algebras and, dually, all finite posets are clearly étale-finite.
2. (Boolean algebras, Stone spaces) If X is a Stone space, seen as an Esakia space with the discrete order, then the function $! : X \rightarrow \bullet$ to the one-element space $\bullet$ is a continuous strict p-morphism. Therefore, every Stone space is étale-finite, and every Boolean algebra is étale-finite.
3. (The space $\alpha\mathbb{N}$ with a new isolated top, the algebra $\mathcal{P}_{\text{fin}, \text{cof}}(\mathbb{N})$ with a new bottom) Now consider the Alexandroff compactification $\alpha\mathbb{N} = \mathbb{N} \sqcup \{\infty\}$ of the set of natural numbers. It is well known (see, e.g., [167, Example 1.9]) that the dual of this space is the Boolean algebra $\mathcal{P}_{\text{fin}, \text{cof}}(\mathbb{N})$ of all subsets of $\mathbb{N}$ that are finite or cofinite. Let $(\alpha\mathbb{N})^\top = (\alpha\mathbb{N}) \cup \{\top\}$, where $\{\top\}$ is added as a clopen, and order the space by declaring every point in $\alpha\mathbb{N}$ to be below $\top$ (see Figure 7.3). Then $(\alpha\mathbb{N})^\top$ is an Esakia space; its dual Heyting algebra adds a new bottom element to $\mathcal{P}_{\text{fin}, \text{cof}}(\mathbb{N})$.

Let $\mathbf{2} = 0 < 1$ be the discrete two-element chain. Consider the map $\acute{e} : (\alpha\mathbb{N})^\top \rightarrow \mathbf{2}$ that sends $\top$ to 1 and every other point to 0. Then $\acute{e}$ is a continuous strict p-morphism. Therefore, $(\alpha\mathbb{N})^\top$ is an *étale-finite Esakia space*.

Figure 7.3: The space $(\alpha\mathbb{N})^\top$ and its dual Heyting algebra.

4. More generally, let X be an étale-finite Esakia space, say with $\acute{e}: X \rightarrow X_0$ a continuous strict p-morphism, and let Y_0 be a finite poset. We can consider $X \oplus Y_0$, whose underlying topological space is the disjoint union $X \sqcup Y_0$ of X and the discrete space Y_0 , and whose partial order is defined by letting every point of X be below every point of Y_0 (see e.g., [54, Def. 4.1.12]). Then, $X \oplus Y_0$ is an étale-finite Esakia space. Indeed, the map $\acute{e}^{Y_0}: X \oplus Y_0 \rightarrow X_0 \oplus Y_0$ sending $x \in X$ to $\acute{e}(x)$ and y_0 to y_0 is a continuous strict p-morphism.

Dually, given an étale-finite Heyting algebra H and a finite Heyting algebra H_0 , their *vertical sum* $H_0 \oplus H$ defined by taking the union of the algebras and identifying 1_{H_0} with 0_H (see [54, Def. 4.1.13]), is an étale-finite Heyting algebra; see Figure 7.4.

Figure 7.4: The space $X \oplus Y_0$ and its dual Heyting algebra $H_0 \oplus H$.

5. (Discrete version) Suppose that $\acute{e}: P \rightarrow P_0$ is a strict p-morphism from a poset P to a finite poset P_0 . We show that the inverse-image function $\acute{e}^{-1}[-]: \text{Up}(P_0) \rightarrow \text{Up}(P)$ satisfies Jibladze's law, and hence that $\text{Up}(P)$ is étale-finite.

First, notice that every point $p \in P$ has only finitely many successors, i.e., $\uparrow p$ is finite. For each upset $U \in \text{Up}(P)$, we show that, for each $p \in P$, there is an upset $U_0 \subseteq P_0$ such that

$$p \in (U \leftrightarrow \acute{e}^{-1}[U_0]),$$

where $\leftrightarrow$ is the biimplication in the Heyting algebra $\text{Up}(P)$. Set

$$U_0 := \acute{e}[\uparrow p \cap U];$$

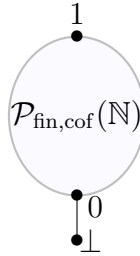
this is an upset because $\uparrow p \cap U$ is an upset and $\acute{e}$ is a p-morphism. Then we show that $p \in (U \leftrightarrow \acute{e}^{-1}[U_0])$. To prove $p \in (U \rightarrow \acute{e}^{-1}[U_0])$, let $p' \geq p$ be such that $p' \in U$; then $p' \in \uparrow p \cap U$, and so $p' \in \acute{e}^{-1}[\acute{e}[\uparrow p \cap U]] = \acute{e}^{-1}[U_0]$. To prove $p \in (\acute{e}^{-1}[U_0] \rightarrow U)$, let $p' \geq p$ be such that $p' \in \acute{e}^{-1}[U_0] = \acute{e}^{-1}[\acute{e}[\uparrow p \cap U]]$; then, there is $p_0 \in \uparrow p \cap U$ such that $\acute{e}(p') = \acute{e}(p_0)$; because $\acute{e}$ is a strict p-morphism, $p' = p_0$ and so $p' \in U$. Since p was arbitrary, we have

$$1_{\text{Up}(P)} = \bigcup_{U_0 \in \text{Up}(P_0)} U \leftrightarrow \acute{e}^{-1}[U_0].$$

This shows Jibladze's law for the map $\acute{e}^{-1}[-]$.

6. (Gödel KM Heyting algebras of bounded depth) It can be shown that all étale-finite Heyting algebras are KM Heyting algebras. Moreover, the KM Gödel algebras of finite depth provide a natural class of examples of étale-finite Heyting algebras. For details, see Appendix A.2.4.

Remark 7.2.42 ($\{\text{étale-finite}\} \not\subseteq \text{HIP}(\{\text{Boolean}\} \cup \{\text{complete}\})$). We show that the étale-finite Heyting algebra H from Example 7.2.41(3)



is not the product of a Boolean algebra and a homomorphic image of a complete Heyting algebra. This will show that our main result (Theorem 7.3.39 below, which produces a topos for each étale-finite Heyting algebra) strictly extends the class of algebras known to be topos-admissible.

Towards a contradiction, suppose that there is an isomorphism $p: B \times C \rightarrow H$, where B is a Boolean algebra, and C is a quotient of a complete Heyting algebra. Since the elements $(1, 0), (0, 1) \in B \times C$ are complements, their images $p(1, 0)$ and $p(0, 1)$ are complements. Since the only complemented elements of H are $\perp$ and 1 , either $p(1, 0) = 1$ and $p(0, 1) = \perp$, or $p(1, 0) = \perp$ and $p(0, 1) = 1$; i.e., H is isomorphic to either B or C .

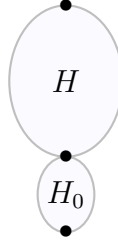
Since H is not Boolean, it is not isomorphic to B .

Then, H must be isomorphic to C . It is known that, if $\pi: L \twoheadrightarrow M$ is a surjective lattice homomorphism between lattices and M is countable, then π has

a section as posets, i.e., there is an order-preserving function $i: M \hookrightarrow L$ such that the composite $M \xrightarrow{i} L \xrightarrow{\pi} M$ is the identity (see the “weaving map” in [232, p. 98], which goes back to [95, Thm. 3, p. 35]); it follows that, if L is complete, then M is complete as well. So, since C is a quotient of a complete Heyting algebra and H is countable, H must be complete. However, H is not complete, a contradiction.

This concludes the proof that H is not a product of a Boolean algebra and a quotient of a complete Heyting algebra.

More generally, the same proof shows that the vertical sum $H_0 \overline{\oplus} H$ of a finite bounded distributive lattice H_0 with at least two elements and a non-complete countable étale-finite algebra H (for example, the free Boolean algebra on countably many generators) is not the product of a Boolean algebra and a quotient of a complete Heyting algebra.



Remark 7.2.43 (Closure properties of étale-finite Heyting algebras / Esakia spaces).

1. It follows from Proposition 7.2.33 that, for a fixed finite Heyting algebra H_0 , H_0 -étale Heyting algebras are closed under arbitrary products and homomorphic images, as well as under subalgebras containing the image of H_0 . Dually, a closed upset of an étale-finite Esakia space is an étale-finite Esakia space, and, for X_0 a fixed finite poset, an arbitrary coproduct (in **Esa**, which is computed as in **Pries**) of X_0 -étale Esakia spaces is X_0 -étale.
2. Given two étale-finite Esakia spaces X and Y , their topological and order-theoretic disjoint union $X \sqcup Y$ is étale-finite. Indeed, if $\acute{e}: X \rightarrow X_0$ and $\acute{e}': Y \rightarrow Y_0$ are continuous strict p-morphisms to finite Esakia spaces, then the sum $\acute{e} \sqcup \acute{e}': X \sqcup Y \rightarrow X_0 \sqcup Y_0$ is a continuous strict p-morphism. Dually, this shows that the finite product of étale-finite Heyting algebras is étale-finite.
3. Given two étale-finite Esakia spaces X and Y , their product $X \times Y$ computed in **Pries** is étale-finite. Indeed, let $\acute{e}: X \rightarrow X_0$ and $\acute{e}': Y \rightarrow Y_0$ be continuous strict p-morphisms to finite Esakia spaces. First, $X \times Y$ is an Esakia space (see Proposition 2.1.69). Second, the map $(\acute{e} \times \acute{e}') : X \times Y \rightarrow X_0 \times Y_0$ is continuous and order-preserving by general facts. Finally, we note that it is a strict p-morphism: if $(\acute{e} \times \acute{e}')(x, y) \leq (x_0, y_0)$ then $\acute{e}(x) \leq x_0$, and so

there is a unique $x' \geq x$ such that $\acute{e}(x') = x_0$, and similarly a unique $y' \geq y$ such that $\acute{e}'(y') = y_0$. Then $(x', y') \geq (x, y)$ is the unique pair such that $(\acute{e} \times \acute{e}')(x', y') = (x_0, y_0)$.

In general, an *infinite* product of étale-finite Heyting algebras need *not* be étale-finite. To see this, we first record the following facts.

Lemma 7.2.44 (Jibladze's law + join-irreducible top $\Rightarrow$ finite). Let H be a Heyting algebra whose top element 1 is finitely join-irreducible. Let S be a finite subset of H such that, for every $x \in H$,

$$\bigvee_{x_0 \in S} (x \leftrightarrow x_0) = 1. \quad (7.3)$$

Then, $S = H$ (and hence H is finite).

Proof. The inclusion $S \subseteq H$ is obvious, so let us prove $H \subseteq S$. Let $x \in H$. Since 1 is finitely join-irreducible, from (7.3) we deduce that there is $x_0 \in S$ such that $x \leftrightarrow x_0 = 1$, i.e., such that $x = x_0$. Then, $x = x_0 \in S$. ■

Corollary 7.2.45 (Étale-finite + join-irreducible top $\Rightarrow$ finite). *Every finitely subdirectly irreducible étale-finite Heyting algebra is finite.*

Proof. This follows immediately from Remark 7.2.32(2) and Lemma 7.2.44. ■

We can now prove the claim made before Lemma 7.2.44.

Remark 7.2.46 (Infinite product of étale-finite Heyting algebras need not be étale-finite). The following provides an example of an *infinite* product of étale-finite Heyting algebras that is not étale-finite. For each $n \in \mathbb{N} \setminus \{0\}$, let $[n] := 1 < \dots < n$ be the n -element linearly ordered Heyting algebra, and let $\prod_{n \in \mathbb{N} \setminus \{0\}} [n]$ be the product of these chains. Towards a contradiction, suppose that H_0 is a finite Heyting algebra and $f: H_0 \rightarrow \prod_{n \in \mathbb{N} \setminus \{0\}} [n]$ a homomorphism satisfying Jibladze's law. Note that, for each $n \in \mathbb{N} \setminus \{0\}$, letting $\pi_n: \prod_{n \in \mathbb{N} \setminus \{0\}} [n] \rightarrow [n]$ denote the projection, for each $x \in [n]$ we have

$$\bigvee_{h \in H_0} (x \leftrightarrow \pi_n f(h)) = 1.$$

For every $n \geq 2$, the chain $[n]$ has a finitely join-irreducible top element and so, by Lemma 7.2.44,

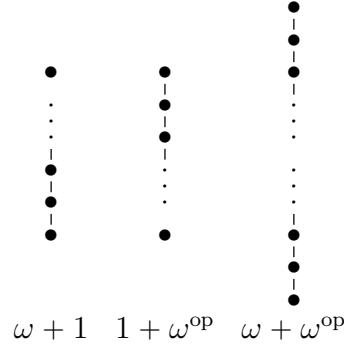
$$n = |[n]| \leq |\pi_n f[H_0]| \leq |H_0|.$$

Since this holds for every $n \geq 2$, we have a contradiction. Thus, $\prod_{n \in \mathbb{N} \setminus \{0\}} [n]$ is not étale-finite.

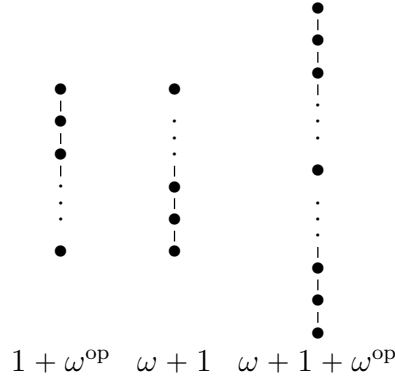
We provide some non-examples of étale-finite Heyting algebras and Esakia spaces.

Example 7.2.47 (Non-examples of étale-finite Heyting algebras and Esakia spaces).

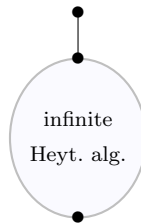
1. (Infinite bounded chains) Every infinite bounded chain (such as $\omega + 1$, $1 + \omega^{\text{op}}$, or $\omega + \omega^{\text{op}}$) is not étale-finite; this follows immediately from Corollary 7.2.45, since the top element of an infinite bounded chain is finitely join-irreducible.



Dually, every infinite linearly ordered Esakia space (such as $1 + \omega^{\text{op}}$, $\omega + 1$, or $\omega + 1 + \omega^{\text{op}}$) is not étale-finite.

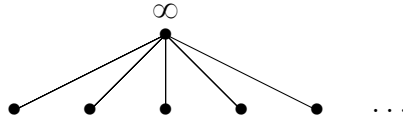


2. (Infinite Heyting algebras with greatest non-top element) Every infinite Heyting algebra with a greatest non-top element is not étale-finite. This follows immediately from Corollary 7.2.45. So, for every infinite Heyting algebra H , the Heyting algebra obtained by adjoining a new top element above the original top element of H is not étale-finite.



It follows that any infinite subdirectly irreducible Heyting algebra is not étale-finite (Proposition 2.1.32).

3. (The space $\alpha\mathbb{N}$ with ∞ as top, and its dual algebra $\mathcal{P}_{\emptyset, \text{cof}}(\mathbb{N})$) The Esakia space $X = \alpha\mathbb{N}$, with the accumulation point ∞ as its top element and $\mathbb{N}$ discretely ordered, is not étale-finite.



Indeed, there is no continuous strict p-morphism to a finite poset, since otherwise $\{\infty\}$ would be the preimage of the set of maximal elements of the finite poset and hence $\{\infty\}$ would be clopen, a contradiction. Dually, the Heyting algebra $\mathcal{P}_{\emptyset, \text{cof}}(\mathbb{N})$ is not étale-finite.

4. (Free Heyting algebras over nonempty sets) The free Heyting algebra over a nonempty set is not étale-finite. To prove this, it suffices to show that the free Heyting algebra $\mathcal{F}_{\mathbf{HA}}(1)$ on one generator is not étale-finite, because, by Remark 7.2.43(1), étale-finite Heyting algebras are closed under surjective homomorphisms. By Remark 7.2.34, if $\mathcal{F}_{\mathbf{HA}}(1)$ were étale-finite, it would be locally finite—a contradiction since it is 1-generated, yet infinite.

For the interested reader, in Appendix A.2.3 we characterise étale-finite Esakia spaces as those Esakia spaces where principal upsets are finite and locally constant; see Proposition A.2.9.

7.3 A topos for étale-finite Heyting algebras

In this section we develop the theory of étale-finite Esakia spaces with a view towards showing that every étale-finite Heyting algebra is topos-admissible. For the reader less familiar with topos theory, we recall the basic definition here:

Definition 7.3.1 (Power object). Let $\mathbf{C}$ be a category with finite limits, and $A \in \mathbf{C}$. The *power object* of A is a triple $(P(A), \in_A, i)$ such that $P(A)$ and $\in_A$ are objects, $i: \in_A \hookrightarrow A \times P(A)$ is a monomorphism, and:

- Whenever B is an object, and $f: C \hookrightarrow A \times B$ is a monomorphism, there is a unique morphism $\tilde{f}: B \rightarrow P(A)$ such that there is a morphism $C \rightarrow \in_A$ making the square in Figure 7.5 a pullback.

Definition 7.3.2 ([157, Corollary A.2.3.4]). An *elementary topos* is a category that has all finite limits and power objects.

Remark 7.3.3. An alternative, equivalent definition of an elementary topos is that it is a finitely complete, Cartesian closed category with a subobject classifier.

$$\begin{array}{ccc}
C & \longrightarrow & \in_A \\
f \downarrow & & \downarrow i \\
A \times B & \xrightarrow{id_A \times f} & A \times P(A)
\end{array}$$

Figure 7.5: Power Object Base Diagram

7.3.1 Spectral local homeomorphisms

Definition 7.3.4 (Spectral topology). The *spectral topology* on a Priestley space is the topology of all open upsets.

The clopen upsets of a Priestley space form a basis for its spectral topology [88, Lem. 2.1]. We stress that by default, continuity refers to continuity in the Priestley topology.

Recall that a function $f: Y \rightarrow X$ between topological spaces is said to be a *local homeomorphism* if, for each $y \in Y$, there is an open neighbourhood A of y such that $f[A]$ is open, and the restriction $f|_A: A \rightarrow f[A]$ of f is a homeomorphism. It is well known that this is equivalent to being a continuous open map that is *locally injective*, i.e., such that for each $y \in Y$ there is an open neighbourhood A of y on which f is injective.

Definition 7.3.5 (Spectral local homeomorphism). A *spectral local homeomorphism* $f: Y \rightarrow X$ between Priestley spaces is a local homeomorphism with respect to the spectral topologies.

In Theorem 7.3.10 below, we will characterise spectral local homeomorphisms purely in terms of the Priestley orders and topologies.

For convenience, we recall the following well-known fact.

Lemma 7.3.6 (Spectral continuity $\Rightarrow$ order-preservation). A function between Priestley spaces that is continuous with respect to the spectral topologies (in particular, any spectral local homeomorphism) is order-preserving.

Proof. Let $f: Y \rightarrow X$ be one such map. Let $y, y' \in Y$ be such that $f(y) \not\leq f(y')$, and let us prove $y \not\leq y'$. By the Priestley separation axiom, there is a clopen upset U such that $f(y) \in U$ and $f(y') \notin U$. Since f is continuous with respect to the spectral topology, $f^{-1}[U]$ is an open upset. Combining the facts that $f^{-1}[U]$ is an upset and that $y \in f^{-1}[U]$ and $y' \notin f^{-1}[U]$, we obtain $y \not\leq y'$. ■

It is likewise known that the clopen upsets in a Priestley space are exactly the compact opens in its spectral topology [88, Lem. 2.1]. Moreover, a function between Priestley spaces is continuous and order-preserving (i.e., a morphism in **Pries**) if and only if it is a spectral map (i.e., a continuous map $f: X \rightarrow Y$ with respect to the spectral topologies such that, for every spectral compact open U of

Y , $f^{-1}[U]$ is spectral compact open). As an immediate consequence, we obtain the following characterisation of homeomorphisms.

Lemma 7.3.7 (Spectral homeomorphisms $\Leftrightarrow$ Priestley order-homeomorphisms). The following are equivalent for a function $f: Y \rightarrow X$ between Priestley spaces.

1. f is an order-homeomorphism (with respect to the Priestley orders and topologies);
2. f is a homeomorphism with respect to the spectral topologies.

Recall from Proposition 2.1.69 that a closed set of a Priestley space is again a Priestley space in the induced order and topology.

Lemma 7.3.8 (Spectral topology on closed subspaces). Let C be a closed subspace of a Priestley space X . The spectral topology of C coincides with the subspace topology obtained by restricting to C the spectral topology of X .

Proof. Clearly, the restriction to C of a spectral open of X is a spectral open of C . Let us show the reverse inclusion. Let U be a clopen upset of C . For each $x \in U$ and $y \in C \setminus U$, we have $x \not\leq y$ and, by the Priestley separation property in X , there is a clopen upset U_{xy} of X such that $x \in U_{xy}$ and $y \notin U_{xy}$. Then, for each $y \in C \setminus U$, we have $U \subseteq \bigcup_{x \in U} U_{xy}$; by compactness of U , there is a finite subcover $U \subseteq \bigcup_{i=1}^{n_y} U_{x_i y}$. We set $V_y := \bigcup_{i=1}^{n_y} U_{x_i y}$; then $C \setminus U \subseteq \bigcup_{y \in C \setminus U} (C \setminus V_y)$. Since each $C \setminus V_y$ is open in C , by compactness there is a finite cover $C \setminus U \subseteq \bigcup_{j=1}^m (C \setminus V_{y_j})$. Let $W := \bigcap_{j=1}^m V_{y_j}$. Then W is a clopen upset of X such that $W \cap C = U$. ■

Lemma 7.3.9 (Local injectivity near a closed set). Let $f: Y \rightarrow X$ be a local homeomorphism, where Y is compact Hausdorff and X is Hausdorff. Let $K \subseteq Y$ be closed. If $f|_K$ is injective, then there is an open neighbourhood $U \supseteq K$ such that $f|_U$ is injective.

Proof. Set

$$R := \{(y_1, y_2) \in Y \times Y : f(y_1) = f(y_2)\} = (f \times f)^{-1}(\Delta_X).$$

Since X is Hausdorff, Δ_X is closed, and hence R is closed in $Y \times Y$. Let

$$E := R \setminus \Delta_Y = \{(y_1, y_2) \in Y \times Y : f(y_1) = f(y_2), y_1 \neq y_2\}.$$

Because f is a local homeomorphism, it is locally injective. Hence Δ_Y is open in R : for every $y \in Y$ there is an open neighbourhood W_y of y such that

$$R \cap (W_y \times W_y) = \Delta_Y \cap (W_y \times W_y).$$

Therefore, $E = R \setminus \Delta_Y$ is closed in R , and hence closed in $Y \times Y$.

Since $f|_K$ is injective, we have

$$E \cap (K \times K) = \emptyset.$$

By Wallace's theorem, also known as the "Tube Lemma" [205, Lem. 26.8] (see also [Stacks, Tag 005N]), we have open neighbourhoods U_1 and U_2 of K such that $U_1 \times U_2 \subseteq (Y \times Y) \setminus E$; thus, by taking $U := U_1 \cap U_2$, we obtain an open neighbourhood U of K such that $(U \times U) \cap E = \emptyset$.

Now if $u, v \in U$ and $f(u) = f(v)$, then $(u, v) \in R$. Since $(u, v) \notin E$, it follows that $u = v$. Thus $f|_U$ is injective. ■

The following characterises spectral local homeomorphisms purely in terms of Priestley topology and order.

Theorem 7.3.10 (Characterisations of spectral local homeomorphisms). *Let $f: Y \rightarrow X$ be a function between Priestley spaces. The following are equivalent:*

1. f is a spectral local homeomorphism;
2. for every $y \in Y$, there is a clopen upset $V \subseteq Y$ with $y \in V$ such that $f[V]$ is a clopen upset of X and

$$f|_V: V \longrightarrow f[V]$$

is an order-homeomorphism (with respect to the Priestley order and topology);

3. there are $n \in \mathbb{N}$ and clopen upsets $U_1, \dots, U_n$ of Y such that $Y = U_1 \cup \dots \cup U_n$ and, for each $i \in \{1, \dots, n\}$, $f[U_i]$ is a clopen upset and $f|_{U_i}: U_i \rightarrow f[U_i]$ is an order-homeomorphism;
4. f is a strict p -morphism with respect to the Priestley orders and a local homeomorphism with respect to the Priestley topologies.

Proof. (1) $\Rightarrow$ (2) Let $y \in Y$. By (1), there is an open upset $A \subseteq Y$ with $y \in A$ such that $f[A]$ is an open upset in X and $f|_A: A \rightarrow f[A]$ is a homeomorphism with respect to the spectral topologies. Since clopen upsets form a basis for the spectral topology, there is a clopen upset V with $y \in V \subseteq A$. Then, V is compact and open in the spectral topology on Y , and so $f[V]$ is compact by continuity of f with respect to the spectral topologies; since V is open in the subspace A , the set $f[V]$ is open in $f[A]$, and hence spectral open in X . Thus $f[V]$ is compact and open in the spectral topology of X , and hence a clopen upset. Moreover, the restriction $f|_V: V \rightarrow f[V]$ is a homeomorphism with respect to the spectral topologies. Now, since V and $f[V]$ are clopen, they are closed Priestley subspaces, and so (2) follows easily from Lemma 7.3.7.

(2) $\Rightarrow$ (3) By (2), for each $y \in Y$ there is a clopen upset U_y containing y such that $f[U_y]$ is a clopen upset and f restricts to an order-homeomorphism on U_y . Then, $Y = \bigcup_{y \in Y} U_y$. By compactness, we get a finite cover, which shows (3).

(3) $\Rightarrow$ (4) The fact that f is a local homeomorphism with respect to the Priestley topologies is immediate. Let us prove that f is a strict p-morphism. If $y \in U_i$ and $f(y) \leq x$, then $x \in f[U_i]$ because $f[U_i]$ is an upset; the unique preimage of x under $f|_{U_i}$ is the candidate $y' \geq y$; the uniqueness follows from the fact that U_i is an upset, and so every $z \geq y$ is still in U_i .

(4) $\Rightarrow$ (1) Fix $y \in Y$. Since Y is a Priestley space, the principal upset $\uparrow y$ is closed in Y [222, Prop. 2.6]. Since f is a strict p-morphism, by Remark 7.2.36, the restriction

$$f|_{\uparrow y}: \uparrow y \longrightarrow \uparrow f(y)$$

is an order-isomorphism; in particular, it is injective. By Lemma 7.3.9, there is an open neighbourhood U of $\uparrow y$ such that $f|_U$ is injective.

Using a standard fact for Priestley spaces [94, Lem. 11.21], choose an open upward-closed neighbourhood V of y such that

$$\uparrow y \subseteq V \subseteq U.$$

Then $f|_V$ is still injective. Since f is a local homeomorphism for the Priestley topologies, it is an open map. Hence, $f[V]$ is open in X , and

$$f|_V: V \longrightarrow f[V]$$

is a homeomorphism.

Since V is upwards closed and f is a p-morphism, the set $f[V]$ is an upset (see Remark 2.1.54.(1)). We show that f moreover reflects the order. Let $v, w \in V$, and assume that $f(v) \leq f(w)$. Since $f(w) \in \uparrow f(v)$ and

$$f|_{\uparrow v}: \uparrow v \longrightarrow \uparrow f(v)$$

is surjective, there is $u \in \uparrow v$ such that $f(u) = f(w)$. As V is upward closed and $v \in V$, we get $u \in V$. Since $f|_V$ is injective, it follows that $u = w$, and hence $v \leq w$.

Thus, for all $v, w \in V$,

$$v \leq w \iff f(v) \leq f(w),$$

and so $f|_V$ is an order-homeomorphism onto the open upset $f[V]$. Since $y \in Y$ was arbitrary, f is a spectral local homeomorphism. $\blacksquare$

Corollary 7.3.11 (Uniform finite fibres over principal upsets). *Let $f: Y \rightarrow X$ be a spectral local homeomorphism between Priestley spaces. If X is an étale-finite Esakia space, then there is $m \in \mathbb{N}$ such that, for each $x \in X$, $f^{-1}[\uparrow x]$ is a finite set of cardinality at most m .*

Proof. Choose a continuous strict p-morphism $\acute{e}: X \rightarrow X_0$ with X_0 finite. By the implication (1) $\Rightarrow$ (3) in Theorem 7.3.10, there are $n \in \mathbb{N}$ and clopen upsets $U_1, \dots, U_n$ of Y such that $Y = U_1 \cup \dots \cup U_n$ and, for each $i \in \{1, \dots, n\}$, $f[U_i]$ is a clopen upset and $f|_{U_i}: U_i \rightarrow f[U_i]$ is an order-homeomorphism. Then, for each $x \in X$, the set $f^{-1}[\uparrow x]$ has at most $n|X_0|$ elements. ■

The following corollary shows that the existence of a spectral local homeomorphism to an Esakia space forces the domain to be Esakia.

Proposition 7.3.12 (Étale over Esakia is Esakia). Let $f: Y \rightarrow X$ be a spectral local homeomorphism, where Y is a Priestley space and X is an Esakia space. Then Y is also an Esakia space.

Proof. By Theorem 7.3.10(3), there are $n \in \mathbb{N}$ and clopen upsets $U_1, \dots, U_n$ of Y such that $Y = U_1 \cup \dots \cup U_n$ and, for each $i \in \{1, \dots, n\}$, $f[U_i]$ is a clopen upset and $f|_{U_i}: U_i \rightarrow f[U_i]$ is an order-homeomorphism. Since a closed upset of an Esakia space is an Esakia space, $f[U_i]$ is an Esakia space; therefore, U_i is also an Esakia space. Let $U_1 \sqcup \dots \sqcup U_n$ be the coproduct of the U_i , whose elements are of the form (x, i) for $x \in U_i$. We thus have a natural map

$$q: U_1 \sqcup \dots \sqcup U_n \twoheadrightarrow Y,$$

which sends (x, i) to x . For each clopen $W \subseteq Y$, we have $q^{-1}[W] = \{(x, i) : x \in W\} = \bigsqcup_{i=1}^n (W \cap U_i)$, which is clopen; thus q is continuous. Moreover, q is obviously surjective and a p-morphism. Thus Y is an Esakia space. ■

To conclude this section, recall that the category $\mathbf{v}$ of topological spaces and local homeomorphisms has an interesting property with respect to the ambient category **Top** of topological spaces and continuous functions (see e.g. [72, Lem. 6.4.5]):

Definition 7.3.13 (Triangle property). Let **C** be a category. We say that a (not necessarily full) subcategory **L** of **C** has the *triangle property* if, for all **C**-morphisms $Z \xrightarrow{f} Y \xrightarrow{g} X$ between objects of **L**, if g and gf are **L**-morphisms then f is an **L**-morphism.

$$\begin{array}{ccc} Z & \xrightarrow{\quad f \quad} & Y \\ & \searrow gf \quad \swarrow g & \\ & X & \end{array}$$

By the implication (1) $\Rightarrow$ (4) in Theorem 7.3.10, every spectral local homeomorphism between Priestley spaces is continuous and order-preserving; hence, we may regard the category of Priestley spaces and spectral local homeomorphisms between them as a subcategory of **Pries**. We now have:

Proposition 7.3.14 (Triangle property for spectral local homeomorphisms). The category of Priestley spaces and spectral local homeomorphisms has the triangle property in **Pries**.

Proof. Let $f: Z \rightarrow Y$ and $g: Y \rightarrow X$ be continuous order-preserving maps between Priestley spaces, and suppose that g and gf are spectral local homeomorphisms. The function f is continuous in the spectral topology because it is continuous and order-preserving. Since local homeomorphisms satisfy the triangle property in the category of topological spaces and continuous maps (see e.g. [72, Lem. 6.4.5]), f is a spectral local homeomorphism. ■

7.3.2 Esakia étale spaces

Let $\mathbf{Esa}_{\text{SLH}}$ be the category of Esakia spaces with spectral local homeomorphisms between them.

Definition 7.3.15 (Esakia étale space). Let X be an Esakia space. We denote by $\mathbf{Esa}\acute{\text{Et}}(X)$ the slice category $\mathbf{Esa}_{\text{SLH}}/X$, which we refer to as the category of *Esakia étale spaces over X* .

Concretely,

1. An object of $\mathbf{Esa}\acute{\text{Et}}(X)$ is a spectral local homeomorphism $Y \rightarrow X$, where Y is an Esakia space,
2. A morphism from $Y_1 \rightarrow X$ to $Y_2 \rightarrow X$ is a spectral local homeomorphism $Y_1 \rightarrow Y_2$ such that the following diagram commutes.

$$\begin{array}{ccc} Y_1 & \xrightarrow{\quad} & Y_2 \\ & \searrow & \swarrow \\ & X & \end{array}$$

Our goal is to prove that, for every étale-finite Esakia space X ,

$$\mathbf{Esa}\acute{\text{Et}}(X)$$

is a topos whose lattice of truth values is isomorphic to $\mathbf{ClopUp}(X)$.

First, we will prove that, for every Esakia space X , the category $\mathbf{Esa}\acute{\text{Et}}(X)$ is finitely complete; then, we will show that, if X is *étale-finite*, then $\mathbf{Esa}\acute{\text{Et}}(X)$ is a topos. In the next section, we will also prove the converse: if $\mathbf{Esa}\acute{\text{Et}}(X)$ is a topos, then the Esakia space X is étale-finite (Theorem 7.4.7).

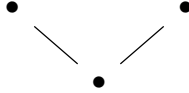
Example 7.3.16 (The Boolean/Freyd case). Let X be a Stone space. Then $\mathbf{Esa}\acute{\text{Et}}(X)$ is isomorphic to the slice category $\mathbf{Stone}_{\text{LH}}/X$, where $\mathbf{Stone}_{\text{LH}}$ is the category of Stone spaces and local homeomorphisms between them. Indeed, given

an object of $\mathbf{Esa}\acute{\mathbf{E}}\mathbf{t}(X)$, i.e., a spectral local homeomorphism $f: Y \rightarrow X$ with Y an Esakia space, if the order on X is discrete, then so must be Y (because any spectral local homeomorphism is a strict p-morphism). It was shown by Freyd (see [159, Exercise 9.11]) that this category is a topos.

7.3.3 Finite limits

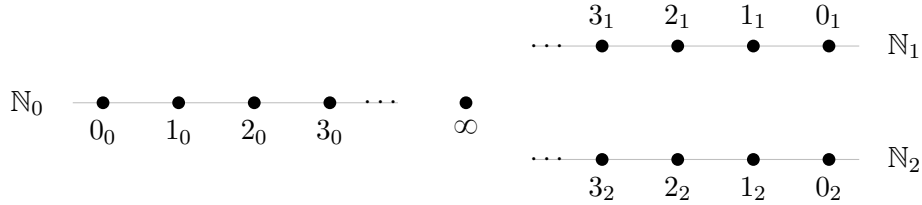
We note that, while calculations of limits will turn out to be easy in slice categories, the same need not be true in the ambient category $\mathbf{Esa}_{\text{SLH}}$:

Example 7.3.17 (Lack of finite limits in the ambient category). $\mathbf{Esa}_{\text{SLH}}$ is easily seen not to have a terminal object. Indeed, suppose there is a terminal object T , and let us derive a contradiction. Then, there is a unique morphism $!_V: V \rightarrow T$, where V is the V-shaped three-element poset below.



Since a strict p-morphism is injective on principal upsets, $!_V$ is injective. Let $\sigma: V \rightarrow V$ be the function that swaps the two maximal elements of V . Then, $!_V \sigma$ and $!_V$ are two different morphisms from V to T , a contradiction.

We next show that, unlike $\mathbf{Top}_{\text{LH}}$,⁵ $\mathbf{Esa}_{\text{SLH}}$ does not have equalisers, either. To see this, let X be the following Stone space:



It consists of three copies of the discrete space $\mathbb{N}$, which we call $\mathbb{N}_i := \{n_i : n \in \mathbb{N}\}$ for $i \in \{0, 1, 2\}$, together with a limit point ∞ . That is, X is the Alexandroff compactification of the discrete set $\mathbb{N}_0 \sqcup \mathbb{N}_1 \sqcup \mathbb{N}_2$. The Stone space X is an Esakia space when considered with equality as a partial order. The identity id_X is moreover easily seen to be a spectral local homeomorphism, and the map $f: X \rightarrow X$ that permutes the two copies $\mathbb{N}_1$ and $\mathbb{N}_2$ (by mapping n_1 to n_2 and vice versa) is likewise a spectral local homeomorphism. Assume that $\mathbf{Esa}_{\text{SLH}}$ had an equaliser $i: \mathbf{Eq}(f, \text{id}_X) \rightarrow X$ of f and id_X , and let us reach a contradiction. We denote the image of i by U ; the set U is contained in $\mathbb{N}_0 \cup \{\infty\}$ since the equaliser equalises. Moreover, for every $n \in \mathbb{N}$, the inclusion $\{n_0\} \hookrightarrow X$ is a

⁵The equaliser in $\mathbf{Top}_{\text{LH}}$ is well known to be the interior of the usual equaliser (see [157, Example A.1.2.7]).

morphism, since it is an injective spectral local homeomorphism that equalises f and id_X , and so it factors through i . Hence, $n_0 \in U$ and therefore $\mathbb{N}_0 \subseteq U$. Thus, $\mathbb{N}_0 \subseteq U \subseteq \mathbb{N}_0 \cup \{\infty\}$. Note that U is a clopen subset of X (because the map i is closed and, by the implication (1) $\Rightarrow$ (4) in Theorem 7.3.10, open). However, no subset S of X satisfying $\mathbb{N}_0 \subseteq S \subseteq \mathbb{N}_0 \cup \{\infty\}$ is clopen.

We now study how limits behave in slice categories. For this, we use the following category-theoretic fact:

Theorem 7.3.18 (Finite limits in the slice under the triangle property). *Let $\mathbf{C}$ be a category and $\mathbf{L}$ a (not necessarily full) subcategory with the triangle property. Suppose that $\mathbf{C}$ has pullbacks and that the pullback cone in $\mathbf{C}$ of a cospan belonging to $\mathbf{L}$ belongs to $\mathbf{L}$. Then,*

1. $\mathbf{L}$ has pullbacks, which are computed as in $\mathbf{C}$; that is, the inclusion $\mathbf{L} \hookrightarrow \mathbf{C}$ creates pullbacks⁶.
2. For every $X \in \mathbf{L}$, the category $\mathbf{L}/X$ is finitely complete:

- (a) A pullback of two objects of $\mathbf{L}/X$ is computed as the pullback in $\mathbf{C}$, i.e., the composite

$$\mathbf{L}/X \longrightarrow \mathbf{L} \hookrightarrow \mathbf{C}$$

creates pullbacks.

- (b) A terminal object of $\mathbf{L}/X$ is the identity $\text{id}_X: X \rightarrow X$.
- (c) A product of two objects of $\mathbf{L}/X$ is computed as the pullback in $\mathbf{C}$.
- (d) Whenever $a: Y \rightarrow X$ and $b: Z \rightarrow X$ are two objects in $\mathbf{L}/X$, and $f, g: Y \rightarrow Z$ two parallel morphisms between them, the equaliser of f and g in $\mathbf{C}$ exists and is also the equaliser in $\mathbf{L}/X$.

Proof. (1) Let $f: Y \rightarrow X$ and $g: Z \rightarrow X$ be morphisms in $\mathbf{L}$ and let

$$\begin{array}{ccc} Y \times_X Z & \xrightarrow{\pi_Z} & Z \\ \pi_Y \downarrow & \lrcorner & \downarrow g \\ Y & \xrightarrow{f} & X \end{array}$$

be the pullback diagram in $\mathbf{C}$. By assumption, $Y \times_X Z$ is an object of $\mathbf{L}$, and π_Y and π_Z are $\mathbf{L}$ -morphisms. In order to conclude that the diagram is also a pullback

⁶A functor $F: \mathbf{C} \rightarrow \mathbf{D}$ is said to *create limits of shape $\mathbf{J}$* if, whenever $D: \mathbf{J} \rightarrow \mathbf{C}$ is a diagram and $\{\beta_j: M \rightarrow FD(j) : j \in \mathbf{J}\}$ is a limit cone for FD , there is a cone $\{\alpha_j: L \rightarrow D(j) : j \in \mathbf{J}\}$ over D whose image under F is isomorphic to $\{\beta_j: M \rightarrow FD(j) : j \in \mathbf{J}\}$, and any such cone is a limit for D . It is well known that if $F: \mathbf{C} \rightarrow \mathbf{D}$ creates limits of shape $\mathbf{J}$ and $\mathbf{D}$ has limits of shape $\mathbf{J}$, then F preserves and reflects limits of shape $\mathbf{J}$.

in $\mathbf{L}$, we need to show that, given another cone in $\mathbf{L}$, the induced morphism also belongs to $\mathbf{L}$; this clearly follows from the triangle property.

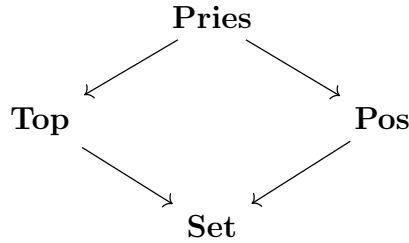
(2) Pullbacks in $\mathbf{L}/X$ are computed as in $\mathbf{L}$, and the identity $\text{id}_X: X \rightarrow X$ is a terminal object of $\mathbf{L}/X$. Since $\mathbf{L}/X$ has pullbacks and a terminal object, it is finitely complete. Moreover, it is easy to check that products in $\mathbf{L}/X$ are the same thing as pullbacks in $\mathbf{L}$. Finally, we prove the assertion about equalisers. The composite functor

$$\mathbf{L}/X \longrightarrow \mathbf{L} \hookrightarrow \mathbf{C}$$

preserves pullbacks. By [157, Lem. A.1.2.9] (see also [212, p. 742]), any pullback-preserving functor whose domain is finitely complete preserves equalisers. Since $\mathbf{L}/X$ is finitely complete, the composite functor above preserves equalisers; thus, the equaliser in $\mathbf{L}/X$, which exists because $\mathbf{L}/X$ is finitely complete, is also the equaliser in $\mathbf{C}$. ■

From this, we can deduce the following description of limits in $\mathbf{EsaÉt}(X)$; similar facts were proved, with the base category being $\mathbf{Esa}$, in [177, Cor. 4.14].

Remark 7.3.19 (Limits in $\mathbf{Pries}$). Recall that the category $\mathbf{Pries}$ has limits, which are computed as in the category $\mathbf{Set}$ of sets and functions (see e.g. [124, p. 69]), as well as in the categories $\mathbf{Top}$ of topological spaces and continuous functions, and $\mathbf{Pos}$ of posets and order-preserving maps. In fact, the forgetful functors in the following commutative diagram preserve limits.



Lemma 7.3.20 (Pullbacks of Esakia spaces and spectral local homeomorphisms). The pullback cone in $\mathbf{Pries}$ of a cospan in $\mathbf{Esa}_{\text{SLH}}$ is in $\mathbf{Esa}_{\text{SLH}}$. That is, given a pullback in $\mathbf{Pries}$

$$\begin{array}{ccc}
 Y \times_X Z & \xrightarrow{\pi_Z} & Z \\
 \pi_Y \downarrow & \lrcorner & \downarrow g \\
 Y & \xrightarrow{f} & X
 \end{array}$$

with X , Y and Z Esakia spaces and $f: Y \rightarrow X$ and $g: Z \rightarrow X$ spectral local homeomorphisms, we have that $Y \times_X Z$ is an Esakia space and π_Y and π_Z are spectral local homeomorphisms.

Proof. The pullback is also a pullback in $\mathbf{Top}$. Note that π_Y and π_Z are local homeomorphisms because local homeomorphisms are stable under pullback in $\mathbf{Top}$ (see [157, Example A.1.2.7]).

We now prove that π_Y is a strict p-morphism. Assume that $\pi_Y(y, z) \leq y'$; then $y \leq y'$, and so $f(y) \leq f(y')$, and so $g(z) \leq f(y')$; by the strict p-morphism condition there is a unique $z' \geq z$ such that $g(z') = f(y')$. Hence, $(y, z) \leq (y', z')$, and this element is unique as well. So, π_Y is a strict p-morphism. Similarly, π_Z is a strict p-morphism. Hence π_Y and π_Z are spectral local homeomorphisms by the implication (4) $\Rightarrow$ (1) in Theorem 7.3.10.

Since π_Y is a spectral local homeomorphism, and Y is an Esakia space, we can therefore conclude that $Y \times_X Z$ is an Esakia space by Proposition 7.3.12. ■

Theorem 7.3.21 (Finite limits in $\mathbf{Esa\acute{E}t}(X)$). *For every Esakia space X , the category $\mathbf{Esa\acute{E}t}(X)$ is finitely complete:*

1. (Terminal object) *The identity $\text{id}_X: X \rightarrow X$ is the terminal object in $\mathbf{Esa\acute{E}t}(X)$.*
2. (Binary product) *If $f: Y \rightarrow X$ and $g: Z \rightarrow X$ are two objects in $\mathbf{Esa\acute{E}t}(X)$, then the pullback $Y \times_X Z$ as ordered spaces with the two projection maps is the product in $\mathbf{Esa\acute{E}t}(X)$.*
3. (Equaliser) *If $f: Y \rightarrow X$ and $g: Z \rightarrow X$ are two objects of $\mathbf{Esa\acute{E}t}(X)$, and $p_1, p_2: Y \rightarrow Z$ are two spectral local homeomorphisms making the two triangles in the diagram commute,*

$$\begin{array}{ccc} Y & \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} & Z \\ & \searrow f \quad \swarrow g & \\ & X & \end{array}$$

then

$$\text{Eq}(p_1, p_2) = \{y \in Y : p_1(y) = p_2(y)\},$$

together with the inclusion into Y , is the equaliser of p_1 and p_2 in $\mathbf{Esa\acute{E}t}(X)$.

4. (Pullbacks) *The forgetful functor $\mathbf{Esa\acute{E}t}(X) \rightarrow \mathbf{Pries}$ (i.e., the domain functor) creates pullbacks.*

Proof. First, it clearly follows from Proposition 7.3.14 that the category $\mathbf{Esa}_{\text{SLH}}$ satisfies the triangle property in $\mathbf{Pries}$. Moreover, $\mathbf{Pries}$ has limits (see Remark 7.3.19). By Remark 7.3.20, the pullback cone in $\mathbf{Pries}$ of a cospan in $\mathbf{Esa}_{\text{SLH}}$ is in $\mathbf{Esa}_{\text{SLH}}$. The result now follows from Theorem 7.3.18. ■

Corollary 7.3.22 (Mono $\Leftrightarrow$ injective). *Let X be an Esakia space. A morphism in $\mathbf{Esa\acute{E}t}(X)$ is a monomorphism if and only if it is injective.*

Proof. By Theorem 7.3.21(4), the forgetful functor $\mathbf{Esa\acute{E}t}(X) \rightarrow \mathbf{Pries}$ preserves pullbacks. Moreover, the forgetful functor $\mathbf{Pries} \rightarrow \mathbf{Set}$ preserves limits and hence the composite $\mathbf{Esa\acute{E}t}(X) \rightarrow \mathbf{Pries} \rightarrow \mathbf{Set}$ preserves pullbacks. Therefore, the forgetful functor $\mathbf{Esa\acute{E}t}(X) \rightarrow \mathbf{Set}$ preserves monomorphisms, and it also reflects them because it is faithful. ■

Corollary 7.3.23 (Subobject = clopen upset). *Let X be an Esakia space and $f: Y \rightarrow X$ an object of $\mathbf{Esa\acute{E}t}(X)$. The posets $\mathbf{ClopUp}(Y)$ and $\mathbf{Sub}_{\mathbf{Esa\acute{E}t}(X)}(f)$ are order-isomorphic, via the map sending $V \in \mathbf{ClopUp}(Y)$ to the subobject of f represented by*

$$\begin{array}{ccc} V & \hookrightarrow & Y \\ & \searrow f|_V & \swarrow f \\ & X & \end{array}$$

Proof. If $V \in \mathbf{ClopUp}(Y)$, then V is an Esakia space and the inclusion $V \hookrightarrow Y$ is an injective spectral local homeomorphism, hence a subobject of f .

Conversely, let $m: (Z \xrightarrow{g} X) \rightarrow (Y \xrightarrow{f} X)$ represent a subobject of f . By Lemma 7.3.22, m is injective. Its image $m[Z]$ is closed because Z is compact and Y is Hausdorff, open because m is a local homeomorphism, and an upset because m is a p-morphism. Thus $m[Z] \in \mathbf{ClopUp}(Y)$. Moreover, $m: Z \rightarrow m[Z]$ is an isomorphism in $\mathbf{Esa\acute{E}t}(X)$: it is a homeomorphism onto its image and reflects the order by strictness and injectivity. Hence, m represents the same subobject as $m[Z] \hookrightarrow Y$.

These two assignments are inverse of each other and preserve inclusions. $\blacksquare$

Corollary 7.3.24 (Truth value = clopen upset of the base). *Let X be an Esakia space. Then*

$$\mathbf{Sub}_{\mathbf{Esa\acute{E}t}(X)}(1) \cong \mathbf{ClopUp}(X).$$

Proof. By Theorem 7.3.21(1), the terminal object of $\mathbf{Esa\acute{E}t}(X)$ is $\text{id}_X: X \rightarrow X$. Apply Proposition 7.3.23 to $f = \text{id}_X$. $\blacksquare$

Remark 7.3.25 (Priestley spectral local homeomorphisms). The results of this subsection hold already at the level of Priestley spaces, with a straightforward adaptation of the proofs. Indeed, let $\mathbf{Pries}_{\text{SLH}}$ be the category of Priestley spaces and spectral local homeomorphisms. Then, for every Priestley space X , the slice category $\mathbf{Pries}_{\text{SLH}}/X$ is finitely complete, the monomorphisms are precisely the injective morphisms, and $\mathbf{Sub}_{\mathbf{Pries}_{\text{SLH}}/X}(1) \cong \mathbf{ClopUp}(X)$. Note that, by Proposition 7.3.12, whenever X is an Esakia space, $\mathbf{Pries}_{\text{SLH}}/X$ coincides with the category of our interest $\mathbf{Esa\acute{E}t}(X)$.

7.3.4 Power objects

We now describe power objects, which require a more intricate construction. Here, we will need to assume étale-finiteness.

Definition 7.3.26 (The candidate power object). Let $f: Y \rightarrow X$ be a spectral local homeomorphism between Esakia spaces, with X étale-finite. We define the set

$$P_f(Y) := \{(x, K) : x \in X, K \text{ an upset of } Y \text{ with } K \subseteq f^{-1}[\uparrow x]\}.$$

On $P_f(Y)$, we define the partial order $\leq$ by setting

$$(x, K) \leq (x', K') \iff x \leq x' \text{ and } K' = K \cap f^{-1}[\uparrow x'].$$

Note that $f^{-1}[\uparrow x]$ is finite, and hence, for every $(x, K) \in P_f(Y)$, K is closed. Then, by the definition of the Vietoris space,

$$P_f(Y) \subseteq X \times \mathcal{V}(Y).$$

We equip $P_f(Y)$ with two topologies, which we will show coincide.

- (The topology τ_{Viet}) We denote by τ_{Viet} the subspace topology on $P_f(Y)$ inherited from $X \times \mathcal{V}(Y)$.
- (The topology $\tau_{\Pi, \Lambda}$) For every clopen $W \subseteq X$, we define

$$\Pi(W) := \{(x, K) \in P_f(Y) : x \in W\}.$$

For every clopen upset $V \subseteq Y$, we define

$$\Lambda(V) := \{(x, K) \in P_f(Y) : K = V \cap f^{-1}[\uparrow x]\}.$$

We define $\tau_{\Pi, \Lambda}$ as the topology on $P_f(Y)$ with subbasis consisting of the sets of the form $\Lambda(V)$ and $P_f(Y) \setminus \Lambda(V)$ for a clopen upset $V \subseteq Y$, as well as the sets of the form $\Pi(W)$ for a clopen $W \subseteq X$.⁷

Finally, we define the (restriction of the) projection map

$$\begin{aligned} p: P_f(Y) &\longrightarrow X \\ (x, K) &\longmapsto x. \end{aligned}$$

To develop some intuition about these constructions, we give some basic examples of power objects:

Example 7.3.27 (Examples of power objects).

1. (Subobject classifier) Let X be an étale-finite Esakia space, and let $\acute{e}: X \rightarrow X_0$ be a continuous strict p-morphism, where X_0 is finite. The object $P_{\text{id}_X}(X)$ is simply

$$\{(x, K) : x \in X, K \subseteq \uparrow x, K \text{ an upset}\}.$$

The intuition for this is that the pair (x, K) is interpreted as saying that K consists of the points in $\uparrow x$ that are “true”. Note that, since $\acute{e}$ is a strict p-morphism, the only possible truth values are, up to isomorphism, the upsets in X_0 .

The topology $\tau_{\Pi, \Lambda}$ on $P_{\text{id}_X}(X)$ can be motivated from this point of view:

⁷Mnemonics: Π should be reminiscent of the projection $(x, K) \mapsto x$, while the symbol Λ is similar to the intersection symbol $\cap$ appearing in $V \cap f^{-1}[\uparrow x]$.

- Since we want $(P_{\text{id}_X}(X), p)$ to be an object in $\mathbf{Esa}\acute{\mathbf{E}}\mathbf{t}(X)$, we need $\Pi(W)$ to be clopen, for each clopen $W \subseteq X$;
- Since we want the possible “truth values” to form clopen subsets, we need to add, for each $x_0 \in X_0$, the upset

$$\begin{aligned} \Lambda(\acute{e}^{-1}[\uparrow x_0]) &= \{(x, K) \in P_{\text{id}_X}(X) : K = \acute{e}^{-1}[\uparrow x_0] \cap \uparrow x\} \\ &= \{(x, K) \in P_{\text{id}_X}(X) : \acute{e}[K] = \uparrow \acute{e}(x) \cap \uparrow x_0\}; \end{aligned}$$

a natural generalisation of this is to take any clopen upset $V \subseteq X$.

On the other hand, we want $P_{\text{id}_X}(X)$ to be an Esakia space, which is easiest to achieve by seeing it as a subspace of $X \times \mathcal{V}(X)$, since the latter is always compact. This motivates the introduction of the topology τ_{Viet} .

2. (Boolean subobject classifier) As a special case of item (1), if X is a Stone space, then the space $P_{\text{id}_X}(X)$ can equivalently be described as

$$\{(x, 1) : x \in X\} \sqcup \{(x, 0) : x \in X\},$$

with the disjoint union topology and disjoint union partial order, where we identify $1 = \{x\}$ and $0 = \emptyset$; each summand is order-homeomorphic to X . It follows that $P_{\text{id}_X}(X) \cong X \sqcup X$.

In this section, we prove that, if $f: Y \rightarrow X$ is a spectral local homeomorphism between Esakia spaces and X is étale-finite, then

$$\tau_{\Pi, \Lambda} = \tau_{\text{Viet}}.$$

The two topologies are useful for different purposes: $\tau_{\Pi, \Lambda}$ makes Priestley separation transparent, while τ_{Viet} makes compactness transparent. Their coincidence then allows us to show that $P_f(Y)$ is a Priestley space, and that the projection $p: P_f(Y) \rightarrow X$ is a spectral local homeomorphism, and hence that $P_f(Y)$ is an Esakia space. We will then prove that p is the power object of f in $\mathbf{Esa}\acute{\mathbf{E}}\mathbf{t}(X)$.

Lemma 7.3.28 (In principal fibres: upset = restriction of clopen). Let $f: Y \rightarrow X$ be a spectral local homeomorphism between Esakia spaces, with X étale-finite. For each $(x, K) \in P_f(Y)$, there is a clopen upset $V \subseteq Y$ such that $K = V \cap f^{-1}[\uparrow x]$.

Proof. Let $(x, K) \in P_f(Y)$. The set K is a closed upset of Y contained in $f^{-1}[\uparrow x]$. The set $f^{-1}[\uparrow x]$ is finite by Corollary 7.3.11. Therefore, $f^{-1}[\uparrow x] \setminus K$ is finite and hence closed in Y . Since $f^{-1}[\uparrow x] \setminus K$ is disjoint from the upset K , also its down-closure $\downarrow(f^{-1}[\uparrow x] \setminus K)$ is disjoint from K . By standard facts on Priestley spaces [94, Lem. 11.21], there is a clopen upset V of Y such that $K \subseteq V$ and $V \cap (\downarrow(f^{-1}[\uparrow x] \setminus K)) = \emptyset$. Therefore, $K = V \cap f^{-1}[\uparrow x]$. ■

Remark 7.3.29 (Π and Λ of clopen upsets are clopen upsets). If $W \subseteq X$ is a clopen upset, then $\Pi(W)$ is clearly a clopen upset. If $V \subseteq Y$ is a clopen upset, then $\Lambda(V)$ is clearly clopen by definition of the topology on $P_f(Y)$, and we show that it is an upset. Let $(x, K), (x', K') \in P_f(Y)$ be such that $(x', K') \geq (x, K) \in \Lambda(V)$. Then, by definition, $K = V \cap f^{-1}[\uparrow x]$, $x \leq x'$, and $K' = K \cap f^{-1}[\uparrow x']$. Therefore,

$$K' = K \cap f^{-1}[\uparrow x'] = V \cap f^{-1}[\uparrow x] \cap f^{-1}[\uparrow x'] = V \cap f^{-1}[\uparrow x'],$$

i.e., $K' \in \Lambda(V)$.

From these results, we can deduce the following:

Lemma 7.3.30 (Priestley separation for $P_f(Y)$). Let $f: Y \rightarrow X$ be a spectral local homeomorphism between Esakia spaces, with X étale-finite. Then, $(P_f(Y), \leq, \tau_{\Pi, \Lambda})$ satisfies the Priestley separation axiom.

Proof. Assume that $(x, K) \not\leq (x', K')$. If $x \not\leq x'$, then there is a clopen upset V of X such that $x \in V$ and $x' \notin V$, and thus

$$(x, K) \in \Pi(V) \text{ and } (x', K') \notin \Pi(V);$$

note that $\Pi(V)$ is a clopen upset by Remark 7.3.29. Otherwise, $x \leq x'$, and so $K' \neq K \cap f^{-1}[\uparrow x']$. By Lemma 7.3.28, there is a clopen upset $V \subseteq Y$ such that $V \cap f^{-1}[\uparrow x] = K$. Then, $(x, K) \in \Lambda(V)$ and $(x', K') \notin \Lambda(V)$ (since, otherwise, $K' = V \cap f^{-1}[\uparrow x'] = V \cap f^{-1}[\uparrow x] \cap f^{-1}[\uparrow x'] = K \cap f^{-1}[\uparrow x']$, a contradiction); note that $\Lambda(V)$ is a clopen upset by Remark 7.3.29. ■

Lemma 7.3.31 (Compactness of $P_f(Y)$). Let $f: Y \rightarrow X$ be a spectral local homeomorphism between Esakia spaces, with X étale-finite. Then the space $(P_f(Y), \tau_{\text{Vét}})$ is a closed subspace of $X \times \mathcal{V}(Y)$, and hence it is compact.

Proof. First, note that, by definition, $P_f(Y) \subseteq X \times \mathcal{V}^\uparrow(Y)$; that is, for each $(x, K) \in P_f(Y)$, the set K is assumed to be an upset. Because Y is an Esakia space, by Lemma 2.1.81(4) we have that $\mathcal{V}^\uparrow(Y)$ is a closed subset of $\mathcal{V}(Y)$. Thus, it suffices to show that for every $(x, K) \in (X \times \mathcal{V}^\uparrow(Y)) \setminus P_f(Y)$ there is a neighbourhood containing (x, K) entirely outside of $P_f(Y)$.

Let $(x, K) \in (X \times \mathcal{V}^\uparrow(Y)) \setminus P_f(Y)$. Then, $K \not\subseteq f^{-1}[\uparrow x]$, i.e., there is $y \in K$ such that $x \not\leq f(y)$. Let $V \subseteq X$ be a clopen upset such that $x \in V$ and $f(y) \notin V$. Consider the set

$$V \times \langle f^{-1}[X \setminus V] \rangle,$$

which is open since it is the product of basic opens. Certainly, (x, K) belongs to this set; moreover, whenever (z, K') belongs to it, we cannot have $K' \subseteq f^{-1}[\uparrow z]$. This shows the result. ■

For the following two statements, we fix:

$$\begin{array}{ccccccc}
 Y & \xrightarrow{f} & X & \xrightarrow{\acute{e}f} & X_0 \\
 \uparrow & & \uparrow & & \uparrow \\
 \text{Esakia} & & \text{spectral} & & \text{étale-finite} & & \text{continuous} \\
 \text{space} & & \text{local} & & \text{Esakia} & & \text{strict} \\
 & & \text{homeomorphism} & & \text{space} & & \text{p-morphism} \\
 & & & & & & \text{finite} \\
 & & & & & & \text{poset}
 \end{array}$$

Moreover, by the implication (1) $\Rightarrow$ (3) in Theorem 7.3.10, we fix a decomposition

$$Y = U_1 \cup \dots \cup U_n$$

of Y , where, for each $i \in \{1, \dots, n\}$, U_i is a clopen upset of Y , and $f|_{U_i}: U_i \rightarrow f[U_i]$ is an order-homeomorphism.

For each n -tuple $(T_1, \dots, T_n)$ of upsets of X_0 such that there is $x_0 \in X_0$ with $T_i \subseteq \uparrow x_0$ for all $i \in \{1, \dots, n\}$, consider the following subsets of $P_f(Y)$:

$$\mathbf{Typ}(T_1, \dots, T_n) = \{(x, K) \in P_f(Y) : \acute{e}f[K \cap U_i] = T_i \text{ for all } i \in \{1, \dots, n\}\}.$$

Lemma 7.3.32 ($\mathbf{Typ}$ is clopen). Let $f: Y \rightarrow X$ be a spectral local homeomorphism between Esakia spaces, with X étale-finite. For each n -tuple $(T_1, \dots, T_n)$ of upsets as above, the set $\mathbf{Typ}(T_1, \dots, T_n)$ is clopen in $(P_f(Y), \tau_{\mathbf{V}\acute{e}t})$.

Proof. Since $\acute{e}f: Y \rightarrow X_0$ is a continuous p-morphism, by Lemma 2.1.81 we have a continuous map:

$$\begin{aligned}
 \acute{e}f[-]: \mathcal{V}(Y) &\longrightarrow \mathcal{V}(X_0) \\
 C &\longmapsto (\acute{e}f)[C].
 \end{aligned}$$

Moreover, by Lemma 2.1.81(3), for each $i \in \{1, \dots, n\}$ the map $r_{U_i}: \mathcal{V}(Y) \rightarrow \mathcal{V}(Y)$ sending C to $C \cap U_i$ is continuous.

For each $i \in \{1, \dots, n\}$, let α_i be defined by

$$\begin{aligned}
 \alpha_i: P_f(Y) &\longrightarrow \mathcal{V}(X_0) \\
 (x, K) &\longmapsto \acute{e}f[K \cap U_i];
 \end{aligned}$$

note that α_i is continuous, since it can be factored as the composite of the following morphisms:

$$P_f(Y) \xrightarrow{\pi_{\mathcal{V}(Y)}} \mathcal{V}(Y) \xrightarrow{r_{U_i}} \mathcal{V}(Y) \xrightarrow{\acute{e}f[-]} \mathcal{V}(X_0)$$

Moreover, note that

$$\mathbf{Typ}(T_1, \dots, T_n) = \bigcap_{i=1}^n \alpha_i^{-1}[\{T_i\}];$$

indeed, given any $(x, K) \in P_f(Y)$, we have $\alpha_i(x, K) = (\acute{e}f[-])r_{U_i}\pi_{\mathcal{V}(Y)}(x, K) = \acute{e}f[K \cap U_i]$. This coincides with the definition of $\mathbf{Typ}(T_1, \dots, T_n)$, and, since $\{T_i\}$ is clopen in $\mathcal{V}(X_0)$ (since the latter is discrete), we have that $\mathbf{Typ}(T_1, \dots, T_n)$ is clopen, as desired. $\blacksquare$

Using this lemma, we can prove the equality of the topologies:

Theorem 7.3.33 (The two topologies on $P_f(Y)$ coincide). $\tau_{\Pi, \Lambda} = \tau_{\text{Viet}}$.

Proof. First note that by Lemma 7.3.31 the topology τ_{Viet} is compact, and by Lemma 7.3.30 the topology $\tau_{\Pi, \Lambda}$ is Hausdorff. Therefore, if the inclusion $\tau_{\Pi, \Lambda} \subseteq \tau_{\text{Viet}}$ holds, then the identity map $\text{id}: (P_f(Y), \tau_{\text{Viet}}) \rightarrow (P_f(Y), \tau_{\Pi, \Lambda})$ is a continuous bijection from a compact space to a Hausdorff space, and so it is a homeomorphism. Thus, it suffices to establish this inclusion.

To show $\tau_{\Pi, \Lambda} \subseteq \tau_{\text{Viet}}$, we will show (i) that every set of the form $\Pi(W)$ for W a clopen of X is clopen in the topology τ_{Viet} , and (ii) that every set of the form $\Lambda(V)$ for V a clopen upset of Y is clopen in the topology τ_{Viet} (and so is its complement $P_f(Y) \setminus \Lambda(V)$, too).

For every clopen W of X , we have

$$\Pi(W) = \{(x, K) \in P_f(Y) : x \in W\} = P_f(Y) \cap (W \times \mathcal{V}(Y)),$$

which is clearly clopen in τ_{Viet} .

Now fix a clopen upset $V \subseteq Y$ and, for each $i \in \{1, \dots, n\}$, set $V_i := V \cap U_i$. Consider:

$$\mathbf{Types}(V) := \{(T_1, \dots, T_n) : \exists x \in X, \forall i \leq n, T_i \subseteq \uparrow \acute{e}(x), \acute{e}f[V_i \cap f^{-1}[\uparrow x]] = T_i\}.$$

Note also that, for all subsets W and W' of Y , we have

$$W \cap f^{-1}[\uparrow x] \subseteq W' \text{ iff } f^{-1}[\uparrow x] \subseteq (W \Rightarrow W') \text{ iff } x \in \Box \forall_f [W \Rightarrow W']. \quad (7.4)$$

Now, letting $\pi: P_f(Y) \rightarrow \mathcal{V}(Y)$ denote the restriction of the projection function $X \times \mathcal{V}(Y) \rightarrow \mathcal{V}(Y)$, we show:

$$\Lambda(V) = \bigcup_{(T_1, \dots, T_n) \in \mathbf{Types}(V)} \mathbf{Typ}(T_1, \dots, T_n) \cap \pi^{-1}[\llbracket V \rrbracket] \cap \bigcap_{i=1}^n \Pi(\Box \forall_f [V_i \Rightarrow f^{-1} \acute{e}^{-1}[T_i]]). \quad (7.5)$$

By Lemma 7.3.32, the sets of the form $\mathbf{Typ}(T_1, \dots, T_n)$ appearing in this finite union are clopen in τ_{Viet} . Moreover, the sets of the form $\forall_f [V_i \Rightarrow f^{-1} \acute{e}^{-1}[T_i]]$ are clopen because f is closed and open; since X is Esakia, it follows that also $\Box \forall_f [V_i \Rightarrow f^{-1} \acute{e}^{-1}[T_i]]$ is clopen. Therefore, the set on the right-hand side of (7.5) is clopen in τ_{Viet} .

To prove (7.5), we first prove the inclusion $\supseteq$. Let $(T_1, \dots, T_n) \in \mathbf{Types}(V)$ and let $(x, K) \in \mathbf{Typ}(T_1, \dots, T_n) \cap \pi^{-1}[\llbracket V \rrbracket] \cap \bigcap_{i=1}^n \Pi(\Box \forall_f [V_i \Rightarrow f^{-1} \acute{e}^{-1}[T_i]])$. We need to show that

$$K = V \cap f^{-1}[\uparrow x].$$

Since $Y = U_1 \cup \dots \cup U_n$, for this it suffices to show that, for each $i \in \{1, \dots, n\}$, upon setting $K_i := K \cap U_i$, we have $K_i = V_i \cap f^{-1}[\uparrow x]$. Let $i \in \{1, \dots, n\}$.

The inclusion $K_i \subseteq V_i \cap f^{-1}[\uparrow x]$ is immediate because $(x, K) \in \pi^{-1}[\llbracket V \rrbracket]$, $K_i = K \cap U_i \subseteq U_i$ and $K_i \subseteq f^{-1}[\uparrow x]$. To prove the inclusion $V_i \cap f^{-1}[\uparrow x] \subseteq K_i$, let $w \in V_i \cap f^{-1}[\uparrow x]$. Since $x \in \Box \forall_f([V_i \Rightarrow f^{-1}e^{-1}[T_i]])$, by (7.4) we have $V_i \cap f^{-1}[\uparrow x] \subseteq f^{-1}e^{-1}[T_i]$, and so $ef[V_i \cap f^{-1}[\uparrow x]] \subseteq T_i = ef[K_i]$ (since $(x, K) \in \mathbf{Typ}(T_1, \dots, T_n)$). Hence, there is $w' \in K_i$ such that $ef(w) = ef(w')$. Since $w, w' \in V \cap f^{-1}[\uparrow x]$, it follows that $x \leq f(w), f(w')$; since e is a strict p-morphism and $f(w), f(w') \in \uparrow x$, from $ef(w) = ef(w')$ we deduce $f(w) = f(w')$. Since $w, w' \in U_i$, and f is an order-homeomorphism on this component, we have $w = w'$, i.e., $w \in K_i$, as desired.

Conversely, assume that $(x, K) \in \Lambda(V)$, i.e., $K = V \cap f^{-1}[\uparrow x]$. For each $i \in \{1, \dots, n\}$, set $K_i := K \cap U_i$ and $T_i := ef[K_i]$. Then, $(T_1, \dots, T_n) \in \mathbf{Types}(V)$. Clearly, $(x, K) \in \mathbf{Typ}(T_1, \dots, T_n)$ and $(x, K) \in \pi^{-1}[\llbracket V \rrbracket]$. Moreover, for every $i \in \{1, \dots, n\}$,

$$\begin{array}{ccc} V_i \cap f^{-1}[\uparrow x] = U_i \cap V \cap f^{-1}[\uparrow x] = U_i \cap K = K_i & \subseteq & f^{-1}e^{-1}[T_i], \\ & \uparrow & \uparrow \\ & (x, K) \in \Lambda(V) & ef[K_i] = T_i \end{array}$$

and so, by (7.4), $x \in \Box \forall_f[V_i \Rightarrow f^{-1}e^{-1}[T_i]]$. ■

In light of Theorem 7.3.33, we will simply write $(P_f(Y), \leq, \tau)$ to refer to the topology on $P_f(Y)$. We conclude with the following:

Theorem 7.3.34 ($P_f(Y)$ is an Esakia étale space). *Let $f: Y \rightarrow X$ be a spectral local homeomorphism between Esakia spaces, with X étale-finite. Then $(P_f(Y), \leq, \tau)$ is an Esakia space, and the map $p: P_f(Y) \rightarrow X$ is a spectral local homeomorphism.*

Proof. Because of Lemma 7.3.30 we know that $(P_f(Y), \leq, \tau_{\Pi, \Lambda})$ satisfies the Priestley separation axiom, and by Lemma 7.3.31, that $(P_f(Y), \tau_{\mathbf{Viet}})$ is compact. By Theorem 7.3.33, we know that $\tau_{\Pi, \Lambda} = \tau_{\mathbf{Viet}}$, and so $(P_f(Y), \leq, \tau)$ is a Priestley space.

We now show that p is a spectral local homeomorphism. From this and Proposition 7.3.12 it will follow that $(P_f(Y), \leq)$ is an Esakia space.

By definition, p is order-preserving, and it is continuous, since it is the restriction of the projection $\pi_X: X \times \mathcal{V}(Y) \rightarrow X$, which is continuous.

Let $(x, K) \in P_f(Y)$ be arbitrary. By Lemma 7.3.28, let V be a clopen upset such that $V \cap f^{-1}[\uparrow x] = K$. The set $\Lambda(V)$ is a clopen upset by Remark 7.3.29. Then $(x, K) \in \Lambda(V)$, and note that $p[\Lambda(V)] = X$ because for each $x' \in X$ the element $(x', V \cap f^{-1}[\uparrow x'])$ is mapped to x' by p . Moreover, $p|_{\Lambda(V)}$ is order-reflecting: if $p(x, K) \leq p(x', K')$ then $x \leq x'$ and

$$K' = V \cap f^{-1}[\uparrow x'] = V \cap f^{-1}[\uparrow x] \cap f^{-1}[\uparrow x'] = K \cap f^{-1}[\uparrow x'],$$

and so by definition, $(x, K) \leq (x', K')$. Thus, the function $p|_{\Lambda(V)}: \Lambda(V) \rightarrow X$ is a continuous bijection between compact Hausdorff spaces, and hence a

homeomorphism; since it is order-preserving and order-reflecting, it is an order-homeomorphism.

We thus conclude that the set $p[\Lambda(V)] = X$ is a clopen upset, and $p|_{\Lambda(V)}$ is an order-homeomorphism. By the implication (2) $\Rightarrow$ (1) in Theorem 7.3.10, the map p is a spectral local homeomorphism as desired. ■

In light of Theorem 7.3.34, we have that, for X an étale-finite Esakia space and $f: Y \rightarrow X$ a spectral local homeomorphism, $(P_f(Y), p)$ is an object in $\mathbf{Esa}\dot{\mathbf{E}}\mathbf{t}(X)$. We now show that it is indeed the power object of $f: Y \rightarrow X$.

Definition 7.3.35 ($\in_Y$). By Theorem 7.3.21(2), we can compute the product of (Y, f) and $(P_f(Y), p)$ in $\mathbf{Esa}\dot{\mathbf{E}}\mathbf{t}(X)$, which is

$$Y \times_X P_f(Y) = \{(y, (x, K)) \in Y \times P_f(Y) : f(y) = x\},$$

together with the composite $Y \times_X P_f(Y) \xrightarrow{\pi_Y} Y \xrightarrow{f} X$. We define its subset

$$\in_Y := \{(y, (x, K)) \in Y \times_X P_f(Y) : y \in K\}.$$

Lemma 7.3.36 ($\in_Y$ is a clopen upset). Let $f: Y \rightarrow X$ be a spectral local homeomorphism between Esakia spaces, with X étale-finite. The set $\in_Y$ is a clopen upset of $Y \times_X P_f(Y)$.

Proof. We first show that $\in_Y$ is an upset. Let $(y, (x, K))$ be an element of $\in_Y$, and let $(y', (x', K')) \in Y \times_X P_f(Y)$ be such that $(y, (x, K)) \leq (y', (x', K'))$. Then $y \leq y'$ and $(x, K) \leq (x', K')$; since K is an upset and $y \in K$, it follows that $y' \in K$. Since

$$K' = K \cap f^{-1}[\uparrow x'],$$

and $f(y') = x'$, we have $y' \in K \cap f^{-1}[\uparrow x']$, and so $y' \in K'$.

Let us now prove that $\in_Y$ is open. Let $(y, (x, K))$ be an element of $\in_Y$, and let us find an open neighbourhood (in $Y \times_X P_f(Y)$) of $(y, (x, K))$ contained in $\in_Y$. By Lemma 7.3.28, there is a clopen upset V of Y such that $K = V \cap f^{-1}[\uparrow x]$. We claim that the open set

$$V \times_X \Lambda(V)$$

contains the element $(y, (x, K))$ and is a subset of $\in_Y$. Since $y \in K = V \cap f^{-1}[\uparrow x]$, we have $y \in V$. Moreover, (x, K) belongs to $\Lambda(V)$ by definition of $\Lambda(V)$. Therefore, $(y, (x, K)) \in V \times_X \Lambda(V)$. We now prove that $V \times_X \Lambda(V)$ is a subset of $\in_Y$. Let $(y', (x', K')) \in V \times_X \Lambda(V)$. Since $f(y') = x'$ (and hence $y' \in f^{-1}[x'] \subseteq f^{-1}[\uparrow x']$) and $y' \in V$, we have $y' \in V \cap f^{-1}[\uparrow x'] = K'$. Therefore, $(y', (x', K'))$ belongs to $\in_Y$. This proves that $V \times_X \Lambda(V)$ is a subset of $\in_Y$.

It remains to show that $\in_Y$ is closed. So, let

$$(y, (x, K)) \in (Y \times_X P_f(Y)) \setminus \in_Y,$$

and let us find an open neighbourhood (in $Y \times_X P_f(Y)$) of $(y, (x, K))$ disjoint from $\in_Y$. By Lemma 7.3.28, there is a clopen upset V of Y such that

$$K = V \cap f^{-1}[\uparrow x].$$

We claim that the open set

$$(Y \setminus V) \times_X \Lambda(V)$$

contains $(y, (x, K))$ and is disjoint from $\in_Y$. Let us first prove that it contains $(y, (x, K))$. Recall that $K = V \cap f^{-1}[\uparrow x]$, that $f(y) = x$ (and hence $y \in f^{-1}[x] \subseteq f^{-1}[\uparrow x]$) and that $y \notin K$. Thus, $y \notin V$. Moreover, (x, K) belongs to $\Lambda(V)$ by definition of $\Lambda(V)$. Therefore, $(y, (x, K)) \in (Y \setminus V) \times_X \Lambda(V)$. Let us now prove that $(Y \setminus V) \times_X \Lambda(V)$ is disjoint from $\in_Y$. Let $(y', (x', K')) \in (Y \setminus V) \times_X \Lambda(V)$. Since $f(y') = x'$ (and hence $y' \in f^{-1}[x'] \subseteq f^{-1}[\uparrow x']$), we have $y' \notin V \cap f^{-1}[\uparrow x'] = K'$. Therefore, $(y', (x', K'))$ does not belong to $\in_Y$. This proves that $(Y \setminus V) \times_X \Lambda(V)$ is disjoint from $\in_Y$. This proves our claim, and hence $\in_Y$ is closed.

So, $\in_Y$ is clopen, as desired. ■

Consequently, $\in_Y$ (with the composite map $\in_Y \hookrightarrow Y \times_X P_f(Y) \xrightarrow{\pi_Y} Y \xrightarrow{f} X$) is an object of **EsaÉt**(X). To verify that it is the power object, we make use of the following lemma.

Lemma 7.3.37 (Subobject pullbacks as inverse images). Let $g: S \rightarrow T$ be a morphism in **Pries**, and let A and B be closed subsets of S and T , respectively (and so Priestley spaces). There is a morphism $i: A \rightarrow B$ making the following square a pullback

$$\begin{array}{ccc} A & \xrightarrow{i} & B \\ \downarrow & & \downarrow \\ S & \xrightarrow{g} & T \end{array}$$

if and only if $g^{-1}[B] = A$. The same holds when **Pries** is replaced by **EsaÉt**(X) and A and B are assumed to be clopen upsets.

Proof. Let $g: S \rightarrow T$ be a morphism in **Pries**, and let A and B be closed subsets of S and T , respectively. Suppose first that there is a morphism $i: A \rightarrow B$ making the square in the statement a pullback. Since the functor **Pries** $\rightarrow$ **Set** preserves limits, the square is also a pullback in **Set**. Since pullbacks along inclusions in **Set** are given by inverse image, the conclusion follows.

Conversely, assume that $g^{-1}[B] = A$. Then, the morphism g restricts to a map $i := g|_A: A \rightarrow B$. Since A carries the subspace topology and induced order from S , and B carries the subspace topology and induced order from T , the restriction i is continuous and order-preserving, and hence a morphism in **Pries** making the square in the statement commute. Moreover, this square is a pullback in **Set**. To prove that the square is a pullback in **Pries**, let C be a Priestley space, and

let $h: C \rightarrow S$ and $k: C \rightarrow B$ be continuous order-preserving maps such that $gh = i_B k$. Then, there is a unique map $p: C \rightarrow A$ in **Set** such that $ip = k$ and $i_A p = h$, as in the following diagram:

$$\begin{array}{ccccc}
 C & & & & \\
 \swarrow p & & k \searrow & & \\
 & A & \xrightarrow{i} & B & \\
 \downarrow i_A & & & & \downarrow i_B \\
 S & \xrightarrow{g} & T & &
 \end{array}$$

(Note: In the original image, there is also a curved arrow from C to S labeled h, and a curved arrow from C to B labeled k. The diagram is a pullback square with an additional map p from C to A.)

It remains to check that p is a morphism in **Pries**. The subspace topology on A is the initial topology with respect to the inclusion $i_A: A \hookrightarrow S$. Therefore, the map p is continuous because $i_A p = h$ is continuous. Similarly, since the order on A is induced from the order on S , the inclusion i_A is an order-embedding. Hence, for $c, c' \in C$ with $c \leq c'$, we have

$$i_A p(c) = h(c) \leq h(c') = i_A p(c'),$$

and therefore $p(c) \leq p(c')$ in A . Thus p is order-preserving, and so p is a morphism in **Pries**.

The uniqueness of p as a morphism in **Pries** follows from its uniqueness as a map in **Set**. Hence, the square is a pullback in **Pries**.

Finally, let us prove the statement for **EsaÉt**(X). Let $g: S \rightarrow T$ be a morphism in **EsaÉt**(X), and let $A \subseteq S$ and $B \subseteq T$ be clopen upsets, equipped with the induced structure maps to X . By the first part, the corresponding square is a pullback in **Pries** if and only if $g^{-1}[B] = A$. Since the forgetful functor

$$\mathbf{EsaÉt}(X) \longrightarrow \mathbf{Pries}$$

creates pullbacks by Theorem 7.3.21(4), the square is a pullback in **EsaÉt**(X) if and only if its image is a pullback in **Pries**. Therefore, the same equivalence holds in **EsaÉt**(X). ■

Theorem 7.3.38 ($P_f(Y)$ is the power object). *Let $f: Y \rightarrow X$ be a spectral local homeomorphism between Esakia spaces, with X étale-finite. In **EsaÉt**(X), the object $p: P_f(Y) \rightarrow X$ is the power object of $f: Y \rightarrow X$.*

Proof. Let $m: Z \rightarrow X$ be an object in **EsaÉt**(X). By Proposition 7.3.23, the subobjects of $Y \times_X Z \rightarrow X$ are the clopen upsets of $Y \times_X Z$. So, let $R \subseteq Y \times_X Z$ be a clopen upset. We shall show that there is a unique morphism $k: Z \rightarrow P_f(Y)$ such that there is a morphism $R \rightarrow \epsilon_Y$ making the following square a pullback.

$$\begin{array}{ccc}
 R & \longrightarrow & \epsilon_Y \\
 \downarrow & & \downarrow \\
 Y \times_X Z & \xrightarrow{\text{id}_Y \times k} & Y \times_X P_f(Y)
 \end{array}$$

By Lemma 7.3.37, this amounts to showing that there is a unique morphism $k: Z \rightarrow P_f(Y)$ such that $(\text{id}_Y \times k)^{-1}[\in_Y] = R$, where $\text{id}_Y \times k$ is the product function $Y \times_X Z \rightarrow Y \times_X P_f(Y)$. In turn, $(\text{id}_Y \times k)^{-1}[\in_Y] = R$ means: for all $(y, z) \in Y \times_X Z$, setting $(x_z, K_z) := k(z)$,

$$y \in K_z \iff (y, z) \in R. \quad (7.6)$$

We first prove the uniqueness of k . Let $k: Z \rightarrow P_f(Y)$ be a morphism such that $(\text{id}_Y \times k)^{-1}[\in_Y] = R$. For each $z \in Z$, we write $(x_z, K_z) := k(z)$. Since k is a morphism, the following diagram commutes.

$$\begin{array}{ccc} Z & \xrightarrow{k} & P_f(Y) \\ & \searrow m & \swarrow p \\ & X & \end{array}$$

Therefore, $x_z = p(x_z, K_z) = pk(z) = m(z)$. This shows that we have at most one possible value for x_z . We are left to show that there is at most one possible value for K_z . Recall that, since $(x_z, K_z) \in P_f(Y)$, we have $K_z \subseteq f^{-1}[\uparrow x_z] = f^{-1}[\uparrow m(z)]$ (the equality following from what was just shown). Therefore, we shall determine which elements of $f^{-1}[\uparrow m(z)]$ belong to K_z . If we take $y' \in f^{-1}[\uparrow m(z)]$, we have $f(y') \geq m(z)$. Then, by the strictness of m , there is a unique $z' \in \uparrow z$ such that $m(z') = f(y')$, which means that $(y', z') \in Y \times_X Z$. By (7.6), $y' \in K_z$ if and only if $(y', z') \in R$. Since k is order-preserving and $z \leq z'$, we have $k(z) \leq k(z')$. Therefore, by the definition of the partial order $\leq$ on $P_f(Y)$, we have $K_{z'} = K_z \cap f^{-1}[\uparrow m(z')]$. Since $y' \in f^{-1}[\uparrow m(z')]$, we have

$$y' \in K_z \iff y' \in K_{z'} \iff (y', z') \in R.$$

In conclusion, for every $z \in Z$, $x_z = m(z)$, and K_z is the set of $y' \in f^{-1}[\uparrow m(z)]$ such that, for the unique $z' \in \uparrow z$ with $f(y') = m(z')$, we have $(y', z') \in R$. This proves the uniqueness of k .

Having shown that there is at most one possible candidate, let us prove the existence of the morphism k with the desired properties.

For each z , set $x_z := m(z)$, and define K_z as the set of elements $y' \in f^{-1}[\uparrow m(z)]$ such that, for the unique $z' \in \uparrow z$ with $(y', z') \in Y \times_X Z$, we have $(y', z') \in R$. In other words:

$$K_z = f^{-1}[\uparrow m(z)] \cap \pi_Y[R \cap \pi_Z^{-1}[\uparrow z]].$$

We show that the function

$$\begin{aligned} k: Z &\longrightarrow P_f(Y) \\ z &\longmapsto (x_z, K_z) \end{aligned}$$

is well-defined. This is done by showing that, for each $z \in Z$, $k(z)$ is indeed an element of $P_f(Y)$, i.e., that $K_z \subseteq f^{-1}[\uparrow x_z]$ and that K_z is an upset of Y . Clearly,

$K_z \subseteq f^{-1}[\uparrow m(z)] = f^{-1}[\uparrow x_z]$; we prove that K_z is an upset of Y . Let $y' \in K_z$ and $y' \leq y''$. Let z' , resp. z'' , be the unique element in $\uparrow z$ such that $f(y') = m(z')$, resp. $f(y'') = m(z'')$. Since $y' \in K_z$, we have $(y', z') \in R$. Moreover, since $m|_{\uparrow z}$ is an order-isomorphism and $m(z') = f(y') \leq f(y'') = m(z'')$, we have $z' \leq z''$. Since R is an upset of $Y \times_X Z$ and $(y'', z'') \geq (y', z') \in R$, we have $(y'', z'') \in R$, and hence $y'' \in K_z$. This shows that K_z is an upset. Thus, k is a well-defined function.

It is immediate that the following diagram of functions commutes.

$$\begin{array}{ccc} Z & \xrightarrow{k} & P_f(Y) \\ & \searrow m & \swarrow p \\ & X & \end{array}$$

To prove that k is a morphism in $\mathbf{Esa}\hat{\mathbf{Et}}(X)$, by the triangle property of $\mathbf{Esa}_{\text{SLH}}$ in $\mathbf{Pries}$ (Proposition 7.3.14) it suffices to show that k is order-preserving and continuous.

Let us prove that k is order-preserving. Let $z, z' \in Z$ with $z \leq z'$, and let us prove that $k(z) \leq k(z')$, i.e., that $(m(z), K_z) \leq (m(z'), K_{z'})$, where $\leq$ is the order on $P_f(Y)$. By monotonicity of m , we have $m(z) \leq m(z')$. We prove the equality $K_{z'} = K_z \cap f^{-1}[\uparrow m(z')]$. The inclusion $K_{z'} \subseteq K_z \cap f^{-1}[\uparrow m(z')]$ holds because $z \leq z'$: indeed, if $y \in K_{z'}$, the unique point of $\uparrow z'$ mapping to $f(y)$ is also the unique point of $\uparrow z$ mapping to $f(y)$. Conversely, let $y \in K_z \cap f^{-1}[\uparrow m(z')]$. Let $\bar{z}$ be the unique point in $\uparrow z$ such that $m(\bar{z}) = f(y)$. Since $z' \in \uparrow z$, $\bar{z} \in \uparrow z$, and

$$m(z') \leq f(y) = m(\bar{z}),$$

the restriction $m: \uparrow z \rightarrow \uparrow m(z)$ reflects the order by strictness of m . Hence $z' \leq \bar{z}$. Therefore, $\bar{z} \in \uparrow z'$, and so $\bar{z}$ is also the unique point of $\uparrow z'$ mapping to $f(y)$. Since $y \in K_z$, we have $(y, \bar{z}) \in R$, and hence $y \in K_{z'}$. Thus

$$K_{z'} = K_z \cap f^{-1}[\uparrow m(z')],$$

as required. Therefore, k is order-preserving.

Next, we prove continuity of k by showing that preimages of subbasic clopens for the topology τ_{Viet} are clopen; by Theorem 7.3.33, this establishes continuity. Consider first a clopen $W \subseteq X$; then,

$$k^{-1}[\Pi(W)] = \{z \in Z : m(z) \in W\} = m^{-1}[W],$$

which is clopen. Now let V be a clopen subset of Y ; then

$$k^{-1}[\llbracket V \rrbracket] = \{z \in Z : K_z \subseteq V\}.$$

Let $\pi_Y: Y \times_X Z \rightarrow Y$ and $\pi_Z: Y \times_X Z \rightarrow Z$ denote the projections. To show that $k^{-1}[\llbracket V \rrbracket]$ is clopen, we claim

$$k^{-1}[\llbracket V \rrbracket] = \Box \forall_{\pi_Z} [R \Rightarrow \pi_Y^{-1}[V]].$$

(Note that $\Box \forall_{\pi_Z}[R \Rightarrow \pi_Y^{-1}[V]]$ is clopen because R is clopen, π_Z is an open map and Z is Esakia.) We prove the claim. Let $z \in Z$. Then $z \in \Box \forall_{\pi_Z}[R \Rightarrow \pi_Y^{-1}[V]]$ if and only if, for every $z' \geq z$ and every $y \in Y$ such that $f(y) = m(z')$, we have

$$(y, z') \in R \text{ implies } y \in V.$$

We show that this is equivalent to $K_z \subseteq V$. Suppose first that $z \in \Box \forall_{\pi_Z}[R \Rightarrow \pi_Y^{-1}[V]]$, and let $y' \in K_z$. Let z' be the unique point in $\uparrow z$ such that $m(z') = f(y')$. By definition of K_z , we have $(y', z') \in R$. Since $z' \geq z$, the assumption gives $y' \in V$. Thus $K_z \subseteq V$. Conversely, suppose that $K_z \subseteq V$. Let $z' \geq z$, and let $y' \in Y$ be such that $f(y') = m(z')$. Assume that $(y', z') \in R$. Since $z' \geq z$, we have $f(y') = m(z') \geq m(z)$, and so $y' \in f^{-1}[\uparrow m(z)]$. Moreover, by strictness of m , the point z' is the unique element of $\uparrow z$ mapped by m to $f(y')$. Hence, by definition of K_z , from $(y', z') \in R$ we get $y' \in K_z$. Since $K_z \subseteq V$, it follows that $y' \in V$. Therefore

$$z \in \Box \forall_{\pi_Z}[R \Rightarrow \pi_Y^{-1}[V]].$$

This proves

$$k^{-1}[\llbracket V \rrbracket] = \Box \forall_{\pi_Z}[R \Rightarrow \pi_Y^{-1}[V]].$$

This proves that $k^{-1}[\llbracket V \rrbracket]$ is clopen. Clearly, also the set $k^{-1}[\langle V \rangle]$, equal to $Z \setminus k^{-1}[\llbracket Y \setminus V \rrbracket]$, is clopen. We thus have that the preimage of subbasic clopens in $P_f(Y)$ is clopen, and hence k is continuous.

Since $pk = m$ and k is continuous and order-preserving, by the triangle property (Proposition 7.3.14) k is a spectral local homeomorphism.

Finally, let $(y, z) \in Y \times_X Z$. We prove that

$$y \in K_z \iff (y, z) \in R.$$

Suppose first that $(y, z) \in R$. Since $z \in \uparrow z$ and $m(z) = f(y)$, the point z is the unique element of $\uparrow z$ mapped by m to $f(y)$. Hence, by the definition of K_z , we have $y \in K_z$. Conversely, suppose that $y \in K_z$. By definition of K_z , if $\bar{z}$ is the unique point in $\uparrow z$ such that $m(\bar{z}) = f(y)$, then $(y, \bar{z}) \in R$. But $(y, z) \in Y \times_X Z$, so $m(z) = f(y)$; hence z is also an element of $\uparrow z$ mapped by m to $f(y)$. By uniqueness, $\bar{z} = z$. Therefore, $(y, z) \in R$. ■

We arrive at the main result of this chapter:

Theorem 7.3.39 (Étale-finite $\Rightarrow$ topos-admissible). *For every étale-finite Heyting algebra H , there is an elementary topos $\mathcal{E}$, namely*

$$\mathcal{E} := \mathbf{Esa}\dot{\mathbf{E}}\mathbf{t}(\mathrm{Spec}(H)),$$

such that $\mathrm{Sub}_{\mathcal{E}}(1) \cong H$.

Proof. Set $\mathcal{E} := \mathbf{Esa}\dot{\mathbf{E}}\mathbf{t}(\mathrm{Spec}(H))$. By Theorem 7.3.21, this category has all finite limits, and, by Theorem 7.3.38, every object has a power object. Thus, it is an elementary topos. By Proposition 7.3.24, $\mathrm{Sub}_{\mathcal{E}}(1) \cong \mathbf{ClopUp}(\mathrm{Spec}(H))$, and, by Esakia duality, this is isomorphic to H . ■

7.4 Finitely propositional toposes

In this section, we exhibit an obstruction to extending our use of compact étale spaces beyond the étale-finite case. For this purpose, we introduce the notion of a *finitely propositional* topos and show that the étale-finite Heyting algebras are precisely those Heyting algebras that arise as the lattice of truth values of such toposes.

Definition 7.4.1 ((Finitely) propositional). We say that a category with a terminal object is *(finitely) propositional* if every object admits a jointly epimorphic (finite) family of morphisms from subterminal objects; that is, for every object Y there is a jointly epimorphic (finite) family

$$\{V_i \longrightarrow Y\}_{i \in I}$$

with each V_i subterminal.

Our interest in (finite) propositionality is, roughly speaking, that it is a property of categories of (compact) étale spaces.

Remark 7.4.2 (Propositionality for toposes). Note that a category with a terminal object is propositional precisely when its subterminal objects form a separating family (see [157, p. 12]). For Grothendieck toposes, propositionality is characterised geometrically: it holds precisely for those Grothendieck toposes that are localic (i.e., equivalent to the topos of sheaves on a locale), see [157, Thm. C1.4.7]. Thus, for instance, the presheaf topos $[\mathbf{C}^{\bar{p}}, \mathbf{Set}]$ is non-propositional whenever $\mathbf{C}$ is not a preorder (see [81, Cor. 3.2]).

Outside the Grothendieck setting, the notion of propositionality for elementary toposes has been studied in [163].

Remark 7.4.3 (Propositional $\not\Rightarrow$ Fin. Propositional). The conditions of propositionality and finite propositionality do not coincide: an example of a propositional topos that is not finitely propositional is the category $\mathcal{E} := \mathbf{Sh}(\omega + 1)$ of sheaves on the locale $\omega + 1$. Indeed, since $\mathbf{Sub}_{\mathcal{E}}(1) \cong \omega + 1$ and the Heyting algebra $\omega + 1$ is not étale-finite, it will follow from Corollary 7.4.9 below that $\mathcal{E}$ is not finitely propositional.

Theorem 7.4.4 (Esakia étale spaces are finitely propositional). *For every Heyting algebra H , the category $\mathbf{Esa}\dot{\mathbf{Et}}(\mathbf{Spec}(H))$ is finitely propositional.*

Proof. Let $X := \mathbf{Spec}(H)$ and let $f: Y \rightarrow X$ be an object of $\mathbf{Esa}\dot{\mathbf{Et}}(X)$. By the implication (1) $\Rightarrow$ (3) in Theorem 7.3.10, there are clopen upsets $U_1, \dots, U_n$ of Y such that $Y = U_1 \cup \dots \cup U_n$ and, for each $i \in \{1, \dots, n\}$, the restriction $f|_{U_i}: U_i \rightarrow f[U_i]$ is an order-homeomorphism. Set $V_i := f[U_i]$. Then, each V_i is a clopen upset of X , and hence determines a subterminal object $V_i \rightarrowtail X$ of $\mathbf{Esa}\dot{\mathbf{Et}}(X)$. Let

$$g_i: V_i \longrightarrow Y$$

be the composite of $(f \upharpoonright_{U_i})^{-1}$ with the inclusion $U_i \hookrightarrow Y$. Since fg_i is the inclusion $V_i \hookrightarrow X$, each g_i is a morphism in the slice category $\mathbf{Esa\acute{E}t}(X)$.

The family $\{g_i : i \in \{1, \dots, n\}\}$ is jointly epimorphic. Indeed, any faithful functor reflects jointly epimorphic families, and so, by considering the composite

$$\mathbf{Esa\acute{E}t}(\mathrm{Spec}(H)) \longrightarrow \mathbf{Esa}_{\mathrm{SLH}} \longrightarrow \mathbf{Set},$$

it suffices to check that $\{g_i : i \in \{1, \dots, n\}\}$ is jointly surjective. But this follows immediately from $Y = U_1 \cup \dots \cup U_n$. $\blacksquare$

It is reasonable to ask whether one could extend our approach to Heyting algebras that are not étale-finite—such as the chains $\omega + 1$ or $\omega + \omega^{\mathrm{op}}$. We will now show that, for every finitely propositional topos $\mathcal{E}$, the Heyting algebra $\mathrm{Sub}_{\mathcal{E}}(1)$ is étale-finite.

Proposition 7.4.5 (Finitely propositional $\Rightarrow$ Jibladze’s law over a finite subset). Let $\mathcal{E}$ be a finitely propositional elementary topos, and set $H := \mathrm{Sub}_{\mathcal{E}}(1)$. There is a finite subset $S \subseteq H$ such that, for every $a \in H$,

$$\bigvee_{s \in S} (a \leftrightarrow s) = 1.$$

Proof. By finite propositionality applied to the subobject classifier Ω , let $V_1, \dots, V_n$ in $\mathrm{Sub}_{\mathcal{E}}(1)$ be such that the family $a_i : V_i \rightarrow \Omega$ is jointly epimorphic. Consider the subobject $U_i \rightarrowtail V_i$ classified by a_i , i.e., appearing in the following pullback diagram

$$\begin{array}{ccc} U_i & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow t \\ V_i & \xrightarrow{a_i} & \Omega. \end{array} \quad (7.7)$$

For each $i \in \{1, \dots, n\}$, the composite $U_i \rightarrowtail V_i \rightarrowtail 1$ of monomorphisms shows that U_i is also a subterminal object. We claim that $S = \{U_1, \dots, U_n\}$ works.

Let $W \rightarrowtail 1$ be any subterminal and let $p : 1 \rightarrow \Omega$ be its classifying morphism, i.e., the morphism for which there is a pullback diagram as follows.

$$\begin{array}{ccc} W & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow t \\ 1 & \xrightarrow{p} & \Omega \end{array} \quad (7.8)$$

For each $i \in \{1, \dots, n\}$, let $p^*(V_i)$ denote the pullback of $a_i : V_i \rightarrow \Omega$ along $p : 1 \rightarrow \Omega$.

$$\begin{array}{ccc} p^*(V_i) & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow p \\ V_i & \xrightarrow{a_i} & \Omega \end{array} \quad (7.9)$$

We now construct the following two diagrams:

$$\begin{array}{ccccc} p^*(V_i) \wedge U_i & \rightharpoonup & U_i & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow t \\ p^*(V_i) & \rightharpoonup & V_i & \xrightarrow{a_i} & \Omega \end{array}$$

$$\begin{array}{ccccc} p^*(V_i) \wedge W & \rightharpoonup & W & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow t \\ p^*(V_i) & \rightharpoonup & 1 & \xrightarrow{p} & \Omega \end{array}$$

Both left squares are pullback squares by the definition of the meet in subobject lattices. The right squares are pullback squares because of (7.7) and (7.8), respectively. Hence, both outer squares are pullbacks. But note that the lower composites of both lower squares are equal by (7.9). Hence, both subobjects $p^*(V_i) \wedge U_i$ and $p^*(V_i) \wedge W$ of $p^*(V_i)$ are classified by the same morphism—i.e., $p^*(V_i) \wedge U_i = p^*(V_i) \wedge W$. It follows that

$$p^*(V_i) \leq U_i \leftrightarrow W.$$

Finally, construct the pullback

$$\begin{array}{ccc} p^*(V_1 \sqcup \cdots \sqcup V_n) & \twoheadrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow p \\ V_1 \sqcup \cdots \sqcup V_n & \twoheadrightarrow & \Omega \end{array}$$

Since in a topos (regular) epimorphisms are pullback-stable, the morphism $p^*(V_1 \sqcup \cdots \sqcup V_n) \rightarrow 1$ in the diagram is an epimorphism. Moreover, recall that also coproducts are pullback-stable (see [159, Lem. 1.51]), and so we have an epimorphism $p^*(V_1) \sqcup \cdots \sqcup p^*(V_n) \twoheadrightarrow 1$. But joins in subobject lattices in a topos are computed by taking the monomorphic part of the image factorisation of the map from the coproduct (see [157, p. 31]); since this map is already an epimorphism, it follows that $p^*(V_1) \vee \cdots \vee p^*(V_n)$ is the top element of H . Hence,

$$1 = \bigvee_{i=1}^n p^*(V_i) \leq \bigvee_{i=1}^n U_i \leftrightarrow W.$$

■

Proposition 7.4.6 (Jibladze’s law over a finite *subset* $\Rightarrow$ étale-finite). Let H be a Heyting algebra and $S \subseteq H$ a finite set such that, for all $x \in H$,

$$\bigvee_{s \in S} (x \leftrightarrow s) = 1.$$

Then H is locally finite, and so the Heyting subalgebra generated by S is finite. Therefore, H is étale-finite.

Proof. Let n be the cardinality of S . The first-order sentence

$$\exists s_1 \dots \exists s_n \forall x \left(\bigvee_{i=1}^n (x \leftrightarrow s_i) = 1 \right) \quad (7.10)$$

holds in H . Moreover, note that this sentence is positive (in the first-order sense: it is built from atomic formulas without use of negation) and hence, by [83, Prop. 5.2.12], it is preserved under surjective homomorphisms. Recall that the top element of a subdirectly irreducible Heyting algebra is completely join-irreducible [102, Prop. A.1.1]; thus, by Lemma 7.2.44, every subdirectly irreducible Heyting algebra satisfying (7.10) has at most n elements. Therefore, every subdirectly irreducible homomorphic image of H has cardinality at most n . Therefore, H has, up to isomorphism, finitely many subdirectly irreducible homomorphic images. By Birkhoff's subdirect representation theorem, H embeds into a product of its subdirectly irreducible homomorphic images. Therefore:

$$\mathbf{Var}(H) = \mathbf{Var}(\{H_0 : H \twoheadrightarrow H_0, H_0 \text{ is s.i.}\}),$$

and we conclude that $\mathbf{Var}(H)$ is finitely generated, i.e., generated by a finite family of finite algebras.

It is a classical result [74, Theorem 10.16] that finitely generated varieties are locally finite. Therefore, the Heyting subalgebra $\langle S \rangle$ of H generated by the finite set S is finite, as well. But note that, for all $x \in H$,

$$\bigvee_{s \in \langle S \rangle} (x \leftrightarrow s) = 1,$$

since the join is taken over a set that extends S . So, by Remark 7.2.32(2), the inclusion $\langle S \rangle \hookrightarrow H$ satisfies Jibladze's law, and so H is étale-finite. ■

We can now conclude the following:

Theorem 7.4.7 (Finitely propositional $\leftrightarrow$ étale-finite). *For a Heyting algebra H , the following are equivalent:*

1. $\mathbf{EsaÉt}(\mathbf{Spec}(H))$ is a topos;
2. H is isomorphic to the lattice of truth values of some finitely propositional topos;
3. H is étale-finite.

Proof. (1) $\Rightarrow$ (2) By Theorem 7.4.4, the category $\mathbf{Esa}\acute{\mathbf{E}}\mathbf{t}(\mathrm{Spec}(H))$ is finitely propositional. By (1), it is also a topos. By Proposition 7.3.24, $\mathrm{Sub}_{\mathbf{Esa}\acute{\mathbf{E}}\mathbf{t}(\mathrm{Spec}(H))}(1) \cong \mathbf{ClopUp}(\mathrm{Spec}(H))$. By Esakia duality, $\mathbf{ClopUp}(\mathrm{Spec}(H)) \cong H$. Thus, H is isomorphic to the lattice of truth values of a finitely propositional topos.

(2) $\Rightarrow$ (3) This follows from Proposition 7.4.5 and 7.4.6.

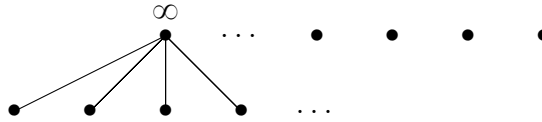
(3) $\Rightarrow$ (1) If H is étale-finite, then by Theorem 7.3.39 $\mathbf{Esa}\acute{\mathbf{E}}\mathbf{t}(\mathrm{Spec}(H))$ is an elementary topos. ■

Remark 7.4.8 (Functoriality?). We have shown that étale-finite Heyting algebras give rise to finitely propositional toposes, and, conversely, the Heyting algebra of truth values of every finitely propositional topos is étale-finite. We leave open whether these assignments can be made functorial, and what categorical relationship, if any, they induce.

From Theorem 7.4.7 we can deduce the following:

Corollary 7.4.9 (Heyting algebras with no finitely propositional topos). *None of the following algebras arises as the lattice of truth values of a finitely propositional topos.*

1. Any infinite bounded chain (such as $\omega + 1$ or $\omega + \omega^{\bar{p}}$).
2. The free Heyting algebra $\mathcal{F}_{\mathbf{HA}}(X)$ over any nonempty set X .
3. The dual of the following Esakia space:



that is, the subalgebra of $\mathcal{P}_{\emptyset, \mathrm{cof}}(\mathbb{N}) \times \mathcal{P}_{\mathrm{fin}, \mathrm{cof}}(\mathbb{N})$ given by the subset

$$\{(A, B) : A = \emptyset \Leftrightarrow B \text{ is finite}\}.$$

Proof. By Example 7.2.47, all infinite bounded chains and all free Heyting algebras over a nonempty set are not étale-finite. Therefore, by Theorem 7.4.7, they cannot arise as the lattices of truth values of a finitely propositional topos. The algebra H in item (3) is not étale-finite, since the projection onto the first coordinate is a surjective homomorphism to $\mathcal{P}_{\emptyset, \mathrm{cof}}(\mathbb{N})$, which is not étale-finite; since étale-finite Heyting algebras are closed under quotients (Remark 7.2.43(1)), it follows that H is not étale-finite. ■

The examples in Corollary 7.4.9 are quite different in nature:

1. The chain $\omega + 1$ is complete, which implies that there is a (localic) topos $\mathcal{E}$ such that $\mathrm{Sub}_{\mathcal{E}}(1) \cong \omega + 1$. For any non-complete bounded chain C (such as $\omega + \omega^{\bar{p}}$), we do not know whether C arises as the lattice of truth values of some elementary topos.

2. A solution for all free algebras would imply a solution for all algebras (because of the filter-quotient construction and the fact that every Heyting algebra is the quotient of a free Heyting algebra).
3. The algebra $\mathcal{P}_{\emptyset, \text{cof}}(\mathbb{N}) \times \mathcal{P}_{\text{fin}, \text{cof}}(\mathbb{N})$ is not étale-finite; however, since it is the product of a complete Heyting algebra and a Boolean algebra, we know that it is topos-admissible.⁸ On the other hand, we do not know whether its subalgebra $\{(A, B) : A = \emptyset \Leftrightarrow B \text{ is finite}\}$ is topos-admissible, although it is similar to the examples studied in this paper (for example, every principal upset in the dual space is finite).

7.5 Étale Heyting algebras and topos-admissible beyond étale-finite

Given the strong results obtained in the previous section, and the limitations outlined in Section 7.4, the reader might want to know what alternative directions can be pursued in studying the Heyting-to-topos problem, and what limitations lie ahead.

We return here to the setting of Section 7.2.3. There we provided a characterization of duals of free algebras, which as we have seen in Example 7.3.27, can be thought of as corresponding to subobject classifiers. On the other hand, Theorem 7.2.29 opens the door for several counterexamples.

Proposition 7.5.1 (Negative results on free étale-Heyting algebras). There exist Heyting algebras H , such that:

1. The dual of the map $\iota: H_0 \rightarrow H_0\langle\iota\rangle$ is a local homeomorphism but not a strict p-morphism;
2. The dual of the map $\iota: H_0 \rightarrow H_0\langle\iota\rangle$ is not an open map. Indeed, it fails to have a right adjoint.

Proof. (1). Consider again the space from Example 7.2.27.

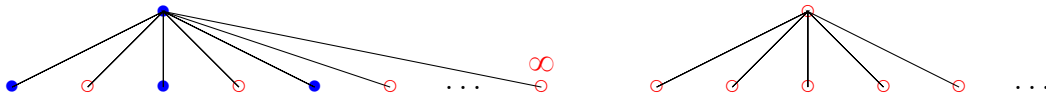
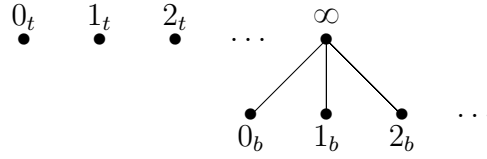


Figure 7.6: The space Ω .

⁸Note that, given two toposes $\mathcal{E}_0$ and $\mathcal{E}_1$, the lattice of truth values of $\mathcal{E}_0 \times \mathcal{E}_1$ is isomorphic to $\text{Sub}_{\mathcal{E}_0}(1) \times \text{Sub}_{\mathcal{E}_1}(1)$ (since the terminal object is simply the pair $(1_{\mathcal{E}_0}, 1_{\mathcal{E}_1})$ and subobjects are computed pointwise).


 Figure 7.7: Space Y

By Theorem 7.2.29 we know that this is the dual of $(\mathcal{P}_{\text{cof}}(\omega))_{\langle \iota \rangle}$. The root map is obviously a local homeomorphism, since one sees that it carries $\llbracket \top \rrbracket$, $\llbracket -\top \rrbracket$ and $\langle \top \rangle \setminus \llbracket \top \rrbracket$ homeomorphically. However it is not a strict p-morphism, since the red point ∞ is not mapped strictly.

(2). Consider the space Y from Figure 7.7. The topology is the finite and cofinite topology, where cofinite subsets include ∞ , and the order is that displayed: points in $\{n_t : n \in \omega\}$ have no successors, whilst points in $\{n_b : n \in \omega\}$ have ∞ as their only successor. Note that Y is a **Sme**-space, since each point has height at most 2.

Computing $\text{CC}_2^*(Y)$, we have that all chains $C_{n_b} = \{(n_b, 0), (\infty, 1)\}$ will be in $\text{Sep}_2(Y)$, and the chain $C_* = \{(\infty, 0), (\infty, 1)\}$ will be their limit, so $C_* \in \text{CC}_2^*(Y)$, and moreover, $C_* \in \langle \top \rangle \setminus \llbracket \top \rrbracket$. Note that $\langle \top \rangle \setminus \llbracket \top \rrbracket$ is a clopen downset. Note that also:

$$r[\langle \top \rangle \setminus \llbracket \top \rrbracket] = \{\infty\} \cup \{n_b : n \in \omega\},$$

since none of the points n_t can be in this image. However this set is certainly not clopen, given it is cofinite. Moreover, this image set is a downset, and so we obtain:

$$Y \downarrow r[\langle \top \rangle \setminus \llbracket \top \rrbracket] = \{n_t : t \in \omega\},$$

which is not a clopen set. Recalling the definition of right adjoints to Heyting homomorphisms (Proposition 2.1.77), we obtain the result. $\blacksquare$

Should Proposition 7.5.1 be taken as spelling doom for continuing the analysis of étale Heyting algebras, and of the Heyting-to-topos problem through them, in geometric terms? The situation appears to be slightly more subtle. As a final note we present two potential lines of analysis which are influenced by the studies done in this thesis.

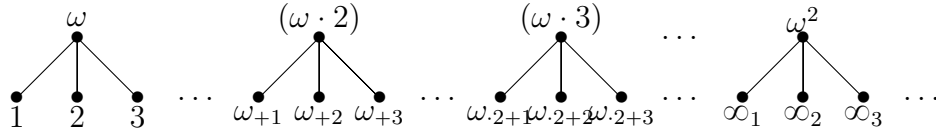
7.5.1 Identifying rich spaces

Noticing that the algebra $\text{ClopUp}(Y)$ is not known to be topos admissible (see Appendix A.5.3), we may wonder whether this could provide a counterexample to the Heyting-to-topos question. However, this does not seem, at first sight, very easy to achieve, for the following reason: even if the algebra $\text{ClopUp}(Y)$ may fail to admit an obvious transformation in a topos.

Taking the space Y as a base, one can construct a larger space Y^+ as depicted in Figure 7.8; this space contains infinitely many copies of the space X , with ω^2 compactifying the points $\omega \cdot n$ on top, and the points ω_n compactifying the sequences $(\omega \cdot k) + n$. Such a space is still an Esakia space, indeed, a **Sme**-space, and Y is a closed upset of Y^+ (as can be seen by the discussion of convergence above). Unlike with the case for Y , $r: \mathbf{CC}_2^*(Y^+) \rightarrow Y^+$ is a local homeomorphism; this follows essentially from the fact that the three subsets $\llbracket \top \rrbracket$, $\llbracket -\top \rrbracket$ and $\langle \top \rangle$ form a partition of $\mathbf{CC}_2^*(Y^+)$, such that r is injective inside of each of them, and it is always easy to verify that r is surjective on the first two subsets. For the third, note that each chain of the form $\{(\omega \cdot n, 0), (\omega \cdot n, 1)\}$ will be the limit of the chains $\{(\omega \cdot n_{-1+k}, 0), (\omega \cdot n, 1)\}$, and hence will appear in the closure of $\text{Sep}_2(Y^+)$. Thus $r[\langle \top \rangle \setminus \llbracket \top \rrbracket]$ is the whole space, and hence, clearly open.

The upshot of this is the following: perhaps the most natural construction does not allow us to obtain a topos containing $\text{ClopUp}(Y)$ as its lattice of truth-values; however, if the same construction works for $\text{ClopUp}(Y^+)$, then by the filter-quotient construction [183, pp.256], we can obtain a topos $\mathcal{E}$ which is still well-understandable geometrically such that $\text{Sub}_{\mathcal{E}}(1) \cong \text{ClopUp}(Y)$.

The precise connection between Y and Y^+ , in the form of a more general result, may therefore be an interesting line of further analysis.

Figure 7.8: Space Y^+

7.5.2 Locally Esakia spaces

In a different line of analysis, note that considering the space $(X, \leq)$ of Example 7.2.41.(3), with the identity map $1_X: X \rightarrow X$, the construction of Definition 7.3.26 can be formulated in exactly the same way, regardless of whether X is étale-finite: namely $(P_{1_X}(X), \leq, \tau)$, still forms an ordered topological space and by a similar argument to that given in Lemma 7.3.30 it satisfies the Priestley separation axiom. More holds:

Proposition 7.5.2 (Power object is locally Esakia). The space $(P_{1_X}(X), \leq, \tau)$ is a locally Esakia space.

Proof. By definition the sets of the form $U \setminus V$ for U, V clopen upsets, form a basis for the topology, so $P_{1_X}(X)$ is an order-zero-dimensional space. It is continuously ordered: to see this, consider the map $t: P_{1_X}(X) \rightarrow \mathbf{CC}_2^*(X)$ defined as follows:

1. If $K = \emptyset$, $t(x, K) = \uparrow x \times \{0\}$;

2. If $K = \uparrow x$, $t(x, K) = \uparrow x \times \{1\}$;
3. If K is a singleton element y above x , $t(x, K) = \{(x, 0), (y, 1)\}$.

Then t defines an homeomorphism onto its image, which is $\mathbf{Sep}_2(X)$ (see Definition 7.2.26). To see why, first partition $P_{1_X}(X)$ into three subsets $\Xi(X)$, $\Xi(\emptyset)$ and $P_{1_X}(X) \setminus (\Xi(X) \cup \Xi(\emptyset))$, which are obviously disjoint and exhaustive. Their images correspond, respectively, to the subsets $\llbracket \top \rrbracket$, $\llbracket -\top \rrbracket$ and $\langle \top \rangle \setminus \llbracket \top \rrbracket$, and are therefore open in $\mathbf{Sep}_2(X)$. The map t is moreover easily seen to be injective, and surjective onto $\mathbf{Sep}_2(X)$. Finally, given a clopen upset of the form $\llbracket (\top \cap W) \cup (-\top \cap V) \rrbracket$ for $W, V \subseteq X$ clopens, $t^{-1}[\llbracket (\top \cap W) \cup (-\top \cap V) \rrbracket] = (\Xi(X) \cap \Theta(W)) \cup (\Xi(\emptyset) \cap \Theta(V))$, which is clopen. Since every clopen upset of $\mathbf{CC}^*(X)$ can be written in this form, and the preimage is a clopen subset, we have that t is continuous. The map preserves the order: whenever $(x, K) \leq (x', K')$, then $K' = \uparrow x' \cap K$; if K is empty, then so is K' , so $t(x, K) \leq t(x', K')$; if $K = \uparrow x$, then $K' = \uparrow x'$ as well, and again $t(x, K) \leq t(x', K')$; and if K is a singleton element y above x , then $K' = \{y\}$ necessarily, and so $t(x, K) = \{(x, 0), (y, 1)\} \supseteq \{(y, 1)\} = t(x', K')$, and the order is restricted in the appropriate way. The order-reflection follows by exactly the same arguments. Thus t is an order-homeomorphism onto $\mathbf{Sep}_2(X)$, which is a dense subset of $\mathbf{CC}_2^*(X)$.

We conclude that $\mathbf{CC}_2^*(X)$ is a Priestley order-compactification of $P_{1_X}(X)$ (Definition 3.3.12).

Now note that t is moreover a p-morphism: by the arguments given for order-preservation, if $t(x, K) \leq (x', K')$, we can determine the shape of (x', K') purely on the basis of (x, K) , and force it to be in the image of t . Thus $(\mathbf{CC}_2^*(X), t)$ is in fact an Esakia order-compactification. It follows from [12, Theorem 4.17] that $P_{1_X}(X)$ is continuously ordered, and it is obvious that it is image-compact, since each upset is finite. Thus it is a locally Esakia space as desired. ■

In light of this fact, and the results of Section 7.4, another possible avenue of study would thus be to drop compactness, working instead with locally Esakia spaces, and to find a suitable strengthening of spectral local homeomorphisms – since in the absence of compactness, these are no longer closed or proper maps, and in fact properness [157, pp. C. 3.2] would be too strong, as it would force compactness on the domain of the local homeomorphism. We leave such questions for further research.

Chapter 8

Conclusions and Further Work

The center of the analysis of this thesis has been the study of free Heyting algebras, their representations, and model theoretic questions which follow from such representations. Though in this thesis we have strived to advance the research into such representations and their model theoretic consequences. Throughout, we have shown how free algebras feature prominently, both directly and indirectly, in many questions of definability in intuitionistic logic.

The work of the previous chapters has outlined many potential “suspects” in our presumed conspiracy: locally Esakia spaces (Chapter 3), Ghilardi complexes (Chapter 4) and étale Heyting algebras (Chapter 7) being the most extensively discussed. Through their analysis, we may say that some smaller level offenders have been cleared from the streets: in Chapters 6 we answered Shehtman’s problem on uniform local tabularity, and reframed Kuznetsov’s problem on local tabularity in these terms, and in 7 we expanded the class of Heyting algebras known to be topos-admissible.

Nevertheless, it is clear we have not gathered enough to take down the *upper-echelons* of such a conspiracy – the open problems on general completeness, topos-admissibility, and local tabularity, outlined in the Introduction, stand unsolved.

Throughout the rest of this chapter, we proceed to summarise concrete problems in this area, and make several remarks on the possible questions and directions taken, as well as potential lines of research following from this.

8.1 Free Heyting algebra

Ghilardi complex constructions

In Chapter 4 the theory of Ghilardi complexes was introduced for several purposes: describing free Heyting algebras; representing other free constructions dually, like products of Esakia spaces and duals of polynomial Heyting algebras; and to define a structural interpretation and an adjoint translation from intuitionistic

logic to its $\rightarrow$ -free fragment. However, we noted that many basic questions about such structures are both unknown and have quite far-reaching consequences.

The following to us is one of the most basic questions in this line of research:

Problem 8.1.1. Given $g: X \rightarrow Y$ a continuous and order-preserving map, when is $\mathcal{V}_g(X)$ an Esakia space? For instance, is this the case assuming that X is an Esakia space?

We note that if the answer to this problem was positive, a simple proof of uniform interpolation of IPC could be carried out by showing that the dual map to the inclusion $i: H \rightarrow H[x]$ has a right adjoint (see Appendix A.3.1). This motivated just as well the interest of the following, broader question:

Problem 8.1.2. Give more general sufficient conditions for an Esakia space X to be G -conservative, or for a pair $g: X \rightarrow Y$ of an Esakia space and an order-preserving continuous map to be G_g -conservative.

8.1.1 Free constructions in other varieties

In Section 5.1.3 we saw how Carai's construction manages to simplify the general construction of free algebras quite substantially. It would therefore be reasonable to ask whether the general representation problem can be improved for any other varieties, and which ones admit such simplifications¹. This appears to be a difficult problem, since even for arbitrary finitely generated varieties it is not clear how one can give this simplified presentation (see Chapter 7 for more discussion on this). This problem can be seen as a form of generalized correspondence problem:

Problem 8.1.3. Does every finitely generated variety admit a simplified representation of free algebras, as in the case of LC and its subvarieties? What if we assume that the logic L corresponding to the variety is axiomatized by Sahlqvist equations?

Problem 8.1.4. Can such a representation moreover be done for n -uniform varieties? If not, are there reasonable conditions ensuring that these representations are possible?

A more theoretical question on this nature would be to try to explain the nature of these representations through some general theory. One direction worth pursuing here lies in the following observation: in the representation of free objects over **S5** due to Bass [29] and Horn [147], the representation boils down to a *Grothendieck fibration*, namely the classical family fibration, over the category of finite sets and surjective maps (see [72, Ch. 5] for general discussion, and [10] for

¹In unpublished work (personal communication), Carai has represented the algebras for **BD**₂ and its subvarieties.

specific discussion of the **S5** case). The case of **LC** shares some, but not all, of these properties, through the construction of “unraveling” of chains. Thus we are tempted to pose the following:

Problem 8.1.5. Provide a theoretical construction which explains the calculation of limits in subvarieties of Heyting algebras, analogous to Grothendieck fibrations.

8.1.2 Coequivalence relations

One domain where Ghilardi complexes could be of especial use is the study of *coequivalence relations*. As laid out in [130, pp. 101], for a given superintuitionistic logic L , these correspond to formulas $\rho(\bar{p}_1, \bar{p}_2)$ which L can prove to be equivalences (possibly in a context $\tau(\bar{p})$). Typical examples of such formulas are the formulas of the form $\theta(\bar{p}_1) \leftrightarrow \theta(\bar{p}_2)$, which are called *separating*, in that they allow defining explicitly the equivalence. In **CPC**, all coequivalence relations are separating. It was shown by Ghilardi and Zawadowski that there exist in **IPC** coequivalence relations which are not separating, which corresponds, categorically, to the fact that $\mathbf{HA}_{\text{fp}}^{\text{op}}$ is not a Barr-exact category.

Problem 8.1.6. Characterize the coequivalence relations over **IPC** which are separating. Categorically, characterize the effective equivalence relations over $\mathbf{HA}_{\text{fp}}^{\text{op}}$.

Beyond the interest of studying coequivalence relations more generally in superintuitionistic logics (see Appendix A.4 for some remarks and basic results) a study of what Ghilardi and Zawadowski called “regular⁺” – the condition that all epimorphisms are of effective descent type – is likewise open. Recent work in this direction [10] in the context of modal logic employed a representation of irreducible **S5**-frames to establish this property. Key to this is the fact that using such representations with irreducible elements, the conditions for effectiveness of equivalence relations coincide with some natural closure conditions on the representation: closure under non-empty unions and downward closure of bisimulations. Thus the following seems like a natural question:

Problem 8.1.7. Is every (regular) epimorphism in $\mathbf{HA}_{\text{fp}}^{\text{op}}$ of effective descent type? If not, what is a characterization of this class of epimorphisms?

We expect that the representations of Ghilardi complexes can be useful in deriving the necessary and sufficient conditions for effectiveness, along the lines of closure under a union-like condition, and might be paired with the flexible sheaf-theoretic representation to prove positive results.

8.2 First-order theory of free Heyting algebras

In this thesis we did not touch on much of the existing theory studying the first-order theory of free Heyting algebras. Nevertheless, we point here to several interesting open questions, and their connections to the questions of this thesis.

Admissibility and Unification

As noted in the introduction, it has been known since the work of Grzegorzcyk [142] and Rybakov [236] that the full first-order theory of the free Heyting algebras is undecidable. Nevertheless, over the years fragments of such a theory have been shown to be decidable: the essential result of [234] is that the universal Horn theory is decidable, as well as an extension to the full universal theory [152]. Typically this kind of problem manifests logically in the study of “admissible rules”, which through the work of Rybakov correspond to the quasi-equations or universal formulas valid in the free algebra (see [235] for a general discussion).

In [17], the author and Silvio Ghilardi studied a unification problem closely related to the admissibility of so-called Π_2 -rules; such rules trace their origin to the work of Gabbay [121] in axiomatizing classes of Kripke models, and Takeuti-Titani [248] in axiomatizing classes of Gödel algebras, and were subsequently extended as a framework by several authors [254, 42, 55, 5]. The main result of [17] reduces the problem of decidability of the formulas associated to a Π_2 -rule to the problem of decidability of usual rules, exploiting the uniform interpolation of IPC. In the context of that work, the problem of unification that arose turned out to be a special instance of the general problem of unification with linear constant restrictions (see e.g. [19]). It would therefore be interesting to understand whether the full first-order positive Horn theory of the algebra $\mathcal{F}_{\mathbf{HA}}(\omega)$ can be described, whether it is decidable, and to formulate it as a unification problem with linear constant restrictions.

Problem 8.2.1. Is the first-order positive Horn theory of $\mathcal{F}_{\mathbf{HA}}(\omega)$ decidable?

We note that such questions concerning Π_2 -rules have connection to the theory of étale Heyting algebras of Chapter 7: as noted there, the natural closure conditions on classes of Heyting algebras axiomatized by Π_2 -rules correspond closely to $\forall$ -homomorphisms. Thus the following appears as a related problem:

Problem 8.2.2. Give general conditions for a homomorphism $f: H \rightarrow H'$ to preserve the $\forall$ -theory (or the Π_n -theory more broadly) of the Heyting algebra H .

8.2.1 First-order completions

In a different direction it would be interesting to study how the first-order theory of free Heyting algebras can be characterized amongst the first-order theories of

Heyting algebras. It is well-known that the theory of atomless Boolean algebras corresponds to the theory of the model completion of Boolean algebras. In [130, Ch. 7], Ghilardi and Zawadowski studied the model completion of Heyting algebras, proving the surprising result that such a theory has two countable, locally finite and finitely subdirectly irreducible models². Darnière and Junker [93] and Darnière [92] have provided axiomatizations of the model-completions of all varieties of Heyting algebras which have such a model completion – however, these are still somewhat unexplored. The following seems to us a basic problem in this direction:

Problem 8.2.3. Characterize the ω -categorical theories of Heyting algebras (or locally finite Heyting algebras). Contrast them with the complete theories in the same varieties.

8.3 Uniform local tabularity

In Chapter 6, we introduced the study of uniform local tabularity, and remarked its connections to the study of local tabularity, a longstanding open problem in intuitionistic logic. One natural line of research here would be to try to understand the extent to which uniform local tabularity is a simpler concept than local tabularity. In particular, it is interesting in this respect to study the relationship between this and Maksimova’s ω -operation studied in [189]. It might be informative to try to axiomatize the n -uniform logics for $n > 3$, and the example of LFC can serve for this as a guiding light, though we expect that such a task might be combinatorially quite challenging. In a structural sense, the most interesting problem would be to analyze pre-uniform local tabularity above logics larger than KG – for example, BW₂, the logic of bounded width 2:

Problem 8.3.1. Is uniform local tabularity above BW₂ decidable? What about over IPC? Is the logic Box the unique pre-uniformly locally tabular extension of IPC?

8.4 Étale algebras and toposes

The connection between Heyting algebras and toposes has been at the heart of the logical uses of toposes, since the constructive nature of Heyting algebras leads to toposes allowing general models of mathematics beyond set theoretic ones (see e.g. [183, Sec. 7.8, Ch. 10]). Nevertheless, the Heyting-to-topos question reveals that, in many ways, this connection is less well-understood than we would hope.

²The authors use the term “prime” to refer to subdirectly irreducible algebras, but this unfortunately clashes with the notion of a prime model in model theory.

The Heyting-to-topos question remaining wide open, it would be interesting to obtain more general results concerning topos-admissible Heyting algebras along the lines of those obtained in Chapter 7: to understand what are the precise conditions necessary to have a Heyting algebra arise as the lattice of truth-values of a topos. A potentially valuable avenue of research in this sense would be to study the model-theoretic content of topos-admissibility:

Problem 8.4.1. Following the known positive answers, find model-theoretic conditions (similar to the existence of model-completions or uniform elimination of imaginaries) which are necessary and sufficient for a Heyting algebra H in general, or a free Heyting algebra $\mathcal{F}_L(\omega)$ for L a superintuitionistic logic to be topos-admissible.

In this context, we expect that the framework of étale Heyting algebras in general, and internal H -calculi can prove to be a useful context to discuss such problems, especially when dealing with general Heyting algebras.

8.5 Definability and Geometry

A running thread throughout the present thesis is that intrinsic to Heyting algebras and to non-classical logical systems more broadly, there is a form of *geometry* which corresponds to definability. Duality theory is one way in which we can explore such a geometry, and its tradition witnesses the way in which one can give precise geometric meaning to algebraic axioms, using ordered structures. But the questions which we answered in this thesis, and those which remain open above, reveal that the nature of this geometry is much finer. The exact nature of this geometry, and the way in which can be fleshed out logically, is to us the heart of the “conspiracy” which we alluded to in the prologue of this thesis.

In this final section, we outline some broader and more speculative directions, drawn from ongoing work with Matteo de Berardinis and Lingyuan Ye. We hope the reader may be inspired to pursue some of these ideas, and to try to develop a solid theory with which one can advance the study of superintuitionistic logics and non-classical logics more broadly.

At the heart of the question of definability and geometry, in our view, lie a few mysteries, which we briefly outline:

1. *Spectral coincidences*: It is not clear *why* distributive lattices and commutative rings should both be so intimately related to spectral spaces (see e.g. [96]); and consequently, why Heyting algebras, which are deeply asymmetric structures, should carry such a wealth of geometric structure inherent in them, which often bridges logical meaning with spatial intuition. This was discussed at length in [96], and some research seeking to precisely untangle the nature of the connections has been initiated by Guram Bezhanishvili and Marcus Tressl [53].

2. *Overlapping combinatorics*: consider (1) the definability of second-order quantifiers as in uniform interpolation, (2) Beth definability, (3) the definability of bi-Heyting structure on Heyting algebras, (4) the definability of KM-structures, and (5) the conservativity of infinitary connectives over finitary ones. All of these have some claim to being issues of definability, yet to our knowledge no framework exists which would explain in what way they are all definable. The work of Ghilardi and Zawadowski [130] on one hand, Esakia, Patariaia and Jibladze [108], as well as many others, point in the direction of studying such definability questions using deeper methods borrowed from algebraic geometry and K -theory.
3. *Geometry and topology out of sight*: in addition to the above points, it is an empirical observation that a number of methods pursued in the study of superintuitionistic logics coincide with techniques appearing in the algebraic topology of posets [170]: beat points reductions are exactly α -reductions, the class of Boolean sums and the notion of “stack” studied in Chapter 6 appear verbatim [170, Def. 10.7], and the natural classes of posets considered for intuitionistic logic, such as locally linear posets, appear naturally also in such settings (as face posets of simplicial complexes, for instance).

In our view, understanding which of these are mathematical accidents and which are deep connections, leads us to two natural and interrelated directions.

The first, seeks to flesh out a way of explaining the overlapping combinatorics as part of a central phenomenon, integral to the Grothendieck analysis of commutative rings, and contemporary algebraic geometry and algebraic topology: placing homotopical, homological and cohomological invariants at the heart of the meaning of *definable*. We thus phrase the following problem:

Problem 8.5.1. Develop a theory of definability which equates definability with a homological, cohomological or homotopical criterion.

Note that developing such a theory is not a trivial task, since Heyting algebras are quite far from the usual homological categories [71], lacking anything like a zero object, and most crucially, being strongly non-abelian; it also lacks enough injectives in as strong terms as possible [22]. The failures of exactness showed by [130] also point to the need for careful adaptation of such techniques from abelian settings. In related settings, recent work by Reggio [228] has shown that there is promise in considering homotopical methods in modal logic.

Ultimately, we believe that such theories could provide a unifying explanation for many phenomena discussed in this thesis – like uniform interpolation, interpolation, or topos-admissibility – as well as others which have not been discussed, such as Ruitenburg’s theorem of fixed point elimination, or construction of solutions for unification problems. All of these rely on special normal forms of IPC, and on cleverly chosen combinatorics of k -bisimulations, which from the point of view of this thesis coincide.

In place of chain complexes indexed by simplices, constructions like Ghilardi complexes – though ideally with reasonable simplifications – which carry both a strong geometric and a strong game-theoretic nature, can hopefully serve the role of the fundamental invariants out of which one builds structures analogous to fundamental groups. A concrete avenue for this research is to take a Grothendieck approach³, identifying formulas with “affine varieties” or “affine schemes”, and find criteria analogous to Serre’s affineness criteria [143, Thm. 3.7] out of the sheaf cohomology present in such Grothendieck toposes. In a logical framework, that would provide a general set of conditions that characterize definability. Note also that in this setting, hypotheses like uniform local tabularity become analogous to Noetherianity assumptions. Such a setting also has the interest of allowing us to uniformly discuss *undefinability* results – including failures of uniform interpolation [131], fixed-point elimination [180], or undefinability of higher-order connectives [249, 221, 165].

Such a study is closely related to the context of the *categorical Galois theory* of Heyting algebras [72]. Thus we pose the following problem:

Problem 8.5.2. Develop the Galois theory of finite posets with p-morphisms, and construct Galois-theoretic invariants for the study of intuitionistic logic.

This relates to the question raised above in Section 8.1.2, about effective descent morphisms, since for such a study, the effective descent epimorphisms constitute the basic class for which a Galois theoretic analysis can be pursued. The interest of such a study would be in providing new powerful invariants for studying varieties, quasivarieties and other classes of algebras: similarly to how Galois theory characterizes the existence of closed-form solutions by radicals in terms of certain purely group-theoretic properties, one could hope that it should be possible to characterize the existence of logical solutions to formulas in extended languages (such as with quantifiers, fixed points, or additional operations) by way of characterizing the Galois groupoids of the algebras at hand.

The hope would be that making precise these geometric meanings, in whatever form they may assume, can bring to light enough data to ultimately solve the problems which we discussed in this thesis: most notably, the Heyting-to-topos problem, and Kuznetsov’s problem on the general completeness of complete Heyting algebras.

³We should disclaim for the expert reader that whilst in [130, Ch. 7] Ghilardi and Zawadowski remarked that the usual covering Grothendieck topology on the site $\mathbf{HA}_{\text{fp}}^{\text{op}}$ is not subcanonical [130, Ex. 8], if one restricts to *finite covers*, this forms a subcanonical topology, as remarked in the same exercise. Moreover, the spectral local homeomorphisms discussed in Chapter 7, and the usual open immersion maps typical of the Zariski topology all form natural Grothendieck topologies which can be studied on this site.

In this appendix we collect a number of results, observations and remarks. Many of these results collected are of a folklore nature, despite seemingly never having appeared in print. Some are small results which whilst interesting would distract a bit from the nature of the main discussion. Others are simplified proofs of known results, which we include here to make such results more widely known and accessible.

A.1 Free distributive lattices

It is a folklore fact that free distributive lattices are bi-Heyting algebras. The following algebraic proof originally appeared in [11, Lem. 2.7]:

Proposition A.1.1 (Free distributive lattices are bi-Heyting). $\mathcal{F}_{\mathbf{DL}}(X)$ is a (bi-)Heyting algebra for each set X . Thus, the variety $\mathbf{DL}$ is $\mathbf{HA}$ -($\mathbf{biHA}$ -)anarchic.

Proof. Let $P, Q \in \mathcal{F}_{\mathbf{DL}}(X)$. We show that the implication $P \rightarrow Q$ exists in $\mathcal{F}_{\mathbf{DL}}(X)$. By [21, p. 89, Thm. 7], $P = \bigvee_{i=1}^n S_i$, where each S_i is join-irreducible, and $Q = \bigwedge_{j=1}^m T_j$, where each T_j is meet-irreducible (see Definition 2.1.18). It is sufficient to show that each $S_i \rightarrow T_j$ exists in $\mathcal{F}_{\mathbf{DL}}(X)$ since then by basic properties of the relative pseudocomplement, we have:

$$\begin{aligned} P \rightarrow Q &= (\bigvee_{i=1}^n S_i) \rightarrow (\bigwedge_{j=1}^m T_j) \\ &= \bigwedge_{i=1}^n (S_i \rightarrow \bigwedge_{j=1}^m T_j) \\ &= \bigwedge_{i=1}^n \bigwedge_{j=1}^m (S_i \rightarrow T_j). \end{aligned}$$

We show that

$$S_i \rightarrow T_j = \begin{cases} 1 & \text{if } S_i \leq T_j; \\ T_j & \text{otherwise.} \end{cases}$$

Note that if $S_i \leq T_j$ certainly $S_i \rightarrow T_j = 1$. So assume that $S_i \not\leq T_j$. We need to show that for each $R \in \mathcal{F}_{\mathbf{DL}}(X)$, $S_i \wedge R \leq T_j$ if and only if $R \leq T_j$. One direction is obvious: if $R \leq T_j$ then $S_i \wedge R \leq T_j$. So assume that $S_i \wedge R \leq T_j$. Since T_j is meet-irreducible, and we assume that $S_i \not\leq T_j$, this implies that $R \leq T_j$. Thus, $S_i \rightarrow T_j = T_j$, showing that $\mathcal{F}_{\mathbf{DL}}(X)$ is a Heyting algebra.

To see that **DL** is **HA** (even **biHA**) anarchic, we recall that all Heyting algebras are distributive lattices (Theorem 2.1.15), and distributive lattices admit unique Heyting algebra structures (by uniqueness of adjoints). The facts for bi-Heyting algebras follows by dualizing all statements with respect to the order. ■

A.2 Topological and combinatorial structures

A.2.1 Vietoris spaces

The following lemma shows that beyond being Priestley spaces, the Vietoris spaces carry an additional Esakia structure. Such a lemma was used in [8] to prove some key facts about the functor $\mathcal{V}_{\text{Max}}$, constructing the dual representation of the adjoint to the Booleanisation functor, as was described in Section 5.3.2.

Lemma A.2.1 (Vietoris is Esakia). Let $(X, \leq)$ be a Priestley space. Then $(\mathcal{V}(X), \supseteq)$ is an Esakia space.

Proof. Given a clopen subset of $\mathcal{V}(X)$, note that because of the subbasis of this space, we can write such a clopen as a union of sets of the form:

$$W = \llbracket U \rrbracket \cap \langle V_0 \rangle \cap \cdots \cap \langle V_n \rangle,$$

where U, V_i are clopen subsets of X . To show that $\mathcal{V}(X)$ is an Esakia space, it thus suffices to show that $\downarrow W$ is clopen for any W in the above shape (since $\downarrow$ commutes with unions). Supposing that W is as given, we consider the following subset:

$$W_0 := \bigcap_{i=1}^n \langle U \cap V_i \rangle.$$

If $C \in \mathcal{V}(X)$ belongs to W_0 , then it contains a subset $D = \{x_1, \dots, x_n\}$, where each $x_i \in V_i \cap U$; so $D \subseteq U$, it is a closed subset, and intersects each of the sets, and so $C \in \downarrow W$. Conversely, if $C \in \downarrow W$, then there exists some $C \supseteq D$, where $D \subseteq U$, and $D \in \langle V_i \rangle$. This means that for each i there is some $x_i \in D$, which by the assumption will be in $U \cap V_i$. Thus $C \in W_0$.

Since $\langle U \cap V_i \rangle$ are subbasic clopen subsets of X , we get that $\downarrow W$ is a clopen, as desired. ■

A.2.2 Combinatorics of the universal model

It was observed by Bellissima [30] that the combinatorial structure of the n -universal model for $n > 1$ is quite involved. Most of these difficulties seem to arise out of the existence of infinite antichains. We give an example of constructing such antichains, with a slightly different argument to that given in [30, Lemma 4.1].

Notation A.2.1 (Right combinatorial properties). To simplify the following statements we will simply write *the right combinatorial properties* for a logic L which is closed under the kinds of frames employed by the proof. This is always met by IPC, but often is much more general.

Lemma A.2.2 (Infinite antichain in universal model). Let $n > 1$. Let L be a logic with the FMP with the right combinatorial properties. Then the poset $\mathcal{U}(n)$ contains an infinite antichain.

Proof. Let x, y be two maximal points such that $c(x) = 1^n$ and $c(y) = 1^{n-1}0$. Then we can recursively construct a poset isomorphic to the Rieger-Nishimura ladder such that x, y are the two maximal points, and each point has color equal to $c(y)$. Note that in $\mathcal{U}_L(n)$, the points in such a recursive construction are precisely those which validate the formula $\bigwedge_{i=1}^{n-1} p_i$, and hence this upset, call it P is definable.

For each level k , let a_k, b_k be the two elements in P . Since $n > 1$, for each k , there exists w_k which has a_k as its only immediate successor (note that $c(a_k) > 0^n$). Then we can consider the set:

$$\{w_k : k \in \omega\}.$$

This is an antichain, since each of the w_k has as its immediate successors elements which never see w_i for any i . Moreover, it is infinite because P itself is infinite. ■

The following pair of lemmas is specific to IPC:

Lemma A.2.3 (Completely meet-irreducible are definable and join-irreducible). For each $n > 1$, and $x \in \mathcal{U}(n)$, the set $\mathcal{X}(n) \setminus \downarrow x$ is a rooted clopen upset.

Proof. By [30, Corollary 2.8], the subsets $\mathcal{U}(n) \setminus \downarrow x$ are definable, and are completely meet-irreducible. Thus the same formula defines in $\mathcal{X}(n)$ the subset $\mathcal{X}(n) \setminus \downarrow x$.

The fact that completely meet-irreducible elements are join-irreducible was observed in [30, Thm. 3.1]. By Proposition 2.1.66, such a join-irreducible element corresponds to a rooted clopen upset in $\mathcal{X}(n)$. ■

We will need the following lemma originally due to Urquhart [251]; this can be proved easily by employing the results of Chapter 4:

Lemma A.2.4 (Finite minimum lemma). Let U be a clopen upset in $\mathcal{X}(n)$. Then $\min(U)$ is finite.

Proof. Note that by Proposition 4.3.4, the algebra $\mathcal{F}_{\mathbf{HA}}(n)$ is generated by join-irreducible elements. Hence, given a clopen upset $U \subseteq \mathcal{X}(n)$, we have that $U = V_1 \cup \dots \cup V_n$ where V_i is a clopen upset which is join-irreducible. By Proposition 2.1.66, V_i is rooted. Thus, $\min(U)$ corresponds to the roots of the clopen upsets V_i , and is finite. ■

Proposition A.2.5 (Wicked combinatorics). If $n > 1$ and L is a logic with the FMP with the right combinatorial properties, we have:

1. The algebra $\mathbf{Up}(\mathcal{U}_L(n))$ is uncountable, thus $\mathcal{F}_L(n) \not\cong \mathbf{Up}(\mathcal{U}_L(n))$.
2. There exist two elements $a, b \in U(n)$ such that $\downarrow a \cap \downarrow b$ has infinitely many maximal elements.
3. The free algebra $\mathcal{F}_L(n)$ has an infinite descending chain.
4. The n -universal model $\mathcal{U}_{\mathbf{PC}}(n)$ contains an infinite ascending chain.

Proof. In light of Lemma A.2.2, fix $W = \{w_n : n \in \omega\}$ an infinite antichain, as constructed there.

(1) This it follows easily: given $S \subseteq W$, we can consider $\uparrow S$, and obtain that for $S \neq S' \subseteq W$, $\uparrow S \neq \uparrow S'$. Since W is infinite, there are uncountably many upsets, whilst $\mathcal{F}_L(n)$ is countable.

For (2), look at $S_0 = \downarrow w_0$ and $S_1 = \downarrow w_1$. We claim that $S_0 \cap S_1$ has infinitely many elements in the maximum. To see this, note that by construction of $\mathcal{U}_L(n)$, for each $i > 1$, there is at least one point v_i such that $v_i < \{w_0, w_1, w_i\}$. Now note that $v_i \in S_0 \cap S_1$ for each i . Moreover, for each successor m of v_i , $m \notin S_0 \cap S_1$, and $v_i \not\leq v_j$ whenever $i \neq j > 1$.

(3) is [30, Corollary 4.3]; we prove it for completeness. Consider the following sequence: $S_0 = \mathcal{X}(n)$, and $S_{n+1} = S_n \setminus \downarrow w_n$. Then $S_0 \supseteq S_1 \supseteq S_2 \supseteq \dots$ is an infinite descending chain, since $w_{n+1} \in S_n - S_{n+1}$ for each n .

(4). Consider the sequence $S_0 \supseteq S_1 \supseteq S_2 \supseteq \dots$. It is well-known that $\mathcal{X}(n) = S_0$ is rooted, and moreover, that S_n are all rooted by Lemma A.2.3. Now consider

$$M = \bigcup_{i \in I} \min(S_i);$$

Note that since the chain is proper, and by Lemma A.2.4, $\min(S_i)$ is finite, M must be infinite: otherwise, if M was finite, for some k onwards, $\min(S_k) = \min(S_{k+1})$, which would be a contradiction, since then $S_k = S_{k+1}$.

Since S_0 is rooted, in particular, we must have that r_0 , the root, cannot be in $\min(S_1)$, which is finite. Since $M \supseteq \uparrow \min(S_1)$, and the latter is finite, there must

be some element x_1 which sees infinitely many elements from M . Note that $\uparrow x_1$ is clopen as well, so we can consider:

$$S'_n = \uparrow x \cap S_n;$$

and we will obtain that this is still an infinite descending chain, all the elements are rooted and clopen. Hence we can proceed with the construction as just outlined, constructing an infinite ascending chain. ■

As a simple corollary of these structural results, we obtain the following result by Esakia (announced without proof in [104]):

Proposition A.2.6 (Free algebras for $n > 1$ are not KM). For each $n > 1$, the algebra $\mathcal{F}_{\mathbf{HA}}(n)$ is not a KM Heyting algebra.

Proof. By Proposition 2.1.61, it suffices to show that there is a clopen upset U in $\mathcal{X}(n)$ such that $U \cup \max(\mathcal{X}(n) \setminus U)$ is not clopen. Consider the upset P constructed in Lemma A.2.2, which is definable (by the formula $\bigwedge_{i=1}^{n-1} p_i$). Thus, the clopen upset representing the same formula in $\mathcal{X}(n)$ contains P . If $\mathcal{X}(n)$ was a KM-space (Definition 2.1.60), we would have that

$$\tau(P) = P \cup \max(\mathcal{X}(n) \setminus P),$$

would be a definable upset; however, note that:

$$\{w_k : k \in \omega\} \subseteq \min(\tau(P)),$$

since each point w_k has no predecessors which lie in the same set. This means that $\min(\tau(P))$ is infinite; by the finite minimum Lemma A.2.4, this is a contradiction to $\tau(P)$ being clopen. ■

Remark A.2.7 (bi-Heyting and KM). For a locally finite variety $\mathbf{V}$, all the free n -generated algebras will be KM Heyting (since they are finite, Example 2.1.40). Except for $L = \mathbf{CPC}$, the algebra $\mathcal{F}_L(\omega)$ is never a KM Heyting algebra: otherwise, since all countable L -algebras are homomorphic images of it, by Proposition 2.1.41, all countable L -algebras would be KM Heyting algebras, which we noted only holds for $\mathbf{CPC}$ (see Remark 2.1.42).

By contrast, we have seen in Section 4.3 that even ω -generated Heyting algebras are bi-Heyting algebras. It was shown by Carai [79, Sec. 5] that despite all n -generated Gödel algebras of bounded depth k being bi-Heyting (as they are finite), for $k > 1$, the free Gödel algebras of depth k on ω -many generators are not bi-Heyting.

It would be interesting to understand for which superintuitionistic logics L the algebras $\mathcal{F}_L(n)$ are KM, for which they are bi-Heyting, and what is the general structure of such properties.

We finish with a different, lattice-theoretic, negative result:

Proposition A.2.8 (Free algebra for $n > 1$ is not meet-irreducibly generated). For $n > 1$ the algebra $\mathcal{F}_{\mathbf{HA}}(n)$ is not meet-irreducibly generated.

Proof. Assume towards a contradiction that every element of $\mathcal{F}_{\mathbf{HA}}(n)$ is equivalent to a meet of meet-irreducible elements. Then:

$$\mathcal{F}_{\mathbf{HA}}(n) \subseteq \langle \{\mathcal{U}(n) \setminus \downarrow x : x \in \mathcal{U}(n)\} \rangle \subseteq \text{Def}(\mathcal{U}(n)),$$

that is the free algebra is generated by the meet-irreducible elements. By Lemma 3.4.18.(1) we have that $\mathcal{U}(n) \setminus \downarrow x$ can be generated, by one layer of implications, from principal upsets of the n -universal model. Consequently:

$$\langle \{\mathcal{U}(n) \setminus \downarrow x : x \in \mathcal{U}(n)\} \rangle \subseteq \text{Bel}(n);$$

however, this contradicts the fact that the Bellissima algebra does not coincide with $\mathcal{F}_{\mathbf{HA}}(n)$ (Proposition 3.4.13). ■

A.2.3 A characterisation of étale-finite Esakia spaces

We provide here a general characterization of étale-finite Esakia spaces which does not rely on the existence of strict p-morphisms, and hence, can be verified directly on the spaces themselves.

Proposition A.2.9 (Characterisation of étale-finite Esakia spaces). An Esakia space X is étale-finite if and only if the following conditions hold:

1. (Locally finite) for every $x \in X$, the principal upset $\uparrow x$ is finite.
2. (Types are clopen) for every finite poset P , the set

$$X_P := \{x \in X : \uparrow x \cong P\}$$

is clopen.

Proof. ($\Rightarrow$) Suppose $\acute{e}: X \rightarrow X_0$ is a continuous strict p-morphism with X_0 finite. For each $x \in X$, the restriction of $\acute{e}$ gives $\uparrow x \cong \uparrow \acute{e}(x)$; hence, each principal upset is finite. Moreover, for a finite poset P ,

$$X_P = \bigcup_{\{x_0 \in X_0 : \uparrow x_0 \cong P\}} \acute{e}^{-1}[\{x_0\}],$$

which is clopen because X_0 is finite (and hence discrete) and $\acute{e}$ is continuous.

($\Leftarrow$) For each $n \in \mathbb{N} \setminus \{0\}$, set

$$X_n := \{x \in X : |\uparrow x| = n\}.$$

Each X_n is clopen: indeed, it is the finite union of the sets X_P , with P ranging over representatives of the finitely many isomorphism classes of posets of cardinality n .

Claim A.2.10. There is a finite clopen partition

$$B_1, \dots, B_r$$

of X such that, for every $x \in X$, distinct points of $\uparrow x$ lie in distinct blocks of the partition.

Proof of the claim. Let $x \in X$, and write $n := |\uparrow x|$. Since the underlying topological space of a Priestley space is a Stone space, we may choose pairwise disjoint clopen neighbourhoods

$$U_y^x \quad (y \in \uparrow x)$$

such that $y \in U_y^x$ for each $y \in \uparrow x$. Define

$$\mathcal{O}_x := X_n \cap \square \left(\bigcup_{y \in \uparrow x} U_y^x \right) \cap \bigcap_{y \in \uparrow x} \downarrow U_y^x.$$

Then $\mathcal{O}_x$ is clopen. Indeed, X_n is clopen, the set $\bigcup_{y \in \uparrow x} U_y^x$ is clopen, and both $\square W$ and $\downarrow W$ are clopen whenever W is clopen in an Esakia space.

Moreover, $x \in \mathcal{O}_x$. Thus the family $(\mathcal{O}_x)_{x \in X}$ covers X , and hence, by compactness, there are $x^1, \dots, x^m \in X$ such that

$$X = \mathcal{O}_{x^1} \cup \dots \cup \mathcal{O}_{x^m}.$$

Collect the finitely many clopen subsets

$$U_y^{x^a} \quad (1 \leq a \leq m, y \in \uparrow x^a),$$

and let $B_1, \dots, B_r$ be the finite clopen partition of X generated by them.

We show that this partition has the desired property. Let $z \in X$, and choose a such that $z \in \mathcal{O}_{x^a}$. Set

$$n := |\uparrow x^a|.$$

Since $z \in X_n$, we have $|\uparrow z| = n$. Moreover, by the definition of $\mathcal{O}_{x^a}$, the set $\uparrow z$ is contained in

$$\bigcup_{y \in \uparrow x^a} U_y^{x^a}$$

and meets each of the pairwise disjoint clopens $U_y^{x^a}$. Since both $\uparrow z$ and $\uparrow x^a$ have cardinality n , the set $\uparrow z$ meets each $U_y^{x^a}$ in exactly one point. Hence, any two distinct points of $\uparrow z$ are separated by one of the chosen clopens, and therefore lie in distinct blocks of the generated partition. This proves the claim. $\blacksquare$

We now construct the finite target poset. For $x \in X$, define

$$S_x := \{i \in \{1, \dots, r\} : \uparrow x \cap B_i \neq \emptyset\}.$$

By Claim A.2.10, for each $i \in S_x$ there is a unique point

$$x_i \in \uparrow x \cap B_i.$$

Define a partial order on S_x by

$$i \leq_x j \iff x_i \leq x_j.$$

Let

$$q_x := (S_x, \leq_x).$$

Thus, q_x is a finite poset, and the map

$$\begin{aligned} \uparrow x &\longrightarrow q_x \\ y &\longmapsto \text{the unique } i \text{ such that } y \in B_i \end{aligned}$$

is an order-isomorphism.

Let

$$Q := \{q_x : x \in X\}.$$

This is finite, since there are only finitely many partial orders on subsets of $\{1, \dots, r\}$.

If $q = (S, \leq_q) \in Q$ and $i \in S$, write

$$q_{\geq i} := \{j \in S : i \leq_q j\},$$

with the induced order. Notice that $q_{\geq i} \in Q$: indeed, if $q = q_x$ and $y \in \uparrow x$ is the unique point lying in B_i , then

$$q_y = (q_x)_{\geq i}.$$

Define an order on Q by declaring, for $q = (S, \leq_q)$,

$$q \leq_Q q' \iff q' = q_{\geq i} \text{ for some } i \in S.$$

This is a partial order. Reflexivity follows because each q_x has a least element, namely the block containing x . Transitivity is immediate. For antisymmetry, if $q \leq_Q q'$ and $q' \leq_Q q$, then the underlying set of q' is contained in the underlying set of q , and conversely. Hence, the two underlying sets are equal; since principal upsets carry the induced order, it follows that $q = q'$.

We regard Q as a finite Esakia space with the discrete topology. Define

$$\begin{aligned} \acute{e} : X &\longrightarrow Q, \\ x &\longmapsto q_x. \end{aligned}$$

We prove that $\acute{e}$ is continuous. Since Q is finite and discrete, it suffices to show that each fibre of $\acute{e}$ is clopen. For $i \in \{1, \dots, r\}$, put

$$D_i := \{x \in X : \uparrow x \cap B_i \neq \emptyset\} = \downarrow B_i.$$

This is clopen because X is Esakia. For $i, j \in \{1, \dots, r\}$, put

$$E_{ij} := \{x \in X : \text{there are } y \in \uparrow x \cap B_i \text{ and } z \in \uparrow x \cap B_j \text{ with } y \leq z\}.$$

Equivalently,

$$E_{ij} = \downarrow(B_i \cap \downarrow B_j).$$

Hence E_{ij} is clopen, since B_j is clopen, $\downarrow B_j$ is clopen, and X is Esakia.

Fix $q = (S, \leq_q) \in Q$. Then

$$\acute{e}^{-1}[\{q\}] = \bigcap_{i \in S} D_i \cap \bigcap_{i \notin S} (X \setminus D_i) \cap \bigcap_{\substack{i, j \in S \\ i \leq_q j}} E_{ij} \cap \bigcap_{\substack{i, j \in S \\ i \not\leq_q j}} (X \setminus E_{ij}).$$

Indeed, the first two intersections express exactly that the set of colours appearing above a point is S , while the last two intersections express exactly that the order between those colours is $\leq_q$. Therefore, each fibre of $\acute{e}$ is clopen, and so $\acute{e}$ is continuous.

It remains to show that $\acute{e}$ is a strict p-morphism. Let $x \in X$, and set $q := \acute{e}(x)$. Consider the map

$$\begin{aligned} \alpha_x: \uparrow x &\longrightarrow q \\ y &\longmapsto \text{the unique } i \text{ such that } y \in B_i. \end{aligned}$$

By construction, α_x is an order-isomorphism. Also, the map

$$\begin{aligned} \beta_q: q &\longrightarrow \uparrow_Q q \\ i &\longmapsto q_{\geq i} \end{aligned}$$

is an order-isomorphism. It is surjective by the definition of $\leq_Q$. It is injective because, if $q_{\geq i} = q_{\geq j}$, then $i \in q_{\geq j}$ and $j \in q_{\geq i}$, hence $j \leq_q i$ and $i \leq_q j$, and therefore $i = j$.

We now check that β_q preserves and reflects the order. If $i \leq_q j$, then $q_{\geq j} = (q_{\geq i})_{\geq j}$, and hence $q_{\geq i} \leq_Q q_{\geq j}$. Conversely, if $q_{\geq i} \leq_Q q_{\geq j}$, then

$$q_{\geq j} = (q_{\geq i})_{\geq k}$$

for some $k \in q_{\geq i}$. Thus $q_{\geq j} = q_{\geq k}$, and by injectivity $j = k$. Since $k \in q_{\geq i}$, it follows that $i \leq_q j$.

For every $y \in \uparrow x$ we have

$$\acute{e}(y) = \beta_q(\alpha_x(y)).$$

Hence

$$\acute{e}|_{\uparrow x}: \uparrow x \longrightarrow \uparrow_Q \acute{e}(x)$$

is an order-isomorphism.

By Remark 7.2.36, $\acute{e}$ is a strict p-morphism. Thus $\acute{e}: X \rightarrow Q$ is a continuous strict p-morphism to a finite Esakia space. Therefore, X is étale-finite. $\blacksquare$

Remark A.2.11 (Étale-finite $\Leftrightarrow$ principal upsets are finite and locally constant). The characterisation in Proposition A.2.9 immediately entails the following similar characterisation:

An Esakia space X is étale-finite if and only if, for every $x \in X$, $\uparrow x$ is finite and there is an open neighbourhood U of x such that, for every $y \in U$, the posets $\uparrow x$ and $\uparrow y$ are isomorphic.

Indeed, it is clear that the condition in Proposition A.2.9 implies that principal upsets are finite and locally constant. Conversely, if principal upsets are finite and locally constant, then all the sets of the form X_P (P a finite poset) are open and, since they partition X , also closed.

A.2.4 KM Heyting algebras and étale-finiteness

In Section 7.2.4 we gave a number of examples of étale-finite Heyting algebras. In this appendix we show how KM Heyting algebras appear naturally in this setting. In particular, we will show that those KM Heyting algebras that satisfy linearity and bounded-depth axioms (see e.g. [179]) are in fact étale-finite.

We begin with the following:

Proposition A.2.12 (Étale-finite $\Rightarrow$ KM). For every étale-finite Esakia space X , if $W \subseteq X$ is clopen, then $\max(W)$ is clopen. Dually, every étale-finite Heyting algebra is a KM Heyting algebra.

Proof. Let $\acute{e}: X \rightarrow X_0$ be a continuous strict p-morphism. By Proposition 2.1.61, it suffices to show that, for each clopen subset W of X , $\max(W)$ is clopen.

Let W be a clopen subset of X , and $x \in \max(W)$. Then, $W \cap \Box(W \Rightarrow \acute{e}^{-1}[\acute{e}(x)])$ is a clopen subset (since X_0 is discrete, $\acute{e}$ is continuous and X is an Esakia space), and $x \in W \cap \Box(W \Rightarrow \acute{e}^{-1}[\acute{e}(x)])$ since x is maximal. Moreover, $W \cap \Box(W \Rightarrow \acute{e}^{-1}[\acute{e}(x)])$ is contained in $\max(W)$: if $y \in W \cap \Box(W \Rightarrow \acute{e}^{-1}[\acute{e}(x)])$ and $y \leq y' \in W$, then, by definition, $\acute{e}(y) = \acute{e}(y') = \acute{e}(x)$, and, since $y \leq y'$, we have $y = y'$. Thus, $\max(W)$ is open. It is also closed because X is an Esakia space: by [102, Thm. 3.2.6], W is an Esakia space with the induced topology and order, and so, by [102, Thm. 3.2.3], $\max(W)$ is closed in the subspace topology; since W is clopen, $\max(W)$ is closed in X . Thus $\max(W)$ is clopen, as desired. ■

We can prove a converse to this, when we restrict to *tabular Gödel algebras*, in the following sense:

Definition A.2.13 (KM-tabular Gödel algebra). A KM-tabular Gödel algebra is a KM Heyting algebra for which there is a finite chain $[k] := 1 < \dots < k$ for $k \geq 1$ such that $H \in \mathbf{Var}_{\mathbf{HA}}([k])$.

Every KM-tabular Gödel algebra is in particular a Gödel algebra. We will now show that all such algebras are étale-finite:

Proposition A.2.14 (For Gödel algebras, étale-finite $\Leftrightarrow$ KM-tabular). For every Heyting algebra H , the following are equivalent:

1. H is a Gödel algebra and is étale-finite;
2. H is a KM-tabular Gödel algebra.

Proof. By Proposition 2.3.19, we know that a Heyting algebra H is a Gödel algebra if and only if $\text{Spec}(H)$ is locally linear, and it validates the bounded depth k axiom – equivalently, belongs to the variety $\text{Var}([k])$ – if and only if $\text{Spec}(H)$ has depth less than or equal to k . Set $X := \text{Spec}(H)$.

(1) $\Rightarrow$ (2) Since H is a Gödel algebra, every principal upset in X is a chain. If H is étale-finite, then there is a finite poset X_0 and a strict p-morphism $\acute{e}: X \rightarrow X_0$. Let X'_0 be the quotient of X_0 obtained by identifying points of the same depth. Since all principal upsets of X_0 are finite chains, the quotient $\pi: X_0 \twoheadrightarrow X'_0$ is a strict p-morphism, and so the composite $X \xrightarrow{\acute{e}} X_0 \xrightarrow{\pi} X'_0$ is a continuous strict p-morphism. Then, by Proposition 7.2.33, H is the Heyting reduct of an algebra in $\text{Var}(H_0)_{h \in H_0}$, where $H_0 = \text{ClopUp}(X'_0)$; thus, H belongs to the variety generated by a finite chain. Moreover, H is a KM Heyting algebra by Proposition A.2.12. Thus H is a KM-tabular Gödel algebra as desired.

(2) $\Rightarrow$ (1) As noted above, there is $k \geq 0$ such that every principal upset in X is a chain of cardinality at most k . We show that X is étale-finite. For this purpose, note that to each point $x \in X$ we can assign its depth $d(x) := |\uparrow x|$. Let $k^* = k < \dots < 1$ be the k -element chain with the reverse of the natural order. Then the map

$$\begin{aligned} d: X &\longrightarrow k^* \\ x &\longmapsto d(x) \end{aligned}$$

is a continuous strict p-morphism. The fact that it is a strict p-morphism follows from simple cardinality considerations, since, for each $y \in \uparrow x$, one has $|\uparrow y| = |\uparrow x| - |[x, y)|$ where $[x, y) = \{z \in X : x \leq z < y\}$.

We finally show that the map d is continuous: for each $j \in \{1, \dots, k\}$, consider the set $L_j := \{x \in X : d(x) = j\}$. Note that $L_1 = \max(X)$, since maximal elements are exactly the elements such that their upset is a singleton. Since X is a clopen subset of itself, Proposition 2.1.61 gives that $\max(X)$ is clopen. For each $1 \leq j < k$ we have $L_{j+1} = \max(X \setminus \bigcup_{i=1}^j L_i)$; so, by induction, all such subsets are clopen. This shows that d is continuous and concludes the proof. ■

A.3 Ghilardi complexes

A.3.1 Adjoints in Ghilardi complexes

In Section 4.4.1 we provided a representation of finitely presented Heyting algebras using Ghilardi complexes of the form $\mathcal{V}_G^g(P)$. In this appendix, we make

remarks on this structure.

Adjoints and uniform interpolation

In this brief section, we provide a sketch of the claim made in Chapter 8 – that a general theory of G-conservative objects is connected to uniform interpolation.

Recall that uniform interpolation follows, for IPC, from the adjoint definability property (Definition 2.3.29). To show this, it is sufficient to show that the inclusion $i: H \rightarrow H[x]$ from a Heyting algebra H to the Heyting algebra of polynomials (Definition 4.4.22) has a right adjoint.

So given an arbitrary Esakia space X , look at $X \times \mathbf{2}_{\leq}$, and the map $\pi_X: X \times \mathbf{2}_{\leq} \rightarrow X$. By what was shown in Section 4.4.3, the inverse limit of the polynomial Ghilardi complex $\mathcal{G}^{\pi_X}(X \times \mathbf{2}_{\leq})$, $\mathcal{V}_G^{\pi_X}(X \times \mathbf{2}_{\leq})$, is dual to $H[x]$. The following is moreover easy to observe:

Lemma A.3.1 (Projection dual to inclusion). The projection map $\pi_X \pi_0: \mathcal{V}_G^{\pi_X}(X \times \mathbf{2}_{\leq}) \rightarrow X$ is dual to the inclusion $i: H \rightarrow H[x]$.

Note that since the projection π_X is an open map, by the characterization of adjoints of Heyting algebra homomorphisms (Proposition 2.1.77), it has a left adjoint. In this case, however, such a left adjoint *is a map in the dual Esakia space*. It is the map

$$\begin{aligned} l: X &\rightarrow X \times \mathbf{2}_{\leq} \\ x &\mapsto (x, 0). \end{aligned}$$

This is the left adjoint of π_X , since $l(x) \leq (y, i)$ if and only if $x \leq \pi_X(y, i)$.

Now assume that one of the following holds:

1. Each $(X \times \mathbf{2}_{\leq})_n$ is an Esakia space;
2. For each n , the iterated adjoint map $\uparrow^n: (X \times \mathbf{2}_{\leq}) \rightarrow (X \times \mathbf{2}_{\leq})_n$ is continuous.

Note that the first hypothesis implies the following by Lemma 2.1.80, since each of these spaces being Esakia would imply that the local adjoint $\uparrow: (X \times \mathbf{2}_{\leq})_n \rightarrow (X \times \mathbf{2}_{\leq})_{n+1}$ would be continuous.

In either case, we would have that the map $\uparrow^\omega: X \times \mathbf{2}_{\leq} \rightarrow \mathcal{V}_G^{\pi_X}(X \times \mathbf{2}_{\leq})$ – which is order-theoretically the left adjoint of the projection $\pi_0: \mathcal{V}_G^{\pi_X}(X \times \mathbf{2}_{\leq}) \rightarrow X \times \mathbf{2}_{\leq}$ – would be continuous (see Remark 4.5.5).

In this situation, we would have, by composition:

$$l \uparrow^\omega \dashv \pi_X \pi_0,$$

and such an adjunction would be true on the level of continuous maps. This would then prove that the map i has a right adjoint by duality. It would also explain immediately the join-distributivity property (Proposition 2.3.34), as a simple corollary of such an adjoint being the dual of a Priestley morphism.

Stability maps

In addition to having, for each n , up-set maps $\uparrow: P_n \rightarrow P_{n+1}$ which are definable (i.e., continuous) there are *stability maps* $\delta_n: P_n \rightarrow P_{n+2}$ which are injective p-morphisms:

$$\delta_n(x) := \{\uparrow y : x \leq y\}.$$

The name is connected to the following, which was observed by Butz [75], as well as by D. Patariaia, in unpublished work:

Proposition A.3.2 (Stability in g -Ghilardi complex). Let $g: X \rightarrow Y$, be a continuous and order-preserving map, and $\mathcal{G}^g(X) = (X_n, r_n)$ be the g -Ghilardi complex. Then $\delta_n: X_n \rightarrow X_{n+2}$ defined by $\delta_n(x) = \{\uparrow y : y \in \uparrow x\} = \uparrow[\uparrow(x)]$ is an injective p-morphism. Moreover, δ_n is continuous if and only if $\uparrow: X_n \rightarrow X_{n+1}$ is continuous.

Proof. To see that it is injective, note that if $x \neq y$, then either $x \not\leq y$ or $y \not\leq x$, so either $\uparrow x \notin \delta_n(y)$ or $\uparrow y \notin \delta_n(x)$. To see that it is a p-morphism, assume that $\delta_n(x) \supseteq C$ for some $C \in X_{n+2}$; by construction, C is rooted at some element $\uparrow y$ for $x \leq y$. If $y \leq z$, then $\uparrow y \supseteq \uparrow z$, so because C is r_n -open, there is some $D \subseteq \uparrow y$ such that $D \in C$, and $r_n(D) = z$; but since $C \subseteq \delta_n(x)$, it follows that the root of D must be $\uparrow z$. It follows that $C = \delta_n(y)$, which shows that δ_n is a p-morphism, as desired.

Note that if $\uparrow: X_n \rightarrow X_{n+1}$ is continuous, then we have that X_n is actually an Esakia space; consequently no concern over X_n being g -Esakia for any g is necessary. Moreover, $\uparrow: X \rightarrow X_{n+1}$ is r_n -open: whenever $\uparrow x \supseteq C$, then by taking $r_n(C)$, we have that $\uparrow x \supseteq \uparrow r_n(C)$. So the map δ_n is in fact the unique, and therefore continuous, lifting of $\uparrow$ to X_{n+1} . Conversely, if δ_n is continuous, we note that for each $x \in X_n$, $r_{n+1}\delta_n(x) = \uparrow x$, and since r_{n+1} is always continuous, we have that $\uparrow: X_n \rightarrow X_{n+1}$ will be continuous. ■

The above Proposition shows that in most cases we cannot expect the δ_n maps to be continuous. But for complexes over finite posets, this is of course never an issue, and the maps δ_n serve a very natural purpose: they connect the universal models $\mathcal{U}(n)$ with the Ghilardi complexes.

Definition A.3.3 (Stable points). Let $g: P \rightarrow Q$ be an order-preserving map between finite posets, and $\mathcal{G}^g(P) = (P_n, r_n)$ the g -Ghilardi complex over P . We say that a point $x \in P_n$ is *stable* if $r_{n+2}^{-1}[x]$ is a singleton for each $y \in \uparrow x$.

Proposition A.3.4 (Stable points from stability maps). For each $x \in P_n$ $\delta_n(x)$ is stable. Moreover, if $x \in P_n$ is stable, then $\uparrow x$ is likewise stable, and $\uparrow x \in P_{n+1}$ is the unique point mapping to x .

Proof. The fact that $\delta_n(x)$ is stable follows from Proposition A.3.2: assume that $\delta_n(x) \supseteq C$; by what was observed, this is of the form $\delta_n(y)$ for a unique $y \geq x$.

Now assume that $D_0, D_1 \in P_{n+3}$ are such that $r_{n+2}(D_0) = r_{n+2}(D_1) = \delta_n(y)$. Let us show that then $D_0 = D_1$. Suppose that $F \in D_0$; then by construction, we have that $F = \delta_n(z)$ for some $z \geq y$. Since D_1 is r_{n+1} -open, and $\delta_n(y) \supseteq \delta_n(z)$, there is some $F' \in D_1$ such that $r_{n+1}(F') = r_{n+1}(\delta_n(z)) = \uparrow z$. But then $\delta_n(y) \supseteq F'$, so there is some $z' \geq y$ such that $\delta_n(z') = F'$; thus $\uparrow z' = r_{n+1}(\delta_n(z')) = r_{n+1}(F') = \uparrow z$. By injectivity of δ_n , we get that $z = z'$, and so $F' = F$. This shows that $D_0 \subseteq D_1$, with the converse inclusion being exactly the same.

The fact that $\uparrow x$ is stable whenever x is stable is clear: certainly $\uparrow x$ has x as a root, so if x is stable, it is the unique such point. Moreover, if $\uparrow x \supseteq C$, we show that C is stable: indeed, since then $x \leq r_n(C)$, then $r_n(C)$ is also stable, meaning that $C = \uparrow r_n(C)$; it follows that if D_0, D_1 are arbitrary in P_{n+2} , and $r_n(D_0) = r_n(D_1) = C$, then by the same arguments as above, we can show that $D_0 = D_1$, using that all the extensions of C are precisely upsets above x . ■

A consequence of Proposition A.3.4 we can connect our Ghilardi complex construction $\mathcal{V}_G$, the discrete Ghilardi complex from Section 5.2, and the n -universal model from Chapter 3:

Proposition A.3.5 (Finite approximability from step by step). Let $g: P \rightarrow Q$ be an order-preserving map, and $\mathcal{G}^g(P)$ the g -Ghilardi complex over P . Then:

1. For each n , P_n is an upset of $\mathcal{V}_G^g(P)$.
2. If $\bar{x} \in \mathcal{V}_G^g(P)$ is a point of finite depth, then there exists some n such that $\uparrow \bar{x} \cong \uparrow \bar{x}(n)$.
3. The points of finite depth (precisely $\mathcal{P}_G^g(P)$) are dense in $\mathcal{V}_G^g(P)$.

Proof. (1) Let $p \in P_n$ be a minimal element. By Proposition A.3.4, $\delta_n(p)$ is a stable point, and by Proposition A.3.2, we have that $\uparrow p \cong \uparrow \delta_n(p)$, given this is an injective p-morphism. Note moreover that then considering the point $l_{n+2}(\delta_n(p))$, where $l_k: P_k \rightarrow \mathcal{V}_G^g(P)$ is the adjoint to the projection from 4.3.2, namely, by defining $l_{n+2}(p)(n+2) = \delta_n(p)$ and for each $k \geq n+2$, $l_{n+2}(p)(k) = \uparrow l_{n+2}(p)(k)$. It follows from Proposition A.3.4 that then $l_n: \uparrow \delta_n(p)$ is an injective p-morphism. Since this holds for each minimal $p \in P_n$, we obtain that P_n is an upset of $\mathcal{V}_G^g(P)$.

(2) We prove this by induction on the depth of $\bar{x}$. If $\bar{x}$ is maximal, then note that we can take $n = 1$: if $\bar{x}(1)$ had any successors other than itself, say $\bar{x}(1) \supseteq C$, since $\bar{x}(2)$ is r_0 -open, we could take $C' \in \bar{x}(2)$ such that $r_0(C) = r_0(C')$, and proceeding recursively construct $\bar{C} \geq \bar{x}$, which cannot be equal, since $C \neq \bar{x}(1)$.

Now assume that $\bar{x}$ has depth $k+1$. Let n be large enough that each point of depth k has $\uparrow \bar{y} \cong \uparrow \bar{y}(n)$, and such that for each successor $\bar{y}$ of $\bar{x}$, $\bar{x}(n) \neq \bar{y}(n)$. Then we claim that $\bar{x}(n+1)$ works. Indeed, we consider the map:

$$\pi_{n+1}: \uparrow \bar{x} \rightarrow \uparrow \bar{x}(n+1);$$

Such a map is order-preserving. It is moreover order reflecting: if $\bar{x}(n+1) \leq \bar{y}(n+1)$, then we show that $\bar{x}(n+2) \leq \bar{y}(n+2)$. Indeed, if $\bar{x} = \bar{y}$ we are done. Otherwise if $C \in \bar{y}(n+2)$, by assumption, $C = \bar{z}(n+1)$ for some unique $\bar{z} \geq \bar{y}$, by induction hypothesis (recalling that we assume the roots to be distinct, and $\uparrow \bar{y} \cong \uparrow \bar{y}(n)$, which is preserved at each further step). Then $\bar{x} \leq \bar{z}$ as well, so $\bar{x}(n+2) \leq \bar{z}(n+2)$, and so $\bar{z}(n+1) \in \bar{x}(n+2)$. Inductively we then prove that $\bar{x} \leq \bar{y}$. This shows that π_{n+1} is order-reflecting.

Finally, assume that $\bar{x}(n+1) \leq C$. Since $\bar{x}(n+2)$ is r_{n+1} -open, there is some $C' \in \bar{x}(n+2)$ such that $\bar{x}(n+1) \leq C'$, and $r_{n+1}(C) = r_{n+1}(C')$. Recursively we can again construct $\bar{C} \geq \bar{x}$. This shows that π_{n+1} is moreover surjective above $\bar{x}$. We thus have that the induction hypothesis is preserved.

(3) Given any clopen upset of $\mathcal{V}_G^q(P)$, this will be of the form $\pi_n^{-1}[U]$ for $U \subseteq P_n$. By (1), we have that there is some $\bar{x}$ such that $\uparrow \bar{x}$ will be isomorphic to $\uparrow \bar{x}(n)$, and $\bar{x}(n) \in U$. Thus $\pi_n^{-1}[U]$ contains a point of finite depth. This shows such points to be dense. ■

The above yields an independent (if perhaps overly elaborated) proof of the FMP of IPC, and it connects the step-by-step construction with the universal model: by taking $P = \mathbf{2}_{\leq}^n$, one can define a coloring on the points of finite depth of $\mathcal{V}_G(P)$, and this coloring will yield unique irreducible models.

Remark A.3.6 (Normal forms and De Jongh formulas). As an upshot of the above Proposition, we can explain the seeming similarity between the normal forms from Proposition 4.3.10 and the De Jongh formulas characterizing the points of $\mathcal{U}(n)$ (see e.g. [54, Ch. 3]) associated with the point $x \in \mathcal{U}(n)$: they coincide, up to equivalence, whenever one takes x to be a stable point.

A.3.2 Connections with the sheaf duality

For the reader familiar with [130, Chapter 4], we briefly outline here the construction of the Ghilardi complex, and the the sheaf-theoretic representation of finitely presented Heyting algebras. The precise connection is the following: denoting by

$$h_P(Q) = \{v: Q \rightarrow P : v \text{ order preserving map}\}$$

where Q is finite and rooted, we get a (contravariant) functor from the category **FinPos**_r of finite rooted posets to the category of sets, in fact a sheaf. For each Q , $(Q, v) \in h_P(Q)$ is a “general model” over P . Such models are subject to a bisimulation game played with labels in P , in complete analogy to that described in Section 2.3.3.

By imposing a natural subcanonical topology, each h_P becomes a sheaf, and its subsheaves will correspond exactly to the collections of models closed under bisimulation. These correspond to “infinitary formulas” as recently studied in [32]. If one asks for classes furthermore closed under n -bisimulation for P – looks

for *sheaves of finite index*, denoted $\text{Sub}_{f.i.}(h_P)$, then these such sheaves are in 1-1 correspondence with $\text{CloUp}(V_G(P))$, that is, with the “formulas over P ”. More concretely, the class $\pi_n^{-1}[U]$ for U a finite subset will correspond to the subsheaf specified by all the models which are n -bisimilar (in the P -game) with elements of U . In the special case where P is the free distributive lattice on n -many generators, we obtain a representation of free Heyting algebras.

In that sense, the convenience of the Ghilardi complex representation lies in the fact that all elements considered are precisely the irreducible models (in the sense of Definition 3.2.3). This allows the presence of additional structure which the existence of bisimulations might make less visible. On the other hand, the advantage of the sheaf theoretic representations are clear in settings like the study of uniform interpolation, see [130, Chapter 4.5]: one has more flexibility in doing constructions.

From such a perspective one can also clarify the difference between Carai’s construction for duals of Gödel algebras and the one we presented in Section 5.1.3. Given a finite poset P , let us consider the following classes of P -models over chains, that is, maps $f: C \rightarrow P$ from a chain to a poset P :

1. $\text{Chain}(P)$ is the class of all P -models which are chains.
2. $\text{IrrChain}(P)$ is the class of all P -models which are irreducible: whenever $x < y$ then $f(x) < f(y)$.

These structures can be in turn equipped with several orders:

1. Given two chain models (C, f) and (C', f') we write $(C, f) \leq_\infty (C', f')$ if there is a world $x \in C$ such that $(C_x, f) \sim_\infty (C', f')$, the full bisimulation for the P -game;
2. Given two chain models (C, f) and (C', f') we write $(C, f) \leq_0 (C', f')$ if $f(r(C)) \leq f(r(C'))$, and for each k , we write $(C, f) \leq_{k+1} (C', f')$ if there is a world $x \in C$ such that $(C_x, f) \sim_n (C', f')$, the full bisimulation for the P -game.

From this point of view, by taking P a finite poset, the structure $(\mathcal{V}_n^{\text{LC}}(P), \supseteq)$ can be identified with the structure $(\text{IrrChain}(P), \leq_n)$, that is, equipping the irreducible chain models with the $\leq_n$ -order. By contrast, Carai’s order on this structure is $(\text{IrrChain}(P), \leq_\infty)$. We can then see that Theorem 5.1.12 truly connects to the framework of uniform local tabularity discussed in Chapter 5.1.12: such a result is simply showing that the order $\leq_2$ already coincides with $\leq_\infty$.

A.3.3 General theory of Ghilardi complexes

In Section 4.1.3 we described the step-by-step procedure for general varieties $\mathbf{K} \subseteq \mathbf{K}_0$. Here we track our main assumptions on this construction, returning

to the setting of varieties $\mathbf{K}$ and $\mathbf{K}_0$ from Section 4.1.3. Note that the algebraic equivalent of g -Ghilardi complexes is entirely reasonable: one considers chains of algebras parametrized by one initial homomorphism. Then our assumptions can be listed as follows:

1. (Finitely presented objects of $\mathbf{K}_0$) To obtain the fact that finitely presented algebras can be constructed by applying the left adjoint to *finite* algebras, local finiteness is employed. Having local finiteness, one can ensure that the construction at each step is always finite. To obtain the fact that quotients correspond to formulas, the filter-congruence correspondence is used, which corresponds to a form of congruence extension property (see [201, Section 5]).
2. (Coproducts and colimits of $\mathbf{K}_0$) Very little was assumed of the relationship between the two varieties. The key facts seem to be: if $f: A \rightarrow A_1$ is an $\mathcal{L}_0$ -homomorphism and so is $g: A \rightarrow A_2$, then the pushout inclusions $i_{A_i}: A_i \rightarrow A_1 \oplus_{A_0} A_2$ have the property that the pushout is i_{A_i} - $\mathbf{K}_0$.
3. (Amalgamation lifting) The amalgamation lifting proof (Theorem 4.4.21) seems to be quite general, assuming crucially the property (One-step inclusion is injective), together with the fact that $\mathbf{K}$ is closed under amalgamation, to obtain amalgamation in $\mathbf{K}_0$. This can be seen as a categorical basis for the fact that amalgamation is typically proven by “basing it” on a previously known to be amalgamable class. It also gives a different lens from which to look at the scarcity of classes of amalgamation, from the point of view of which varieties admit a step-by-step construction of left adjoints. Since the latter has a strongly syntactic structure, this may provide a new line of attack for the general theory of amalgamation.
4. (Polynomial algebras) to obtain the fact that polynomial algebras can be constructed out of the finite generating objects in $\mathbf{K}$, we use local finiteness.

As a further remark on the one-step inclusion being injective, note that in light of Theorem 4.4.21, such an assumption must be rather strong: after all, by Maksimova’s theorem on interpolation (Theorem 2.3.27), there are only varieties of Heyting algebras with amalgamation. In light of all of this, one can also ask for the converse: whether amalgamation of a variety $\mathbf{K}_0$ always imply the property that one-step inclusions are injective. At present we do not have any general result on this front.

A.3.4 Logic of the structural interpretation

Here we characterize the logic of the interpretation $\mathfrak{S}$, discussed in Section 4.5.1. As explained in Example 4.5.2, the logic of such interpretation corresponds precisely to the logic of $\mathcal{F}_{\mathbf{DL}}(\omega)$ seen as a Heyting algebra. Hence it suffices to

characterize what such a logic is, which will be precisely the logic associated to the variety generated by this algebra.

Proposition A.3.7 (Logic of free distributive lattice). The superintuitionistic logic corresponding to $\mathbb{V}(\mathcal{F}_{\mathbf{DL}}(\omega))$ is precisely **KC**.

Proof. As noted in Example 4.5.2, the algebra $\mathcal{F}_{\mathbf{DL}}(\omega)$ validates the Yankov axiom. We will show that such an algebra generates the variety of **KC**-algebras.

First, as shown in Lemma 4.5.16, $\mathcal{F}_{\mathbf{DL}}(\omega)$ embeds all the finite free distributive lattices as Heyting subalgebras, and it is their limit. Since every variety is closed under directed colimits and subalgebras, it suffices then to characterize the logic of $\{\mathcal{F}_{\mathbf{DL}}(n) : n \in \omega\}$. For this, we use duality. So first, recall by Proposition 4.1.1, that the dual of $\mathcal{F}_{\mathbf{DL}}(n)$ is precisely $\mathcal{P}(n)$.

From the results of [188] (see also [117, Claim 9]) that each finite rooted poset P with a greatest element is a p-morphic image of some frame of the form $\mathcal{P}(n) - \{[n]\}$, where we remove the maximal node, say via $p: \mathcal{P}(n) - \{[n]\} \rightarrow P$. We can then define a p-morphism $p^*: \mathcal{P}(n) \rightarrow P$ by extending p , sending $[n]$ to the top node in P . Thus, every finite rooted poset with a maximal element can be obtained as a p-morphic image of $\mathcal{P}(n)$ for some n . By duality, this means that each finite **KC**-algebra is obtained from $\{\mathcal{F}_{\mathbf{DL}}(n) : n \in \omega\}$. Since by Proposition 2.3.26, **KC** has the finite model property, we conclude that all **KC**-algebras are generated from these ones, obtaining the result. ■

The reader may compare the above result with Yankov's theorem concerning **KC** [265], which shows one can eliminate negations obtaining “implicative-positive formulas”, and proves that **KC** is the strongest extension of **IPC** that is conservative over implicative positive formulas. The above interpretation gives a sharper reduction, since it removes all implications; on the other hand, Yankov's result stands as a more general fact about conservativity of formulas over superintuitionistic logics.

A.3.5 Conservative translations and uniform interpolation

In Section 4.5, we showed that every locally finite Heyting algebra is dual to a G -conservative object – that is, there exists a naturally defined right adjoint to the unit η_H . As an application of this categorical fact, we provided an explicit conservative translation between intuitionistic logic and positive logic, using the fact that free distributive lattices are locally finite Heyting algebras.

In this section we show how one can derive the existence of this translation from the existence of uniform interpolants in intuitionistic logic, and some of their special properties. It is not clear to us at the moment whether such methods can be extended to prove the G -conservativeness of countable locally finite Heyting algebras.

One key observation we need on uniform interpolation is the following folklore fact:

Lemma A.3.8 (Uniform interpolants of positive formulas). Let $\varphi(\bar{p}, q)$ be a formula in the language of PL. Then the uniform interpolant $\exists q \varphi(\bar{p}, q)$ is a positive formula; it is simply equivalent to:

$$\varphi(\bar{p}, \top).$$

Proof. Note that since $\varphi(\bar{p}, q)$ is a strictly positive formula, it can be written as a disjunction of join-irreducible formulas over $\bar{p}, q$ (Proposition 4.1.3). Thus by applying distributivity, we can write it as $(q \wedge \chi(\bar{p})) \vee \mu(\bar{p})$, where $\chi(\bar{p}), \mu(\bar{p})$ are joins of join-irreducible formulas over $\bar{p}$.

Thus consider the formula $\chi(\bar{p}) \vee \mu(\bar{p})$. First notice that certainly $\varphi(\bar{p}, q)$ implies this formula. Moreover, assume that $\varphi(\bar{p}, q) \leq \psi(\bar{p})$. Then we can prove that $(q \wedge \chi(\bar{p})) \vee \mu(\bar{p}) \leq \psi(\bar{p})$. By uniform substitution on q , replacing it by $\top$, we obtain $\chi(\bar{p}) \vee \mu(\bar{p})$. This shows that $\chi(\bar{p}) \vee \mu(\bar{p}) \leq \psi(\bar{p})$, proving the result. ■

With this in hand, we first can show a uniform positive interpolation theorem:

Proposition A.3.9 (Uniform positive interpolation). Whenever $\vdash_{\text{IPC}} \psi(\bar{p}, \bar{q}) \rightarrow \varphi(\bar{q}, \bar{r})$ and ψ is a positive formula, there exists some positive formula $\chi(\bar{q})$ such that $\vdash_{\text{IPC}} \psi(\bar{p}, \bar{q}) \rightarrow \chi(\bar{q})$ and $\vdash_{\text{IPC}} \chi(\bar{q}) \rightarrow \varphi(\bar{q}, \bar{r})$.

Proof. Assuming that $\vdash_{\text{IPC}} \psi(\bar{p}, \bar{q}) \rightarrow \varphi(\bar{q}, \bar{r})$, and $\psi(\bar{p}, \bar{q})$ is a positive formula, we have by uniform interpolation that $\vdash_{\text{IPC}} \exists \bar{p} \psi(\bar{p}, \bar{q}) \rightarrow \varphi(\bar{q}, \bar{r})$. Since by Lemma A.3.8, the uniform interpolant is a positive formula, we are done. ■

From this we can deduce the following, in a different way from Theorem 4.5.27:

Theorem A.3.10 (Uniform positive approximation). For each positive formula $\varphi \in \mathcal{L}_{\text{IPC}}$ there exists a uniform positive approximation $P(\varphi)$: for each positive formula ψ , $\vdash_{\text{IPC}} \psi \rightarrow \varphi$ if and only if $\vdash_{\text{IPC}} \psi \rightarrow P(\varphi)$.

Proof. For each formula $\varphi(\bar{p})$, we define:

$$P(\varphi) := \bigvee \{ \psi(\bar{p}) : \psi(\bar{p}) \in \mathcal{L}_{\text{PL}}, \vdash_{\text{IPC}} \psi(\bar{p}) \rightarrow \varphi(\bar{p}) \},$$

that is, we take the join over all positive approximations of φ in the same language. Note that since the language is finite, this join is finite, and hence definable. It is clear that $P(\varphi)$ is itself a positive approximation, since it is the join of formulas implying φ . Hence if $\vdash_{\text{IPC}} \psi \rightarrow P(\varphi)$, then $\vdash_{\text{IPC}} \psi \rightarrow \varphi$. Conversely, assume that $\psi(\bar{p}, \bar{q})$ is a positive approximation of φ . By Proposition A.3.9, there is some positive approximation of φ , $\chi(\bar{p})$ such that $\vdash_{\text{IPC}} \psi(\bar{p}, \bar{q}) \rightarrow \chi(\bar{p})$. Hence $\vdash_{\text{IPC}} \psi(\bar{p}, \bar{p}) \rightarrow P(\varphi)$. This shows the result. ■

By exploiting further properties of uniform interpolation we can obtain that $P(-)$ is actually a distributive lattice homomorphism:

Lemma A.3.11 (Join-irreducibility of formulas). Let $\varphi(\bar{p}) \in \mathcal{L}_{\text{IPC}}$ be a formula which is join-irreducible over $\bar{p}$. Then whenever $\vdash_{\text{IPC}} \varphi(\bar{p}) \rightarrow (\mu_0 \vee \mu_1)$, either $\vdash_{\text{IPC}} \varphi(\bar{p}) \rightarrow \mu_0$ or $\vdash_{\text{IPC}} \varphi(\bar{p}) \rightarrow \mu_1$.

Proof. This was shown in Theorem 4.3.7. ■

Proposition A.3.12 (Positive approximations induce translation). The assignment $P(-)$ sends $\top$ to $\top$, $\perp$ to $\perp$, and commutes with conjunction and disjunction.

Proof. The facts for $\top$ and $\perp$ are obvious. For conjunction, note that $\vdash_{\text{IPC}} \psi \rightarrow (\varphi_1 \wedge \varphi_2)$ if and only if $\vdash_{\text{IPC}} \psi \rightarrow \varphi_1$ and $\vdash_{\text{IPC}} \psi \rightarrow \varphi_2$. This immediately implies that the construction commutes with conjunction.

Finally, for disjunctions, note that $P(\varphi \vee \psi) \leq P(\varphi) \vee P(\psi)$, directly by the definitions. Conversely, assume that $\vdash_{\text{IPC}} \chi \rightarrow (\varphi \vee \psi)$, for χ a positive formula. Then χ is a join of join-irreducible formulas. For each μ_i , a join-irreducible positive formula such that $\vdash_{\text{IPC}} \mu_i \rightarrow \chi$, we have then that $\vdash_{\text{IPC}} \mu_i \rightarrow (\varphi \vee \psi)$. By Lemma A.3.11, we have that then $\vdash_{\text{IPC}} \mu_i \rightarrow \varphi$ or $\vdash_{\text{IPC}} \mu_i \rightarrow \psi$. Thus, either $\vdash_{\text{IPC}} \mu_i \rightarrow P(\varphi)$ or $\vdash_{\text{IPC}} \mu_i \rightarrow P(\psi)$. In both cases we obtain $\vdash_{\text{IPC}} \mu_i \rightarrow P(\varphi) \vee P(\psi)$. Since this holds for any such μ_i , we conclude that $\vdash_{\text{IPC}} \chi \rightarrow P(\varphi) \vee P(\psi)$. ■

Jointly, all of these facts show that $P(-)$ indeed specifies the conservative translation studied in Section 4.5.3.

A.4 Coequivalence relations

In this appendix, we include some basic results concerning coequivalence relations in IPC, including a self-contained proof that not every coequivalence relation in IPC *separates variables*. This was shown in [130] through a complex argument involving sheaves. We provide a more elementary proof, using Kripke models, in hope of making the result by Ghilardi and Zawadowski more widely accessible. We also provide a similar proof for KC. Together with the previous work of De Berardinis [31], we conclude that there are exactly two superintuitionistic logics amongst having both interpolation and CoSP.

The results of this appendix were obtained with Matteo De Berardinis, and some of the basic discussion was contained in [10], though the characterization of exactness is new.

A.4.1 Coequivalence separation property

Definition A.4.1 (Coequivalence relation). Let L be a superintuitionistic logic. Given a tuple $\bar{p}$, and a formula $\tau(\bar{p})$, we say that a formula $\rho(\bar{p}_1, \bar{p}_2)$ (where $|\bar{p}_i| = |\bar{p}|$) is a *coequivalence relation over τ* if in the context $\tau(\bar{p})$, $\rho(\bar{p}_1, \bar{p}_2)$ defines an equivalence relation, that is:

1. (Domain) $\rho(\bar{p}_1, \bar{p}_2) \vdash_L \tau(\bar{p}_1) \wedge \tau(\bar{p}_2)$;
2. (Reflexivity) $\tau(\bar{p}) \vdash_L \rho(\bar{p}, \bar{p})$;
3. (Symmetry) $\rho(\bar{p}_1, \bar{p}_2) \vdash_L \rho(\bar{p}_2, \bar{p}_1)$;
4. (Transitivity) $\rho(\bar{p}_1, \bar{p}_2) \wedge \rho(\bar{p}_2, \bar{p}_3) \vdash_L \rho(\bar{p}_1, \bar{p}_3)$.

Example A.4.2 (Example of coequivalence relation). For each set $\varphi_1(\bar{p}), \dots, \varphi_n(\bar{p})$ of formulas, the formula,

$$\bigwedge_{i=1}^n \varphi_i(\bar{p}_1) \leftrightarrow \varphi_i(\bar{p}_2)$$

is a coequivalence relation. This follows easily from the properties of the equivalence $\leftrightarrow$.

Definition A.4.3 (Coequivalence separation). We say that a coequivalence relation $\rho(\bar{p}_1, \bar{p}_2)$ over $\tau(\bar{p})$ *separates variables* or simply that it is *separating* if there exists a sequence of formulas $(\theta_1(\bar{p}), \dots, \theta_m(\bar{p}))$ such that:

$$\rho(\bar{p}_1, \bar{p}_2) \dashv\vdash_L \left(\tau(\bar{p}_1) \wedge \tau(\bar{p}_2) \wedge \bigwedge_{i=1}^m \theta_i(\bar{p}_1) \leftrightarrow \theta_i(\bar{p}_2) \right).$$

We say that a superintuitionistic logic L has the *coequivalence separation property* (CoSP) if every coequivalence relation over L if for each $\tau(\bar{p})$, and each $\rho(\bar{p}_1, \bar{p}_2)$ a coequivalence relation over $\tau(\bar{p})$, ρ is separating.

Remark A.4.4 (Categorical equivalence relations). For the reader familiar with the notion of an *equivalence relation* over an arbitrary category (with pullbacks), note that the notion of a coequivalence relation corresponds precisely with the notion of an equivalence relation over a finitely presented algebra. The notion of a separating coequivalence relation is then nothing more than an *effective* equivalence relation. We refer the reader to [130, pp.100] (see also [10, Sec. 3]) for a detailed discussion of this connection.

The reader is possibly wondering what are examples of non-trivial coequivalence relations, and how one shows to be separating. The following example is instructive:

Example A.4.5 (Example coequivalence relation). Over p_1, p_2 we can consider the coequivalence relation:

$$\rho(p_1, p_2) = \neg\neg(p_1 \wedge p_2) \vee (p_1 \leftrightarrow p_2);$$

it is not difficult to verify algebraically that this is a coequivalence relation (over $\top$) for IPC. Over KC, it is a separating coequivalence relation. First one notices that in IPC, by distributing the double negation and disjunction, we obtain:

$$(\neg\neg p_1 \vee (p_1 \leftrightarrow p_2)) \wedge (\neg\neg p_2 \vee (p_1 \leftrightarrow p_2));$$

we claim that this implies:

$$(\neg p_1 \leftrightarrow \neg p_2).$$

Certainly $p_1 \leftrightarrow p_2$ implies $\neg p_1 \leftrightarrow \neg p_2$. On the other hand, if $\neg\neg(p_1 \wedge p_2) = 1$, then $\neg\neg p_1 = 1$ and $\neg\neg p_2 = 1$, so $\neg p_1 = \neg p_2$.

On the other hand, if one takes H a finite, subdirectly irreducible Heyting algebra for **KC**, it has both a second greatest and a second least element. Assume that $\neg a = \neg b$ holds, we will derive $\rho(a, b)$. Note that for each $a \in H$, either $a = 0$, and then $\neg a = 1$, or since $\neg a \vee \neg\neg a = 1$, and there is a second largest element, $\neg\neg a = 1$, so $\neg a = 0$; thus if $\neg a = \neg b$, either $a = b = 0$ or $\neg\neg a = \neg\neg b = 1$.

A.4.2 Coequivalence relations and elimination of imaginaries

To have some more intuition about exact logics, it will be interesting to briefly remark on some of their model theoretic content. There is a clear connection between the above notion of coequivalence separation, and the framework of *elimination of imaginaries* [220]. This was introduced by Bruno Poizat as a generalization of classical Galois theory, with the intention of providing a model-theoretic framework which allows the fundamental theorem of Galois theory to hold –namely, that intermediate field extensions L/K correspond to subgroups of the Galois group $[L : K]$. We follow the presentation as given in [146, p. 4.4]:

Definition A.4.6 (Imaginary elements). Let M be an $\mathcal{L}$ -structure. An *equivalence formula* is a first-order formula $\theta(\bar{x}_1, \bar{x}_2)$ such that the relation

$$\{(\bar{a}, \bar{c}) : M \models \theta(\bar{a}, \bar{c})\},$$

is a non-empty (partial) equivalence relation E_θ . The *domain* $D_\theta = \{\bar{a} : M \models \theta(\bar{a}, \bar{a})\}$. We write $\bar{a}/\theta$ for the E_θ -equivalence class of the tuple $\bar{a} \in D_\theta$. The element $\bar{a}/\theta$ is called an *imaginary element*.

In this language, our coequivalence relations can be seen as follows:

Proposition A.4.7 (Coequivalence relations and equivalence formulas). Let L be a logic. Then $\rho(\bar{p}_1, \bar{p}_2)$ is a coequivalence relation over $\tau(\bar{p})$ for L , if and only if for each $A \in \text{Alg}(L)$, the first-order formula,

$$\rho(\bar{x}_1, \bar{x}_2) = 1$$

is an equivalence formula for A , with domain $\tau(A) = \{\bar{a} \in A^n : A \models \tau(\bar{a})\}$.

Definition A.4.8 (Elimination of imaginaries). Given a first-order theory T , a model M of T has *elimination of imaginaries* if for every equivalence formula

$\theta(\bar{x}_1, \bar{x}_2)$ of M and every tuple $\bar{a} \in A^n$, and formula $\varphi(\bar{x}_1, \bar{z})$, there exists a unique tuple $\bar{b}$ such that:

$$\bar{a}/\theta = \{\bar{c} : M \models \varphi(\bar{c}, \bar{b})\}.$$

We say that M *uniformly eliminates the imaginary* $\bar{a}/\theta$ if the formula φ can be chosen independently of $\bar{a}$. We say that T has *uniform elimination of imaginaries* if every model M of T uniformly eliminates imaginaries.

The following is the connection between our setting and elimination of imaginaries:

Proposition A.4.9 (CoSP implies special uniform elimination of imaginaries). Let L be a logic, and let $\rho(\bar{p}_1, \bar{p}_2)$ be a coequivalence relation for L . Assume that ρ separates variables, and $\psi_1(\bar{p}), \dots, \psi_m(\bar{p})$ witnesses this. Then for each $A \in \text{Alg}(L)$, and any imaginary element $\bar{a}/\rho$, A uniformly eliminates $\bar{a}/\rho$.

Proof. Consider the formula in variables $z_1, \dots, z_n$

$$\bigwedge_{i=1}^m \psi_i(\bar{x}) \leftrightarrow y_i = 1.$$

Given any imaginary element $\bar{a}/\rho$, by definition of separating variables, we have

$$\bar{a}/\rho = \{\bar{c} : A \models \rho(\bar{a}, \bar{c}) = 1\} = \{\bar{c} : A \models \bigwedge_{i=1}^m \psi_i(\bar{c}) \leftrightarrow \psi_i(\bar{a}) = 1\};$$

so the tuple $\psi_i(\bar{a})$ provides the elimination. Now supposing that $(b_1, \dots, b_m)$ is any other tuple, note that certainly, by definition $b_i \leftrightarrow \psi_i(\bar{a}) = 1$, so $b_i = \psi_i(\bar{a})$. ■

Remark A.4.10 (Full elimination of imaginaries fails). In light of Proposition A.4.9, the property CoSP can thus be seen as a special instance of uniform elimination of imaginaries. Full elimination of imaginaries cannot hold for any first-order theory of a variety of Heyting algebras in its usual language: it is well-known (see [220]) that a theory which eliminates imaginaries and interprets at least two constants must interpret infinitely many constants, whilst the first-order theory of any variety of Heyting algebras can only interpret $\top$ and $\perp$. The situation may be different with extensions of KM, since this theory does interpret infinitely many constants, namely $\Box^n \perp$.

A.4.3 Coequivalences and exact logics

We take a look at a special kind of logic with the CoSP:

Definition A.4.11 (Exact logics). A superintuitionistic logic L is said to be an *exact logic* if L has the Craig interpolation property and the CoSP.

To understand the flavour of this property, we can give an explicit proof of it for classical logic:

Proposition A.4.12 (Classical logic is exact). The logic CPC is an exact logic.

Proof. We know (Theorem 2.3.27) that CPC has interpolation. We will show that every coequivalence relation over CPC is separating, through elementary methods to illustrate the logical flavour of the property (in Proposition A.4.15 a more general result will encompass this one).

Assume that $\rho(\bar{p}_1, \bar{p}_2)$ is a coequivalence relation (over $\top$). The conditions of being a coequivalence relation means three things:

- (reflexivity) for any truth-assignment $v: \bar{p}_1, \bar{p}_2 \rightarrow \{0, 1\}$ where for each $p \in \bar{p}$, $v(p_1) = v(p_2)$, then v makes $\rho(\bar{p}_1, \bar{p}_2)$ true;
- (symmetry) if v makes $\rho(\bar{p}_1, \bar{p}_2)$ true, then if $v^\dagger$ is defined by setting, for each $p \in \bar{p}$, $v^\dagger(p_1) = v(p_2)$ and $v^\dagger(p_2) = v(p_1)$, we have that $v^\dagger$ also makes $\rho(\bar{p}_1, \bar{p}_2)$ true;
- (transitivity) if v makes $\rho(\bar{p}_1, \bar{p}_2)$ true and $v': \bar{p}_2, \bar{p}_3 \rightarrow \{0, 1\}$ is another valuation which makes $\rho(\bar{p}_2, \bar{p}_3)$ true, and such that for each $p \in \bar{p}$, $v(p_2) = v'(p_2)$, then the valuation $v'' = v \cup v'$ makes $\rho(\bar{p}_1, \bar{p}_3)$ true.

Given such a coequivalence relation, define an equivalence relation on the set $\text{Val}(\bar{p})$ of valuations v over $\bar{p}$ as follows: $v \equiv v'$ if and only if there is a valuation v_0 over $\bar{p}_1, \bar{p}_2$ such that $v_0 \upharpoonright_{\bar{p}_1} = v$ and $v_0 \upharpoonright_{\bar{p}_2} = v'$ and v_0 validates $\rho(\bar{p}_1, \bar{p}_2)$. This is an equivalence relation by the properties just listed. For each equivalence class C_i , let ψ_i be the formula given by the disjunction of the (atoms corresponding to the) valuations in C_i , so that a valuation v over $\bar{p}$ validates ψ_i if and only if $v \in C_i$. Then we claim that:

$$\rho(\bar{p}_1, \bar{p}_2) \dashv\vdash \bigwedge_{i=1}^m \psi_i(\bar{p}_1) \leftrightarrow \psi_i(\bar{p}_2).$$

Notice that a valuation v on $\bar{p}_1$ and $\bar{p}_2$ validating the conjunction means that “ $v \upharpoonright_{\bar{p}_1} \in C_i$ if and only if $v \upharpoonright_{\bar{p}_2} \in C_i$ ” for all i ; this is equivalent to say that $v \upharpoonright_{\bar{p}_1} \equiv v \upharpoonright_{\bar{p}_2}$, since $v \upharpoonright_{\bar{p}_1}$ must belong to some C_i . Then, by construction any valuation that validates ρ validates the conjunction. Vice versa, if v is a valuation validating the conjunction, then $v \upharpoonright_{\bar{p}_1} \equiv v \upharpoonright_{\bar{p}_2}$. This means by assumption that there is some v_0 satisfying ρ , such that $v_0 \upharpoonright_{\bar{p}_1} = v \upharpoonright_{\bar{p}_1}$ and $v_0 \upharpoonright_{\bar{p}_2} = v \upharpoonright_{\bar{p}_2}$. This forces $v = v_0$, hence v validates ρ . $\blacksquare$

Remark A.4.13 (Exactness is a sheaf condition). The proof of Proposition A.4.12 is instructive: one can understand exactness for a logic as corresponding to a sheaf-like condition. Indeed, by interpolation, corresponding to amalgamation, one can form amalgamations of models; the CoSP forces such amalgamations to

be unique. We do not pursue this connection further here, though we signal it for further work.

As noted before, Ghilardi and Zawadowski [130, pp.100] showed that IPC is not exact. This was done by exhibiting a specific sheaf in their sheaf-theoretic duality, and proving that a given equivalence relation for it is not effective. This has the slight disadvantage of being rather indirect. To make such a result more well-known, we provide the following, more direct proof; we also prove a similar result for KC, for completeness:

Proposition A.4.14 (Failure of CoSP for IPC and KC). The logics IPC and KC do not have the CoSP.

Proof. First we do this for (IPC). For that purpose, recall the coequivalence relation from Example A.4.5:

$$\rho(p_1, p_2) = \neg\neg(p_1 \wedge p_2) \vee (p_1 \leftrightarrow p_2);$$

As shown in the example, this is a coequivalence relation over IPC. Now assume that $\theta_1(p), \dots, \theta_n(p)$ are a sequence of formulas, such that:

$$\vdash_{\text{IPC}} \rho(p_1, p_2) \leftrightarrow \bigwedge_{i=1}^n \theta_i(p_1) \leftrightarrow \theta_i(p_2).$$

Note that then any model of ρ will satisfy this equivalence relation. So consider the model $\mathfrak{M}$ from Figure A.1, with valuations as labelled, taking values over p_1 and p_2 . Note that by computing its irreducible quotient (Definition 3.2.4), when projecting to p_1 and p_2 , we obtain *the same model*: this is so for distinct reasons, namely because upon projecting to the first coordinate the left point labelled with 10 disappears being identified with the top one, forming a diamond, whilst when projecting on the right, the same point gets identified with its immediate predecessor labelled with 00.

We then claim that $\mathfrak{M} \models \bigwedge_{i=1}^n \theta_i(p_1) \leftrightarrow \theta_i(p_2)$. By the Kripke semantics, this holds if and only if at the root of $\mathfrak{M}$, denoted r , it can be forced. By definition this holds if at each successor $r \leq x'$, $x' \Vdash \theta_i(p_1)$ if and only if $x' \Vdash \theta_i(p_2)$. Now, note that for each proper successor $r < x \in \mathfrak{M}$, the model $\mathfrak{M}_x$ is actually a model of $\rho(p_1, p_2)$: on the left pentagon, the root of the pentagon satisfies this because it satisfies $\neg\neg(p_1 \wedge p_2)$, whilst the right point labelled with 00 satisfies $p_1 \leftrightarrow p_2$. For the root r , assume that $\mathfrak{M}, r \Vdash \theta_i(p_1)$. Note that since $\mathfrak{M}^{p_1} \cong \mathfrak{M}^{p_2}$ (their restrictions to these variables are isomorphic as far as the root is concerned), then by switching p_1 by p_2 , the two models must satisfy exactly the same formulas – they are the same model. Moreover, $\mathfrak{M}, r \Vdash \theta_i(p_1)$ if and only if $(\mathfrak{M})^{p_1}, r \Vdash \theta(p_1)$. But by what we just discussed, then also $\mathfrak{M}, r \Vdash \theta_i(p_2)$. The converse, assuming that $\mathfrak{M}, r \Vdash \theta_i(p_2)$, goes exactly the same way.

We thus have that $\mathfrak{M} \models \bigwedge_{i=1}^n \theta_i(p_1) \leftrightarrow \theta_i(p_2)$. But we can see that $\mathfrak{M} \not\models \rho(p_1, p_2)$: it does not validate $\neg\neg(p_1 \wedge p_2)$, since it sees the right node labelled

with 00, which is maximal, and it does not validate $p_1 \leftrightarrow p_2$, since it sees a node labelled with 01. Since $(\theta_1(p), \dots, \theta_n(p))$ was a generic sequence of formulas, we conclude that $\rho(p_1, p_2)$ is not a separating coequivalence relation.

Now we move on to **KC**. The essence of the proof is exactly the same, however, one cannot use the same coequivalence relation, since ρ as above is separating over **KC**. For that purpose, we take the following formula

$$\chi(p_1, q_1, p_2, q_2) = ((p_1 \leftrightarrow p_2) \wedge (q_1 \leftrightarrow q_2)) \vee ((p_1 \leftrightarrow q_2) \wedge (p_2 \leftrightarrow q_1)).$$

It is easy to see that χ defines a reflexive and symmetric relation. One can also check that it is transitive, by verifying the composition table in Table A.1: indeed, given a model (P, x) over $(pq)_{1,2,3}$, satisfying χ , we can ask which particular configuration of the disjuncts it satisfies – whether the p_i -match and the q_i -match, or whether it is the extremal points in the four tuple of colors which match. When put in this table form, such verification is straightforward.

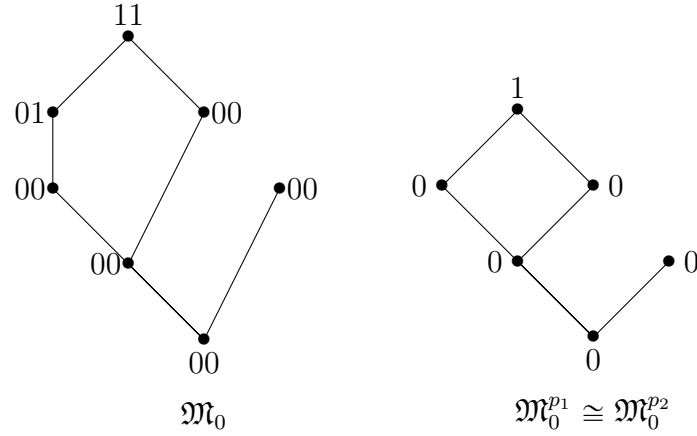
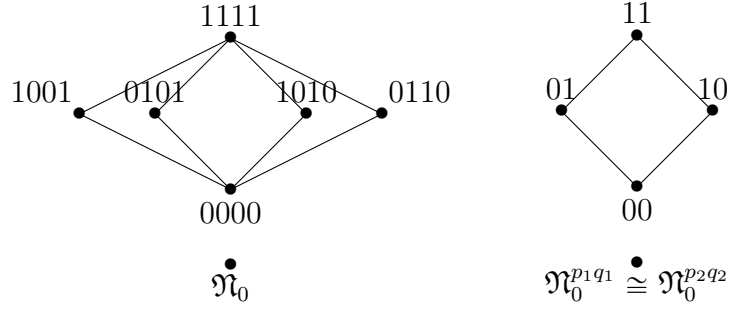
Table A.1: Composition table for χ

$(pq)_{1,2} \ (pq)_{2,3}$	1111	1010	1001	0110	0101	0000
1111	1111	-	-	-	-	-
1010	-	1010	1001	-	-	-
1001	-	-	-	1010	1001	-
0110	-	0110	0101	-	-	-
0101	-	-	-	0110	0101	-
0000	-	-	-	-	-	0000

So now consider the model $\mathfrak{M}_0$ from Figure A.2. Note again that for each node except for the root, it is a model of χ , but $\mathfrak{M}$ itself is not a model of χ . Moreover, the two projections – to p_1, q_1 and to p_2, q_2 – yield isomorphic models. Thus, by the exact same arguments as in the case for **IPC**, we conclude that there can be no tuple of formulas which are consequences of χ which also prove χ . ■

To conclude, we turn to the locally tabular logics with interpolation. For this we rely on the work from [31]. There, De Berardinis characterized exactness in the locally finite frames for some natural extensions of normal modal logic, discussed also in [32]. Explicitly, one considers a system L^∞ (see [32]): replace the category of finitely presented L -algebras with the category $\text{Pro-Alg}(L)_f$ of profinite L -algebras, whose dual is the category $L\text{Fr}_{l,f}$ of locally finite posets that validate L . The results in [31] concerning the CoSP for L^∞ can be transferred to the finitary fragment for locally tabular logics¹. The proof assumes the reader to be familiar with this work, and the notion of equivalence relation therein discussed.

¹We note that the setting considered in [32] was normal modal logic; however, as far as the frames considered, there is no difference: in both cases they are posets, and the maps between them are p-morphisms.

Figure A.1: Model $\mathfrak{M}_0$ and its (isomorphic) projectionsFigure A.2: Model $\mathfrak{N}_0$ and its (isomorphic) projections

Proposition A.4.15 (Failure of CoSP for some locally tabular logics). Let L be a locally tabular superintuitionistic logic. The following hold:

1. If L^∞ has the CoSP, then L has it, too.
2. A sufficient condition for L not to have the CoSP is that it be contained in the logic of one of the following Kripke frames:



3. If L has the CIP, then it has the CoSP if and only if $L = \text{CPC}$ or $L = \text{Sme}$.

Proof. (1) Since L is locally tabular, we have that $\text{Alg}(L)_f = \text{Alg}(L)_{\text{fp}}$. Moreover, $\text{Alg}(L)_f$ is a full subcategory of $\text{Pro-Alg}(L)_f$ and the embedding preserves finite limits (by [158, Chapter VI, Theorem 1.6]) and all existing colimits (by (ii) of [158, Chapter VI, Proposition 1.7]), among which we have all finite colimits, since $\text{Alg}(L)_f = \text{Alg}(L)_{\text{fp}}$. As a consequence, an r -equivalence relation in $\text{Alg}(L)_{\text{fp}}^{\text{op}}$, is

an r -equivalence relation in $\text{Pro-Alg}(L)_f^{\text{op}}$, too. The vice versa holds, provided that the r -equivalence relation in $\text{Pro-Alg}(L)_f^{\text{op}}$ is based on the dual of a finite (=finitely presented) L -algebra. Moreover, an r -equivalence relation in $\text{Alg}(L)_{fp}^{\text{op}}$ is effective in $\text{Alg}(L)_{fp}^{\text{op}}$ if and only if it is so in $\text{Pro-Alg}(L)_f^{\text{op}}$ (to be effective means to be the kernel pair of one's own quotient).

(2) Using the arguments from (1), in combination with [31, Proposition 5.6] – where examples of failure of effectiveness for equivalence relations were given which used the finite frames discussed – proves the claim.

(3) It is well-known that there are only six superintuitionistic logics with interpolations which are locally tabular; those excluding the frames from (2) are just CPC and Sme. The fact that these logics have the CoSP follows from (1), and [31, Thm. 6.10], where it was shown that the infinitary extensions of CPC and Sme have the CoSP: indeed, their locally finite frames form a topos: these are respectively, up to equivalence, the topos **Set**, and the topos $\mathbf{B}\mathbb{Z}_2$ of M -sets of the two element multiplicative monoid $(\mathbb{Z}_2, \times)$. ■

Theorem A.4.16 (Exact superintuitionistic logics). *A superintuitionistic logic L with the CIP is exact if and only if $L = \text{CPC}$ or $L = \text{Sme}$.*

Proof. By Proposition A.4.15.(3), if L is locally tabular and has the CIP, it is exact if and only if it is one of the two logics. The only other two superintuitionistic logics with CIP are IPC and KC, which by Proposition A.4.14 fail to have the CoSP. ■

A.5 Toposes and first-order model theory

In this appendix, we briefly discuss two questions on topos-admissible Heyting algebras. First, we discuss a key part of Pataria's approach to solving the Pitts-Pataria problem, which is public, yet folklore, knowledge. This consists of a presentation of a first-order theory of algebras which encodes the structure of an elementary topos. We do not present this specific axiomatization, but discuss how it relates to Freyd's classical elementary presentation of the theory of toposes [119]. We then elaborate on the closure conditions on topos-admissible Heyting algebras discussed in Chapter 7.

A.5.1 First-order elementary toposes

The key idea of Pataria, as it is known, is that he intended to solve the problem in the positive through using two ingredients:

1. An algebraic presentation of toposes, using so-called “Higher-order cylindric Heyting algebras”;

2. And a step-by-step procedure, inspired by the work of Ghilardi [127], namely, the construction of the free Heyting algebra as given in Chapter 4, to add quantifiers to such an algebra.

Whilst the details of the second step are entirely lost², the first step has been studied extensively: in a different setting, Andrew Pitts proved that all toposes arise out of “first-order hyperdoctrines with comprehension”, as described in [219], and Peter Johnstone, in unpublished work, showed that Pataia’s hochas (whose definitions did survive) do provide a representation of so called “minimal toposes”³.

Pataia’s hochas are generally seen, like many cylindric algebras, as quite complex objects. In this sense, it is worthwhile to look at them in connection with Freyd’s [119] elementary axiomatization of toposes: such an axiomatization makes use of two sorts, one for morphisms and one for maps, and essentially encodes all of the basic axioms of elementary toposes using their natural definitions in category theory. We briefly summarise this here:

Definition A.5.1 (First-order language of categories). The *first-order language of categories* $\mathcal{L}_{\text{Cat}}$ is specified as follows:

1. Two sorts: Ob , the objects of the category, and Mor , the morphisms;
2. Three unary operations: $\text{dom}: \text{Mor} \rightarrow \text{Ob}$ and $\text{cod}: \text{Mor} \rightarrow \text{Ob}$ and an identity operation $\text{Id}: \text{Ob} \rightarrow \text{Mor}$.
3. A ternary relation $C \subseteq \text{Mor} \times \text{Mor} \times \text{Mor}$.

We extend this with several additional symbols according to the needs of the theories; we refer to some of these as operations, though it is clear they are only defined when objects satisfy some conditions (such as being in the range of another operation):

1. (Terminal objects) Add a constant 1 together with a unary operation $t: \text{Ob} \rightarrow \text{Mor}$.
2. (Products) Add four binary (partial) operations: $\times_{\text{Ob}}: \text{Ob} \times \text{Ob} \rightarrow \text{Ob}$, $\text{proj}_i: \text{Ob} \rightarrow \text{Mor}$ for $i \in \{1, 2\}$ and $\times_{\text{Mor}}: \text{Mor} \times \text{Mor} \rightarrow \text{Mor}$.
3. (Equalizers) Add a binary operation $\text{Eq}: \text{Mor} \times \text{Mor} \rightarrow \text{Ob}$ and $\text{Eq}_{\text{Mor}}: \text{Mor} \times \text{Mor} \rightarrow \text{Mor}$.

²We do point the reader to the recent work [118], where a step-by-step construction of toposes is given by adding power objects. Crucially, in the case of the construction presented there, the power objects are added to generate a free topos, and no requirements are made of the lattice of subterminals. It is not clear to us whether such a construction could be employed to solve the problem, or whether this is connected to the intended solution by Pataia.

³Slides of a presentation by Johnstone can found in [156]

4. (Power objects) Add an operation $\text{Pow}_{\text{Ob}}: \text{Ob} \rightarrow \text{Ob}$, and $\text{Pow}_\varepsilon: \text{Ob} \rightarrow \text{Mor}$.

The axioms for categories are all the obvious first-order translations of their usual specification:

1. The operations dom and cod specify the *domain* and *codomain* of morphisms, which are unique objects. Hence we simply write $f: A \rightarrow B$ for a morphism.
2. The operation of Id picks the *identity* map, which is likewise unique. We simply write Id_A for the identity on an object.
3. The binary relation C identifies *composition*, and is a functional operation whenever the domain and codomain match appropriately. We write fg for the composition of g with f , when the domains match appropriately.
4. The terminal object map t picks a unique map from any object to the terminal object, denoted t_A for A an object, such that $t_1 = \text{Id}_1$, and such that if $f: A \rightarrow B$ is a morphism, the composition ft is equal to t_A .

For all other operations, the encoding procedure is entirely similar, and the reader is referred to Freyd for a detailed outline of the equations involved. The upshot is the following:

Theorem A.5.2 (Freyd’s axiomatization of elementary toposes). *There is a first-order theory Topos whose models are elementary toposes, and whose homomorphisms are logical functors between elementary toposes.*

Usually, size issues, and consistency issues, lead to difficulties in seeing toposes in this way – the category of sheaves over any category is always a proper class. Pataia’s hochas do not suffer from this limitation, by correctly identifying the precise algebraic core of a topos; similarly, first-order hyperdoctrines [219] can be used to solve this issue. The reader who is more happy of accepting large cardinal axioms, Grothendieck universes, or countable models, can work up to their favourite formalism. In any of these cases, we do obtain a way of translating elementary toposes into models of Freyd’s first-order theory.

Now in this language, one can define a predicate Sub_1 , which picks out exactly the subobjects of the terminal object. Given a topos $\mathcal{E}$, seen as a first-order model, we then have definable operations $(\wedge, \vee, \rightarrow, 0, 1)$ on Sub_1 which make it a Heyting algebra. Since all of these operations are first-order definable, we may then consider Topos to carry such operations as primitive symbols, yielding a definable expansion of the language. Using this the following is easily shown:

Proposition A.5.3 (Elementary embeddings of toposes give elementary embeddings of Heyting algebras). Let $F: \mathcal{E}_0 \rightarrow \mathcal{E}_1$ be an elementary embedding of elementary toposes, seen in the first-order language of categories. Then the map $F \upharpoonright_{\text{Sub}_1}: \text{Sub}_1(\mathcal{E}_0) \rightarrow \text{Sub}_1(\mathcal{E}_1)$ is an elementary embedding of Heyting algebras.

Proof. Since all Heyting operations are definable, and $\text{Sub}_1(\mathcal{E}_0)$ is a Heyting algebra under these operations, the induced map will be a homomorphism of Heyting algebras, and an embedding since it is an embedding on the whole topos. It will be elementary since any first-order sentence can be relativized to the definable subset of the subterminals, and will therefore be preserved under the embedding. ■

A.5.2 Closure conditions on topos-admissible Heyting algebras

As we remarked at the opening of Chapter 7, topos-admissible Heyting algebras are closed under the operations of products and homomorphic images. For less familiar reader, we give a short proof of this fact:

Proposition A.5.4 (Product and Homomorphic closure). The class of topos-admissible Heyting algebras is closed under products and homomorphic images.

Proof. For products, note that if $(H_i)_{i \in I}$ is a family of Heyting algebras, and $\mathcal{E}_i$ is a family of elementary toposes, such that $\text{Sub}_{\mathcal{E}_i}(1) \cong H_i$, then if we denote by $\mathcal{E} = \prod_{i \in I} \mathcal{E}_i$, the product of the toposes, we have:

$$\text{Sub}_{\mathcal{E}}(1) \cong \prod_{i \in I} H_i,$$

which follows simply because subobjects in product categories are calculated coordinatewise.

For homomorphic images, this follows from the filter-quotient construction (see e.g. [183, Sec. 4.9]): given an elementary topos $\mathcal{E}$, and F a filter of subobjects of $\text{Sub}_{\mathcal{E}}(1)$, there is an elementary topos $\mathcal{E}/F$ whose lattice of subterminal objects is precisely $\text{Sub}_{\mathcal{E}}(1)/F$. ■

A consequence of this is for instance that the class of topos admissible Heyting algebras is closed under ultraproducts. Whether it is closed under ultraroots – and therefore, forms an elementary class – is not clear at this point. Nevertheless, such elementary considerations can obtain us one more closure condition for topos admissibility of Heyting algebras, albeit in a very restrictive setting:

Definition A.5.5 (ω -categorical Heyting algebra). A Heyting algebra H is said to be ω -categorical, if it's first-order theory is ω -categorical: it's first-order theory contains exactly one countable model up to isomorphism.

The study of ω -categorical theories is of course a classical topic of model theory; the reader can consult for example [146, Ch. 7]. Several intrinsic characterizations of ω -categoricity exist, in terms of type-spaces, oligomorphic groups,

or the existence of prime and saturated models. In the setting of Heyting algebras, to the best of our knowledge, the examples of ω -categorical Heyting algebras are very sparse⁴:

1. In his work on linear orders, Rosenstein categorized all ω -categorical linear orders [233, Thm. 13.91], as assembled from finite linear orders and $\mathbb{Q}$ -like structures by sums and “shuffles”. Linear orders with endpoints can all be given the structure of Heyting algebras, in a definable way depending only on their order. So their first-order theory as Heyting algebras and as linear orders is the same, meaning that all of these are ω -categorical linear orders.
2. An analysis of ω -categorical Heyting algebras in a collection of special classes C_n was studied by Pal’chunov and Touraille [210]. In our terms, the algebras in C_n ([210, Def. III, pp. 9]) are dual to the following Esakia spaces:
 - (a) Locally directed;
 - (b) Each level Lev_k of the dual Esakia space is clopen, and each point has depth at most n ;
 - (c) Whenever $\text{Lev}_k \subseteq U \subseteq \text{Lev}_{k+1}$ where U is a clopen upset, there is a clopen regular upset V such that $U = (V \cap \text{Lev}_{k+1}) \cup \text{Lev}_k$.

It is not hard to see that in fact such Esakia spaces force the spaces to be étale-finite, in fact, to be dual to **KM**-tabular Gödel algebras (Definition A.2.13). The only missing condition is local linearity, which is easily established⁵.

Thus, of studied classes of ω -categorical Heyting algebras, the only ones which present new examples for topos-admissibility are the ω -categorical chains. Our elementarity arguments allow us to conclude that they are in fact topos-admissible, as proved by the following, the key idea of which comes from the proof of [119, Prop 2.71]:

Proposition A.5.6 (Omega-categorical topos-admissible). *Let H be such that:*

1. H is ω -categorical

⁴It was noted in [130, Ch. 7] (see also [261]) that existentially closed Heyting algebras do not provide.

⁵If local linearity fails, given we have local directedness, we can find two points at the same depth which both see the same final point, since if two such points do not exist, any two points seeing the same final point must be comparable. Assuming we have $x, y \in \text{Lev}_k$ in these condition, let U be a clopen upset such that $x \in U$ and $y \notin U$; we can assume that $\text{Lev}_k \subseteq U$, given the depth of y . By assumption, there is some regular clopen upset V such that $U = (V \cap \text{Lev}_k) \cup \text{Lev}_{k-1}$. Since x has depth k , $x \in V \cap \text{Lev}_k$. Since V is regular, it contains all points which eventually see something there, and thus must contain y – a contradiction

2. There is a complete Heyting algebra H_0 such that H embeds elementarily into a complete Heyting algebra

Then H is topos-admissible.

Proof. Let H be an ω -categorical Heyting algebra, and let H_0 be the complete Heyting algebra into which it embeds elementarily. Let $\mathcal{E}_0$ be an elementary topos such that $\text{Sub}_{\mathcal{E}_0}(1) \cong H_0$, seen as a first-order model. By the Downwards Löwenheim-Skölem theorem, there is a countable elementary submodel $\mathcal{E} \subseteq \mathcal{E}_0$. By Proposition A.5.3, we know that the elementary embedding of toposes restricts to an elementary embedding of the Heyting algebras of subterminal objects. Since $\mathcal{E}$ is countable, then $\text{Sub}_{\mathcal{E}}(1)$ is countable, embedding elementarily in H_0 ; but since H by assumption was ω -categorical, we obtain that $H \cong \text{Sub}_{\mathcal{E}}(1)$. ■

As a Corollary we get:

Corollary A.5.7 (Rationals are topos-admissible). *The algebra $1 + \mathbb{Q} + 1$ is topos-admissible.*

Proof. It is well-known (see e.g. [233, Ch. 2]) that $1 + \mathbb{Q} + 1$ is up to isomorphism the unique countable dense linear order with endpoints. Moreover, it embeds into $1 + \mathbb{R} + 1 \cong [0, 1]$, which is a complete dense linear order, and the embedding is elementary. ■

A.5.3 Boundaries of topos-admissible Heyting algebras

In Chapter 7 we noted that the class of étale-finite Heyting algebras is not contained in the class of products of a homomorphic image of a complete Heyting algebra and a Boolean algebra. If one introduces chains, which are certainly not closed under products, hypothetically, the proof could fall through. We show that this is not the case:

Proposition A.5.8 (Étale-finite not contained in class extended with chains). The algebra $2 \oplus \mathcal{P}_{\text{fin}, \text{cof}}(\mathbb{N})$ is not in the class:

$$\text{HIP}(\{\text{complete, Boolean, t.a. chains}\}),$$

where “t.a. chains” denotes the topos-admissible chains.

Proof. Similar to the argument of Remark 7.2.42, we can show that if $H = 2 \oplus \mathcal{P}_{\text{fin}, \text{cof}}(\mathbb{N})$ was in this class, it would have to be a homomorphic image of $B \times C \times P_L$, where B is a Boolean algebra, C is a complete Heyting algebra, and P_L is a product of chains which are topos admissible. Note that by Proposition 2.1.68, this would imply H to be a product of homomorphic images. Dually, this would imply it is a disjoint union of components from each of these algebras – and

since there is a single connected component in $\text{Spec}(H)$, it must be a homomorphic image of one of B , C and P_L .

As argued in Remark 7.2.42, it cannot be a homomorphic image of B or C . For P_L , we employ Corollary 3.4.9. Assume that H is a homomorphic image of P_L by the Corollary, since H is dual to an order-connected space, then $H \in \mathbb{P}_U(\{L_i : i \in I\})$, where L_i are topos-admissible chains. However, the ultraproduct of chains must be a chain, and H is certainly not a chain – a contradiction. ■

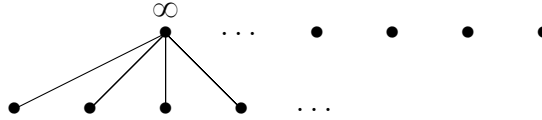
We also remarked towards the end of the chapter that the present class, containing the $\mathbb{HIP}$ -closure of étale-finite Heyting algebras and complete Heyting algebras, does not exhaust all Heyting algebras, and delegated the proof to this Appendix. Even if we add to it the topos-admissible chains, we show that this does not exhaust all Heyting algebras:

Proposition A.5.9 (Still some Heyting algebras to go). Let $\mathbf{K}$ be the following class of Heyting algebras:

$$\mathbb{HIP}(\{\text{complete, étale-finite, t.a. chains}\}),$$

where “t.a. chains” denotes the topos-admissible chains. Then the following algebras do not belong to $\mathbf{K}$:

1. The algebra $\mathcal{F}_{\mathbf{HA}}(n)$ for $n > 1$;
2. The algebra H_0 , dual of the following Esakia space:



that is, the subalgebra of $\mathcal{P}_{\emptyset, \text{cof}}(\mathbb{N}) \times \mathcal{P}_{\text{fin}, \text{cof}}(\mathbb{N})$ given by the subset

$$\{(A, B) : A = \emptyset \Leftrightarrow B \text{ is finite}\}.$$

Proof. The argument is similar for both of the algebras, and similar to Proposition A.5.8: if we assume that H is a homomorphic image of $C \times P_E \times P_L$ where C is complete, P_E is a product of étale-finite Heyting algebras, and P_L is a product of topos-admissible chains, then since H has no complemented elements, it must be a homomorphic image of one of them.

For both algebras, they cannot be homomorphic images of a complete algebra, since they are countable, but not complete: this is obvious for the subalgebra H_0 of $\mathcal{P}_{\emptyset, \text{cof}}(\mathbb{N}) \times \mathcal{P}_{\text{fin}, \text{cof}}(\mathbb{N})$, and for $\mathcal{F}_{\mathbf{HA}}(n)$ was shown by Bellissima [30, Thm. 4.2].

For both of them they cannot be homomorphic images of P_L : for H_0 , if it was, then also $\mathcal{P}_{\emptyset, \text{cof}}(\mathbb{N})$ would be a homomorphic image of P_L . $\mathcal{P}_{\emptyset, \text{cof}}(\mathbb{N})$ is dual to an order-connected space, and so again by Corollary 3.4.9, we have that H_0

would need to be a chain. Similarly, $\mathcal{F}_{\mathbf{HA}}(n)$ is finitely subdirectly irreducible (as follows easily from Proposition 4.3.4, and corresponds to the disjunction property of IPC), and is not a chain.

Finally, both of them cannot be homomorphic images of P_E : whilst we showed that products of étale-finite Esakia spaces may fail to be étale-finite (see Remark 7.2.46), we showed in Proposition A.2.12 that all étale-finite Heyting algebras are KM Heyting algebras. The fact that KM Heyting algebras are closed under products and homomorphic image 2.1.41 implies however that then H_0 or $\mathcal{F}_{\mathbf{HA}}(n)$ would have to be a KM Heyting algebra. By Proposition 2.1.61, noting that the maximum of $\mathbf{Spec}(H_0)$ is certainly not clopen, we get that H_0 cannot be a KM Heyting algebra. The fact that for $n > 1$, $\mathcal{F}_{\mathbf{HA}}(n)$ is not a KM-algebra follows from Proposition A.2.6. ■

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List of Symbols

Universal algebra

$\langle S \rangle$	Subalgebra generated by S .
$\mathbb{H}$	Homomorphic image closure.
$\mathbb{I}$	Isomorphism closure.
$\mathbb{P}$	Product closure.
$\mathbb{P}_S$	Subdirect product closure.
$\mathbb{P}_U$	Ultraproduct closure.
$\mathbb{S}$	Subalgebra closure.
$\mathbb{V}$	Variety closure.
$\mathbf{A}$	Generic variety.
$\mathbf{K}_{\text{si}}$	Subdirectly irreducibles in $\mathbf{K}$.
$\mathbf{V}$	Generic variety of $\mathbf{HA}$.
$\mathcal{F}_{\mathbf{K}}(X)$	Free $\mathbf{K}$ -algebra on X .
$\text{Alg}(L)$	Algebras of logic L .
$\text{Con}(A)$	Congruence lattice
$\text{Pro-Alg}(L)_f$	Profinite L -algebras.
$\text{E}(A, \mathbf{A}_0, g)$	One-step g -extension of $A \in \mathbf{A}$ to an $\mathbf{A}_0$ -algebra.
$\prod_{i \in I} A_i / U$	Ultraproduct of A_i over U .

$\Theta(F)$	Congruence associated to filter F .
$\Theta(S)$	Congruence generated by S .
$A[x]$	Algebra of polynomials over A .
$A \upharpoonright_{\mathcal{L}}$	$\mathcal{L}$ -reduct
F_{Θ}	Filter associated to congruence Θ .
$H_0\langle \iota \rangle$	Free étale 1-generated Heyting algebra.

Logic

3Uni	Logic of the 3-uniformity.
BD_n	Bounded depth n logic.
biIPC	Bi-intuitionistic logic.
Box	Logic of width 2 stacks.
BW_n	Bounded width n logic.
CPC	Classical logic.
Form	Set of formulas.
IK	Fischer-Servi intuitionistic modal logic.
IPC	Intuitionistic logic.
KC	Yankov logic, Logic of the Weak Excluded.
KG	Kuznetsov-Gerciu logic.
KM	Kuznetsov-Muravitsky logic.
LC	Gödel-Dummett logic, Linear calculus.
LFC	Logic of finite combs.
LTab	Class of locally tabular logics.
PL	Positive logic.
RN	Logic of the Rieger-Nishimura lattice.
Sme	Smetanich logic, logic of Here-and-There.

Tab	Class of tabular logics.
ULTab	Class of uniformly locally tabular logics.
wPL	Weak Peirce's logic.
$\vdash$	Generic Implicative logic.

Lattices

0	Bottom element in a lattice.
1	Top element in a lattice.
$\bigvee / \bigwedge$	Arbitrary supremum/infimum in a lattice.
$\exists_f / \forall_f$	Left adjoint to lattice homomorphism.
$\text{Fil}(S)/\text{Id}(S)$	Filter/Ideal generated by S .
$\text{Bel}_L(n)$	Bellissima algebra of the n -universal model.
$\text{CloUp}(X)$	Lattice of clopen upsets of Priestley/Esakia space.
$\text{Def}_L(n)$	Definable upsets of n -universal model.
$\text{Fil}(L)$	Filters of lattice L .
$\text{Up}(P)$	Lattice of upsets of poset P .
$\mathcal{P}_{\text{cof}, \emptyset}(\omega)$	Lattice of cofinite subsets of ω and the emptyset.
$a \wedge b / a \vee b$	Meet/Join of a and b in a lattice.

Categories

Esa_{SLH}	Esakia spaces with spectral local homeomorphisms.
EsaÉt (X)	Esakia étale-spaces over X .
Ét_{H₀}	Variety of H_0 -étale algebras.
biERS	bi-Esakia locally linear spaces.
biEsa	bi-Esakia spaces and continuous bi-p-morphisms.
biGD	bi-Gödel algebras.
biHA	bi-Heyting algebras.

biKC	bi-DeMorgan Heyting algebras.
CABA	Complete and atomic Boolean algebras.
CoAlg(F)	Coalgebras for an endofunctor F .
C_∞DL	Profinite distributive lattices and complete homomorphisms.
DL	Distributive lattices.
Esa	Esakia spaces and continuous p-morphisms.
FinBA	Finite Boolean algebras.
FinHA	Finite Heyting algebras.
FinSet	Finite sets and functions.
GA	Gödel algebras.
HA	Heyting algebras.
HA_{H_0}	Variety of all H_0 -algebras.
HA_{lf}	Locally finite Heyting algebras.
HA_{lf}^n	Variety of n-uniform Heyting algebras.
KC	DeMorgan Heyting algebras.
KM	KM-algebras.
LocFinK	Locally finite modal intuitionistic frames.
LocFinPos	Locally finite posets with p-morphisms.
OZD	Order-zero-dimensional spaces.
Pos	Posets with order-preserving maps.
Pos_p	Posets with p-morphisms.
Pries	Priestley spaces and continuous order preserving map.
ProHA	Profinite HA and complete Heyting homomorphism.
Sme	Smetanich algebras.
Stone	Stone spaces.
Top	Topological spaces.

$\mathbf{Top}_{\text{LH}}$	Topological spaces with local homeomorphisms.
$\mathcal{E}$	Generic elementary topos.

Logical Symbols

$(\mathfrak{M}, x)$	Kripke model.
χ_w	Normal form associated to point w .
Γ, Δ	Sets of formulas.
$\Gamma \vdash \varphi$	Rule.
$\leftrightarrow$	Bi-implication.
$\leq_{\infty, \bar{p}} / \leq_{n, \bar{p}}$	(n-)Order-bisimilarity relation.
$\mathcal{J}(Q)$	Yankov formula of rooted poset Q .
$\mathcal{M}_{\bar{p}}(\mathbf{P})$	Class of K. models over $\mathbf{P}$.
$\mathfrak{M}^{\bar{p}}$	Restriction of a Kripke model to variables $\bar{p}$.
$\mathfrak{M}^{\sim}$	Bisimulation quotient of Kripke model.
$\text{Ext}(\vdash)$	Lattice of extensions of a logic.
$\text{Fm}_{\mathcal{L}}(\text{Prop})$	Formulas.
$P(\varphi)$	Uniform positive approximation of φ .
$\text{Th}_n(S)$	Depth n theory of class of models S .
$\bar{p}$	Tuple of distinct variables.
φ	Formula/term.
$\rho(\bar{p}_1, \bar{p}_2)$	Coequivalence relation over $\bar{p}$.
$\sim_{\bar{p}} / \sim_{n, \bar{p}}$	(n-)Bisimilarity relation.
$\tilde{\forall}q, \tilde{\exists}q$	Bisimulation quantifiers.
$\Vdash$	Forcing relation on a Kripke model model.
$L \oplus S$	Smallest logic extending L containing the axioms from S .
p, q, r	Propositional letters.

Miscellaneous

$(\mathbf{RP}(L), \leqslant)$	Rooted posets of L ordered by $\mathbb{HS}$ order.
$(A_n, \iota_{i,j})$	Directed system (of algebras)
$(X_n, p_{i,j})$	Inverse system (of spaces)
$\mathcal{P}_{\emptyset, \text{cof}}(\mathbb{N})$	Cofinite subsets of naturals and empty set.
δ_n	n -th stability map in Ghilardi complex.
η/ε	Unit/counit of adjunction.
$\mathcal{P}_{\text{fin}, \text{cof}}(\mathbb{N})$	Finite and cofinite subsets of naturals.
$\langle V \rangle$	Subsets C such that $C \cap V \neq \emptyset$.
$\mathbf{V}_{\text{ref}}$	Epi-subcategory reflector.
$\mathcal{D}_a$	Dense elements over a .
$\mathfrak{A}$	Adjoint translation of IPC in PL.
$\mathfrak{S}$	Structural interpretation of IPC in PL.
$\text{Cen}(H)$	Complemented elements of Heyting algebra H .
$\text{Reg}(H)$	Regular elements of Heyting algebra H .
μ	Multiplication of a monad.
$\omega/\mathbb{N}$	Natural numbers.
$\bar{a}/\theta$	Imaginary element over θ .
$\prod_{i \in I} A_i$	Cartesian product of algebras.
τ	Successor/Kuznetsov-Muravitsky operator.
$\llbracket U \rrbracket$	Subsets C such that $C \subseteq U$.
A_ω	Directed colimit directed system
$d_p(x)$	Depth of a point x in a poset.
$F \dashv G$	F left adjoint to G .
$f \upharpoonright_U$	Restriction of map along subset
l_n	Iterated adjoint, adjoint to $\pi_n: \mathcal{V}_G(X) \rightarrow X_n$.

$w_p(x)$	Width of a point x in a poset.
X_ω	Inverse limit inverse system
$\uparrow^\omega$	Adjoint to the projection π_0 in Ghilardi complex.
$\uparrow_n$	Upset map, adjoint of root $r_n: X_n \rightarrow X_{n+1}$ to in Ghilardi complex.

Topological Symbols

$(P, \leq)$	Poset.
$(X, \leq, \tau)$	Ordered space/Priestley Space/Esakia space.
β_0	Banaschewski compactification.
$\square C$	Dual of downwards closure: $X \downarrow (X \setminus C)$.
$\text{Disc}(X)$	One-step discrete extension of X .
η/η_0	Nachbin/Order-zero-dimensional order-compactification.
$\mathbf{2}_{\leq}$	Two element chain $\{0 < 1\}$.
$\mathcal{C}(F)$	Closed upset determined by filter F .
$\mathcal{G}(X)/\mathcal{G}^g(X)$	Ghilardi/ g -Ghilardi complex over X .
$\mathcal{G}_{\mathbf{V}}(X)$	Subvariety relativised Ghilardi complex over X .
$\mathcal{I}$	Family of closed downsets.
$\mathcal{K}_0(X)/\mathcal{K}_E(X)$	Poset of (Priestley/Esakia) order-compactifications.
$\mathcal{P}(X)$	Discrete Ghilardi complex over X .
$\mathcal{P}_G(X)$	Inverse limit of discrete Ghilardi complex over X .
$\mathcal{X}(n)$	Esakia dual of the free n -generated free Heyting algebra.
$\mathfrak{F}$	Family of closed upsets.
$\text{Lev}_k(P)$	k -th level of the poset P .
Rep_n	Representable points in n -th step of internal Ghilardi complex.
$\mathbf{B}_n$	Dual of the Bellissima n -algebra.

$\text{CC}(X)$	Carai construction on Priestley space X .
$\text{CC}(X)$	Closed chains of X .
$\text{CC}_2(X)$	Carai construction of depth 2.
$\text{CC}_2^*(X)$	Dual of Smetanich étale algebra.
$\text{CC}_2^{\text{pol}}(X)$	Polynomial Smetanich space.
$\text{Sep}_2(X)$	Smetanich separable upsets.
$\text{Spec}(D)$	Prime spectrum of distributive lattice.
$\text{Spec}(D)_S$	Subset of $\text{Spec}(D)$ determined by a set of pairs $S \subseteq D \times D$.
$\max(C)/\min(C)$	Maximal/minimal points in C .
$\text{Nerve}(X)$	One-step extension of chains of X .
$\overline{A}$	Topological closure of A .
$\preceq$	Total covering relation.
$\prod_{i \in I} X_i$	Cartesian product of spaces/posets.
$\tau(U)$	Topological KM operation: $U \cup \max(X \setminus U)$.
$\mathcal{U}(n)$	n -universal model.
$\mathcal{V}(X)$	Vietoris space of X .
$\mathcal{V}^\uparrow(X)$	Vietoris of upsets of X .
$\mathcal{V}_G(X)$	Inverse limit of Ghilardi-complex.
$\mathcal{V}_g(X)$	Vietoris of g -open rooted subsets of X .
$\mathcal{V}_r(X)$	Vietoris of rooted subsets of X .
${}^rP$	Poset P with new root adjoined.
P^{op}	Dual order on a poset.
r_W	Endomorphism on $\mathcal{V}(X)$ sending C to $C \cap W$.
$U \Rightarrow V$	Classical implication on subsets: $X \setminus U \cup V$.
$U \rightarrow V$	Heyting implication on subsets: $\Box(U \Rightarrow V)$.
$X \times_Y Z$	Cartesian pullback of spaces/posets.
$\uparrow C / \downarrow C$	Upwards/Downwards closure of a subset C .

Samenvatting

In dit proefschrift bestuderen we representaties en de algemene modeltheorie van vrije Heyting-algebra's, vrije producten van Heyting-algebra's en verscheidene verwante constructies en concepten. In het bijzonder gebruiken we systematisch de dualiteitstheorie om de geometrische structuur van deze eigenschappen te onthullen, met een focus op de logische analyse van de intuïtionistische logica en haar uitbreidingen. De meest directe resultaten van een dergelijk onderzoek betreffen de theorie van normale vormen en haar toepassingen, de studie van vertalingen tussen intuïtionistische logica en haar fragmenten, en de ontwikkeling van instrumenten voor andere verwante systemen zoals bi-intuïtionistische logica of intuïtionistische modale logica. Dit vindt verdere toepassingen in de theoretische analyse van deze eigenschappen zoals interpolatie, of in de uniforme ontwikkeling van co-algebraïsche semantiek.

De in het eerste deel van het proefschrift ontwikkelde representaties worden vervolgens gebruikt om enkele openstaande problemen in het vakgebied op te lossen, met betrekking tot de modeltheorie van vrije Heyting-algebra's. Deze zijn tweeledig: de studie van uniforme grenzen aan de complexiteit van vrije algebra's leidt ons tot het concept van uniforme lokale tabulariteit, waar we een vraag van Shehtman over de equivalentie tussen dit concept en lokale tabulariteit beslechten. Aan de andere kant leidt de studie van hogere-orde uitbreidingen van de intuïtionistische logica ons naar een vraag van Andrew Pitts over welke Heyting-algebra's optreden als tralies van waarheidswaarden van elementaire topoi, waarbij we de klasse van algebra's waarvoor een oplossing bekend is, uitbreiden.

Abstract

In this thesis we study representations and general model theory of free Heyting algebras, free products of Heyting algebras, and several related constructions and concepts. In particular, we systematically employ duality theory to uncover the geometric structure of those properties, with a focus on the logical analysis of intuitionistic logic, and its extensions. The most direct outcomes of such a study concern the theory of normal forms and its applications, the study of translations between intuitionistic logic and its fragments, and the development of tools for other related systems such as bi-intuitionistic logic, or intuitionistic modal logic. This finds further uses in the theoretical analysis of properties like interpolation, or in the uniform development of coalgebraic semantics.

The representations developed in the first part of the thesis are then used to settle some open problems in the field, concerning the model theory of free Heyting algebras. These are twofold: the study of uniform bounds on the complexity of free algebras leads us to the concept of uniform local tabularity, where we settle a question by Shehtman over the equivalence between this concept and local tabularity. On the other hand, in studying higher-order extensions of intuitionistic logic, we are lead to a question of Andrew Pitts about which Heyting algebras arise as lattices of truth-values of elementary toposes, where we extend the class of algebras for which a solution is known.

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