

On elementary classifications of perfectoid fields

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Introduction

This thesis investigates the model theory of *perfectoid fields*, and revolves around the general project of classifying perfectoid fields, i.e. identifying an elementary class whose members match as closely as possible properties of perfectoid fields. We will investigate the question working in the language of valued fields, and also in the language of rings. Perfectoid fields are a class of valued fields whose original *raison d'être* was to transfer across characteristics. They have however proven extremely useful since, and have a number of applications, the most important one being a crucial advance in the *weight-monodromy conjecture*, a major problem of arithmetic geometry. See [18] for a survey.

Perfectoid fields may be defined in a number of different ways, depending on the choice of underlying language, most notably as *valued fields*, or as *topological rings*. These definitions are materially equivalent, in the sense that they describe the same class of structures, but they differ slightly in the restricted point of view of a working logician, as they are given in two different languages. For the sake of completeness, and in order to underline the richness of perfectoid fields, we will provide both points of view here, but we may already notice that the topological point of view shall be of little use to us - other than moral - as the language of topology is pretty adverse to first-orderisation attempts.

Our goal here is to find first-order analogues to the definitional properties of perfectoid fields, and axiomatise a class of models satisfying the analogue properties. A clear criterion of success will be that : 1. every perfectoid field should be a model of our theory, 2. every model of our theory should be elementarily equivalent to a perfectoid field. The characterisation up to elementary equivalence, rather than isomorphism, is a forced consequence of fundamental limitative results of model theory, for e.g. Löwenheim-Skolem theorems. At the cost of a relative lack of specificity, i.e. the existence of non-standard models, we aim to give an elementary theory, whose models still capture the most important properties of perfectoids.

The general plan of this thesis is as follows. The first chapter is a gentle introduction to perfectoid fields, mostly motivated as a bridge between characteristics. Chapters 2 and 3 present more model theoretic aspects of valued fields. The following chapters, which make the core of the thesis introduce a class of fields almost capturing perfectoids in the language of valued fields. The last chapter, the only one with a claim to originality, resolves a modified version of this classification problem, in the language of rings.

Notations

- (Kv) , occasionally $(K, \mathcal{O})$ or K : a valued field K with valuation v (or valuation ring $\mathcal{O}$)
- $\mathcal{O}_K$ or $\mathcal{O}_v$: the valuation ring of some valued field (K, v)
- $\mathfrak{m}$ or $\mathfrak{m}_K$: the maximal ideal of the ring $\mathcal{O}_K$
- vK : the value group of a valued field (K, v)
- Kv : the residue field of a valued field (K, v)
- ϖ : (reads as π) a pseudo-uniformiser for the valuation
- Γ, Δ, Ξ : always denotes an ordered abelian group
- $\widehat{K}$: the topological completion of K
- K^{alg} : the algebraic closure of K
- K^s : the separable closure of K
- K^h : the henselization of K
- $K(t^{\frac{1}{p^\infty}})$: the field obtained by adjoining all p^n -th roots of an element t of K
- $|\cdot|_p$: the p -adic absolute value
- $|\cdot|_\infty$: the archimedean absolute value, i.e. the usual absolute value
- $\varprojlim K_i$ or $\varprojlim_{\varphi_j} K_i$: the projective limit of an inverse system (with morphisms φ_j)
- $\flat, \sharp$: the tilting map
- $K^\flat$: the perfectoid tilt of a field K
- $\equiv$ or $\equiv_?$: elementary equivalence (with respect to the language $\mathcal{L}_?$)
- $\preceq$ or $\preceq_?$: elementary embedding (with respect to the language $\mathcal{L}_?$)
- $\mathcal{C}$: always denotes a class of structures of interest
- K^* or $K^{\mathcal{U}}$: an ultrapower over K , occasionally, an ultraproduct of fields K_i (with respect to a non-principal ultrafilter $\mathcal{U}$)
- x^* : the image of an element x by an ultrapower diagonal map, or occasionally simply an element of an ultrapower/ultraproduct (the $*$ is sometimes omitted)
- v^* : the natural valuation on an ultrapower (K, v)

I Transfer principles, valuation theory, perfectoid fields

1.1 Lefschetz principle

A first concrete motivation for our endeavour may be found in the following

Theorem 1.1 (Lefschetz principle). Let $\varphi \in \mathcal{L}_{rng}$ be a statement in the language of rings. Then the following are equivalent:

1. $\mathbb{C} \models \varphi$.
2. $k \models \varphi$ for k an arbitrary algebraically closed field of characteristic 0.
3. $k \models \varphi$ for k of characteristic p *arbitrarily* high.
4. $k \models \varphi$ for k of characteristic p *sufficiently* high.

Proof. $1 \Leftrightarrow 2$ is an application of completeness of the theory ACF_0 .

$2 \Rightarrow 4$ Suppose φ holds in a characteristic 0 algebraically closed field k . By Gödel's completeness theorem and finiteness of proofs of φ , there is a finite subset $\Delta \subsetneq \text{ACF}_0$ such that $\Delta \models \varphi$. Such a Δ may contain only a finite number of claims about the characteristic of k , and there are models of $\text{ACF} \cup \Delta$ of any characteristic past this bound.

$3 \Rightarrow 2$ By contraposition. Suppose no field of characteristic 0 satisfy φ . Then, by completeness $\text{ACF}_0 \models \neg\varphi$, implying by the previous claim that any field of sufficiently high characteristic satisfies $\neg\varphi$, which is a contradiction.

$4 \Rightarrow 3$ is obvious. ■

Now morally, the Lefschetz principle allows one to transfer information between characteristic 0 and positive characteristic, but in an *asymptotic* way, i.e. by letting p go to infinity. One of the most interesting features of perfectoid fields is that they provide a way to transfer such information for a fixed value of p , in other word they offer a non-asymptotic transfer principle. More precisely, this transfer principle, called *tilting*, defines a functor between the category of fields of mixed characteristic and positive characteristic, while the Lefschetz principle does not offer anything remotely as concrete.

1.2 Numbers and functions, and a first sketch of the tilting correspondence

Another point of view on the idea of bridging phenomena in characteristic 0 and positive characteristic is the duality between elements of $\mathbb{Q}$ and $\mathbb{F}_p(t)$ (numbers and function), which can be thought of as fractions over primes or irreducible polynomials over some finite field $\mathbb{F}_p[t]$. This motivates the terminology classifying global fields between *function fields*, and *number fields*. Number fields are finite extensions of $\mathbb{Q}$, while function fields are fields of power series over a prime field $\mathbb{F}_p$. A parent analogy relates local fields, i.e. completions of $\mathbb{Q}$ with respect to a topology to a function field.

A first sketch of a formal bridge between characteristics may be given by noticing the syntactic resemblance between elements of function fields and number fields (given a choice of numerary basis). In both cases, one may write such an element as a formal sum $\sum_{\Gamma} \alpha_{\gamma} x^{\gamma}$ where Γ is some group of indices, and x is a designated element of the underlying field. The difference between polynomials (or formal series) and numbers resides in the phenomenon of *carrying* which occurs when computing addition on numbers, but not on polynomials.

Let us show this starting from the ring of integers, and the ring of polynomials over $\mathbb{F}_p$, and see how the correspondence is preserved through various completions, namely, taking the fraction fields, taking "perfectisation", and completing topologically. The diagram below¹ presents the steps to construct "simplest" pair of perfectoid fields, starting from the analogy between $\mathbb{Z}$ and $\mathbb{F}_p[t]$.

Example 1 (A construction of the simplest perfectoid pair).

$$\begin{array}{ccccc}
\mathbb{Z} & \sum_{n=0}^N a_n p^n & \sum_{n=0}^N a_n t^n & & \mathbb{F}_p[t] \\
\cap & & & & \cap \\
\mathbb{Z}_p & \sum_{n=0}^{\infty} a_n p^n & \sum_{n=0}^{\infty} a_n t^n & & \mathbb{F}_p[[t]] \\
\cap & & & & \cap \\
\mathbb{Q}_p & \sum_{-\infty \ll n}^{+\infty} a_n p^n & \sum_{-\infty \ll n}^{+\infty} a_n t^n & & \mathbb{F}_p((t)) \\
\cap & & & & \cap \\
\mathbb{Q}_p(p^{\frac{1}{p^{\infty}}}) & \sum_{n \in \mathbb{Z}[\frac{1}{p}]} a_n p^n & \sum_{n \in \mathbb{Z}[\frac{1}{p}]} a_n t^n & & \mathbb{F}_p((t^{\frac{1}{p^{\infty}}})) \\
\cap & & & & \cap \\
\widehat{\mathbb{Q}_p(p^{\frac{1}{p^{\infty}}})} & \sum_{i \in \mathbb{N}} \left(\sum_{n_i \in \mathbb{Z}[\frac{1}{p}]} a_{n_i} p^{n_i} \right) & \sum_{i \in \mathbb{N}} \left(\sum_{n_i \in \mathbb{Z}[\frac{1}{p}]} a_{n_i} t^{n_i} \right) & & \widehat{\mathbb{F}_p((t^{\frac{1}{p^{\infty}}}))}
\end{array}$$

Starting from the number ring $\mathbb{Z}$ and the polynomial ring $\mathbb{F}_p[t]$, one constructs the Laurent series field $\mathbb{F}_p[[t]]$ whose analogue on the number side is the p -adic integers. In this regard the p -adic integers may be seen as analogous to considering formal series in the "indeterminate" p . However, the construction is in fact somewhat asymmetric at this stage, in so far as $\mathbb{Z}_p$ supposes a choice of topology, the p -adic topology, for which $\mathbb{Z}_p$ is the completion

¹This way of presenting the construction is essentially borrowed from presentations by Bernard Le Stum, and Bhargav Bhatt

of $\mathbb{Z}$. In a sense, the analogy between both sides is becoming less nominal now that it is not a mere syntactic resemblance. Passing to fraction fields, one gets the p -adic numbers and Hahn series in the variable t . The next step is "perfection", i.e. adding p -th roots to every elements of the fields $\mathbb{Q}_p$, and $\mathbb{F}_p((t))$. Finally, one may complete those fields to obtain the first pair of perfectoid fields $\widehat{\mathbb{Q}_p(p^{\frac{1}{p^\infty}})}$ and $\widehat{\mathbb{F}_p(t^{\frac{1}{p^\infty}})}$.

At each step, the pair of rings or fields under consideration is presented together with a way of representing an element. The form of the elements should make abundantly clear that the syntactic analogy is preserved. In fact, much more is true, and a class of statements about perfectoid fields can be transferred to the tilt.

The first remarkable such statement was given by the theorem of Fontaine-Wintemberger, and is largely the starting point of the interest in tilting.

Theorem 1.2 (Fontaine-Wintemberger [6]). The absolute Galois groups of the fields $\mathbb{Q}_p(t^{\frac{1}{p^\infty}})$ and $\mathbb{F}_p(t^{\frac{1}{p^\infty}})$ are isomorphic.

This theorem was greatly generalised by the theory of perfectoid fields by Scholze in [17], which we will present in section 1.6.3.

In the second half of this thesis, we will investigate elementary classes arising from perfectoid fields. Instead, we will use elementary replacements, which may be thought of as the shadow of second-order properties on elementary logic.

1.3 Valued fields

Definition 1.1 (Valuation). A *valuation* on a field K is a morphism $v : K \rightarrow \Gamma \cup \{\infty\}$ for Γ an ordered abelian group, called the *value group*, such that

1. $v(x) = \infty \Leftrightarrow x = 0$
2. $v(xy) = v(x) + v(y)$
3. $v(x + y) \geq \min(v(x), v(y))$

Valuations generalise the notion of absolute value in a rather interesting way, since while absolute value is intuitively a geometric concept, i.e. the distance to zero, the most intuitive examples of valuations come from arithmetic. In this regard, this common generalisation of geometric and arithmetic concepts may be thought of as part of a bridge between mathematical settings.

Now let us introduce the central algebraic object of our study.

Definition 1.2 (Valued field). A *valued field* (K, v) is a field K equipped with a valuation v .

Let us now define some associated *auxiliary structures*. One of the key features of valued field is the transfer of properties between these related components.

Definition 1.3 (Valuation ring, maximal ideal, residue field). If (K, v) is a valued field, the *valuation ring* $\mathcal{O}_v$, or $\mathcal{O}_K$, of K is the ring obtained by restricting K to elements of positive

valuation, i.e. $\mathcal{O}_v = \{x \in K : v(x) \geq 0\}$. Equivalently, it is a subring of K containing at least one of x or x^{-1} , for every $x \in K \setminus \{0\}$, the elements of valuation 0 being exactly the invertible elements of $\mathcal{O}_v$.

The *maximal ideal* of $\mathcal{O}_v$ is the ideal of all strictly positive valuation : $\mathfrak{m}_K = \{x \in K : v(x) > 0\}$. Equivalently, $\mathfrak{m}_K = \mathcal{O}_K \setminus \mathcal{O}_K^\times$.

The *residue field* Kv is the quotient $\mathcal{O}_K/\mathfrak{m}_K$.

Notice that the residue field is by construction a field whose elements are elements K of valuation 0. It may fail to be a subfield of K . In fact, it may be of positive characteristic, even if K is of characteristic 0.

Definition 1.4 (Characteristic of a valued field). A valued field (K, v) is

1. of *equicharacteristic* $(0, 0)$ if $\text{char}(K) = \text{char}(Kv) = 0$
2. of *equicharacteristic* $(0, p)$ if $\text{char}(K) = \text{char}(Kv) = p$
3. of *mixed characteristic* $(0, p)$ if $\text{char}(K) = 0$ and $\text{char}(Kv) = p$

Remark that a "mixed characteristic $(p, 0)$ " is impossible, as the residue field is obtained from a quotient of a subring of the field.

Example 2. The p -adic valuation, which we will observe in some details later on (see definition 1.10).

Example 3 (Polynomial valuation). Let $f \in \mathbb{A}[t]$ be a polynomial over a ring. The t -adic valuation of f in $\mathbb{A}[t]$ is the minimum non-zero exponent of f .

$$v_t \left(\sum_{i=0}^n a_i t^i \right) = \min\{j : a_j \neq 0\}$$

Furthermore, it extends naturally to a valuation on other "polynomial-like" such as the Laurent series

$$v_t \left(\sum_{i=0}^{\infty} a_i t^i \right) = \min\{j : a_j \neq 0\}$$

and the Hahn series field

$$v_t \left(\sum_{i \in \Gamma} a_i t^i \right) = \min\{j : a_j \neq 0\}$$

Furthermore, in the spirit of the construction from example 1, one may notice that this reduces to p -adic valuation if one considers a ring of numbers, whose elements are seen as "polynoms in the variable p ". On the other hand, the t -adic valuation does not depend on the higher order terms of the element, so it also extends to larger classes of rings of formal series, such as Laurent series, Hahn Series etc.

Let us finish with a less algebraic example (for the sake of completeness).

Example 4 (Hardy Field). A *Hardy field* (over an archimedean field, typically $\mathbb{R}$) is a field whose elements are *germs* of functions at infinity (i.e. equivalence classes for the relation "being equal in a neighbourhood of $+\infty$ "). The valuation is given by asymptotic comparison: two elements have the same valuation if they are asymptotically comparable, i.e. $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} \neq 0$, and $v(0) := \infty$.

1.3.1 Topology, completion, henselisation

Several convenient languages may be used to study valued fields. Let us first mention a non elementary characterisation. Since a valuation induces a topology on a field, valued fields may be seen as instances of topological fields, and studied as such. Note that distinct valuations may induce the same topology – in which case they are called *equivalent*, but the topology carries the relevant information.

Definition 1.5 (Topological field). A *topological field* is a field K equipped with a topology such that the addition and multiplication (seen as maps $K \times K \rightarrow K$) and inverse map are continuous.

Proposition 1.1. Valued fields are topological fields.

Proof. Since the balls are a basis for the topology, it suffices to show that the preimages of an open ball under the addition and multiplication maps are open. This is essentially an argument in terms of the " ε, δ " definition of a limit. ■

As such, concepts from the realm of topology or normed fields transfer to valued fields. The topology will of course be the topology induced by the valuation.

Definition 1.6 (Cauchy sequence on a valued field). Let (K, v) be a valued field. A *Cauchy sequence* on (K, v) is a sequence (x_n) such that

$$\forall \varepsilon \in \Gamma. \exists N \in \mathbb{N}. \forall n, m > N. v(x_n - x_m) > \varepsilon$$

Definition 1.7 (Complete valued field). A complete valued field (K, v) is a valued field which is topologically complete, i.e. every Cauchy sequence converges.

Lemma 1.1 (Non-zero Cauchy sequences over non-archimedean valued fields are stationary). Let (K, v) be a non-archimedean valued field. Any non-zero Cauchy sequence (x_n) is stationary, i.e. constant after a certain rank.

Proof. Let (x_n) be a non-zero Cauchy sequence over a field (K, v) . By the Cauchy property for sequences on valued fields, it verifies :

$$\forall \varepsilon \in \Gamma \cup \{\infty\}. \exists N \in \mathbb{N}. \forall m, n > N. [v(x_n - x_m) > \varepsilon]$$

Furthermore, from the valuation axioms, it follows that $v(x_n - x_m) \geq \min(v(x_n), v(x_m))$ with equality if $v(x_n) \neq v(x_m)$.

Now, we consider ε such that :

$$v(x_n - x_m) > \varepsilon \geq \min(v(x_n), v(x_m))$$

By the assumption that our series does not tend to 0, the valuation does not tend to ∞ , so the right ε must eventually exist. Now, the inequality implies in particular that $v(x_n - x_m)$ is different from $\min(v(x_n), v(x_m))$, and therefore $v(x_m) = v(x_n)$. In other words, our series eventually reaches a stable value. ■

For a very elementary, but fundamental, consequence of this result, the halting of Euclid's division algorithm could for example be derived from the fact that this algorithm defines a Cauchy sequence over the field of the rationals equipped with the p -adic valuation (for p a prime divisor).

More important to our purpose is that this allows us to define the topological completion of a valued field.

Theorem 1.3 (Completion of a valued field). Any valued field (K, v) has a *completion* $(\widehat{K}, \widehat{v})$, i.e. an extension of K , with a valuation prolongating v , which is topologically complete, and minimal among complete extensions of (K, v) . In other words, it satisfies the following universal property (where (L, w) denotes an arbitrary complete valued field):

$$\begin{array}{ccc} & & (L, w) \\ & \nearrow & \uparrow \exists! \\ (K, v) & \hookrightarrow & (\widehat{K}, \widehat{v}) \end{array}$$

Proof. Consider the set of Cauchy sequences with value in K . It admits a field structure by defining addition and multiplication point-wise. Valuations on constant sequences are given by valuation of the corresponding element in K . Furthermore, since by lemma 1.1, any Cauchy sequence is equivalent to a constant one, one can extend the valuation through the equivalences. ■

Definition 1.8 (Hensel's lemma in a valued field). For any polynomial of the valuation ring $f \in \mathcal{O}_v[X]$, $a \in \mathcal{O}_v$, with $v(f(a)) > 2v(f'(a))$, there is a root $b \in \mathcal{O}_v$, $f(b) = 0$ such that $v(a - b) > v(f'(a))$.

Definition 1.9 (Henselian field). A valued field $(K, \mathcal{O})$ is *henselian* if it satisfies one of the following equivalent conditions

1. Its valuation ring prolongs uniquely to any algebraic extension L/K .
2. Its valuation ring prolongs uniquely to finite extensions L/K .
3. Its valuation ring prolongs uniquely to the separable closure K^s .
4. Its valuation ring satisfies Hensel's lemma.

Proof. See ([4], section 4.1) for a proof of the equivalence. ■

Henselianity interacts quite nicely with notions of algebraicity, as we may see from the two following preservation properties. Let us recall that a *relatively algebraically closed* subfield $K \subset L$ is a subfield such that any algebraic element of K in L is already in K , i.e. $K = K^{alg} \cap L$.

Proposition 1.2. Henselianity passes to algebraic extensions.

Proof. This follows directly from the definition, and from the fact that the composition of two algebraic extensions is algebraic. ■

Proposition 1.3 (Relative algebraic closed subfield and henselianity). Let (K, v) be a henselian valued field, $L \subseteq K$ a relatively algebraically closed subfield of K . Then $(L, v|_L)$ is henselian.

Proof. A polynomial in $\mathcal{O}_L$ is in particular a polynomial of $\mathcal{O}_K$, so if it meets the conditions of Hensel's lemma, one can apply henselianity in K to obtain a root, but by algebraicity of L , this root already lies in L . ■

Theorem 1.4 (Henselization). Any valued field K has an henselian extension K^h , which is unique up to isomorphism, as the "smallest" such extension (i.e. K^h is initial in the category of henselian field extensions of K).

$$\begin{array}{ccc} & & (L, w) \\ & \nearrow & \uparrow \exists! \\ (K, v) & \longrightarrow & (K^h, v^h) \end{array}$$

Proof. See [14], chapter 5, theorem 2. ■

Let us observe that the henselianity property is weaker than completeness, in the sense that completeness entails henselianity.

Proposition 1.4. Any complete valued field (K, v) , with an archimedean value group, is henselian.

Proof. One may show that finite extensions over complete fields extend uniquely (to complete fields), and then simply apply our definition of henselian fields. ■

1.3.2 Languages and settings to study valued fields

As topological fields, as was seen in the previous section.

As fields with a valuation map. Now, there are a number of elementary languages available to work with this map. The first one is the language of rings $\mathcal{L}_{rng} = \{+, \cdot, 0, 1\}$. Alternatively, extensions of $\mathcal{L}_{rng}$ may capture phenomena in the auxiliary structures. First is the language of valued fields, which adds a sort for the valuation group $\mathcal{L}_{val} = \mathcal{L}_{rng} \cup$

$\{v : K \rightarrow \Gamma, 0_\Gamma, +_\Gamma, \leq_\Gamma\}$. Another one is the language of Denef-Pas $\mathcal{L}_{DP} = \mathcal{L}_{val} \cup \{\text{res}, +_k, \cdot_k, 0_k, 1_k\}$ which furthermore adds a sort and symbols for the residue field.

As the data of a field and a (specific) affine scheme. If K is a valued field with valuation ring $\mathcal{O}_K$, then the prime ideals of $\mathcal{O}_K$ are in correspondence with the convex subgroups of Γ . Now, the affine scheme over $\mathcal{O}_K$ has the prime ideals as its points, and the elements of $\mathcal{O}_K$ as its regular functions. This is not terribly practical for our purpose, but does suggest a nice connection between valued fields and a class of schemes when asking the question: what schemes are obtained from the affine schemes defined in this way?

1.4 p -adic valuations and fields

p -adic fields are obtained by considering the completion of $\mathbb{Q}$ with respect to the topology induced by so-called p -adic valuations.

Definition 1.10 (p -adic valuation). Let $p \in \mathbb{N}$ be a prime integer. The p -adic valuation on the integers is the following application:

$$\begin{aligned} v_p : \mathbb{Z} &\rightarrow \mathbb{N} \cup \{\infty\} \\ 0 &\mapsto \infty \\ 0 \neq n &\mapsto \max\{m : p^m \mid n\} \end{aligned}$$

It extends to a valuation over $\mathbb{Q}$, by setting, for $a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\}$:

$$v_p\left(\frac{a}{b}\right) := v_p(a) - v_p(b)$$

The valuation properties follow easily from properties from basic properties of divisibility.

Now, let us discuss two ways of defining the p -adic numbers, the first one through a concrete topological construction, the other one *via* a universal property.

By adopting the topological point of view on valuation discussed earlier (theorem 1.3), one may construct the field of p -adic numbers in a way that is analogous to the most standard construction of $\mathbb{R}$,² i.e. as the set of all Cauchy sequences over $\mathbb{Q}$, relative to the p -adic topology, instead of the order topology.

Definition 1.11 ($\mathbb{Q}_p$). The field of p -adic numbers $\mathbb{Q}_p$ is the field obtained as the Cauchy completion of $\mathbb{Q}$ with respect to the p -adic topology.

Proof. Let $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ be two p -adic numbers, i.e. Cauchy sequences of $\mathbb{Q}$. The multiplication $(a_n) \cdot (b_n)$ is defined as (the class of) $(a_n b_n)_{n \in \mathbb{N}}$. To see that it is well defined, it suffices to show that $(a_n b_n)_{n \in \mathbb{N}}$ is indeed Cauchy relative to v_p . ■

²Note that the other most well-known construction of $\mathbb{R}$, namely by Dedekind cuts would not have a p -adic analogue, because the p -adic numbers do not form an ordered field.

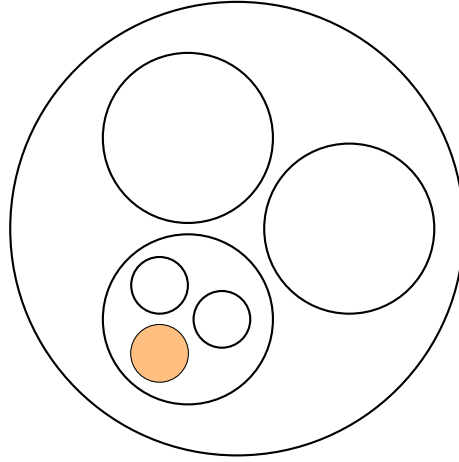


Figure 1: A view of a 3-adic number

Definition 1.12 (*p*-adic absolute value). Let p be a prime number. The *p*-adic absolute value on $\mathbb{Q}$ is the absolute value defined, for $x \in \mathbb{Q}$ by

$$\begin{cases} |x|_p = p^{-v_p(x)} & \text{if } x \neq 0 \\ |0|_p = 0 \end{cases}$$

The celebrated following theorem by Ostrowski [13] classifies ultrametric absolute values on $\mathbb{Q}$.

Theorem 1.5 (Ostrowski 1916). The only norms on $\mathbb{Q}$, up to equivalence, are the real norm $|\cdot|_\infty$ and the norms $|\cdot|_p$ induced by p -adic valuations.

The proof is done by case disjunction on the archimedeanity of the norm. Any archimedean norm is equivalent to $|\cdot|_\infty$, and any non-archimedean norm is equivalent to a unique $|\cdot|_p$. An immediate corollary is the following, which is morally interesting as it shows how natural the p -adics are.

Corollary 1.1 (Completions of $\mathbb{Q}$). The only topological completions of the field of rational numbers $\mathbb{Q}$ are the p -adic fields $\mathbb{Q}_p$ and the field of real numbers $\mathbb{R}$.

A more formal point of view on p -adic fields is to see them as an inverse limit.

Definition 1.13 (Projective system of sets). Let $(I, \leq)$ be an ordered set, $(E_i)_{i \in I}$ be a I -indexed family of sets, and for any pair $i \leq j \in I$, $f_i^j : E_i \rightarrow E_j$, be a map, verifying the conditions

1. $\forall i \in I, f_i^i = \text{Id}_{E_i}$
2. $\forall (i, j, k) \in I^3, i \leq j \leq k \Rightarrow f_j^k \circ f_i^j = f_i^k$

The projective limit is by definition the set of all sequences of elements satisfying the constraints given by the maps.

Definition 1.14 (Limit of a projective system). Let $((E_i)_I, (f_i^j)_I)$ be a projective system of sets. The *projective limit* of this system, $\varprojlim E_i$ is the set

$$\left\{ (a_i) \in \prod_I E_i : \forall i \leq j, a_i = f_i^j(a_j) \right\}$$

The projective limit is the limit represented by the following commutative diagram. (It is straightforward to see that it is indeed a limit.) It generalises to arbitrary categories. In particular, the E_i may have inner structure, e.g. groups or rings, in which case the f_i^j should be morphisms in the appropriate category.

$$\begin{array}{ccccccc} & & & \varprojlim (E_i)_I & & & \\ & \swarrow \pi_i & & \downarrow \pi_j & \searrow \pi_{j+1} & & \\ \cdots & \longrightarrow & E_i & \longrightarrow & \cdots & \longrightarrow & E_j \xrightarrow{f_j^{j+1}} E_{j+1} \longrightarrow \cdots \end{array}$$

Note that the case where I is well-ordered (for example the natural numbers or the primes) is particularly interesting for our purposes, but this is not a definitional requirement. Now we may formulate the following definition for the p -adics.

Definition 1.15 (p -adic integers). The ring $\mathbb{Z}_p$ of p -adic integers is the following projective limit of rings

$$\varprojlim \mathbb{Z}/p^n \mathbb{Z}$$

for $n \in \mathbb{N}$, equipped with the product of discrete topologies in each component.

Then one defines the p -adic numbers as the fraction field of this ring (which is equivalent to the localisation of the ring by its invertible elements. Note that the underlying ring has to be an integral domain, which is the case here).

Definition 1.16 (p -adic numbers). $\mathbb{Q}_p := \text{Frac}(\mathbb{Z}_p)$

1.5 Perfectoid fields and tilting

Perfectoid fields are a special class of valued fields, with some remarkable features giving them an important role in arithmetic geometry.

Definition 1.17 (Perfectoid field). A *perfectoid field* is a valued field (K, v) such that

1. K is Cauchy-complete.
2. v is a rank 1, non-discrete valuation, i.e. the codomain Γ of v is isomorphic to a subgroup of $\mathbb{R}$, distinct from $\mathbb{Z}$.
3. The residue ring $\mathcal{O}_v/(p)$ is semi-perfect, i.e. the Frobenius map $\mathcal{O}_v/(p) \rightarrow \mathcal{O}_v/(p), x \mapsto x^p$ is surjective.

1.6 Tilting

The $\flat$ map, called *tilt*, defines a functor between the categories of valued fields of residual characteristic p (i.e. fields of characteristic $(0, p)$ or (p, p)) and valued fields of equicharacteristic p . Intuitively, one changes the characteristic by forcing the Frobenius endomorphism to be surjective. The following diagram (borrowed from a note by Bharghav Bhatt) summarises the relations between K , $K^\flat$ and their auxiliary structures:

$$\begin{array}{ccc}
 & & \mathcal{O}_K \xrightarrow{\text{loc}(p)} K \\
 & & \downarrow / (p) \\
 \mathcal{O}_{K^\flat} & \xrightarrow{/(t)} & \mathcal{O}_{K^\flat}/(t) \cong \mathcal{O}_K/(p) \\
 \downarrow \text{loc}(t) & & \\
 K^\flat & &
 \end{array}
 \quad \begin{array}{c}
 \nearrow \text{b} \\
 \searrow \flat
 \end{array}$$

K is the fraction field of $\mathcal{O}_K$, or equivalently, the localisation by p . The central structures are obtained by quotienting by (p) and (t) , and the tilted field $K^\flat$ is the fraction ring of $\mathcal{O}_{K^\flat}$.

A way to make sense of this construction is to look back at the analogy from chapter 1, and recalling that the obvious obstruction in seeing $\mathbb{Z}$ and $\mathbb{F}_p[t]$ as similar objects was the existence of carrying in addition. Now, the moding out in the central part may viewed as a step where such an obstruction is removed, and both rings are of the same characteristic.

Let us first present the definition of tilting as a projective limit (see definition 1.14), before expanding it in a more concrete construction.

1.6.1 Abstract construction

Definition 1.18 (Set of the tilt). Let (K, v) be a perfectoid field of residue characteristic p . The set of the *tilt* $K^\flat$ is defined as the projective limit (see definition 1.14)

$$\varprojlim_{\text{Frob}} K$$

i.e. the E_i are all instances of K , and the maps are all instances of the Frobenius map $K \rightarrow K, x \mapsto x^p$.

Let us add the valued field structure. This can be done by specifying the valuation ring.

Definition 1.19 (Valuation ring of the tilt). Let $(K, \mathcal{O}_K)$ be a perfectoid field, and $K^\flat$ be the set of the tilt. The *valuation ring of the tilt* is the following projective limit

$$\mathcal{O}_{K^\flat} = \varprojlim_{x \mapsto x^p} \mathcal{O}_K/(p)$$

Now, the ring of the tilt is a ring of characteristic p because it is a limit of rings of characteristic p .

Furthermore, we can check that $\text{Frac}(\mathcal{O}_{K^b}) = K^b$ (as a set): since multiplication is defined component wise, we need to add inverses in each component, so K^b is the only possibility. The existence of a field structure on K^b follows from the existence of fields of fraction over integral domains, and the existence of projective limits.

1.6.2 Concrete construction

Let us have a closer look at a concrete realisation of the above construction.

Definition 1.20 (Set of the tilt). Let K be a perfectoid field. The set of the field K^b is defined as follows

$$K^b := \{(x_i)_{i \in \mathbb{N}} \in K^{\mathbb{N}} \mid x_{n+1}^p = x_n\}$$

The multiplication is defined term-wise and the addition is more intricate (as one should expect, since it has to account for the change in characteristic).

Definition 1.21 (Field structure in the tilt). Multiplication is defined by

$$(x_n)_{n \in \mathbb{N}} \cdot (y_n)_{n \in \mathbb{N}} = (x_n \cdot y_n)_{n \in \mathbb{N}}$$

which is well defined because $x^p y^p = (xy)^p$. Addition is defined by

$$(x_n)_{n \in \mathbb{N}} + (y_n)_{n \in \mathbb{N}} = (z_n)_{n \in \mathbb{N}}$$

where

$$z_n = \lim_{m \rightarrow \infty} (x_{n+m} + y_{n+m})^{p^m}$$

To understand the field structure, let us concentrate on the addition, and show why it is well-defined, how fields properties are satisfied, and how characteristic changes. Note the unusual definition of addition: addition cannot be defined component-wise in this context because the components are not independent, since they must satisfy the inverse system condition. Furthermore, if we consider the tilt as a means of changing characteristic, a component-wise addition simply won't do. Instead, we use this limit formula, which "restores" compatibility by pushing the addition deep into the sequence, and then pulling the result back via Frobenius.

Lemma 1.2. The components in the tilting addition are well defined.

Proof. The field K is complete with respect to the valuation v , so every Cauchy sequence has a limit. Now, in K , a sequence of the form $(x)^{p^m}$ for $m \in \mathbb{N}$ and $v(x) > 0$ tends towards 0, since its valuation diverges, so the non-zero elements of K^b will be sequences of elements of valuation 0, i.e. elements of the residue field, in the case. This ensures that the terms $z_i = \lim_{m \rightarrow \infty} (x_{i+m} + y_{i+m})^{p^m}$ are well defined. ■

Lemma 1.3. Addition respects the inverse system condition, i.e. $z_{i+1}^p = z_i$.

Proof. Let us unravel the definition and do a useful rewriting of the index:

$$z_{i+1} = \lim_{m \rightarrow \infty} (x_{i+1+m} + y_{i+1+m})^{p^m} = \lim_{m \rightarrow \infty} (x_{i+m+1} + y_{i+m+1})^{p^m}$$

and recall that by assumption $w_{i+m+1}^p = w_{i+m}$ for any $w_{i+m} \in K^\flat$.

Let us raise z_{i+1} to the p -th power:

$$z_{i+1}^p = \left(\lim_{m \rightarrow \infty} (x_{i+m+1} + y_{i+m+1})^{p^m} \right)^p$$

The exponent map is continuous in K (because multiplication is), so we can rewrite this as

$$z_{i+1}^p = \lim_{m \rightarrow \infty} (x_{i+m+1} + y_{i+m+1})^{p^{m+1}}$$

and since we were already considering a limit for $m \rightarrow \infty$, we get the final equality by a change of indices $m \mapsto m + 1$

$$z_{i+1}^p = z_i$$

so the addition of the tilt is well-defined. ■

Lemma 1.4 (Associativity of addition). Let $(x_i)_{i \in \mathbb{N}}, (y_i)_{i \in \mathbb{N}}, (z_i)_{i \in \mathbb{N}} \in K^\flat$. Then for all $n \in \mathbb{N}$,

$$x_n + (y_n + z_n) = (x_n + y_n) + z_n$$

Proof. We unravel the definition and extract the inner limit (using the continuity of addition in K). From there, we can do the following computation. The next l steps are using the inverse system condition on x , in order to homogenise the inner terms, until we are able to extract the p^l exponent. Finally, we can apply associativity from K , and rearrange our terms to conclude properly.

$$\begin{aligned}
& x_n + (y_n + z_n) \\
&= \lim_{m \rightarrow \infty} (x_{n+m} + (\lim_{l \rightarrow \infty} (y_{n+m+l} + z_{n+m+l})^{p^l}))^{p^m} \\
&= \lim_{m \rightarrow \infty} \lim_{l \rightarrow \infty} (x_{n+m} + (y_{n+m+l} + z_{n+m+l})^{p^l})^{p^m} \\
&= \lim_{m \rightarrow \infty} \lim_{l \rightarrow \infty} ((x_{n+m+l})^p + (y_{n+m+l} + z_{n+m+l})^{p^l})^{p^m} \\
&= \dots \\
&= \lim_{m \rightarrow \infty} \lim_{l \rightarrow \infty} ((x_{n+m+l})^{p^l} + (y_{n+m+l} + z_{n+m+l})^{p^l})^{p^m} \\
&= \lim_{m \rightarrow \infty} \lim_{l \rightarrow \infty} ((x_{n+m+l}) + (y_{n+m+l} + z_{n+m+l}))^{p^{m+l}} \\
&= \lim_{m \rightarrow \infty} \lim_{l \rightarrow \infty} ((x_{n+m+l} + y_{n+m+l}) + z_{n+m+l})^{p^{m+l}} \\
&= \lim_{m \rightarrow \infty} \lim_{l \rightarrow \infty} ((x_{n+m+l} + y_{n+m+l})^{p^l} + z_{n+m+l}^{p^l})^{p^m} \\
&= \lim_{m \rightarrow \infty} \lim_{l \rightarrow \infty} ((x_{n+m+l} + y_{n+m+l})^{p^l} + z_{n+m})^{p^m} \\
&= \boxed{(x_n + y_n) + z_n}
\end{aligned}$$

■

Lemma 1.5 (Commutativity of addition). Let $x, y \in K^b$. Then $x + y = y + x$.

Proof. Develop a n th component, and note that the equality

$$\lim_{m \rightarrow \infty} (x_{n+m} + y_{n+m})^{p^m} = \lim_{m \rightarrow \infty} (y_{n+m} + x_{n+m})^{p^m}$$

is commutativity in K .

■

Lemma 1.6 (Distributivity). Let $x, y, z \in K^b$, then $x \cdot (y + z) = (xy) + (xz)$.

Proof. We consider a given component $s_n = \lim_{m \rightarrow \infty} (y_{n+m} + z_{n+m})^{p^m}$. Then

$$\begin{aligned}
(x \cdot s)_n &= x_n \cdot s_n \\
&= x_n \cdot \lim_{m \rightarrow \infty} (y_{n+m} + z_{n+m})^{p^m} \\
&= \lim_{m \rightarrow \infty} x_n \cdot (y_{n+m} + z_{n+m})^{p^m} \\
&= \lim_{m \rightarrow \infty} (x_n y_{n+m} + x_n z_{n+m})^{p^m} \\
&= \boxed{(xy)_n + (xz)_n}
\end{aligned}$$

comes by distributivity inside K and the fact that multiplication is continuous in K (by definition of a topological field). Therefore $x(y + z) = (xy) + (xz)$. ■

Theorem 1.6 ($K^\flat$ is a field). The tilt $K^\flat$ of a perfectoid field K of mixed characteristic $(0, p)$ is a field of characteristic p .

Proof. The previous lemmas established the field properties. Commutativity and associativity of multiplication clearly lift from the corresponding properties in K . Let us focus on the characteristic of $K^\flat$.

We want to show that the sum $p = 1 + 1 + \dots$ vanishes. Recall that in $K^\flat$, the multiplicative neutral element 1 is the sequence $(\dots, 1, 1, 1)$. Now, let us consider a single component of the sum, which is of the form:

$$\sum_{i=1}^p (1) = \lim_{m \rightarrow \infty} (1 + \dots + 1)^{p^m}$$

and notice that since $1 + 1 + 1 + \dots = p$ has positive valuation in K by assumption, the limit of the sequence when taking p -th powers is 0, so the sum is 0 in each component, so it is 0, and the resulting field $K^\flat$ is indeed of characteristic p . ■

Later on, we will encounter an elementary avatar of the tilt construction, namely the residue field of valued fields in the class we investigate. More precisely, pairs consisting of a valued field with its associated residue field in the classes studied will be elementary extensions of pairs consisting of a perfectoid field together with its tilt.

1.6.3 Tilting equivalence

The fundamental theorem of perfectoid fields is the following result, allowing transfer of information between a field and its tilt.

Theorem 1.7 (Tilting equivalence [17]). If K is a perfectoid field, and $K^\flat$ is the tilt of K , then the categories of K -algebras and $K^\flat$ -algebras are equivalent.

As an example of the power of the tilting correspondence, let us prove the following simple case.

Theorem 1.8 (Algebraic closure survives tilting). A perfectoid field K is algebraically closed if and only if its tilt $K^\flat$ is.

Proof. This comes directly from tilting equivalence. By tilting equivalence there is an equivalence of categories between K -algebras and $K^\flat$ -algebras. In particular, this means there is an equivalence between the categories of finite extensions over K and $K^\flat$. But a field is algebraically closed if and only if it has no non-trivial finite extensions. So if K is algebraically closed, both categories are trivial, and $K^\flat$ is algebraically closed as well. The other direction is exactly the same. ■

Something worth noting is that the map $\flat : K \rightarrow K^\flat$ is not entirely determined: since an element of K may have more than one p -th root, the image $x^\flat$ of an element $x \in K$ is in fact only defined up to a choice of p -th roots.

1.6.4 Examples

We have already presented the pair of perfectoid fields given by the Fontaine-Wintemberger theorem. Let us now present other classes of perfectoid fields.

Complex p -adics

From the field $\mathbb{Q}_p$ of p -adic numbers, we can define another rich example of valued field, namely the *complex p -adic*, a p -adic analogue for the complex numbers. Interestingly enough, simply taking the algebraic closure of $\mathbb{Q}_p$ does not yield a complete field, so one needs to take completion a second time to obtain a nice analogue of $\mathbb{C}$. (See [10], chapter III, for the construction of $\mathbb{C}_p$)

Definition 1.22 ($\mathbb{C}_p$). The field $\mathbb{C}_p$ of *complex p -adic numbers* is the topological completion of an algebraic closure of $\mathbb{Q}_p$:

$$\mathbb{C}_p := \widehat{\mathbb{Q}_p^{alg}}$$

Notice that this is well-defined because algebraic closures are unique up to isomorphism, $\mathbb{Q}_p$ is henselian so its valuation extends uniquely to $\mathbb{Q}_p^{alg}$ and the completion is unique. Now, an odd consequence is that $\mathbb{C}_p$ is an uncountable algebraically closed field of characteristic 0, so it is in particular isomorphic to $\mathbb{C}$ as fields.

Incidentally, this gives us our first "faithful" class of perfectoids. The valuation on $\mathbb{C}_p$ is complete, and of rank 1, and the residue ring is semi-perfect (this follows from $\mathbb{C}_p$ being algebraically closed). Now completeness of ACVF gives us faithfulness for free: since algebraically closed valued fields are elementary equivalent, in particular, they models are equivalent to $\mathbb{C}_p$, which is perfectoid. However, note that this is a very restrictive theory, as perfectoid fields are in general not algebraically closed.

Cyclotomic fields

The cyclotomic perfectoid field is the field

$$\mathbb{Q}_p(\mu_{p^\infty}) = \bigcup_{n \in \mathbb{N}} \mathbb{Q}_p(\mu_{p^n})$$

of the p -adics, extended with the p^n th primitive roots of unity for all n .

2 Ordered abelian groups : archimedeanity and regularity

Among the defining properties of perfectoid fields is having a so-called *rank 1* valuation, which is in particular equivalent to "being valued over a subgroup of $\mathbb{R}$ ". An important step towards providing a first-order axiomatisation for generalised perfectoid fields is therefore to find an elementary replacement for this property.

In this chapter, we introduce ordered abelian groups. We then review some important properties and study their first-order expressibility. Finally, we prove an important embedding theorem: any regular ordered abelian group is elementarily equivalent to a subgroup of the reals.

2.1 Ordered abelian groups

Definition 2.1 (Ordered abelian group). An *ordered abelian group* (henceforth OAG) $(G, \leq, +)$ is an abelian group whose group operation $+$ is compatible with the order relation $\leq$ as follows :

$$\forall a, b, c. a \leq b \rightarrow a + c \leq b + c$$

Let us furthermore call *positive* and *negative* elements greater and less than the neutral element of the group, with respect to the order $<$.

Definition 2.2 (Discreteness). An ordered abelian group G is *discrete* if it has a minimal element x , i.e. if it satisfies the following elementary property

$$(\exists x)(\forall y)[y > 0 \Rightarrow y = x \vee y > x].$$

Such an element is called the *unit* of the group. Otherwise G is *non-discrete*, or *dense*.

Remark. We may already remark that OAGs are axiomatisable in first-order logic by appending the above formula to the (elementary) theory of abelian groups. Clearly, any non-trivial ordered abelian group is torsion free. It will therefore be useful to consider only non-trivial OAG, and include a non-triviality axiom in our elementary axiomatisation.

Let us observe some easy but fundamental preservation properties.

Proposition 2.1 (Preservation by substructure). Any subgroup of an abelian group is abelian. Any subgroup of an ordered group is an ordered group, by restricting the order to the subgroup.

Proposition 2.2 (Preservation by product). Any product of OAGs is an OAG, by using the lexicographic order.

Example 5 (Ordered abelian groups).

- $(\mathbb{Z}, +, \leq)$, $(\mathbb{Q}, +, \leq)$, $(\mathbb{R}, +, \leq)$ are clearly OAG, under the usual order.
- Free abelian groups with n generators and the (anti)-lexicographic order.

Let us now quickly review some mundane but useful algebraic properties of ordered abelian groups.

Proposition 2.3 (Composition of inequations). In an OAG, if $x \leq y$ and $w \leq z$ then $x + w \leq y + z$.

Proof. $x \leq y$ gives $x + w \leq y + w$ by the compatibility condition. Similarly $w \leq z$ gives $w + y < z + y$. Now, by transitivity we obtain $x + w \leq y + w \leq z + y$, which is our result. ■

Proposition 2.4 (Inverses reverse inequalities). If $x \leq y$, then $-y \leq -x$.

Proof. In the inequality $x \leq y$, add $-y$ to the right to get $x - y \leq y - y$ and $-x$ to the left $-x + x - y \leq -x$, then simplify to get $-y \leq -x$. ■

Theorem 2.1 (Characterisation of orderable abelian groups). An abelian group can be totally ordered iff it is torsion-free.

Proof. $\Rightarrow$ comes from antisymmetry : suppose M non torsion-free. Wlog, assume $x, y > 0$ and $x + y = 0$. This contradicts $x + y > 0 + y$.

$\Leftarrow$ We first prove that any finitely generated subgroup G of a torsion-free group M is totally orderable. Since G is a finitely generated, abelian, torsion-free group, it is isomorphic to $\mathbb{Z}^n$ for some $n \in \mathbb{N}$, and can therefore be ordered. We conclude by compactness. ■

2.2 Archimedeanity & Hölder's embedding theorem

A fundamental property for OAG is *archimedeanity*, which corresponds intuitively to the absence of infinitesimal or infinite elements.

Definition 2.3 (Archimedeanity). An OAG is *archimedean* if for all non-zero x, y , there exists $n \in \mathbb{N}$ such that $n \cdot x \geq y$ where $n \cdot x := \underbrace{x + \dots + x}_{n \text{ times}}$

The discrete case of the following theorem is immediate. For the general case, we follow the proof strategy deployed in [14].

Proposition 2.5 (Hölder, Discrete case). Any linearly ordered discrete archimedean group is abelian and embeds in $(\mathbb{R}, +, \leq)$.

Proof. Let $(M, +, \leq)$ be a linearly ordered discrete archimedean group. Then it is generated by its unit, and is therefore isomorphic to $(\mathbb{Z}, +, \leq)$, which is a subgroup of $(\mathbb{R}, +, \leq)$. ■

Theorem 2.2 (Hölder). Any linearly ordered dense archimedean group is abelian and embeds in $(\mathbb{R}, +, \leq)$.

Proof. ([14]) Let $(M, +, \leq)$ be an archimedean OAG. We define a mapping to the reals. First we fix $\theta(0) = 0$. Then, choose a positive element $\alpha \in M$, and let $\theta(\alpha) = 1$ and for $\beta > 0$, we consider the set

$$S_\beta = \left\{ \frac{m}{n} \in \mathbb{Q} : n > 0 \text{ and } m\alpha \leq n\beta \right\}$$

By archimedeanity, S_β is neither empty, nor $\mathbb{Q}$. Furthermore, it is downwards closed, in the sense that if $\frac{m'}{n'} \leq \frac{m}{n}$ with $\frac{m'}{n'} \in \mathbb{Q}$ and $\frac{m}{n} \in S_\beta$, then $\frac{m'}{n'} \in S_\beta$.

The sets S_β define real numbers by considering their least upper bounds, so we can complete our mapping θ by sending any positive $\beta \in M$ to the least upper bound of S_β . Furthermore, negative elements are also mapped by mapping them to the opposites of their positive group inverse : $\theta(\beta) = -\theta(-\beta)$.

By construction, positive elements are mapped to positive reals, and our mapping is an order-preserving group homomorphism. ■

Archimedeanity is not an elementary property, i.e. there is no first-order formula logically equivalent to it. Let us show it in two ways, relying on different model-theoretic tools.

Theorem 2.3 (First-order inexpressibility of archimedeanity). Archimedeanity is not a first-order property.

Proof. For the contradiction, let φ be an elementary formula expressing archimedeanity. Now, let us consider a discrete OAG, with minimal positive element 1. We define sentences of the form

$$\psi_n := \exists x. x > \underbrace{1 + \dots + 1}_{n \text{ times}}$$

Let Δ denote the set of all ψ . Clearly the theory $\text{DiscOAG} \cup \{\varphi\} \cup \Delta$ is finitely satisfiable. By compactness, this theory has a model, but this model is clearly not archimedean, which contradicts our earlier assumption. ■

Note that the proof strategy easily generalises to the dense case by choosing any strictly positive element (since the minimal non-zero element is no longer defined), and making the analogue construction over it.

Proof. For the contradiction, let φ be an elementary formula expressing archimedeanity. We remark that the theory of archimedean ordered abelian groups is then a countable first-order theory. Therefore, Löwenheim-Skolem theorems allow us to obtain models of $\text{OAG} \cup \{\varphi\}$ of any cardinality. However, the Hölder embedding theorem restricts the cardinality of an archimedean OAG below $\aleph_1$, which is a contradiction. ■

2.3 Divisibility

Divisibility is an important related property. Let us just remark that all divisible OAGs are regular, but not all divisible OAGs are archimedean (for an intuitive counterexample, one may consider the ordered field of the hyperreals).

Definition 2.4 (n -divisibility). A group $(G, +)$ is n -divisible if any element of G is divisible by n , i.e. the following elementary condition:

$$(\forall x \in G)(\exists y \in G)[n \cdot y = x]$$

(It is important to recall that the multiplicative notation is shorthand for iterated group addition, so we can express it in the language of our structure G , with no reference to $\mathbb{N}$.)

Definition 2.5 (Divisibility). A group $(G, +)$ is *divisible* if it is n -divisible for every n , i.e. if for any $y \in G$ and $n \in \mathbb{N}$, there is x such that $n \cdot x = y$.

Divisibility can be expressed as an axiom scheme, by using the n -divisibility formula for each $n \in \mathbb{N}$. Every OAG embeds into a divisible extension, and there is a minimal such extension, the so-called divisible hull.

Definition 2.6 (Divisible hull). If Γ is an ordered abelian group, then there is a divisible ordered abelian group $\text{Div}(\Gamma)$, called the *divisible hull* of Γ , such that any embedding $\Gamma \hookrightarrow \Delta$ to a divisible group factors uniquely through $\text{Div}(\Gamma)$, i.e.

$$\begin{array}{ccc} & & \Delta \\ & \nearrow & \uparrow! \\ \Gamma & \longrightarrow & \text{Div}(\Gamma) \end{array}$$

Proof. Define $\text{Div}(\Gamma)$ as the set of formal fractions $\{\frac{\gamma}{n} : \gamma \in \Gamma, n \in \mathbb{Z}\} / \sim$ with $\frac{\gamma}{n} \sim \frac{\gamma'}{n'} : \Leftrightarrow n\gamma' = n'\gamma$. The divisible ordered abelian group structure is clear, by using the obvious group operation, and the order $\frac{\gamma}{n} \leq \frac{\gamma'}{n'} : \Leftrightarrow n\gamma' \leq n'\gamma$.

Minimality among divisible extensions comes from the observation that for any embedding $f : \Gamma \hookrightarrow \Delta$, and any element $\gamma \in \Gamma$, Δ must contain divisors of $f(\gamma)$ by divisibility, so one may simply define $\tilde{f} : \text{Div}(\Gamma) \rightarrow \Delta$ by $\frac{\gamma}{1} \mapsto f(\gamma)$ and $\frac{\gamma}{n} \mapsto \frac{f(\gamma)}{n}$. ■

Theorem 2.4 (Classification of divisible ordered abelian groups). Models of the theory DOAG of divisible ordered abelian groups are exactly ordered $\mathbb{Q}$ -vector spaces.

Proof. Any rational vector space is a model of DOAG. Just consider the underlying additive group, and verify divisibility as a property of the scalar multiplication.

Likewise, a model of DOAG has a unique vector space structure, because divisibility determines the scalar multiplication from $\mathbb{Q}$. ■

The rank of a divisible ordered abelian group coincide with the dimension of the corresponding $\mathbb{Q}$ -vector space. Note that this doesn't hold for more general classes of divisible groups, because of torsion.

From this characterisation, we may directly infer the following *model completeness* property:

Theorem 2.5 (Elimination of quantifiers). DOAG eliminates quantifiers in the language $\mathcal{L}_{oag}$.

Proof. Following a proof by Martin Hils [7], we use the following test for quantifier elimination: any elementary embedding $f : A \rightarrow \Delta$ from a substructure $A \subseteq \Gamma$ of a countable model, to an $\aleph_1$ saturated model Δ extends to an elementary embedding $\tilde{f} : \Gamma \rightarrow \Delta$.

First, we may extend $f : A \rightarrow \Delta$ to $f' : A' \rightarrow \Delta$ where A' is the divisible hull of the subgroup of Γ generated by A . The embedding to Δ is then elementary by the universal property of divisible hulls.

Then, we can extend f' to an embedding $\langle A' \cup \{\gamma\} \rangle \rightarrow \Delta$. The group structure is obvious, and the order depends on the cut of γ over A' . Since Δ is $\aleph_1$ -saturated, the cut is also realised in the image. By iterating this argument over every $\gamma \in \Gamma$, and passing repeatedly to divisible hull, we can extend f to an embedding $\Gamma \preceq \Delta$. Since Γ is countable, there is no need to consider limit steps in the induction. ■

As a result, we obtain immediately the following properties.

Corollary 2.1 (Model completeness of DOAG). Every embedding of non-trivial divisible ordered abelian groups is elementary.

Corollary 2.2 (Completeness). DOAG is complete.

Proof. The theory DOAG is model-complete and $(\mathbb{Q}, +, <)$ is a prime model, therefore, it is complete. ■

2.4 Regularity

Regularity provides us with a convenient substitute for archimedeanity, as it is an elementary property, and as we will see, any regular OAG is elementarily equivalent to an archimedean OAG. We summarise the proof from [16].

Let us first give some useful definitions. The prime invariants serve as a tool to classify regular OAGs. It is easy to check that these definitions generalise usual properties of the integers.

Definition 2.7 (p -invariant). Given a prime p and an abelian group M , the *prime invariant* $[p]$, or $[p]M$ is the maximal number in $\mathbb{N} \cup \{\infty\}$ of elements pairwise incongruent modulo p .

Definition 2.8 (prime rank). The p -th *prime rank* of an abelian group A is a (possibly infinite) number of elements in a maximal p -independent subset. It is precisely $\log_p [p]A$.

Definition 2.9 (Regularly discrete). M is *regularly discrete* if it is discretely ordered and $[p]M = p$ for every $p \in \mathbb{P}$.

Definition 2.10 (Regularly dense). M is *regularly dense* if for any $p \in \mathbb{N}$ and $a < b \in M$ there is $x \in M$ such that $a < x < b$ and p divides x .

The first important model-theoretic result is the classification theorem. For the discrete case, one shows model completeness and remarks that the integers are a prime model, so the theory of discrete regular OAG is complete. For the dense case, consider any choice of prime invariants (given as a sequence of positive integers or infinity), and remark that if a theory of dense regular ordered abelian groups satisfying these invariants is consistent, then it is complete. This is done by showing model completeness and joint-embedding property.

Theorem 2.6 (Archimedeanity entails regularity). Every archimedean OAG is regular.

Proof. By case disjunction on whether the group is discrete or dense.

Lemma 2.1 (discrete case).

Proof. By archimedeanity, every element is a multiple of the unit, and the group is isomorphic to $(\mathbb{Z}, +)$ which is regular. ■

Lemma 2.2 (dense case).

Proof. Consider M an archimedean OAG, and S an interval from M (in additive notation, using left multiplication as shorthand for iterated addition). We want to find a "multiple" of n in S . Let us first remark that if $0 \in S$, this is trivial, so let us assume S only has positive elements. For any $n \in \mathbb{N}$ and $y > 0$, there exists $0 < n \cdot x < y$.

Now take $a, b \in S$, wlog, $a < b$. We consider the minimum $\min(a, b - a)$. In both cases, for any $n \in \mathbb{N}$, there exists x such that $0 < nx < \min(a, b - a)$.

Fixing nx and letting a new factor m vary gives us a discrete subgroup. By archimedeanity, there exists m such that $mnx > \min(a, b - a)$. Taking the least such m must give us a solution for $a < mnx < b$, and therefore a solution for S . ■

Conversely, every class of regular OAG contains an archimedean model, as shown in [22].

Theorem 2.7 (Existence of archimedean models). For any acceptable choice of prime ranks $(m_i)_{i \in \mathbb{N}}$, there exists a dense archimedean ordered abelian group M satisfying $[p_n]M = p_n^{m_n}$.

Let us outline the proof strategy. Let $(m_i)_{i \in \mathbb{N}}$ be the sequence of prime invariants. The

idea is to construct our archimedean group explicitly as a subgroup of $\mathbb{R}$. We fix a transcendental number e and define M as a subgroup of $\mathbb{R}$ generated by all fractions of the form a_{nj}/r with r any integer not divisible by p_n , and a_{nj} defined as 0 when $j > m_n$ and otherwise $a_{nj} = e^{k_{nj}}$ where k_{nj} are arbitrarily chosen distinct integers. Now this ensures archimedeanity and dense ordering. Now, by transcendence of e , the sets of non-zero a_{nj} (fixing n and letting j vary) is a maximal p_n -independent subset of M , containing exactly m_n elements. Now, it can be shown that $[p_n]M = p_n^{m_n}$.

2.5 Embedding theorem for regular ordered abelian groups

We are now ready to state our theorem.

Theorem 2.8 (Embedding theorem). Any regular ordered abelian group is elementarily equivalent to a subgroup of $(\mathbb{R}, +)$.

Proof. This follows directly from Hölder's embedding theorem (2.2), completeness and existence lemma (2.7). ■

Remark. Note that this gives us an analogon for archimedeanity, but only modulo the choice of the prime ranks. In particular, this does not quite give us a theory of "OAGs elementary equivalent to an archimedean OAG", but rather a family of theories, indexed by the prime ranks. Crucially, the property "being regularly dense for some (non-fixed) choice of prime ranks" does not seem to be elementary. In particular, one could take an ultraproduct of groups satisfying $[p_n]M = p_n^{m_n}$ for different prime ranks, and obtain a OAG satisfying none of the equalities.

3 Valuation rings

We recall from the first chapter that a *valuation ring* is a local ring whose maximal ideal is given by the strictly positively valued elements of a field.

A major property of valuation rings is that their prime ideals correspond uniquely to convex subgroups of the valuation groups. Looking back to the definition of a perfectoid field, this means that control on the Krull dimension, i.e. the maximal length of a chain of prime ideals, would allow us to force the valuation to be of rank one. As is to be expected, this is not feasible. However, one may ask for sufficient first order conditions to control Krull dimension. A trivial such property would be “being a field”. Of course, this is of no use to us, as if $\mathcal{O}_K$ is a field, the valuation is trivial.

3.1 Perfection

Definition 3.1 (Semi-perfection). A valuation ring $\mathcal{O}_K$ is *semi-perfect* if the Frobenius map $x \mapsto x^p$ is surjective.

Proposition 3.1 (Perfection in characteristic p). A valued field K of characteristic p is perfect if and only if $\mathcal{O}_v$ is semi-perfect.

Proof. $\Rightarrow$ Let K be perfect, and let $x \in \mathcal{O}_v$. Then by assumption, since x has an antecedent $y \in K$ for the Frobenius map, just notice that $y \in \mathcal{O}_v$. $\Leftarrow$ For the converse property, let $x \in \mathcal{O}_K$ and $y^p = x$. Then $\frac{1}{x} = \frac{1}{y^p} = \left(\frac{1}{y}\right)^p$ gives a root for x^{-1} . Therefore K is perfect. ■

In characteristic 0, however we don’t have such an equivalence, because the valuation ring $\mathcal{O}_v$ doesn’t coincide with the residue ring $\mathcal{O}_v/(p)$.

Proposition 3.2 (Semi-perfection in characteristic 0). Let us consider (K, v) of characteristic $(0, p)$. Surjectivity of the map $x \mapsto x^p$ in K still implies semi-perfection of $\mathcal{O}_K$, and therefore of the residue ring $\mathcal{O}_K/p$.

However, the converse property doesn’t necessarily hold, i.e. $\mathcal{O}_K/p$ may be semi-perfect even though $x \mapsto x^p$ is not surjective on K . To express semi-perfection of the residue ring in K , we need an “error term” accounting for the quotient:

$$(\forall x)(\exists y, z)[x = y^p + zp]$$

Now if this holds in K , consider $x \in \mathcal{O}_K$, and observe that $v(x) = v(y^p + zp) \geq \min\{pv(y), v(z)+v(p)\}$. Suppose $pv(y) \neq v(z)+v(p)$. Then $v(x) = \min\{pv(y), v(z)+v(p)\}$. This implies that $y \in \mathcal{O}_K$ and that $v(z) \geq -1$.

3.2 Krull dimension & valuation

Theorem 3.1 (Prime ideals & convex subgroups). If $\mathcal{O}_K$ is a valuation ring on K , then the prime ideals of $\mathcal{O}_K$ are in bijection with the convex subgroups of vK .

Proof. Let $\Delta \subset vK$ be a convex subgroup, consider the set

$$I_\Delta = \{x \in \mathcal{O}_K : v(x) \notin \Delta\}$$

and let us show it is a prime ideal. If x, y are in I_Δ , then $x + y \in I_\Delta$ follows from $v(x), v(y) > 0 \notin \Delta$, which implies $v(x + y) \notin \Delta$. If x is in I_Δ and $y \in \mathcal{O}_K$, then $v(xy) = v(x) + v(y)$ (both positive) is not in Δ . Finally, if $xy \in I_\Delta$, then $v(xy) \notin \Delta$. Wlog, suppose $x \in \mathcal{O}_K \setminus I_\Delta$. Then $v(x) \in \Delta$ implies $v(y) \notin \Delta$, so $y \in I_\Delta$.

Now let $I \subset \mathcal{O}_K$ be a prime ideal. Let us show that

$$\Delta_I := \{\gamma \in vK : -v(I) < \gamma < v(I)\}$$

is a convex subgroup. Clearly the sum of two elements of Δ_I is in Δ_I because otherwise, there would be elements $i, j \in \mathcal{O} \setminus I$ whose product ij would be in I , contradicting the hypothesis that I is a prime ideal. Therefore Δ_I is a subgroup of vK . Furthermore, it is convex by construction. ■

We immediately get the following.

Corollary 3.1 (Group rank & Krull dimension). The Krull dimension of a valuation ring is equal to the rank of the value group.

3.3 (Non-)noetherianity

In the later part of this thesis, we show that a (“nice”) valued field of our class, with a valuation of finite rank is elementary equivalent to a nice field of rank 1, which is by definition a perfectoid field. However, this does not suffice to define an elementary class, because fields of infinite rank might not have a rank 1 analogue. One therefore needs to make sure either that the rank may not become infinite when passing to ultraproducts and equivalences, or that models of infinite rank still have finite rank analogues.

From chapter 2, we may recall that being of finite rank is not an elementary property. A natural strategy is to try and find a first-order replacement for this condition, as we did for completeness earlier. Such first-order analogue properties could also come in the form of strictly sufficient conditions, at the cost of generality. However, this turns out to be tricky, even for sufficient conditions.

Let us note that rank finiteness in Γ may be reformulated as a property of the valuation ring $\mathcal{O}_K$.

Now, in principle, the language of rings might provide new ways to control rank. this does unfortunately not quite work, as most immediately relevant ring properties are not elementary either.

Now, a strategy could be to find an elementary condition ensuring that $\mathcal{O}_K$ has finite Krull dimension. In its usual formulation, Krull dimension is clearly a second order statement, so we have to make more assumptions about the ring, or use the properties we know from its status as a valuation ring (locality etc).

As we shall see, there is a way to control the Krull dimension (and hence valuation rank) of a noetherian ring with an elementary formula, but this restriction does not pass to ul-

trapowers in general, because noetherianity doesn't. At any rate, we are interested in the non-noetherian case.

3.3.1 Properties in the noetherian case

Definition 3.2 (Noetherian). A ring is *noetherian* if any chain of ideals is eventually stationary.

A trivial example of an elementary criterion of noetherianity is *being a field*, because there is only one non-zero ideal. Fields are remarkable in that they are an elementary class of noetherian rings. However, they are somewhat useless in the context of valuation rings.

A *non-example* is the class of local rings. Locality is a first-order condition, and one could hope that a chain of ideals in a local ring would eventually reach its maximal ideal and become stationary. However, if the ideal is infinitely generated, one defines an infinite chain by taking ideals over successive subsets of the generators. The canonical example is a ring $k[x_1, x_2, x_3, \dots]$ of polynomials in infinitely many variables.

Proposition 3.3 (Noetherian rings and DVR). A noetherian valuation ring is either a field or a discrete valuation ring.

In this case, Krull dimension 1 corresponds to having the group be isomorphic to $\mathbb{Z}$.

Definition 3.3 (PID). A *principal ideal domain* is an integral domain in which every ideal is principal.

Lemma 3.1. Valuation rings are integral domains.

Proof. This is direct from the axiom of valuation. If $\mathcal{O}$ has zero-divisors $ab = 0$, $a, b \neq 0$ the valuation of a or b must be ∞ since $v(0) = v(ab) = v(a) + v(b)$. But this contradicts the first axiom of valuations. ■

Proposition 3.4 (Krull dimension in PID). A PID which is not a field has Krull dimension 1.

Let us consider the following notion of so-called *stable range*. We follow [19] for the definition and elementarity. We refer to (the online version of) [20] for the proof of the dimension theorem.

Definition 3.4 (Stable range). A commutative ring has *stable range* n if n is the minimal integer such that if a $m + 1$ -tuple $a_0, \dots, a_m$ generate the unit ideal for $m \geq n$, then there is a m -tuple $b_1, \dots, b_m$ such that the differences $a_i - b_i a_0$ also generates the unit ideal.

Theorem 3.2 ((Bass) Stable range is a lower bound for Krull dimension on noetherian rings). If A is a noetherian ring of Krull dimension d , then its stable range is $\leq d + 1$.

Proof. See [20], chapter II, corollary 2.3.2 ■

furthermore,

Proposition 3.5. The formula for a higher bound for stable range is elementary.

Proof. Let A be a ring. The following formula limits stable range:

$$\begin{aligned} \varphi_{\leq m} : (\forall a_0, \dots, a_m) & \left[(\forall x. \exists b_0, \dots, b_m) \left(\prod a_i b_i = x \right) \dots \right. \\ & \left. \dots \Rightarrow (\exists c_1, \dots, c_m. \forall x. \exists d_1, \dots, d_m) \left(\prod_{>1} (a_i - c_i a_0) = x \right) \right] \end{aligned}$$

Elementarity is clear, as well as the equivalence to the condition mentioned earlier. ■

As a result, we can see that finite Krull dimension does not pass to ultraproducts in the noetherian case.

Theorem 3.3 (Ultraproduct of noetherian rings of finite Krull dimension). Ultraproducts do not preserve finite Krull dimension and noetherianity together.

Proof. Let the $(A_i)_{i \in \mathbb{N}}$ be a family of noetherian rings satisfying formulas of the form $\varphi_{\leq i+1}$, and $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Then the ultraproduct $\prod_{\mathcal{U}} A_i / \sim_{\mathcal{U}}$ is either a ring of infinite Krull dimension or it is non-noetherian. ■

and together with

Lemma 3.2 (Krull dimension of local noetherian rings). The Krull dimension of noetherian local rings is finite.

and

Lemma 3.3 (Locality is elementary). Being a local ring is an elementary property.

Proof. The maximal ideal of a local ring is exactly the set of invertible elements, i.e.

$$\{x : (\exists y)[xy = 1]\}$$

therefore a ring is local if it satisfies non triviality and the sum of non-invertible element is non invertible. Being invertible is defined by $I(x) := (\exists y)[xy = 1]$. Therefore the following is an axiom for locality:

$$0 \neq 1 \wedge (\forall x, y)[\neg I(x) \wedge \neg I(y) \Rightarrow \neg I(x + y)]$$

■

this gives us a proof of non-elementarity of noetherianity:

Theorem 3.4 (Ultraproducts of noetherian rings). Ultraproducts do not preserve noetherianity.

Proof. Consider the ultraproduct in theorem 3.3. An ultraproduct of local rings must be local by the previous lemma and Łoś's theorem, but if it is noetherian, it must be of infinite dimension by stable range lower bound, and this contradicts the previous lemma. ■

Another concrete example of this phenomenon of non-preservation: an ultrapower of $\mathbb{Z}$ has infinite Krull dimension, even though $\dim \mathbb{Z} = 1$.

Now, stable range is not that helpful as a way to control the dimension of a local ring (and in particular a valuation ring) because it only gives higher bounds, although

Proposition 3.6 (Stable range of a local ring). Any local ring has stable range 1.

As a result, we may see immediately the following.

Corollary 3.2. Any valuation of rank > 1 has a non-noetherian valuation ring.

and by

Corollary 3.3. Any noetherian valuation ring is a discrete valuation ring.

we can conclude that

Corollary 3.4. A valuation is non-discrete if and only if the valuation ring is non-noetherian.

In other words, we are interested in the exceptional case of rank 1 valuations with a non noetherian ring. Therefore, the case which interests us for elementary perfectoids is that of non-noetherian rings, but unfortunately the relation between stable range and Krull dimension doesn't seem to generalise to this case.

4 Model theory of henselian valued fields

4.1 Completeness and henselianity

In this section, we prove that a henselian valued field of mixed characteristic $(0, p)$, with a valuation of rank 1, is elementarily equivalent to its completion. Let us recall definitions here.

Theorem 4.1 (Ax-Kochen-Ershov principle (semi-perfect quotiented residue ring)). If $(K, v) \subseteq (L, w)$ are henselian fields of mixed characteristic $(0, p)$, such that $\mathcal{O}_v/(p)$ and $\mathcal{O}_w/(p)$ are semi-perfect, then $(K, v) \preceq (L, w)$ if and only if $\mathcal{O}_v/(p) \preceq \mathcal{O}_w/(p)$ and $(vK, v(p)) \preceq (wL, w(p))$.

Proof. See [8] for the proof. ■

This theorem, an instance of so-called Ax-Kochen-Ershov principle, allows one to reduce the theory of a valued field to the theories of its valuation group and residue field or residue ring. This is often useful, as these objects tend to be simpler, by way of various quotientings.

Let us now use it to show that a sufficiently nice field (K, v) is equivalent to its completion $(\widehat{K}, \widehat{v})$, where by "sufficiently nice", we mean any class of henselian valued field satisfying an Ax-Kochen/Ershov principle.

Now, as a preparation step for an AK/E style argument, let us use the following lemma to reduce the valuations of Cauchy sequences to that of constant sequences.

Theorem 4.2 (Structure of $\mathcal{O}_{\widehat{v}}/(t)$). Let t be a non-zero element of $\mathfrak{m}_{\widehat{K}}$. Then any Cauchy sequence (a_n) is equivalent to a constant Cauchy sequence (b_n) in $\mathcal{O}_{\widehat{v}}/(t)$, i.e. $v(a_n - b_n) \geq v(t)$.

Proof. Any Cauchy sequence (a_n) which converges to 0 is equivalent to the constant 0 sequence. Indeed, for any $r \in \Gamma$, there is a rank m at which $v(a_n - 0) \geq r$ for any $n \geq m$, so in particular, $(a_n) \sim (0)$.

Now, let (a_n) be an arbitrary non-zero Cauchy sequence. By lemma 1.1, (a_n) is eventually stationary. We find a constant sequence (b_n) such that (a_n) and (b_n) are equivalent modulo (t) . Consider the sequence defined as the constant sequence of the maximum of a_N and t (with the order lifted from the valuation, i.e. we choose the element with the greatest valuation), for a_N the eventual value of (a_n) .

$$(b_n) = (\max(a_N, t))$$

Now, by the axioms of a valuation, $(a_n - b_n)$ has valuation $\geq \min(a_N, t)$. But note that the case where the minimum is a_N implies that $(a_n - b_n) = (a_N - a_N) = \infty$. Therefore $v(a_n - b_n) \geq v(t)$. ■

As a result, considering the case $\mathcal{O}_v/(p)$, we obtain that $\mathcal{O}_{\widehat{v}}/(p) = \mathcal{O}_v/(p)$, and we may apply the AK/E principle to conclude that $(K, v) \preceq (\widehat{K}, \widehat{v})$.

Theorem 4.3 (Elementary equivalence of the completion). Let (K, v) be a henselian valued field of rank 1 and $(\widehat{K}, \widehat{v})$ its completion. Then $(K, v) \preceq (\widehat{K}, \widehat{v})$.

Proof. This is a direct application of the previous AK/E principle, in theorem 4.1, and the equivalences between the associated valuation rings and value groups. ■

4.2 Standard decomposition

Given a valued field of characteristic $(0, p)$, one may decompose the map to the residue field into three morphisms induced by successive valuations on coarsened value groups. This technique is known as *standard decomposition* is a central tool in the model theory of valued fields (see for e.g. [1]).

$$\begin{array}{ccccccc} & & & f_v & & & \\ & & \swarrow & & \searrow & & \\ K & \longrightarrow & K_0 & \longrightarrow & K_p & \longrightarrow & Kv \end{array}$$

Here K is a field with a valuation $v : K \rightarrow \Gamma \cup \{\infty\}$, inducing a map f_v on the residue field Kv . We consider coarsenings $v_0 : K \rightarrow (\Gamma/\Delta_0) \cup \{\infty\}$ and $v_p : K \rightarrow (\Gamma/\Delta_p) \cup \{\infty\}$ where Δ_0 and Δ_p are the smallest convex subgroup of Γ including $v(p)$ and the largest convex subgroup excluding $v(p)$ respectively. This gives us:

$$v_0(x) = \begin{cases} v(x) & \text{if } v(x) \notin \Delta_0 \\ 0 & \text{otherwise} \end{cases}$$

$$v_p(x) = \begin{cases} v(x) & \text{if } v(x) \notin \Delta_p \\ 0 & \text{otherwise} \end{cases}$$

We define K_0 and K_p as the residue fields derived from v_0 and v_p .

We furthermore consider valuations $\bar{v}_0 : K_0 \rightarrow \Gamma$ and $\bar{v}_p : K_p \rightarrow \Gamma$ as the restrictions of v to these fields.

Characteristic of the intermediate fields

Now, let us observe how the coarsenings affect the structure of the intermediate residue fields K_0 and K_p . We first make the following crucial observation:

Lemma 4.1 (Maximal ideal and characteristic). If the maximal ideal $\mathfrak{m}_v$ of a valuation contains p , then it contains $p\mathcal{O}_v$, and the corresponding residue field Kv is of characteristic p . Otherwise, Kv is of characteristic 0.

Proof. The main implication follows from the valuation axioms : $v(p) > 0$ implies $v(n \cdot p) > 0$ for $n \in \mathbb{N}$ and $v(p \cdot x) = v(p) + v(x) \geq v(p) > 0$. The second reflects the fact that p then acts as 0 in the residue field. ■

As a result, we may concentrate on the value $v(p)$ in the different valuations, to describe the standard decomposition in details.

(K, v_0) is of characteristic $(0, 0)$: indeed note that the value of p is forgotten by the coarsening, $v_0(p) = 0$, so $p \notin \mathfrak{m}_v$, and we conclude from our lemma. Likewise (K, v_p) is of characteristic $(0, p)$ as $v(p)$ is now preserved, so K_p is of characteristic p . Now let us consider the intermediate $(K_0, \bar{v}_p)$, corresponding to the central map of our decomposition. Its value group is Δ_0/Δ_p . If the maximal ideal $\mathfrak{m}_v$ of a valuation contains p , then it contains pv , and the corresponding residue field Kv is of characteristic p . Otherwise, Kv is of characteristic 0, so the value of p is preserved, and $(K_0, \bar{v}_p)$ is of bicharacteristic $(0, p)$. Finally $(K_p, \bar{v})$ is of bicharacteristic (p, p) by the earlier remarks and the same lemma.

Corollary 4.1. We may immediately derive a few corollaries from lemma 4.1.

1. A valuation is positive in at most 1 prime number.
2. A valuation of residue characteristic 0 satisfies $v(p) = 0$ for all p .

Proof. 1. By Bézout's theorem, $\mathbb{Z}$ -linear combinations of elements of principal ideals over distinct prime numbers cover $\mathbb{Z}$, and since positive valuation of a prime extends to its principal ideal, this implies that every integer has positive valuation, which contradicts the axiom $v(1) = 0$.

2. is just a contrapositive form of the lemma. ■

4.3 AK/E principles

The Ax-Kochen / Ershov principles are a family of theorems reducing questions about valued fields to questions about their associated structures (namely the valuation group and residue field). These are a central tool in the model theory of valued field, and we will make a heavy use of them in the later parts of this thesis. Proofs of the AK/E theorems are out of scope of the current work.

The original AK/E theorem was found independently by Ax and Kochen in [3] and Ershov in [5].

Theorem 4.4 (Ax-Kochen / Ershov 1965). If (K, v) and (L, w) are henselian fields of equicharacteristic 0, then

$$(K, v) \equiv_{\mathcal{L}_{val}} (L, w) \Leftrightarrow vK \equiv_{grp} wL \wedge Kv \equiv_{rng} Lw$$

Theorem 4.5 (AK/E mixed,). If $(K, v) \subseteq (L, w)$ are henselian fields of characteristic $(0, p)$, with $\mathcal{O}_v/p, \mathcal{O}_w/p$ semi-perfect. Then

$$(K, v) \preceq (L, w) \Leftrightarrow \mathcal{O}_v/p \preceq \mathcal{O}_w/p \text{ and } (vK, v(p)) \preceq (wL, w(p))$$

Theorem 4.6 (Jahnke-Kartas '23). If $(K, v) \subseteq (L, w)$ are henselian fields of positive characteristic. $t \in \mathfrak{m}_v$ is such that $\mathcal{O}_v[\frac{1}{t}]$ and $\mathcal{O}_w[\frac{1}{t}]$ are algebraically maximal, $\mathcal{O}_v/t$ and $\mathcal{O}_w/t$ are semi-perfect and K, L are perfect, then

$$(K, v) \preceq (L, w) \Leftrightarrow \mathcal{O}_v/t \preceq \mathcal{O}_w/t \text{ and } (vK, v(t)) \preceq (wL, w(t))$$

5 Approximating a classification in the language of valued fields

Let us now present a class of valued fields, generalising $(0, p)$ perfectoid fields. The overall strategy in this section is to define a class of fields satisfying elementary analogons of the perfectoid properties, and which is tame enough that we are able to use the AK/E principle to fully classify its models.

More precisely, in this chapter, we will mostly consider valued fields which are henselian, mixed characteristic, semi-perfect quotient ring, with a pseudo-uniformiser $t \in \mathfrak{m}_v$ making $\mathcal{O}_v[t^{-1}]$ algebraically maximal. We occasionally call them *nice*.

Ideally, we would like such a class to generalise perfectoids (i.e. every perfectoid field should be a model), and to be generated by perfectoids and elementary equivalence (i.e. every model should be equivalent to a perfectoid field).

While we are able to recover every perfectoid, the second requirement remains open. The main obstruction here is that taking ultraproducts of valued fields doesn't preserve rank, so one ideally needs an elementary condition capturing "being elementary equivalent to a rank one model".

As we have seen in section two, such a condition can be partially defined, modulo a specific choice of prime invariants for the valuation group. This may give us a "faithful" class, in the sense that every model is equivalent to a perfectoid, but not a "full one", as some perfectoid fields won't be in the class (namely those whose value group is regular for a different choice of invariants).

On the other hand, omitting the rank restriction condition can give us a full class, but it won't be faithful anymore, as we shall see later on.

The first part of the chapter ensures we are able to apply an AK/E principle. Then we want to use this principle as a classification tool, by reducing the theory of the models to their auxiliary structures.

We solve the problem of rank reduction in the finite case by ensuring that models of our class are all elementary equivalent to their Hahn series fields.

5.1 Taming and Ramification theory

In this section, we study in some details conditions for tameness. Ensuring tameness is important as it will allow us to make use of the tame AK/E principle introduced before.

We want to show here that algebraic maximality of $(K, \mathcal{O}_v[t^{-1}])$ is elementary in the language of valued fields, with a symbol for t . The first section of this chapter is a condensed introduction to ramification, and associated concepts. As we will see, in the finitely ramified case, algebraic maximality is simply given by henselianity, and therefore elementary, but in the case of interest, namely the deeply ramified case, this does not hold, so we consider different approaches to ensure elementarity of this property.

Definition 5.1 (Algebraic maximality). A valued field is *algebraically maximal* if it has no non-trivial algebraic immediate extension.

5.2 Ramification, defect, fundamental inequality

We present two related notions of ramification, namely ramification of an extension, and an intrinsic notion of ramification. By the fundamental inequality, the degree of a valued field extension is given by the product of the ramification index, the residue degree, and the defect. Therefore, ramification describes how much changes in the value group contribute to the extension, as opposed to the residue field. Loosely, non-trivial ramification corresponds to extensions adding positively valued elements, while non-trivial residue degree corresponds to extensions adding new elements of valuation 0. If (K, v, p) is a valued field with a fixed pseudo-uniformiser p , the (intrinsic) ramification of K counts the number of values in Γ_K between 0 and $v(p)$.

Ramification of extensions

Definition 5.2 (Injection of value groups, Ramification index). Let $(K_1, \mathcal{O}_1) \subseteq (K_2, \mathcal{O}_2)$ be an extension of valued fields. Then $\Gamma_1 \subseteq_{\text{OAG}} \Gamma_2$. The *ramification index* is the subgroup index $e(\mathcal{O}_2/\mathcal{O}_1) := [\Gamma_2 : \Gamma_1]$.

Definition 5.3 (Injection of value groups, Residue degree). Let $(K_1, \mathcal{O}_1) \subseteq (K_2, \mathcal{O}_2)$ be an extension of valued fields. Then the residue fields form a field extension: $k_1 \subseteq_{\text{Fields}} k_2$. The *residue degree* is the extension degree of the residue fields $f(\mathcal{O}_2/\mathcal{O}_1) := [k_2 : k_1]$.

Definition 5.4 (Immediate extension). An extension $(K_1, \mathcal{O}_1) \subseteq (K_2, \mathcal{O}_2)$ of valued fields is *immediate* if it is of ramification index and residue degree both 1.

Theorem 5.1 (Fundamental inequality). Let $(K_1, \mathcal{O}_1) \subseteq (K_2, \mathcal{O}_2)$ be an extension of valued fields of degree n . Then

$$n \geq e(v'/v) \cdot f(v'/v)$$

Definition 5.5 (Defect). The *defect* of an extension is the multiplicative error term of the fundamental inequality. When it is 1, the extension is called *defectless*.

Intrinsic ramification

As mentioned earlier, intrinsic ramification is a property of a valued field with a designated element.

Let us first consider the finitely ramified case.

Definition 5.6 (Finitely ramified). A valued field $(K, \mathcal{O}_v)$ is *finitely ramified* if it is either of residue characteristic 0, or it is of residue characteristic p and there are finitely many values between 0 and $v(p)$.

In the finitely ramified case, henselianity and algebraic maximality are equivalent.

Theorem 5.2 (Algebraic maximality of finitely ramified henselian fields). If $(K, \mathcal{O})$ is a finitely ramified valued field, then it is algebraically maximal if and only if it is henselian.

Proof. See [4], theorem 4.1.10 ■

As a corollary, algebraic maximality is elementary in this case.

Observe furthermore that finite ramification is an elementary property. Indeed, the only way infinite ramification can occur is if Γ has a dense subset between 0 and $v(p)$, which is elementary, via the following formula (in the three-sorted language):

$$(\exists \alpha, \beta \in \Gamma)(0 < \alpha < \beta < v(p)) \wedge (\forall \gamma, \delta \in [\alpha, \beta](\gamma < \delta) \Rightarrow (\exists \varepsilon. \gamma < \varepsilon < \delta))$$

5.2.1 Deep ramification

The general definition of deep ramification relies on so-called Kähler differentials, which we will avoid here, and instead use the following elementary characterisation, found in [12], as our definition:

Theorem 5.3 ((Gabber-Ramero) Deep ramification in rank 1, positive equal characteristic). A valued field of rank 1 and equal characteristic is deeply ramified if and only if its completion is perfect.

Theorem 5.4 ((Gabber-Ramero) Deep ramification in rank 1, mixed characteristic). A valued field of rank 1 and mixed characteristic $(0, p)$ is deeply ramified if and only if the value $v(p)$ is not the smallest positive value in vK and the Frobenius endomorphism $x \mapsto x^p$ is surjective in $\mathcal{O}_K/p\mathcal{O}_K$.

Proposition 5.1 (Deep ramification in higher rank). The Gabber-Ramero characterisation of deep ramification extends to higher ranks, with the additional condition that the value group does not have discrete archimedean components.

As we can see, we can reformulate our notion of perfectoid field as "a complete, deeply ramified valued field of rank 1".

Let us now show that algebraic maximality of $\mathcal{O}_K[t^{-1}]$ is elementary in the language $\mathcal{L}_{val} \cup \{t\}$.

5.3 Algebraic maximality of the localised valuation ring

Let us show that the property that " $\mathcal{O}_v[t^{-1}]$ is algebraically maximal" is elementary in (K, v) . We first consider two distinct proof strategies, corresponding to two ways of controlling algebraic maximality of a field. In both case, the difficulty lies with the fact that $\mathcal{O}_v[t^{-1}]$ is not interpretable in K , so we will also need some machinery to access $\mathcal{O}_v[t^{-1}]$, which we will consider in the next section.

5.3.1 Algebraic maximality via extremality

Let us present a characterisation of algebraic maximality, due to Ershov, and exposed in [11], which uses extremality relative to a class of polynomials.

Definition 5.7 (Relative K -extremality, Relative $\mathcal{O}$ -extremality). A valued field (K, v) is K -extremal (resp. $\mathcal{O}$ -extremal) with respect to a polynomial $f \in K[\bar{x}]$ of n variables, for some n , if the set

$$vf[\mathcal{I}] := \{vf(\vec{a}) | \vec{a} \in \mathcal{I}\} \subseteq vK \cup \{\infty\}$$

has a maximum, where $\mathcal{I}$ denotes K or $\mathcal{O}$ respectively.

Definition 5.8 (Extremality). A valued field (K, v) is *extremal* if it is K extremal with respect to every polynomial f in any number of variables.

Proposition 5.2 (Algebraic closure and extremality). Let (K, v) be a valued field. If (K, v) is algebraically closed, then it is extremal.

Proof. If (K, v) is algebraically closed, then the valuation of the image of any polynomial contains ∞ , which is maximal. ■

Proposition 5.3 (Elementarity of relative extremality). Relative extremality of a valued field (K, v) with respect to a single polynomial f can be expressed as a first-order statement in the language of valued fields:

$$(\exists \vec{a})(\forall \vec{x})[vf(\vec{x}) \leq vf(\vec{a})]$$

Proposition 5.4 (Elementarity of extremality). Extremality of a valued field can be expressed as a countable scheme of first-order statements in the language of valued fields.

Proof. Follows directly from the former, and from countability of the polynomials in each number of variables. ■

Algebraic maximality can be expressed as extremality with respect to a specific class of polynomials.

Proposition 5.5 (Algebraic maximality via extremality). A valued field is algebraically maximal if and only if it is henselian and K -extremal with respect to every polynomial in one variable.

Proof. See [11], chapter 14, proposition 14.40. ■

Corollary 5.1 (Elementarity of algebraic maximality). Algebraic maximality is elementary in the language of valued fields.

Now, this suggests a second strategy to control algebraic maximality of $\mathcal{O}_v[t^{-1}]$, namely to ensure that it is extremal for every polynomial in one variable. However, this may not be done easily, working over K , as $\mathcal{O}_v[t^{-1}]$ is not interpretable in K in general, so we will use another strategy.

5.3.2 Algebraic maximality via defectlessness

Let us consider the following two results from [8] and [1].

Lemma 5.1 ([8]). Let (K, v) be a valued field of residue characteristic p , $\mathcal{O}_v/(p)$ semi-perfect. If t is a pseudo-uniformiser, then the following are equivalent

1. $\mathcal{O}_v[t^{-1}]$ is defectless.

2. $\mathcal{O}_v[t^{-1}]$ is algebraically maximal.

Lemma 5.2 ([1], lemma 2.4). The class of defectless valued fields is first-order axiomatisable in the language of valued fields.

Proof. Normal extensions of a fixed degree are interpretable in the base field by using the tuples of coefficients of the generating polynomials. (note : by normality, any tuple gives only one extension.)

For a fixed coefficient tuple in K , define (K, L, v) such that L is the corresponding field extension, equipped with an appropriate embedding from K .

Next, one shows that the collection of valuation rings extending v is definable from the same parameters. (See [9], 9.4.8).

Finally, one may separate those rings which satisfies the fundamental equality, which is clearly elementary. ■

Note that this result gives us a characterisation for defectlessness, but not yet defectlessness of the ring $\mathcal{O}_v[t^{-1}]$, so we are not quite done. Let us first adapt the proof method of lemma 2.4 in [1] to express algebraic maximality directly instead of defectlessness. Then we will try to adapt it further to work on the ring $\mathcal{O}_v[t^{-1}]$.

5.4 Interpreting normal extensions of valued fields

Let (K, v) be a nice valued field. In order to show that an extension of K is algebraically maximal, we first show that extensions are interpretable in K , and then show that algebraic maximality of a field extension is elementary in the language of the extension.

Consider the following algebraic fact.

Theorem 5.5 (Extensions by quotient of the polynomial ring). Let K be a field, $P \in K[X]$ be an irreducible polynomial over K , $L = K(x)$ be an extension generated by a root of P . Then $K[X]/(P) \cong L$.

Now, a crucial aspect of this construction is that elements of $K[X]$ are definable from tuples in K . Fixing a degree n , this hints at a way to interpret sufficiently nice extensions of K inside K .

Proposition 5.6 (Interpretation of normal extensions). Let K be a field. The class of normal extensions of a fixed degree n of K is interpretable in K^n .

Proof. For each normal extension L/K of degree n , let P_L be a polynomial such that $P(\xi) = 0$ and $L \cong K(\xi)$. Now, define $(\vec{c})_L$ to be the tuple of coefficients of P_L . Now each extension L has a corresponding tuple in the set K^n . ■

Theorem 5.6 (Interpretation of a fixed normal extension). Let (K, v) be a valued field. Algebraic extensions of K are interpretable with parameters in K .

Proof. (Wlog, we just consider the case of one generator). Let L/K be such an extension

of degree n , generated by a polynomial $P \in K[x]$ of degree n . Let $\vec{c}$ be the coefficients of P . We may write $P_{\vec{c}}(X) := \sum_{i=0}^n c_i X^i$ for the polynomial with coefficients $\vec{c}$. Now suppose wlog that α is a generator, i.e. a root of P such that $L \cong K(\alpha)$.

Then the interpretation of L in K^n is obtained by evaluating polynomials of degree $\leq n$ at α , i.e. $f(\vec{c}) = P_{\vec{c}}(\alpha)$. In particular, equality in the interpretation is defined modulo P . ■

Note that this approach requires L/K to be a finite normal extension as opposed to any algebraic extension. The reason is that L needs to be uniquely determined by the tuple $(\vec{c})$. Now, more general classes of extensions may be interpretable using a different method of interpretation, and in particular, in more parameters.

Definition 5.9 (The interpreted sort). Let (K, v) be a valued field. From the previous proposition, we may define (K, v, L) to be the valued field (K, v) , together with the (interpretation of) the extension L . L is a structure, interpreting formulas in a sort $\mathbb{L}$, in the language of rings. We furthermore enrich our language with a predicate for the canonical inclusion $K \hookrightarrow L$.

Since L is interpretable in K , the previous definition does not require stronger assumptions than the usual three-sorted setting. Now, we naturally want to work on valuations over L . There is a canonical extension of v .

Theorem 5.7 (Extension of valuations). Let L/K be a normal extension of K , and $v : K \rightarrow \Gamma \cup \{\infty\}$. Then there exists valuations of L which restrict to v on K .

Proof. Using the interpretation defined earlier, an extension of the valuation v is given by its value at the generating elements. ■

Now, let us show that the extended valuation v' is itself interpretable. The argument here is adapted from [9].

Lemma 5.3 (Polynomial characterisation of the ring extension). Let K/L be a valued fields extension, and $\vec{a} \in L^n$ be a tuple. Now, the following are equivalent.

1. $\{a_1, \dots, a_k\}$ is not contained in any extension of $\mathcal{O}_L \cap K$.
2. $P(\vec{a}) = 1$ for some $P \in \mathfrak{m}[X_1, \dots, X_k]$

Proof. $1 \Rightarrow 2$. Let $\{a_1, \dots, a_k\}$ be a tuple contained in no extension of $\mathcal{O}_L \cap K$. This implies that at least one a_i of $a_1, \dots, a_k$ is non-zero, and of negative valuation. Now the polynomial is simply given by $P = a_i^{-1} X$. Since $v(a_i) < 0$, we have $v(a_i^{-1}) > 0$, so $P \in \mathfrak{m}[X_1, \dots, X_k]$, and the evaluation at a_i is 1.

$2 \Rightarrow 1$. Let $P(\vec{a}) = 1$ with $P \in \mathfrak{m}[X_1, \dots, X_n]$. Now, since all coefficients have valuation strictly positive, and by the axioms of valuations, $v(a_i) < 0$ for at least one of the a_i , so the tuple does not live in an extension of $\mathcal{O}_L \cap K$. ■

In order to use this characterisation, let us furthermore show that the degree of P is bounded here. To do so, we show yet another equivalence.

Lemma 5.4. Let K/L be a valued fields extension, and $\vec{a} \in L^n$ be a tuple. Now, the following are equivalent:

1. $P(\vec{a}) = 1$ for $P \in \mathfrak{m}[X_1, \dots, X_k]$
2. $\{a_1, \dots, a_k\} \not\subseteq \sigma(\mathcal{O}_L)$ for $\sigma \in \text{Aut}(L/K)$

Proof. $\Rightarrow$ Let $P(\vec{a}) = 1$. Then $\{a_1, \dots, a_k\}$ is not in any extension of $\mathcal{O}_L \cap K$. It follows that $\vec{a} \notin \mathcal{O}_L$. As a valued field automorphism, σ should fix K (point-wise) and preserves $\mathcal{O}_L$ (set-wise). Therefore $\vec{a} \notin \sigma(\mathcal{O}_L)$.

$\Leftarrow$ Let $\{a_1, \dots, a_k\} \not\subseteq \sigma(\mathcal{O}_L)$ for any σ . Then there is at least one element $x = a_i$ with $v(x) < 0$. Define the polynomial $P_{\vec{a}}$ with coefficients 0 everywhere except in $c_i := \frac{1}{x}$, which is in $\mathfrak{m}$ by hypothesis. ■

Lemma 5.5 (Degree bound). The degree of the polynomial P in the previous lemma is bounded.

Proof. By compactness. ■

Theorem 5.8 (Interpreting the extended valuation). Let (K, v, L) be a normal valued field extension. Then v' is interpretable in K^n using $\vec{c}$ and K .

Proof. From the previous two lemmas, we may now interpret $\mathcal{O}_{v'}$ using $\vec{c}$ and the polynomial characterisation. ■

5.5 Elementary condition for maximality

Using the interpretation framework defined in the previous section, we show how to control maximality of the ring $\mathcal{O}_v[t^{-1}]$ from K .

Definition 5.10 (Different). Let $a \in L/K$ be an element of a finite extension. The *different* $\delta(a)$ of a is the evaluation of the formal derivative of the minimal polynomial of a , at a .

$$\delta(a) := P'_a(a)$$

Let us now consider the structure $\mathcal{O}_v[t^{-1}]$. This is the subring of K generated by $\mathcal{O}_v$ and t^{-1} . If $t \in Kv$, then $t, t^{-1} \in \mathcal{O}_v$ and $\mathcal{O}_v = \mathcal{O}_v[t^{-1}]$, so we can assume $v(t) > 0$. Intuitively, since $t \in \mathcal{O}_v$, the ring $\mathcal{O}_v[t^{-1}]$ is a localisation of $\mathcal{O}_v$ at t . Since $K \cong \text{Frac}(\mathcal{O}_v)$, we have $\mathcal{O}_v \subseteq \mathcal{O}_v[t^{-1}] \subseteq K$. Furthermore any localisation of a valuation ring is a valuation ring, and the maximal ideal is given by $\mathfrak{m}_K \setminus \langle t \rangle$

Theorem 5.9 (Elementary analogon for algebraic maximality, henselian, semi-perfect setting ([8], lemma 4.1.4)). If (K, v) is a henselian valued field with residue characteristic $p > 0$, $\mathcal{O}_v/(p)$ semi-perfect, and $t \in \mathfrak{m}_K$. The following are equivalent:

1. $\mathcal{O}_v[t^{-1}]$ is algebraically maximal

2. For any finite extension $(K', v')/(K, v)$ with $e(v', v) = 1$, there exists $a \in \mathcal{O}_{v'}$ with $0 \leq v(\delta(a)) \leq v(t)$, and $K' = K(a)$.

Theorem 5.10 (Interpreting algebraic maximality). The second condition in the previous theorem is an elementary statement, using the interpretability of extensions.

5.6 Taming

Let us present tameness, show that valued fields in our class are indeed tame. Once this is done, we will be able to classify these models using the theories of their value group and residue field as invariants.

Definition 5.11. A finite valued fields extension $(K, v) \subseteq (K', v')$ is *tame* if

1. if positive, the characteristic of k does not divide the ramification index, i.e. $p \nmid [\Gamma' : \Gamma]$
2. $k \subseteq k'$ is a separable extension of fields.
3. $(K, v) \subseteq (K', v')$ is defectless.

Definition 5.12 (Tame valued field). A valued field (K, v) is *tame* if all its finite extensions are tame.

Proposition 5.7 (Characterisation of henselian tame fields). A henselian valued field (K, v) is tame if and only if it is algebraically maximal, Γ is p divisible, and k is perfect.

5.7 The Hahn series field $(K((\mathbb{Q})), v_{\mathbb{Q}})$

Definition 5.13 (Hahn series field). Let K be a field. The *Hahn series field* $K((\mathbb{Q}))$ associated to K is the field of series of the form

$$\sum_{i \in \mathbb{Q}} \alpha_i t^i$$

such that the support is well ordered.

Addition is defined point-wise by

$$\sum_{i \in \mathbb{Q}} \alpha_i t^i + \sum_{i \in \mathbb{Q}} \beta_i t^i = \sum_{i \in \mathbb{Q}} (\alpha_i + \beta_i) t^i$$

Multiplication is defined by :

$$\left(\sum_{i \in \mathbb{Q}} a_i t^i \right) \cdot \left(\sum_{j \in \mathbb{Q}} b_j t^j \right) = \sum_{h \in \mathbb{Q}} \left(\sum_{i+j=h} a_i b_j \right) t^h$$

which this is well defined since the sum $\sum_{i+j=h} a_i b_j t^h$ is finite (as a consequence of well-order).

Proposition 5.8 (Embedding to the Hahn series field). Let K be a field, then K embeds into its Hahn series field $K((\mathbb{Q}))$ as follows :

$$\begin{aligned} K &\rightarrow K((\mathbb{Q})) \\ x &\mapsto x \end{aligned}$$

Proof. This is well defined by considering $x \in K((\mathbb{Q}))$ as a sum with zero coefficient everywhere except at the 0 exponent. ■

5.7.1 Series valuation

The Hahn series field of a valued field admits a *series valuation* similar to the polynomial valuation, as well as a valuation capturing information about the underlying valued field.

Definition 5.14 (Valuation on the Hahn series field). A Hahn series field $K((\mathbb{Q}))$ may be equipped with the valuation

$$v_{\mathbb{Q}} : K((\mathbb{Q})) \rightarrow \mathbb{Q}$$

$$\sum_{i \in \mathbb{Q}} \alpha_i t^i \mapsto \min\{i : \alpha_i \neq 0\}$$

which is well defined for non-zero sequences because we use a well-order on $\mathbb{Q}$ here. Furthermore, the identically 0 sequence is conventionally given valuation ∞ .

Proof. Let us check that this construction verifies the valuation axioms. $v(0) = \infty$ is conventional. $v(a + b) \geq \min(v(a), v(b))$ follows from the point-wise definition of the addition. Finally $v(ab) = v(a) + v(b)$. Consider the series $\sum_h \sum_{i+j=h} a_i b_j t^h$, and let $h < v(a) + v(b)$. Then clearly in each component $a_i b_j$ of the sum $\sum_{i+j=h} a_i b_j t^j$ there must be a zero term, since either $a_i < v(a)$ or $b_j < v(b)$. On the other hand, for $h = v(a) + v(b)$, there is a non-zero component, and this is the minimal such index, therefore $v(ab) = v(a) + v(b)$, which concludes our proof. ■

By definition of $v_{\mathbb{Q}}$, we can see that elements of K have valuation 0 when viewed inside $K((\mathbb{Q}))$. In fact, let us show that K is exactly the residue field of $K((\mathbb{Q}))$.

Theorem 5.11 (Residue field of the Hahn series field). If K is a valued field and $(K((\mathbb{Q})), v_{\mathbb{Q}})$ is its Hahn series field, equipped with the series valuation, then K is the residue field of $(K((\mathbb{Q})), v_{\mathbb{Q}})$.

Proof. The valuation ring of $K((\mathbb{Q}))$ is the ring of Hahn series without negative exponents. The maximal ideal is given by series with strictly positive exponents. Quotienting by the maximal ideal, any series in the residue field of $K((\mathbb{Q}))$ is considered up to an additive factor of a positively valued series. As a result, only the coefficients at exponent 0 matter, but coefficients are taken in K by definition. Furthermore, every element of K gives a zero-valued element of $K((\mathbb{Q}))$ by the embedding defined earlier. Therefore, K is the residue field of $(K((\mathbb{Q})), v_{\mathbb{Q}})$ ■

Remark. Note that the reference to $\mathbb{Q}$ in the definition is not really essential, although it is convenient if we want to think of the elements as actual series of a field rather than purely formal objects. In particular, one may replace $\mathbb{Q}$ with another value group, so we are still within the realm of first-order logic, and safe from Gödelian phenomena.

5.7.2 Composed valuation

By considering the composition of the map $K((\mathbb{Q})) \rightarrow K$ induced by the valuation $v_{\mathbb{Q}}$ with the map $K \rightarrow Kv$ induced by the valuation v , we can equip $K((\mathbb{Q}))$ with a new valuation w , such that Kv is the residue field of $(K((\mathbb{Q})), w)$.

Let us check that $\mathcal{O}_v/p = \mathcal{O}_w/p$

Proposition 5.9 (induced valuation on the Hahn series field). Let (K, v) be a valued field, Kv be its residue field, and $K((\mathbb{Q}))$ be its Hahn series field. There exists a valuation on $K((\mathbb{Q}))$ with residue field Kv .

Proof. We consider the following decomposition of induced residue morphisms.

$$\begin{array}{ccccc} & & f_w & & \\ & \swarrow & & \searrow & \\ K((\mathbb{Q})) & \xrightarrow{f_{v_{\mathbb{Q}}}} & K & \xrightarrow{f_v} & Kv \end{array}$$

We define a valuation satisfying the decomposition: our *composed* valuation on $K((\mathbb{Q}))$ is defined by

$$\begin{aligned} w : K((\mathbb{Q})) &\rightarrow \Gamma_K \oplus \mathbb{Q} \\ x &\mapsto v_K(x_{v_{\mathbb{Q}}(x)}) \oplus v_{\mathbb{Q}}(x) \end{aligned}$$

Now, first note that w is a well-defined valuation.

Proof. By definition of $v_{\mathbb{Q}}$, 0 is the only element of $K((\mathbb{Q}))$ with an empty support. The axioms of valuation are preserved by the component-wise definitions of operations, and antilexicographic order. ■

Furthermore the residue field is Kv .

Proof. Recall that the residue field of $v_{\mathbb{Q}}$ is K itself. Now, by taking the v_K valuation of $x_{v_{\mathbb{Q}}(x)}$, one has positive valuation whenever either $v_{\mathbb{Q}}$ is positive or v_K is positive. As a result, the only elements with 0 valuation are the residues of v_K . ■

5.8 Equivalence to Hahn series

Let us show that fields in our class are always elementarily equivalent to their Hahn series field. Using the AK/E principle from Jahnke and Kartas (theorem 4.6), this amounts to proving elementary equivalences at the level of the quotient ring and the valuation group.

5.8.1 Quotient rings isomorphism

Theorem 5.12 (Quotient rings isomorphism). Let (K, v) be a valued field and $(K((\mathbb{Q})), w)$, where $w = v \circ v_{\mathbb{Q}}$. Then $\mathcal{O}_w/(p) \cong \mathcal{O}_v/(p)$.

Proof. Recall that $\mathcal{O}_w = \{x \in K((\mathbb{Q})) : w(x) \geq 0\}$ so in particular $v(x) \geq 0$ and $w(x) \geq 0$. Explicitly, these are Hahn series with positive exponents, quotiented by the ideal (p) . Clearly, $\mathcal{O}_v(p)$ embeds in $\mathcal{O}_w/(p)$ via $x \mapsto (x, 0, 0, \dots)$. This is a ring embedding: if $x \in \mathcal{O}_v/(p)$, then $p - x \in \mathcal{O}_v(x)/p$ as well, because $v(p - x) \geq \min(v(p), v(x)) \geq 0$.

Now, in the Hahn series, quotienting by (p) means the constant coefficients are quotiented by p , and together with the positivity of w , this means the constant coefficients are exactly $\mathcal{O}_v(p)$. However, note that the coefficients at higher degrees need not be in $\mathcal{O}_v$ at all, but are still taken in K . In other words, $\mathcal{O}_w \not\cong \mathcal{O}_v((\mathbb{Q}))$.

As a result, since K is a field, quotienting by p simply trivialise non-minimal degrees of Hahn series, so at this stage there only remains monomials with coefficients in $\mathcal{O}_v/(p)$. Let us show these are all of degree 0. Let P be a monomial of non-zero degree. Now, $1 + P = 1$ by quotienting out the coefficients in the higher-degree terms, and by associativity, we get:

$$P = (-1 + 1) + P = -1 + (1 + P) = -1 + 1 = 0$$

Therefore, the only elements left are the constants $\mathcal{O}_v/(p)$. ■

5.8.2 Elementary embedding

We already saw a way to embed K into $K((\mathbb{Q}))$. Let us show that it is actually an elementary embedding of valued fields for our class of valued fields. This will be an important step in controlling the rank of valued fields.

We now consider a nice (henselian, positive residue characteristic, semi-perfect value ring, with some pseudo-uniformiser $t \in \mathfrak{m}_v$ making $\mathcal{O}_v[t^{-1}]$ algebraically maximal) valued field K . Let us show $K \equiv K((\mathbb{Q}))$.

We use the following powerful theorem from [8].

Theorem 5.13 (Tame AK/E principle). If (K, v) , (L, w) are nice valued field then

$$(K, v) \preceq_{val} (L, w)$$

if and only if

$$\mathcal{O}_v/t \preceq_{rng} \mathcal{O}_w/t \quad \text{and} \quad vK \preceq_{oag} wL$$

Now, considering (K, v) and $(K((\mathbb{Q})), v \circ v_{\mathbb{Q}})$, since Kv is the residue field for both K and $(K((\mathbb{Q})))$ with the composed valuation, our problem reduces to

$$vK \preceq_{grp} vK \oplus \mathbb{Q}$$

Let us first consider the case where vK is divisible, and then try to extend the argument for the regular case.

Theorem 5.14 (Group embedding). With the previous conditions, there is an elementary embedding $vK \preceq_{grp} vK \oplus \mathbb{Q}$.

Proof. Let us first reduce the question of finding an elementary embedding to that of finding an embedding.

Lemma 5.6 (Divisibility passes to direct sums). If vK and $\mathbb{Q}$ are divisible, then $vK \oplus \mathbb{Q}$ is as well.

Consider the trivial embedding from $vK \hookrightarrow vK \oplus \mathbb{Q}$ given by $x \mapsto x \oplus 0$. By lemma 2.1, the theory DOAG_2 of divisible ordered abelian group which at least two elements is model complete, so this embedding is elementary, which concludes our proof. ■

5.8.3 General case

For the general case, we need to use a standard decomposition argument in order to reduce to the divisible case. The general strategy is to get rid of every value infinitesimal or infinite relative to a given value, which is done by quotienting out the right coarsenings.

Definition 5.15 (Almost divisible). An ordered abelian group Γ is *almost divisible* if a quotient Γ/Δ by any proper convex subgroup is divisible.

Lemma 5.7 (Regularly dense implies almost divisible).

For our argument to work, we need to ensure that the quotient Δ^+/Δ of coarsenings around a designated value ϖ is non-trivial. To do so, we don't build the divisible subgroup directly in $\Gamma = vK$, but instead we make the argument for an $\aleph_1$ -saturated ultraproduct $\Gamma^* := \prod_{\mathcal{U}} \Gamma / \sim$.

Lemma 5.8 (Standard decomposition). We consider the standard decomposition of (K^*, v^*) to extract the right convex subgroup.

Proof. Choose a pseudo-uniformiser $\varpi \in \mathfrak{m}_v$ (more precisely, take p in characteristic $(0, p)$, and t in positive characteristic). Abusing notation, let us also call ϖ its image through the embedding $K \hookrightarrow K^*$.

Now define Δ^+ to be the minimal convex subgroup of Γ^* containing $v^*\varpi$, and Δ the maximal convex subgroup not containing $v^*\varpi$. Let K_0^*, K_p^*, K^*v the residue fields of the valuations induced by the groups Γ^*/Δ^+ , Δ^+/Δ and Δ on the appropriate fields. This gives us the decomposition

$$K^* \xrightarrow{\Gamma_v/\Delta^+} K_0^* \xrightarrow{\Delta^+/\Delta} K_p^* \xrightarrow{\Delta} K^*v$$

Notice how the quotient dispels non-Archimedean phenomena. Here, Δ^+/Δ is intuitively a convex subgroup whose non-zero elements are valuations close to $v^*\varpi$, in the sense that there are no infinitesimals relative to $v^*\varpi$ (by quotienting Δ out), and there are no infinite elements relative to $v^*\varpi$ (because Δ^+ is convex).

Furthermore, Δ^+/Δ is nontrivial, as from saturation of Γ^* , and positivity of the valuation of ϖ , we can find some point dividing $v^*\varpi$, by satisfying any type containing a division formula $\sum x = v^*\varpi$.

Therefore, Δ^+/Δ must still be divisible, as $v^*\varpi$ is non-zero and Δ^+ is divisible. So in particular any divisor of $v^*\varpi$ in Δ^+ must be in $\Delta^+ \setminus \Delta$, and therefore in Δ^+/Δ . ■

Write $\Gamma = vK$, and let us now show that we do have the required equivalence.

$$\Gamma \equiv \Gamma \oplus \mathbb{Q}$$

Proof. Let Γ^* be a saturated ultrapower of Γ . We have $\Gamma^* \cong \Delta \oplus \mathbb{Q}$, for Δ a divisible group.

Lemma 5.9 (Sum decomposition). Γ^* may be written as a direct sum of divisible groups, i.e. $\Gamma^* \simeq \bigoplus_i \Delta_i$

Now, consider the last term D in $\bigoplus_i \Delta_i$. D is a divisible group. By model completeness, $D \leq D \oplus \mathbb{Q}$, so in particular $\Gamma^* \equiv (\bigoplus_i \Delta_i \oplus \mathbb{Q}) \leq \Delta \oplus (\mathbb{Q} \oplus \mathbb{Q}) \equiv (\Delta \oplus \mathbb{Q}) \oplus \mathbb{Q} = \Gamma \oplus \mathbb{Q}$ (from the antilexicographic order)

$$\begin{aligned} \Gamma &\preceq \Gamma^* \\ &= \bigoplus_{i \leq n} \Delta_i \\ &\preceq \bigoplus_{i \leq n-1} \Delta_i \oplus (\Delta_n \oplus \mathbb{Q}) \\ &= \left(\bigoplus_{i \leq n-1} \Delta_i \oplus \Delta_n \right) \oplus \mathbb{Q} \\ &= \Gamma^* \oplus \mathbb{Q} \\ &\preceq \Gamma^* \oplus \mathbb{Q}^* \\ &\equiv (\Gamma \oplus \mathbb{Q})^* \\ &\equiv \Gamma \oplus \mathbb{Q} \end{aligned}$$

where the first embeddings are the diagonal embedding into the ultrapower, lifting the rank finiteness of Γ , model-completeness of divisible groups (lemma 2.1), and the final non-trivial equivalence is a property of ultraproducts of ordered abelian groups (lemma 5.10), shown below.

Lemma 5.10 (Distributivity of ultraproducts on direct sums). Let G and H be ordered abelian groups. Let $\mathcal{U}$ be a non principal ultrafilter on $\mathbb{N}$. Then

$$\prod_{\mathcal{U}} (G \oplus H / \sim) \equiv \left(\prod_{\mathcal{U}} G / \sim \right) \oplus \left(\prod_{\mathcal{U}} H / \sim \right)$$

in the language $\mathcal{L}_{\text{oag}, G}$ of order abelian groups with a predicate for G . ■

5.9 Rank reduction

Theorem 5.15 (Finite rank reduction). *If K is a henselian valued field of finite rank, with vK dense, $\mathcal{O}_v/(p)$ semi-perfect, then K is elementarily equivalent to a field of rank 1.*

We already saw that a valued field in $\mathcal{C}$ embeds elementarily into its Hahn series field. Now, valuation of finite rank can be decomposed as iterated rank one valuation, and applying AK/E principles recursively allows one to reduce the theory of the outer fields to the inner ones.

However, while this method allows us to reduce fields of finite rank to fields of rank 1, it will not help for fields of infinite rank, so the earlier reduction is only a partial result.

5.9.1 Aside : non-elementary approaches to rank reduction

There are actually some alternative, non-elementary ways to conduct a model-theoretic study of perfectoid fields, or generally, valued fields of rank 1, without fully giving in to second-order logic. Let us briefly consider two alternative ways to "solve" our rank reduction problem by using two stronger logics.

An infinitary logic of the form $\Lambda_{\omega_1, \omega}$ would directly let us express rank 1, using an infinite disjunction, for e.g. $v(x) = \infty \vee \bigvee_{n \in \mathbb{N}^*} v(x) < n$. This is obviously not an elementary approach, since by the same token, we would be able to express finiteness or other strong properties.

Another approach, which is at the core of an active research programme, is to work in *continuous* logic. In this context, valuations are abstracted in the logic itself, so one would not use a multi-sorted language, but rather define the valuation as a real-valued predicate satisfying relevant axioms. In this setting, every valuation is of rank 1 by definition (or equivalently, the definition of rank loses its meaning since there is no group to consider, but only a truth value set). This approach to valuation was introduced in the context of so-called globally valued fields (see [21]), but was also used to study perfectoid tilting ([15]).

5.9.2 Classes in the language of valued fields

From previous study, we are therefore able to define elementary classes containing perfectoid fields in two ways.

- The class of nice perfectoids we defined in this chapter does contain every perfectoid field of characteristic $(0, p)$, but may contain inadequate non-standard models, in the form of models of infinite rank with no rank one counterpart.
- Replacing the group conditions here with a (complete) theory of a regularly dense group, with a given choice of invariant, we obtain a class such that every model is elementary equivalent to a perfectoid field, but this class will not capture every perfectoid field of a given characteristic.

6 Classification in the language of rings

We consider another elementary classification of perfectoid fields, in the language of rings. From a practical perspective, this change in language means two things, namely that it will be harder to prove elementarity but also that the theory will now not distinguish between models differing only by formulae in the language of valued fields. This is useful, as a massive obstruction when working in the language of valued field is that the proof of rank reduction for the valuation is hard in the infinite case ; working over the language of rings allow us to circumvent this obstruction. Let us define the following elementary class:

Definition 6.1 (Perfectoid fields in the language of rings). The class $\mathcal{C}_{0,p}$ is the class of fields k admitting a valuation $v : k \rightarrow \Gamma \cup \{\infty\}$, such that the valued field (K, v) is of characteristic $(0, p)$, is henselian, has a p -divisible group and a semi-perfect quotiented valuation ring, and such that $\Gamma / \langle v(p) \rangle$ is divisible.

Now, we claim the following

Theorem 6.1 (Main theorem).

1. $\mathcal{C}_{0,p}$ is elementary.
2. every perfectoid field belongs to $\mathcal{C}_{0,p}$.
3. every model of $\mathcal{C}_{0,p}$ is elementary equivalent to a perfectoid field.

Let us outline our strategy for this proof. For the remainder of this chapter, let K be an model of $\mathcal{C}_{0,p}$, and L be a field which is elementarily equivalent to K . Let us show that $L \in \mathcal{C}_{0,p}$. To do so, we need to show that L admits a valuation satisfying the properties listed in the definition of $\mathcal{C}_{0,p}$. In general, a good starting point to transfer properties between structure would be to have some map between them, but our assumption only provides an "abstract" equivalence, so it is up to us to obtain it.

As it turns out, general model theory does provide a nice framework to define a map from an equivalence relation, namely, we want to embed K and L into a common superstructure, elementary equivalent to both, and go "up and then down". The relevant maps should be as canonical as possible. The best way to do this is by use of the Keisler-Shellah theorem, applied to the fields K and L .

Our proof strategy may be summarised by the following diagram:

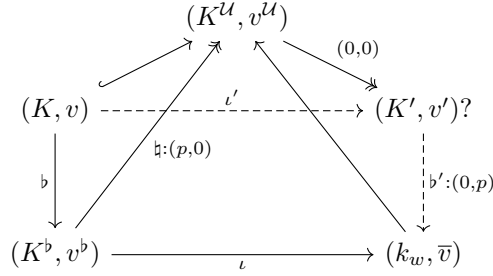
$$\begin{array}{ccc} K^* & \cong & L^* \\ \uparrow \iota & & \downarrow \delta \\ K & \equiv & L \end{array}$$

Intuitively, we lift the valuation structure from K to K^* using the elementary embedding ι to the ultrapower, we transfer it from K^* to L^* through the Keisler-Shelah isomorphism, and we pull it down to L using the section. The non-trivial part is of course showing that relevant properties are preserved along this path.

Let $K \in \mathcal{C}_{0,p}$ be a suitable field. We may endow it with a henselian valuation v . Let us show that the properties of v are preserved under ultrapowers of K . Let us first show that the ultrapower $(K, v)^{\mathcal{U}} =: (K^{\mathcal{U}}, v^{\mathcal{U}})$ is of characteristic $(0, p)$ as well.

6.1 By untilting $\mathcal{C}_{p,p}$?

In the context of the model-theoretic study of perfectoid fields, our current endeavour can be interpreted as an attempt at constructing the following object (K', v') in the following construction from [8] (section 6)



Explicitly, (K, v) is a standard perfectoid field, (K^b, v^b) is its tilt, (K^u, v^u) an ultrapower. Jahnke and Kartas explicitly construct an elementary extension of the tilt (K^b, v^b) , namely $(k_w, \bar{v})$. Our current proof relies a similar construction at the mixed characteristic level.

Our question amounts to recovering extensions of (K, v) , which may appear as (K', v') in our diagram. Could one construct models of $\mathcal{C}_{0,p}$ directly by constructing untilts of the structure $(k_w, \bar{v})$?

6.2 Transfer over ultraproducts and ultrapowers

In this section, we consider an infinite family $(K_i)_{i \in I}$ of fields in $\mathcal{C}_{0,p}$, and show that an ultraproduct $K^* = \prod_{\mathcal{U}} K_i / \mathcal{U}$ (for some non-principal ultrafilter $\mathcal{U}$ on I) also belongs to $\mathcal{C}_{0,p}$. Of particular interest is the case where K is an ultrapower, i.e. $K_i = K$ for every $i \in I$, as this will help us later to deal with the proof of preservation under elementary equivalence.

6.2.1 Characteristic

The field characteristic 0 is an elementary property in $\mathcal{L}_{rng}$ so it passes to K^* easily by the ultraproduct (resp. ultrapower) construction and Łoś's theorem.

For the residue characteristic, let us construct a valuation on K^* from valuations v_i of its components. This is simply done by working in the language of valued fields, and taking the ultraproduct $\prod_{\mathcal{U}} (K_i, v_i) / \mathcal{U}$. Residue characteristic being elementary in $\mathcal{L}_{val}$, we again conclude by Łoś's theorem.

The ultraproduct of valued fields $\prod_{\mathcal{U}} (K, v)_i / \mathcal{U}$ has K_i , and $K v_i$, as its components so characteristic $(0, p)$ transfers, via the sentence $v(p) > 0$ and an argument similar to 4.1.

6.2.2 Henselianity

Likewise, we can lift henselianity directly by elementarity in the language of valued fields and Łoś's theorem.

6.2.3 Not-unramified

By the assumption that $K_i \in \mathcal{C}_{0,p}$ we have

$$K_i \models (\exists x_i)[0 < v_i(x_i) < v_i(p)]$$

so by Łoś's theorem, we can lift this to

$$K^* \models (\exists x)[0 < v(x) < v(p)]$$

in the ultraproduct.

6.2.4 Semi-perfection of the residue ring

Lemma 6.1 (Semi-perfection lifts to ultraproducts). Let $(K, v) \preceq (K^*, v^*)$ be a valued field and an associated ultrapower. If the residue ring $\mathcal{O}_K/(p)$ is semi-perfect, then $\mathcal{O}_{K^*}/(p)$ is semi-perfect as well.

Proof. Let us first note that semi-perfection of a ring is elementary in the language of rings via the formula $(\forall x \exists y)[x = y^p]$ where the exponent p is shorthand for iterated product. However, we also need to take the quotient into account. This is done by adding an additive "error term" in the above property. So we can express semi-perfection of $\mathcal{O}_K/(p)$ as an elementary property of $\mathcal{O}_K$, by

$$\mathcal{O}_K \models (\forall x)(\exists y, z)[x = y^p + zp]$$

And this gives us a property of K in the language of valued fields, by defining the predicate $\mathcal{O}_K(x) :\Leftrightarrow v(x) \geq 0$. We denote $\mathcal{O}_K(x)$ by $x \in \mathcal{O}_K$.

$$K \models (\forall x \in \mathcal{O}_K)(\exists y, z \in \mathcal{O}_K)[x = y^p + zp]$$

By Łoś's theorem, this property is preserved by ultrapowers of (K, v) , which concludes our proof. ■

(Note that the proof would be entirely similar for ultraproducts.)

6.2.5 Rough divisibility

Definition 6.2 (Rough divisibility). A valued field (K, v) is *roughly divisible*, denoted (RD) , if any element of $\Gamma/\langle v(p) \rangle = [-v(p), v(p)]$ is divisible, i.e. $[-v(p), v(p)] \subseteq n\Gamma$ for any n

Lemma 6.2 (Rough divisibility lifts). Let (K_i, v_i) be a family of valued fields satisfying (RD) . Then the ultraproduct (K^*, v^*) also has the (RD) property.

Proof. Consider the following $\mathcal{L}_{grp}$ formula:

$$\forall g. -v(p) < g < v(p) \exists h. (nh = g)$$

Now, let us translate it to a condition in $\mathcal{L}_{val}$, a sufficient condition is the following

$$(\forall x)[v(p^{-1}) < v(x) < v(p) \Rightarrow \exists y.y^n = x]$$

Note, however, that this is too strong, since we only want such a y to exist up to valuation, so we can weaken it as follows:

$$K_i \models (\forall x)[v(p^{-1}) < v(x) < v(p) \Rightarrow \exists y.v(y^n) = v(x)]$$

By elementarity and Łoś's Theorem, this implies that (K^*, v^*) has (RD) . ■

6.3 Transfer over elementary equivalences

In this section, we consider a field $K \in \mathcal{C}_{0,p}$, and a field L such that $L \equiv_{rng} K$. Using the Keisler-Shelah theorem, we can lift the equivalence between K and L to an isomorphism between ultrapowers $K^* \cong L^*$. Now, from the previous section, and the isomorphism, we can transfer properties from K to L^* , and in particular, assume that $L^* \in \mathcal{C}_{0,p}$. More specifically, K^* is an elementary extension of K . Working in the language of valued fields, we lift the valuation v to a valuation v^* on K^* (and by isomorphism, on L^*), and obtain an elementary embedding $(K, v) \preceq (K^*, v^*)$.

We now seek to transfer our properties downwards from L^* to L . For the remainder of this section, we define the following *restriction valuation* on L , $w := v^*|_L$, namely the restriction of w along the canonical injection $\iota : L \hookrightarrow L^*$.

Definition 6.3 (Restricted valuation on L). Consider (L^*, v^*) an ultrapower $L^{\mathcal{U}}$ of L , endowed with a valuation v_{L^*} . We define the *restriction valuation* $w = v_{L^*}|_L$ as the valuation given by the functional restriction of the domain v_{L^*} from L^* to L .

Proof. The restriction of a valuation is again a valuation, because the axioms 1.1 are either universal statements or defined on constants of the field. ■

Let us show that (L, w) has the desired properties to ensure that $L \in \mathcal{C}_{0,p}$.

6.3.1 Characteristic

Consider (L, w) where w is defined by restriction $v^{\mathcal{U}}|_L$. Let us show that (L, w) is a henselian valued field of characteristic $(0, p)$. The field characteristic already transfers from K to L by elementary equivalence, so let us focus on the residue characteristic.

Theorem 6.2 (Residual characteristic is preserved by the restriction). The restriction $w = v_{L^*}|_L$ valuation preserves positive characteristic.

Proof. Since w is a restriction map from a valuation of positive residual characteristic, and p is an element of L , we have $v(p) > 0$ and by lemma 4.1, this is equivalent to $\text{char}(Lw) = 0$. ■

6.3.2 Henselianity

Let us show that "admitting an henselian valuation of mixed characteristic" is a property in $\mathcal{L}_{rng}$.

Following [2], we define the so-called canonical henselian valuation on a field. This valuation has nice transfer properties, and also plays a role as a tool in the classification of henselian valuations over a given field. We consider all henselian valuations over K , and distinguish between the class $H_1(K)$ of henselian valuations making Kv separably closed and the class $H_2(K)$ of henselian valuation with Kw non separably closed. Of particular interest is the question of whether K admits valuations with a separably closed residue field, i.e. whether $H_2(K) = \emptyset$.

Definition 6.4 (Canonical henselian valuation). Let K be a field. The *canonical henselian valuation* is either

1. the finest henselian valuation on K , if $H_2(K) = \emptyset$, or
2. the coarsest valuation in $H_2(K)$, otherwise.

Now, let us first show some definability conditions on canonical valuations, which will allow us to transfer the broader class of henselian, mixed characteristic valuation.

Theorem 6.3 (Henselianity, mixed characteristic). Having a henselian valuation of mixed characteristic is an elementary property in $\mathcal{L}_{rng}$.

Proof. By case disjunction over the characteristic of the canonical valuation v_K . If $\text{char}(K, v_K) = (0, p)$, and $K \equiv_{rng} L$, then the residual characteristic transfers by restriction to $w = v^*|_L$. If $w_L \in H_1(L)$, then v_L is a refinement of w , so it preserves the characteristic. In other words $(Kv_K) = p$ is elementary in $\mathcal{L}_{rng}$.

Otherwise, $\text{char}(K, v_K) = (0, 0)$, v refines v_K (by definition of the canonical henselian valuation) and Kv_K and Kv are separably closed. Furthermore, L_{v_L} is also a separably closed field of characteristic $(0, 0)$. But such a field admits mixed characteristic henselian valuations, so we can equip L_{v_L} with a mixed characteristic valuation w' . Finally, the composed valuation $(L, w \circ v_L)$ is a characteristic $(0, p)$ valuation, and it is henselian because a composition of henselian valuations is henselian. ■

6.3.3 Not-unramified

Let $K \in \mathcal{C}_{0,p}$, with a valuation v , and $L \equiv_{rng} K$. v lifts to a valuation v^* on K^* , where $K^* \cong L^*$ is a common ultrapower of K and L (obtained by the Keisler-Shelah theorem).

By assumption, $v(p)$ is not minimal in K , and hence $v^*(p)$ is not minimal in K^* . We want to show that $v^*|_L(p)$ cannot be minimal in L either. By contradiction. Suppose $w(p)$ is minimal positive in L . Now, a discrete valuation is definable in the language of rings.

Theorem 6.4 (Definability of discrete valuation rings). A discrete valuation ring $\mathcal{O}_K$ of a henselian valued field K is $\mathcal{L}_{rng}$ -definable in K .

Proof. Consider the following set.

$$R = \{x : (\exists y)[1 + px^2 = y^2]\}$$

We claim this is a definition of $\mathcal{O}_K$ if it is discrete in K .

Let us unravel the arithmetic part of the formula. The first sum $1 + px^2$ may be seen as a way to conduct a case disjunction on the valuation sign: if $v(x) \geq 0$ then $v(1 + px^2) = v(1) = 0$. In this case, one may apply Hensel's lemma to show that the polynom defining x has roots. This gives us $R \subseteq \mathcal{O}_K$

Otherwise, $v(x) < 0$ and the argument relies on the fact that the valuation of a n -th root is n -th divisible (obvious from the valuation axioms). Now suppose $v(p)$ minimal positive, i.e. wlog $vK = \mathbb{Z}$ and $v(p) = 1$ and $v(x) < 0$, and observe that the valuation $v(px^2) = 1 + 2v(x)$ is necessarily odd, so it cannot have a square root y satisfying the equation. Therefore $\mathcal{O}_K \subseteq R$.

Note that the coefficient/exponent 2 may not be used when $p = 2$, in this case one may simply use 3 instead. ■

Theorem 6.5. If $K \in \mathcal{C}_{0,p}$ with witness a valuation v , $K \equiv_{rng} L$, $K^* \cong L^*$ and $w = v^*|_L$, then w cannot be unramified.

Proof. The valuation w induces a definable valuation w' on L^* with $w'(p)$ minimal positive, and we can compare the valuations v^* and w' to arrive at a contradiction.

- We may not have $\mathcal{O}_{w'} \subseteq \mathcal{O}_{v^*}$. Indeed if $0 < v^*(q) < v^*(p)$ then $w'(q) = 0$ so $q^{-1} \in \mathcal{O}_{w'}$ but not $\mathcal{O}_{v^*}$.
- $\mathcal{O}_{v^*} \subseteq \mathcal{O}_{w'}$ is also a contradiction, as p -divisibility of vK breaks minimality of $w(p)$.
- Finally, if $\mathcal{O}_{v^*} \not\subseteq \mathcal{O}_{w'}$ and $\mathcal{O}_{w'} \not\subseteq \mathcal{O}_{v^*}$, then K^* has two independent henselian valuations, and is the separably closed. Since K^* is in characteristic 0, this means that K^* is algebraically closed, so K and L already were. But then in particular, the polynomial $X^p - p$ has roots in the fields K , L and K^* . More generally, this implies that an algebraically closed field does not admit an unramified valuation, so the assumption that $w(p)$ was minimal positive was wrong.

Therefore, if (K, v) is not-unramified, and $L \equiv K$, then $(L, v^*|_L)$ cannot be unramified either. In other words, the property "admitting an henselian, mixed characteristic, not unramified valuation" transfers. ■

6.3.4 Residual semi-perfection

We let $K \in \mathcal{C}_{0,p}$, $L \equiv_{rng} K$ and v be a valuation on K as before. Suppose that $\mathcal{O}_v/(p)$ is semi-perfect but $\mathcal{O}_w/(p)$ is not. Let us show that v and w induce non-comparable valuations on K^* , a contradiction. The valuations v and w lift to valuations v^* and w^* on K^* .

w^* may be coarsened to a valuation with an imperfect residue field.

Lemma 6.3 (Residual semi-imperfection / residual imperfection). If (L, w) is a non residually-perfect henselian valued field, then there is a coarsening w' of w such that Lw' is imperfect.

Proof. Let $x \in \mathcal{O}_w$ be such that for any $y, z \in \mathcal{O}_w$ $y^p + zp \neq x$.

If $w(x) = 0$, then $x \in Lw$, and in particular, taking $z = 0$ means Lw is already imperfect. Otherwise, consider any coarsening such that $w'(x) = 0$. Since $y^p + z = x$ (as a polynomial in the variables y and z) has no solution in $\mathcal{O}_w$, it won't have solutions in a coarsening $\mathcal{O}_{w'}$ either. ■

As a result, our ultrapower L^* admits two valuations v^* and w' such that L^*v^* is perfect and L^*w' is imperfect.

In the situation above, the finest common subring of $\mathcal{O}_{v^*}$ and $\mathcal{O}_{w'}$ is $\mathcal{O}_w$.

But now, this means that L^* has independent valuations, so as before, we can conclude that it must be separably closed, contradicting the assumption that it admitted an imperfect residue field. Therefore, w must be residually semi-perfect, in other words the property transfers by elementary equivalence.

6.3.5 Rough divisibility

Let v be a roughly divisible valuation on K , $K \equiv_{rng} L$, define v^* and w^* the lifts of v and w on $K^* \cong L^*$, the common ultrapower obtained by Keisler-Shelah.

Now, we consider the lifts v^* and w^* , and remark that the quotient groups $v^*K^*/\langle v^*(p) \rangle$ and $w^*K^*/\langle w^*(p) \rangle$ are precisely the valuation groups of the equicharacteristic coarsenings.

- If v_0^* and w_0^* are not comparable, then L^* is separably closed, so it only admits divisible non-trivial valuations.
- If $\mathcal{O}_{v_0^*} \subseteq \mathcal{O}_{w_0^*}$, then, $w_0^*K^*$ is a quotient of the divisible group $v_0^*K^*$, and it should be divisible as well.
- Then $\mathcal{O}_{w_0^*} \subseteq \mathcal{O}_{v_0^*}$ and $v_0^*K^* = w_0^*K^*/\Delta$. Now, wlog, Δ is convex subgroup of w^*K^* . Write $\Delta = \langle w^*(q) \rangle$, with $q \in L^*$, and $w^*(q) > w^*(p)$ (otherwise, $w_0^*L^* = v_0^*L^*$). Note that we may not have $q \in L$. Indeed $w(q) = v^*|_L(q)$, but $v_0^*(q)$ and $v^*(q)$ should agree since $v^*(q) > v^*(p)$, but $v_0^*(q) = 0$ by $v_0^*L^* = w_0^*L^*/\langle w(q) \rangle$. Contradiction. Therefore, $q \in L^* \setminus L$. Now, $v^*(q) > v^*(p)$ by assumption (otherwise the quotient $v_0^*K^* = w_0^*K^*/\Delta$ would be meaningless). Since $v_0K^* = v^*K^*/\langle v^*(p) \rangle$, $v_0^*(q) = v^*(q)$. Contradiction. So, let us assume $v^*(q) = 0$. v^* and w^* should be comparable (otherwise, L^* is separably closed), so $v^*L^* = w^*L^*/\Delta$, for Δ and $w(q) \in \Delta$. But by $w(p) < w(q)$, but this contradicts v^* being a mixed characteristic valuation.

6.4 Perfectoids

Theorem 6.6 (Perfectoids are in $\mathcal{C}_{0,p}$). Any perfectoid field (K, v) of characteristic $(0, p)$ is in $\mathcal{C}_{0,p}$.

Proof. This follows directly from the definition of $\mathcal{C}_{0,p}$, an application of the AK/E principle and the definition of a perfectoid. ■

Let us now show that any field in $\mathcal{C}_{0,p}$ is equivalent to a perfectoid field. (In other words, perfectoid fields form a skeleton for the category given by the set $\mathcal{C}_{0,p}$ with arrows elementary maps.) This results from the fact that the logic doesn't detect the valuation rank anymore, so the pathological models are no longer distinguished from the others.

Theorem 6.7 (Perfectoids generate $\mathcal{C}_{0,p}$). Any element of $\mathcal{C}_{0,p}$ is $\mathcal{L}_{rng}$ elementary equivalent to a perfectoid field.

Proof.

Let K be a field of $\mathcal{C}_{0,p}$. Let v be a henselian valuation on K , satisfying the axioms. Take the completion $\widehat{K}$ and recall that $K = K^h \equiv \widehat{K}$ (lemma 4.3). Now, restrict this equivalence to the sublanguage $\mathcal{L}_{rng} \subset \mathcal{L}_{val}$. ■

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