

Towards a Joyal-Tierney Theorem for Arrow Algebras

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written by

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Abstract

The Joyal-Tierney theorem states that morphisms into a given locale are equivalent to internal locales of its sheaf topos. This is useful in the study of topoi and can even be used in algebraic topology. Since arrow algebras generalize locales, we wanted to see if this theorem also holds for them. In this thesis, we will look at different approaches of defining the internalization functor for arrow algebras, which gives one direction of the Joyal-Tierney theorem. These approaches have varying degrees of maturity and we will discuss their strengths and weaknesses.

On the way to these results, we will also go on a journey through categorical logic with a special focus on our main tool: triposes. We will explore the tripos to topos construction and some characterizations in it. In particular, we will look at assemblies and proto-assemblies to define the evolution functor.

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Chapter 1

Introduction

The starting point for this thesis was to see which facts and theorems we can generalize to arrow algebras from locales. Arrow algebras were introduced by Benno van den Berg and Marcus Briët to capture localic and realizability triposes under one framework [BB26]. To do so, they weakened the notion of an implicative algebra which was introduced by Alexandre Miquel [Miq20b]. This way, they also covered all partial combinatory algebras and thus also various realizability triposes. Then Umberto Tarantino added a notion of morphisms to arrow algebras, turning them into a category [Tar25]. He showed that this notion is the correct one in the sense that these (geometric) morphisms give rise to geometric morphisms between the respective arrow triposes and that we can understand morphisms between the arrow triposes at the level of arrow algebras. Most recently, Zhaorui Hu used arrow algebras to construct a topos for monotone modified realizability [Hu26]. Since arrow algebras are a comparatively new structure, there is still a lot we have to understand about them. So for this thesis, we set out to see what else we can show about arrow algebras and focused especially on the Joyal-Tierney theorem.

Since there are multiple theorems called the “Joyal-Tierney theorem”, we want to clarify that we are referring to the one stating that for any locale X , the slice category $\mathbf{Loc}/X$ is equivalent to the category $\mathbf{Loc}(\mathrm{Sh}(X))$ of internal locales in the sheaf topos over X [JT84]. So our idea was to prove:

Conjecture 1.1. Let $\mathcal{A}$ be an arrow algebra and $\mathrm{AT}[\mathcal{A}]$ its associated arrow topos. Then we have an equivalence of categories

$$\mathbf{ArrAlg}/\mathcal{A} \simeq \mathbf{ArrAlg}(\mathrm{AT}[\mathcal{A}]).$$

The arrow topos is the canonical construction of a topos from an arrow algebra and comparable to the sheaf topos over a locale. Indeed, it is easy to check that if $\mathcal{A}$ is a locale, then $\mathrm{AT}[\mathcal{A}] \cong \mathrm{Sh}(\mathcal{A})$. The motivation for the idea came from the fact that similar machinery is used in proving the Gleason cover for a topos [Joh81b]. Originally, our goal was to prove a Gleason cover for arrow algebras, but after realizing that its proof needs a statement similar to the Joyal-Tierney theorem, we changed direction.

Another application of the Joyal-Tierney theorem is the study of bundles, which are used in algebraic topology. A *bundle* over a locale X is just a morphism $p: E \rightarrow X$. We can also consider its pullback along a global element $x: 1 \rightarrow X$, giving the fibre $E_x := p^{-1}(x)$. Since the category of locales is complete, every E_x is itself a locale and we can also view p as a family of locales E_x indexed by and varying continuously in x . The Joyal-Tierney theorem now tells us that we can identify p with a locale internal to the sheaf topos $\mathrm{Sh}(X)$, notably the one whose corresponding internal frame is given by the

map $V \mapsto \mathcal{O}(p^{-1}(V))$. Since every theorem proven constructively holds with respect to the internal logic of a topos, every constructive theorem about locales also holds inside $\mathrm{Sh}(X)$. This can now be translated into a theorem of bundles over X by introducing the fibre-wise notions, e.g. products are replaced by fibre products over X [Joh02, Thm. C1.6.3]. Since sober topological spaces are a full subcategory of **Loc**, the same idea applies to parametrized spaces, which are just spaces over a base.

To prove [Conjecture 1.1](#), we have to show the equivalence of two categories. One usually first constructs an internalization functor, which in our case would be a functor $\mathbf{ArrAlg}/\mathcal{A} \rightarrow \mathbf{ArrAlg}(\mathrm{AT}[\mathcal{A}])$. The main concern of this thesis are different ways one could define this internalization functor and the problems we encountered. We obtained one positive result and one negative result. Unfortunately, the positive result is likely too restrictive for a Joyal-Tierney-like equivalence, so we also discuss a possible modification. This also means that we cannot present a full theory of the internalization of arrow algebras, but instead give potential approaches other people can follow, as well as warnings which approaches are not worth pursuing.

This thesis is structured in the following way: after introducing the preliminaries and background in [Chapter 2](#), we will have a look at the tripos to topos construction in [Chapter 3](#). Both of these chapters contain mostly known results from the literature. We will give a self-contained description of the tripos to topos construction in great detail. Besides the necessary conditions to conclude that we indeed get a topos, we also make a few characterizations we will need in the following chapters. Additionally, we make two observations, we have never seen before (namely the idea to avoid the axiom of choice in the meta-logic in [Section 3.2](#) and [Proposition 3.31](#)).

In [Chapter 4](#), we focus on proto-assemblies. Besides their definition and the fact that they form a category **pAsm**, this entire chapter contains only new results. Among these is a way to transform proto-assemblies into assemblies, which we call the *evolution functor*. We also show that **pAsm** has finite products and that the evolution functor preserves these. Lastly, we look into an ex/reg-completion for proto-assemblies and discuss why this is likely not possible.

The next chapter, [Chapter 5](#), discusses arrow algebras and the theory around them. We just state it for future reference, since most of the results we need have already been developed. For convenience, we make some parts of the construction of the arrow topos more explicit. We also prove equivalent conditions for cartesian morphisms of arrow algebras.

[Chapter 6](#) is the heart of this thesis, where we discuss the Joyal-Tierney theorem and introduce different approaches to defining an internalization functor of arrow algebra morphisms. After an introduction to the theorem and the construction of the internalization functor for locale morphisms, we prove our first positive result, namely that we can internalize arrow algebra morphisms as locales ([Theorem 6.5](#)). The remainder of the chapter is concerned with generalizations of this functor. We pursue two directions: on the one hand, we discuss how one may internalize arrow algebra morphisms as arrow algebras. This idea will be discussed in [Section 6.2](#) and [6.3](#). On the other hand, in [Section 6.4](#), we will look at localic morphisms between topoi and if we can define a similar notion between arrow algebras. This is because the functor of [Theorem 6.5](#) is likely too restrictive to be an equivalence. Thus we will try to restrict the arrow algebra morphisms in the domain to obtain an equivalence.

Finally, I want to add my two cents about the plural of the word topos. In what follows, I will use the Greek plural *topoi*; simply because in my opinion it sounds the nicest and to keep my beverages warm in the winter, I use my *thermoi* [Joh77]. I also see the motivation for other plurals like the

English *toposes* or the French *topos* [Joh77; Oos21], but the main remark I want to make is something a friend made me aware of: how “topoi” is actually pronounced. Reading it like an English word /'tɒp.ɔɪ/ defeats in my opinion the point of using the Greek plural. Some people may argue that they are using the ancient Greek pronunciation [All68]. After all, a lot of Greek terms in mathematics have their origin in ancient Greece. But I would argue that the modern pronunciation has the nicest ring to it. It would be pronounced similar to *topi*, /'to.pɪ/ [Wil98].

Chapter 2

Preliminaries

To start, we assume basic familiarity with algebraic structures such as preorders, posets and lattices. The first non-standard definition is that of Heyting prealgebra. Intuitively, it works like a Heyting algebra, which we do not require to be anti-symmetric. Formally, we define:

Definition 2.1 (Heyting prealgebras). A *Heyting prealgebra* is a tuple $(H, \preceq, \wedge, \vee, \rightarrow, \top, \perp)$, where H is a set, $\preceq$ is a preorder on that set, $\perp, \top \in H$ should give the top and bottom element and $\wedge, \vee, \rightarrow: H \times H \rightarrow H$ are the meet, join and implication respectively. This means that for all $a, b, c \in H$, we have

- $\perp \preceq a$ and $a \preceq \top$,
- $a \wedge b \preceq a$ and $a \wedge b \preceq b$ as well as $d \preceq a \wedge b$ for all $d \in H$ with $d \preceq a$ and $d \preceq b$,
- $a \preceq a \vee b$ and $b \preceq a \vee b$ as well as $a \vee b \preceq d$ for all $d \in H$ with $a \preceq d$ and $b \preceq d$ and
- $c \preceq a \rightarrow b$ iff $c \wedge a \preceq b$. We call $\rightarrow$ the *Heyting implication* and the equivalence the *Heyting adjunction*. ┘

If additionally $\preceq$ is anti-symmetric, we would simply get a Heyting algebra. In that case, we could also identify the binary operations and constants uniquely by the properties in [Definition 2.1](#). In the case of Heyting prealgebras, we cannot do this as one can see with the following example:

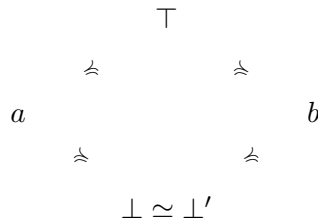


Diagram 2.1

We write $\perp \simeq \perp'$ to denote logical equivalence, i.e. that $\perp \preceq \perp'$ and $\perp' \preceq \perp$. Both $\perp$ and $\perp'$ satisfy the properties to be the meet of a and b . So we have to select one of them to be the designated meet. We can also force anti-symmetry by quotienting a Heyting prealgebra by logical equivalence, i.e. the equivalence relation given by $a \sim b$ iff $a \preceq b$ and $b \preceq a$. We call this quotient its *poset reflection*. One can easily check that this quotient yields a Heyting algebra. This also means that there is no immediate advantage of considering a Heyting prealgebra over its poset reflection, but in

most cases, our constructions will simply result in a Heyting prealgebra and it is more elegant to keep it rather than using the poset reflection. In a way, $\simeq$ replaces $=$. We will also use $\leftrightarrow$ to denote logical equivalence.

A *morphism* of Heyting prealgebras is a set-function that preserves the operations and constants given in [Definition 2.1](#) up to logical equivalence. Together, Heyting prealgebras and their morphisms form a category **preHeyt**.

Definition 2.2 (Locale). A *locale* is a complete poset $(L, \preceq)$, such that the *infinite distributivity law* holds. This means that for all $B \subseteq L$ and $a \in L$, we have

$$a \wedge \bigvee_{b \in B} b = \bigvee_{b \in B} a \wedge b. \quad \lrcorner$$

To define locale morphisms, we have to go on a slight detour as the category of locales is defined as the opposite of the category of frames. So as objects, frames and locales are the same. A *frame morphism* $f: L \rightarrow L'$ is a function that preserves finite meets and all joins. A *locale morphism* is then the opposite of this. This also defines a category **Loc**.

Since complete Heyting algebras always satisfy the infinite distributivity law, they are also locales. Conversely, if we define a binary operation on a locale via

$$a \rightarrow b := \bigvee \{c \in L : c \wedge a \preceq b\},$$

we see that this is a Heyting implication. So on the object level, locales and complete Heyting algebras are the same. But locale morphisms do not have to preserve the implication and are thus weaker. Furthermore, locale morphisms are defined to be in the opposite direction of Heyting morphisms. For more information on locales, we refer the interested reader to [\[Joh86; MM94\]](#).

Lastly, we will look at the notion of a *partial combinatory algebras*. These will play an important role in constructing triposes in the next section and an even more important role in constructing arrow algebras in [Chapter 5](#). We will follow the notion of a pca as given in [\[BB26; Zoe22\]](#).

Definition 2.3 (PCA). A *partial combinatory algebra* or *pca* is a quadruple $(P, \cdot, \leq, P^\#)$ where

- $(P, \cdot, \leq)$ is a *partial applicative poset*, meaning that $(P, \leq)$ is a poset and $\cdot$ is a partial binary application $P \times P \rightarrow P$ which has downwards closed domain and is monotone. This means that for all $a', a, b', b \in P$, such that $a' \cdot b'$ is defined, $a \leq a'$ and $b \leq b'$, then $a \cdot b$ is also defined and we have $a \cdot b \leq a' \cdot b'$. We will often write application as ab instead of $a \cdot b$,
- $P^\# \subseteq P$ is a *filter* on P , i.e. it is closed upwards and under application,
- there are elements $\mathbf{k}, \mathbf{s} \in P^\#$, such that for all $a, b, c \in P$, we have that $\mathbf{k}ab$ is defined and $\mathbf{k}ab \leq a$; $\mathbf{s}ab$ is always defined and whenever $ac(bc)$ is defined, then so is $\mathbf{s}abc$ and $\mathbf{s}abc \leq ac(bc)$.

When writing these expressions, we assume that the application associates to the left as usual meaning that abc should be read as $(ab)c$. To say that the application ab is defined, we write $ab\downarrow$. We say a pca is *absolute* if $P^\# = P$ and *discrete* if $\leq$ coincides with $=$. $\lrcorner$

Sometimes pcas are also referred to as *Schönfinkel algebras* since Moses Schönfinkel mentions a similar structure around 1920 [\[Sch24\]](#). However, his notions only covers the case where $\leq$ is the

identity and the application is total. 50 years later, Feferman considered the generalization of only using a partial application and proved that these are combinatory complete [Fef75]. This means that we can interpret λ -terms in them, which is quite powerful given such a simple structure. This notion of a pca is also extensively researched in [Oos08] and its extension using the partial order in [Zoe22].

2.1 Basic categorical notions

We assume familiarity with basic categorical notion and especially limits, subobjects and exponentials. We refer to the standard resources for more information [Bor94a; Mac98; MM94]. We will prove special cases of these in Chapter 3, but without explicit definitions. As usual, we will assume that if a category has limits or exponentials, there are canonical choices of these. For example, if we say that a category has binary products, we assume that we are given an assignment, which for any two objects gives us their product. Accordingly, if we say that a functor preserves e.g. binary products, we mean that it does so up to isomorphism, i.e. it maps the terminal product cone to a terminal product cone in the codomain. We do not require it to send the canonical choice of the product to the canonical choice.

We also assume that the reader is familiar with the categories **Set** of sets and set-functions, **Preord** of preorders and monotone maps and **Poset** of posets and monotone maps as we will make frequent use of them.

In this section, we will look at some less standard definitions in category theory needed for this thesis. We start with the notion of adjoint functors. These are in a sense a very weak notion of inverses between two categories:

Definition 2.4 (Adjoint functors). Let $\mathbb{C}$ and $\mathbb{D}$ be two categories with functors $F^+ : \mathbb{C} \rightarrow \mathbb{D}$ and $F_+ : \mathbb{D} \rightarrow \mathbb{C}$. We say that they form an *adjunction* with F^+ being *left adjoint* to F_+ and symmetrically F_+ being *right adjoint* to F^+ , written $F^+ \dashv F_+$, if for all $C \in \mathbb{C}$ and $D \in \mathbb{D}$, we have an isomorphism

$$\mathrm{Hom}_{\mathbb{D}}(F^+C, D) \cong \mathrm{Hom}_{\mathbb{C}}(C, F_+D),$$

which is natural in C and D . ⌋

The notation using $^+$ and $_+$ follows the convention of geometric morphisms, which we will see later. We want to highlight the case where $\mathbb{C}$ and $\mathbb{D}$ are preorders considered as categories. In that case, the Hom-sets are either empty or contain one element, depending whether the two objects are in relation or not. In that case, the condition above reduces to

$$F^+C \leq D \quad \Longleftrightarrow \quad C \leq F_+D.$$

By choosing $D = F^+C$, we get from reflexivity that $C \leq F_+F^+C$ for all C . Similarly we have $F^+F_+D \leq D$. Furthermore, these two conditions together imply the equivalence above. If we have $F^+C \leq D$, then since F_+ is monotone, we have $C \leq F_+F^+C \leq F_+D$. The other direction is analogous.

2.1.1 Preorder-enriched categories

To define triposes, we need the notion of a pseudo-functor. These can be defined between 2-categories in general, but we will focus on the case of preorder-enriched categories. This will suffice for our case

and makes some definitions easier since any parallel 2-cells are equal.

Definition 2.5 (Preorder-enriched category). A *preorder-enriched category* $\mathbb{C}$ is a category, where for every $X, Y \in \mathbb{C}$, the collection of morphisms $\mathbb{C}(X, Y)$ has the structure of a preorder, and composition respects this order. This means that if f and g are morphisms $X \rightarrow Y$ with $f \leq g$, then for all morphisms $h: Z \rightarrow X$ and $k: Y \rightarrow W$, we have $fh \leq gh$ and $kf \leq kg$. $\lrcorner$

Every category can be trivially preorder-enriched by taking the relation to be the identity. The category **preHeyt** is also preorder-enriched by defining $f \leq g$ iff for all x , we have $f(x) \leq g(x)$. One can also enrich categories over other structures, giving the general notion of *2-categories*, which have “morphisms between morphisms” (these are then called *transformations*). For us, preorder-enriched categories will suffice and we refer the interested reader to [Joh02, Sec. B1.1]. There one can also find the more general definition of a pseudo-functor and a pseudo-natural transformation, which we will state only for preorder-enriched categories.

Definition 2.6 (Pseudo-functor). Given two preorder-enriched categories $\mathbb{C}$ and $\mathbb{D}$, a *pseudo-functor* between them $F: \mathbb{C} \rightarrow \mathbb{D}$ consists of a mapping on objects $F_0: \mathbb{C}_0 \rightarrow \mathbb{D}_0$ and a mapping on morphisms $F_1: \mathbb{C}_1 \rightarrow \mathbb{D}_1$ so that the usual functoriality laws hold up to isomorphism and F preserves the order. This means that

- for every element $C \in \mathbb{C}$, we have $F(\text{id}_C) \simeq \text{id}_{FC}$,
- for every pair of morphisms $f: C \rightarrow D, g: D \rightarrow E$ in $\mathbb{C}$, we have $F(g \circ f) \simeq F(g) \circ F(f)$ and
- for $f, g: C \rightarrow D$ with $f \leq g$ in $\mathbb{C}$, we have $F(f) \leq F(g)$ in $\mathbb{D}$.

Here $\simeq$ again means that both inequalities hold. $\lrcorner$

Clearly, pseudo-functors between categories seen as preorder-enriched categories with the discrete order on the collection of morphisms are simply functors. Similarly, we have

Definition 2.7 (Pseudo-natural transformation). Given two pseudo-functors $F, G: \mathbb{C} \rightarrow \mathbb{D}$ a *pseudo-natural transformation* $\alpha: F \Rightarrow G$ between them is a family of maps $\alpha_C: FC \rightarrow GC$ indexed by the objects $C \in \mathbb{C}$, such that for all morphisms $f: C \rightarrow C'$ in $\mathbb{C}$, the following diagram commutes up to isomorphism:

$$\begin{array}{ccc} FC & \xrightarrow{\alpha_C} & GC \\ \downarrow Ff & & \downarrow Gf \\ FC' & \xrightarrow{\alpha_{C'}} & GC' \end{array}$$

Diagram 2.2

This means that $\alpha_{C'} \circ Ff \simeq Gf \circ \alpha_C$. If every α_C is an isomorphism, we will also write $F \cong G$. $\lrcorner$

Similar to the case of pseudo-functors, a pseudo-natural transformation between categories seen as preorder-enriched categories with the discrete order is just a natural transformation.

2.1.2 Topoi

Johnstone famously listed many different ways one could see a topos and all the sketches one could make of it [Joh02, Introduction]. For us, two viewpoints are particularly important:

(ii) “A topos is a category with finite limits and power-objects.”

(iii) “A topos is (the embodiment of) an intuitionistic higher-order theory.”

The viewpoint (ii) is what characterizes topoi as universes to do mathematics in. These two conditions guarantee that the category is nice in a certain sense, i.e. similar to the category **Set**. This already enables us to do all mathematics we are familiar with. By this definition, we have defined it via its objects. As a by-product of this definition, we can do logic in a topos, which is always intuitionistic in the sense that it is given by a Heyting algebra [Joh02, Lem. A1.6.3], see also [Joh02, Sec. D].

Condition (iii) gives another view to topoi, suggesting we can start with the logic and derive the objects in some other way. This will become important when we talk about triposes. We use (ii) as our definition of a topos.

Definition 2.8 (Topos). A *elementary topos* or just *topos* is a category $\mathcal{E}$ that has finite limits and power objects. ┘

Before concluding the categorical preliminaries, we want to mention that we can also define Heyting prealgebras categorically. Some readers may be more comfortable with such presentations, so we want to include it as well:

Definition 2.9 (Heyting prealgebras, categorically). A *Heyting prealgebra* is a preorder that is cartesian closed and has finite coproducts.

A *morphism* of Heyting prealgebras is a functor that preserves this structure. ┘

2.2 Triposes

The motivating idea for a tripos is to define a mathematical universe by its logic and a way to construct a topos (hence also the name “topos-representing indexed pre-ordered set” [HJP80]). This is what we hinted at in the previous section with the viewpoint (iii). This way, triposes and topoi try to achieve the same thing with different approaches. In both cases, we want to model mathematics in some way. With Definition 2.8 for a topos, we are going the more classical route analogous to **Set**: We construct a category that has certain nice properties and get as a by-product that we can do logic in it. With a tripos, we first define what our logic should look like and then derive the objects from that.

The original definition of a tripos is due to Hyland, Johnstone and Pitts [HJP80]. We will largely follow their paper, [Oos08] and also refer to the development in Pitts’ Ph.D. thesis [PCS81].

Definition 2.10 (Tripos). Let $\mathbb{C}$ be a category with finite products. A $\mathbb{C}$ -*tripos* is a pseudo-functor $P: \mathbb{C}^{\text{op}} \rightarrow \mathbf{preHeyt}$, satisfying the following conditions:

- i) For every morphism $f: X \rightarrow Y$ in $\mathbb{C}$, the map $P(f): P(Y) \rightarrow P(X)$ has a left adjoint $\exists_f$ and a right adjoint $\forall_f$ in **Preord**, satisfying the *Beck-Chevalley condition*: if Diagram 2.3 is a pullback square in $\mathbb{C}$, then in **Preord**, the composites $\forall_f \circ P(g)$ and $P(h) \circ \forall_k$ are isomorphic, as are the composites $\exists_f \circ P(g)$ and $P(h) \circ \exists_k$.

- ii) For every object $X \in \mathbb{C}$, there is an *object of predicates* $\pi(X) \in \mathbb{C}$ and a *membership predicate* $\in_X \in P(X \times \pi(X))$, such that:

For every object $Y \in \mathbb{C}$ and every $\phi \in P(X \times Y)$, there is a morphism $\{\phi\}: Y \rightarrow \pi(X)$ in $\mathbb{C}$, such that in the Heyting prealgebra $P(X \times Y)$, we have $\phi \simeq P(\text{id}_X \times \{\phi\})(\in_X)$. ┘

Notice that the adjoints $\exists_f$ and $\forall_f$ of $P(f)$ are just morphisms of preorders and not of Heyting prealgebras in general. This means that we cannot assume them to preserve conjunction, disjunction or implication.

The adjunctions not only have suggestive names, but serve as the universal and existential quantifier once we define a tripos logic. The condition $\exists_f \dashv P(f) \dashv \forall_f$ guarantees that the introduction and elimination rules of the quantifiers work as usual while the Beck-Chevalley condition is for managing the quantified variables (cf. [Subsection 2.3.4](#)). For notational convenience, we shall often write f^* for $P(f)$ if the tripos is clear from context.

The second condition enables us to construct exponentials and the subobject classifier in [Section 3.5](#) and [3.6](#). Since condition [ii\)](#) is sometimes hard to check, we often use an equivalent condition. For this, we need $\mathbb{C}$ to be cartesian closed. Then we stipulate

- ii') There is a *generic element* $\Sigma \in \mathbb{C}$ and a *generic predicate* $\sigma \in P(\Sigma)$, such that for all $X \in \mathbb{C}$ and all $\phi \in P(X)$, there is a morphism $[\phi]: X \rightarrow \Sigma$ in $\mathbb{C}$, such that in $P(X)$, we have $\phi \simeq P([\phi])(\sigma)$.

Lemma 2.11. Let $\mathbb{C}$ be a cartesian closed category and $P: \mathbb{C}^{\text{op}} \rightarrow \mathbf{preHeyt}$ a pseudo-functor. Then P satisfies [ii\)](#) iff it satisfies [ii'\)](#).

Proof. For the only if, we notice that since $\mathbb{C}$ has finite products, it has a terminal object 1 . We take $\Sigma := \pi(1)$ and $\sigma := \epsilon_1 \in P(1 \times \pi(1)) \cong P(\pi(1))$. If we have an $X \in \mathbb{C}$ and $\phi \in P(X) \cong P(1 \times X)$, there is a $\{\phi\}: X \rightarrow \pi(1) = \Sigma$. So by taking $[\phi] := \text{id}_1 \times \{\phi\}$, we have

$$\phi \simeq P(\text{id}_1 \times \{\phi\})(\epsilon_1) \simeq P([\phi])(\sigma).$$

For the reverse direction, let $\pi(X) := \Sigma^X$ and let $\epsilon_X := P(\text{ev})(\sigma) \in P(X \times \Sigma^X)$. If $\phi \in P(X \times Y)$, there is a $[\phi]: X \times Y \rightarrow \Sigma$ and hence by cartesian closure, there is a map $\widetilde{[\phi]}: Y \rightarrow \Sigma^X = \pi(X)$, such that $\text{ev} \circ (\text{id}_X \times \widetilde{[\phi]}) = [\phi]$. If we let $\{\phi\} := \widetilde{[\phi]}$, we get

$$P(\text{id}_X \times \{\phi\})(\epsilon_X) \simeq P(\text{id}_X \times \widetilde{[\phi]})P(\text{ev})(\sigma) \simeq P(\text{ev} \circ (\text{id}_X \times \widetilde{[\phi]}))(\sigma) \simeq P([\phi])(\sigma) \simeq \phi$$

as desired. □

Example 2.12. An easy example of triposes are the *localic triposes*. Given a locale $(L, \preccurlyeq)$, we can define a functor $P_L: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{preHeyt}$ by setting $P_L(X) := (L^X, \vdash)$, where

$$\varphi \vdash \psi \iff \varphi(x) \preccurlyeq \psi(x) \text{ for all } x \in L,$$

noting that this gives L^X the structure of a Heyting prealgebra (this follows essentially from [Lemma 5.9](#) viewing $(L, \preccurlyeq)$ as an arrow algebra in the canonical sense with $S = \{\top\}$). On morphisms, P_L acts by

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & & \downarrow h \\ Z & \xrightarrow{k} & W \end{array}$$

Diagram 2.3

precomposition. The quantifiers are given by sending an element $\alpha \in P_L(X)$ to

$$\begin{aligned}\forall_f(\alpha): Y \rightarrow A, y \mapsto \bigwedge_{f(x)=y} \alpha(x), \\ \exists_f(\alpha): Y \rightarrow A, y \mapsto \bigvee_{f(x)=y} \alpha(x).\end{aligned}$$

We can take $\Sigma = L$ and $\sigma = \text{id}_L \in P_L(\Sigma) = L^L$ as the generic element and generic predicate. This is a tripos, which we call the *localic tripos*. $\lrcorner$

Furthermore, the associated topos to P_L is equivalent to the sheaf topos $\text{Sh}(L)$. This equivalence is usually ascribed to [Hig73], which remained unpublished. The author mentions it again in [Hig84] and a proof can be found in [FS79].

Another class of triposes are of course the arrow triposes, which we will see in Section 5.2. Using pcas, we can also construct the realizability triposes. We need the following proposition:

Proposition 2.13 ([Oos08, Prop. 1.2.1]). Let $\mathcal{P}$ be an absolute and discrete pca and X a set. Consider the collection $\text{Pow}(\mathcal{P})^X$ of $\text{Pow}(\mathcal{P})$ -valued predicates on X . For $\varphi, \psi \in \text{Pow}(\mathcal{P})^X$, we set $\varphi \leq \psi$ if there is a realizer $a \in \mathcal{P}$, such that for all $x \in X$ and all $b \in \varphi(x)$, we have $ab \downarrow$ and $ab \in \psi(x)$. Then $(\text{Pow}(\mathcal{P})^X, \leq)$ is a Heyting prealgebra. $\square$

Example 2.14. Let $\mathcal{P}$ be an absolute and discrete pca. Define a functor $P_{\mathcal{P}}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{preHeyt}$ by sending a set X to the Heyting prealgebra $(\text{Pow}(\mathcal{P})^X, \leq)$ and acting on morphisms by precomposition. The quantifiers are given by sending $\alpha \in P_{\mathcal{P}}(X)$ to

$$\begin{aligned}\forall_f(\alpha): Y \rightarrow A, y \mapsto \{p \in P : \forall p' \in P, x \in X. (f(x) = y \Rightarrow pp' \downarrow \wedge pp' \in \alpha(x))\}, \\ \exists_f(\alpha): Y \rightarrow A, y \mapsto \{p \in P : \exists x \in X. (f(x) = y \wedge p \in \alpha(x))\}.\end{aligned}$$

Similar to before, we take $\Sigma = \text{Pow}(\mathcal{P})$ and $\sigma = \text{id}_{\text{Pow}(\mathcal{P})}$. Then $P_{\mathcal{P}}$ is a tripos, which is called the *realizability tripos* on $\mathcal{P}$. $\lrcorner$

Definition 2.15 (Morphisms of triposes). Let P and Q be two triposes over the same base category $\mathbb{C}$. A *cartesian morphism* $\Phi: P \rightarrow Q$ is a pseudo-natural transformation, such that every component $\Phi_X: P(X) \rightarrow Q(X)$ preserves finite meets.

A *geometric morphism* $\Phi: P \rightarrow Q$ is a pair of cartesian morphisms $(\Phi^+: Q \rightarrow P, \Phi_+: P \rightarrow Q)$, such that $(\Phi^+)_X \dashv (\Phi_+)_X$ for all X . To save space we will sometimes omit the X and write $\Phi^+ \dashv \Phi_+$ keeping in my that this should hold componentwise. $\lrcorner$

We will give geometric morphisms in the direction of the right adjoint, i.e. the “spatial” direction. This means that given a geometric morphism $\Phi = (\Phi^+, \Phi_+): P \rightarrow Q$, we have $\Phi_+: P \rightarrow Q$ and $\Phi^+: Q \rightarrow P$. We call the right adjoint Φ_+ the *direct image functor* and the left adjoint Φ^+ the *inverse image functor*. One way to remember this is to see that for the inverse image functor, the plus (sometimes also denotes with an asterisk) is at the top, where one would usually write the $^{-1}$ for the inverse of a function.

2.3 Logic in a tripos

To do logic in a tripos, we first need to define a language, which we will interpret. This language will be dependent on the base category $\mathbb{C}$. This $\mathbb{C}$ -*typed relational language* is a set of relation symbols R with certain types $\langle X_1, \dots, X_n \rangle$, where the X_i are objects of $\mathbb{C}$. We also allow the empty type $\langle \rangle$ and shall write X for the type $\langle X \rangle$. We proceed in the usual way of defining formulas by first defining terms of the language and then its formulas. We will follow [Pit00, Ch. 5] and also make use of contexts, while the general structure is taken from [Oos08]. As usual, we assume that we have a countable list of variables $x_1, x_2, \dots$.

Definition 2.16 (Context). Let $\mathbb{C}$ be a category. A *context* is a finite list $\Gamma = \langle x_1 : X_1, \dots, x_n : X_n \rangle$ of distinct variables with types $X_i \in \mathbb{C}$. $\dashv$

2.3.1 Syntax

Given such a $\mathbb{C}$ -typed relational language $\mathcal{L}$, we define the judgement $\Gamma \vdash t : X$. This should be read as “ t is an $\mathcal{L}$ -term of type X in context Γ ”.

- i) For every $x : X$ occurring in Γ , we have $\Gamma \vdash x : X$.
- ii) If we have $\Gamma \vdash t_1 : X_1, \dots, \Gamma \vdash t_n : X_n$ and a morphism $f : X_1 \times \dots \times X_n \rightarrow Y$ in $\mathbb{C}$, then $\Gamma \vdash f(t_1, \dots, t_n) : Y$.

From this, we define the sequent $\Gamma \vdash \phi$ meaning “ ϕ is a well-formed $\mathcal{L}$ -formula in context Γ ”.

- i) $\Gamma \vdash \perp$ and $\Gamma \vdash \top$.
- ii) If R is a relation symbol of type $\langle X_1, \dots, X_n \rangle$ and $\Gamma \vdash t_1 : X_1, \dots, \Gamma \vdash t_n : X_n$, then $\Gamma \vdash R(t_1, \dots, t_n)$.
- iii) If $\Gamma \vdash \phi$ and $\Gamma \vdash \psi$, then $\Gamma \vdash \phi \wedge \psi$, $\Gamma \vdash \phi \vee \psi$, $\Gamma \vdash \phi \rightarrow \psi$ and $\Gamma \vdash \neg \phi$.
- iv) If $\Gamma, x : X \vdash \phi$, then $\Gamma \vdash \forall x : X. \phi$ and $\Gamma \vdash \exists x : X. \phi$.

Of course, negation is not explicitly needed since it is inter-definable via $\neg \phi := \phi \rightarrow \perp$, but it is still nice to have it separately. We will often omit the types of variables if these are clear from the context (that is, the text around it).

2.3.2 Semantics

Given a language $\mathcal{L}$ and a tripos $\mathbf{P}$ (with corresponding domain), we want to interpret $\mathcal{L}$ in $\mathbf{P}$. We do this inductively. First, we need an *interpretation* of the relation symbols. This is a map assigning to each relation symbol R of type $\langle X_1, \dots, X_n \rangle$ an element $[R] \in \mathbf{P}(X_1, \dots, X_n)$. With slight abuse of notation, we will sometimes identify $\Gamma = \langle x_1 : X_1, \dots, x_n : X_n \rangle$ with the product $X_1 \times \dots \times X_n$. It will always be clear from the context, which of the two we refer to. From this, we can inductively extend this interpretation to terms in a context and then to formulas in a context. For every $\mathcal{L}$ -term $\Gamma \vdash t : X$, we define a map $[\Gamma \vdash t : X] : \Gamma \rightarrow X$:

- i) If $\Gamma \vdash x_i : X_i$, let $[\Gamma \vdash x_i : X_i] := \pi_i$ be the i -th projection.

- ii) If $\Gamma \vdash f(t_1, \dots, t_n) : Y$, where $\Gamma \vdash t_1 : X_1, \dots, \Gamma \vdash t_n : X_n$ and $f : X_1 \times \dots \times X_n \rightarrow Y$, we can inductively assume that we have maps $[\Gamma \vdash t_i : X_i] : \Gamma \rightarrow X_i$. Then we have

$$\Gamma \xrightarrow{\langle [\Gamma \vdash t_1 : X_1], \dots, [\Gamma \vdash t_n : X_n] \rangle} X_1 \times \dots \times X_n \xrightarrow{f} Y$$

Diagram 2.4

We define $[\Gamma \vdash f(t_1, \dots, t_n) : Y] := f \circ \langle [\Gamma \vdash t_1 : X_1], \dots, [\Gamma \vdash t_n : X_n] \rangle$ to be the composite.

For a $\mathcal{L}$ -formulas $\Gamma \vdash \phi$, we define $[\Gamma \vdash \phi] \in \mathbf{P}(\Gamma)$:

- i) Let $[\Gamma \vdash \perp] := \perp$ and $[\Gamma \vdash \top] := \top$ be the bottom and top element of $\mathbf{P}(\Gamma)$.
- ii) For $\Gamma \vdash R(t_1, \dots, t_n)$, where R is of type $\langle X_1, \dots, X_n \rangle$, we get similarly to above a map $\langle [\Gamma \vdash t_1 : X_1], \dots, [\Gamma \vdash t_n : X_n] \rangle : \Gamma \rightarrow X_1 \times \dots \times X_n$ and define

$$[\Gamma \vdash R(t_1, \dots, t_n)] := \langle [\Gamma \vdash t_1 : X_1], \dots, [\Gamma \vdash t_n : X_n] \rangle^* [R].$$

- iii) For $\phi \star \psi$ with $\star \in \{\wedge, \vee, \rightarrow\}$, we assume that we have interpreted $\Gamma \vdash \phi$ and $\Gamma \vdash \psi$. Then we set $[\Gamma \vdash \phi \star \psi] := [\Gamma \vdash \phi] \star [\Gamma \vdash \psi]$, where $\star$ on the right side refers to the operation in the Heyting prealgebra. Further we set $[\Gamma \vdash \neg \phi] := \neg [\Gamma \vdash \phi]$.
- iv) For the quantifiers assume we have interpreted $[\Gamma, x : X \vdash \phi] \in \mathbf{P}(\Gamma \times X)$. Let $\pi : \Gamma \times X \rightarrow \Gamma$ be the projection. Then we define

$$[\Gamma \vdash \exists x : X. \phi] := \exists_\pi [\Gamma, x : X \vdash \phi] \quad \text{and} \quad [\Gamma \vdash \forall x : X. \phi] := \forall_\pi [\Gamma, x : X \vdash \phi],$$

where $\exists_\pi$ and $\forall_\pi$ are the left and right adjoint of π^* .

As usual, we will omit brackets whenever possible and assume that quantifiers bind the strongest, followed by conjunction and disjunction and implication binding the weakest. We will not use negation, so we need no convention for it and as usual we will also not define a conventional precedence between conjunction and disjunction.

In what follows, we will use the π -notation quite liberally as the pairing of multiple projections. This way, we will occasionally also include the swapping of components or the duplication (similar to $\delta : X \rightarrow X \times X$) in this notation. If we have for example the product $X_1 \times \dots \times X_5$, then π_{3141} denotes the morphism

$$\pi_{3141} = \langle \pi_3, \pi_1, \pi_4, \pi_1 \rangle : X_1 \times \dots \times X_5 \rightarrow X_3 \times X_1 \times X_4 \times X_1.$$

If the context Γ is clear from the context (that is, the text around it), we will omit it.

2.3.3 Validity, soundness and completeness

As usual, we call a formula ϕ a *sentence* if it has no free variables, i.e. $\emptyset \vdash \phi$. Since 1 is the empty product, $[\phi]$ (that is, $[\emptyset \vdash \phi]$) is an element of $\mathbf{P}(1)$. For these we can define what it means to be true:

Definition 2.17 (Validity). Let $\mathbf{P} : \mathbb{C}^{\text{op}} \rightarrow \mathbf{preHeyt}$ be a tripos, ϕ a sentence in a $\mathbb{C}$ -typed relational language $\mathcal{L}$ and $[\cdot]$ an interpretation of $\mathcal{L}$ in $\mathbf{P}$. Then we say that ϕ is *true* or *valid* in $\mathbf{P}$ relative to $[\cdot]$ if $[\phi] \simeq \top$ in $\mathbf{P}(1)$. In this case, we write $\mathbf{P} \models \phi$. $\dashv$

With these definitions, we can prove soundness with respect to first order intuitionistic logic:

Theorem 2.18 (Soundness Theorem). Suppose ϕ is a sentence in a $\mathbb{C}$ -typed relational language $\mathcal{L}$. If ϕ is an intuitionistic tautology without equality, then we have $\mathsf{P} \models \phi$ for all $\mathbb{C}$ -triposes P and all interpretations $[\cdot]$ of $\mathcal{L}$ in P . $\square$

The proof is by induction on proofs in a formal system [HJP80; Oos08]. A corresponding completeness result also holds, but is less interesting, since it includes a Lindenbaum-Tarski construction where satisfaction coincides with provability [Jac99, Lem. 4.3.4], see also [Pit02]. The nice thing about Theorem 2.18 is that it allows us to abandon the more tedious equational way of proving statements in the logic of triposes. Instead, we can use intuitionistically valid statements, making a lot of proofs easier to read and shorter. In long strings containing many $\exists_\pi$ and f^* , it is hard to follow what is going on and the idea of the proof is not understandable anymore.

We would like to use Theorem 2.18 all the time, but as we will see in Chapter 4 and 6, this will not always be possible. In the first case, it is because we will have existentials of the form $\exists_f$ where f is not a projection. This means that the expression does not correspond to a formula we can write in our language. However, Theorem 2.18 only makes a statement about these. There is a workaround using projections and the predicate $\sim_\top$ we will see in Lemma 2.23 [Oos08, Sec. 2.1.4.5], but this makes the expressions even longer. Thus we will have to use the equational style in Chapter 4. In the second case, we will be using maps between triposes and interpret predicates from one tripos in another. In that case, we cannot always apply Theorem 2.18 because formulae containing maps between triposes can also not be formulated in our language. In this case, we will use the interpretation of the logic in the arrow topos, but more on that in Chapter 6. For this and the following chapter, we will use the easier style.

2.3.4 Motivation for the adjoints

With this at hand, we can explain the idea behind f^* having both adjoints. Suppose we have the formula $\forall x.\phi(x)$. This corresponds to the term $\forall_![\phi]$, where $! : X \rightarrow 1$. The adjunction property now tell us that $\top \leq \forall_![\phi]$ iff $\top \leq [\phi]$ since $!^*\top \simeq \top$. The inequality $\top \leq \forall_![\phi]$ says that $\forall x.\phi(x)$ is valid and $\top \leq [\phi]$ tells us that $\phi(x)$ is true for all x . Thus the if and only if directions of this equivalence are the rules for introduction and instantiation of the universal quantifier respectively. The advantage of this is that we need no constraints on which variables are allowed to appear freely since this is encoded in the application of $!^*$.

The case for the existential quantifier is similar: consider the sequent $\phi(x) \vdash \psi$, where x is not free in ψ . This entailment corresponds to the inequality $[\phi] \leq !^*[\psi]$, where $! : X \rightarrow 1$. Consider additionally the sequent $\exists x.\phi(x) \vdash \psi$, which can be written as $\exists_![\phi] \leq [\psi]$. Since x is not free in ψ , these two sequents are equivalent by the introduction and instantiation rule of the existential quantifier. Thus the two inequalities should also be equivalent, which is exactly what the adjunction property for $\exists$ tells us.

2.3.5 Some useful lemmata

Lastly, we want to collect some properties for future reference. These are taken from [Oos08, Sec. 2.1.4]. We fix a $\mathbb{C}$ -tripos P .

Lemma 2.19. If $X \in \mathbb{C}$, then we have for all $\phi \in \mathbf{P}(X)$ that $\exists_{\text{id}_X} \phi \simeq \phi$.

Proof. By reflexivity and the $\exists$ adjunction, we have $\exists_{\text{id}_X} \phi \leq \exists_{\text{id}_X} \phi$ and $\phi \leq \text{id}_X^* \exists_{\text{id}_X} \phi \simeq \exists_{\text{id}_X} \phi$. The other inequality follows directly from the $\exists$ adjunction. $\square$

Lemma 2.20. If $i: U \rightarrow X$ is mono in $\mathbb{C}$, then for any $\phi \in \mathbf{P}(U)$, we have $i^* \exists_i \phi \simeq \phi$.

Proof. By basic category theory, [Diagram 2.5](#) is a pullback iff i is mono, so we get

$$i^* \exists_i \phi \simeq \exists_{\text{id}_U} \text{id}_U^* \phi \simeq \phi$$

by the Beck-Chevalley condition and the previous lemma. $\square$

Lemma 2.21 (Frobenius Condition). For any map $f: X \rightarrow Y$, $\psi \in \mathbf{P}(X)$ and $\phi \in \mathbf{P}(Y)$, we have $\exists_f(\psi \wedge f^* \phi) \simeq \exists_f \psi \wedge \phi$.

Proof. Since $\exists_f \dashv f^*$, we have $\psi \leq f^* \exists_f \psi$ and hence $\psi \wedge f^* \phi \leq f^* \exists_f \psi \wedge f^* \phi \simeq f^*(\exists_f \psi \wedge \phi)$, which gives us “ $\leq$ ”. For the converse, we use that f^* commutes with the implication and get

$$\begin{aligned} \exists_f \psi \wedge \phi \leq \exists_f(\psi \wedge f^* \phi) &\iff \exists_f \psi \leq \phi \rightarrow \exists_f(\psi \wedge f^* \phi) \iff \psi \leq f^* \phi \rightarrow f^* \exists_f(\psi \wedge f^* \phi) \\ &\iff \psi \wedge f^* \phi \leq f^* \exists_f(\psi \wedge f^* \phi). \end{aligned}$$

The last term is clearly true since $\exists_f \dashv f^*$. $\square$

Convention 2.22 (Tripos reasoning). For readability purposes, we will refer to a combination of the Beck-Chevalley condition and the Frobenius condition as *tripos reasoning*. $\lrcorner$

Lemma 2.23. Let $X \in \mathbb{C}$, $\delta: X \rightarrow X \times X$ be the diagonal and $\phi \in \mathbf{P}(X)$. For $\sim_\phi := \exists_\delta \phi$, we have

$$\mathbf{P} \models \forall x, y. (x \sim_\phi y \rightarrow y \sim_\phi x) \quad \text{and} \quad \mathbf{P} \models \forall x, y, z. (x \sim_\phi y \wedge y \sim_\phi z \rightarrow x \sim_\phi z).$$

Proof. For the first property, we need to prove $\top \simeq [\forall x, y. (x \sim_\phi y \rightarrow y \sim_\phi x)] = \forall_!(\sim_\phi \rightarrow \text{tw}^* \sim_\phi)$, which is equivalent to just showing $\top \leq \forall_!(\sim_\phi \rightarrow \text{tw}^* \sim_\phi)$. Here $\text{tw}: X \times X \rightarrow X \times X$ is the twist map. Using the $\forall$ adjunction, we need to prove $\top \leq \sim_\phi \rightarrow \text{tw}^* \sim_\phi$ since $!^* \top \simeq \top$. By the Heyting adjunction, this is equivalent to $\sim_\phi \leq \text{tw}^* \sim_\phi$. Using the definition of $\sim_\phi$ and the $\exists$ adjunction, we need to show $\phi \leq \delta^* \text{tw}^* \exists_\delta \phi$. This is now immediate from the $\exists$ adjunction since $\text{tw} \circ \delta = \delta$.

For the second property we can use a similar treatment of the universal quantifier and implication to see that it is equivalent to showing

$$\pi_{12}^* \exists_\delta \phi \wedge \pi_{23}^* \exists_\delta \phi \leq \pi_{13}^* \exists_\delta \phi.$$

$$\begin{array}{ccc} U & \xrightarrow{\text{id}_U} & U \\ \downarrow \text{id}_U \lrcorner & & \downarrow i \\ U & \xrightarrow{i} & X \end{array}$$

Diagram 2.5

$$\begin{array}{ccc} X \times X & \xrightarrow{\pi_{112}} & X \times X \times X \\ \downarrow \pi_1 \lrcorner & & \downarrow \pi_{12} \\ X & \xrightarrow{\delta} & X \times X \end{array}$$

Diagram 2.6

By tripos reasoning using [Diagram 2.6](#) the left side becomes

$$\pi_{12}^* \exists_\delta \phi \wedge \pi_{23}^* \exists_\delta \phi \simeq \exists_{\pi_{112}} \pi_1^* \phi \wedge \pi_{23}^* \exists_\delta \phi \simeq \exists_{\pi_{112}} (\pi_1^* \phi \wedge \pi_{112}^* \pi_{23}^* \exists_\delta \phi) \simeq \exists_{\pi_{112}} (\pi_1^* \phi \wedge \exists_\delta \phi).$$

By the $\exists$ adjunction, this is less than $\pi_{13}^* \exists_\delta \phi$ iff $\pi_1^* \phi \wedge \exists_\delta \phi \leq \pi_{112}^* \pi_{13}^* \exists_\delta \phi$. This is clearly true since $\pi_{13} \circ \pi_{112} = \text{id}_{X \times X}$. $\square$

The previous proof may be a bit unconventional in the sense that we start with the statement we want to prove and use equivalence transformations to find an equivalent statement that is clearly true. Such a proof can of course easily be translated to a direct proof by essentially reading it from the bottom up. However, we think that this way of writing better represents what is going on and how one usually approaches such problems. It is not always clear from the beginning which tautology one has to use to get the desired statement. Thus this way of writing also serves as an educational device to present a way of thinking about these problems. We shall also extend this method in the following by mentioning when something is implied by another statement, but not necessarily equivalent. This is of course valid, but has the danger of using an antecedent that is too weak. So that extension should be used with care.

Chapter 3

The tripos to topos construction

When triposes were introduced in [HJP80], the authors also proved that triposes give rise to topoi:

Theorem 3.1 (Hyland, Johnstone, Pitts). For any tripos $P: \mathbb{C}^{\text{op}} \rightarrow \mathbf{preHeyt}$, the category $\mathbb{C}[P]$ is a topos.

In this chapter, we will define the category $\mathbb{C}[P]$ and prove Theorem 3.1. This result is of course standard by now [HJP80; Oos08; PCS81]. We will still do it explicitly for future reference and to give more detailed proofs. Theorem 3.1 already follows from the results of Section 3.2 and 3.5. The other sections contain characterizations we need in those proofs and in later chapters.

We start by explaining what the category $\mathbb{C}[P]$ is. Let $P: \mathbb{C}^{\text{op}} \rightarrow \mathbf{preHeyt}$ be a tripos. The category $\mathbb{C}[P]$ of *partial equivalence relations over P* is defined as follows. Its objects are pairs $(X, \sim)$, where X is an object of $\mathbb{C}$ and $\sim \in P(X \times X)$, such that $\sim$ is an *equality predicate* on X , i.e. it is symmetric and transitive:

$$P \models \forall x, y. (x \sim y \rightarrow y \sim x) \quad \text{and} \quad P \models \forall x, y, z. (x \sim y \wedge y \sim z \rightarrow x \sim z).$$

We have seen a special case of this in Lemma 2.23: The predicate $\sim_\phi = \exists \delta \phi$ is an equality predicate. We will often use the case where $\phi = \top$. This case is particularly interesting since it gives us the least reflexive equality predicate. An equality predicate is said to be *reflexive* if $P \models \forall x. x \sim x$, i.e. $\top \leq \delta^* \sim$. By the $\exists$ adjunction, it is easy to see that $\sim_\top$ is then the least such. We will see more advantages of $\sim_\top$ after introducing the morphisms of $\mathbb{C}[P]$.

Now one might wonder why we do not require equality predicates to be reflexive. After all, the usual equality is reflexive. The answer is easy: If we required this, then for $\mathbb{C}[P]$ to be a topos, $\mathbb{C}$ would need to satisfy more conditions. For example, we do not need $\mathbb{C}$ to have equalizers because we can just restrict the domain of the parallel morphisms to those elements on which the two morphisms are equal (cf. Lemma 3.7). This way, one element of $\mathbb{C}$ can be turned into many elements of $\mathbb{C}[P]$. It also comes in handy in other cases, when we want to exclude some variables. For example, when constructing the power object in Section 3.5, we start with the object of predicates $\pi(X)$, but restrict to those that behave like subsets. We can make “behaving like a subset” precise using first-order logic and exclude all elements of $\pi(X)$ that do not satisfy it.

This restriction of the equality predicate is similar to the axiom of comprehension. We will usually use an additional predicate to construct an object that only contains some of the original variables.

This is like using a formula to define a subset of an existing set. We will make this idea more precise in [Section 3.4](#). So we do not require equality predicates to be reflexive so that we have more examples of triposes.

A morphism $(X, \sim) \rightarrow (Y, \sim)$ in $\mathbb{C}[\mathbf{P}]$ is an equivalence class of functional relations in $\mathbf{P}(X \times Y)$. Two functional relations are equivalent iff they are as predicates in $\mathbf{P}(X \times Y)$. A predicate $F \in \mathbf{P}(X \times Y)$ is a functional relation if

$$\begin{array}{ll} \mathbf{P} \models \forall x, y. (F(x, y) \rightarrow x \sim x \wedge y \sim y) & F \text{ is } \textit{strict} \\ \mathbf{P} \models \forall x, x', y, y'. (F(x, y) \wedge x \sim x' \wedge y \sim y' \rightarrow F(x', y')) & F \text{ is } \textit{relational} \\ \mathbf{P} \models \forall x, y. (F(x, y) \wedge F(x, y') \rightarrow y \sim y') & F \text{ is } \textit{single-valued} \\ \mathbf{P} \models \forall x. (x \sim x \rightarrow \exists y. F(x, y)) & F \text{ is } \textit{total} \end{array}$$

The identity on an object $(X, \sim)$ is then simply represented by $\sim$ and if we have predicates F and G representing morphisms $(X, \sim) \rightarrow (Y, \sim)$ and $(Y, \sim) \rightarrow (Z, \sim)$ respectively, then the composition is represented by

$$[(G \circ F)(x, z)] = [\exists y. (F(x, y) \wedge G(y, z))].$$

The motivation for totality and single-valuedness are clear if we want to represent morphisms. Relationality is needed since we may consider more elements to be equal and this should be respected by morphisms. Strictness on y already follows from single-valuedness and strictness on x is needed to make sure that we get an interesting topos:

Proposition 3.2. Let $\mathbf{P}: \mathbb{C}^{\text{op}} \rightarrow \mathbf{preHeyt}$ be a tripos, such that $\mathbb{C}[\mathbf{P}]$ is a topos, but morphisms are not required to be strict on their domain. Then all elements of $\mathbb{C}[\mathbf{P}]$ are isomorphic.

Proof. In the first step, we will proof that in this case, there can be at most one morphism between any two elements. First let $(X, \sim)$ be an arbitrary element in $\mathbb{C}[\mathbf{P}]$. Any topos is finitely cocomplete [\[Joh77, Cor. 1.36\]](#), so it especially has an initial object 0 . Next, consider the element $(X, \perp)$. It is easy to check that $\perp$ represents a morphism $(X, \perp) \rightarrow 0$. Thus $(X, \perp)$ is also initial since 0 is strict [\[MM94, Prop. IV.7.4\]](#).

Let $F: (X, \sim) \rightarrow (Y, \sim)$ represent any morphism. Then especially $F \in \mathbf{P}(X \times Y)$. Since it does not need to be strict on its domain, F satisfies all the conditions to be a morphism $(X, \perp) \rightarrow (Y, \sim)$. Similarly, any other $G: (X, \sim) \rightarrow (Y, \sim)$ is also a morphism $(X, \perp) \rightarrow (Y, \sim)$. Since $(X, \perp)$ is initial, we must have $F \simeq G$, i.e. they are equal.

Next, we will show that we have a morphism $1 \rightarrow 0$. As in any topos, 0 is the pullback of true $t: 1 \rightarrow \Omega$ and false $f: 1 \rightarrow \Omega$ by definition [\[Joh77, Ch. 5.1\]](#). But since there can only be one morphism $1 \rightarrow \Omega$, we must have $t = f$ and the identity on 1 makes the pullback commute. By the universal property of the pullback, we get a map $1 \rightarrow 0$.

Now by terminality of 1 and the map $1 \rightarrow 0$, we have a map $(X, \sim) \rightarrow 0$ for every object $(X, \sim)$ in $\mathbb{C}[\mathbf{P}]$. Since 0 is strict, $(X, \sim)$ must be isomorphic to 0 and thus also to any other element. $\square$

One advantage of reflexive equality predicates is that we do not need to check strictness of morphisms with respect to them. Additionally, every predicate is relational with respect to $\sim_{\top}$: Let $\Pi \in \mathbf{P}(X)$. We need to show that $[\Pi(x) \wedge x \sim_{\top} x'] \leq [\Pi(x')]$, which is just $\pi_1^* \Pi \wedge \sim_{\top} \leq \pi_2^* \Pi$. Since

$\sim_{\top} = \exists_{\delta} \top$, we can use the Frobenius condition to get

$$\pi_1^* \Pi \wedge \exists_{\delta} \top \simeq \exists_{\delta} (\delta^* \pi_1^* \Pi \wedge \top) \simeq \exists_{\delta} \Pi$$

since $\pi_1 \circ \delta = \text{id}_X$ by definition. This is less than $\pi_2^* \Pi$ by the $\exists$ adjunction since also $\pi_2 \circ \delta = \text{id}_X$. A similar argument works if Π also has more free variables.

For the remainder of this chapter, we let $\mathbf{P}: \mathbb{C}^{\text{op}} \rightarrow \mathbf{preHeyt}$ be a fixed tripos.

3.1 Equality of morphisms

We want to find easier conditions when two morphisms in $\mathbb{C}[\mathbf{P}]$ are equal, i.e. equivalent. The first one is well-known and taken from [Oos08], while the second one states that two morphisms are equal if for a given input, their outputs are related, i.e. equal.

Lemma 3.3. Let F and G represent morphisms $(X, \sim) \rightarrow (Y, \sim)$, such that $F \leq G$. Then they are equivalent.

Proof. We just have to show that in this case, we also have $G \leq F$. By strictness of G and totality of F , we have

$$[G(x, y)] \leq [G(x, y) \wedge x \sim x] \leq [G(x, y) \wedge \exists y'. F(x, y')] \simeq [\exists y'. (G(x, y) \wedge F(x, y'))].$$

By assumption we have $F \leq G$, i.e. $[F(x, y)] \leq [G(x, y)]$, G is single-valued and F is relational:

$$[\exists y'. (G(x, y) \wedge F(x, y'))] \leq [\exists y'. (G(x, y) \wedge F(x, y') \wedge G(x, y'))] \leq [\exists y'. (F(x, y') \wedge y \sim y')] \leq [F(x, y)].$$

Thus we also have $G \leq F$. □

Lemma 3.4. Let F and G represent morphisms $(X, \sim) \rightarrow (Y, \sim)$. They are equivalent if we have

$$\mathbf{P} \models \forall x y y'. (F(x, y) \wedge G(x, y') \rightarrow y \sim y').$$

Proof. We will use the previous lemma and only show that $F \leq G$. We have

$$[F(x, y)] \leq [F(x, y) \wedge x \sim x] \leq [\exists y'. (F(x, y) \wedge G(x, y'))] \leq [\exists y'. (G(x, y') \wedge y \sim y')] \leq [G(x, y)]. \quad \square$$

3.2 Finite Limits

We show that $\mathbb{C}[\mathbf{P}]$ has finite limits by showing that it has a terminal object, binary products and equalizers. These three are sufficient for constructing all other finite limits. At the end, we will also mention the construction of a pullback for future reference.

Lemma 3.5. Let 1 be terminal in $\mathbb{C}$. Then $(1, \top)$ is terminal in $\mathbb{C}[\mathbf{P}]$.

Proof. First, note that $\mathbb{C}$ must have a terminal object since it has finite products by assumption. Clearly, $\top$ is an equality predicate. For any other element $(X, \sim)$, define $[(x, *)] := [x \sim x]$. This is strict, relational and single-valued. It is total since we have a morphism $\phi: X \rightarrow 1$ by assumption so

that $[x \sim x] \leq [x \sim x] \simeq [!(x, \phi(x))]$, because $!$ says nothing about the codomain. It follows directly from [Lemma 3.4](#) that $!$ is unique. $\square$

Lemma 3.6. Let $(X, \sim)$ and $(Y, \sim)$ be elements in $\mathbb{C}[\mathbf{P}]$. Their product is given by $(X \times Y, \approx)$ with projections p_1, p_2 , where

$$\begin{aligned} [(x, y) \approx (x', y')] &:= [x \sim x' \wedge y \sim y'], \\ [p_1((x, y), x')] &:= [x \sim x' \wedge y \sim y], \\ [p_2((x, y), y')] &:= [x \sim x \wedge y \sim y']. \end{aligned}$$

Proof. It is easy to see that $\approx$ is an equality predicate and that p_1, p_2 represent morphisms. Now let $F: (Z, \sim) \rightarrow (X, \sim)$ and $G: (Z, \sim) \rightarrow (Y, \sim)$ be arbitrary morphisms. Define

$$[\langle F, G \rangle(z, (x, y))] := [F(z, x) \wedge G(z, y)].$$

Since F and G represent morphisms, it is clear that $\langle F, G \rangle$ also represents one. By relationality of F , we have

$$[(p_1 \circ \langle F, G \rangle)(z, x)] \simeq [\exists x', y'. (x \sim x' \wedge y \sim y \wedge F(z, x') \wedge G(z, y'))] \leq [F(z, x)].$$

And similarly for G . Since they are all morphisms by [Lemma 3.3](#) this shows that $\langle F, G \rangle$ factors as wanted. For uniqueness, we assume that there is another morphism $H: (Z, \sim) \rightarrow (X \times Y, \approx)$ that factors F and G . By strictness of H , we have

$$\begin{aligned} [H(z, (x, y))] &\leq [H(z, (x, y)) \wedge x \sim x \wedge y \sim y] \leq [\exists x', y'. (H(z, (x', y')) \wedge x' \sim x \wedge y' \sim y)] \\ &\simeq [(p_1 \circ H)(z, x)] \simeq [F(z, x)]. \end{aligned}$$

It is also easy to see that $[\langle F, G \rangle(z, (x, y))] \leq [F(z, x)]$, whence by single-valuedness of F ,

$$[H(z, (x_1, y_1)) \wedge \langle F, G \rangle(z, (x_2, y_2))] \leq [F(z, x_1) \wedge F(z, x_2)] \leq [x_1 \sim x_2].$$

By a symmetric argument on y , we can see that

$$[H(z, (x_1, y_1)) \wedge \langle F, G \rangle(z, (x_2, y_2))] \leq [(x_1, y_1) \approx (x_2, y_2)].$$

By [Lemma 3.4](#), this shows that $H \simeq \langle F, G \rangle$, which is thus unique. $\square$

Lemma 3.7. Let $(X, \sim)$ and $(Y, \sim)$ be elements in $\mathbb{C}[\mathbf{P}]$ with morphisms $F, G: (X, \sim) \rightarrow (Y, \sim)$. The equalizer of F and G is given by $(X, \sim')$ and $\sim': (X, \sim') \rightarrow (X, \sim)$, where

$$[x \sim' x'] := [x \sim x' \wedge \exists y. (F(x, y) \wedge G(x', y))].$$

Proof. It is easy to see that $\sim'$ is an equality predicate and that it also represents a morphism $(X, \sim') \rightarrow (X, \sim)$. It equalizes F and G by [Lemma 3.4](#) because

$$[(F \circ \sim')(x, y_1) \wedge (G \circ \sim')(x, y_2)]$$

$$\begin{aligned}
&\leq [\exists x_1, x_2. (F(x_1, y_1) \wedge x \sim x_1 \wedge \exists y'_1. (F(x, y'_1) \wedge G(x, y'_1)) \wedge G(x_2, y_2) \wedge x \sim x_2)] \\
&\leq [\exists x_1, x_2, y'_1. (F(x, y_1) \wedge F(x, y'_1) \wedge G(x, y'_1) \wedge G(x, y_2))] \\
&\leq [\exists x_1, x_2, y'_1. (y_1 \sim y'_1 \wedge y'_1 \sim y_2)] \leq [y_1 \sim y_2].
\end{aligned}$$

Now assume we have another $H: (Z, \sim) \rightarrow (X, \sim)$ that equalizes F and G . Thus it is also strict for $\sim'$ and also represents a morphism $(Z, \sim) \rightarrow (X, \sim')$. Similar to the previous proof, it is also easy to see that $\sim' \circ H \simeq H$. If there is an $H': (Z, \sim) \rightarrow (X, \sim')$, such that $H \simeq \sim' \circ H'$, we have

$$[H'(z, x')] \leq [H'(z, x') \wedge x' \sim' x'] \leq [\exists \hat{x}. (H'(z, \hat{x}) \wedge \hat{x} \sim' x')] \simeq [(\sim' \circ H')(z, x')] \simeq [H(z, x')].$$

Since both are morphisms, we have $H' \simeq H$ by [Lemma 3.3](#) and hence uniqueness. $\square$

Combining [Lemma 3.6](#) and [3.7](#), we can also construct pullbacks. This construction is canonical, but still nice to have given explicitly.

Corollary 3.8. Let $(X, \sim), (Y, \sim)$ and $(Z, \sim)$ be elements in $\mathbb{C}[\mathbf{P}]$ with morphisms $F: (X, \sim) \rightarrow (Z, \sim)$ and $G: (Y, \sim) \rightarrow (Z, \sim)$. The pullback is given by $(X \times Y, \approx)$ with projections p_1, p_2 where

$$\begin{aligned}
[(x, y) \approx (x', y')] &:= [x \sim x' \wedge y \sim y' \wedge \exists z. (F(x, z) \wedge G(y, z))], \\
[p_1((x, y), x')] &:= [(x, y) \approx (x, y) \wedge x \sim x'], \\
[p_2((x, y), y')] &:= [(x, y) \approx (x, y) \wedge y \sim y'].
\end{aligned}$$

$\square$

In the proof of [Lemma 3.7](#), we have secretly used the axiom of choice. The problem lies in identifying a morphism (which is an equivalence class of functional predicates) with one of its elements. We are just given the morphisms, but to construct $\sim'$, we need a representative, which is a textbook example of using the axiom of choice. So if we want to stay fully intuitionistic also in our meta-logic, we cannot prove that $\mathbb{C}[\mathbf{P}]$ has equalizers (cf. [HJP80](#), Rmk. 2.7).

We propose a way to avoid this by changing the definition of objects in $\mathbb{C}[\mathbf{P}]$ to be equivalence classes of pairs $(X, \sim)$, where we say that $(X, \sim)$ and $(Y, \sim')$ are equivalent iff $X = Y$ and $\sim \simeq \sim'$, i.e. the equality predicates are logically equivalent. This does not change any of the other properties of $\mathbb{C}[\mathbf{P}]$ since if two elements are equivalent in this sense, then they are isomorphic as elements of $\mathbb{C}[\mathbf{P}]$. Furthermore, this solves the problem of using the axiom of choice in the proof of [Lemma 3.7](#). We just need to show that for different representatives F_1 and F_2 of F and G_1 and G_2 of G respectively, the constructed equality predicates $\sim'_1$ and $\sim'_2$ are equivalent. This is clear since

$$[x \sim'_1 x'] = [x \sim x' \wedge \exists y. (F_1(x, y) \wedge G_1(x, y))] \leq [x \sim x' \wedge \exists y. (F_2(x, y) \wedge G_2(x, y))] = [x \sim'_2 x']$$

by $F_1 \leq F_2$ and $G_1 \leq G_2$. The other inequality follows symmetrically.

We want to emphasize that our construction does not make $\mathbb{C}[\mathbf{P}]$ skeletal, since we require the first component of the pair to be identically the same. If we consider $\mathbb{C} = \mathbf{Set}$ for example, there are isomorphic elements that are not identical. So it is not hard to transform $(X, \sim)$ into some isomorphic $(X', \sim')$ by taking an X' with $X \cong X'$, but $X \neq X'$ and appropriate application of the isomorphism to $\sim$.

3.3 Properties of morphisms

In topoi, we can do logic and thus characterize morphisms by logical expressions. We will prove that the characterizations of monos, isos and epis coincide with the usual conditions we know from **Set** for the case of $\mathbb{C}[\mathbf{P}]$.

Lemma 3.9. A morphism $F: (X, \sim) \rightarrow (Y, \sim)$ is mono iff we have

$$\mathbf{P} \models \forall x x' y. (F(x, y) \wedge F(x', y) \rightarrow x \sim x'),$$

which can also be written as $\mathbf{P} \models \forall x x'. (\exists y. (F(x, y) \wedge F(x', y)) \rightarrow x \sim x')$.

Proof. For the if direction, let $G, H: (Z, \sim) \rightarrow (X, \sim)$ be arbitrary morphisms, such that $F \circ G \simeq F \circ H$. Then we have

$$\begin{aligned} [G(z, x) \wedge H(z, x')] &\leq [\exists y, y'. (G(z, x) \wedge F(x, y) \wedge H(z, x') \wedge F(x', y'))] \\ &\leq [\exists y, y'. ((F \circ G)(z, x) \wedge (F \circ H)(z, x') \wedge F(x, y) \wedge F(x', y'))] \\ &\simeq [\exists y, y'. ((F \circ G)(z, x) \wedge (F \circ G)(z, x') \wedge F(x, y) \wedge F(x', y'))] \\ &\leq [\exists y, y'. (y \sim y' \wedge F(x, y) \wedge F(x', y'))] \\ &\leq [\exists y. (F(x, y) \wedge F(x', y))] \leq [x \sim x'], \end{aligned}$$

which shows $G \simeq H$ by [Lemma 3.4](#). For the only if direction recall that a morphism $f: A \rightarrow B$ is mono iff the analogous diagram to [Diagram 2.5](#) is a pullback. By [Corollary 3.8](#), we know that the pullback of F along itself is given by $(X \times X, \approx)$ where

$$[(x_1, x_2) \approx (x'_1, x'_2)] \simeq [x_1 \sim x'_1 \wedge x_2 \sim x'_2 \wedge \exists y. (F(x_1, y) \wedge F(x_2, y))].$$

Now this must be isomorphic to $(X, \sim)$, so we have maps $\phi: (X \times X, \approx) \rightarrow (X, \sim)$ and $\phi^{-1}: (X, \sim) \rightarrow (X \times X, \approx)$ whose compositions are the respective identities. Using the description from the previous section, one can easily check that $[\phi^{-1}(x, x_1, x_2)] = [x \sim x_1 \wedge x \sim x_2]$. If we assume $[\exists y. (F(x, y) \wedge F(x', y))]$, then we also have $[(x, x') \approx (x, x')]$. Thus there is a $\hat{x}$, such that $[\phi(x, x', \hat{x})]$. Since $\phi^{-1} \circ \phi \simeq \text{id}$, we have that $[\phi^{-1}(\hat{x}, x, x')]$ must hold. But by the description we gave earlier, we must have $[\hat{x} \sim x \wedge \hat{x} \sim x']$ and thus $[x \sim x']$, giving us the claim. $\square$

Lemma 3.10. A morphism $F: (X, \sim) \rightarrow (Y, \sim)$ is iso iff we have

$$\mathbf{P} \models \forall x x' y. (F(x, y) \wedge F(x', y) \rightarrow x \sim x') \quad \text{and} \quad \mathbf{P} \models \forall y. (y \sim y \rightarrow \exists x. F(x, y)).$$

Proof. For the if direction, define $[\overline{F}(y, x)] := [F(x, y)]$. It is easy to see that $\overline{F}$ is strict and relational since F is. Furthermore, the two conditions on F guarantee that $\overline{F}$ is single-valued and total respectively. So we just need to check that the compositions are the respective identities. However, these follow easily from single-valuedness of F and $\overline{F}$.

For the only if direction, assume that F is iso and let $\overline{F}$ be its inverse. We have

$$[y \sim y] \leq [\exists x. \overline{F}(y, x)] \leq [\exists x, y'. (\overline{F}(y, x) \wedge F(x, y'))] \leq [\exists x, y'. (y \sim y' \wedge F(x, y'))] \leq [\exists x. F(x, y)]$$

and

$$[F(x, y) \wedge F(x', y)] \leq [\exists \hat{x}. (F(x, y) \wedge F(x', y) \wedge \overline{F}(y, \hat{x}))] \leq [\exists \hat{x}. (x \sim \hat{x} \wedge x' \sim \hat{x})] \leq [x \sim x']$$

as desired. $\square$

Lemma 3.11. A morphism $F: (X, \sim) \rightarrow (Y, \sim)$ is epi iff we have

$$\mathbf{P} \models \forall y. (y \sim y \rightarrow \exists x. F(x, y)).$$

Proof. Assume that F satisfies this condition and that $G, H: (Y, \sim) \rightarrow (Z, \sim)$ are morphisms such that $G \circ F = H \circ F$. We will use [Lemma 3.4](#) to conclude the statement:

$$\begin{aligned} [G(y, z) \wedge H(y, z')] &\leq [G(y, z) \wedge H(y, z') \wedge y \sim y] \\ &\leq [\exists x. (F(x, y) \wedge G(y, z) \wedge H(y, z'))] \\ &\leq [\exists x. ((\exists y. F(x, y) \wedge G(y, z)) \wedge (\exists y. F(x, y) \wedge H(y, z')))] \\ &\simeq [\exists x. ((G \circ F)(x, z) \wedge (H \circ F)(x, z'))] \\ &\simeq [\exists x. ((G \circ F)(x, z) \wedge (G \circ F)(x, z'))] \leq [z \sim z'] \end{aligned}$$

by single-valuedness.

For the only if direction let F be epi. Consider $(Y, \sim')$ with $[y \sim' y'] := [y \sim y' \wedge \exists x. F(x, y)]$. Since F is relational, this is an equality predicate. It is easy to see that F also represents a morphism $(X, \sim) \rightarrow (Y, \sim')$. Furthermore, $\sim'$ is a morphism $(Y, \sim') \rightarrow (Y, \sim)$ and we have that [Diagram 3.1](#) commutes by relationality of F and using [Lemma 3.3](#). Thus $\sim'$ is epi since F is. It is also easily seen to be mono and thus an iso since topoi are balanced [[Joh77](#), Cor 1.22]. Let $i: (Y, \sim) \rightarrow (Y, \sim')$ be its inverse. First we observe that by strictness of i we have

$$[i(y, y')] \leq [i(y, y') \wedge y' \sim' y'] \leq [\exists \hat{y}. (i(y, \hat{y}) \wedge \hat{y} \sim' y')] \simeq [(\sim' \circ i)(y, y')] \simeq [y \sim y'],$$

where the last step holds because i is the inverse of $\sim'$, so their composition is the identity on $(Y, \sim)$, which is represented by $\sim$. Now we can show that F has the property in the statement of the lemma. By totality and strictness of i and relationality of F , we have

$$[y \sim y] \leq [\exists y'. i(y, y')] \leq [\exists y'. (i(y, y') \wedge y' \sim' y')] \leq [\exists y'. (y \sim y' \wedge \exists x. F(x, y'))] \leq [\exists x. F(x, y)].$$

It follows from the Heyting adjunction and the $\forall$ adjunction that this inequality is equivalent to $\mathbf{P} \models \forall y. (y \sim y \rightarrow \exists x. F(x, y))$. $\square$

Notice that in this proof, we used that topoi are balanced and especially that $\mathbb{C}[\mathbf{P}]$ is a topos. Thus we can use this characterization only *a posteriori* after [Section 3.5](#). We will not need it for any of the

$$\begin{array}{ccc} (X, \sim) & \xrightarrow{F} & (Y, \sim) \\ & \searrow F & \nearrow \sim' \\ & (Y, \sim') & \end{array}$$

Diagram 3.1

proofs until after [Section 3.5](#), but it is nice to have this characterization.

3.4 Subobjects and strict predicates

The goal of this section is to prove [Theorem 3.18](#), i.e. that in the tripos to topos construction, subobjects correspond to strict predicates. A 1-to-1 correspondence is well-known and mentioned [[HJP80](#); [Oos08](#)]. The authors were undoubtedly aware of the isomorphism as well, but did not mention it explicitly. It is an important result for us, so we will explicitly mention it and give a full proof.

Definition 3.12 (Subobject functor). Given an elementary topos $\mathcal{E}$, we can define the subobject functor $\text{Sub}_{\mathcal{E}}: \mathcal{E}^{\text{op}} \rightarrow \mathbf{Preord}$, sending an object to the preorder of its subobjects. Formally we define the mappings

$$\begin{aligned} X &\mapsto \text{Sub}_{\mathcal{E}}(X) \\ f: X \rightarrow Y &\mapsto f^*: \text{Sub}_{\mathcal{E}}(Y) \rightarrow \text{Sub}_{\mathcal{E}}(X), \end{aligned}$$

where f^* maps a subobject $m: Y' \rightarrowtail Y$ to $m': X \times_Y Y' \rightarrowtail X$, i.e. its pullback along f in [Diagram 3.2](#). ⌋

$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{m'} & X \\ \downarrow & \lrcorner & \downarrow f \\ Y' & \xrightarrow{m} & Y \end{array}$$

Diagram 3.2

If $\mathbb{C}$ is well-powered, i.e. there is only a set of equivalence classes of subobjects, then $\text{Sub}_{\mathcal{E}}$ a hyperdoctrine [[Pit00](#), Ex. 5.2.5]. In most cases we will see, $\mathbb{C}$ is well-powered, so we can assume this. If our topos comes from a tripos, i.e. is of the form $\mathbb{C}[\mathbf{P}]$, we can also define the functor of strict predicates:

Definition 3.13 (Strict predicate functor). The *functor of strict predicates* $\text{SPred}_{\mathbb{C}[\mathbf{P}]}: \mathbb{C}[\mathbf{P}]^{\text{op}} \rightarrow \mathbf{Preord}$ sends an object (X, E) to the preorder of its strict predicates by setting

$$\text{SPred}_{\mathbb{C}[\mathbf{P}]}(X, E) := \{\Pi \in \mathbf{P}(X) : \Pi \text{ is strict and relational for } E\},$$

where we say

$$\begin{aligned} \mathbf{P} \models \forall x. (\Pi(x) \rightarrow E(x, x)) & \quad \Pi \text{ is } \textit{strict}, \\ \mathbf{P} \models \forall x, y. (\Pi(x) \wedge E(x, y) \rightarrow \Pi(y)) & \quad \Pi \text{ is } \textit{relational}. \end{aligned}$$

The order on the strict predicates is inherited from $\mathbf{P}(X)$. On morphisms, $\text{SPred}_{\mathbb{C}[\mathbf{P}]}$ acts as

$$f: (X, E) \rightarrow (Y, F) \mapsto f^*: \text{SPred}_{\mathbb{C}[\mathbf{P}]}(Y, F) \rightarrow \text{SPred}_{\mathbb{C}[\mathbf{P}]}(X, E),$$

where $[f^*\Pi(x)] := [\exists y. (f(x, y) \wedge \Pi(y))]$. ⌋

Our goal is to show that these Sub and SPred are pseudo-naturally isomorphic. We will start by

proving that $\mathbf{SPred}_{\mathbb{C}[\mathbf{P}]}$ is indeed a functor.

Proposition 3.14. $\mathbf{SPred}_{\mathbb{C}[\mathbf{P}]}$ is a functor.

Proof. We easily see that f^* is well-defined since f is a functional predicate: Since f is strict and relational, so must be $f^*\Pi$. Thus we just need to check the identity and composition laws. We know that the identity on (X, E) is represented by E . Since Π is relational and strict, we get

$$[\exists x'.(E(x, x') \wedge \Pi(x'))] \leq [\exists x'.\Pi(x)] \leq [\Pi(x)] \quad \text{and} \quad [\Pi(x)] \leq [\Pi(x) \wedge E(x, x)] \leq [\exists x'.(E(x, x') \wedge \Pi(x'))].$$

This shows that E^* is the identity. For $f: (X, E) \rightarrow (Y, F)$ and $g: (Y, F) \rightarrow (Z, G)$, we have

$$\begin{aligned} [f^*g^*\Pi(x)] &= [\exists y.(f(x, y) \wedge \exists z.(g(y, z) \wedge \Pi(z)))] \simeq [\exists y, z.(f(x, y) \wedge g(y, z) \wedge \Pi(z))] \\ &\simeq [\exists z.(\exists y.(f(x, y) \wedge g(y, z)) \wedge \Pi(z))] \\ &\simeq [\exists z.((g \circ f)(x, z) \wedge \Pi(z))] \\ &= [(gf)^*\Pi(x)]. \end{aligned}$$

This shows that composition is preserved and $\mathbf{SPred}_{\mathbb{C}[\mathbf{P}]}$ is thus a functor. $\square$

Now let (X, E) be an arbitrary element of $\mathbb{C}[\mathbf{P}]$. We define the maps $\sigma_{(X, E)}: \mathbf{SPred}_{\mathbb{C}[\mathbf{P}]}(X, E) \rightarrow \mathbf{Sub}_{\mathbb{C}[\mathbf{P}]}(X, E)$ and $\sigma_{(X, E)}^{-1}: \mathbf{Sub}_{\mathbb{C}[\mathbf{P}]}(X, E) \rightarrow \mathbf{SPred}_{\mathbb{C}[\mathbf{P}]}(X, E)$ by

$$\begin{aligned} \sigma_{(X, E)}(\Pi) &:= (X, [\Pi(x_1) \wedge E(x_1, x_2)]) \quad \text{and} \\ \sigma_{(X, E)}^{-1}(X', E') &:= [\exists x'.i(x', x)], \end{aligned}$$

where $i: (X', E') \rightarrow (X, E)$ is the inclusion. It is easy to see that $\sigma_{(X, E)}$ is well-defined: Since E is an equality predicate and Π is relational, $[\Pi(x_1) \wedge E(x_1, x_2)]$ is an equality predicate as well. Furthermore, $(X, [\Pi(x_1) \wedge E(x_1, x_2)])$ is clearly a subobject of (X, E) with the inclusion being represented by $[\Pi(x_1) \wedge E(x_1, x_2)]$. Since i is a functional predicate, it is strict and relational so that $[\exists x'.i(x', x)]$ is as well. Thus both maps are indeed well-defined.

To show that they are maps in **Preord**, we need to check that they preserve the order. If $\Pi \leq \Pi'$ in $\mathbf{P}(X)$, we need to show that $i: (X, [\Pi(x_1) \wedge E(x_1, x_2)]) \rightarrow (X, E)$ is less than $i': (X, [\Pi'(x_1) \wedge E(x_1, x_2)]) \rightarrow (X, E)$ as subobjects. This means that we have to find a monomorphism $j: (X, [\Pi(x_1) \wedge E(x_1, x_2)]) \rightarrow (X, [\Pi'(x_1) \wedge E(x_1, x_2)])$, such that $i = i'j$. Recall that $i = [\Pi(x_1) \wedge E(x_1, x_2)]$ and $i' = [\Pi'(x_1) \wedge E(x_1, x_2)]$. Then it is easy to see that $j = i$ represents this morphism. It is mono because it is mono into (X, E) and that condition makes no claim about the equality predicate in the codomain. We need to check that $i = i'j$. By relationality of i , we get

$$[(i' \circ j)(x, z)] = [\exists y.(j(x, y) \wedge i'(y, z))] = [\exists y.(i(x, y) \wedge \Pi'(y) \wedge E(y, z))] \leq [\exists y.i(x, z)] \leq [i(x, z)].$$

Since both sides are morphisms, this shows that they must be equal. Thus the order is preserved.

Conversely, if we have subobjects $i: (Y, F) \rightarrow (X, E)$ and $j: (Z, G) \rightarrow (X, E)$, such that (Z, G) is less than (Y, F) , we have a monomorphism $h: (Z, G) \rightarrow (Y, F)$, such that $j = ih$. We now need to show that $[\exists z.j(z, x)] \leq [\exists y.i(y, x)]$. But this is clear since

$$[\exists z.j(z, x)] = [\exists z.(ih)(z, x)] \leq [\exists y.i(y, x)],$$

because by totality of h , we can choose the y such that $h(z, y)$.

Remark 3.15. Now we can explain better what we meant earlier when we said that we can create subobjects in $\mathbb{C}[\mathbb{P}]$ similarly to the axiom of comprehension. The map σ tells us that we can turn strict predicates into subobjects. In particular, this subobject will only contain those elements that satisfy the predicate. This is similar to the axiom of comprehension since there we use a logical formula to obtain the subset of a given set by restricting to those elements that satisfy the formula. $\lrcorner$

Next, we want to show that the suggestive notation is justified and σ and σ^{-1} are each other's inverses. We have

Lemma 3.16. Let $\mathbb{P}$ be a tripos and (X, E) an object of the associated topos. Then we have

$$\sigma_{(X,E)}^{-1} \circ \sigma_{(X,E)} = \text{id}_{\text{SPred}_{\mathbb{C}[\mathbb{P}]}(X,E)} \quad \text{and} \quad \sigma_{(X,E)} \circ \sigma_{(X,E)}^{-1} = \text{id}_{\text{Sub}_{\mathbb{C}[\mathbb{P}]}(X,E)}$$

Proof. If we start with a strict predicate $\Pi \in \text{SPred}_{\mathbb{C}[\mathbb{P}]}(X, E)$, it gets mapped to $(X, [\Pi(x_1) \wedge E(x_1, x_2)])$, whose inclusion into (X, E) is represented by $[\Pi(x_1) \wedge E(x_1, x_2)]$. Thus it then gets mapped to $[\exists x'. (\Pi(x') \wedge E(x', x))]$. Since Π is relational, we get

$$[\exists x'. (\Pi(x') \wedge E(x', x))] \leq [\exists x'. \Pi(x)] \leq [\Pi(x)],$$

since we can disregard quantified variables we do not use. For the other inequality note that by strictness of Π we have

$$[\Pi(x)] \leq [\Pi(x) \wedge E(x, x)] \leq [\exists x'. (\Pi(x') \wedge E(x', x))]$$

since we can choose $x' = x$. Thus $[\exists x'. (\Pi(x') \wedge E(x', x))] \simeq [\Pi(x)]$.

To show $\sigma_{(X,E)} \circ \sigma_{(X,E)}^{-1} = \text{id}_{\text{Sub}_{\mathbb{C}[\mathbb{P}]}(X,E)}$, let $i: (Y, F) \rightarrow (X, E)$ be a subobject of (X, E) . It gets mapped to the predicate $[\exists y. i(y, x)]$, which then gets mapped to $(X, [\exists y. i(y, x_1) \wedge E(x_1, x_2)])$. We want to show that this element is isomorphic to (Y, F) . To do so, we claim that i also represents a morphism $(Y, F) \rightarrow (X, [\exists y. i(y, x_1) \wedge E(x_1, x_2)])$, which is iso. Totality still holds since it makes no claim about the equality predicate in the codomain. For strictness, we only have to show it for $[\exists y. i(y, x) \wedge E(x, x)]$ since i is strict for E by assumption. The second conjunct also follows from strictness of i for E , so we just need to show that $[i(y, x)] \leq [\exists y'. i(y', x)]$, which is clear since we can choose $y' = y$. Relationality follows from $[\exists y. i(y, x_1) \wedge E(x_1, x_2)] \leq [E(x_1, x_2)]$. For single-valuedness, we have to show that $[i(y, x) \wedge i(y, x')] \leq [\exists y'. i(y', x) \wedge E(x, x')]$. The first conjunct is trivial and the second one follows from single-valuedness of i for E . Lastly, it remains to check that it is iso. By [Lemma 3.10](#), we need to check that $[i(x, y) \wedge i(x', y)] \leq [E(x, x')]$ and $[\exists y. i(y, x) \wedge E(x, x)] \leq [\exists y. i(y, x)]$. The second one is trivial and the first one follows by [Lemma 3.9](#) since i is mono. $\square$

Lemma 3.17. Let $\mathbb{P}: \mathbb{C}^{\text{op}} \rightarrow \mathbf{preHeyt}$ be a tripos. Then the maps σ form a pseudo-natural transformation $\text{Sub}_{\mathbb{C}[\mathbb{P}]} \Rightarrow \text{SPred}_{\mathbb{C}[\mathbb{P}]}$ and the σ^{-1} form a pseudo-natural transformation $\text{SPred}_{\mathbb{C}[\mathbb{P}]} \Rightarrow \text{Sub}_{\mathbb{C}[\mathbb{P}]}$.

Proof. Since we have already seen that σ and σ^{-1} are families of maps between the correct objects, it just remains to show that they make the naturality squares commute up to isomorphism. So let $f: (X, E) \rightarrow (Y, F)$ be an arbitrary map. We want to show that the following two diagrams commute:

$$\begin{array}{ccc}
\text{Sub}_{\mathbb{C}[\mathbf{P}]}(X, E) & \xrightarrow{\sigma_{(X, E)}^{-1}} & \text{SPred}_{\mathbb{C}[\mathbf{P}]}(X, E) \\
\text{Sub}_{\mathbb{C}[\mathbf{P}]}(f) \uparrow & & \text{SPred}_{\mathbb{C}[\mathbf{P}]}(f) \uparrow \\
\text{Sub}_{\mathbb{C}[\mathbf{P}]}(Y, F) & \xrightarrow{\sigma_{(Y, F)}^{-1}} & \text{SPred}_{\mathbb{C}[\mathbf{P}]}(Y, F)
\end{array}$$

Diagram 3.3

$$\begin{array}{ccc}
\text{SPred}_{\mathbb{C}[\mathbf{P}]}(X, E) & \xrightarrow{\sigma_{(X, E)}} & \text{Sub}_{\mathbb{C}[\mathbf{P}]}(X, E) \\
\text{SPred}_{\mathbb{C}[\mathbf{P}]}(f) \uparrow & & \text{Sub}_{\mathbb{C}[\mathbf{P}]}(f) \uparrow \\
\text{SPred}_{\mathbb{C}[\mathbf{P}]}(Y, F) & \xrightarrow{\sigma_{(Y, F)}} & \text{Sub}_{\mathbb{C}[\mathbf{P}]}(Y, F)
\end{array}$$

Diagram 3.4

Since we have already shown that σ and σ^{-1} are component-wise isomorphisms, it suffices to check one, so we will only show that [Diagram 3.3](#) commutes. We start with a subobject $i: (Y', F') \multimap (Y, F)$. Via the left upper path, we first go to the pullback $(X, E) \times_{(Y, F)} (Y', F')$ with the inclusion into (X, E) represented by

$$[\iota(x, y, \hat{x})] = [E(x, \hat{x}) \wedge E(x, x) \wedge F(y, y) \wedge \exists \hat{y}. f(x, \hat{y}) \wedge i(y, \hat{y})] \simeq [E(x, \hat{x}) \wedge \exists \hat{y}. (f(x, \hat{y}) \wedge i(y, \hat{y}))].$$

Thus it gets mapped to the predicate

$$[\exists x', y'. \iota(x', y', x)] = [\exists x', y'. (E(x', x) \wedge \exists \hat{y}. (f(x', \hat{y}) \wedge i(y', \hat{y})))].$$

Via the lower right path, it first gets mapped to the predicate $[\exists y'. i(y', y)]$ and then to $[\exists y. (f(x, y) \wedge \exists y'. i(y', y))]$. By strictness and relationality of f and choosing $x' = x$, we see that these two predicates are equivalent and thus equal. Thus [Diagram 3.3](#) commutes and then also [Diagram 3.4](#). $\square$

The previous two lemmata combined give us:

Theorem 3.18. For a tripos $\mathbf{P}: \mathbb{C}^{\text{op}} \rightarrow \mathbf{preHeyt}$, the functors $\text{Sub}_{\mathbb{C}[\mathbf{P}]}$ and $\text{SPred}_{\mathbb{C}[\mathbf{P}]}$ are pseudonaturally isomorphic. $\square$

Since $\text{Sub}_{\mathbb{C}[\mathbf{P}]}$ is a hyperdoctrine if $\mathbb{C}$ is well-powered, this also shows that $\text{SPred}_{\mathbb{C}[\mathbf{P}]}$ is one in that case. Since we can do logic in both, this isomorphism also tells us that they have the same logic: If we look at how we defined the interpretation of the logic in a tripos in [Section 2.3](#), it basically just states that certain diagrams in the associated topos commute or factor in a certain way. Since the functors are isomorphic, we can freely move between them and a formula holds in one iff it holds in the other.

3.5 Power objects

Our next goal is to prove that $\mathbb{C}[\mathbf{P}]$ has power objects. Given an element (X, E) of $\mathbb{C}[\mathbf{P}]$, consider its element of predicates $\pi(X) \in \mathbb{C}$ and the membership predicate $\in_X \in \mathbf{P}(X \times \pi(X))$. We define $\mathcal{P}(X, E) := (\pi(X), \asymp)$, where

$$\begin{aligned}
[R \asymp S] &:= [R \in \mathcal{P}(X, E) \wedge \forall x. (x \in_X R \leftrightarrow x \in_X S)], \\
[R \in \mathcal{P}(X, E)] &:= [\forall x, x'. (x \in_X R \wedge E(x, x') \rightarrow x' \in_X R) \wedge \forall x. (x \in_X R \rightarrow E(x, x))].
\end{aligned}$$

We use “ $R \in \mathcal{P}(X, E)$ ” simply suggestive notation. It is a unary predicate on $\mathcal{P}(X, E)$. It is easy to see that $\asymp$ is an equality predicate. We claim that $\mathcal{P}(X, E)$ acts as the power object on (X, E) . So we need to prove that $\text{Sub}_{\mathbb{C}[\mathbf{P}]}((X, E) \times -) \cong \text{Hom}_{\mathbb{C}[\mathbf{P}]}(-, \mathcal{P}(X, E))$. We already know that $\text{Sub}_{\mathbb{C}[\mathbf{P}]} \cong \text{SPred}_{\mathbb{C}[\mathbf{P}]}$, so we just need to prove $\text{Hom}_{\mathbb{C}[\mathbf{P}]}(-, \mathcal{P}(X, E)) \cong \text{SPred}_{\mathbb{C}[\mathbf{P}]}((X, E) \times -)$. First we define mappings $\sigma_{(Y, F)}: \text{Hom}_{\mathbb{C}[\mathbf{P}]}((Y, F), \mathcal{P}(X, E)) \rightarrow \text{SPred}_{\mathbb{C}[\mathbf{P}]}((X, E) \times (Y, F))$ and $\sigma_{(Y, F)}^{-1}: \text{SPred}_{\mathbb{C}[\mathbf{P}]}((X, E) \times$

$(Y, F)) \rightarrow \text{Hom}_{\mathbb{C}[\mathbf{P}]}((Y, F), \mathcal{P}(X, E))$ by

$$\begin{aligned} [\sigma_{(Y, F)}(f)(x, y)] &:= [\exists R. (f(y, R) \wedge x \in_X R)], \\ [\sigma_{(Y, F)}^{-1}(\Pi)(y, R)] &:= [F(y, y) \wedge \{\Pi\}(y) \asymp R], \end{aligned}$$

where $\{\Pi\}$ is the map $Y \rightarrow \pi(X)$ we get by condition [ii](#)) of [Definition 2.10](#). We have that $(\text{id}_X \times \{\Pi\})^* \in_X \simeq \Pi$, which states $[x \in \{\Pi\}(y) \leftrightarrow \Pi(x, y)]$. It is easy to see that the images of σ and σ^{-1} are predicates of the correct types respectively. It also follows immediately that $\sigma_{(Y, F)}(f)$ is strict and relational for y , while strictness and relationality for x follow from the fact that f represents a morphism $(Y, F) \rightarrow \mathcal{P}(X, E)$, whence for the witness R we have $[R \in \mathcal{P}(X, E)]$.

We see that $\sigma_{(Y, F)}^{-1}(\Pi)$ is strict and single-valued. Relationality follows from the defining property of $\{\Pi\}$ and relationality of Π , so we just need to check that it is total. This means showing that $[F(y, y) \rightarrow \exists R. \{\Pi\}(y) \asymp R]$, which follows if we show $[\{\Pi\}(y) \asymp \{\Pi\}(y)]$. Since the second conjunct of $\asymp$ is clearly true in this case, we just have to check validity of $[\{\Pi\}(y) \in \mathcal{P}(X, E)]$. We show both conjuncts independently. The first one states $[x \in_X \{\Pi\}(y) \wedge E(x, x') \rightarrow x' \in_X \{\Pi\}(y)]$, which follows from the defining property of $\{\Pi\}$ and relationality of Π . Similarly, the second one $[x \in_X \{\Pi\}(y) \rightarrow E(x, x)]$ follows from strictness. This shows that also $\sigma_{(Y, F)}^{-1}$ is well-defined.

Lemma 3.19. Let $\mathbf{P}$ be a tripos and (Y, F) an object of the associated topos. Then we have

$$\sigma_{(Y, F)}^{-1} \circ \sigma_{(Y, F)} = \text{id}_{\text{Hom}_{\mathbb{C}[\mathbf{P}]}(Y, F)} \quad \text{and} \quad \sigma_{(Y, F)} \circ \sigma_{(Y, F)}^{-1} = \text{id}_{\text{SPred}_{\mathbb{C}[\mathbf{P}]}(Y, F)}$$

Proof. If we start with a strict predicate Π , it first gets mapped to $[F(y, y) \wedge \{\Pi\}(y) \asymp R]$ and then to

$$[\exists R. (F(y, y) \wedge \{\Pi\}(y) \asymp R \wedge x \in_X R)] \simeq [F(y, y) \wedge x \in_X \{\Pi\}(y)].$$

We know that $[\{\Pi\}(y) \asymp \{\Pi\}(y)]$ always holds by the previous paragraph. We can transform this term further to get

$$[F(y, y) \wedge x \in_X \{\Pi\}(y)] \simeq [F(y, y) \wedge \Pi(x, y)] \simeq [\Pi(x, y)]$$

by strictness of Π .

To show $\sigma_{(Y, F)}^{-1} \circ \sigma_{(Y, F)} = \text{id}_{\text{Hom}_{\mathbb{C}[\mathbf{P}]}(Y, F)}$, we start with a map $f: (Y, F) \rightarrow \mathcal{P}(X, E)$. It gets mapped to $\Pi_f := [\exists R. (f(y, R) \wedge x \in_X R)]$ and then to $[F(y, y) \wedge \{\Pi_f\}(y) \asymp R] \simeq [F(y, y) \wedge R \asymp \{\Pi_f\}(y)]$. We switched the arguments of $\asymp$ since it is symmetric and it makes the argument later easier. We will show that this is equivalent to f by using [Lemma 3.4](#). We need to show that

$$[f(y, R) \wedge F(y, y) \wedge R' \asymp \{\Pi_f\}(y)] \leq [R \asymp R'].$$

We unfold the left side:

$$[f(y, R) \wedge F(y, y) \wedge R' \asymp \{\Pi_f\}(y)] \leq [f(y, R) \wedge R' \in \mathcal{P}(X, E) \wedge \forall x. (x \in_X R' \leftrightarrow x \in_X \{\Pi_f\}(y))].$$

Substituting $[x \in_X \{\Pi_f\}(y)] \simeq [\Pi_f(x, y)] = [\exists R''. (f(y, R'') \wedge x \in_X R'')]$ into the previous equation, we

get

$$\begin{aligned}
& [f(y, R) \wedge R' \in \mathcal{P}(X, E) \wedge \forall x. (x \in_X R' \leftrightarrow x \in_X \{\Pi_f\}(y))] \\
& \simeq [f(y, R) \wedge R' \in \mathcal{P}(X, E) \wedge \forall x. (x \in_X R' \leftrightarrow \exists R''. (f(y, R'') \wedge x \in_X R''))] \\
& \simeq [f(y, R) \wedge R' \in \mathcal{P}(X, E) \wedge \forall x. (x \in_X R' \leftrightarrow \exists R''. (f(y, R) \wedge f(y, R'') \wedge x \in_X R''))] \\
& \simeq [f(y, R) \wedge R' \in \mathcal{P}(X, E) \wedge \forall x. (x \in_X R' \leftrightarrow \exists R''. (f(y, R) \wedge R \asymp R'' \wedge x \in_X R''))] \\
& \simeq [f(y, R) \wedge R' \in \mathcal{P}(X, E) \wedge \forall x. (x \in_X R' \leftrightarrow (x \in_X R \wedge f(y, R)))] \\
& \simeq [f(y, R) \wedge R' \in \mathcal{P}(X, E) \wedge \forall x. (x \in_X R' \leftrightarrow x \in_X R)] \\
& \simeq [f(y, R) \wedge R' \asymp R] \leq [R' \asymp R].
\end{aligned}$$

This gives us the claim by [Lemma 3.4](#). □

Finally, we want to conclude that σ and σ^{-1} are pseudo-natural isomorphisms so that we have an pseudo-isomorphism of functors between $\mathbf{Sub}_{\mathbb{C}[\mathbf{P}]}((X, E) \times -)$ and $\mathbf{Hom}_{\mathbb{C}[\mathbf{P}]}(-, \mathcal{P}(X, E))$ and not just a 1-to-1 correspondence.

Lemma 3.20. The maps σ and σ^{-1} are pseudo-natural isomorphisms.

Proof. We have already shown that they are well-defined maps and each others inverses. Thus it just remains to show pseudo-naturality. So let (X, E) and $f: (Y, F) \rightarrow (Z, G)$ be arbitrary. For pseudo-naturality of σ , we need to show commutativity of the following diagram:

$$\begin{array}{ccc}
\mathbf{Hom}_{\mathbb{C}[\mathbf{P}]}((Y, F), \mathcal{P}(X, E)) & \xrightarrow{\sigma_{(Y, F)}} & \mathbf{SPred}_{\mathbb{C}[\mathbf{P}]}((X, E) \times (Y, F)) \\
\mathbf{Hom}_{\mathbb{C}[\mathbf{P}]}(-, \mathcal{P}(X, E))(f) \uparrow & & \mathbf{SPred}_{\mathbb{C}[\mathbf{P}]}((X, E) \times -)(f) \uparrow \\
\mathbf{Hom}_{\mathbb{C}[\mathbf{P}]}((Z, G), \mathcal{P}(X, E)) & \xrightarrow{\sigma_{(Z, G)}} & \mathbf{SPred}_{\mathbb{C}[\mathbf{P}]}((X, E) \times (Z, G))
\end{array}$$

Diagram 3.5

We start with a $g \in \mathbf{Hom}_{\mathbb{C}[\mathbf{P}]}((Z, G), \mathcal{P}(X, E))$. Going up and then right, it first gets mapped to $g \circ f$ and then to $[\exists R. ((g \circ f)(y, R) \wedge x \in_X R)]$. First going right and then up, it first gets mapped to $[\exists R. (g(z, R) \wedge x \in_X R)]$ and then to $[\exists z. (f(y, z) \wedge \exists R. (g(z, R) \wedge x \in_X R))]$. Using the definition of composition in the tripes to topos construction, it is not hard to see that this is equivalent to the previous term.

Since we already showed that σ and σ^{-1} are each others inverses, naturality of σ^{-1} follows automatically. □

This lemma in combination with [Theorem 3.18](#) shows:

Theorem 3.21. Let $\mathbf{P}: \mathbb{C}^{\text{op}} \rightarrow \mathbf{preHeyt}$ be a tripes and (X, E) an object in the associated topos. Its power object is given by $\mathcal{P}(X, E) = (\pi(X), \asymp)$ where

$$\begin{aligned}
[R \asymp S] &:= [R \in \mathcal{P}(X, E) \wedge \forall x. (x \in_X R \leftrightarrow x \in_X S)], \\
[R \in \mathcal{P}(X, E)] &:= [\forall x, x'. (x \in_X R \wedge E(x, x') \rightarrow x' \in_X R) \wedge \forall x. (x \in_X R \rightarrow E(x, x))].
\end{aligned}$$
□

3.6 The Subobject Classifier

From [MM94, Sec. IV.1], we know that we can recover the subobject classifier as the power object of the terminal object. In our case, we get from [Theorem 3.21](#) and [Lemma 3.5](#) that $\Omega = \mathcal{P}(1, \top) = (\pi(1), \asymp)$. Although this is sufficient, we want to simplify this. First, we look at the equality predicate $\asymp$. We claim that the predicate $[R \in \mathcal{P}(X, E)]$ is trivial, i.e. always true. Indeed, for the second component this is trivial since $E = \top$ in this case. For the first component, we have to show validity of

$$[\forall x, x'. (x \in_1 R \rightarrow x' \in_1 R)].$$

In the equational style, this can be written as $\forall_{\pi_3}(\pi_{13}^* \in_1 \rightarrow \pi_{23}^* \in_1)$, whose validity is equivalent to the claim $\pi_{13}^* \in_1 \leq \pi_{23}^* \in_1$ by an easy application of the $\forall$ and Heyting adjunction. Notice that $\mathcal{P}(1 \times \pi(1)) \cong \mathcal{P}(\pi(1))$. Thus $\in_1 \in \mathcal{P}(1 \times \pi(1))$ is equivalent to some $\in'_1 \in \mathcal{P}(\pi(1))$. One can check that $\in'_1 \simeq \langle !, \text{id}_{\pi(1)} \rangle^* \in_1$ and $\in_1 \simeq \pi_2^* \in'_1$. Then it is easy to see that

$$\pi_{13}^* \in_1 \simeq \pi_{13}^* \pi_2^* \in'_1 \simeq \pi_3^* \in'_1 \leq \pi_3^* \in'_1 \simeq \pi_{23}^* \pi_2^* \in'_1 \simeq \pi_{23}^* \in_1.$$

Thus $[R \asymp S] = [\forall x. (x \in_1 R \leftrightarrow x \in_1 S)]$ in this case. We will suggestively write $\Leftrightarrow$ for $\asymp$ in this case so that the subobject classifier is $(\pi(1), \Leftrightarrow)$. We will see the reason for this suggestive notation in [Section 5.2](#).

Secondly, we want to see what truth $t: 1 \rightarrow \Omega$ looks like. It is an easy exercise in topos theory that this is the morphism corresponding to the subobject $1 \rightarrowtail 1$, which is given by the identity id_1 . To transform this into the morphism $1 \rightarrowtail \Omega$, we have to go in multiple steps. First we turn $1 \rightarrowtail 1$ into a subobject of 1×1 . Using the result from [Section 3.4](#), we can turn this then into a strict predicate on 1×1 . Finally using the [Section 3.5](#), we can turn it into a morphism $1 \rightarrow \mathcal{P}1 = \Omega$. In our case, the terminal object is the pair $(1, \top)$, whose identity is $\top$. Of course, $\top$ also represents the morphism which makes $(1, \top)$ a subobject of $(1, \top) \times (1, \top)$. By the results from [Section 3.4](#), this corresponds to the strict predicate $[\exists x'. \top]$ on 1×1 , which can be written as $\exists_{\pi_{23}} \top$ with the projection $\pi_{23}: 1 \times 1 \times 1 \rightarrow 1 \times 1$. Since the domain and codomain are both isomorphic to 1, this must be an isomorphism. By a similar argument to [Lemma 2.19](#), this shows that the predicate is logically equivalent to $\top$. In [Section 3.5](#) we saw that this gets turned into the morphism $1 \rightarrow \Omega$ represented by

$$[t(*, R)] = [\top(*, *) \wedge \{\top\}(*, R) \asymp R] \simeq [\forall x. (x \in_1 \{\top\}(*, R) \leftrightarrow x \in_1 R)]$$

Notice that the first term is $[x \in_1 \{\top\}(*, R)] \simeq [\top(x, *)] = \top$. Thus $[t(*, R)] \simeq [\forall x. (\top \leftrightarrow x \in_1 R)] \simeq [\forall x. x \in_1 R]$.

Lastly, we want to see what the classifying morphism looks like in this case. One can easily check that the classifying morphism of a subobject $m: A' \rightarrowtail A$ is given via the isomorphism $\text{Sub}(-) \cong \text{Hom}(-, \Omega)$. With our construction, we have to do this in two steps again: First recall from [Section 3.4](#) that a subobject $i: (X', E') \rightarrowtail (X, E)$ corresponds to the strict predicate $[\Pi(x)] = [\exists x'. i(x', x)]$. The same predicate also corresponds to (X', E') as a subobject of $(X, E) \times (1, \top)$ since there is no condition on the elements of $(1, \top)$. By the results of the [Section 3.5](#), this predicate corresponds to the morphism represented by

$$[\chi_{(X, E)}(x, R)] = [E(x, x) \wedge \{\Pi\}(x, R) \asymp R] \simeq [E(x, x) \wedge \forall x'. (x' \in_1 \{\Pi\}(x, R) \leftrightarrow x' \in_1 R)]$$

$$\simeq [E(x, x) \wedge \forall x'. (\Pi(x', x) \leftrightarrow x' \in_1 R)].$$

The second to last step follows from the discussion above that we can simplify $\asymp$ on $\mathcal{P}(1, \top)$ and the last one by the defining property of $\{\Pi\}$.

3.7 Quotients

Using the pseudo-isomorphism $\mathbf{Sub} \cong \mathbf{SPred}$ from [Section 3.4](#), we want to show that we can identify equivalence relations in the topos sense with internal equivalence relations in terms of predicates. We need a few definitions:

Definition 3.22 (Relation). Let $\mathbb{C}$ be a topos and $X \in \mathbb{C}$. A *relation* R on X is a subobject $i: R \rightarrowtail X \times X$. We say it is

reflexive if the diagonal $X \rightarrow X \times X$ factors through R ,

symmetric if the composition $R \rightarrow X \times X \xrightarrow{\text{tw}} X \times X$ factors through R ,

anti-symmetric if the intersection of R and R^{op} as subobjects of $X \times X$ is the diagonal $X \rightarrow X \times X$, where R^{op} is the subobject consisting of R and the inclusion $i^{\text{op}} := \text{tw} \circ i$, or in other words, [Diagram 3.6](#) a pullback.

This of course requires R to be reflexive, but we will only use this definition for partial orders anyway.

transitive if the map $\langle r_0 s, r_1 t \rangle: R' \rightarrow X \times X$ factors through R , where we assume that we can write $i = \langle r_0, r_1 \rangle$ and s, t are given by [Diagram 3.7](#).

We call a relation an *equivalence relation* if it is reflexive, symmetric and transitive and a *partial order* if it is reflexive, transitive and anti-symmetric. $\lrcorner$

Definition 3.23 (Internal relation). Let $\mathbf{P}$ be a tripos and $(X, \sim)$ an element in its associated topos. An *internal relation* on $(X, \sim)$ is a strict predicate Π on $(X, \sim) \times (X, \sim)$. We say it is

internally reflexive if $\mathbf{P} \models \forall x. (x \sim x \rightarrow \Pi(x, x))$,

internally symmetric if $\mathbf{P} \models \forall x, y. (\Pi(x, y) \rightarrow \Pi(y, x))$,

internally anti-symmetric if $\mathbf{P} \models \forall x, y. (\Pi(x, y) \wedge \Pi(y, x) \rightarrow x \sim y)$,

internally transitive if $\mathbf{P} \models \forall x, y, z. (\Pi(x, y) \wedge \Pi(y, z) \rightarrow \Pi(x, z))$.

We call an internal relation an *internal equivalence relation* if it is internally reflexive, symmetric and transitive and an *internal partial order* if it is internally reflexive, transitive and anti-symmetric. $\lrcorner$

$$\begin{array}{ccc} X & \xrightarrow{\delta} & R \\ \downarrow \delta & \lrcorner & \downarrow i \\ R & \xrightarrow{\text{two}i} & X \times X \end{array}$$

Diagram 3.6

$$\begin{array}{ccc} R' & \xrightarrow{t} & R \\ \downarrow s & \lrcorner & \downarrow r_0 \\ R & \xrightarrow{r_1} & X \end{array}$$

Diagram 3.7

Notice that the definition of internal reflexivity is weaker than the one for reflexivity. We use this one, because this way the internal equivalence relations correspond 1-to-1 to the equivalence relations.

Lemma 3.24. The pseudo-isomorphism $\text{Sub} \cong \text{SPred}$ preserves the properties from [Definition 3.22](#) and [Definition 3.23](#). This means that a relation is respectively reflexive / symmetric / anti-symmetric / transitive iff the corresponding internal relation is respectively internally reflexive / symmetric / anti-symmetric / transitive.

Proof. We will prove this exemplary for reflexivity. Given a reflexive relation $i: R \rightarrow X \times X$, we know that the diagonal factors through it, which means that as a subobject, it is larger than $\delta: X \rightarrow X \times X$. By the pseudo-isomorphism from [Section 3.4](#), this means that $\Pi_\delta \leq \Pi_i$, i.e. $[\exists x'. \delta(x', (x_1, x_2))] \leq [\exists r. i(r, (x_1, x_2))]$. Since $[\delta(a, (b, c))] \simeq [a \sim b \wedge a \sim c]$, we get that

$$[x_1 \sim x_2] \leq [\exists x'. \delta(x', (x_1, x_2))] \leq [\exists r. i(r, (x_1, x_2))],$$

so especially $\delta^* \sim \leq \delta^* \Pi_i$.

For the converse, assume $\delta^* \sim \leq \delta^* \Pi$. The predicates Π and $\sim$ correspond to the subobjects $(X \times X, [\Pi(x_1, x_2) \wedge (x_1, x_2) \approx (x'_1, x'_2)])$ and $(X \times X, [x_1 \sim x_2 \wedge (x_1, x_2) \approx (x'_1, x'_2)])$ respectively. We want to show that the latter is less than the former as subobjects. We see that the inclusion $(X \times X, [x_1 \sim x_2 \wedge (x_1, x_2) \approx (x'_1, x'_2)]) \rightarrow (X \times X, \approx)$ which is represented by $[x_1 \sim x_2 \wedge (x_1, x_2) \approx (x'_1, x'_2)]$ is also a monomorphism into $(X \times X, [\Pi(x_1, x_2) \wedge (x_1, x_2) \approx (x'_1, x'_2)])$ since Π is reflexive. Using [Lemma 3.3](#), it is easy to see that composed with the inclusion $(X \times X, [\Pi(x_1, x_2) \wedge (x_1, x_2) \approx (x'_1, x'_2)]) \rightarrow (X \times X, \approx)$, it is again the inclusion $(X \times X, [x_1 \sim x_2 \wedge (x_1, x_2) \approx (x'_1, x'_2)]) \rightarrow (X \times X, \approx)$. $\square$

Of course, we could also argue that the definitions in [Definition 3.22](#) are given precisely like this so that the corresponding definitions we gave for a tripos hold in the logic in the topos. In that case the statement would follow immediately from [Theorem 3.18](#). But in that case we would first have to prove that the definitions given in terms of the topos correspond to the statements in the internal logic of the topos. Since we are mainly concerned with the tripos logic, we will omit this. We have shown that a relation is an equivalence relation iff the corresponding internal relation is an internal equivalence relation, which gives us the following theorem:

Theorem 3.25. Let $\mathbf{P}$ be a tripos and $(X, \sim)$ an element of the associated topos. There is a 1-to-1 correspondence between the equivalence relations on $(X, \sim)$ and the internal equivalence relations on $(X, \sim)$. $\square$

Now we want to use this correspondence to find an easy representation of the quotient of a relation.

Definition 3.26 (Quotient). Given an equivalence relation R on an object X , the *quotient* X/R of X with respect to R is the coequalizer $q: X \rightarrow X/R$ of $r_0, r_1: R \rightrightarrows X$, where $r_j = p_j \circ i$. $\lrcorner$

We call a quotient *effective*, if R is the pullback of q along itself (i.e. its kernel pair). However, this is always the case in topoi [[Joh77](#), Prop. 1.23], so we will omit this criterion. The following result is already mentioned in [[Pit02](#), Thm. 3.4], so we will just elaborate on the proof.

Theorem 3.27. Let $\mathbf{P}$ be a tripos, $(X, \sim)$ an element of the associated topos and R an equivalence relation on $(X, \sim)$. Then the quotient of $(X, \sim)$ with respect to R is (X, Π) with the quotient map $\Pi: (X, \sim) \rightarrow (X, \Pi)$ where Π is the internal equivalence relation associated to R .

Proof. Let i be the inclusion $(R, \approx') \hookrightarrow (X \times X, \approx)$. We know that $[\Pi(x, x')] \simeq [\exists r. i(r, x, x')]$. Let us collect some properties of Π . Since R is an equivalence relation, Π is an internal equivalence relation, so it is especially an equality predicate on X . Since it is internally reflexive, it is also total as a morphism $(X, \sim) \rightarrow (X, \Pi)$. Strictness, relationality and single-valuedness follow immediately, making Π a morphism $(X, \sim) \rightarrow (X, \Pi)$. It remains to check that it coequalizes r_0 and r_1 and is the initial such.

We will use [Lemma 3.4](#) to conclude that Π equalized r_0 and r_1 . We see

$$\begin{aligned}
& [(\Pi \circ p_0 \circ i)(r, x) \wedge (\Pi \circ p_1 \circ i)(r, x')] \\
& \simeq [\exists x_1, x_2, x'_1, x'_2. (i(r, x_1, x_2) \wedge (\Pi \circ p_0)(x_1, x_2, x) \wedge i(r, x'_1, x'_2) \wedge (\Pi \circ p_1)(x'_1, x'_2, x'))] \\
& \leq [\exists x_1, x_2. (i(r, x_1, x_2) \wedge (\Pi \circ p_0)(x_1, x_2, x) \wedge (\Pi \circ p_1)(x_1, x_2, x'))] \\
& \simeq [\exists x_1, x_2, \bar{x}_1, \bar{x}_2. (i(r, x_1, x_2) \wedge p_0(x_1, x_2, \bar{x}_1) \wedge \Pi(\bar{x}_1, x) \wedge p_1(x_1, x_2, \bar{x}_2) \wedge \Pi(\bar{x}_2, x'))] \\
& \leq [\exists x_1, x_2, \bar{x}_1, \bar{x}_2. (i(r, x_1, x_2) \wedge x_1 \sim \bar{x}_1 \wedge \Pi(\bar{x}_1, x) \wedge x_2 \sim \bar{x}_2 \wedge \Pi(\bar{x}_2, x'))] \\
& \leq [\exists x_1, x_2. (i(r, x_1, x_2) \wedge \Pi(x_1, x) \wedge \Pi(x_2, x'))] \\
& \leq [\exists x_1, x_2. (\Pi(x_1, x_2) \wedge \Pi(x_1, x) \wedge \Pi(x_2, x'))] \leq [\Pi(x, x')].
\end{aligned}$$

Lastly, we need to check that (X, Π) is initial with this property. It suffices to check that every other morphism co-equalizing r_0 and r_1 factors through (X, Π) and that Π is epi. By [Lemma 3.11](#), we see easily that it is epi. For the factoring, we assume there is a $q: (X, \sim) \rightarrow (Y, \sim)$, which also equalizes r_0 and r_1 . We claim that q also represents a morphism $(X, \Pi) \rightarrow (Y, \sim)$, which further factors q . It is clearly strict, single-valued, total and relational on Y . For relationality of q for Π , we need to show that $[q(x, y) \wedge \Pi(x, x')] \leq [q(x', y)]$. First we bound $[\Pi(x, x')]$ from above. We have

$$[\Pi(x, x')] \leq [\Pi(x, x') \wedge x \sim x \wedge x' \sim x'] \simeq [\Pi(x, x') \wedge (x, x') \approx (x, x')] \simeq [(x, x') \approx_{\Pi} (x, x')],$$

where $\approx_{\Pi}$ is the equality predicate on the subobject corresponding to Π . By [Theorem 3.18](#), we have an isomorphism $\phi: (X \times X, \approx_{\Pi}) \rightarrow (R, \approx')$. We also know that $i \circ \phi \simeq \approx_{\Pi} \leq \approx$ since $\approx_{\Pi}$ is also the inclusion. Furthermore, $q \circ r_0$ and $q \circ r_1$ are also maps and thus total. Since q equalizes r_0 and r_1 , their images must be the same. Using all of these facts, we get

$$\begin{aligned}
[(x, x') \approx_{\Pi} (x, x')] & \leq [\exists r. \phi(x, x', r)] \\
& \leq [\exists r, y. (\phi(x, x', r) \wedge (q \circ r_0)(r, y) \wedge (q \circ r_1)(r, y))] \\
& \leq [\exists r, y, x_1, x_2. (\phi(x, x', r) \wedge i(r, x_1, x_2) \wedge (q \circ p_0)(x_1, x_2, y) \wedge (q \circ p_1)(x_1, x_2, y))] \\
& \leq [\exists y, x_1, x_2. ((x, x') \approx_{\Pi} (x_1, x_2) \wedge q(x_1, y) \wedge q(x_2, y))] \\
& \leq [\exists y, x_1, x_2. (x \sim x_1 \wedge x' \sim x_2 \wedge q(x_1, y) \wedge q(x_2, y))] \\
& \leq [\exists y. (q(x, y) \wedge q(x', y))].
\end{aligned}$$

Putting everything together, we get

$$[q(x, y) \wedge \Pi(x, x')] \leq [\exists y'. (q(x, y) \wedge q(x, y') \wedge q(x', y'))] \leq [\exists y'. (y \sim y' \wedge q(x', y'))] \leq [q(x', y)]$$

as desired. After unfolding the definition of $q \circ \Pi$, we see that $q \circ \Pi \leq q$ follows directly from relationality, which shows that they are equal. Thus q factors correctly. $\square$

3.8 From tripos morphisms to topos morphisms

The names for the morphisms between triposes were inspired by those between topoi, so one would expect that cartesian and geometric morphisms between triposes become cartesian and geometric morphisms between topoi respectively.

Definition 3.28 (Morphisms of topoi). Let $\mathcal{E}$ and $\mathcal{F}$ be topoi. A *cartesian morphism* $\mathcal{E} \rightarrow \mathcal{F}$ is a functor $F: \mathcal{E} \rightarrow \mathcal{F}$ that preserves finite limits.

A *geometric morphism* $\mathcal{E} \rightarrow \mathcal{F}$ is a pair of adjoint functors $F^+ \dashv F_+$ with $F^+: \mathcal{F} \rightarrow \mathcal{E}$ and $F_+: \mathcal{E} \rightarrow \mathcal{F}$, such that F^+ preserves finite limits. We call F_+ the *direct image functor* and F^+ the *inverse image functor*. $\lrcorner$

From now on, will focus only on geometric morphisms since those are the ones of most interest (cf. [Chapter 6](#)). So assume we have a geometric morphism of triposes $(\Phi^+, \Phi_+): \mathbf{P} \rightarrow \mathbf{Q}$ and want to construct a geometric morphism between the associated topoi $(\Phi^*, \Phi_*): \mathbb{C}[\mathbf{P}] \rightarrow \mathbb{C}[\mathbf{Q}]$. For the inverse image part, we can simply define it by postcomposition: On objects, we set $\Phi^*(X, \sim) := (X, \Phi_{X \times X}^+(\sim))$. Given a morphism $F: (X, \sim) \rightarrow (Y, \sim)$, one can check that $\Phi_{X \times Y}^+(F)$ represents a morphism $(X, \Phi_{X \times X}^+(\sim)) \rightarrow (Y, \Phi_{Y \times Y}^+(\sim))$.

Unfortunately, the definition is not as easy for the direct image functor. If we were to use the same construction as before on Φ_+ , we would not be able to show totality of the morphisms [[Oos08](#), Sec. 2.5]. Thus we need to look at those objects which are weakly complete:

Definition 3.29 (Weak completeness). Let $\mathbf{P}$ be a tripos. A *partial functional relation* $(X, \sim) \rightarrow (Y, \sim)$ is a predicate $F \in \mathbf{P}(X \times Y)$ that is strict, relational and single-valued, but not necessarily total. We call an element $(X, \sim)$ *weakly complete*, if for any partial functional relation $F: (Y, \sim) \rightarrow (X, \sim)$, there is a $f: Y \rightarrow X$ in the base category $\mathbb{C}$ of $\mathbf{P}$, such that

$$\mathbf{P} \models \exists x. F(y, x) \leftrightarrow F(y, f(y)). \quad \lrcorner$$

The implication $\leftarrow$ is always true, so the definition is equivalent to requiring $\rightarrow$. This definition is the older one of weak completeness [[HJP80](#); [PCS81](#)]. There is a stronger version introduced in [[Oos08](#), Sec. 2.5], but since [Definition 3.29](#) will suffice for us, we will not go into this topic deeply. We just need the following fact:

Proposition 3.30 ([[HJP80](#), Prop. 3.3]). Let $\mathbf{P}$ be any tripos and $(X, \sim)$ be an element of the associated topos. Then the *object of singletons* $(\pi(X), \sim_S)$ is always weakly complete. Its equality predicate is given by

$$[U \sim_S V] := [S(U) \wedge \forall x. (x \in_X U \leftrightarrow x \in_X V)], \quad [S(U)] := [\exists x. (x \in_X U \wedge \forall y. (y \in_X U \leftrightarrow y \sim x))].$$

Furthermore, $(X, \sim)$ and $(\pi(X), \sim_S)$ are isomorphic. $\square$

We can generalize the first part of [Proposition 3.30](#). This proposition will only apply to quite specific elements of $\mathbb{C}[\mathbf{P}]$, but it covers all cases we will see. Besides generalizing [Proposition 3.30](#), it covers power objects ([Section 3.5](#)), the subobject classifier (cf. [Section 3.6](#), [Lemma 3.32](#)) and we will use it for the partial map classifier in [Subsection 6.4.2](#).

Proposition 3.31. Let $\mathbf{P}$ be any tripos and $(\pi(X), \sim)$ any element of the associated topos, such that

$$[U \sim V] = [\forall x. (x \in_X U \leftrightarrow x \in_X V) \wedge \Theta(U)],$$

where Θ is any predicate on $\pi(X)$. Then $(\pi(X), \sim)$ is weakly complete.

Proof. Let $F: (Y, \sim) \rightarrow (\pi(X), \sim)$ be a partial functional relation. Define $[F'(y, x)] := [\exists R. (F(y, R) \wedge x \in_X R)]$. Clearly, $F' \in \mathbf{P}(Y \times X)$. Then by property [ii](#) of [Definition 2.10](#), there is a morphism $f: Y \rightarrow \pi(X)$, such that

$$\mathbf{P} \models F'(y, x) \leftrightarrow x \in_X f(y).$$

We claim that f is the morphism with the property of [Definition 3.29](#). So we need to show $\mathbf{P} \models \exists R. F(y, R) \rightarrow F(y, f(y))$, which clearly follows from $[F(y, R)] \leq [R \sim f(y)]$. Since F is strict and Θ only makes a claim about the first argument, we just need to check $[F(y, R)] \leq [\forall x. (x \in_X R \leftrightarrow x \in_X f(y))]$. By the $\forall$ adjunction, this is equivalent to $[F(y, R)] \leq [x \in_X R \leftrightarrow x \in_X f(y)]$. Since $\leftrightarrow$ is defined to be the conjunction of both implications, the claim follows from $[F(y, R)] \leq [x \in_X R \rightarrow x \in_X f(y)]$ and $[F(y, R)] \leq [x \in_X f(y) \rightarrow x \in_X R]$. For the first one we use the defining property of f :

$$[F(y, R) \wedge x \in_X R] \leq [\exists R. (F(y, R) \wedge x \in_X R)] \simeq [F'(y, x)] \simeq [x \in_X f(y)].$$

The claim follows by the Heyting adjunction. Similarly, for the second one we get

$$[F(y, R) \wedge x \in_X f(y)] \simeq [F(y, R) \wedge F'(y, x)] \leq [\exists R'. (F(y, R) \wedge F(y, R') \wedge x \in_X R')] \leq [x \in_X R].$$

The last step follows from F being single-valued and thus $[\forall x'. (x' \in_X R \leftrightarrow x' \in_X R')]$. By the Heyting adjunction, we get $[F(y, R)] \leq [x \in_X f(y) \rightarrow x \in_X R]$ and thus also the equivalence. $\square$

Since the problem with defining the direct image functor was that the morphisms may not be total, we can make use of [Proposition 3.30](#). We essentially use the same idea as for the inverse image functor, but with $(\pi(X), \sim_S)$ instead of $(X, \sim)$, i.e. we define $\Phi_*: \mathbb{C}[\mathbf{P}] \rightarrow \mathbb{C}[\mathbf{Q}]$ on objects by

$$\Phi_*(X, \sim) = (\pi(X), \Phi_+(\sim_S)).$$

On morphisms, we also use composition with Φ_+ but additionally apply the isomorphism from [Proposition 3.30](#). Then $(\Phi^*, \Phi_*): \mathbb{C}[\mathbf{P}] \rightarrow \mathbb{C}[\mathbf{Q}]$ is a geometric morphism of topoi [[Oos08](#), Thm. 2.5.8].

It is important to note here that we only need the object of singletons, because it is weakly complete. We could also define Φ_* using a case distinction. Then we would use the construction above only on elements that are not weakly complete. For those that are, we can simply set $\Phi_*(X, \sim) = (X, \Phi_+(\sim))$ as for Φ^* . After examining the proof of [[Oos08](#), Thm. 2.5.8], it is not hard to see that with this case distinction we still get the desired geometric morphism of topoi. Of course, we adjust the mapping on morphisms in a similar way using a case distinction and only using the isomorphism where necessary. Unfortunately, this approach needs the law of the excluded middle to check which elements are weakly complete. Nonetheless, we will make use of it where we can since it makes the representation easier. We just have to keep in mind that all results we prove with this case distinction also work without.

This is useful for us, since we will be working a lot with the subobject classifier in [Chapter 6](#) and we have the following Lemma, which is a direct consequence of [Proposition 3.31](#):

Lemma 3.32 ([Joh13, Lem. 1.3]). Given a tripos $\mathbf{P}$, the subobject classifier $(\pi(1), \Leftrightarrow)$ of the associated topos (cf. Section 3.6) is always weakly complete. $\square$

Chapter 4

Assemblies and proto-assemblies

To understand more complex structures, it can be useful to find simpler substructures that describe them sufficiently well, maybe even up to equivalence. This is the idea behind assemblies. They are a class of fairly simple objects in $\mathbb{C}[\mathbf{P}]$, which in a certain sense still generate the entire topos. They can be simplified further to proto-assemblies which are less researched. In this chapter, we will look into what we know about assemblies and what we can extend to proto-assemblies. Our hope was that these would help us with our goal of internalizing arrow algebra morphisms (cf. [Chapter 6](#)), because assemblies look similar to our candidate internal arrow algebra. As we will see in [Section 6.3](#), they did not help that much. We decided to still include our findings in case these results are helpful to others.

For this chapter, let a tripos $\mathbf{P}: \mathbb{C}^{\text{op}} \rightarrow \mathbf{preHeyt}$ be fixed. We will additionally assume that $\mathbb{C}$ has pullbacks because we need them to define composition of morphisms between proto-assemblies.

At the heart of assemblies is the *constant objects functor* $\nabla: \mathbb{C} \rightarrow \mathbb{C}[\mathbf{P}]$. It maps an object $X \in \mathbb{C}$ to the pair $(X, \exists_{\delta} \top)$ and a morphism $f: X \rightarrow Y$ to $\exists_{(\text{id}_X, f)} \top$ [[Oos08](#), Sec. 2.4]. Using this, we can define:

Definition 4.1 (Assembly). An object of $\mathbb{C}[\mathbf{P}]$ is called an *assembly* if it is a sub- ∇ , i.e. a subobject of an object $(X, \exists_{\delta} \top)$. Morphisms between assemblies are just morphisms between them in $\mathbb{C}[\mathbf{P}]$. In other words, the category **Asm** of assemblies is the full category of sub- ∇ 's in $\mathbb{C}[\mathbf{P}]$. $\sqcup$

The category of assemblies is regular, meaning that it is finitely complete and has nice image factorization properties, which it inherits from $\mathbb{C}[\mathbf{P}]$ [[Oos08](#), Sec. 2.4]. However, in general it is not exact, i.e. it does not have quotients. Via the so-called *ex/reg-completion* adding quotients, we can make it exact [[Oos08](#), Sec. 2.4]. This completion **Asm**_{ex/reg} is then equivalent to $\mathbb{C}[\mathbf{P}]$ [[Oos08](#), Cor. 2.4.5], which means we can recover the entire category from assemblies. With proto-assemblies, we can consider objects that are even simpler.

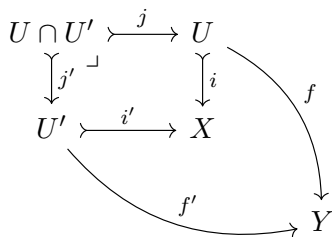


Diagram 4.1

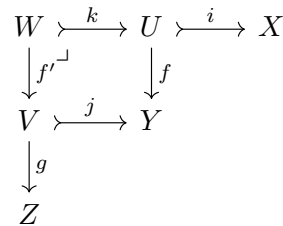


Diagram 4.2

Definition 4.2 (Proto-assemblies). A *proto-assembly* is a pair (X, ϕ) , where X is an object of $\mathbb{C}$ and $\phi \in \mathbf{P}(X)$. A morphism of proto-assemblies $(X, \phi) \rightarrow (Y, \psi)$ is an equivalence class of pairs (U, f) , where $i: U \rightarrow X$ is a subobject and $f: U \rightarrow Y$ is a map, such that the inequality $\phi \leq \exists_i f^* \psi$ holds. Two morphisms (U, f) and (U', f') are equivalent, if f and f' agree on $U \cap U'$ that is if the outer square of [Diagram 4.1](#) also commutes. $\lrcorner$

The identity on a proto-assembly (X, ϕ) is (X, id_X) and composition is given in the following way: Suppose we have morphisms of proto-assemblies $(U, f): (X, \phi) \rightarrow (Y, \psi)$ and $(V, g): (Y, \psi) \rightarrow (Z, \chi)$ with $i: U \rightarrow X$ and $j: V \rightarrow Y$. Then the composition $(V, g) \circ (U, f)$ is given by (W, gf') from the pullback in [Diagram 4.2](#). Proto-assemblies also form a category **pAsm** [[Oos08](#), Sec. 2.1.4.6].

In the next section, we will show that there is a functor turning proto-assemblies into assemblies. We were hoping that we could also define an ex/reg-completion for proto-assemblies, but we were not able to do so. Our attempts did not work and we are not sure whether other attempts could possibly work. We will discuss this further in [Section 4.4](#).

4.1 The evolution functor

To define the evolution functor, we recall from [Lemma 2.23](#) that $\sim_\phi := \exists_\delta \phi$ is an equality predicate. Then we define

Definition 4.3 (Evolution functor). The *evolution functor* $\mathcal{E}v: \mathbf{pAsm} \rightarrow \mathbf{Asm}$ is defined by

$$\begin{aligned} (X, \phi) &\mapsto (X, \sim_\phi), \\ (U, f) &\mapsto \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi), \end{aligned}$$

where $i: U \rightarrow X$ is the inclusion and $(U, f): (X, \phi) \rightarrow (Y, \psi)$. $\lrcorner$

Of course, we can only call it a functor *a posteriori* after checking all the conditions. For simplicity's sake, we will use the name already.

Note that the assignment of (U, f) is equivalent to $\exists_{\langle i, f \rangle} i^* \phi \wedge \exists_{\langle i, f \rangle} f^* \psi$: We see by postcomposition with π_X that [Diagram 4.3](#) is a pullback since i is mono. By tripos reasoning and [Lemma 2.20](#), we get

$$\exists_{\langle i, f \rangle} i^* \phi \wedge \exists_{\langle i, f \rangle} f^* \psi \simeq \exists_{\langle i, f \rangle} (i^* \phi \wedge \langle i, f \rangle^* \exists_{\langle i, f \rangle} f^* \psi) \simeq \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi).$$

$$\begin{array}{ccc} U & \xrightarrow{\text{id}_U} & U \\ \downarrow \text{id}_U \lrcorner & & \downarrow \langle i, f \rangle \\ U & \xrightarrow{\langle i, f \rangle} & X \times Y \end{array}$$

Diagram 4.3

$$\begin{array}{ccc} U \cap U' & \xrightarrow{j} & U \\ \downarrow j' \lrcorner & & \downarrow \langle i, f \rangle \\ U' & \xrightarrow{\langle i', f' \rangle} & X \times Y \end{array}$$

Diagram 4.4

We verify that $\mathcal{E}v$ is a functor by first showing that it is well-defined on objects and then checking the functoriality laws.

Lemma 4.4. The evolution functor is well-defined on objects.

Proof. We need to show that $(X, \sim_\phi)$ is a sub- ∇ and aim to show that $\sim_\phi: (X, \sim_\phi) \rightarrow (X, \exists_\delta \top)$ is

a monomorphism in $\mathbb{C}[\mathbf{P}]$. By definition, it is clear that it is an element of $\mathbf{P}(X \times X)$. Strictness and relationality for $\sim_\phi$ are clear. We already saw that all predicates are relational for $\exists_\delta \top$ in the beginning of [Chapter 3](#). Strictness for $\exists_\delta \top$ as well as single-valuedness follow from $\sim_\phi \leq \exists_\delta \top$ by $\phi \leq \top$. By [Lemma 2.19](#) and [2.20](#), we have

$$\delta^* \exists_\delta \phi \simeq \phi \simeq \exists_{\text{id}_X} \phi \simeq \exists_{\pi_1 \delta} \phi \simeq \exists_{\pi_1} \exists_\delta \phi,$$

which gives us totality. It is mono by [Lemma 3.9](#). Thus we have shown that $(X, \sim_\phi)$ is indeed an assembly and the functor is well-defined on objects. $\square$

For the morphism part, we observe that $\exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi)$ is indeed an element of $\mathbf{P}(X \times Y)$. So we need to show that this assignment is well-defined, i.e. that it is a functional relation and not dependent on the representative.

Lemma 4.5. The mapping of morphisms of the evolution functor gives us functional relations in $\mathbb{C}[\mathbf{P}]$.

Proof. We need to show that the image of (U, f) is indeed a morphism, i.e. a functional relation:

Strict: We will show that $\exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \leq \pi_1^* \delta^* \exists_\delta \phi$. The case with $\exists_\delta \psi$ is similar and together they imply the statement. By [Lemma 2.20](#), we have

$$i^* \phi \wedge f^* \psi \leq i^* \phi \leq (\pi_1 \circ \langle i, f \rangle)^* \phi \simeq \langle i, f \rangle^* \pi_1^* \delta^* \exists_\delta \phi.$$

The statement follows by adjunction of $\exists$.

Single-Valued: We want to show that

$$\pi_{12}^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \wedge \pi_{13}^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \leq \pi_{23}^* \exists_\delta \psi.$$

By tripos reasoning using [Diagram 4.5](#) and [Diagram 4.6](#), we see

$$\begin{aligned} & \pi_{12}^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \wedge \pi_{13}^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \\ & \simeq \exists_{\langle i, f \rangle \times \text{id}_Y} \pi_U^* (i^* \phi \wedge f^* \psi) \wedge \pi_{13}^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \\ & \simeq \exists_{\langle i, f \rangle \times \text{id}_Y} (\pi_U^* (i^* \phi \wedge f^* \psi) \wedge (\langle i, f \rangle \times \text{id}_Y)^* \pi_{13}^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi)) \\ & \simeq \exists_{\langle i, f \rangle \times \text{id}_Y} (\pi_U^* (i^* \phi \wedge f^* \psi) \wedge (i \times \text{id}_Y)^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi)) \\ & \simeq \exists_{\langle i, f \rangle \times \text{id}_Y} (\pi_U^* (i^* \phi \wedge f^* \psi) \wedge \exists_{\langle \text{id}_U, f \rangle} \text{id}_U^* (i^* \phi \wedge f^* \psi)) \\ & \simeq \exists_{\langle i, f \rangle \times \text{id}_Y} \exists_{\langle \text{id}_U, f \rangle} (\langle \text{id}_U, f \rangle^* \pi_U^* (i^* \phi \wedge f^* \psi) \wedge (i^* \phi \wedge f^* \psi)) \\ & \simeq \exists_{\langle i, f, f \rangle} (i^* \phi \wedge f^* \psi). \end{aligned}$$

$$\begin{array}{ccc} U \times Y & \xrightarrow{\langle i, f \rangle \times \text{id}_Y} & X \times Y \times Y \\ \downarrow \pi_U & \lrcorner & \downarrow \pi_{12} \\ U & \xrightarrow{\langle i, f \rangle} & X \times Y \end{array}$$

Diagram 4.5

$$\begin{array}{ccc} U & \xrightarrow{\langle \text{id}_U, f \rangle} & U \times Y \\ \downarrow \text{id}_U & \lrcorner & \downarrow i \times \text{id}_Y \\ U & \xrightarrow{\langle i, f \rangle} & X \times Y \end{array}$$

Diagram 4.6

To see that $\exists_{\langle i, f, f \rangle} (i^* \phi \wedge f^* \psi) \leq \pi_{23}^* \exists_{\delta} \psi$, we use the $\exists$ adjunction and see that

$$\langle i, f, f \rangle^* \pi_{23}^* \exists_{\delta} \psi \simeq \langle f, f \rangle^* \exists_{\delta} \psi \simeq f^* \delta^* \exists_{\delta} \psi \simeq f^* \psi \geq i^* \phi \wedge f^* \psi.$$

Total: By the Frobenius condition and the assumption that (U, f) is a morphism, we get

$$\exists_{\pi_X} \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \simeq \exists_i (i^* \phi \wedge f^* \psi) \simeq \phi \wedge \exists_i f^* \psi \geq \phi \wedge \phi \simeq \phi \simeq \delta^* \exists_{\delta} \phi.$$

Relational: We will prove that $\exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi)$ is relational in two steps by showing that

$$\begin{aligned} \pi_{13}^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \wedge \pi_{12}^* \exists_{\delta} \phi &\leq \pi_{23}^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \text{ and} \\ \pi_{12}^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \wedge \pi_{23}^* \exists_{\delta} \psi &\leq \pi_{13}^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi). \end{aligned}$$

The statement will then follow by applying these identities successively. Indeed, it suffices to show one, since the other one will follow exactly the same way. To show the first one, we use tripos reasoning on [Diagram 4.7](#) to get

$$\begin{aligned} \pi_{13}^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \wedge \pi_{12}^* \exists_{\delta} \phi &\simeq \pi_{13}^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \wedge \exists_{\delta \times \text{id}_Y} \pi_X^* (\phi) \\ &\simeq \exists_{\delta \times \text{id}_Y} ((\delta \times \text{id}_Y)^* \pi_{13}^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \wedge \pi_X^* (\phi)) \\ &\simeq \exists_{\delta \times \text{id}_Y} (\exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \wedge \pi_X^* (\phi)) \\ &\simeq \exists_{\delta \times \text{id}_Y} \exists_{\langle i, f \rangle} ((i^* \phi \wedge f^* \psi) \wedge \langle i, f \rangle^* \pi_X^* (\phi)) \\ &\simeq \exists_{\langle i, i, f \rangle} (i^* \phi \wedge f^* \psi \wedge i^* \phi) \simeq \exists_{\langle i, i, f \rangle} (i^* \phi \wedge f^* \psi) \end{aligned}$$

To see that $\exists_{\langle i, i, f \rangle} (i^* \phi \wedge f^* \psi) \leq \pi_{23}^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi)$, we apply the $\exists$ adjunction and note that the right-hand side becomes

$$\langle i, i, f \rangle^* \pi_{23}^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \simeq \langle i, f \rangle^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi).$$

We get the statement by [Lemma 2.20](#) since $\langle i, f \rangle$ is mono. □

Lemma 4.6. The mapping of morphisms of the evolution functor is independent of the representative.

Proof. Let (U, f) and (U', f') be equivalent morphisms $(X, \phi) \rightarrow (Y, \psi)$. By [Lemma 3.3](#), it suffices to show

$$\exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \leq \exists_{\langle i', f' \rangle} (i'^* \phi \wedge f'^* \psi).$$

We start by noting that [Diagram 4.4](#) is a pull-back since postcomposition with π_X yields the defining pullback of $U \cap U'$, namely [Diagram 4.1](#). Here we use the fact that (U, f) and (U', f') are equivalent, because we need that the diagram commutes, so especially $fj = f'j'$. By tripos reasoning on this and [Diagram 4.1](#), we get

$$\langle i, f \rangle^* \exists_{\langle i', f' \rangle} (i'^* \phi \wedge f'^* \psi) \simeq \exists_{jj'^*} (i'^* \phi \wedge f'^* \psi) \simeq i^* \exists_{i'} (i'^* \phi \wedge f'^* \psi) \simeq i^* \phi \wedge i^* \exists_{i'} f'^* \psi.$$

$$\begin{array}{ccc}
X \times Y & \xrightarrow{\delta \times \text{id}_Y} & X \times X \times Y \\
\downarrow \pi_X \lrcorner & & \downarrow \pi_{12} \\
X & \xrightarrow{\delta} & X \times X
\end{array}$$

Diagram 4.7

$$\begin{array}{ccc}
X \times V & \xrightarrow{\text{id}_X \times \langle j, g \rangle} & X \times Y \times Z \\
\downarrow \pi_V \lrcorner & & \downarrow \pi_{YZ} \\
V & \xrightarrow{\langle j, g \rangle} & Y \times Z
\end{array}$$

Diagram 4.8

Since (U', f') is a morphism, we have $i^* \exists_{i'} f'^* \psi \geq i^* \phi$. Thus

$$\langle i, f \rangle^* \exists_{\langle i', f' \rangle} (i'^* \phi \wedge f'^* \psi) \simeq i^* \phi \wedge i^* \exists_{i'} f'^* \psi \geq i^* \phi \wedge i^* \phi \simeq i^* \phi \geq i^* \phi \wedge f^* \psi.$$

This gives us the desired inequality by the $\exists$ adjunction and the mapping of the morphism is independent of the representative. $\square$

Lastly, we will check the functor laws. The unit law is easy because the identity on (X, ϕ) is simply given by (X, id_X) . It gets mapped to $\exists_{\langle \text{id}_X, \text{id}_X \rangle} (\text{id}_X^* \phi \wedge \text{id}_X^* \phi) \simeq \exists_{\delta} \phi$, which is the identity morphism on $(X, \sim_{\phi})$.

Lemma 4.7. The evolution functor respects the functor laws.

Proof. We just need to check composition. Recall that the composition $(V, g) \circ (U, f)$ is given by (W, gf') from the pullback using [Diagram 4.2](#). Composition of F and G in $\mathbb{C}[\mathbf{P}]$ is given by $[\exists y. F(x, y) \wedge G(y, z)]$. To show that $\mathcal{E}v$ preserves composition, we thus have to show that

$$\exists \pi_{XZ} (\pi_{XY}^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \wedge \pi_{YZ}^* \exists_{\langle j, g \rangle} (j^* \psi \wedge g^* \chi)) \simeq \exists_{\langle ik, gf' \rangle} ((ik)^* \phi \wedge (gf')^* \chi),$$

where again $\leq$ suffices by [Lemma 3.3](#). We apply the $\exists$ adjunction and use tripos reasoning on [Diagram 4.8](#) to get on the left side

$$\begin{aligned}
& \pi_{XY}^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \wedge \pi_{YZ}^* \exists_{\langle j, g \rangle} (j^* \psi \wedge g^* \chi) \\
& \simeq \pi_{XY}^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \wedge \exists_{\text{id}_X \times \langle j, g \rangle} \pi_V^* (j^* \psi \wedge g^* \chi) \\
& \simeq \exists_{\text{id}_X \times \langle j, g \rangle} ((\text{id}_X \times \langle j, g \rangle)^* \pi_{XY}^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \wedge \pi_V^* (j^* \psi \wedge g^* \chi)) \\
& \simeq \exists_{\text{id}_X \times \langle j, g \rangle} ((\text{id}_X \times j)^* \exists_{\langle i, f \rangle} (i^* \phi \wedge f^* \psi) \wedge \pi_V^* (j^* \psi \wedge g^* \chi)) \\
& \stackrel{(*)}{\simeq} \exists_{\text{id}_X \times \langle j, g \rangle} (\exists_{\langle ik, f' \rangle} k^* (i^* \phi \wedge f^* \psi) \wedge \pi_V^* (j^* \psi \wedge g^* \chi)) \\
& \simeq \exists_{\text{id}_X \times \langle j, g \rangle} \exists_{\langle ik, f' \rangle} (k^* (i^* \phi \wedge f^* \psi) \wedge \langle ik, f' \rangle^* \pi_V^* (j^* \psi \wedge g^* \chi)) \\
& \simeq \exists_{\langle ik, jf', gf' \rangle} ((ik)^* \phi \wedge (fk)^* \psi \wedge f'^* (j^* \psi \wedge g^* \chi)) \\
& \simeq \exists_{\langle ik, jf', gf' \rangle} ((ik)^* \phi \wedge (jf')^* \psi \wedge (gf')^* \chi)
\end{aligned}$$

In $(*)$, we used the Beck-Chevalley condition on [Diagram 4.9](#). Thus we need to show

$$\exists_{\langle ik, jf', gf' \rangle} ((ik)^* \phi \wedge (jf')^* \psi \wedge (gf')^* \chi) \leq \pi_{XZ}^* \exists_{\langle ik, gf' \rangle} ((ik)^* \phi \wedge (gf')^* \chi).$$

We can apply the $\exists$ adjunction again. Note that [Diagram 4.10](#) is also a pullback by postcomposition with π_X using the fact that ik is mono. Using the Beck-Chevalley condition on it, the right side

becomes

$$\begin{aligned}
\langle ik, jf', gf' \rangle^* \pi_{XZ}^* \exists_{\langle ik, gf' \rangle} ((ik)^* \phi \wedge (gf')^* \chi) &\simeq \langle ik, gf' \rangle^* \exists_{\langle ik, gf' \rangle} ((ik)^* \phi \wedge (gf')^* \chi) \\
&\simeq \exists_{\text{id}_W} \text{id}_W^* ((ik)^* \phi \wedge (gf')^* \chi) \\
&\simeq (ik)^* \phi \wedge (gf')^* \chi.
\end{aligned}$$

This is clearly greater than $(ik)^* \phi \wedge (jf')^* \psi \wedge (gf')^* \chi$, which gives us the claim. $\square$

$$\begin{array}{ccc}
W & \xrightarrow{\langle ik, f' \rangle} & X \times V \\
\downarrow k & \lrcorner & \downarrow \text{id}_X \times j \\
U & \xrightarrow{\langle i, f \rangle} & X \times Y
\end{array}$$

Diagram 4.9

$$\begin{array}{ccc}
W & \xrightarrow{\text{id}_W} & W \\
\downarrow \text{id}_W & \lrcorner & \downarrow \langle ik, gf' \rangle \\
W & \xrightarrow{\langle ik, gf' \rangle} & X \times Z
\end{array}$$

Diagram 4.10

Proposition 4.8. The evolution functor is indeed a functor. $\square$

4.2 Finite products of proto-assemblies

We want to show that the category of proto-assemblies has finite products by showing that it has a terminal object and binary products. For the terminal object, consider $(1, \top)$, where 1 is terminal in $\mathbb{C}$. For any other proto-assembly (X, ϕ) consider the morphism $(X, !_X)$ where $!_X: X \rightarrow 1$ in $\mathbb{C}$. We clearly have $\phi \leq \top \simeq \exists_{\text{id}_X} !_X^* \top$, which shows that it is a morphism. Thus there is always a morphism $(X, \phi) \rightarrow (1, \top)$. For uniqueness, assume we have another morphism $(U, f): (X, \phi) \rightarrow (1, \top)$. Since f must be a morphism $U \rightarrow 1$, we must have $f = !_U$ by terminality of 1. Now it is clear that [Diagram 4.11](#) is a pullback. The outer diagram also commutes by terminality of 1, whence (U, f) and $(1, !_X)$ are equivalent.

$$\begin{array}{ccc}
U & \xrightarrow{i} & X \\
\downarrow \text{id}_U & \lrcorner & \downarrow \text{id}_X \\
U & \xrightarrow{i} & X \\
& \searrow \text{id}_U & \downarrow !_X \\
& & 1
\end{array}$$

Diagram 4.11

$$\begin{array}{ccccc}
W = U \cap V & \xrightarrow{j_U} & U & \xrightarrow{f} & X \\
\downarrow j_V & \lrcorner & \downarrow \iota_U & & \downarrow \pi_X \\
V & \xrightarrow{\iota_V} & Z & \xrightarrow{h} & X \\
& \searrow g & \downarrow \pi_Y & \swarrow \pi_X & \\
& & Y & \xleftarrow{\pi_Y} & X \times Y
\end{array}$$

Diagram 4.12

Proposition 4.9. The category **pAsm** has all finite products.

Proof. It just remains to check binary products. Let (X, ϕ) and (Y, ψ) be proto-assemblies. We define the product to be

$$(X \times Y, \pi_X^* \phi \wedge \pi_Y^* \psi)$$

with projections $(X \times Y, \pi_X)$ and $(X \times Y, \pi_Y)$. It is easy to see that this is also a proto-assembly, so we check that the morphisms are valid. They are clearly of the right type and we just have to check the additional condition, namely that

$$\pi_X^* \phi \wedge \pi_Y^* \psi \leq \exists_{\text{id}_{X \times Y}} \pi_X^* \phi$$

and similarly for ψ . But this is clear by the definition of the meet since $\exists_{\text{id}_{X \times Y}} \pi_X^* \phi \simeq \pi_X^* \phi$. To confirm that this is indeed the product, we assume that we have another proto-assembly (Z, χ) and morphisms $(U, f): (Z, \chi) \rightarrow (X, \phi)$ and $(V, g): (Z, \chi) \rightarrow (Y, \psi)$. We define an morphism $(W, h): (Z, \chi) \rightarrow (X \times Y, \pi_X^* \phi \wedge \pi_Y^* \psi)$ by first setting $W := U \cap V$. This is a subobject of Z via $\iota := \iota_U j_U = \iota_V j_V$ (cf. [Diagram 4.12](#)). We define h to be the unique morphism we get from the universal property of the product in [Diagram 4.12](#), i.e. $h = \langle f j_U, g j_V \rangle$. To check that it is a morphism we need to confirm that $\chi \leq \exists_{\iota} h^* (\pi_X^* \phi \wedge \pi_Y^* \psi)$. We note that

$$\begin{aligned} \exists_{\iota} h^* (\pi_X^* \phi \wedge \pi_Y^* \psi) &\simeq \exists_{\iota_U} \exists_{j_U} ((\pi_X h)^* \phi \wedge (\pi_Y h)^* \psi) \stackrel{(1)}{\simeq} \exists_{\iota_U} \exists_{j_U} (j_U^* f^* \phi \wedge j_V^* g^* \psi) \stackrel{(2)}{\simeq} \exists_{\iota_U} (f^* \phi \wedge \exists_{j_U} j_V^* g^* \psi) \\ &\stackrel{(3)}{\simeq} \exists_{\iota_U} (f^* \phi \wedge \iota_U^* \exists_{\iota_V} g^* \psi) \\ &\stackrel{(2)}{\simeq} (\exists_{\iota_U} f^* \phi) \wedge (\exists_{\iota_V} g^* \psi), \end{aligned}$$

where we have used the defining property of h in (1), the Frobenius condition in (2) and the Beck-Chevalley condition on [Diagram 4.12](#) in (3). This is clearly greater than χ by assumption that (U, f) and (V, g) are morphisms so that χ is less than both conjuncts. To conclude that we have indeed found the product, we must show that (W, h) makes the triangles in [Diagram 4.13](#) commute and that it is unique with this property.

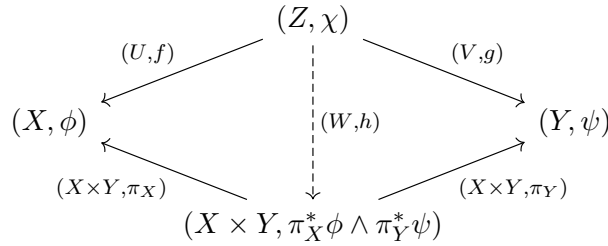


Diagram 4.13

Note that [Diagram 4.14](#) is a pullback. This means that the composition $(X \times Y, \pi_X) \circ (W, h)$ is $(W, \pi_X h)$. We want to show that this is equivalent to (U, f) . We note that [Diagram 4.15](#) is a pullback. To show that $(W, \pi_X h)$ and (U, f) are equivalent, we want to show that $\pi_X h$ and f agree on $W \cap U \cong W$, i.e. that

$$f j_U = \pi_X h \text{id}_W,$$

which is true by the defining property of h . The composition with (V, g) is analogous.

$$\begin{array}{ccc} W & \xrightarrow{\text{id}_W} & W \\ \downarrow h & \lrcorner & \downarrow h \\ X \times Y & \xrightarrow{\text{id}_{X \times Y}} & X \times Y \end{array}$$

Diagram 4.14

$$\begin{array}{ccc} W & \xrightarrow{\text{id}_W} & W \\ \downarrow j_U & \lrcorner & \downarrow \iota \\ U & \xrightarrow{\iota_U} & Z \end{array}$$

Diagram 4.15

It remains to show uniqueness of (W, h) . So we assume that we have another morphism (W', h') that makes the triangles in [Diagram 4.13](#) commute. We want to show that this is equivalent to (W, h) . This amounts to showing that the outer part of [Diagram 4.16](#) commutes, i.e. that $h k = h' k'$. It already suffices to show that the postcompositions with π_X and π_Y are equal, because then by the universal property of the product, there has to be a unique arrow $W \cap W' \rightarrow X \times Y$ making everything commute,

but we have shown that both hk and $h'k'$ make everything commute and thus have to be equal. We will show it for π_X , because the other case is symmetric. Consider [Diagram 4.17](#): by assumption that (W', h') makes the product diagram commute and since the pullback in [Diagram 4.14](#) is trivial, the lower outer diagram commutes, i.e. $\pi_X h' \ell_{W'} = f \ell_U$. Further, we have

$$\iota' k' = \iota k = \iota_U j_U k,$$

where the first equality holds by [Diagram 4.16](#) and the second one by definition of ι . Thus by the universal property of the pullback, we get the indicated arrow $\alpha: W \cap W' \rightarrow W' \cap U$ making the triangles commute. By diagram chasing, this gives us

$$\pi_X h' k' = \pi_X h' \ell_{W'} \alpha = f \ell_U \alpha = f j_U k = \pi_X h k,$$

where the last equality holds because (W, h) makes the product diagram commute. Since we can do the same argument with postcomposition with π_Y , we get that there is a unique arrow $W \cap W' \rightarrow X \times Y$ making everything commute. Since both hk and $h'k'$ make everything commute, we must have $hk = h'k'$ so that (W, h) and (W', h') are equivalent. This finishes the proof that our construction gives indeed the product. $\square$

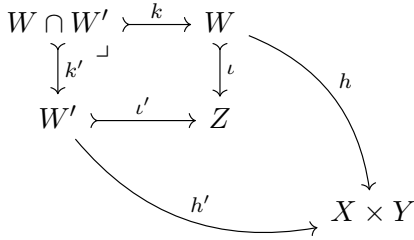


Diagram 4.16

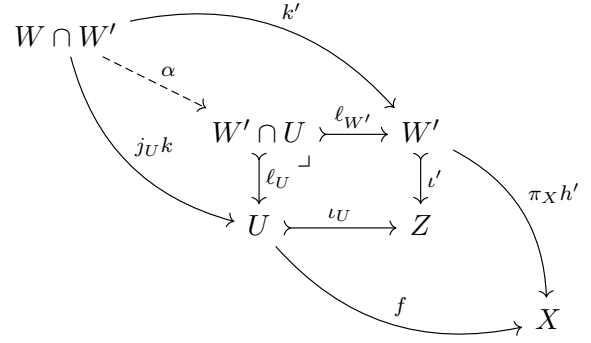


Diagram 4.17

4.3 Preservation of products by the evolution functor

From the previous section we know what finite products of proto-assemblies look like and from [Section 3.2](#), we know what they look like for elements in $\mathbb{C}[\mathbf{P}]$, so especially for assemblies. The terminal proto-assembly is $(1, \top)$ and gets mapped to $\mathcal{E}v(1, \top) = (1, \exists_\delta \top) = \nabla 1$. Since 1 is terminal, the diagonal $\delta: 1 \rightarrow 1 \times 1$ is an isomorphism so that $\exists_\delta \top \simeq \top$ by an analogous argument to [Lemma 2.19](#). Thus $\mathcal{E}v(1, \top) = (1, \top)$, which is the terminal object in $\mathbb{C}[\mathbf{P}]$.

Given some other proto-assembly (X, ϕ) , its unique morphism to $(1, \top)$ is given by $(X, !_X)$. This gets mapped to the morphism

$$\exists_{\langle \text{id}_X, !_X \rangle} (\text{id}_X^* \phi \wedge !_X^* \top) \simeq \exists_{\langle \text{id}_X, !_X \rangle} \phi \simeq \pi_X^* \phi.$$

The last step is analogous to [Lemma 2.19](#) since $\langle \text{id}_X, !_X \rangle: X \rightarrow X \times 1$ is an isomorphism. We saw in [Lemma 3.5](#) that in $\mathbb{C}[\mathbf{P}]$, the unique morphism to the terminal object is given by $\pi_X^* \delta^* \sim$, which in this case is $\pi_X^* \delta^* \exists_\delta \phi \simeq \pi_X^* \phi$, which is the same as the image of $(X, !_X)$ under $\mathcal{E}v$. Thus $\mathcal{E}v$ preserves the

terminal object.

Proposition 4.10. The evolution functor preserves all finite products.

Proof. We just need to check preservation of binary products. We know from [Lemma 3.6](#) that the product of two assemblies $(X, \sim)$ and $(Y, \sim')$ is given by $(X \times Y, \approx)$ where

$$\approx = \pi_{13}^* \sim \wedge \pi_{24}^* \sim'.$$

We aim to show that the evolution functor preserves products. To this end, we note that it does not modify the first component of the pair, so we only need to show that

$$\exists_\delta(\pi_X^* \phi \wedge \pi_Y^* \psi) \simeq \pi_{13}^* \exists_\delta \phi \wedge \pi_{24}^* \exists_\delta \psi.$$

But this follows easily by tripos reasoning on [Diagram 4.18](#) and [Diagram 4.19](#):

$$\begin{aligned} \pi_{13}^* \exists_\delta \phi \wedge \pi_{24}^* \exists_\delta \psi &\simeq \exists_{\pi_{1213}} \pi_1^* \phi \wedge \pi_{24}^* \exists_\delta \psi \\ &\simeq \exists_{\pi_{1213}} (\pi_1^* \phi \wedge \pi_{1213}^* \pi_{24}^* \exists_\delta \psi) \\ &\simeq \exists_{\pi_{1213}} (\pi_1^* \phi \wedge \pi_{23}^* \exists_\delta \psi) \\ &\simeq \exists_{\pi_{1213}} (\pi_1^* \phi \wedge \exists_{\text{id}_X \times \delta} \pi_2^* \psi) \\ &\simeq \exists_{\pi_{1213}} \exists_{\text{id}_X \times \delta} ((\text{id}_X \times \delta)^* \pi_1^* \phi \wedge \pi_2^* \psi) \\ &\simeq \exists_{\pi_{1213} \circ (\text{id}_X \times \delta)} ((\pi_1 \circ (\text{id}_X \times \delta))^* \phi \wedge \pi_2^* \psi) \\ &\simeq \exists_\delta (\pi_1^* \phi \wedge \pi_2^* \psi) \end{aligned}$$

as desired. To check the projections, recall that the projection in the case of proto-assemblies are given by $(X \times Y, \pi_X)$ and $(X \times Y, \pi_Y)$. In the case of assemblies, they are given by $\pi_{13}^* \exists_\delta \phi \wedge \pi_2^* \delta^* \exists_\delta \psi$ and $\pi_1^* \delta^* \exists_\delta \phi \wedge \pi_{23}^* \exists_\delta \psi$. The projection to X gets mapped to

$$\exists_{\langle \text{id}_{X \times Y}, \pi_X \rangle} (\text{id}_{X \times Y}^* (\pi_X^* \phi \wedge \pi_Y^* \psi) \wedge \pi_X^* \phi) \simeq \exists_{\langle \text{id}_{X \times Y}, \pi_X \rangle} (\pi_X^* \phi \wedge \pi_Y^* \psi).$$

To show that it is equivalent to $\pi_{13}^* \exists_\delta \phi \wedge \pi_2^* \delta^* \exists_\delta \psi$, we will use [Lemma 3.3](#) and the $\exists$ adjunction. Then we have to show that

$$\pi_X^* \phi \wedge \pi_Y^* \psi \leq \langle \text{id}_{X \times Y}, \pi_X \rangle^* (\pi_{13}^* \exists_\delta \phi \wedge \pi_2^* \delta^* \exists_\delta \psi).$$

This is clear since we have $\langle \text{id}_{X \times Y}, \pi_X \rangle^* \pi_{13}^* \exists_\delta \phi \simeq \pi_X^* \delta^* \exists_\delta \phi \simeq \pi_X^* \phi$ by [Lemma 2.20](#) and similarly $\langle \text{id}_{X \times Y}, \pi_X \rangle^* \pi_2^* \delta^* \exists_\delta \psi \simeq \pi_Y^* \psi$. It is an analogous argument to see that the projection to Y also is preserved. $\square$

$$\begin{array}{ccc} X \times Y \times Y & \xrightarrow{\pi_{1213}} & (X \times Y) \times (X \times Y) \\ \downarrow \pi_1 & \lrcorner & \downarrow \pi_{13} \\ X & \xrightarrow{\delta} & X \times X \end{array}$$

Diagram 4.18

$$\begin{array}{ccc} X \times Y & \xrightarrow{\text{id}_X \times \delta} & X \times Y \times Y \\ \downarrow \pi_2 & \lrcorner & \downarrow \pi_{23} \\ Y & \xrightarrow{\delta} & Y \times Y \end{array}$$

Diagram 4.19

4.4 An ex/reg-completion for proto-assemblies?

If we could prove an ex/reg-completion for $\mathbf{pAsm}$ and that this is equivalent to $\mathbb{C}[\mathbf{P}]$, we would get an even easier description of this topos. To do that, we would first need to show that $\mathbf{pAsm}$ is regular and in particular that it has all finite limits. By [Section 4.2](#), it only remains to show that it has binary equalizers. Since $\mathbb{C}$ has pullbacks and finite limits, it also has binary equalizers. We think that it may be possible to construct the equalizer of $(U, f), (U', f') : (X, \phi) \rightarrow (Y, \psi)$ by taking the equalizer of the outer square in [Diagram 4.1](#). We get a morphism $e : E \rightarrow U \cap U'$ and can define two morphisms $ije, i'j'e : E \rightarrow X$. Let $p : P \rightarrow E$ be their equalizer. If we define $\iota := ije p = i'j'e p$, then $(P, \iota^*\phi)$ is a proto-assembly. Furthermore, (P, ι) is a morphism $(P, \iota^*\phi) \rightarrow (X, \phi)$. Since the pullbacks for the composition are trivial, (P, ι) equalizes (U, f) and (U', f') . However, we did not manage to prove the universal property of the equalizer for this object, so we are not sure if this would work. When given another morphism $(V, g) : (F, \chi) \rightarrow (X, \phi)$, we can pull back g along ι to obtain a subobject of F with a morphism into P . However, it lacks the inequality morphisms in $\mathbf{pAsm}$ need to satisfy. Furthermore, we are not sure how to show that $\mathbf{pAsm}$ has nice image factorization properties.

One other, more direct attempt to prove the ex/reg-completion would be to show that $\mathbf{pAsm}$ and $\mathbf{Asm}$ are equivalent categories. A clear candidate is the evolution functor. While essential surjectivity is just Proposition 2.4.2 of [\[Oos08\]](#), fullness and faithfulness are not so clear. For fullness the problem is that morphisms of assemblies are functional predicates while morphisms of proto-assemblies require actual morphisms in $\mathbb{C}$ and there is no clear way to obtain them from the former. Of course, for weakly complete objects this is trivial (cf. [Definition 3.29](#)), but not all object of $\mathbb{C}[\mathbf{P}]$ are weakly complete. There is some work done researching when a functional relation arises from a function by the Italian school around Maietti [\[MST26\]](#). They refer to the condition that we can obtain a morphism in $\mathbb{C}$ from a functional predicate in $\mathbb{C}[\mathbf{P}]$ as the *Axiom of Unique Choice* and additionally argue why it does not hold in general. The Axiom of Unique Choice is likely not necessary to conclude fullness, but it seems unlikely that we can conclude fullness without a similar property.

Faithfulness is also problematic, since $\mathcal{E}v$ takes morphisms from $\mathbb{C}$ into $\mathbb{C}[\mathbf{P}]$, where we are considering their images under $\mathbf{P}$. If $\mathcal{E}v(U, f) \simeq \mathcal{E}v(U', f')$, then we can only make statements about them with respect to the internal logic of $\mathbb{C}[\mathbf{P}]$. To conclude faithfulness, we would need to be able to derive a statement about the morphisms in $\mathbb{C}$. We would likely need $\mathbf{P}$ to be faithful or have a similar property, which triposes rarely are. Thus is it unlikely that we can prove $\mathbf{pAsm} \simeq \mathbf{Asm}$; at least not via the evolution functor.

Furthermore, the inclusion of a regular category into its ex/reg-completion is always fully faithful. This means that if an ex/reg-completion for $\mathbf{pAsm}$ exists, it would not factor through $\mathcal{E}v$, because otherwise $\mathcal{E}v$ would be faithful. Thus it is unlikely that an ex/reg-completion exists since $\mathcal{E}v$ is the obvious choice to define a functor $\mathbf{pAsm} \rightarrow \mathbf{Asm}$.

Chapter 5

Arrow algebras

Arrow algebras were introduced by Benno van den Berg and Marcus Briët and extended to a category by Umberto Tarantino [BB26; Tar25]. They generalize locales to give a unified framework to capture localic and realizability triposes. It is well-known that Heyting algebras are sufficient for doing intuitionistic logic. When one wants to move to first-order logic, usually triposes are used. But as it turns out, one does not need full Heyting implication or equivalently the infinite distributivity law to be able to do logic. Indeed, a weaker implication suffices if we make up for it in another way:

Definition 5.1 (Arrow structure). An *arrow structure* is a complete poset $(A, \preceq)$ together with a binary operation $\rightarrow: A \times A \rightarrow A$ that is antitone in the first and monotone in the second argument, i.e. for all $a' \preceq a$, $b \preceq b'$, we have $a \rightarrow b \preceq a' \rightarrow b'$. $\dashv$

As usual, we assume associativity to the right for $\rightarrow$, meaning that $a \rightarrow b \rightarrow c$ should be read as $a \rightarrow (b \rightarrow c)$. Furthermore, we assume that $\rightarrow$ binds stronger than the binary meet $\wedge$ and the infinite meet $\bigwedge$.

We call $\preceq$ the *evidential order* on A . Then $a \preceq b$ means that b contains less evidence than a . This can be understood via realizing: for a to be true, we need to realize all of its evidence. If b contains less evidence than a , then everything realizing a also realizes b . For the trivial elements, we see that we can always realize $\top$, so it must contain the least or no information. If we can realize $\perp$ on the other hand, then we can realize everything. We will later also see the *logical order*, which is closer to the order on a Heyting algebra in the sense that with that order, A becomes a Heyting prealgebra.

Since the implication of (complete) Heyting algebras and locales is anti-/monotone, they also carry an arrow structure. However, this structure alone is not sufficient to be able to do logic. The top-element alone does not suffice anymore to characterize truth. Thus we want to consider more elements of A to be true:

Definition 5.2 (Separator). Let $\mathcal{A} = (A, \preceq, \rightarrow)$ be an arrow structure. A *separator* on $\mathcal{A}$ is a subset $S \subseteq A$, such that the following conditions hold:

- (1) (Upwards closure) If $a \in S$ and $a \preceq b$, then $b \in S$.
- (2) (Closure under modus ponens) If $a, a \rightarrow b \in S$, then also $b \in S$.
- (3) S contains the combinators $\mathbf{k}, \mathbf{s}, \mathbf{a}$, where

$$\mathbf{k} := \bigwedge_{a, b \in A} a \rightarrow b \rightarrow a,$$

$$\begin{aligned} \mathbf{s} &:= \bigwedge_{a,b,c \in A} (a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow (a \rightarrow c), \\ \mathbf{a} &:= \bigwedge_{a \in A, B \subseteq \text{Im}(\rightarrow)} \left(\bigwedge_{b \in B} a \rightarrow b \right) \rightarrow a \rightarrow \left(\bigwedge_{b \in B} b \right). \end{aligned} \quad \lrcorner$$

The separator tell us, which elements we can realize. Notably, S is not a filter, i.e. not closed under $\wedge$. Just because a and b can individually be realized, this does not mean that we can realize them at the same time. Together, they may contain too much evidence. However, S is closed under the logical meet [BB26]. We will introduce the logical order and logical meet properly in the following two sections.

The motivation of the first two requirements is clear: If a can be realized and b contains less evidence than a , then we can also realize b . Similarly, if we can realize a and $a \rightarrow b$, then we can also realize b . The combinators require some more explanation.

The terms under the meet of $\mathbf{k}, \mathbf{s}$ are intuitionistic tautologies, which axiomatize propositional logic [SU06, Thm. 5.1.10]. They enable us to do logic inside an arrow algebra. To understand the motivation for $\mathbf{a}$, we shall first consider a locale L . Given a subset $L' \subseteq L$ and an element $a \in L$, one can easily check that we have

$$\bigwedge_{\ell \in L'} a \rightarrow \ell = a \rightarrow \bigwedge_{\ell \in L'} \ell.$$

Indeed, the inequality $\succcurlyeq$ already follows from monotonicity of $\rightarrow$ in its second argument, which we also required for arrow structures. The other inequality is by the Heyting adjunction equivalent to

$$a \wedge \bigwedge_{\ell \in L'} a \rightarrow \ell \preccurlyeq \bigwedge_{\ell \in L'} \ell.$$

Since we can pull a into the big meet, we see that this is true since we have element-wise $a \wedge (a \rightarrow \ell) \preccurlyeq \ell$. We want to require a similar condition for arrow algebras. Requiring the exact statement gives us exactly implicative algebras [Miq20b, Def. 2.1], which are too restrictive for our cases as we will see later. So we will require it to hold only with respect to $\rightarrow$ instead of $\preccurlyeq$. We will see later that if $a \preccurlyeq b$, we also have $a \rightarrow b \in S$. Furthermore, we will not allow any set L' , but only elements in the image of the implication, hence $B \subseteq \text{Im}(\rightarrow)$. Together, we get

Definition 5.3 (Arrow algebra). We say a quadruple $(A, \preccurlyeq, \rightarrow, S)$ is an *arrow algebra* if $(A, \preccurlyeq, \rightarrow)$ is an arrow structure and S is a separator on it. $\lrcorner$

If we consider a locale as an arrow structure, then since the terms under the meet of $\mathbf{k}$ and $\mathbf{s}$ are intuitionistic tautologies, they are both $\top$. From the argument above, we also see that $\mathbf{a} = \top$. Thus a locale is an arrow algebra with $S = \{\top\}$. One can also easily check that any filter works as the separator. However, we also get more examples. One of the motivating cases to go from implicative algebras to arrow algebras was to cover all partial combinatory algebras. They are also one of the main reason we know that interesting arrow algebras exist:

Example 5.4 ([BB26, Thm. 3.9]). Let $\mathcal{P} = (P, \cdot, \leq, P^\#)$ be a pca and $D\mathcal{P}$ the collection of downwards closed subsets of P ordered by inclusion. We define an implication by

$$X \rightarrow Y := \{z \in P : \forall x \in X. zx \downarrow \text{ and } zx \in Y\}$$

and a separator by $S := \{X \in DP : X \cap P^\# \neq \emptyset\}$. Then $(DP, \subseteq, \rightarrow, S)$ is an arrow algebra. $\square$

Through this example, we get many examples of arrow algebras, including realizability ones. This example is the main motivation for generalizing from implicative algebras to arrow algebras. For the former, this example only works if the application is total, i.e. defined on all elements of $P \times P$ [BB26, Thm. 2.13]. Miquel also proves this for discrete and absolute pcas [Miq20b, Def. 2.30, Fact 2.5(2)]. We will see two more examples of arrow algebras stemming from realizability in Section 5.1.

One of the motivations of defining arrow algebras was to do logic inside them. Indeed, by our choice of the combinators, we can do intuitionistic propositional logic:

Proposition 5.5 ([BB26, Prop. 3.5]). Let $\mathcal{A} = (A, \preceq, \rightarrow, S)$ be an arrow algebra. If ϕ is an implicative propositional formula with free variables $p_1, \dots, p_n$, which is also an intuitionistic tautology, then

$$\bigwedge_{p_1, \dots, p_n \in A} \phi \in S. \quad \square$$

This proposition is very powerful and we shall make frequent use of it. In what follows, we will often have to prove that certain meets are in the separator, which requires a lot of syntactic work. We will not always reference it explicitly, but usually refer to Proposition 5.5 and the following proposition by *intuitionistic reasoning*. The following proposition allows us to perform a modified version of Modus Ponens in an arrow algebra.

Lemma 5.6 ([BB26, Prop. 3.6]). Let $\mathcal{A} = (A, \preceq, \rightarrow, S)$ be an arrow algebra and $(x_i)_{i \in I}, (y_i)_{i \in I}, (z_i)_{i \in I}$ be families of elements in A . If $\bigwedge_{i \in I} x_i \rightarrow y_i \rightarrow z_i \in S$ and $\bigwedge_{i \in I} x_i \in S$, then also $\bigwedge_{i \in I} y_i \rightarrow z_i \in S$. $\square$

5.1 Examples of arrow algebras

Since we can do λ -calculus in pcas, we can especially encode finite lists of elements of $\mathcal{P}$ as an element in $\mathcal{P}$. Given a set $A \subseteq P$, we write $!A$ to denote the collection of (codes of) finite lists of elements in A . This way, we can get more examples stemming from realizability. We will just state the motivation and constructions here, the proofs can be found in [Weh].

Since both sources for these examples assume that the underlying pcas are absolute and discrete, we will do the same. If the pca were not discrete for example, we would not be able to extract elements from lists again. This is because β -reduction of λ -terms only gives us an element that is at least as large, but not necessarily the same [Zoe22, Prop. 2.1.24]. Thus for the remainder of this section, we will assume that all pcas are absolute and discrete.

For the first example, we are looking at the *Herbrand topos* [Ber13]. This is constructed via the *Herbrand tripos*. Given a pca $\mathcal{P} = (P, \cdot)$ we define the set of truth values

$$\Sigma = \{(A_0, A_1) : A_0, A_1 \subseteq \mathcal{P}, A_0 \subseteq !A_1, A_0 \text{ is upwards closed in } !A_1\}.$$

These truth values are *realizers*. To motivate this definition, we recall that the Herbrand topos was introduced as a semantic point of view for Herbrand realizability [BBS12]. In this paper, the authors introduce a quantifier $\exists^{\text{st}}$. To realize $\exists^{\text{st}} n. \varphi(n)$, it suffices to provide a finite list of elements $\langle n_1, \dots, n_m \rangle$, such that for at least one i , we have $\varphi(n_i)$. This is why the first component A_0 consists

of finite lists of elements. We call these the *actual realizers*. The elements of A_1 are the *potential realizers*. This also explains the two conditions: Actual realizers should be lists of potential realizers, hence $A_0 \subseteq !A_1$ and if a finite list contains an element that realizes the statement, then any extension of that list still contains this realizer, hence A_0 should be closed upwards in $!A_1$ [Ber13; BBS12].

The Herbrand tripos then sends a set X to Σ^X with a suitable order. We will not go into the order since it is not important for this discussion. On morphisms, it acts by precomposition. Both components are similar to an arrow tripos (which we will see in the following section), so this motivates to take Σ as the underlying set. Since the Herbrand tripos is a tripos, it also gives a notion of implication where $(C_0, C_1) = (A_0, A_1) \rightarrow (B_0, B_1)$ is given by

$$\begin{aligned} C_1 &:= \{c \in \mathcal{P} : \forall m \in !A_1. cm \downarrow \text{ and } cm \in !B_1\}, \\ C_0 &:= \uparrow_{!C_1} \{\langle c \rangle : c \in C_1 \text{ and } \forall m \in A_0. cm \in B_0\}. \end{aligned}$$

This should remind one of how functions in **Set** work: The elements in C_1 turn finite lists of potential realizers of A into potential realizers of B . By definition, C_0 must be closed upwards under $!C_1$ and each list should contain at least one element that turns actual realizers of A into actual realizers of B . We will also take this notion of implication on our arrow algebra.

Both of these constructions were already given in [Ber13]. To define an arrow algebra, we need to add an order and a separator. With the motivation of realizers, we say that one realizer is less than another if it contains less potential and less actual realizers, i.e. is a subset in both components: $\preceq := \subseteq \times \subseteq$. Similarly, a truth value should be considered true if there is an actual realizer: $S := \{(A_0, A_1) \in \Sigma : A_0 \text{ is inhabited}\}$. With these definitions, we get

Example 5.7 ([Weh, Sec. 2.2, 4.2]). The tuple $(\Sigma, \preceq, \rightarrow, S)$ is an arrow algebra, which we call the *Herbrand arrow algebra*. Furthermore, its arrow tripos coincides identically with the Herbrand topos (and hence its arrow topos with the Herbrand topos). $\dashv$

Here another advantage of arrow algebras becomes apparent: They are easy to define. We did not have to do a lot of creative work to define the Herbrand arrow algebra. The underlying set and the implication were already given to us and we only had to define an order and a separator. Given how the Herbrand topos is defined, these were also obvious first choices. For both of them the first try worked. This is similar in the next example:

In this example, we want to define an arrow algebra motivated by the Diller-Nahm Dialectica tripos [Str]. In this case, given a pca $\mathcal{P} = (P, \cdot)$ we define the set of truth values to be

$$\Sigma_{\text{DN}} := \{(X, Y, R) \in \text{Pow}(\mathcal{P}) \times \text{Pow}(\mathcal{P}) \times \text{Pow}(\mathcal{P} \times \mathcal{P}) : R \subseteq X \times Y\}.$$

For an element $p = (X, Y, R)$, we will also write p^+ for X , p^- for Y and $p(a, b)$ iff $R(a, b)$. For the motivation in this case, we need to go on a slight tangent. The motivating idea behind the Diller-Nahm Dialectica tripos comes from the Diller-Nahm variant of Gödel's Dialectica interpretation. Recall that the Dialectica interpretation is a translation from intuitionistic arithmetic to Σ_2^0 -formulas, i.e. formulas of the form $\exists \bar{x}. \forall \bar{y}. A$, where A is quantifier-free. Then the Diller-Nahm Variant can be understood via game semantics, where first the proponent argues why the statement is true, then the opponent argues why it is false and then an arbiter checks if the argument in favour wins [Str; DN74; Göd58].

Streicher attributes the idea to “define a tripos reflecting as much as possible [of] the original idea of the Diller-Nahm variant of Gödels Dialectica Interpretation” to Hyland, who supposedly suggested this in a talk in the early 1980ies. As his reference for this are “handwritten notes of a talk” [Str], we could not verify this.

Like before, the Diller-Nahm Dialectica tripos sends a set X to Σ_{DN}^X and acts on morphisms by precomposition, motivating to take Σ_{DN} as the underlying set of our arrow algebra. Additionally, we are also given an implication:

$$\begin{aligned}(p \rightarrow q)^+ &:= (p^+ \rightarrow q^+) \times (p^+ \times q^- \rightarrow !p^-) \\ (p \rightarrow q)^- &:= p^+ \times q^- \\ (p \rightarrow q)(\langle e^+; e^- \rangle, \langle a; b \rangle) &:\Leftrightarrow ([\forall c \in e^- \langle a; b \rangle. p(a, c)] \Rightarrow q(e^+ a, b))\end{aligned}$$

Coming from the motivation of game semantics, we understand the functions for the arguments in favour as pre-emptive reactions by the proponent depending on what the opponent plays. This is because in the case of the implication, the order of the moves is changed. Recall that the opponent wants to show that the implication is false, i.e. that the antecedent is true, but the consequent is false. So the opponent starts by choosing an argument in favour of the antecedent in p^+ . Then the proponent chooses an argument in favour of the consequent in q^+ using the function $p^+ \rightarrow q^+$. In the third step the opponent argues why the consequent is false by choosing some q^- . Finally the proponent chooses an argument why the antecedent is false using the function $p^+ \times q^- \rightarrow !p^-$. The finite list of elements in p^- is just needed for technical reasons. The arbiter then says that the proponent won in the following case: if the opponent chose a strong argument in favour of p , meaning that it wins against the proponent's argument against p (keeping in mind that the finite list is just a technicality), then the proponent should be able to play a strong argument in favour of q that wins against the argument of the opponent.

To finish the idea behind the implication, we need to discuss two caveats: First, even though the proponent knows the q^+ played before their second turn, the second function of the proponent does not need this as an argument since it was chosen by them and can be calculated using the first function. More interesting is the case of the second turn of the opponent. They know the p^+ and q^+ that were previously chosen, yet we say their choice of q^- should not depend on them. For the p^+ this makes sense since they chose it. We do not need dependency on q^+ because the two cases are equivalent in terms of the winning condition. If we let $(p \rightarrow q)^{-'} := p^+ \times (q^+ \rightarrow q^-)$, then one can easily check that

$$\forall a, b. ([\forall c \in e^- \langle a; b \rangle. p(a, c)] \Rightarrow q(e^+ a, b)) \iff \forall a, \tau. ([\forall c \in e^- \langle a; \tau e^+ a \rangle. p(a, c)] \Rightarrow q(e^+ a, \tau e^+ a)).$$

The left side says that the proponent chose a winning play using the definition $(p \rightarrow q)^-$ and the second one says the same for $(p \rightarrow q)^{-'}$. The direction from left to right is clear since if the inner statement holds for all b , it must also hold for $\tau e^+ a$. The other direction follows by setting $\tau(-) := b$ for a given b . Thus it does not matter whether the opponent knows the previous state or not, so we choose the representation that is easier.

The underlying set and the implication were again given for free and we just have to define an order and a separator. We set

$$p \preceq q :\Leftrightarrow p^+ \subseteq q^+, p^- \supseteq q^-, \forall a \in p^+, b \in q^-. p(a, b) \Rightarrow q(a, b)$$

because if q contains more arguments in favour, less arguments against and contains the arbiter of p (guarded for technical reasons), then it is a stronger argument. We consider a truth value to be true, if it contains an argument in favour that wins against all counter-arguments:

$$S := \{p \in \Sigma_{\text{DN}} : \exists x \in p^+. \forall y \in p^-. p(x, y)\}.$$

Given the definition of the Diller-Nahm Dialectica tripos, the definition of the order and the separator were again clear. So we again had to do little creative work to get an arrow algebra:

Example 5.8 ([Weh, Sec. 2.3, 4.3]). The tuple $(\Sigma_{\text{DN}}, \preceq, \rightarrow, S)$ is an arrow algebra, which we call the *Diller-Nahm Dialectica arrow algebra*. Its arrow topos coincides exactly with the Diller-Nahm Dialectica tripos. $\dashv$

In [Weh], more properties of these arrow algebras are investigated. We want to highlight one in particular, namely that neither of the Herbrand and Diller-Nahm arrow algebras is an implicative algebra. This gives us further confidence in the generalization from implicative algebras to arrow algebras since we would miss these examples otherwise.

5.2 The arrow tripos and the arrow topos

Arrow algebras were introduced to combine realizability and localic topoi under one framework. The arrow topos $\text{AT}[\mathcal{A}]$ of an arrow algebra $\mathcal{A}$ is actually constructed in two steps: First we construct the arrow tripos $\mathbf{P}_{\mathcal{A}}$ and then make use of the tripos to topos constructions we saw in Chapter 3. We will not go into detail of the proofs and methods used to construct the arrow tripos here for space reasons and simply state the results.

Lemma 5.9 ([BB26, Prop. 5.9]). Let $\mathcal{A} = (A, \preceq, \rightarrow, S)$ be an arrow algebra and $\vdash_S$ the *logical order* on A given by

$$a \vdash_S b \iff a \rightarrow b \in S.$$

Then $(A, \vdash_S)$ is a Heyting prealgebra with $\rightarrow$ as Heyting implication. $\square$

Lemma 5.10. Let $\mathcal{A} = (A, \preceq, \rightarrow, S)$ be an arrow algebra and X an arbitrary set. Then $\mathcal{A}^X := (A^X, \vdash_{S^X}, \rightarrow, S^X)$ is also an arrow algebra where $S^X := \{\varphi \in A^X : \bigwedge_{x \in X} \varphi(x) \in S\}$.

Proof. This is shown in [BB26, Sec. 5.2]. $\square$

If we are working with the logical order on an arrow algebra $\mathcal{A}$ and the separator is clear, we will often just write $\vdash_X$ for $\vdash_{S^X}$.

Now the combination of these two lemmata tells us that for any set X , the pair $(A^X, \vdash_X)$ is a Heyting prealgebra. For future reference, it is nice to explicitly mention that the order is given by

$$\varphi \vdash_X \psi \iff \bigwedge_{x \in X} \varphi(x) \rightarrow \psi(x) \in S.$$

We define the *arrow tripos* $\mathbf{P}_{\mathcal{A}}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{preHeyt}$ to send a set X to this Heyting prealgebra and to act on morphisms by precomposition. This already gives us a functor, but to get a tripos, we also need adjoints and a generic element. The latter is easy as $\mathbf{Set}$ is cartesian closed and we can choose

$\Sigma := A$ and $\sigma := \text{id}_A \in P_{\mathcal{A}}(A) = A^A$. Given a morphism $f: X \rightarrow Y$ in **Set**, we define the left and right adjoints by sending an element $\alpha \in P_{\mathcal{A}}(X)$ to

$$\begin{aligned}\forall_f(\alpha): Y \rightarrow A, y \mapsto \bigwedge_{f(x)=y} \alpha(x), \\ \exists_f(\alpha): Y \rightarrow A, y \mapsto \bigwedge_c \left(\bigwedge_{f(x)=y} \alpha(x) \rightarrow c \right) \rightarrow c.\end{aligned}$$

Theorem 5.11 ([BB26, Thm. 5.10]). For an arrow algebra $\mathcal{A}$, the functor $P_{\mathcal{A}}: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{preHeyt}$ is a tripos, which we call the *arrow tripos*. $\square$

Given an arrow algebra $\mathcal{A}$, we can construct the *arrow topos* $\text{AT}[\mathcal{A}]$ from $P_{\mathcal{A}}$ in the canonical way. One important detail we want to mention now already is the subobject classifier, which will become important in Chapter 6. Following the construction in Section 3.6, we want to calculate it explicitly. We saw there that it is given as $(\pi(1), \asymp)$ with

$$[R \asymp S] = [\forall x. (x \in_1 R \leftrightarrow x \in_1 S)].$$

We have defined the arrow tripos via the generic element and generic predicate and not by the object of predicates and the membership predicate. Luckily, we saw in Lemma 2.11 how we can construct the latter from the former. Since **Set** is cartesian closed, we have that $\pi(1) = A^1$ and $\in_1 = P_{\mathcal{A}}(\text{ev})(\text{id}_A) = \text{id}_A \circ \text{ev}$. Thus we see

$$[R \asymp S] = \bigwedge_{* \in 1} \text{id}_A(\text{ev}(R, *)) \leftrightarrow \text{id}_A(\text{ev}(S, *)) = R(*) \leftrightarrow S(*)$$

since in **Set**, the terminal object 1 is a singleton $\{*\}$. Using the isomorphism $A^1 \cong A$, it is then not hard to see that $(\pi(1), \asymp)$ is isomorphic to $(A, \leftrightarrow)$, the underlying set A of $\mathcal{A}$ with logical equivalence as an equality predicate. Using the same isomorphism, we can also see that the truth morphism in this case becomes $[t(*, a)] \simeq [\top \leftrightarrow a] \simeq [a]$.

5.3 The category of arrow algebras

The theory of arrow algebras was extended by Umberto Tarantino by adding a notion of morphisms between them [Tar25]. With a motivation from partial combinatory algebras, he introduced *implicative* and *computationally dense* morphisms. Since they induce cartesian and geometric morphisms between the respective triposes and then also between the respective topoi, we shall adopt that naming for convenience.

Definition 5.12 (Cartesian morphisms of arrow algebras). A *cartesian morphism* between two arrow algebras $\mathcal{A} = (A, \rightarrow, \preceq, S_{\mathcal{A}})$ and $\mathcal{B} = (B, \rightarrow, \preceq, S_{\mathcal{B}})$ is a function $f: A \rightarrow B$, such that

- (i) for every $a \in S_{\mathcal{A}}$, we have $f(a) \in S_{\mathcal{B}}$,
- (ii) we have $\bigwedge_{a, a' \in A} f(a \rightarrow a') \rightarrow f(a) \rightarrow f(a') \in S_{\mathcal{B}}$,
- (iii) we have $\bigwedge_{a \preceq a'} f(a) \rightarrow f(a') \in S_{\mathcal{B}}$. $\lrcorner$

This definition differs slightly from the one given in [Tar25, Def 3.2]. There, the conditions (ii) and (iii) are modified to

- (ii') there exists an element $r \in S_B$, such that $r \preceq f(a \rightarrow a') \rightarrow f(a) \rightarrow f(a')$ for all $a, a' \in A$,
- (iii') for all $X \subseteq A \times A$,

$$\text{if } \bigwedge_{(a,a') \in X} a \rightarrow a' \in S_A, \text{ then } \bigwedge_{(a,a') \in X} f(a) \rightarrow f(a') \in S_B..$$

This discrepancy is no problem since by the definition of the meet, it is easy to see that (ii) and (ii') are equivalent. Assuming the other two conditions, (iii) and (iii') are also equivalent:

Lemma 5.13. A function $f: A \rightarrow B$ satisfying conditions (i) and (ii), satisfies (iii) iff it satisfies (iii').

Proof. For the if direction, note that since $\mathbf{i} = \bigwedge_{a \in A} a \rightarrow a \in S_A$ by Proposition 5.5, we get

$$\bigwedge_{a \in A} a \rightarrow a \preceq \bigwedge_{a \preceq a'} a \rightarrow a' \in S_A$$

by monotonicity of $\rightarrow$ in its second argument and upwards closure of S_A . Then (iii) is a direct consequence of (iii') by choosing $X = \{(a, a') \in A \times A : a \preceq a'\}$.

For the only if direction, let $X \subseteq A \times A$ be arbitrary, such that

$$\bigwedge_{(a,a') \in X} a \rightarrow a' \in S_A. \tag{1}$$

By (iii) since $\hat{X} := \bigwedge_{(a,a') \in X} a \rightarrow a' \preceq a \rightarrow a'$ for all $(a, a') \in X$, we get by upwards closure of S_B that

$$\bigwedge_{a \preceq a'} f(a) \rightarrow f(a') \preceq \bigwedge_{(a,a') \in X} f(\hat{X}) \rightarrow f(a \rightarrow a') \in S_B. \tag{2}$$

By Proposition 5.5, we have $\mathbf{b} = \bigwedge_{a,b,c \in B} (b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow (a \rightarrow c) \in S_B$. By intuitionistic reasoning on (ii) and (2), we obtain

$$\bigwedge_{(a,a') \in X} f(\hat{X}) \rightarrow (f(a) \rightarrow f(a')) \in S_B.$$

Since $f(\hat{X})$ is independent of $(a, a') \in X$, we have $\bigwedge_{(a,a') \in X} f(\hat{X}) = f(\hat{X}) \in S_B$ by (i) and (1). Thus we can apply Lemma 5.6 to get $\bigwedge_{(a,a') \in X} f(a) \rightarrow f(a') \in S_B$. $\square$

We can define an order on the collection of cartesian morphisms between two arrow algebras by setting $f \vdash g$ if there is a $u \in S_B$, such that for all $a \in A$, we have $u \preceq f(a) \rightarrow g(a)$ or equivalently $\bigwedge_{a \in A} f(a) \rightarrow g(a) \in S_B$. Then we get

Proposition 5.14 ([Tar25, Prop. 3.5]). Arrow algebras, cartesian morphisms and their order form a preorder-enriched category. $\square$

Remark 5.15. To define morphisms between arrow algebras, we have not used the evidential order. Indeed, one can state the definition of an arrow algebra morphism purely in terms of the logical order [Tar25, Lem. 3.6]. $\dashv$

If $f: \mathcal{A} \rightarrow \mathcal{B}$ is a cartesian morphism of arrow algebras, then for any set X , the function $f^X: \mathcal{A}^X \rightarrow \mathcal{B}^X$ given by postcomposition is also a cartesian morphism of arrow algebras [Tar25, Lem. 5.2]. Condition (ii) then tells us that we can pull the implication out of its argument. More precisely, for any elements $\varphi, \psi \in \mathbf{P}_{\mathcal{A}}(X)$, it tells us that

$$f^X(\varphi \rightarrow \psi) \vdash_X f^X(\varphi) \rightarrow f^X(\psi).$$

Similarly, condition (iii') tells us that f^X is monotone with respect to the logical order $\vdash_X$: Given again φ, ψ with $\varphi \vdash_X \psi$, i.e. $\bigwedge_{x \in X} \varphi(x) \rightarrow \psi(x) \in S_{\mathcal{A}}$, then $\bigwedge_{x \in X} f(\varphi(x)) \rightarrow f(\psi(x)) \in S_{\mathcal{B}}$, i.e. $f^X(\varphi) \vdash_X f^X(\psi)$. As usual, monotonicity gives us one inequality for preservation of meets. Here it is important to note that we are of course talking about the logical meet $\wedge$, i.e. the meet with respect to the logical order $\vdash_X$. The other inequality holds as well:

Lemma 5.16 ([Tar25, Lem. 5.3]). Let $f: \mathcal{A} \rightarrow \mathcal{B}$ be a cartesian morphism. Then

$$\bigwedge_{a, b \in A} (f(a) \wedge f(b)) \rightarrow f(a \wedge b) \in S_{\mathcal{B}}. \quad \square$$

Taking again some φ, ψ , this tells us that $\bigwedge_{x \in X} (f(\varphi(x)) \wedge f(\psi(x))) \rightarrow f(\varphi(x) \wedge \psi(x)) \in S_{\mathcal{B}}$, which states

$$f^X(\varphi) \wedge f^X(\psi) \vdash_X f^X(\varphi \wedge \psi).$$

Together with monotonicity, we thus get preservation of meets $f^X(\varphi) \wedge f^X(\psi) \simeq f^X(\varphi \wedge \psi)$.

Definition 5.17 (Geometric morphisms of arrow algebras). Given two arrow algebras $\mathcal{A}$ and $\mathcal{B}$, a *geometric morphism* between them is a pair of adjoint cartesian morphisms $f^+ \dashv f_+$ with $f^+: \mathcal{B} \rightarrow \mathcal{A}$ and $f_+: \mathcal{A} \rightarrow \mathcal{B}$. We give the morphism $(f^+ \dashv f_+): \mathcal{A} \rightarrow \mathcal{B}$ the direction of the right adjoint since we view arrow algebras as generalized spaces. $\lrcorner$

The adjunction $f^+ \dashv f_+$ is of course with respect to the preorder-enrichment of Proposition 5.14. Composition of two geometric morphism $f: \mathcal{A} \rightarrow \mathcal{B}$ and $g: \mathcal{B} \rightarrow \mathcal{C}$ is given as usual by $(f^+ g^+ \dashv g_+ f_+): \mathcal{A} \rightarrow \mathcal{C}$.

Given the category of arrow algebras with cartesian morphisms from Proposition 5.14, the author of [Tar25] defines the wide subcategory of arrow algebras with geometric morphisms. For the purpose of this thesis, when we talk about the category of arrow algebras **ArrAlg**, we will take the morphisms to be geometric.

Since one of the main motivations for defining arrow algebras was them giving rise to tripases, one would expect the morphisms between arrow algebras to lift to morphisms between the arrow tripases. As we already mentioned and the naming suggests, this is indeed the case. The family f^X of morphisms indexed by sets X constitute a cartesian morphism of arrow tripases $\mathbf{P}_{\mathcal{A}} \rightarrow \mathbf{P}_{\mathcal{B}}$ [Tar25, Prop. 5.4]. If $f: \mathcal{A} \rightarrow \mathcal{B}$ has a left adjoint $f^+: \mathcal{B} \rightarrow \mathcal{A}$, so that together they form a geometric morphism of arrow algebras, we can do this on both components. For readability, we will use the usual notation of $f_+: \mathcal{A} \rightarrow \mathcal{B}$ for the right adjoint. Then we define

$$(\Phi_+)_X(\varphi) := f_+ \circ \varphi, \quad (\Phi^+)_X(\varphi) := f^+ \circ \varphi.$$

Since in this case, we started with a geometric morphism of arrow algebras, the pair (Φ^+, Φ_+) is a geometric morphism of arrow triposes $\mathbf{P}_{\mathcal{A}} \rightarrow \mathbf{P}_{\mathcal{B}}$ [Tar25, Thm. 5.10].

Together with the results from Section 3.8, this already tells us how the morphism between two arrow topoi looks like, but it is nice to have an explicit description. Let $(f^+, f_+): \mathcal{A} \rightarrow \mathcal{B}$ be a geometric morphism of arrow algebras and $(\Phi^+, \Phi_+): \mathbf{P}_{\mathcal{A}} \rightarrow \mathbf{P}_{\mathcal{B}}$ be the corresponding geometric morphism between arrow triposes. By the description from Section 3.8, this becomes the geometric morphism between arrow topoi $(\Phi^*, \Phi_*): \mathbf{AT}[\mathcal{A}] \rightarrow \mathbf{AT}[\mathcal{B}]$ given by

$$\begin{aligned} \Phi^*: \mathbf{AT}[\mathcal{B}] &\rightarrow \mathbf{AT}[\mathcal{A}], & (X, \sim) &\mapsto (X, f^+ \circ \sim), \\ F: (X, \sim) &\rightarrow (Y, \sim) &\mapsto f^+ \circ F: (X, f^+ \circ \sim) &\rightarrow (Y, f^+ \circ \sim). \\ \Phi_*: \mathbf{AT}[\mathcal{A}] &\rightarrow \mathbf{AT}[\mathcal{B}], & (X, \sim) &\mapsto (\pi(X), f_+ \circ \sim_S), \\ F: (X, \sim) &\rightarrow (Y, \sim) &\mapsto \overline{F}: (\pi(X), f_+ \circ \sim_S) &\rightarrow (\pi(Y), f_+ \circ \sim_S). \end{aligned}$$

Here $\overline{F}$ is given by

$$\overline{F} := f_+(i_Y \circ F \circ i_X^{-1}).$$

The composition is in the arrow topos as described in Chapter 3 and the maps i_X are the isomorphisms between $(X, \sim)$ and $(\pi(X), \sim_S)$ (cf. Proposition 3.30).

For Φ_* we can also do the case distinction mentioned in Section 3.8, where we use the mapping above only for the elements that are not weakly complete and for the ones that are, we set

$$\Phi_*(X, \sim) := (X, f_+ \circ \sim).$$

In that case, the action on morphisms by Φ_* needs four cases on whether the domain and codomain are weakly complete or not and then applying the necessary isomorphisms or not. We trust the reader to fill in these details and will not mention them for space reasons.

Chapter 6

Internalizing arrow algebra morphisms

We have finally arrived at the heart of this thesis. Initially, we set out to generalize the Joyal-Tierney theorem to arrow algebras. It states that for any locale X , the slice category $\mathbf{Loc}/X$ of locales over X is equivalent to the category of internal locales in the sheaf topos $\mathbf{Sh}(X)$ of X :

Theorem 6.1 ([JT84, Prop. VI.3.2]). For any locale X , we have $\mathbf{Loc}/X \simeq \mathbf{Loc}(\mathbf{Sh}(X))$. $\square$

The slice category is a canonical construction in category theory, whose objects are morphisms with codomain X and morphisms making the following triangle commute:

$$\begin{array}{ccc} Y & \xrightarrow{h} & Z \\ & \searrow f \quad \swarrow g & \\ & X & \end{array}$$

Diagram 6.1

The category of internal locales is more involved, but it should not be surprising with everything we have seen so far. Since $\mathbf{Sh}(X)$ is a topos [Joh77, Sec. 3.2], we can do logic inside it. Thus we can also formulate Definition 2.2 inside a topos. The notation is taken from [Bor94b, Def. 6.1.1]. We define

Definition 6.2 (Internal locale). Let $\mathcal{E}$ be a topos. An *internal locale* in $\mathcal{E}$ is a tuple $(L, \preceq)$, where L is an object of $\mathcal{E}$ and $\preceq$ is a subobject of $L \times L$, such that the following conditions are satisfied. For readability, we will write $a \preceq b$ for $(a, b) \in \preceq$. All lowercase variables are of type L and the uppercase ones are of type Ω^L . We require that with respect to the internal logic, $\preceq$ is a partial order in the sense of Definition 3.22. Furthermore, it should be complete. So we require there to be a map $\bigwedge : \Omega^L \rightarrow L$, which gives us the meet for every subobject, i.e. the largest lower bound:

$$\models \forall b. (b \in S \Rightarrow \bigwedge S \preceq b) \wedge \forall c. (\forall b. (b \in S \Rightarrow c \preceq b) \Rightarrow c \preceq \bigwedge S).$$

Lastly, we require $(L, \preceq)$ to satisfy the infinite distributivity law:

$$\models \forall a, B. (a \wedge \bigvee B = \bigvee \{x : \exists b. (b \in B \wedge x = a \wedge b)\}). \quad \lrcorner$$

Although this definition is formulated inside the logic of a topos, we will not go into how this gets interpreted. Instead, we note that we can reduce the logic of $\mathbb{C}[\mathbf{P}]$ to that of $\mathbf{P}$ [Oos08, Sec. 2.3]. Since

we are only interested in arrow topoi, we will make use of this and just use the translated version of this definition whenever necessary. We refer the interested reader to [Bor94b; Oos08]. In most cases, we will rely on other results as the one described below.

In our attempt to generalize Theorem 6.1, we only got so far as considering the internalization functor $\mathcal{I}: \mathbf{Loc}/X \rightarrow \mathbf{Loc}(\mathbf{Sh}(X))$. To motivate our attempts, we will shortly explain the construction of this functor, which is taken from [Joh02, Sec. 1.6], where the interested reader can also find sources for the claims in the following paragraph.

Given a locale morphism $f: Y \rightarrow X$, we consider the subobject classifier Ω_Y in $\mathbf{Sh}(Y)$, which is an internal complete Heyting algebra. The morphism f lifts to a geometric morphism of the sheaf topoi $f: \mathbf{Sh}(Y) \rightarrow \mathbf{Sh}(X)$ in such a way that the right adjoint f_* preserves internal Heyting structure and complete internal posets. Thus $f_*\Omega_Y$ is an internal complete Heyting algebra in $\mathbf{Sh}(X)$. The internalization $\mathcal{I}f$ of f is then defined to be the internal locale corresponding to this internal complete Heyting algebra (cf. the discussion in Definition 2.2).

In the remainder of this chapter, we will present and discuss different approaches for an internalization functor of arrow algebras. We were able to prove one positive result, but also discuss why this will likely not lead to a full equivalence. We could also not get an internalization from the approach via proto-assemblies and are quite sure that it cannot work. We will still present all of our approaches so that other people know what does and does not work and can continue with the open questions.

6.1 Internalizing arrow algebras morphisms as locales

One way to try to generalize the Joyal-Tierney theorem is to internalize morphisms between arrow algebras as locales. As in any topos, the subobject classifier in an arrow topos is an internal Heyting algebra, i.e. an internal locale [Joh02, Lem. A1.6.3]. Starting from there, we can follow the same construction as Johnstone did for locale morphisms. Indeed, we can get an internalization functor that way. But first we need two auxiliary results:

Lemma 6.3. Let the following be a commuting diagram of geometric arrow algebra morphisms:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{h} & \mathcal{C} \\ & \searrow f & \swarrow g \\ & \mathcal{A} & \end{array}$$

Diagram 6.2

Furthermore, consider the elements $(B, \sim)$ and $(C, \sim)$ in $\mathbf{AT}[\mathcal{A}]$, where $\sim := f_+ \circ \leftrightarrow$ and $\sim := g_+ \circ \leftrightarrow$ respectively with $\leftrightarrow$ being the respective logical equivalence. Then we have

$$[b \sim b'] \leq [h_+(b) \sim h_+(b')] \quad \text{and} \quad [c \sim c'] \leq [h^+(c) \sim h^+(c')].$$

Proof. Since the diagram commutes, we have $f_+ = g_+ \circ h_+$ and we have seen in Section 5.3 that cartesian morphisms preserve the logical meet, so by $\bigwedge_a a \rightarrow a \in S_{\mathcal{A}}$ (cf. Proposition 5.5), we have

$$\bigwedge_{b, b'} f_+(b \leftrightarrow b') \rightarrow g_+(h_+(b \rightarrow b') \wedge h_+(b' \rightarrow b)) \in S_{\mathcal{A}}.$$

Furthermore, by condition (ii) of [Definition 5.12](#), we can always pull the implication out of the argument for a cartesian morphism. So by intuitionistic reasoning, we get

$$\bigwedge_{b,b'} f_+(b \leftrightarrow b') \rightarrow g_+(h_+(b) \leftrightarrow h_+(b')) \in S_{\mathcal{A}},$$

which is exactly the first statement. For the second one, we use that $h^+ \dashv h_+$ and hence $c \leq h_+h^+(c)$ for all $c \in C$. Since $\rightarrow$ is monotone in its second argument, we get

$$\bigwedge_{c,c'} g_+(c \leftrightarrow c') \rightarrow g_+h_+h^+(c \leftrightarrow c') \in S_C.$$

Now using again that h^+ distributes over logical meets and we can pull the implication out, we get

$$\bigwedge_{c,c'} g_+(c \leftrightarrow c') \rightarrow f_+(h^+(c) \leftrightarrow h^+(c')) \in S_{\mathcal{A}}$$

since $g_+h_+ = f_+$. □

The following result follows by combining known results about the subobject classifier and the arrow topos. We make it explicit for further reference.

Lemma 6.4. Let $\mathcal{A}$ be an arrow algebra. The order on the subobject classifier of $\text{AT}[\mathcal{A}]$ is given by the implication. This means that $[a \leq a' \leftrightarrow (a \rightarrow a')]$, where $\leq$ is the order on the subobject classifier.

Proof. Let $\Omega_{\mathcal{A}} = (A, \leftrightarrow)$ be the subobject classifier of $\text{AT}[\mathcal{A}]$ (cf. [Section 5.2](#)). By [[Joh77](#), Lem. 3.31], the order on $\Omega_{\mathcal{A}}$ is given as the equalizer of $\wedge_{\mathcal{A}}, p_1: \Omega_{\mathcal{A}} \times \Omega_{\mathcal{A}} \rightarrow \Omega_{\mathcal{A}}$. So we first need to calculate the meet. By [[Joh02](#), Lem. A1.6.3], the internal meet is constructed as the classifying map of $\langle t, t \rangle: 1 \rightarrow \Omega \times \Omega$. We saw in [Section 5.2](#) that truth in the arrow topos is given by $[t(*, a)] = [a]$, so it is easy to see that $[\langle t, t \rangle(*, a, a')] = [a \wedge a']$. We want to use the description of [Section 3.6](#) to find the classifying morphism. Since 1 is terminal, it is easy to see that $[\Pi(a, a')] = [a \wedge a']$, where Π is the predicate we used in [Section 3.6](#). Then the classifying morphism is given by

$$[\wedge_{\mathcal{A}}(a_1, a_2, a)] = [(a_1, a_2) \approx (a_1, a_2) \wedge \forall x. (x \in_1 \{\Pi\}(a_1, a_2) \leftrightarrow x \in_1 a)].$$

By the same argument we saw when calculating $(A, \leftrightarrow)$ in [Section 5.2](#), namely using the isomorphism $A^1 \cong A$, the part $[\forall x. (x \in_1 \{\Pi\} \leftrightarrow x \in_1 a)]$ can be simplified to $[\{\Pi\}(a_1, a_2) \leftrightarrow a]$. Lastly, we want to simplify the term $\{\Pi\}$. In the arrow tripos, this is constructed as $\{\Pi\} := \widetilde{[\Pi]}$. Since $\Pi \in \mathbf{P}_{\mathcal{A}}(A \times A) = A^{A \times A}$, the map $[\Pi]$ is just Π itself viewed as a map. Its exponential transpose is again itself, just viewed as a map in two stages $A \rightarrow A^A$. Thus the above expression for the meet can be simplified to

$$[\wedge_{\mathcal{A}}(a_1, a_2, a)] = [(a_1, a_2) \approx (a_1, a_2) \wedge (a_1 \wedge a_2 \leftrightarrow a)].$$

By [Lemma 3.7](#) the equalizer of $\wedge_{\mathcal{A}}, p_1: \Omega_{\mathcal{A}} \times \Omega_{\mathcal{A}} \rightarrow \Omega_{\mathcal{A}}$ is given by $(A \times A, \approx')$, where

$$\begin{aligned} [(a_1, a_2) \approx' (a'_1, a'_2)] &\simeq [(a_1, a_2) \approx (a'_1, a'_2) \wedge \exists a. (p_1(a_1, a_2, a) \wedge \wedge_{\mathcal{A}}(a_1, a_2, a))] \\ &\simeq [(a_1, a_2) \approx (a'_1, a'_2) \wedge \exists a. ((a_1 \leftrightarrow a) \wedge (a_1 \wedge a_2 \leftrightarrow a))] \\ &\simeq [(a_1, a_2) \approx (a'_1, a'_2) \wedge (a_1 \wedge a_2 \leftrightarrow a_1)] \end{aligned}$$

$$\simeq [(a_1, a_2) \approx (a'_1, a'_2) \wedge (a_1 \rightarrow a_2)].$$

The order is then given as the pairs where $[(a_1, a_2) \approx' (a_1, a_2)]$. Since $\leftrightarrow$ is reflexive, so is $\approx$ and $[(a_1, a_2) \approx' (a_1, a_2)]$ is equivalent to $[a_1 \rightarrow a_2]$. $\square$

Theorem 6.5. Let $\mathcal{A}$ be an arrow algebra. We have an internalization functor $\mathcal{I}: \mathbf{ArrAlg}/\mathcal{A} \rightarrow \mathbf{Loc}(\mathbf{AT}[\mathcal{A}])$.

Proof. On objects, we follow the same idea as the internalization functor for locales. Given a geometric arrow algebra morphism $f: \mathcal{B} \rightarrow \mathcal{A}$, we first consider the subobject classifier $\Omega_{\mathcal{B}} = (B, \leftrightarrow)$ in $\mathbf{AT}[\mathcal{B}]$ and then its image under $f_*: \mathbf{AT}[\mathcal{B}] \rightarrow \mathbf{AT}[\mathcal{A}]$. In the usual definition, this would be mapped to $f_*(\Omega_{\mathcal{B}}) = (\pi(B), f_+ \circ \leftrightarrow_S)$. However, since the subobject classifier is always weakly complete, we can modify the definition using a case distinction, such that $f_*(\Omega_{\mathcal{B}}) = (B, f_+ \circ \leftrightarrow)$ (cf. [Section 3.8](#)). We know from the exposition in the previous section that this is an internal locale in $\mathbf{AT}[\mathcal{A}]$. For morphisms, assume we have a commuting triangle in $\mathbf{ArrAlg}$ like in [Diagram 6.2](#). We want to find a locale morphism $\mathcal{I}(f) \rightarrow \mathcal{I}(g)$, so we will construct a frame morphism $\mathcal{I}(g) \rightarrow \mathcal{I}(f)$. Consider $\mathcal{I}(h) := \bar{h}^+$ given by $[\bar{h}^+(c, b)] := [c \sim c \wedge h^+(c) \sim b]$. It is easily seen to be strict and single-valued. Totality follows from the fact that h^+ is a morphism $C \rightarrow B$ in $\mathbf{Set}$ and [Lemma 6.3](#). Relationality on B is obvious and on C it follows from [Lemma 6.3](#). Thus $\bar{h}^+$ is an morphism $(C, \sim) \rightarrow (B, \sim)$. We still need to check that it is also a morphism of internal locales. But first we check functoriality: It is easy to see that for the identity on $\mathcal{A}$, we have $[\bar{\text{id}}_{\mathcal{A}}^+(a, a')] = [a \sim a \wedge (\text{id}_{\mathcal{A}})^+(a) \sim a'] \simeq [a \sim a']$. Thus $\bar{\text{id}}_{\mathcal{A}}^+ = \sim$, which represents the identity on $(A, \sim)$. If we have $h: \mathcal{B} \rightarrow \mathcal{C}$ and $h': \mathcal{C} \rightarrow \mathcal{D}$ with $\ell: \mathcal{D} \rightarrow \mathcal{A}$ such that [Diagram 6.3](#) commutes, then

$$\begin{aligned} [(\bar{h}^+ \circ \bar{h}'^+)(d, b)] &\simeq [\exists c. (\bar{h}'^+(d, c) \wedge \bar{h}^+(c, b))] \simeq [\exists c. (d \sim d \wedge h'^+(d) \sim c \wedge c \sim c \wedge h^+(c) \sim b)] \\ &\stackrel{(*)}{\leq} [\exists c. (d \sim d \wedge h^+(h'^+(d)) \sim h^+(c) \wedge c \sim c \wedge h^+(c) \sim b)] \\ &\leq [d \sim d \wedge h^+h'^+(d) \sim b] = [\bar{h}' \circ \bar{h}(d, b)], \end{aligned}$$

where $(*)$ is by [Lemma 6.3](#). Since both represent morphisms, they must be equal. Thus $\mathcal{I}$ preserves application and is functorial. We just need to check that $\mathcal{I}(h)$ is a morphism of internal locales, i.e. that it is monotone and preserves finite meets and arbitrary joins.

By [Lemma 6.4](#), the order on the subobject classifier is given by the implication. Thus for monotonicity, we need to show $[\bar{h}^+(c, b) \wedge \bar{h}^+(c', b') \wedge c \rightarrow c'] \leq [b \rightarrow b']$. In what follows, we will omit the conjunct $[c \sim c]$ of $\bar{h}^+$ since we only introduced it for strictness and do not need it otherwise. It does not change our result since $\bar{h}^+$ will only appear in the antecedent and the meet is less than both its arguments. Then checking the inequality $[\bar{h}^+(c, b) \wedge \bar{h}^+(c', b') \wedge c \rightarrow c'] \leq [b \rightarrow b']$ becomes verifying

$$\bigwedge_{c, c', b, b'} f_+(h^+(c) \leftrightarrow b) \wedge f_+(h^+(c') \leftrightarrow b') \wedge g_+(c \rightarrow c') \rightarrow f_+(b \rightarrow b') \in S_{\mathcal{A}}.$$

$$\begin{array}{ccccc} \mathcal{B} & \xrightarrow{h} & \mathcal{C} & \xrightarrow{h'} & \mathcal{D} \\ & \searrow f & \downarrow g & \swarrow \ell & \\ & & \mathcal{A} & & \end{array}$$

Diagram 6.3

Since $f_+ = g_+ h_+$ and g_+ commutes logically with logical meets, the antecedent is logically equivalent to $g_+(h_+(h^+(c) \leftrightarrow b) \wedge h_+(h^+(c') \leftrightarrow b) \wedge (c \rightarrow c'))$. Then we can use the fact that $h^+ \dashv h_+$ and monotonicity of g_+ with respect to the logical order to see that this term is logically less than $g_+(h_+(h^+(c) \leftrightarrow b) \wedge h_+(h^+(c') \leftrightarrow b) \wedge h_+ h^+(c \rightarrow c'))$. Using that h_+ also commutes logically with logical meets and the fact that we can pull the implication out of a cartesian morphism, this term is logically less than $g_+ h_+((h^+(c) \leftrightarrow b) \wedge (h^+(c') \leftrightarrow b) \wedge (h^+(c) \rightarrow h^+(c')))$. It is now an easy application of intuitionistic reasoning and using $g_+ h_+ = f_+$ to see that this term is logically less than $f_+(b \rightarrow b')$ and thus the meet above is indeed contained in $S_{\mathcal{A}}$.

To show preservation of meets, we need to show validity of

$$[\bar{h}^+(c, b) \wedge \bar{h}^+(c_1, b_1) \wedge \bar{h}^+(c_2, b_2) \wedge (c_1 \rightarrow c_2 \rightarrow c) \rightarrow b_1 \rightarrow b_2 \rightarrow b]$$

since the other inequality follows from monotonicity. The idea is that we can instantiate c with the meet of c_1 and c_2 and then this statement gives us $\bar{h}^+(c_1 \wedge c_2) \geq \bar{h}^+(c_1) \wedge \bar{h}^+(c_2)$ since $c_1 \wedge c_2 \leq c_1 \wedge c_2$. (Note however that the last two expressions are informal.) This expression translates to

$$\bigwedge_{c_1, c_2, c, b_1, b_2, b} f_+(h^+(c_1) \leftrightarrow b_1) \wedge f_+(h^+(c_2) \leftrightarrow b_2) \wedge f_+(h^+(c) \leftrightarrow b) \wedge g_+(c_1 \rightarrow c_2 \rightarrow c) \rightarrow f_+(b_1 \rightarrow b_2 \rightarrow b) \in S_{\mathcal{A}}.$$

Again, we use the adjunction $h^+ \dashv h_+$, the fact that cartesian morphisms commute logically with logical meets and we can pull the implication out to see that the antecedent is logically less than

$$f_+((h^+(c_1) \leftrightarrow b_1) \wedge (h^+(c_2) \leftrightarrow b_2) \wedge (h^+(c) \leftrightarrow b) \wedge (h^+(c_1) \rightarrow h^+(c_2) \rightarrow h^+(c))).$$

Again, we easily see using intuitionistic reasoning that the meet above is in $S_{\mathcal{A}}$.

Thus it remains to check that $\bar{h}^+$ preserves arbitrary joins. It suffices to show that it has an internal right adjoint as internal preorder morphisms. For this, we turn to h_+ . We define $[\bar{h}_+(b, c)] := [b \sim b \wedge h_+(b) \sim c]$. One can prove that it is a morphism by the same argument as for $\bar{h}^+$. We again omit the conjunct $[b \sim b]$ in the following. To check that it is a morphism of preorders, we check monotonicity. Analogous to before, we need to check

$$\bigwedge_{c, c', b, b'} g_+(h_+(b) \leftrightarrow c) \wedge g_+(h_+(b') \leftrightarrow c') \wedge f_+(b \rightarrow b') \rightarrow g_+(c \rightarrow c') \in S_{\mathcal{A}}.$$

This can easily be seen to be true similar to before: We write $f_+ = g_+ h_+$, use the g_+ commutes logically with logical meets, pull the implication out of h_+ and use intuitionistic reasoning.

So it remains to check that $\bar{h}^+ \dashv \bar{h}_+$. This is equivalent to checking validity of

$$[(\bar{h}_+ \circ \bar{h}^+)(c, c') \rightarrow c \rightarrow c'] \quad \text{and} \quad [(\bar{h}^+ \circ \bar{h}_+)(b, b') \rightarrow b' \rightarrow b].$$

These are like requiring $c \leq h_+ h^+(c)$ and $h^+ h_+(b) \leq b$, but written with predicates and identifying the order with validity of implication. Their validities are equivalent to the inequalities $[(\bar{h}_+ \circ \bar{h}^+)(c, c')] \leq [c \rightarrow c']$ and $[(\bar{h}^+ \circ \bar{h}_+)(b, b')] \leq [b' \rightarrow b]$. Starting with the first one, we notice

$$[(\bar{h}_+ \circ \bar{h}^+)(c, c')] \simeq [\exists b. (c \sim c \wedge h^+(c) \sim b \wedge b \sim b \wedge h_+(b) \sim c)] \leq [c \sim c \wedge h_+ h^+(c) \sim c'],$$

where the last inequality follows from [Lemma 6.3](#) and transitivity. We will show that this is already less than $[c \rightarrow c']$. Since externally $c \leq h_+h^+(c)$, we have $\bigwedge_c c \rightarrow h_+h^+(c) \in S_{\mathcal{C}}$. By intuitionistic reasoning, this gives us

$$\bigwedge_{c,c'} (h_+h^+(c) \rightarrow c') \rightarrow c \rightarrow c' \in S_{\mathcal{C}}.$$

Now the fact that g_+ is cartesian and condition (iii) with [Lemma 5.13](#) gives $\bigwedge_{c,c'} g_+(h_+h^+(c) \rightarrow c') \rightarrow g_+(c \rightarrow c') \in S_{\mathcal{A}}$, which is $[h_+h^+(c) \rightarrow c'] \leq [c \rightarrow c']$. Using the definition of the meet, we get $[h_+h^+(c) \sim c'] \leq [c \rightarrow c']$, which is the inequality we wanted to show.

The second inequality $[(\bar{h}^+ \circ \bar{h}_+)(b, b')] \leq [b' \rightarrow b]$ follows in a similar way: First we note that $[(\bar{h}^+ \circ \bar{h}_+)(b, b')] \leq [b \sim b \wedge h^+h_+(b) \sim b']$. Similarly, we get from the adjunction that $\bigwedge_b h^+h_+(b) \rightarrow b \in S_{\mathcal{B}}$, which gives us by intuitionistic reasoning

$$\bigwedge_{b,b'} (b' \rightarrow h^+h_+(b)) \rightarrow (b' \rightarrow b) \in S_{\mathcal{B}}.$$

We can apply f_+ in the same way to get the statement. This shows that $\bar{h}_+$ is right adjoint to $\bar{h}^+$, which is thus a continuous locale morphism. $\square$

Of course, it is unlikely that this actually gives us the equivalence we want since $\mathcal{I}$ is quite restrictive. It reduces all morphisms of arrow algebras to internal locales. Furthermore, if $\mathcal{A}$ is a locale, then the arrow tripos coincides with the localic tripos and hence $\text{AT}[\mathcal{A}] = \text{Sh}(\mathcal{A})$. So if this functor gave rise to an equivalence, we could derive together with [Theorem 6.1](#)

$$\mathbf{ArrAlg}/\mathcal{A} \simeq \mathbf{Loc}(\text{AT}[\mathcal{A}]) = \mathbf{Loc}(\text{Sh}(\mathcal{A})) \simeq \mathbf{Loc}/\mathcal{A},$$

which would mean that all geometric arrow algebra morphisms into a locale $\mathcal{A}$ would be essentially locale morphisms. One idea to make this work would be to restrict the morphisms allowed. We will discuss this attempt in [Section 6.4](#).

6.2 The subobject classifier as the quotient of the internal arrow algebra

If we want to generalize [Theorem 6.1](#) in a way that is closer to [Conjecture 1.1](#), we need a notion of internal arrow algebras. We go a similar way as for locales in [Definition 6.2](#) and just formulate the definitions of [Chapter 5](#) inside a topos $\mathcal{E}$.

Definition 6.6 (Internal arrow algebra). Let $\mathcal{E}$ be a topos. An *internal arrow algebra* in $\mathcal{E}$ is a tuple $(A, \preceq, \rightarrow, S)$, where A is an object of $\mathcal{E}$, $\preceq$ and S are subobjects $\preceq \rightrightarrows A \times A$ and $S \rightrightarrows A$ and $\rightarrow$ is a morphism $\rightarrow: A \times A \rightarrow A$, such that the following conditions are satisfied. For readability, we will write $a \preceq b$ for $(a, b) \in \preceq$ and $a \rightarrow b$ for $\rightarrow(a, b)$. All lowercase variables are of type A , while B is of type Ω^A . We require:

1. With respect to the internal logic, $\preceq$ is a complete partial order as stated in [Definition 6.2](#).
2. $\rightarrow$ is antitone in its first and monotone in its second argument:

$$\models \forall a, a', b, b'. (a' \preceq a \wedge b \preceq b') \Rightarrow a \rightarrow b \preceq a' \rightarrow b'.$$

3. S is a separator, i.e.

closed upwards: $\models \forall a, b. (a \in S \wedge a \preceq b) \Rightarrow b \in S$,

closed under MP: $\models \forall a, b. (a \in S \wedge a \rightarrow b \in S) \Rightarrow b \in S$,

contains the combinators:

$$\begin{aligned}
&\models \bigwedge \{x : \exists a, b. x = t_{\mathbf{k}}(a, b)\} \in S, & t_{\mathbf{k}}(a, b) &:= a \rightarrow b \rightarrow a, \\
&\models \bigwedge \{x : \exists a, b, c. x = t_{\mathbf{s}}(a, b, c)\} \in S, & t_{\mathbf{s}}(a, b, c) &:= (a \rightarrow b \rightarrow c) \rightarrow (a \rightarrow b) \rightarrow (a \rightarrow c), \\
&\models \bigwedge \{x : \exists a, B. (x = t_{\mathbf{a}}(a, B) \wedge \varphi_{\rightarrow}(B))\} \in S, & t_{\mathbf{a}}(a, B) &:= \left(\bigwedge \{x : \exists b. (b \in B \wedge x = a \rightarrow b)\} \right) \\
& & & \rightarrow a \rightarrow \bigwedge B, \\
& & \varphi_{\rightarrow}(B) &:= \forall b. (b \in B \Rightarrow \exists x, y. b = x \rightarrow y). \quad \lrcorner
\end{aligned}$$

In [Lemma 5.9](#), we saw that for any arrow algebra $\mathcal{A} = (A, \preceq, \rightarrow, S)$, its underlying set with the logical order $(A, \vdash_S)$ is a Heyting prealgebra. By [Definition 2.1](#), this means that its quotient by logical equivalence $(A/\leftrightarrow, \vdash_S)$ is a Heyting algebra. In [Section 5.2](#), we saw that the subobject classifier in $\text{AT}[\mathcal{B}]$ is the underlying set of $\mathcal{B}$ together with logical equivalence $\Omega_{\mathcal{B}} = (B, \leftrightarrow)$. Since this equality predicates identifies all logically equivalent elements, i.e. makes them equal, this is reminiscent of a quotient by logical equivalence. Furthermore, the internal locale corresponding to $f : \mathcal{B} \rightarrow \mathcal{A}$ in $\text{AT}[\mathcal{A}]$ is (up to isomorphism) given by $(B, f_+ \circ \leftrightarrow)$. As we already discussed in the previous section, taking this as the internalization of f is likely to be restrictive. But with what we just saw, there is another clear attempt: We would have to try to find an object $(X, \sim)$ of $\text{AT}[\mathcal{A}]$, whose (internal) quotient by (internal) logical equivalence is $(B, f_+ \circ \leftrightarrow)$. Then we would have reconstructed the external quotient inside $\text{AT}[\mathcal{A}]$.

Since the internal locale $(B, f_+ \circ \leftrightarrow)$ should represent the quotient $(B/\leftrightarrow, \vdash_S)$ (with an additional application of f_+), the equality predicate $\sim$ should represent external equality as closely as possible (with an additional application of f_+). Thus $(B, f_+ \circ \exists_{\delta} \top)$ is a clear candidate: We already saw that $\exists_{\delta} \top$ is reflexive, which is preserved by f_+ , and it is the least such, so it does not identify more elements than necessary. We thus want to check if we can view $(B, f_+ \circ \leftrightarrow)$ as a quotient of $(B, f_+ \circ \exists_{\delta} \top)$. For this, we will use the characterization from [Section 3.7](#) and need to show that $(B \times B, \approx')$ with

$$[(b_1, b_2) \approx' (b'_1, b'_2)] := [(b_1, b_2) \approx (b'_1, b'_2) \wedge f_+(b_1 \leftrightarrow b_2)]$$

is an equivalence relation on $(B, f_+ \circ \exists_{\delta} \top)$. By [Lemma 3.24](#), this is equivalent to $f_+ \circ \leftrightarrow$ being an internal equivalence relation on $(B, f_+ \circ \exists_{\delta} \top)$, which is clear since $f_+ \circ \leftrightarrow$ is an equality predicate and reflexive. Thus we will use $(B, f_+ \circ \exists_{\delta} \top)$ as the underlying element of our candidate internal arrow algebra.

Of course, the argument why $(B, f_+ \circ \exists_{\delta} \top)$ is a good candidate, also works for $(B, \exists_{\delta} \top)$. Even more, its equality predicate is less than $f_+ \circ \exists_{\delta} \top$, i.e. it quotients by less. However, it has no connection to f and since we are trying to define an internalization of a geometric arrow algebra morphism, this does not seem a useful idea.

The main problem in using $(B, f_+ \circ \exists_{\delta} \top)$ as the candidate internal arrow algebra will be proving completeness. Notice that $(B, f_+ \circ \exists_{\delta} \top)$ is the image of the underlying set B of $\mathcal{B}$ under the constant

object functor ∇ followed by the right adjoint Φ_* of the geometric morphism induced by f . So the direct way to try to internalize f would be to use the composition $\Phi_* \circ \nabla$ also on the order, implication and separator of $\mathcal{B}$. Since both ∇ and Φ_* preserve finite limits and we discussed in [Section 2.3](#) that the logic in a tripos essentially just states that certain diagrams commute and certain morphisms factor, the composition $\Phi_* \circ \nabla$ preserves all logical properties that can be formulated in terms of finite limits. This includes all finitary properties of $\mathcal{B}$, i.e. everything that is not concerned with the infinite meet. So if it were not for completeness, we could easily internalize f . To preserve the properties of the meet as well, we would need preservation of the subobject classifier, which is a property that functors rarely have. Indeed, if ∇ and Φ_* would preserve the subobject classifier, the logic in $\text{AT}[\mathcal{A}]$ would be boolean.

Miquel mentioned a similar problem when proving that all **Set**-triposes arise from implicative algebras [[Miq20a](#), Rmk. 2.8]. One of the main complications in defining the implicative algebra for an arbitrary **Set**-tripos is defining the complete lattice. So it is not surprising that we get a similar problem.

6.3 Via proto-assemblies

In [Chapter 4](#), we saw that assemblies are a useful tools in understanding $\mathbb{C}[\mathbf{P}]$ and that we can get them from even easier to understand elements, the proto-assemblies. Although we do not have a ex/reg-completion for them (cf. [Section 4.4](#)), we wanted to try to derive our candidate internal arrow algebra $(B, f_+ \circ \exists_\delta \top)$ from a proto-assembly. The clear candidate here is of course (B, f_+) . It is easy to see that this is a $\mathbf{P}_{\mathcal{A}}$ -proto-assembly. This approach seemed promising as we can even construct products in **pAsm** and the implication of $\mathcal{B}$ is also a morphism:

As we saw in [Proposition 4.9](#), we have $(B, f_+) \times (B, f_+) = (B \times B, \pi_1^* f_+ \wedge \pi_2^* f_+)$. So to show that the implication from $\mathcal{B}$ gives rise to a morphism $(B, f_+) \times (B, f_+) \rightarrow (B, f_+)$, we consider it as the pair $(B \times B, \rightarrow)$ and have to show that

$$\pi_1^* f_+ \wedge \pi_2^* f_+ \leq \exists_{\text{id}_{B \times B} \rightarrow}^* f_+.$$

We will show that already $\pi_2^* f_+ \leq \rightarrow^* f_+$, which gives us the claim by [Lemma 2.19](#) and the definition of the meet. By the definition of the order in the arrow tripos (cf. [Section 5.2](#)), we have to show that

$$\bigwedge_{a, b \in B} f_+(b) \rightarrow f_+(a \rightarrow b) \in S_{\mathcal{A}}.$$

This is clear from $\mathbf{k} \in S_{\mathcal{B}}$ and condition (iii') of [Definition 5.12](#). Thus $(B \times B, \rightarrow): (B, f_+) \times (B, f_+) \rightarrow (B, f_+)$ is a valid morphism of $\mathbf{P}_{\mathcal{A}}$ -proto-assemblies.

However, if we look at the image of (B, f_+) under $\mathcal{E}v$, we notice that $\mathcal{E}v(B, f_+) = (B, \exists_\delta f_+)$, which is clearly not our candidate internal arrow algebra $(B, f_+ \circ \exists_\delta \top)$. Indeed, they evaluate quite differently:

$$(\exists_\delta f_+)(a, b) = \bigwedge_c \left(\bigwedge_{a=b} f_+(a) \rightarrow c \right) \rightarrow c, \quad (f_+ \circ \exists_\delta \top)(a, b) = f_+ \left(\bigwedge_c \left(\bigwedge_{a=b} \top \rightarrow c \right) \rightarrow c \right),$$

since there is a α , such that $\delta(\alpha) = (a, b)$ iff $a = b$, in which case $\alpha = a = b$. The most prominent difference is the range of c : In the first case, it ranges over all elements in A , in the second only over

all elements of B .

So at least with our definition of the evolution functor, this attempt does not help in achieving our goal. One could argue that a different definition of the evolution functor may help, but then there is still the problem of the missing ex/reg-completion. The arguments of [Section 4.4](#) still largely apply and maybe this is a breaking problem.

6.4 Characterizing localic geometric morphisms

In [Section 6.1](#), we presented an internalization functor and argued that it is likely too restrictive to be turned into an equivalence. We suggested that this problem may be solved by restricting the morphisms between arrow algebras. We suggest a restriction to localic morphisms between arrow algebras. Of course, we first have to define what a localic morphism of arrow algebras should be, which is going to be the goal of this section. We start by giving the definition of a localic morphism of topoi:

Definition 6.7 (Localic morphisms). A geometric morphism $f: \mathcal{E} \rightarrow \mathcal{F}$ of topoi is called *localic*, if every object $\mathcal{E}$ is a *subquotient* of an element of the form f^*B , where $B \in \mathcal{F}$. This means that for every $A \in \mathcal{E}$, there exists a $B \in \mathcal{F}$ and a $C \in \mathcal{E}$, such that there are maps $f^*B \hookleftarrow C \twoheadrightarrow A$. $\lrcorner$

The name is inspired by the fact that locales and their sheaf topoi are closely related to localic morphisms:

Theorem 6.8 ([\[Joh02, Thm. C1.4.7\]](#)). Let $\gamma: \mathcal{E} \rightarrow \mathbf{Set}$ be a topos defined over set. Then γ is localic iff $\mathcal{E}$ is equivalent to $\mathbf{Sh}(X)$ for some locale X . $\square$

This theorem also tells us that $\mathbf{Loc}/\mathbf{Sh}(X) \simeq \mathbf{Loc}/X$, where $\mathbf{Loc}/\mathcal{E}$ is the slice category of Grothendieck topoi over $\mathcal{E}$, where we only consider localic morphisms. Since any sheaf topos is a Grothendieck topos (essentially by definition), the following is a generalization of the Joyal-Tierney theorem:

Theorem 6.9 ([\[Car22, Prop. 4.2\]](#)). For any Grothendieck topos $\mathcal{E}$, we have

$$\mathbf{Loc}(\mathcal{E}) \simeq \mathbf{Loc}/\mathcal{E}.$$

$\square$

Our goal is to define an appropriate notion of localic morphisms between arrow algebras. This definition should ensure that localic morphisms between arrow algebras correspond to localic morphisms between topoi. We call this hypothetical category of arrow algebras with localic morphisms $\mathbf{ArrAlg}_{loc}$. Then our new conjecture is

Conjecture 6.10. Let $\mathcal{A}$ be an arrow algebra and $\mathbf{AT}[\mathcal{A}]$ its associated arrow topos. Then we have an equivalence of categories

$$\mathbf{ArrAlg}_{loc}/\mathcal{A} \simeq \mathbf{Loc}(\mathbf{AT}[\mathcal{A}])$$

between the localic morphisms over $\mathcal{A}$ and the internal locales in $\mathbf{AT}[\mathcal{A}]$. $\lrcorner$

Our first idea was to use [Theorem 6.9](#) and prove that $\mathbf{ArrAlg}_{loc}/\mathcal{A} \simeq \mathbf{Loc}/\mathbf{AT}[\mathcal{A}]$. But the theorem requires $\mathcal{E}$ to be Grothendieck, which arrow topoi are in general not. The easiest counter-example is the effective topos, which arises as an arrow topos using Kleene's first algebra in [Example 5.4](#).

Nonetheless it may be possible to prove [Conjecture 6.10](#) in a different way and localic arrow algebra morphisms add to the theory, so we will still present what we discovered.

6.4.1 Constructing the hyperconnected part in the hyperconnected-localic factorization

Unfortunately, we have not found a suitable definition of what a localic arrow algebra morphism should be. Instead, we will be presenting our progress in finding a necessary condition for when a geometric morphism between arrow algebras induces a localic morphism between the arrow topoi. In this regard, we need the fact that localic morphisms are part of a factorization theorem:

Theorem 6.11 ([\[Joh02, Thm. A4.6.5\]](#)). Every geometric morphism of topoi factors, uniquely up to equivalence, as a hyperconnected morphism followed by a localic one. $\square$

We will not define hyperconnected morphisms since we only need this factorization theorem and its explicit construction. The interested reader can find more information in [\[Joh02, Sec. A4.6\]](#).

This theorem also tells us that if we have a localic morphism of topoi, the hyperconnected part will be an equivalence. This is useful for us, because Johnstone constructs it explicitly in [\[Joh02, Lem. A4.6.3\]](#). We will give this construction to see what we can derive from it in the case where we have a localic morphism $\text{AT}[\mathcal{B}] \rightarrow \text{AT}[\mathcal{A}]$ between arrow topoi which comes from a morphism of arrow algebras.

For the construction, we need the notion of a partial map classifier:

Definition 6.12 (Partial map classifier). Let $\mathcal{E}$ be a topos. A *partial map* $X \rightarrow Y$ consists of a subobject $X' \rightarrowtail X$ and a map $f: X' \rightarrow Y$.

A *partial map classifier* for $X \in \mathcal{E}$ is an element $\tilde{X} \in \mathcal{E}$ together an inclusion $\eta: X \rightarrowtail \tilde{X}$ classifying partial maps. This means that for any partial map $(X', f): X' \rightarrow Y$, there is a unique morphism $\tilde{f}: Y \rightarrow \tilde{X}$, such that [Diagram 6.4](#) is a pullback. $\lrcorner$

Proposition 6.13 ([\[Joh02, Prop. A2.4.7\]](#)). Let $\mathcal{E}$ be a topos. Then every $X \in \mathcal{E}$ has a partial map classifier. $\square$

Now given a geometric morphism of topoi $f: \mathcal{E} \rightarrow \mathcal{F}$, we define the hyperconnected part $G: \mathcal{E} \rightarrow \mathcal{G}$ of its hyperconnected-localic factorization in the following way: Start with an object $B \in \mathcal{E}$ and consider the map to its partial map classifier $\eta: B \rightarrow \tilde{B}$. Since f consists of an adjunction, we have the counit map $\varepsilon_{\tilde{B}}: f^* f_* \tilde{B} \rightarrow \tilde{B}$ and can form the pullback in [Diagram 6.5](#). Then we find the epi-mono factorization of $P \rightarrow B$ and define GB to be the middle part $P \twoheadrightarrow GB \rightarrowtail B$. Since all constructions have been functorial, this also gives the action on morphisms. If F is localic, then G is as well and thus an equivalence [\[Joh81a, Lem. 0.1\(i\), 1.1\(i\)\]](#). The pseudo-inverse is the inverse image G^+ . From [\[Joh02, Thm. A4.6.5\]](#), we know that this is the inclusion. Thus if F is localic, the morphism $GB \rightarrowtail B$ from the construction is an isomorphism. It is also easy to see that the converse also holds: if B and GB are isomorphic for all B , then F is localic. To see what we can derive from this equivalence, we will make these constructions explicit for the case of the arrow topoi.

$$\begin{array}{ccc}
X' & \xrightarrow{\quad} & X \\
\downarrow f & & \downarrow \eta \\
Y & \xrightarrow{\tilde{f}} & \tilde{X}
\end{array}$$

Diagram 6.4

$$\begin{array}{ccc}
P & \xrightarrow{\quad} & B \\
\downarrow & \lrcorner & \downarrow \eta \\
f^* f_* \tilde{B} & \xrightarrow{\varepsilon_{\tilde{B}}} & \tilde{B}
\end{array}$$

Diagram 6.5

6.4.2 The hyperconnected part for a morphism of arrow topoi

For readability purposes, we will give the partial map classifier in $\mathbb{C}[\mathbf{P}]$ and the counit explicitly. Then the construction as described above is easier. In [Oos08, p. 71], it was shown that for an element $(X, \sim)$ in $\mathbb{C}[\mathbf{P}]$, the partial map classifier is given by $\widetilde{(X, \sim)} = (\pi(X), \varpi)$, where

$$[U \varpi V] = [U \asymp V \wedge \forall x, y. (x \in_X U \wedge y \in_X U \rightarrow x \sim y)].$$

The predicate $\asymp$ is from the power object (cf. Section 3.5). This essentially makes elements of $\widetilde{(X, \sim)}$ sub-singletons. The map η is represented by

$$[\eta(x, U)] = [U \asymp U \wedge x \sim x \wedge \forall y. (y \in_X U \leftrightarrow x \sim y)].$$

To make the further construction easier, we will make use of the case distinction described in Section 3.8. The following follows directly from Proposition 3.31:

Corollary 6.14. Let $\mathbf{P}: \mathbb{C}^{\text{op}} \rightarrow \mathbf{preHeyt}$ be a tripos and $(X, \sim)$ an element of $\mathbb{C}[\mathbf{P}]$ with $\widetilde{(X, \sim)}$ its partial map classifier. Then $\widetilde{(X, \sim)}$ is weakly complete. $\square$

Next, we want to construct the counit explicitly. It is given in the proof of [HJP80, Prop. 3.5] for the definition of Φ without the case distinction. With the case distinction, we get the following description:

$$[\varepsilon_{\widetilde{(X, \sim)}}(U, U')] = [f^+ f_+(U \varpi U) \wedge U \varpi U'].$$

This can easily be checked using the isomorphism between an element and its object of singletons or checking the triangle inequalities manually.

With all of this, we can construct the pullback in the construction of the hyperconnected part. We will use Corollary 3.8. It tells us that the pullback of Diagram 6.5 is in this case given by $(\pi(X) \times X, \approx)$, where

$$\begin{aligned}
[(U_1, x_1) \approx (U_2, x_2)] &\simeq [U_1 \varpi U_2 \wedge x_1 \sim x_2 \wedge \exists U. (\varepsilon_{\widetilde{(X, \sim)}}(U_1, U) \wedge \eta(x_1, U))] \\
&\simeq [U_1 \varpi U_2 \wedge x_1 \sim x_2 \wedge \exists U. (f^+ f_+(U_1 \varpi U_1) \wedge U_1 \varpi U \wedge \forall y. (y \in_X U \leftrightarrow x_1 \sim y))] \\
&\simeq [U_1 \varpi U_2 \wedge x_1 \sim x_2 \wedge f^+ f_+(U_1 \varpi U_1) \wedge \forall y. (y \in_X U_1 \leftrightarrow x_1 \sim y)].
\end{aligned}$$

Its projection to $(X, \sim)$ is given by

$$[p_2(U, x, x')] \simeq [(U, x) \approx (U, x) \wedge x \sim x'].$$

Consider the equality predicate $\sim'$ on X given by

$$[x \sim' x'] \simeq [x \sim x' \wedge \exists U, x''. p_2(U, x'', x)] \simeq [x \sim x' \wedge \exists U. p_2(U, x, x)]$$

$$\begin{aligned}
&\simeq [x \sim x' \wedge \exists U.(S(U) \wedge f^+ f_+(U \simeq U) \wedge x \in_X U)] \\
&\simeq [x \sim x' \wedge \exists U.(f^+ f_+(U \simeq U) \wedge U \preceq \{x\})],
\end{aligned}$$

where $\{x\}$ suggestively written for $\{\Pi\}$ with the predicate $[\Pi(y)] := [y \sim x]$. By [Theorem 3.18](#), the predicate $\sim'$ identifies a subobject of $(X, \sim)$, whose inclusion is given by $\sim'$. Furthermore, p_2 is also a map $(\pi(X) \times X, \approx) \rightarrow (X, \sim')$, which is also epi. It is now clear that the epi-mono factorization of $p_2: (\pi(X) \times X, \approx) \rightarrow (X, \sim)$ is given by

$$\begin{array}{ccc}
(\pi(X) \times X, \approx) & \xrightarrow{p_2} & (X, \sim) \\
& \searrow p_2 & \nearrow \sim' \\
& (X, \sim') &
\end{array}$$

Diagram 6.6

This gives us the full description of the hyperconnected part.

So now assume we have a geometric morphism of arrow algebras $f: \mathcal{B} \rightarrow \mathcal{A}$. By the discussion in [Subsection 6.4.1](#), we know that $(X, \sim)$ and $(X, \sim')$ are isomorphic for all objects $(X, \sim)$ in $\mathbb{C}[\mathbf{P}]$. Since this isomorphism is given by the inclusion $\sim'$, we know that

$$[\forall x.(x \sim x \rightarrow \exists U.(f^+ f_+(U \simeq U) \wedge U \simeq \{x\}))].$$

Writing this in terms of the logic of the arrow algebra, we get

Proposition 6.15. A geometric morphism of arrow algebras $f: \mathcal{B} \rightarrow \mathcal{A}$ is localic iff for all sets X and all equality predicates $\sim$ on X we have

$$\bigwedge_{x \in X} x \sim x \rightarrow \bigwedge_{c \in B} \left(\bigwedge_{U \in B^X} f^+ f_+(U \simeq U) \wedge U \simeq \{x\} \rightarrow c \right) \rightarrow c \in S_B. \quad (*)$$

Proof. By the observation at the end of [Subsection 6.4.1](#) a geometric morphism is localic iff $GB \cong B$ for all B via the inclusion $GB \hookrightarrow B$. In our case, we have $B = (X, \sim)$ and $GB = (X, \sim')$ with the inclusion $\sim'$. Since $\sim'$ is always injective and topoi are balanced, it is iso iff it is epi. Thus f is localic iff $\sim'$ is epi for all $(X, \sim)$, which is exactly what $(*)$ states. $\square$

We would like to derive a condition about f from this. One attempt would be to choose X to be a singleton so that we can better control its elements. However, none of our attempts gave us a useful condition. Due to time constraints, we were not able to pursue this direction further. Maybe someone can pick up this work and derive a good necessary condition from this.

Chapter 7

Discussion and Outlook

Da steh' ich nun, ich armer Tor,
Und bin so klug als wie zuvor!¹

Faust, der Tragödie erster Teil
Johann Wolfgang von Goethe

Fortunately, we are a bit more lucky than Dr. Faust and have gotten somewhat wiser. Initially, we set out to prove the equivalence $\mathbf{ArrAlg}/\mathcal{A} \simeq \mathbf{ArrAlg}(\mathbf{AT}[\mathcal{A}])$ and wanted to start by defining an internalization functor $\mathbf{ArrAlg}/\mathcal{A} \rightarrow \mathbf{ArrAlg}(\mathbf{AT}[\mathcal{A}])$. The main result of the thesis is [Theorem 6.5](#), where we showed that we have a internalization functor $\mathbf{ArrAlg}/\mathcal{A} \rightarrow \mathbf{Loc}(\mathbf{AT}[\mathcal{A}])$. Since internal locales are of course also internal arrow algebras, we could claim to have reached our goal. However, we did not just want to define any internalization functor, but one that is part of a Joyal-Tierney-like equivalence. As we discussed in [Section 6.1](#), internalizing arrow algebra morphisms as locales is unlikely to work as the equivalence would entail $\mathbf{ArrAlg}/X \simeq \mathbf{Loc}/X$ for any locale X . This would mean that all arrow algebra morphisms into a locale would essentially be locale morphisms.

Thus in [Section 6.4](#), we looked at a possibility of restricting the admissible morphisms of arrow algebras with an inspiration from [Theorem 6.9](#). We would not be able to use that theorem directly, because arrow topoi are in general not Grothendieck. Instead, the inspiration is supposed to give us a direction to look in. Localic morphisms between topoi are well-studied and we wanted to define a notion of localic morphisms between arrow algebras, which should correspond to localic morphisms between the arrow topoi. We derived a necessary condition at the level of arrow topoi, but did not manage to formulate it at the level of arrow algebras. If we could do that, we could investigate whether this condition is also sufficient for an arrow algebra morphism to become a localic morphism between the arrow topoi. Then we would obtain a full characterization of localic morphisms between arrow topoi at the level of arrow algebras. However, this line of reasoning would internalize (localic) arrow algebra morphisms as locales, not arrow algebras.

We also tried to internalize arrow algebra morphisms as arrow algebras via proto-assemblies in [Section 6.3](#). This approach did not give us any positive result and we are quite confident that it is not fruitful. As we discussed in [Section 6.2](#), internal completeness is the major problem for internalization. We can transport the finitary data (i.e. everything not concerned with the infinite meet) of the arrow algebra $\mathcal{B}$ into $\mathbf{AT}[\mathcal{A}]$ via the composition $\Phi_* \circ \nabla$. But without the infinite meet, we are not even able

¹“And here, poor fool, I stand once more, No wiser than I was before.”

to formulate the definition of an internal arrow algebra.

It is also not clear if trying to transport the external properties of $\mathcal{B}$ into $\text{AT}[\mathcal{A}]$ is the best approach. After all, the internalization of locale morphisms works differently. The internal order and meet are not given by the external properties, but by internal characterizations of the subobject classifier (cf. [Lemma 6.4](#)). Thus this attempt may be flawed from the beginning. On the other hand, we could also try to better understand the characterising morphisms of the internal meet and the internal order so that we can mimic them for our candidate arrow algebra $(B, f_+ \circ \exists_\delta \top)$.

The directions for further research are clear: Due to time constraints, we were not able to fully develop all of our ideas, which thus could be further researched. We think that the approach via localic geometric morphisms in [Section 6.4](#) is very promising as [Conjecture 6.10](#) is close to the formulation of the Joyal-Tierney theorem. As a first step, the characterization of localic morphisms between arrow algebras would need to be finished. Then one would have to find a way to turn a topos with a localic morphism into an arrow topos into a arrow algebra with a corresponding morphism.

To pursue the original [Conjecture 1.1](#), we would first need to understand the internal arrow algebras better. Indeed, we have only given a definition and a candidate internal arrow algebra, but done little further research into this. One could look into how to construct morphisms for our candidate internal arrow algebra that act similarly to the meet and order of the subobject classifier.

On the further end of further research, we still hope that after generalizing the Joyal-Tierney theorem in some way, we can use it to construct and prove the Gleason cover for arrow algebras.

But alas, the clocks grow impatient
and we rest our steed for this night
May tomorrow someone come by
and deliver our toil to greater height.

E2

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