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ON THE COMPLEXITY OF ARITHMETICAL INTERPRETATIONS OF MODAL FORMULAE

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ON THE COMPLEXITY OF ARITHMETICAL INTERPRETATIONS OF MODAL FORMULAE

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ON THE COMPLEXITY OF ARITHMETICAL INTERPRETATIONS OF MODAL FORMULAE

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It is a well-known fact that for any arithmetic sentence A

$$A \in \Sigma_1^{PA} \Rightarrow PA \vdash A \longrightarrow Pr\ulcorner A \urcorner.$$

Here Pr stands for Gödel's formula expressing provability in Peano Arithmetic PA and Σ_1^{PA} denotes the class of sentences PA-equivalent to those in Σ_1-form.

C.Kent [1] showed that the converse implication does not hold. Moreover, he found that for each natural number n there exists an arithmetic sentence A such that

$$PA \vdash A \longrightarrow Pr\ulcorner A \urcorner \text{ and } A \notin \Delta_n^{PA}.$$

D.Guaspari [2] rediscovered (a sharpened version of) this result applying his own techniques of partially conservative sentences. He also showed that arithmetically complex sentences implying their own provability cannot be constructed by some class of restricted means.

D.Guaspari posed a few problems generalizing one solved by Kent and himself, which are formulated in terms of provability interpretations of propositional modal logic.

DEFINITION. Let $\mathcal{L}$ be the language consisting of propositional variables $p,q,\ldots$; boolean connectives $\wedge$,$\vee$,$\longrightarrow$,$\leftrightarrow$,$\neg$ and $\bot$; modal operator $\Box$.

An *arithmetical interpretation* f is a mapping of $\mathcal{L}$-formulae to arithmetic sentences which commutes with boolean connectives and translates $\Box$ as provability, i.e. for every modal formula ϕ

$$f(\Box\phi) = Pr\ulcorner f(\phi) \urcorner.$$

D.Guaspari noted that for any arithmetic sentence A obviously

$$PA \vdash A \longrightarrow Pr\ulcorner A \urcorner \Leftrightarrow \text{there is a sentence } B \text{ s.t. } PA \vdash A \leftrightarrow B \wedge Pr\ulcorner B \urcorner.$$

So, the original question is equivalent to one, whether there exist non Σ_1^{PA} (or even arbitrarily complex) arithmetic interpretations of the modal formula $p\wedge\Box p$. D.Guaspari set the problem of characterizing those formulae of $\mathcal{L}$, which are Σ_1^{PA} under every arithmetical interpretation, and conjectured that they are exactly the formulae provably in the modal logic GL (called PRL in [3]) equivalent to disjunctions of boxed formulae, i.e. those of the form $\Box\phi$. An analogous conjecture was also made for the modal language enriched by witness comparison formulae.

These conjectures have been proved by A.Visser [4] and D.de Jongh & D.Pianigiani [5] respectively. The latter authors also extended the result to arbitrary theories containing arithmetic $I\Sigma_1$.

The aim of present paper is to characterize the formulae of $\mathcal{L}$ having bounded arithmetical complexity, i.e. whose arithmetical interpretations are all Δ_n^{PA} for some fixed n. It turns out, perhaps not very surprisingly, that such formulae are exactly those equivalent in GL to boolean combinations of boxed ones. In other words, each modal formula not equivalent to any such combination admits arbitrarily complex arithmetical interpretations. Thus, we may say that the "modal analog" of the arithmetical hierarchy with respect to PA collapses after the level of boolean combinations of Σ_1-sentences.

It is easily seen that the formula $p\wedge\Box p$ is not equivalent to any boolean combination of boxed formulae, so our result provides us with one more proof of Kent's theorem. However, it will be clear after reading the sequel that, when applied to this particular formula, our construction essentially goes along the lines of that of Kreisel & Lévy (cf also [6], [7]).

Let B_n^{PA} denote the collection of all arithmetic sentences PA-equivalent to boolean combinations of those in Σ_n-form.

THEOREM.

Suppose ϕ is a formula of $\mathcal{L}$ such that for no boolean combination ψ of boxed formulae $GL\vdash \phi \leftrightarrow \psi$. Then for each $n\geq 1$ there exists an arithmetical interpretation f such that

$$f(\phi) \in \Delta_{n+1}^{PA}\backslash B_n^{PA}.$$

In order to prove this theorem first of all we obtain some Kripke-style characterization of modal formulae GL-equivalent to boolean combinations of boxed formulae. We shall call the usual finite strictly partially ordered (not necessary treelike) Kripke models for GL just *models* (cf [3]). The expression $\mathcal{K} \Vdash \phi$ will denote the fact that ϕ is forced at the bottom node of the model $\mathcal{K}$.

DEFINITION. A modal formula ϕ is called *stable* iff for all models $\mathcal{K}$ and $\mathcal{K}'$ with the same frame and the same forcing of propositional variables at all nodes except bottom ones

$$\mathcal{K} \Vdash \phi \Leftrightarrow \mathcal{K}' \Vdash \phi.$$

LEMMA 1. A modal formula ϕ is GL-equivalent to a boolean combination of boxed formulae iff ϕ is stable.

PROOF. It is trivial that boolean combinations of boxed formulae are stable. We check the converse implication.

Suppose ϕ is stable. Clearly ϕ is a boolean combination of boxed subformulae $\Box\phi_1, \ldots, \Box\phi_k$ and propositional variables $p_1, \ldots, p_m$ occurring outside any $\Box$ in ϕ. Let ϕ^* denote the result of substituting in ϕ the constant $\bot$ for all the *occurrences* of variables outside any $\Box$. Clearly, ϕ^* is a boolean combination of boxed formulae. We shall show that $GL \vdash \phi \leftrightarrow \phi^*$. By the completeness of GL with respect to models it is sufficient to show that ϕ and ϕ^* are forced exactly at the same models (cf [3]).

Let $\mathcal{K}$ be an arbitrary model and let $\mathcal{K}^*$ denote the model with the same frame as $\mathcal{K}$ and the same forcing of propositional variables at all nodes except the bottom node, where (in $\mathcal{K}^*$) no variable is forced. Clearly we have

$$\mathcal{K} \Vdash \Box\phi_i \Leftrightarrow \mathcal{K}^* \Vdash \Box\phi_i \text{ and } \mathcal{K}^* \Vdash p_j \Leftrightarrow \mathcal{K} \Vdash \bot, \text{ hence}$$

$$\mathcal{K} \Vdash \phi^* \Leftrightarrow \mathcal{K}^* \Vdash \phi.$$

By our assumption ϕ is stable, hence

$$\mathcal{K} \Vdash \phi \Leftrightarrow \mathcal{K}^* \Vdash \phi.$$

It follows that

$$\mathcal{K} \Vdash \phi \Leftrightarrow \mathcal{K} \Vdash \phi^*.$$

Thus,

$$GL \vdash \phi \leftrightarrow \phi^*.$$

COROLLARY 1. The class of $\mathcal{L}$-formulae equivalent to boolean combinations of boxed ones is decidable.

Suppose now that ϕ is a nonstable formula. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be a pair of models obtained by Lemma 1, such that $\mathcal{K}_1 \Vdash \phi$ and $\mathcal{K}_2 \Vdash \neg\phi$. By identifying the corresponding nodes of $\mathcal{K}_1$ and $\mathcal{K}_2$ except the bottom ones and by adding a new bottom node below them we can construct a model $\mathcal{K} = (K, <, \Vdash)$, such that

(i) $K = \{0,\ldots,k\}$ for some $k \in \mathbb{N}$;

(ii) 0 is the bottom node of $\mathcal{K}$ and 1 and 2 are the only immediate successors of 0;

(iii) 1 and 2 have the same successors;

(iv) for all $i,z,w \in K$, if $1< z,w$ and $z,w <i$ then either $z<w$ or $w\leqslant z$;

(v) $1 \Vdash \phi$ and $2 \Vdash \neg\phi$.

Condition (iv) will be satisfied if we beforehand choose $\mathcal{K}_1$ and $\mathcal{K}_2$ treelike .

Our next goal is to describe a Solovay-style [8] embedding of such models into arithmetic. First of all, let us try to explain it informally. Since we are going to define a very complex interpretation, say f, it is of no importance for us, whether we shall work within PA or within some finite extension of it of low complexity. The new axiom we will add is the statement, that some Solovay-type function has the limit ℓ either in the node 1 or in 2. The words "Solovay-type" mean that for all $i \in K\backslash\{0\}$ and all formulae ψ of $\mathcal{L}$ we will have

$$\text{PA} \vdash \ \ell=i \longrightarrow (i \Vdash \psi \leftrightarrow f(\psi)).$$

Thus, since $1 \Vdash \phi$ and $2 \Vdash \neg\phi$, we shall obtain

$$\text{PA} \vdash \ \ell\in\{1,2\} \longrightarrow (\ell=1 \leftrightarrow f(\phi)).$$

At this point all we need is to define our function in such a way, as it would be an impracticable task not only for PA + $\ell\in\{1,2\}$, but also for any finite consistent extension of PA + $\ell\in\{1,2\}$ of low complexity, to distinguish whether ℓ equals 1 or 2. And it is here that we apply the trick of Kreisel & Lévy. It is worth mentioning that such an idea would not work, had not our frame been fully symmetrical with respect to transpositions of nodes 1 and 2.

Now we turn to explicit definitions. Let for $n\geq 1$ Tr_n denote the standard Δ_{n+1}-definition of truth for $\mathbf{B}_n$-formulae, i.e. the standard arithmetic Δ_{n+1}-formula such that

$$\text{PA} \vdash \ \forall\, u \in \mathbf{B}_n \ \text{Pr}(\ u \stackrel{\cdot}{\longleftrightarrow} \ulcorner \text{Tr}_n(u) \urcorner\ )$$

and

$$\text{PA} \vdash \ \forall\, u \notin \mathbf{B}_n \ \neg\text{Tr}_n(u).$$

Further, let $\text{Prf}_u(m,x)$ denote the primitive recursive formula expressing the predicate " m is (the gödel number of) a proof of the formula x from axioms of PA together with the additional axiom u ".

Given an arithmetic sentence B $\mathrm{Prf}_B(m,x)$ will abbreviate the formula $\mathrm{Prf}_{\ulcorner B\urcorner}(m,x)$ and $\mathrm{Prf}(m,x)$ will denote $\mathrm{Prf}_{0=0}(m,x)$.

By the routine formalization of the definition below with the aid of the Fixed Point Theorem one can define an arithmetic Δ_{n+1}-function h such that PA proves that $h(0) = 0$ and for all m $h(m+1)$ is "computed" by the following instructions:

$$\underline{\text{if}}\ z > h(m),\ z > 1\ \underline{\text{and}}\ \mathrm{Prf}(m+1, \ulcorner \ell \neq z \urcorner) \tag{1}$$

$\underline{\text{then}}$ $h(m+1)=z$

$\underline{\text{else}}$ $\underline{\text{if}}$ $h(m)=0$ $\underline{\text{and}}$ $\exists\, y_1,y_2 \leq m+1\ (\ \mathrm{Prf}(y_1,\ulcorner \ell\neq 1\urcorner) \wedge \mathrm{Prf}(y_2,\ulcorner \ell\neq 2\urcorner)\ )$ (2)

$\underline{\text{then}}$ $\underline{\text{if}}$ $\exists\, z,u \leq m+1\ (\mathrm{Tr}_n(u) \wedge \mathrm{Prf}_u(z,\ulcorner \ell\neq 1\urcorner) \wedge \forall\, t,w < z\ (\neg \mathrm{Tr}_n(w) \vee \neg \mathrm{Prf}_w(t,\ulcorner \ell\neq 2\urcorner)))$ (3)

$\underline{\text{then}}$ $h(m+1)=1$

$\underline{\text{else}}$ $h(m+1)=2$ (4)

$\underline{\text{else}}$ $h(m+1)=h(m)$. (5)

Here $\ell=z$ denotes the formula $\exists N\ \forall\ m>N\ h(m)=z$.

The following Lemma establishes the properties of our function h similar to those of the original Solovay's function (cf [8],[3]).

LEMMA 2.

1. $\mathrm{PA} \vdash\ \forall m\ (\ h(m) \leqslant h(m+1)\ )$; $\mathrm{PA} \vdash\ \ell=0 \vee \ell=1 \vee \ldots \vee \ell=k$;
2. $i,j \in K$ and $i\neq j \Rightarrow \mathrm{PA}\vdash\ \ell=i \longrightarrow \ell\neq j$;
3. $j>i>0 \Rightarrow \mathrm{PA}\vdash\ \ell=i \longrightarrow \neg \mathrm{Pr}\ulcorner \ell\neq j\urcorner$;
4. $i>0 \Rightarrow \mathrm{PA}\vdash\ \ell=i \longrightarrow \mathrm{Pr}\ulcorner \ell>i\urcorner$;
5. PA + $\ell\in\{1,2\}$ is a consistent theory;
6. $\ell=0$ is true;
7. $\mathrm{PA}\vdash\ \ell\in\{1,2\} \longleftrightarrow \mathrm{Pr}\ulcorner \ell\notin\{1,2\}\urcorner \wedge \forall i>1\ \neg\mathrm{Pr}\ulcorner \ell\neq i\urcorner$.

PROOF. Claims 1 and 2 follow immediately from the definition of h. Statement 3 is proved as in Solovay's paper; the following argument can be formalized in PA: "If $\ell\neq j$ is provable then it has PA-proofs with arbitrarily large gödel numbers. So, if $\ell=i$ then h is bound to make a move from i by clause (1) because $j>i>0$.".

The proof of 4 is not quite the same as in [8], because our function h is not Σ_1. Yet for all $i>1$ one can easily show that the sentence $\exists m\ h(m)=i$ is Σ_1^{PA}.

In fact,

$$PA \vdash \exists m\ h(m)=i \leftrightarrow \exists m\ (Prf(m,\ulcorner \ell\neq i\urcorner) \wedge \forall y<m\ \forall j\in K\ (\neg j\leqslant i\ \wedge$$
$$Prf(y,\ulcorner \ell\neq j\urcorner) \longrightarrow \exists z\leqslant i\ \exists x<y\ (\ Prf(x,\ulcorner \ell\neq z\urcorner) \wedge \neg\ z<j\ )\ )).$$

The ($\longrightarrow$) implication follows from the monotonicity of h, for if m is the least (gödel number of a) proof of $\ell\neq i$, $h(y)$ is to be beneath i for all $y<m$. So, by clause (1), if at some stage $y<m$ we receive a proof of $\ell\neq j$, where not $j\leqslant i$, $h(y)$ is already to be not $<j$. And this is possible only if we have earlier received a proof of some sentence $\ell\neq z$, where $z\leqslant i$ but not $z<j$.

In order to prove the converse implication we only have to check that h cannot make a move from beneath i before stage m. Since both 1 and 2 are beneath i, such a move can only be made by clause (1); but this is impossible because if h moved from some $w<i$ to j, where not $j\leqslant i$, we would earlier have obtained a proof of $\ell\neq z$ for some $z\leqslant i$ and z not $<j$. By property (iv) of the model $\mathcal{K}$, either $w<z$ or $z\leqslant w$. The latter is not the case since $w<j$ but z is not. So, h is bound to have moved from w to z earlier than to j. Absurd.

Suppose now that $i>1$. Then

$$\begin{aligned} PA \vdash\ \ell=i &\longrightarrow \exists m\ h(m)=i \\ &\longrightarrow Pr\ulcorner \exists m\ h(m)=i\urcorner \\ &\longrightarrow Pr\ulcorner \ell\geqslant i\urcorner . \end{aligned}$$

But for all $i>0$ trivially

$$PA \vdash\ \ell=i \longrightarrow Pr\ulcorner \ell\neq i\urcorner ;$$

hence

$$PA \vdash\ \ell=i \longrightarrow Pr\ulcorner \ell>i\urcorner .$$

We treat the case $i\in\{1,2\}$ in a different way. By clause (2), h is allowed to make a move to 1 only after having received both proofs of $\ell\neq 1$ and $\ell\neq 2$. It follows that

$$PA \vdash\ \ell=1 \longrightarrow Pr\ulcorner \ell\notin\{1,2\}\urcorner .$$

We also have

$$PA \vdash\ Pr\ulcorner \ell\notin\{1,2\}\urcorner \longrightarrow \ell\neq 0;$$

hence

$$\begin{aligned} PA \vdash\ \ell=1 &\longrightarrow Pr\ulcorner \ell\notin\{1,2\}\urcorner \\ &\longrightarrow Pr\ulcorner Pr\ulcorner \ell\notin\{1,2\}\urcorner\urcorner \\ &\longrightarrow Pr\ulcorner \ell\neq 0\urcorner . \end{aligned}$$

Thus we obtain

$$\begin{aligned} PA \vdash\ \ell=1 &\longrightarrow Pr\ulcorner \ell\notin\{0,1,2\}\urcorner \\ &\longrightarrow Pr\ulcorner \ell>1\urcorner . \end{aligned}$$

The case $i=2$ is fully symmetrical; so statement 4 follows.

Claim 6 is trivial.

To check 5 and 7 notice that if PA proves $\ell \notin \{1,2\}$, then h is bound to make a move from 0.

REMARK. Note that statement 3 of this Lemma is weaker than the corresponding statement of Solovay, because we have not proved that

$$\mathrm{PA} \vdash \ell=0 \longrightarrow \neg \mathrm{Pr}\ulcorner \ell\neq 1 \urcorner .$$

Although it might seem that our function h lacked this property, because h was to stay in 0 until we received *both* proofs of $\ell\neq 1$ and $\ell\neq 2$, nevertheless one can show within PA that once we have received one of these proofs we are guaranteed to receive the other. We shall put off the proof of this statement for it is needed only to improve slightly on a simpler result that we are going to obtain first.

Now we define an arithmetical interpretation f exactly as in Solovay's paper:

$$f(p) := \exists i \ (\ell=i \wedge i \Vdash p).$$

As in [8] we obtain the following Corollary.

COROLLARY 2. For all nodes $i>0$ and all formulae ψ

$$i \Vdash \psi \Rightarrow \mathrm{PA} \vdash \ell=i \longrightarrow f(\psi).$$

LEMMA 3. Suppose $A \in \mathbf{B}_n$ is an arithmetic sentence, such that the theory $\mathrm{PA} + \ell\in\{1,2\} + A$ is consistent. Then $\mathrm{PA} + \ell\in\{1,2\} + A$ does not prove either $\ell=1$ or $\ell\neq 1$.

PROOF. Since $\mathrm{PA} + \ell\in\{1,2\} \vdash \ell=1 \leftrightarrow \ell\neq 2$ and 1 and 2 are symmetrical, we only have to prove that $\mathrm{PA} + \ell\in\{1,2\} + A$ does not prove $\ell\neq 1$.

Suppose for a reductio that it does. Let m be the least (gödel number of a) proof of $\ell\neq 1$ from an arithmetic sentence $B \in \mathbf{B}_n$ consistent with $\mathrm{PA} + \ell\in\{1,2\}$, i.e.

$$\mathrm{Prf}_B(m,\ulcorner \ell\neq 1\urcorner) \ , \ \mathrm{PA} + \ell\in\{1,2\} + B \text{ is a consistent theory}$$
$$\text{and for all } y<m \text{ and all } C \in \mathbf{B}_n$$
$$\text{if } \mathrm{Prf}_C(y,\ulcorner \ell\neq 1\urcorner) \text{ then } \mathrm{PA} + \ell\in\{1,2\} + C \text{ is inconsistent.}$$

Such a B exists because by Lemma 2.7 $\ell\in\{1,2\}$ is $\mathbf{B}_1^{\mathrm{PA}}$, hence $A \wedge \ell\in\{1,2\}$ is $\mathbf{B}_n^{\mathrm{PA}}$. Further we obtain

$$\mathrm{PA} + \ell\in\{1,2\} + B \vdash \mathrm{Tr}\ \ulcorner B\urcorner \wedge \mathrm{Prf}_B(\mathrm{m},\ulcorner \ell\neq 1\urcorner)$$
$$\vdash \ell=1 \vee \exists y<\mathrm{m}\ \exists u<\mathrm{m}\ (\mathrm{Tr}_n(u) \wedge \mathrm{Prf}_u(y,\ulcorner \ell\neq 2\urcorner)).$$

Reason in PA + $\ell\in\{1,2\}$ + B : "Let t be the least stage such that we have received both proofs of $\ell\neq 1$ and $\ell\neq 2$ by the moment t. So, since $\ell\in\{1,2\}$ the function h is not allowed to leave 0 before t. Thus, by clause (2), h is to make a move either to 1 or to 2. If not $\exists y<m\ \exists u<m\ (\mathrm{Tr}_n(u) \wedge \mathrm{Prf}_u(y,\ulcorner\ell\neq 2\urcorner))$ then h will move to 1 by clause (3) because B is true $\mathbf{B}_n$.".

Further

$$\mathrm{PA} + \ell\in\{1,2\} + B \vdash \exists y<m\ \exists u<m\ (\mathrm{Tr}_n(u) \wedge \mathrm{Prf}_u(y,\ulcorner\ell\neq 2\urcorner)),$$

because by our assumption

$$\mathrm{PA} + B \vdash \ell\neq 1.$$

We claim that there is a sentence $C \in \mathbf{B}_n$ such that C is consistent with PA + $\ell\in\{1,2\}$ and for some $y<m$ $\mathrm{Prf}_C(y,\ulcorner\ell\neq 2\urcorner)$.

Otherwise we would have for all $y,u<m$ that

either not $\mathrm{Prf}_u(y,\ulcorner\ell\neq 2\urcorner)$ and then $\mathrm{PA} \vdash \neg\, \mathrm{Prf}_{\mathbf{u}}(\mathbf{y},\ulcorner\ell\neq 2\urcorner)$;

or u is not a gödel number of a $\mathbf{B}_n$-formula and then

$$\mathrm{PA} \vdash \neg\mathrm{Tr}_n(\mathbf{u});$$

or $u = \ulcorner C\urcorner$, $\mathrm{Prf}_u(y,\ulcorner\ell\neq 2\urcorner)$, $C \in \mathbf{B}_n$ and $\mathrm{PA} + \ell\in\{1,2\} \vdash \neg\, C$; then

$$\mathrm{PA} + \ell\in\{1,2\} \vdash \neg\ \mathrm{Tr}_n\ulcorner C\urcorner.$$

In each case we obtain

$$\mathrm{PA} + \ell\in\{1,2\} \vdash \neg\ \mathrm{Tr}_n(\mathbf{u}) \vee \neg\ \mathrm{Prf}_{\mathbf{u}}(\mathbf{y},\ulcorner\ell\neq 2\urcorner).$$

Thus,

$$\mathrm{PA} + \ell\in\{1,2\} \vdash \forall y<m\ \forall u<m\ (\neg\ \mathrm{Tr}_n(u) \vee \neg\ \mathrm{Prf}_u(y,\ulcorner\ell\neq 2\urcorner)).$$

It follows that PA + $\ell\in\{1,2\}$ + B is inconsistent. Absurd.

Applying now similar arguments to C and y instead of B and m respectively we obtain a sentence $D \in \mathbf{B}_n$ such that D is consistent with PA + $\ell\in\{1,2\}$ and for some $x<y<m$ $\mathrm{Prf}_D(x,\ulcorner\ell\neq 1\urcorner))$. This contradicts our assumption that m is the minimal gödel number of a proof of $\ell\neq 1$ from a $\mathbf{B}_n$-sentence consistent with PA + $\ell\in\{1,2\}$.

COROLLARY 3. For no sentence $A \in \mathbf{B}_n$

$$\mathrm{PA} + \ell\in\{1,2\} \vdash A \leftrightarrow \ell=1.$$

PROOF. Suppose that

$$\mathrm{PA} + \ell\in\{1,2\} \vdash A \leftrightarrow \ell=1.$$

Then by Lemma 3 the theory PA + $\ell\in\{1,2\}$ + A is inconsistent. It follows that

$$PA + \ell\in\{1,2\} \vdash \ell\neq 1.$$

Applying Lemma 3 again we obtain that $PA + \ell\in\{1,2\}$ is inconsistent. This contradicts Lemma 2.5.

Now we are in a position to prove the following weaker version of our theorem.

PROPOSITION. Suppose ϕ is a formula of $\mathcal{L}$ such that for no boolean combination ψ of boxed formulae $GL\vdash \phi \leftrightarrow \psi$. Then for each $n\geq 1$ there exists an arithmetical interpretation f such that $f(\phi)\notin \mathbf{B}_n^{PA}$.

PROOF. Let the model $\mathcal{K}$ and the function h be defined as above. By Corollary 2

$$PA \vdash \ell=1 \longrightarrow f(\phi) \quad \text{and} \quad PA \vdash \ell=2 \longrightarrow \neg f(\phi).$$

So,

$$PA + \ell\in\{1,2\} \vdash \ell=1 \leftrightarrow f(\phi).$$

If for some $A \in \mathbf{B}_n$

$$PA + \ell\in\{1,2\} \vdash A \leftrightarrow f(\phi),$$

we shall get

$$PA + \ell\in\{1,2\} \vdash \ell=1 \leftrightarrow A$$

contradicting Corollary 3.

Of course, this Proposition is weaker then the Theorem because it does not give us any upper bound on the complexity of the interpretation constructed.

To improve the result in this respect we have to do an additional piece of work.

LEMMA 4. Suppose ϕ is a nonstable formula. Then there is a model $\mathcal{K} = (K , < , \Vdash)$ such that

(i) $K = \{0,\dots,k\}$ for some $k \in \mathbb{N}$;

(ii) 0 is the bottom node of $\mathcal{K}$;

(iii) 1 and 2 have the same successors and predecessors;

(iv) for all $i,z,w \in K$, if $1< z,w$ and $z,w <i$ then either $z<w$ or $w\leqslant z$;

(v) the set of predecessors of 1 and 2 is linearly ordered;

(vi) for all subformulae ψ of formulae ϕ $0\Vdash \Box\psi \longrightarrow \psi$;

(vii) $1\Vdash \phi$ and $2\Vdash\neg\phi$.

PROOF. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be the given pair of (treelike) models such that $\mathcal{K}_1\Vdash \phi$ and $\mathcal{K}_2\Vdash\neg\phi$. We identify the corresponding nodes of $\mathcal{K}_1$ and $\mathcal{K}_2$ except the roots and add a sufficiently long finite linearly ordered set T below them. The words "sufficently long" mean that T should be longer than the number of all boxed subformulae of ϕ. The forcing relation of propositional variables is defined arbitrarily on T.

By the Pigeon-Hole Principle there is a node $x \in T$ such that the same boxed subformulae of ϕ are forced at x as at the immediate predecessor of x. Clearly we shall have

$$x \Vdash \Box\psi \longrightarrow \psi$$

for all subformulae $\Box\psi$ of formula ϕ. Thus we may take x as the bottom node of the model required.

Let $\mathcal{K} = (K, <, \Vdash)$ be a model as in Lemma 4. We modify the definition of our function h as follows: $h(0)=0$ and for all m

$$
\begin{array}{ll}
\underline{\text{if}}\ z > h(m),\ z \notin \{1,2\}\ \underline{\text{and}}\ \mathrm{Prf}(m+1, \ulcorner \ell \neq z \urcorner) & (1)\\
\underline{\text{then}}\ h(m+1)=z & \\
\underline{\text{else}}\ \underline{\text{if}}\ h(m)<1\ \underline{\text{and}}\ \exists\, y_1, y_2 \leq m+1\ (\mathrm{Prf}(y_1, \ulcorner \ell \neq 1 \urcorner) \wedge & \\
\quad \mathrm{Prf}(y_2, \ulcorner \ell \neq 2 \urcorner)\ ) & (2)\\
\quad \underline{\text{then}}\ \underline{\text{if}}\ \exists\, z,u \leq m+1\ (\mathrm{Tr}_n(u) \wedge \mathrm{Prf}_u(z, \ulcorner \ell \neq 1 \urcorner) \wedge & \\
\qquad \forall\, t,w < z\ (\neg \mathrm{Tr}_n(w) \vee \neg \mathrm{Prf}_w(t, \ulcorner \ell \neq 2 \urcorner))) & (3)\\
\qquad \underline{\text{then}}\ h(m+1)=1 & \\
\qquad \underline{\text{else}}\ h(m+1)=2 & (4)\\
\quad \underline{\text{else}}\ h(m+1)=h(m). & (5)
\end{array}
$$

It is easy to see that all (the proofs of) statements of Lemma 2 except Statement 4 remain true for the modified function h. Now we are going to show that even a stronger version of 4 holds. The following technical claim is a generalized variant of Lemma 4.1.6 of [7] (cf also Corollary 2.23 [9]).

LEMMA 5. For all arithmetic sentences $A \in \mathbf{B}_n$

$$\mathrm{PA} \vdash \forall m\ \mathrm{Pr}\ulcorner \exists x<m\ \exists u<m\ (\mathrm{Prf}_u(x, \ulcorner A \urcorner) \wedge \mathrm{Tr}_n(u)) \longrightarrow A \urcorner$$

PROOF. Since Pr commutes with restricted quantifiers it is sufficient to prove that

$$\mathrm{PA} \vdash \forall m\ \forall u<m\ \mathrm{Pr}\ulcorner \exists x<m\ (\mathrm{Prf}_u(x, \ulcorner A \urcorner) \wedge \mathrm{Tr}_n(u)) \longrightarrow A \urcorner$$

Clearly

$$
\begin{aligned}
\mathrm{PA} \vdash \exists x<m\ \mathrm{Prf}_u(x, \ulcorner A \urcorner) \wedge u \in \mathbf{B}_n &\longrightarrow (\ \mathrm{Pr}(u \dot{\longrightarrow} \ulcorner A \urcorner) \wedge u \in \mathbf{B}_n\ )\\
&\longrightarrow (\ \mathrm{Pr}(u \dot{\longrightarrow} \ulcorner A \urcorner) \wedge \mathrm{Pr}(\ulcorner \mathrm{Tr}_n(u) \urcorner \dot{\longleftrightarrow} u)\ )\\
&\longrightarrow \mathrm{Pr}\ulcorner\ \mathrm{Tr}_n(u) \longrightarrow A \urcorner
\end{aligned}
$$

$$\longrightarrow \mathrm{Pr}\ulcorner \exists x{<}m\ (\mathrm{Prf}_u(x,\ulcorner A\urcorner) \wedge \mathrm{Tr}_n(u)) \longrightarrow A\urcorner.$$

On the other hand, since PA $\vdash u \notin \mathbf{B}_n \longrightarrow \neg\mathrm{Tr}_n(u)$

$$\mathrm{PA} \vdash \forall x{<}m\ \neg\mathrm{Prf}_u(x,\ulcorner A\urcorner) \vee u \notin \mathbf{B}_n \longrightarrow \mathrm{Pr}\ulcorner \forall x{<}m\ \neg\mathrm{Prf}_u(x,\ulcorner A\urcorner) \vee u \notin \mathbf{B}_n\urcorner$$

$$\longrightarrow \mathrm{Pr}\ulcorner \exists x{<}m\ (\mathrm{Prf}_u(x,\ulcorner A\urcorner) \wedge \mathrm{Tr}_n(u)) \longrightarrow \bot\urcorner$$

$$\longrightarrow \mathrm{Pr}\ulcorner \exists x{<}m\ (\mathrm{Prf}_u(x,\ulcorner A\urcorner) \wedge \mathrm{Tr}_n(u)) \longrightarrow A\urcorner.$$

Thus PA $\vdash \forall m\ \forall u\ \mathrm{Pr}\ulcorner \exists x{<}m\ (\mathrm{Prf}_u(x,\ulcorner A\urcorner) \wedge \mathrm{Tr}_n(u)) \longrightarrow A\urcorner$ and Lemma 5 follows.

LEMMA 6. PA $\vdash \mathrm{Pr}\ulcorner \ell\neq 1\urcorner \leftrightarrow \mathrm{Pr}\ulcorner \ell\neq 2\urcorner$.

PROOF. It is easy to see that

$$\mathrm{PA} \vdash \ell>1 \longrightarrow \exists i>1\ \mathrm{Pr}\ulcorner \ell\neq i\urcorner.$$

Besides,

$$\mathrm{PA} \vdash \ell<1 \wedge \mathrm{Pr}\ulcorner \ell\neq 1\urcorner \longrightarrow \neg\mathrm{Pr}\ulcorner \ell\neq 2\urcorner,$$

because clearly h would make a move from beneath 1, if we received both proofs of $\ell\neq 1$ and $\ell\neq 2$. Hence

$$\mathrm{PA} \vdash \mathrm{Prf}(m,\ulcorner \ell\neq 1\urcorner) \longrightarrow \mathrm{Pr}\ulcorner \mathrm{Pr}\ulcorner \ell\neq 1\urcorner\urcorner$$

$$\longrightarrow \mathrm{Pr}\ulcorner \ell<1 \longrightarrow \neg\mathrm{Pr}\ulcorner \ell\neq 2\urcorner\urcorner.$$

On the other hand, by clause (3) of the definition of h

$$\mathrm{PA} \vdash \mathrm{Prf}(m,\ulcorner \ell\neq 1\urcorner) \longrightarrow (\ \ell=2 \longrightarrow \exists x{<}m\ \exists u{<}m\ (\mathrm{Prf}_u(x,\ulcorner \ell\neq 2\urcorner) \wedge \mathrm{Tr}_n(u))\ ),$$

hence

$$\mathrm{PA} \vdash \mathrm{Prf}(m,\ulcorner \ell\neq 1\urcorner) \longrightarrow \mathrm{Pr}\ulcorner \ell=2 \longrightarrow \exists x{<}m\ \exists u{<}m\ (\mathrm{Prf}_u(x,\ulcorner \ell\neq 2\urcorner) \wedge \mathrm{Tr}_n(u))\urcorner.$$

Thus,

$$\mathrm{PA} \vdash \mathrm{Prf}(m,\ulcorner \ell\neq 1\urcorner) \longrightarrow \mathrm{Pr}\ulcorner \ell>1 \vee \ell<1 \vee \ell=2\urcorner$$

$$\longrightarrow \mathrm{Pr}\ulcorner (\exists i>1\ \mathrm{Pr}\ulcorner \ell\neq i\urcorner) \vee \neg\mathrm{Pr}\ulcorner \ell\neq 2\urcorner \vee \exists x{<}m\ \exists u{<}m\ (\mathrm{Prf}_u(x,\ulcorner \ell\neq 2\urcorner) \wedge \mathrm{Tr}_n(u))\urcorner.$$

Obviously

$$\mathrm{PA} \vdash \mathrm{Pr}\ulcorner (\exists i>1\ \mathrm{Pr}\ulcorner \ell\neq i\urcorner) \longrightarrow \ell\neq 2\urcorner$$

and by Lemma 5

$$\mathrm{PA} \vdash \forall m\ \mathrm{Pr}\ulcorner \exists x{<}m\ \exists u{<}m\ (\mathrm{Prf}_u(x,\ulcorner \ell\neq 2\urcorner) \wedge \mathrm{Tr}_n(u)) \longrightarrow \ell\neq 2\urcorner.$$

It follows that

$$\mathrm{PA} \vdash \mathrm{Prf}(m,\ulcorner \ell\neq 1\urcorner) \longrightarrow$$

$$\longrightarrow \mathrm{Pr}\ulcorner \mathrm{Pr}\ulcorner \ell\neq 2\urcorner \longrightarrow (\exists i>1\ \mathrm{Pr}\ulcorner \ell\neq i\urcorner) \vee \exists x{<}m\ \exists u{<}m\ (\mathrm{Prf}_u(x,\ulcorner \ell\neq 2\urcorner) \wedge \mathrm{Tr}_n(u))\urcorner$$

$$\longrightarrow \mathrm{Pr}\ulcorner \mathrm{Pr}\ulcorner \ell\neq 2\urcorner \longrightarrow \ell\neq 2\urcorner.$$

By formalized Löb's theorem we obtain

$$\mathrm{PA} \vdash \mathrm{Prf}(m, \ulcorner \ell \neq 1 \urcorner) \longrightarrow \mathrm{Pr}\ulcorner \ell \neq 2 \urcorner,$$

whence

$$\mathrm{PA} \vdash \exists m\, \mathrm{Prf}(m, \ulcorner \ell \neq 1 \urcorner) \longrightarrow \mathrm{Pr}\ulcorner \ell \neq 2 \urcorner.$$

The converse implication is symmetrical, so Lemma 6 follows.

COROLLARY 5. For all $i, j \in \mathrm{K}$

$$i < j \;\Rightarrow\; \mathrm{PA} \vdash \ell = i \longrightarrow \neg \mathrm{Pr}\ulcorner \ell \neq j \urcorner.$$

PROOF. This is trivial for $j \notin \{1,2\}$. Suppose $j=1$. The following argument can be formalized in PA:

"If $\mathrm{Pr}\ulcorner \ell \neq 1 \urcorner$ then by Lemma 6 $\mathrm{Pr}\ulcorner \ell \neq 2 \urcorner$, hence we are to receive both proofs of $\ell \neq 1$ and $\ell \neq 2$ by some stage m. Since $\ell < 1$ condition (2) will be satisfied, so h is bound to make a move either to 1 or to 2. Absurd."

We define the arithmetical interpretation f just as before and as in the proof of Solovay's second theorem (cf [8]) obtain the following

COROLLARY 6. For all subformulae ψ of formula ϕ

$$\mathrm{PA} \vdash f(\psi) \leftrightarrow \exists i\ (\ell = i \wedge i \Vdash \psi).$$

Consequently $f(\phi)$ is equivalent to a boolean combination of formulae of the form $\ell = i$, because the quantifier $\exists i$ actually ranges over the finite set K. For $i \notin \{1,2\}$ it is easily seen that the formula $\ell = i$ is $\mathrm{B}_1^{\mathrm{PA}}$. Now for $i \in \{1,2\}$.

It follows immediately from the definition of h that

$$\mathrm{PA} \vdash \ell = 1 \leftrightarrow \ell \in \{1,2\} \wedge \exists m\, (\exists u < m\, (\mathrm{Tr}_n(u) \wedge \mathrm{Prf}_u(m, \ulcorner \ell \neq 1 \urcorner) \wedge \forall y < m\, \forall w < m\, (\neg \mathrm{Tr}_n(w) \vee \neg \mathrm{Prf}_w(y, \ulcorner \ell \neq 2 \urcorner)))).$$

We have already seen that $\ell \in \{1,2\}$ is $\mathrm{B}_1^{\mathrm{PA}}$, hence $\ell = 1$ is $\Sigma_{n+1}^{\mathrm{PA}}$. But clearly

$$\mathrm{PA} \vdash \ell = 1 \leftrightarrow \ell \neq 2 \wedge \ell \in \{1,2\},$$

whence by symmetry of nodes 1 and 2 $\ell = 1$ is also Π_{n+1}^{PA}. Thus $\ell = 1$ and $f(\phi)$ both are $\Delta_{n+1}^{\mathrm{PA}}$.

We have received the desired upper bound on the arithmetical complexity of $f(\phi)$. To obtain the lower one we proceed just as in the proof of Proposition, because evidently Lemma 3 still holds for the modified function h.

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