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USING THE UNIVERSAL MODALITY: GAINS AND QUESTIONS

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USING THE UNIVERSAL MODALITY:
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1. Introduction

The paper suggests a simple and natural enrichment of the usual modal language $\mathcal{L}=\mathcal{L}(\Box)$: we add an auxiliary "universal" modality ⊡, interpreted in the usual Kripke semantics for $\mathcal{L}$ (on frames $\langle W,R\rangle$ with $R\subseteq W^2$) by the Cartesian square W^2 of the universe, i. e. for each $x\in W$: $x\models⊡\varphi$ iff $\forall y\in W(y\models\varphi)$.

Of course, this is definitely not a novelty. On the contrary, a number of authors have, explicitly or not, introduced the universal modality under different names and in different contexts, e.g. in tense logic, cf. [Cre], [Bul], [Gol2]; in dynamic logic, cf. [PT1], [PT2]; just technically, cf. [Koy]. In this paper we propose a systematic and purposeful investigation of this idea.

Indeed, taken in isolation, ⊡ is nothing more than the well-known old S5-modality. The point of the paper, however, is to consider ⊡ just as an *auxiliary* modality, enriching the classical modal language.

The enriched language $\mathcal{L}_⊡=\mathcal{L}(\Box,⊡)$ turns out to be fairly different from the classical one. In particular the notions of satisfiability, validity and consequence in models become interreducible as well as local and global first-order definability. These peculiarities are sketched in Section 3.

Section 4 deals with *modal definability* in $\mathcal{L}_⊡$. The universal modality considerably strengthens the expressiveness of the language. Model-theoretic characterizations of modal definability (cf. [Ben]) in $\mathcal{L}_⊡$ of classes of modal algebras, general frames, arbitrary and Δ-elementary classes of frames are obtained based on the notion of *modally definable closure* (see [Gor]). $\mathcal{L}_⊡$-definability is proved to be equivalent with *sequential definability* (see [Kap]) in $\mathcal{L}$. A number of first- and second-order frame conditions, which are definable in $\mathcal{L}_⊡$ but not in $\mathcal{L}$ are adduced.

In section 5 the minimal normal $\mathcal{L}_⊡$-logic $K_⊡$ is axiomatized and a general completeness theorem (with respect to models) for the family of normal extensions of $K_⊡$ is proved. Special attention is paid to the so called *minimal extensions* – $\mathcal{L}_⊡$-logics axiomatized with schemata of $\mathcal{L}$ over $K_⊡$. Conservativeness of all minimal extensions is shown.

Section 6 promotes a general study on possible *transfer of properties* of $\mathscr{L}$-logics to their minimal extensions. In particular the problems of transferring completeness, finite completeness and decidability are investigated and several partial but representative results are obtained. For a large class of $\mathscr{L}$-logics, completeness is shown to transfer to their minimal extensions, including all canonical ones as well as all complete logics having a theorem of the form $\Box^m p \to \Box^n p$, for m<n (e. g. all complete extensions of K4). Also the filtration technique is proved to be transferable. However, the general transferring problems remain still open.

In section 7 several concrete completeness and decidability results for logics having essentially $\mathscr{L}_{\boxdot}$-axiomatics are stated and some other elegant applications of $\boxdot$ are sketched, including the axiomatization of the so called "*proper names*" *for possible worlds* (cf. [PT1], [PT2]) and the axiomatization of the modality $\boxtimes$ "*necessary and sufficient*" (cf. Humberstone, [Hum] who calls it "*all and only*") having semantics $x \models \boxtimes\varphi$ iff $\forall y(xRy \Leftrightarrow y \models \varphi)$. In general it seems that $\boxdot$ can fairly well play the role of the so-called *admissible forms* (cf. [Gol2]) for axiomatizations.

2. Preliminaries

2.1 Throughout the paper we fix a propositional modal language $\mathscr{L}=\mathscr{L}(\Box)$ of one modality $\Box$ and its dual $\Diamond =_{DF} \neg\Box\neg$. We assume familiar the notions of *frame*, *model*, *general frame*, *modal algebra* and *validity* in them as well as the basic *frame constructions* – *subframe* (this will mean *generated* subframe), *p-morphic image*, *disjoint union* and the basic *algebraic constructions* – *subalgebra*, *homomorphic image*, *direct product* (for exact definitions see e.g. [HC], [Ben], [Grä]). Some notation: $\mathfrak{M} \models \varphi[x]$ will mean that the formula φ is true at the world x of the model $\mathfrak{M}$; $\mathfrak{M} \models \varphi$ will mean that φ is valid in $\mathfrak{M}$. Notation for truth and validity in frames, general frames and modal algebras will be analogous.

Also we use the categorical connections between general frames and modal algebras (see [Gol1]): to each general frame $\mathfrak{F}$ there corresponds a modal algebra $\mathfrak{F}^+$ and to each modal algebra $\mathfrak{A}$ the general frame $\mathfrak{A}^*$ which is its *Stone representation*.

If F is a frame, then the underlying frame of $(F^+)^*$ is called an *ultrafilter extension* of F, denoted by ue(F). F is called an *ultrafilter contraction* of ue(F).

If $\mathfrak{F}$ is a general frame then $(\mathfrak{F}^+)^*$ is called a *Stone representation* of $\mathfrak{F}$, denoted also $Sr(\mathfrak{F})$.

Another construction to be used is *ultraproduct of general frames* (see [Gol1] or [Ben]). Note that this construction, applied to a family of frames, yields a general frame (unlike the ordinary ultraproduct of frames) since it regards frames as full general frames. That is why it is called a *general ultraproduct*.

2.2 Now we enrich the language $\mathcal{L}$ with a new, call it *universal*, modality $⊡$ (and its dual $\Diamond$), interpreted in the Kripke semantics on a frame ⟨W,F⟩ by the Cartesian square W^2 of the universe W. Denote the language thus obtained by $\mathcal{L}_{⊡}$ and the set of formulae of $\mathcal{L}_{⊡}$ by $FOR_{⊡}$. Here are some basic notions for the new language.

A *frame* (or *standard frame*) for $\mathcal{L}_{⊡}$ ($\mathcal{L}_{⊡}$*-frame*) is a frame $\langle W,R,W^2\rangle$ which will be identified with ⟨W,R⟩. Operators □ and ⊡ on subsets of the universe are defined in a frame F=⟨W,R⟩ as follows:

$\Box X = \{x\in W \ / \ R(x)\subseteq X\}$ and $⊡X = \begin{cases} W \text{ if } X=W \\ \emptyset \text{ otherwise} \end{cases}$.

$\mathcal{L}_{⊡}$*-model* is a pair ⟨F,V⟩ where F is a frame and V is a valuation in $FOR_{⊡}$, obeying both the familiar conditions for an $\mathcal{L}$-valuation and the conditions for □ and ⊡: $V(\Box\varphi) = \Box V(\varphi)$ and $V(⊡\varphi) = ⊡V(\varphi)$.

The notions of *general* $\mathcal{L}_{⊡}$*-frame* is also defined in due standard way as a pair $\langle F,\mathbb{W}\rangle$ where F=⟨W,R⟩ is a frame and $\mathbb{W}\subseteq P(W)$ is closed under the Boolean operations, □ and ⊡. Clearly, the operator ⊡ does not impose extra closure conditions and so we can identify /general/ $\mathcal{L}$-frames with /general/ $\mathcal{L}_{⊡}$-frames.

An $\mathcal{L}_{⊡}$*-algebra* is a non-trivial modal algebra with an additional unary operator ⊡, satisfying the condition: $⊡a = \begin{cases} 1 & \text{if } a=1 \\ 0 & \text{otherwise} \end{cases}$ for each element a of the algebra.

It is easy to see that the $\mathcal{L}_{⊡}$-algebras are exactly those bimodal algebras which are isomorphic to $\mathfrak{F}^+$ for some general $\mathcal{L}_{⊡}$-frame $\mathfrak{F}$.

The notions of *validity* ($\models$) in $\mathcal{L}_{⊡}$-models, general $\mathcal{L}_{⊡}$-frames, $\mathcal{L}_{⊡}$-frames and $\mathcal{L}_{⊡}$-algebras are also defined in the standard way.

Closed formulae in $\mathcal{L}_{⊡}$ are the Boolean combinations of formulae beginning with $\boxdot$. (this notion is borrowed from [PT2])

3. Some effects of the universal modality.

The universal modality makes possible to express global properties (for the whole model or frame) by means of local ones. This is grounded on the obvious fact that truth of a closed formula at a point (local validity) is equivalent to validity of this formula in the whole model (global validity). Here are some issues of this effect:

Proposition 3.1 1) Global validity of any $\mathscr{L}_{\boxdot}$-formula φ is equivalent to local validity of $\boxdot\varphi$.

2) Global consequence $\Gamma\models\varphi$ is equivalent to local (point-wise) consequence $\boxdot(\Gamma)\models_1\varphi$, where $\boxdot(\Gamma) = \{\boxdot\gamma \ / \ \gamma\in\Gamma\}$.

3) When $\Gamma = \{\gamma_1,\dots,\gamma_n\}$ then $\Gamma\models\varphi$ is equivalent to the validity $\models\boxdot(\gamma_1\wedge\dots\wedge\gamma_n)\to\varphi$. #

Analogous effect appears in *first-order definability* (cf. [Ben]): an $\mathscr{L}_{\boxdot}$-formula φ is (globally) first-order definable iff $\boxdot\varphi$ is locally first-order definable.

As it follows from [Cha], first-order definability in $\mathscr{L}$ is undecidable, so we cannot rely on effective syntactical characterization of this property but only on sufficient conditions like Sahlqvist theorem (see [Sahl]) and some generalizations (cf. [Ben]). As a rule, these criteria are transferable here in the following sense: if $\varphi\in$FOR satisfies a concrete syntactical criterion for first-order definability (say Sahlqvist's) and φ' is obtained from φ by replacing some arbitrary occurrences of $\Box$ by $\boxdot$, then φ' is FOD too. However, one should be cautious in rising more general conjectures of this kind. An example: $(\Box p\to\Box\Box p)\wedge(\Box\Diamond p\to\Diamond\Box p)$ is FOD whereas $(\boxdot p\to\boxdot\boxdot p)\wedge(\Box\Diamond p\to\Diamond\Box p)$, which is equivalent to $(\Box\Diamond p\to\Diamond\Box p)$, is not (cf. [Ben, 7.4, 10.2]).

Likewise, first-order definability in $\mathscr{L}_{\boxdot}$ is not decidable. Anyway, a question rising:

Question 1: Is first-order definability in $\mathscr{L}_{\boxdot}$ decidable modulo first-order definability in $\mathscr{L}$?

4. Modal definability.

4.1 Classes of frames modally definable in $\mathcal{L}_{\boxdot}$.

If C is a class of $\mathcal{L}_{\boxdot}$-frames then the *modal theory of* C, $MT_{\boxdot}(C)$, is the set of all $\mathcal{L}_{\boxdot}$-formulae valid in C. If Γ is a set of $\mathcal{L}_{\boxdot}$-formulae then $FR(\Gamma)$ is the class of frames in which the formulae of Γ are valid.

Definition. A class of frames C is *modally definable* in $\mathcal{L}_{\boxdot}$ (*$\mathcal{L}_{\boxdot}$-definable*) if there exists a set $\Gamma\subseteq FOR_{\boxdot}$ such that for each frame F: $F\in C$ iff $F\models\Gamma$. The class of the modally definable classes of frames in $\mathcal{L}_{\boxdot}$ will be denoted by $MD(\mathcal{L}_{\boxdot})$. We will describe the $\mathcal{L}_{\boxdot}$-definability in a model-theoretic fashion, by means of closure under certain constructions.

Now we will define some operators on classes of algebras and frames. Let A be a class of algebras of some signature Ω. Then we denote by I(A) /S(A), H(A), P(A), U(A)/ the class of all isomorphic copies /subalgebras, homomorphic images, direct prodicts, ultraproducts/ of algebras from A.

Analogously, let C be a class of frames. Then we denote by $I_f(C)$ /$H_f(C)$, $U_f(C)$, SR(C), CU(C)/ the class C extended with all isomorphic copies / p-morphic images, ultraproducts, Stone representations, ultrafilter contractions/ of frames from C.

The same notation will be used for classes of general frames.

Fact 4.0 *All of the operators defined above preserve the validity of modal formulae.* # (see e.g. [Gol1])

Definition. *Modally definable closure* (MDC) of a class C of frames in $\mathcal{L}_{\boxdot}$ is the smallest $\mathcal{L}_{\boxdot}$-definable class [C] containing C.

Here is an explicit definition of [C]: $[C]=FR(MT_{\boxdot}(C))$.

The definitions and notations for modally definable classes and modally definable closures of classes of general frames, models and modal algebras are in the same spirit.

The next results are obtained as analogs to those in [Gor], where definability in the bimodal language $\mathcal{L}(R,-R)$ (with modalities both over a relation and its complement) is studied. The universal modality is explicitly definable in $\mathcal{L}(R,-R)$ by the formula $\boxdot p\leftrightarrow([R]p\wedge[-R]p)$.

Let $\mathbb{M}_{\boxdot}$ be the class of all $\mathcal{L}_{\boxdot}$-algebras.

Lemma 4.1 $\mathbb{M}_{\boxdot}$ *consists of simple algebras (without proper congruences).* # (cf. [Gor, 3.3])

Lemma 4.2. *If* $K \subseteq \mathbb{M}_{\boxdot}$ *then* $[K] = HSP(K) \cap \mathbb{M}_{\boxdot} = ISU(K)$. # (cf. [Gor, 3.5])

The above result says that a class of $\mathcal{L}_{\boxdot}$-algebras is modally definable iff it is closed under isomorphisms, subalgebras and ultraproducts.

Now we shall define specifically for $\mathcal{L}_{\boxdot}$, a simpler version of the notion of SA-construction introduced by Goldblatt and Thomason in [GT] and used to characterize $\mathcal{L}$-definability.

Definition. (cf. [Gor]) Let $\mathcal{F}=\langle W,R,\mathbb{W}\rangle$ and $F'=\langle W',R'\rangle$. F' is said to be a $\boxdot$-*collapse* of $\mathcal{F}$ iff there exists a complete atomic subalgebra of $\mathcal{F}^+$, $\mathcal{F}_1^+=\langle W,R,\mathbb{W}_1\rangle^+$ such that W' is the set of atoms of $\mathcal{F}_1^+$,

for each $a,b \in W'$: $R'ab$ iff $a \leq \Diamond b$ (i.e. $\forall x \in a \exists y \in b Rxy$)

and the following condition holds:

$$\forall a \in W' \forall X \in \mathbb{W}_1 (\forall b \in W' (R'ab \Rightarrow b \leq X) \Rightarrow R[a] \leq X).$$

Let C be a class of general frames. The class of all $\boxdot$-collapses of C will be denoted by $C_{\boxdot}(C)$.

Theorem 4.3 1) *If* C *is a class of frames then* $[C] = I_f C_{\boxdot} U_f(C)$.

2) *A class of frames* C *is in* $MD(\mathcal{L}_{\boxdot})$ *iff it is closed under isomorphisms and* $\boxdot$-*collapses of general ultraproducts.* # (cf. [Gor, 3.11])

The essential difference between this characterization and the classical case of Goldblatt & Thomason is due to the fact that in the enriched language the notions of [generated] subframe and disjoint union of frames are trivialized.

Definition A class of frames C is Δ-*elementary* iff there is a set $\Sigma \subseteq FOR_0$ such that for each frame F, $F \in C$ iff $F \models \Sigma$.

Corollary 4.5 i) *If* C *is a class of frames closed under ultraproducts then* $[C]=I_f C_{\boxdot}(C)$. # (cf. [Gor, 4.12, 4.13])

Following the scheme from [Gor] one could obtain another, more convenient characterization of the Δ-elementary classes in $MD(\mathcal{L}_{\boxdot})$. However, in the next section, we will have this characterization for free as well as other results concerning modal definability in $\mathcal{L}_{\boxdot}$.

4.2 Definability in $\mathcal{L}_{⊡}$ and sequential definability.

Kapron in [Kap] considers definability by means of sequents in the usual modal language as follows.

Definition

1) A *modal sequent* in $\mathcal{L}$ (*$\mathcal{L}$-sequent*) is a pair $\langle\Gamma,\Delta\rangle$ of finite sets of formulae of $\mathcal{L}$;

2) An $\mathcal{L}$-sequent $\langle\Gamma,\Delta\rangle$ is *valid in a model* $\mathfrak{M}$, notation $\mathfrak{M}\models\langle\Gamma,\Delta\rangle$, if $(\forall\varphi\in\Gamma)(\mathfrak{M}\models\varphi)$ implies $(\exists\psi\in\Delta)(\mathfrak{M}\models\psi)$;

3) $\langle\Gamma,\Delta\rangle$ is *valid in a frame* F, notation $F\models\langle\Gamma,\Delta\rangle$, if $\langle\Gamma,\Delta\rangle$ is valid in each model on F;

4) A *set of modal sequents* Ξ *is valid in a frame* F, $F\models\Xi$, if each member of Ξ is.

5) A *class of frames* C *is modally sequentially-definable* (MSD) (we do not use the notions of "axiomatic" and "sequent-axiomatic" class (cf. [GT] and [Kap]) because a class of frames can be defined by a set or formulas or sequents but not axiomatized by this set) if there exists a set Ξ of modal sequents such that for each frame F: $F\models\Xi$ iff $F\in C$. The class of modally sequent-definable classes in $\mathcal{L}$ will be denoted by MSD($\mathcal{L}$).

Lemma 4.6 MSD($\mathcal{L}$) $\subseteq$ MD($\mathcal{L}_{⊡}$).

Proof: Let a class C be defined by a set of sequents Ξ. For each sequent $\sigma=\langle\Gamma,\Delta\rangle\in\Xi$ define $\varphi_\sigma\in FOR_{⊡}$:

$$\varphi_\sigma =_{DF} \bigwedge_{\psi\in\Gamma} ⊡\psi \rightarrow \bigvee_{\theta\in\Delta} ⊡\theta.$$

It is easy to see, using 3.1, that for each model $\mathfrak{M}$: $\mathfrak{M}\models\sigma$ iff $\mathfrak{M}\models\varphi_\sigma$. So C is defined by the set of formulae $\{\varphi_\sigma \ / \ \sigma\in\Xi\}$. #

Now we are going to prove the opposite inclusion. This will be done using a kind of normal forms of the formulae of $\mathcal{L}_{⊡}$.

Definition. 1) *An elementary conjunction /disjunction/* is any formula of the type $\chi\wedge⊡\chi_0\wedge\Diamond\chi_1\wedge\ldots\wedge\Diamond\chi_s$ $/\chi\vee\Diamond\chi_0\vee⊡\chi_1\vee\ldots\vee⊡\chi_s/$, where $\chi,\chi_i\in\mathcal{L}(\Box)$;

2) *A conjunctive form, CF for short /disjunctive form, DF/*, is any conjunction /disjunction/ of elementary disjunctions (conjunctions).

By a standard propositional modal argument, each CF is equivalent to a DF and vice versa. So, by *form* we will mean either CF or DF.

Proposition 4.7 *For each formula* φ *and closed formula* ψ,

a) $\models\Box(\varphi\vee\psi)\leftrightarrow(\Box\varphi\vee\psi)$;

b) $\models\boxdot(\varphi\vee\psi)\leftrightarrow(\boxdot\varphi\vee\psi)$;

Proof: Standard semantic arguments, using 3.1. #

Theorem 4.8 *For each $\varphi\in FOR_{\boxdot}$ there is a form equivalent to φ.*

Proof: By induction on φ. The Boolean steps are standard. Let $\varphi = \Box\psi$ and $\psi' = \psi_1\wedge\ldots\wedge\psi_n$ be a CF of ψ. Then $\varphi \equiv \Box\psi_1\wedge\ldots\wedge\Box\psi_n$ and 5.7.a guarantees that all $\Box\psi_i$'s have equivalent CF's, and so does φ. For $\varphi = \boxdot\psi$, the proof is the same, using 5.7.b. #

Now let φ' be some CF, equivalent to φ and $\chi\vee\Diamond\chi_0\vee\boxdot\chi_1\vee\ldots\vee\boxdot\chi_s$ be an elementary disjunction in φ'. For each $\mathcal{L}_{\boxdot}$-model $\mathfrak{M}$: $\mathfrak{M}\models\chi\vee\Diamond\chi_0\vee\boxdot\chi_1\vee\ldots\vee\boxdot\chi_s$ iff $\mathfrak{M}\models\boxdot(\chi\vee\Diamond\chi_0\vee\boxdot\chi_1\vee\ldots\vee\boxdot\chi_s)$ iff $\mathfrak{M}\models\boxdot\chi\vee\Diamond\chi_0\vee\boxdot\chi_1\vee\ldots\vee\boxdot\chi_s$ iff $\mathfrak{M}\models\boxdot\neg\chi_0\rightarrow(\boxdot\chi\vee\ldots\vee\boxdot\chi_s\vee\boxdot\chi_1)$.

Let us denote by $\tau(\varphi)$ the conjunction of such transformed elementary disjunctions of φ'. So, $\tau(\varphi)$ is a formula without nested occurrences of $\boxdot$, such that for each $\mathcal{L}_{\boxdot}$-model $\mathfrak{M}$: $\mathfrak{M}\models\tau(\varphi)$ iff $\mathfrak{M}\models\varphi$. Obviously each conjunctive member of the transformed type is equivalent, with respect to validity in an $\mathcal{L}_{\boxdot}$-model (hence in a frame), to the corresponding sequent $\langle\neg\chi_0,\{\chi,\chi_1\ldots,\chi_s,\}\rangle$. This observation and theorem 4.8 yield an equivalence between sequential definability in $\mathcal{L}$ and definability in $\mathcal{L}_{\boxdot}$:

Theorem 4.9 $MSD(\mathcal{L}) = MD(\mathcal{L}_{\boxdot})$. #

Moreover, definability and sequential definability coincide in each polymodal language having $\boxdot$ explicitly definable.

Now some nice results from [Kap] about sequent definability are directly translated into $\mathcal{L}_{\boxdot}$:

Corollary 4.10 *A Δ-elementary class of frames is MD in $\mathcal{L}_{\boxdot}$ iff it is closed under p-morphisms and ultrafilter contractions.* # ([Kap, theorem 7])

In particular, a first-order condition is definable in $\mathcal{L}_{\boxdot}$ iff it is preserved under p-morphisms and ultrafilter contractions. For instance, $\forall x\neg Rxx$ and $\exists xRxx$ are not definable in $\mathcal{L}_{\boxdot}$ since the former fails after an appropriate p-morphism (e.g. the only mapping from $\langle\{x,y\},\{\langle x,y\rangle,\langle y,x\rangle\}\rangle$ onto $\langle\{u\},\{\langle u,u\rangle\}\rangle$) and the latter fails after an appropriate ultrafilter contraction (e.g. if $\mathbb{N}$ is the set of natural number then $\langle\mathbb{N},<\rangle\not\models\exists xRxx$ but $ue(\langle\mathbb{N},<\rangle)\models\exists xRxx$).

Corollary 4.11 *If* C *is a Δ-elementary class of frames then* $[C]=C_u H_f(C)$. # (cf. [Gor, 4.14])

This last result is preserved in the enrichments of $\mathcal{L}$ having $[u]$ explicitly definable. It has a methodological value: as a rule such enrichments yield non-standard semantics; let a first-order definable logic of such an enrichment of $\mathcal{L}$ is proved to be complete with respect to this semantics (this is usually done using classical techniques as canonical model, filtration etc.) Now the problem arises how to prove "standard completeness". For the purpose of completeness of such a logic, only the frames carrying descriptive general frames are sufficient; these frames are p-morphic images of their ultrafilter extensions. So the class of non-standard frames of this logic consists of all p-morphic images of standard frames. Therefore, if the logic is complete with respect to the standard semantics then this should be proved using the "copying" technique (i.e. construction of a standard p-morphic inverse image of all non-standard frames; cf. e.g. [GPT]).

Corollary 4.12 *A class of general frames* C *is MD in* $\mathcal{L}_{[u]}$ *iff* C *is closed under p-morphisms, ultraproducts* and *Stone representations, and the complement of* C *is closed under Stone representations.* # ([Kap, th. 6])

Actually, the proof of ([Kap, th. 6]) gives something more:

Corollary 4.13 *If* C *is a class of general frames then* $[C]=SR^{-1}H_f SRU_f(C)$.

4.3 Some examples of conditions that are $\mathcal{L}_{[u]}$-definable but not $\mathcal{L}$-definable.

SEMANTIC CONDITION	MODAL FORMULA
$R = W^2$	$\Box p \to [u]p$
$\exists x \forall y \neg Rxy$	$\langle u\rangle \Box \bot$
$\exists x \exists y Rxy$ /$R \neq \emptyset$/	$\langle u\rangle \Diamond \top$
$\lvert W \rvert = 1$	$p \to [u]p$
$\lvert W \rvert \leq n$	$\bigwedge_{i=1}^{n+1} \langle u\rangle p_i \to \bigvee_{i \neq j} \langle u\rangle (p_i \wedge p_j)$
$\forall x \forall y \forall z (Rxz \to Ryz)$	$\Box p \to [u]\Box p$
$\forall x \forall y \forall z (Rxy \to Rxz)$	$\Box p \to \Box [u] p$

$\forall x\forall y\forall z(Rxz\&x\neq z\to Ryz)$	$(p\wedge\Diamonddot\Box p)\to\Box p$
$\forall x\forall y\forall z(Rxy\to(x=z\vee Ryz)$	$p\wedge\Diamond\Box p\to\boxdot p$
$\forall x\forall y\forall z(x=z\vee Ryz)$	$p\wedge\Diamonddot\Box p\to\boxdot p$
$\forall x\exists yRyx$	$\boxdot\Box p\to p$
R^{-1} is well-founded	$\boxdot(\Box p\to p)\to p$

One could easily prove non-definability in $\mathcal{L}$ of these conditions, using the criteria of Goldblatt and Thomason ([GT, th.3, th.8]).

5. Axiomatization and proof theory of $\mathcal{L}_{\boxdot}$-logics

In this section we consider normal modal logics in $\mathcal{L}_{\boxdot}$.

5.1 The minimal normal $\mathcal{L}_{\boxdot}$-logic $K_{\boxdot}$. Standard and non-standard models. Minimal extensions of $\mathcal{L}$-logics in $\mathcal{L}_{\boxdot}$.

The first question arising here is: what will the analog of K in $\mathcal{L}_{\boxdot}$ be?. In order to obtain this analog we have to add to K some axiom schemata which would axiomatize the additional universal modality. Some schemata, coming at first sight are:

$(\boxdot)$	$\boxdot(p\to q)\to(\boxdot p\to\boxdot q)$
$(ref_{\boxdot})$	$\boxdot p\to p$
$(sym_{\boxdot})$	$p\to\boxdot\Diamonddot p$
$(trans_{\boxdot})$	$\boxdot p\to\boxdot\boxdot p$
$(incl)$	$\boxdot p\to\Box p$.

These schemata determine that $\boxdot$ is an S5-modality with corresponding equivalence relation U containing the relation R corresponding to $\Box$. This does not guarantee that U is a universal relation but this property can not be expressed by means of modal formulae since it is not preserved in disjoint unions. Indeed, we shall see (as a consequence of the completeness theorem) that the above schemata are all we can say about $\boxdot$. The extension of the minimal normal modal logic K with these schemata and the rule

$(NEC_{\boxdot})$: $\dfrac{\varphi}{\boxdot\varphi}$ will be called $K_{\boxdot}$.

Note that the rule $(NEC_{\boxdot})$, combined with (incl), makes the rule $(NEC_{\Box})$: $\dfrac{\varphi}{\Box\varphi}$ redundant.

So we have another semantics, larger than the one envisaged thus far, semantics, also correct for $\mathcal{L}_{\boxdot}$, viz. models over frames $\langle W,R,U\rangle$ where U is an equivalence relation containing R. These frames, when $U \neq W^2$, will be said to be *non-standard frames for* $\mathcal{L}_{\boxplus}$ and the frames $\langle W,R,W^2\rangle$ will be *standard* ones. Analogous terminology will be accepted for general frames and models over standard and non-standard frames. From now on the notion of $\mathcal{L}_{\boxplus}$-frame ($\mathcal{L}_{\boxplus}$-general frame, $\mathcal{L}_{\boxplus}$-model) will include both standard and non-standard cases.

Definition *A simple extension of* $K_{\boxplus}$, *or* $\mathcal{L}_{\boxplus}$*-logic*, is any extension of $K_{\boxplus}$ by means of schemata of $\mathcal{L}_{\boxplus}$.

Definition Given an *$\mathcal{L}$-logic* L, the *minimal extension of* L *in* $\mathcal{L}_{\boxplus}$ is the simple extension $L_{\boxplus}$ of $K_{\boxplus}$ with the schemata of L, taken over $\mathcal{L}_{\boxplus}$.

Now two general notions of completeness arise: completeness with respect to the general semantics and completeness with respect to the standard one. Of course we are interested in the latter, but, with S5 in mind, it is clear that these two notions are equivalent since each generated subframe (as a bi-relational frame) of a $\mathcal{L}_{\boxplus}$-frame is a standard $\mathcal{L}_{\boxplus}$-frame and each formula refuted in a frame is refuted in some of its generated subframes. Combining the above observations with the usual canonical model technique we obtain the *general completeness theorem for* $\mathcal{L}_{\boxplus}$*-logics:*

Theorem 5.1 *All* $\mathcal{L}_{\boxplus}$*-logics are complete with respect to the class of their standard* $\mathcal{L}_{\boxplus}$*-models.* #

In particular $K_{\boxplus}$ proves to be complete with respect to the standard $\mathcal{L}_{\boxplus}$-frames i.e. it is actually the minimal normal $\mathcal{L}_{\boxplus}$-logic, analogous to K.

5.2 Conservativity of the minimal extensions.

If V is a valuation on a frame and Γ a set of formulae, denote $V[\Gamma] = \{V(\varphi) \ / \ \varphi\in\Gamma\}$.

Lemma 5.2 *Let* $\mathfrak{M}=\langle W,R,V\rangle$ *be an* $\mathcal{L}_{\boxplus}$*-model. Then* $V[FOR] = V[FOR_{\boxplus}]$.

Proof: Immediately follows from the observation that, in each model, each formula, beginning with $\boxplus$ is equivalent either to $\top$ or to $\bot$. #

Corollary 5.3 *Let* L *be an* $\mathcal{L}$*-logic and* $\mathfrak{M}=\langle F,V\rangle$ *be an* L*-model. Let* $\mathfrak{M}'$ *be obtained from* $\mathfrak{M}$ *by extending* V *over* $FOR_{\boxplus}$ *accordingly. Then* $\mathfrak{M}'$ *is a standard* $L_{\boxplus}$*-model.*

Proof: Let ψ be an axiom schema of L and $\psi(\psi_1/p_1,\ldots,\psi_k/p_k)$ be a substitution instance of ψ in $\mathcal{L}_{\boxplus}$. Then there exist $\psi_1',\ldots,\psi_k'$ $\in$

FOR such that $V(\psi_i') = V(\psi_i)$ for $i=1,\ldots,k$ according to 5.2. Then $V(\psi(\psi_1/p_1,\ldots,\psi_k/p_k)) = V(\psi(\psi_1'/p_1,\ldots,\psi_k'/p_k)) = W$ since $\psi(\psi_1'/p_1,\ldots,\psi_k'/p_k)$ is an axiom of $\mathcal{L}$ and $\mathfrak{M}$ is an L-model. Obviously $\mathfrak{M}$ is a model for the schemata concerning ⊡, hence $\mathfrak{M}$ is an $L_{⊡}$-model.#

Note. In virtue of the above assertion we can consider each model $\langle W,R,V\rangle$ both as an $\mathcal{L}$-model and as a standard $\mathcal{L}_{⊡}$-model (extending the valuation V over $FOR_{⊡}$).

Corollary 5.4 *Each minimal extension* $L_{⊡}$ *of an* $\mathcal{L}$*-logic* L *is conservative over* L.

Proof: let $\varphi \in FOR$ and $L \nvdash \varphi$. Then by the general completeness theorem for L $\mathfrak{M} \not\models \varphi$ for some L-model $\mathfrak{M}=\langle W,R,V\rangle$. Then $\mathfrak{M}'=\langle W,R,W^2,V\rangle$, with V spread over $FOR_{⊡}$, is an $L_{⊡}$-model refuting φ. #

6. Transfer of results in minimal extensions.

When an enrichment $\mathcal{L}''$ of a (poly-)modal language $\mathcal{L}'$ is considered, one has a natural notion of *minimal extension*: if K' and K" are the correspon- ding minimal normal logics in $\mathcal{L}'$ and $\mathcal{L}''$ and L' is an $\mathcal{L}'$-logic (a simple extension of K') then the minimal extension in $\mathcal{L}''$ of L' is the $\mathcal{L}''$-logic L", axiomatized over K" by the axioms of L' over K'. Now a general question about translations arises: Let $\mathfrak{P}$ be some property of logics. Prove that if the $\mathcal{L}'$-logic L' enjoys the property $\mathfrak{P}$ then so does the minimal extension L". As a rule it is not difficult to prove such results for particular logics but the general transferring problems seem quite puzzling.

6.1 Transfer of completeness. Strong completeness of $\mathcal{L}$-logics.

Definition. An $\mathcal{L}_{⊡}$-logic L is *complete* if for each $\varphi \in FOR_{⊡}$ such that $L \nvdash \varphi$ there exists a frame F such that $F \models L$ and $F \not\models \varphi$.

The problems to be overcome while proving completeness of $\mathcal{L}_{⊡}$-logics seem to be the same as those for $\mathcal{L}$-logics (the universal modality is not expected to introduce new difficulties) so the methods will be the same too. Anyway one should surely prefer not to re-create here all familiar completeness achievements in the usual modal logics but to effortlessly transfer as many of them as possible to the enriched language. At least it seems quite plausible and desirable:

Conjecture 1. If an $\mathcal{L}$-logic L is complete, then its minimal extension $L_{\boxdot}$ is complete too.

As a first step toward attacking this conjecture (we hasten to warn the reader that in this paper it will not be completely decided) we will make some digression from $\mathcal{L}_{\boxdot}$ in order to translate the problem into an equivalent one in the original language $\mathcal{L}$.

Definition Let L be an $\mathcal{L}$-logic and $\varphi,\psi\in$FOR.

1) The *normal φ-theory over* L, denoted by $Th_L(\varphi)$, is the set of formulae derivable from $L\cup\{\varphi\}$ using MP and NEC; each $\psi\in Th_L(\varphi)$ is said to be *derivable in* $Th_L(\varphi)$, denoted $\varphi\vdash_L\psi$;

2) *ψ is a semantic consequence of φ over* L, notation $\varphi\models_L\psi$, if for each L-model $\mathfrak{M}$: $\mathfrak{M}\models\varphi$ implies $\mathfrak{M}\models\psi$;

3) *ψ is a normal semantic consequence of φ over* L, notation $\varphi\models_{nL}\psi$, if for each normal (i.e. based on an L-frame) L-model $\mathfrak{M}$: $\mathfrak{M}\models\varphi$ implies $\mathfrak{M}\models\psi$.

Lemma 6.1 (Deduction lemma for normal φ-theories)
Let L *be an $\mathcal{L}$-logic and $\varphi,\psi\in$FOR. Then $\varphi\vdash_L\psi$ iff there exists some formula θ of the type $\Box^{k_1}\varphi\wedge\ldots\wedge\Box^{k_s}\varphi$, such that $L\vdash\theta\to\psi$.*

Proof: An easy induction on the inference $\varphi\vdash_L\psi$. #

Proposition 6.2 (General completeness theorem - generalized version)
Let L *be an $\mathcal{L}$-logic and $\varphi,\psi\in$FOR. Then $\varphi\vdash_L\psi$ iff $\varphi\models_L\psi$.*

Proof: Since validity in a model is preserved under MP and NEC, we obtain the soundness-direction. Suppose $\varphi\vdash_L\psi$ does not hold. Then the set $X=\{\Box^k\varphi\ /\ k\in\mathbb{N}\}\cup\{\neg\psi\}$ is L-consistent: otherwise some finite subset should be inconsistent hence $\varphi\vdash_L\psi$ by 6.1. So there exists a maximal L-theory x containing X. Then φ is valid in the x-generated submodel $\mathfrak{M}^L_x$ of the canonical L-model while ψ is refuted in the root, whence $\varphi\models_L\psi$ fails. #

In particular, when φ is T, one obtains the usual general completeness theorem for $\mathcal{L}$-logics. Now a question arises: what will be the general version of the completeness theorem with respect to frames? A natural candidate for an answer is the following

Conjecture 2. An $\mathcal{L}$-logic L is complete iff it satisfies the condition: (*) for each $\varphi,\psi\in$FOR: $\varphi\vdash_L\psi$ iff $\varphi\models_{nL}\psi$. #

Notes.

1) (*), when φ is T, expresses the completeness of L.

2) $\varphi \vdash_{L} \psi$ implies $\varphi \models_{nL} \psi$ in virtue of the soundness of L and the preservation of validity in a model under MP and NEC.

Definition. An $\mathcal{L}$-logic will be called *strongly complete* if it satisfies the condition (*).

So conjecture 2 asserts that the strong completeness is not stronger than the ordinary completeness. Now we are going to prove that actually conjecture 1 and conjecture 2 claim the same.

Theorem 6.3 *An $\mathcal{L}$-logic* L *is strongly complete iff its minimal extension* $L_{\boxdot}$ *is complete.*

Proof: 1) Let $L_{\boxdot}$ be complete, $\varphi,\psi \in FOR$ and $\varphi \nvdash_{L} \psi$. Then there exists an $\mathcal{L}$-model $\mathfrak{M}$ such that $\mathfrak{M} \models L$, $\mathfrak{M} \models \varphi$ and $\mathfrak{M} \not\models \psi$. Now regarding $\mathfrak{M}$ as an $\mathcal{L}_{\boxdot}$-model we have $\mathfrak{M} \models L_{\boxdot}$ and $\mathfrak{M} \not\models \boxdot\varphi \to \boxdot\psi$, so $L_{\boxdot} \nvdash \boxdot\varphi \to \boxdot\psi$, and hence there exists a normal $L_{\boxdot}$-model $\mathfrak{M}'$ such that $\mathfrak{M}' \not\models \boxdot\varphi \to \boxdot\psi$, hence $\mathfrak{M}' \models \varphi$ and $\mathfrak{M}' \not\models \psi$, i.e. $\varphi \not\models_{nL} \psi$.

2) Let L be strongly complete, $\varphi \in FOR_{\boxdot}$ and $L_{\boxdot} \nvdash \varphi$. Then $L_{\boxdot} \nvdash \tau(\varphi)$, so there exists a conjunctive member $\boxdot\chi \to (\boxdot\chi_1 \vee \ldots \vee \boxdot\chi_s)$ of $\tau(\varphi)$ for some $\chi, \chi_1, \ldots, \chi_s \in FOR$, which is not derivable in $L_{\boxdot}$. We shall find normal L-models $\mathfrak{M}_i$ such that $\mathfrak{M}_i \models \chi$ and $\mathfrak{M}_i \not\models \chi_i$ for $i=1,\ldots,s$. Assume that for some i no such a model exists i.e. $\chi \models_{nL} \chi_i$. Then, by the strong completeness of L, $\chi \vdash_{L} \chi_i$, whence by 6.1. $L \vdash \theta \to \chi_i$ for some formula $\theta = \Box^{k_1}\chi \wedge \ldots \wedge \Box^{k_r}\chi$. Therefore $L_{\boxdot} \vdash (\Box^{k_1}\chi \wedge \ldots \wedge \Box^{k_r}\chi) \to \chi_i$; also $L_{\boxdot} \vdash \boxdot\chi \to (\Box^{k_1}\chi \wedge \ldots \wedge \Box^{k_r}\chi)$ hence $L_{\boxdot} \vdash \boxdot\chi \to \chi_i$ so $L_{\boxdot} \vdash \boxdot\chi \to \boxdot\chi_i$ and $L_{\boxdot} \vdash \boxdot\chi \to (\boxdot\chi_1 \vee \ldots \vee \boxdot\chi_s)$ - a contradiction. So, let $\mathfrak{M}_1, \ldots, \mathfrak{M}_s$ be the normal L-models with the desired property. Let $\mathfrak{M}$ be their disjoint union. Considered as an $\mathcal{L}_{\boxdot}$-model $\mathfrak{M}$ is a normal $L_{\boxdot}$-model such that $\mathfrak{M} \models \boxdot\chi$ and $\mathfrak{M} \models \Diamond\!\!\!\cdot\,\neg\chi_i$ for $i=1,\ldots,s$, so $\mathfrak{M} \not\models \boxdot\chi \to (\boxdot\chi_1 \vee \ldots \vee \boxdot\chi_s)$, therefore $\mathfrak{M} \not\models \tau(\varphi)$, and so $\mathfrak{M} \not\models \varphi$. This shows that $L_{\boxdot}$ is complete. #

Corollary 6.4

1) Each canonical $\mathcal{L}$-logic is strongly complete.

2) If L *is a canonical $\mathcal{L}$-logic, then* $L_{\boxdot}$ *is complete.*

Proof: Immediately by (the proof of) 6.2. #

In particular, in virtue of a Fine's result [Fin], all first-order definable complete axiomatics are strongly complete and hence their minimal extensions in $\mathcal{L}_{\boxdot}$ are complete.

Corollary 6.5 *If a complete $\mathcal{L}$-logic* L *contains a theorem of the type* $\Box^k p \to \Box^m p$ *for some m,k such that m>k, then* L *is strongly complete and hence* L_ω *is complete.*

Proof: Let $L \vdash \Box^k p \to \Box^m p$, m>k. Then the set $\{\Box^i \varphi \,/\, i \in \mathbb{N}\} \cup \{\chi\}$ is L-consistent iff $\{\Box^i \varphi \,/\, i=0,1,\ldots m-1\} \cup \{\chi\}$ is L-consistent, since for each $n \geq m$ a formula $\Box^r p \to \Box^n p$ is a theorem of L for some integer r such that $k \leq r < m$. (More exactly we can choose $r =_{DF} k + r_0$ where r_0 is the remainder of n-k modulo m-k.) Therefore for each $\varphi, \chi \in$ FOR, $\varphi \vdash_L \chi$ iff $L \vdash (\varphi \wedge \Box\varphi \wedge \ldots \wedge \Box^{m-1}\varphi) \to \chi$. On the other hand for each generated L-model $\mathfrak{M}$: $\mathfrak{M} \models \varphi$ iff $\mathfrak{M} \models \varphi \wedge \Box\varphi \wedge \ldots \wedge \Box^{m-1}\varphi[x]$ where x is the root of $\mathfrak{M}$. Hence L is strongly complete. #

E. g. each complete extension of K4 is strongly complete.

Question 2: Are the conjectures 1/2 true?

The above assertions show that, if our conjecture 1-2 is not true, a counter-example should be a relatively weak, complete but not canonical (even not compact) extension of K.

Question 2': Isn't KM = K + $\Box\Diamond p \to \Diamond\Box p$ such a counter-example?

Let L be an $\mathcal{L}$-logic and $\varphi, \psi \in$ FOR. Let us note that, as a consequence of 6.1, $\varphi \vdash \psi$ is equivalent to L-consistency of the set $\{\neg\psi, \varphi, \Box\varphi, \ldots, \Box^n\varphi, \ldots\}$. So, an equivalent definition of strong completeness is: L is strongly complete if for each $\varphi, \psi \in$ FOR, if the set $S = \{\neg\psi, \varphi, \Box\varphi, \ldots, \Box^n\varphi, \ldots\}$ is L-consistent then S is satisfiable in a world of a normal L-model. This condition is a particular case of the notion of compactness: L is compact if each L-consistent set is satisfiable in a normal L-model (see [HC]). So, the above condition could be called *weak compactness*. Thus: an $\mathcal{L}$-logic L is strongly complete iff it is weakly compact.

Here is a sufficient model-theoretic condition for compactness, hence for strong completeness of L-logics.

Proposition 6.6 *If* L *is a complete* $\mathcal{L}$-*logic and* FR(L) *is closed under ultrapowers, then* L *is compact.*

Proof: FR(L) is closed under generated subframes, disjoint unions and isomorphisms, therefore closedness under ultrapowers implies closeness under ultraproducts (see [Ben, S.2]). Now let S be an L-consistent set and S_f be the set of all finite subsets of S.

For each $\Gamma\in S_f$ there exists a normal L-model $\mathfrak{M}_\Gamma = \langle W_\Gamma, R_\Gamma, V_\Gamma\rangle$ and $x_\Gamma\in W_\Gamma$, such that $\mathfrak{M}_\Gamma \models \left(\bigwedge_{\varphi\in\Gamma}\varphi\right)[x_\Gamma]$ i. e. $\mathfrak{M}_\Gamma \models \left(\bigwedge_{\varphi\in\Gamma} ST(\varphi)\right)[x_\Gamma]$ where $\mathfrak{M}_\Gamma$ is considered as a model for the first-order language L_1 having a binary predicate symbol R and unary predicate symbols (corresponding to the propositional variables) $P_1, P_2, \ldots$ and $ST(\varphi)$ is the standard translation φ in L_1 (cf. [Ben]). Let for each $\Gamma\in S_f$ $X_\Gamma = \{\Delta\in S_f \ / \ \Gamma\subseteq\Delta\}$. The family $X = \{X_\Gamma \ / \ \Gamma\in S_f\}$ is centered, hence it is included in a ultrafilter D. Let $\langle\mathfrak{M}, x\rangle = \left(\prod_{\Gamma\in S_f}\langle\mathfrak{M}_\Gamma, x_\Gamma\rangle\right)/D$. $\mathfrak{M}$ is a normal L-model (the underlying frame for $\mathfrak{M}$ being an L-frame) and $\mathfrak{M}\models S[x]$. #

Here is an example of a weakly compact (being transitive) but not compact (cf. [HC]) logic:

$K4.3W = K + \Box(\Box p\to p)\to\Box p + \Box((\Box p\wedge p)\to q)\vee\Box((\Box q\wedge q)\to p)$

Warning. ([Vak]) The results about completeness transfer do not carryover to completeness results with respect to classes of frames, defined through additional semantic conditions, inexpressible syntactically. For instance the logic S4.3 is complete w.r.t. the class of all linear orderings LO (cf. [Seg]) but is characterized by the class of weak linear orderings WLO. However, $S4.3_{\boxdot}$ (which is characterized by WLO too, thanks to 6.2, 6.4 and the canonicity of S4.3) is not complete w.r.t. LO since the formula $\boxdot(\Box p\to q)\vee\boxdot(\Box q\to p)$ is true in LO and not true in WLO, hence not a theorem of $S4.3_{\boxdot}$. #

6.2 Transfer of finite completeness in $\mathcal{L}_{\boxdot}$.

Now we are interested in showing finite completeness of $\mathcal{L}_{\boxdot}$-logics. Of course we can confine ourselves to the class of standard models. Let us first note the following analog of Segerberg's theorem, proved just as in the classical case (see e.g. [HC]):

Fact 6.7 *An $\mathcal{L}_{\boxdot}$-logic has the finite model property iff it has the finite frame property.* #

We can translate the problem into $\mathcal{L}$, too:

Definition An $\mathcal{L}$-logic L is *strongly finitely complete* if for each $\varphi,\psi\in FOR$ such that $\varphi\nvdash_L\psi$ there exists a finite normal L-model $\mathfrak{M}$ such that $\mathfrak{M}\models\varphi$ and $\mathfrak{M}\not\models\psi$.

In virtue of the proof of the equivalence between FMP and FFP in $\mathcal{L}$ the requirement of normality of the refuting model can be dropped.

Here we suggest the next series of progressively harder problems:

Question 3: Do canonicity and finite completeness in $\mathcal{L}$ imply strong finite completeness?

Question 4: Do finite completeness and strong completeness imply strong finite completeness?

Question 5: Does finite completeness imply strong completeness?

Question 6: Does finite completeness imply strong finite completeness?

Theorem 6.8 *An $\mathcal{L}$-logic* L *is strongly finitely complete iff the minimal extension* $L_{\boxdot}$ *is finitely complete.*

Proof: The same as the proof of 6.3, since a finite disjoint union of finite models is a finite model, too. #

Unfortunately for the time being we have no universal means to prove strong finite completeness, so 6.8 still is not of a great use. However, we can easily ascertain the transferring of the stronger but most frequently used property for proving FMP, viz. admitting filtration.

Theorem 6.9 *If an $\mathcal{L}$-logic* L *admits filtration then* $L_{\boxdot}$ *does so, too.*

Proof: Let $\Gamma\subseteq FOR_{\boxdot}$ be closed under subformulae and $\mathfrak{M}=\langle F,V\rangle$ be an $L_{\boxdot}$-model. For each $\varphi\in\Gamma$ we take a formula $\varphi'\in FOR$ obtained from φ by replacing all occurrences of subformulae of the sort $\boxdot\psi$ by $\top$ or $\bot$ in accordance with $V(\boxdot\psi)$ (as in the proof of 5.2). Obviously $V(\varphi)=V(\varphi')$. Thus we obtain a set $\Gamma'\subseteq FOR$ which is closed under subformulae, too. The sets Γ and Γ' will lead to the same filtrations since $\boxdot$ does not add new conditions. We can obtain by filtration on Γ (hence on Γ') an L-model hence an $L_{\boxdot}$-model by 5.3.#

6.3 Transfer of decidability in $\mathcal{L}_{\boxdot}$.

The next general problem is:

Question 7: Does decidability of an $\mathcal{L}$-logic L imply decidability of

$L_{\boxdot}$?

Proposition 6.10

The disjunction property $\dfrac{\vdash \boxdot\varphi\rightarrow(\boxdot\psi\vee\boxdot\chi)}{\vdash \boxdot\varphi\rightarrow\boxdot\psi \text{ or } \vdash \boxdot\varphi\rightarrow\boxdot\chi}$, $\varphi,\psi,\chi \in$ FOR, *holds in* $L_{\boxdot}$, *for each* $\mathcal{L}_{\boxdot}$*-logic* L.

Proof: Let us assume $L_{\boxdot} \nvdash \boxdot\varphi\rightarrow\boxdot\psi$ and $L_{\boxdot} \nvdash \boxdot\varphi\rightarrow\boxdot\chi$. Then there exist $L_{\boxdot}$-models $\mathfrak{M}_1=\langle W_1,R_1,V_1\rangle$ and $\mathfrak{M}_2=\langle W_2,R_2,V_2\rangle$ and points $x_1\in W_1$ and $x_2\in W_2$ such that $\mathfrak{M}_1\models\varphi$, $\mathfrak{M}_1\not\models\psi[x_1]$, $\mathfrak{M}_2\models\varphi$ and $\mathfrak{M}_2\not\models\chi[x_2]$. Let $\mathfrak{M}$ be the disjoint union of $\mathfrak{M}_1$ and $\mathfrak{M}_2$, considered as L-models. $\mathfrak{M}\models L \Rightarrow \mathfrak{M}\models L_{\boxdot}$ (as an $L_{\boxdot}$-model) by 5.3. Moreover $\mathfrak{M}\models\varphi$ hence $\mathfrak{M}\models\boxdot\varphi$ but $\mathfrak{M}\not\models\psi[x_1] \Rightarrow \mathfrak{M}\not\models\boxdot\psi$ and $\mathfrak{M}\not\models\chi[x_2] \Rightarrow \mathfrak{M}\not\models\boxdot\chi$ hence $\mathfrak{M}\not\models\boxdot\varphi\rightarrow(\boxdot\psi\vee\boxdot\chi)$. Therefore $L_{\boxdot}\nvdash\boxdot\varphi\rightarrow(\boxdot\psi\vee\boxdot\chi)$. #

Though the translation τ, proposition 6.10 reduces the decision problem for $L_{\boxdot}$ to the problem of deciding provability in $L^{*}_{\boxdot}$ of formulae of the form $\boxdot\varphi\rightarrow\psi$ (equivalent to $\boxdot\varphi\rightarrow\boxdot\psi$) where $\varphi,\psi \in$ FOR. Here we hazard a positive answer of question 7, raising the next

Conjecture 3: For each [decidable] $\mathcal{L}$-logic L there exists an effective function f_L: FOR×FOR $\rightarrow \mathbb{N}$ such that for each $\varphi,\psi\in$FOR $\varphi\vdash_L\psi$ iff $\vdash_L(\varphi\wedge\Box\varphi\wedge\ldots\wedge\Box^n\varphi)\rightarrow\psi$ where $n=f_L(\varphi,\psi)$. #

We finish this section with a strengthening of question 7:

Question 8: Do minimal extensions preserve complexity?

And one more question:

Question 9: Is the interpolation property preserved by minimal extensions?

7. Some uses of the universal modality.

The universal modality can a be fairly useful tool for axiomatization. Here we sketch some examples demonstrating its merits.

7.1 Let us first mention that the standard techniques for proving completeness and finite model property in $\mathcal{L}$ (canonical model, filtrations etc.) works as well in $\mathcal{L}_{\boxdot}$. As we have already noticed, the canonical model technique will cause no additional complications, connected with the non-standard models, since all $\boxdot$-rooted models are standard, which is sufficient for the purposes of the completeness. For instance it is a routine task to prove that all but the last conditions, adduced in 4.3, are *axiomatized* by the corresponding formulae, added to $K_{\boxdot}$. Indeed, all of them but the last are canonical (note that $\boxdot\Box$ corresponds to the composition $W^2 \circ R$ and $\boxdot\Box p \to p$ says that this relation is reflexive which is equivalent to the condition given in section 5.3). All these examples axiomatize logics admitting filtration and hence having the finite model property and being decidable. (The proof for the logic of finite paths $K_{\boxdot} + \boxdot(\Box p \to p) \to p$ goes through a minimal filtration and is a slight modification of the well-known proof of completeness for GL.)

Another curious example is due to Dimiter Vakarelov, [Vak]. The condition $\exists x Rxx$ is definable neither in $\mathcal{L}$ nor in $\mathcal{L}_{\boxdot}$ as we have already known. This condition is axiomatized in $\mathcal{L}$ by K, i.e. no part of it can be expressed there. In $\mathcal{L}_{\boxdot}$ however, it is axiomatized over $K_{\boxdot}$ by the infinite set of axioms $\{\theta_n\}_{n\in\mathbb{N}}$ where $\theta_n = \Diamond((\Box p_1 \to p_1) \wedge \ldots \wedge (\Box p_n \to p_n))$. First, all frames with an reflexive world satisfy all θ_n. Actually, validity of θ_n in $F=\langle W,R\rangle$ means that for every n subsets $P_1,\ldots,P_n$ of W there exists a world x which has R-successors in all P's containing x. In particular, if F is finite and $W=\{x_1,\ldots,x_n\}$ then $F \models \theta_n$ implies (taking $\{x_1\},\ldots,\{x_n\}$) that $F \models \exists x Rxx$. So, the axioms $\{\theta_n\}_{n\in\mathbb{N}}$ guarantees existence of an R-reflexive world in all finite frames satisfying them though not in all such infinite frames. The proof of completeness uses the standard canonical model technique: observe that if $L = K_{\boxdot} + \{\theta_n\}_{n\in\mathbb{N}}$ and $L \nvdash \varphi$ then $\{\varphi\}\cup\{\Box\alpha\to\alpha \ / \ \alpha\in FOR_{\boxdot}\}$ is inconsistent and hence included in a maximal L-consistent set which is reflexive.

7.2 The finitely axiomatized $\mathcal{L}_{\boxdot}$-logics form a lattice (unlike the finitely axiomatized $\mathcal{L}$-logics, cf. [Ben, ch. 5]) as follows from the next proposition.

Proposition 7.1 *If* $L_1 = K_{\boxdot} + \varphi_1$ *and* $L_2 = K_{\boxdot} + \varphi_2$ *are* $\mathcal{L}_{\boxdot}$*-logics then* $L_1 \cap L_2 = K_{\boxdot} + \boxdot\varphi_1 \vee \boxdot\varphi_2$.

Proof: It is clear that $L_1 \cap L_2 \vdash \boxdot\varphi_1 \vee \boxdot\varphi_2$, hence $K_{\boxdot} + \boxdot\varphi_1 \vee \boxdot\varphi_2 \subseteq L_1 \cap L_2$. Vice versa, a standard deduction lemma for $\mathcal{L}_{\boxdot}$-logics shows that $L_1 \vdash \psi$ iff $K_{\boxdot} \vdash \boxdot\varphi_1^1 \wedge \ldots \wedge \boxdot\varphi_1^k \to \psi$ for certain substitution instances $\varphi_1^1, \ldots \varphi_1^k$ of φ_1; analogously $L_2 \vdash \psi$ iff $K_{\boxdot} \vdash \boxdot\varphi_2^1 \wedge \ldots \wedge \boxdot\varphi_2^m \to \psi$ for some $\varphi_2^1, \ldots \varphi_2^m$. But $K_{\boxdot} + t\boxdot\varphi_1 \vee \boxdot\varphi_2 \vdash (\boxdot\varphi_1^1 \wedge \ldots \wedge \boxdot\varphi_1^k) \vee (\boxdot\varphi_2^1 \wedge \ldots \wedge \boxdot\varphi_2^m)$, whence

$L_1 \cap L_2 \le K_{\boxdot} + \boxdot\varphi_1 \vee \boxdot\varphi_2$. #

The above fact is certainly not surprising; an analogous property is proved by analogous arguments, for the normal extensions of S4 in [MR].

7.3 The prime stimulus for considering the universal modality has come up in the context of the *proper names for the possible worlds* (see [PT1], [PT2]). They are special kind of propositional variables evaluated in the Kripke semantics in single worlds which, added to modal and dynamic languages, strongly increase their expressiveness and deductive power. A complete axiomatization of the minimal normal logic K_N in the modal language with names is given in [GPT] using special kinds of axiom schemata, called by Goldblatt [Gol2] *admissible forms*. The names are axiomatized by the scheme $M(c \wedge A) \to L(c \to A)$, where c is a name, A is a formula, M is a possibility form and L is a necessity form.

After adding the universal modality to the language the need of forms disappears because the form scheme is replaced by the axiom scheme $\Diamond(c \wedge A) \to \boxdot(c \to A)$. In addition we can already say that each name has a denotation by means of the schema $\Diamond c$.

7.4 Using $\boxdot$ one could elegantly axiomatize puzzling non-classical modalities. Just an example:

Let us consider a modality $\boxtimes$ with the following semantics in ordinary Kripke model $\mathfrak{M} = \langle W, R, V \rangle$:

(*) $\mathfrak{M} \models \boxtimes\varphi[x]$ iff $\forall y (Rxy \Leftrightarrow \mathfrak{M} \models \varphi[y])$ i.e. $R(x) = V(\varphi)$.

We shall call $\boxtimes$ the "iff-modality" having in mind a natural interpretation as "necessary and sufficient" (see [GPT]) or "all and only" (cf. Humberstone [Hum]). This is a fairly strange modality: neither monotonic, nor anti-monotonic, but extensional; no formula of the kind $\boxtimes\varphi$ or its negation is universally valid.

Humberstone has axiomatized $\boxtimes$ in [Hum] by means of an infinite set of schemata and an infinite set of rules. Adding the universal modality to the language we can replace this really ingenious axiomatics by the following transparent one in the language $\mathcal{L}(\boxtimes, \boxdot)$:

Axiom schemata of the logic IFF:

1) all propositional tautologies;

2) S5 axioms for $\boxdot$;

3) $(\boxtimes_1)$: $(\boxtimes p \wedge \boxtimes q) \to \boxdot(p \leftrightarrow q)$;

4) $(\boxtimes_2)$: $\boxdot(p \leftrightarrow q) \to (\boxtimes p \leftrightarrow \boxtimes q)$;

Rules: MP and $NEC_{\boxdot}$.

Theorem 7.2 The logic IFF is sound and complete.

Proof: Soundness is straightforward. Let φ be an IFF-consistent formula and w be a maximal IFF-consistent set containing φ. (There are no problems in the Lindenbaum lemma.) Denote $W_0 = \{y \,/\, y$ is a maximal IFF-consistent set and $\boxdot w \subseteq y\}$. Let w' be a copy of w and $W = W_0 \cup \{w'\}$. It is clear that for each $x, y \in W$ $\boxdot x \subseteq y$. Now we define a relation R in W:

$$Rxy \text{ iff } \Big[(\boxtimes\psi \in x \ \&\ \psi \in y \text{ for some } \psi) \text{ or } (\boxtimes\psi \notin x \ \&\ y = w' \text{ for each } \psi)\Big].$$

Obviously $R(w) = R(w')$. Consider the model $\langle W,R,V\rangle$ with the canonical valuation V: $V(p) = \{x \in W \,/\, \psi \in x\}$ for each propositional variable p. Extend V to a valuation on all formulae through the standard semantics of $\boxdot$ and (*). Now we shall prove the *truth lemma*: For each formula ψ, $V(\psi) = \{x \in W \,/\, \psi \in x\}$.

The only non-trivial case in the induction on ψ is that $V(\boxtimes\psi) = \{x \,/\, \boxtimes\psi \in x\}$.

1) Let $\boxtimes\psi \in x$.

a) if $y \in V(\psi)$ then by IH $\psi \in y$ and Rxy by definition;

b) if Rxy then there exists χ such that $\boxtimes\chi \in x$ and $\chi \in y$. Then $\boxtimes\psi \wedge \boxtimes\chi \in x$ hence by $(\boxtimes_1)$ $\boxdot(\psi \leftrightarrow \chi) \in x$, so $\psi \leftrightarrow \chi \in y$ and therefore $\psi \in y$.

2) Let $\boxtimes\psi \notin x$.

case a) for each χ, $\boxtimes\chi \notin x$. Therefore Rxw'. If $\psi \notin w'$ then $w' \in R(x) \setminus V(\psi)$; if $\psi \in w'$ then $\psi \in w$ and so $w \in V(\psi) \setminus R(x)$;

case b) $\boxtimes\chi \in x$ for some χ. Then $\boxtimes\chi \to \boxtimes\psi \notin x$ hence by $(\boxtimes_2)$ $\boxdot(\chi \leftrightarrow \psi) \notin x$ so $\dot{\Diamond}((\psi \wedge \neg\chi) \vee (\chi \wedge \neg\psi)) \in x$ i.e. there exists y such that $(\psi \wedge \neg\chi) \in y$ or $(\chi \wedge \neg\psi) \in y$.

subcase b1) $(\psi \wedge \neg\chi) \in y$. Then $\neg Rxy$: otherwise, since $\boxtimes\chi \in x$, there exists θ such that $\boxtimes\theta \in x$ and $\theta \in y$ and hence $\boxtimes\theta \wedge \boxtimes\psi \in x$, so $\boxdot(\theta \leftrightarrow \chi) \in x$, whence $\theta \leftrightarrow \chi \in y$ and so $\chi \in y$: a contradiction.

subcase b2) $\neg\psi \wedge \chi \in y$. Then $\boxtimes\chi \in x$ and $\chi \in y$ imply Rxy and so $y \in R(x) \setminus V(\psi)$.

The proof of the lemma is finished. So the theorem is proved, too.#

Note. Both modalities $\boxdot$ and $\boxtimes$ are expressible in the bimodal language $\mathcal{L}(R,-R)$: $\boxdot p = \boxplus p \wedge \boxminus p$ and $\boxtimes p = \boxplus p \wedge \boxminus\neg p$ where $\boxplus$ and $\boxminus$ are the modalities corresponding to R and −R. The minimal logic of $\mathcal{L}(R,-R)$,

$\tilde{K}$ is axiomatized (see [GPT]) just by S5-axioms for @ thus expressed and is proved to have the FMP and hence to be decidable. Since $\tilde{K}$ is conservative over IFF (by an easy semantic argument) we have the FMP and the decidability of IFF.

The last two examples suggest that the universal modality can fairly well play the role of the admissible forms and more precisely, that the admissible forms are devised as rough approximations of @.

8. Epilogue.

In this paper one of the simplest and most natural enrichments of the classical modal languages was investigated. Several advantages were shown and a series of naturally arising problems of "transferring properties" (most of them, more or less, left open) concerning the universal modality was raised. The advantages pointed out are specific for the case while the problems are typical: they suggest a general approach to a large class of enrichments of this kind. In our opinion, as far as most of the applications of modality, arising at the present time, need and use such enrichments, these problems should lay down an important direction of advance.

Finally, we think the paper gives enough grounds to advertise the universal modality as a natural and helpful modal tool, providing a better medium for the mission of modality.

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