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PROVABILITY LOGICS FOR NATURAL TURING PROGRESSIONS OF ARITHMETICAL THEORIES

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PROVABILITY LOGICS FOR NATURAL TURING PROGRESSIONS OF ARITHMETICAL THEORIES

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Abstract

Provability logics with many modal operators for progressions of theories obtained by iterating their consistency statements are introduced. The corresponding arithmetical completeness theorem is proved.

Received May 1990

PROVABILITY LOGICS FOR NATURAL TURING PROGRESSIONS OF ARITHMETICAL THEORIES

We shall deal here with the usual arithmetical interpretation of propositional language with several modal operators, i.e. modal operators will be interpreted as provability predicates in certain recursively enumerable (r.e.) theories. We shall assume for simplicity that all the theories considered are true extensions of Peano Arithmetic (PA) in the language of PA.

Let $(\mathcal{T}_i)$, $i=1,\dots,n$ be a provably increasing sequence of theories. Carlson's logic $PRL(n)^+$ axiomatizes the collection of all modal formulas which are universally provable in PA under the interpretation w.r.t. $(\mathcal{T}_i)$, provided each $\mathcal{T}_{i+1}$ is "much stronger" than $\mathcal{T}_i$ (i.e. $\mathcal{T}_{i+1}$ proves the local reflection principle $Rfn(\mathcal{T}_i)$ for $\mathcal{T}_i$) (cf. [1]). Of course the analogous result still holds for infinite monotone recursive progressions of theories $(\mathcal{T}_\alpha)$, where α is a constructive ordinal notation (cf. [2]). For every ordinal λ the provability logic $PRL(\lambda)^+$ in the language with λ operators is common for all the progressions of this kind of length λ.

We shall describe the provability logics associated with recursive progressions $(\mathcal{T}_\alpha)$, $\alpha<\varepsilon_0$ being a "natural" ordinal notation, where each theory $\mathcal{T}_{\alpha+1}$ is obtained from $\mathcal{T}_\alpha$ by adding the consistency statement for $\mathcal{T}_\alpha$ as a new axiom. The reason we restrict ourselves to "natural" ordinal notations is that, in contrast with Carlson's result, provability logics in this situation depend essentially on the choice of ordinal notation system.

The provability logics introduced below turn out to be decidable and admit natural Kripke-like semantics.

1. Arithmetical interpretation.

Let $\leqslant\cdot$ denote the canonical primitive recursive (p.r.) well-ordering described in [3, 5]. $\leqslant\cdot$ is a p.r. formula which provably linearly orders the set $\mathbb{N}$ of natural numbers and has order type ε_0 in the standard model of PA. We also have p.r. terms $x \oplus y$, $x \otimes y$, $\underline{\omega}^x$, $sc(x)$, $pd(x)$ representing (oridnal) addition, multiplication, exponentiation, successor, and predecessor functions respectively ($pd(x) = x$ if x has no predecessors) and p.r. formulas $Lim(z) :=$ " z is a limit ordinal" and $Sc(z) :=$ " z is a successor". For any ordinal $\alpha < \varepsilon_0$ let $\underline{\alpha}$ denote the standard closed arithmetical term representing α ; $\bar{n}$ will denote the term $0\underbrace{''\ldots'}_{n \text{ times}}$ representing the natural number n (of course we generally will not have $\bar{n} = \underline{n}$).

We know (cf. [3, 5]) that all "natural" properties of functions and predicates mentioned above are provable in PA. Moreover for every arithmetic formula $\varphi(u, \vec{x})$ and every ordinal $\alpha < \varepsilon_0$,

$$\text{(TI)} \quad \vdash_{PA} \forall w <\cdot \underline{\alpha} \, (\forall u <\cdot w \, \varphi(u, \vec{x}) \rightarrow \varphi(v, \vec{x})) \rightarrow \forall w <\cdot \underline{\alpha} \, \varphi(w, \vec{x}),$$

where $u <\cdot w$ abbreviates the formula $u \leqslant\cdot w \wedge u \neq w$.

Let $\tau(z; x)$ be an r.e. formula s.t. for all $n \in \mathbb{N}$, $\tau(\bar{n}; x)$ is a numeration of a theory $\mathcal{T}_n$ in PA. The formula $Prf_{\tau}(z;x,y)$ constructed from $\tau(z;x)$ in a natural way denotes the predicate " y is a proof of the formula x in the the theory $\mathcal{T}_z$ " (cf.[2]). For any arithmetic formula $\varphi(x)$ $\ulcorner\varphi(\dot{x})\urcorner$ denotes the natural p.r. term representing the p.r. function $\lambda n.\ \ulcorner\varphi(n)\urcorner$. Define: $[z]_{\tau}(x):=$

$$=\exists y\, Prf_{\tau}(z;x,y)\quad,\ Con_{\tau}(z):=\neg[z]_{\tau}(\ulcorner\bar{0}=\bar{1}\urcorner)\ ,$$
$$[z]_{\tau}\varphi(x):=[z]_{\tau}(\ulcorner\varphi(\dot{x})\urcorner)\ .$$

Let $\tau_0(x)$ be an r.e. numeration of a given theory $\mathcal{T}_0$. An r.e. formula $\tau(z;x)$ is called a natural Turing numeration for $\tau_0(x)$ iff the following conditions hold:

(N1) $\vdash_{PA} \tau(\underline{0};x)\leftrightarrow\tau_0(x)$,

(N2) $\vdash_{PA} \tau(sc(z);x)\leftrightarrow\tau(z;x)\vee x=\ulcorner Con_{\tau}(\dot{z})\urcorner$,

(N3) $\vdash_{PA} Lim(z)\rightarrow(\tau(z;x)\leftrightarrow\exists w<\cdot z\,\tau(w;x))$.

This definition is analogous to that of [2] when restricted to a certain natural path of length ε_0 within $\mathcal{O}$. As in [2] one can easily show, via the arithmetical fixed point theorem, the existence of a natural Turing numeration for arbitrary $\tau_0(x)$.

Given a natural Turing numeration $\tau(z;x)$, let $\mathcal{T}_{\alpha}$ denote the theory numerated by $\tau(\underline{\alpha};x)$. Of course we have $\mathcal{T}_{\alpha+1}=\mathcal{T}_{\alpha}+Con_{\tau}(\underline{\alpha})$ and $\mathcal{T}_{\alpha}=\bigcup_{\beta<\alpha}\mathcal{T}_{\beta}$, if α is a limit ordinal.

Lemma 1. Let $\tau(z;x)$ be a natural Turing numeration for $\tau_0(x)$. Then for all $\lambda < \varepsilon_0$

1. $\vdash_{PA} \forall u, w\, (u \leq\cdot\, w <\cdot\, \underline{\lambda} \to ([u]_\tau(x) \to [w]_\tau(x)),$

2. $\vdash_{PA} \forall z <\cdot\, \underline{\lambda}\, ([sc(z)]_\tau(x) \leftrightarrow [z]_\tau(\ulcorner Con_\tau(\dot{z})\urcorner \dot{\to} x)$
$\leftrightarrow [\underline{0}]_\tau(\ulcorner Con_\tau(\dot{z})\urcorner \dot{\to} x)),$

3. $\vdash_{PA} \forall z <\cdot\, \underline{\lambda}\, (Lim(z) \to ([z]_\tau(x) \leftrightarrow \exists w <\cdot\, z\, [w]_\tau(x)$
$\leftrightarrow \exists w <\cdot\, z\, [\underline{0}]_\tau(\ulcorner Con_\tau(\dot{w})\urcorner \dot{\to} x))).$

Proof: Statement 1 and the first part of 2 and 3 are easy. To check the second part of 2 and 3 note that the following argument can be formalized in PA using (TI) and 1: "The axioms of $\mathcal{T}_{\alpha+1}$ not in $\mathcal{T}_0$ are those of the form $Con_\tau(\underline{\beta})$, $\beta \leq \alpha$. Among them $Con_\tau(\underline{\alpha})$ is the strongest one. Hence $\mathcal{T}_{\alpha+1} = \mathcal{T}_0 + Con_\tau(\underline{\alpha})$ ". ∎

Corollary 1. Let $\tau(z;x)$ and $\tau'(z;x)$ be natural Turing numerations for $\tau_0(x)$. Then for all $\lambda < \varepsilon_0$

$$\vdash_{PA} \forall z <\cdot\, \underline{\lambda}\, ([z]_\tau(x) \leftrightarrow [z]_{\tau'}(x)).$$ ∎

By this Corollary the natural Turing progression of theories $(\mathcal{T}_\lambda)_{\lambda<\varepsilon_0}$ actually depends only on the choice of $\tau_0(x)$. From now on we fix $\tau_0(x)$ and an arbitrary natural Turing numeration $\tau(z;x)$ for $\tau_0(x)$.

Let L_{ε_0} be the language consisting of
propositional variables $p, q, \ldots$;
boolean connectives $\to, \bot$;
modal operators $[\alpha]$ for all $\alpha < \varepsilon_0$.

Define: $\langle\alpha\rangle := \neg[\alpha]\neg$, $\Box := [0]$, $\Diamond := \langle 0\rangle$.

Let f be an arbitrary assignment of arithmetic sentences to propositional variables. An interpretation $f_\tau(\varphi)$ of a formula φ in L_{ε_0} induced by f is defined inductively as follows:

(I1) $f_\tau(\varphi) = f(\varphi)$, if φ is a propositional variable,

(I2) $f_\tau(\bot) = (\bar{0} = \bar{1})$, $f_\tau(\varphi \to \psi) = (f_\tau(\varphi) \to f_\tau(\psi))$,

(I3) $f_\tau([\alpha]\varphi) = [\underline{\alpha}]_\tau f_\tau(\varphi)$.

TL_{ε_0} is the logic in the language L_{ε_0} with the following axiom schemata:

(i) Tautologies,

(ii) $[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$,

(iii) $\Box(\Box\varphi \to \varphi) \to \Box\varphi$,

(iv) $[\alpha]\varphi \to \Box[\alpha]\varphi$,

(v) $[\alpha]\varphi \to [\lambda]\varphi$, if $\alpha < \lambda$,

(vi) $[\alpha]\varphi \to \Box([\alpha]\bot \vee \varphi)$,

(vii) $[\lambda]\varphi \to [\lambda]\langle\alpha\rangle(\varphi \wedge \bigwedge_{i=1}^{n}([\alpha]\varphi_i \to \varphi_i))$ if $\alpha < \lambda$,

(viii) $[\alpha + n + 1]\varphi \leftrightarrow \Box(\neg\Box^n[\alpha]\bot \to \varphi)$,

where α, λ are either 0 or limit ordinals $< \varepsilon_0$, $n \in \mathbb{N}$, and the rules of inference of TL_{ε_0} are Modus Ponens and Necessitation: $\varphi \vdash \Box\varphi$. $TL^+_{\varepsilon_0}$ is the logic in L_{ε_0} generated from all the theorems of TL_{ε_0} and the scheme $[\alpha]\varphi \to \varphi$ and α a limit ordinal, using Modus Ponens only.

Lemma 2. $TL_{\varepsilon_0} \vdash \varphi \Rightarrow \vdash_{PA} f_\tau(\varphi)$ for every assignment f .

Proof: The statement is trivial for formulas φ of the form (i)-(iv) and for those of the form (v), (vi) and (viii) it follows immediately from Lemma 1. We only derive (vii).

Notice that (vii) is a generalized variant of Goryachev's result about interpretability of the theory $\mathcal{T}+Rfn(\mathcal{T})$ in $\mathcal{T}_\omega$ [12].

Let GL be the provability logic for PA (called PRL in [4]). It is known (see e.g. [11]) that for all $n\in\mathbb{N}$

$GL\vdash \neg\Box^{n+1}\bot\to\Diamond\bigwedge_{i=1}^{n}(\Box p_i\to p_i)$. It follows that

$PA\vdash \forall z(\neg[z]_\tau^{n+1}\bot\to\langle z\rangle_\tau\bigwedge_{i=1}^{n}([z]_\tau A_i\to A_i))$ for any tuple of arithmetic sentences $A_1,\dots,A_n$. Since $PA\vdash\forall z(\underline{\alpha}\leq\cdot z<\cdot\underline{\lambda}\to$ $\to([\underline{\alpha}]_\tau A_i\to[z]_\tau A_i))$ we have $PA\vdash\underline{\alpha}\leqslant\cdot z<\cdot\underline{\lambda}\to(\neg[z]_\tau^{n+1}\bot\to$ $\to\langle z\rangle_\tau\bigwedge_{i=1}^{n}([\underline{\alpha}]_\tau A_i\to A_i))$. But for all sentences A and B $PA\vdash\underline{\alpha}\leqslant\cdot z<\cdot\underline{\lambda}\wedge[z]_\tau A\to(\langle z\rangle_\tau B\to\langle\underline{\alpha}\rangle_\tau(A\wedge B))$. Hence

$$PA\vdash\underline{\alpha}\leqslant\cdot z<\cdot\underline{\lambda}\wedge[z]_\tau A\to(\neg[z]_\tau^{n+1}\bot\to\langle\underline{\alpha}\rangle_\tau(A\wedge\bigwedge_{i=1}^{n}([\underline{\alpha}]_\tau A_i\to A_i)))$$

It follows that

$$\vdash_{PA}\underline{\alpha}\leqslant\cdot z<\cdot\underline{\lambda}\wedge[z]_\tau A\to[\underline{0}]_\tau(\underline{\alpha}\leqslant\cdot z<\cdot\underline{\lambda}\wedge[z]_\tau A)$$

$$\to[\underline{0}]_\tau(\neg[z]_\tau^{n+1}\bot\to\langle\underline{\alpha}\rangle_\tau(A\wedge\bigwedge_{i=1}^{n}([\underline{\alpha}]_\tau A_i\to A_i)))$$

$$\to[\underline{\lambda}]_\tau\langle\underline{\alpha}\rangle_\tau(A\wedge\bigwedge_{i=1}^{n}([\underline{\alpha}]_\tau A_i\to A_i)),$$

by Lemma 1, because λ is a limit ordinal. Using Lemma 1 again we obtain:

$$\vdash_{PA}[\underline{\lambda}]_\tau A\to\exists z(\underline{\alpha}\leqslant\cdot z<\cdot\underline{\lambda}\wedge[z]_\tau A)$$

$$\to [\underline{\lambda}]_\tau \langle \underline{\alpha} \rangle_\tau (A \wedge \bigwedge_{i=1}^{n} ([\underline{\alpha}]_\tau A_i \to A_i)), \text{ q.e.d.} \qquad \blacksquare$$

Lemma 3. $TL^+_{\varepsilon_D} \vdash \varphi \Rightarrow f_\tau(\varphi)$ is true in the standard model of PA for every assignment f.

Proof: Trivial. $\blacksquare$

For the sake of convenience we restrict ourselves to a language with a finite number of modal operators. Let Λ be a finite sequence of limit ordinals and let max Λ denote the maximum of Λ.

L_Λ is the propositional language with modal operators $\Box$ and $[\lambda]$ for all $\lambda \in \Lambda$. TL_Λ (TL^+_Λ) is the logic in L_Λ whose axioms and inference rules are precisely those of TL_{ε_D} ($TL^+_{\varepsilon_D}$) formulated in L_Λ. In particular no axiom of TL_Λ has the form (viii).

2. Models for CDC_Λ.

CSM_Λ is the logic in L_Λ whose axiom schemata are (i)-(v) and whose inference rules are modus ponens and necessitation. This logic has been studied in [1, 4, 6] under several different names. The present one is a modification of that from [9].

Let CDC_Λ be CSM_Λ together with scheme (vi).

A C-model $\mathcal{K}$ is a structure $(K, \{K_\lambda \mid \lambda \in \Lambda\}, \prec, \Vdash, b)$, where

(C1) $(K, \prec, b)$ is a finite irreflexive tree with bottom node b,

(C2) $K_\lambda \subseteq K$, $b \in K_\lambda$ for all λ,

(C3) $K_\alpha \supseteq K_\lambda$ for all $\alpha < \lambda$, (C4) $\Vdash$ is a forcing relation on $\mathcal{K}$ s.t. for all $x \in K$ and formulas φ and ψ :

a) $x \nVdash \bot$, $x \Vdash (\varphi \to \psi) \Leftrightarrow (x \nVdash \varphi$ or $x \Vdash \psi)$;

b) $x \Vdash [\lambda]\varphi \Leftrightarrow \forall y \succ x\, (y \in K_\lambda \Rightarrow y \Vdash \varphi)$, where $\lambda \in \Lambda \cup \{0\}$, $K_0 := K$.

We write $\mathcal{K} \Vdash \varphi$ iff $b \Vdash \varphi$.

It is well known (cf. [1]) that CSM_Λ is complete w.r.t. C -models, i.e.

$$CSM_\Lambda \vdash \varphi \Leftrightarrow (\mathcal{K} \Vdash \varphi \text{ for any } C\text{-model } \mathcal{K}).$$

We shall show the completeness of CDC_Λ w.r.t. C -models of a special kind.

A C -model $\mathcal{K}$ is called a DC -model iff

(C6) K_λ is downward closed for all $\lambda \in \Lambda$, i.e.

$$x \prec y \in K_\lambda \Rightarrow x \in K_\lambda .$$

Theorem 1. $CDC_\Lambda \vdash \varphi \Leftrightarrow (\mathcal{K} \Vdash \varphi$ for any DC -model $\mathcal{K}$).

Proof: $(\Rightarrow)$ Easy.

$(\Leftarrow)$ We apply the Henkin Construction. In doing this we follow the presentation of [9].

Let X be a set of formulas of L_Λ . We write $X \vdash \varphi$ for: there is a finite $X_0 \subseteq X$ s.t. $CDC_\Lambda \vdash \bigwedge X_0 \to \varphi$.

Suppose $CDC_\Lambda \nvdash \varphi$ and Γ is the set of all subformulas of formulas in $\{\varphi\} \cup \{[\lambda]\bot \mid \lambda \in \Lambda\}$. A set X is Γ -saturated iff $X \nvdash \bot$ and for all Θ and ψ in Γ :

$X \vdash \theta \vee \psi \Rightarrow \theta \in X$ or $\psi \in X$.

Define $W := \{X \subseteq \Gamma \mid X$ is Γ-saturated$\}$. Further for every $\lambda \in \Lambda \cup \{0\}$ and every $X, Y \in W$ define:

$X\ R_\lambda\ Y$ iff (W1) for all $\alpha \leq \lambda, ([\alpha]\psi \in X \Rightarrow \psi, [\alpha]\psi \in Y)$,

(W2) for all $\alpha > \lambda, ([\alpha]\psi \in X \Rightarrow ([\alpha]\psi \in Y$ and $(\psi \in Y$ or $[\alpha]\bot \in Y)))$

(W3) for some ψ and α , $[\alpha]\psi \in Y$ and $[\alpha]\psi \notin X$.

It is a matter of routine to check that for every $\lambda \in \Lambda \cup \{0\}$

(R1) R_λ is irreflexive and transitive,

(R2) $X R_\lambda Y \Rightarrow X R_\alpha Y$ for all $\alpha \leq \lambda$,

(R3) $X R_\alpha Y R_\lambda Z \Rightarrow X R_\lambda Y$ for all $\alpha \leq \lambda$.

Let $\Vdash$ be a forcing relation defined on W inductively as follows:

a) $X \Vdash p \Longleftrightarrow p \in X$, where p is a propositional variable;

b) $X \Vdash \theta \rightarrow \psi \Longleftrightarrow (X \nVdash \theta$ or $X \Vdash \psi$), $X \nVdash \bot$;

c) $X \Vdash [\lambda]\psi \Longleftrightarrow \forall Y\ (X R_\lambda Y \Rightarrow Y \Vdash \psi)$.

Lemma 4. For all $\psi \in \Gamma$, $X \in W$,

$$X \Vdash \psi \Longleftrightarrow \psi \in X.$$

Proof: By induction on φ in Γ. We only consider the case $\varphi = [\lambda]\theta$.

($\Leftarrow$) Suppose $[\lambda]\theta \in X$. Then $\forall Y, (X R_\lambda Y \Rightarrow \theta \in Y)$ by (W1). By I.H. $\forall Y, (X R_\lambda Y \Rightarrow Y \Vdash \theta)$. Hence $X \Vdash [\lambda]\theta$.

($\Rightarrow$) Suppose $[\lambda]\theta \notin X$. Define

$$V_\lambda := \{[\alpha]\chi, \chi \mid \alpha \leq \lambda, [\alpha]\chi \in X\} \cup \{[\alpha]\chi, \chi \vee [\alpha]\bot \mid \alpha > \lambda, [\alpha]\chi \in X\} \cup \{[\lambda]\theta\}$$

We claim: $V_\lambda \not\vdash \theta$. Otherwise we would have

$$\{[\alpha]\chi, \chi \mid \alpha \leq \lambda, [\alpha]\chi \in X\} \cup \{[\alpha]\chi, \chi \vee [\alpha]\bot \mid \alpha > \lambda, [\alpha]\chi \in X\} \vdash [\lambda]\theta \to \theta,$$

ergo $\{[\alpha]\chi \mid [\alpha]\chi \in X\} \vdash [\lambda]([\lambda]\theta \to \theta)$.

Since for any λ $CSM_\Lambda \vdash [\lambda]([\lambda]\theta \to \theta) \to [\lambda]\theta$ (cf. [6], p.183), it would follow that $X \vdash [\lambda]\theta$, quod non.

Let $\hat{V}_\lambda$ be Γ-saturated s.t. $V_\lambda \subseteq \hat{V}_\lambda$ and $\hat{V}_\lambda \not\vdash \theta$. Put $Y_\lambda := \hat{V}_\lambda \cap \Gamma$. It is easily seen that $X R_\lambda Y_\lambda$. Since $\theta \notin Y$ we obtain $Y_\lambda \not\Vdash \theta$ by I.H., q.e.d. ∎

Since $CDC_\Lambda \not\vdash \varphi$ by Lemma 4 we can find a Γ-saturated $B \in W$ s.t. $B \not\Vdash \varphi$. Define the desired DC-model $\mathcal{K} = (K, \{K_\lambda \mid \lambda \in \Lambda\}, <, \Vdash, b)$ as follows:

$$K := \{(X_1, \ldots, X_m) \mid X_1 = B,\ m \geq 1 \text{ and } \forall i\ X_i \in W,\ X_i\ R_0\ X_{i+1}\};$$

$$K_\lambda := \{(X_1, \ldots, X_m) \in K \mid X_{m-1} R_\lambda X_m,\ m \geq 2\} \cup \{(B)\};$$

$(X_1,\dots,X_k) \prec (Y_1,\dots,Y_m) :\Leftrightarrow (k<m$ and $X_i = Y_i$ for $i=1,\dots,k$);

$(X_1,\dots,X_m) \Vdash p :\Longleftrightarrow X_m \Vdash p$;

$b := (B)$.

It is not difficult to see that $\mathcal{K}$ is indeed a DC-model. Notice that property (C6) follows from (R3). As in [9] for every formula ψ we obtain $(X_1,\dots,X_m) \Vdash \psi \Longleftrightarrow X_m \Vdash \psi$. Hence $b \not\Vdash \varphi$ and $\mathcal{K} \not\Vdash \varphi$, q.e.d. ∎

Corollary 2. CDC_Λ is decidable. ∎

Proof: This follows from the proof of Theorem 1.

Remark. Let CDC_n be the logic in the language with modal operators $[0],\dots,[n]$ whose axioms and inference rules are obtained from those of CDC_Λ by substituting natural numbers for ordinals preserving the ordering. CDC_n is complete w.r.t. the following provability interpretation. Consider a (finite) sequence of theories $(\mathcal{T}_i)$, $i=0,\dots,n$ s.t. for every i, $\mathcal{T}_{i+1} = \mathcal{T}_i + A_i$ where A_i is a (true) Π_1-sennetce. For any assignment f let $f_\pi(\varphi)$ denote the usual arithmetical interpretation of a modal formula φ induced by f , where π is the natural r.e. numeration of the sequence $(\mathcal{T}_i)$. Then

$$CDC_n \vdash \varphi \Longleftrightarrow \forall f \quad \forall \pi \ \vdash_{PA} f_\pi(\varphi).$$

The proof, which we omit, is based on Theorem 1 together with the usual arithmetical completeness theorem for GL (cf.[6], p.201-204).

3. Models for TL_λ .

Let $\mathcal{K} = (K, \prec, b)$ be a countable upwards well-founded tree with bottom node b . Depth d is the function from K to the ordinals uniquely determined by the following conditions:

(D1) $d(x) = 0$, if $\forall y \in K \quad x \not\prec y$

(D2) $d(x) = \sup\limits_{y \succ x} (d(y)+1)$, otherwise.

We write $h(\mathcal{K})$ for $d(b)$.

Let $\lambda \in \Lambda \cup \{0\}$. A T_λ -model $\mathcal{K}$ is a structure $(K, \{K_\gamma \mid \gamma > \lambda, \gamma \in \Lambda\}, \prec, \Vdash, b)$, where

(T1) $(K, \prec, b)$ is a countable upwards well-founded tree with bottom node b ,

(T2) $h(\mathcal{K}) < \lambda + \omega$,

(T3) $K_\gamma \subseteq K$, $b \in K_\gamma$ for all $\gamma > \lambda$, $\gamma \in \Lambda$,

(T4) $K_\gamma \supseteq K_\delta$ for all $\delta > \gamma$,

(T5) K_γ is downward closed,

(T6) $\Vdash$ is a forcing relation on $\mathcal{K}$ s.t. for all $x \in K$ and all formulas φ and ψ :

a) $x \not\Vdash \bot$, $x \Vdash \varphi \to \psi \Longleftrightarrow$ ($x \not\Vdash \varphi$ or $x \Vdash \psi$),

b) $x \Vdash \Box\varphi \Leftrightarrow \forall y \succ x \quad y \Vdash \varphi$,

c) for all $\gamma \in \Lambda$, $\gamma \leq \lambda$

$$x \Vdash [\gamma]\varphi \Leftrightarrow \exists \alpha < \gamma \ \forall y \succ x \ (d(y) \geq \alpha \Rightarrow y \Vdash \varphi),$$

(Note that α is not assumed to be in Λ).

d) for all $\gamma \in \Lambda$, $\gamma > \lambda$

$$x \Vdash [\gamma]\varphi \Leftrightarrow \forall y \succ x\ (y \in K_\gamma \Rightarrow y \Vdash \varphi).$$

As usual $\mathcal{K} \Vdash \varphi$ stands for $b \Vdash \varphi$.

Clearly a T_0-model $\mathcal{K}$ is in fact a DC-model provided K is finite. $T_{max\,\Lambda}$ -models are also called T--models. Note that no formula of L_Λ is forced via (T6d) at any T-model $\mathcal{K}$.

Lemma 5. Let $\mathcal{K}$ be a T-model, $\lambda \in \Lambda$. Suppose $(x_n)_{n \in \mathbb{N}}$ is a sequence of nodes of $\mathcal{K}$ s.t. $d(x_0) \geq \lambda$ and for all n $x_{n+1} \prec x_n$. Then 1) for any formula φ there is at most one x_m s.t. $x_m \nVdash [\lambda]\varphi \to \varphi$;

2) for any $\varphi_1, \dots, \varphi_n$ there are at most n different nodes $x_{m_1}, \dots, x_{m_n}$ s.t. for all i ,

$$x_{m_i} \nVdash \bigwedge_{i=1}^{n} ([\lambda]\varphi_i \to \varphi_i).$$

Proof: 2) follows from 1) by the Pigeon-hole Principle. 1) is obvious. ∎

Lemma 6. Let $\mathcal{K}$ be a T-model. Then for all φ in L_Λ

$$TL_\Lambda \vdash \varphi \Rightarrow \mathcal{K} \Vdash \varphi.$$

Proof: We only consider the case that φ is an axiom of the form (vii), say $[\lambda]\psi \to [\lambda]\langle\alpha\rangle(\psi \wedge H_n^\alpha)$, where $H_n^\alpha :=$ $= \bigwedge_{i=1}^{n} ([\alpha]\varphi_i \to \varphi_i)$. Suppose $b \nVdash \varphi$; then $\forall z \succ b\ (d(z) \geq$ $\geq \gamma \Rightarrow z \Vdash \psi)$ for some $\gamma < \lambda$. Put $\delta := max(\alpha, \gamma)$; clearly $\delta < \lambda$.

Since $b \not\Vdash [\lambda]\langle\alpha\rangle(\psi \wedge H_n^\alpha)$, there is a sequence $(x_k)_{k\in N}$ s.t.

1) $\forall k \in N \; x_k \in K$ and $x_k \succ b$,

2) $\sup_k d(x_k) \geq \lambda$,

3) $\forall k \in N \quad x_k \Vdash [\alpha]\neg(\psi \wedge H_n^\alpha)$.

Since λ is a limit ordinal, by 2) we may choose an x_k s.t. $d(x_k) \geq \delta + n + 1$. Hence there is a sequence $a_0 \succ a_1 \succ \dots \dots \succ a_n \succ x_k$ s.t. for all i, $d(a_i) \geq \delta + i$. Since $x_k \Vdash [\alpha]\neg(\psi \wedge H_n^\alpha)$ and $d(a_i) \geq \delta \geq \alpha$ we have $a_i \not\Vdash \psi \wedge H_n^\alpha$ for all i. But $\delta \geq \gamma$; hence $a_i \Vdash \psi$ and ergo $a_i \not\Vdash H_n^\alpha$ for $i = 0, \dots, n$. This contradicts Lemma 5.

Before proving the completeness of TL_Λ w.r.t. T-models, we introduce some abbreviations. Suppose Γ is a finite set of formulas of L_Λ closed under subformulas and containing $\{[\lambda]\bot \mid \lambda \in \Lambda\}$. For $\lambda \in \Lambda \cup \{0\}$, define: $\Gamma_\lambda := \{\varphi \mid [\lambda]\varphi \in \Gamma\}$ and $H^\lambda(\Gamma) := \bigwedge\{[\lambda]\psi \to \psi \mid \psi \in \bigcup_{\gamma \leq \lambda} \Gamma_\gamma\}$. For any $S \subseteq \Gamma_\lambda$, $\lambda < \max \Lambda$ define

$$I_S^\lambda(\Gamma) := [\lambda^+] \bigwedge S \to [\lambda^+]\langle\lambda\rangle(\bigwedge S \wedge H^\lambda(\Gamma)) \qquad \text{, where}$$

λ^+ is the minimum of $\{\gamma \in \Lambda \mid \gamma > \lambda\}$;

$$I^\lambda(\Gamma) := \bigwedge\{I_S^\lambda(\Gamma) \wedge \Box I_S^\lambda(\Gamma) \mid S \subseteq \Gamma\}.$$

Clearly $I^\lambda(\Gamma)$ is a theorem of TL_Λ. Note that if S is empty then

$$CSM_\Lambda \vdash I_S^\lambda(\Gamma) \leftrightarrow [\lambda^+]\langle\lambda\rangle H^\lambda(\Gamma).$$

We are now in a position to describe a procedure transforming T_λ -models into T_{λ^+} -models, which (under certain conditions) preserves the forcing relation.

Lemma 7. Let $\mathcal{K}$ be a T_λ -model, $\lambda < max\,\Lambda$. Suppose $\mathcal{K} \Vdash I^\lambda(\Gamma)$. Then there is a T_{λ^+} -model $\mathcal{K}'$ s.t. for all $\varphi \in \Gamma$

$$\mathcal{K} \Vdash \varphi \iff \mathcal{K}' \Vdash \varphi.$$

Proof: Define $Z := \{z \in K_{\lambda^+} \mid \forall x \succ z \;\; x \notin K_{\lambda^+}\}$, $\mathcal{U} := \{u \in K_{\lambda^+} \mid \exists z \in Z \;\; (z \succ u$ and $\forall t \succ u \;\; t \not\prec z \;\;)\}$.

If $\mathcal{U}$ is empty then $\mathcal{K}_\gamma = \{b\}$ for all $\gamma \geqslant \lambda^+$. It follows by (T2) that $\mathcal{K}' := (K, \{K_\gamma \mid \gamma > \lambda^+, \gamma \in \Lambda\}, \prec, \Vdash, b)$ is the desired T_{λ^+} -model.

Suppose $\mathcal{U}$ is not empty. For every $z \in Z$ we shall now define a node $g(z)$. Let $u \in \mathcal{U}$ be the unique node s.t. $u \prec z$ and $\forall t \succ u \;\; t \not\prec z$. Define $S := \{\varphi \in \Gamma_{\lambda^+} \mid u \Vdash [\lambda^+]\varphi\}$. Since $u \Vdash I^\lambda_S(\Gamma)$ and $u \Vdash [\lambda^+] \wedge S$, we have $u \Vdash [\lambda^+] \langle\lambda\rangle (\wedge S \wedge H^\lambda(\Gamma))$; thus $z \nVdash [\lambda] \neg (\wedge S \wedge H^\lambda(\Gamma))$. Hence there is an $x \succ z$ s.t. $x \Vdash \wedge S \wedge H^\lambda(\Gamma)$. Put $g(z) :=$ any such x . Note that $d(g(z)) \geqslant \lambda$ because $\neg[\lambda]\bot$ occurs in $H^\lambda(\Gamma)$ if $\lambda \in \Lambda$.

Furthermore, without loss of generality we may assume that for no $y \in K \;\; z \prec y \prec g(z)$. If not we add to our model an isomorphic copy of the structure $\mathcal{K}_{g(z)} := (\{x \in K \mid x \succcurlyeq g(z)\}, \prec, \Vdash, g(z))$ right over z (cf. [11], p.925). Define $V := \{g(z) \mid z \in Z\}$ and $C := \{x \in K \mid \exists z \in Z \;\; z \preccurlyeq x\}$. Let be the standard increasing sequence of ordinals s.t. $\lambda+\nu_n \to \lambda^+$.

The desired T_{λ^+}-model $\mathcal{K}'$ is defined as follows:

(U) $K' := \{(x,m,\gamma) \mid (x\in C, m\in\mathbb{N}, \gamma=0)$ or $(x\in V, m\in\mathbb{N}, \gamma\leq\nu_m)$ or $(x\in K\setminus C, m=0, \gamma=0)\}$;

$b' := (b, 0, 0)$;

$(x_1, m_1, \gamma_1) \prec' (x_2, m_2, \gamma_2) :\Leftrightarrow$ (01) $x_1 \preccurlyeq x_2$,

(02) $x_1 \in C \Rightarrow m_1 = m_2$,

(03) $x_1 = x_2 \Rightarrow \gamma_1 > \gamma_2$.

For any $x' = (x,m,\gamma)\in K'$ define $f(x') := x$. Thus f is a surjective function from K' to K . For all propositional variables p and all $x'\in K'$ define $x' \Vdash p :\Leftrightarrow f(x') \Vdash p$.

Finally for all $\gamma > \lambda^+$ put $K'_\gamma := \{x'\in K' \mid f(x')\in K_\gamma\}$.

Lemma 8. 1. For all $x', y'\in K'$, $(x' \succ' y' \Rightarrow f(x') \succcurlyeq f(y'))$.

2. For all $z\in K$, $x'\in K'$, $(z \succ f(x') \Rightarrow \exists y' \succ' x' \; (f(y') = z))$.

3. For all $x', y'\in K'$, $(x' \succ' y'$ and $f(x') = f(y') \Rightarrow f(x')\in V)$.

Proof: Straightforward. ∎

Lemma 9. $\prec'$ is transitive, treelike and upwards well-founded.

Proof: We verify only the well-foundedness. Let P be a non-empty subset of K' . First consider $f(P)\subseteq K$. Since K is upwards well-founded we may choose an $x'\in P$, say (x,m,γ) , s.t. $\forall y \succ f(x') \; y\notin f(P)$. If $P_1 := \{y'\in P \mid y' \succ' x'\}$ is empty, we are done. Otherwise for all $y' \in P_1$ $f(y') = f(x')$. Suppose $(x, m_1, \gamma_1) \in P_1$, then by (03) $\gamma_1 < \gamma$ and by Lemma 8.3 and (02) $m_1 = m$. Thus for all $y_1' := (x, m, \gamma_1) \in P_1$ and $y_2' := (x, m, \gamma_2) \in P_1$, we have:

$$y_1' <' y_2' \Longleftrightarrow \gamma_1 > \gamma_2 .$$

Put $\alpha := \inf\{\delta \mid (x, m, \delta) \in P_1\}$. Clearly (x, m, α) is a maximal element of P .

Since $<'$ is upwards well-founded the depth function is defined properly on $\mathcal{K}'$. Let d_λ denote the depth function defined on the tree $(K_{\lambda^+} \setminus Z, <)$. By the construction of $g(z)$ we have $d(z) \geq d(g(z)) \geq \lambda$ for all $z \in Z$. Hence, by (T2) $d_\lambda(x) < \omega$ for all $x \in K_{\lambda^+} \setminus Z$.

Lemma 10. Suppose $z' = (z, m, \gamma) \in K'$. Then

1) $z \in Z \Rightarrow d(z') = \max(d(g(z)) + \nu_m + 1, d(z))$,

2) $z \in V \Rightarrow d(z') = d(z) + \gamma$,

3) $z \in K_{\lambda^+} \setminus Z \Rightarrow d(z') = \lambda^+ + d_\lambda(z)$,

4) For all remaining z , $d(z') = d(z)$.

Proof: Long and trivial. ∎

Lemma 11. For all $z' \in K'$

1) $d(z') \geq \lambda^+ \Longleftrightarrow f(z') \in K_{\lambda^+} \setminus Z$,

2) $d(z') \geq d(f(z'))$,

3) $d(z') > d(f(z')) \Rightarrow d(f(z')) \geq \lambda$,

4) $h(\mathcal{K}') < \lambda^+ + \omega$.

Proof: Follows immediately from Lemma 10. ∎

It follows from Lemma 8 that $\mathcal{K}'$ satisfies (T3), (T4) and (T5). By Lemma 11.4) we also have (T2), thus $\mathcal{K}'$ is indeed a T_{λ^+} -model.

Lemma 12. For all $x' \in K'$ and $\varphi \in \Gamma$

$$x' \Vdash \varphi \iff f(x') \Vdash \varphi.$$

Proof: By induction on φ in Γ. The cases of propositional variables and boolean connectives are trivial.

I. Suppose $\varphi = \Box\psi$.

a) Suppose $x' \Vdash \Box\psi$. Then $\forall y' \succ' x' (f(y') \Vdash \psi)$ by I.H.
It follows by Lemma 8.2 that $\forall z \succ f(x')\ (z \Vdash \psi)$. Thus $f(x') \Vdash \Box\psi$.

b) Suppose $x' \nVdash \Box\psi$. Then $\exists y' \succ' x'\ (f(y') \nVdash \psi)$.
If $f(y') \succ f(x')$ then trivially $f(x') \nVdash \Box\psi$. Otherwise by Lemma 8.3 $f(y') = f(x')$ and $f(x') \in V$.
Note that if $x \in V$ and $x \Vdash \Box\psi$ then $x \Vdash \psi$, because $\psi \in \Gamma_0$ and $x \Vdash H^{\lambda}(\Gamma)$. Hence $f(x') = f(y') \nVdash \Box\psi$.

II. Suppose $\varphi = [\gamma]\psi$, $0 < \gamma \leq \lambda$.

a) Suppose $x' \Vdash [\gamma]\psi$. Then $\forall y' \succ' x' (d(y') \geq \delta \Rightarrow$ $\Rightarrow f(y') \Vdash \psi)$ for some $\delta < \gamma$. By Lemma 8.2 for every $y \succ f(x')$ one can find an $y' \succ' x'$ s.t. $f(y') = y$.
By Lemma 11.2) $d(y') \geq d(f(y')) = d(y)$, hence we obtain $\forall y \succ f(x')\ (d(y) \geq \delta \Rightarrow y \Vdash \psi)$, q.e.d.

b) Suppose $x' \nVdash [y]\psi$. Then there is a sequence $(y'_n)_{n \in \mathbb{N}}$ s.t. for all n $y'_n \succ' x'$, $f(y'_n) \nVdash \psi$ and

$\sup_n d(y_n') \geq \gamma$.

b1) Suppose $\forall n \;\; f(y_n') \succ f(x')$. By Lemma 11.3) either

$$\exists n \;\; d(f(y_n')) \geq \lambda \quad \text{or} \quad \forall n \;\; d(f(y_n')) = d(y_n') \, .$$

In any case $\sup_n d(f(y_n')) \geq \gamma$. Thus $(f(y_n'))_{n\in\mathbb{N}}$ is the desired sequence.

b2) Suppose $\exists n \;\; f(y_n') = f(x')$. Then by Lemma 8.3 $f(x') \in V$ and, as in I.b), we obtain $f(x') \not\Vdash [\gamma]\psi$.

III. Suppose $\varphi = [\gamma]\psi$, $\gamma > \lambda^+$.

a) Suppose $x' \Vdash [\gamma]\psi$. Then $\forall y' \succ' x'$, $(y' \in K_\gamma' \Rightarrow f(y') \Vdash \psi)$.

If $y \succ f(x')$ and $y \in K_\gamma$ then by Lemma 8.2 $f(y') = y$ for some $y' \succ' x'$. By the definition of K_γ' $y' \in K_\gamma'$ and hence $y = f(y') \Vdash \psi$ q.e.d.

b) Suppose $x' \not\Vdash [\gamma]\psi$. Then there is an $y' \in K_\gamma'$ s.t. $y' \succ' x'$ and $f(y') \not\Vdash \psi$. We claim: $f(y') \succ f(x')$. Otherwise we would have $f(y') = f(x')$, ergo $f(y') \in V$ and $f(y') \notin K_\gamma$, quod non.

We have $f(y') \in K_\gamma$, $f(y') \succ f(x')$ and $f(y') \not\Vdash \psi$. Thus $f(x') \not\Vdash [\gamma]\psi$.

IV. Suppose $\varphi = [\lambda^+]\psi$.

a) Suppose $f(x') \Vdash [\lambda^+]\psi$. If $f(x') \notin K_{\lambda^+} \setminus Z$ then by Lemma 11.1) $d(x') < \lambda^+$. Hence $x' \Vdash [\lambda^+]\bot$ and clearly $x' \Vdash [\lambda^+]\psi$. Suppose $f(x') \in K_{\lambda^+} \setminus Z$. Define $A := \{y \in V \mid y \succ f(x')\} \cup \{y \in K_{\lambda^+} \mid y \succ f(x')\}$. Then by the construction of V, $y \Vdash \psi$ for all $y \in A$. By I.H. $\forall y' \in K'$, $(f(y') \in A \Rightarrow y' \Vdash \psi)$. Suppose $y' \succ' x'$ and $f(y') \notin A$. Then $f(y') \notin V \cup K_{\lambda^+}$

and hence by Lemma 10.4) $d(y') = d(f(y')) \leq h(\mathcal{K}) < \lambda + \omega \leq \lambda$

Put $\gamma := h(\mathcal{K})$. It follows that $\forall y' \succ' x'$,

($y' \Vdash \psi$ or $d(y') \leq \gamma$), q.e.d.

b) Suppose $f(x') \not\Vdash [\lambda^+]\psi$. Clearly $f(x') \in K_{\lambda^+} \setminus Z$.

Let $y \in K_{\lambda^+}$ be a node s.t. $y \succ f(x')$ and $y \not\Vdash \psi$.

b1) Suppose $y \notin Z$. Then for some $y' \in K'$, $f(y') = y$

and $d(y') \geq \lambda^+$. By I.H. $y' \not\Vdash \psi$. Hence $x' \not\Vdash [\lambda^+]\psi$.

b2) Suppose $y \in Z$. Consider the sequence $(y'_n)_{n \in \mathbb{N}}$,

where $y'_n := (y, n, 0)$. Clearly for all n, $y'_n \succ' x'$

and $y'_n \not\Vdash \psi$. By Lemma 10.1) $\sup_n d(y'_n) \geq$

$\geq \sup_n (d(g(y)) + \nu_n + 1) \geq \sup_n (\lambda + \nu_n + 1) = \lambda^+$.

It follows that $\forall \gamma < \lambda^+ \exists y' \succ' x'$, ($d(y') \geq \gamma$ and

$y' \not\Vdash \psi$), q.e.d.

This completes the proof of Lemma 12 together with Lemma 7.■

Theorem 2. Suppose φ is a formula of L_Λ and $TL_\Lambda \not\Vdash \varphi$.
Then there is a T-model $\mathcal{K}$ s.t. $\mathcal{K} \not\Vdash \varphi$.

Proof: Let Γ be the set of all subformulas of formulas in $\{\varphi\} \cup \{[\lambda]\bot \mid \lambda \in \Lambda\}$. Since $TL_\Lambda \not\vdash \varphi$ we have

$CDC_\Lambda \not\vdash I(\varphi) \to \varphi$, where

$$I(\varphi) := \bigwedge \{ I^\lambda(\Gamma) \mid \lambda \in \{0\} \cup \Lambda,\ \lambda < \max \Lambda \}$$

By Theorem 1 there is a finite DC-model $\mathcal{K}_0$ s.t. $\mathcal{K}_0 \not\Vdash I(\varphi) \to \varphi$. Apply now Lemma 7 several times to obtain a sequence of models $\{\mathcal{K}_\lambda \mid \lambda \in \Lambda\}$ s.t. for all λ $\mathcal{K}_\lambda$ is a T_λ-model and for all ψ in Γ $\mathcal{K}_\lambda \Vdash \psi \Leftrightarrow \mathcal{K}_0 \Vdash \psi$. Clearly $\mathcal{K}_{\max \Lambda}$ is the desired T-model.

Corollary 3. For all φ in L_Λ ,

$$TL_\Lambda \vdash \varphi \Longleftrightarrow CDC_\Lambda \vdash I(\varphi) \to \varphi .$$

Corollary 4. TL_Λ is decidable.

4. Arithmetical completeness of TL_{ε_0} and $TL^+_{\varepsilon_0}$.

Our proof of the arithmetical completeness is very close to the usual one for GL due to R.Solovay [8]. Therefore we only sketch it, omitting some boring details of formalization. However in addition to the usual proof we have to treat some quantifiers "on ordinals" occuring in the definition of forcing relation and in Lemma 1.3. The reason why the Solovay construction still works is the extreme simplicity of the countermodels constructed via Theorem 2. Somewhat similar idea has been exploited by C.Smoryński in [7].

Theorem 3. Suppose φ is a formula in L_Λ and $TL_\Lambda \nvdash \varphi$. Then there is an arithmetical assignment f s.t. $\nvdash_{PA} f_\tau(\varphi)$.

Proof: First apply Theorem 2 to produce a T-model $\mathcal{K}$ s.t.

(K1) K is a p.r. subset of $\mathbb{N}$; $0, 1 \in K$;

(K2) The relations $x \prec y$ and $x \Vdash \psi$, for every formula in L_Λ , are primitive recursive;

(K3) $0 \in K$ is the bottom node and $\{x \in K \mid x \succcurlyeq 1\} = K \setminus \{0\}$;

(K4) The depth function $d(x)$ is primitive recursive;

(K5) $1 \nVdash \varphi$.

To satisfy requirements (K1)-(K5), note that the countermodel for φ produced via the proof of Theorem 2 is constructed from a finite model applying several simple p.r. procedures. In particular p.r. definitions of K and $\prec$ can be read off from (U) and (O1)-(O3) (of course we should apply them several times). P.r. definitions of $x \Vdash \varphi$ and $d(x)$ can be read off from Lemma 12 and Lemma 10 respectively.

Further, all the p.r. functions and predicates mentioned above can be represented in PA by natural p.r. terms and formulas. As soon as this is done, we formalize the proofs of Lemmas 7-12 in PA to verify the (formalizations of) recursive clauses of the usual definitions of $d(x)$ and $x \Vdash \varphi$ such as (D1), (D2), (T1), (T2), (T6) a)-c).

The upwards well-foundedness of $\prec$ is expressed by the scheme

$$\text{(WF)}\quad \exists x \in K\ \varphi(x, \vec{u}) \to \exists x \in K\ (\forall y \succ x\ \neg\varphi(y, \vec{u}) \wedge \varphi(x, \vec{u})),$$

whose proof in PA can be obtained by formalization of (the proof of) Lemma 9. Using (WF) all the "natural" properties of introduced formulas are easily verifiable in PA.

We turn to the Solovay type construction. By formalization of the Recursion Theorem, a p.r. function $h(m)$ is defined s.t. provably in PA:

$$h(0) = 0;$$
$$h(m+1) = \begin{cases} z, & \text{if } z \in K,\ z \succ h(m) \text{ and } Prf_\tau(\underline{0}; \ulcorner \ell \neq \bar{z} \urcorner, m), \\ h(m), & \text{otherwise.} \end{cases}$$

Here $\ell = z$ abbreviates the formula $\exists n\ \forall m > n\ \ h(m) = z$. By Craig's Theorem we assume without loss of generality that Prf_τ is a p.r. relation.

Lemma 13. 1. $\vdash_{PA} \exists!\, z \in K\ (\ell = z)$,

2. $\vdash_{PA} \forall u\ (\ell = u \neq 0 \rightarrow [\underline{0}]_\tau \exists z\ (\ell = z \succ u))$,

3. $\vdash_{PA} \forall u, v\ (\ell = u \prec v \rightarrow \neg [\underline{0}]_\tau\ \ell \neq v)$,

4. $\ell = 0$ is true in the standard model of PA,

5. $PA + \ell = \bar{1}$ is consistent.

Proof: As in [8]. To check 1 use (WF) and the provable monotonicity of h.

Lemma 14. For all $\lambda < \varepsilon_0$ $\vdash_{PA} \forall x\ \forall z <\cdot \underline{\lambda}\ [\ell = x \neq 0 \rightarrow (d(x) \succcurlyeq 1 \oplus z \leftrightarrow Con_\tau(z))]$. Note that provably $1 + \alpha = \alpha$ if α is an infinite ordinal.

Proof: By transfinite induction up to λ in PA.

($\rightarrow$) Define $\varphi_1(z) := \forall x\ (d(x) \succcurlyeq \underline{1} \oplus z \wedge \ell = x \rightarrow Con_\tau(z))$. We apply (TI) to the formula $\Phi_1(z) := \varphi_1(z) \wedge [\underline{0}]_\tau \varphi_1(z)$, i.e. we show that

$$\vdash_{PA} z <\cdot \underline{\lambda} \wedge \forall u <\cdot z\, \Phi_1(u) \rightarrow \Phi_1(z). \qquad (*)$$

It is easily seen using (D1) and Lemma 13.3 that

a) $\vdash_{PA} z = \underline{0} \rightarrow \Phi_1(z)$.

Further using (D2), Lemma 13.3, Lemma 1.2 and some simple properties of functions $sc(x)$ and $pd(x)$, we obtain

b) $\vdash_{PA} Sc(z) \wedge \forall u <\cdot z\, \Phi_1(u) \wedge z <\cdot \underline{\lambda} \rightarrow \Phi_1(z)$.

We treat the case when z is a limit ordinal more formally.

c) $\vdash_{PA} Lim(z) \wedge \forall u <\cdot z\, \Phi_1(u) \wedge z <\cdot \underline{\lambda} \rightarrow \Phi_1(z)$.

Let A abbreviate the formula $Lim(z) \wedge \forall u \prec z\, \Phi_1(u) \wedge z \prec \underline{\lambda} \wedge \wedge \ell = x \wedge d(x) \succcurlyeq \underline{1} \oplus z$. Then we obtain by (D2)

$$\vdash_{PA} A \to \underline{1} \oplus z = z \wedge \forall u \prec z\, \exists y \succ x\, (d(y) \succcurlyeq \underline{1} \oplus u).$$

On the other hand

$$\vdash_{PA} A \wedge u \prec z \wedge d(y) \succcurlyeq \underline{1} \oplus u \wedge y \succ x \to \Phi_1(u)$$

$$\to [\underline{0}]_\tau (\ell = y \wedge d(y) \succcurlyeq \underline{1} \oplus u \to Con_\tau(u))$$

$$\to [\underline{0}]_\tau (\ell = y \to Con_\tau(u))$$, since

$\vdash_{PA} d(y) \succcurlyeq \underline{1} \oplus u \to [\underline{0}]_\tau\, d(y) \succcurlyeq \underline{1} \oplus u$, because $d(y)$ is a p.r. term. Therefore $\vdash_{PA} A \wedge u \prec z \wedge d(y) \succcurlyeq \underline{1} \oplus u \wedge y \succ x \to (\neg [\underline{0}]_\tau\, \ell \neq y \to$
$\to \neg [\underline{0}]_\tau\, Con_\tau(u)) \to \langle \underline{0} \rangle_\tau\, Con_\tau(u)$, by Lemma 13.3.
Hence $\vdash_{PA} A \wedge u \prec z \wedge \exists y \succ x\, (d(y) \succcurlyeq \underline{1} \oplus u) \to \langle \underline{0} \rangle_\tau\, Con_\tau(u)$,

$\vdash_{PA} A \wedge \forall u \prec z\, \exists y \succ x\, (d(y) \succcurlyeq \underline{1} \oplus u) \to \forall u \prec z\, \langle \underline{0} \rangle_\tau\, Con_\tau(u)$ and

$\vdash_{PA} A \to \forall u \prec z\, \langle \underline{0} \rangle_\tau\, Con_\tau(u)$.
Consequently $\vdash_{PA} A \to Con_\tau(z)$ by Lemma 1.3.
Of course, a), b) and c) together imply (~~x~~), q.e.d.

($\leftarrow$) Define $\varphi_2(z) := \forall x\, (d(x) \prec \underline{1} \oplus z \wedge \ell = x \neq 0 \to \neg Con_\tau(z))$,

$$\Phi_2(z) := \varphi_2(z) \wedge [\underline{0}]_\tau\, \varphi_2(z).$$

We only treat the case

c) $\vdash_{PA} Lim(z) \wedge \forall u \prec z\, \Phi_2(u) \wedge z \prec \underline{\lambda} \to \Phi_2(z)$.

Let A abbreviate the formula $Lim(z) \wedge z \prec \underline{\lambda} \wedge \forall u \prec z\, \Phi_2(u) \wedge \ell =$
$= x \neq 0 \wedge d(x) \prec \underline{1} \oplus z$. Then we obtain by (D2)

$$\vdash_{PA} A \to \exists u \mathrel{<\cdot} z\, \forall y \succ x\, (d(y) \mathrel{<\cdot} \underline{1} \oplus u)$$

$$\to \exists u \mathrel{<\cdot} z\, [\underline{0}]_\tau (\forall y \succ x\; d(y) \mathrel{<\cdot} \underline{1} \oplus u)$$, since

$\vdash_{PA} \forall y \succ x\, d(y) \mathrel{<\cdot} \underline{1} \oplus u \leftrightarrow d(x) \mathrel{\leq\cdot} \underline{1} \oplus u$ and $d(x)$ is a p.r. term.

Therefore $\vdash_{PA} A \to \forall u \mathrel{<\cdot} z\, [\underline{0}]_\tau (\forall y\, (l = y \neq 0 \wedge d(y) \mathrel{<\cdot} \underline{1} \oplus u \to \neg Con_\tau(u)))$

$$\to \exists u \mathrel{<\cdot} z\, [\underline{0}]_\tau (\forall y\, (y \succ x \wedge l = y \to \neg Con_\tau(u)))$$

$$\to \exists u \mathrel{<\cdot} z\, ([\underline{0}]_\tau (\exists y \succ x\, l = y) \to [\underline{0}]_\tau \neg Con_\tau(u))$$

$$\to ([\underline{0}]_\tau (\exists y \succ x\; l = y) \to \exists u \mathrel{<\cdot} z\, [\underline{0}]_\tau \neg Con_\tau(u)).$$

By Lemma 13.2 $\vdash_{PA} A \to \exists u \mathrel{<\cdot} z\, [\underline{0}]_\tau \neg Con_\tau(u)$ and by Lemma 1.3.

$$\vdash_{PA} A \to \neg Con_\tau(z)$$, q.e.d.

The desired arithmetical assignment f is defined in the following way: for every propositional variable p, $f(p) := (\exists x\, (l = x \wedge x \Vdash p))$

Lemma 15. Suppose φ is a formula of L_Λ . Then

1. $\vdash_{PA} l = x \neq 0 \wedge x \Vdash \varphi \to f_\tau(\varphi)$,
2. $\vdash_{PA} l = x \neq 0 \wedge \neg (x \Vdash \varphi) \to \neg f_\tau(\varphi)$.

Proof: By induction on φ . We consider only the case that φ has the form $[\lambda]\theta$, $\lambda \in \Lambda$.

1. First of all by (T6)c) we have

$$\vdash_{PA} x \Vdash [\lambda]\theta \leftrightarrow \exists z \mathrel{<\cdot} \underline{\lambda}\, \forall y \succ x\, (d(y) \mathrel{\geq\cdot} z \to y \Vdash \theta).$$

By I.H. $\vdash_{PA} z \mathrel{<\cdot} \underline{\lambda} \wedge y \succ x \neq 0 \wedge l = y \wedge y \Vdash \theta \to f_\tau(\theta)$.

On the other hand by Lemma 14,

$$\vdash_{PA} \forall x \forall z \lessdot \underline{\lambda}\, (d(x) \lessdot z \wedge \ell = x \neq 0 \rightarrow \neg\, Con_\tau(z)).$$

It is not difficult to see that the formula $\forall y \succ x\, (d(y) \gtrdot z \rightarrow y \Vdash \theta)$ is in fact equivalent to a p.r. formula, because, by the construction of $\mathcal{K}$, the quantifier $\forall y$ can be reduced to one ranging over a finite set. Therefore, we obtain

$$\vdash_{PA} z \lessdot \underline{\lambda} \wedge \forall y \succ x\, (d(y) \gtrdot z \rightarrow z \Vdash \theta) \rightarrow [\underline{0}]_\tau (z \lessdot \underline{\lambda} \wedge \forall y \succ x\, (d(y) \gtrdot z \rightarrow z \Vdash \theta))$$

$$\rightarrow [\underline{0}]_\tau\, \forall y \succ x\, (\ell = y \wedge Con_\tau(z) \rightarrow f_\tau(\theta))$$

$$\rightarrow ([\underline{0}]_\tau (\exists y \succ x\, \ell = y) \rightarrow [\underline{0}]_\tau (Con_\tau(z) \rightarrow f_\tau(\theta))).$$

Hence, by Lemma 13.2 $\vdash_{PA} x \Vdash [\lambda]\theta \wedge \ell = x \neq 0 \rightarrow [\underline{0}]_\tau (\exists y \succ x\, \ell = y)$

$$\rightarrow \exists z \lessdot \underline{\lambda}\, [\underline{0}]_\tau\, (Con_\tau(z) \rightarrow f_\tau(\theta)).$$

Consequently, by Lemma 1.3

$$\vdash_{PA} x \Vdash [\lambda]\theta \wedge \ell = x \neq 0 \rightarrow [\underline{\lambda}]_\tau f_\tau(\theta) \qquad \text{, q.e.d.}$$

2. By (T6)c) we have

$$\vdash_{PA} x \not\Vdash [\lambda]\theta \leftrightarrow \forall z \lessdot \underline{\lambda}\; \exists y \succ x\, (d(y) \gtrdot z \wedge z \not\Vdash \theta).$$

By I.H. $\vdash_{PA} \ell = y \neq 0 \wedge y \not\Vdash \theta \rightarrow \neg f_\tau(\theta)$ and by Lemma 14

$$\vdash_{PA} z \lessdot \underline{\lambda} \wedge d(y) \gtrdot z \wedge \ell = y \neq 0 \longrightarrow Con_\tau(z).$$

Therefore, $\vdash_{PA} y \succ x \neq 0 \wedge z \lessdot \underline{\lambda} \wedge d(y) \gtrdot z \wedge y \not\Vdash \theta \rightarrow$

$$\rightarrow [\underline{0}]_\tau (y \succ x \neq 0 \wedge z \lessdot \underline{\lambda} \wedge d(y) \gtrdot z \wedge y \not\Vdash \theta)$$

$$\rightarrow [\underline{0}]_\tau (\ell = y \rightarrow Con_\tau(z) \wedge \neg f_\tau(\theta)).$$

Hence $\vdash_{PA} x \neq 0 \wedge z < \underline{\lambda} \wedge \exists y \succ x\, (d(y) \geqslant z \wedge y \Vdash\!\!\!/\; \theta) \rightarrow$

$$\rightarrow \exists y \succ x [\underline{0}]_\tau ((Con_\tau(z) \rightarrow f_\tau(\theta)) \rightarrow \ell \neq y)$$

$$\rightarrow \exists y \succ x (\neg [\underline{0}]_\tau \ell \neq y \rightarrow \neg [\underline{0}]_\tau (Con_\tau(z) \rightarrow f_\tau(\theta))).$$

It follows by Lemma 13.3 that $\vdash_{PA} \ell = x \neq 0 \wedge x \Vdash\!\!\!/\; [\lambda]\theta \rightarrow$

$$\rightarrow \forall z < \underline{\lambda}\; \neg [\underline{0}]_\tau (Con_\tau(z) \rightarrow f_\tau(\theta)).$$

Consequently by Lemma 1.3

$$\vdash_{PA} \ell = x \neq 0 \wedge x \Vdash\!\!\!/\; [\lambda]\theta \rightarrow \neg [\underline{\lambda}]_\tau f_\tau(\theta)$$, q.e.d. ■

To derive the statement of Theorem 3 note that by Lemma 15

$$\vdash_{PA} \bar{1} \Vdash\!\!\!/\; \varphi \wedge \ell = \bar{1} \succ 0 \rightarrow \neg f_\tau(\varphi).$$

Clearly by (K3) and (K5) $\vdash_{PA} \bar{1} \Vdash\!\!\!/\; \varphi \wedge \bar{1} \succ 0$; hence $\vdash_{PA} \ell = \bar{1} \rightarrow \neg f_\tau(\varphi)$

By Lemma 13.5 $\vdash_{PA} f_\tau(\varphi)$. ■

Theorem 4. Suppose $TL^+_\Lambda \nvdash \varphi$. Then there is an arithmetical assignment f s.t. $f_\tau(\varphi)$ is false in the standard model of PA.

Proof: Let $H(\varphi)$ denote the formula $H^{max\Lambda}(\Gamma)$, where Γ is the set of all subformulas of formulas in $\{\varphi\} \cup \cup \{[\lambda]\bot \mid \lambda \in \Lambda\}$. Since $TL^+_\Lambda \nvdash \varphi$ we have $TL_\Lambda \nvdash H(\varphi) \rightarrow \varphi$. Let $\mathcal{K}$ be a countermodel for $H(\varphi) \rightarrow \varphi$ as in the proof of Theorem 3. Of course $1 \Vdash H(\varphi)$ and $1 \Vdash\!\!\!/\; \varphi$. Since $\neg [max\,\Lambda]\bot$

occurs in $H(\varphi)$, we obtain $d(1) \geqslant \max\Lambda$.

It is easily seen now that for all ψ in Γ , $0 \Vdash \psi \Leftrightarrow 1 \Vdash \psi$. Consequently $0 \Vdash H(\varphi)$ and $0 \nVdash \varphi$.

Lemma 16. For every formula φ in Γ ,

1. $\vdash_{PA} \ell = 0 \wedge 0 \Vdash \psi \rightarrow f_\tau(\psi)$;

2. $\vdash_{PA} \ell = 0 \wedge 0 \nVdash \psi \rightarrow \neg f_\tau(\psi)$.

Proof: 2 is analogous to Lemma 15.2.

1. We only consider the case where φ has the form $[\lambda]\theta$, $\lambda \in \Lambda$.

a) Suppose $0 \nVdash [\lambda]\theta$. Then $\vdash_{PA} 0 \nVdash [\lambda]\theta$ and trivially

$$\vdash_{PA} \ell = 0 \wedge 0 \Vdash \psi \rightarrow f_\tau(\psi).$$

b) Suppose $0 \Vdash [\lambda]\theta$. Pick $\alpha < \lambda$ s.t. $\forall x \succ 0 \ (d(x) \geqslant 1+\alpha \Rightarrow x \Vdash \theta)$. Clearly $\vdash_{PA} \forall x \succ 0 \ (d(x) \geqslant \underline{1+\alpha} \rightarrow x \Vdash \theta)$.

On the other hand by Lemma 14, $\vdash_{PA} Con_\tau(\underline{\alpha}) \wedge \ell = x \neq 0 \rightarrow d(x) \geqslant \underline{1+\alpha}$.

Hence by Lemma 15.1

$$\vdash_{PA} Con_\tau(\underline{\alpha}) \wedge \ell = x \neq 0 \rightarrow x \Vdash \theta \rightarrow f_\tau(\theta),$$

ergo $\vdash_{PA} Con_\tau(\underline{\alpha}) \wedge \exists x \ \ell = x \neq 0 \rightarrow f_\tau(\theta)$.

Since $0 \Vdash [\lambda]\theta$ and $0 \Vdash H(\varphi)$, clearly $0 \Vdash \theta$ and $\vdash_{PA} 0 \Vdash \theta$.

By I.H. we obtain $\vdash_{PA} \ell = 0 \rightarrow f_\tau(\theta)$. It follows that

$\vdash_{PA} Con_\tau(\underline{\alpha}) \wedge (\exists x \ \ell = x \neq 0 \vee \ell = 0) \rightarrow f_\tau(\theta)$, and by Lemma 13.1

$\vdash_{PA} Con_\tau(\underline{\alpha}) \rightarrow f_\tau(\theta)$.

Since $\alpha < \lambda$, we obtain $\vdash_{PA} [\underline{\lambda}] f_\tau(\theta)$ and trivially

$$\vdash_{PA} \ell = 0 \wedge 0 \Vdash \theta \rightarrow f_\tau(\psi) \quad \text{, q.e.d.}$$ ■

To derive Theorem 4 note that by Lemma 16

$$\vdash_{PA} \ell = 0 \wedge 0 \Vdash \varphi \rightarrow \neg f_\tau(\varphi).$$

By Lemma 13.4 $\ell = 0 \wedge 0 \Vdash \varphi$ is true; hence $f_\tau(\varphi)$ is false in the standard model of PA. ■

Corollary 5. 1. $TL^+_\Lambda \vdash \varphi \Leftrightarrow TL_\Lambda \vdash H(\varphi) \rightarrow \varphi.$

2. TL^+_Λ is decidable.

Proof: Follows from the proof of Theorem 4 and Corollary 4. ■

Corollary 6. Let φ be a formula in L_{ε_0}. Then

1. If $TL_{\varepsilon_0} \nvdash \varphi$ then there is an assignment f s.t.

$$PA \nvdash f_\tau(\varphi).$$

2. If $TL^+_{\varepsilon_0} \nvdash \varphi$ then there is an assignment f s.t. $f_\tau(\varphi)$ is false in the standard model of PA.

Proof: Suppose φ is any formula in L_{ε_0}. Every modal operator occuring in φ has the form either $\Box$ or $[\lambda + n]$, where λ is a limit ordinal and $n < \omega$. Let Λ denote $\{\lambda \mid \lambda$ is a limit ordinal and for some $n < \omega$ $[\lambda + n]$ occurs in $\varphi\}$ and let φ^0 denote the result of substituting the formula $\Box(\neg \Box^n [\lambda] \bot \rightarrow \psi)$ for each subformula of φ of the form $[\lambda + n + 1]\psi$. Clearly φ^0 is a formula of L_Λ and $TL_{\varepsilon_0} \vdash \varphi \leftrightarrow \varphi^0$.

By Theorem 3 (respectively 4) there is an assignment f s.t. $f_\tau(\varphi^0)$ is unprovable in PA (false). On the other hand by Lemma

2, $\vdash_{PA} f_\tau(\varphi^0) \leftrightarrow f_\tau(\varphi)$. It follows that $f_\tau(\varphi^0)$ is unprovable (resp. false) whenever $f_\tau(\varphi)$ is, q.e.d. ∎

Corollary 7. TL_{ε_0} and $TL^+_{\varepsilon_0}$ are decidable. ∎

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