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ON ROSSER'S PROVABILITY PREDICATE

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ON ROSSER'S PROVABILITY PREDICATE

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In their paper [4] Guaspari and Solovay investigate the system R of modal provability logic extended with witness comparison operators $\Box A \prec \Box B$ and $\Box A \preccurlyeq \Box B$ (see also Smoryński [9] and de Jongh [5]). These are intended to express that there is a proof of A whose gödelnumber is smaller than (resp. smaller than or equal to) the gödelnumber of any proof of B. They prove an arithmetical completeness theorem which states that R is precisely all that can be generally said (i.e. proved in arithmetic) about $\prec$ and $\preccurlyeq$.

In this paper we restrict our attention to witness comparison formulae of the form $\Box A \prec \Box \neg A$. This is abbreviated by $\Box^R A$ because "to have a proof smaller than any refutation" is, in essence, the provability concept used by Rosser [7] to strengthen Gödel's First Incompleteness Theorem. Some modal principles valid for $\Box$, which stands for the usual provability formula, and $\Box^R$ were listed in Visser [11].

Section 1 introduces a system GR of propositional modal logic with operators $\Box$ and $\Box^R$ and constructs a Kripke semantics for it. This system of ours is in fact nothing but a fragment og Guaspari and Solovay's R. In Section 2 we discuss the provability interpretation of GR. Section 3 is devoted to the proof of the uniform arithmetical completeness of GR which is the main result of the present paper.

One of the difficulties the arithmetical completeness theorem of [4] meets is that R is not complete with respect to any single proof predicate and to prove completeness one also has to vary proof predicates. A reason for this lies in the fact that any proof predicate will either validate

$$\Box(T\wedge T)\preccurlyeq\Box(T\vee T) \quad \text{or} \quad \Box(T\vee T)\prec\Box(T\wedge T)$$

and it is not clear why we should prefer one of these to the other. Of course, neither is derivable in R . Formulas like these are absent in the language of GR and the completeness proof for GR employs only suitable proof predicate (although some proof predicates validate more than GR). It should however be noted that the trivial examples like the one cited above do by no means exhaust what is being lost by restricting witness comparisons to $\Box^{R}$. The weaker language also enables another improvement. The completeness theorem for R requires the use of proof predicates that view each proof as a proof of not just one but possibly many theorems unless one imposes severe and otherwise unjustified restrictions on the kind of interpretations he takes into account. This shortcoming just vanishes when working with GR.

Section 4 is also inspired by Guaspari and Solovay's paper. It is shown in [4] that provable uniqueness of Rosser fixed points (i.e. fixed points of $\neg\Box^{R}$) depends on the choice of a particular proof predicate. We take a look at those predicates none of whose Rosser fixed points are provably equivalent. It turns out that such proof predicates not only do exist but are in a sense inseparable from those possessing a provably unique Rosser fixed point.

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1. The system GR

We work with a propositional language with two modal operators $\Box$ and $\Box^R$. Let $\Diamond$ abbreviate $\neg\Box\neg$ and let $\bot$ denote falsehood. $\boxdot$ stands for $A \wedge \Box A$. The following defines the system GR.

Axiom schemata.

A1. Those of GL for $\Box$ (cf. Solovay [10])

A2. $\Box^R A \to \Box A$

A3. $\Box A \to \Box\Box^R A$

A4. $\Box A \to (\Box\bot \vee \Box^R A)$

A5. $\Diamond\Box^R A \to \Diamond A$.

Rules of inference.

R1. $A, A \to B \vdash B$

R2. $A \vdash \Box A$

R3. $\Box A \vdash A$.

The system GR^- is obtained from the above list by dropping R3.

1.1. D e f i n i t i o n. A model K is a tuple $(K, \prec, 0, \Vdash)$ where K is a non-empty set; $\prec$ is a reversely well-founded partial order on K with $0 \in K$ as the lowermost element; and $\Vdash$ is a forcing relation for formulae of the modal language. $\prec$ is the accessibility relation for $\Box$ and each node forces every instance of A2-5. Write $K \Vdash A$ to mean that $a \Vdash A$, all $a \in K$.

1.2. L e m m a. If $GR^{-} \vdash A$ and K is a model then $K \Vdash A$.

P r o o f. Trivial. $\Box$

Let K be a model and a be a top node of K . The relation $\subset_a$ on modal formulas is defined as follows.

$$\subset_a = \{(A, \neg A) \mid a \Vdash \Box A\} \cup \{(\neg A, A) \mid a \nVdash \Box^R A\}.$$

In other words,

$$A \subset_a \neg A \Leftrightarrow a \Vdash \Box^R A \quad , \text{ and } \quad \neg A \subset_a A \Leftrightarrow a \nVdash \Box^R A.$$

Finally let R_a be the transitive closure of $\subset_a$.

1.3. L e m m a. R_a is an irreflexive partial order.

P r o o f. Suppose the contrary. There are then modal formulas $A_1, \dots, A_n$ $(n \geq 1)$ s.t.

$$A_1 \subset_a A_2 \subset_a \dots \subset_a A_n \subset_a A_1 .$$

But this implies that there exists $i \in \{1, \dots, n\}$ with

$$A_i \subset_a \neg A_i \subset_a A_i \quad or \quad \neg A_i \subset_a A_i \subset_a \neg A_i$$

which is impossible. $\Box$

1.4. D e f i n i t i o n. A top node a of a model K is compact if every R_a -chain possessing an uppermost element is finite. The model K is compact if so is each one of its nodes.

1.5. D e f i n i t i o n. A finite set α of modal formulae is said to be adequate if it is closed under subformulas. $K = (K, \prec, a, \Vdash)$ is an α -model for α adequate if $K, \prec, 0$ and $\Vdash$ are as in Definition 1.1. except that $\Vdash$ is only defined for those modal formulas all of whose variables and subformulas of the form $\Box^R A$ are in α. In particular, if the forcing of some instance B of schemas A2-5 is defined in K then $K \Vdash B$. An α-model is called C-model if α is the set of subformulas of the modal formula C.

1.6. L e m m a. Let α be an adequate set and $K = (K, \prec, 0, \Vdash)$ an α-model. Then there exists a forcing relation $\Vdash'$ extending $\Vdash$ s.t. $K' = (K, \prec, 0, \Vdash')$ is a compact model.

P r o o f. Let $a \in K$. Define $a \Vdash'$ inductively as follows.

(i) $a \Vdash' p_i$ if p_i is a propositional variable and $p_i \notin \alpha$;

(ii) the induction step for Boolean connectives and for $\Box$ is as usual;

(iii) if $\Box^R A$ is not in α let

$$a \Vdash' \Box^R A \Leftrightarrow \begin{cases} a \Vdash' A & \text{if a is a top node of } K \\ a \Vdash' \Box A & \text{otherwise} \end{cases}$$

We show that K' is a model, that is every instance of A2-5 is forced in K'. We only consider A3 and A5. Suppose $\Box^R A \notin \alpha$ (otherwise the corresponding instances of axiom schemas are forced in K' because they are forced in K).

Ad A3. In case a is a top node of K we have $a \Vdash' \Box B$ for every formula B and so $a \Vdash' \Box \Box^R A$. If a is not a top node then $a \Vdash' \Box^R A$ implies $a \Vdash' \Box A$. Ergo for all $b \in K$ with $a \prec b$ one has $b \Vdash' \Box A$ and $b \Vdash' A$, hence $b \Vdash' \Box^R A$ and therefore $a \Vdash' \Box \Box^R A$. In either case $a \Vdash' \Box^R A \rightarrow \Box \Box^R A$.

Ad A5. Assume $a \Vdash' \Diamond \Box^R A$ so there is a node b s.t. $a \prec b$ and $b \Vdash' \Box^R A$. If b is a top node we get $b \Vdash' A$ whence $a \Vdash' \Diamond A$. Otherwise $b \Vdash' \Box A$ and any $c \in K$ with $b \prec c$ will force A. Such c exists by the assumption on b and we have $a \prec c$ which implies $a \Vdash' \Diamond A$. Conclude $a \Vdash' \Diamond \Box^R A \to \Diamond A$.

Next it has to be shown that K' is compact. Let n be the cardinality of α. Consider a top node a of K'. Suppose for a contradiction that there exists a R_a-chain of length $k = 3n+4$, that is there are formulas $A_1, \dots, A_k$ s.t. $A_1 \subset_a \dots \subset_a A_k$.

It is easily seen that one either has $A_i = \neg^{i-1} A_1$ for every $i \in \{1, \dots, k\}$ or $A_i = \neg^{k-i} A_k$ for every $i \in \{1, \dots, k\}$ (we take $\neg^0 B = B$ and $\neg^{m+1} B = \neg^m \neg B$). The choice of k guarantees the existence of $i \in \{1, \dots, k-2\}$ s.t. $A_i, A_{i+1} \notin \alpha$. Suppose $A_{i+2} = \neg A_{i+1} = \neg\neg A_i$. This implies $A_i \subset_a A_i \subset_a \neg\neg A_i$ and therefore $a \Vdash' A_i$ and $a \Vdash' \neg A_i$ which is absurd. The situation $A_i = \neg A_{i+1} = \neg\neg A_{i+2}$ is treated similarly. $\Box$

1.7. D e f i n i t i o n. A-model $K = (K, \prec, 0, \Vdash)$ is finite if so is K. K is nontrivial if there is a node $a \in K$ s.t. $0 \prec a$.

1.8. T h e o r e m. The following are equivalent

(1) $GR^- \nvdash A$;

(2) There exists a finite A-model K s.t. $K \nVdash A$;

(3) There exists a finite compact model K' s.t. $K' \nVdash A$.

P r o o f. (2) $\Longrightarrow$(3) follows at once from Lemma 1.6; (3) $\Longrightarrow$(1) is in fact Lemma 1.2.

(1) $\Longrightarrow$(2). Let $\Box^R B_1, \dots, \Box^R B_n$ be all graphically distinct subformulas of A of the form indicated. Fix a string $\vec{q} = q_1, \dots, q_n$ of distinct variables not occurring in A. We define a translation $^+$ of subformulas of A into the modal language not containing $\Box^R$.

(i) $p_i^+ \equiv p_i$;

(ii) $^+$ distributes over Booleans and $\Box$;

(iii) $(\Box^R B_i)^+ \equiv q_i$.

Trivially, $GL \vdash B$ implies $GR^- \vdash B$. Thus assuming $GR^- \nvdash A$ one gets

$$GL \nvdash (\bigwedge_{i=1}^{k} \boxdot ((q_i \to \Box B_i^+) \wedge (q_i \to \Box q_i) \wedge$$

$$\wedge (\Box B_i^+ \to (\Box\bot \vee q_i)) \wedge (\Diamond q_i \to \Diamond B_i^+))) \to A^+$$

because the antecedent of the formula above is a conjunction of translations of axioms of GR^-. The modal completeness theorem for GL (cf. Solovay [10]) provides a finite $\{\vec{p}, \vec{q}\}$ -model $K^0 = (K, \prec, 0, \Vdash^0)$ (elements of $\vec{p}$ are propositional variables occuring in A) with $K^0 \Vdash \bigwedge \boxdot (\dots)$ and $K^0 \nVdash A$. We turn it into an A-model $K = (K, \prec, 0, \Vdash)$ by letting

$$a \Vdash \Box^R B_i \Longleftrightarrow a \Vdash^0 q_i$$

That K is an A-model is now implied by $K^0 \Vdash \bigwedge \boxdot (\dots)$. $\Box$

1.9. T h e o r e m. (a) If $GR \nvdash A$ then there exists a finite compact nontrivial model K s.t. $K \nVdash A$.

(b) If $GR \vdash A$ then $K \Vdash A$, all nontrivial K .

P r o o f. (a). It follows from $GR \nvdash A$ that $GR^- \nvdash \Box A$. Theorem 1.8 supplies a finite compact K s.t. $K \nVdash \Box A$ and hence $K \nVdash A$. Clearly K can not be trivial.

(b). Use induction on the length of proof of A in GR . We only consider the step corresponding to $R3$. Assume $K \Vdash \Box A$ for all nontrivial K . Let $M = (M, \prec, 0, \Vdash)$ be an arbitrary nontrivial model. Define $M' = M \cup \{0'\}$; $\prec' = \prec \cup \{(0', a) \mid a \in M\}$ and let $\Vdash'$ extend $\Vdash$ so that

$$0' \Vdash' p_i \quad \text{,all variables } p_i \text{ , and}$$

$$0' \Vdash' \Box^R B \Leftrightarrow 0' \Vdash' \Box B (\Leftrightarrow M \Vdash B).$$

We prove that $M' = (M', \prec', 0', \Vdash')$ is a model. It will certainly suffice to check that $0'$ forces every axiom of GR^-. The only interesting case is A5. Suppose $0' \Vdash' \Diamond \Box^R B$ whence $a \Vdash' \Box^R B$ for some $a \in M$. If $a = 0$ then $a \Vdash' \Box\Box^R B$ by A3 and since M is nontrivial there is a node b , $a \prec' b$, $b \Vdash' \Box^R B$. Hence $0 \Vdash' \Diamond \Box^R B$. In case $a \neq 0$ this obviously is also true. Now $0 \Vdash' \Diamond \Box^R B$ implies $0 \Vdash' \Diamond B$ because M forces A5. Recalling $0' \prec' 0$ we get $0' \Vdash' \Diamond B$.

By our assumption on A and because M' is nontrivial we have $M' \Vdash \Box A$. So $M \Vdash A$. $\Box$

1.10. C o r o l l a r y. $GR \vdash A \Leftrightarrow GR^- \vdash \Box A$.

1.11. E x a m p l e. The formula $\neg \Box^R \bot$ being a theorem of GR is not derivable in GR^- .

2. The provability interpretation

If $\varphi(x_1,\dots,x_n)$ is a formula of first order arithmetic then $\overline{\varphi(\bar{x}_1,\dots,\bar{x}_n)}$ denotes the usual primitive recursive (p.r.) term representing substitution of the x_i th numeral (i.e. 0 followed by x_i strokes) for the variable x_i in φ. PA is Peano Arithmetic.

2.1. D e f i n i t i o n. A formula $Thm(x,y)$ ("x is a proof of y") with just x and y free is a standard proof predicate (s.p.p.) if letting $Th(y)$ stand for $\exists x\ Thm(x,y)$ one has

P1. $Th(y)$ numerates theorems of PA in PA;

P2. Let $Pr_\alpha(y)$ be the formula expressing in a natural way that y is provable from $\{x \mid \alpha(x)\}$ in the first order logic (cf. e.g. Feferman [2]). Then

$$PA \vdash \forall y\,(Pr_{Th}(y) \leftrightarrow Th(y))$$

P3. There exists a p.r. term $g(x)$ s.t.

$$PA \vdash \forall xy\ (g(x) = y \leftrightarrow Thm(x,y)).$$

2.2. R e m a r k. All results of this paper remain valid if one replaces the property P2 by a longer list of its weaker consequences such as

$$PA \vdash Th(x \to y) \wedge Th(x) \to Th(y)$$

$$\sigma(x_1,\dots,x_n) \text{ is } \Sigma_1 \Rightarrow PA \vdash \sigma(x_1,\dots,x_n) \to$$
$$\to Th(\overline{\sigma(\bar{x}_1,\dots,\bar{x}_n)})$$

etc. Also the property P3 is only given its present form for reasons of simplicity. Results below could do with the following condition which is met by most of "usual" provability predicates.

P3'. There exist p.r. terms $g(x)$ and $m(x)$ s.t.

$$PA \vdash \forall zy\,(g(z)=y \leftrightarrow Thm\,(m(z),y))$$

$$PA \vdash \forall z\,(m(z)< m\,(z+1))$$

$$PA \vdash \forall xy\,(Thm\,(x,y) \leftrightarrow \exists z\; m(z)=x).$$

In cases when we want to stress that a term $g(x)$ witnesses the fact that a formula is a s.p.p. we write $Thm_{(g)}(x,y)$ for this formula. For the sequel we fix one such term $g(x)$. If $f(x)$ is a p.r. term we sometimes write $Thm_f\,(x,y)$ for $f(x)=y$ even if we do not know at the moment whether $f(x)=y$ is a s.p.p.

2.3. D e fi n i t i o n. A s.p.p. $Thm_f\,(x,y)$ is called g-like if

(i) $PA \vdash \forall y\,(Th_f\,(y) \leftrightarrow Th_{(g)}(y))$ and

(ii) $\omega \models \forall x\,(f(x)=g(x))$.

Following Rosser [7] we define

$$Thm^R(x,y) \equiv Thm\,(x,y) \wedge \forall z \leq x \neg\, Thm\,(z, \neg\, y)$$

$$Th^R(y) \equiv \exists x\, Thm^R(x,y).$$

2.4. D e f i n i t i o n. Let $Thm(x,y)$ be a s.p.p. A function $*$ mapping modal formulae to arithmetic sentences is said to be a Thm -translation if

(i) $\perp * \equiv (0=1)$;

(ii) $*$ distributes over Boolean connectives;

(iii) $(\Box A)^* \equiv Th(\overline{A^*})$;

(iv) $(\Box^R A)^* \equiv Th^R(\overline{A^*})$.

$*$ is a translation if it is a Thm-translation for some s.p.p. $Thm(x,y)$.

Let GR^ω denote the modal system axiomatized by theorems of GR (or GR^-) and formulas of the form $\Box A \to A$ and with R1 the only rule of inference.

2.5. L e m m a. (a) $GR \vdash A \Rightarrow PA \vdash A^*$ for all translations $*$.

(b) $GR^\omega \vdash A \Rightarrow \omega \models A^*$ for all translations $*$.

P r o o f. We only check A5: $\Diamond \Box^R A \to \Box A$. Reason in PA . Assume $(\neg \Diamond A)^*$, that is, $Th(\overline{\neg A^*})$. There is an x s.t. $Thm(x, \overline{\neg A^*})$. If $\forall z \leq x \neg Thm(z, \overline{A^*})$ then $\neg Th^R(\overline{A^*})$. Formalizing this we get $Th(\overline{\neg Th^R(\overline{A^*})})$. Otherwise if $\exists z \leq x\, Thm(z, \overline{A^*})$ we have $Th(\overline{\neg A^*})$ and $Th(\overline{A^*})$ whence $Th(\overline{0=1})$ and so $Th(\overline{\neg Th^R(\overline{A^*})})$. We have proved $(\Box \neg A \to$ $\to \Box \neg \Box^R A)^*$ which is equivalent to $(\Diamond \Box^R A \to \Diamond A)^*$. $\Box$

2.6. E x a m p l e. The following theorem of $GR (GR^-)$ is an approximation to (formalized) Rosser's theorem [7].

$$\Box(p \leftrightarrow \neg \Box^R p) \to (\Box p \vee \Box \neg p \to . \Box \perp).$$

2.7. R e m a r k. The converse of Lemma 2.5 need not generally hold true. For example one can easily construct a g-like s.p.p. $Thm(x,y)$ s.t. for every Thm -translation $*$

$$PA \vdash (\Box^R(p \to q) \to . \Box^R p \to \Box^R q)^*$$

whereas

$$GR \nvdash \Box^R(p \to q) \to . \Box^R p \to \Box^R q \ .$$

3. Uniform arithmetical completeness of GR

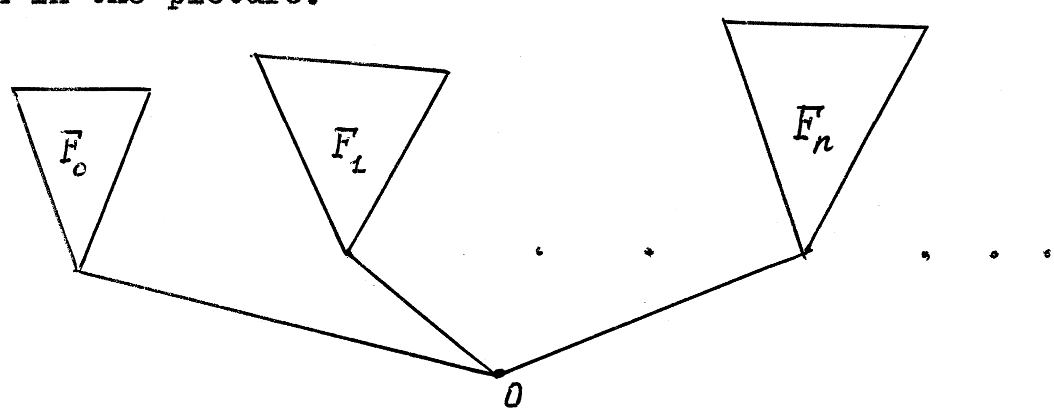
3.1. T h e o r e m. There exist a p.r. term $f(x)$ and a Thm_f -translation $*$ s.t.

(1) $Thm_f(x,y)$ is a g -like s.p.p.

(2) $PA \vdash A^*$ implies $GR \vdash A$, all modal formulas A .

Fix a "natural" gödelnumbering $\ulcorner \ \urcorner$ of modal formulae. By $\dot{\to}$ we denote a p.r. term representing the function $(\ulcorner A \urcorner, \ulcorner B \urcorner) \mapsto \ulcorner A \to B \urcorner$, similarly for the other connectives of the modal language.

Looking at the proof of semantic completeness theorem for GR (see Section 1) and the proof of the underlying semantic completeness theorem for GL (see Solovay [10]) one can see that given a modal formula A we can in a p.r. way associate with it a finite compact nontrivial model $F_{\ulcorner A \urcorner}$ with the property $GR \vdash A \Leftrightarrow F_{\ulcorner A \urcorner} \Vdash A$. Following Artemov [1] and de Jongh and Montagna [6] we combine F_i's into a single model X shown in the picture.



Letting $0 \Vdash p_i$ for all propositional variables p_i and extending $0 \Vdash$ to all modal formulae in the usual way we have that X is a compact nontrivial model s.t. $X \Vdash A \Leftrightarrow GR \vdash A$ for all modal formulas A. Assume without loss of generality that the domain of X is ω.

Next we choose p.r. formulas $x \prec y$ and $x \Vdash y$ binumerating the accessibility and forcing relations in X respectively (see Artemov [1] and de Jongh and Montagna [6]) and also a p.r. term $\gamma(x,y)$ representing the height of a modal formula (with the gödelnumber) y in the partial order R_x in case x is a top node of X (see Section 1). Since all F_i 's are compact this height is finite. By formalizing results of Section 1 in PA we can assume that the said formulas and term are chosen so that PA proves the following conditions.

M1. $\forall x \ (x \neq 0 \rightarrow 0 \prec x)$

M2. $\forall xy \ (x \prec y \rightarrow \neg\, y \prec x)$

M3. $\forall xyz \ (x \prec y \wedge y \prec z \rightarrow . \ x \prec z)$

M4. $\forall xyz \ (x \Vdash y \dot{\wedge} z \leftrightarrow . \ x \Vdash y \wedge x \Vdash z)$

and similarly for other Boolean connectives

M5. $\forall xy \ (x \Vdash \dot{\Box} y \leftrightarrow \forall z \ (x \prec z \rightarrow z \Vdash y))$

M6. $\forall xy \ (x \Vdash \dot{\Box}^R y \dot{\rightarrow} \dot{\Box} y)$ and similarly for schemas A3-5

M7. $\forall x \ (0 \Vdash \dot{\Box} x \rightarrow 0 \Vdash x)$

M8. $\forall xy \ (x \Vdash \ulcorner \Box \bot \urcorner \wedge x \Vdash \dot{\Box}^R y \rightarrow . \ \gamma(x,y) < \gamma(x, \dot{\neg} y))$

M9. $\forall xy \ (x \Vdash \ulcorner \Box \bot \urcorner \wedge x \nVdash \dot{\Box}^R y \rightarrow . \ \gamma(x, \dot{\neg} y) < \gamma(x,y))$.

It is well-known (see e.g. Solovay [10], Artemov [1] and de Jongh and Montagna [6]) that there are a p.r. term $h(x)$ and a ι-term ℓ s.t.

S1. $\omega \models \ell = 0$

S2. $PA \vdash \forall z\,(h(x) = h(x+1) \vee h(x) \prec h(x+1))$

S3. $PA \vdash \exists x\, \forall y \geq x\, h(y) = \ell$

S4. $PA \vdash \forall x\,(h(x) \neq 0 \rightarrow Th_{(g)}\,(\overline{h(x) \prec \ell}))$

S5. $PA \vdash \forall x\,(\ell \prec x \rightarrow \neg Th_{(g)}\,(\overline{\ell \neq \bar{x}}))$

S6. $PA \nvdash \ell \neq n$, all $n \in \omega$.

Our arithmetical completeness proof will now follow that of Guaspari and Solovay [4]. First we define the value of $*$ on propositional variables p_i .

$p_1^* \equiv \ell \Vdash \ulcorner p_i \urcorner$.

We proceed to construct the desired term $f(x)$ and a p.r. term x^* to represent a Thm_f-translation $*$. This is done with the help of auxiliary finite sets Y_x and sets S_x which are constructed parallel to f and $*$. These sets are formally represented by p.r. formulas. Think of each number as a plea that $g(x)$ be made a value of f . The set Y_x will consist of those pleas $\leq x$ that have not been satisfied by $f(y)$ with $y < x$. A plea is allowed to be satisfied by $f(x)$ iff it is in S_x . Invoke the (formalized) recursion theorem to insure that the following clauses are theorems of PA .

$$Y_0 = \{0\}$$

$$S_x = \{z \mid \forall y\, (g(z) = y^* \rightarrow (h(x) \Vdash \underset{\cdot}{\Box} y \wedge (h(x) \not\Vdash \underset{\cdot}{\Box}^R y \rightarrow$$
$$\rightarrow \exists w < x\, f(w) = \neg y^*) \wedge \forall v\, ((y = \underset{\cdot}{\neg} v \wedge h(x) \Vdash \underset{\cdot}{\Box}^R v) \rightarrow$$
$$\rightarrow \exists w < x\, f(w) = v^*)))\}$$

$$f(x) = \begin{cases} g(\min(Y_x \cap S_x)) \text{ if } Y_x \cap S_x & \text{is non-empty} \\ \overline{0=0} & \text{otherwise} \end{cases}$$

$$Y_{x+1} = \begin{cases} \{x+1\} \cup Y_x \setminus \{\min(Y_x \cap S_x)\} \text{ if } Y_x \cap S_x & \text{is non-empty} \\ \{x+1\} \cup Y_x & \text{otherwise} \end{cases}$$

x^* = the value of x under the unique Thm_f -translation which assigns $\overline{1 \Vdash \ulcorner p_i \urcorner}$ to p_i, all variables p_i

3.2. **Lemma.**

$$PA \vdash \forall x\, (Th_f(x) \rightarrow Th_{(g)}(x)).$$

Proof. Trivial. □

3.3. **Lemma.**

$$PA \vdash \forall x\, ((\exists y\, \forall z \geq y\ x \in S_z) \rightarrow Th_f(g(x))).$$

Proof. Reason in PA. Fix an arbitrary x and assume $\exists y\, \forall z \geq y\ x \in S_z$. In particular, this implies that $\forall z \geq y\ S_z \neq \emptyset$. If it were the case that $g(x)$ is not output by f then we would have $x \in Y_z$, all $z \geq x$, and $\min(Y_z \cap S_z) < x$, all $z > \max(x, y)$. Therefore

$$|Y_z \cap \{0, \dots, x-1\}| = |Y_{z+1} \cap \{0, \dots, x-1\}| + 1$$

for all large enough z . This however can not be true. The contradiction proves $\exists z\ f(z) = g(x)$. $\square$

3.4. L e m m a.

$$PA \vdash \ell \Vdash \ulcorner \Box \bot \urcorner \leftrightarrow \forall y\ Th_f(y^*).$$

P r o o f (PA). $(\leftarrow)$. Assume $\forall y\ \exists z\ f(z) = y^*$ whence $\exists z\ f(z) = \ulcorner \bot \urcorner^*$. By Lemma 3.2 this implies $Th_{(g)}(\overline{0 = \ell})$ and hence by P2 $Th_{(g)}(\overline{\ell \neq \bar{z}})$, all z . Supposing $\ell \not\Vdash \ulcorner \Box \bot \urcorner$ provides a z with $\ell \prec z$ (see M4, M5). Now apply S5 to obtain a contradction.

$(\rightarrow)$. As in Solovay [10], $\ell \Vdash \ulcorner \Box \bot \urcorner$ implies

(I) $\forall x\ Th_{(g)}(x)$

(use M1, M4, M5, S3, S4 and P2). Consider an arbitrary modal formula A . Assume

(II) $\forall z\ (\gamma(\ell, z) < \gamma(\ell, \ulcorner A \urcorner) \rightarrow \exists y\ f(y) = z^*)$

in the right of induction hypothesis. Let x_0 be s.t.

(III) there exists y_0 s.t. $y_0 < x_0$ and $g(y_0) = \overline{A^*}$

(IV) there exists y_1 s.t. $y_1 < x_0$ and $g(y_1) = \overline{\neg A^*}$

(V) if A is of the form $\neg B$ then there exists y_2 s.t. $y_2 < x_0$ and $g(y_2) = \overline{B^*}$

(VI) $\forall y \geq x_0\ h(y) = \ell$

(VII) for all modal formulas C ,

$$(\exists y \leq max(y_0, y_1, y_2)\ g(y) = \overline{C^*}) \wedge \exists y\ f(y) = \overline{C^*} \rightarrow$$

$$\rightarrow . \exists y < x_0\ f(y) = \overline{C^*}$$

Such x_c exists because all large enough x_c satisfy (III) - (V) as granted by (I) and all large enough x_c satisfy (VI) because of S3. The property (VII) for large enough x_0 is established by induction on $max\ (y_0, y_1, y_2)$.

Let $y \geq x_c$. Note that $h(y) = \ell$ by (VI). If $h(y) \Vdash\!\!\!\!/_{R} \Box A$ then $\gamma(\ell, \ulcorner \neg A \urcorner) < \gamma(\ell, \ulcorner A \urcorner)$ by M9, whence $\exists z\ f(z) = \overline{\neg A^*}$ by virtue of (II). Therefore by (VII) one has $\exists z < y\ f(z) = \overline{\neg A^*}$. In case when A is of the form $\neg B$ we can prove $\exists z < y\ f(z) = \overline{B^*}$ in a similar way (use M8, (II), (V), (VII)). All this amounts to $y_0 \in S_y$. Lemma 3.3 implies now that $\exists z\ f(z) = \overline{A^*}$. $\Box$

3.5. L e m m a. (a) For all modal formulas A ,

$PA \vdash \ell \Vdash \ulcorner A \urcorner \leftrightarrow A^*$

(b) $PA \vdash \forall x\ Th_{(g)}(\overline{\ell \Vdash \bar{x} \leftrightarrow x^*})$

P r o o f. (a). Use induction on the structure of A . The induction step is straightforward for variables and Boolean connectives.

Treat $\Box$. Reason in PA . Let $\ell \Vdash \Box A$. If one has $\ell \Vdash \Box \bot$ then $(\Box A)^*$ follows from Lemma 3.4. We therefore assume $\ell \Vdash\!\!\!\!/\ \Box \bot$. In case $\ell = 0$ we get $0 \Vdash \Box A$ from M7 which implies $Th_{(g)}(\overline{0 \Vdash \ulcorner \Box A \urcorner})$ by P2. In view of M1 and M5 we conclude $Th_{(g)}(\overline{\forall x\ x \Vdash \ulcorner A \urcorner})$ and subsequently

$Th_{(g)}(\overline{\ell \Vdash \ulcorner A \urcorner})$. In case $\ell \neq 0$ we choose an x s.t. $h(x) = \ell$ and use S4 to obtain $Th_{(g)}(\overline{\bar{\ell} \prec \ell})$ whence $Th_{(g)}(\overline{\ell \Vdash \ulcorner A \urcorner})$ by M5 as in the previous case. Formalizing the induction hypothesis (see P1) yields $Th_{(g)}(\overline{\ell \Vdash \ulcorner A \urcorner \leftrightarrow A^*})$ ergo $Th_{(g)}(\overline{A^*})$. Now let $\overline{A^*} = g(z)$. We show that for all y with $y \geq x$ there holds $z \in S_y$. Indeed, since ℓ forces A4 (see M6), $\ell \nVdash \Box\bot$ and $h(y) = \ell$ (use S2, M2, M3 and the assumption $h(x) = \ell$) it is easily seen from M4 that $h(y) \Vdash \Box^R A$ and $h(y) \nVdash \Box^R B$ in case A is of the form $\neg B$. Apply Lemma 3.3 to conclude $Th_f(\overline{A^*})$.

Conversely, let $\ell \Vdash \Box A$. There exists an x s.t. $\ell \prec x$ and $x \nVdash A$. By S5 we get $\neg Th_{(g)}(\overline{\ell \neq \bar{x}})$. Therefore $\neg Th_{(g)}(\overline{\ell \Vdash \ulcorner A \urcorner})$ whence $\neg Th_{(g)}(\overline{A^*})$ implying $(\neg \Box A)^*$ by Lemma 3.2.

We now turn to the case of $\Box^R$. Again reason in PA. Assume $\ell \Vdash \Box^R A$. Because of A2 and the equivalence of $\ell \Vdash \Box A$ and A^* proved earlier it suffices to check that

$$\forall z\ (f(z) = \overline{\neg A^*} \longrightarrow \exists y < z\, f(y) = \overline{A^*})$$

Let us assume $Thm_f(z, \overline{\neg A^*})$. If $h(z) = \ell$ then it is seen from the costruction of f and S that this could not be the case unless there were y s.t. $y < z$ and $Thm_f(y, \overline{A^*})$. Otherwise, if one has $h(z) \prec \ell$, one also has $h(z) \Vdash \Diamond \Box^R A$ (because of S2). Conclude by A5 that $h(z) \Vdash \Diamond A$, that is $h(z) \nVdash \Box \neg A$ which implies $y \notin S_z$ whenever $g(y) = \overline{\neg A^*}$ contradicting the assumption $f(z) = \overline{\neg A^*}$.

Next let $\ell \Vdash\!\!\!/\, \Box^R A$. With the help of A3 it is not difficult to see that $\forall z\, h(z) \Vdash\!\!\!/\, \Box^R A$. By the construction of f the equality $f(z) = \overline{A^*}$ would imply $\exists y < z\; f(y) = \overline{\neg A^*}$ entailing $(\neg \Box^R A)^*$.

(b). Formalize the proof of (a) in PA. $\Box$

3.6. Lemma.

$$PA \vdash \forall x\,(Th_{(g)}(x^*) \rightarrow \ell \Vdash \dot{\Box} x)$$

Proof (PA). Consider a modal formula A. By Lemma 3.5(b) $Th_{(g)}(\overline{A^*})$ implies $Th_{(g)}\overline{(\ell \Vdash \ulcorner A \urcorner)}$ whence $Th_{(g)}(\overline{\ell \neq \bar{x}})$, all $x \Vdash\!\!\!/\, A$. In view of S5 this implies $x \Vdash A$, all x satisfying $\ell < x$. Conclude $\ell \Vdash \Box A$ by M5. $\Box$

3.7. Lemma.

$$PA \vdash \forall x\,(Th_{(g)}(x) \longrightarrow Th_f(x))$$

Proof (PA). In case $g(x) \notin rng^*$ this is an immediate consequence of Lemma 3.3. Therefore assume $g(x) = \overline{A^*}$ for some modal formula A. If $\ell \Vdash \Box\bot$ then by Lemma 3.4 we are done. Assume $\ell \Vdash\!\!\!/\, \Box\bot$ and let $h(z) = \ell$. From Lemma 3.6 one has $\ell \Vdash \Box A$. We show $\forall y \geq max(x, z)\; z \in S_y$. Indeed, A4 yields $h(y) \Vdash \Box^R A$ and also $h(y) \Vdash\!\!\!/\, \Box^R B$ in case A is $\neg B$ (see A2). The proof is completed by applying Lemma 3.3. $\Box$

3.8. Proof of Theorem 3.1 concluded. (2). Suppose $GR \vdash A$ for A a modal formula. There is then a $n \in \omega$ with $n \Vdash\!\!\!/\, A$ and subsequently $PA \vdash n \Vdash\!\!\!/\, \ulcorner A \urcorner$. By Lemma 3.5(a) we have $PA \vdash A^* \rightarrow \ell \neq n$ whence $PA \nvdash A^*$ by S6.

(1). First we check that for every $n \in \omega$ there holds $f(n)=g(n)$. We proceed by induction on n. We assume $Y_n = \{n\}$ and calculate $f(n)$. In case $g(n)$ is not in rng^* we clearly have $f(n)=g(n)$. Suppose $g(n)=\overline{A^*}$. By Lemma 3.6 this implies $\ell \Vdash \Box A$ whence by S1 $X \Vdash A$ and subsequently $GR \vdash A$. Since $h(n)$ $(= \ell = 0)$ is not a top node of X we also have $h(n) \Vdash \Box^R A$ and $h(n) \not\Vdash \Box^R B$ in case A is $\neg B$. From this it follows that $n \in S_n$ and therefore $f(n)=g(n)$. Note that in both cases $Y_{n+1} = \{n+1\}$. Finally combine Lemmas 3.2 and 3.7 to see that Thm_f is a g-like s.p.p. $\Box$

3.9. Remark. The set of modal formulae that are true in the standard model under every translation is given by GR^ω. Moreover, the following holds true.

There exists a p.r. term $f(x)$ s.t.

(1) Thm_f is a g-like s.p.p.

(2) For every modal formula A, if $\omega \models A^*$ for every Thm_f-translation $*$ then $GR^\omega \vdash A$.

4. Rosser fixed points

4.1. Definition. Let $Thm(x,y)$ be a s.p.p. A sentence ρ is said to be a Rosser fixed point (R.f.p.) for Thm if

$$PA \vdash \rho \leftrightarrow \neg Th^R(\bar{\rho})$$

4.2. Remark. The notation of a Rosser fixed point is sometimes given a "dual" definition, that is

$$PA \vdash \rho \leftrightarrow \exists x \, (Thm(x, \overline{\neg\rho}) \wedge \forall z \leq x \neg Thm(z, \bar{\rho}))$$

(see Guaspari and Solovay [4]). It turns out that quite a number of properties of R.f.p.'s in our sense is shared by negations of representatives of this alternative class. The same holds true for the result of this section.

4.3. R e m a r k. Each s.p.p. is easily seen to possess an infinite number of graphically distanct R.f.p.'s. Indeed, the well-known proof of Gödel's Self-Reference Lemma provides the formula $n = n \wedge \neg Th^R(\cdot)$ with distinct fixed points for distinct $n \in \omega$ which are evidently Rosser for Thm.

Guaspari and Solovay [4] construct two g-like s.p.p.'s $Thm_1(x,y)$ and $Thm_2(x,y)$ s.t. all R.f.p.'s for Thm_1 are provably equivalent while there are at least two non-equivalent R.f.p.'s for Thm_2.

Consider two conditions on an arbitrary s.p.p. .

(1) If ρ_1 and ρ_2 are R.f.p.'s for Thm then $PA \vdash \rho_1 \leftrightarrow \rho_2$

(∞) If ρ_1 and ρ_2 are R.f.p.'s for Thm and $PA \vdash \rho_1 \leftrightarrow \rho_2$ then $\bar{\rho}_1 = \bar{\rho}_2$.

The corners $\ulcorner \; \urcorner$ will now denote a gödelnumbering of p.r. terms with just x free. Define

$$A_1 = \{ \ulcorner f \urcorner \mid Thm \text{ is a } g\text{-like s.p.p. satisfying (1)} \}$$

$$A_\infty = \{ \ulcorner f \urcorner \mid Thm \text{ is a } g\text{-like s.p.p. satisfying } (\infty) \}$$

In view of Remark 4.3 if $\ulcorner f \urcorner \in A_\infty$ then the number of non-equivalent R.f.p.'s for Thm is infinite.

For the remainder of the paper we shall allow ourselves "modal" abbreviations in arithmetic contexts. For example we write

$\Box_g \bot$ for $Th_{(g)}(\overline{0=1})$

$\Box_f^R \varphi$ for $Th_f^R(\overline{\varphi})$ etc.

The following theorem strengthens Theorem 6.2 of Guaspari and Solovay [4] by saying that A_∞ can not be separated from A by any Σ_1^0 set.

4.4. Theorem. Let S be $a\Sigma_1^0$ set. Then there exists a p.r. term $f(x)$ s.t.

$$\ulcorner f \urcorner \in (A_\infty \setminus S) \cup (A_1 \cap S).$$

Moreover f can be constructed effectively from (an r.e. index for) S.

The proof begins with the construction of f and then assumes the form of a sequence of lemmas.

Let $\sigma(z) \equiv \exists x \; \sigma_0(x, z)$ be a formula numerating S in PA and in $PA + \Box_g \bot$ with σ_0 p.r. The existence of $\sigma_0(x, z)$ with the said properties follows from §3 in Smoryński [8] and it can also be seen that σ_0 can be constructed effectively from S .

Next we turn to the (formalized) recursion theorem to produce (within PA) the function $f(x)$ which is constructed by stages. As in Guaspari and Solovay [4] we compile a list R which is empty before Stage 0 and keeps record of the R.f.p.'s for Thm_f . At every stage R will only contain a finite number of sentences. For simplicity assume that every number is the gödelnumber of some arithmetic sentence.

Stage x. Let $\rho_1, \ldots, \rho_n$ denote the sentences constituting the list R compiled heretofore.

C a s e A. The other cases do not apply.

In this case we set $f(x)=g(x)$. Next find out whether $g(x)$ has the form $\overline{\rho \leftrightarrow \neg \Box_f^R \rho}$ for some sentence ρ. If so and neither $\bar{\rho}$ nor $\overline{\neg \rho}$ nor $\bar{\nu}$ s.t. $\overline{\neg \nu}$ is ρ is in R then put $\bar{\rho}$ in R.

C a s e B. $g(x)=\bar{\varphi}$, $\bar{\varphi}$ is in R and $\exists y \leq x\, \sigma_0(y, \ulcorner f \urcorner)$

Set $f(x), \ldots, f(x+2n+1)$ equal to $\bar{\varphi}, \overline{\neg \varphi}, \bar{\rho}_1, \ldots, \bar{\rho}_n, \overline{\neg \rho_1}, \ldots, \overline{\neg \rho_n}$ respectively and set $f(x+2n+2+k)=k$, all k.

C a s e C. $g(x)=\overline{\neg \varphi}$, $\bar{\varphi}$ is in R and $\exists y \leq x\, \sigma_0(y, \ulcorner f \urcorner)$.

Set $f(x), \ldots, f(x+2n+1)$ equal to $\overline{\neg \varphi}, \bar{\varphi}, \overline{\neg \rho_1}, \ldots, \overline{\neg \rho_n}, \bar{\rho}_1, \ldots, \bar{\rho}_n$ respectively and $f(x+2n+2+k)=k$, all k.

For θ_1 and θ_2 sentences, call a triple (θ_1, θ_2, z) critical (at Stage x) if

(i) $\bar{\theta}_1 \neq \bar{\theta}_2$;

(ii) $z \leq x$;

(iii) at Stage z both $\bar{\theta}_1$ and $\bar{\theta}_2$ are in R ;

(iv) there exists an y s.t. $y \leq z$ and $g(y) = \overline{\theta_1 \leftrightarrow \theta_2}$;

(v) for no $y < z$ is there a triple (χ_1, χ_2, y) satisfying (i) - (iv).

In the remaining two cases we suppose that (θ_1, θ_2, z) is critical.

C a s e D. $g(x)=\bar{\varphi}$, $\bar{\varphi}$ is in R and $\forall y \leq x \neg \sigma_0(y, \ulcorner f \urcorner)$.

Set $f(x)$ and $f(x+1)$ equal to $\bar{\varphi}$ and $\overline{\neg\varphi}$ respectively. In case $\bar{\varphi} \neq \bar{\theta}_2$ let $f(x+2), \ldots, f(x+5)$ be $\bar{\theta}_1, \overline{\neg\theta_1}, \overline{\neg\theta_2}, \bar{\theta}_2$ respectively, otherwise $\overline{\neg\theta_1}, \bar{\theta}_1, \bar{\theta}_2, \overline{\neg\theta_2}$ respectively. If θ_i 's are undefined we may give arbitrary values to $f(x+2), \ldots, f(x+5)$. Further define $f(x+6+k)=k$, all k .

C a s e E. $g(x)=\overline{\neg\varphi}$, $\bar{\varphi}$ is in R and $\forall y \leq x \neg \sigma_0(y, \ulcorner f \urcorner)$.

Define $f(x)=\overline{\neg\varphi}$, $f(x+1)=\bar{\varphi}$

$$f(x+2), \ldots, f(x+5) = \begin{cases} \text{anything you like if there is no critical triple} \\ \overline{\neg\theta_1}, \bar{\theta}_1, \bar{\theta}_2, \overline{\neg\theta_2} & \text{if } \bar{\varphi} \neq \bar{\theta}_2 \\ \bar{\theta}_1, \overline{\neg\theta_1}, \overline{\neg\theta_2}, \bar{\theta}_2 & \text{otherwise} \end{cases}$$

$f(x+6+k)=k$.

If Case A was the case then we go to the next stage. Otherwise f is already total and Stage x was the last stage.

The construction of f is now complete. Please note that since the proof of the formalized recursion theorem is effective the p.r. term f is constructed effectively from $\sigma_0(x,z)$. •

4.5. L e m m a (PA). (a) Each element of R at any stage is a R.f.p. for Thm_f.

(b) If one of the Cases B - E applies at Stage x then Case A applied at every preceeding stage.

(c) At no stage are there sentences ρ and ν both in s.t. $\bar{\rho}=\overline{\neg\nu}$.

(d) If (θ_1, θ_2, z) is critical at Stage x then so it is at every succeeding stage.

P r o o f. Easy. □

4.6. L e m m a (PA). If one of the Cases B - E applies at a stage then $\operatorname{rng} g = \omega$.

P r o o f. Suppose Case B or Case D applies at Stage x , that is $g(x) = \bar{\varphi}$ for $\bar{\varphi}$ in . By Lemma 4.5(a) we have $\varphi \longleftrightarrow \neg \Box_f^R \varphi \in \operatorname{rng} g$. By virtue of P2 this implies $\Box_g \neg \Box_f^R \varphi$. In the case when for some y with $y < x$ there holds $g(y) = \overline{\neg\varphi}$ one has $\Box_g \varphi$ and $\Box_g \neg \varphi$, ergo $\Box_g \bot$. If on the contrary $\forall y < x \neg Thm_{(g)}(y, \overline{\neg\varphi})$ then $f(x) = \bar{\varphi}$ by the construction of f. Since by Lemma 4.5(b) $\forall y < x\, f(y) = g(y)$, we get $\Box_f^R \varphi$ which implies $\Box_g \Box_f^R \varphi$ and so $\Box_g \bot$ again. The Cases C and E are treated similarly. □

4.7. L e m m a (PA). $\operatorname{rng} f = \operatorname{rng} g$. .

P r o o f. If Case A applies at every stage this is obvious. Otherwise by the construction of f we have $f = \omega$ and by Lemma 4.6 also $g = \omega$. □

4.8. L e m m a. (a) Case A applies at every stage.

(b) $\omega \models \forall x\, (f(x) = g(x))$.

P r o o f. (a) follows from P1 and the consistency of PA by Lemma 4.6.

(b) is an immediate consequence of (a). □

4.9. L e m m a. If ϱ is a R.f.p. for Thm_f then ρ is eventually put in $\mathcal{R}$.

P r o o f. As in Guaspari and Solovay [4] (use Lemma 4.7).

4.10. L e m m a (PA). If $rng = \omega$ then there is a stage at which one of the Cases B - E applies.

P r o o f. Choose some R.f.p. ρ for Thm_f and a Stage n s.t. ρ is in R at this stage (see Lemma 4.9). We have that $\forall x \leq n \neg Thm_{(g)}(x, \bar{\rho})$. Now $rng\, g = \omega$ implies $\exists x > n\, Thm_{(g)}(x, \bar{\rho})$ activating Case B or Case D at Stage x unless one of the Cases B - E applied earlier. $\Box$

4.11. L e m m a. If $\ulcorner f \urcorner \in S$ then all R.f.p.'s for Thm_f are provably equivalent.

P r o o f. Let $\ulcorner f \urcorner \in S$. Since $\sigma(z)$ numerates S in PA there axists a number n s.t. $PA \vdash \sigma_0(n, \ulcorner f \urcorner)$. Therefore Cases D and E can not apply at stages $\geq n$ and this fact is formalizable in PA. Let ρ_1 and ρ_2 be R.f.p.'s for Thm_f s.t. at Stage m with $m \geq n$ both $\bar{\rho}_1$ and $\bar{\rho}_2$ are in R. Note that for no $k \leq n$ do we have $(f(k) =)\ g(k) = \bar{\rho}_1, \overline{\neg\rho_1}, \bar{\rho}_2, \overline{\neg\rho_2}$ because R.f.p.'s are independent of PA. Reason in PA. If Case A applies at every stage then $(f(x) =)\ g(x) = \rho_i, \neg\rho_i$ for no x. In particular, $\neg \Box_f^R \rho_i$. If Case B happens at Stage x then $f(y) \neq \overline{\neg\rho_1}, \overline{\neg\rho_2}$ all $y < x$ by Lemma 4.5(b) and by the definition of Casr A. In view of Lemma 4.5(c) and by the construction of f (Case B) this implies $\Box_f^R \rho_1$ and $\Box_f^R \rho_2$. Treat Case C similarly to obtain $\neg \Box_f^R \rho_1$ and $\neg \Box_f^R \rho_2$. The Cases D and E were excluded earlier. We have obtained $\Box_f^R \rho_1 \leftrightarrow \Box_f^R \rho_2$ in every possible case. Infer $\rho_1 \leftrightarrow \rho_2$. $\Box$

4.12. L e m m a. Let θ_1 and θ_2 be graphically distinct R.f.p.'s for Thm_f and $PA \vdash \theta_1 \leftrightarrow \theta_2$. Then $PA \vdash \Box\bot \rightarrow \sigma(\ulcorner f \urcorner)$.

P r o o f. By Lemmas 4.5(d) and 4.9 and the assumption of the present lemma there exists a triple (χ_1, χ_2, n) which is critical at every large enough stage. Without loss of generality assume $\overline{\chi}_1 = \overline{\theta}_1$. The criticality of (θ_1, θ_2, n) is clearly verifiable in PA. Reason in $PA + \Box_g\bot$. At a Stage x one of the Cases B - E applies (see Lemma 4.10). For all y s.t. $y < x$ there holds $f(y) \neq \overline{\theta_i}, \overline{\neg\theta_i}$ as in Lemma 4.11. Therefore in Cases D and E one has $\Box_f^R \theta_1 \leftrightarrow \neg\Box_f^R \theta_2$ by the construction of f whence $\neg\theta_1 \leftrightarrow \theta_2$ which is impossible because $\theta_1 \leftrightarrow \theta_2$. Conclude that only Cases B and C may happen which is impossible unless $\sigma(\ulcorner f \urcorner)$. $\Box$

4.13. P r o o f of Theorem 4.4 concluded. Combining Lemmas 4.7 and 4.8(b) we see that Thm_f is a g-like s.p.p. In case $\ulcorner f \urcorner \in S$ Lemma 4.11 guarantees that Thm_f satisfies (1), that is $\ulcorner f \urcorner \in A_1$. If $\ulcorner f \urcorner \notin S$ then $PA + \Box_g\bot \nvdash \sigma(\ulcorner f \urcorner)$ because $\sigma(z)$ numerates S in $PA + \Box_g\bot$. In view of Lemma 4.12 this implies $\ulcorner f \urcorner \in A_\infty$. $\Box$

4.14. C o r o l l a r y. The sets A_1 and A_∞ are effectively inseparable.

4.15. R e m a r k. It can easily be seen that Σ_1^0 can not be replaced by Π_1^0 in the statement of Theorem 4.4.

4.16. Q u e s t i o n. I am not aware of a construction which, given a s.p.p. $Thm_{(f)}$ produces unprovable sentences φ and ψ s.t.

$PA \vdash \neg (\Box_f \varphi \prec \Box_f \psi)$.

In fact it seems very unlikely that we should generally know anything about the order in which the proofs of unprovable sentences appear. It is therefore natural to ask whether there exists a s.p.p. Thm_f s.t. for any string $\varphi_1, \ldots, \varphi_n$ of graphically distinct sentences

$$PA \vdash \neg \bigwedge_{i=1}^{n-1} (\Box_f \varphi_i \prec \Box_f \varphi_{i+1})$$

implies that one of φ_i 's is provable.

In [3] Goryachev investigates the local reflection principle based on $\Box^R$ and shows that there exists a s.p.p. Thm_f s.t. the local reflection principle based on $\Box_f^R$ is equivalent to the usual one (i.e. based on $\Box_f$). A positive answer to Question 4.16 would provide an example of a s.p.p. Thm_f s.t. the local reflection principle for $\Box_f^R$ is strictly weaker than that for $\Box_f$.

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