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# BI-UNARY INTERPRETABILITY LOGIC

Maarten de rijke

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# BI-UNARY INTERPRETABILITY LOGIC

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# Bi-Unary Interpretability Logic

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## 1 Introduction

In recent years several modal systems have been introduced to study the relation of relative interpretability between arithmetical theories. The intepretability principles of several important classes of arithmetical theories have been axiomatised. In [6] the system *ILP* is shown to be the interpretability logic of all $\Sigma_1^0$-sound finitely axiomatised sequential theories that extend $I\Delta_0 + \text{SupExp}$; in [1] it is shown that *ILM* is the interpretability logic of *PA*. Montagna and Hájek [2] show that *ILM* is also the logic of $\Pi_1^0$-conservativity of all $\Sigma_1^0$-sound extensions of $I\Sigma_1$. (As is well-known, in the case of *PA* the two relations of relative interpretability and of $\Pi_1^0$-conservativity coincide).

Given the above results it is only natural to consider a modal logic with two binary modal operators, one of which is to be interpreted arithmetically as the relation of $\Pi_1^0$-conservativity between extensions of some given finitely axiomatized sequential extension $T$ of $I\Sigma_1$, while the other operator is to be interpreted as relative interpretability over the same theory $T$. Such a system, called *ILM/P*, has been introduced by Dick de Jongh and Albert Visser, and is conjectured to be the logic of relative interpretability and $\Pi_1^0$-conservativity of all $\Sigma_1^0$-sound finitely axiomatized sequential extensions of $I\Sigma_1$. Both the modal and arithmetical completeness of *ILM/P* are still open.

Interpretability may also be viewed as a unary predicate over extensions of a fixed theory $T$. The modal analysis of the interpretability predicate has been undertaken in [3], using, of course, a unary modal operator. In this note we axiomatize the bi-unary subsystem of *ILM/P*. That is, we introduce two unary operators $\mathbf{I}_M$, $\mathbf{I}_P$ with the following interpretations: $\mathbf{I}_M A$ stands for '$T + A$ is a $\Pi_1^0$-conservative extension of $T$', and $\mathbf{I}_P A$ stands for '$T + A$ is interpretable in

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$T$', and we axiomatize all formulas $A$ in the language with only $\Box$, $\mathbf{I}_M$, $\mathbf{I}_P$ that are provable in $ILM/P$.

## 2 Axioms and models

The *provability logic L* is propositional logic plus the axiom schemas $\Box(A \to B) \to (\Box A \to \Box B)$, $\Box A \to \Box\Box A$ and $\Box(\Box A \to A) \to \Box A$, and the rules Modus Ponens ($\vdash A, \vdash A \to B \Rightarrow \vdash B$) and Necessitation ($\vdash A \Rightarrow \vdash \Box A$). We use $\mathcal{L}(\Box)$ to denote the language of $L$. $\mathcal{L}(\Box)$ is extended with a binary operator $\rhd$ to obtain the language $\mathcal{L}(\Box, \rhd)$ of binary interpretability logic. The *binary interpretability logic IL* is obtained from $L$ by adding the axioms

| | | | |
|---|---|---|---|
| $(J1)$ | $\Box(A \to B) \to A \rhd B$ | $(J4)$ | $A \rhd B \to (\Diamond A \to \Diamond B)$ |
| $(J2)$ | $(A \rhd B) \wedge (B \rhd C) \to (A \rhd C)$ | $(J5)$ | $\Diamond A \rhd A$ |
| $(J3)$ | $(A \rhd C) \wedge (B \rhd C) \to (A \vee B) \rhd C$ | | |

where $\Diamond \equiv \neg\Box\neg$. $ILM$ is $IL + M$ and $ILP$ is $IL + P$, where $M \equiv A \rhd B \to A \wedge \Box C \rhd B \wedge \Box C$ and $P \equiv A \rhd B \to \Box(A \rhd B)$.

The system $ILM/P$ is defined in a language $\mathcal{L}(\Box, \rhd_M, \rhd_P)$ which contains the operator $\Box$ as well as two binary interpretability operators: $\rhd_M$ and $\rhd_P$. For the operator $\rhd_M$ we assume the axioms $J1$–$J5$ and $M$; for the operator $\rhd_P$ we assume the axioms $J1$–$J5$ and $P$. In addition there is one mixed axiom:

$$A \rhd_M B \to A \wedge (C \rhd_P D) \rhd_M B \wedge (C \rhd_P D).$$

Define in $\mathcal{L}(\Box, \rhd)$ the unary interpretability operator '$\mathbf{I}$' by $\mathbf{I}A := \top \rhd A$, and let $\mathcal{L}(\Box, \mathbf{I})$ extend $\mathcal{L}(\Box)$ with $\mathbf{I}$. The *unary interpretability logic il* is obtained from $L$ by adding the axioms

| | | | |
|---|---|---|---|
| $(I1)$ | $\mathbf{I}\Box\bot$ | $(I3)$ | $\mathbf{I}(A \vee \Diamond A) \to \mathbf{I}A$ |
| $(I2)$ | $\Box(A \to B) \to (\mathbf{I}A \to \mathbf{I}B)$ | $(I4)$ | $\mathbf{I}A \wedge \Diamond\top \to \Diamond A$. |

We use $ilm$ to denote $il + m$ and $ilp$ to denote $il + p$, where $m \equiv \mathbf{I}A \to \mathbf{I}(A \wedge \Box\bot)$ and $p \equiv \mathbf{I}A \to \Box\mathbf{I}A$.

In $\mathcal{L}(\Box, \rhd_M, \rhd_P)$ we define the unary interpretability operators $\mathbf{I}_M$ and $\mathbf{I}_P$ by $\mathbf{I}_M A := \top \rhd_M A$ and $\mathbf{I}_P A := \top \rhd_P A$ respectively. It is sometimes convenient to assume that the unary system $ilm$ is defined in the language $\mathcal{L}(\Box, \mathbf{I}_M)$ with $\Box$ and $\mathbf{I}_M$ as the only modal operators, and similarly for $ilp$ and $\mathcal{L}(\Box, \mathbf{I}_P)$. The system $ilm/p$ is defined in $\mathcal{L}(\Box, \mathbf{I}_M, \mathbf{I}_P)$ as follows; it contains the axioms $I1$–$I4$ and $m$ for the operator $\mathbf{I}_M$, and the axioms $I1$–$I4$ and $p$ for the operator $\mathbf{I}_P$; it has no mixed axioms. (Note that $ilp \vdash m$, so in $ilm/p$ we also have axiom $m$ for the operator $\mathbf{I}_P$.)

Recall that an $L$-frame is a pair $\langle W, R\rangle$ with $R \subseteq W^2$ transitive and conversely well-founded, and that an $L$-model is given by an $L$-frame $\mathcal{F}$ together with a forcing relation $\Vdash$ that satisfies the usual clauses for $\neg$ and $\wedge$, while $u \Vdash A$ iff $\forall v\,(uRv \Rightarrow v \Vdash A)$. A *(Veltman-) frame* for $IL$ is a triple $\langle W, R, S\rangle$, where $\langle W, R\rangle$ is an $L$-frame, and $S = \{S_w : w \in W\}$ is a collection of binary relations on $W$ satisfying

1. $S_w$ is a relation on $wR$
2. $S_w$ is reflexive and transitive
3. if $w', w'' \in wR$ and $w'Rw''$ then $w'S_w w''$.

An *IL-model* is given by a Veltman-frame $\mathcal{F}$ for $IL$ together with a forcing relation $\Vdash$ that satisfies the above clauses for $\neg$, $\wedge$ and $\Box$, while

$$u \Vdash A \rhd B \Leftrightarrow \forall v\,(uRv \,\text{and}\, v \Vdash A \Rightarrow \exists w\,(vS_u w \,\text{and}\, w \Vdash B)).$$

An $ILP$-model is an $IL$-model that satisfies the extra condition: if $wRw'RuS_w v$ then $uS_{w'}v$. An $ILM$-model is an $IL$-model satisfying the extra condition: if $uS_w vRz$ then $uRz$.

An $ILM/P$-frame is a tuple $\langle W, R, S^M, S^P\rangle$ such that $\langle W, R, S^M\rangle$ is an $ILM$-frame, and $\langle W, R, S^P\rangle$ is an $ILP$-frame, while the following extra condition connecting $S^M$ and $S^P$ holds:

$$\forall xyzuv\,(xRyS^M_x zRuS^P_y v \rightarrow uS^P_z v).$$

An $ILM/P$-model is a tuple $\langle W, R, S^M, S^P, \Vdash\rangle$ such that $\langle W, R, S^M, S^P\rangle$ is an $ILM/P$-frame, and such that the semantics of the operator $\rhd_M$ is based on the relation $S^M$, while the semantics of the operator $\rhd_P$ is based on the relation $S^P$.

The truth definition for $\mathbf{I}_K$ ($K \in \{M, P\}$) follows from the above definitions:

$$x \Vdash \mathbf{I}_K A \text{ iff } \forall y\,(xRy \rightarrow \exists z\,(yS^K_x z \wedge z \Vdash A)).$$

# 3 Preliminaries

In this Section we introduce the tools needed to prove the modal completeness of *ilm/p*. We start with some definitions.

**Definition 3.1** Let $K \in \{M, P\}$, and let $\Gamma$, $\Delta$ be two maximal *ilm/p*-consistent sets.

1. $\Delta$ is called a *successor* of $\Gamma$ ($\Gamma \prec \Delta$) if
   (a) $A, \Box A \in \Delta$ for each $\Box A \in \Gamma$
   (b) $\Box A \in \Delta$ for some $\Box A \notin \Gamma$.
2. $\Delta$ is called an $(\mathbf{I}_K, C)$-*critical successor* of $\Gamma$ if

(a) $\Gamma \prec \Delta$

(b) $\mathbf{I}_K C \notin \Gamma$

(c) $\neg C, \Box\neg C \in \Delta$.

Note that if $\Delta$ is a successor $\Gamma$ then it is both an $(\mathbf{I}_M, \bot)$-critical and an $(\mathbf{I}_P, \bot)$-critical successor of $\Gamma$.

**Proposition 3.2** *Let $\Gamma$ be a maximal ilm/p-consistent set such that $\Diamond C \in \Gamma$. Then there is a maximal ilm/p-consistent successor $\Delta$ of $\Gamma$ with $C, \Box\neg C \in \Delta$.*

*Proof.* Well-known (or cf. [4]). QED.

**Proposition 3.3** *Let $K \in \{M, P\}$, and let $\Gamma$ be a maximal ilm/p-consistent set such that $\neg\mathbf{I}_K C \in \Gamma$. The there exists a maximal ilm/p-consistent $(\mathbf{I}_K, C)$-critical successor $\Delta$ of $\Gamma$ with $\Box\bot \in \Delta$.*

*Proof.* Cf. [3, Proposition 2.4]. QED.

**Proposition 3.4** *Let $K \in \{M, P\}$, and let $\mathbf{I}_K C \in \Gamma$, where $\Gamma$ is a maximal ilm/p-consistent set. If there exists a maximal ilm/p-consistent $(\mathbf{I}_K, E)$-critical successor $\Delta$ of $\Gamma$, then there exists a maximal ilm/p-consistent $(\mathbf{I}_K, E)$-critical successor $\Delta'$ of $\Gamma$ such that $C, \Box\bot \in \Delta'$.*

*Proof.* By axiom *m*, $\mathbf{I}_K C$ implies $\mathbf{I}_K(C \wedge \Box\bot)$. By [3, Proposition 2.5] the result follows. QED.

Here is one more definition:

**Definition 3.5** A set of formulas $\Phi$ is called *adequate* if

1. if $B \in \Phi$ and $C$ is a subformula of $B$ then $C \in \Phi$
2. if $B \in \Phi$ and $B$ is no negation then $\neg B \in \Phi$

It is clear that every formula is contained in a *finite* adequate set.

# 4 The main theorem

Given some maximal *ilm/p*-consistent set $\Gamma$ and a finite adequate set $\Phi$, we define the structure $\langle W_\Gamma, R, S^M, S^P \rangle$, which consists of pairs $\langle \Delta, \tau \rangle$, where $\Delta$ is a maximal *ilm/p*-consistent set needed to handle the truth definition for formulas in $\Gamma$, and $\tau$ is a sequence of pairs we use to index the pairs we put into $W_\Gamma$.

For the time being, we fix a maximal *ilm/p*-consistent set $\Gamma$ and a finite adequate set $\Phi$. We use $\bar{w}, \bar{v}, \ldots$ to denote pairs $\langle \Delta, \tau \rangle$. If $\bar{w} = \langle \Delta, \tau \rangle$, then $(\bar{w})_0 = \Delta$, $(\bar{w})_1 = \tau$. We write $\sigma \subseteq \tau$ for $\sigma$ is an initial segment of $\tau$, and $\sigma \subset \tau$ if $\sigma$ is a proper initial segment of $\tau$. Finally, $\sigma^\frown\tau$ denotes the concatenation of $\sigma$ and $\tau$.

**Definition 4.1** Define $W_\Gamma$ to be a minimal set of pairs $\langle \Delta, \tau \rangle$ such that

1. $\langle \Gamma, \langle\langle\rangle\rangle \rangle \in W_\Gamma$;
2. if $\langle \Delta, \tau \rangle \in W_\Gamma$, $\Diamond B \in \Delta \cap \Phi$, and if there exists a successor $\Delta'$ of $\Delta$ with $B, \Box\neg B \in \Delta'$, then $\langle \Delta', \tau^\frown\langle\langle \Diamond B, \bot \rangle\rangle \rangle \in W_\Gamma$ for *one* such $\Delta'$;
3. if $\langle \Delta, \tau \rangle \in W_\Gamma$, $\neg \mathbf{I}_M B \in \Delta \cap \Phi$, and if there exists an $(\mathbf{I}_M, B)$-critical successor $\Delta'$ of $\Delta$ with $\Box\bot \in \Delta'$, then $\langle \Delta', \tau^\frown\langle\langle \neg\mathbf{I}_M B, B \rangle\rangle \rangle \in W_\Gamma$ for *one* such $\Delta'$;
4. if $\langle \Delta, \tau \rangle \in W_\Gamma$, $\neg \mathbf{I}_P B \in \Delta \cap \Phi$, and if there exists an $(\mathbf{I}_P, B)$-critical successor $\Delta'$ of $\Delta$ with $\Box\bot \in \Delta'$, then $\langle \Delta', \tau^\frown\langle\langle \neg\mathbf{I}_P B, B \rangle\rangle \rangle \in W_\Gamma$ for *one* such $\Delta'$;
5. if $\langle \Delta, \tau \rangle \in W_\Gamma$, $\mathbf{I}_M B \in \Delta \cap \Phi$, $C \in \Phi$, and if there exists an $(\mathbf{I}_M, C)$-critical successor $\Delta'$ of $\Delta$ with $B, \Box\bot \in \Delta'$, then $\langle \Delta', \tau^\frown\langle\langle \mathbf{I}_M B, C \rangle\rangle \rangle \in W_\Gamma$ for *one* such $\Delta'$;
6. if $\langle \Delta, \tau \rangle \in W_\Gamma$, $\mathbf{I}_P B \in \Delta \cap \Phi$, $C \in \Phi$, and if there exists an $(\mathbf{I}_P, C)$-critical successor $\Delta'$ of $\Delta$ with $B, \Box\bot \in \Delta'$, then $\langle \Delta', \tau^\frown\langle\langle \mathbf{I}_P B, C \rangle\rangle \rangle \in W_\Gamma$ for *one* such $\Delta'$.

Define $R$ on $W_\Gamma$ by putting $\bar{w} R \bar{v}$ if $(\bar{w})_1 \subset (\bar{v})_1$.
Define $S^M$ on $W_\Gamma$ by putting $\bar{v} S^M_{\bar{w}} \bar{u}$ iff for some $B$, $B'$, $C$, $C'$, $\sigma$ and $\sigma'$:

$$(\bar{v})_1 = (\bar{w})_1{}^\frown\langle\langle B, C\rangle\rangle^\frown\sigma \text{ and } (\bar{u})_1 = (\bar{w})_1{}^\frown\langle\langle B', C'\rangle\rangle^\frown\sigma'$$

and *either* $(\bar{v})_1 \subseteq (\bar{u})_1$, *or* $B$ is not of the form $\mathbf{I}_M D$ or $\neg\mathbf{I}_M D$, and then $B' \equiv \mathbf{I}_M D'$ and $C' \equiv \bot$ for some $D'$, *or* $B$ is of the form $\mathbf{I}_M D$ or $\neg\mathbf{I}_M D$, and then $C' \equiv C$ and $B' \equiv \mathbf{I}_M D'$ for some $D'$.

Define $S^P$ on $W_\Gamma$ by putting $\bar{v} S^P_{\bar{w}} \bar{u}$ iff for some $B$, $B'$, $C$, $C'$, $\tau$, $\tau'$ and $\sigma$:

$$(\bar{v})_1 = (\bar{w})_1{}^\frown\tau^\frown\langle\langle B, C\rangle\rangle \text{ and } (\bar{u})_1 = (\bar{w})_1{}^\frown\tau^\frown\langle\langle B', C'\rangle\rangle^\frown\sigma$$

and *either* $(\bar{v})_1 \subseteq (\bar{u})_1$, *or* $B$ is not of the form $\mathbf{I}_P D$ or $\neg\mathbf{I}_P D$, and then $B' \equiv \mathbf{I}_P D'$ and $C' \equiv \bot$ for a $D'$, *or* $B$ is of the form $\mathbf{I}_P D$ or $\neg\mathbf{I}_P D$, and then $C' \equiv C$ and $B' \equiv \mathbf{I}_P D'$ for some $D'$.

**Proposition 4.2** *1. $\langle W_\Gamma, R, S^M, S^P \rangle$ is finite.*

*2. If $\bar{w} \in W_\Gamma$, and $(\bar{w})_1 = \tau^\frown\langle\langle(\neg)\mathbf{I}_K B, C\rangle\rangle^\frown\sigma$, where $K \in \{M, P\}$, then $\bar{w}$ is an $R$-endpoint, $\Box\bot \in (\bar{w})_0$, and $\sigma = \langle\cdot\rangle$.*

*3. If $\bar{u} \in W_\Gamma$, $(\bar{u})_1 = \tau^\frown\langle\langle \Diamond B, \bot\rangle\rangle$, and if we have $\bar{v} S^M_{\bar{w}} \bar{u}$ or $\bar{v} S^P_{\bar{w}} \bar{u}$, then $\bar{w} R \bar{v} \underline{R} \bar{u}$.*

*4. If $(\bar{w})_1 = (\bar{v})_1$ then $\bar{w} = \bar{v}$.*

*5. If $\bar{w} R \bar{v}$ then $(\bar{w})_0 \prec (\bar{v})_0$.*

*6. $\langle W_\Gamma, R \rangle$ is a tree.*

*7. $\langle W_\Gamma, R, S^M \rangle$ is an ILM-frame.*

*8. $\langle W_\Gamma, R, S^P \rangle$ is an ILP-frame.*

*9. $\langle W_\Gamma, R, S^M, S^P \rangle$ is an ILM/P-frame.*

*Proof.* Left to the reader. QED.

**Theorem 4.3** *Let $A \in \mathcal{L}(\Box, \mathbf{I}_M, \mathbf{I}_P)$. Then $ilm/p \vdash A$ iff for all finite ILM/P-models $\mathcal{M}$ we have $\mathcal{M} \models A$.*

*Proof.* We only prove completeness. Assume $ilm/p \nvdash A$. Let $\Gamma$ be a maximal $ilm/p$-consistent set with $\neg A \in \Gamma$, and let $\Phi$ be a finite adequate set with $\neg A \in \Phi$. Construct $\langle W_\Gamma, R, S^M, S^P \rangle$ as in 4.1. We complete the proof by putting $\bar{w} \Vdash p$ iff $p \in (\bar{w})_0$, and by proving that for all $F \in \Phi$ and $\bar{w} \in W_\Gamma$, we have $\bar{w} \Vdash F$ iff $F \in (\bar{w})_0$. The proof is by induction on $F$. We only consider the cases $F \equiv \Diamond C$, $\mathbf{I}_M D$ and $\mathbf{I}_P D$.

If $F \equiv \Diamond C \in (\bar{w})_0$, then we have to show that $\exists \bar{v}\,(\bar{w} R \bar{v} \wedge B \in (\bar{v})_0)$. Now, by 3.2 there exists a successor $\Delta$ of $(\bar{w})_0$ with $B, \Box\neg B \in \Delta$. We may assume that $\bar{v} := \langle \Delta, (\bar{w})_1{}^\frown\langle\langle \Diamond B, \bot\rangle\rangle\rangle \in W_\Gamma$. Obviously, $\bar{w} R \bar{v}$ and $B \in (\bar{v})_0$, as required.

The case $F \equiv \Diamond C \notin (\bar{w})_0$ is trivial.

Assume that $\mathbf{I}_M D \in (\bar{w})_0$. We have to show that $\forall \bar{v}\,(\bar{w} R \bar{v} \rightarrow \exists \bar{u}\,(\bar{v} S^M_{\bar{w}} \bar{u} \wedge D \in (\bar{u})_0))$. So assume that $\bar{w} R \bar{v}$; then for some $B$, $C$ and $\sigma$, $(\bar{v})_1 = (\bar{w})_1{}^\frown\langle\langle B, C\rangle\rangle^\frown\sigma$. If $B$ is not of the form $(\neg)\mathbf{I}_M B'$, then we consider $(\bar{v})_0$ to be an $(\mathbf{I}_M, \bot)$-critical successor of $(\bar{w})_0$. By 3.4 there exists an $(\mathbf{I}_M, \bot)$-critical successor $\Delta$ of $(\bar{w})_0$ with $D, \Box\bot \in \Delta$. Put $\bar{u} := \langle \Delta, (\bar{w})_1{}^\frown\langle\langle \mathbf{I}_M D, \bot\rangle\rangle\rangle$. We may assume that $\bar{u} \in W_\Gamma$. It is clear that $\bar{v} S^M_{\bar{w}} \bar{u}$ and $D \in (\bar{u})_0$, as required. Next we suppose that $B$ is of the form $(\neg)\mathbf{I}_M B'$. Then $(\bar{v})_0$ is an $(\mathbf{I}_M, C)$-critical successor of $(\bar{w})_0$. By 3.4 there exists an $(\mathbf{I}_M, C)$-critical successor $\Delta$ of $(\bar{w})_0$ with $D, \Box\bot \in \Delta$. Put $\bar{u} := \langle \Delta, (\bar{w})_1{}^\frown\langle\langle \mathbf{I}_M D, C\rangle\rangle\rangle$. Then we may assume that $\bar{u} \in W_\Gamma$. Moreover, we have $\bar{v} S^M_{\bar{w}} \bar{u}$ and $D \in (\bar{u})_0$, as required.

Assume that $\mathbf{I}_M D \notin (\bar{w})_0$. Then $\neg\mathbf{I}_M D \in (\bar{w})_0$. We have to prove that $\exists \bar{v}\,(\bar{w} R \bar{v} \wedge \forall \bar{u}\,(\bar{v} S^M_{\bar{w}} \bar{u} \rightarrow D \notin (\bar{u})_0))$. Now, by 3.3 there exists an $(\mathbf{I}_M, D)$-critical successor $\Delta$ of $(\bar{w})_0$ with $\Box\bot \in \Delta$. We may assume that $\bar{v} := \langle \Delta, (\bar{w})_1{}^\frown\langle\langle \neg\mathbf{I}_M D, D\rangle\rangle\rangle \in W_\Gamma$. Now suppose that for some $\bar{u}$, $\bar{v} S^M_{\bar{w}} \bar{u}$. By definition $(\bar{u})_1 = (\bar{w})_1{}^\frown\langle\langle B', C'\rangle\rangle^\frown\sigma'$, for some $B'$, $C'$ and $\sigma'$. Since $\Box\bot \in (\bar{v})_0$, we can not have $\bar{v} R \bar{u}$. Hence, we have either $\bar{u} = \bar{v}$ and then $D \notin (\bar{u})_0$, or $C' \equiv D$ and $B' \equiv \mathbf{I}_M D'$ for some $D'$. But then $(\bar{u})_0$ must be an $(\mathbf{I}_M, D)$-critical successor of $(\bar{w})_0$—and so $D \notin (\bar{u})_0$.

Assume that $\mathbf{I}_P D \in (\bar{w})_0$. We have to show that $\forall \bar{v}\,(\bar{w} R \bar{v} \rightarrow \exists \bar{u}\,(\bar{v} S^P_{\bar{w}} \bar{u} \wedge D \in (\bar{u})_0))$. So assume that $\bar{w} R \bar{v}$. Since $\langle W_\Gamma, R\rangle$ is a tree, we can find a unique immediate $R$-predecessor $\bar{w}'$ of $\bar{v}$. By axiom $p$ (for $\mathbf{I}_P$) we must have $\mathbf{I}_P D \in (\bar{w}')_0$, and so, by axiom $m$ for $\mathbf{I}_P$, also $\mathbf{I}_P(D \wedge \Box\bot) \in (\bar{w}')_0$. By construction $(\bar{v})_1 = (\bar{w}')_1{}^\frown\langle\langle B, C\rangle\rangle$ for some $B$ and $C$. If $B$ is not of the form $(\neg)\mathbf{I}_P B'$, then we consider $(\bar{v})_0$ to be an $(\mathbf{I}_P, \bot)$-critical successor of $(\bar{w}')_0$. By 3.4 there exists an $(\mathbf{I}_P, \bot)$-critical successor $\Delta$ of $(\bar{w}')_0$ with $D, \Box\bot \in \Delta$. We may assume that $\bar{u} := \langle \Delta, (\bar{w}')_1{}^\frown\langle\langle \mathbf{I}_P D, \bot\rangle\rangle\rangle \in W_\Gamma$. Moreover it is clear that $\bar{v} S^P_{\bar{w}} \bar{u}$ and $D \in (\bar{u})_0$, as required. If, on the other hand, $B$ is of the form $(\neg)\mathbf{I}_P B'$, then $(\bar{v})_0$ is an $(\mathbf{I}_P, C)$-critical successor of $(\bar{w}')_0$. By 3.4 there exists an $(\mathbf{I}_P, C)$-critical successor $\Delta$ of $(\bar{w}')_0$ with $D, \Box\bot \in \Delta$. As before we may assume that $\bar{u} := \langle \Delta, (\bar{w})_1 {}^\frown \langle\langle \mathbf{I}_P D, C \rangle\rangle \rangle \in W_\Gamma$. Moreover, we have $\bar{v} S^P_{\bar{w}} \bar{u}$ and $D \in (\bar{u})_0$, as required.

The last case we have to consider is the case that $\mathbf{I}_P D \notin (\bar{w})_0$. But this case is entirely analogous to the case $\mathbf{I}_M D \notin (\bar{w})_0$. QED.

**Proposition 4.4** *Let $A \in \mathcal{L}(\Box, \mathbf{I}_M, \mathbf{I}_P)$. Then $ilm/p \vdash A$ iff $ILM/P \vdash A$.*

*Proof.* If $ilm/p \vdash A$ then, by a simple induction on derivations, $ILM/P \vdash A$. If $ilm/p \nvdash A$ then by 4.3 there is a finite $ILM/P$-model $\mathcal{M}$ with $\mathcal{M} \not\models A$. By the soundness of $ILM/P$ w.r.t. $ILM/P$-models it follows that $ILM/P \nvdash A$. QED.

**Proposition 4.5** *Let $A \in \mathcal{L}(\Box, \mathbf{I}_M)$. Then $ilm/p \vdash A$ iff $ilm \vdash A$ iff $ILM \vdash A$.*

*Proof.* The second equivalence is [3, Proposition 2.15]. If $ilm \vdash A$ then obviously $ilm/p \vdash A$. And if $ilm \nvdash A$ then by [3, Theorem 2.14] there is an $ILM$-model $\mathcal{M}$ with $\mathcal{M} \not\models A$. $\mathcal{M}$ may be turned into an $ILM/P$-model $\mathcal{M}'$ by defining $y S^P_x z$ iff $xRy\underline{R}z$. Obviously, $\mathcal{M}' \not\models A$. So by 4.3 $ilm/p \nvdash A$. QED.

**Proposition 4.6** *Let $A \in \mathcal{L}(\Box, \mathbf{I}_P)$. Then $ilm/p \vdash A$ iff $ilp \vdash A$ iff $ILP \vdash A$.*

*Proof.* Similar to the proof of 4.5—using [3, Proposition 2.25 and Theorem 2.23]. QED.

Fix $T$ to be a $\Sigma^0_1$-sound finitely axiomatized sequential extension of $I\Sigma_1$, and define the arithmetical interpretation $(\cdot)^*$ of $\mathcal{L}(\Box, \mathbf{I}_M, \mathbf{I}_P)$ into the language of $T$ as usual for proposition letters, Boolean connectives and $\Box$, while

$$
\begin{array}{lcl}
(\mathbf{I}_P A)^* & := & \text{`}T + A^* \text{ is interpretable in } T\text{'} \\
(\mathbf{I}_M A)^* & := & \text{`for all } \Pi^0_1\text{-sentences } \varphi, \text{ if } \varphi \text{ is provable in } T + A^*, \\
 & & \text{then } \varphi \text{ is provable in } T\text{'}.
\end{array}
$$

**Proposition 4.7** *1. Let $A \in \mathcal{L}(\Box, \mathbf{I}_M)$. Then $ilm/p \vdash A$ iff for all $(\cdot)^*$, $T \vdash A^*$.*

*2. Let $A \in \mathcal{L}(\Box, \mathbf{I}_P)$. Then $ilm/p \vdash A$ iff for all $(\cdot)^*$, $T \vdash A^*$.*

*Proof.* To prove (1) use 4.5 and the fact that by [5, Theorem 10.1], $ILM \vdash A$ iff for all interpretations $(\cdot)^*$ of $\mathcal{L}(\Box, \mathbf{I}_M)$ into the language of $T$, $T \vdash A^*$. To prove (2) use 4.6 and the fact that by [6, Theorem 8.2], $ILP \vdash A$ iff for all interpretations $(\cdot)^*$ of $\mathcal{L}(\Box, \mathbf{I}_P)$ into the language of $T$, $T \vdash A^*$. QED.

According to Propositions 4.4 and 4.7 what $ILM/P$ says about unary interpretability and unary $\Pi^0_1$-conservativity considered *separately* is precisely what it should say about these predicates. This lends additional support to the conjecture that $ILM/P$ is the logic of the relations of relative interpretability and $\Pi^0_1$-conservativity (taken together) of all $\Sigma^0_1$-sound finitely axiomatized sequential extensions of $I\Sigma_1$.

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