Institute for Language, Logic and Information

DZHAPARIDZE'S POLYMODAL LOGIC:

ARITHMETICAL COMPLETENESS, FIXED POINT PROPERTY, CRAIG'S PROPERTY

K.N. Ignatiev

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(Department of Mathematics and Computer Science)
Plantage Muidergracht 24
1018TV Amsterdam

Faculteit der Wijsbegeerte
(Department of Philosophy)
Nieuwe Doelenstraat 15
1012CP Amsterdam

DZHAPARIDZE'S POLYMODAL LOGIC:

ARITHMETICAL COMPLETENESS, FIXED POINT PROPERTY, CRAIG'S PROPERTY

K.N. Ignatiev
Steklov Mathematical Institute
Vavilova Str. 42, Moscow 117966,GSP-1, USSR

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Ignatiev K.N.
Steklov Mathematical Institute,
Vavilova str.,42, Moscow,
117966, GSP-1, USSR.

ABSTRACT: In [1] G. Boolos considered the notion of ω-provability in Peano Arithmetic PA. Later in [2] G. Dzhaparidze introduced a polymodal logic GP (referred to below as GLP) for iterated ω-provability and obtained its arithmetical completeness. In this paper we prove the fixed point property and the Craig's interpolation property for GLP. This allows us to give a simpler proof of the arithmetical completeness of GLP and to obtain some generalizations.

§1. Introduction.

Consider an arbitrary r.e. theory T in the language of arithmetic. A formula A is provable in T if and only if the theory $T+\neg A$ is inconsistent. Analogously, one can also consider the following notion of ω-provability. The formula A is ω-provable iff $T+\neg A$ is ω-inconsistent. The formalized Σ_3-predicate of ω-provability $Pr^{\omega}_T(\cdot)$ is considered in [1] where it is shown that the modal properties of this predicate are identical to those of the usual provability predicate $Pr_T(\cdot)$, i.e.the provability logic for the predicate $Pr^{\omega}_T(\cdot)$ along with that for the $Pr_T(\cdot)$ is logic GL (Gödel-Löb).

On the other hand, the set T^{ω} of all formulae ω-provable in T is closed under the usual inferences of predicate calculus and thus forms an arithmetical theory which in general will not be r.e. T^{ω} will be referred below to as ω-extension of T. The theory T^{ω} is given by the axioms of T together with all the formulae of the form $\forall x\, Q(x)$ such that $T\vdash Q(\underline{1})$, $T\vdash Q(\underline{2}),\ldots,$ $T\vdash Q(\underline{n}),\ldots$ (i.e.

formula $\forall x\ Q(x)$ is derivable in T through just one application of ω-rule).

We can iterate the construction leading from T to T^{ω}, i.e. starting with theory T we can construct a sequence of theories $T_0, T_1, \ldots, T_n, \ldots$ .Moreover, we can attach a natural provability predicate to each one of these theories provided we are given one for T (see example 1 below).In [2] the joint provability logic of all the infinite family of theories $T_0, T_1, \ldots, T_n, \ldots$ with T_0 an arbitrary sound r.e. arithmetical theory is considered .(This logic is introduced below as GLP).The language of logic GLP , apart from boolean connectives, contains infinitely many modal operators [0],[1],...,[n],... translated as the provability in the theories $T_0, T_1, \ldots, T_n, \ldots$ .Logics formulated in this language are called polymodal.

However, the semantic for GLP used in [2] is too complicated to be of help in the investigation of the modal properties of GLP. E.g. the questions if GLP possesses fixed point property and Craig's property was left open.

In the present paper we prove the properties of GLP mentioned above. For this we provide a simpler semantic for GLP which allows us to give a simplified presentation of the arithmetical completeness theorem for GLP. Moreover, we make use of this new semantic for GLP to prove some other properties of GLP and other polymodal logics (some of these properties were already obtained in [2])

When presenting the arithmetical properties of GLP we consider a sequence of theories of more general form than the sequence of ω-extensions of a given theory T (in fact, this sequence should only satisfy the requirements imposed by the soundness of GLP).

So, to interpret the modal language we are to associate with each theory its provability predicate and a natural number characterizing the arithmetical complexity of this predicate.

Now let us turn to the formal definition.

<u>Definition</u> 1. A *theory T* is a triple $\langle T, Pr_T(\cdot), n\rangle$, where T is a set of arithmetical formulae, $Pr_T(\cdot)$ is an arithmetical formula with one free variable and n is a positive integer (referred to as the

degree of theory T, deg(T)),possessing the following properties:

1. $Pr(\cdot)\in\Sigma_n$.
2. $PA\vdash Pr_T\ulcorner\phi\rightarrow\psi\urcorner\rightarrow(Pr_T\ulcorner\phi\urcorner\rightarrow Pr_T\ulcorner\psi\urcorner)$.
3. If $A\in\Sigma_n^{PA}$, then $PA\vdash A\rightarrow Pr_T\ulcorner A\urcorner$.
4. $T\vdash A \Leftrightarrow\ \vDash Pr_T\ulcorner A\urcorner$.
5. $A\in\Sigma_n$, $T\vdash A\Longrightarrow\ \vDash A$.

("$\vDash A$" stands for "A is true in the standard model of arithmetic".)

A theory T is *correct*, iff for any arithmetical sentence $T\vdash A\Longrightarrow\ \vDash A$.

Note. Actually property 5 of the above definition implies that for all $A\in\Pi_{n+1}^{PA}$ $T\vdash A\Longrightarrow\ \vDash A$.

Definition 2. A sequence of theories $T_0,T_1,\ldots,T_n,\ldots$ (finite or infinite) is an *increasing sequence*, if it satisfies the following conditions:

1. $\deg(T_0)<\deg(T_1)<\ldots<\deg(T_n)<\ldots$.
2. $n<k\Longrightarrow PA\vdash Pr_{T_n}\ulcorner\phi\urcorner\rightarrow Pr_{T_k}\ulcorner\psi\urcorner$.

Note. In the following examples we will assume that all theories under consideration have a stronger property than property 2 of definition 1:

2^*. $PA\vdash\forall x,y(x=\ulcorner\phi\urcorner\wedge y=\ulcorner\psi\urcorner\wedge Pr_T\ulcorner\phi\urcorner\wedge Pr\ulcorner\phi\rightarrow\psi\urcorner\rightarrow Pr_T\ulcorner\psi\urcorner)$.

If a theory T does not satisfy 2^*, we should replace in th following examples $Pr_T(\cdot)$ by

$$\exists\psi_1,\ldots,\psi_n(\forall k\leq n\ Pr_T\ulcorner\psi_k\urcorner\ \wedge\ Pr_{PA}\ulcorner\textstyle\bigwedge_{k\leq n}\psi_k\rightarrow\phi\urcorner),$$

where $Pr_{PA}\ulcorner\phi\urcorner$ is the natural formalization of "ϕ is provable in PA".

Examples.

1.Let T=<T,$Pr_T(\cdot)$,n> be a correct theory. Consider the triple

$$T^{\omega}:=\langle T^{\omega},Pr_T^{\omega}(\cdot),n+2\rangle,$$

where

$$T^{\omega}:=T\cup\{\forall x\ \phi(x)\mid\forall n\ T\vdash\phi(\underline{n})\},$$

$$Pr_T^{\omega}\ulcorner\phi\urcorner:=\exists\psi(\cdot)\ (\forall n\ Pr_T\ulcorner\psi(\underline{n})\urcorner\wedge Pr_T\ulcorner(\forall x\ \psi(x))\rightarrow\phi\urcorner).$$

It can be easily shown that T^{ω} is a correct theory.

Let T_0 be a correct theory. Define T_n by induction on n: $T_{n+1}:=T_n^{\omega}$. Then $T_0,T_1,\ldots,T_n,\ldots$ is infinite increasing sequence.

2.Let $Tr_n(\cdot)$ denote the Σ_n-definition of truth for Σ_n-formulae, i.e. if $\phi\in\Sigma_n$, then $PA\vdash\phi\leftrightarrow Tr_n\ulcorner\phi\urcorner$. We also require that if $\phi\notin\Sigma_n$, then $PA\vdash\neg Tr_n\ulcorner\phi\urcorner$, and

$$PA \vdash \forall\phi,\psi(Tr_n\ulcorner\phi\wedge\psi\urcorner \leftrightarrow Tr_n\ulcorner\phi\urcorner\wedge Tr_n\ulcorner\psi\urcorner),$$
$$PA \vdash \forall\phi(Tr_n\ulcorner\phi\urcorner \rightarrow Tr_{n+1}\ulcorner\phi\urcorner).$$

Consider the following construction:

$$T_n := \langle T, Pr_T(\cdot), n \rangle,$$

where

n is arbitrary, n>0,

T is the set of all true Σ_n-sentence,

$Pr_T\ulcorner\phi\urcorner := \exists\psi(Tr_n\ulcorner\psi\urcorner\wedge Pr_{PA}\ulcorner\psi\rightarrow\phi\urcorner)$.

Clearly T_n is a minimal theory of degree n (by property 3 of definition 1), $T_1=PA$, where $PA := \langle PA, Pr_{PA}(\cdot), 1 \rangle$, and $T_1, T_2, \ldots, T_n, \ldots$ is an increasing sequence.

3. Let T_1 and T_2 be correct theories of the same degree. The theory $T_1 \cup T_2$ is defined as follows:

$$T_1 \cup T_2 := \langle T_1 \cup T_2, Pr_{T_1 \cup T_2}(\cdot), n \rangle,$$

where

$n = deg(T_1) = deg(T_2)$,

$Pr_{T_1 \cup T_2}\ulcorner\phi\urcorner := \exists\psi,\theta(Pr_{T_1}\ulcorner\psi\urcorner\wedge Pr_{T_2}\ulcorner\theta\urcorner\wedge Pr_{PA}\ulcorner\psi\wedge\theta\rightarrow\phi\urcorner)$.

It can be easily shown that $T_1 \cup T_2$ is a correct theory.

Definition 3. Let $\mathcal{L}$ be the language consisting of propositional variables p,q,... ; boolean connectives $\rightarrow,\bot$; modal operators [i],i=0,1,... . Working with the language $\mathcal{L}$ we use the standard abbreviation for $\wedge,\vee,\neg,\leftrightarrow$ and the following abbreviations:

$\langle i\rangle\phi := \neg[i]\neg\phi$,

$\Box\phi := [0]\phi$,

$\Diamond\phi := \langle 0\rangle\phi$,

$\boxdot\phi := \phi\wedge\Box\phi$.

We will often write "modal formula" instead of "$\mathcal{L}$-formulae".

Consider an arbitrary sequence of theories $T_0, T_1, \ldots, T_n, \ldots$. An *arithmetical interpretation* f is a mapping of $\mathcal{L}$-formulae to arithmetical sentences which commutes with the boolean connectives and translates [n] as provability in T_n, i.e. for every modal formula ϕ

$$f([n]\phi) := Pr_{T_n}\ulcorner f(\phi)\urcorner.$$

Note that $f(\phi)$ depends not only on f,ϕ, but also on the sequence $T_0, T_1, \ldots, T_n, \ldots$.

Using the standard Solovay's framework one can show the following fact:

Fact. Let ϕ be a modal formula, containing only boolean connectives and modal operator [n]. Then for any interpretation f PA $\vdash$ f(ϕ) iff GL $\vdash \phi$ (where GL is the Gödel-Löb's provability logic, formulated in terms of [n]).

Definition 4. The logic GLP is given as the minimal set of $\mathcal{L}$-formulae containing the following axioms and closed under the following rules:

(in all axioms the statement "for all $n \geq 0$" is supposed)

Axioms:

0. All tautologies of propositional logic.
1. $[n](\phi \rightarrow \psi) \rightarrow ([n]\phi \rightarrow [n]\psi)$.
2. $[n]([n]\phi \rightarrow \phi) \rightarrow [n]\phi$.
3. $[n]\phi \rightarrow [n+1]\phi$.
4. $\langle n\rangle\phi \rightarrow [n+1]\langle n\rangle\phi$.

Inference rules:

1. *Modus ponens.*
2. $\dfrac{\phi}{\Box\phi}$ (*[0]-necessitation*)

Some theorems of GLP are:

5. $[k]\phi \rightarrow [n]\phi$.
6. $[k]\phi \rightarrow [n][k]\phi$

} $k \leq n$

7. $\langle k\rangle\phi \rightarrow [n]\langle k\rangle\phi$
8. $[n]([k]\phi \rightarrow \phi)$

} $k < n$

Definition 5. The logic GLP^{ω} is given as the minimal set of $\mathcal{L}$-formulae closed under MP and containing the following axioms:

1. All theorems of GLP.
2. $[n]\phi \rightarrow \phi$, $n \geq 0$.

Theorem 1. *Arithmetical completeness of GLP.*

Let $T_0, T_1, \ldots, T_n, \ldots$ be an increasing sequence of theories. Then fo any modal formula ϕ GLP $\vdash \phi$ iff for any interpretation f PA $\vdash$ f(ϕ).

Theorem 2. *Arithmetical completeness of GLP^{ω}.*

Let $T_0, T_1, \ldots, T_n, \ldots$ be an increasing sequence of correct theories Then for any modal formula ϕ $GLP^{\omega} \vdash \phi$ iff for any interpretation f $\models$ f(ϕ).

Note. One can see that if an increasing sequence $T_0, T_1, \ldots, T_n, \ldots$ is infinite, then all theories $T_0, T_1, \ldots, T_n, \ldots$ are correct.

Theorem 3. *Fixed point property for GLP.*

Let $A(p;q_1,q_2,\dots,q_n)$ be modalized in p (i.e. every occurrence of p in A lies in the scope of [k] for some k). Then there exists a formula $F(q_1,q_2,\dots,q_n)$ such that

$$GLP \vdash \Box(p \leftrightarrow A) \leftrightarrow \Box(p \leftrightarrow F)$$
$$GLP \vdash \boxdot(p \leftrightarrow A) \leftrightarrow \boxdot(p \leftrightarrow F)$$

Theorem 4. The logics GLP and GLP^{ω} possess Craig's interpolatio property.

In the sequel we assume that the language $\mathcal{L}$ contains modal operators [n] only for $0 \le n \le N$ for some fixed $N>0$. Obviously, this bound does not affect theorems 1-4.

In order to prove theorems 1-4 we need to investigate a certain polymodal logic LN and prove the fixed point theorem and Craig's property for LN. We will then be able to investigate the relationship between LN and GLP.

We now formulate our basic results.

Definition 6. The logic LN is given as the minimal set of $\mathcal{L}$-formulae containing the following axioms and closed under the following rules:

Axioms:

0. All tautologies of propositional logic.
1. $[n](\phi \longrightarrow \psi) \longrightarrow ([n]\phi \longrightarrow [n]\psi)$. } $0 \le n \le N$
2. $[n]([n]\phi \longrightarrow \phi) \longrightarrow [n]\phi$ } $0 \le n \le N$
3. $[k]\phi \longrightarrow [n][k]\phi$ } $0 \le k < n \le N$
4. $\langle k\rangle\phi \longrightarrow [n]\langle k\rangle\phi$ } $0 \le k < n \le N$

Inference rules:

1. *Modus ponens.*
2. $\dfrac{\phi}{[n]\phi}$ (*[n]-necessitation*)

We define operators $\Delta\phi$ and $\Delta^{+}\phi$ as follows:

$$\Delta\phi := \bigwedge_{0 \le i_1 < i_2 < \dots < i_n \le N} [i_1][i_2]\dots[i_n]\phi$$
$$\Delta^{+}\phi := \phi \wedge \Delta\phi$$

Note that $GLP \vdash \Delta\phi \leftrightarrow \Box\phi$, $GLP \vdash \Delta^{+}\phi \leftrightarrow \boxdot\phi$.

Theorem 5. *Fixed point property for LN.*

Let $A(p;q_1,q_2,\dots,q_n)$ be modalized in p. Then there exists a formula

$F(q_1,q_2,\ldots,q_n)$ such that

$$LN \vdash \Delta(p \leftrightarrow A) \leftrightarrow \Delta(p \leftrightarrow F)$$

$$LN \vdash \Delta^+(p \leftrightarrow A) \leftrightarrow \Delta^+(p \leftrightarrow F)$$

Theorem 6. The logic LN possesses Craig's property.

For any modal formula ϕ we define $M(\phi)$ by the following way:

$$M(\phi) := \bigwedge_{[k]\psi\subseteq\phi,\ k<n\leq N} \Delta\,(\,[k]\psi \longrightarrow [n]\psi\,),$$

where "$\theta_1\subseteq\theta_2$" stands for "$\theta_1$ is a subformula of θ_2".

Theorem 7. *A reduction of GLP to LN.*

For any modal formula ϕ

$$GLP \vdash \phi \Leftrightarrow LN \vdash M(\phi) \longrightarrow \phi$$

Theorem 8. *A reduction of LN to GLP.*

Assume that ϕ does not contain the sentence letters $p_0,p_1,\ldots,p_N$. We define a translation ψ^* for $\psi\subseteq\phi$ as follows:

1. * commutes with boolean connectives.
2. $([n]\psi)^* := [n](p_n \longrightarrow \psi^*)$

Then

$$LN \vdash \phi \Leftrightarrow LN \vdash \phi^* \Leftrightarrow GLP \vdash \phi^*.$$

Theorem 9. *Relationship between GLP and GLP^ω.*

a) Define:

$$H(\phi) := \bigwedge_{[n]\psi\subseteq\phi} ([n]\psi \longrightarrow \psi).$$

Then for any ϕ

$$GLP^\omega \vdash \phi \Leftrightarrow GLP \vdash H(\phi) \longrightarrow \phi.$$

b) For any ϕ

$$GLP \vdash \phi \Leftrightarrow GLP \vdash \Box\phi \Leftrightarrow GLP^\omega \vdash \Box\phi.$$

c) Assume that ϕ does not contain [N]. Then

$$GLP^\omega \vdash \phi \Leftrightarrow GLP \vdash [N]\phi \Leftrightarrow GLP^\omega \vdash [N]\phi.$$

We assume the reader to be acquainted with the basic facts about Kripke semantic for polymodal logics.

Definition 7. An *LN-model* $\mathcal{K}$ is a N+3-tuple $\langle K,R^0,R^1,\ldots,R^N,\Vdash\rangle$, where K is a nonempty finite set (*support* of $\mathcal{K}$), R^i is the accessibility relation for [i], $\Vdash$ is a forcing relation, possessing the following properties:

1. $\forall n\ R^n$ is irreflexive and transitive.

2. $\forall k,n$: $k<n \wedge xR^k y \wedge (xR^n z \vee zR^n x) \longrightarrow zR^k y$.

<u>Definition</u> 8. An LN-model $\mathcal{K}$ is *ϕ-complete*, where ϕ is a modal formula, iff

$$\forall x\in K\ \forall\psi\colon [k]\psi\subseteq\phi\ \forall n\colon k<n\le N\ (\ x\Vdash [k]\psi \longrightarrow [n]\psi\ ).$$

<u>Theorem</u> 10. $LN\vdash \phi$ iff ϕ is valid in every LN-model.

<u>Theorem</u> 11. $GLP\vdash \phi$ iff ϕ is valid in every ϕ-complete LN-model

Note that by theorems 7,9,10 the logics GLP, GLP^{ω} and LN are decidable.

§2. LN-models.

<u>Proof of theorem 10</u>.

Soundness is evident; thus, we only need to prove the completeness.

Let Γ be a finite set of modal formulae. We say that Γ is *LN-consistent*, iff $LN\nvdash\neg\bigwedge_{\phi\in\Gamma}\phi$.

Let $LN\nvdash\phi$, and $W:=\{\psi,\neg\psi\,|\,\psi\subseteq\phi\}$. Now we define an LN-model $\mathcal{K}$, in which ϕ is not valid:

$\mathcal{K}:=\langle K,R^0,R^1,\ldots,R^N,\Vdash\rangle$,

$K:=\{x\subseteq W\,|\,x$ is maximal LN-consistent set$\}$,

$xR^n y :\Leftrightarrow$

1. $\forall\psi$ ($[n]\psi\in x \longrightarrow \psi,[n]\psi\in y$);
2. $\forall\psi\ \forall k<n$ ($[k]\psi\in x \leftrightarrow [k]\psi\in y$);
3. $\exists\psi$: $[n]\psi\in y \wedge [n]\psi\notin x$;

$x\Vdash p :\Leftrightarrow p\in x$.

It is evident that $\mathcal{K}$ is LN-model.

<u>Lemma.</u> For any $\psi\subseteq\phi$, $x\in K$ $\ x\Vdash\psi \Leftrightarrow \psi\in x$.

<u>Proof.</u> The only interesting case is $\psi = [n]\theta$.

"$\Leftarrow$": Using condition 1 of the definition of R^n.

"$\Rightarrow$": Assume $[n]\theta\notin x$, i.e. $\neg[n]\theta\in x$. Let x have the following form:

$x=\{\neg[n]\theta;[0]\Gamma_0,[1]\Gamma_1,\ldots,[n-1]\Gamma_{n-1};\neg[0]\Sigma_0,\ldots,\neg[n-1]\Sigma_{n-1};[n]\Gamma_n;\ldots\}$.

Consider the set of formulae:

$y':=\{\neg\theta,[n]\theta;[0]\Gamma_0,[1]\Gamma_1,\ldots,[n-1]\Gamma_{n-1};\neg[0]\Sigma_0,\ldots,\neg[n-1]\Sigma_{n-1};$
$[n]\Gamma_n,\Gamma_n\}$

and suppose it to be LN-inconsistent. Then

$$LN \vdash \bigwedge_0^{n-1}([i]\Gamma_i \wedge \neg[i]\Sigma_i) \wedge [n]\Gamma_n \wedge \Gamma_n \longrightarrow ([n]\theta \longrightarrow \theta).$$

Using the [n]-necessitation rule, GL theorems for [n] and axioms 3,4 by the definition of LN we obtain:

$$LN \vdash \bigwedge_0^{n-1}([i]\Gamma_i \wedge \neg[i]\Sigma_i) \wedge [n]\Gamma_n \longrightarrow [n]\theta,$$

i.e. x is inconsistent.

Then denote by y a maximal consistent extension of y'. We have: $xR^n y$, $y \nVdash \theta$, hence $x \nVdash [n]\theta$. The lemma is thus proved.

Let u be a node of $\mathcal{K}$ such that $\neg\phi \in u$. By the lemma, $u \nVdash \phi$. QED.

<u>Note.</u> One could think that we could use this proof to give a normalization theorem for the calculus LN (as for GL) using the following sequel rule:

$$\frac{[0]\Gamma_0, [1]\Gamma_1, \ldots, [n]\Gamma_n, \Gamma_n, [n]\phi \implies \phi, [0]\Sigma_0, \ldots, [n-1]\Sigma_{n-1}}{[0]\Gamma_0, [1]\Gamma_1, \ldots, [n]\Gamma_n, \implies [n]\phi, [0]\Sigma_0, \ldots, [n-1]\Sigma_{n-1}}$$

The Gentzen system obtained by adding the above rule (GLN) will be adequate for LN, but unlike GL, we cannot exploit the previous proof to prove a cut-elimination theorem for GLN. The reason for this is that: if we substitute in the proof of the theorem underivable in GLN^- "saturated" sequents for the maximal LN-consistent set, thus in saturating y' new formulae (which were not in x) of the form $[k]\psi$ can appear. Thus the right to the left implication in the second condition of the definition of R^n is the cause of the fact that the cuts cannot be eliminated.

Construction of LN-models.

1.Submodels.

First a trivial remark: the restriction of an LN-model to a subset yields an LN-model.

2.A cone restriction.

This is a standard idea that can be used for any Kripke model. For $x \in K$ the cone restriction of x is defined as follows:

$$\tilde{W}_x := \{x\} \cup W_x,$$

$$W_x := \{t \in K \mid \text{there exists a chain } x = x_0 R^{i_0} x_1 \ldots R^{i_n} t\}.$$

One can easily see that for LN-models we can assume $i_0 < i_1 < \ldots < i_n$ in the definition of W_x. Thus,

$$x \Vdash \Delta\phi \Leftrightarrow \forall y \in W_x \ y \Vdash \phi$$

$$x \Vdash \Delta^+\phi \Leftrightarrow \forall y \in \tilde{W}_x \ y \Vdash \phi$$

Define xRy to be $y \in W_x$. Then R is transitive and irreflexive.

As for arbitrary Kripke models, passing to the cone restriction of LN-models preserves the forcing relations on formulae.

3. LN-closure.

If $\mathcal{K}$ is an arbitrary Kripke model we say that $\mathcal{K}_1$ is an LN-closure of $\mathcal{K}$ iff $\mathcal{K}$ and $\mathcal{K}_1$ have the same support and $\mathcal{K}_1$ is the minimal LN-model which contains $\mathcal{K}$. Note that if $\mathcal{K}_1$ exists, then it is unique.

4. "Link" of LN-models.

Let s LN-models $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_s$ be given, and their supports do not intersect. Let a node x_i be fixed in each $\mathcal{K}_i$. Also, let n be a natural number and Γ be a set of formulae that is closed under subformulae. Assume that the following property is fulfilled:

(*) $\forall \phi, k$: $[k]\phi \in \Gamma$, $k<n$ $\forall i,j \le s$ ($x_i \Vdash [k]\phi \leftrightarrow x_j \Vdash [k]\phi$).

Define an LN-model $\mathcal{K}$:

$K := K_1 \cup K_2 \cup \ldots \cup K_s \cup \{h\}$;

before introducing R^i we define the sets E^i_j, $1 \le j \le s$, $0 \le i < n$, $E^i_j \subseteq K_j$ as minimal sets satisfying the following conditions:

1. $x_j \in E^i_j$.
2. $x \in E^i_j$, (($xR^k_j y \wedge k>i$) $\vee$ ($yR^k_j x \wedge k \ge i$)) $\Longrightarrow y \in E^i_j$

Note that

(**) $x_j R^i_j x \Longrightarrow \forall y \in E^i_j \ \ yR^i_j x$.

Define now R^i as the minimal binary relation on $\mathcal{K}$ such that:

1. $x_j R^i_j x$, $y \in E^i_j \Longrightarrow yR^i x$, $i<n$.
2. $x_j R^i_j x \Longrightarrow hR^i x$, $i \le n$.
3. $hR^n x_j$.
4. $xR^i_j y \Longrightarrow xR^i y$.

The forcing relation on $\mathcal{K}$ is that induced naturally by $\mathcal{K}_1, \ldots, \mathcal{K}_s$. We take an arbitrary forcing relation in h. It is easy to prove that $\mathcal{K}$ is an LN-model. (In fact, we can obtain $\mathcal{K}$ by another way: use conditions 3,4 in the above definition and then take the LN-closure of the obtained Kripke model.)

Lemma. $\forall\phi\in\Gamma\ \forall x\in K_j\quad x \Vdash_{\mathcal{K}_j} \phi \Leftrightarrow x \Vdash_{\mathcal{K}} \phi$.

Proof. The only interesting case is $\phi=[k]\theta$, where $k<n$.

"$\Leftarrow$" is trivial.

"$\Rightarrow$". Assume $x \nVdash_{\mathcal{K}} [k]\theta$. Then there exists a node y such that $y \nVdash \theta$, $xR^k y$ (by induction hypothesis, $\Vdash_{\mathcal{K}} \theta$ and $\Vdash_{\mathcal{K}_i} \theta$ are equivalent), where either y lies in K_j or the relation $xR^k y$ is the result of applying rule 1 from the definition of R^k . In the last case $y\in K_m$, $x_m R^k y$, $x\in E^k_j$, but since $y \nVdash_{\mathcal{K}_m} \theta$, then $x_m \nVdash_{\mathcal{K}_m} [k]\theta$. By (*), $x_j \nVdash_{\mathcal{K}_j} [k]\theta$, and therefore there is z such that $x_j R^i_j z$, $z \nVdash_{\mathcal{K}_j} \theta$. But $x\in E^k_j$, and thus by (**) $xR^k z$. So, $x \nVdash_{\mathcal{K}_j} [k]\theta$.

The lemma is thus proved.

§3.Fixed Point Property and Craig's Property for LN.

The properties of LN-models established in the previous paragraph have are of an independent interest. Now we consider some technical concepts which will be used in proving theorems 5 and 6.

Define the set:

$$\mathbb{V}:=\{<x_0,x_1,\ldots,x_N> \mid x_0,x_1,\ldots,x_N\in\omega\}.$$

We impose on $\mathbb{V}$ the following structure: vector sum **x+y** (**x+y**$\in\mathbb{V}$), partial (component) order **x<y** ($:\Leftrightarrow\ \forall i\ \ x_i<y_i$) and linear lexicographical order **x<y** ($:\Leftrightarrow\ \exists i:\ x_i<y_i\ \wedge\ \forall j<i\ \ x_j=y_j$). We also use a reflexive orders **x≤y**, **x⩽y**.

Define $\sigma(\mathbf{x},\mathbf{y})$ for $\mathbf{x},\mathbf{y}\in\mathbb{V}$ as the maximal number n for which there exists a sequence:

$$\mathbf{x}=\mathbf{x}_0>\mathbf{x}_1>\mathbf{x}_2>\ldots>\mathbf{x}_n;\ \forall i\ \ \mathbf{x}_i\le\mathbf{y}.$$

Let ϕ be a modal formula. Define $\rho(\phi)\in\omega$ and $\nu(\phi)\in\mathbb{V}$ as follows:

$\nu(\phi):=<x_0,\ldots,x_n>$, where x_i is the number of all subformulae of ϕ of the form [i]D;

$\rho(\phi):=\sigma(2\nu(\phi),2\nu(\phi))$.

In the following reasoning we will follow [3].

Let S be a finite set of sentence letters, $\mathcal{K}$ be an LN-model and

$x \in K$. We define by induction on n the *n-S-character of x* to be the conjunction of all formulae of the form:

1. $p_i, \neg p_i \mid p_i \in S$;

2. $<k>C$, $\neg <k>C$, where C is (n-1)-S-character.

(a modal formula C is *n-S-character*, iff C is an n-S-character of x for some LN-model $\mathcal{K}$ and $x \in K$), which are true in x (if n=0 the definition consists only of clause 1).

Note that if x_1 and x_2 have the same n-S-character, then for any y_1 such that $x_1 R^k y_1$ there exists y_2 such that $x_2 R^k y_2$ and y_1 and y_2 have the same (n-1)-S-character.

Proof of theorem 5.

Lemma 1. Let ϕ, ψ be modal formulae. The following statements are equivalent:

1). For any LN-model $\mathcal{K}$ $\mathcal{K} \Vdash \phi \Longrightarrow \mathcal{K} \Vdash \psi$;

(from now on $\mathcal{K} \Vdash \phi$ denotes $\forall x \in K\ x \Vdash \phi$),

2). $LN \vdash \Delta\phi \longrightarrow \Delta\psi$;

3). $LN \vdash \Delta^+\phi \longrightarrow \Delta\, \psi$.

Proof.

3) $\Longrightarrow$ 2). $LN \vdash \phi\wedge\Delta\phi \longrightarrow \psi\wedge\Delta\psi$

$LN \vdash \Delta(\phi\wedge\Delta\phi) \longrightarrow \Delta(\psi\wedge\Delta\psi)$

$LN \vdash \Delta\phi \longrightarrow \Delta\psi$

(Using the model completeness of LN (theorem 10) it is easy to see that LN contains all theorems of GL, formulated in terms of Δ and that LN is closed under Δ-necessitation).

1) $\Longrightarrow$ 3). Assume $LN \nvdash \Delta^+\phi \longrightarrow \Delta^+\psi$. Then there exists an LN-model $\mathcal{K}$ such that $\mathcal{K} \nVdash (\Delta^+\phi \longrightarrow \Delta^+\psi)$, i.e. for some $x \in K$ $x \Vdash \Delta^+\phi$, $x \nVdash \Delta^+\psi$. Consider the cone $\tilde{W}_x$. Then $\tilde{W}_x \Vdash \phi$, but there is $y \in \tilde{W}_x$ such that $y \nVdash \psi$.

2) $\Longrightarrow$ 1). Assume there exists an LN-model $\mathcal{K}$ such that $\mathcal{K} \Vdash \phi$ $\mathcal{K} \nVdash \psi$. Adjoin a bottom node h to $\mathcal{K}$ such that for any $x \in K$ hR^0x. Then $h \Vdash \Delta\phi$, $h \nVdash \Delta\psi$, i.e.2) does not hold.

This proves lemma 1.

Fix now a formula $A(p;q_1,\ldots q_m)$ modalized in p, and let $n:=\rho(A)$, $S:=\{q_1,\ldots,q_m\}$.

Lemma 2. Suppose that $\mathcal{K}_1$ and $\mathcal{K}_2$ are LN-models in which $p \leftrightarrow A$ is valid, and let x_1, x_2 be nodes of $\mathcal{K}_1, \mathcal{K}_2$ respectively which have the same n-S-character. Then x_1 and x_2 agree on p.

<u>Proof</u>. Suppose not. Define the function μ: $K_1, K_2 \longrightarrow \mathbb{V}$ as follows: $\mu(x) := \langle y_0, \ldots, y_N \rangle$ where y_i is the number of those subformulae of A of the form [i]D that are false in x.

Define by induction two finite sequences:

$$x_i^0, x_i^1, \ldots, x_i^{n+1},\ x_i^j \in K_i,\ i \in \{1,2\},$$

such that (for every step of the inductive definition):

1) x_1^j, x_2^j have the same (n-j)-S-character, $j \le n$

2) $\mu(x_1^j) + \mu(x_2^j) > \mu(x_1^{j+1}) + \mu(x_2^{j+1})$, $j \le n$

3) x_1^j, x_2^j differ on some $\psi \subseteq A$.

Basis. $x_i^0 := x_i$, $i \in \{1,2\}$: x_1, x_2 differ on p.

Induction. Because x_1^j, x_2^j differ on ψ and $\psi \subseteq A$, one of three cases holds:

1. x_1^j, x_2^j differ on some sentence letter from S.
2. x_1^j, x_2^j differ on p.
3. x_1^j, x_2^j differ on some subformula A of the form [k]D.

Case 1 is impossible according to the assumption 1); case 2 implies cases 1 or 3, because $p \longleftrightarrow A$ holds in $\mathcal{K}_1$ and $\mathcal{K}_2$. Thus 3 must be the case.

Let, for example, $x_1^j \nVdash [k]D$, $x_2^j \Vdash [k]D$. Since $LN \vdash \neg[k]D \longrightarrow \langle k \rangle(\neg D \wedge [k]D)$ (Löb's axiom), there exists x_1^{j+1} s.t. $x_1^j R_1^k x_1^{j+1}$ and $x_1^{j+1} \Vdash \neg D \wedge [k]D$. It is easy to see that

(*) 1. $\forall i \le k \quad x_1^j \Vdash [i]\phi \Longrightarrow x_1^{j+1} \Vdash [i]\phi$

2. $x_1^j \nVdash [k]D, \quad x_1^{j+1} \Vdash [k]D.$

Choose x_2^{j+1} s.t. $x_2^j R_2^k x_2^{j+1}$ and x_1^{j+1}, x_2^{j+1} have the same (n-j-1)-S-character. Assumption 1) holds, 3) holds because $x_1^{j+1} \nVdash D$, $x_2^{j+1} \Vdash D$ ($\Leftarrow x_2^j \Vdash [k]D$). Further (as for (*))

(**) $\forall i \le k \quad x_2^j \Vdash [i]\phi \Longrightarrow x_2^{j+1} \Vdash [i]\phi.$

(*) and (**) imply that assumption 2) holds. The construction is finished.

Let now $\mathbf{y}_j := \mu(x_1^j) + \mu(x_2^j) \in \mathbb{V}$. Then $\mathbf{y}_0 > \mathbf{y}_1 > \ldots > \mathbf{y}_{n+1}$ and $\forall j\ \mathbf{y}_j \le 2\nu(A)$. But this contradicts the definition on n. Lemma 2 is thus proved.

Let F be the disjunction of all n-S-characters C with th following property: there exists an LN-model $\mathcal{K}$ and a node $x \in K$ such that $p \leftrightarrow A$ is valid in $\mathcal{K}$, $x \Vdash p$ and C is the n-S-character of x.

Suppose that $p \leftrightarrow A$ is valid in an arbitrary LN-model $\mathcal{K}_1$; we show that $p \leftrightarrow F$ is also valid in $\mathcal{K}$. Let $x_1 \in K_1$ and C be the n-S-character of x_1. If $x_1 \Vdash p$, then C is among the n-S-characters disjoined to form F; hence $x_1 \Vdash F$. Conversely, if $x_1 \Vdash F$ then C must be a disjunct of F; hence there exists an LN-model $\mathcal{K}_2$ in which $p \leftrightarrow A$ is valid and a node $x_2 \in K_2$ such that $x_2 \Vdash p$ and x_1, x_2 have the same n-S-character C. By lemma 2, x_1 and x_2 agree on p. Thus $x_1 \Vdash p$.

Let now $\mathcal{K} \Vdash p$ be given. We show that $p \leftrightarrow A$ is valid in $\mathcal{K}$. Suppose not. Then there exists $x \in K$, $x \nVdash p \leftrightarrow A$; by the properties of LN-models, there exists $x_1 \in K$ such that $x_1 \Vdash \neg(p \leftrightarrow A) \wedge \Delta(p \leftrightarrow A)$. Consider an LN-model $\mathcal{K}_1$ which is different from $\tilde{W}_{x_1}$ only in that $x_1 \Vdash_{\mathcal{K}_1} p :\Leftrightarrow x_1 \nVdash_{\mathcal{K}} p$. The forcing of A and F is not being changed, because A is modalized in p and F does not to contain p. But then $\mathcal{K}_1 \Vdash p \leftrightarrow A$ and $\mathcal{K}_1 \nVdash p \leftrightarrow F$, which contradicts to the above considerations.

Theorem 5 is thus proved by applying lemma 1.

<u>Proof of theorem 6</u>.

Let A and B be arbitrary modal formulae, and S be a set of common sentence letters of A and B. For an LN-model $\mathcal{K}$ and a node $x \in K$ let $\mu_A(x)$ be $\langle y_0, y_1, \ldots, y_N \rangle \in \mathbb{V}$, where y_i is the number of subformulae of A of the form [i]D, which are false in x; let also A(x) be a conjunction of all formulae of the form $p_i, \neg p_i, [k]D, \neg[k]D$ (where $p_i, [k]D \subseteq A$) which are true in x. The notations $\mu_B(x)$, B(x) are similar. Let $m := \mu(A) + \mu(B) \in \mathbb{V}$.

<u>Lemma 1</u>. Let x_1, x_2 be nodes of LN-models $\mathcal{K}_1, \mathcal{K}_2$ respectively, that have the same k-S-character, where $k \geq \sigma(m, \mu_A(x_1) + \mu_B(x_2))$. Then $A(x_1) \wedge B(x_2)$ is *satisfiable*, i.e. there exists an LN-model $\mathcal{K}$ and $h \in K$ such that $h \Vdash A(x_1) \wedge B(x_2)$.

<u>Proof</u>. Induction on k.

Basis. Let k=0. Then $\mu_A(x_1) = \mu_B(x_2) = 0$. We take an LN-model $\mathcal{K}$ with only one node h and define the forcing in h as follows: $h \Vdash p :\Leftrightarrow$ p is the conjunct $A(x_1) \wedge B(x_2)$. (Note that x_1 and x_2 agree on all common sentence letters of A and B).

Induction. k>0. Let

$$X:=\{[m]D \mid ([m]D\subseteq A \wedge x_1 \nVdash [m]D) \vee ([m]D\subseteq B \wedge x_2 \nVdash [m]D)\}$$
$$l:=\max\{d \mid [d]D\in X \text{ for some } D\}.$$

Now for each D s.t. $[l]D\in X$ make the following construction:

Assume, for example, $x_1 \nVdash [l]D$. Then there exists u such that $x_1R_1^lu$, $u\Vdash \neg D\wedge[l]D$. Let v be chosen such that $v\in K_2$, $x_2R_2^lv$, u and v have the same (k-1)-S-character. Like in the previous proof one can easily see that

$$\mu_A(u)+\mu_B(v)<\mu_A(x_1)+\mu_B(x_2),$$

hence

$$\sigma(m,\mu_A(u)+\mu_B(v))<\sigma(m,\mu_A(x_1)+\mu_B(x_2))\le k.$$

So, the induction hypothesis holds for LN-models $\tilde{W}_u$ and $\tilde{W}_v$. Then there exists an LN-model $\mathcal{K}$ and $y\in K$ s.t. $y\Vdash A(u)\wedge B(v)$. The construction is finished.

Having done this for each formula D of the appropriate form we have nodes $u_1,\dots,u_s\in K_1$, $x_1R_1^lu_1,\dots,x_1R_1^lu_s$; $v_1,\dots,v_s\in K_2$, $x_2R_2^lv_1,\dots,x_2R_2^lv_s$; LN-models $\mathcal{K}^1,\dots,\mathcal{K}^s$ with fixed nodes $y_1,\dots,y_s$, $y_1\in K^1,\dots,y_s\in K^s$. It is easy to show that if $d<l$ then $v_i \Vdash [d]\phi \Leftrightarrow v_j \Vdash [d]\phi$ ($\Leftrightarrow x_2 \Vdash [d]\phi$), $u_i \Vdash [d]\phi \Leftrightarrow u_j \Vdash [d]\phi$ ($\Leftrightarrow x_1 \Vdash [d]\phi$). By the construction, $y_i \Vdash A(u_i)\wedge B(v_i)$, hence if $[d]\phi\subseteq A\vee B$ then

$$y_i \Vdash [d]\phi \Leftrightarrow y_j \Vdash [d]\phi.$$

Thus we see that the link procedure can be used for LN-models $\mathcal{K}^1,\dots,\mathcal{K}^s$ and nodes $y_1,\dots,y_s$ (where $n:=l$, $\Gamma:=\{\phi \mid \phi\subseteq A\vee\phi\subseteq B\}$). It gives an LN-model $\mathcal{K}$ with bottom h. The forcing in h is defined as in the case k=0.

The LN-model $\mathcal{K}$ and the node h are sufficient for our goals. Proof: let (for example) $[k]D\subseteq A$. We show that

$$h \Vdash [k]D \Leftrightarrow x_1 \Vdash [k]D.$$

Case 1. Let $k>l$. Then $[k]D\notin X$, i.e. $x_1 \Vdash [k]D$. On the other hand $h \Vdash [k]\phi$ for any ϕ.

Case 2. Let $k\le l$ and $h \nVdash [k]D$.

Subcase 2.1. $k<l$. Then $y_i \nVdash [k]D$

Subcase 2.2. $k=l$. Then $y_i \nVdash D\wedge[k]D$

for some i. Since $y_i \Vdash A(u_i)$, y_i and u_i agree on every boolean combination of subformulae of A. So, the same formulae are true in u_i, implying $x_1 \nVdash [k]D$ in both subcases.

Case 3. Let $k\le l$ and $x_1 \nVdash [k]D$.

Subcase 3.1. $k<l$. Then $u_1 \nVdash [k]D$, hence $y_1 \nVdash [k]D$ and $h_1 \nVdash [k]D$.

Subcase 3.2. $k=l$. Then there exists j s.t. $u_j \nVdash D$, hence $y_j \nVdash D$ and $h \nVdash [l]D$.

Lemma 1 is thus proved.

Lemma 2. Let $n=\sigma(m,m)$ (where $m=\nu(a)+\nu(B)$). Let $\mathcal{K}_1, \mathcal{K}_2$ be LN-models, $x_i \in K_i$, $i=1,2$, x_1 and x_2 have the same n-S-character and $x_1 \Vdash A$, $x_2 \Vdash B$. Then there exists an LN-model $\mathcal{K}$ and $x \in K$ such that $x \Vdash A \wedge B$.

(Equivalent formulation: if there exists an n-S-character C such that $A \wedge C$ and $B \wedge C$ are each satisfiable then $A \wedge B$ is satisfiable).

Proof. The lemma is a consequence of lemma 1 and the following inequalities:

$$\mu_A(x_1)+\mu_B(x_2) \le \nu(A)+\nu(B)=m,$$
$$n=\sigma(m,m) \ge \sigma(m,\mu_A(x_1)+\mu_B(x_2)).$$

Now we prove Craig's property for LN. Assume that $LN \vdash A \rightarrow C$, and let B be a disjunction of all the formulae D such that there exists an LN-model $\mathcal{K}$ and $x \in K$ such that $x \Vdash A$ and D is an n-S-character of x, where S is a set of common sentence letters of A and C, $n=\sigma(\nu(A)+\nu(B),\nu(A)+\nu(B))$. It is clear that $LN \vdash A \rightarrow C$. Assume that $LN \nvdash B \rightarrow C$, and let $\mathcal{K}_1$ be an LN-model such that for some $x_1 \in K_1$ $x_1 \Vdash B \wedge \neg C$. Let D be the disjunct of B which is true in x_1. Then: 1) D is the n-S-character of x_1; 2) there exists an LN-model $\mathcal{K}_2$ and $x_2 \in K_2$ such that $x_2 \Vdash A$ and D is the n-S-character of x_2. Hence, x_1 and x_2 have the same n-S-character. By lemma 2, there exists an LN-model $\mathcal{K}$ and $x \in K$ such that $x \Vdash A \wedge \neg C$. This contradicts to $LN \vdash A \rightarrow C$. QED.

It is well-known that both finite irreflexive partial orders an finite irreflexive trees can be considered as GL-models. Like treelike GL-models, we introduce the concept of *simple* LN-models.

Definition. An LN-model $\mathcal{K}$ is *simple*, iff the following does not hold ($x,y,z \in K$):

$$xR^i z,\ yR^j z,\ i \ne j.$$

(In fact we could give a stronger definition, including into it the analogue of treelike structure, but this is not necessary for our goals).

Theorem. If $LN \nvdash \phi$ then there exists a simple LN-model $\mathcal{K}$ such that $\mathcal{K} \nVdash \phi$.

Proof. From the proof of lemma 1 one can see that the LN-model $\mathcal{K}$ constructed in the proof is simple (it follows from the fact that a link of simple models by their bottom nodes gives us a simple model). If $LN \nvdash \phi$ then by theorem 10 there exists a model $\mathcal{K}$ and $x \in K$ s.t. $x \Vdash \neg\phi$. Use now lemma 2 in the case $\mathcal{K}_1=\mathcal{K}_2=\mathcal{K}$, $x_1=x_2=x$, $A=B=\neg\phi$.

§4. Arithmetical Completeness of GLP.

We will prove theorems 1,7,11 together:

Theorem. Consider an increasing sequence of theories $T_0, T_1, \ldots, T_N$. For any modal formula ϕ the following statements are equivalent:

1) $GLP \vdash \phi$.
2) For any arithmetical interpretation f $PA \vdash f(\phi)$.
3) for any ϕ-complete LN-model $\mathcal{K}$ $\mathcal{K} \Vdash \phi$.
4) $LN \vdash M(\phi) \rightarrow \phi$, where $M(\phi)$ was defined above. (Before theorem 7).

Proof.

1) $\Rightarrow$ 2). We have to prove the arithmetical soundness of GLP.

By induction on (the length of) the proof of ϕ.

Case 1. ϕ has the form:

$$\phi = [n](\psi \rightarrow \theta) \rightarrow ([n]\psi \rightarrow [n]\theta).$$

Then 2) follows from condition 2 from the definition on the theory.

Case 2. ϕ is Löb's axiom:

$$\phi = [n]([n]\psi \rightarrow \psi) \rightarrow [n]\psi.$$

Let $T := T_n$. Note that by definition on the theory for any arithmetical sentences A and B one has:

(*) If $PA \vdash A$, then $PA \vdash Pr_T \ulcorner A \urcorner$.

(**) If $PA \vdash A \rightarrow B$, then $PA \vdash Pr_T \ulcorner A \urcorner \rightarrow Pr_T \ulcorner B \urcorner$.

(***) $PA \vdash Pr_T \ulcorner A \urcorner \rightarrow Pr_T \ulcorner Pr_T \ulcorner A \urcorner \urcorner$.

Let A be such that:

(1) $PA \vdash A \leftrightarrow \neg Pr_T \ulcorner \neg f(\psi) \rightarrow A \urcorner$.

We have:

(2) $PA \vdash Pr_T\ulcorner\neg f(\psi) \to A\urcorner \to Pr_T\ulcorner Pr_T\ulcorner\neg f(\psi) \to A\urcorner\urcorner$ by (***)

$PA \vdash Pr_T\ulcorner\neg f(\psi) \to A\urcorner \to \neg A$ by (1)

$PA \vdash Pr_T\ulcorner Pr_T\ulcorner\neg f(\psi) \to A\urcorner\urcorner \to Pr_T\ulcorner\neg A\urcorner$ by (**)

$PA \vdash Pr_T\ulcorner\neg f(\psi) \to A\urcorner \to Pr_T\ulcorner\neg A\urcorner$ by (2)

(3) $PA \vdash Pr_T\ulcorner\neg f(\psi) \to A\urcorner \to Pr_T\ulcorner f(\psi)\urcorner$

$PA \vdash \neg A \to Pr_T\ulcorner f(\psi)\urcorner$ by (1)

$PA \vdash (Pr_T\ulcorner f(\psi)\urcorner \to f(\psi)) \to (\neg A \to f(\psi))$

$PA \vdash Pr_T\ulcorner Pr_T\ulcorner f(\psi)\urcorner \to f(\psi)\urcorner \to Pr_T\ulcorner\neg A \to f(\psi)\urcorner$ by (***)

$PA \vdash f([n]([n]\psi \to \psi) \to [n]\psi)$ by (3).

Case 3. ϕ has the form:

$$\phi = [k]\psi \to [n]\psi,\ k<n.$$

Then 2) follows from condition 2 of the definition of increasing sequence.

Case 4. ϕ has the form:

$$\phi = \langle n\rangle\psi \to [n+1]\langle n\rangle\psi.$$

Let f be an arbitrary arithmetical interpretation, $s:=\deg(T_n)$. By condition 1 of the definition of the theory,

$$f(\langle n\rangle\psi) = \neg Pr_{T_n}\ulcorner\neg f(\psi)\urcorner \in \Pi^{PA}_s \subseteq \Sigma^{PA}_{s+1}.$$

By condition 1 of the definition of increasing sequence, $\deg(T_{n+1}) \geq s+1$. The third condition of the definition of the theory states that

$$PA \vdash f(\langle n\rangle\psi) \to Pr_{T_{n+1}}\ulcorner f(\langle n\rangle\psi)\urcorner.$$

Case 5. ϕ is the result of an application of MP. This case is trivial.

Case 6. ϕ is the result of an application of the necessitation rule, i.e. $\phi = \Box\psi$. Then by the induction hypothesis, for any arithmetical interpretation f $PA \vdash f(\phi)$. By (*), $PA \vdash Pr_{T_0}\ulcorner f(\psi)\urcorner$, i.e. $PA \vdash f(\Box\psi)$.

4) $\Rightarrow$ 1). This is trivial: $GLP \vdash M(\phi)$ for any ϕ.

3) $\Rightarrow$ 4). Assume $LN \nvdash M(\phi) \to \phi$. Then for some LN-model $\mathcal{K}$ and for some $x \in K$ $x \Vdash M(\phi) \wedge \neg\phi$. It is easy to see that $\tilde{W}_x$ is a ϕ-complete LN-model in which ϕ does not hold.

2) $\Rightarrow$ 3). Let $\mathcal{K}_1$ be a ϕ-complete LN-model in which ϕ does not hold. Adjoin a bottom node 0 to the model $\mathcal{K}_1$ such that $0R^0x$ for any $x \in K_1$; call this LN-model $\mathcal{K}$. (Note that $\mathcal{K}$ is also ϕ-complete.).

Define the Solovay function h: $\omega \longrightarrow K$ in the following way:

(We assume that $Pr_{T_n}(\cdot)$ is of the form:

$$Pr_{T_n}(x) = \exists y\, Prf_{T_n}(y,x)$$

where $Prf_{T_n}(\cdot,\cdot) \in \Pi_{deg(T_n)-1}$ and

$$PA \vdash Prf_{T_n}(m,x) \wedge Prf_{T_n}(m,y) \longrightarrow x=y. \text{)}$$

h(0):=0;

<u>if</u> there is a triple <z,n,y> ($z \in K$, $n,y \in \omega$) such that:

1. $h(m) R^n z$

2. $Prf_{T_n}(y, \ulcorner l \neq z \urcorner)$

<u>then</u> choose such triple to be minimal with respect to n, and then minimal with respect to y; say $\langle z_0, n_0, y_0 \rangle$. Put

$h(m+1) := z_0$.

<u>else</u>

$h(m+1) := h(m)$,

where $l=z$ denotes an arithmetic formula "$\lim_{m \to \infty} h(m)=z$" as usual.

<u>Lemma</u> <u>1</u>. (PA). Let

$$0=h(0) R^{i_1} h(1) R^{i_2} h(2) \ldots R^{i_n} h(n) = \lim_{m \to \infty} h(m).$$

Then $i_1 \leq i_2 \leq i_3 \leq \ldots \leq i_n$.

<u>Proof</u>. (PA). Let, for example, $i_1 > i_2$. Then $h(0) R^{i_2} h(2)$ and $T_{i_2} \vdash l \neq h(2)$. This contradicts "n-minimality" in the definition of h and $h(0) R^{i_1} h(1)$, $i_1 > i_2$.

Introduce the binary relation S^n, $0 \leq n \leq N$ as follows:

$uS^n v$ iff there exists a chain:

$u Q_1 u_1 Q_2 u_2 \ldots Q_s u_s R^m v$,

$m \geq n$

for any $i \leq s$ $Q_i = R^k$ for $k \geq n$ or $\tilde{R}^k$ for $k > n$

$x \tilde{R}^k y :\Leftrightarrow y R^k x$.

<u>Note</u>. 1. If $u_s R^n v$ (i.e. m=n), then $uR^n v$.

2. $S^0 \supseteq S^1 \supseteq S^2 \supseteq \ldots \supseteq S^N$; $S^n \supseteq R^n$, $0 \leq n \leq N$.

3. S^n need not be irreflexive. Moreover, if xR^ny then $yS^{n-1}y$ ($n\ge 1$), because we have the chain $y\tilde{R}^n xR^n y$.

Let $s:=\deg(T_N)$, B_n^{PA} denotes the set of all arithmetic sentences PA-equivalent to boolean combinations of Σ_n-sentences.

Lemma 2.

1. $PA \vdash \bigvee_{z\in K} l=z$.
2. $PA \vdash l=z_1 \wedge l=z_2 \rightarrow z_1=z_2$.
3. If uR^nv then $PA \vdash l=u \rightarrow \neg Pr_{T_n}\ulcorner l\ne v\urcorner$.
4. If not uS^nv and $u\ne 0$, $PA \vdash l=u \rightarrow Pr_{T_n}\ulcorner l\ne v\urcorner$.
5. For any $z\in K$ $l=z\in B_s^{PA}$.
6. $l=0$ is true (i.e. $\models l=0$).
7. For any $z\in K$ $PA \nvdash l\ne z$.

Proof.

1,2,3 are entirely routine.

4. We reason in PA.

Let $l=u\ne 0$. Consider the Solovay function from the point of view of T_n. It is clear that T_n knows everything about the jumps of h through R^k for $k\le n$. (In fact, h jumps of h through R^k iff something is provable in T_k and something else is unprovable in T_{k-1}. Both facts can be stated by means of T_n if they are true).

Let e be the last node on which h jumps through R^k for $k\le n$. Thus, T_n knows that the Solovay function went through e:

$$\ldots R^{i_{-1}}h(m-1)R^{i_0}\underset{\substack{\|\\ e}}{h(m)}R^{i_1}h(m+1)\ldots R^{i_l}u \qquad \ldots i_{-1}\le i_0\le n<i_1\le i_2\le\ldots\le i_l$$

$e\ne 0$, because $i_1>0$.

(Perhaps $e=u$).

By the definition of h, $T_{i_0}\vdash l\ne e$. So, T_n knows that $l\ne e$.

The function h goes from e through R^{i_1}, where $i_1>n$. Thus, it cannot go from e through R^k for $k<n$, and obviously T_n knows it. So, from T_n's point of view all the jumps of the Solovay function after e (they must exist!) proceed only through R^k for $k\ge n$. Now it is clear that from T_n's point of view , if $l=v$, then uS^nv. QED.

5. It is sufficient to prove that "$h(m)=z$"$\in\Sigma_s^{PA}$ for any $z\in K$.

By the definition of h (PA proves that)

$h(m)=z \Leftrightarrow$ there exists a chain $z_0=0, z_1, \ldots, z_m=z$ s.t. for any $0<i\leq m$ $\bigvee_{k: z_{i-1} R^k z_i} \{ \exists y (Prf_{T_k}(y, \ulcorner 1\neq z_i \urcorner) \wedge \forall y_1<y \ \forall z': z_{i-1} R^k z' \ \neg Prf_{T_k}(y, \ulcorner 1\neq z' \urcorner)) \wedge \forall j<k \ \forall z': z_{i-1} R^j z' \ \neg Pr_{T_j} \ulcorner 1\neq z' \urcorner \}$.

6. Assume for some $z\neq 0$ $1=z$ is true. Then for some $e\leq N$ $T_e \vdash 1\neq z$, then $T_N \vdash 1\neq z$. By the previous property, $1\neq z \in B_s^{PA} \subseteq \Pi_{s+1}^{PA}$, hence $\models 1\neq z$. This is a contradiction.

7. Assume $PA \vdash 1\neq z$, $z\neq o$. Then $T_0 \vdash 1\neq z$ and $0R^0 z$, i.e. $1=0$ cannot be true.

This proves lemma 2.

Lemma 3. Consider the Kripke model $<K, S^0, S^1, \ldots, S^N, \Vdash'>$, where $\Vdash'$ coincides with $\Vdash$ on all sentence letters. Then $\Vdash$ and $\Vdash'$ coincide on every subformula of ϕ.

Proof. The only interesting case is

$$x \Vdash [k]\psi \Rightarrow x \Vdash' [k]\psi.$$

(The inverse statement is evident because $R^n \subseteq S^n$).

Assume $x \nVdash' [k]\psi$. Then there exists a chain:

$$xQ_1 x_1 Q_2 x_2 \ldots Q_n x_n R^m y, \quad m\geq k, \quad y \nVdash \psi.$$

By ϕ-completeness, $x_n \Vdash [k]\psi \rightarrow [m]\psi$, hence $x_n \nVdash [k]\psi$. Thus, there exists y' s.t. $y' \nVdash \psi$, $x_n R^k y'$. It is clear that $xR^k y'$, hence $x \nVdash [k]\psi$. This proves lemma 3.

Define now an arithmetical interpretation f as follows:

$$f(p) := \exists z (1=z \wedge z \Vdash p).$$

Lemma 4. Let $z\neq 0$. Then for any $\psi \subseteq \phi$

$z \Vdash \psi \Rightarrow PA \vdash 1=z \rightarrow f(\psi)$

$z \nVdash \psi \Rightarrow PA \vdash 1=z \rightarrow \neg f(\psi)$

Proof. As usual, the only interesting case is $\psi = [n]\theta$.

1. Let $z \Vdash [n]\theta$. By lemma 3, for any u $zS^n u \Rightarrow u \Vdash \theta$, hence (using induction hypothesis)

$$PA \vdash (1=u \rightarrow zS^n u) \rightarrow f(\theta).$$

Using the definition of the theory, we have

$$PA \vdash Pr_{T_n} \ulcorner 1=u \rightarrow zS^n u \urcorner \rightarrow Pr_{T_n} \ulcorner f(\theta) \urcorner.$$

By claim 4 of lemma 2, ($z\neq 0$),

$$PA \vdash 1{=}z \rightarrow Pr_{T_n}\ulcorner 1{=}u \rightarrow zS^n u\urcorner .$$

So,

$$PA \vdash 1{=}z \rightarrow Pr_{T_n}\ulcorner f(\theta)\urcorner .$$

2. Let $z \not\Vdash [n]\theta$. Then there exists u such that $zR^n u$, $u \not\Vdash \theta$. By induction hypothesis,

$$PA \vdash 1{=}u \rightarrow \neg f(\theta),$$

hence,

$$PA \vdash Pr_{T_n}\ulcorner f(\theta)\urcorner \rightarrow Pr_{T_n}\ulcorner 1{\neq}u\urcorner$$

$$PA \vdash \neg Pr_{T_n}\ulcorner 1{\neq}u\urcorner \rightarrow \neg Pr_{T_n}\ulcorner f(\theta)\urcorner .$$

By claim 3 of lemma 2,

$$PA \vdash 1{=}z \rightarrow \neg Pr_{T_n}\ulcorner 1{\neq}u\urcorner .$$

So,

$$PA \vdash 1{=}z \rightarrow \neg f([n]\theta).$$

Lemma 4 is thus proved.

Now it is easy to show that $PA \nvdash f(\phi)$. Suppose not. We know that there exists $x \in K$, $x \neq 0$ such that $x \not\Vdash \phi$. By lemma 4,

$$PA \vdash 1{=}x \rightarrow \neg f(\phi).$$

Since $PA \vdash f(\phi)$, $PA \vdash 1 \neq x$. This contradicts claim 7 of lemma 2. QED.

Theorems 1,7,11 are proved. Moreover, theorem 5 implies theorem 3, because $GLP \supset LN$ and

$$GLP \vdash \Delta p \leftrightarrow \Box p, \quad GLP \vdash \Delta^{+} p \leftrightarrow \boxdot p.$$

§5. Arithmetical Completeness of GLP^{ω}.

We will prove theorems 2, 9a,c together in the following formulation:

<u>Theorem</u>. Let $T_0, T_1, \ldots, T_N$ be an increasing sequence of correct theories. Then for any modal formula ϕ which does not contain [N] the following statements are equivalent:

1) $GLP^{\omega} \vdash \phi$.
2) $GLP \vdash H(\phi) \rightarrow \phi$.
3) $GLP \vdash [N]\phi$.
4) for any interpretation f $\models f(\phi)$.

<u>Proof</u>.

1) $\Rightarrow$ 4) is trivial.

(Use the assumption that the theories $T_0,\ldots,T_N$ are correct).

2) $\Rightarrow$ 1) is trivial: $GLP^{\omega} \vdash H(\phi)$ for any ϕ.

3) $\Rightarrow$ 2). Assume that $GLP \nvdash H(\phi) \rightarrow \phi$. By theorem 11, there exists $(H(\phi) \rightarrow \phi)$-complete LN-model $\mathcal{K}$ and $x \in K$ such that $x \Vdash H(\phi) \wedge \neg\phi$. Adjoin to model $\mathcal{K}$ a bottom node h such that $hR^N x$. (Note that $\mathcal{K}$ does not contain R^N!).

We claim that the obtained model $\mathcal{K}_1$ is $[N]\phi$-complete. It is sufficient to show that

$$\forall\psi\colon [n]\psi \subseteq \phi \qquad h \Vdash [n]\psi \rightarrow [N]\psi.$$

Let $h \nVdash [N]\psi$. Then $x \nVdash \psi$; but $x \Vdash [n]\psi \rightarrow \psi$ (because $x \Vdash H(\phi)$), hence $x \nVdash [n]\psi$. It is clear that $h \nVdash [n]\psi$ also.

So, we obtain a $[N]\phi$-complete LN-model $\mathcal{K}_1$ in which $[N]\phi$ does not hold. ($h \nVdash [N]\phi$). Thus, $GLP \nvdash [N]\phi$.

4) $\Rightarrow$ 3). Fix a $[N]\phi$-complete LN-model $\mathcal{K}$ such that for some $x \in K$ $x \nVdash [N]\phi$, i.e. there is w such that $xR^N w$ and $w \nVdash \phi$.

Define a Solovay function h analogously to that defined in §4. Because 0 is not defined here, it must be replaced by w in each case (e.g. h(0)=w). The proof is to be modified as follows:

1. Remove all the restriction to u from claim 4 of lemma 2 and add the restriction n<N. In the proof of the lemma 2 it is necessary to deal also with the case e=w, using the existence of the chain $w\tilde{R}^N x R^N w$.

2. Claims 5,7 of lemma 2 are to be removed; claim 6 ($\vDash$ 1=w is to be proved using the correctness of theories $T_0, T_1, \ldots, T_N$.

3. In the statement of lemma 4 $z \neq w$ is to be removed. In the proof of lemma 4 one must use the fact that ϕ does not contain [N] and therefore the restriction n<N of claim 4 of lemma 2 is irrelevant.

4. To prove lemma 4 it is sufficient to note that $PA \vdash 1=w \rightarrow \neg f(\phi)$, 1=w is true and hence $f(\phi)$ is false. QED.

The proof of theorems 2, 9a,c is complete.

<u>Proof of theorem 9b</u>.

$GLP \vdash \phi \Rightarrow GLP \vdash \Box\phi$ by the definition of GLP.

$GLP \vdash \Box\phi \Rightarrow GLP^{\omega} \vdash \Box\phi$ by the definition of GLP^{ω}.

$GLP^{\omega} \vdash \Box\phi \Longrightarrow GLP \vdash \phi$.

Let $T_0, T_1, \ldots, T_n$ be an arbitrary increasing sequence of correct theories such that $T_0 = PA$ (see e.g. examples 1,2 after definition 2). Assume $GLP \nvdash \phi$. Then there exists an interpretation f such that $PA \nvdash f(\phi)$, i.e. $\models \neg f(\Box\phi)$. Thus $GLP^{\omega} \vdash \Box\phi$ is impossible.

<u>Proof of theorem 4</u>.

First consider Craig's property for GLP. Let $GLP \vdash A \longrightarrow C$. By theorem 7,

$$LN \vdash M(A \longrightarrow C) \longrightarrow (A \longrightarrow C).$$

it is clear that $M(A \longrightarrow C) = M(A) \wedge M(B)$; so,

$$LN \vdash M(A) \wedge A \longrightarrow (M(C) \longrightarrow C).$$

By theorem 6, there exists a formula B, containing only common sentence letters A and C such that

$$LN \vdash M(A) \wedge A \longrightarrow B \quad \text{and} \quad LN \vdash B \longrightarrow (M(C) \longrightarrow C).$$

But $GLP \vdash M(\phi)$ for any ϕ, hence

$$GLP \vdash A \longrightarrow B \quad \text{and} \quad GLP \vdash B \longrightarrow C,$$

i.e. Craig's interpolation property is proved.

Craig's property for GLP^{ω} can be proved in the same way by using Craig's property for GLP and theorem 9a.(Note that $GLP^{\omega} \vdash \phi \Leftrightarrow LN \vdash M(\phi) \wedge H(\phi) \longrightarrow \phi$).

§6. Provability Semantic for LN.

<u>Proof of theorem 8</u>.

$LN \vdash \phi \Longrightarrow LN \vdash \phi^*$.

Suppose $LN \nvdash \phi^*$ and $\mathcal{K}$ be an LN-model such that for some $x \in K$ $x \nVdash \phi^*$. Without loss of generality we can assume that $\mathcal{K} = \tilde{W}_x$ and $\mathcal{K}$ is simple. Define the function π: $W_x \longrightarrow \omega$ as follows: for any u if there is y s.t. $yR^n u$ then $\pi(u) = n$ (by the definition of simple LN-model, this definition is correct). Consider the set

$$K_1 := \{z \in W_x \mid z \Vdash p_{\pi(z)}\} \cup \{x\}$$

as a submodel of $\mathcal{K}$.

<u>Lemma</u>. Let ψ not to contain $p_0, p_1, \ldots, p_N$. Then for every $z \in K_1$

$$z \Vdash_{\mathcal{K}_1} \psi \Leftrightarrow z \Vdash_{\mathcal{K}} \psi^*.$$

<u>Proof</u>. As usual, let $\psi = [n]\theta$; then $\psi^* = [n](p_n \longrightarrow \theta^*)$.

"$\Longrightarrow$".

Assume $z \Vdash\!\!\!\!/_{\mathcal{K}} [n](p_n \rightarrow \theta^*)$. Hence, for some $y \in K$ $y \Vdash_{\mathcal{K}} p_n \wedge \neg\theta^*$, $zR^n y$. By the definition of π and $\mathcal{K}_1$, $\pi(y)=n$ and $y \in K_1$. By induction hypothesis, $y \Vdash\!\!\!\!/_{\mathcal{K}_1} \theta$. Thus $z \Vdash\!\!\!\!/_{\mathcal{K}_1} [n]\theta$.

"$\Longleftarrow$".

Assume $z \Vdash\!\!\!\!/_{\mathcal{K}_1} [n]\theta$, hence for some $y \in K_1$ $y \Vdash\!\!\!\!/_{\mathcal{K}_1} \theta$, $zR^n y$ (note that $y \neq x$!). By the definition of K_1, $y \Vdash_{\mathcal{K}} p_n$. By the induction hypothesis, $y \Vdash\!\!\!\!/_{\mathcal{K}} \theta^*$. Hence, $y \Vdash\!\!\!\!/_{\mathcal{K}} p_n \rightarrow \theta^*$. Thus, $z \Vdash\!\!\!\!/_{\mathcal{K}} [n](p_n \rightarrow \theta^*)$.

So, since $x \Vdash\!\!\!\!/_{\mathcal{K}} \phi^*$, $x \Vdash\!\!\!\!/_{\mathcal{K}_1} \phi$, hence $LN \nvdash \phi$.

$GLP \vdash \phi^* \Longrightarrow LN \vdash \phi$.

Assume that $LN \nvdash \phi$ and $\mathcal{K}$ is a simple LN-model such that $\mathcal{K}=\tilde{W}_x$, $x \Vdash\!\!\!\!/ \phi$. Using the function π defined earlier define the forcing of sentence letters $p_0, p_1, \ldots, p_N$ in W_x as follows:

$$y \Vdash p_n \;:\Leftrightarrow\; \pi(y)=n.$$

It is clear now that if $n \neq k$ then

(*) $$\mathcal{K} \Vdash [n]\neg p_k.$$

Using the previous lemma, one can see that $x \Vdash\!\!\!\!/ \phi^*$. We need to prove that $\mathcal{K}$ is ϕ^*-complete. Let $[n]\psi \subseteq \phi^*$. Clearly, ϕ has the form: $\psi = p_n \rightarrow \theta$; but if $k>n$ then by (*) $\mathcal{K} \Vdash [n]\psi \rightarrow [k]\psi$; by theorem 11, this implies statement of the theorem 8.

<u>Theorem</u>. There exists a sequence of theories $T_0, T_1, \ldots, T_N$ such that for any modal formula ϕ $LN \vdash \phi$ iff for any interpretation f $PA \vdash f(\phi)$.

<u>Proof</u>. Fix an arbitrary increasing sequence $T_0, T_1, \ldots, T_N$. Using a well-known trick due to Montagna (cf.[4],[5],[6],[7]), it is easy to show that there exists a "uniform" interpretation f such that

$$GLP \vdash \phi \Leftrightarrow PA \vdash f(\phi).$$

(We use: decidability of GLP, r.e. of PA, the effectiveness of the construction of interpretation f ("counter-interpretation") from §4 and its bounded complexity). Now it is clear that $f(\phi^*)$ differ from $f(\phi)$ only in the substitution of $T_0+f(p_0)$, $T_1+f(p_1)$,..., $T_N+f(p_N)$ for $T_0,T_1,...,T_N$. So, theories $T_0+f(p_0)$, $T_1+f(p_1)$,..., $T_N+f(p_N)$ are sufficient for own goals, and the theorem is proved.

(Note that $f(p_0),f(p_1),...,f(p_N)\in\Delta^{PA}_{n+1}$, where $n=deg(T_N)$).

Note. This is the only place in this paper where the finiteness the modal language is essential. We cannot answer the question whether the theorem could be generalized to the infinite case, for the uniform version of the arithmetical completeness theorem for GLP fails ([2]).

REFERENCES:

1.Boolos G. "ω-consistency and the diamonds", Studia Logica, v.39, 1980, 237-243.

2.Dzhaparidze G."Модально-логические средства исследования доказуемости", Ph. D. Thesis Dissertation, Moscow, MSU, 1986 /Russian.

3.Gleit Z., Warren G. "Characters and fixed point in provability logic",Notre Dame Jour. of Form. Logic.

4.Montagna F. "On diagonalizable algebra of Peano arithmetic", Boll. della Unione Math. Ital.,1979, 66-B, pp.795-812.

5.Artemov S.,"Extensions of arithmetics and connected with them modal theories", sixth LMPS Congress, Hannover, 1979, Abstracts, Sections 1-4, pp.15-19.

6.Visser A.,"Aspects of diagonalization and provability", Ph. D. Thesis, Utrecht, 1981.

7.Boolos G.,"Extremely undecidable sentences", J. Symbolic Logic, 47(1982), 191-196.

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