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UNDECIDABILITY OF THE DISJUNCTION PROPERTY OF INTERMEDIATE PROPOSITIONAL LOGICS

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UNDECIDABILITY OF THE DISJUNCTION PROPERTY OF INTERMEDIATE PROPOSITIONAL LOGICS

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UNDECIDABILITY OF THE DISJUNCTION PROPERTY OF PROPOSITIONAL LOGICS AND OTHER RELATED PROBLEMS

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'How can we recognize, given axioms and inference rules of a calculus, whether it has such-and-such property?', a question of that kind arises whenever we deal with a new logic system. For large families of logics, this question may be considered as an algorithmic problem, and a property is called *decidable* if there exists an algorithm which is capable of deciding, for a finite axiomatics of a calculus (in a given family), whether or not it has the property.

In the class of intermediate propositional logics, for instance, decidable are such non-trivial properties as the tabularity, pretabularity and interpolation property (Maksimova [1972, 1977]). However, for many other important properties - the decidability, finite model property, disjunction property, Halldén-completeness, etc. - in spite of considerable efforts effective criteria were not found.

In this paper we show that difficulties in investigating these properties in the classes of intermediate logics and normal modal logics containing **S4** are of principal nature, since all of them turn out to be algorithmically undecidable. In other words, there are no algorithms which can recognize, given a finite set of axioms of an intermediate or modal calculus, whether or not it is decidable, Halldén-complete, has the finite model or disjunction property.

The first results concerning the undecidability of properties of calculi seems to be obtained by Linial and Post [1949] who proved the undecidability of the problem of equivalence to the classical calculus in the class of *all* propositional calculi with the same language as the classical one and two inference rules: modus ponens and substitution. Kuznetsov [1963] generalized this result having proved the undecidability of the problem of equivalence to any fixed intermediate calculus (for instance, to the intuitionistic calculus or even inconsistent). However, these results will not hold if we confine ourselves only to the class of intermediate logics, though the problem of equivalence to the undecidable intermediate calculus of Shekhtman [1978] is clearly undecidable in this class as well.

Thomason [1982] proved the undecidability of Kripke completeness in the class of all normal modal logics. Chagrova [1990, 1992] established the undecidability of the problem of first-order definability of intuitionistic formulas. We will use her method of simulating the Minsky machine behavior for obtaining our undecidability results. Chagrov [1991] proposed a general scheme for proving the undecidability of properties of calculi with the help of which he established the undecidability of many properties in the classes of normal and arbitrary extensions of the Gödel-Löb provability logic **GL**. We will take advantage of this scheme below too.

The examples of decidable properties above show that from the algorithmic point of view there is a fundamental difference between properties of calculi and functional properties of enumerations of computable functions: the undecidability of the latter is provided actually only by their non-triviality whereas for proving the undecidability of some property, say, of modal calculi it is necessary to construct rather complicated calculi with and without the property, i.e. we have to know something non-trivial about the property itself.[1]

In this respect, there are some problems concerning the disjunction property and Halldén-completeness which are studied much worse than the finite model property or decidability. We hope this paper will help to improve the situation to some extent: here we not only construct concrete logics (used for proving the undecidability) which are Halldén-complete and have the disjunction property, but prove a few syntactical sufficient conditions for these properties. We also obtain a number of results characterizing the relationship between the disjunction property and Halldén-completeness of intermediate and modal logics.

Our study of the disjunction property and Halldén-completeness is based essentially on the canonical formulas which were introduced by Zakharyashchev [1983, 1984, 1988, 1989]. Special cases of the canonical formulas are the subframe formulas of Fine [1985] and the frame (or Jankov-Fine) formulas (see Jankov [1963], Fine [1974]). However, in contrast to these two kinds of formulas the canonical ones can axiomatize all intermediate logics and all modal logics containing **S4**.

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[1] However, Rice's Theorem holds for properties of *recursively axiomatizable logics*: each non-trivial property of such logics is undecidable. This fact was discovered by A.V. Kuznetsov; we are grateful to L.L. Maksimova for this information.

The plan of the paper is as follows. In §0 we collect all necessary preliminaries. §1 describes a general scheme for proving the undecidability of properties of calculi. It is partially realized in §2, where the behavior of Minsky machines is simulated by means of modal logics. In §3 we use this scheme for proving the undecidability of the finite model property and decidability of normal modal calculi containing the Grzegorczyk system **S4Grz**. §4 and §6 are brief introductions to the canonical formulas of Zakharyashchev [1988, 1989] which are applied in §5 and §7 for obtaining syntactical sufficient conditions for the disjunction property of intermediate logics and Halldén-completeness of normal extensions of **S4Grz**. Moreover, in §7 we show that unlike the disjunction property, Halldén-completeness need not be inherited by any modal companion (with respect to the Gödel translation) of an intermediate logic and that there are a continuum of logics containing **S4Grz** with each of the four possible combinations of the disjunction property, Halldén-completeness and the negations of these properties (a proof of the continuality of the class of Halldén-incomplete modal logics with the disjunction property can be found in Chagrov [1991a] and so is omitted here). In §8 we prove the undecidability of the disjunction property, Halldén-completeness and some other properties of intermediate and modal calculi. In §9 we consider another property concerning the disjunction. It is so called Maksimova-completeness (or the variable separation principle) which was introduced by Maksimova [1976, 1979] for relevant and intermediate logics. A few open problems are discussed in §10.

§0. **Preliminaries.** In this paper we deal with two kinds of propositional logics: intermediate logics and normal extensions of the Lewis modal system **S4**.

An *intermediate logic* L is a set of formulas (constructed from propositional variables p, q, r, ... and the constant $\bot$ (falsehood) by means of the connectives &, $\vee$, and $\supset$) containing all the axioms of the intuitionistic propositional logic **Int** and closed under modus ponens and substitution. (Usually, L is also required to be contained in the classical propositional logic **Cl** which justifies the term "intermediate logic".[2]) Similarly, a *normal extension* of **S4** is a set of modal formulas (differing from the intuitionistic ones only by the unary connective $\Box$ - "necessarily") containing all the axioms of **S4** (i.e. the axioms of Cl and the formulas $\Box(p\supset q)\supset(\Box p\supset\Box q)$, $\Box p\supset\Box\Box p$ and $\Box p\supset p$ as well) and closed under modus ponens, substitution and necessitation $A/\Box A$. As usual, $\neg A$ is the abbreviation for $A\supset\bot$, $\Diamond A$ ("possibly" A) is the abbreviation for $\neg\Box\neg A$ and $\top = \bot \supset \bot$.

Each normal modal logic M containing **S4** can be represented as the closure (under the inference rules) of some set of formulas Γ which is added to the axioms of **S4**. In this case we write

M = **S4** + Γ.

If Γ is a finite set, say, $\Gamma = \{A_1, \dots, A_n\}$, then we write

M = **S4** + $A_1 + \dots + A_n$

and call M a *calculus*. (It would be more exact to say the logic M is *given* by a calculus, i.e. by the axioms of **S4**, the additional

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[2]In the USSR more preferable is the term "*superintuitionistic logic*" covering the inconsistent logic too.

axioms $A_1, \ldots, A_n$ and the inference rules mentioned above. Note by the way that the same finitely axiomatizable logic can be given by infinitely many calculi and that the equivalence problem for logics represented by calculi is undecidable, as it follows from Shekhtman [1978].) In this sense, a calculus is, for instance, the Grzegorczyk logic (see Segerberg [1971])

$$\mathbf{S4Grz} = \mathbf{S4} + \Box(\Box(p\supset\Box p)\supset p)\supset p),$$

extensions of which we shall often deal with in what follows.

Similar notation will be used for intermediate logics and calculi.

An intermediate logic L is said to have *the disjunction property* (DP) if $L \vdash A\vee B$ implies $L \vdash A$ or $L \vdash B$, for any formulas A and B. A modal logic M has *the (modal) disjunction property* if, for any modal formulas A and B, $M \vdash \Box A\vee\Box B$ implies $M \vdash \Box A$ or $M \vdash \Box B$. A (modal or intermediate) logic L is called *Halldén-complete* (HC) if, for any A and B having no variables in common, $L \vdash A\vee B$ implies $L \vdash A$ or $L \vdash B$.

As is known (see Rasiowa and Sikorski [1963]), each normal extension of **S4** is determined by a suitable class of topological Boolean (or interior) algebras. Relational representations of these algebras, viz. general frames for **S4**, will be our main semantic tools.

Remind that a *general frame* (or simply *frame*) for **S4** (see, for instance, Goldblatt [1976]) is a triple $\mathfrak{F} = \langle W,R,S\rangle$, where W is a non-empty set (of worlds), R is a reflexive and transitive relation (of accessibility) on W and S is a set of subsets of W which contains $\emptyset$ and is closed under the set-theoretic operations $\cap$, $\cup$, $-$ and the (interior) operation $\mathbf{I}$:

$$\mathbf{I}V = \{a\in W \mid \forall b\in W\ (aRb \Rightarrow b\in V)\},\quad \text{for all } V\subseteq W.$$

The ordinary Kripke frames for **S4**, considered in the context of general frames, have the form $\mathfrak{F} = \langle W, R, 2^W \rangle$. However, we will keep the conventional notation and write $\mathfrak{F} = \langle W, R \rangle$ instead of $\mathfrak{F} = \langle W, R, 2^W \rangle$.

Valuations on a frame $\mathfrak{F} = \langle W, R, S \rangle$ are defined in the ordinary way (the truth set of every variable must be contained in S, of course). As usual, $a \models A$ means that (under a given valuation on $\mathfrak{F}$) A is *true* at the world $a \in W$. If A is *valid* in $\mathfrak{F}$, i.e. A is true at all the worlds in $\mathfrak{F}$ under every valuation, then we write $\mathfrak{F} \models A$. $\mathfrak{F}$ is a *frame for a logic* M (notation: $\mathfrak{F} \models M$) if every formula in M is valid in $\mathfrak{F}$. M is *determined* (or *characterized*) by a class $\mathscr{C}$ of frames if

$$M = \{A \mid \forall \mathfrak{F} \in \mathscr{C}\ \mathfrak{F} \models A\}.$$

In the case when $\mathscr{C}$ consists of only Kripke frames or finite frames, M is said to be *Kripke complete* or have the *finite model property* (FMP), respectively. For example, **S4Grz** is determined by the class of all finite partially ordered frames (Segerberg [1971]).

The equality in the previous paragraph may be used also as a method for defining logics: for a class $\mathscr{C}$ of frames, the set of formulas which are valid in all frames in $\mathscr{C}$ is a logic, and we call it the *logic of* $\mathscr{C}$.

We use the following notation. For a frame $\mathfrak{F} = \langle W, R, S \rangle$, $U, V \subseteq W$ and $a \in W$, let

$$V\uparrow = \{a \in W \mid \exists b \in V\ bRa\}, \quad a\uparrow = \{a\}\uparrow,$$

$$V\downarrow = \{a \in W \mid \exists b \in V\ aRb\}, \quad a\downarrow = \{a\}\downarrow,$$

$$U \to V = \mathbf{I}(-U \cup V).$$

$\mathfrak{F}$ is called *sharp* if there is a point $a \in W$ such that $a\uparrow = W$; in this case we say a is the *origin* of $\mathfrak{F}$.

The relational counterparts of homomorphic images, direct products and subalgebras of topological Boolean algebras are generated subframes, disjoint unions and reductions (or p-morphic images), respectively. A general frame $\mathfrak{F}_1 = \langle W_1, R_1, S_1\rangle$ is a *generated subframe* of $\mathfrak{F} = \langle W,R,S\rangle$ if W_1 is an upwards closed subset of W (i.e. $W_1 = W_1\uparrow$ in $\mathfrak{F}$), R_1 is the restriction of R to W_1 and $S_1 = \{V\cap W_1 \mid V\in S\}$. The *disjoint union* $\mathfrak{F} = \mathfrak{F}_1 + \mathfrak{F}_2$ of frames $\mathfrak{F}_1 = \langle W_1, R_1, S_1\rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2, S_2\rangle$ with disjoint W_1 and W_2 is the frame $\mathfrak{F} = \langle W,R,S\rangle$, where $W = W_1\cup W_2$, $R = R_1\cup R_2$ and $S = \{V_1\cup V_2 \mid V_1\in S_1, V_2\in S_2\}$. Finally, a mapping f from W_1 onto W is a *reduction* (or *p-morphism*) from $\mathfrak{F}_1 = \langle W_1, R_1, S_1\rangle$ onto $\mathfrak{F} = \langle W,R,S\rangle$ if

$aR_1 b \Rightarrow f(a)Rf(b)$,

$cRd \Rightarrow \forall a\in f^{-1}(c)\ \exists b\in f^{-1}(d)\ aR_1 b$,

$\forall V\in S\ f^{-1}(V)\in S_1$.

If f is a partial (i.e. not completely defined, in general) mapping from W_1 onto W satisfying these three conditions (the first one must hold only for $a,b\in f^{-1}(W)$, of course) then we call f a *subreduction* (Fine [1985]) or a *partial p-morphism* (Zakharyashchev [1984]).

The relational semantics for intermediate logics can easily be derived from the relational semantics for modal logics if we recall that each intermediate logic is determined by a suitable class of pseudo-Boolean (or Heyting) algebras and that each pseudo-Boolean algebra is the algebra of open elements of some topological Boolean algebra (see Rasiowa and Sikorski [1963]).

A general frame for **Int** is a triple $\mathfrak{F} = \langle W,R,S\rangle$, where R is a partial ordering on W and S is a collection of upwards closed subsets of W which contains $\emptyset$ and is closed under $\cup$, $\cap$ and the operation $\rightarrow$ defined above. If S contains all upwards closed subsets of W then $\mathfrak{F} = \langle W,R,S\rangle$ is in effect the ordinary Kripke frame for **Int**; in this case we use a more simple notation: $\mathfrak{F} = \langle W,R\rangle$.

The definitions of truth, validity, generated subframe, disjoint union and reduction remain the same as for the modal general frames and in the definition of subreduction only the last of the three conditions is changed: it is replaced by

$\forall V\in S\ -(f^{-1}(-V)\downarrow)\in S_1$

(note that $\downarrow$ is the closure operation which is dual to the interior operation I).

The relationship between algebras and general frames mentioned above guarantees that each modal or intermediate logic is determined by an appropriate class of general frames. Moreover, by the Generation Theorem of Segerberg [1971], we may use only sharp general frames.

We say that a set $V\subseteq W$ is a *cover for* the frame $\mathfrak{F} = \langle W,R,S\rangle$ if $W = V\downarrow$.

The following Proposition (which will be used in §5 and §7) is proved only for intermediate logics though it can be readily generalized to the modal case too.

PROPOSITION. *Each intermediate logic L is determined by some class of general frames having finite covers.*

Proof. We show that L is determined by the class of all the general frames for L having finite covers.

Let $L \nvdash A$. Then there is a frame $\mathfrak{F} = \langle W,R,S\rangle$ for L such that $\mathfrak{F} \nvDash A$. Fix some valuation under which A is refuted in $\mathfrak{F}$. Let $p_1,\dots,p_n$ be all the variables in A and let

$V_a = \{p_i \mid a\models p_i,\ 1\le i\le n\}$, for any $a\in W$.

We say a is *stable* (in $\mathfrak{F}$) if $V_a = V_b$, for all $b\in a\uparrow$. Since aRb

implies $V_a \subseteq V_b$, there exists a stable $a \in c^{\uparrow}$, for each $c \in W$.

We define an equivalence relation $\equiv$ on W by taking $a \equiv b$ iff either a and b are stable with $V_a = V_b$ or $a = b$. Consider the quotient frame $\langle W_1, R_1 \rangle$, where $W_1 = W/_{\equiv}$ and $a/_{\equiv} \; R_1 \; b/_{\equiv}$ iff $a/_{\equiv} \subseteq b/_{\equiv}{\downarrow}$. It is clear that this frame has a finite cover (the different equivalence classes generated by stable elements in W) and the canonical mapping $f : W \to W/_{\equiv}$ is a reduction from $\langle W, R \rangle$ onto $\langle W_1, R_1 \rangle$.

Define a truth-relation $\models$ on $\langle W_1, R_1 \rangle$ by taking

$a/_{\equiv} \models p_i$ iff $a \models p_i$, for $i = 1, \ldots, n$.

According to the well-known P-morphism Theorem of Segerberg [1971],

$a \models B$ iff $f(a) \models B$,

for each formula B containing only the variables $p_1, \ldots, p_n$. Let S_1 be generated (as a pseudo-Boolean algebra, i.e. by operations $\cup$, $\cap$, $\to$ and $\emptyset$) by the upwards closed sets

$\{a/_{\equiv} \mid a/_{\equiv} \models p_i\}$, for $i = 1, \ldots, n$,

that is $V \in S_1$ iff $\{a/_{\equiv} \mid a/_{\equiv} \models B\}$, for some formula B constructed from $p_1, \ldots, p_n$. Then $f^{-1}(V) \in S$, for every $V \in S_1$. Therefore f is a reduction from $\mathfrak{F}$ onto $\mathfrak{F}_1 = \langle W_1, R_1, S_1 \rangle$, and so, by the P-morphism Theorem, $\mathfrak{F}_1 \not\models A$ and $\mathfrak{F}_1 \models L$. ■

To prove the undecidability of a property of calculi we will simulate the behavior of Minsky machines by means of modal logics.

A *Minsky machine* (see Minsky [1961]) has two left-bounded tapes, the machine heads (one on each tape) write or erase nothing and information on a tape is the number of cells to the left of the head.

A program for a Minsky machine is a finite set of instructions of the form:

$$q_\alpha \to q_\beta\, T_0 T_1, \quad q_\alpha \to q_\beta\, T_1 T_0,$$
$$q_\alpha \to q_\beta\, T_0\, T_{-1}(q_\gamma T_0 T_0), \quad q_\alpha \to q_\beta\, T_{-1} T_0 (q_\gamma T_0 T_0).$$

The last of them, for instance, means: if the machine is in the state q_α and there are cells to the left of the head on the first tape then move this head one cell to the left and then pass to the state q_β; but if the machine is in the state q_α and there are no cells to the left of the head on the first tape then, changing nothing on both tapes, pass to the state q_γ. We shall identify a Minsky machine with its program.

A *configuration* of a Minsky machine is a triple $\mathfrak{a}=(\alpha,m,n)$, where q_α is a state, m is information on the first tape and n is information on the second tape. Notation $P:\mathfrak{a}\to\mathfrak{b}$ means that the Minsky program P passes from the configuration $\mathfrak{a}$ to the configuration $\mathfrak{b}$ by some computation; otherwise we write $P:\mathfrak{a}\not\to\mathfrak{b}$.

According to the Minsky theorem, for any partial recursive function φ there is a program P such that the value $\varphi(x)$ is defined iff $P:(\alpha,2^x,0)\to(\beta,2^{\varphi(x)},0)$, where q_α and q_β are the initial and terminal states, respectively. Thus, the following *configuration problem* is undecidable: for a program P and configurations $\mathfrak{a}$ and $\mathfrak{b}$, determine whether $P:\mathfrak{a}\to\mathfrak{b}$. Moreover, it is not difficult to prove (see Chagrov and Zakharyashchev [1989]) that there are P and $\mathfrak{a}$ for which the *problem of the second configuration* is undecidable, i.e. there is no algorithm that is capable of deciding, given a configuration $\mathfrak{b}$, whether $P:\mathfrak{a}\to\mathfrak{b}$.

§1. **A general scheme for proving the undecidability of properties of calculi.** Suppose we deal with extensions of a logic L_o and are facing the following problem: is it possible to determine, given a formula A, whether the logic $L_o + A$ has some property $\mathcal{P}$. Probably at first we will enthusiastically try to construct an algorithm for recognizing $\mathcal{P}$, and only after exhausting and futile efforts to do this the brilliant idea will strike us: what if the property $\mathcal{P}$ is algorithmically undecidable? We will take then some undecidable problem, say, the configuration problem for Minsky machines or the problem of the second configuration, and try to reduce it to the problem of recognizing $\mathcal{P}$. Here is one of possible schemes for such a reduction.

First we construct a formula F such that

(1) $L_o + F$ has the property $\mathcal{P}$.

Then with a Minsky program P and configurations $\mathfrak{a}$ and $\mathfrak{b}$ we associate formulas AxP and $C(\mathfrak{a},\mathfrak{b})$ satisfying

(2) $L_o + AxP \vdash C(\mathfrak{a},\mathfrak{b})$ *iff* $P:\mathfrak{a}\to\mathfrak{b}$.

In addition, F and AxP are chosen so that

(3) $L_o + F \vdash AxP$.

Now, consider the calculus

$L(P,\mathfrak{a},\mathfrak{b}) = L_o + AxP + C(\mathfrak{a},\mathfrak{b}) \supset F + G,$

where G is some formula which also satisfies

(4) $L_o + F \vdash G$.

If $P:\mathfrak{a}\to\mathfrak{b}$ then, by (2) - (4), $L(P,\mathfrak{a},\mathfrak{b}) = L_o + F$, and so, by (1), $L(P,\mathfrak{a},\mathfrak{b})$ has $\mathcal{P}$; but if $P:\mathfrak{a}\not\to\mathfrak{b}$ then the fact that $L(P,\mathfrak{a},\mathfrak{b})$ does not have $\mathcal{P}$ must be ensured by an appropriate choice of G.

If we succeed in realizing this plan then we shall obtain

(5) $L(P,\mathfrak{a},\mathfrak{b})$ has $\mathcal{P}$ *iff* $P:\mathfrak{a}\to\mathfrak{b}$.

Since $L(P,\mathfrak{a},\mathfrak{b})$ is effectively constructed of $\mathcal{P}$, $\mathfrak{a}$ and $\mathfrak{b}$, it follows at once that the property $\mathcal{P}$ is undecidable.

In §2 we will show how to construct the formulas AxP and $C(\mathfrak{a},\mathfrak{b})$ when L_0 is **S4Grz**, and we will find also some semantic characteristics of the formula F.

§2. **Simulation of Minsky machines.** Let P be a Minsky program and $\mathfrak{a} = (\alpha,m,n)$ be a configuration. Beginning our simulation of P which starts with $\mathfrak{a}$, it is useful to keep in mind the frame $\mathfrak{F} = \langle W,R\rangle$ shown in Fig.1. Intuitively, its left vertical stripe consisting of elements with the superscript 1 is intended for representing the states of P, its middle and right stripes whose elements have the superscripts 2 and 3 represent the tapes of P while the elements $t(\beta,k,l)$ represent the configurations $\mathfrak{b} = (\beta,k,l)$ for which $P:\mathfrak{a}\to\mathfrak{b}$ holds.

The formal definition of $\mathfrak{F} = \langle W,R\rangle$ is presented below:

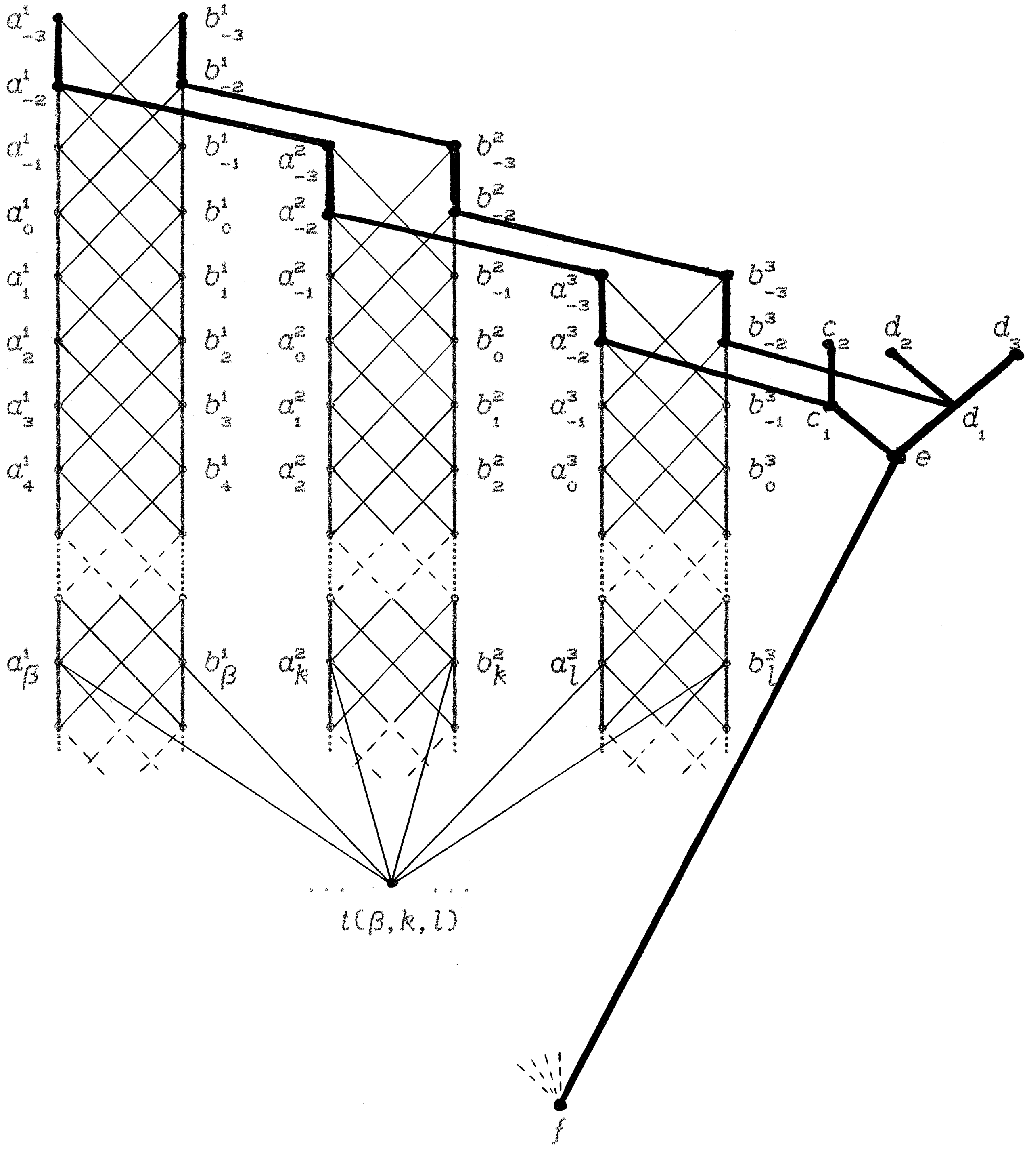
$$W = \{f,\ e,\ c_1,\ c_2,\ d_1,\ d_2,\ d_3\} \cup \{a^i_j,\ b^i_j \mid 1\le i\le 3,\ j\ge -3\} \cup \cup \{t(\beta,k,l) \mid P:(\alpha,m,n)\to(\beta,k,l)\},$$

and R is the transitive and reflexive closure of the following binary relation R':

$$xR'y \text{ iff } \exists i,j,k,l,\beta\ (x=f \vee (x=a^i_j \,\&\, y=a^i_k \,\&\, j\ge k) \vee (x=b^i_j \,\&\, y=b^i_k \,\&\, j\ge k) \vee (x=a^i_j \,\&\, y=b^i_k \,\&\, j\ge k+2) \vee (x=b^i_j \,\&\, y=a^i_k \,\&\, j\ge k+2) \vee (x=a^i_{-3} \,\&\, y=a^{i-1}_{-2}) \vee (x=b^i_{-3} \,\&\, y=b^{i-1}_{-2}) \vee (x=c_1 \,\&\, (y=a^3_{-2} \vee y=c_2)) \vee (x=d_1 \,\&\, (y=b^3_{-2} \vee y=d_2 \vee y=d_3)) \vee (x=e \,\&\, (y=c_1 \vee y=d_1)) \vee (x=t(\beta,k,l) \,\&\, y\in\{a^1_\beta,b^1_\beta,a^2_k,b^2_k,a^3_l,b^3_l\})\ ).$$

By $\mathfrak{F}_0 = \langle W_0,R_0\rangle$ we denote the subframe of $\mathfrak{F}$ whose diagram is shown in Fig. 2 (in Fig. 1 it is depicted by the bold-face lines).

The formula F is constructed so that $\mathfrak{F} \nvDash F$ and there are formulas C_1, D_1, A^i_j, B^i_j ($j=-2,-3$, $i=1,2,3$) for which, under every



a^1_{-3} a^1_{-2} a^1_{-1} a^1_0 a^1_1 a^1_2 a^1_3 a^1_4 a^1_β
b^1_{-3} b^1_{-2} b^1_{-1} b^1_0 b^1_1 b^1_2 b^1_3 b^1_4 b^1_β
a^2_{-3} a^2_{-2} a^2_{-1} a^2_0 a^2_1 a^2_2 a^2_k
b^2_{-3} b^2_{-2} b^2_{-1} b^2_0 b^2_1 b^2_2 b^2_k
a^3_{-3} a^3_{-2} a^3_{-1} a^3_0 a^3_l
b^3_{-3} b^3_{-2} b^3_{-1} b^3_0 b^3_l
c_2 c_1 d_2 d_3 d_1 e f
$t(\beta, k, l)$

Fig.1.

valuation refuting F in $\mathfrak{F}$, the following equalities hold

(6) $\{\alpha \mid \alpha \not\models F\} = \{f\}$,

(7) $\{\alpha \mid \alpha \models C_1\} = \{c_1\}$, $\{\alpha \mid \alpha \models D_1\} = \{d_1\}$

and, for $j=-2,-3$, $i=1,2,3$,

(8) $\{\alpha \mid \alpha \models A^i_j\} = \{a^i_j\}$, $\{\alpha \mid \alpha \models B^i_j\} = \{b^i_j\}$.

The conditions (6) – (8) are satisfied, for instance, by the negation of Fine's subframe formula for $\mathfrak{F}_0$ (see Fine [1985]). However, we should remember that F must also satisfy the condition (1) in §1.

Using A^i_j, B^i_j, for $i=1,2,3$, $j=-2,-3$, C_1 and D_1, we construct formulas A^i_j, B^i_j, for $i=1,2,3$, $j\geq -1$, and $T(\beta, A^2_k, A^3_l)$ so that, for every valuation refuting F in $\mathfrak{F}$, (8) holds for all $j\geq -3$ and

$$(9) \quad \{\alpha \mid \alpha \models T(\beta, A^2_k, A^3_l)\} = \begin{cases} \{t(\beta,k,l)\} & \text{if } P\colon (\alpha,m,n) \to (\beta,k,l), \\ \emptyset & \text{if } P\colon (\alpha,m,n) \not\to (\beta,k,l). \end{cases}$$

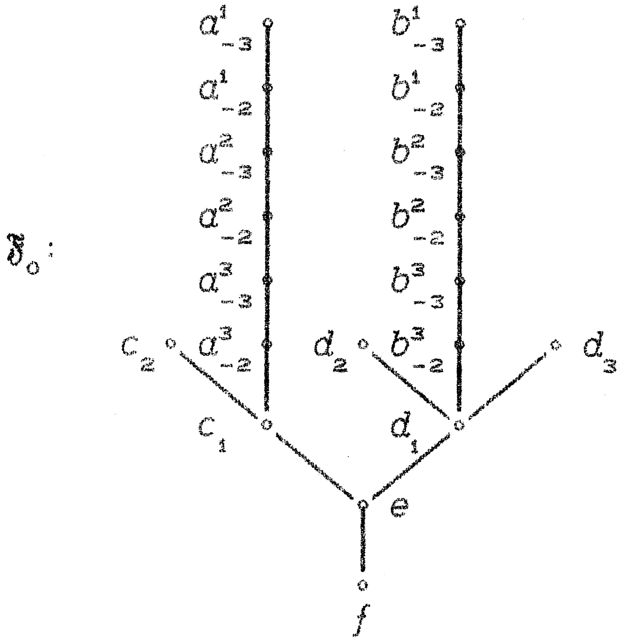


Fig. 2.

Thus, let

$$A^1_{n+1} = \Diamond A^1_n \,\&\, \Diamond B^1_{n-1} \,\&\, \neg\Diamond B^1_n, \quad B^1_{n+1} = \Diamond B^1_n \,\&\, \Diamond A^1_{n-1} \,\&\, \neg\Diamond A^1_n \qquad (n\geq -2);$$

$$Q_{-2} = r,\quad Q'_{-2} = s,\quad Q_{-1} = p,\quad Q'_{-1} = q,$$

$$Q_{n+1} = \Diamond A^2_{-3} \,\&\, \Diamond B^2_{-3} \,\&\, \neg\Diamond A^3_{-3} \,\&\, \neg\Diamond B^3_{-3} \,\&\, \Diamond Q_n \,\&\, \Diamond Q'_{n-1} \,\&\, \neg\Diamond Q'_n,$$
$$Q'_{n+1} = \Diamond A^2_{-3} \,\&\, \Diamond B^2_{-3} \,\&\, \neg\Diamond A^3_{-3} \,\&\, \neg\Diamond B^3_{-3} \,\&\, \Diamond Q'_n \,\&\, \Diamond Q_{n-1} \,\&\, \neg\Diamond Q_n, \qquad (n\geq -1);$$

$$A^2_{n+1} = \Diamond A^2_{-3} \,\&\, \Diamond B^2_{-3} \,\&\, \neg\Diamond A^3_{-3} \,\&\, \neg\Diamond B^3_{-3} \,\&\, \Diamond A^2_n \,\&\, \Diamond B^2_{n-1} \,\&\, \neg\Diamond B^2_n,$$
$$B^2_{n+1} = \Diamond A^2_{-3} \,\&\, \Diamond B^2_{-3} \,\&\, \neg\Diamond A^3_{-3} \,\&\, \neg\Diamond B^3_{-3} \,\&\, \Diamond B^2_n \,\&\, \Diamond A^2_{n-1} \,\&\, \neg\Diamond A^2_n, \qquad (n\geq -2);$$

$$R_{-2} = r',\quad R'_{-2} = s',\quad R_{-1} = p',\quad R'_{-1} = q',$$

$$R_{n+1} = \Diamond A^3_{-3} \,\&\, \Diamond B^3_{-3} \,\&\, \neg\Diamond C_1 \,\&\, \neg\Diamond D_1 \,\&\, \Diamond R_n \,\&\, \Diamond R'_{n-1} \,\&\, \neg\Diamond R'_n,$$
$$R'_{n+1} = \Diamond A^3_{-3} \,\&\, \Diamond B^3_{-3} \,\&\, \neg\Diamond C_1 \,\&\, \neg\Diamond D_1 \,\&\, \Diamond R'_n \,\&\, \Diamond R_{n-1} \,\&\, \neg\Diamond R_n, \qquad (n\geq -1);$$

$$A^3_{n+1} = \Diamond A^3_{-3} \,\&\, \Diamond B^3_{-3} \,\&\, \neg\Diamond C_1 \,\&\, \neg\Diamond D_1 \,\&\, \Diamond A^3_n \,\&\, \Diamond B^3_{n-1} \,\&\, \neg\Diamond B^3_n,$$
$$B^3_{n+1} = \Diamond A^3_{-3} \,\&\, \Diamond B^3_{-3} \,\&\, \neg\Diamond C_1 \,\&\, \neg\Diamond D_1 \,\&\, \Diamond B^3_n \,\&\, \Diamond A^3_{n-1} \,\&\, \neg\Diamond A^3_n, \qquad (n\geq -2);$$

$$T(n, Q_i, R_j) = \Diamond A^1_n \,\&\, \Diamond B^1_n \,\&\, \neg\Diamond A^1_{n+1} \,\&\, \neg\Diamond B^1_{n+1} \,\&\, \Diamond Q_i \,\&\, \Diamond Q'_i \,\&$$
$$\&\, \neg\Diamond Q_{i+1} \,\&\, \neg\Diamond Q'_{i+1} \,\&\, \Diamond R_j \,\&\, \Diamond R'_j \,\&\, \neg\Diamond R_{j+1} \,\&\, \neg\Diamond R'_{j+1} \qquad (n, i, j\geq 0).$$

Since under substitution of A^2_{i-3}, B^2_{i-3}, A^2_{i-2}, B^2_{i-2} instead of r, s, p, q and A^3_{i-3}, B^3_{i-3}, A^3_{i-2}, B^3_{i-2} instead of r', s', p', q', respectively, the formulas Q_j, Q'_j and R_j, R'_j, for $i,j\geq 0$, turn into A^2_{i+j-1}, B^2_{i+j-1} and A^3_{i+j-1}, B^3_{i+j-1}, we use the following notation:

$$T(\alpha, A^2_{i+j-1}, \Phi_1) = T(\alpha, Q_j, \Phi_1)\{A^2_{i-3}/r,\ B^2_{i-3}/s,\ A^2_{i-2}/p,\ B^2_{i-2}/q\},$$
$$T(\alpha, \Phi_2, A^3_{i+j-1}) = T(\alpha, \Phi_2, R_j)\{A^3_{i-3}/r',\ B^3_{i-3}/s',\ A^3_{i-2}/p',\ B^3_{i-2}/q'\},$$

where Φ_1 does not contain r, s, p, q and Φ_2 does not contain r', s', p', q'.

LEMMA 1. *If under some valuation on* $\mathfrak{F}$ *formulas* C_1, D_1, A^i_j, B^i_j, *for* $i=1,2,3$, $j=-2,-3$, *satisfy* (7) *and* (8) *then* (8) *holds for all* $j\geq -3$ *and* (9) *is also true.*

Proof. (8) is proved by induction on j, and (9) follows from (8). ■

The formulas simulating instructions of a Minsky machine are defined as follows:

if $I = q_\alpha \to q_\beta T_1 T_0$ then

$$AxI = \Diamond T(\alpha, Q_1, R_1) \supset \Diamond T(\beta, Q_2, R_1) \vee F;$$

if $I = q_\alpha \to q_\beta T_0 T_1$ then

$$AxI = \Diamond T(\alpha, Q_1, R_1) \supset \Diamond T(\beta, Q_1, R_2) \vee F;$$

if $I = q_\alpha \to q_\beta T_0 T_{-1} (q_\gamma T_0 T_0)$ then

$$AxI = (\Diamond T(\alpha, Q_1, R_2) \supset \Diamond T(\beta, Q_1, R_1) \vee F) \;\&\; (\Diamond T(\alpha, Q_1, A_0^3) \supset \Diamond T(\gamma, Q_1, A_0^3) \vee F).$$

if $I = q_\alpha \to q_\beta T_{-1} T_0 (q_\gamma T_0 T_0)$ then

$$AxI = (\Diamond T(\alpha, Q_2, R_1) \supset \Diamond T(\beta, Q_1, R_1) \vee F) \;\&\; (\Diamond T(\alpha, A_0^2, R_1) \supset \Diamond T(\gamma, A_0^2, R_1) \vee F).$$

For a Minsky program P, we let

$$AxP = \mathop{\&}_{I \in P} AxI.$$

LEMMA 2. *If* $P\colon (\alpha, m, n) \to (\beta, k, l)$ *then*

$$S4 + AxP \vdash \Diamond T(\alpha, A_m^2, A_n^3) \supset \Diamond T(\beta, A_k^2, A_l^3) \vee F.$$

Proof. By induction on the length of the computation transforming the first configuration to the second. Here we confine ourselves to consideration of only one case: when the configuration (β, k, l) is obtained from (α, m, n) as a result of applying the instruction $q_\alpha \to q_\beta T_1 T_0$. We have $k = m + 1$, $l = n$,

$$S4 + AxP \vdash \Diamond T(\alpha, Q_1, R_1) \supset \Diamond T(\beta, Q_2, R_1) \vee F,$$

and so, by substituting A_{m-3}^2, B_{m-3}^2, A_{m-2}^2, B_{m-2}^2 in place of r, s, p, q and A_{n-3}^3, B_{n-3}^3, A_{n-2}^3, B_{n-2}^3 in place of r', s', p', q', we obtain

$$S4 + AxP \vdash \Diamond T(\alpha, A_m^2, A_n^3) \supset \Diamond T(\beta, A_k^2, A_l^3) \vee F. \qquad \blacksquare$$

LEMMA 3. *Let* $\mathfrak{F}$ *be the frame shown in Fig. 1 and corresponding to a program* P *and a configuration* $\mathfrak{a} = (\alpha, m, n)$. *Then*

(i) $\mathfrak{F} \models AxP$ *and*

(ii) *if* $P\colon (\alpha, m, n) \not\to (\beta, k, l)$ *then*

$$\mathfrak{F} \nvDash \Diamond T(\alpha, A^2_m, A^3_n) \supset \Diamond T(\beta, A^2_k, A^3_l) \vee F.$$

Proof. A detailed proof of (i) which contains no conceptual difficulties but is rather cumbersome can be found in Appendix, and (ii) follows from (6) and (9). ∎

It follows from Lemma 3 and the obvious relation $\mathfrak{F} \vDash$ **S4Grz** that if $P: (\alpha, m, n) \nrightarrow (\beta, k, l)$ then

$$\mathbf{S4Grz} + AxP \nvdash \Diamond T(\alpha, A^2_m, A^3_n) \supset \Diamond T(\beta, A^2_k, A^3_l) \vee F.$$

Thus, taking into account Lemma 2, for a program P and configurations $\mathfrak{a} = (\alpha, m, n)$ and $\mathfrak{b} = (\beta, k, l)$, as a formula $C(\mathfrak{a}, \mathfrak{b})$ satisfying (2) we may take

(10) $C(\mathfrak{a}, \mathfrak{b}) = \Diamond T(\alpha, A^2_m, A^3_n) \supset \Diamond T(\beta, A^2_k, A^3_l) \vee F.$

The choice of F and G depends, of course, on the property $\mathcal{P}$. It is not difficult to find suitable F and G for proving the undecidability of FMP and decidability. We will demonstrate this in the next section. However, the proof of the undecidability of DP and HC requires somewhat greater efforts and will be completed in §8.

§3. **The undecidability of the finite model property and decidability.** Let us consider the frame $\mathfrak{F}_0 = \langle W_0, R_0 \rangle$ shown in Fig. 2 and give its elements new names, viz., the integers from 0 to 18 so that the origin f obtains the name 0. Construct the subframe formula $B_{\mathfrak{F}_0}$ (of Fine [1985]) for $\mathfrak{F}_0$ which is the conjunction of the following formulas:

p_0;

$\Box(p_i \supset \neg p_j)$, $0 \le i < j \le 18$;

$\Box(p_i \supset \Diamond p_j)$, for $iR_0 j$;

$\Box(p_i \supset \neg\Diamond p_j)$, for $i, j \le 18$ and $\neg iR_0 j$.

By B_i we denote the formula which is obtained from $B_{\mathfrak{F}_0}$ by replacing its first conjunct with p_i. It follows from Lemma 1 in Fine [1985, §3] that a frame $\mathfrak{F}'$ satisfies $B_{\mathfrak{F}_0}$ iff $\mathfrak{F}'$ is subreducible to $\mathfrak{F}_0$; moreover, if $B_{\mathfrak{F}_0}$ is true at some world in $\mathfrak{F}'$ under some valuation then a subreduction f from $\mathfrak{F}'$ onto $\mathfrak{F}_0$ can be constructed by taking

$$f(a) = \begin{cases} i \text{ if } a \models B_i, \\ \text{undefined otherwise.} \end{cases}$$

THEOREM 1. *There is no algorithm which is capable of deciding, for a modal formula A, whether the logic* **S4Grz** + A *is decidable.*

Proof. Take a Minsky program P and a configuration $\mathfrak{a}$ for which the problem of the second configuration is undecidable and construct the frame $\mathfrak{F}$ according to Fig. 1. Let

$$F = \neg B_{\mathfrak{F}_0},\ C_1 = B_{i_1},\ D_1 = B_{i_2},\ A^i_j = B_{i_3},\ B^i_j = B_{i_4},$$

where i_1, i_2, i_3, i_4 are the new names (numbers) of c_1, d_1, a^i_j, b^i_j, respectively (i=1,2,3, j=-2,-3). There is essentially only one subreduction from $\mathfrak{F}$ onto $\mathfrak{F}_0$, viz., the embedding of $\mathfrak{F}_0$ in $\mathfrak{F}$ shown in Fig. 1 by the bold-face lines. So, by Fine's result mentioned above, the constructed formulas satisfy (6) - (8).

As for G, we may let $G = \top$, that is in this case we can do without G at all.

With each configuration $\mathfrak{b}$ we associate the calculus

$$L(P,\mathfrak{a},\mathfrak{b}) = \mathbf{S4Grz} + AxP + C(\mathfrak{a},\mathfrak{b}) \supset F,$$

where $C(\mathfrak{a},\mathfrak{b})$, remind, has the form (10).

Now, if $P{:}\,\mathfrak{a}\rightarrow\mathfrak{b}$ then, by Lemma 2,

$$L(P,\mathfrak{a},\mathfrak{b}) = \mathbf{S4Grz} + F.$$

Therefore, by Theorem 5 of Fine [1985, §4], $L(P,\mathfrak{a},\mathfrak{b})$ has FMP, and so is decidable. Thus, it remains to show that if $P{:}\,\mathfrak{a}\not\rightarrow\mathfrak{b}$ then $L(P,\mathfrak{a},\mathfrak{b})$ is undecidable.

Let $P{:}\,\mathfrak{a}\not\rightarrow\mathfrak{b}$ and let $\mathfrak{c}$ be an arbitrary configuration. If $P{:}\,\mathfrak{a}\rightarrow\mathfrak{c}$ then, by Lemma 2,

$$L(P,\mathfrak{a},\mathfrak{b}) \vdash C(\mathfrak{a},\mathfrak{c}).$$

But if $P{:}\,\mathfrak{a}\not\rightarrow\mathfrak{c}$ then, by Lemma 3, (6) and (9),

$$\mathfrak{F} \models L(P,\mathfrak{a},\mathfrak{b}),\ \mathfrak{F} \not\models C(\mathfrak{a},\mathfrak{c}),$$

and so

$$L(P,\mathfrak{a},\mathfrak{b}) \nvdash C(\mathfrak{a},\mathfrak{c}).$$

We have proved that $L(P,\mathfrak{a},\mathfrak{b}) \vdash C(\mathfrak{a},\mathfrak{c})$ iff $P{:}\,\mathfrak{a}\rightarrow\mathfrak{c}$, and the undecidability of $L(P,\mathfrak{a},\mathfrak{b})$ follows now from the undecidability of the problem of the second configuration for P and $\mathfrak{a}$. ■

Since the undecidability of a finitely axiomatizable logic implies that the logic does not have FMP, we have simultaneously proved the following

THEOREM 2. *There is no algorithm which is capable of deciding, for a formula A, whether the logic* **S4Grz** *+ A has FMP.* ■

The proof above, as was to be expected, does not enrich us too much with knowledge about the nature of the properties proved to be undecidable. In this connection, it is worth to remember the well-known Rice Theorem from the theory of algorithms which is proved actually by the same scheme. However, unlike the Rice Theorem in which a property is required to be only non-trivial and invariant, these two conditions are clearly insufficient for proving the undecidability of properties of calculi, the examples of decidable (non-trivial and invariant) properties mentioned in the introduction being the witnesses. Proving Theorem 1 we used Fine's results on FMP and decidability for constructing the formula F with the desirable properties. In exactly the same way in order to prove the undecidability of DP and HC we need some results concerning these properties themselves. It is our next goal to obtain them.

Our investigation of DP and HC is essentially based on the semantic sufficient conditions of Maksimova [1986] and van Benthem and Humberstone [1983] which presuppose certain knowledge of the construction of (general) frames for a logic. This is why we prefer to deal with logics whose axioms are frame based formulas such as the frame or subframe formulas of Fine [1974, 1985]. Both these kinds of formulas are special cases of the canonical formulas introduced by Zakharyaschev [1983, 1984, 1988, 1989]. It is these formulas that will be considered as axioms of modal and intermediate logics. Note that doing this we do not lose generality because all such logics can be axiomatized by the canonical formulas.

§4 and §6 contain all the necessary definitions and theorems concerning the canonical formulas and in §5 and §7 we obtain a few general results on DP and HC.

§4. **Canonical formulas for Int.** We begin with the canonical formulas for **Int** because it is the intuitionistic case that shows most clearly a deep connection between syntactical parameters of formulas and the construction of their countermodels. We will explain the origin of the canonical formulas by the following

EXAMPLE. Let us consider the Scott Axiom (see Kreisel and Putnam [1957]):

$A = ((\neg\neg p \supset p) \supset p \vee \neg p) \supset \neg p \vee \neg\neg p.$

(Remind that the Scott logic **SL** = **Int** + A was one of the first examples of proper extensions of **Int** having DP, and the formula A was invented just for producing such an example.)

Now imagine that we want to find out the construction of frames refuting A. Simple calculations (e.g. with the help of the semantic tableaux) show that the elementary frame which refutes A is the frame $\mathfrak{F} = \langle W,R \rangle$ depicted in Fig. 3. Moreover, every (general) frame $\mathfrak{F}'$ such that $\mathfrak{F}' \not\models A$ must be subreducible to $\mathfrak{F}$, i.e. it must contain a subframe which is reducible to $\mathfrak{F}$. However, this is only a necessary condition, since the frame $\mathfrak{F}_1$ shown in Fig. 4 is reducible to $\mathfrak{F}$ but $\mathfrak{F}_1 \models A$ (to refute A it is required that $a_3 \models p$ and $a_2 \models \neg p$, but then $a_4 \models p \& \neg p$). There are different ways to forbid such elements as a_4 to appear. One of them is to request that a subreduction from $\mathfrak{F}'$ onto $\mathfrak{F}$ should be confinal, i.e., roughly speaking, that each world in $\mathfrak{F}'$ should "see" at least one pre-image of a world in $\mathfrak{F}$. The exact definition of this notion is as follows.

A subreduction f from $\mathfrak{F}' = \langle W',R',S' \rangle$ onto $\mathfrak{F} = \langle W,R,S \rangle$ is called *confinal* if, for all $a \in W'$,

(1) $a \in f^{-1}(W)\uparrow \;\Rightarrow\; a \in f^{-1}(W)\downarrow.$

If in our Example a subreduction f from $\mathfrak{F}_1$ onto $\mathfrak{F}$ was confinal then $f(a_4)$ would be an element in $\mathfrak{F}$ which is impossible.

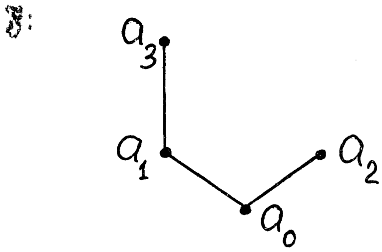


Fig. 3.

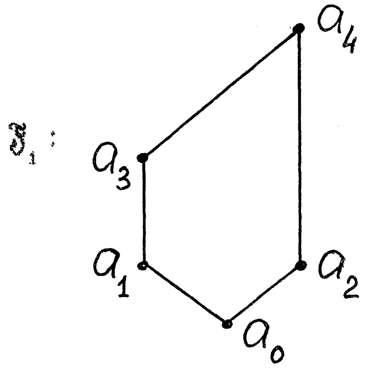


Fig. 4.

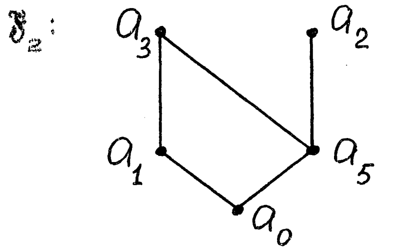


Fig. 5.

However, the existence of a confinal subreduction from $\mathfrak{F}'$ onto $\mathfrak{F}$ does not guarantee the refutability of A in $\mathfrak{F}'$ either. Indeed, the frame $\mathfrak{F}_2$ shown in Fig. 5 is confinally subreducible to $\mathfrak{F}$, but $\mathfrak{F}_2 \models A$. This time the world a_5 is to blame: it "sees" pre-images of a_3 and a_2 where (under a valuation refuting A) $\neg p$ and p are false, and so $p \vee \neg p$ must be false at a_5; but then $a_5 \not\models \neg\neg p \supset p$, and so a_5 must "see" also a pre-image of a_1 which is "responsible" for refuting $\neg\neg p \supset p$.

In general, it is impossible to forbid elements like a_5 to appear using such a "global" restriction as (1) and not making a desired refutability criterion sufficient but not necessary. (It is possible in the case of the Scott Axiom because it axiomatizes the same logic as the negation of the frame formula for $\mathfrak{F}$.)

A "local" restriction may be as follows. We declare that a pair $\delta = (\{a_2, a_3\}, \{a_1\})$ is a *disjunctive domain* (one can read it like this: if some world "sees" all the worlds in the first component of the pair then it "sees" at least one world in the second component), denote by $\mathfrak{D}$ the set of all disjunctive domains (in our Example $\mathfrak{D} = \{\delta\}$) and request that a subreduction f from $\mathfrak{F}'$ onto $\mathfrak{F}$ should satisfy not only ($\bot$) but also one more condition

(*d*) *if* $(\bar{a}, \bar{b}) \in \mathfrak{D}$ *and* $c \in f^{-1}(W)\uparrow$ *then*

$$c \in \bigcap_{a \in \bar{a}} (f^{-1}(a)\downarrow) \Rightarrow c \in \bigcup_{b \in \bar{b}} (f^{-1}(b)\downarrow);$$

As a final result we obtain the necessary and sufficient condition for refutability of the Scott formula in a frame $\mathfrak{F}'$, viz., the existence of a confinal subreduction f from $\mathfrak{F}'$ onto $\mathfrak{F}$ satisfying (d). ■

By this special example we have tried to illustrate a general principle which was discovered by Zakharyashchev [1983]: one can characterize the construction of countermodels for an arbitrary intuitionistic formula starting from some finite set of its finite elementary countermodels and using only the notion of subreduction and conditions ($\bot$) and (d). Moreover, Zakharyashchev [1983, 1989] showed that if a formula is positive (i.e. it contains no $\bot$) then the requirement of confinality may be not imposed on a subreduction, and if a formula contains no $\vee$ then its elementary countermodels do not have disjunctive domains and so the condition (d) becomes degenerate. Roughly speaking, the role of $\supset$ is characterized (on the semantic level) in terms of subreduction, the role of $\bot$ (or $\neg$) is characterized by the condition ($\bot$) and the role of $\vee$ is characterized by (d).

Now we will go in reverse direction. Beginning with a finite frame $\mathfrak{F}$ and a given set $\mathfrak{D}$ of disjunctive domains in $\mathfrak{F}$ we will construct a formula so that it will be refuted by a general frame $\mathfrak{F}'$ if and only if there is a confinal subreduction from $\mathfrak{F}'$ onto $\mathfrak{F}$ satisfying (d).

So, let $\mathfrak{F} = \langle W, R \rangle$ be a finite sharp partially ordered frame, with $a_0, a_1, \dots, a_n$ being all the distinct elements of W and a_0 being the origin. A pair $\delta = (\bar{a}, \bar{b})$ of non-empty sets $\bar{a}, \bar{b} \subseteq W$ is called a *disjunctive domain* (*d-domain*, for short) in $\mathfrak{F}$ if the following three conditions are satisfied:

(i) $\bar{a}$ and $\bar{b}$ are anti-chains in $\mathfrak{F}$ and $\bar{a}$ has at least two elements;

(ii) $\forall a \in \bar{a}\ \forall b \in \bar{b}\ \neg aRb$;

(iii) $\forall c \in W\ (c \in \bigcap_{a \in \bar{a}} a{\downarrow} \Rightarrow c \in \bigcup_{b \in \bar{b}} b{\downarrow})$.

Let $\mathfrak{D}$ be some (possibly empty) set of d-domains in $\mathfrak{F}$. With $\mathfrak{F}$ and $\mathfrak{D}$ we associate the formula

$$X(\mathfrak{F}, \mathfrak{D}, \bot) = \mathop{\&}_{a_i R a_j} A_{ij} \ \&\ \mathop{\&}_{\delta \in \mathfrak{D}} B_\delta \ \&\ C \supset p_0,$$

where

$$A_{ij} = (\mathop{\&}_{\neg a_j R a_k} p_k \supset p_j) \supset p_i,$$

$$C = \mathop{\&}_{i=0}^{n} (\mathop{\&}_{\neg a_i R a_k} p_k \supset p_i) \supset \bot$$

and if $\delta = (\bar{a}, \bar{b})$ then

$$B_\delta = \mathop{\&}_{a_i \in \bar{b}} (\mathop{\&}_{\neg a_i R a_k} p_k \supset p_i) \supset \bigvee_{a_i \in \bar{a}} p_i.$$

We denote by $X(\mathfrak{F}, \mathfrak{D})$ the formula which is obtained from $X(\mathfrak{F}, \mathfrak{D}, \bot)$ by deleting the conjunct C. The formulas of the form $X(\mathfrak{F}, \mathfrak{D}, \bot)$ and $X(\mathfrak{F}, \mathfrak{D})$ are called *canonical formulas* and *positive canonical formulas for* **Int**, respectively.

The following criterion was proved by Zakharyashchev [1983, 1989].

REFUTABILITY CRITERION FOR THE INTUITIONISTIC CANONICAL FORMULAS. (i) $\mathfrak{F}' \not\models X(\mathfrak{F}, \mathfrak{D}, \bot)$ *iff there is a confinal subreduction* f

from $\mathfrak{F}'$ *onto* $\mathfrak{F}$ *satisfying* (d).

(ii) $\mathfrak{F}' \not\models \mathcal{X}(\mathfrak{F},\mathfrak{D})$ *iff there is a subreduction* f *from* $\mathfrak{F}'$ *onto* $\mathfrak{F}$ *satisfying* (d). ■

The next theorem, also proved by Zakharyashchev [1983, 1989], shows that the set of the canonical formulas is complete in the sense that every extension of **Int** (i.e. every intermediate logic) can be axiomatized by canonical formulas.

COMPLETENESS THEOREM FOR THE INTUITIONISTIC CANONICAL FORMULAS. (i) *There exists an algorithm which, for each formula* A, *constructs a finite number of canonical formulas* $\mathcal{X}(\mathfrak{F}_1,\mathfrak{D}_1,\bot), \ldots, \mathcal{X}(\mathfrak{F}_n,\mathfrak{D}_n,\bot)$ *such that*

$$\mathbf{Int} + A = \mathbf{Int} + \mathcal{X}(\mathfrak{F}_1,\mathfrak{D}_1,\bot) + \ldots + \mathcal{X}(\mathfrak{F}_n,\mathfrak{D}_n,\bot).$$

Moreover, if A *has no occurrences of* $\vee$ *then* $\mathfrak{D}_i = \emptyset$, *for all* $i=1,\ldots,n$.

(ii) *There exists an algorithm which, for each positive formula* A, *constructs a finite number of positive canonical formulas* $\mathcal{X}(\mathfrak{F}_1,\mathfrak{D}_1), \ldots, \mathcal{X}(\mathfrak{F}_n,\mathfrak{D}_n)$ *such that*

$$\mathbf{Int} + A = \mathbf{Int} + \mathcal{X}(\mathfrak{F}_1,\mathfrak{D}_1) + \ldots + \mathcal{X}(\mathfrak{F}_n,\mathfrak{D}_n).$$

Moreover, if A *has no occurrences of* $\vee$ *then* $\mathfrak{D}_i = \emptyset$, *for all* $i=1,\ldots,n$. ■

For example, the Scott logic can be represented in the form

$$\mathbf{SL} = \mathbf{Int} + \mathcal{X}(\mathfrak{F},\mathfrak{D},\bot),$$

where $\mathfrak{F}$ is the frame shown in Fig. 3 and $\mathfrak{D}$ contains only one d-domain $\delta = (\{a_2,a_3\},\{a_1\})$.

It is worth to note that the two boundary cases of the canonical formulas - the formulas of the form $\mathcal{X}(\mathfrak{F},\emptyset)$ and $\mathcal{X}(\mathfrak{F},\mathfrak{D}^*,\bot)$, where $\mathfrak{D}^*$ is the set of all d-domains in $\mathfrak{F}$ - are similar to the negations of Fine's subframe and frame modal formulas for

$\mathfrak{F}$, respectively, because $\mathfrak{F}' \nvDash X(\mathfrak{F},\emptyset)$ iff $\mathfrak{F}'$ is subreducible to $\mathfrak{F}$ and $\mathfrak{F}' \nvDash X(\mathfrak{F},\mathfrak{D}^*,\bot)$ iff there is a generated subframe of $\mathfrak{F}'$ which is reducible to $\mathfrak{F}$.

§5. **Two sufficient conditions for the disjunction property of intermediate logics.** The main tool for proving DP is the semantic criterion which, for Kripke frames, seems to have been first used explicitly by Gabbay and de Jongh [1974]. In the most general form, as an algebraic equivalent of DP, it was proved by Maksimova [1986]. We need only the sufficient condition of this criterion. It is formulated below for general frames in order to escape the effect of Kripke incompleteness.

SEMANTIC SUFFICIENT CONDITION FOR THE DISJUNCTION PROPERTY. *Let a logic L be determined by a class $\mathcal{C}$ of general frames. Then L has DP if, for every two frames $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathcal{C}$, there is a sharp frame $\mathfrak{F}_0$ such that*

(*i*) $\mathfrak{F}_0 \vDash L$

and

(*ii*) *the frame $\mathfrak{F}_1 + \mathfrak{F}_2$ is isomorphic to a generated subframe of $\mathfrak{F}_0$.* ■

Remark. This sufficient condition will be also necessary if $\mathcal{C}$ contains only *descriptive* general frames (for definition consult Goldblatt [1976]). ■

Thus, to prove DP (and many other properties as well) of intermediate logics it is desirable to conceive well enough the construction of frames for them. The employment of the canonical formulas makes this problem much easier, and so we will suppose that each intermediate logic L is represented by its canonical axioms:

$$L = \mathbf{Int} + \{\chi(\mathfrak{F}_i, \mathfrak{D}_i, \bot)\}_{i\in Q}.$$

First we note that the necessary condition for L to have DP is non-emptiness of all the sets $\mathfrak{D}_i$ in every such representation of L (or, which is equivalent, the presence of $\vee$ in each formula in L-**Int**). This result was independently obtained by Minari [1986] and Zakharyashchev [1987]. (Much easier and more elegant proof, using the semantic necessary condition for DP, can be found in Zakharyashchev [1990].)

Each intermediate (or modal) logic L is known to be determined by an appropriate class of sharp general frames. Given disjoint sharp frames $\mathfrak{F}_1 = \langle W_1, R_1, S_1\rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2, S_2\rangle$, the simplest way of constructing a frame $\mathfrak{F}_0 = \langle W_0, R_0, S_0\rangle$ satisfying (ii) is to add origin a_0 to the frame $\langle W_1 \cup W_2, R_1 \cup R_2\rangle$ (which yields us the frame $\langle W_0, R_0\rangle$) and take as S_0 the system generated (as pseudo-Boolean algebra) by the elements from S_1 and S_2; see Fig. 6. (It is easy to prove by induction on the construction of $V \in S_0$

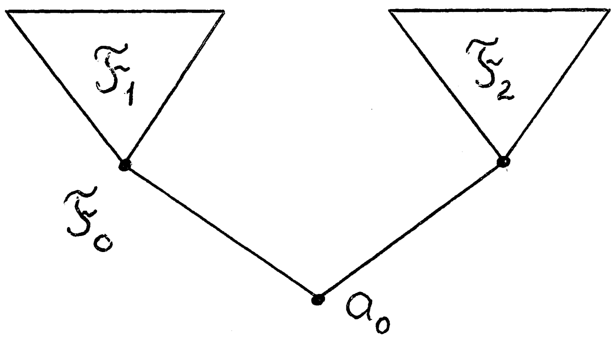


Fig. 6.

that, for i=1,2, $V \cap W_i \in S_i$, and so $\mathfrak{F}_i$ is a generated subframe of $\mathfrak{F}_0$.) In general, $\mathfrak{F}_0$ will not be a frame for L, but for a rather large family of logics this very simple method succeeds. Indeed, as axioms for L we can take canonical formulas $\chi(\mathfrak{F},\mathfrak{D},\bot)$ in which the origin of $\mathfrak{F}$ has at least three immediate successors and d-domains in $\mathfrak{D}$ forbid, roughly speaking, the refutation of $\chi(\mathfrak{F},\mathfrak{D},\bot)$ at worlds having less than three immediate successors.

THEOREM 3. *Let L be axiomatized by canonical formulas $\chi(\mathfrak{F},\mathfrak{D},\bot)$ such that the set V_0 of the immediate successors of the origin in $\mathfrak{F}$ contains at least three elements and*

$$\forall V \subseteq V_0 \ (\lceil \overline{\overline{V}}_0/2 \rceil \le \overline{\overline{V}} \le \overline{\overline{V}}_0 - 1 \ \Rightarrow \ \exists \bar{a} \subseteq V\uparrow \ \exists \bar{b} \subseteq V_0 - V \ (\bar{a},\bar{b}) \in \mathfrak{D})$$

(see Fig. 7). Then L has DP. (Here $\lceil x \rceil$ is the least integer which is greater than or equal to x.)

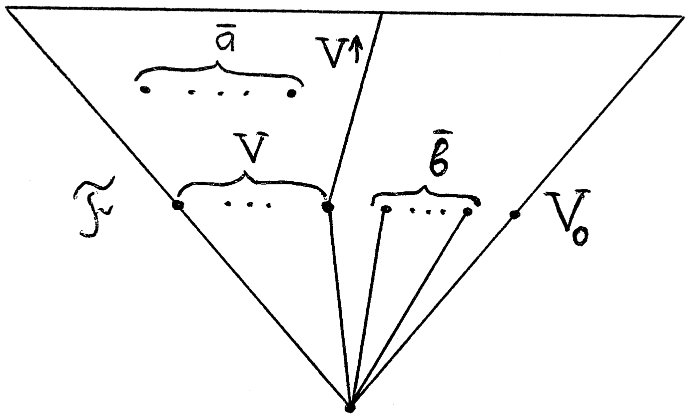


Fig. 7.

Proof. We show that the frame $\mathfrak{F}_0 = \langle W_0, R_0, S_0 \rangle$, constructed of sharp frames $\mathfrak{F}_1$ and $\mathfrak{F}_2$ for L by the method discussed above, satisfies (i). Suppose otherwise. Then $\mathfrak{F}_0 \not\models \chi(\mathfrak{F},\mathfrak{D},\bot)$, for some axiom $\chi(\mathfrak{F},\mathfrak{D},\bot)$ of L, and so there is a confinal subreduction f from $\mathfrak{F}_0$ onto $\mathfrak{F} = \langle W, R \rangle$ satisfying (d). Note that the origin of $\mathfrak{F}_0$ belongs to $f^{-1}(W)$, for otherwise $f(W_i) = W$, for some $i \in \{1,2\}$, and so the restriction of f to W_i is a confinal subreduction from $\mathfrak{F}_i$

onto $\mathfrak{F}$ satisfying (d) for $X(\mathfrak{F},\mathfrak{D},\bot)$ which contradicts $\mathfrak{F} \models X(\mathfrak{F},\mathfrak{D},\bot)$. Moreover, for any $i = 1,2$, there is a world $a \in V_0$ such that $f^{-1}(a) \cap W_i = \emptyset$. For otherwise, when, say, $f^{-1}(a) \cap W_1 \neq \emptyset$, for all $a \in V_0$, we can extend f by mapping the set $\bigcap_{a \in V_0} (f^{-1}(a){\downarrow})$ onto the origin of $\mathfrak{F}$. Since

$$-\bigcap_{a \in V_0} (f^{-1}(a){\downarrow}) = \bigcup_{a \in V_0} - f^{-1}(a){\downarrow} \in S_0,$$

the new mapping f is clearly a confinal subreduction from $\mathfrak{F}_0$ onto $\mathfrak{F}$, with $f^{-1}(a) \cap W_1 \neq \emptyset$, for all $a \in W$. Therefore, the restriction of f to W_1 is a confinal subreduction from $\mathfrak{F}_1$ onto $\mathfrak{F}$ satisfying (d) for $X(\mathfrak{F},\mathfrak{D},\bot)$, and so $\mathfrak{F}_1 \not\models X(\mathfrak{F},\mathfrak{D},\bot)$ which is a contradiction.

Take now that i for which W_i contains f-pre-images of all elements in some $V \subseteq V_0$ such that $\lceil \overline{\overline{V}}_0 / 2 \rceil \leq \overline{\overline{V}}$ and does not contain any pre-images of the other elements in V_0. (As we have just proved, $V_0 - V \neq \emptyset$.) According to the condition of our Theorem, there are $\bar{a} \subseteq V{\uparrow}$ and $\bar{b} \subseteq V_0 - V$ such that the d-domain $\delta = (\bar{a},\bar{b})$ belongs to $\mathfrak{D}$. Thus, by (d), W_i contains a pre-image of some element in $\bar{b}$ which is impossible. ■

Remark. The proof will not change if we take as (some) axioms for L positive canonical formulas of the form $X(\mathfrak{F},\mathfrak{D})$. ■

Theorem 3 covers the Gabbay and de Jongh [1974] logics

$$\mathbf{T}_n = \mathbf{Int} + \mathop{\&}_{i=0}^{n} ((p_i \supset \bigvee_{j \neq i} p_j) \supset \bigvee_{j \neq i} p_j) \supset \bigvee_{i=0}^{n} p_i \quad (n > 1),$$

which can be differently represented as

$$\mathbf{T}_n = \mathbf{Int} + X(\mathfrak{F}_n, \mathfrak{D}_n),$$

where $\mathfrak{F}_n$ is the frame shown in Fig. 8 and $\mathfrak{D}_n$ consists of all d-domains of the form $\delta = (\{a_i, a_j\}, \{a_k\})$.

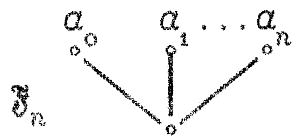


Fig. 8.

Moreover, the conditions of Theorem 3 are satisfied by the Ono [1972] logics

$$\mathrm{Int} + B_n \quad (n>1),$$

where

$$B_n = \mathop{\&}_{i=0}^{n} ((p_i \supset \bot) \equiv \bigvee_{j\neq i} p_j) \supset \bigvee_{i=0}^{n} p_i,$$

which otherwise can be represented as

$$\mathrm{Int} + B_n = \mathrm{Int} + X(\mathfrak{F}_n, \mathfrak{D}_n, \bot),$$

where $\mathfrak{F}_n$ and $\mathfrak{D}_n$ are the same as for T_n. It is easy to show that all logics of Wronski [1973] (remind that there are a continuum of them) also satisfy the conditions of Theorem 3.

However, for the Scott logic

$$\mathrm{SL} = \mathrm{Int} + X(\mathfrak{F}, \mathfrak{D}, \bot),$$

where $\mathfrak{F}$ is the frame shown in Fig. 3 and $\mathfrak{D}$ contains only $\delta = (\{a_2, a_3\}, \{a_1\})$, this construction does not work: the frame $\mathfrak{F}_0 = \langle W_0, R_0, S_0\rangle$ built out of frames $\mathfrak{F}_1 = \langle W_1, R_1, S_1\rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2, S_2\rangle$ for SL by adding an origin to $\langle W_1 \cup W_2, R_1 \cup R_2\rangle$ is not in general a frame for SL. Now our goal is as follows. If $X(\mathfrak{F}, \mathfrak{D}, \bot)$ is refuted by $\mathfrak{F}_0$, i.e. there is a confinal subreduction f from $\mathfrak{F}_0$ onto $\mathfrak{F}$ satisfying (d) for $X(\mathfrak{F}, \mathfrak{D}, \bot)$ then, putting into W_0 new elements below some elements in W_1 and W_2, we should try to violate (d) and, of course, obtain no new confinal subreductions. It is not difficult to realize this idea for SL because the first component of δ consists of the maximal elements in $\mathfrak{F}$, and so, by $(\bot)$, the maximal elements in W_1 and W_2 are f-pre- images of a_2 and

a_3; therefore we may put new elements only below each pair of maximal elements in W_0 (see Fig. 9).

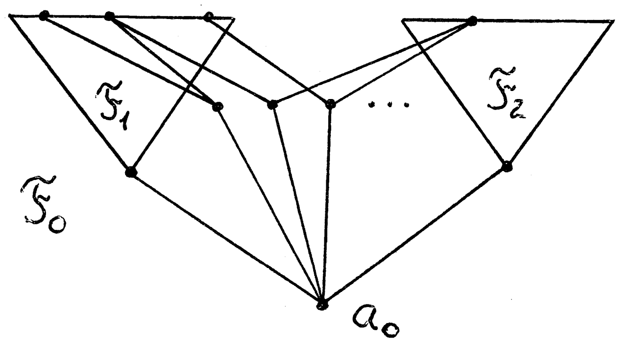


Fig. 9.

We prove now a theorem which shows that this method of constructing $\mathfrak{F}_0$ is suitable for another wide family of intermediate logics. We shall say an element b is a *focus* for a d-domain $\delta = (\bar{a},\bar{b})$ in a frame $\mathfrak{F}$ if $\bar{a} \subseteq b\uparrow$ and, for each $c \in b\uparrow$, either $c=b$ or $c \in \bar{a}\uparrow$. (In other words, the world b "sees" itself and only those worlds that can be "seen" from $\bar{a}$.) According to the definition of d-domain, if b is a focus for $(\bar{a},\bar{b})$ then $b \in \bar{b}$. By $h(\mathfrak{F})$ we denote the "height" of $\mathfrak{F}$, i.e. the number of elements in its longest chain.

THEOREM 4. *Let L be axiomatized by canonical formulas $\chi(\mathfrak{F},\mathfrak{D},\bot)$ for which $h(\mathfrak{F}) \geq 3$ and $\mathfrak{D}$ contains a d-domain $\delta = (\bar{a},\bar{b})$ such that it has no focus and its first component $\bar{a}$ consists of some maximal elements in $\mathfrak{F}$. Then L has DP.*

Proof. Applying the sufficient condition for DP, we will use Proposition from §0 according to which every intermediate logic is determined by a class of sharp frames with finite covers. Let $\mathfrak{F}_1 = \langle W_1, R_1, S_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2, S_2 \rangle$ be sharp frames for L having finite covers. With each set $\bar{a}$ consisting of at least two maximal elements in the frame $\langle W_1 \cup W_2, R_1 \cup R_2 \rangle$ we associate a new element $c_{\bar{a}}$; the set of all such elements we denote by V. Now we construct the frame $\mathfrak{F}_0 = \langle W_0, R_0, S_0 \rangle$ where $W_0 = \{a_0\} \cup W_1 \cup W_2 \cup V$,

$$xR_0y \text{ iff } x=a_0 \vee (x,y \in W_1 \ \& \ xR_1y) \vee (x,y \in W_2 \ \& \ xR_2y) \vee$$
$$(x=c_{\bar{a}} \ \& \ (y=c_{\bar{a}} \vee y \in \bar{a}))$$

and S_0 is generated (as a pseudo-Boolean algebra) by the elements in S_1 and S_2. $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are generated subframes of $\mathfrak{F}_0$ (the proof for the previous construction may be used for the new one without changes), and so only (i), i.e. $\mathfrak{F}_0 \models L$, requires a justification.

Suppose that $X(\mathfrak{F},\mathfrak{D},\bot)$ is an axiom of L and $\mathfrak{F}_0 \not\models X(\mathfrak{F},\mathfrak{D},\bot)$. Since $\mathfrak{F}_i \models X(\mathfrak{F},\mathfrak{D},\bot)$, for $i = 1,2$, and $h(\mathfrak{F}) \geq 3$, there is a confinal subreduction f from $\mathfrak{F}_0$ onto $\mathfrak{F} = \langle W,R \rangle$ satisfying (d), with $a_0 \in f^{-1}(W)$. Take a d-domain $\delta = (\bar{c},\bar{b})$ in $\mathfrak{D}$ having no focus, with $\bar{c}$ consisting of some maximal elements in W, and consider the set $\bar{a}$ of maximal elements in W_0 such that $f(\bar{a}) = \bar{c}$ (the existence of $\bar{a}$ is provided by ($\bot$)). Since $\bar{a} \subseteq c_{\bar{a}}\!\uparrow$, we must have, by (d), $c_{\bar{a}} \in f^{-1}(\bar{b})\!\downarrow$ which is possible only when $f(c_{\bar{a}}) = b \in \bar{b}$. But then b is a focus for δ which is a contradiction. ■

Remark 1. Another interesting use of the notion of focus can be found in Zakharyashchev [1990a] where it is proved that among arbitrary (not necessarily closed under necessitation) extensions of S4Grz there is a greatest logic in which **Int** can be embedded by the Gödel translation (see §6). ■

Remark 2. The sufficient condition of Theorem 4 is used in Chagrov and Zakharyashchev [1989] for constructing incomplete and undecidable intermediate calculi with DP starting from well-known incomplete and undecidable calculi. ■

The conditions of Theorem 4 are clearly satisfied by the

Scott logic, since the d-domain $(\{\alpha_2,\alpha_3\},\{\alpha_1\})$ has no focus in the frame shown in Fig. 3. However, using only Theorems 3 and 4 it is impossible to prove DP of the Kreisel and Putnam [1957] logic

$$\mathbf{KP} = \mathbf{Int} + (\neg p \supset q \vee r) \supset (\neg p \supset q) \vee (\neg p \supset r),$$

which can be represented as

$$\mathbf{KP} = \mathbf{Int} + X(\mathfrak{F}_2, \mathfrak{D}, \bot) + X(\mathfrak{F}, \mathfrak{D}, \bot)$$

where $\mathfrak{F}_2$ and $\mathfrak{F}$ are depicted in Fig. 8 and Fig. 10, respectively, and $\mathfrak{D}$ contains only one d-domain $\delta = (\{\alpha_0,\ \alpha_1\},\ \{\alpha_2\})$. To prove DP

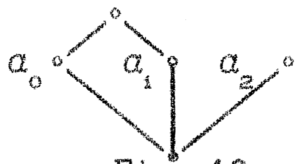


Fig. 10.

of this and other logics which do not satisfy our sufficient conditions new elements violating (d) should be put not only below maximal elements in $\mathfrak{F}_1$ and $\mathfrak{F}_2$, but below some other elements as well. It is not difficult to do this, but we must also guarantee that no new confinal subreductions satisfying (d) for some axiom will appear. The proofs of DP of **KP** (Gabbay [1970]) and its sublogics $\mathbf{ND}_k$ (Maksimova [1986]) show a way for strengthening our criteria, but this question is beyond the limits of the present paper.

§6. **Canonical formulas for S4.** According to the Blok-Esakia theorem (Blok [1976], Esakia [1979]), the lattice of extensions of **Int** is isomorphic to the lattice of normal extensions of **S4Grz**, with the isomorphism preserving such properties of logics as the tabularity, decidability, FMP, DP (the preservation theorems were proved by Maksimova and Rybakov [1974], Gudovshchikov and Rybakov [1982] and Zakharyashchev [1989a]). So, having proved the decidability or undecidability of one of these properties for intermediate calculi, we automatically obtain the same result for normal extensions of **S4Grz** and vice versa (if the isomorphism and its conversion are effective, of course). Unfortunately, the list of properties preserved by the isomorphism does not contain Halldén-completeness. Unlike the intermediate logics where HC follows from DP (which may be used for proving the undecidability of HC), in the modal case HC need an individual approach. We will study HC in the next section and meanwhile we give a brief introduction to the modal canonical formulas of Zakharyashchev [1984, 1988] which will be used in what follows.

The canonical formulas for **S4** are defined similarly to the canonical formulas for **Int**. The only difference is that they are associated with quasi-ordered frames $\mathfrak{F} = \langle W,R\rangle$ which may contain proper clusters, i.e. non-trivial equivalence classes under the following relation $\approx$: $a \approx b$ iff aRb & bRa. The disjunctive domains on the frame $\mathfrak{F}$ are defined in the same way as in the intuitionistic case.

Let $\mathfrak{F} = \langle W,R\rangle$ be a finite sharp quasi-ordered frame, with $a_0, \ldots, a_n$ being all the distinct elements in W and a_0 being an origin. Let $\mathfrak{D}$ be some (possibly empty) set of d-domains in $\mathfrak{F}$. Then

$$\mathcal{Y}(\mathfrak{F},\mathfrak{D},\bot) = \mathop{\&}_{a_iRa_j} A_{ij} \;\&\; \mathop{\&}_{i=0}^{n} A_i \;\&\; \mathop{\&}_{\delta\in\mathfrak{D}} B_\delta \;\&\; C \supset p_0,$$

where

$$A_{ij} = \Box(\Box p_j \supset p_i),$$

$$A_i = \Box((\&\Gamma_i \supset p_i) \supset p_i),$$

$$\Gamma_i = \{p_k, \Box p_l \mid k \neq i,\ \neg a_iRa_l\},$$

$$C = \Box(\mathop{\&}_{i=0}^{n} \Box p_i \supset \bot),$$

and if $\delta=(\bar{a},\bar{b})$ then

$$B_\delta = \Box(\mathop{\&}_{a_i\in\bar{b}} \Box p_i \supset \mathop{\vee}_{a_i\in\bar{a}} \Box p_i).$$

Here $\&\Gamma$ is the conjunction of all formulas in Γ (if $\Gamma=\emptyset$ then $\&\Gamma$ is $\top$).

The formula $\mathcal{Y}(\mathfrak{F},\mathfrak{D})$ is obtained from $\mathcal{Y}(\mathfrak{F},\mathfrak{D},\bot)$ by deleting the conjunct C. The formulas of the form $\mathcal{Y}(\mathfrak{F},\mathfrak{D},\bot)$ and $\mathcal{Y}(\mathfrak{F},\mathfrak{D})$ are called the *canonical* and *positive canonical formulas for* **S4**, respectively.

REFUTABILITY CRITERION FOR THE MODAL CANONICAL FORMULAS (Zakharyashchev [1984, 1988]):

(i) *$\mathfrak{F}' \not\models \mathcal{Y}(\mathfrak{F},\mathfrak{D},\bot)$ iff there is a confinal subreduction f from $\mathfrak{F}'$ onto $\mathfrak{F}$ satisfying (d).*

(ii) *$\mathfrak{F}' \not\models \mathcal{Y}(\mathfrak{F},\mathfrak{D})$ iff there is a subreduction f from $\mathfrak{F}'$ onto $\mathfrak{F}$ satisfying (d).* ■

Remark. If $\mathcal{Y}(\mathfrak{F},\mathfrak{D},\bot)$ is refuted by a general frame $\mathfrak{F}'$ under some valuation then a confinal subreduction f from $\mathfrak{F}'$ onto $\mathfrak{F}$ satisfying (d) can be constructed by taking

$$f(a) = \begin{cases} a_i & \text{if } a \not\models p_i \text{ and } a \models A \text{ where} \\ & A \text{ is the premise of } \mathcal{Y}(\mathcal{F},\mathfrak{D},\bot); \end{cases}$$

undefined otherwise. ■

COMPLETENESS THEOREM FOR THE MODAL CANONICAL FORMULAS (Zakharyashchev [1984, 1988]):

(i) *There exists an algorithm which, for each formula A, constructs a finite number of canonical formulas* $\mathcal{Y}(\mathfrak{F}_1,\mathfrak{D}_1,\bot), \ldots, \mathcal{Y}(\mathfrak{F}_n,\mathfrak{D}_n,\bot)$ *such that*

$$\mathbf{S4} + A = \mathbf{S4} + \mathcal{Y}(\mathfrak{F}_1,\mathfrak{D}_1,\bot) + \ldots + \mathcal{Y}(\mathfrak{F}_n,\mathfrak{D}_n,\bot).$$

(ii) *There exists an algorithm which, for each positive formula A (which contains only* $\supset$, &, $\vee$ *and* $\Box$*), constructs a finite number of positive canonical formulas* $\mathcal{Y}(\mathfrak{F}_1,\mathfrak{D}_1), \ldots, \mathcal{Y}(\mathfrak{F}_n,\mathfrak{D}_n)$ *such that*

$$\mathbf{S4} + A = \mathbf{S4} + \mathcal{Y}(\mathfrak{F}_1,\mathfrak{D}_1) + \ldots + \mathcal{Y}(\mathfrak{F}_n,\mathfrak{D}_n).$$ ■

As we already know, each intermediate logic which is axiomatizable by disjunctionless formulas can be represented in the form

$$\mathbf{Int} + \{\mathcal{X}(\mathfrak{F}_i,\emptyset,\bot)\}_{i\in Q}.$$

By the Diego-McKay theorem (see McKay [1968]), all such logics have FMP. Using the Refutability Criterion and Completeness Theorem above, it is easy to prove (see Zakharyashchev [1988]) that each modal logic of the form

$$\mathbf{S4} + \{\mathcal{Y}(\mathfrak{F}_i,\emptyset,\bot)\}_{i\in Q}$$

has FMP, and so is decidable if finitely axiomatizable. There are a continuum of such logics as it follows, e.g. from Theorem 8 below.

It is worth noting that all normal extensions of **S4**, which appeared in literature and were not created for obtaining another "negative" result, can be axiomatized only by formulas of the form $\mathcal{Y}(\mathfrak{F},\emptyset,\bot)$ or even $\mathcal{Y}(\mathfrak{F},\emptyset)$. For example, S4Grz = S4 + $Y(C_2, \emptyset)$,

where C_2 is the simplest proper cluster, i.e. the frame with two elements "seeing each other".

Now, a few words about the correspondence between extensions of Int and S4 which is given by the Godel translation T (remind that $T(A)$ is the result of prefixing $\Box$ to every subformula of an intuitionistic formula A).

A modal logic M is called a *(modal) companion* of an intermediate logic L if, for any intuitionistic formula A,

$L \vdash A$ iff $M \vdash T(A)$.

Using canonical axiomatization, the relationship between axioms of an intermediate logic and axioms of its modal companions can be represented as follows (see Zakharyashchev [1984, 1989]).

MODAL COMPANION THEOREM. *A logic M is a modal companion of an intermediate logic*

L = Int + $\{X(\mathfrak{F}_i, \mathfrak{D}_i, \bot)\}_{i \in Q}$

iff M can be represented in the form

M = S4 + $\{Y(\mathfrak{F}_i, \mathfrak{D}_i, \bot)\}_{i \in Q}$ + $\{Y(\mathfrak{F}_j, \mathfrak{D}_j, \bot)\}_{j \in P}$,

where each of the frames $\mathfrak{F}_j$, for $j \in P$, contains at least one proper cluster. ∎

It follows from this theorem and the Refutability Criterion that among the set of modal companions of each intermediate logic L = Int + $\{X(\mathfrak{F}_i, \mathfrak{D}_i, \bot)\}_{i \in Q}$ there always exist the least and greatest companions, viz. the logics τL = S4 + $\{Y(\mathfrak{F}_i, \mathfrak{D}_i, \bot)\}_{i \in Q}$ and $\sigma L = \tau L + Y(C_2, \emptyset)$ = S4Grz + τL, respectively. (These results were first obtained by Maksimova and Rybakov [1974], Blok [1976] and Esakia [1979]).

We need also the following results concerning the relationship between frames for logics L, τL and σL (see Zakharyashchev [1989a]). If L is characterized by a general frame $\langle W,R,S\rangle$ then, forming the Boolean closure S' of S, we obtain the general frame $\langle W,R,S'\rangle$ which characterizes σL; in particular, if L is the logic of a finite frame $\langle W,R\rangle$ then the same frame will characterize σL (this is not in general true for infinite frames $\langle W,R\rangle$; moreover, Shekhtman [1980] constructed a logic L which is Kripke complete but σL is not). It follows from this result and Proposition proved in §0 that each normal extension of **S4Grz** is characterized by some class of general frames having finite covers. As for τL, we will use only the Dummett and Lemmon [1959] conjecture which was proved by Zakharyashchev [1989a]: if L is characterized by a Kripke frame $\mathfrak{F}$ then τL is characterized by the Kripke frame $\omega\mathfrak{F}$ obtained from $\mathfrak{F}$ by replacing each of its elements with the cluster containing ω points. Thus, L is Kripke complete iff τL is.

The mapping σ preserves such properties of logics as the tabularity, FMP, decidability, DP, while τ, in addition to this list, preserves Kripke completeness but does not preserve the tabularity (consult Zakharyashchev [1989a]). Note also that all mentioned properties are preserved while passing from any modal companion of an intermediate logic L to L itself. These results will be used below under the name of Preservation Theorems.

To apply the Preservation Theorems for transferring results on the decidability or undecidability of properties from modal calculi containing **S4Grz** to intermediate ones and vice versa, we must justify the effectiveness of the transitions $L \leftrightarrow \sigma L$. The transition from L to σL is effective by the definition: if L =

= **Int**+ A then σL = **S4Grz** + $T(A)$. The effectiveness of the converse transition follows from the Completeness Theorem and Modal Companion Theorem. Indeed, if σL = **S4Grz** + A then first we can effectively find a canonical representation

$$\sigma L = \mathbf{S4} + Y(\mathfrak{F}_1, \mathfrak{D}_1, \bot) + \ldots + Y(\mathfrak{F}_n, \mathfrak{D}_n, \bot) + Y(C_2, \emptyset),$$

with intuitionistic frames $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$, and after that we obtain

$$L = \mathbf{Int} + X(\mathfrak{F}_1, \mathfrak{D}_1, \bot) + \ldots + X(\mathfrak{F}_n, \mathfrak{D}_n, \bot).$$

In the same way, given a modal formula A, we can effectively find an intermediate calculus with the modal companion **S4** + A.

§7. Some theorems on Halldén-completeness.

THEOREM 5. *There is a Halldén-complete intermediate logic having no Halldén-complete modal companions.*

Proof. Let us consider the intermediate logic L of the frame $\mathfrak{F}$ shown in Fig. 11. If this frame refutes some intuitionistic formulas A_1 and A_2 having no variables in common then the disjunction $A_1 \vee A_2$ is refuted at the point a. Therefore, L is Halldén-complete. Note by the way that L (as well as any other tabular logic) does not have DP.

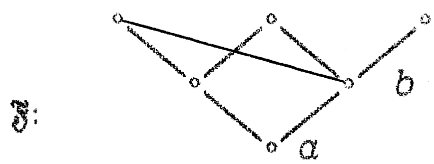


Fig. 11.

Now take an arbitrary modal companion M of L and show that it is Halldén-incomplete. For this end we construct two canonical formulas (without common variables) $A = Y(\mathfrak{F}_1, \emptyset, \bot)$ and $B = Y(\mathfrak{F}_2, \mathfrak{D}, \bot)$

where $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are depicted in Fig. 12 and $\mathfrak{D}$ contains all

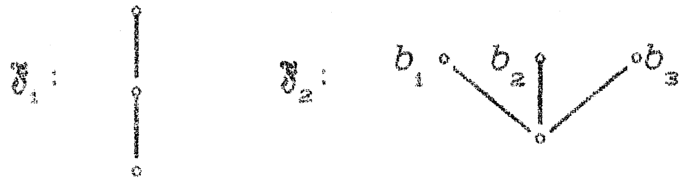


Fig. 12.

d-domains of the form $(\{b_i, b_j\}, \{b_k\})$. (To refute A at x in some frame, x has to "see" a chain containing three different clusters, and to refute B at x, the subframe generated by x has to be reducible to $\mathfrak{F}_2$.) A and B are evidently refuted in $\mathfrak{F}$ (A at a and B at b), and since $\mathfrak{F}$ characterizes the logic $\sigma L \supseteq M$, we have $M \nvdash A$ and $M \nvdash B$.

Let $\omega\mathfrak{F}$ be the frame obtained from $\mathfrak{F}$ by replacing its elements with the clusters containing ω points (in other words, $\omega\mathfrak{F}$ is the direct product of $\mathfrak{F}$ and the cluster with ω points). As was mentioned in §6, $\omega\mathfrak{F}$ characterizes the least modal companion $\tau L \subseteq M$ of L. It remains to note that $\tau L \vdash A \vee B$. Indeed, A may be refuted in $\omega\mathfrak{F}$ only at elements of the cluster generated by a, while B only at elements of the cluster generated by b, and so $\omega\mathfrak{F} \models A \vee B$. ■

Remark. The proof above demonstrates only one example of a Halldén-complete intermediate logic having no Halldén-complete modal companions. However, one can show that there are a continuum of them, the witnesses are the logics of the frames shown in Fig. 13, where each + may be replaced by a point • or just omitted (the proof above can be adapted for these logics too). We leave details to the reader. ■

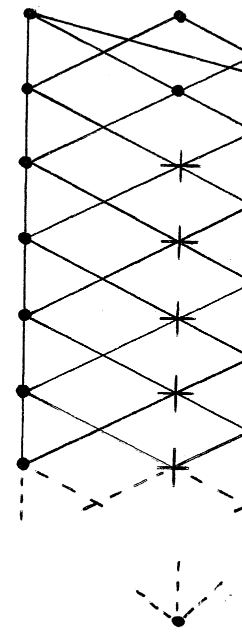


Fig. 13

In the class of intermediate logics, DP implies HC. In the modal case these two properties turn out to be practically independent of each other. The next theorem presents a Halldén-incomplete modal logic with DP, and afterwards we will show that there are a continuum of Halldén-complete modal logics with and without DP.

THEOREM 6. *There is a Halldén-incomplete modal logic having the disjunction property.*

Proof. Take the class of all partially ordered frames having the form of finite binary trees and join a copy of the frame $\mathfrak{F}$ in Fig. 11 to every maximal element of each such frame (see Fig. 14). Consider the modal logic M of the resulting class $\mathcal{C}$. If formulas $\Box A_1$ and $\Box A_2$ do not belong to M, and so are refuted in some frames $\mathfrak{F}_1$ and $\mathfrak{F}_2$ from $\mathcal{C}$, then the disjunction $\Box A_1 \vee \Box A_2$ is refuted at the least element of the frame shown in Fig. 6 which is also in $\mathcal{C}$. Thus, M has DP.

In order to prove that M is Halldén-incomplete, we use the formulas A and B from the proof of Theorem 5 once again. Both these formulas are refuted by all frames in $\mathscr{C}$ and so do not belong to M. However, B is refuted only at b in the joined copies of $\mathfrak{F}$ where A is always true. Therefore, the disjunction $A \vee B$ is valid in all frames in $\mathscr{C}$ and so belongs to M. ■

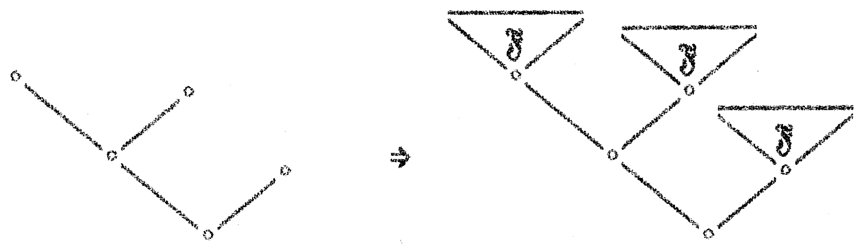


Fig. 14.

Remark. Actually, there are a continuum of Halldén-incomplete modal logics with DP; for a proof see Chagrov [1991a]. ■

Now we obtain two sufficient conditions for HC of normal extensions of **S4Grz** which are formulated in the same spirit as Theorems 3 and 4. It follows from the Refutability Criterion that **S4Grz** $\vdash \mathcal{Y}(\mathfrak{F},\mathfrak{D},\bot)$, for any frame $\mathfrak{F}$ having a proper cluster and any set $\mathfrak{D}$ of d-domains. So each logic $M \supseteq$ **S4Grz** is represented in the form

(11) M = **S4** + $\mathcal{Y}(C_2,\emptyset)$ + $\{\mathcal{Y}(\mathfrak{F}_i,\mathfrak{D}_i,\bot)\}_{i\in Q}$ = **S4Grz** + $\{\mathcal{Y}(\mathfrak{F}_i,\mathfrak{D}_i,\bot)\}_{i\in Q}$,

where every frame $\mathfrak{F}_i$, for $i \in Q$, is partially ordered.

THEOREM 7. *If a logic M of the form (11) is Kripke complete and in every frame $\mathfrak{F}_i$, for $i \in Q$, the least element has only one immediate successor then M is Halldén-complete.*

Proof. Suppose formulas A and B have no common variables, $M \nvdash A$ and $M \nvdash B$. We show that in this case $M \nvdash A \vee B$. Since M is Kripke complete, there are partially ordered sharp Kripke frames $\mathfrak{F}^1$ and

$\mathfrak{F}^2$ for M, with a_1 and a_2 being their least elements, respectively, such that A is refuted at a_1 and B at a_2. Note that $\mathfrak{F}^1$ and $\mathfrak{F}^2$ contain no infinite ascending chains, for otherwise they will refute $Y(C_2,\emptyset)$, i.e. they will not validate M. From $\mathfrak{F}^1$ and $\mathfrak{F}^2$ we construct a new frame $\mathfrak{F}$ by gluing a_1 and a_2 into one point a (see Fig. 15) and show that $\mathfrak{F} \models M$ and $\mathfrak{F} \not\models A \vee B$.

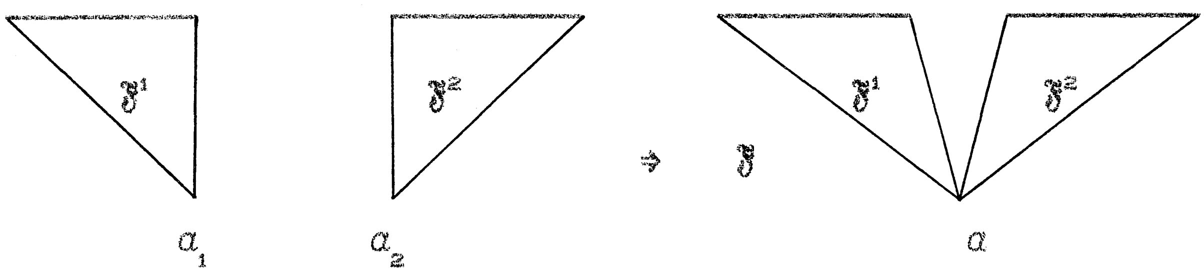


Fig. 15.

Let $\mathfrak{F} \not\models Y(\mathfrak{F}_i,\mathfrak{D}_i,\bot)$, for some $i\in Q$, i.e. there is a confinal subreduction f from $\mathfrak{F}$ onto $\mathfrak{F}_i$ satisfying (d) for $Y(\mathfrak{F}_i,\mathfrak{D}_i,\bot)$. The element a must be mapped by f onto the least element a' in $\mathfrak{F}_i$. For otherwise, taking the restriction of f to the frame $\mathfrak{F}^j$ ($j\in\{1,2\}$) which contains a point from $f^{-1}(a')$ different from a, we would obtain a confinal subreduction from $\mathfrak{F}^j$ onto $\mathfrak{F}_i$ satisfying (d) for $Y(\mathfrak{F}_i,\mathfrak{D}_i,\bot)$, and so $\mathfrak{F}^j \not\models Y(\mathfrak{F}_i,\mathfrak{D}_i,\bot)$ which is impossible. Suppose now that a pre-image of the immediate successor of a' is contained in $\mathfrak{F}^j$, for some $j\in\{1,2\}$. Then again we take the restriction of f to $\mathfrak{F}^j$ and obtain a confinal subreduction from $\mathfrak{F}^j$ onto $\mathfrak{F}_i$ satisfying (d), i.e. $\mathfrak{F}^j \not\models Y(\mathfrak{F}_i,\mathfrak{D}_i,\bot)$ which is a contradiction.

It remains to show that $A \vee B$ is refutable in $\mathfrak{F}$. As we know, A is refuted at a_1 in $\mathfrak{F}^1 = \langle W_1, R_1\rangle$. Let b be some maximal element in $\mathfrak{F}^1$ (its existence is guaranteed by the absence of infinite ascending chains in $\mathfrak{F}^1$). Define a mapping g from $\mathfrak{F}$ onto $\mathfrak{F}^1$ by taking

$$g(a) = a_1,$$

and, for $c \neq a$,

$$g(c) = \begin{cases} c \text{ if } c \in W_1, \\ b \text{ if } c \notin W_1. \end{cases}$$

It is not difficult to see that this mapping is a reduction from $\mathfrak{F}$ onto $\mathfrak{F}^1$. So, by the P-morphism Theorem, A is refuted at a in $\mathfrak{F}$. By the same arguments, B is also refuted at a. Since A and B have no common variables, we can always do so that both these formulas are refuted at a simultaneously. ■

As a consequence of this sufficient condition, we obtain

THEOREM 8. *There are a continuum of Halldén-complete normal extensions of* **S4Grz** *without the disjunction property.*

Proof. Let us consider the sequence of frames $\mathfrak{F}_n$ shown in Fig. 16 (cf. Fine [1985, p.631]). It is not difficult to see that $\mathfrak{F}_i$ validates $\mathcal{Y}(\mathfrak{F}_j, \emptyset, \bot)$, for all $i, j < \omega$, $j \neq i$, (and refutes $\mathcal{Y}(\mathfrak{F}_i, \emptyset, \bot)$, of course). Thus, different sets of formulas $\mathcal{Y}(\mathfrak{F}_i, \emptyset, \bot)$ will give us a continuum of different logics which have FMP (see §6), hence are Kripke complete, and so, by Theorem 7, are Halldén-complete.

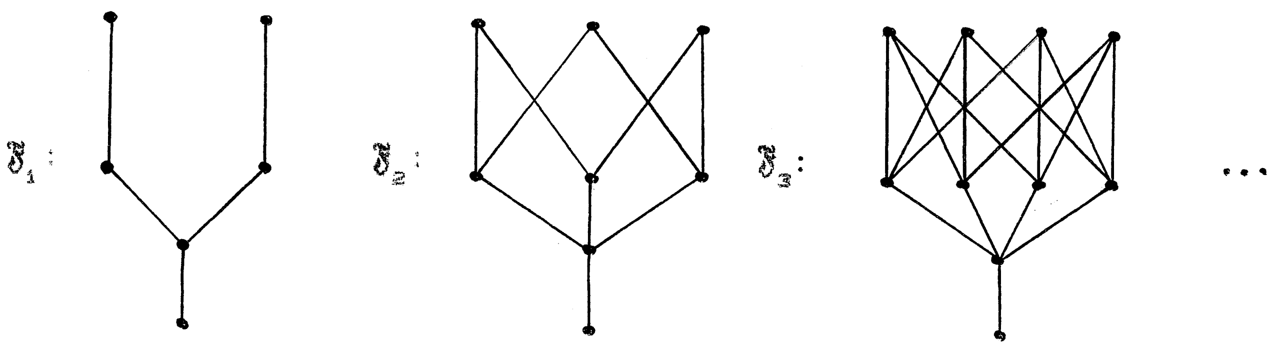


Fig. 16.

These logics, however, do not have DP, since they are the greatest modal companions of intermediate logics axiomatizable by disjunctionless formulas which do not have DP, as was shown by Minari [1986] and Zakharyashchev [1987]. ■

The following sufficient condition for HC is (by Theorem 3, the Modal Companion Theorem and the corresponding Preservation Theorem) a sufficient condition for DP as well.

THEOREM 9. *Let M be a normal extension of* **S4Grz** *which is axiomatizable by canonical formulas $\gamma(\mathfrak{F},\mathfrak{D},\bot)$ (with partially ordered $\mathfrak{F}$) such that the set V_0 of the immediate successors of the origin in $\mathfrak{F}$ contains at least three elements and*

$$\forall V \subseteq V_0 \; (\lceil \overline{\overline{V}}_0/2 \rceil \leq \overline{\overline{V}} \leq \overline{\overline{V}}_0 - 1 \;\Rightarrow\; \exists \bar{a} \subseteq V\!\uparrow \exists \bar{b} \subseteq V_0 - V \; (\bar{a},\bar{b}) \in \mathfrak{D} \;).$$

Then M is Hallden-complete.

Proof. Suppose that formulas A and B have no variables in common, $M \nvDash A$ and $M \nvDash B$. Then there are sharp frames $\mathfrak{F}_1 = \langle W_1, R_1, S_1 \rangle$ and $\mathfrak{F}_2 = \langle W_2, R_2, S_2 \rangle$ for M which have finite covers and refute A and B at their origins a_1 and a_2, respectively. Construct from them a frame $\mathfrak{F}_0 = \langle W_0, R_0, S_0 \rangle$, as it is shown in Fig. 6, where S_0 is generated (as a topological Boolean algebra) by the elements of S_1 and S_2. An easy inspection may convince us of that $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are generated subframes of $\mathfrak{F}_0$. In exactly the same way as in the proof of Theorem 3 we can show that $\mathfrak{F}_0 \models M$. Thus, it remains to prove that both formulas A and B are refuted at the origin a_0 in $\mathfrak{F}_0$.

Choose some maximal element b in $\mathfrak{F}_1$ (it exists, since $\mathfrak{F}_1$ has a finite cover) and construct a mapping f from W_0 onto W_1 by taking

$$f(a_0) = a_1,$$

and, for $a \neq a_0$,

$$f(a) = \begin{cases} a \text{ if } a \in W_1, \\ b \text{ if } a \in W_2. \end{cases}$$

This mapping is clearly a reduction from $\mathfrak{F}_0$ onto $\mathfrak{F}_1$, and so, by the P-morphism Theorem, A is refuted at a_0. B is considered analogously. ■

Note once again that modal logics satisfying the conditions of Theorem 9 have DP, since they are the greatest modal companions of intermediate logics with DP. There are a continuum of such logics, as it is easy to show by considering the logics with canonical axioms of the form $\gamma(\mathfrak{F}_i^*, \mathfrak{D}_i^*, \bot)$, for $i \geq 2$, where $\mathfrak{F}_i^*$ is obtained from the frame $\mathfrak{F}_i$, used in the proof of Theorem 8, by throwing out its first element (i.e. the ith frame in the sequence of Fine [1985, p.631]) and $\mathfrak{D}_i^*$ is the set of all d-domains in $\mathfrak{F}_i^*$. Thus, we have the following

THEOREM 10. *There are a continuum of Halldén-complete normal extensions of* **S4Grz** *having the disjunction property.* ■

To complete the picture we present two more theorems.

THEOREM 11. *There are a continuum of Halldén-incomplete normal extensions of* **S4Grz** *without the disjunction property.*

Proof. Follows from Theorem 12 (iii) and the obvious fact that both the absence of DP and Halldén-incompleteness are preserved when passing from an intermediate logic to its arbitrary modal companion. ■

The first statement of the next theorem was obtained by Wronski [1973] and the others by Galanter [1988].

THEOREM 12. *There are a continuum of*

(i) (Halldén-complete) intermediate logics with DP;

(ii) Halldén-complete intermediate logics without DP;

(iii) Halldén-incomplete intermediate logics (without DP).

Proof. (i) and (ii) are consequences of Theorems 10 and 8 and the corresponding Preservation Theorems. So we must prove only (iii).

By Cn_k we denote the frame which is the chain of k (reflexive) points. It is easy to see that there are a continuum of different logics of the form

$L = \mathbf{Int} + X(Cn_4, \emptyset) + \{X(\mathfrak{F}_i^*, \emptyset, \bot)\}_{i \in Q}$,

where Q is some set of natural numbers and $\mathfrak{F}_i^*$ is the ith frame of Fine's sequence. Each of these logics is incomparable (by inclusion) with the logic of the frame Cn_4, and so their intersection is Halldén-incomplete. It remains to note that there are a continuum of such intersections, since $X(\mathfrak{F}_j^*, \emptyset, \bot)$ belongs to the intersection of L and the logic of Cn_4 iff $j \in Q$. ■

§8. **The undecidability of the disjunction property and Halldén-completeness.**

THEOREM 13. *There is no algorithm which is capable of deciding, given a formula A, whether* **S4Grz** + *A has the disjunction property.*

Proof. We will use the scheme for proving the undecidability which was discussed in §1 and the constructions of §2.

Let $\mathfrak{F}_0$ be the frame shown in Fig. 2. Take $\delta = (\{c_2, d_2\}, \{d_3\})$, $\mathfrak{D}_0 = \{\delta\}$ and construct the canonical formula $\mathcal{Y}(\mathfrak{F}_0, \mathfrak{D}_0, \bot)$. Since, by Theorem 4, the intermediate logic

$$\mathbf{Int} + \mathcal{X}(\mathfrak{F}_0, \mathfrak{D}_0, \bot)$$

has DP, by the Preservation Theorem, its greatest modal companion

$$\mathbf{S4Grz} + \mathcal{Y}(\mathfrak{F}_0, \mathfrak{D}_0, \bot)$$

has DP too.

Thus, we may take $F = \mathcal{Y}(\mathfrak{F}_0, \mathfrak{D}_0, \bot)$. Let $C_1 = A \,\&\, \neg p_{i_1}$, $D_1 = A \,\&\, \neg p_{i_2}$, $A^i_j = A \,\&\, \neg p_{i_3}$, $B^i_j = A \,\&\, \neg p_{i_4}$, where A is the premise of $\mathcal{Y}(\mathfrak{F}_0, \mathfrak{D}_0, \bot)$ and p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4} are the variables in $\mathcal{Y}(\mathfrak{F}_0, \mathfrak{D}_0, \bot)$ corresponding to the elements c_i, d_i, a^i_j, b^i_j in $\mathfrak{F}_0$, respectively (i=1,2,3, j=-2,-3).

With a Minsky program P and a configuration $\mathfrak{a}$ we associate the frame $\mathfrak{F}$ shown in Fig. 1. We have already noted that there is only one subreduction from $\mathfrak{F}$ onto $\mathfrak{F}_0$. It is clearly confinal and satisfies (d) for F, hence $\mathfrak{F} \not\models F$. Therefore, by the Remark to the Refutability Criterion from §6, the formulas F, C_1, D_1, A^i_j, B^i_j satisfy the conditions (6) - (8) of §2.

Let us now choose G. Let $\mathfrak{F}_1$ be the frame shown in Fig. 17. Construct the formula $\mathcal{Y}(\mathfrak{F}_1, \emptyset, \bot)$ having no common variables with F and take

$G = \Box F \vee \Box Y(\mathfrak{F}_1, \emptyset, \bot)$.

It is clear that G satisfies (4) of §1.

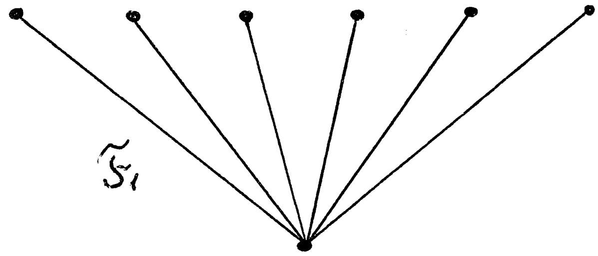


Fig. 17.

Suppose $\mathfrak{b}$ is an arbitrary configuration. Consider the logic

$L(P, \mathfrak{a}, \mathfrak{b}) = \mathbf{S4Grz} + AxP + C(\mathfrak{a}, \mathfrak{b}) \supset F + G$,

If $P: \mathfrak{a} \to \mathfrak{b}$ then, by (2) - (4),

$L(P, \mathfrak{a}, \mathfrak{b}) = \mathbf{S4Grz} + F$,

and so $L(P, \mathfrak{a}, \mathfrak{b})$ has DP.

Suppose $P: \mathfrak{a} \not\to \mathfrak{b}$. According to Lemma 3, (6) and (9),

$\mathfrak{F} \models AxP$, $\mathfrak{F} \models C(\mathfrak{a}, \mathfrak{b}) \supset F$.

Moreover, since $\mathfrak{F}$ has only five maximal elements, $\mathfrak{F} \models Y(\mathfrak{F}_1, \emptyset, \bot)$, and so $\mathfrak{F} \models G$. Thus, $\mathfrak{F} \models L(P, \mathfrak{a}, \mathfrak{b})$, $\mathfrak{F} \not\models \Box F$, i.e. $L(P, \mathfrak{a}, \mathfrak{b}) \nvdash \Box F$.

Now consider $\mathfrak{F}_1$. It is clear that $\mathfrak{F}_1 \models F$, hence $\mathfrak{F}_1 \models L(P, \mathfrak{a}, \mathfrak{b})$, and so $L(P, \mathfrak{a}, \mathfrak{b}) \nvdash \Box Y(\mathfrak{F}_1, \emptyset, \bot)$, since $\mathfrak{F}_1 \not\models Y(\mathfrak{F}_1, \emptyset, \bot)$.

Thus, we have $L(P, \mathfrak{a}, \mathfrak{b}) \vdash \Box F \vee \Box Y(\mathfrak{F}_1, \emptyset, \bot)$, $L(P, \mathfrak{a}, \mathfrak{b}) \nvdash \Box F$ and $L(P, \mathfrak{a}, \mathfrak{b}) \nvdash \Box Y(\mathfrak{F}_1, \emptyset, \bot)$, i.e. $L(P, \mathfrak{a}, \mathfrak{b})$ does not have DP.

Our Theorem follows now from the undecidability of the configuration problem for Minsky machines. ■

Using the preservation of DP when passing from an intermediate logic to its greatest modal companion and vice versa and also the fact that, given a modal formula A, an intuitionistic formula B can be effectively constructed so that $\mathbf{S4Grz} + A =$ $= \mathbf{S4Grz} + T(B)$ (see the end of §6), as a consequence of Theorem 13 we obtain the following

THEOREM 14. *There is no algorithm which can recognize, given a formula A, whether* Int + A *has the disjunction property.* ■

The next theorem is a consequence of the proof of Theorem 13.

THEOREM 15. *There is no algorithm which can recognize, given a formula A, whether* Int + A *is Halldén-complete.*

Proof. Let P be a Minsky program, $\mathfrak{a}$ and $\mathfrak{b}$ be configurations and $L(P,\mathfrak{a},\mathfrak{b})$ be the modal logic constructed in the proof of Theorem 13. Consider the intermediate logic $L'(P,\mathfrak{a},\mathfrak{b})$ having $L(P,\mathfrak{a},\mathfrak{b})$ as its greatest modal companion. Note that $L'(P,\mathfrak{a},\mathfrak{b})$ can be effectively constructed of P, $\mathfrak{a}$ and $\mathfrak{b}$.

If $P:\mathfrak{a}\to\mathfrak{b}$ then $L'(P,\mathfrak{a},\mathfrak{b})$, as well as $L(P,\mathfrak{a},\mathfrak{b})$, has DP, and so is Halldén-complete.

Suppose $P:\mathfrak{a}\not\to\mathfrak{b}$. We will show that in this case $L'(P,\mathfrak{a},\mathfrak{b})$ is Halldén-incomplete.

Let

$B = X(\mathfrak{F}_0,\mathfrak{D}_0,\bot)$, $\quad C = X(\mathfrak{F}_1,\emptyset,\bot)$,

$B^T = Y(\mathfrak{F}_0,\mathfrak{D}_0,\bot)$, $\quad C^T = Y(\mathfrak{F}_1,\emptyset,\bot)$

where $\mathfrak{F}_0$, $\mathfrak{D}_0$, $\mathfrak{F}_1$ are those defined in the proof of Theorem 13. We may assume that B and C, as well as B^T and C^T, have no variables in common. By the Modal Companion Theorem,

$$\mathbf{S4} + B^T = \mathbf{S4} + T(B)$$

and

$$\mathbf{S4} + C^T = \mathbf{S4} + T(C).$$

According to the proof of Theorem 13, we have

(12) $L(P,\mathfrak{a},\mathfrak{b}) \vdash \Box B^T \vee \Box C^T$,

(13) $L(P,\mathfrak{a},\mathfrak{b}) \nvdash \Box B^T$ and $L(P,\mathfrak{a},\mathfrak{b}) \nvdash \Box C^T$.

From (13) we obtain $L'(P,\mathfrak{a},\mathfrak{b}) \nvdash B$ and $L'(P,\mathfrak{a},\mathfrak{b}) \nvdash C$. Therefore, it suffices now to show that $L'(P,\mathfrak{a},\mathfrak{b}) \vdash B \vee C$.

Suppose otherwise, i.e. $L'(P,\mathfrak{a},\mathfrak{b}) \nvdash B \vee C$. Then $L(P,\mathfrak{a},\mathfrak{b}) \nvdash T(B \vee C)$ and hence $L(P,\mathfrak{a},\mathfrak{b}) \nvdash T(B) \vee T(C)$. So there is a sharp general frame $\mathfrak{F}$ for $L(P,\mathfrak{a},\mathfrak{b})$ refuting $T(B) \vee T(C)$. Then $\mathfrak{F} \nvDash T(B)$, $\mathfrak{F} \nvDash T(C)$, and so $\mathfrak{F} \nvDash B^T$ and $\mathfrak{F} \nvDash C^T$. But then $\mathfrak{F} \nvDash \Box B^T \vee \Box C^T$ (since $\mathfrak{F}$ is sharp and $\Box B^T$ and $\Box C^T$ have no common variables) which contradicts (12). ■

THEOREM 16. *There is no algorithm which is capable of deciding, given a formula A, whether* **S4Grz** + *A is Halldén-complete.*

Proof. Take $F = \chi(\mathfrak{F}_0,\emptyset,\bot)$, $G = \chi(\mathfrak{F}_0,\emptyset,\bot) \vee \chi(\mathfrak{F}_1,\emptyset,\bot)$ where $\mathfrak{F}_0$ and $\mathfrak{F}_1$ are the frames shown in Fig. 2 and Fig. 17, respectively. As was noted in §6, the logic

S4Grz + $\chi(\mathfrak{F}_0,\emptyset,\bot)$

has FMP, hence it is Kripke complete, and so, by Theorem 7, is Halldén-complete.

The remaining part of the proof is similar to the proof of Theorem 13. ■

Having proved Theorems 1, 13 and 16, we incidentally established the undecidability of some other properties of intermediate and modal calculi. Here are a few of them.

THEOREM 17. *There is no algorithm which can recognize, given a formula A, whether the logic* **S4Grz** + *A is axiomatizable by*

(i) canonical formulas of the form $\chi(\mathfrak{F},\emptyset)$ *(i.e. whether it is a subframe logic);*

(ii) canonical formulas of the form $\chi(\mathfrak{F},\emptyset,\bot)$.

Proof. Indeed, the logic $L(P,\mathfrak{a},\mathfrak{b})$ constructed in the proof of Theorem 1 is a subframe logic iff $P:\mathfrak{a}\to\mathfrak{b}$. Moreover, in that proof we could use as F the formula $\chi(\mathfrak{F}_0,\emptyset,\bot)$ instead of $\neg B_{\mathfrak{F}_0}$.

This readily gives us (ii) if we remember that all logics axiomatizable by formulas of the form $\gamma(\mathfrak{F}, \emptyset, \bot)$ have FMP. ■

The following theorem is a consequence the previous one, the Modal Companion Theorem and Completeness Theorem for the intuitionistic canonical formulas.

THEOREM 18. *There is no algorithm which can recognize, given a formula A, whether the logic* **Int** + *A is axiomatizable by*

(i) implicative formulas;

(ii) disjunctionless formulas. ■

Finally, the next theorem follows from Theorems 1, 2 and the preservation of the decidability and FMP when passing from an intermediate logic to its greatest modal companion and vice versa (see Zakharyashchev [1989a]).

THEOREM 19. *There is no algorithm which can recognize, given a formula A, whether the logic* **Int** + *A*

(i) has the finite model property;

(ii) is decidable. ■

It is interesting to compare the undecidability results above with those obtained by Chagrov [1991] for extensions of the Gödel-Löb provability logic **GL**. The finite model property, decidability and disjunction property are also undecidable. Moreover, the interpolation property turns out to be undecidable too which contrasts with its decidability in the classes of intermediate logics and normal extensions of **S4** (see Maksimova [1977, 1979a]). As for Halldén-completeness, it is decidable in the class of normal extensions of **GL** but undecidable in the class of arbitrary extensions of **GL** and even of the Solovay logic **S**.

§9. **On Maksimova-completeness.** The following property – so called *variable separation principle* – was considered by Maksimova [1976, 1979] for relevant and intermediate logics: if $L \vdash A \& B \supset C \vee D$, with $A \supset C$ and $B \supset D$ having no variables in common, then $L \vdash A \supset C$ or $L \vdash B \supset D$. This property is clearly related to Halldén-completeness, and we will call it *Maksimova-completeness* (MC).

For modal logics, there is no difference between the two properties.

THEOREM 20. *Each modal logic is Maksimova-complete iff it is Halldén-complete.*

Proof. Trivial, since $A \& B \supset C \vee D$ is classically equivalent to $(A \supset C) \vee (B \supset D)$. ■

In the case of intermediate logics we have only the obvious half of the previous theorem:

THEOREM 21. *Each Maksimova-complete intermediate logic is Halldén-complete.* ■

Indeed, the following is true.

THEOREM 22. *There is a Halldén-complete intermediate logic which is Maksimova-incomplete.*

Proof. Consider the logic L of the frame $\mathfrak{F}$ in Fig. 11 constructed in the proof of Theorem 5 (see §7) and show that it is Maksimova-incomplete. Take the canonical formulas $\chi(\mathfrak{F}_1, \emptyset, \bot)$ and $\chi(\mathfrak{F}_2, \mathfrak{D}, \bot)$ having no common variables, where $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are the frames depicted in Fig. 12 and $\mathfrak{D}$ consists of all d-domains of the form $(\{b_i, b_j\}, \{b_k\})$. Let $A = \top$, $C = \chi(\mathfrak{F}_1, \emptyset, \bot)$, B be the premise of $\chi(\mathfrak{F}_2, \mathfrak{D}, \bot)$ and D its conclusion. Then it is easy to verify that $\mathfrak{F} \nvDash A \supset C$, $\mathfrak{F} \nvDash B \supset D$, but $\mathfrak{F} \vDash A \& B \supset C \vee D$. ■

Remark. In the proof above we might take the logic determined by any of the frames shown in Fig. 13, and so there are a continuum of intermediate logics with the properties mentioned in Theorem 22. Note that these logics do not have DP, since all of them are the logics of width 3 (and contain, say, $X(\mathfrak{F}_1, \emptyset, \bot)$, for $\mathfrak{F}_1$ depicted in Fig. 17). Using constructions of Chagrov [1991a], one can show that there are a continuum of Maksimova-incomplete intermediate logics with DP. ■

The following two criteria for Maksimova-completeness are proved in exactly the same way as the criteria for Halldén-completeness in §6.

THEOREM 23. *If an intermediate logic* $L = \mathbf{Int} + \{X(\mathfrak{F}_i, \mathfrak{D}_i, \bot)\}_{i \in Q}$ *is Kripke complete and in every frame* $\mathfrak{F}_i$, *for* $i \in Q$, *the least element has only one immediate successor then* L *is Maksimova-complete.* ■

THEOREM 24. *Let* L *be an intermediate logic which is axiomatizable by canonical formulas* $X(\mathfrak{F}, \mathfrak{D}, \bot)$ *such that the set* V_0 *of immediate successors of the least element in* $\mathfrak{F}$ *contains at least three elements and*

$$\forall V \subseteq V_0 \; (\lceil \bar{\bar{V}}_0 / 2 \rceil \leq \bar{\bar{V}} \leq \bar{\bar{V}}_0 - 1 \;\Rightarrow\; \exists \bar{a} \subseteq V{\uparrow} \; \exists \bar{b} \subseteq V_0 - V \; (\bar{a}, \bar{b}) \in \mathfrak{D} \;).$$

Then L *is Maksimova-complete.* ■

THEOREM 25. *There are a continuum of Maksimova-complete intermediate logics with the disjunction property and as many without the disjunction property.*

Proof. Similar to the proofs of Theorems 8 and 10. ■

Thus, the relationship between the disjunction property, Halldén-completeness and Maksimova-completeness in the class IL of all intermediate logics may be represented as in Fig. 18, with the cardinality of each depicted set of logics being of continuum.

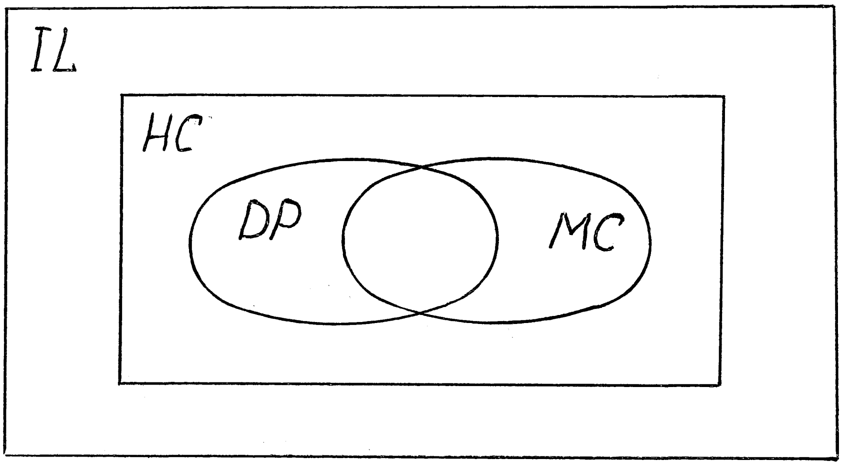


Fig. 18.

THEOREM 26. *There is no algorithm which is capable of deciding, given a formula A, whether* **Int** + *A is Maksimova-complete.*

Proof. Let $L(P,\mathfrak{a},\mathfrak{b})$ be the modal logic constructed in the proof of Theorem 16 and $L'(P,\mathfrak{a},\mathfrak{b})$ be the intermediate logic having $L(P,\mathfrak{a},\mathfrak{b})$ as its greatest modal companion.

If $P{:}\,\mathfrak{a}\to\mathfrak{b}$ then $L(P,\mathfrak{a},\mathfrak{b}) = \mathbf{S4Grz} + Y(\mathfrak{F}_0,\emptyset,\bot)$, where $\mathfrak{F}_0$ is the frame shown in Fig. 2. Hence, by the Modal Companion Theorem, $L'(P,\mathfrak{a},\mathfrak{b}) = \mathbf{Int} + X(\mathfrak{F}_0,\emptyset,\bot)$. But then $L'(P,\mathfrak{a},\mathfrak{b})$ has FMP, and so, by Theorem 23, it is Maksimova complete.

If $P{:}\,\mathfrak{a}\not\to\mathfrak{b}$ then, taking

$B = X(\mathfrak{F}_0,\emptyset,\bot)$, $\quad C = X(\mathfrak{F}_1,\emptyset,\bot)$,

$B^{\mathrm{T}} = Y(\mathfrak{F}_0,\emptyset,\bot)$, $\quad C^{\mathrm{T}} = Y(\mathfrak{F}_1,\emptyset,\bot)$,

by the same arguments as in the proof of Theorem 15, we show that $L'(P,\mathfrak{a},\mathfrak{b})$ is Halldén-incomplete, and so, by Theorem 21, Maksimova-incomplete. ■

§10. **Open problems.** Thus, many important properties of modal and intermediate calculi proved to be undecidable. It is worth, however, noting that logics, used for obtaining these as well as many other "negative" results, are too cumbersome to be regarded as "natural". So, it would be interesting to look for effective criteria for the properties of calculi from more simple and "transparent" classes of logics. There are already some encouraging results in this direction.

For instance, Anderson [1972] proved that DP is decidable in the class of intermediate logics with additional axioms containing only one variable. (Actually, Anderson determined which of the Nishimura [1960] formulas F_n, for $n \leq \omega$, $n \neq 13$, axiomatize logics with DP and which without it. DP of **Int** + F_{13} have been recently established by Professor Hosoi's student Sasaki [1990].) On the other hand, Sobolev [1977] showed that all such logics have FMP, and so are decidable (their finite axiomatizability was proved earlier by Nishimura [1960]).

PROBLEM 1. *Whether it is true that all intermediate logics with additional axioms containing one variable are Halldén-complete and Maksimova-complete ?*

We conjecture that this problem has a positive solution. In any case, such are the logics **Int** + F_n, for $0 \leq n \leq 10$, which can be verified by a straightforward inspection.

PROBLEM 2. *Whether the finite model property, decidability, disjunction property, Halldén-completeness, Maksimova-completeness are decidable for intermediate calculi with additional axioms containing two variables?*

PROBLEM 3. *Whether these properties are decidable for* *extensions of* **S4** *or* **S4Grz** *with additional axioms containing one variable?*

Another positive example is the class of intermediate logics with additional axioms containing no $\vee$. All these logics, as we have already noted, have FMP and, with the exception of **Int**, do not have DP. However, HC and MC are not so trivial in this class.

PROBLEM 4. *Give effective criteria for Halldén-completeness and Maksimova-completeness in the class of intermediate logics axiomatizable by disjunctionless (or even implicative) formulas.*

PROBLEM 5. *Give an effective criterion for Halldén-completeness in the class of extensions of* **S4** *axiomatizable by canonical formulas of the form* $Y(\mathfrak{F},\emptyset,\bot)$ *or* $Y(\mathfrak{F},\emptyset)$.

An effective criterion for DP in this class was found by Zakharyashchev [1987].

Theorem 5 from §7 shows that HC may be not preserved while passing from an intermediate logic L to its least and greatest modal companions τL and σL. However, the following questions remain open.

PROBLEM 6. *Is it true that, for any intermediate logic* L, τL *is HC iff* σL *is HC?*

PROBLEM 7. *Is it true that, for any intermediate logic* L, L *is MC iff* τL *is MC iff* σL *is MC?*

We conclude our paper with the following fundamental question which naturally arises after establishing the undecidability.

PROBLEM 8. *Whether the set of modal or intermediate calculi having FMP (DP, being decidable, HC, MC, etc.) is recursively enumerable?*

Appendix. *Proof* of Lemma 3 (i). We should show that $\mathfrak{F} \models AxI$, for any instruction $I \in P$. Since all kinds of instruction are considered in the same way, we will deal with only one case, when

$$I = q_\delta \to q_\varepsilon T_{-1} T_0 (q_\zeta T_0 T_0).$$

Thus, our aim is to prove that

(14) $\mathfrak{F} \models \Diamond T(\delta, Q_2, R_1) \supset \Diamond T(\varepsilon, Q_1, R_1) \vee F$

and

(15) $\mathfrak{F} \models \Diamond T(\delta, A^2_0, R_1) \supset \Diamond T(\zeta, A^2_0, R_1) \vee F$.

First, we prove (14). Suppose that the formula $\Diamond T(\delta, Q_2, R_1)$ is true at a point x in $\mathfrak{F}$ under some valuation and show that $x \models \Diamond T(\varepsilon, Q_1, R_1) \vee F$. Let $x \not\models F$. According to (6), $x=f$, and there is $y \in W$ such that $y \models T(\delta, Q_2, R_1)$, i.e.

(16) $y \models \Diamond A^1_\delta \,\&\, \Diamond B^1_\delta$,

(17) $y \models \neg\Diamond A^1_{\delta+1} \,\&\, \neg\Diamond B^1_{\delta+1}$,

(18) $y \models \Diamond Q_2 \,\&\, \Diamond Q'_2$,

(19) $y \models \neg\Diamond Q_3 \,\&\, \neg\Diamond Q'_3$,

(20) $y \models \Diamond R_1 \,\&\, \Diamond R'_1$,

(21) $y \models \neg\Diamond R_2 \,\&\, \neg\Diamond R'_2$.

From (16) and (17), taking into account (8), we obtain yRa^1_δ, yRb^1_δ, $\neg yRa^1_{\delta+1}$ and $\neg yRb^1_{\delta+1}$. Therefore, by the construction of $\mathfrak{F}$, $y=t(\delta, m', n')$, for some m' and n' such that $P: (\alpha, m, n) \to (\delta, m', n')$.

Consider now (18) and (19). It follows from (18) that there are points z_1 and z_2 for which $t(\delta, m', n')Rz_1$, $t(\delta, m', n')Rz_2$ and $z_1 \models Q_2$, $z_2 \models Q'_2$, i.e.

(22) $z_1 \models \Diamond A^2_{-3} \,\&\, \Diamond B^2_{-3}$,

(23) $z_1 \models \neg\Diamond A^3_{-3} \,\&\, \neg\Diamond B^3_{-3}$,

(24) $z_1 \models \Diamond Q_1 \,\&\, \Diamond Q'_0$,

(25) $z_1 \models \neg\Diamond Q'_1$,

(26) $z_2 \models \Diamond A^2_{-3} \& \Diamond B^2_{-3}$,

(27) $z_2 \models \neg\Diamond A^3_{-3} \& \neg\Diamond B^3_{-3}$,

(28) $z_2 \models \Diamond Q'_1 \& \Diamond Q_0$,

(29) $z_2 \models \neg\Diamond Q_1$,

From (22), (23), (26) and (27), taking into account (8), we obtain $z_i R a^2_{-3}$, $z_i R b^2_{-3}$, $\neg z_i R a^3_{-3}$, $\neg z_i R b^3_{-3}$, for $i=1,2$, hence $z_1, z_2 \in \{a^2_k, b^2_k | k \geq -1\}$.

It follows from (24), (25), (28) and (29) that the points z_1 and z_2 are incomparable in $\mathfrak{F}$; moreover, there are points z_{11}, z_{12}, z_{21}, z_{22} such that $z_1 R z_{11}$, $z_1 R z_{12}$, $z_2 R z_{21}$, $z_2 R z_{22}$ and $z_{11} \models Q_1$, $z_{12} \models Q'_0$, $z_{21} \models Q'_1$, $z_{22} \models Q_0$, i.e.

(30) $z_{11} \models \Diamond A^2_{-3} \& \Diamond B^2_{-3}$,

(31) $z_{11} \models \neg\Diamond A^3_{-3} \& \neg\Diamond B^3_{-3}$,

(32) $z_{11} \models \Diamond Q_0 \& \Diamond Q'_{-1}$,

(33) $z_{11} \models \neg\Diamond Q'_0$,

(34) $z_{12} \models \Diamond A^2_{-3} \& \Diamond B^2_{-3}$,

(35) $z_{12} \models \neg\Diamond A^3_{-3} \& \neg\Diamond B^3_{-3}$,

(36) $z_{12} \models \Diamond Q'_{-1} \& \Diamond Q_{-2}$,

(37) $z_{12} \models \neg\Diamond Q_{-1}$,

(38) $z_{21} \models \Diamond A^2_{-3} \& \Diamond B^2_{-3}$,

(39) $z_{21} \models \neg\Diamond A^3_{-3} \& \neg\Diamond B^3_{-3}$,

(40) $z_{21} \models \Diamond Q'_0 \& \Diamond Q_{-1}$,

(41) $z_{21} \models \neg\Diamond Q_0$,

(42) $z_{22} \models \Diamond A^2_{-3} \& \Diamond B^2_{-3}$,

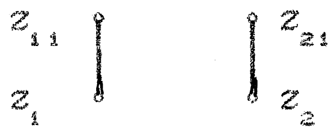
(43) $z_{22} \models \neg\Diamond A^3_{-3} \& \neg\Diamond B^3_{-3}$,

(44) $z_{22} \models \Diamond Q_{-1} \& \Diamond Q'_{-2}$,

(45) $z_{22} \models \neg\Diamond Q'_{-1}$.

From (30), (31), (34), (35), (38), (39), (42), (43), taking into account (8), we obtain that, for $i,j\in\{1,2\}$, $z_{ij}Ra^2_{-3}$, $z_{ij}Rb^2_{-3}$, $\neg z_{ij}Ra^3_{-3}$, $\neg z_{ij}Rb^3_{-3}$, hence $z_{ij}\in\{a^2_k, b^2_k \mid k\geq -1\}$.

By (25) and (29) and the fact that $z_{11}\models Q_1$ and $z_{21}\models Q'_1$, the accessibility relations between z_1, z_2, z_{11} and z_{21} can be completely represented by the following diagram:



Thus, by the construction of $\mathfrak{F}$, without loss of generality we may assume that $z_1=a^2_s$, $z_2=b^2_s$, for some $s\geq 0$, and so $z_{11}=a^2_{s-1}$, $z_{21}=b^2_{s-1}$. Let us see now what are the elements z_{12} and z_{22}. By (37), (32) and the definition of Q_0, $\neg z_{12}Rz_{11}$, and by (45), (40) and the definition of Q'_0, $\neg z_{22}Rz_{21}$. Using (33), (36), (37) and the definition of Q'_0, we obtain $\neg z_{11}Rz_{12}$, while using (41), (44), (45) and the definition of Q_0, we obtain $\neg z_{21}Rz_{22}$. Therefore, $z_{12}=b^2_{s-2}$, $z_{22}=a^2_{s-2}$, and since $z_{j2}\in\{a^2_k, b^2_k \mid k\geq -1\}$, we have

(46) $\quad s\geq 1.$

Note now that

(47) $\quad a^2_{s+1}\models Q_3$

and

(48) $\quad b^2_{s+1}\models Q'_3.$

We will establish only (47), because (48) is proved in exactly the same way.

In view of the fact that $a^2_{s+1}Ra^2_s$, $a^2_{s+1}Rb^2_{s-1}$, $a^2_s\models Q_2$ and $b^2_{s-1}\models Q'_1$ (since $a^2_s=z_1$, $b^2_{s-1}=z^2_{21}$), we have

(49) $\quad a^2_{s+1}\models \Diamond Q_2 \,\&\, \Diamond Q'_1,$

and, by the construction of $\mathfrak{F}$ and (8),

(50) $\quad a^2_{s+1} \models \Diamond A^2_{-3} \,\&\, \Diamond B^2_{-3} \,\&\, \neg\Diamond A^3_{-3} \,\&\, \neg\Diamond B^3_{-3}$.

Thus, it remains to prove that

(51) $\quad a^2_{s+1} \models \neg\Diamond Q'_2$.

Suppose otherwise. Then $a^2_{s+1} \models \Diamond Q'_2$ and there is a point u such that $a^2_{s+1} Ru$ and

(52) $\quad u \models \Diamond A^2_{-3} \,\&\, \Diamond B^2_{-3}$,

(53) $\quad u \models \neg\Diamond A^3_{-3} \,\&\, \neg\Diamond B^3_{-3}$,

(54) $\quad u \models \Diamond Q'_1 \,\&\, \Diamond Q_0$,

(55) $\quad u \models \neg\Diamond Q_1$.

By the construction of $\mathfrak{F}$, (8), (52), (53), we have $u \in \{a^2_i \mid -1 \le i \le s+1\} \cup \{b^2_i \mid -1 \le i \le s-1\}$. If $u \in \{a^2_{s+1}, a^2_s, a^2_{s-1}\}$ then uRa^2_{s-1} which contradicts (55), since $a^2_{s-1} \models Q_1$. If $u \in \{a^2_i \mid -1 \le i \le s-1\}$ then $a^2_s Ru$, and now (54) and (25) contradict each other. Finally, if $u \in \{b^2_i \mid -1 \le i \le s-1\}$ then $b^2_{s-1} Ru$, and so (54) and (41) are inconsistent.

Thus, we have proved (51) and hence (47). From (47), (48) and (19) we obtain $\neg t(\delta,m',n')Ra^2_{s+1}$, $\neg t(\delta,m',n')Rb^2_{s+1}$. Therefore, $m'=s$, i.e. $a^2_s = a^2_{m'}$, $b^2_s = b^2_{m'}$.

In the same way we consider (20) and (21). We obtain that $a^3_{n'} \models R_1$, $b^3_{n'} \models R'_1$, $a^3_{n'-1} \models R_0$, $b^3_{n'-1} \models R'_0$, $a^3_{n'+1} \models R_2$, $b^3_{n'+1} \models R'_2$, with these points being the only ones having such properties.

Now we use the instruction $I \in P$. Since, by (46), $m'=s>0$, the first part of the instruction is applicable to (δ,m',n'), and so $t(\varepsilon,m'-1,n')$ is a point in $\mathfrak{F}$. According to (9), we have

(56) $\quad t(\varepsilon,m'-1,n') \models \Diamond A^1_\varepsilon \,\&\, \Diamond B^1_\varepsilon \,\&\, \neg\Diamond A^1_{\varepsilon+1} \,\&\, \neg\Diamond B^1_{\varepsilon+1}$,

and since, by the construction of $\mathfrak{F}$, $t(\varepsilon,m'-1,n')Ra^2_{m'-1}$ and

$t(\varepsilon,m'-1,n')Rb^2_{m'-1}$, using $a^2_{m'-1}=a^2_{s-1}=z_{11}\models Q_1$ and $b^2_{m'-1}=a^2_{s-1}=z_{21}\models Q'_1$, we obtain $t(\varepsilon,m'-1,n')\models\Diamond Q_1\ \&\ \Diamond Q'_1$. Similarly (using the consequences of (20) and (21) mentioned above) we can prove that $t(\varepsilon,m'-1,n')\models\Diamond R_1\ \&\ \Diamond R'_1$, and so

(57) $t(\varepsilon,m'-1,n')\models\Diamond Q_1\ \&\ \Diamond Q'_1\ \&\ \Diamond R_1\ \&\ \Diamond R'_1$.

Now we show that

(58) $t(\varepsilon,m'-1,n')\models\neg\Diamond Q_2\ \&\ \neg\Diamond Q'_2$.

Suppose otherwise. This means that at least one of the conjuncts is false at $t(\varepsilon,m'-1,n')$. Let

(59) $t(\varepsilon,m'-1,n')\models\Diamond Q_2$.

(The second conjunct is considered in the same way.) Then there is a point v such that $t(\varepsilon,m'-1,n')Rv$ and $v\models Q_2$, i.e.

(60) $v\models\Diamond A^2_{-3}\ \&\ \Diamond B^2_{-3}$,

(61) $v\models\neg\Diamond A^3_{-3}\ \&\ \neg\Diamond B^3_{-3}$,

(62) $v\models\Diamond Q_1\ \&\ \Diamond Q'_0$,

(63) $v\models\neg\Diamond Q'_1$.

It follows from (60), (61) and (8) that $v\in\{a^2_k,b^3_k\mid -1\le k\le m'-1\}$. If $v\in\{a^2_k\mid -1\le k\le m'-2\}\cup\{b^2_k\mid -1\le k\le m'-1\}$ then $z_2=b^2_s=b^2_{m'}Rv$, and so (62) and (29) contradict each other. Therefore, $v=a^2_{m'-1}=a^2_{s-1}=z_{11}$, but then (62) and (33) are inconsistent.

Thus, we have proved (58). Similarly we can prove that

(64) $t(\varepsilon,m'-1,n')\models\neg\Diamond R_2\ \&\ \neg\Diamond R'_2$.

From (56), (57), (58) and (64) we obtain $t(\varepsilon,m'-1,n')\models T(\varepsilon,Q_1,R_1)$, hence $f\models\Diamond T(\varepsilon,Q_1,R_1)$, and so (14) is justified.

Let us consider (15). Suppose again that under some valuation $T(\delta,A^2_0,R_1)$ is true and F is false at a point x in $\mathfrak{F}$. Then, as we know, $x=f$, and there is a point y such that fRy and

(65) $y \models \Diamond A^1_\delta \& \Diamond B^1_\delta$,

(66) $y \models \neg\Diamond A^1_{\delta+1} \& \neg\Diamond B^1_{\delta+1}$,

(67) $y \models \Diamond A^2_0 \& \Diamond B^2_0$,

(68) $y \models \neg\Diamond A^2_1 \& \neg\Diamond B^2_1$,

(69) $y \models \Diamond R_1 \& \Diamond R'_1$,

(70) $y \models \neg\Diamond R_2 \& \neg\Diamond R'_2$.

It follows from (65) - (68) that $y=t(\delta,0,n')$, for some n'. Using the second part of the instruction I (since the second component of the configuration is 0), we obtain that $t(\zeta,0,n')$ is a point in $\mathfrak{F}$. Then, by (9), we must have

(71) $t(\zeta,0,n') \models \Diamond A^1_\zeta \& \Diamond B^1_\zeta$,

(72) $t(\zeta,0,n') \models \neg\Diamond A^1_{\zeta+1} \& \neg\Diamond B^1_{\zeta+1}$,

(73) $t(\zeta,0,n') \models \Diamond A^2_0 \& \Diamond B^2_0$,

(74) $t(\zeta,0,n') \models \neg\Diamond A^2_1 \& \neg\Diamond B^2_1$.

Show that

(75) $t(\zeta,0,n') \models \Diamond R_1 \& \Diamond R'_1$,

(76) $t(\zeta,0,n') \models \neg\Diamond R_2 \& \neg\Diamond R'_2$.

By (69), there are points w_1 and w_2 such that $t(\delta,0,n')Rw_1$, $t(\delta,0,n')Rw_2$, $w_1 \models R_1$ and $w_2 \models R'_1$, hence

(77) $w_1 \models \Diamond A^3_{-3} \& \Diamond B^3_{-3} \& \neg\Diamond C \& \neg\Diamond D \& \Diamond R_0 \& \Diamond R'_{-1} \& \neg\Diamond R'_0$,

(78) $w_2 \models \Diamond A^3_{-3} \& \Diamond B^3_{-3} \& \neg\Diamond C \& \neg\Diamond D \& \Diamond R'_0 \& \Diamond R_{-1} \& \neg\Diamond R_0$.

Using (77), (78) and taking into account (8), we obtain that $w_1, w_2 \in \{a^3_i, b^3_i \mid -1 \le i \le n'\}$. But then, by the construction of $\mathfrak{F}$, $t(\zeta,0,n')Rw_1$, $t(\zeta,0,n')Rw_2$ which implies (75).

Let us prove (76). Suppose otherwise. Then there is a point w

such that $t(\zeta,0,n')Rw$ with $w\nvDash R_2$ or $w\nvDash R'_2$. So

$$w \vDash \Diamond A^3_{-3} \,\&\, \Diamond B^3_{-3} \,\&\, \neg\Diamond C \,\&\, \neg\Diamond D \,\&\, \Diamond R_1 \,\&\, \Diamond R'_0 \,\&\, \neg\Diamond R'_1$$

or

$$w \vDash \Diamond A^3_{-3} \,\&\, \Diamond B^3_{-3} \,\&\, \neg\Diamond C \,\&\, \neg\Diamond D \,\&\, \Diamond R'_1 \,\&\, \Diamond R_0 \,\&\, \neg\Diamond R_1 .$$

Each of these gives us $w \in \{a^3_i,\ b^3_i \mid -1 \le i \le n'\}$, hence $t(\delta,0,n')Rw$, and so $t(\delta,0,n') \vDash \Diamond R_2 \vee \Diamond R'_2$ which contradicts (70).

From (71) - (76) we obtain $t(\zeta,0,n') \vDash T(\zeta, A^2_0, R_1)$, and therefore $f \vDash \Diamond T(\zeta, A^2_0, R_1)$. ■

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