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PARTIAL CONSERVATIVITY AND MODAL LOGICS

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PARTIAL CONSERVATIVITY AND MODAL LOGICS.

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ABSTRACT: PA is Peano arithmetic. Let Γ be an arbitrary decidable set of arithmetical formulas; the Π_2-formula $\alpha \rhd_\Gamma \beta$ (where α, β range over codes of arithmetical sentences), is a formalization of the assertion that the theory PA+β is Γ-conservative over PA+α, i.e. any sentence γ in Γ which is provable in PA+β is also provable in PA+α. We extend Solovay's modal analysis of the formalized provability predicate of PA to the formalized conservative relation. Namely, for $\Gamma=\Pi_n$, $n \geq 2$, $\Gamma=\Sigma_n$, $n \geq 3$, we give an axiomatization and a decision procedure for the class of those modal formulas that express arithmetically valid principles of Γ-conservativity.

CONTENTS

§1. Introduction.

In [3] Guaspari considered a notion of Γ-conservativity, (where $\Gamma=\Sigma_n$, Π_n, $n\geq 1$): theory T_2 is Γ-conservative over theory T_1 if any Γ-sentence provable in T_2 is also provable in T_1 ("theory" means "r.e. theory in arithmetical language containing Peano Arithmetic PA"). Guaspari proved some "obvious" properties of partial conservativity: for example, each consistent theory T has a proper extension which is Γ-conservative over T , and the set of all sentences A such that PA+A is Γ-conservative over PA is not r.e. .

Guaspari also noted, using Orey-Hàjek results ([1], [2]), that Π_1-conservativity (over reflexive theories like PA,ZF, etc.) coincides with relative interpretability, and gave a model--characterization of the notion of relative interpretability.

Later Lindström ([4],[5]) proved that the set of all sentences A s.t. PA+A is Γ-conservative over PA is Π^0_2-complete (here $\Gamma\neq\Sigma_1$). Bennet ([6]) proved some other facts connected with the partial conservativity.

The notion of Γ-conservativity (as well as relative interpretability) can be used for an arithmetical interpretation of modal language with unary modal operator $\Box$ and binary modal operator $\rhd$, where $\Box$ is translated as provability in PA, and $A\rhd B$ is translated as "PA+B is Γ-conservative over PA+A" (or "PA+A interprets the theory PA+B"). Provability logic for Γ-conservativity (respectively, provability logic for relative interpretability) is, by definition, a set of all modal formulas, any arithmetical translation of which is provable in PA; we denote this set by $CL(\Gamma)$.

In [9],[10] it was proved, (with a use of some results from [7],[8]) that the modal system ILM is the logic of relative interpretability over PA; thus, ILM is also the logic of Π_1-conservativity (in [11] this result was extended generalized to more rich class of basic theories).

The aim of this paper is to give an axiomatization and a Kripke-like semantic for the logic of Π_n-,Σ_n-conservativity for $n\geq 3$ (this logics coincide). In appendix we also prove the arithmetical completeness of the logic of Π_2-conservativity.

If we consider axioms of ILM:

A1. $A \rhd B \longrightarrow (\Diamond A \longrightarrow \Diamond B)$

A2. $A \rhd C \wedge B \rhd C \longrightarrow (A \vee B) \rhd C$

A3. $\Box (A \longrightarrow B) \longrightarrow A \rhd B$

A4. $A \rhd B \wedge B \rhd C \longrightarrow A \rhd C$

A5. $\Diamond A \rhd A$

M. $A \rhd B \longrightarrow (A \wedge \Box C) \rhd (B \wedge \Box C)$

from the point of view of the translation $\rhd$ as Γ-conservativity, we can note that axioms A2,A3,A4 are always arithmetically valid, and axioms A1, A5, M express, respectively, that $\bot \in \Gamma$; $\forall \gamma \in \Gamma$ $PA \vdash \Diamond_{PA} \gamma \longrightarrow \gamma$; $\forall \gamma \in \Gamma, \pi \in \Pi_1$ $\gamma \vee \pi \in \Gamma$. Thus, one can see that for $\Gamma = \Pi_n$, Σ_n, $n \geq 2$, only A5 fails. On the other hand, for such Γ, the principle M has an obvious generalization: for example, for $\Gamma = \Pi_2$ it is

Sb. $A \rhd B \longrightarrow (A \wedge \neg (C \rhd D)) \rhd (B \wedge \neg (C \rhd D))$.

Unfortunately, the absence of A5 create some difficulties in the proof of modal completeness theorem for conservativity logics.

In modal completeness proof we use Veltman models ([7]), Visser's simplified models ([8]), and some technical concepts from [7]; in proving the arithmetical completeness we use some ideas from [5] and [10].

We believe that the technique developed in this paper will be useful in investigation of other logics of partial conservativity.

We are special thanks to V. Shavrukov for his substantial support and helpful discussions.

§2. Modal Systems and Arithmetic.

Definition 2.1. The modal language $\mathcal{L}(\Box, \rhd)$ consists of an infinite set of propositional variables p,q,...; boolean connectives $\neg, \vee, \wedge, \bot, \longrightarrow, \longleftrightarrow$, and two modal operators: unary operator '$\Box$' and binary operator '$\rhd$'; '$\Diamond$' is abbreviation for '$\neg \Box \neg$'.

We write "modal formula" instead of "$\mathcal{L}(\Box, \rhd)$-formula".

Definition 2.2. The logic CL is the minimal set of modal formulas containing the following axioms and closed under following rules:

L0. All tautologies of propositional logic.

L1. $\Box(A \to B) \to (\Box A \to \Box B)$

L2. $\Box A \to \Box\Box A$

L3. $\Box(\Box A \to A) \to \Box A$

A1. $A \rhd B \to (\Diamond A \to \Diamond B)$

A2. $A \rhd C \wedge B \rhd C \to (A \vee B) \rhd C$

A3. $\Box(A \to B) \to A \rhd B$

A4. $A \rhd B \wedge B \rhd C \to A \rhd C$

R1. $\vdash A,\ \vdash A \to B \Rightarrow\ \vdash B$ (*modus ponens*)

R2. $\vdash A \Rightarrow\ \vdash \Box A$ (*necessitation*)

<u>Definition</u> 2.3. The logic CLM is the minimal set of modal formulas closed under R1,R2 and containing L0-L3,A1-A4, plus

M. $A \rhd B \to (A \wedge \Box C) \rhd (B \wedge \Box C)$

<u>Definition</u> 2.4. The logic SCL is the minimal set of modal formulas closed under R1,R2 and containing L0-L3,A1-A4, plus

Sa. $A \rhd B \to (A \wedge (C \rhd D)) \rhd (B \wedge (C \rhd D))$

Sb. $A \rhd B \to (A \wedge \neg (C \rhd D)) \rhd (B \wedge \neg (C \rhd D))$

<u>Proposition</u> 2.5. SCL$\supseteq$CLM.

<u>Proof</u>. CL$\vdash \Box A \leftrightarrow (\neg A) \rhd \bot$ by A1,A3; hence, CL+Sa$\vdash$ M.

Let $Proof_{PA}(n,x)$ be the Δ_0-arithmetical formula representing the relation: n is the Gödelnumber of the PA-proof of the formula with Gödelnumber x; $Prov_{PA}(x) := \exists n\ Proof_{PA}(n,x)$; $\Box_{PA} Q$ will stand for $Prov_{PA}(\ulcorner Q \urcorner)$.

<u>Definition</u> 2.6. Let Γ be a decidable set of arithmetical formulas closed under disjunctions such that $\bot \in \Gamma$, and $\Gamma(x)$ be Δ_1^{PA}-formula representing the relation "$x = \ulcorner Q \urcorner \wedge Q \in \Gamma$"; then we define:

$$A \rhd_\Gamma B := \forall Q\ (\Gamma(\ulcorner Q \urcorner) \wedge \Box_{PA}(B \to Q) \to \Box_{PA}(A \to Q))$$

It is clear that $A \rhd_\Gamma B \in \Pi_2^{PA}$.

<u>Definition</u> 2.7. Let Γ be as above. An *arithmetical interpretation* f_Γ is a mapping of $\mathcal{L}(\Box,\rhd)$-formulas to arithmetical sentences which commutes with the boolean connectives and translates $\Box$ as provability in PA and $\rhd$ as Γ-conservativity:

$$f_\Gamma(\Box A) := \Box_{PA} f_\Gamma(A)$$
$$f_\Gamma(A \rhd B) := f_\Gamma(A) \rhd_\Gamma f_\Gamma(B)$$

Definition 2.8. Let Γ be as above. We define *provability logic for Γ-conservativity* $CL(\Gamma)$ and *minimal provability logic for Γ-conservativity* $CL^+(\Gamma)$ (shortly, *logic of Γ-conservativity* and *minimal logic of Γ-conservativity*) as follows:

$$CL(\Gamma):=\{\ A\in\mathcal{L}(\square,\rhd)\ \mid \text{ for any arithmetical interpretation } f_\Gamma\ \ PA\vdash f_\Gamma(A)\ \}$$

$$CL^+(\Gamma):=\bigcap_{\Gamma'\supseteq\Gamma}CL(\Gamma')$$

(It is supposed that Γ' as well as Γ is closed under disjunctions)

Theorem 1. For $n\geq 3$,

$$CL(\Pi_n) = CL(\Sigma_n) = CL^+(\Pi_n) = CL^+(\Sigma_n) = SCL.$$

Theorem 2. $CL=CL^+(\{\bot\})$.

Theorem 3. $CLM=CL^+(\Pi_1)$.

§3. **Kripke Semantics.**

Definition 3.1. A *model* $\mathcal{K}=\langle K,R,\{S_x\},\Vdash\rangle$ contains a nonempty set K, binary relation R, a family of binary relations $\{S_x\}$ for each $x\in K$ and a forcing relation $\Vdash$ such that:

1. R is transitive and (converse) wellfounded;
2. $\Vdash$ commutes with boolean connectives;
3. $x\Vdash \square A \iff \forall y(xRy \longrightarrow y\Vdash A)$

   $x\Vdash A\rhd B \iff \forall y(xRy\wedge y\Vdash A \longrightarrow \exists z(yS_xz\wedge xRz\wedge z\Vdash B))$

We say that model $\mathcal{K}$ is *simplified* iff S_x does not depend on x, and that modal formula A is *valid* in $\mathcal{K}$, ($\mathcal{K}\vDash A$), iff $\forall x\in K\ x\Vdash A$.

Proposition 3.2. A formula $\square A\leftrightarrow(\neg A)\rhd\bot$ is valid in every model $\mathcal{K}$.

Definition 3.3.

1. A *CL-model* is a model $\mathcal{K}=\langle K,R,\{S_x\},\Vdash\rangle$ such that for each x S_x is transitive and reflexive.
2. A *CLM-model* is a CL-model $\mathcal{K}=\langle K,R,\{S_x\},\Vdash\rangle$ such that for any $x,y,z,t\in K$ $yS_xzRt \Rightarrow yRt$.
3. A *SCL-model* is a CLM-model $\mathcal{K}=\langle K,R,\{S_x\},\Vdash\rangle$ such that for any x,y,z $yS_xz \Rightarrow K_y=K_z \wedge S_y=S_z$, where for any $u\in K$ K_u denotes $\{v \mid uRv\}$.

Definition 3.4.

1. A *simplified CL-model* is a simplified model $\mathcal{K}=\langle K,R,S,\Vdash\rangle$ such that:

1. K is finite or countable.
2. S is an equivalence relation.

2. A *simplified CLM-model* is a simplified model $\mathcal{K}=\langle K,R,S,\Vdash\rangle$ such that:

1. K is finite or countable.
2. S is reflexive and transitive.
3. $xSyRz \Rightarrow xRz$.
4. There exists a natural number N and a mapping $\mu: K \longrightarrow \{1,2,...N\}$ such that:
   a) $xSy \Rightarrow \mu(x)=\mu(y)$;
   b) $xRy \Rightarrow \mu(x)<\mu(y)$.

3. A *simplified SCL-model* is a simplified model $\mathcal{K}=\langle K,R,S,\Vdash\rangle$ such that:

1. K is finite.
2. S is an equivalence relation.
3. $xSyRz \Rightarrow xRz$.

Lemma 3.5 (*Soundness*). For l=CL,CLM,SCL respectively,

for any modal formula A if $l \vdash A$ then A is valid in each l-model $\mathcal{K}$.

Proof. Entirely routine.

Theorem 4. (*First modal completeness theorem*).
For l=CL,CLM,SCL respectively,

for any modal formula A if $l \nvdash A$, then A is not valid in some finite l-model $\mathcal{K}$.

Theorem 5. (*Second modal completeness theorem*).
For l=CL,CLM,SCL respectively,

for any modal formula A if $l \nvdash A$, then A is not valid in some simplified l-model $\mathcal{K}$.

§4. Modal Completeness: Preliminaries.

In this paragraph let l be an arbitrary extension of CL, clo under modus ponens and necessitation.

Our goal is to prove a modal completeness of l with respect to some class of CL-models (i.e. S_x must be reflexive and transitive). We will act by the usual way: to use l-consistent subsets of an "adequate" set Φ as nodes of a countermodel; it is supposed that Φ contains all subformulas of the refuting formula, is closed under negations, subformulas and some special operations which will be explained later.

Definition 4.1. An *adequate* set of formulas is a finite set Φ which fulfills the following conditions:

1. Φ is closed under subformulas.
2. If $A\in\Phi$ and A is not a negation, then $\neg A\in\Phi$.
3. $\bot\rhd\bot\in\Phi$.
4. If A as well as B is an antecedent or a consequent of some $\rhd$-formula in Φ, then $A\rhd B\in\Phi$.
5. If $A\rhd B, A_1\rhd B_1, A_2\rhd B_2, \ldots, A_n\rhd B_n\in\Phi$, then $\Box(A\to A_1\vee\ldots\vee A_n)\in\Phi$.

Definition 4.2. $\Phi_\rhd := \{A \mid A\rhd X\in\Phi$ or $X\rhd A\in\Phi$ for some $X\}$.

Proposition 4.3. Each finite set of formula can be extended to an adequate set Φ.

In the following reasoning we consider a fixed adequate set Φ.

Definition 4.4. $W := \{x\subseteq\Phi \mid x$ is maximal l-consistent set$\}$

Further it will be necessary to define a binary relation $<$ on W (we will write '$<$' instead of 'R' for convenience) and a family of binary relations $\{S_x\}$, $x\in W$, such that

(*) for any $A\in\Phi$, $x\in W$ $\quad x\Vdash A \iff A\in x$.

(Where as usual we have defined $x\Vdash p :\iff p\in x$).

By l-consistency of x and propositions 2.5, 3.2 it is enough to prove condition (*) for formulas of the form $B\rhd C$ and $\neg(B\rhd C)$.

The definition of '$<$' is natural:

Definition 4.5. For $x,y\in W$

$$x<y \;:\iff\; 1)\ \forall\Box D\in\Phi:\ \Box D\in x \Rightarrow D,\Box D\in y.$$

(y is a *successor* of x) $\quad$ 2) $\exists\Box D\in\Phi:\ \Box D\notin x,\ \Box D\in y$.

In the definition of S_x the following concept of "C-critical successor" is essential:

Definition 4.6. For $x,y \in W$, $C \in \Phi_{\rhd}$

$$x <_C y \ :\Leftrightarrow \ 1)\ x<y.$$

(y is a *C-critical successor* of x) $\quad 2)\ \forall A \rhd C \in x \quad \neg A \in y$.

To explain why C-critical successors must be used we note that

Proposition 4.7.

$$CL \vdash \neg(A \rhd C) \wedge B_1 \rhd C \wedge \ldots \wedge B_n \rhd C \longrightarrow \neg((A \wedge \neg B_1 \wedge \ldots \wedge \neg B_n) \rhd (C \vee B_1 \vee \ldots \vee B_n)).$$

Proof. Let $B := B_1 \vee \ldots \vee B_n$. We have:

$CL \vdash B_1 \rhd C \wedge \ldots \wedge B_n \rhd C \longrightarrow B \rhd C$

$CL \vdash B \rhd C \longrightarrow (A \wedge B) \rhd (C \vee B)$

$CL \vdash B \rhd C \wedge (A \wedge \neg B) \rhd (C \vee B) \longrightarrow A \rhd (C \vee B)$ (A2)

$\longrightarrow (C \vee B) \rhd C$ (A2)

$\longrightarrow A \rhd C$ (A4)

$CL \vdash B \rhd C \wedge \neg(A \rhd C) \longrightarrow \neg((A \wedge \neg B) \rhd (C \vee B))$. QED.

Suppose it is necessary to provide $x \Vdash \neg(A \rhd C)$ in the model that we want to construct. Let $\{B_1 \rhd C, \ldots, B_n \rhd C\} := \{X \rhd C \mid X \rhd C \in x\}$. By proposition 4.7, because x is 1-consistent,

$$x \Vdash \neg((A \wedge \neg B_1 \wedge \ldots \wedge \neg B_n) \rhd (C \vee B_1 \vee \ldots \vee B_n))$$

Thus, by definition 4.6, we obtain:

(**) $\exists y$: 1) $x <_C y$, $y \Vdash A$

2) $\forall z \quad y S_x z, x<z \longrightarrow$ a) $x <_C z$

b) $z \nVdash C$.

(In fact, a) implies b), because by condition 4 of definition 4.1, $C \rhd C \in \Phi$)

Condition (**)1) can always be satisfied:

Lemma 4.8. Let $x \in W$, $\neg(A \rhd C) \in x$. Then there exists a C-critical successor y of x such that $A \in y$.

Proof. Assume that x has the form:

$$x = \{\neg(A \rhd C);\ B_1 \rhd C, \ldots, B_n \rhd C;\ \Box D_1, \ldots, \Box D_m, \ldots\}$$

(Since $\bot \rhd \bot \in \Phi$ and $CL \vdash \bot \rhd C$, $\bot \rhd C \in x$ and $n>0$).

Consider a set

$$y' := \{A;\ \neg B_1, \ldots, \neg B_n;\ \Box D_1, \ldots, \Box D_m;\ D_1, \ldots D_m;\ \Box(A \longrightarrow B_1 \vee \ldots \vee B_n)\}.$$

By condition 5 of the definition of an adequate set, $y' \subseteq \Phi$. We show that y' is 1-consistent. Suppose not. We have:

$$1 \vdash \Box D_1 \wedge \ldots \wedge \Box D_m \wedge D_1 \wedge \ldots \wedge D_m \longrightarrow (\Box(A \longrightarrow B_1 \vee \ldots \vee B_n) \longrightarrow (A \longrightarrow B_1 \vee \ldots \vee B_n)).$$

Using the necessitation rule, Löb's axiom (L3), and L2, we obtain:

$$l \vdash \Box D_1 \wedge \ldots \wedge \Box D_m \longrightarrow \Box(A \longrightarrow B_1 \vee \ldots \vee B_n).$$

On the other hand, by A2,

$$l \vdash \bigwedge\!\!\!\bigwedge x \longrightarrow (B_1 \vee \ldots \vee B_n) \rhd C.$$

Thus, by A3,A4 we obtain:

$$l \vdash \bigwedge\!\!\!\bigwedge x \longrightarrow (A \rhd C),$$

i.e. x is inconsistent.Contradiction.

Now it is sufficient to put y:=maximal l-consistent extension of y'. QED.

As to the definition of S_x , it is clear from (**) that ideally we were to demand S_x to maintain the status of C-critical successor for any C (it means that if $x <_C y$, $yS_x z$, then $x <_C y$). Unfortunately it is impossible in a general case, however, it becomes possible, if C is fixed.

<u>Lemma</u> 4.9 Let $x,y \in W$, $A \rhd B \in x$, $A \in y$ and y is C-critical successor of x, where $C \in \Phi_{\rhd}$. Then there exists a C-critical successor z of x such that $B \in z$.

<u>Proof</u>. By condition 4 of the definition of an adequate set, $B \rhd C \in \Phi$. Therefore, $\neg(B \rhd C) \in x$ or $B \rhd C \in x$. In the last case by A4 $A \rhd C \in x$ and, since $x <_C y$, $\neg A \in y$; it contradicts l-consistency of y. Hence, $\neg(B \rhd C) \in x$. Now we use lemma 4.8. QED.

To counteract these difficulties we will, following [7], multiply the nodes of W such that for any node there would be only one C such that the relations S_x are to maintain the C-critical status of x for just this C.

This idea along with lemmas 4.8, 4.9 is sufficient for the proof of the modal completeness of CL, which will be shown in the following paragraph.

§5.Modal Completeness of CL.

<u>Proof of theorem 4 for l=CL</u>.

Fix a modal formula ϕ such that $CL \nvdash \phi$, an adequate set Φ such that $\phi \in \Phi$, a set W of maximal CL-consistent subsets of Φ and an element x_0 of W such that $\neg\phi \in x_0$. Now we define a countermodel for ϕ (we use some concepts from the previous paragraph):

<u>Definition</u> 5.1. Let $x \in W$. The *depth* of x is the maximal n such

that there exists a chain:

$$x=y_0<y_1<\dots<y_n.$$

Proposition 5.2.

a). Let $x<y$. Then $\mathrm{depth}(x)>\mathrm{depth}(y)$.

b). Let $\{\Box D \mid \Box D\in x\}\subseteq\{\Box D \mid \Box D\in y\}$. Then $\mathrm{depth}(x)\geq\mathrm{depth}(y)$.

Definition 5.3. Let $\mathcal{K}:=\langle K,R,\{S_x\},\Vdash\rangle$, where

$K:=\{\langle x,\tau\rangle$, where $x\in W$, τ is a sequence of formulas from $\Phi_{\rhd}$, and $|\tau|\leq\mathrm{depth}(x_0)-\mathrm{depth}(x)\}$,

$\langle x_1,\tau_1\rangle R\langle x_2,\tau_2\rangle \quad :\Leftrightarrow \quad x_1<x_2$ and $\tau_1\subset\tau_2$,

(i.e. τ_1 is a proper initial segment of τ_2)

$\langle x_1,\tau_1\rangle S_{\langle y_0,\tau_0\rangle}\langle x_2,\tau_2\rangle \quad :\Leftrightarrow$ if $\tau_1\supseteq\tau_0*\langle E\rangle$ and $y_0<_E x_1$, then $\tau_2\supseteq\tau_0*\langle E\rangle$ and $y_0<_E x_2$,

$\langle x,\tau\rangle\Vdash p \ :\Leftrightarrow\ p\in x$.

Of course, $S_{\langle x,\tau\rangle}$ is reflexive and transitive, hence we have defined a finite CL-model.

Lemma 5.4. For any $A\in\Phi$ and $\langle x,\tau\rangle\in K$

$$\langle x,\tau\rangle\Vdash A \quad\Leftrightarrow\quad A\in x.$$

Proof. Induction on the structure of A. We only need to consider the case $A=B\rhd C$.

1. Suppose $\neg(B\rhd C)\in x$. By lemma 4.8, there exists y such that $x<_C y$ and $B\in y$. By the induction hypothesis, $\langle y,\tau*\langle C\rangle\rangle\Vdash B$. (Note that by proposition 5.2 a), $\langle y,\tau*\langle C\rangle\rangle\in K$). On the other hand, let

$$\langle y,\tau*\langle C\rangle\rangle S_{\langle x,\tau\rangle}\langle z,\sigma\rangle,\quad \langle x,\tau\rangle R\langle z,\sigma\rangle.$$

By the definition of $S_{\langle x,\tau\rangle}$, $x<_C z$, hence, (because $C\rhd C\in x$), $\neg C\in z$ and $\langle z,\sigma\rangle\nVdash C$.

2. Suppose $B\rhd C\in x$, and $\langle y,\sigma\rangle\Vdash B$, where $\langle x,\tau\rangle R\langle y,\sigma\rangle$. By the induction hypothesis, $B\in y$. Since $\tau\subset\sigma$, there exists a unique E s.t. $\sigma\supseteq\tau*\langle E\rangle$. There are two cases to consider:

Case 1. $x<_E y$. Then, by lemma 4.9, there exists z such that $x<_E z$ and $C\in z$. So, we have: $\langle z,\tau*\langle E\rangle\rangle\in K$, $\langle y,\sigma\rangle S_{\langle x,\tau\rangle}\langle z,\tau*\langle E\rangle\rangle$, $\langle z,\tau*\langle E\rangle\rangle\Vdash C$.

Case 2. $x<_E y$ does not hold. Note that each successor is $\bot$-critical successor. So, we can apply lemma 4.9 to obtain z s.t. $x<z$, $C\in z$ and use the construction from case 1 with any $\Phi_{\rhd}$-formula instead of E. QED.

Now it is enough to note that by the last lemma and definition of x_0, $<x_0,<>> \nVdash \phi$. This completes the proof.

To finish the paragraph we discuss the principal difficulties arising in consideration of richer logics of conservativity and the basic ways to overcome them.

First, to provide the condition $yS_x zRt \Rightarrow yRt$ it is necessary to demand the condition $\tau_1 \subseteq \tau_2$ for S to be fulfilled. So, the problem arises how to provide for the pair $<z,\sigma>$ obtained in part 2 of the proof of lemma 5.4 to belong to K, i.e. for the depth of z not to be more that the depth of y. The natural way to solve this problem – to transfer all boxes from y to z – cannot be applied because the necessary adequate set became infinite.

We propose the following way to approach this problem. Imagine that our model is graduated by levels (by the length of τ). Now restrict the validity of the C-critical-maintain condition in the definition of S only to the level immediately above $<x,\tau>$. It is enough to preserve the reasoning in part 1 of the proof of lemma 5.4, because the "counterexample" constructed there lays immediately above the node $<x,\tau>$. On the other hand, one can easily see that the difficulty mentioned above (sufficiently small depth of the node z) does not arise if we deal with $<y,\sigma>$ laying immediately above $<x,\tau>$. Considering the higher levels one needn't to worry about C-critical status, and so it becomes possible to transfer a sufficient quantity of boxes from y to z.

Second, for the condition $zS_x y$, $zRt \Rightarrow yRt$ to be fulfilled, we cannot use the above method (to transfer the sufficient quantity of boxes' negations from y to z), because adequate sets expand too rapidly. The solution of this problem (described in §7) formally uses Visser's construction of obtaining the simplified models (cf.[8]), but really it is the idea that "counterexample" for the formula $A \rhd B$ (i.e. , if $x \nVdash A \rhd B$, such y that xRy, $y \Vdash A$, $\forall z(yS_x z, xRz \longrightarrow z \nVdash B)$) must not be an end of an S-arrow.

§6. Modal Completeness of CLM.

Definition 6.1. A *CLM-adequate set* of modal formulas is a finite set Φ s.t. there exist set $\Phi_{\rhd}$ and set $\Phi_{\Box}$ containing only boxed formulas which satisfy the following conditions:

1. $\Phi_{\rhd}=\{A \mid A\rhd X\in\Phi$ or $X\rhd A\in\Phi$ for some $X\}$;
2. $\Phi_{\Box}\subseteq\Phi$;
3. Φ is closed under negations and subformulas (in sense of definition 4.1);
4. if $A,B\in\Phi_{\rhd}$, then $A\rhd B\in\Phi$;
5. $\bot\in\Phi_{\rhd}$;
6. if $A,B_1,B_2,\ldots B_n\in\Phi_{\rhd}$, then $\Box(A\longrightarrow B_1\vee\ldots\vee B_n)\in\Phi$;
7. if $A\in\Phi_{\rhd},\Box D_1,\ldots,\Box D_n\in\Phi$, then there is a formula A' which is GL-equivalent to $A\wedge\Box D_1\wedge\ldots\wedge\Box D_n$ such that $\Box\neg A'\in\Phi_{\Box}$;
8. if $B\in\Phi_{\rhd}$, $\Box D\in\Phi_{\Box}$, then there is a formula B' which is GL-equivalent to $B\wedge\Box D$ such that $B'\in\Phi_{\rhd}$.

Note that each CLM-adequate set is an adequate set.

We will show that every finite set of modal formulas is contained in some CLM-adequate set.

Definition 6.2. Consider an arbitrary set of modal formulas X. The *LR-closure* of X is the minimal pair <L,R> (with respect to each component) of sets of modal formulas such that $L\supseteq X$ and:

(L) $\forall A\in L,\ B\in L\cup R\ \ A\wedge\Box\neg B\in L$;

(R) $\forall A\in L\cup R\ \forall B,C_1,\ldots,C_n\in L\ \ A\wedge\Box(B\longrightarrow C_1\vee\ldots\vee C_n)\in R$.

Lemma 6.3. Let X be a finite set of modal formulas, and <L,R> be the LR-closure of X. Then L as well as R consists of finitely many equivalence classes with respect to GL-provable equivalence.

Proof. We can assume that $X=\{\bot,p_1,\ldots,p_m\}$. The proof proceeds by induction on m. First, we note that formula D belongs to R iff D has the form:

(*) $\quad D=A\wedge\Box(B^{(1)}\longrightarrow C_1^{(1)}\vee\ldots\vee C_n^{(1)})\wedge\ldots\wedge\Box(B^{(k)}\longrightarrow C_1^{(k)}\vee\ldots\vee C_n^{(k)})$,

where $A,B^{(i)},C_j^{(i)}\in L$, and D belongs to L, iff

(**) $\quad D=p_i\wedge\Box\neg B_1\wedge\ldots\wedge\Box\neg B_n$, or $D=\bot$,

where $p_i\in X$ and $B_1,\ldots B_n\in L\cup R$.

By (*), it is sufficient to prove that L consists of finitely many equivalence classes. By (**), it is enough to prove that there modulo to GL-provable equivalence there are only finitely many formulas of the form $\Box\neg B$ with $B\in L\cup R$. By (*) and (**), B has the form

$$B=p_i\wedge\Box D$$

where D is a boolean combinations of formulas from $L\cup R$. But it is well-known that

$$GL\vdash \Box(\neg p_i\vee\neg\Box D)\leftrightarrow\Box(\neg p_i\vee\neg\Box D'),$$

where D' is obtained from D by replacing all the occurrences of p_i in D by $\bot$. By the induction hypothesis, there are only finitely many such D' modulo to GL-equivalence. QED.

Corollary 6.4. Let X be a finite set of modal formulas. Then there exist finite sets $\bar{L}$ and $\bar{R}$ such that $\bar{L}\supseteq X$ and:

(L) $\forall A\in\bar{L},\ B\in\bar{L}\cup\bar{R}\ \ \exists A'\in\bar{L}:\ \ GL\vdash A'\leftrightarrow A\wedge\Box\neg B$;

(R) $\forall A\in\bar{L}\cup\bar{R}\ \forall B,C_1,\ldots,C_n\in\bar{L}\ \exists A'\in\bar{R}:\ GL\vdash A'\leftrightarrow A\wedge\Box(B\rightarrow C_1\vee\ldots C_n)$;

($\Box$) if $\Box D$ is a subformula of some formula from $\bar{L}\cup\bar{R}$, then either $\Box D$ is a subformula of some formula from X or $D=\neg B$ for $B\in\bar{L}\cup\bar{R}$ or $D=B\rightarrow C_1\vee\ldots\vee C_n$ for $B,C_1,\ldots,C_n\in\bar{L}$;

($\rhd$) if $D\rhd E$ is a subformula of some formula from $\bar{L}\cup\bar{R}$, then $D\rhd E$ is a subformula of some formula from X.

Proof. We define two sequences: $L_0,L_1,\ldots,L_n,\ldots$; $R_0,R_1,\ldots,R_n,\ldots$:

$L_0:=X$, $R_0:=\varnothing$;

$L_{n+1}:=L_n\cup\{D=A\wedge\Box\neg B$ s.t. there is no formula in L_n GL-equivalent to D, where $A\in L_n, B\in L_n\cup R_n\}$;

$R_{n+1}:=R_n\cup\{D=A\wedge\Box(B\rightarrow C_1\vee\ldots\vee C_k)$ s.t. there is no formula in R_n GL-equivalent to D, where $A\in L_n\cup R_n$, $B,C_1,\ldots,C_n\in L_n\}$.

Define now $\bar{L}:=\bigcup_{n=0}^{\infty}L_n$, $\bar{R}:=\bigcup_{n=0}^{\infty}R_n$. By lemma 6.3, $\bar{L}$ and $\bar{R}$ are finite and have all necessary properties.

Lemma 6.5. Each finite set Φ_0 of modal formulas can be extended to a CLM-adequate set Φ.

Proof. We can assume that $\bot\in\Phi_0$. Let Φ_1 be the closure of Φ_0 under subformulas and negations, $\bar{L}$ and $\bar{R}$ be sets defined in corollary 6.4 (where $X=\Phi_1$). We define:

$\Phi_{\rhd}:=\bar{L}$;

$\Phi_{\Box}:=\{\Box\neg A\,|\,A\in\bar{L}\cup\bar{R}\}$;

$\Phi_2:=\{\Box\neg A\,|\,A\in\bar{L}\cup\bar{R}\}\cup\{A\rhd B\,|\,A,B\in\bar{L}\}\cup\{\Box(A\rightarrow B_1\vee\ldots\vee B_n)\,|\,A,B_1,\ldots,B_n\in\bar{L}\}$;

Φ is closure of Φ_2 under subformulas and negations.

Now we check conditions 1-8 of the definition 6.1.

1. Obviously, $A\in\Phi_{\rhd}$ implies $A\rhd A\in\Phi$. Conversely, if $A\rhd X\in\Phi$ or $X\rhd A\in\Phi$ then by corollary 6.4($\rhd$) $A\in\Phi_1$ (because Φ_1 is closed under subformulas).

2,3,4,5,6 are trivial.

7. Suppose $\Box D_1,\ldots,\Box D_n\in\Phi$. For any $i\le n$ by corollary 6.4($\Box$) one of three cases holds:

a) $D_i=\neg A$ for $A\in\overline{L}\cup\overline{R}$;

b) $D_i=B\longrightarrow C_1\vee\ldots\vee C_n$, $B,C_1,\ldots C_n\in\overline{L}$;

c) $D_i\in\Phi_1$.

Case c) can be easily reduced to case a), because $D_i\in\Phi_1$ implies $\neg D_i\in\Phi_1\subseteq\overline{L}$. Let now $\{\Box D_1,\ldots,\Box D_k\}$ be the set of boxes satisfying condition (a) and $\{\Box D_{k+1},\ldots,\Box D_n\}$ be the set of boxes satisfying condition (b). Let also $A\in\Phi_{\rhd}$. By 6.4(L),

there exists $A'\in\overline{L}$ such that $GL\vdash A'\longleftrightarrow A\wedge\Box D_1\wedge\ldots\wedge\Box D_k$;

by 6.4(R),

there exists $A''\in\overline{R}$ such that $GL\vdash A''\longleftrightarrow A'\wedge\Box D_{k+1}\wedge\ldots\wedge\Box D_n$.

By the definition of $\Phi_{\Box}$, $\Box\neg A''\in\Phi_{\Box}$.

8. If $B\in\overline{L}$ and $\Box D=\Box\neg A$ for $A\in\overline{L}\cup\overline{R}$, then by corollary 6.4(L) there exists $B'\in\Phi_{\rhd}$ such that $GL\vdash B'\longleftrightarrow B\wedge\Box D$.

This completes proof of lemma 6.5.

Proof of theorem 4 for l=CLM.

As usual, we fix a modal formula ϕ such that $CLM\nvdash\phi$, an CLM-adequate set Φ (with the corresponding sets $\Phi_{\rhd},\Phi_{\Box}$) such that $\phi\in\Phi$, the set W of maximal CLM-consistent subsets of Φ and an element x_0 of W such that $\neg\phi\in x_0$.

Definition 6.6. For $x,y\in W$

$x\preccurlyeq^W y :\Leftrightarrow \forall\Box D\in\Phi_{\Box}\quad \Box D\in x\longrightarrow\Box D\in y$;

$x<^W y :\Leftrightarrow \forall\Box D\in\Phi_{\Box}\quad \Box D\in x\longrightarrow D,\Box D\in y$.

Proposition 6.7. $\preccurlyeq^W,<^W$ are transitive and $x\preccurlyeq^W y<^W z\Rightarrow x<^W z$.

Definition 6.8. Let $\mathcal{K}=\langle K,R,S,\Vdash\rangle$, where

$K=\{\langle x,\tau\rangle$, where $x\in W$, τ is a sequence of formulas from $\Phi_{\rhd}$ and $|\tau|\le depth(x_0)-depth(x)\}$;

$\langle x_1,\tau_1\rangle R\langle x_2,\tau_2\rangle :\Leftrightarrow x_1<^W x_2$ and $\tau_1\subset\tau_2$;

$\langle x_1,\tau_1\rangle S_{\langle y_0,\tau_0\rangle}\langle x_2,\tau_2\rangle :\Leftrightarrow$

1) $\tau_1=\tau_2$;

2) $x_1\preccurlyeq^W x_2$;

3) if $\tau_1=\tau_2=\tau_0*\langle E\rangle$ and $y_0<_E x_1$, then $y_0<_E x_2$;

$\langle x,\tau\rangle \Vdash p :\Leftrightarrow p\in x$.

Proposition 6.9. $\mathcal{K}$ is a CLM-model.

Proof. Use proposition 6.7.

Proposition 6.10. Let $x\in W$ and $\Diamond D\in x$ (i.e $\neg\Box\neg D\in x$). Then there exists $y\in W$ such that $x<y$, $D\in y$.

Proof. Use lemma 4.9 for $A\rhd C = D\rhd\bot$.

Lemma 6.11. Let A be a subformula of ϕ. Then for any $\langle x,\tau\rangle\in K$

$$\langle x,\tau\rangle \Vdash A \Leftrightarrow A\in x.$$

Proof. Let $A=B\rhd C$. Of course, $B,C\in\Phi_{\rhd}$.

1. Let $\neg(B\rhd C)\in x$. By lemma 4.8, there exists y such that $x<_C y$ and $B\in y$. Of course, $\langle y,\tau*\langle C\rangle\rangle\in K$, and if $\langle y,\tau*\langle C\rangle\rangle S_{\langle x,\tau\rangle}\langle z,\sigma\rangle$, $\langle x,\tau\rangle R\langle z,\sigma\rangle$, then by definition of $S_{\langle x,\tau\rangle}$ $x<_C z$, hence $\neg C\in z$.

2. Let $B\rhd C\in x$, and $\langle y,\sigma\rangle \Vdash B$, where $\langle x,\tau\rangle R\langle y,\sigma\rangle$. We consider two cases:

Case 1. $\sigma=\tau*\langle E\rangle$ and $x<_E y$.

Let $\{\Box D_1,\ldots,\Box D_n\}:=y\cap\Phi_\Box$. By condition 8 of the definition of a CLM-adequate set there exist B',C' which are GL-equivalent to $B\wedge\Box D_1\wedge\ldots\wedge\Box D_n$ and $C\wedge\Box D_1\wedge\ldots\wedge\Box D_n$ respectively such that $B'\rhd C'\in\Phi$. Evidently, (by axiom M) $B'\rhd C'\in x$, $B'\in y$. Therefore by lemma 4.9 there exists z such that $x<_E z$ and $C'\in z$. So, we have $\langle z,\sigma\rangle\in K$ (because $|\sigma|=|\tau|+1$ and depth(z)<depth(x)), $y\preccurlyeq^W z$ and $z\Vdash C$ (because $z\Vdash C\wedge\Box D_1\wedge\ldots\wedge\Box D_n$) and thus $\langle y,\sigma\rangle S_{\langle x,\tau\rangle}\langle z,\sigma\rangle$, $\langle z,\sigma\rangle \Vdash C$.

Case 2. $|\sigma|\geq|\tau|+2$ or $x<_E y$ does not hold.

Let $\{\Box D_1,\ldots,\Box D_n\}:=\{\Box D\,|\,\Box D\in\Phi\cap y\}$. By condition 7 of the definition of CLM-adequate set, there exist B',C' which are GL-equivalent to $B\wedge\Box D_1\wedge\ldots\wedge\Box D_n$ and $C\wedge\Box D_1\wedge..\wedge\Box D_n$ respectively such that $\Box\neg B',\Box\neg C'\in\Phi_\Box$. By axioms (A1),(M), $CLM\vdash \bigwedge\!\!\!\bigwedge x\longrightarrow(\Diamond B'\longrightarrow\Diamond C')$.

We show that $\Diamond B'\in x$. Suppose not. Then $\Box\neg B'\in x$ and $x<^W y$ implies $\neg B'\in y$ (because $\Box\neg B'\in\Phi_\Box$). Contradiction. Thus, $\Diamond B'\in x$ and hence $\Diamond C'\in x$. By proposition 6.10, there exists z such that $C'\in z$ and $x<z$. By proposition 5.2b), depth(z)$\leq$depth(y), therefore $\langle z,\sigma\rangle\in K$. So, we have: $\langle y,\sigma\rangle S_{\langle x,\tau\rangle}\langle z,\sigma\rangle$, $\langle x,\tau\rangle R\langle z,\sigma\rangle$, $\langle z,\sigma\rangle\Vdash C$. QED.

As usual, we note that $\langle x_0,\langle\rangle\rangle \nVdash \phi$. This completes the proof of theorem 4 for l=CLM.

§7. Modal Completeness of SCL. Second Modal Completeness Theorem.

Note that any simplified SCL-model is also finite SCL-model. Thus, it is enough to prove modal completeness of SCL w.r.t. simplified models.

First, we define for each CL-model a "pattern" which will be used for definition of corresponded simplified model (this definitions will be different for different considered logics).

<u>Definition</u> 7.1. Let $\mathcal{K}=\langle K,R,\{S_x\},\Vdash\rangle$ be a CL-model. Assume that $0\notin K$. A *heir of the first type* of $\mathcal{K}$ is the tuple

$$\mathcal{H} = \langle H,\subset,<,R(\cdot),end(\cdot)\rangle,$$

where

$H:=\{\langle\Gamma,\Delta\rangle$: 1) Γ is a finite sequence of elements from K;

2) Δ is a finite sequence of elements from $K\cup\{0\}$;

3) $|\Delta|=|\Gamma|-1$;

So, let $\Gamma=\langle x_0,\dots,x_n\rangle$, $\Delta=\langle y_0,\dots,y_{n-1}\rangle$.

4) $\forall i<n\quad y_i=0 \Rightarrow x_iRx_{i+1}$,

$y_i\neq 0 \Rightarrow x_iS_{y_i}x_{i+1}$. $\}$

$\langle\Gamma_1,\Delta_1\rangle\subset\langle\Gamma_2,\Delta_2\rangle :\Leftrightarrow \Gamma_1\subset\Gamma_2 \wedge \Delta_1\subset\Delta_2$

$\langle\Gamma_1,\Delta_1\rangle<\langle\Gamma_2,\Delta_2\rangle :\Leftrightarrow$ 1) $\langle\Gamma_1,\Delta_1\rangle\subset\langle\Gamma_2,\Delta_2\rangle$;

So, let $\Gamma_1=\langle x_0,\dots,x_n\rangle$, $\Delta_1=\langle y_0,\dots,y_{n-1}\rangle$

$\Gamma_2=\langle x_0,\dots,x_m\rangle$, $\Delta_2=\langle y_0,\dots,y_{m-1}\rangle$

$m>n$

2) $\exists i\ (n\le i<m \wedge y_i=0 \wedge$

$\wedge\ \forall j(i<j<m \longrightarrow \exists k(y_j=x_k \wedge n\le k\le i))\ )$

$R(\langle\langle x_0,\dots,x_n\rangle,\langle y_0,\dots,y_{n-1}\rangle\rangle):=\langle\langle x_0,\dots,x_k\rangle,\langle y_0,\dots,y_{k-1}\rangle\rangle$,

where $k:=min\{\ j\ |\ \forall i\ (\ j\le i<n \longrightarrow y_i\neq 0\ \}$

$end(\langle\langle x_0,\dots,x_n\rangle,\langle y_0,\dots,y_{n-1}\rangle\rangle):=x_n$.

A *heir of the second type* of the $\mathcal{K}$ is the tuple

$$\mathcal{H}_1 = \langle H_1,\subset_1,<_1,R_1(\cdot),end_1(\cdot)\rangle,$$

where

$H_1:=\{t\in H|\forall \mathbf{x}\subseteq t\ \forall \mathbf{y}(R(\mathbf{x})\subseteq\mathbf{y}\subset\mathbf{x}\longrightarrow end(\mathbf{x})\neq end(\mathbf{y}))\}$,

and $\subset_1,<_1,R_1(\cdot),end_1(\cdot)$ are restrictions of $\subset,<,R(\cdot),end(\cdot)$ on H_1. ('$\subseteq$' is the reflexive closure of $\subset$).

We left to the reader verification of the following simple facts (it is supposed that $\mathcal{H}=\langle H,\subset,<,R(\cdot),end(\cdot)\rangle$ is a heir of an arbitrary type of $\mathcal{K}=\langle K,R,S,\Vdash\rangle$):

Proposition 7.2 (For any $\mathbf{x},\mathbf{y},\mathbf{z},\ldots\in H$)

a) $\subset$ is transitive and irreflexive;

b) $R(R(\mathbf{x}))=R(\mathbf{x})$;

c) $R(\mathbf{x})\subseteq\mathbf{x}$;

d) $\neg R(\mathbf{x})<\mathbf{x}$;

e) $\mathbf{x}<\mathbf{y} \Rightarrow \mathbf{x}\subset\mathbf{y}$;

f) $\mathbf{x}\subseteq\mathbf{y}<\mathbf{z} \Rightarrow \mathbf{x}<\mathbf{z}$;

g) $<$ is transitive and irreflexive;

h) $\mathbf{x}<\mathbf{y},\ R(\mathbf{y})\subseteq\mathbf{z}\subseteq\mathbf{y} \Rightarrow \mathbf{x}<\mathbf{z}$.

Proposition 7.3. If $\mathcal{K}$ is a CLM-model, then

a) the heir of $\mathcal{K}$ of the 2-nd type is finite;

b) $\forall\mathbf{x},\mathbf{y}\in H\quad \mathbf{x}<\mathbf{y} \Rightarrow end(\mathbf{x})R\,end(\mathbf{y})$.

Proof. a) Let $\mathcal{H}_1$ be the heir of $\mathcal{K}$ of the second type; $\langle\Gamma,\Delta\rangle\in H_1$, where $\Gamma=\langle x_0,\ldots,x_n\rangle$. We claim that for any $i\neq j$ $x_i\neq x_j$, hence there are only finitely many such Γ; on the other hand, since $|\Delta|=|\Gamma|-1$, for each sequence $\Gamma=\langle x_0,\ldots,x_n\rangle$ there are only finitely many such Δ that $\langle\Gamma,\Delta\rangle\in H$.

Suppose not: $\langle\Gamma,\Delta\rangle=\langle\langle x_0,\ldots,x_n\rangle,\langle y_0,\ldots,y_{n-1}\rangle\rangle\in H_1$, $i<j$, $x_i=x_j$. Let $\mathbf{x}:=\langle\langle x_0,\ldots,x_j\rangle,\langle y_0,\ldots,y_{j-1}\rangle\rangle$, $\mathbf{t}:=\langle\Gamma,\Delta\rangle$, $\mathbf{y}:=\langle\langle x_0,\ldots,x_i\rangle,\langle y_0,\ldots,y_{i-1}\rangle\rangle$; $\langle\langle x_0,\ldots,x_k\rangle,\langle y_0,\ldots,y_{k-1}\rangle\rangle:=R(\mathbf{x})$.

Case 1. $k\leq i$. Then $\mathbf{x}\subseteq\mathbf{t}$, $R(\mathbf{x})\subseteq\mathbf{y}\subset\mathbf{x}$, $end(\mathbf{y})=end(\mathbf{x})$. It contradicts the definition of H_1.

Case 2. $k>i$. Then by definition of $R(\cdot)$, $y_{k-1}=0$, hence

$$x_iQ_ix_{i+1}\ldots x_{k-2}Q_{k-2}x_{k-1}Rx_kQ_k\ldots x_{j-1}Q_{j-1}x_j=x_i,$$

where $Q_s=(R \text{ or } S_{y_s})$. Because $\mathcal{K}$ is a CLM-model, xQ_syRz implies xRz, and thus x_iRx_i. Contradiction.

b) Let $\mathbf{y}:=\langle\langle x_0,\ldots,x_n\rangle,\langle y_0,\ldots,y_{n-1}\rangle\rangle$, $\mathbf{x}:=\langle\langle x_0,\ldots,x_k\rangle,\langle y_0,\ldots,y_{k-1}\rangle\rangle$, $\mathbf{x}<\mathbf{y}$. We must show that x_kRx_n. Indeed, by the definition of $<$, for some i, $k\leq i<n$,

$$x_kQ_k\ldots x_{i-1}Q_{i-1}x_iRx_{i+1}S_{y_{i+1}}\ldots S_{y_{n-1}}x_n.$$

By the property of CLM-models x_kRx_{i+1}. If $i+1=n$, we have done; else, there is j, $n\leq j\leq i$, such that $y_{n-1}=x_j$, and x_jRx_n; as above, we obtain x_kRx_n.

Lemma 7.4. Let $end(\mathbf{x})Ry$. Then there exists $\mathbf{y}$ such that

1) $end(\mathbf{y})=y$;

2) $\mathbf{x}<\mathbf{y}$;

3) $R(\mathbf{y})=\mathbf{y}$;

4) for any $\mathbf{z}$ $(R(\mathbf{z})=\mathbf{y} \wedge \mathbf{x}<\mathbf{z} \longrightarrow end(\mathbf{y})S_{end(x)}end(\mathbf{z}) \wedge$
$\wedge\ end(\mathbf{x})Rend(\mathbf{z})\ )$.

Proof. Let $\mathbf{x}=<<x_0,\dots,x_n>,<y_0,\dots,y_{n-1}>>$, x_nRy. We define $\mathbf{y}:=<<x_0,\dots,x_n,y>,<y_0,\dots,y_{n-1},0>>$. The properties 1), 2), 3) are trivial. We check property 4): fix $\mathbf{z}$ such that $R(\mathbf{z})=\mathbf{y}$, $\mathbf{x}<\mathbf{z}$; let

$$\mathbf{z}=<<x_0,\dots,x_n,y,z_0,z_1,\dots,z_l>,<y_0,\dots,y_{n-1},0,y_{n+1},\dots,y_{n+l+1}>>.$$

Since $R(\mathbf{z})=\mathbf{y}$, $y_{n+1},\dots,y_{n+l+1}\neq 0$; because $\mathbf{x}<\mathbf{z}$, we have $i=n$ in the definition of $<$ and thus for any $j>n$ $y_j=x_n$, i.e.

$$yS_{x_n}z_0S_{x_n}z_1\dots S_{x_n}z_l,\quad x_nRz_0,\dots,x_nRz_l.$$

By the transitivity of S_{x_n}, $yS_{x_n}z_l$ and x_nRz_l. QED.

Lemma 7.5. Let $\mathbf{x}<\mathbf{y}$, $end(\mathbf{y})S_{end(x)}z$, $end(\mathbf{x})Rz$. Then there exists $\mathbf{z}$ such that:

1) $end(\mathbf{z})=z$;

2) $\mathbf{x}<\mathbf{z}$;

3) $R(\mathbf{y})=R(\mathbf{z})$;

4) $\mathbf{y}\subseteq\mathbf{z}$ or $\mathbf{z}\subseteq\mathbf{y}$.

If our heir has the 1-st type, we also can require

4*) $\mathbf{y}\subseteq\mathbf{z}$.

Proof. Let $\mathbf{y}=<<x_0,\dots,x_n>,<y_0,\dots,y_{n-1}>>$, $\mathbf{x}=<<x_0,\dots,x_k>,<y_0,\dots,y_{k-1}>>$, ($k<n$), $\mathbf{x}<\mathbf{y}$. Let also

$$<<x_0,\dots,x_j>,<y_0,\dots,y_{j-1}>> := R(\mathbf{y}).$$

(Note that $k<j\le n$). Fix z s.t. $x_nS_{x_k}z$, x_kRz.

Case 1. $z\notin\{x_j,\dots,x_n\}$ or our heir has the 1-st type. Then $\mathbf{z}:=<<x_0,\dots,x_n,z>,<y_0,\dots,y_{n-1},x_k>>$ has all necessary properties.

Case 2. $z=x_s$, $j\le s\le n$, and our heir has the the 2-nd type. Then it is sufficient to define $\mathbf{z}:=<<x_0,\dots,x_s>,<y_0,\dots,y_{s-1}>>$. Because $R(\mathbf{y})\subseteq\mathbf{z}\subseteq\mathbf{y}$ and $\mathbf{x}<\mathbf{y}$, by proposition 7.2 h), $\mathbf{x}<\mathbf{z}$.

Lemma is proved.

We have proved all necessary properties of a heirs and continue with the following definition:

Definition 7.6. Let l=CL,CLM,SCL and ϕ be a modal formula such that $l\nvdash\phi$; *l-countermodel for* ϕ is

a) (for l=CL,CLM) finite l-model $\mathcal{K}$ such that $\mathcal{K}\nvDash\phi$;

b) (for l=SCL) finite CLM-model $\mathcal{K}$ such that $\mathcal{K}\nvDash\phi$ and for any subformula ϕ of the type $A\rhd B$ or $\Box D$, for any $x,y,z\in K$, if $yS_x z$, then y and z agree on this subformula.

We will reduce a l-countermodel for ϕ to a simplified l-countermodel for ϕ; however, before it we must show that if $SCL\nvdash\phi$, then there exists a SCL-countermodel for ϕ.

Definition 7.7. For any modal formula ϕ let $X(\phi)$ be the set of all $\rhd$-subformulas of ϕ and all formulas of the form $(\neg D)\rhd\bot$, where $\Box D$ is a subformula of ϕ; $\neg X$ is the set of negations of all formulas from X.

Definition 7.8. For any modal formula ϕ

$$S(\phi):=\Box^{+}\bigwedge_{D=C_1\wedge\ldots\wedge C_n,\ C_1,\ldots,C_n\in X\cup\neg X,\ A\rhd B\in X} A\rhd B\longrightarrow(A\wedge D)\rhd B\wedge D)$$

where $\Box^{+}E := E\wedge\Box E$.

Lemma 7.9. If $SCL\nvdash\phi$, then there exists a SCL-countermodel for ϕ.

Proof. Assume that $SCL\nvdash\phi$, then $CLM\nvdash S(\phi)\longrightarrow\phi$; let $\mathcal{K}=\langle K,R,\{S_x\},\Vdash\rangle$ be a finite CLM-countermodel for $S(\phi)\longrightarrow\phi$. We can assume that $x_0\nVdash S(\phi)\longrightarrow\phi$ and $K=\{x_0\}\cup\{x|x_0Rx\}$. Define S as an equivalence relation on K: xSy iff x and y agree on each formula from X. We claim that the model $\langle K,R,\{S_x\cap S\},\Vdash'\rangle$ (where $\Vdash'$ coincides with $\Vdash$ on propositional variables) has all properties required.

It is sufficient to show that the restriction of S_x to $S_x\cap S$ preserves forcing of formulas from $X(\phi)$. Indeed, let $x\Vdash A\rhd B$, xRy, $y\Vdash A$, and D be a conjunction of all formulas from $X(\phi)$ and its negations which are true in y. Since $\mathcal{K}\models S(\phi)$, $x\Vdash (A\wedge D)\rhd(B\wedge D)$ and there exists z s.t. $yS_x z$, xRz, $z\Vdash B\wedge D$; evidently, ySx. QED.

Definition 7.10. Suppose that l=CL,CLM,SCL; ϕ is a modal formula such that $l\nvdash\phi$; $\mathcal{K}$ is a l-countermodel for ϕ; $\mathcal{H}$, $\mathcal{H}_1$ are heirs of $\mathcal{K}$. We define a simplified l-model $\mathcal{K}'=\langle K',R',S',\Vdash'\rangle$ by the following table:

l	K′	xR′y	xS′y	x ⊩′ p
CL	H	$x<y\wedge end(x)\,R\,end(y)$	$R(x)=R(y)$	$end(x)\Vdash p$ in any case.
CLM	H	$x<y$	$R(x)=R(y)\wedge x\subseteq y$	
SCL	H_1	$R(x)<y$	$R(x)=R(y)$	

Proposition 7.11. $\mathcal{K}'$ is, indeed, a simplified l-model.

Proof.

l=CL.

1. *R′ is transitive and wellfounded*: by proposition 7.2g)

2. *S′ is an equivalence relation*. It is trivial.

l=CLM.

1. *R′ is transitive and wellfounded*: by propositions 7.2g) and 7.3b).

2. *S′ is reflexive and transitive*. It is trivial.

3. $\mathbf{x}S'\mathbf{y}R'\mathbf{z} \Rightarrow \mathbf{x}R'\mathbf{z}$: by proposition 7.2f)

4. *There exists a natural number* N *and a mapping* $\mu: K' \longrightarrow \{1,\dots,N\}$ *such that*

   a) $\mathbf{x}S'\mathbf{y} \Rightarrow \mu(\mathbf{x})=\mu(\mathbf{y})$;

   b) $\mathbf{x}R'\mathbf{y} \Rightarrow \mu(\mathbf{x})<\mu(\mathbf{y})$:

   Define for $\mathbf{x}=\langle\langle x_0,\dots,x_n\rangle,\langle y_0,\dots,y_{n-1}\rangle\rangle$ $\mu(\mathbf{x}):=|\{i:y_i=0\}|+1$.

l=SCL.

1. *K′ is finite*: by proposition 7.3a).

2. *R′ is transitive*.

   Let $R(\mathbf{x})<\mathbf{y}$ and $R(\mathbf{y})<\mathbf{z}$. By proposition 7.2h), $R(\mathbf{x})<R(\mathbf{y})$. By proposition 7.2g), $R(\mathbf{x})<\mathbf{z}$.

3. *R′ is irreflexive*: by proposition 7.2 d).

4. *S′ is an equivalence relation*. It is trivial.

5. $\mathbf{x}S'\mathbf{y}R'\mathbf{z} \Rightarrow \mathbf{x}R'\mathbf{z}$. It is trivial.

Lemma 7.12. For any $\mathbf{x}\in K'$ and modal formula A (in the case l=SCL it is necessary to require that A is a subformula of ϕ),

$$\mathbf{x} \Vdash' A \iff end(\mathbf{x}) \Vdash A.$$

Proof. Induction on the complexity of A. We consider the case $A=B\rhd C$. Let $x:=end(\mathbf{x})$; $\mathbf{t}:=R(\mathbf{x})$, if l=SCL, $\mathbf{t}:=\mathbf{x}$ otherwise; $t:=end(\mathbf{t})$. In any case (by definition 7.6b)), $\mathbf{t} \Vdash A \iff \mathbf{x} \Vdash' A$.

1. Suppose that $\mathbf{x} \Vdash' B\rhd C$. We will show that $t \Vdash B\rhd C$. Indeed, let tRy, $y \Vdash B$. We use lemma 7.4 to obtain $\mathbf{y}\in K'$; by the induction hypothesis, $\mathbf{y} \Vdash' B$, hence there is $\mathbf{z}$ such that $\mathbf{z} \Vdash' C$, $\mathbf{t}R'\mathbf{z}$, $\mathbf{y}S'\mathbf{z}$. In any case, $\mathbf{y}=R(\mathbf{y})=R(\mathbf{z})$ and $\mathbf{t}<\mathbf{z}$ (for l=SCL it follows from $\mathbf{t}=R(\mathbf{t})$), hence by a property of $\mathbf{y}$ (claim 4) of lemma 7.4), yS_tz, tRz, where $z:=end(\mathbf{z})$. By the induction hypothesis, $z \Vdash C$. Thus, we proved that $t \Vdash B\rhd C$.

2. Suppose that $t \Vdash B \rhd C$. We will show that $\mathbf{x} \Vdash' B \rhd C$. Indeed, let $\mathbf{x}R'\mathbf{y}$, ($y:=end(\mathbf{y})$), $\mathbf{y} \Vdash' B$. One can see (using proposition 7.3b)) that tRy; by the induction hypothesis, $y \Vdash B$, hence there is z s.t. tRz, $yS_t z$, $z \Vdash C$. Now we can use lemma 7.5 and obtain $\mathbf{z}$ s.t. $\mathbf{z} \Vdash' C$ ($end(\mathbf{z})=z$ and induction hypothesis), $\mathbf{y}S'\mathbf{z}$ (in the case l=CLM we use property 4 of $\mathbf{z}$ from lemma 7.5), $\mathbf{x}R'\mathbf{z}$ (we use that tRz and $t<\mathbf{z}$). Thus, $\mathbf{x} \Vdash' B \rhd C$.

Thus, if $x \nVdash \phi$, then $\langle\langle x\rangle,\langle\rangle\rangle \nVdash' \phi$ and $\mathcal{K}' \nvDash \phi$. This complete the proof of the second modal completeness theorem.

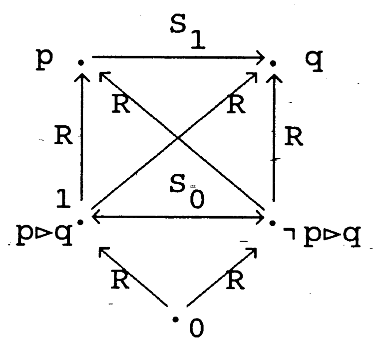
<u>Corollary</u> 7.13. For any modal formula ϕ,

$$\mathrm{SCL} \vdash \phi \iff \mathrm{CLM} \vdash S(\phi) \longrightarrow \phi.$$

<u>Remark</u>. The author doesn't know whether we can use only finite simplified model for l=CL, but for l=CLM we cannot. Indeed, the following formula

$$(\Diamond\top)\rhd(\Diamond\top\wedge(p\rhd q)) \wedge (\Diamond\top)\rhd(\Diamond\top\wedge\neg(p\rhd q)) \longrightarrow \Box\Box\bot$$

is not provable in CLM:



and is valid in every finite simplified CLM-model.

§8. Arithmetical Completeness of SCL.

We will prove theorem 1 in more general formulation:

Theorem 8.1. Let Γ be decidable set of arithmetical formulas closed under conjunctions such that $\Sigma_2 \cup \Pi_2 \subseteq \Gamma \subseteq \Sigma_N$ for some N. Then SCL is the provability logic for Γ-conservativity (i.e. SCL = CL(Γ) = $CL^+(\Gamma)$)

Proof. Arithmetical soundness of SCL is evident. Assume that $SCL \nvdash \phi$, and let $\mathcal{K}=\langle K,R,S,\Vdash\rangle$ be a simplified SCL-countermodel for ϕ. Without a loss of generality we can assume that

(*) $\exists w \in K \quad \forall x \in K \quad (x=w \vee wRx)$

(**) $\exists u \in K \quad (u \neq w \wedge u \nVdash \phi)$

Define now a usual Solovay function h:

Definition 8.2.

$h(0)=w$;

if $Proof_{PA}(n, \ulcorner l \neq z \urcorner)$ and $h(n)Rz$

then $h(n+1):=z$;

else $h(n+1):=h(n)$.

"$l=z$" stands for (Σ_2-formula) "$\lim_{n\to\infty} h(n)=z$".

and establish its usual properties:

Lemma 8.3. (PA proves that)

1. There exists unique z such that $l=z$.
2. If xRy, then $l=x \longrightarrow \neg\Box_{PA} l\neq y$.
3. If $x\neq w$, then $l=x \longrightarrow \Box_{PA}(l=y \longrightarrow xRy)$.

In the following considerations we will use inside PA notion "truth" for some formulas defined by their Gödelnumbers. It is admitable, because one can check (using assumption $\Gamma \subseteq \Sigma_N$) that complexity of all such formulas bounded by Π_N.

Definition 8.4. For all $z \in K$ we define a formula $L=z$ (by Diagonal Lemma):

$L=z$:⇔ "For any x such that $l=x$

if there exists n such that

$Proof_{PA}(n, \ulcorner L=z \longrightarrow Q \urcorner)$;

$Q \in \Gamma$, Q is false;

xSz, h(n)Rz

then our z must be minimal with respect to such n

else z=x."

Lemma 8.5. (PA proves that)

1. There exists unique z such that L=z.
2. If xRy,xRz,ySz, then L=x $\longrightarrow$ L=y$\rhd_\Gamma L$=z.
3. If x≠w, then L=x $\longrightarrow$ $\Box_{PA}(L\text{=y}\longrightarrow\text{xRy})$.
4. If xRy, then L=x $\longrightarrow$ $\neg\Box_{PA}L\neq$y.

Proof.

1. is trivial.
2. Reason in PA:

Let L=x and $\Box_{PA}(L\text{=z}\longrightarrow\text{Q})$, where Q∈Γ. Let also l=t, and $\text{Proof}_{PA}(\text{n},\ulcorner L\text{=z}\Longrightarrow\text{Q}\urcorner)$, where n so great that h(n)=t; evidently, $\Box_{PA}(\text{h}(\underline{\text{n}})\text{=t})$.

Reason in PA+L=y:

Let l=v, then vSy, hence vSz. Suppose that Q is false, and note that t=h($\underline{\text{n}}$)Rz.

Why $L\neq$z? By definition 8.4, we have the only reason: $\exists\text{m}\leq\underline{\text{n}}$ $\text{Proof}_{PA}(\text{m},\ulcorner L\text{=y}\longrightarrow\text{Q}_1\urcorner)$, where Q_1 is false. Of course, it implies that $L\text{=y}\longrightarrow\text{Q}_1$ is true and L≠y. Contradiction. Thus, Q is true.

So, $\Box_{PA}(L\text{=y}\longrightarrow\text{Q})$.

3. Reason in PA:

Let L=x≠w, and suppose l=v, where vSx and h(n)=v. By claim 3 of lemma 8.3, $\Box_{PA}(l\text{=t}\longrightarrow\text{vRt})$.

Reason in PA:

Let L=y and l=t. We have vRt, tSy and xRt; hence, we can assume t≠y.

By definition 8.4, fix a number m and a sentence Q∈Γ s.t.:

$\text{Proof}_{PA}(\text{m},\ulcorner L\text{=y}\longrightarrow\text{Q}\urcorner)$;

Q is false;

h(m)Ry.

Consider two cases:

Case 1. m≤$\underline{\text{n}}$. Then, as above, L=y is false. Contradiction.

Case 2. m>$\underline{\text{n}}$. Then h(m)=h($\underline{\text{n}}$)=v or vRh(m); hence, vRy and xRy.

Thus, we proved that xRy.

So, $\Box_{PA}(L\text{=y}\longrightarrow\text{xRy})$.

4. Reason in PA+L=x.

Let l=t. Assume that $\Box_{PA}(L\neq y)$. We claim that $\Box_{PA}(l\neq y)$. Indeed, consider an arbitrary z such that xRy, ySz. By claim 2 of present lemma, $L=z \rhd_{\Gamma} L=y$, hence $\Box_{PA}(L\neq z)$. We have:

$\Box_{PA}(L=z \wedge ySz \longrightarrow \neg xRz)$ (as we proved above)

$\Box_{PA}(L=z \longrightarrow xRz)$ (by claim 3 of present lemma)

Thus, $\Box_{PA}(L=z \longrightarrow \neg ySz)$, hence by definition 8.4 $\Box_{PA}(l\neq y)$. But tSxRy, hence tRy and we have a contradiction with claim 2 of lemma 8.3. So, our assumption ($\Box_{PA} L\neq y$) is false.

This completes the proof of lemma 8.5.

Now we can define an arithmetical interpretation f_{Γ}:

$$f_{\Gamma}(p) := \exists z \ (L=z \wedge z \Vdash p).$$

<u>Lemma</u> 8.6. Let x≠w and A be an arbitrary modal formula. Then

$x \Vdash A \Rightarrow PA \vdash L=x \longrightarrow f_{\Gamma}(A)$

$x \nVdash A \Rightarrow PA \vdash L=x \longrightarrow \neg f_{\Gamma}(A)$

<u>Proof</u>. As usual, we consider only the case A=B$\rhd$C.

1. Suppose $x \Vdash B\rhd C$. Reason in PA+L=x:

Let $\Box_{PA}(f_{\Gamma}(C)\longrightarrow Q)$, where $Q\in\Gamma$.

Reason in PA+$f_{\Gamma}(B)$:

Let L=y. By the induction hypothesis for B, $y \Vdash B$. By claim 3 of lemma 8.5, xRy, hence there exists z such that ySz, xRz, $z \Vdash C$.

Here we interrupt our reasoning and note that by the induction hypothesis for C, $\Box_{PA}(L=z\longrightarrow f_{\Gamma}(C))$, hence $\Box_{PA}(L=z\longrightarrow Q)$; by claim 2 of lemma 8.5, $\Box_{PA}(L=y\longrightarrow Q)$.

We continue reasoning in PA+$f_{\Gamma}(B)$:

Since L=y, Q is true.

So, we have proved $\Box_{PA}(f_{\Gamma}(B)\longrightarrow Q)$, therefore $f_{\Gamma}(B)\rhd_{\Gamma} f_{\Gamma}(C)$.

(Note that in this proof the finiteness of K is essential.)

2. Suppose $x \nVdash B\rhd C$, then there exists y such that xRy, $y \Vdash B$, $\forall z(ySz\wedge xRz \longrightarrow z \nVdash C)$. Reason in PA+$L$=x:

Let Q denote the arithmetical formula $\exists v(l=v\wedge ySv)$. Of course, $Q\in\Sigma_2$. Note that by definition 8.4, $PA \vdash Q \leftrightarrow \exists v(L=v\wedge ySv)$; by the induction hypothesis (for C) $PA \vdash Q \longrightarrow \neg f_{\Gamma}(C)$.

Assume that $f_\Gamma(B) \rhd_\Gamma f_\Gamma(C)$. Since $\Box_{PA}(f_\Gamma(C) \longrightarrow \neg Q)$ and $\Gamma \supseteq \Pi_2$, $\Box_{PA}(f_\Gamma(B) \longrightarrow \neg Q)$. By the induction hypothesis (for B), $\Box_{PA}(L{=}y \longrightarrow f_\Gamma(B))$. Thus, $\Box_{PA}(L{=}y \longrightarrow \neg Q)$ and $\Box_{PA}(L{\neq}y)$. It contradicts claim 4 of lemma 8.5. Thus, $\neg f_\Gamma(B) \rhd_\Gamma f_\Gamma(C)$.

Using standard PA-soundness argument one can obtain a following proposition:

<u>Proposition</u> 8.7. $L{=}w$ is true (i.e. $\models L{=}w$).

<u>Corollary</u> 8.8. For any $x \in K$ PA does not prove $L{\neq}x$.

<u>Proof</u>. If $x=w$, it is trivial. If wRx, claim 4 of lemma 8.5 shows that $L{=}w \longrightarrow \neg\Box_{PA}(L{\neq}x)$ is true and by the previous proposition $\Box_{PA}(L{\neq}x)$ is false. QED.

Recall now that $u \neq w$ and $u \Vdash \neg\phi$ (see (**) in the beginning of this paragraph). By lemma 8.6, if $PA \vdash f_\Gamma(\phi)$, then $PA \vdash L{\neq}u$. It contradicts corollary 8.8. Thus, $PA \nvdash f_\Gamma(\phi)$.

This completes the proof of theorem 1.

In conclusion of this paragraph we consider a "truth variant" of provability logic for Γ-conservativity.

<u>Definition</u> 8.10. The logic SCL^ω is the minimal set of modal formulas closed under *modus ponens* and containing all theorems of SCL and all formulas of the form $\Box A \longrightarrow A$.

<u>Lemma</u> 8.11. Assume that $SCL^\omega \nvdash \phi$. Then there exists a simplified SCL-model $\mathcal{K}=\langle K,R,S,\Vdash\rangle$ and a node $w \in K$ such that:

1. $\forall x \in K \quad wRx \vee wSx$;
2. $w \nVdash \phi$;
3. if $B \rhd C$ is a subformula of ϕ, $w \Vdash B \rhd C$, and for some x wSx and $x \Vdash B$, then there exists y such that wSy and $y \Vdash C$;
4. if $\Box D$ is a subformula of ϕ and $w \Vdash \Box D$, then for any $x \in K$ $x \Vdash D$.

<u>Proof</u>. Assume $SCL^\omega \nvdash \phi$. Let X be the set of all formulas of the form $\Box A \longrightarrow A$, then by definition of SCL^ω $X \cup \{\neg\phi\}$ is SCL-consistent (i.e. every finite subset of $X \cup \{\neg\phi\}$ is SCL-consistent). Define a set Φ as the maximal SCL-consistent extension of $X \cup \{\neg\phi\}$, and ψ as the conjunction of all subformulas of ϕ and their negations which belong to Φ. Obviously, $\psi \in \Phi$ and $\psi \wedge \Diamond\psi \in \Phi$ (because $(\Box\neg\psi \longrightarrow \neg\psi) \in X$), hence $SCL \nvdash \neg(\psi \wedge \Diamond\psi)$ and there exist a simplified SCL-model $\mathcal{K}=\langle K,R,S,\Vdash\rangle$ and nodes $v,w \in K$ such that $K=\{v\} \cup \{x \mid vRx\}$, vRw, $v \Vdash \psi$, $w \Vdash \psi$.

We claim that SCL-model $\mathcal{K}$ and node w satisfy conditions 2,3,4 from the formulation of present lemma. Indeed, $\neg\phi$ is a conjunct of ψ, hence $w \Vdash \neg\phi$. We check condition 3 (condition 4 is analogous). Let $w \Vdash B \rhd C$ and $x \Vdash B$, wSx. Since $B \rhd C$ is a subformula of ϕ, $B \rhd C$ is a conjunct of ψ and $v \Vdash B \rhd C$. Obviously, vRx and hence there exists y such that wSy, $y \Vdash C$.

Thus, the model $\mathcal{K}'=\langle K',R',S',\Vdash'\rangle$, where $K':=\{x\in K \mid wRx \text{ or } wSx\}$ and R', S', $\Vdash'$ are restrictions of R,S,$\Vdash$ on K', has all necessary properties.

Remark. As in the proof of modal completeness of SCL, we note that if $\Box D$ is a subformula of ϕ, $(\neg D)\rhd\bot$ needs not to be among subformulas of ϕ. But applying condition 4 of the previous lemma, we claim that *we can assume* that $(\neg D)\rhd\bot$ is a subformula of ϕ whenever $\Box D$ is. So, in the following reasoning, as usual, we will consider only $\rhd$-subformulas of ϕ.

Theorem 8.12. (*Arithmetical completeness of* SCL^{ω}).
Let Γ be as above (see theorem 8.1). For any modal formula ϕ

$SCL^{\omega} \vdash \phi$ iff for any arithmetical interpretation f_{Γ} $\models f_{\Gamma}(\phi)$.

Proof. Arithmetical soundness of SCL^{ω} is evident.

Let $SCL^{\omega} \nvdash \phi$. Fix a simplified SCL-model described in lemma 8.11. Define a binary relation R' on K by the following:

$$xR'y \;:\Leftrightarrow\; xRy \vee (xSwSy).$$

Proposition 8.13. If $B\rhd C$ and $\Box D$ are subformulas of ϕ, then for any $x\in K$

$x \Vdash B\rhd C \Leftrightarrow \forall y(xR'y \wedge y \Vdash B \rightarrow \exists z(xR'z \wedge ySz \wedge z \Vdash C))$,

$x \Vdash \Box D \Leftrightarrow \forall y(xR'y \rightarrow y \Vdash D)$.

Proof. Using conditions 3,4 from lemma 8.11.

We change definition 8.4 by replacing R with R'. (So, l=w does not imply L=w). The proof is to be changed as follows:

Lemma 8.14. (PA proves that)

1. There exists an unique z s.t. L=z.
2. If xR'y,ySz,xR'z then $L=x \rightarrow L=y \rhd_{\Gamma} L=z$.
3. For any x $L=x \rightarrow \Box_{PA}(L=y \rightarrow xR'y)$.
4. For any x,y, if xRy, then $L=x \rightarrow \neg\Box_{PA}(L\neq y)$.

Proof

1. is trivial.
2. The only interesting case is wSxSySz. Reason in PA+L=y:

Let $Proof_{PA}(n,\ulcorner L=z\to Q\urcorner)$, $Q\in\Gamma$; of course, $l=w$, $\square_{PA}(h(\underline{n})=w)$.
Reason in $PA+L=y$:

Assume that Q is false. Because $h(\underline{n})=wR'z$, the only reason why $L\neq z$ is $\exists m\leq\underline{n}\ Proof_{PA}(m,\ulcorner L=y\to Q_1\urcorner)$. Like earlier, $L=y$ is false. Contradiction. So, Q is true.

Thus, $\square_{PA}(L=y\to Q)$, hence $L=y\rhd_\Gamma L=z$.

3. It is a trivial modification of claim 3 of lemma 8.5.
4. Proof does not differ from the proof of lemma 8.5.

We define Solovay-like interpretation f_Γ as above.

Lemma 8.15. Let a be a subformula of ϕ, $x\in K$. Then

$$x\Vdash A \implies PA\vdash L=x\to f_\Gamma(A),$$
$$x\nVdash A \implies PA\vdash L=x\to\neg f_\Gamma(A).$$

Proof. We assume that $A=B\rhd C$.

In the case $x\nVdash B\rhd C$ proof does not differ from the proof of the similar case of lemma 8.6.

In the case $x\Vdash B\rhd C$ one must replace R by R′ in the proof of lemma 8.6 and use claims 2,3 of lemma 8.14 and proposition 8.13.

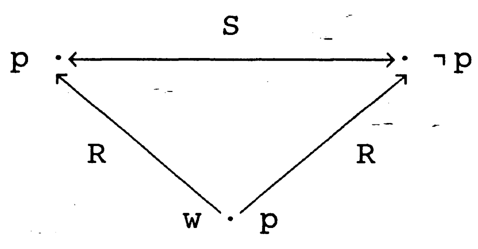
The proof of proposition 8.7 is not changed. Therefore, since $PA\vdash L=w\to\neg f_\Gamma(\phi)$ and $L=w$ is true, $f_\Gamma(\phi)$ is false.

We have proved arithmetical completeness of SCL^ω.

Remark. In fact, we proved that the existence of a SCL-model described in lemma 8.11, *is equivalent* to $SCL^\omega\nvdash\phi$. Thus, the logic SCL^ω is decidable.

Examples:

1. Consider a formula $\square(\top\rhd p)\to\square p$. We claim that this formula is not derivable in SCL^ω. Indeed, there is a "countermodel" for this formula (in sense of lemma 8.11):



(Note that we must define $w\Vdash p$, because $w\Vdash\top\rhd p$ and $w\Vdash\top$).

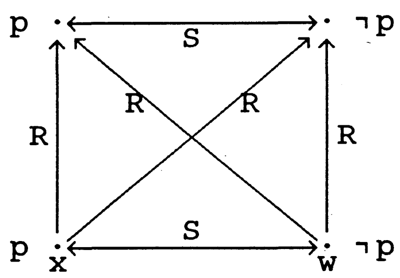
Thus, we proved that there exists a sentence Q such that (for fixed Γ)

(*) $PA \nvdash Q$, but $PA \vdash \top \rhd_{\Gamma} Q$.

2. If we want to find a *false* sentence Q satisfying (*), we are to consider a formula

$$\neg p \wedge \Box(\top \rhd p) \longrightarrow \Box p$$

and a following countermodel:



We have: $w \Vdash \top \rhd p$, $w \Vdash \top$ and $x \Vdash p$.

§9. Arithmetical Completeness of CL and CLM.

Proof of theorem 2.

An arithmetical soundness is evident. We only need to prove a completeness.

Assume that $CL \nvdash \phi$. By the second modal completeness theorem, there exists a simplified CL-model $\mathcal{K}=\langle K,R,S,\Vdash\rangle$ such that:

1). There is $w,u \in K$ such that:

$\forall x \in K \quad wRx \vee w=x;$

$u \neq w; \quad u \nVdash \phi.$

This condition is not strong enough to work in PA with $\mathcal{K}$, because K may be infinite. So, we need also the following properties of $\mathcal{K}$ (these properties can be checked by inspection of the modal completeness proof):

2). The relations $R,S,\Vdash$ are p.r.; moreover, the p.r. definitions of $R,S,\Vdash$ are provable in PA.

3). All properties of $\mathcal{K}$ as a simplified CL-model are provable in PA (more precisely, we must demand, instead of "R is wellfounded", that there exists $n \in \omega$ such that (PA proves that) $\mathcal{K}$ does not consist any R-chain of the length more that n)

Let for $X \subseteq K$ $[X]$ denote $\{y \in K \mid \exists x \in X: xSy\}$.

4). If $X \subseteq K$ is finite and C is a modal formula, then the arithmetical formula $A(x,X) := \exists z(xRz \wedge z \in [X] \wedge z \Vdash C)$ is PA-equivalent to

Δ_0-formula.

We define a usual Solovay function h (see definition 8.2). For an infinite K we need the following lemma (instead of lemma 8.3):

Lemma 9.1. (PA proves that)

1. There exists a unique x such that $l=x$.
2. $l=x \longrightarrow \forall y(xRy \longrightarrow \diamond_{PA} l=\underline{y})$;
3. $l=x\neq w \longrightarrow \Box_{PA}(l=y \longrightarrow \underline{x}Ry)$.

For any finite $X\subseteq K$ we define also a formula $\gamma_X :=l\notin[X]$ (i.e. $\gamma_X:=\exists x(l=x\wedge x\notin[X])$). Let Γ be the set of formulas

$$\Gamma:=\{\gamma_X | X\subseteq K,\ X \text{ is finite}\}\cup\{\bot\}.$$

Of course, $PA\vdash \gamma_X\wedge\gamma_Y\leftrightarrow\gamma_{X\cup Y}$; $PA\vdash \gamma_X\vee\gamma_Y\leftrightarrow\gamma_{X\cap Y}$. So, working in PA we can assume that Γ is closed under disjunctions.

We introduce an arithmetical interpretation f_Γ as usual:

$$f_\Gamma(p) := \exists x(l=x\wedge x\Vdash p).$$

We will prove that $PA\nvdash f_\Gamma(\phi)$.

Lemma 9.2. For any modal formula A PA proves that for any $x\neq w$

$$l=x \longrightarrow (\ x\Vdash A \ \leftrightarrow\ f_\Gamma(A)\ ).$$

Proof. We consider the case $A=B\rhd C$. Reason in PA:

Let $l=x$, where $x\neq w$.

1. Suppose $x\Vdash B\rhd C$. Consider an arbitrary finite subset X of K and assume that $\Box_{PA}(f_\Gamma(C)\longrightarrow\gamma_X)$. We claim that

(*) there is no $z\in K$ s.t. xRz, $z\in[X]$, $z\Vdash C$.

Indeed, if such z exists, then by the induction hypothesis for C, $\Box_{PA}(l=\underline{z}\longrightarrow f_\Gamma(C))$, hence $\Box_{PA}(l=\underline{z}\longrightarrow\gamma_X)$ and $\Box_{PA}(l\neq\underline{z})$. It contradicts claim 2 of lemma 9.1. Thus, by property 4) of model $\mathcal{K}$ PA proves (*).

We reason in $PA+f_\Gamma(B)$:

Let $l=y$. By claim 3 of lemma 9.1, $\underline{x}Ry$. We knew that $x\Vdash B\rhd C$. By the induction hypothesis for B, $y\Vdash B$. So, there exists z s.t. $z\Vdash C$, $\underline{x}Rz$, ySz. By (*), $z\notin[X]$, hence $y\notin[X]$. So, we proved that $l\notin[X]$, i.e. γ_X.

Thus, we proved that $\Box_{PA}(f_\Gamma(C)\longrightarrow\gamma_X)$ implies $\Box_{PA}(f_\Gamma(B)\longrightarrow\gamma_X)$ and left to the reader to prove that $\Box_{PA}(f_\Gamma(C)\longrightarrow\bot)$ implies $\Box_{PA}(f_\Gamma(B)\longrightarrow\bot)$.

2. Suppose $x\nVdash B\rhd C$. Then there exists y such that xRy,

$y \Vdash B$,

(**) there is no $z \in K$ s.t. xRz, ySz, $z \Vdash C$.

Let $Y := \{y\}$. Obviously, $x \in [Y] \Leftrightarrow xSy$. As above, $PA \vdash$ (**).

Claim 1. $\Box_{PA}(f_\Gamma(C) \rightarrow \gamma_Y)$. Proof: Reason in $PA + \neg\gamma_Y$.

Let $l=z$, then ySz; of course, $\underline{x}Rz$; by (**), $t \nVdash C$;

by the induction hypothesis for C, $\neg f_\Gamma(C)$.

Claim 2. $\neg\Box_{PA}(f_\Gamma(B) \rightarrow \gamma_Y)$. Suppose not. Then (by the induction hypothesis for B) $\Box_{PA}(l=\underline{y} \rightarrow \gamma_Y)$, i.e. $\Box_{PA}(l \neq \underline{y})$. Contradiction.

So, since $\gamma_Y \in \Gamma$, $f_\Gamma(B) \rhd_\Gamma f_\Gamma(C)$ does not hold.

As usual, we conclude that $PA \vdash l=u \longrightarrow \neg f_\Gamma(\phi)$ and $PA \nvdash f_\Gamma(\phi)$.

<u>Proof of theorem 3</u>

First, we recollect several facts connected with the logic of Π_1-conservativity (see also §1):

<u>Theorem</u> 9.3. (Hàjek-Guaspari, cf.[2],[3]). If T_1, T_2 are r.e. extensions of PA in the language of PA, then T_2 is Π_1-conservative over T_1 iff T_1 interprets T_2.

<u>Definition</u> 9.4. The logic ILM is given by all axioms of CLM (i.e. L0-L3, A1-A4, M , see §2) plus

A5. $\Diamond A \rhd A$

and usual inference rules (modus ponens and necessitation).

<u>Definition</u> 9.5. A *simplified ILM-model* $\mathcal{K}$ is a simplified model $\langle K,R,S,\Vdash\rangle$ which fulfills the following conditions:

1. K is finite or countable.
2. $S \supseteq R$.
3. $xSyRz \Rightarrow xRz$.

<u>Theorem</u> 9.6. (Visser [8], cf. also [7]). Logic ILM is complete w.r.t. simplified ILM-models.

<u>Theorem</u> 9.7. (Shavrukov [9], Berarducci [10]). ILM is provability logic for relative interpretability (over PA). So, by theorem 9.3., ILM is also the logic of Π_1-conservativity.

We begin to prove the theorem. As above, we only need to prove a completeness of CLM.

Assume that $CLM \nvdash \phi$ and $\mathcal{K} = \langle K,R,S,\Vdash\rangle$ is simplified CLM-model s.t. $\mathcal{K} \nVdash \phi$. We remind that by definition 3.4.2 there exists a mapping μ: $K \longrightarrow \{1,\dots,N\}$ such that $\forall x,y \in K$ ($xSy \longrightarrow \mu(x)=\mu(y)$, $xRy \longrightarrow$

$\mu(x)<\mu(y)$). Let $\mathbb{S}=\{q_1,\ldots,q_N\}$ be a set of new propositional variables which are not contained in ϕ.

Definition 9.8. A simplified model $\mathcal{K}^*$ is $\langle K^*,R^*,S^*,\Vdash^*\rangle$, where

$K^*:=K$;

$R^*:=R$;

$xS^*y \quad :\Leftrightarrow \quad xSy \vee \exists z(xRzSy)$;

$x\Vdash^* p_i$, where $p_i\notin\mathbb{S}$, $\quad :\Leftrightarrow \quad x\Vdash p_i$;

$x\Vdash^* q_i$, where $q_i\in\mathbb{S}$, $\quad :\Leftrightarrow \quad \mu(x)\neq i$.

One can easily check that $\mathcal{K}^*$ is a simplified ILM-model.

Definition 9.9. For any modal formula A we define a translation A^* by the following way:

1. for any propositional variable p $p^*:=p$;
2. * commutes with boolean connectives and $\Box$;
3. $$(A\rhd B)^*:=\bigwedge_{q_{i_1},\ldots,q_{i_n}\in\mathbb{S}} (A^*\wedge\neg q_{i_1}\wedge\ldots\wedge\neg q_{i_k})\rhd(B^*\wedge\neg q_{i_1}\wedge\ldots\wedge\neg q_{i_k}).$$

Lemma 9.10. For any $x\in K$ and modal formula A which does not contain $q_1,\ldots,q_n$,

$$x\Vdash A \quad\Leftrightarrow\quad x\Vdash^* A^*.$$

Proof. We consider the case $A=B\rhd C$.

1. Suppose $x\Vdash B\rhd C$. We show that for any $q_{i_1},\ldots,q_{i_k}\in\mathbb{S}$

$$x\Vdash (B^*\wedge\neg q_{i_1}\wedge\ldots\wedge\neg q_{i_k})\rhd(C^*\wedge\neg q_{i_1}\wedge\ldots\wedge\neg q_{i_k}).$$

Indeed, assume that xRy, $y\Vdash^* B^*\wedge\neg q_{i_1}\wedge\ldots\wedge\neg q_{i_k}$. By the induction hypothesis, $y\Vdash B$, then there is z s.t. xRz, ySz, $z\Vdash C$. Since $\mu(y)=\mu(z)$, $\forall q_i\in\mathbb{S}$ $y\Vdash^* q_i\Leftrightarrow z\Vdash^* q_i$, hence (by the induction hypothesis) $z\Vdash^* C^*\wedge\neg q_{i_1}\wedge\ldots\wedge\neg q_{i_k}$. By definition of S^*, yS^*z.

2. Suppose $x\nVdash B\rhd C$. Then there exists y s.t. xRy, $y\Vdash B$, $\forall z(xRz,\ ySz \longrightarrow z\Vdash\neg C)$. Let $n:=\mu(y)$. We claim that

$$x\nVdash^*(B^*\wedge\neg q_n)\rhd(C^*\wedge\neg q_n),$$

hence $x\nVdash^*(B\rhd C)^*$. Suppose not. Since $y\Vdash^* B^*\wedge\neg q_n$ (because $y\Vdash B$ and $\mu(y)=n$), there is z s.t. xRz, yS^*z, $z\Vdash^* C\wedge\neg q_n$. Since $z\Vdash^*\neg q_n$, $\mu(z)=n=\mu(y)$, and by definition of S^* and properties of μ, ySz. Thus, ySz, xRz, $z\Vdash C$. Contradiction.

Lemma 9.10 implies that $\mathcal{K}^*\nvDash\phi^*$, hence by theorems 9.6, 9.7 there exists an arithmetical interpretation f_{Π_1} s.t. $PA\nvdash f_{\Pi_1}(\phi^*)$. Define a set Γ as closure under disjunctions of the set

$$\{f_{\Pi_1}(q_1),\ldots,f_{\Pi_1}(q_n)\} \cup \Pi_1.$$

To conclude the proof it remains to check the following simple fact:

Lemma 9.11. For any modal formula A,

$$PA \vdash f_{\Pi_1}(A^*) \leftrightarrow f_\Gamma(A),$$

where f_{Π_1} is defined above, f_Γ coincides with f_{Π_1} on propositional variables.

Proof. Induction on A; for $A=B\rhd C$ we use the following proposition:

Proposition 9.11.1. Let Γ be an arbitrary set of arithmetical formulas (as usual, we assume that Γ is closed under disjunctions), Δ be an arbitrary finite set of arithmetical formulas. Define Γ' as closure of $\Gamma\cup\Delta$ under disjunctions, i.e.

$$\Gamma'=\{\gamma\vee q_1\vee\ldots\vee q_n \mid \gamma\in\Gamma,\ q_1,\ldots,q_n\in\Delta\}.$$

Then for any arithmetical formulas A,B

$$PA \vdash A\rhd_{\Gamma'}B \leftrightarrow \bigwedge_{q_1,\ldots,q_n\in\Delta} (A\wedge\neg q_1\wedge\ldots\wedge\neg q_n)\rhd_\Gamma(B\wedge\neg q_1\wedge\ldots\wedge\neg q_n).$$

Proof is trivial.

So, $PA \nvdash f_\Gamma(\phi)$.

§10. Conclusion Remarks.

Generalization. We can consider any r.e. theory T as an "internal" theory instead of PA (i.e. translate $A\rhd_\Gamma B$ as "T+B is Γ-conservative over T+A", and $\Box A$ as "A is provable in T"); if T is Σ_1-sound (i.e. each Σ_1-sentence which is provable in T is true) theorem 1 holds (in fact, it is sufficient to demand $T \nvdash \Box_T^n\bot$ for any n).

Unsolved problems.

1. The main unsolved problem in this area is, of course, the logic of Σ_1-,Σ_2-conservativity. The logic of Σ_1-conservativity is, evidently, an extension of CL+M*, where $M^*:=A\rhd B\rightarrow(A\wedge\Diamond C)\rhd(B\wedge\Diamond C)$, but it is a proper extension (for example, $CL(\Sigma_1)\vdash \Box\bot\rhd(\Box^2\bot\wedge\neg\Box\bot)\rightarrow\Box\bot$). So, we have not now any interesting hypothesis about axiomatisation of $CL(\Sigma_1)$.

As to Σ_2-conservativity, the author does not knew any principle which extend CL+M*+Sa. However, there is a non-trivial *truth* principle:

$$A \rhd B \wedge \Box(B \longrightarrow C \rhd D) \longrightarrow (A \longrightarrow C \rhd D).$$

(This principle is also valid for Σ_n-conservativity for any n, but for n≥3 it is derivable from Sb and $\Box A \longrightarrow A$).

3. Does theorem 1 holds for $T = I\Delta_0 + EXP$, etc.? (Note that the proof of claim 1 of lemma 8.5 uses $T \vdash I\Sigma_n$, where $\Gamma = \Pi_n$).

4. What is the *truth* provability logic for Π_2-conservativity (i.e. the set of all modal formulas whose arithmetical interpretations are truth) ?

5. Is it sufficient to consider only finite simplified models for CL?

6. Has the logic of Π_2-conservativity a simple finite models ? (A variant: is there a modal formula ϕ such that for any n $CL(\Pi_2) \vdash \phi \longrightarrow \neg\Box^n\bot$, but $CL(\Pi_2) \nvdash \neg\phi$?).

APPENDIX

The Logic of Π_2-conservativity.

Definition A.1. The logic SbCLM is given by axioms L0-L3, A1-A4,M,Sb and usual inference rules (modus ponens and necessitation)

Theorem A.2. SbCLM is the logic of Π_2-conservativity (i.e. $SbCLM=CL(\Pi_2)=CL^+(\Pi_2)$).

Proof. Arithmetical soundness of SbCLM is evident. Fix an arbitrary formula ϕ s.t. SbCLM $\nvdash \phi$.

Definition A.3.

$X:=\{A\rhd B \mid A\rhd B\subseteq\phi\}\cup\{(\neg D)\rhd\bot \mid \Box D\subseteq\phi\}$;

$Sb(\phi):=\Box^+\bigwedge_{A\rhd B\in X,\ C_1,\dots,C_n\in\neg X} (A\rhd B \longrightarrow (A\wedge C_1\wedge\dots\wedge C_n)\rhd(B\wedge C_1\wedge\dots\wedge C_n))$.

(cf. definitions 7.7, 7.8)

Lemma A.4. There exists CLM-model $\mathcal{K}=\langle K,R,\{S_x\},\Vdash\rangle$ such that:

1) K is finite;

2) for any $A\rhd B\in X$, $x,y,z\in K$, if yS_xz and $z\Vdash A\rhd B$, then $y\Vdash A\rhd B$;

3) $\mathcal{K}\nVdash\phi$.

Proof. One can see that CLM $\vdash Sb(\phi)\longrightarrow\phi$. Let $\mathcal{K}=\langle K,R,\{S_x\},\Vdash\rangle$ be a finite CLM-model with a bottom node x_0 such that $x_0\Vdash Sb(\phi)\wedge\neg\phi$. We define a binary relation S on K: xSy iff for any $A\rhd B\in X$ if $y\Vdash A\rhd B$, then $x\Vdash A\rhd B$. The model $\mathcal{K}_1:=\langle K,R,\{S_x\cap S\},\Vdash\rangle$ has all properties required (cf. proof of lemma 7.9).

Lemma A.5. There exists a simplified CLM-model $\mathcal{K}=\langle K,R,S,\Vdash\rangle$ and a set $K_0\subseteq K$ such that:

1) K is finite;

2) for any $A\rhd B\in X$, $x,y\in K$, if xSy, and $y\Vdash A\rhd B$, then $x\Vdash A\rhd B$;

3) a) ySx, yRz, $z\in K_0$ $\Rightarrow$ xRz

b) if $A\rhd B\in X$, $x\in K$, $x\nVdash A\rhd B$, then there is $y\in K_0$ such that xRy, $y\Vdash A$, $\forall z(xRz\wedge ySz\longrightarrow z\nVdash B)$

c) $x,y\in K_0$, xSy $\Rightarrow$ x=y

4) $\mathcal{K}\nVdash\phi$.

Proof Fix an CLM-model $\mathcal{K}=\langle K,R,\{S_x\},\Vdash\rangle$ defined in lemma A.4. We define a binary relation S_1 as a transitive closure of $\bigcup S_x$ ($x\in K$), and equivalence relation $S:=S_1\cap S_1^*$ (i.e. $xSy:\Leftrightarrow xS_1y\wedge yS_1x$).

Proposition A.5.1.

a) $xSy \Rightarrow \forall z(xRz \leftrightarrow yRz)$;

b) $xSy \Rightarrow \forall C\in X(x \Vdash C \Leftrightarrow y \Vdash C)$;

c) xRy, yS_1x is impossible.

Proof a) use definition of CLM-model;

b) use property 2 of $\mathcal{K}$ (from lemma A.4);

c) use definition of CLM-model (cf. proof of proposition 7.3)

Definition A.5.2.

$\mathcal{H}=\langle H,\subset,<,R(\cdot),end(\cdot)\rangle$ is the heir of the second type of $\mathcal{K}$ (see definition 7.1)

for any $\mathbf{x}\in H$, $S(\mathbf{x}):=min\{\mathbf{y}\subseteq\mathbf{x} \mid end(\mathbf{x})S end(\mathbf{y})\}$

$\mathcal{K}'=\langle K',R',S',\Vdash\rangle$ is the simplified model:

$K' := H$

$\mathbf{x}R'\mathbf{y} \;:\Leftrightarrow\; S(\mathbf{x})<\mathbf{y}$

$\mathbf{x}S'\mathbf{y} \;:\Leftrightarrow\; R(\mathbf{x})=R(\mathbf{y}) \wedge S(\mathbf{x})\subseteq S(\mathbf{y})$

$\mathbf{x}\Vdash p \;:\Leftrightarrow\; end(\mathbf{x}) \Vdash p$

K_0 is subset of K':

$K_0 := \{\mathbf{x}\in K' \mid R(\mathbf{x})=\mathbf{x}\}$.

Lemma A.5.3

a) $end(S(\mathbf{x}))S end(\mathbf{x})$;

b) $S(S(\mathbf{x}))=S(\mathbf{x})$;

c) $\mathbf{x}R'\mathbf{y} \Rightarrow end(\mathbf{x})R end(\mathbf{y})$;

d) $S(\mathbf{x})\supseteq R(\mathbf{x})$;

e) R' is transitive;

f) K' is finite;

g) $\mathcal{K}'$ is a finite simplified CLM-model;

h) $\mathbf{x}S'\mathbf{y} \wedge C\in X \wedge end(\mathbf{y}) \Vdash C \Rightarrow end(\mathbf{x}) \Vdash C$;

i) $\mathbf{y}S'\mathbf{x}$, $\mathbf{y}R'\mathbf{z}$, $\mathbf{z}\in K_0 \Rightarrow \mathbf{x}R'\mathbf{z}$;

k) $R(\mathbf{x})=R(\mathbf{y}) \wedge \mathbf{x}\supseteq\mathbf{y} \Rightarrow \mathbf{x}S'\mathbf{y}$;

l) $R(\mathbf{x})=R(\mathbf{y}) \wedge \mathbf{x}\subseteq\mathbf{y} \wedge end(\mathbf{y})S_1 end(\mathbf{x}) \Rightarrow \mathbf{y}S'\mathbf{x}$.

Proof

a) immediately from the definition of $S(\cdot)$;

b) is trivial;

c) use a) and propositions A.5.1a) and 7.3b);

d) use definition of $R(\cdot)$ and proposition A.5.1c);

e) if $S(\mathbf{x})<\mathbf{y}$, $S(\mathbf{y})<\mathbf{z}$, then by d) and proposition 7.2h), $S(\mathbf{x})<S(\mathbf{y})$; by proposition 7.2g), $S(\mathbf{x})<\mathbf{z}$;

f) use proposition 7.3a);

g) use e), f) and some obvious properties of R' and S';

h) let $end(\mathbf{y}) \Vdash C$. By a) and proposition A.5.1.b). $end(S(\mathbf{y})) \Vdash C$. Since $R(\mathbf{y}) \subseteq S(\mathbf{x}) \subseteq S(\mathbf{y})$, $end(S(\mathbf{x})) \Vdash C$; and as above $end(\mathbf{x}) \Vdash C$.

i) use the following simple property of heirs:

(*) if $R(\mathbf{z})=\mathbf{z}$, $\mathbf{y}<\mathbf{z}$, $R(\mathbf{x})=R(\mathbf{y})$, then $\mathbf{x}<\mathbf{z}$.

k), l) immediately from the definition of S'.

<u>Lemma</u> A.5.4. If A is a subformula of ϕ, and $\mathbf{x}\in K'$, then

$$\mathbf{x} \Vdash A \iff end(\mathbf{x}) \Vdash A.$$

<u>Proof</u>. We consider the case $A=B\rhd C$; $x:=end(\mathbf{x})$; $\mathbf{t}:=S(\mathbf{x})$; $t:=end(\mathbf{t})$. By lemma A.5.3a) and proposition A.5.3b), $\mathbf{t} \Vdash B\rhd C \iff x \Vdash B\rhd C$.

1. Suppose $\mathbf{x} \Vdash B\rhd C$. We will show that $t \Vdash B\rhd C$. Indeed, let tRy, $y \Vdash B$. We use lemma 7.4 to obtain $\mathbf{y}\in K'$ (moreover, $\mathbf{y}\in K_0$); by induction hypothesis, $\mathbf{y} \Vdash B$; since $S(\mathbf{x})=\mathbf{t}<\mathbf{y}$, there is $\mathbf{z}$ s.t. $\mathbf{z} \Vdash C$, $\mathbf{x}R'\mathbf{z}$, $\mathbf{y}S'\mathbf{z}$. By definition of S', $R(\mathbf{z})=R(\mathbf{y})=\mathbf{y}$, and by definition of R' and lemma A.5.3b), $\mathbf{t}=S(\mathbf{t})<\mathbf{z}$; lemma 7.4 implies that $yS_t z$, tRz, where $z:=end(\mathbf{z})$. By induction hypothesis, $z \Vdash C$. Thus, we proved that $t \Vdash B\rhd C$. (In fact, we proved that $t \nVdash B\rhd C$ implies $\exists \mathbf{y}\in K_0$ ($\mathbf{x}R'\mathbf{y}$, $\mathbf{y} \Vdash B$, $\forall \mathbf{z}(\mathbf{x}R'\mathbf{z} \wedge \mathbf{y}S'\mathbf{z} \rightarrow \mathbf{z} \nVdash C)$) .)

2. Suppose $t \Vdash B\rhd C$. We will show that $\mathbf{x} \Vdash B\rhd C$. Indeed, let $\mathbf{x}R'\mathbf{y}$, $y=end(\mathbf{y})$, $\mathbf{y} \Vdash B$. By definition of R' and lemma A.5.3c), tRy. Since $y \Vdash B$, there is z such that tRz, $yS_t z$, $z \Vdash C$. Now we can use lemma 7.5 and obtain $\mathbf{z}$ such that $\mathbf{z} \Vdash C$, $R(\mathbf{y})=R(\mathbf{z})$, $\mathbf{x}R'\mathbf{z}$ (because $S(\mathbf{x})=\mathbf{t}<\mathbf{z}$). By lemma 7.5, we consider two cases:

Case 1. $\mathbf{z}\supseteq\mathbf{y}$. Then by lemma A.5.3k), $\mathbf{y}S'\mathbf{z}$.

Case 2. $\mathbf{y}\supseteq\mathbf{z}$. Then by lemma A.5.3l), because $yS_t z$, we obtain $\mathbf{y}S'\mathbf{z}$.

Thus, in any case $\mathbf{y}S'\mathbf{z}$, and we proved that $\mathbf{x} \Vdash B\rhd C$. QED.

We have proved that $\mathcal{K}' \nVdash \phi$. Set K_0 was defined in definition A.5.2; as to properties a)-c) of K_0, a) was be stated in lemma A.5.3.i); b) can be easily obtained from part 1 of the proof of lemma A.5.4, and c) is trivial. It remained to show that for any $\mathbf{x},\mathbf{y}\in K'$, if $\mathbf{x}S'\mathbf{y}$ and $\mathbf{y} \Vdash C$, then $\mathbf{x} \Vdash C$, where $C\in X$. Indeed, by lemma A.5.4 $end(\mathbf{y}) \Vdash C \iff \mathbf{y} \Vdash C$, $end(\mathbf{x}) \Vdash C \iff \mathbf{x} \Vdash C$; now it is enough to use lemma A.5.3h).

Lemma A.5 is thus proved.

We continue with the arithmetical part of our proof.

We will use the following fact which is a simple modification of Goldfarb's theorem:

Lemma A.6. Let $T_0, \ldots, T_n$ be r.e. extensions of PA, $S(x) \in \Sigma_1$. Then there exists a formula $\sigma(x) \in \Sigma_1$ such that PA + "for any i T_i is consistent" proves that for any x

1) if $S(x)$, then $PA \vdash \sigma(\underline{x})$;

2) if not $S(x)$, then for any i $T_i \nvdash \sigma(\underline{x})$.

Proof (sketch). Let $S(x) = \exists y \delta(x,y)$ where $\delta(x,y) \in \Delta_0$. The following formula (defined by Diagonal Lemma),

$$\sigma(x) := \exists y (\delta(x,y) \wedge \forall z<y \forall i \neg \mathrm{Proof}_{T_i}(z, \ulcorner \sigma(\underline{x}) \urcorner))$$

has all necessary properties.

Fix a model $\mathcal{K}' = \langle K', R', S', \Vdash' \rangle$ and a set K'_0 which fulfills conditions of lemma A.5; we can suppose that $v \nVdash \phi$, where v is a bottom node of $\mathcal{K}'$. Let w be a new node, and $K := K \cup \{w\}$, $R := R' \cup \{\langle w,x \rangle \mid x \in K'\}$, $S := S' \cup \{\langle w,w \rangle\}$, $\Vdash$ is an arbitrary extension of $\Vdash'$ on K, $K_0 := K'_0 \cup \{w,v\}$.

We claim that the model $\mathcal{K}$ satisfies all conditions of lemma A.5. The only interesting case is 3b).

Let $x \in K$, $A \rhd B \in X$, $x \nVdash A \rhd B$.

Case 1. $x \neq w$. Use the property 3b) of $\mathcal{K}'$ from lemma A.5.

Case 2. $x = w$, $v \Vdash A \wedge \neg B$. We can put $y := v$.

Case 3. $x = w$, $v \Vdash A \rightarrow B$. One can see that $v \nVdash A \rhd B$; so, we can reason as in case 1.

Consider a set K_0 as a submodel of $\mathcal{K}$ and define a Solovay function h on this submodel; thus, we will use lemma 8.3 w.r.t. $x, y, \ldots \in K_\sigma$. $l := \lim h$; of course, $l \in K_0$.

Definition A.7.

1) $Tr(\cdot)$ is Σ_2-definition of truth for Σ_2-formulas;

2) let $Tr(x) = \exists y\, tr(y,x)$, where $tr(\cdot,\cdot) \in \Pi_1$; it is supposed that

$$PA \vdash Tr(x) \rightarrow \exists^\infty y\, tr(y,x);$$

3) $\{Q,n,m\}$ will stand for

$$\exists A \in \Sigma_2 (\mathrm{Proof}_{PA}(n, \ulcorner A \rightarrow Q \urcorner) \wedge tr(m, \ulcorner A \urcorner)).$$

(This formula is Π_1, because the first quantifier can be bounded by p.r. function on n).

Proposition A.8. $\forall Q\ \forall n_0\quad PA \vdash \exists m \exists n<\underline{n_0}\{Q,n,m\} \longrightarrow Q$.

Proof. It is enough to show that

$$\forall Q\ \forall n\quad PA \vdash \exists m\{Q,\underline{n},m\} \longrightarrow Q,$$

or

$$\forall Q\ \forall n\ PA \vdash \forall A\in\Sigma_2\ (\ Proof_{PA}(\underline{n},\ulcorner A\rightarrow Q\urcorner) \wedge \exists m tr(m,\ulcorner A\urcorner) \longrightarrow Q\ ).$$

We noted above that the quantifier on A is bounded by n, hence we can assume that this quantifier is "external"; on the other hand, $\exists m tr(m,\ulcorner A\urcorner)\leftrightarrow A$. Thus, we must show that

$$\forall Q\ \forall n\ \forall A\in\Sigma_2\quad PA \vdash Proof_{PA}(\underline{n},\ulcorner A\rightarrow Q\urcorner) \longrightarrow (A\rightarrow Q).$$

But it is trivial.

Definition A.9. We will define (in PA) a function $H:\omega\longrightarrow K\times(\omega+1)$ (H_0 and H_1 will stand for left and right components of H) and a constants L and Rk by induction on $\mu(l)$; we will define them such that the formula $Q(m,n,z) := H(m)=\langle z,n\rangle$ will be Σ_2.

Basis. If l=w, $H(m)\equiv\langle w,\omega\rangle$.

Induction. Assume that we have defined H in the case $\mu(l)<n$.

1. Let $A_x(y):=\neg(l=x\wedge y\in \mathcal{R}ange(H_0))$, where $\mu(x)<n$; by the induction hypothesis, $A_x(\cdot)\in\Pi_2$. Let $A_x(y)=\forall m\ S_x(m,y)$, where $S_x(\cdot,\cdot)\in\Sigma_1$ and it is supposed that $PA\vdash(\ \neg A_x(y)\longrightarrow\exists^\infty m\neg S_x(m,y)\ )$. We use lemma A.6 (where $\{T_0,\dots,T_n\}:=\{PA+L=z\mid xRz, z\in K_0\}$) to define a formula $\sigma_x(m,y)$ such that

$PA + \forall z(xRz\wedge z\in K_0\longrightarrow\diamond_{PA}(L=z))$ proves that for any y,m

1. if $\neg S_x(m,y)$, then $\forall z\in K_0\ \ (xRz\longrightarrow \neg\Box_{PA}(L=z\longrightarrow\sigma_x(\underline{m},y)))$,
2. if $S_x(m,y)$, then $\Box_{PA}\sigma_x(\underline{m},y)$.

or, by the definition of $A_x(\cdot)$,

$PA + \forall z(xRz\wedge z\in K_0\longrightarrow\diamond_{PA}(L=z))$ proves that for any y

1. if l=x and $y\in\mathcal{R}ange(H_0)$, then
$$\exists^\infty m\ \forall z\in K_0\ \ (xRz\longrightarrow\neg\Box_{PA}(L=z\longrightarrow\sigma_x(\underline{m},y))),$$
2. if $l\neq x$ or $y\notin\mathcal{R}ange(H_0)$, then $\forall m\ \Box_{PA}\sigma_x(\underline{m},y)$.

2. $H(0):=\langle x,\alpha\rangle$, where

l=x (it is supposed that $\mu(x)=n$);

if $\Box_{PA}(L\neq x)$, then

$\alpha:=min\{k\mid Proof_{PA}(k,\ulcorner L\neq x\urcorner)$

else

$\alpha:=\omega$.

3. For any $m\geq 0$,

if there exists a pair <z,k> such that

1) $H_0(m)Sz$;
2) $H_1(m)>k$;
3) $h(k)Rz$;
4) $\{L\neq z,k,m\}$;
5) $\forall i<k\ \forall b\ (h(i)Sb \longrightarrow \sigma_{h(i)}(i,b) \lor bRz)$

then

$H(m+1):=<z_0,k_0>$,

where $<z_0,k_0>$ is the minimal pair $<z,k>$ satisfying conditions 1)-5) (w.r.t. k).

else

$H(m+1):=H(m)$.

4. $L:=\lim_{m\to\infty} H_0(m)$.
5. $Rk:=\lim_{m\to\infty} H_1(m)$.

One can see using condition 2) from the definition of H, that H can "jump" only finitely many times, hence these limits exist. Note also that by condition 1) for any m $\mu(H_0(m))=\mu(l)=n$, and by condition 3) $\mu(h(k))<n$; thus, we can use the formula $\sigma_{h(i)}$, where $i\leq k$, in the definition of H.

Lemma A.10 (PA)

a) if $L=y$, $y\in K_0$, then $l=y$;
b) for any n, $\Box_{PA}(Rk>\underline{n})$;
c) if xRy, $y\in K_0$, then $l=x\longrightarrow\neg\Box_{PA}L\neq y$;
d) $l=x\neq w \longrightarrow \Box_{PA}(L=y\longrightarrow xRy)$.

Proof.

a) Use property c) of K_0 (lemma A.5), and an obvious fact lSL.

b) Fix $n\geq 0$ and reason in PA:

Let $l=t$. Suppose that $Rk\leq\underline{n}$ and $m_0:=min\{m|H(m)=<L,Rk>\}$.

Case 1.$m_0=0$. By the definition of H, $Proof_{PA}(Rk,\ulcorner L\neq t\urcorner)$, hence $L\neq t$. Contradiction.

Case 2.$m_0>0$. By the definition of H, $\{L\neq x,Rk,m_0\}$; by proposition A.8, $L\neq x$. Contradiction.

Thus, $Rk>\underline{n}$.

c) One can see from the definition of H that (PA proves that) for any y

(*) $\quad l=y \wedge \mathrm{Proof}_{PA}(n,\ulcorner L\neq y\urcorner) \longrightarrow Rk\leq n.$

We have:

$$\begin{aligned}
PA \vdash \mathrm{Proof}_{PA}(n,\ulcorner L\neq y\urcorner) &\longrightarrow \Box_{PA}(\mathrm{Proof}_{PA}(\underline{n},\ulcorner L\neq y\urcorner)) \\
&\longrightarrow \Box_{PA}(l=y\longrightarrow Rk\leq\underline{n}) \qquad \text{by (*)} \\
&\longrightarrow \Box_{PA}(l\neq y) \qquad \text{by claim b)}
\end{aligned}$$

$PA \vdash \Box_{PA}(L\neq y) \longrightarrow \Box_{PA}(l\neq y)$

$PA \vdash l=x \longrightarrow \neg\Box_{PA}(L\neq y)$ by claim 2 of lemma 8.3.

d) Let $l=x=h(n)$. Reason in PA:

Let $L=y$

Case 1. $l=y$. Then by claim 3 of lemma 8.3, xRy.

Case 2. $l\neq y$. Then by definition of H, $h(Rk)Ry$; by claim a), $Rk>\underline{n}$, hence

$$x=h(\underline{n})Rh(Rk) \quad \text{or} \quad h(\underline{n})=h(Rk);$$

in both cases xRy.

<u>Corollary</u> A.11. For any $x\in K$,

PA + $l=x$ proves that for any y

1. if $y\in \mathcal{R}ange(H_0)$, then
$$\exists^{\infty}m\ \forall z\in K_0\ (xRz\longrightarrow \neg\Box_{PA}(L=z\longrightarrow\sigma_x(\underline{m},y))),$$
2. if $y\notin \mathcal{R}ange(H_0)$, then $\forall m\ \Box_{PA}\sigma_x(\underline{m},y)$.

<u>Proof</u>. Use definition of σ_x and lemma A.10.c).

We define arithmetical interpretation f (henceforward f denote f_{Π_2}, $\rhd$ (in the arithmetical context) denote $\rhd_{\Pi_2}$) as usual: $f(p):=L\Vdash p$.

<u>Lemma</u> A.12. For any subformula A of ϕ and $x\neq w$,

$x\Vdash A \Rightarrow PA\vdash L=x \longrightarrow f(A)$

$x\nVdash A \Rightarrow PA\vdash L=x \longrightarrow \neg f(A)$.

<u>Proof</u>. We consider the case $A=B\rhd C\in X$.

1. Suppose $x\Vdash B\rhd C$. Reason in PA+$L=x$.

Let $l=t$; of course, tSx and $t\neq w$.

Let $F:=\mathcal{R}ange(H_0)$, $m_0:=min\{m|H(m)=\langle L,Rk\rangle\}$ and for any $a,b\in F$
$$a<b :\Leftrightarrow max\{i\leq m_0|H_0(i)=a\} < max\{i\leq m_0|H_0(i)=b\}.$$

Set F is finite, and we can work with F inside PA, using the following properties of F:

1. < is a linear order with maximal element x;
2. $x\leqslant y \Rightarrow xSy$;
3. $t\in F$.

(But, of course, PA does not knew that $F=\mathcal{R}ange(H_0)$).

Note also that by corollary A.11, if $b\notin F$, then

(*) $$\forall m\ \square_{PA}\sigma_t(\underline{m},b)$$

Suppose that $\square_{PA}(f(C)\longrightarrow Q)$, $Q\in\Pi_2$.

Reason in PA+f(B):

Let L=y. Assume that Q is false, i.e. for some m $tr(m,\ulcorner\neg Q\urcorner)$; by definition A.7, we can choose m so great that $H(m)=\langle y,Rk\rangle$.

By the induction hypothesis, $y\Vdash B$.

Let u be the maximal element of the set $\{a\in F|aRy\}$ w.r.t. order < (such u exists, because by lemma A.10d), tRy, and $t\in F$). Since $u\leqslant x$, uSx, (by the properties 1-3 of F, see above), by the property 2 of $\mathcal{K}$, (lemma A.5) we have $u\Vdash B\rhd C$; on the other hand, by the definition of u, uRy. Thus, there exists z such that uRz, ySz, $z\Vdash C$. Note also that tSuRz implies tRz.

Interrupting our reasoning, we note that $\square_{PA}(L=z\longrightarrow f(C))$, hence $\square_{PA}(L=z\longrightarrow Q)$. Let n be so large that h(n)=t and $Proof_{PA}(n,\ulcorner\neg Q\longrightarrow L\neq z\urcorner)$. We continue:

By definition A.7, $\{L\neq z,\underline{n},m\}$. We claim that the pair $\langle z,\underline{n}\rangle$ satisfies conditions 1)-5) from the definition of H; it implies that $H(m+1)=\langle z',n'\rangle$, where $n'\leq\underline{n}$. It is a contradiction, because $Rk>\underline{n}$. Indeed, using $Rk>\underline{n}$, we check these conditions:

1),2),3) are trivial, because ySz, $Rk>\underline{n}$ and $h(\underline{n})$=tRz.

4): see above.

5). Fix $i<\underline{n}$, $b\in K$, where h(i)Sb (note that $h(\underline{n})=t$).

Case 1. $h(i)\neq t$. Therefore, h(i)Rt and, since $t\in K_0$, h(i)Sb implies bRt; but we have tRy, hence bRy.

Case 2. h(i)=t, $b\notin F$. Then by (*) $\sigma_t(i,b)$.

Case 3. h(i)=t, $b\in F$, l=y (hence, $y\in K_0$). Then by the property of K_0 (lemma A.5) tRy and tSb imply bRy.

Case 4. $h(i)=t$, $b\in F$, $l\neq y$, $b\preccurlyeq u$. Then bSu and uRy imply bRy.

Case 5. $h(i)=t$, $b\in F$, $l\neq y$, $u<b$. We claim that $\sigma_t(i,b)$. Suppose not. Because $i<Rk$, $L=y\neq l$ implies bRy (it is enough to apply condition 5) from the definition of H to the "jump" H to $\langle y,Rk\rangle$). However, it contradicts the definition of u.

Thus, in any case $\sigma_t(i,b)$ or bRy.

We proved that Q is true.

Thus, $\Box_{PA}(f(B)\longrightarrow Q)$, hence $f(B)\rhd f(C)$.

2. Suppose $x\nVdash B\rhd C$; by the property of K_0, there is y such that $y\in K_0$, $y\Vdash B$, xRy, $\forall z(xRz\wedge ySz \longrightarrow z\nVdash C)$.

Reason in PA+L=x.

Let $l=t$, tSx. Since $x\in \mathcal{R}ange(H_0)$, by corollary A.11, there exists m such that $\neg\Box_{PA}(L=y\longrightarrow\sigma_t(\underline{m},x))$ and $h(m)=t$. Let $Q := l=y\longrightarrow\sigma_t(m,x)$. Of course, $Q\in\Pi_2$.

Claim 1. $\neg\Box_{PA}(f(B)\longrightarrow Q)$. Proof: Suppose not. Because $y\Vdash B$, $\Box_{PA}(L=y\longrightarrow f(B))$ and $\Box_{PA}(L=y\longrightarrow Q)$, hence by lemma A.10.a) $\Box_{PA}(L=y\longrightarrow\sigma_t(\underline{m},x))$. Contradiction.

Claim 2. $\Box_{PA}(f(C)\longrightarrow Q)$. Proof: Reason in PA+$\neg$Q:

Let L=z. Since l=y, ySz. We claim that xRz; hence, by the definition of y, $z\nVdash C$, and by the induction hypothesis, $\neg f(C)$.

Since xRy, we can assume that $y\neq z$. Consider a "jump" H to $\langle z,Rk\rangle$. Since $\neg\sigma_t(\underline{m},x)$, $t=h(\underline{m})$, tSx and $Rk>\underline{m}$, by the condition 5) from the definition of H, we have xRz. QED.

We proved that $f(B)\rhd f(C)$ does not hold.

As usual, we finish our proof by $PA\vdash L=v \longrightarrow\neg f(\phi)$, hence $PA\nvdash f(\phi)$ (we used that $v\in K_0$, hence by lemma A.10.c) $PA\nvdash L\neq v$).

This concludes the proof of the arithmetical completeness of SbCLM. As in the case SCL^{ω}, we in fact have proved some more:

<u>Corollary</u> A.13. For any modal formula ϕ

$$SbCLM \vdash \phi \iff CLM\vdash Sb(\phi) \longrightarrow \phi.$$

In particular, the logic SbCLM is decidable.

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