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TEMPORAL LOGIC

Johan van Benthem

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TEMPORAL LOGIC

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A chapter for Volume III in the forthcoming
Handbook of Logic in Artificial Intelligence and Logic Programming
D. Gabbay, Chr. Hogger and J. Robinson,eds.
Oxford University Press
(second revised version)

Received February 1991

Contents

I Origins of Temporal Logic

There are various historical sources for the discipline of Temporal Logic. The insight that temporal discourse and temporal argument show significant logical structures arises naturally in the empirical study of human reasoning, but it has recently become a necessity in the design of mechanical reasoning systems as well. A few specific examples may serve to start off our survey.

In Philosophy, there has been a long-standing interest in the structure of temporal argument, ever since Antiquity, when an influential philosophical opinion held that reasoning about change was bound to be contradictory. Examples of famous temporal arguments from the tradition are the 'Master Argument' of Diodorus Cronos (cf. Prior 1967), purporting to derive the so-called 'Principle of Plenitude' stating that all possible events are bound to occur in the actual course of history, or Aristotle's 'Sea Battle Argument' (see Kneale & Kneale 1962), deriving Determinism concerning the Future from the principle of Excluded Middle as applied to statements in the future tense. And as late as the eighteenth century, Immanuel Kant presented temporal paradoxes in his 'Antinomien der reinen Vernunft' as showing that reasoning concerning the global structure of time is bound to lead to logical problems. Thus, part of the philosophical motivation for the development of temporal logic in this century was to create a precise apparatus for stating and analyzing such lore from the tradition.

On the syntactic side, this means having a formalism enabling us to state precise argument patterns, while forcing us to bring out potential sources of ambiguity in their formulation. For instance, in Aristotle's argument, a key ambiguity turns out to be that between the two 'scope orderings' in a sentence like

"if it is true now that a sea battle will take place to-morrow,
then it is necessarily true that a sea battle takes place to-morrow".

With wide scope for the "necessarily", this is indeed true: but nothing dramatic follows, while the small scope reading for "necessarily", though yielding the dramatic deterministic conclusion, already begs the question at the start. As a more mundane benefit, a precise logical formalism enables one to make a more systematic study of temporal argument patterns, whether or not employed in the tradition. For instance, there are obvious questions as to how the temporal operators interact with propositional connectives:

Does future tense commute with conjunctions of propositions?
And does it commute with disjunctions?

Further precision may be added on the semantic side too, by looking more systematically at the temporal models underlying such formalisms. For instance, Kant's problems concerning beginnings or endings of Time evaporate once mathematical distinctions are made between various order types that can be chosen for temporal precedence. At this point, the semantics of temporal reasoning touches upon physical intuitions concerning the structure of time, such as Asymmetry (there is 'no return' in the flow of time) or Homogeneity (time has 'the same pattern throughout').

Another source of temporal logic has been in Linguistics. One of the most pervasive features of human languages is their inclination to put every statement in some temporal perspective: past, present or future. Moreover, the syntax and semantics of temporal expressions in natural languages show many recurrent patterns, which can be brought out again for systematic abstract investigation, using convenient logical forms. There is a major descriptive question here, namely to chart the rich lexical and grammatical apparatus of temporality in natural language, including such diverse mechanisms as tense, aspect, temporal quantification, temporal adverbials, temporal connectives, etcetera. Another, more difficult issue, however, is to add explanatory power to the description. Why is it that we find certain temporal patterns lexicalized universally? Can this be related to a certain optimal efficiency in the structure, or at least the cognitive functioning of natural languages? Thus, questions of temporal 'design' come in, and the border line with research interests in Computer Science and Artificial Intelligence becomes blurred.

In Computer Science, the description of program execution naturally involves reasoning about the passage of time. Therefore, temporal logic formalisms for proving assertions of correctness at the end of a computational process, or intermediate assertions about some computation itself, such as fairness or absence of dead-lock, have become widely employed. There are already various different strands in this movement, with some modelings involving 'global time' in which computational events take place, and others rather reflecting 'process time' generated directly by the computational actions themselves. Also, logical techniques employed in the area range widely, from semantic model checking to pure syntactic deduction. In the process, surprising new uses have been found for existing temporal formalisms: such as the automatic generation of programs meeting certain specifications of desired temporal behaviour. (See the surveys Goldblatt 1987, Manna & Pnueli 1989, Emerson & Srinivasan 1988.)

Finally, as the latest comer in this field, Artificial Intelligence has joined in with both old and new concerns. There is an obvious interest in designing computationally convenient temporal formalisms, as well as a more fundamental understanding of temporal representation, once tasks are considered such as planning rational action in a changing environment, or building common sense reasoning into a moving robot. Maintaining temporal knowledge, as well as making temporal predictions are then of the essence (see McDermott 1982, Shoham 1986). Moreover, a prominent AI research program like the development of a 'common sense physics' has issues of temporal modelling as one of its favourite testing grounds (cf. Allen & Hayes 1985).

Such manifestations of temporal logic in Artificial Intelligence are not intrinsically different from those in the other fields mentioned above. For instance, there are various similarities between common sense physics and earlier philosophical concerns with the logical connections between the world of sensory experience and the world of theoretical science (cf. Russell 1926). Also, issues of finding a most convenient temporal representation of computational processes may be very similar in AI and in computer science proper (cf. Lamport 1985). And likewise, reasoning about temporal knowledge and ignorance of a cognitive agent is not vastly different from epistemic reasoning about the behaviour of a multi-processor distributed protocol (cf. Halpern & Moses 1985). Therefore, we shall take a rather free-wheeling attitude in the following toward 'computational uses' of temporal logic, drawing upon diverse sources.

Now, the purpose of this Chapter is twofold.

- On the one hand, we want to present a compact up-to-date survey of the technical state of temporal logic. This will be done by stressing themes and methods, rather than long inventories of theorems. Moreover, we shall point at a number of technical open questions which arise in retrospect, even with the best-established systems of temporal logic. This survey will start from the basic paradigm invented in the fifties, and then chronicle some of its subsequent developments and modifications, including the model theory and proof theory of a ladder of ever stronger formalisms. In the process, we shall point at analogies with standard extensional logical systems throughout. All this is relatively known territory, and the work can be done while staying within the same broad logical research program.

- But also, we want to stress the importance of stepping back from existing logical machinery, and asking some general questions arising from the descriptive area of temporal reasoning as well as certain phenomena encountered in computational applications. In particular, much traditional work has been directed towards specific logics or models by themselves, whereas perhaps the more interesting task ahead is to study what might be called 'logical variety'. This variety, as it emerges in computational applications, has various aspects. Semantically, one becomes interested in relating temporal representations at different 'grain levels' of the same reality, shifts in interpretation as one passes from one temporal model to another, and even varying mechanisms of interpretation for one single standard formalism in one kind of model. More deductively, 'variety' means an interest in locating deductive relations linking different logical calculi for temporal reasoning, and indeed even in charting different varieties of what counts as valid temporal reasoning. We shall illustrate a number of such recent developments, thereby displaying what we take to be one of the main virtues of logical work in Artificial Intelligence: its unconventional fresh look at presuppositions and established practices in standard logic. For instance, we shall even try to undermine the usual identification of 'logical' with 'declarative' approaches, pointing at possible imperative, procedural versions of temporal logic.

II The Basic System

Temporal logic as a rigorous field of investigation was started by Arthur Prior, starting from the fifties (Prior 1967 is the best overview). In this Section, we describe his basic system of temporal logic, which has served as a point of departure for most subsequent work in the field.

II.1 Propositional Tense Logic

In an ordinary propositional language, formulas are interpreted in unchanging environments to denote truth values, whose combinatorics are reflected by the Boolean connectives. Now we introduce a temporal perspective: henceforth, formulas denote statements whose truth value may change over situations across time, such as "it is raining", "the block is lying on the table" or "the current value of register x is 3".

II.1.1 Language

The most fundamental additional operations, over and above the standard propositional formalism, then describe the temporal environment of an instantaneous situation, corresponding (roughly) to the future and past tense of natural languages:

$F\phi$ at least once in the future, ϕ will be the case

$P\phi$ at least once in the past, ϕ has been the case.

These are 'existential' notions, so to speak, and derived from these, we have two dual 'universal' operators:

$G\phi$ always in the future from now ϕ

$H\phi$ always in the past up until now ϕ

The latter are definable via $\neg F \neg$ and $\neg P \neg$, respectively. (There is an obvious analogy here with quantifiers in standard logic, that we shall develop more systematically in Section II.7 below.) This simple formalism already generates a lot of interesting temporal forms, via iteration among temporal operators themselves as well as interaction with Boolean connectives:

$GF\phi$	ϕ is always going to be true at some later stage
$PH\phi$	once upon a time, ϕ had always been the case
$F\phi \wedge F\psi$	ϕ will be the case and so will ψ
$F(\phi \wedge \psi)$	ϕ and ψ will be the case simultaneously
$\phi \rightarrow G\psi$	if ϕ then ψ will always be the case from now on
$G(\phi \rightarrow \psi)$	ϕ will always 'guarantee' ψ
$G(\phi \rightarrow F\psi)$	ϕ will always 'enable' ψ to become true afterwards.

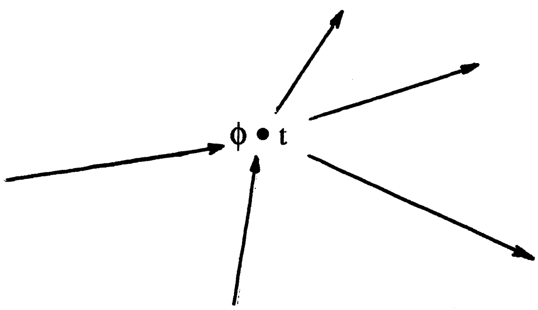
With operators F , G only, one speaks of 'pure future' formulas, and with P , H of 'pure past'.

This extremely simple formalism has an interesting 'grammar' all the same. For instance, we shall encounter various special 'fragments', defined by restrictions on occurrences of operators or proposition letters that lead to special temporal behaviour. Moreover, the following syntactic measure of expressive complexity will turn out useful. The *temporal depth* $d(\phi)$ of a formula ϕ is the maximum length of a nest of temporal operators occurring in ϕ .

II.1.2 <u>Models</u>

Interpretation for this language takes place in temporal *frames* $\mathbb{F} = (T, <)$, consisting of non-empty sets T of 'points in time' ordered by a binary relation < of precedence ('earlier than'). Moreover, a 'valuation' V maps proposition letters p to the sets V(p) of those points in time where they hold (their 'life times'). Triples $\mathbb{M} = (T, <, V)$ are called temporal *models*, that may be thought of as a flow of time decorated with a history over it. Such flows may be of arbitrary kinds, including both linear and branching patterns.

Then, the basic truth definition explains the notion of 'truth of a formula ϕ at a moment t in a model $\mathbb{M}$ ', where $\mathbb{M}$ supplies the total temporal environment:



- $\mathbb{M}, t \models p$ iff $t \in V(p)$
- $\mathbb{M}, t \models \neg\phi$ iff *not* $\mathbb{M}, t \models \phi$

  $\mathbb{M}, t \models \phi \wedge \psi$ iff $\mathbb{M}, t \models \phi$ *and* $\mathbb{M}, t \models \psi$

  and analogously for the other Boolean connectives
- $\mathbb{M}, t \models F\phi$ iff *for some* t' *with* $t<t'$, $\mathbb{M}, t' \models \phi$

  $\mathbb{M}, t \models P\phi$ iff *for some* t' *with* $t'<t$, $\mathbb{M}, t' \models \phi$

So far, these temporal models are completely general, and consist of just any binary relation on a carrier set of points over which certain unary predicates are defined. But intuitively, 'real time' satisfies additional constraints, inducing certain mathematical properties of the ordering. Here are some well-known examples. The simplest of these are expressible in *first-order* predicate logic:

transitivity	$\forall x \forall y \forall z\, ((x<y \wedge y<z) \rightarrow x<z)$
irreflexivity	$\forall x\, \neg x<x$
linearity	$\forall x \forall y\, (x<y \vee y<x \vee x=y)$

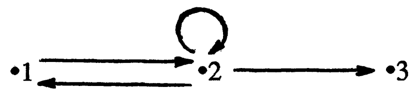
Some interesting candidates are essentially *second-order*, however, involving quantification over sets of points in time, such as:

Continuity	'every subset with an upper bound has a supremum'
Homogeneity	'every point can be mapped onto any other one by some order automorphism of the temporal frame'.

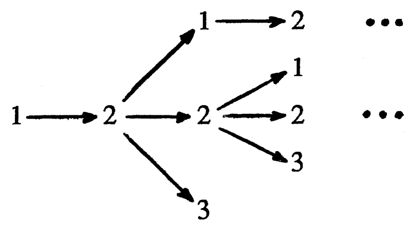
In general, no unique set of constraints has emerged valid for all cases. For, one wants to leave options in temporal representation for specific applications, e.g., whether to have time dense or discrete, with or without an ending, etcetera.

Another source of variety has arisen here in a computational perspective. In the original more philosophical way of thinking, models stood for actual temporal patterns, along which histories of some system may develop. But in more recent applications, one has tended to view temporal frames also as 'state diagrams' for machines producing those histories in their evolution. Formal constraints on the 'temporal pattern' need not be the same in these two perspectives. E.g., a machine diagram may contain loops, even when its associated unfolding time is acyclic.

Example: Histories versus Machines.
The following machine diagram



can be 'unfolded' to a tree of possible histories produced by it:



Given this interest in the pattern of 'temporal flow' as such, one also defines a notion of truth for tense-logical formulas solely by virtue of the temporal order only. Thus, we introduce truth of a formula at a point in a *frame*:

$\mathbb{F} = (T, <), t \models \phi$ iff $(T, <, V), t \models \phi$ *for all valuations* V.

$\mathbb{F} = (T, <) \models \phi$ iff $\mathbb{F}, t \models \phi$ *for all points* $t \in T$.

Truth in models is a *first-order* notion, as we shall demonstrate precisely in Section II.7. By contrast, its quantification over valuations for proposition letters makes truth in frames a *second-order* notion.

II.1.3 Validity and Consequence

Using these semantic structures, a notion of *valid consequence* may now be introduced ('conclusion ψ follows from assumptions Σ'):

$\Sigma \models \psi$ iff for each model $\mathbb{M}$ and each point $t \in T$,
if $\mathbb{M}, t \models \phi$ for all $\phi \in \Sigma$, then also $\mathbb{M}, t \models \psi$.

This notion of inference $\models$ depends on 'local' truth of formulas, at single points in time. A reasonable variant $\models^*$ would use 'global' truth of the relevant formulas ϕ , ψ (i.e., truth at all points in the model $\mathbb{M}$). The latter is reducible to the former, however:

Fact. $\Sigma \models^* \psi$ iff $\{ \underline{A}\phi \mid \text{all } \phi \in \Sigma, \text{all } \underline{A} \} \models \psi$,
where $\underline{A}$ is any sequence of temporal operators G , H of length at most $d(\psi)$.

In the special case without premises, both notions reduce to *universal validity* of formulas ψ in all models at all points ('$\models \psi$') . For instance, returning to some earlier examples, $F\phi \wedge F\psi$ follows from $F(\phi \wedge \psi)$ in the above sense, while the converse does not hold, witness this semantic counter-example:

$$\begin{tikzcd}
 & \phi, \neg\psi \bullet 2 \\
\neg F(\phi\wedge\psi), F\phi, F\psi \bullet 1 \arrow[ur] \arrow[dr] & \\
 & \neg\phi, \psi \bullet 3
\end{tikzcd}$$

By contrast, a simple argument shows that $F\phi \vee F\psi \leftrightarrow F(\phi \vee \psi)$ ís universally valid.

Universal validity has a number of useful general properties, of which we list a few without proof.

- 'Mirror Image Property' for Future versus Past:
  if $\models \phi$ (F, P, G, H) , then also $\models$ [P/F, F/P, H/G, G/H] ϕ .

- 'Disjunction Property' for pure future formulas:
  if $\models G\phi \vee G\psi$, then $\models \phi$ or $\models \psi$.
  The full language lacks this feature: witness the counter-example of
  $\models G\neg p \vee GPFp$, $\not\models \neg p$, $\not\models PFp$.

- 'Interpolation Property':
  if $\models \phi \vee \psi$, then there exists a formula χ whose atomic vocabulary is the intersection of that for ϕ and ψ (together with $\perp$ and $\top$) such that $\models \phi \vee \chi$ and $\models \neg\chi \vee \psi$.

II.2 Model Theory

Let us now investigate some general logical properties of the above temporal semantics.

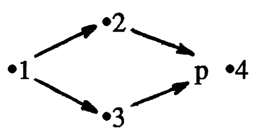
II.2.1 Basic Invariance: Zigzags

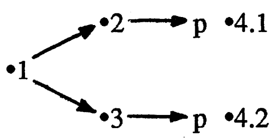
Perhaps the most fundamental measure of expressive power of a formalism is to locate the 'criterion of identity' induced by it on models. For the basic tense logic, the answer involves the following 'sieve of indistinguishability':

Definition. A binary relation C between two temporal models $\mathbb{M}_1 = (T_1, <_1, V_1)$ and $\mathbb{M}_2 = (T_2, <_2, V_2)$ is a *zigzag* , or 'bisimulation', if it relates points in T_1 to points in T_2 where the same atomic propositions hold under V_1, V_2 , respectively, in such a manner that the following back and forth clauses obtain:

- if $t_1 \, C \, t_2$ and $t_1 <_1 t_1'$, then there exists t_2' in T_2 with $t_2 <_2 t_2'$ and $t_1' \, C \, t_2'$
  if $t_1 \, C \, t_2$ and $t_1' <_1 t_1$, then there exists t_2' in T_2 with $t_2' <_2 t_2$ and $t_1' \, C \, t_2'$
- likewise in the opposite direction.

What this says intuitively is that tracing any history or computation path in $\mathbb{M}_1$ can be matched step by step by some path in $\mathbb{M}_2$, and vice versa, with continuations freely chosen on either side. For instance, the earlier unfolding map between a machine diagram and the tree of its potential histories was a zigzag. Another illustration is the following 'unraveling' of a diamond into a tree:





By a simple induction on temporal formulas, our formalism cannot distinguish between such situations:

Proposition. Temporal formulas ϕ are *invariant* under zigzags, in the sense that,

if $t_1 \, C \, t_2$, then $\mathbb{M}_1, t_1 \models \phi$ if and only if $\mathbb{M}_2, t_2 \models \phi$.

Special cases of the preceding result arise with specific choices of the zigzag relation C. With the identical inclusion map from $\mathbb{M}_1$ to $\mathbb{M}_2$, one gets the so-called 'Generation Theorem' from the modal literature, while a surjective map from $\mathbb{M}_1$ onto $\mathbb{M}_2$ gives the well-known 'p-Morphism Theorem'. More difficult to prove is the converse result (cf. van Benthem 1985, 1991), which says that, in a sense, our tense-logical formalism is captured precisely by this invariance:

Proposition. If $\mathbb{M}_1, t_1 \models \phi$ iff $\mathbb{M}_2, t_2 \models \phi$ for all tense-logical formulas ϕ, then there exists some zigzag C between two *elementary extensions* $\mathbb{M}_1{}^*$ of $\mathbb{M}_1$ and $\mathbb{M}_2{}^*$ of $\mathbb{M}_2$ such that t_1 C t_2.

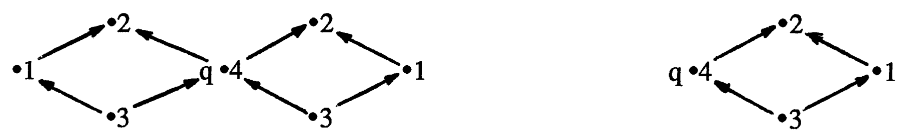
For *finite* temporal models, these elementary extensions must be identical to $\mathbb{M}_1, \mathbb{M}_2$ themselves, and we obtain total agreement between 'existence of a bisimulation' and 'equality of temporal theories'.

By way of contrast, zigzag invariance is no longer guaranteed for richer temporal statements, referring, e.g., to topological *betweenness* in the temporal order.

Example. Progressive Tense versus Bisimulation.
A natural temporal operator beyond the P, F formalism is the progressive tense (as in the sentence "Mary *is* cry*ing*") :

$$\mathbb{M}, t \models \Pi\phi \qquad \text{iff} \qquad \exists t_1{<}t\ \exists t_2{>}t\ \forall u\ (t_1{<}u{<}t_2 \Rightarrow \mathbb{M}, u \models \phi).$$

This statement does not survive the following bisimulation, where corresponding numbers indicate points to be identified ('fold the left-hand model'):



Set V(q) = {4} in both cases. Then, Πq will be true on the left in the point 4 (consider some upper 2 and its diagonally opposite 3); but, it fails in the point 4 on the right-hand side. ◆

Remark. 'Locality' of Evaluation.

In a sense, zigzags are still too coarse, in that they preserve truth for all formulas of the language at once. For specific formulas ϕ, at any given point t in a model $\mathbb{M}$, it suffices to consider only those points in $\mathbb{M}$ which can be reached via at most $d(\phi)$ steps along $>$ and/or $<$. For, only this 'environment' can be relevant to the evaluation of ϕ at t. This upper bound on semantic complexity will be used repeatedly.

II.2.2 Lindström Properties

Now we turn to more general Model Theory. By Lindström's characterization of first-order predicate logic (cf. Hodges 1983), the latter's two characteristic properties are the Compactness and Löwenheim-Skolem theorems. These also hold here, with respect to truth in temporal models:

Compactness

If every *finite* subset of a set of formulas Σ is satisfiable (at some point t in some model $\mathbb{M}$), then so is the whole set Σ.

Löwenheim-Skolem

If a set Σ is satisfiable at all, it is satisfied in some *countable* model.

For truth in temporal frames, however, the picture changes. Compactness fails (Thomason 1972), and so does the Löwenheim-Skolem Theorem (a rather involved counter-example may be constructed). These failures reflect general features of second-order logic. We shall return to such standard perspectives in Section II.7 below.

II.2.3 Preservation Behaviour on Models

Typical for the logical way of thinking is the systematic interplay between the syntactic form of statements and their semantic properties. Now, besides the general semantic behaviour of our language explained so far, there is also special semantic behaviour, useful under special circumstances, signalled by restricted syntactic forms of expression. Important examples arise with phenomena of 'persistence' of temporal statements inside or across semantic situations.

- Temporal Persistence.

Even though temporal statements may change their truth values arbitrarily in the course of time, there is a special interest to those which are more stable in certain temporal directions. For instance, let us call a statement ϕ *forward persistent* if always

$$\mathbb{M}, t \models \phi \textit{ and } t<t' \quad \textit{imply that} \quad \mathbb{M}, t' \models \phi .$$

Forward persistence is decidable for arbitrary formulas ϕ, since it amounts to the universal validity of the implication $\phi \rightarrow G\phi$ (and universal validity is decidable, as will be seen in Section II.5). Nevertheless, its explicit syntactic description is not quite straightforward. We merely list two simple observations to show the flavour:

Fact. On *transitive* models, all formulas constructed from arbitrary formulas $\bot$, $\top$, $P\phi$, $G\phi$ using P , G , $\wedge$, $\vee$ are forward persistent.

Fact. If ϕ is forward persistent on *arbitrary* models, then it implies $G^{d(\phi)+1}\bot$. (Here, '$G^n\phi$' stands for ϕ prefixed by n occurrences of the operator G).

Question. What are necessary and sufficient syntactic conditions for forward persistence on transitive models? On arbitrary models?

- Informational Persistence.

A second kind of persistence arises, not with time-travel inside models, but with changes in temporal models themselves: for instance, when obtaining further information about temporal objects to be represented. Let us say that model $\mathbb{M}_2$ *extends* $\mathbb{M}_1$ if T_2 contains T_1 and $<_2$ contains $<_1$, while V_2 and V_1 agree on all points in T_1 . Again, there is an obvious notion of *informational persistence* here, and a simple induction establishes the following

Fact. All 'positive existential' formulas constructed from propositional atoms and their negations using only F , P , $\wedge$, $\vee$ are informationally persistent.

This time, a converse holds too, which will be proved in Section II.7.1 below, using some techniques from standard model theory.

• Recursive Queries and Monotonicity.

Finally, consider a somewhat different case again. Patterns of events may sometimes be described by explicit definition, as in the equivalence

$p \leftrightarrow \phi\,(q_1, ..., q_n)$ with p not among the parameters q_i.

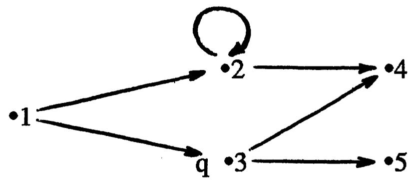
But it also happens quite frequently that only an implicit definition is available. For instance, when querying a data base about some predicate programmed 'recursively' (say, by means of some logic program), one may get a description of the form:

$p \leftrightarrow \phi\,(p, q_1, ..., q_n)$ in which p itself occurs in its own description.

When will this stabilize to some fixed denotation for p, starting from the empty set as a first approximation to its extension? Here is an illustration.

Example. Computing a Fixed Point.

Let a predicate p of temporal points be given by the recursion $p \leftrightarrow (F(p \wedge q) \vee Gp)$ on the following initial model:



Successive approximations for $V(p)$ may be computed as follows:

$\emptyset$ $\{5, 4\}$ $\{5, 4, 3\}$ $\{5, 4, 3, 1\}$.

The general mechanism here is similar to one for ordinary predicate logic. The natural condition for stabilization is that ϕ define a 'monotonic' operation from old approximations P to new approximations of p (cf. Stirling 1990 on the 'μ-calculus'):

$$\lambda P \bullet \{ t \in T \mid (T, <, V[p \mapsto P]), t \models \phi \}.$$

So, let us call a formula ϕ *monotone in* p if its truth at any point is never lost when passing from a model to a new model differing only in having a larger extension for p. Then we have this characterization, comparable to 'Lyndon's Theorem' in standard logic:

Theorem. A formula ϕ is monotone in p if and only if it is semantically equivalent to one in which p has only *positive* syntactic occurrences.

II.2.4 <u>Correspondence and Definability on Frames</u>

So far, all topics had to do with truth on temporal models. Now, let us look at temporal frames, where tense-logical formulas express pure ordering properties of time. Again, characteristic strengths and weaknesses of the basic formalism then come to light.

Example: Frame Correspondence.

Here are some first-order properties of temporal patterns, taken from an earlier list:

- *transitivity*
  $\mathbb{F}, t \models \forall y \forall z\ ((x<y \wedge y<z) \rightarrow x<z)$ iff
  $\mathbb{F}, t \models FFp \rightarrow Fp$
- *irreflexivity* has no tense-logical counterpart
- 'rightward *linearity*' is expressible as follows
  $\mathbb{F}, t \models \forall y \forall z\ ((x<y \wedge x<z) \rightarrow (y<z \vee z<y \vee z=y))$ iff
  $\mathbb{F}, t \models (Fp \wedge Fq) \rightarrow (F(p \wedge q) \vee F(p \wedge Fq) \vee F(q \wedge Fp))$,
  and likewise leftward into the past.

Not only first-order properties can be expressed, however:

- 'Löb's Axiom' $H(Hp \rightarrow p) \rightarrow Hp$ is a temporal principle which corresponds, at each point t of any frame $\mathbb{F}$, to the conjunction of the above transitivity and *well-foundedness*:
  i.e., no downward chain $t = t_1 > t_2 > t_3 > \ldots$ starts from t.

The first-order cases are of special interest here for several reasons. First, as to representation, they make do with what is a simple and perspicuous medium for describing temporal structures. But also computationally, unlike the general second-order case, they allow the use of well-known complete proof systems.

Phenomena of correspondence raise two kinds of more systematic question (see van Benthem 1984A, 1985 for more extensive theory).

In one direction, one can ask which temporal principles define first-order frame properties, and whether the latter can be obtained effectively. There is an abstract model-theoretic answer to the former question, that will be presented in Section II.7. As to more concrete algorithmic information, indeed, certain special forms of tense-logical principles guarantee pleasant behaviour. One ubiquitous useful example is that of 'Sahlqvist forms':

$\phi \to \psi$

with the antecedent ϕ constructed from propositional atoms, possibly prefixed by a number of operators G, H, using only F, P, $\wedge$, $\vee$, and the consequent ψ a syntactically positive formula (which may also contain G, H).

Theorem. There exists an effective algorithm computing first-order frame equivalents for all Sahlqvist forms.

Note that the above axioms for transitivity and linearity had Sahlqvist forms.

In the opposite direction, one can also start from first-order frame conditions, asking which of these are tense-logically definable. Necessary conditions here involve certain 'preservation properties' on frames falling out of the earlier invariance under zigzags between models:

Fact. Frame truth of tense-logical formulas is preserved under the formation of *generated subframes*, *disjoint unions* and *p-morphic images* of frames.

Details are omitted here (see van Benthem 1983, 1984A, Stirling 1990, or Section II.7 below). One illustration of this line of thinking may suffice, demonstrating a well-known peculiarity of the basic Priorean formalism:

Example. Temporal Undefinability of Irreflexivity.

Contraction to one single point is a 'p-morphism' from the irreflexive frame of the integers $(\mathbf{Z}, <)$ to the reflexive one-point frame.

There are still further frame operations preserving truth, such as returning to a frame from its *ultrafilter extension*. As before, we omit details (cf. Stirling 1990, van Benthem 1989B). Again, there is also an algorithmic aspect here, concerning explicit description of syntactic first-order forms admitting of temporal definition. Amongst others, it may be shown that (van Benthem 1983, 1985)

First-order sentences enjoying all tense-logical preservation properties must have their quantifiers restricted to $<$-successors or -predecessors throughout, while only positive atoms are allowed otherwise.

See also Kracht 1990 on more systematic translations from first-order frame conditions to tense-logical principles.

Another measure of expressive power for our formalism in the realm of frames consists in locating temporal frames having the same 'tense-logical theory'

$$Th_{tense}(\mathbb{F}) = \{ \phi \mid \mathbb{F} \models \phi \}.$$

Example. Comparing Tense Logics of Frames.
Here is a simple pilot case. On temporal *well-orders*, all ordinal frames $(\alpha, <)$ have distinct theories for $\alpha < \omega.\omega + \omega$. After that, the theories of $\omega.\omega + n$ $(n \in \omega)$ recur. (Cf. van Benthem 1989B, De Jongh, Veltman & Verbrugge 1988.)

In practice, however, comparisons between temporal frames may require more subtlety. For instance, the discrete integer frame $(\mathbf{Z}, <)$ and the dense frame $(\mathbf{Q}, <)$ of the rationals have obviously different theories: the tense-logical formula $F\phi \rightarrow FF\phi$ in fact defines 'density' of the binary relation, and hence it only holds in the latter frame. Nevertheless, when changing modellings for a physical phenomenon, one might want to pass from a discrete to a dense temporal perspective, or vice versa. But then, the question arises whether the old insights survive in one way or another, not necessarily directly, but at least by way of *translation.*

Question. Does there exist some (compositional) translation τ such that, for all tense-logical formulas ϕ, $\phi \in Th_{tense}((\mathbf{Z}, <))$ if and only if $\tau(\phi) \in Th_{tense}((\mathbf{Q}, <))$? And vice versa?

On this view, an answer is less evident, although we suspect it to be negative both ways. A simple positive illustration is the following temporal 'modelling shift':

Fact. $Th_{tense}((\mathbf{Z}, <))$ and $Th_{tense}((\mathbf{N}, <))$ are effectively translatable into each other.

II.3 Proof Theory

Temporal logic may also be approached more proof-theoretically, as a field of reasoning. There are various ways of organizing a deductive apparatus for the above Prior system, in order to describe its valid inferences (cf. Fitting 1983). And all these formats have their peculiarities of design, making them more or less suited for different computational tasks. We shall merely present a sketch of the main possibilities.

II.3.1 Axiomatic Calculus and Natural Deduction

• Perhaps the oldest format for deduction is axiomatic. The *minimal tense logic* K_t consists of the following principles:

Axioms

	all instances of propositional tautologies		
	$G(\phi \rightarrow \psi) \rightarrow (G\phi \rightarrow G\psi)$	$H(\phi \rightarrow \psi) \rightarrow (H\phi \rightarrow H\psi)$	Distribution
	$F\phi \leftrightarrow \neg G \neg \phi$	$P\phi \leftrightarrow \neg H \neg \phi$	Duality
	$\phi \rightarrow GP\phi$	$\phi \rightarrow HF\phi$	Conversion

Rules

	from ϕ and $\phi \rightarrow \psi$ infer ψ	Modus Ponens
	if ϕ is provable, then so are $G\phi$, $H\phi$	Temporalization

This system is well-known from other areas of Intensional Logic. In fact, it is a rather standard bi-modal calculus, be it with one peculiarity. In its most general guise, there would be *two* alternative relations R_F and R_P for the two operators. But the effect of the two Conversion axioms is to tie the two directions in time together, by making these two relations set-theoretic converses of each other.

We present an illustration of a theorem in this proof-theoretic format, that will serve as a running example throughout this Section.

Example. Conjunctive Distribution.
The following distribution principle is universally valid on temporal models:

$$G(\phi \wedge \psi) \leftrightarrow (G\phi \wedge G\psi) .$$

From left to right, this expresses 'monotonicity' of the universal future tense - from right to left, its 'conjunctivity'. Here is an outline of an axiomatic derivation:

1	$(\phi \wedge \psi) \rightarrow \phi$	propositional tautology
2	$G((\phi \wedge \psi) \rightarrow \phi)$	Temporalization (1)
3	$G((\phi \wedge \psi) \rightarrow \phi) \rightarrow (G(\phi \wedge \psi) \rightarrow G\phi)$	Distribution
4	$(G(\phi \wedge \psi) \rightarrow G\phi)$	Modus Ponens (2, 3)

Note how this 'deductive subroutine' really shows that, whenever some implication $\alpha \rightarrow \beta$ is derivable, then so is its temporalized form $G\alpha \rightarrow G\beta$. Thus, $(G(\phi \wedge \psi) \rightarrow G\psi)$ must be a theorem too. Moreover, starting from the tautology $(\phi \rightarrow (\psi \rightarrow (\phi \wedge \psi)))$, a similar proof establishes the other direction of the desired equivalence.

• A second approach to tense-logical inference is by way of *natural deduction*, constructing geometric proof trees. Then, the usual propositional rules of elimination and introduction for the classical Boolean connectives apply, and we have in addition:

> if T is a derivation of ψ from premises $\phi_1, ..., \phi_n$,
> then T induces a derivation for $G\psi$ from $G\phi_1, ..., G\phi_n$,
> and likewise for H.

Note how this generalizes over both the Distribution axioms and the Temporalization rule.

Example. Distribution II.
The relevant natural deduction trees might look as follows:

$$\frac{\phi\wedge\psi}{\phi} \quad \rightarrow \quad \frac{\dfrac{G(\phi\wedge\psi)}{G\phi} \qquad\qquad \dfrac{G(\phi\wedge\psi)}{G\psi}}{G\phi\wedge G\psi} \quad \leftarrow \quad \frac{\phi\wedge\psi}{\psi}$$

$$\frac{\phi \qquad \psi}{\phi\wedge\psi} \rightarrow \frac{G\phi \qquad G\psi}{G(\phi\wedge\psi)} \qquad : \qquad \frac{\dfrac{G\phi\wedge G\psi}{G\phi} \qquad\qquad \dfrac{G\phi\wedge G\psi}{G\psi}}{G(\phi\wedge\psi)}$$

Here and henceforth, we omit the obvious final 'conditionalization steps' leading to the literal implicational form of our equivalence.

II.3.2 Sequent Calculus and Semantic Tableaus

• Another approach to deduction employs a *calculus of sequents* $\Sigma \Rightarrow \Delta$, whose intended interpretation is that the conjunction of all assumptions Σ implies the disjunction of all conclusions Δ. Now there will be left and right introduction rules for the logical operators, starting from 'axiomatic sequents' one of whose conclusions already appears among the premises. For the propositional operators, these introduction rules are as usual, while the temporal ones again require a analogue of the earlier Distribution-cum-Temporalization (note the one-formula conclusion here):

> from $\phi_1, ..., \phi_n \Rightarrow \psi$ infer $G\phi_1, ..., G\phi_n \Rightarrow G\psi$,
> and likewise for H.

Example. Distribution III.

1 $\underline{\phi, \psi \Rightarrow \phi}$

$\underline{G\phi, G\psi \Rightarrow G\phi}$

$G\phi \wedge G\psi \Rightarrow G\phi$ and likewise for $G\phi \wedge G\psi \Rightarrow G\psi$.

2 $\underline{\phi, \psi \Rightarrow \phi \qquad\qquad \phi, \psi \Rightarrow \psi}$

$\underline{\phi, \psi \Rightarrow \phi \wedge \psi}$

$\underline{G\phi, G\psi \Rightarrow G(\phi \wedge \psi)}$

$G\phi \wedge G\psi \Rightarrow G(\phi \wedge \psi)$

- This approach is quite similar to a more semantically oriented one, namely that of *semantic tableaux*. Here, looking in the opposite direction along possible proof trees, sequents are analyzed for the possible constructibility of a semantic counter-example. Again, propositional rules in the relevant reduction process are standard, be it that every node in the tableau must now be marked for some point in time being investigated. The two rules for temporal operators are then as demonstrated here for the case of G :

$G\phi$ true at node t :	make ϕ true at all nodes t' in the construction that are to be 'later than' t
$G\phi$ false at node t :	create a new node t' 'later than' t and make ϕ false there.

Closed tableaus, reflecting totally failed attempts at constructing a counter-example, may then be defined in an obvious fashion. They will be in one-to-one correspondence with succesful derivations of their top sequents in the earlier sense.

Example. Distribution IV.

Semantic tableaus for conjunctive distribution might look as follows. First, we have:

$G(\phi \wedge \psi)$	• 1	$G\phi$	
			$1<2$
	• 2	ϕ	
$\phi \wedge \psi$	• 2		
ϕ (!) , ψ	• 2		

resulting in *closure* in world 2 .

And likewise for the conclusion $G\psi$ from the same premise.

In the opposite direction of our equivalence, we have:

$G\phi \wedge G\psi$	• 1	$G(\phi \wedge \psi)$	
$G\phi$, $G\psi$	• 1		
			$1<2$
	• 2	$\phi \wedge \psi$	
ϕ, ψ	• 2		

Now comes a propositional option for the false conjunction:

• 2 ϕ (!)	• 2 ψ (!)

both of whose branches turn out to *close*.

Further details of these proof techniques may be found in Melvin Fitting's Chapter 'Basic Modal and Temporal Logics' in Volume I of this Handbook.

II.3.3 Proof-Theoretic Equivalences

All formats of deduction reviewed here support the same valid inferences.

Theorem. There is an effective correspondence between axiomatic proofs, natural deductions and closed semantic tableaus for any given tense-logical formula.

A direct combinatorial proof of this result is not a trivial matter. Axiomatic proofs and natural deductions are indeed directly related, and so are closed semantic tableaus and derivations in a sequent calculus. But between the two families, there lies an interesting transition. In particular, showing that provable sequents, for which a derivation exists by the above introduction rules only, satisfy the seemingly innocuous principle of Modus Ponens requires the full Gentzen procedure of 'Cut Elimination' (cf. Prawitz 1965, Schwichtenberg 1977).

Despite this 'extensional' equivalence in provable transitions generated, these various proof formats show many 'intensional' differences in logical behaviour. For instance, sequent derivations have the advantage that they contain more constructive information, as their conclusion is built up progressively from its subformulas only. One useful corollary is 'Conservativity':

Valid pure future inferences can always be proved without detours using rules involving P or H . And likewise for pure past inferences.

From a computational viewpoint, however, a practical choice between such different proof formats may again involve quite different criteria (cf. the Chapter by Luis Fariñas and Andreas Herzig on modal theorem proving in the present Volume of this Handbook). For instance, at least in the usual systems of logic, cut-free sequent proofs tend to be quite expensive, involving a combinatorial explosion due to extensive copying of identical parts, whereas natural deductions can be much faster in this respect (cf. Boolos 1984).

II.3.4 The Lattice of Tense Logics

What has been described so far is merely the minimal deductive apparatus for tense logic, without imposing any special structure on the underlying models, or any special features on the temporal operators. But in fact, the literature has a whole variety of weaker and stronger 'tense logics', depending on which further principles are adopted for F , P , G and H . In particular, the minimal tense logic does not identify any two different strings of temporal operators, whereas stronger systems usually do. For instance, working on the real numbers, there is a simple result due to Hamblin, showing that only fifteen distinct 'temporalities' survive. As a result, there is a whole landscape of temporal logics, forming a lattice under inclusion, which represent different 'inference engines' for different intended applications (cf. van Benthem 1983, Bull & Segerberg 1984, Blok 1980).

This deductive landscape can still be organized in various ways. The standard description of temporal logics is by mere adoption of additional *axioms* on top of the above minimal logic. But recently, there has also been a growing interest in experimentation with various *rules of inference*. For instance, the minimal tense logic K_t also has the following derived rule for pure future formulas:

"if $G\phi$ is a theorem, then so is ϕ itself" .

But imposing this rule throughout would clearly change the family of admissible temporal logics: e.g., the system axiomatized by the earlier Löb Axiom for the future operator G fails to satisfy this principle. Further more complex rules have been proposed for axiomatizing irreflexive frame classes in Gabbay 1981A (Venema 1989 even shows their indispensability). Likewise, options will be much reduced by imposing some Gentzen-style sequent regimentation on admissible proof formats (cf. Fitting 1983, Dunn 1986).

Proof-theoretic viewpoints with an emphasis on structuring of arguments are coming to the fore these days, especially because of a growing interest in 'resource-based' styles of inference, such as categorial logics (van Benthem 1991) or linear logic (Girard 1987), which abandon such 'structural rules' as Monotonicity or Contraction on premises.

II.4 **Axiomatic Completeness**

II.4.1 General Completeness

Provability in the minimal tense logic and universal validity are co-extensive notions:

Theorem. For all tense-logical formulas ϕ, $K_t \vdash \phi$ iff $\models \phi$.

The soundness part is immediate here: all provable formulas are valid.

As for the completeness part, there are many well-known proofs of this result. We sketch one particularly easy route. Consider a fixed finite universe of formulas, namely ϕ together with all its subformulas. Define a *consistent* set Σ of formulas to be one whose conjunction is not refutable in K_t (i.e., $\neg(\wedge\Sigma)$ is not a theorem). *Maximally consistent* sets Σ are then defined as usual (always for formulas inside our restricted universe). Now, in the familiar propositional manner, these satisfy the following decomposition:

- $\phi \in \Sigma$ iff $K_t \vdash \bigwedge\Sigma \rightarrow \phi$
- $\phi\wedge\psi \in \Sigma$ iff $\phi \in \Sigma$ *and* $\psi \in \Sigma$
- $\neg\phi \in \Sigma$ iff *not* $\phi \in \Sigma$.

Next, define a binary relation $<$ among such sets Σ, Σ' by stipulating that

$\Sigma < \Sigma'$ iff $F\phi \in \Sigma$ whenever $\phi \in \Sigma'$ and also
$P\phi \in \Sigma'$ whenever $\phi \in \Sigma$.

Then we have a further decomposition (again within the restricted formula universe):

- $F\phi \in \Sigma$ iff some Σ' exists with $\Sigma < \Sigma'$ which contains ϕ,
- and likewise for formulas $P\phi$.

As a result, one may defined a finite model $\mathbb{M}$ whose points are all maximally consistent sets, with the ordering $<$, and a valuation on the relevant proposition letters read off from the Σ themselves. By an easy induction on the relevant formulas ϕ, this model is 'canonical' in the following sense:

$\phi \in \Sigma$ iff $\mathbb{M}, \Sigma \models \phi$.

In particular then, if some temporal formula ϕ is not derivable in K_t, its negation forms a consistent set $\{\neg\phi\}$, which can be extended to a maximally consistent one: at which point ϕ gets refuted in $\mathbb{M}$, whence it cannot be universally valid.

A side-effect of this construction is the so-called *Finite Model Property*:

Fact. A formula is universally valid if and only if it holds on all finite models.

By performing some surgery on models (or alternatively, by analyzing counter-examples to validity via the above-mentioned semantic tableaus), these finite models may even be taken to be of the special form where the precedence relation $<$ is intransitive without cycles or confluences. The latter results may also be obtained by direct semantic analysis, using the so-called *filtration* technique, with respect to the finite universe of subformulas of the formula at issue (cf. Gabbay 1976, Bull & Segerberg 1984, Goldblatt 1987, de Jongh & Veltman 1990).

On top of the general completeness theorem, the literature on temporal logic knows many special completeness results for systems in the earlier deductive landscape, whose proof requires more sophisticated mathematical argumentation:

II.4.2 From Logics to Frames

In a first direction, some particular set of axioms for temporal reasoning is given, and we want to find out whether its theorems characterize validity in some special class of temporal frames. In other words, we want to give an adequate modelling for some 'style of temporal reasoning'. There is an immense amount of results of this kind, of which we mention merely the following:

- The tense logic consisting of the minimal calculus K_t together with the earlier transitivity principle is complete with respect to universal validity on
  - i the class of all *transitive* frames,
  - ii the class of all *transitive irreflexive* frames.
- The tense logic which adds the two earlier linearity axioms to the preceding system is complete with respect to the class of all *strict linear orders*.
- If also Löb's Axiom is added in the earlier-mentioned form, the resulting logic becomes complete with respect to all *well-orders*.
- In general, further axioms often have their semantic effect predicted by the earlier frame correspondences of Section II.2.4. For instance, adding 'Hamblin's Axiom' $\phi \rightarrow FH(\phi \vee F\phi)$ will impose (forward) *discreteness* on temporal frames.

II.4.3 From Frames to Logics

In the opposite direction, one starts from a certain class of temporal frames, and wants to find some perspicuous axiomatization of its set of valid principles. Thus, the issue is now to determine the complete proof theory of some prior ontological 'view of Time'. Here, well-known examples are the complete theories of such linear structures as the Integers, Rationals or Reals (cf. Burgess 1984), or of branching ones like Minkowski space-time (Goldblatt 1980, Shehtman 1983). For instance,

- The complete tense logic of the rational number line **Q** is given by the above axioms for strict linear order plus
  - i the earlier 'density' principle $Fp \rightarrow FFp$
  - ii two axioms stating the existence of predecessors and successors PT, FT .
- The tense logic of the reals **R** extends that of **Q** by the further principle of 'Dedekind Continuity':
  $(FHp \wedge F\neg p \wedge G(\neg p \rightarrow G\neg p)) \rightarrow F((p \wedge G\neg p) \vee (\neg p \wedge Hp))$.

Proof techniques here may be described as follows. Non-theorems of the logic are refuted via some syntactic method like that sketched already for general completeness, and then a counter-example falling within the target class of frames is obtained, either by transforming the initially obtained model in some suitable fashion, or by building the required structural characteristics into the initial construction to begin with. (For examples of various strategies, see Burgess 1984, Doets 1987, Goldblatt 1987, De Jongh & Veltman 1990.)

II.4.4 Pathology: Incompleteness and Non-Axiomatizability

Although the completeness industry has enjoyed an immense success in temporal logic (and in Intensional Logic generally), this is not due to any special predestination. For, in both of the preceding directions, there are counter-examples to its goals.

- On the one hand, there exist not too unnatural *incomplete* axiomatic tense logics which fail to match the tense-logical theory of any frame class.

Example. An Incomplete Tense Logic.
Consider the earlier 'Löb Axiom' $H(Hp \rightarrow p) \rightarrow Hp$ together with the following principle of 'Future Stabilization': $GFp \rightarrow FGp$. On transitive frames, the latter corresponds to the existence of ***end-points***: $\forall x\, \exists y\, (x<y \wedge \forall z\, (y<z \rightarrow z=y))$. Together, these two principles turn out to form a consistent logic, but they hold on no temporal frame at all (Thomason 1973, van Benthem 1989D).

Still, there are some general results guaranteeing completeness of this kind at least for tense logics involving only special forms of axioms. For instance,

Theorem. The earlier-mentioned 'Sahlqvist forms' all define tense logics that are deductively complete with respect to the class of frames obeying their associated first-order condition.

Sambin & Vaccaro 1989 has a modern presentation of this result and its proof.

- In the other direction, there is no guarantee that tense-logical theories of natural temporal frames will turn out to be effectively axiomatizable.

In fact, since only countably many effective axiomatizations are available, and many more non-isomorphic (even countable) temporal frames, a mismatch between the two is bound to happen. Nevertheless, here too, there are some general reasons why many natural frame classes have turned out tractable. For instance, tense logics of *first-order definable* frame classes are effectively axiomatizable (van Benthem 1989A). Moreover, for many specific countable structures, the tense logic may even be reduced to the *decidable* monadic second-order theory of the so-called 'Rabin Structure' of finite sequences over the natural numbers (a technique first introduced in Gabbay 1976).

II.5 **Decidability and Complexity**

Although the present survey is mainly of a semantic, representational slant, there is an obvious, and in the end unavoidable, computational interest to the actual algorithmic complexity of temporal reasoning in our various calculi. For a start, by the lights of the average logician, the complexity of basic temporal reasoning is not very high:

Fact. Universal validity of temporal formulas, or equivalently, theoremhood in the minimal tense logic, is a *decidable* notion.

The reason turns on the earlier 'Finite Model Property'. The universal validities form a recursively enumerable class of formulas, by the above Completeness Theorem: but then, so do the non-theorems, being enumerable by checking all finite models. Therefore, Post's Theorem gives decidability.

Conversely, not all decidable tense logics possess the finite model property: Gabbay 1976 presents a counter-example.

As for computational complexity below the level of decidability, the minimal tense logic does move one step up from the purely propositional case (which is co-NP-complete). For convenience, we follow usual practice in considering satisfiability rather than validity in what follows, recording some results from Spaan 1991 (which has been written as a complexity-theoretic supplement to the present Survey; cf. also Ladner 1977):

- Satisfiability in K_t is PSPACE-complete.

The latter complexity is quite frequent among temporal logics. For instance, referring to some earlier examples, we have also

- Satisfiability in K_t4 (transitive irreflexive orders) is PSPACE-complete,
- Satisfiability in Löb's Logic (well-founded orders) is PSPACE-complete.

But certain further restrictions may restore lower complexity:

- Satisfiability in $K_t4.3$ (strict linear orders) is NP-complete.

So far, higher complexities have only been found for other kinds of temporal logic, viz. branching calculi running in EXP-TIME (Emerson & Srinivasan 1989).

Of course, what is crucial here is not so much absolute complexity of temporal logics as such, but rather an insight into the interplay of expressive resources of a formalism and special frame structure that lead to certain complexity behaviour. For instance, more expressive formalisms over linear orders may lead to higher complexity:

- Satisfiability in the full monadic first-order language over linear orders is at least PSPACE-complete (this follows from Sistla & Clarke 1982).

In this connection, there are many obvious open systematic questions concerning the 'composition' of temporal logics out of their future and past components. For instance, there is this natural

Question. Given certain complexity classes for two single operator temporal logics, what will be the complexity of their obvious 'minimal tensed combination'? In particular, will it always be the maximum of those for the components?

Now, let us consider some other issues besides theoremhood where computational complexity may play a role. As a consequence of our initial observation, many further semantic properties of the basic tense logic become decidable, even in cases where their counterparts from first-order logic are not (compare Gurevich 1985). For instance, unlike what happens in full first-order predicate logic, where monotonicity is undecidable, we have here this

Example. Decidability of Monotonicity.
The earlier semantic *monotonicity* of a temporal formula ϕ in some propositional atom p (cf. Section II.2.3) is a decidable notion. The reason is as follows. Monotonicity amounts to semantic validity of the following inference, which runs from a finite set of premises to its conclusion:

$$\{ \underline{A}\,(p \rightarrow p^+) \mid \underline{A} \text{ any sequence of operators } G, H \text{ up to length } d(\phi) \}$$
$$\models \ \phi(p) \rightarrow \phi(p^+) ,$$

and hence to universal validity of the obvious associated implication.

Considerations of effective computability may enter tense logic at other spots too. For instance, the Finite Model Property suggests that there might be an interesting restriction to finite models, being the prime example of concretely computable models. And indeed, there has been a good deal of research into 'model checking' of temporal formulas on such structures (cf. Stirling 1990, as well as Section V below). Moreover, there might be a *Finite Model Theory* for our formalism, operating on concretely computable models, as has turned out to be the case for standard first-order logic in general (Gurevich 1985). Here are two typical model-theoretic questions in this vein, returning to some of our earlier concerns:

- Does Compactness still hold when restricted to the universe of finite models?

For certain special frame classes, such as transitive irreflexive orders, the answer is clearly negative. A counter-example is the finitely satisfiable set of formulas $\{ F^nT \mid n \in N \}$, which has no finite satisfying model within that class. Another example concerns the earlier phenomenon of Preservation. For instance, in the Finite Model Theory of first-order predicate logic, the well-known equivalence between existential definability and preservation under extensions turns out to break down on the universe of finite models. But our special case might be better-behaved:

- Do the positive existential tense-logical formulas still capture all forms of semantic preservation under model extensions within the universe of all finite models?

The two questions are related. A positive answer to the first may be seen to imply one to the second. But in fact, the first has remained open so far, whereas the second will be settled positively by the methods of Section II.7 below.

II.6 Temporal Algebras

There is also another mathematical perspective for tense logic, which ties up syntax and semantics in a somewhat different way. Our language can be interpreted in *temporal algebras*(Thomason 1972), being Boolean algebras $A = (A, 0, 1, +, \bullet, -)$ having two additional unary operators f and p satisfying the following conditions, corresponding to the axioms of the minimal tense logic:

$$f(x+y) = f(x) + f(y) \qquad p(x+y) = p(x) + p(y)$$
$$f(0) = 0 \qquad p(0) = 0$$
$$f(-p-(x)) \leq x \qquad p(-f-(x)) \leq x\,.$$

Interpretation in temporal algebras starts from an *assignment* from proposition letters to elements of the algebra, after which the Boolean operators take care inductively of Boolean compounds in formulas, and the additional operations f, p of compounds formed using the temporal operators F, P. 'Truth' of a formula in an algebra under an assignment will then mean its receiving value 1 under this computation. We shall merely outline some features of this alternative approach to temporal semantics.

It is easy to show that a formula is provable in the minimal tense logic K_t if and only if it receives value 1 in all temporal algebras under all assignments. For, soundness is immediate, and as to Completeness, one proceeds via a straightforward construction of a temporal 'Lindenbaum algebra', by identifying formulas modulo provable equivalence.

The more interesting issue is how this algebraic perspective relates to the earlier model-theoretic one. One direction here is obvious. The prime examples of temporal algebras are *power set algebras* over the earlier temporal frames, provided with additional set-theoretic operations f , p defined as follows:

$$f(X) = \{ t \in T \mid \exists t' \in X\ t < t' \} \qquad p(X) = \{ t \in T \mid \exists t' \in X\ t' < t \} .$$

But also in the opposite direction, temporal algebras **A** may be represented as temporal frames, via the well-known Stone *Ultrafilter Representation*. That is, there exists a frame $\mathbb{F}(\mathbf{A})$ whose points T are the ultrafilters over A , ordered by a binary precedence relation < defined as in the earlier completeness proof, as well as a special family P of subsets of T which forms a temporal algebra that is isomorphic to **A** . (For general computational uses of this representation method, see Abramsky 1989.)

What is actually obtained here, then, are not in general full power set algebras, but rather frames with a 'distinguished range' of subsets, closed under the set-theoretic Boolean operations as well as the above operations f and p . Let us call such structures $(T , < , P)$ *general frames*. Evidently, in the reverse direction again, general frames **F** are already rich enough to generate corresponding temporal algebras $\mathbb{A}(\mathbf{F})$.

In all then, there turns out to be a full categorial duality between all general frames, when equipped with an appropriate version of the earlier-mentioned 'p-morphisms', and that of temporal algebras, with their appropriate algebraic homomorphisms (cf. Goldblatt 1976, van Benthem 1984, Sambin & Vaccaro 1988). Such mathematical connections have been exploited in the literature for transferring basic results and methods from Universal Algebra to Intensional Logic, and hence also to temporal logic. An example is Birkhoff's well-known characterization of equational varieties, which has been applied to obtain a model-theoretic description of those classes of general frames that are 'tense-logically definable' as the class of all frames validating some set of tense-logical formulas (Goldblatt & Thomason 1975, van Benthem 1984A; see also Section II.7 below).

General frames are also of interest by themselves. They may be regarded as a kind of 'two-sorted' version of temporal frames, having both a domain of temporal 'points' and one of admissible temporal 'propositions'. Truth of a temporal formula at some point in a general frame amounts to its truth in all models over that frame evaluating propositional atoms by sets in the distinguished range P . Such structures have an independent interest in applications, where the relevant temporal propositions usually satisfy some constraints (cf. van Benthem 1986B, Blackburn 1990): specific examples will be found in Section III below. Moreover, by a slight adaptation of earlier completeness arguments, they may be seen to provide a complete semantics for the following plausible extended notion of minimal deduction:

> ϕ is derivable from Σ using all the principles of the minimal tense logic K_t , with an added rule of arbitrary *Substitution* of formulas for proposition letters.

The model theory of general frames is close to that of first-order logic. In fact, they may be compared to Henkin's well-known 'general models' for higher-order logic (Enderton 1972, Doets & van Benthem 1983), having prescribed ranges for predicate quantification, which make the latter system essentially into a many-sorted first-order logic.

II.7 Perspectives from Standard Logic

II.7.1 Temporal Logic as First-Order Logic over Models

Various analogies in the preceding exposition suggest a 'first-order reduction' for our tense-logical formalism. This may be implemented via the following *standard translation* into a predicate logic having variables over points in time, one binary relation symbol $<$ as well as unary predicates P , Q , ... corresponding to the earlier proposition letters p , q , ... Here, each tense-logical formula ϕ turns into a first-order formula $\tau(\phi)$ with one free variable t_0 representing the 'current point of evaluation':

$$\begin{array}{lll} \tau(p) & = & Pt_0 \\ \tau(\neg\phi) & = & \neg\tau(\phi) \\ \tau(\phi\#\psi) & = & \tau(\phi)\ \#\ \tau(\psi) \qquad \text{for all binary Boolean connectives } \# \\ \tau(F\phi) & = & \exists t'\ (\ t_0<t' \wedge [t'/t_0]\ \tau(\phi)\ ) \\ \tau(P\phi) & = & \exists t'\ (\ t'<t_0 \wedge [t'/t_0]\ \tau(\phi)\ ) \end{array}$$

Temporal models may be viewed directly as structures for this first-order language too, and then we have an evident equivalence at each point between a tense-logical formula ϕ and its translation $\tau(\phi)$ evaluated in the standard way. This translation has been a role model for many similar ones in intensional logic: it has been rediscovered several times in the computational literature.

Remark. Object-Language and Meta-Language.
In the computational literature, this translation is sometimes described as follows. The temporal formalism is an 'object language', while the first-order formalism is its semantic 'meta-language'. Accordingly, one often finds more baroque notations for the latter, such as TRUE (p, t) or AT (t, p) . Note however, that nothing is gained by the latter notation, which even suggests a Tarskian mystery that is not there.

This translation transfers a number of results about first-order logic directly to tense logic. In particular, one obtains the earlier-mentioned Compactness and Löwenheim-Skolem properties for the language, as well as the recursive axiomatizability of universal validity. What it does not yield automatically, however, is a result like the earlier decidability of universal validity: since that is not a property of the full first-order language as such, but rather a peculiarity of its smaller 'tense-logical fragment'.

A more subtle case arises with the earlier preservation results, which neatly illustrate the peculiarities of working with restricted first-order fragments. For instance, if a tense-logical formula ϕ is preserved under model extensions, then so is its predicate-logical counterpart $\tau(\phi)$ (and that in the standard model-theoretic sense). Therefore, the usual preservation result of Los applies: and $\tau(\phi)$ must be logically equivalent to some positive existential first-order formula. But, in general, there is no guarantee that the latter will itself be the translation of some tense-logical formula! Hence, there is still work to be done, in order to show that the characteristic forms can be found within the tense-logical fragment. One illustration of the distinction between the two formalisms is this.

Example. Preservation under Expansion.
Positive existential forms in tense logic have another preservation property too, due to the 'positive occurrence' of their operators F and P :

If $\mathbb{M}, t \models \phi$, and some *expansion* $\mathbb{M}^+$ arises from $\mathbb{M}$
by merely adding pairs to its relation $<$, then $\mathbb{M}^+, t \models \phi$.

This is not an automatic consequence of preservation under extensions. E.g., the first-order formula $\exists y\ (\ \neg t<y \wedge Py\ )$ is preserved in the latter sense, but not in the former. But then, it is not inside the tense-logical fragment .

Nevertheless, the following result does hold:

Theorem. A tense-logical formula is preserved under model extensions if and only if it is equivalent to some positive existential tense-logical form.

We shall elaborate the argument, both for its intrinsic interest, and because it demonstrates some peculiarities of working with restricted formalisms rather nicely: sometimes, one can appeal to general features of first-order logic, at other times, special behaviour of tense-logical formulas is to be invoked.

Proof. First, positive existential temporal forms clearly have the stated preservation property.

The converse starts by a standard model-theoretic route. Let $PE(\phi)$ be the set of all positive existential consequences of ϕ. We show that $PE(\phi) \models \phi$. Hence, by Compactness, some finite subset of $PE(\phi)$ must already imply ϕ, whose conjunction will then be the required positive existential equivalent.

Let $\mathbb{M}, t \models PE(\phi)$. Then the following set of formulas will be finitely satisfiable: $\{ \phi \} \cup \{ \neg\psi \mid \psi$ is positive existential with $\mathbb{M}, t \models \psi \}$. By Compactness then, it is even simultaneously satisfiable: say, in some model $\mathbb{N}$ at a point t'. In particular, every positive existential formula which holds at t' in $\mathbb{N}$ is also true at t in $\mathbb{M}$.

Next, take any *ω-saturated* elementary extension $\mathbb{M}^*$ of $\mathbb{M}$. (Cf. Chang & Keisler 1973. The technical notion of saturation is not essential here, but it obviates a longer argument via a special 'diagram' for $\mathbb{N}$.) Then, the binary relation C defined by

Cxy iff 'y verifies all positive existential formulas true at x'

connects the model $\mathbb{N}$ with $\mathbb{M}^*$ in the following manner:

- $Ct't$
- the domain of C in $\mathbb{N}$ is closed under <-successors and <-predecessors,

so that one half of the zigzag condition holds for C: namely, from $\mathbb{N}$ to $\mathbb{M}^*$.

Next, we pass from $\mathbb{N}$ to its *unraveling* from t': i.e., the structure $\mathbb{N}^\$$ whose domain consists of all finite sequences of points in $\mathbb{N}$ starting with t', such that each next point in the sequence is either a <-successor or <-predecessor of the previous one, with the obvious associated ordering and valuation over such sequences. There is an evident zigzag from $\mathbb{N}^\$$ to $\mathbb{N}$, obtained by mapping sequences to their last element. Also, by recursion on the length of sequences in $\mathbb{N}^\$$, one can easily define a zigzag function Z from $\mathbb{N}^\$$ to $\mathbb{M}^*$.

Now, we *extend* $\mathbb{N}^\$$ to a new model $\mathbb{N}^£$ having a *two-way* zigzag connection with $\mathbb{M}^*$ which extends Z. The idea is as follows. Join an isomorphic copy of $\mathbb{M}^*$ to

$\mathbb{N}^\$$, and then extend Z by the obvious isomorphism on this copy. Moreover, provide the following new $<$ links between points x in $\mathbb{N}^\$$ and points y in the copy of $\mathbb{M}^*$:

if $Zxy', y'<y$, then put $x<y$ if $Zxy', y<y'$, then put $y<x$.

It is easy to check that this defines a zigzag between the extended model $\mathbb{N}^£$ and $\mathbb{M}^*$.

Putting all this together, one can clinch the argument:

$\mathbb{N}, t' \models \phi$ (construction) $\mathbb{N}^\$, t' \models \phi$ (zigzag invariance)

$\mathbb{N}^£, t' \models \phi$ (preservation under extensions) $\mathbb{M}^*, t \models \phi$ (zigzag invariance)

$\mathbb{M}, t \models \phi$ (elementary submodel).

II.7.2 Characterizing the Tense-Logical Fragment

What fragment of the full first-order language derives from tense-logical formulas? One answer may be found in van Benthem 1977, 1984A, showing that the earlier semantic analysis hit the mark.

Theorem. A first-order formula $\phi = \phi(t_0)$ in the above language is equivalent to the translation of a tense-logical formula if and only if it is invariant for zigzag relations between temporal models.

Proof. The invariance itself was already shown above. Conversely, suppose that $\phi(t_0)$ is any first-order formula having this invariance property. Let $TL(\phi)$ be the set of all τ-translations of tense-logical formulas semantically implied by it. We show that $TL(\phi) \models \phi$, from which the desired equivalence follows by Compactness. The argument is reminiscent of the preceding one for preservation under extensions.

So, let $\mathbb{M}_1, t_1 \models TL(\phi)$. Then, it is easy to see that the following set of formulas must be finitely satisfiable: $\{ \phi \} \cup \{ \tau(\psi) \mid \psi$ any tense-logical formula which is true at t_1 in $\mathbb{M}_1 \}$. Therefore, by Compactness, this set has a model $\mathbb{M}_2$ with a point t_2 where ϕ holds, and which agrees completely with t_1 on all tense-logical formulas. Now, take any two *ω-saturated* elementary extensions $\mathbb{M}_1{}^*$, $\mathbb{M}_2{}^*$ of $\mathbb{M}_1$, $\mathbb{M}_2$, respectively. Then, by a straightforward argument involving Saturation, the relation of agreeing on all tense-logical formulas must be a *zigzag* between $\mathbb{M}_1{}^*$ and $\mathbb{M}_2{}^*$ relating t_1 to t_2 . Therefore, we have:

$\mathbb{M}_2, t_2 \models \phi$ (construction) $\mathbb{M}_2{}^*, t_2 \models \phi$ (elementary extension)

$\mathbb{M}_1{}^*, t_1 \models \phi$ (invariance) $\mathbb{M}_1, t_1 \models \phi$ (elementary descent).

But we can also analyze the situation from a different point of view, which stresses semantic complexity of temporal operators, via the number of variables or 'semantic registers' involved in formulating their truth conditions. Re-examining the above translation, it may be seen that its complexity is low for the basic temporal formalism:

> Every tense-logical formula may be translated into a first-order one having only *two variables*, through judicious 're-cycling'.

For instance, the iteration GFPq may be translated into $\forall t>t_0\ \exists t_0>t\ \exists t<t_0\ Qt$.

Now, Gabbay 1981 has made the following general observation:

Fact. There is an effective correspondence between propositional temporal logics with a finite number of operators having a first-order definition, and so-called k-*variable fragments* of our first-order language, employing only a fixed finite amount of k variables (whether bound or free).

Thus, for instance, having 'functional completeness' for a temporal formalism may be seen as the existence of some k-variable fragment which already generates the whole first-order language. We shall return to this matter in Section III below.

What we have seen for the moment is that the basic tense logic lives at the low level $k = 2$. Still, this cannot be the whole story yet: for, there are also formulas in the 2-variable fragment of our predicate logic which are not invariant for zigzags. But we have at least determined the right level of 'expressive complexity', so to speak.

Remark. The Complete Two-Variable Fragment.
To obtain the full 2-variable fragment, one would have to add at least an operator I of 'temporal indifference' (see van Benthem 1989A), defined by the schema:

$$Iq = \exists t\ (\neg t<t_0 \wedge \neg t_0<t \wedge Qt)\ .$$

There are several semantic characterizations of k-variable fragments. Notably, Immerman & Kozen 1987 use the model-theoretic technique of 'Ehrenfeucht Games' of model comparison, suitably modified by the addition of 'pebbling' to mark objects selected in the course of play. Section III will present another analysis, extending the earlier zigzags to a more general kind of simulation found in Abstract Model Theory.

II.7.3 What is a Good Fragment?

In the philosophical and computational literature, a certain tension may be observed between users of temporal operator formalisms and those preferring a standard predicate-logical formalism having explicit quantification over points in time. Without adjudicating the issue here, some points may be noted.

First, the temporal operator formalism is quite perspicuous as a means of formalization of temporal reasoning. Moreover, one cannot exclude the possibility that it might find other reasonable interpretations beyond those encoded in its current first-order translation, thus enhancing its intrinsic interest. Next, from a model-theoretic perspective, the retreat into the above fragment has both advantages and disadvantages. The advantages are a somewhat nicer behaviour as regards various important semantic properties, such as monotonicity or other forms of preservation. Disadvantages include the absence of such useful techniques as Skolemization: the language does not give us prenex forms, pulling temporal operators out front, and then displaying functional dependencies. For instance, in the earlier example of preservation under model extensions, all translations of existential positive tense-logical forms are evidently existential in the standard sense, but most of the relevant prenex forms have no tense-logical counterpart, witness the case of

$$Fp \wedge Fq \qquad \exists t\,(t_0<t \wedge Pt) \wedge \exists t\,(t_0<t \wedge Qt) \qquad \exists t\,\exists t'\,(t_0<t \wedge t_0<t' \wedge Pt \wedge Qt').$$

(Various methods for alleviating this have been proposed, however: cf. Fitting 1989.) More generally, the issue is to which extent k-variable fragments are closed under classical theorems that hold for predicate logic as a whole. For many standard results, answers are not known as yet.

As concerns proof theory, the picture is diverse again. For the case of predicate logic in general, it is known that complete deductive systems for k-variable fragments may have essential need of 'detours' via higher fragments. Part of the content of the earlier Completeness Theorems is then that this is not going to happen to us in the basic tense logic. Nevertheless, staying inside the fragment does deprive us of useful deductive techniques, such as general Resolution (compare the Chapter by Luis Fariñas and Andreas Herzig in this Volume on modal theorem proving). On the other hand, there is a high perspicuity to operator manipulation in deduction too, as many people have found in practice (cf. Boolos 1979 for the case of modal 'provability logic'). So again, there does not seem to be any clear-cut outcome. Probably, both research perspectives should remain available in tandem: as they have already been for quite a while.

II.7.4 Tense Logic as a Higher-Order Logic

When changing over from temporal models to temporal frames, tense-logical truth acquires a *second-order* flavour. For, the earlier notion $\mathbb{F}, t \models \phi(p_1, ..., p_n)$ amounts to truth of the following 'monadic universal' second-order formula

$$\forall P_1 \dots \forall P_n \ \tau(\phi) .$$

This perspective is as fundamental as the previous first-order one, since many discussions of expressive power and axiomatizability are concerned primarily with frame classes. Again, through this 'standard transcription', existing positive knowledge about second-order logic can also be enlisted for the purposes of temporal logic. But, there is also an additional source of uncertainty here, given the known complexity of second-order logic (cf. Doets & van Benthem 1983), whose expressive power is bought at the price of having, e.g., a non-effectively axiomatizable notion of consequence. Therefore, there is often a subtle question as to whether and when temporal logic is given to this hereditary ancestral sin.

We start with model theory, namely, the earlier notions of definability. On the positive side, Van Benthem 1985 shows (amongst others) by general model-theoretic reasoning about second-order logic that one direction of the earlier correspondences is characterizable as follows:

Proposition. A tense-logical formula defines a first-order property of frames if and only if it is preserved under the formation of *ultrapowers*.

In the other direction, there is a positive result too (Goldblatt & Thomason 1975):

Theorem. A first-order definable class of frames is definable by means of some set of tense-logical formulas if and only if it closed under the formation of *disjoint unions*, *generated subframes* and p-*morphic images*, while its complement is closed under the formation of *ultrafilter extensions*.

On the negative side, e.g., first-order definability of second-order formulas is an *undecidable* notion (van Benthem & Doets 1983), and Chagrova 1990 implies the same for even the basic Priorean tense logic.

Next, we consider semantic consequence and axiomatic deduction. As for general complexity, Thomason 1975 has shown that tense-logical consequence on frame classes is fully as complex as second-order consequence in general. Nevertheless, there are many positive results too, concerning validity on special frames or frame classes. A good illustration comes from Doets 1987. Although the full second-order logic of such structures as the integers $\mathbf{Z} = (Z, <)$ and the reals $\mathbf{R} = (R, <)$ is exceedingly complex, its *universal monadic fragment* turns out to be effectively axiomatizable in a natural way, involving the complete first-order theory of these frames plus obvious schematic forms of 'Induction' and 'Continuity' principles (as well as a 'Suslin Property' for $\mathbf{R}$). This provides another explanation of the earlier-mentioned remarkable success in axiomatizing tense logics for natural temporal frames. Perhaps the best available result of this kind is to be found in Burgess & Gurevich 1985, who even prove decidability of the full monadic second-order theories of such frame classes as 'all elementary classes of linear frames', 'all continuous linear frames'.

Thus, temporal logic may also be viewed as the study of certain judiciously chosen fragments of higher-order logic over temporal frames, which, although reasonably expressive, manage to escape from the general intricacy and scarcity of pleasant logical properties besetting the latter formalism in its full generality.

III Extensions of the Paradigm

III.1 Additional Temporal Operators

The basic tense logic may be a natural one from several points of view, yet it still suffers from a severe lack of expressive power in many applications. Therefore, various possible strengthenings will be reviewed here, starting from some relatively modest ones.

- Difference

A first addition to the basic formalism, proposed independently by various authors (cf. Goranko 1989, Koymans 1989) for the purpose of creating smoother specification languages, refers to truth 'in a different world':

$\mathbb{M}, t \models D\phi$ iff *there exists some* $t' \neq t$ *with* $\mathbb{M}, t' \models \phi$.

This overcomes many limitations of the old basic formalism. For instance, on frames, *irreflexivity* of the temporal order now becomes definable by means of the principle $F\phi \rightarrow D\phi$. In fact, it is easy to see that even *all universal* first-order conditions on the temporal order become frame-definable in temporal difference logic.

The resulting model theory changes, but not beyond comprehension (see De Rijke 1990 for a first exploration). As for axiomatics, the minimal logic now adds a number of principles making the associated relation R_D as much like real inequality as possible:

$D\neg D\neg\phi \rightarrow \phi$	Symmetry
$DD\phi \rightarrow (D\phi \vee \phi)$	Pseudo-Transitivity
$F\phi \rightarrow (D\phi \vee \phi)$	$\forall x \forall y$: $x<y \rightarrow (R_D xy \vee x=y)$

When axiomatizating further logics over this base, it often turns out necessary to employ a new rule of inference, namely

If $(p \wedge \neg Dp) \rightarrow \phi$ is a theorem (with p not occurring in ϕ), then so is ϕ itself.

This is a general point: enriching temporal formalisms invites broadening our former deductive apparatus too.

- **Progressive Tense**

Even in natural language, there are other tenses than just Past and Future. A well-known example is the English progressive ("be –ing" ; cf. Section II.2.1), whose meaning may be approximated via an operator of 'topological interior':

$\mathbb{M}, t \models \Pi\phi$ iff *there exist* $t'<t$, $t''>t$ *such that, for all* x *in between* t' *and* t'' , $\mathbb{M}, x \models \phi$.

This operator lies typically one step up in the earlier k-variable hierarchy, since it requires *three variables* for its statement. What becomes expressible now are temporal patterns of 'betweenness' that were disregarded by the earlier zigzags. A complete axiomatization for this temporal logic on linear orders has been found recently in Shehtman 1989.

- **Next Time**

Another useful operator at this same level of complexity is 'next' (N) , on discrete linear orders having immediate successsors t+1 for any point in time t :

$\mathbb{M}, t \models N\phi$ iff $\mathbb{M}, t+1 \models \phi$.

When written out in pure < notation, again, three variables are needed essentially here. Note how this addition cuts across traditional linguistic schemes of classification: "next" or "to-morrow" is not a tense, but a so-called 'temporal adverb'. Such adverbs are also involved in our final illustration:

- **Since and Until**

Perhaps the best-known example of a strengthened tense logic arises with the *binary* temporal operators 'Since' (S) and 'Until' (U) introduced in Kamp 1966:

$\mathbb{M}, t \models S\phi\psi$ iff *for some* $t'<t$, $\mathbb{M}, t' \models \phi$ *and for all* x *in between* t' *and* t , $\mathbb{M}, x \models \psi$

$\mathbb{M}, t \models U\phi\psi$ iff *for some* $t'>t$, $\mathbb{M}, t' \models \phi$ *and for all* x *in between* t *and* t' , $\mathbb{M}, x \models \psi$.

One reason for their ubiquity lies in a Functional Completeness result to be discussed later on. In a sense, these two operators mark the end of the road: by now, we have gained the full strength of the first-order language over temporal models - at least, as long as we stick with continuous linear orders. For complete axiomatizations of the { S , U } logic of various frame classes, see Burgess 1982, Goldblatt 1987, Gabbay & Hodkinson 1989.

Other variations in the basic tense-logical framework are possible too, not having to do with temporal operators as such. Notably, one may consider certain special classes of temporal propositions, enjoying privileged semantic properties. For instance, in describing physical events, one might restrict attention to propositions whose life times are either convex intervals, or at most finite unions of these. Tense logic with the latter restriction on valuations for proposition letters has been studied in van Benthem 1986B. Also, Blackburn 1990 introduces 'nominals' into tense-logical languages, being special proposition letters whose denotation can only be a singleton set of the temporal domain.

III.2 Logical Theory of Temporal Formalisms

The above extensions can be studied by the same model-theoretic and proof-theoretic techniques that were developed before. Of course, specific theorems for the basic case may or may not carry over. We merely give some examples of what may happen.

Example. Sahlqvist's Theorem.
The earlier Sahlqvist Theorem of Sections II.2.4, II.4.4 generalizes to D-logic in its correspondence part. But, it fails in its completeness part. For instance, the logic with the single Sahlqvist axiom $\phi \rightarrow D\phi$ turns out to be deductively consistent without having any frames in which it holds (De Rijke 1990; Venema 1991 proves a generalization when the earlier 'irreflexivit rule' is added).

Moreover, some formalisms may just be ill-suited for bringing out the content of such an earlier result. For instance, the theorem does not generalize to an obvious statement in terms of the above operators S and U . Nevertheless, many new first-order definable temporal operators may be used in Sahlqvist forms besides the original ones. Further consideration of its proof shows that in fact:

> In the antecedent, any *continuous* m-ary modality is admissible, instead of just F, P , which commutes with arbitrary unions of its propositional arguments – while the consequent may contain any *monotone* modality besides F , P , G , H .

Thus, Sahlqvist forms will turn up in many enriched modal formalisms, whose additional operators are often monotone or even continuous. A typical temporal operator of the latter kind would be an 'existential' binary notion like

$$\mathbb{M}, t \models \phi + \psi \quad \text{iff} \quad t \textit{ is the supremum of some } t_1, t_2 \textit{ such that } \mathbb{M}, t_1 \models \phi \textit{ and } \mathbb{M}, t_2 \models \psi .$$

What this example demonstrates is in fact a more general research program. In many cases, classical results concerning basic modal or temporal logics turn out to have a 'mathematical core' that can be stated independently of the original formalism. And then, generalization to further modal systems becomes relatively straightforward. (Another instance of the same program is the general analysis of fixed-point theorems for modal provability logic found in van Benthem 1987, which turn out to depend on very little except 'forward persistence' of operators on well-founded orderings.) This research program still awaits development in its full generality.

Perhaps the most famous result in our general setting concerns a 'limiting point' of the process of enrichment, namely *functional completeness* of temporal formalisms.

Kamp's Theorem. On *continuous* linear orders, every first-order statement with one free variable is definable in the {S , U} formalism.

For a proof, see Kamp 1966. This result has been extended since by Yonathan Stavi (cf. Gabbay 1981B) who provided two additional temporal operators which make propositional temporal logic complete for *arbitrary* linear orders.

What would be a more systematic perspective on the variety of temporal operators that arise within the setting of our general first-order description language? There are several possible view-points here.

One illuminating semantic way of analyzing progressively stronger fragments of the full first-order language starts from the earlier notion of *zigzag* or 'bisimulation', and its induced invariance on temporal formulas. There exists a natural hierarchy of ever finer notions of 'simulation', respecting ever more structure of the temporal models being compared, such as 'betweenness' for triples of points, or suprema and infima in the precedence ordering. On the syntactic side, there is an accompanying ladder of ever more complex temporal operators for which there is still invariance under suitably sensitive kinds of simulation. This perspective is developed in more technical detail for general Modal Logic in van Benthem 1989A, but it also applies here.

Another useful perspective uses the earlier observations on *finite-variable fragments* corresponding to temporal operator formalisms (cf. Section II.7.2), which involve only semantic computation over configurations up to some fixed finite number of temporal points. The basic temporal logic of P and F is located at the *two-variable* level, while S and U involve essentially *three* variables. Kamp's Theorem may then be understood as saying that, under favourable circumstances, three variables suffice for defining the whole temporal first-order language. We shall analyze the finite-variable hierarchy in somewhat more detail to understand this situation better.

For full first-order languages, we have the well-known notion of *partial isomorphism* between two models, being the existence of a non-empty family of finite partial isomorphisms between their domains, satisfying the Back and Forth extension properties with respect to addition of individual objects on either side (cf. Hodges 1983). The partial isomorphisms in such a family will match finite sequences of objects in the two models being compared that verify the same first-order formulas from the full description language. Now, when describing k-variable fragments of the latter, this notion may be restricted to the use of 'k-*partial isomorphism*', being the existence of a non-empty family of partial isomorphisms of length *at most* k which has the Back and Forth properties only under extension up to length k , while also being closed under restriction of its isomorphisms to sub-isomorphisms of smaller length.

Example. Partial Isomorphism Between Linear Orders.

Matching all finite sequences of equal length and relative position gives a partial isomorphism between the rational numbers and the reals. Comparing the rationals with the integers, however, only 2-partial isomorphism can be established (through matching of 'similar pairs' and single points). No 3-partial isomorphism exists: one runs into problems with the characteristic difference between the two models, being the mathematical property of *density*. Of course, in general, outcomes may also be affected by the pattern of atomic statements over such linear orders.

Invariance under k-partial isomorphism for varying k provides a finer sieve for types of statement inside the full first-order language. By way of illustration, the basic tense-logical formalism in Section II needs only partial isomorphisms of length at most 2: where the maximum length is not even involved in the back-and-forth process. This explains, essentially, why its characteristic notion of zigzag could get by with matching individual points in time only. Now, here are some general model-theoretic results about these notions, demonstrated for the conventional case $k = 3$ (but our outcomes are completely general).

Proposition. Formulas $\phi = \phi(x_1, x_2, x_3)$ constructed using only the variables x_1, x_2, x_3 are *invariant for 3-partial isomorphism* in the following sense:
Let **P** be a family of partial isomorphisms of length at most three establishing 3-partial isomorphism between two models $\mathbb{M}_1$, $\mathbb{M}_2$. Then, any pair of matching sequences in **P** will give such formulas the same truth value in both models.

Proof. A straightforward induction on ϕ suffices. Suppose that the partial isomorphism $a_i \mapsto b_i$ $(1 \leq i \leq 3)$ belongs to **P**. Here is one direction of the central case in the induction.

$\mathbb{M}_1 \vDash \exists x_1 \phi(x_1, x_2, x_3)[a_1, a_2, a_3]$ implies that $\mathbb{M}_1 \vDash \phi(x_1, x_2, x_3)[a, a_2, a_3]$ for some object $a \in A$ (by the truth definition). Since the restriction $a_i \mapsto b_i$ $(2 \leq i \leq 3)$ also belongs to **P**, the Back-and-Forth property applied to a provides a partial isomorphism $\{(a, b), (a_2, b_2), (a_3, b_3)\} \in \mathbf{P}$ for some $b \in B$ - and so (by the inductive hypothesis) $\mathbb{M}_2 \vDash \phi(x_1, x_2, x_3)[b, b_2, b_3]$. Again by the truth definition, $\mathbb{M}_2 \vDash \exists x_1 \phi(x_1, x_2, x_3)$ $[b, b_2, b_3]$, whence also $\mathbb{M}_2 \vDash \exists x_1 \phi(x_1, x_2, x_3)[b_1, b_2, b_3]$.

This analysis provides a perfect fit, thanks to the following *converse*:

Theorem. Any formula $\phi = \phi(x_1, x_2, x_3)$ in the full first-order language (possibly employing other bound variables besides x_1, x_2, x_3) which is invariant for 3-partial isomorphism is logically equivalent to a formula constructed using these three variables only.

Proof. This Theorem may be proved essentially like the characterization of the basic tense-logical fragment given in Section II.7.2. The crux is again the introduction of a suitable zigzag relation at the end. What works here is the observation that a family **P** of partial isomorphisms $a_i \mapsto b_i$ of length up to 3 may be defined between the saturated elementary extensions $\mathbb{M}_1^*, \mathbb{M}_2^*$ as follows (we display only the longest case for sequences of points):

for all 3-variable formulas ψ,

$\mathbb{M}_1^* \models \psi[a_1, a_2, a_3]$ iff $\mathbb{M}_2^* \models \psi[b_1, b_2, b_3]$.

Together, the preceding two propositions provide a complete model-theoretic characterization of the 3-variable fragment of a full first-order language, and of k-variable fragments in the general case. Moreover, our analysis also has the following

Corollary. The following condition is sufficient for expressive completeness of a three-variable fragment with respect to its full first-order language:

'If two models are 3-partially isomorphic via some family **P** ,
then they are also partially isomophic via some extension of **P** ' .

Proof. Any formula in the full first-order language (having free variables at most x_1, x_2, x_3) was invariant under finite partial isomorphism. But then, it must even be invariant for 3-partial isomorphism, as any mapping in a family **P** for the 3-case will also belong to a full family of partial isomorphisms. Hence, the earlier Theorem applies.

Thanks to this analysis, the earlier-mentioned functional completeness of the 3-variable fragment of a monadic first-order language over *linear orders* may now be understood as follows.

Proposition. On linear orders, 3-partial isomorphism implies genuine partial isomorphism.

Proof. Let **P** be a family establishing 3-partial isomorphism between two linear models $\mathbb{M}$ and $\mathbb{N}$. Define a new family $\mathbf{P}^*$ as follows:

Take all finite partial matchings $a_i \mapsto b_i$ between $\mathbb{M}, \mathbb{N}$ having the property that,
whenever a_i , a_j are immediate <-neighbours in the a -sequence,
then $\{(a_i, b_i), (a_j, b_j)\}$ belongs to **P** .

To get the desired conclusion, it suffices to observe that, on linear orders, $\mathbf{P}^*$ has the Back and Forth properties.

Here are two more general aspects of the above characterization. On the 'positive' side, more concrete information about temporal formalisms corresponding to finite-variable fragments is provided by the various 'extension patterns' needed to induce back-and-forth properties up to the desired length. These suggest an obvious choice for a functionally complete set of operators in the corresponding variable-free temporal notation. On the 'negative' side, on general temporal models allowing *branching* patterns, no functional completeness theorem can hold for the whole first-order language. The reason is that, for each k, branching with at least k+1 incomparable successors is expressible in the first-order language, but not in its k-variable fragment. (A top node with k incomparable successors and one with k+1 incomparable successors form two frames that admit of an obvious k-partial isomorphism.) That is,

Proposition. No finite set of temporal operators is functionally complete for the full first-order language on arbitrary transitive irreflexive orders.

Thus, once branching time is admitted, the general picture in temporal logic over all frames becomes an open-ended one: there exists a genuinely infinite *Temporal Hierarchy* of possible operators, involving ever more complex configurations of points.

Remark. Higher-Order Temporal Operators.
Of course, there is still a restriction here to first-order definable temporal operators. The logical picture becomes even more diverse when we consider *higher-order* extensions. An interesting illustration of the latter possibility is Wolper 1983, which enriches the basic temporal logic via operators computable by suitable finite automata over models. ◆

III.3 Multi-Dimensional Tense Logic

So far, our extensions of the basic system were concerned with strengthened operators. But, the above perspective also provides another option for setting up temporal logic, having to do rather with the mechanism of interpretation. From the point of view of the general first-order description language, there is no compelling reason to stick with formulas having only one free variable. One can just as well have any finite number, and accordingly, evaluate statements at *sequences of points* in time:

$$\mathbb{M}, t_1, \ldots, t_n \models \phi .$$

There are various areas of application where this makes sense. For instance, already in the linguistic study of tenses, there have been systems of evaluation employing so-called 'auxiliary points of reference', for which more-dimensional tense logics have been proposed. Reichenbach 1947 was a pioneering study in this vein, in which evaluation of temporal statements involves three points, being one of 'speech' (S), one of the 'event described' and one of 'reference':

"I am sinning"	E, R, S		
"I have sinned"	E	R, S	
"I sinned"	E	R, S	
"I had sinned"	E	R	S .

Another instance of at least 'double-indexing' occurs in the study of indexicals like the temporal expression "now" (Kamp 1971), whose semantics refers both to some running point of evaluation and to a fixed 'present' perspective. Moreover, in studies of time as based on intervals (see Section IV below), it has also turned out convenient to work with interpretation of tense-logical formalisms in pairs of points 'beginning / end'. And finally, this perspective also arises when we describe similar logics of direction in Space (Segerberg 1973, Venema 1989), where pairs of spatial coordinates (x, y) form natural units of evaluation.

In the process, new temporal operators will emerge for operating with sequences, or at least ordered pairs, of points. For instance, Kamp's "now" evaluates as follows:

$$\mathbb{M}, t_1, t_2 \models \mathbb{N}\phi \qquad \text{iff} \qquad \mathbb{M}, t_1, t_1 \models \phi .$$

And the earlier Progressive might now be read as a two-dimensional operator too:

$$\mathbb{M}, t_1, t_2 \models \Pi\phi \qquad \text{iff} \qquad \textit{for all } t \textit{ in between } t_1 \textit{ and } t_2,\ \mathbb{M}, t, t \models \phi .$$

Or in space, there are geographical movements like 'moving up north':

$$\mathbb{M}, t_1, t_2 \models \Uparrow\phi \qquad \text{iff} \qquad \textit{for some } t, t_2<t \textit{ and } \mathbb{M}, t_1, t \models \phi .$$

More technical combinatorial operators have been proposed too, such as the 'permutation'

$$\mathbb{M}, t_1, t_2 \models \otimes\phi \qquad \text{iff} \qquad \mathbb{M}, t_2, t_1 \models \phi .$$

The latter auxiliary notion lacks any evident temporal meaning. And indeed, what is really happening here is a technical move toward a variable-free reformulation of the full predicate logic over temporal models along the lines of Quine 1966, via auxiliary operators manipulating arguments of predicates, such as of 'conversion' and 'identification'. The latter system is indeed an alternative to our Prior-style temporal logics for compact notation of predicate logic (be it that the conventions governing its combinatorial operators put it outside of the above finite-variable analysis).

Multi-dimensional formalisms can be analyzed by the same model-theoretic techniques as the above 'one-dimensional' systems (although no systematic generalization of the earlier theory has been published so far). Indeed, a multi-dimensional approach is already implicit in Section II . For instance, the model-theoretic treatment of 'k-partial isomorphism' clearly suggests that the more natural k-variable fragments are those which allow up to k free variables, rather than just one. Hence, van Benthem 1989A, 1990B argue that this leads to a more natural generalization of such semantic notions as temporal 'zigzag' or 'bisimulation', as relating sequences of points, rather than single points in models. If one takes this track seriously, then further changes in the background logic will become advisable. Notably, with pairs of points, the natural underlying logic is no longer the *Boolean Algebra* of propositional logic, but rather some form of *Relational Algebra*. Further independent motivation for this move will be found in Section V.1 .

III.4 Linear Time versus Branching Time

Our final extension takes its point of departure in the temporal models themselves, rather than in temporal languages or mechanisms of interpretation. So far, both linear and branching structures have been considered for time, but the sense of 'branching' involved has been a conservative one. Temporal frames can branch out into the past and future, without any significant repercussions for our formalisms so far. But in the computational literature, the choice between 'linear time' and 'branching time' has often referred to a deeper decision, namely, whether to stay with a pure time axis, or to introduce a more 'modal' picture of branching histories along which temporal propositions can be evaluated. In the latter case, a richer language is needed too, since genuine modalities should be available for comparing what happens along different histories (see Thomason 1984, Stirling 1990). We shall present a few possible semantic formats in this field, which is known for a certain technical complexity.

Temporal *frames* can now be identified with the familiar branching patterns of 'possible histories', formalized, for instance, in triples

$$\mathbb{F} = (S, R, C),$$

where S is a set of temporal *states*, R a relation of temporal *succession*, and C is a set of 'computation paths' or *possible histories* σ of some system being described: that is, a set of countable sequences of states where each state σ_{i+1} is an R-successor of σ_i. (For convenience, we assume that tails of histories in C are always in C as well.)

With a valuation assigning sets of states to proposition letters, then, branching temporal *models* $\mathbb{M}$ arise allowing a truth definition as follows. We only demonstrate its workings for a future time formalism, having one 'temporal' and one 'modal' operator, whose interaction provides for much of the interest of the system:

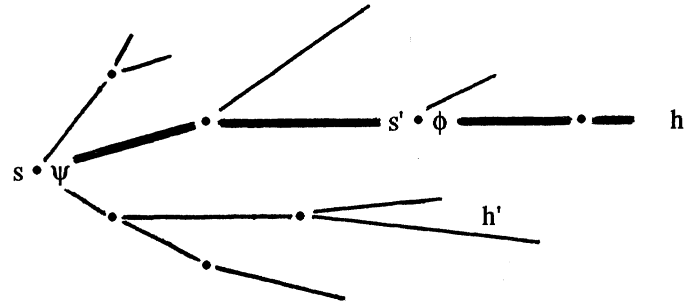
- $\mathbb{M}, \sigma \models p$ iff $\sigma_0 \in V(p)$
- Boolean connectives have the usual truth conditions
- $\mathbb{M}, \sigma \models G\phi$ iff $\mathbb{M}, \sigma' \models \phi$ for all proper tails σ' of σ
- $\mathbb{M}, \sigma \models \Box\phi$ iff $\mathbb{M}, \sigma' \models \phi$ for all histories σ' sharing their first state with σ

For complete axiomatizations of some branching time logics validated by such models, see Stirling 1989, 1990, Zanardo 1985, 1990, 1991, as well as the references therein.

For the purpose of comparison with the earlier semantic analysis of Section II, however, it is more convenient to redefine branching time structures as 'two-sorted' frames, having a domain of *states* ordered by a binary relation of possible precedence, as well as *histories* – where the two sorts interact as follows: states can *occur* in histories. In general, there might be atomic propositions here referring to states but also to histories: an option of which the above system only took the former. Then, given a suitable valuation to form models $\mathbb{M}$, the truth definition follows the pattern

$\mathbb{M}, s, h \models \phi$: 'in model $\mathbb{M}$, formula ϕ is true at state s in history h',

with key clauses (again displayed in the future direction only):



- $\mathbb{M}, s, h \models F\phi$ iff *there exists some* $s'>s$ *occurring in* h *such that* $\mathbb{M}, s', h \models \phi$
- $\mathbb{M}, s, h \models \Diamond\psi$ iff *there exists some* h' *on which* s *occurs such that* $\mathbb{M}, s, h' \models \psi$.

There are obvious downward duals P and $\Diamond^*$ for these two operators. In this modified format, branching temporal logic becomes more amenable to the earlier style of analysis. In particular, 'frame correspondences' defined essentially as in Section II.2.4 will record the effects of temporal principles on the pattern of histories and states.

Example. Branching Frame Correspondence.
For a start, pure F, P principles will express conditions on single histories. For instance, the earlier linearity axioms will force them to become linear sets of states. Then, a 'mixing principle' in the combined language like

$$(\Diamond Fq \wedge \Diamond Fr) \rightarrow \Diamond F(q \wedge \Diamond F\Diamond^* Pr)$$

expresses 'Confluence' for the web of histories as seen from any state s on a branch h :

$$\forall s_1 \forall s_2 \forall h_1 \forall h_2 \; ((Osh_1 \wedge Osh_2 \wedge Os_1h_1 \wedge Os_2h_2 \wedge s<s_1 \wedge s<s_2) \rightarrow \\ \exists s' \, (s_1<s' \wedge s_2<s' \wedge \exists h_3 \exists h_4 \, (Os_1h_3 \wedge Os'h_3 \wedge Os_2h_4 \wedge Os'h_4))) .$$

Conversely, an interesting frame property like the 'fusion closure' of Stirling 1989 – which states that, for any state occurring in two histories, its past in the one and its future in the other may be glued together so as to form a new history – turns out to be expressed by the following temporal Sahlqvist form:

$$(\Diamond(Hp \wedge Gq) \wedge \Diamond(Hr \wedge Gs)) \rightarrow \Diamond(Hp \wedge Gs)$$

There has been a continuing controversy in the literature on the relative merits of 'linear' versus 'branching' time for computational purposes. We do not want to enter into this debate here, but refer the reader to some stimulating discussions in Lamport 1980, Emerson & Srinivasan 1989, Manna & Pnueli 1989. What tends to confuse matters at times is a certain lack of terminological precision. 'Branching time' is often a name for what is really a joint temporal-modal or temporal-epistemic framework, whose underlying 'pure time' might well be linear after all. Also, 'linear time' is sometimes meant to stand for all linear orders, and then again just for one particular frame, usually the natural numbers. Whatever the merits of the case (if there is one) from an applied standpoint, it cannot be the business of temporal logic to support a priori taboos here.

IV Changes in Temporal Representation

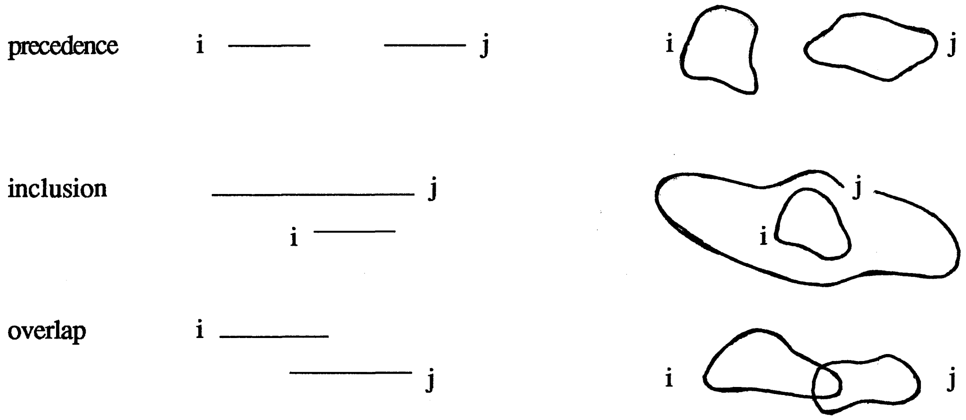
IV.1 Interval Structures

Although the picture of durationless mathematical points has been the prevalent image of Time, there have been continuing attempts at developing an alternative intuition, viewing time as consisting of extended 'periods' or 'intervals' as its primary stuff. The motivation for this is partly philosophical: human beings first come to experience time via extended events, and the point-based picture of non-extended temporal units seems a rather late abstraction arising out of this primary ontology. Thus, philosophers have considered both, as well as their interaction (see Russell 1926, Wiener 1914). Moreover, a move toward intervals has also been advocated in linguistics, as providing intuitively and technically more appropriate 'indices of evaluation' for assertions in natural language (see Dowty 1979, Kamp 1979). For instance, the earlier progressive tense is more naturally understood as describing properties of intervals, rather than points in time. And such linguistic properties need not have any obvious reduction to distribution of corresponding 'instantaneous properties' at points in time. Finally, the computational literature has seen various proposals for interval-based temporal logics. An example is Lamport 1985, who uses event structures reflecting durations of subprocesses in parallel computation, as a more appropriate qualitative model for different 'views' of a distributed process. Another example is the 'Naive Physics' of Hayes 1979, Hobbs 1985, where more common sense oriented models for physical phenomena are developed to serve as a basis for computation by simple algorithms, rather than the usual 'scientific world image' with its extensive mathematical apparatus. Temporal structures have been a prime example for this well-known AI enterprise.

Therefore, we now want to introduce *interval frames* whose objects are extended temporal intervals, connected by suitable relations. As to the latter, a number of options arises, involving both temporal order and temporal inclusion, such as:

$i < j$	i wholly *precedes* j
$i \subseteq j$	i is *included* in j
$i \, O \, j$	i *overlaps* with j .

The familiar pictures that go with this intuition show intervals as linear stretches, or sometimes also as extended spatial regions:



Against this background, one can also introduce more complex relations, such as one interval being the exact 'sum' of two others, or even corresponding interval *operations*, such as 'union' of overlapping intervals. At present, there seems to be no uniformly accepted choice of primitive relations or operations in the field. One systematic perspective is that of representing at least all possible relative positions between bounded linear intervals: of which there are exactly *thirteen*, as may be shown by listing the possible positions for i and j in the above pictures of line segments.

Concrete examples of such structures arise as families of intervals on point frames, taken as *convex* sets X in the ordering, that is:

$$\forall t_1 \in X\ \forall t_2 \in X\ \forall t \in T\ \ ((t_1 < t \wedge t < t_2) \rightarrow t \in X)\ .$$

One can think of intervals on linear orders here, but also of convex sets in a two-dimensional plane, etcetera. On the other hand, 'interval frames' need not be defined by reference to such underlying point frames at all: they may also be taken to stand for primary temporal pictures, on a par with point frames.

As for structural conditions to be imposed on such frames, to make them count as genuine temporal structures, there is again a variety of accounts in the literature. Systems of axioms for interval frames may be found in Russell 1926, Allen&Hayes 1985, Lamport 1985, Ladkin&Maddux 1987, Schulz 1987, van Benthem 1983, Thomason 1979 or Kamp 1979. Here, we shall merely formulate a number of plausible candidates, showing how various primitive relations between intervals might interact:

$<, O$ $\quad \forall x \; \neg x<x$

$\forall x \; xOx$

$\forall x \forall y \; (xOy \rightarrow yOx)$

$\forall x \forall y \; (xOy \rightarrow \neg x<y)$

$\forall x \forall y \forall z \forall u \; (x<yOz<u \rightarrow x<u)$

add $\subseteq$ $\quad \forall x \; x \subseteq x$

$\forall x \forall y \forall z \; (x \subseteq y \subseteq z \rightarrow x \subseteq z)$

$\forall x \forall y \; (x \subseteq y \subseteq x \rightarrow x=y)$

$\forall x \forall y \forall z \forall u \; (x \subseteq y<z \supseteq u \rightarrow x<u)$

$\forall x \forall y \; (x \supseteq yOz \subseteq u \rightarrow xOu)$

Many of these conditions are universal Horn clauses, driving the 'composition table' for the primitive relations involved. In addition, there are some more negotiable requirements on interval frames, such as

Convexity	$\forall x \forall y \forall z \forall u \; (u \supseteq x<y<z \subseteq u \rightarrow y \subseteq u)$
Linearity	$\forall x \forall y \; (x<y \vee y<x \vee xOy)$

These are all first-order constraints. There are also higher-order intuitions concerning interval frames, however, such as the earlier Homogeneity making all 'vantage points' in time equivalent, or 'Reflection' of larger intervals in smaller ones via suitable order/inclusion automorphisms (see van Benthem 1983).

Logical model theory in this area has been concerned with model comparisons between induced interval frames of point orderings, as well as with complete axiomatizations for full first-order interval theories of specific structures: such as all integer intervals, or all real intervals (see van Benthem 1983, Ladkin&Maddux 1987).

IV.2 Temporal Interval Logic

The earlier tense logic may be extended to these new interval structures (provided with appropriate valuations for proposition letters), while adding suitable operators taking advantage of the new extended structure. For instance, with two primitive relations $<$ and $\subseteq$, one can supplement the old G and H with two new 'modalities':

$\Box_{down}\phi$ $\quad\quad$ ϕ *holds in in all subintervals*

$\Box^{up}\phi$ $\quad\quad$ ϕ *holds in all superintervals* .

Of course, there are obvious existential duals $\Diamond^{up}$ and $\Diamond_{down}$ too. Then, a polymodal logic arises which can be studied by the usual techniques. For instance, standard properties of inclusion from the above list will be reflected as follows:

- *reflexivity* $\quad$ $\Box_{down}p \to p$
- *transitivity* $\quad$ $\Box_{down}p \to \Box_{down}\Box_{down}p$
- *anti-symmetry* has no counterpart here
  (it would have one in a version with a Difference operator, as in Section III.1)
- and optionally, *atomicity* of the interval ordering would be reflected by
  $\Box_{down}\Diamond_{down}p \to \Diamond_{down}\Box_{down}p$.

As for the interaction with temporal precedence, e.g.,

$Fp \to \Box_{down}Fp$	expresses	*right monotonicity*
$Pp \to \Box_{down}Pp$	expresses	*left monotonicity* .

Again, the earlier theory returns. There exist appropriate semantic notions of 'zigzag' or 'frame correspondence', while also the usual axiomatic methods of proof, as well as their corresponding completeness arguments remain valid (see van Benthem 1983).

In practice, however, this system does raise a number of interesting new issues. For instance, one common theme in the linguistic and computational literature is that of possible 'persistence' of temporal information, not just into the temporal past and future, but also along inclusion or extension of intervals. Dowty 1979 discusses various kinds of 'aspectual' behaviour for verbs in natural language, while Kowalski&Sergot 1985 consider similar phenomena in maintaining temporal data bases. For instance, in a temporal knowledge base, statements may have been stored initially referring to specific intervals, that need not be the ones that are going to be queried afterwards. In that case, one wants to know which statements true at some interval will continue to hold at later intervals, or will persist down to subintervals. (E.g., are employees over a period employees over subperiods, or at later periods?). Persistence may be partly a lexical matter (consider the temporal behaviour of "alive" versus "dead"), but it is also often triggered by certain syntactic forms of temporal assertion. For instance, on the above interval frames, it is easy to show the following common form of 'downward persistence':

Fact. Truth of all formulas constructed from arbitrary formulas $P\phi$, $F\phi$, $\Box_{down}\phi$ and $\Diamond^{up}\phi$, using $\wedge$ and $\vee$, is propagated downward along inclusion.

One way of interpreting results like this is as follows. In natural language, there are certain 'aspectual operators' which can change the 'temporal constitution' of linguistic expressions. For instance, the *Progressive* "be –ing" turns an 'event description' into a 'state description', whose meaning is captured, to a first approximation, by the above operator $\Diamond^{up}$: and hence, it creates 'downward persistent' propositions. Likewise, a *Perfective* operator "have –ed" turns an event description into a downward persistent state description, which fact is reflected in the behaviour of the above operator P . Thus, an interval tense logic like the present one provides a formal apparatus for what linguists have sought for, namely, a precise 'calculus of aspect'. Similar calculi have been proposed for computational purposes too (cf. again Kowalski & Sergot 1985; and also Section VI below for logical refinements). Of course, in this calculus, compound statements may also lose persistence behaviour of their components (think of negations) or at least modify it: the intersection of a future-persistent and a past-persistent assertion is merely 'convex'.

Despite all this smooth generalization of earlier point-based approaches, there is also a new technical feature to many proposed interval tense logics. This shows through the earlier perspective of 'translation' into *standard logic.* An appropriate first-order language for describing the present kind of models has variables ranging over intervals, and which includes the primitive relations of precedence and inclusion. Then, for instance,

$$Fq \rightarrow \Box_{down}Fq$$

will translate into the following statement about some current interval i :

$$\exists j\,(i<j \wedge Qj) \rightarrow \forall k\,(k \subseteq i \rightarrow \exists j\,(k<j \wedge Qj)) .$$

Now, many authors restrict attention here to those special interval frames which are induced by underlying point frames: assuming, e.g., that intervals can be identified with ordered pairs (t_1, t_2) of points for which $t_1 \leq t_2$. But then, the above translation can be 'unpacked' further to one couched wholly in terms of the point frame. For instance,

$$Fq \to \Box_{down}Fq$$

will come to say that some ambient interval $[t_1, t_2]$ has the following property:

$$\exists t_3 \exists t_4 (t_2<t_3 \wedge Q(t_3, t_4)) \to \forall t_5 \forall t_6 (t_1 \le t_5 \le t_6 \le t_2 \to \exists t_3 \exists t_4 (t_6<t_3 \wedge Q(t_3, t_4))).$$

Thus, proposition letters, formerly denoting unary properties of intervals, will come to stand for binary relations between points, whence our standard translation leads no longer into monadic second-order logic, but into *dyadic second-order* logic. The latter is much more complex than the former: e.g., fewer completeness theorems are forthcoming, even for simple frames like the integers. Moreover, in contrast to the situation in Section III.2, no functional completeness can hold for any finite set of temporal operators, even with linear intervals only (cf. Venema 1989).

Digression. Point-Based versus Primitive Intervals.
The interplay between interval frames and underlying point frames raises some interesting logical questions. For instance, any first-order definable property of general interval frames will also become a first-order property of point frames through the above transcription. But the converse seems to be open so far:

> Do formulas of a temporal interval logic which express first-order properties of point frames also express first-order properties of interval frames?

Next, temporal logic over intervals invites experimentation with a much greater range of operators than those appearing so far. For a start, logical constants may acquire new temporal shades of meaning in the present setting. Thus, Humberstone 1979 has proposed a more sweeping reading of *negation* as 'absence of truth in all subintervals': i.e., $\Box_{down}\neg p$ in our formalism. And Cresswell 1985 claims that in an interval setting, ordinary *conjunctions* $\phi \wedge \psi$ get a flavour of temporal succession: "the current interval is a directed sum of one in which ϕ holds and one in which ψ holds". Thus, there is an interest to studying various new operators too.

One rich system of this kind is the interval logic of Halpern & Shoham 1986, which works on point-based intervals, and then creates a whole aspectual calculus by introducing such operators as the following:

BEGINϕ is true at $[t_1, t_2]$ iff there exists $t_3<t_2$ such that ϕ is true at $[t_1, t_3]$
STARTϕ is true at $[t_1, t_2]$ iff there exists $t_3>t_2$ such that ϕ is true at $[t_1, t_3]$
BEFOREϕ is true at $[t_1, t_2]$ iff there exists $t_3 \leq t_1$ such that ϕ is true at $[t_3, t_1]$

as well as their obvious converses. One sign of the expressive power of the latter system is that it can define the four Stavi operators on linear frames (Section III.2): whence it is at least as expressive as the strongest point-based tense logic there. (But Venema 1988 also shows how it exceeds the latter in power of distinguishing between countable linear frames.) The price for this expressive power, as was noted already, is a scarcity of completeness results for well-known temporal frames like the integers or reals, whose temporal point logic was effectively axiomatizable or even decidable. Indeed, Halpern&Shoham show that their logic is $\Pi^1{}_1$-hard over the reals, and $\Pi^1{}_1$-complete over the integers. These somewhat daunting outcomes are characteristic of many interval logics over point-based intervals (cf. the earlier comments on this approach; Spaan 1991 shows that even the pure inclusion logic over intervals is PSPACE-complete).

Even so, over suitable larger classes of interval frames, the situation may improve: the last-mentioned reference axiomatizes a reasonable base logic in this vocabulary capturing universal validity in a more abstract sense. One useful auxiliary trick here is a topological re-interpretation of the logic, as describing directions of travel in a two-dimensional plane, with various 'compass operators'. And then, most relevant axioms turn out to belong to a more tractable fragment of the full language, namely a poly-modal analogue of the 'Sahlqvist forms' of Section II . For instance, geographical directions obey natural axioms of 'confluence', similar to those encountered before.

There are still useful temporal operators over interval models that are beyond the Halpern-Shoham system. Venema 1989 extends it with a binary 'Chop' operator, stating that an interval can be divided into a left part where one assertion holds, and an adjoining right part where another assertion holds (cf. Cresswell's proposal above). The latter system demonstrates an interesting analogy between interval tense logic and another area of mathematics, namely *Relational Algebra* (cf. Németi 1990). As was observed above, with propositions standing for intervals, viewed as sets of ordered pairs, formulas come to denote binary relations among points. But then, the operator structure of relations comes into play: for instance, the Chop operator is nothing but the well-known operation of *composition* of relations. The latter kind of structure is well-known from the semantics of programs: it is interesting to see that it also emerges in a temporal computational setting.

Such binary operators seem quite appropriate to many applications of interval tense logics. For instance, in the semantics of temporal constitution in natural language, one very common condition is 'additivity': certain propositions have temporal denotations that are closed under the formation of *sums* of intervals. An example are so-called 'activities', like writing: if I am writing over an interval, as well as over another one, then I am writing over the union of those two – and the same holds for 'states', like being in love. (The importance of this condition is enhanced by its occurrence in other linguistic fields. For instance, 'mass nouns' like "water", "sadness" – or indeed "time" itself! – have similar additive behaviour.) Therefore, a further binary temporal operator

$\phi \oplus \psi$, true at all intervals which are sums of a ϕ interval and a ψ interval,

seems useful. A proposition ϕ's being 'additive' then means that the inference $\phi \oplus \phi \models \phi$ is valid. Now, few propositions will have the latter property in general. But an aspectual calculus can again tell us at least how this property is *preserved* when we start from basic lexical expressions already having it. For instance,

If ϕ , ψ are cumulative statements, then so are their compounds
$\phi \wedge \psi$, $\Diamond_{down}\phi$, $\Diamond^{up}\phi$, $P\phi$, $F\phi$; but not necessarily $\phi \vee \psi$.

Finally, we mention the "IQ" system of temporal interval logic proposed in Richards & Bethke 1987, which comes with a somewhat different angle on temporal interval logic. IQ mixes temporal operators in the above spirit with deictical expressions referring directly to specific intervals. Thus, it may be viewed as combining ideas both from standard tense-logical formalisms and from their underlying first-order languages allowing direct reference to points or intervals in time.

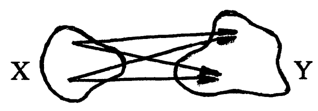
IV.3 Different Views and Representations

The two different temporal paradigms can be related to each other by mathematical means. There are various motivations for doing this, theoretical and applied. In philosophy, the relation between interval-based common sense time and point-based scientific time by itself forms a focus of interest. (Smith 1982 provides connections with Brentano's phenomenology, Thomason 1979, 1987 has a reconstruction of Bertrand Russell's views on comparing 'private' and 'public' time in these terms.) And similar dual viewpoints have arisen in linguistics, where interval models serve as temporary representational structures for discourse processing, which are eventually related to physical point time in the real world around us (Kamp 1979, Kamp & Rohrer 1988). Finally, in computer science and AI too, there is often a need for different views of the same system, from higher level descriptions in terms of events having duration to last details of actual physical processes in the machine. (See Allen & Hayes 1985, Lamport 1985, Joseph & Goswami 1988.)

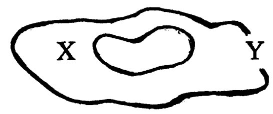
- Point frames directly induce interval frames.

Each point frame $(T, <)$ yields its associated family of *non-empty convex subsets*, with obvious set-theoretic definitions for precedence and inclusion:

$X<Y \qquad \forall t_1 \in X\ \forall t_2 \in Y:\ t_1<t_2$



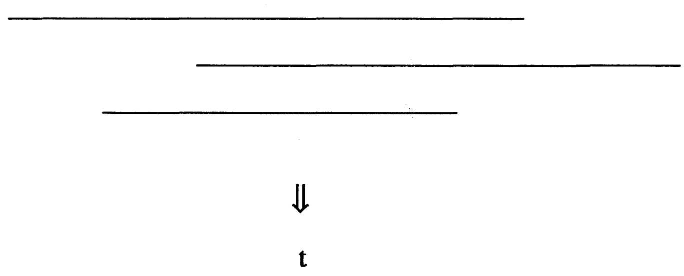
$X \subseteq Y \qquad \forall t \in X:\ t \in Y$



Van Benthem 1983 axiomatizes the complete first-order theory of such interval frames over transitive irreflexive orders. Note how all earlier Horn principles are valid on this point-set account of intervals.

There is also a converse transformation between the two perspectives:

- Each interval frame may be *represented* mathematically as a family of convex intervals over some underlying point frame:



This may be achieved by introducing 'points' via any one of a number of mathematical constructions, such as 'filters' or 'maximal filters' (van Benthem 1983, 1984B), 'Dedekind cuts' (Burgess 1984B, Thomason 1979) or other methods, some of them reviewed in Whitrow 1980. With suitably 'discrete' interval frames, even a very straightforward approach exists, namely to identify points with atomic indivisible periods.

Here is a small illustration of the workings of such representations. Suppose that we have an interval frame with three primitive relations $<$, $\subseteq$ and O . Now, let temporal 'points' t be all 'filters' over this frame, defined as follows:

> A *filter* is a set of intervals which is upward closed under $\subseteq$ and in which every two intervals overlap via O .

A precedence ordering among filters may then be defined as follows:

$$t_1 < t_2 \qquad \text{iff} \qquad \exists i \in (!) t_1 \ \exists j \in t_2 \ i<j \, .$$

Now, there is a natural map π sending intervals i to the set of all points 'in' them, i.e., to $\{ t \mid i \in t \}$, which has the following properties:

Fact. π is a *strong homomorphism* with respect to all three primitive relations on intervals and their set-theoretic analogues among point sets.

Proof. Inclusion. If $i \subseteq j$, then $\pi(i) \subseteq \pi(j)$ (by upward closure of filters). Conversely, if $\pi(i) \subseteq \pi(j)$, then in particular, the principal filter $UP(i) = \{ k \mid i \subseteq k \}$ belongs to $\pi(j)$: whence $i \subseteq j$.

Precedence. If $i<j$, then $\pi(i)<\pi(j)$, by the definition of $<$ among points. Conversely, if $\pi(i)<\pi(j)$, then $UP(i)<UP(j)$: whence some $i' \supseteq i$ precedes some $j' \supseteq j$, and therefore $i<j$, by Monotonicity.

Overlap. If iOj , then the filter $\{ k \mid i \subseteq k \text{ or } j \subseteq k \}$ is in the set-theoretic intersection of $\pi(i)$ and $\pi(j)$. If, on the other hand, $\pi(i) \cap \pi(j) \neq \varnothing$, then the intervals i, j occur together in some filter: whence iOj .

If one wants the induced order $<$ among points to have further special properties under this representation, then additional features of the original interval frame will have to be exploited. For instance, a combination of transitivity for intervals and monotonicity will make the point ordering *transitive*, while the earlier principle $\forall x \forall y \ (xOy \rightarrow \neg x<y)$ ensures its *irreflexivity*.

Finally, the two perspectives on temporal modelling can also be brought together, ensuring their co-existence in one logical system:

- There exists a complete *categorial duality* between
  suitably defined categories of point frames and interval frames.

For the latter purpose, suitable *morphisms* between frames are to be introduced, and connected via our representations. For instance, van Benthem 1983 correlates:

i *positive extension* among period frames:
  i.e., extension of domains and relations (both on new and old intervals)

ii *anti-morphic surjections* between point frames:
  i.e., partial maps f from F_1 onto F_2 such that $\forall x \forall y\, (f(x)<f(y) \rightarrow x<y)$
  which also satisfy a suitable continuity condition on convex subsets.

Intuitively, the former notion describes growth of information about some temporal situation, introducing both new events and new temporal connections between already available events. The latter then describes the obvious restriction map from temporal 'points' (that is, filters in the above sense) which arise in the resulting richer structure to those already constructed in the old one. Note how, in this process, all old points are related to new ones (though not always conversely!), but sometimes even to more than one: 'splittings' may occur, and we have to keep track of diverging 'histories' of temporal points along successive stages of the construction. (Mathematically, the process creates an 'inverse limit'.)

This categorial perspective becomes inevitable if one wants to model the same computational process with different 'grain sizes' as it were. An example is the theory of events in Lamport 1985, which describes intervals with three primitives:

'total precedence' $<$ 'overlap' O

as well as 'partial precedence': $X<<Y$ iff $\exists t_1 \in X\ \exists t_2 \in Y\ t_1<t_2$.

Complete axiomatizations of these notions over convex intervals are given in Anger 1986, van Benthem 1989D, Ben-David 1987. Plausible morphisms here include:

i *embeddings* from one system view into another,
  respecting both total and partial precedence of events,

ii *higher level views* induced by surjections f from one level to another,
  satisfying the implications $\forall x \forall y\, (x<y \rightarrow f(x)<<f(y))$, $\forall x \forall y\, (f(x)<f(y) \rightarrow x<y)$.

These morphisms may be studied again from a model-theoretic point of view, noting which structural properties of temporal frames are 'transferred' via these relations from one view of a system to another (see van Benthem 1989D).

Finally, as has been implicit in much of the terminology so far, there is often a vague border line between recent theories of intervals and theories of *events*. In principle, of course, the latter notion is the richer one, involving not just temporal structure, but also spatial extent, and perhaps even causal powers. Richer theories of events, extending the above patterns, have been developed both in the more linguistic tradition (see Krifka 1989, Link 1987) and in the computational literature (see Winskel 1989). What events tend to bring in at least is a notion of *measurement*: one constructs temporal substrata for events having a certain measurable duration, which can be referred to by such expressions as "for a whole night" or "within a few seconds" . Van Benthem 1983 explores representations from interval frames to point frames from the point of view of Measurement Theory, Michon 1985 has a more general psychological perspective. This more quantitative perspective leads eventually from the purely *topological* temporal logic of the present Survey to a *metric* temporal logic. At present, very little is known about the latter topic (see Burgess 1984 for what little history there is, and Koymans 1989 for an interesting applied system).

V Changes in Temporal Procedure

V.1 Dynamics of Interpretation

There is a noticeable computational slant to much recent work on temporal semantics, coming from various angles. For instance, in linguistics, there has been a number of 'dynamic' proposals which assume that human processing of temporal quantifiers or adverbials involves something like the action of an automaton surveying an ordered sequence of events (see Löbner 1986, de Swart 1991). And a similar perspective from automata theory has been employed in technical investigations of decidability for temporal logics on linear time (cf. Wolper 1983, Thomas 1989, Stirling 1990). Moreover, Gabbay 1989 has recently emphasized the possibility of a non-standard imperative reading of the earlier temporal formalisms, as executable specification of desired behaviour of systems. A statement $F\phi$ may not only describe the future, but it may also serve as an instruction for bringing it about that ϕ , while a combination like $GF\phi$ amounts to a standing order to see to it that ϕ keeps recurring. These various proposals are instances of a more general phenomenon: the idea of 'dynamic processing' is in the logical air to-day. There have been many designs in recent years for logical systems modelling certain dynamic aspects of interpretation and information flow, where propositions no longer function as declarative statements, but rather as instructions for changing cognitive and/or physical states (see Gärdenfors 1988, or the survey in van Benthem 1990B, 1991). The present Section explores how such ideas coming from a more computational setting may affect our understanding of the basic temporal logic itself, in its options for, successively: interpretation, inference, model structures, and even syntactic design.

One interesting system of dynamic interpretation arises by transferring certain ideas from the logico-linguistic literature on anaphora in natural language (cf. Heim 1982, Barwise 1987), themselves inspired by analogies with the semantics of imperative programming languages, to the realm of temporal expressions. Intuitively, evaluation of tense-logical formulas in a model is a process that takes us along various points of evaluation (compare the 'multi-dimensional' systems of Section II , which recorded its 'traces'). In the simplest analysis, this suggests viewing the denotations of formulas no longer as sets of points in time, i.e., as unary *properties* of temporal points, but rather as binary transition *relations* among them: i.e., sets of ordered pairs of points. Here is a simple implementation of this idea, taking a cue from Groenendijk & Stokhof 1989.

Fix a model $\mathbb{M} = (T, <, V)$. Each formula ϕ denotes a binary relation $[[\phi]]$ on T, constructed via the following induction:

- *atomic propositions* function as instantaneous tests:
  $[[p]] \quad = \quad \{ (t, t) \mid t \in V(p) \}$

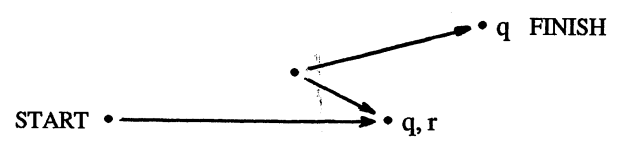
- *conjunction* becomes sequential composition of two successive tasks:
  $[[\phi \wedge \psi]] \quad = \quad \{ (t, t') \mid \textit{for some } t'', (t, t'') \in [[\phi]] \textit{ and } (t'', t') \in [[\psi]] \}$

- *futurity* involves making a step to the right and then starting again from there:
  $[[F\phi]] \quad = \quad \{ (t, t') \mid \textit{for some } t'', t<t'' \textit{ and } (t'', t') \in [[\phi]] \}$,
  the explanation for the *past* operator P is analogous toward the left

- one reasonable form of *negation* is this test for 'strong failure':
  $[[\neg\phi]] \quad = \quad \{ (t, t) \mid \textit{for no } t', (t, t') \in [[\phi]] \}$.

On this account, 'procedural' differences will emerge between formulas which used to be equivalent in the basic logic. For instance, $Fp \wedge q$ will now be read as an instruction to move first to some future point where p is the case, and then test whether q holds there. The net effect is a transition to some future point where both p and q hold. This outcome is similar to that of the formerly non-equivalent instruction $F(p \wedge q)$, which tells us to move to some future point where successive tests for p and q are succesful. Thus, it seems as if the operator F had acquired a wider scope toward the right in the former formula. On the other hand, negations do not induce this scope shift. For instance, $\neg Fp \wedge q$ tells us to first test whether no excursion into the future yields p, and then upon one's return, to check whether q holds. This is not equivalent to $\neg F(p \wedge q)$, which says that no excursion into the future should yield a point where both p and q are true. The difference with the earlier case is that testing for strong failure, rather than truth, of the formula Fp does not change the current point in time. Here is a more complex example, with eventual transitions indicated:

$Fq \wedge r \wedge PFq$:



Two new logical features are conspicuous about this system.

Varieties of Logical Consequence

There is no single preferred candidate for an appropriate notion of *valid inference* from premises $\phi_1, ..., \phi_n$ to a conclusion ψ in the dynamic setting. Perhaps the most direct extrapolation of the classical Tarskian notion would be as follows. A dynamic proposition α may be said to 'hold' at a point in time t in some model if 'testing' for α at t is succesful: $(t, t) \in [[\alpha]]$. (In other words, t is a 'fixed point' of the relation $[[\alpha]]$). Then we can require that

In each model, if all premises 'hold' at some point, then so does the conclusion.

But more genuinely dynamic notions of consequence would reflect the intuition that, in evaluation of an inference, premises are processed in their proper order, followed by some kind of check for the conclusion. Perhaps the most straightforward candidate reads:

$$\phi_1, ..., \phi_n \models \psi \qquad \text{if} \qquad \textit{in all models}, \ [[\phi_1 \wedge ... \wedge \phi_n]] \subseteq [[\psi]].$$

What this means is that processing the premises successively will give a transition which is also acceptable for the conclusion.

The latter notion is quite different from the standard one. This shows already in the basic *structural rules*, defining its 'style of reasoning'. Standard consequence obeys such structural rules as *Permutation* of premises, or *Monotonicity* under addition of premises. Dynamic consequence has neither. First, the order of premises matters now:

$$Fp, q \models F(p \wedge q) \qquad \text{but} \qquad q, Fp \not\models F(p \wedge q) .$$

And addition of premises yields a new process whose effects may differ from the old one:

$$Fp \models Fp \qquad \text{but} \qquad Fp, F\neg p \not\models Fp .$$

This does not mean that no structural regularities hold at all. For instance, a strong transitivity principle like the standard *Cut Rule* remains valid, in patterns such as:

$$\text{if} \ \Phi \models \psi \ \text{and} \ \Sigma, \psi \models \chi, \qquad \text{then} \quad \Sigma, \Phi \models \chi$$

Proliferation of Logical Constants

The second noticeable feature in dynamic interpretation is the emergence of new *options* for introducing *logical constants*, reflecting various 'modes of control' for procedures of evaluation. For instance, besides the above 'sequential' conjunction, there is also a plausible 'parallel' one, taking all those transitions which are succesful for two actions at the same time. This is indeed just Boolean intersection:

$$[[\phi \cap \psi]] \quad = \quad [[\phi]] \cap [[\psi]] \,.$$

The emergence of new logical constants in a dynamic setting is discussed at length in van Benthem 1990B, 1991, which address the question what are natural candidates. Their variety may be understood as follows. What we have been doing here is to consider a switch from standard semantics to what is sometimes called in a modern slogan: 'Propositions as Programs'. That is, our former temporal specification languages now also serve as *programming languages*: a view which is indeed defensible, even for natural language itself. The simplest logic appropriate for the latter view is that of *Relational Algebra* (or more generally, 'Dynamic Logic'). And the latter is indeed known for its greater arsenal of logical operators than the Boolean Algebra of standard propositions.

Accordingly, a reasonable more extensive system of 'Dynamic Tense Logic' might have the following repertoire:

Syntax	proposition letters	$p, q, \ldots$	
	temporal operators	F, P	
	connectives		
	Boolean Algebra	$-, \wedge, \vee$	
	Relational Algebra	o	(composition)
		$\check{}$	(converse)
		Δ	('diagonal')

Semantics proposition letters denote atomic tests, as above,
Boolean operators denote the corresponding set-theoretic operations on binary relations: complement, intersection, union,

o is the binary operation of relational composition

$\check{}$ is the unary operation taking the converse of a relation

Δ is the distinguished identity relation.

As an example of its expressive power, the earlier strong failure $\neg\phi$ is definable in the form $-\Diamond\phi$, where the 'modality' $\Diamond\phi$ denotes the 'existential test'

$$\{\ (t\ ,\ t)\mid \textit{for some}\ \ t' \in T\ ,\ (t\ ,\ t') \in [[\phi]]\ \}\ .$$

But, the latter can be defined as follows in relational algebra: $\Delta \cap (1 \circ [[\phi]]^{\vee})$.

Eventually, one might also add further natural operators to the framework, such as the infinitary *transitive closure* of relations (i.e., the well-known 'Kleene star' *).

What is the complexity of the change from Boolean Algebra to Relational Algebra? For instance, the original dynamic system presented above is decidable (van Benthem 1989D effectively reduces it to the standard tense logic K_t). But the present calculus would embed the complete algebra over a binary relation, and the latter system is known to be undecidable. Hence, the issue is what becomes of its various 'temporal fragments'.

Remark. Syntactic Reformulation.
A more elegant reformulation of this system would arise by changing its syntax. Note that the earlier clause for formulas $F\phi$ really has the effect of a sequential conjunction '$F{\wedge}\phi$' . So, one could also work with the following repertoire. Atomic actions will include atomic tests, named by proposition letters, as well as three fixed binary relations

$<$ (denoted by F) $>$ (denoted by P) $=$ (denoted by Δ) .

Program instructions will be the following operations of control:

complement intersection choice composition inversion.

V.2 Connections with Standard Logic

Despite its perhaps exotic flavour, the above system may still be studied by the techniques of Section II . In particular, van Benthem 1989D, 1991 give a simple *translation* from formulas ϕ of our dynamically interpreted tense logic into formulas $\tau(\phi)(t_0, t_1)$ of the earlier first-order predicate language over models having *two free* variables, explicitly describing their corresponding first-order definable transition relations. Therefore, semantic complexity of dynamic formalisms can be measured in terms of an earlier hierarchy (Section II.7.2): all translations may be constructed so as to end up in the *three variable* (free or bound) fragment of the full first-order description language over models.

(Moreover, one can formalize all relevant unary statements concerning these transitions in the temporal Since / Until formalism of Section II.1.)

Example. Static Translation of Dynamic Propositions.
Here are two inductive clauses:

$$\tau(q) \quad := \quad Qt_0 \wedge t_0{=}t_1$$

$$\tau(\phi\wedge\psi) \quad := \quad \exists t\, (\tau(\phi)(t_0, t) \wedge \tau(\psi)(t, t_1)) .$$

Thus, the earlier formula $Fq\wedge r\wedge PFq$ will translate eventually into

$$\exists t\, (t_0{<}t \wedge Qt \wedge Rt \wedge \exists t'\, (t'{<}t \wedge t'{<}t_1 \wedge Qt_1))) ,$$

which may be rewritten to the three-variable form

$$\exists t\, (t_0{<}t \wedge Qt \wedge Rt \wedge \exists t_0\, (t_0{<}t \wedge t_0{<}t_1 \wedge Qt_1))) .$$

Likewise, e.g., the formula $\neg Fq\wedge r$ will turn into

$$\neg\, \exists t\, (t_0{<}t \wedge Qt) \wedge Rt_0 \wedge t_0{=}t_1.$$

What this translation establishes is at least recursive axiomatizability for dynamic consequence, as well as various other standard logical properties. For special results, however, such as decidability for specific fragments, further argument will be needed.

But there are also obvious questions of logical model theory. For instance, in line with the earlier interest in *Preservation*, one obvious concern would be to relate computation of relations $[[\phi]]$ across different models. For instance, if a model M_2 extends M_1, for which formulas ϕ will their denotation as computed in the submodel be the matching restriction of their denotation in the extended model? ('Generated submodels' in the earlier sense will guarantee this connection in general: arbitrary submodels only for very simple 'a-temporal' cases.)

In the end, 'reduction' to standard formalisms does not seem to be the most fruitful perspective on what is going on here. What we rather want is a framework for *co-existence*. We conclude with one possible mode, again taken from a computational analogy in the semantics of programs.

Definition. Strongest Postconditions and Weakest Postconditions.
Consider any temporal model $\mathbb{M}$. For each formula ϕ of dynamic tense logic, and each standard formula A, the *strongest postcondition* of ϕ with respect to A in $\mathbb{M}$ denotes the image of the set $[[A]]$ of all points where A holds in the standard declarative sense under the transition relation $[[\phi]]$. Generalizing over all models $\mathbb{M}$, we write

$SP(A, \phi)$.

This is the strongest statement that can be made about states where one ends up after having processed ϕ starting from a state satisfying the 'precondition' A. Likewise, there exists a *weakest precondition* of ϕ with respect to some 'postcondition' B: $WP(\phi, B)$, which denotes the inverse image of $[[B]]$ under $[[\phi]]$.

Now, the following inductive recipes compute these 'declarative projections' of dynamic processes effectively:

Fact. Weakest preconditions obey the following recursion:

$$
\begin{array}{lcl}
WP(p, B) & = & p \wedge B \\
WP(\phi \wedge \psi, B) & = & WP(\phi, WP(\psi, B)) \\
WP(F\phi, B) & = & F\, WP(\phi, B) \\
WP(P\phi, B) & = & P\, WP(\phi, B) \\
WP(\neg\phi, B) & = & B \wedge \neg WP(\phi, TRUE) .
\end{array}
$$

For instance,

$$WP(Fp \wedge q, B) = WP(Fp, WP(q, B)) = F\, WP(p, q \wedge B) = F(p \wedge q \wedge B) .$$

The analogous recursion for strongest postconditions is as follows:

$$
\begin{array}{lcll}
SP(A, p) & = & A \wedge p & \\
SP(A, \phi \wedge \psi) & = & SP(SP(A, \phi), \psi) & \\
SP(A, F\phi) & = & SP(PA, \phi) & (!) \\
SP(A, P\phi) & = & SP(FA, \phi) & (!) \\
SP(A, \neg\phi) & = & A \wedge \neg WP(\phi, TRUE) & (!) .
\end{array}
$$

Remark. Conversion.

The preceding two notions become duals, by the operation of conversion $^{\vee}$:

$$
\begin{array}{lcl}
SP(A, \phi) & = & WP(\phi^{\vee}, A) , \\
WP(\phi, A) & = & SP(A, \phi^{\vee}) .
\end{array}
$$

This provides another method of computation, via the equivalences:

$$
\begin{array}{lll}
p^{\vee} = p & (\phi \wedge \psi)^{\vee} = \psi^{\vee} \wedge \phi^{\vee} & \\
(\neg\phi)^{\vee} = \neg\phi & (F\phi)^{\vee} = \phi^{\vee} \wedge PT & (P\phi)^{\vee} = \phi^{\vee} \wedge FT
\end{array}
$$

These notions create a 'two-level' system of temporal logic, performing dynamic interpretation via sequential procedures, but also keeping track of more classical static information content, that remains even after detailed instructions have been forgotten.

Example. Computing Successive Information States.
Here are the successive stages for SP computation from some precondition A , taking advantage of the associativity of conjunction to disregard bracketing:

	(F	q)	$\wedge$	r
A:	PA	PA$\wedge$q		PA$\wedge$q$\wedge$r

	r	$\wedge$	(F	q)
A:	A$\wedge$r		P(A$\wedge$r)	P(A$\wedge$r)$\wedge$q

	(¬	F	q)	$\wedge$	r
A:	via	A $\wedge$ ¬WP (Fq , TRUE) $\wedge$ r			
	to	A $\wedge$ ¬Fq $\wedge$ r			

	F	q	$\wedge$	r	$\wedge$	P	F	q
	(compare the earlier picture for the associated transitions)							
A:	PA	PA$\wedge$q		PA$\wedge$q$\wedge$r		F(PA$\wedge$q$\wedge$r)	PF(PA$\wedge$q$\wedge$r)	PF(PA$\wedge$q$\wedge$r)$\wedge$q

This recursion for weakest preconditions and strongest postconditions does not extend very easily to other program operators, such as the earlier Boolean conjunction. But for richer formalisms, the above general translation into standard predicate logic will still work, and it could be used as an alternative way for keeping track of 'static content'.

Finally, richer dynamic semantics are equally feasible. For instance, one could associate propositions with finite 'traces' of all reference points encountered during their processing. The above semantic clauses would then carry over in an obvious sense. This would be closer to actual understanding of temporal discourse, where a growing finite 'sample' of points from the temporal domain is 'visited' during evaluation. Moreover, in the end, one might follow usual computational practice, and even include procedural information about 'procedural control' into such 'temporal states'.

V.3 Varieties of Inference

The dynamic perspective is not confined to semantic interpretation of logical formalisms. It applies equally well to processing of information, with its characteristic phenomena of inference, revision, etcetera. This time, there will be cognitive transitions, not between points (or more generally, assignments) in models, but rather between 'information states' in some suitable cognitive space.

One example is the mechanics of maintenance of temporal knowledge found in Allen 1983. Information states are states of a network of intervals and arrows carrying information about (thirteen possible) temporal relations between them. For instance, we may know that interval i either precedes, or is properly included in interval j , etcetera. Now, an updating procedure is defined which, for each new temporal fact, adds that fact in its proper position, and then computes how this additional information gets propagated to other nodes and arrows.

Example. A Temporal Network.

Let the following be known about a story having four events i , j , k and l .

" i either wholly precedes or overlaps with j , which itself wholly precedes k ;
i either wholly succeeds or contains l " :

In a diagram,

$$\begin{array}{ccccc} i & \text{----}\{<, O\}\text{----} & j & \text{----}\{<\}\text{----} & k \\ \{>, \supseteq\} & & & & \\ l & & & & \end{array}$$

Given any two temporal relations R_1 , R_2 , one can compute the set $R_1{}^oR_2$ of all possible relative positions for pairs of intervals in their relational composition. For instance, applying this to the above l , i , j , one may compute the following label of possibilities for the arrow l ---- ---- j :

$$\{ >^o< , >^oO , \supseteq^o< , \supseteq^oO \} \qquad = \qquad \{ < , O , \subseteq \} .$$

Then, adding any new 'arrow' x ---- A ---- y to the network will have an effect described by the following non-deterministic algorithm:

Choose any 'fitting' link y ---- B ---- z (or z ---- B ---- x).
Compute the image A^oB as a new label for x ---- ---- z (or z ---- ---- y).
Intersect the latter with its already available label (if any).
Repeat until no new labels appear.

This is a terminating algorithm which updates network states, so as to reveal new information. For instance, adding the new link

l ----$\{\subseteq, \supseteq\}$---- j

to the above network will replace its formerly computed label $\{<, O, \subseteq\}$ by $\{\subseteq\}$, after which further propagation yields that, e.g., i ----$\{O\}$ ---- j. ■

Again, updating of temporal networks shows many of the earlier dynamic features: it produces transition relations between states, and its characteristic operations are those of Relational Algebra, such as composition, intersection or choice.

But there are many further varieties of inference proposed in Artificial Intelligence that are of relevance to temporal reasoning. For instance, reflection on the 'Frame Problem' of maintaining information across changes of time has engendered various new 'non-monotonic logics', as recorded extensively in Volume II of this Handbook. One obvious candidate is John McCarthy's *Circumscription*, which allows one to impose 'minimal life-times' on certain exceptional propositions, thus supporting certain defeasible inferences about future temporal behaviour of a system. Again we return to the basic tense logic, which remains a useful 'laboratory' for new ideas. The following is a relatively simple-minded form of propositional circumscription, merely intended to demonstrate the feasibility of this transplant.

Definition. Propositional Minimization.

Consider models $\mathbb{M}$ with a distinguished vantage point t. $(\mathbb{M}, t)$ is a p-*minimal* model for a tense-logical formula ϕ if

- $\mathbb{M}, t \models \phi$
- for no $\mathbb{M}'$ differing from $\mathbb{M}$ only in that the extension $V'(p)$ is properly contained in $V(p)$, $\mathbb{M}', t \models \phi$.

The definition is analogous for the case of several proposition letters simultaneously. ■

For instance, let p express the exceptional circumstance that "there is severe frost in the Netherlands" and q the statement that " the Dutch nation is skating". Models for the formula $G(q \rightarrow p)$ allow any future in which occasions of general skating are occasions of severe frost, while allowing the latter also without skating. But p-minimal models would make the skating occasions the *only* instances of severe frost: a sufficient condition has become a necessary one.

Now, let us say that circumscriptive inference takes place as follows:

$$\Sigma \models_{p\text{-min}} \psi \qquad \text{iff} \qquad \psi \text{ is true in all p-minimal models of } \Sigma .$$

This validates all earlier standard consequences, but also new ones like

$$G(q \rightarrow p) \models_{p\text{-min}} G(p \rightarrow q) \qquad \text{or even} \qquad F(q \wedge p) \models_{p\text{-min}} G(p \rightarrow q) .$$

Again, the resulting system is unlike standard ones, in that even basic structural rules like Monotonicity, or this time also the Cut Rule, will fail:

$$G(q \rightarrow p) \models_{p\text{-min}} G(p \rightarrow q) , \qquad \text{but} \quad G(q \rightarrow p),\ G(r \rightarrow p) \not\models_{p\text{-min}} G(p \rightarrow q)$$
$$G(q \rightarrow p),\ G(r \rightarrow p) \models_{p\text{-min}} G(q \rightarrow p) \models_{p\text{-min}} G(p \rightarrow q)$$
$$\text{but again} \quad G(q \rightarrow p),\ G(r \rightarrow p) \not\models_{p\text{-min}} G(p \rightarrow q) .$$

Nevertheless, tensed Circumscription can be analyzed by the logical techniques of Section II. For instance, there is an obvious analogue of the standard translation here: which will now take circumscribed formulas $\mu p \bullet \phi(p)$ to *second-order* ones explicitly stating the minimality of the previous definition for the unary predicate P corresponding to p . (As in Lifschitz 1985, one may then study those simpler cases where a 'first-order reduction' is feasible after all: one notable example being all ϕ having only syntactically *positive* occurrences of p . Nevertheless, such a reduction does not work generally (van Benthem 1989C). For instance, there is no first-order circumscription for the simple tense-logical formula $Fp \wedge G(p \rightarrow Fp) \wedge G(p \rightarrow Hp)$.)

One obvious question in this setting concerns complexity for circumscriptive inference in the basic tense logic: is this notion at least *decidable*? As this system is intermediate between monadic predicate logic (where Circumscription is decidable) and general dyadic predicate logic (where it is not), the matter is non-trivial. Since the original version of this paper was written, the question has been settled in the negative by Alexander Chagrov (personal communication).

Temporal minimization in the above style is still a crude first approximation. In actual temporal reasoning, it might be more realistic to minimize, not over extensions for propositions p , but rather over temporal *changes* from p to $\neg p$ or vice versa. Moreover, many temporal default rules would seem to trigger some explicit dynamic procedure, which does not merely select 'minimal models' in some fixed pattern of preference, but rather *changes* those preferences over possible courses of history. (Spohn 1988 and Veltman 1990 present concrete dynamic systems for changing preferences over general possible worlds models.) And in the end, Shoham 1988 argues convincingly for the necessity of passing to a more complex temporal-*epistemic* account of minimization, involving both courses of events and our knowledge or ignorance about them, illustrated by the famous 'Yale Shooting Problem'. Models must now involve parallel temporal structures across different epistemic alternatives (cf. the discussion of 'branching time' in Section III.4), which allow one to render maximal ignorance at some point in time on some history by having as many epistemic alternatives as possible continuing from there. A full treatment goes beyond the confines of this Survey.

V.4 Partial Models and Information

An emphasis on processing of information presupposes an interest in the structure of information states on which the relevant procedures are to operate. One pervasive tendency in current semantics has been to switch to a more information-oriented conception of models for this purpose, now seen as partial records of what we know or ignore about a certain situation. In partial models, predicates will be described by a finer-grained grid of options, distinguishing between 'positive', 'negative' and 'unknown / undefined' parts. Accordingly, formulas may now become 'true', 'false' or 'undecided'. Eventually, this grid may be extended with other 'truth values' too. In particular, many authors also use a fourth 'over-defined' case, representing a state of contradictory information about a predicate. Blamey 1985 presents various philosophical, linguistical and mathematical motivations for such models. Computational motivation may be found in Fitting 1985 on logic programming, or Tan & Treur 1990 on expert systems. A general survey of partiality in Intensional Logic may be found in van Benthem 1988.

An early temporal use for partial models has been proposed in Kamp 1979, who assumes that these are the appropriate structures for recording the temporal information that comes in during the processing of natural language. One interesting result of his analysis is the following. Suppose that we have a class $\mathbb{K}$ of frames in the ordinary sense, defined by some set Σ of first-order formulas. Its obvious partial counterpart is the class $\mathbb{K}_{part}$ of all frames having disjoint partial extensions $<^+$, $<^-$ for precedence that can still be completed to a standard relation '$<$, $(T \times T) - <$' so as to end up in $\mathbb{K}$. $\mathbb{K}_{part}$ then also turns out to have a first-order definition, effectively obtainable from Σ.

Example. Axiomatizing Partial Frame Classes.

The partial counterpart of the class of transitive irreflexive frames is defined by:

$$\forall x \ \neg\, x <^- x$$

$$\forall x \forall y \forall z \ ((x <^+ y \wedge y <^+ z) \rightarrow \neg\, x <^- z).$$

Not only temporal models can be partialized: the same holds for the interpretation of temporal languages over them. By way of illustration, here is a partial variant of the basic tense logic.

Models will now assign positive extensions $V^+(p)$ as well as negative extensions $V^-(p)$ to proposition letters. For convenience, the precedence relation $<$ will remain total here (although this could be partialized as well in our treatment of temporal operators). Following this initial pattern for atoms, arbitrary formulas can then be 'verified' or 'refuted' at points in time, where these 'positive' and 'negative' notions are now treated on a par in the following simultaneous induction:

•	$\mathbb{M}, t \models^+ p$	iff	$t \in V^+(p)$
	$\mathbb{M}, t \models^- p$	iff	$t \in V^-(p)$
•	$\mathbb{M}, t \models^+ \neg\phi$	iff	$\mathbb{M}, t \models^- \phi$
•	$\mathbb{M}, t \models^+ \phi\wedge\psi$	iff	$\mathbb{M}, t \models^+ \phi$ *and* $\mathbb{M}, t \models^+ \psi$
	$\mathbb{M}, t \models^- \phi\wedge\psi$	iff	$\mathbb{M}, t \models^- \phi$ *or* $\mathbb{M}, t \models^- \psi$
•	$\mathbb{M}, t \models^+ \phi\vee\psi$	iff	$\mathbb{M}, t \models^+ \phi$ *or* $\mathbb{M}, t \models^+ \psi$
	$\mathbb{M}, t \models^- \phi\vee\psi$	iff	$\mathbb{M}, t \models^- \phi$ *and* $\mathbb{M}, t \models^- \psi$

For the temporal operators, here are two out of several possible options:

•	$\mathbb{M}, t \models^+ F\phi$	iff	*for some* $t'>t$, $\mathbb{M}, t' \models^+ \phi$
	$\mathbb{M}, t \models^- F\phi$	iff	*for each* $t'>t$, $\mathbb{M}, t' \models^- \phi$
••	$\mathbb{M}, t \models^+ F\phi$	iff	*for some* $t'>t$, $\mathbb{M}, t' \models^+ \phi$
	$\mathbb{M}, t \models^- F\phi$	iff	*for no* $t'>t$, $\mathbb{M}, t' \models^+ \phi$

The case of the past operator P is analogous.
Choosing the first alternative has the following effect:

Fact. All formulas in our language are *persistent*, in that replacing any model by a 'more informative' one (in which there are only further decisions in former 'truth value gaps' $T - (V^+(p) \cup V^-(p))$) will preserve all previous judgments $\models^+$ and $\models^-$.

Persistence is an obvious concern, reminiscent of earlier Sections. Nevertheless, what it excludes is the possibility that certain statements in the language would express a certain degree of *ignorance* rather than knowledge. The latter will be reflected by introducing another negation $\sim$, expressing 'absence of positive information':

$\mathbb{M}, t \models^+ \sim\phi$	iff	*not* $\mathbb{M}, t \models^+ \phi$
$\mathbb{M}, t \models^- \sim\phi$	iff	$\mathbb{M}, t \models^+ \phi$.

Then the non-persistent reading •• of the future operator becomes definable as $\sim\sim F\phi$.

Partial Logic has a somewhat ambivalent status, in that it can be *effectively reduced* to classical logic. For instance, if we replace all proposition letters p in the above setting by pairs p^+ , p^- , then the following translation (due to Paul Gilmore) will reduce partial evaluation of formulas ϕ in the first sense given above to standard evaluation of positive and negative counterparts $(\phi)^+$ and $(\phi)^-$:

$(p)^+$	=	p^+	$(p)^-$	=	p^-
$(\neg\phi)^+$	=	$(\phi)^-$	$(\neg\phi)^-$	=	$(\phi)^+$
$(\phi\wedge\psi)^+$	=	$(\phi)^+ \wedge (\psi)^+$	$(\phi\wedge\psi)^-$	=	$(\phi)^- \vee (\psi)^-$
$(\phi\vee\psi)^+$	=	$(\phi)^+ \vee (\psi)^+$	$(\phi\vee\psi)^-$	=	$(\phi)^- \wedge (\psi)^-$
$(F\phi)^+$	=	$F(\phi)^+$	$(F\phi)^-$	=	$G(\phi)^-$
$(P\phi)^+$	=	$P(\phi)^+$	$(P\phi)^-$	=	$H(\phi)^-$

Further details of such reductions may be found in Langholm 1988, Muskens 1989.

Nevertheless, Partial Logic raises questions of intrinsic interest too, many of them having to do with persistence of information. Van Benthem 1988, Langholm 1988, Thijsse 1990 offer various 'functional completeness theorems' for persistent operators.

V.5 Appendix: Changing Models

The preceding Subsections have by no means exhausted the dynamics inherent in current computational uses of logical semantics. For instance, one intriguing recent development has been the use of evaluation in models ('model checking') as a source of information in addition to deductive inference ('theorem proving'). Often, useful information may be stored in model diagrams as well as in premise sets, and in many cases, the former process is more effective than the latter (cf. Stirling 1990, as well as the stimulating polemics in Halpern & Vardi 1990). The general picture then becomes one which has also been advocated in the philosophical literature by authors like Jaakko Hintikka. Actual information processing does not strive for methodological purity: humans avails themselves of any technique at hand, both querying models where possible (comparable to 'experimentation' in Nature) and deducing useful consequences from higher-order information ('rules', 'laws', 'constraints'). 'Seeing' with one's physical eyes is often just as good as seeing with the mind's eye! Evidently, this kind of mixed activity cuts across most of the convenient methodological distinctions made so far.

Finally, we draw attention to yet another aspect of 'dynamics' arising in practice, which has remained an undercurrent in our survey so far. In standard semantics of formal or natural languages, and also that of many programming languages, there is an exclusive emphasis on defining what it means for one single model to verify some completed piece of text. But in reality, this is only where the work begins. For instance, the 'universe of discourse' keeps changing in uses of natural language. And also, intended models keep changing as one extends a program, or embeds it into larger programming environments. Therefore, what is just as important is the global structure of the universe of models. Which natural forms of 'extension', 'contraction' or 'representation' connect different models for the same language, or related languages, and how is truth or falsity of their assertions affected from one to another? This theme has emerged at various places in our exposition. It occurred in the Preservation theorems in Section II, but also in the above notion of Persistence, or in the possible transfer of statements across different 'views' of a system in temporal Representation (Section IV). Nevertheless, it seems fair to say that this topic has been relatively neglected in logic so far.

Merely as a convenient focus, here is one 'meta-model' which embodies both local and global semantic perspectives. Consider the following structure $\mathbf{M}$. Its domain consists of all tense-logical models $\mathbb{M}$ of Section II, ordered by the earlier relation $\subseteq$ of model extension (allowing both new individuals and new facts). We introduce a bi-modal language, having both the earlier temporal operators F , P as well as modalities $\Diamond^{up}$ and $\Diamond_{down}$, interpreted as follows in models at some temporal point occurring in them:

- $\mathbf{M} \models p\ [\mathbb{M}, t]$ iff $\mathbb{M} \models p\,[t]$
- Boolean connectives are interpreted as usual
- $\mathbf{M} \models F\phi\ [\mathbb{M}, t]$ iff $\mathbf{M} \models \phi\ [\mathbb{M}, t']$ *for some point* $t'>t$ *in* $\mathbb{M}$,
- the past temporality is interpreted likewise
- $\mathbf{M} \models \Diamond^{up}\phi\ [\mathbb{M}, t]$ iff $\mathbf{M} \models \phi\ [\mathbb{M}', t]$ *for some model extension* $\mathbb{M}' \supseteq \mathbb{M}$
- the downward modality is interpreted likewise.

Note the obvious formal similarity with interpretation in the branching temporal structures of Section III . The modal logic of this structure has a certain independent interest. For instance, restricting attention to the extension pattern only, we have an S4.2 modal logic for $\Diamond^{up}$ and an S4.1 logic for $\Diamond^{down}$. So far, there are no known complete axiomatizations for such natural meta-models:

Question. What is the complete bi-modal logic of $\mathbf{M}$?

Moreover, in the interaction between the modalities and the temporal operators, many properties of model extension and persistence are reflected. For instance, the following principle of 'confluence' is valid in $\mathbf{M}$:

$$(Fq \wedge Fr) \rightarrow \Diamond^{up} F\,(Pq \wedge Pr)\ .$$

Question. To axiomatize this modal logic of temporal persistence completely.

VI Richer Temporal Frameworks

The standard languages of temporal logic have their own habitual design, with its inevitable blind spots. Therefore, independent sources of temporal expression are worth attention too, of which we shall consider two.

- A Linguistic Perspective

Independent intuitions abound in the temporal system of *natural languages*. In the past decade, interesting logical systems have appeared taking more cues from the latter field.

One noticeable example is the *aspectual calculus* of Galton 1984. The ontological picture behind natural language is lush, unlike the Spartan spirit behind most logical formalisms. We are living in a rich common sense world populated not just by individuals and events, but also 'processes', 'states', etcetera. In Galton's formalism, *states* and *events* appear on a par as basic temporal entities. In the resulting two-level system, the earlier tenses F , P become operators from states to states, whereas the Progressive (PROG) as well as the Perfect (PERF) change events into states. But there are also operators changing states into events, such as INGR ("begin to") or PO ("spend a while"). Galton proceeds axiomatically, stating a number of plausible intuitions. A fair sample is the following:

$$\begin{array}{lcl} \text{PERF INGR } q & \leftrightarrow & (P(P\neg q \wedge q) \vee (P\neg q \wedge q)) \\ \text{PROG PO } q & \leftrightarrow & (P\neg q \wedge q \wedge F\neg q) \\ \text{PERF } q & \rightarrow & G \text{ PERF } q\ . \end{array}$$

In subsequent work, a matching model-theoretic semantics has been provided as well.

Another noteworthy, linguistically more faithful temporal calculus is the system of *temporal discourse representation* found in Kamp 1979, Hinrichs 1981, Kamp & Rohrer 1988. Here a special representation formalism serves as an intermediary between linguistic texts and eventual real situations where these texts can be true or false. Essentially, discourse representations are annotated patterns of atomic statements concerning events, processes or states, which contain partial information about their temporal relationships (as well as certain anaphoric connections between them). Such relationships enter the picture gradually when a discourse or text is being processed: either implicitly, as a side-effect of certain tenses which drive the narrative forward, or explicitly, following instructions embodied in such temporal connectives as "before", "since", "during" .

One virtue of the latter approach is that it explains certain vexing linguistic subtleties. Sometimes, a natural language contains two tenses without any difference in temporal meaning, which give different instructions for viewing a situation. For instance, French 'imparfait' is a past tense presenting events as open intervals, during which many other things can happen: "elle chanteait 'La Vie en Rose' " . By contrast, the 'passé simple' may present that same past event as an indivisible whole, serving as a marker around which other events may be grouped: "elle chanta 'La Vie en Rose' " . Here, the difference cannot be that the latter process is instantaneous, whereas the former is not: we are referring to the same event, which takes an extended interval of real physical time. But, the two instructions for organizing further temporal information around it are different, so to speak. And similar remarks can be made about linguistic 'aspect', reflecting temporal constitution of events. The same event may be described as an 'activity' which takes time, or as an 'achievement', which is construed as being indivisible. Technically, the latter option makes the events described atomic intervals in the discourse representation: even though these may correspond eventually to extended point set intervals when the latter is represented in actual time. Such observations may provide cues as to the actual mechanisms that humans use in processing temporal information, where these distinctions probably carry some computational advantage. Nevertheless, actual human temporal interpretation remains largely mysterious at present. For instance, quite closely related languages like English and German already show considerable divergences in their tense system. So, at the surface, the linguistic temporal system is quite 'vulnerable' to historical accidents, and the art is to locate its deeper 'invariants'. More computational research in the field is currently directed toward defining direct inferential algorithms on discourse representations, that are expected to be more efficient than those obtained via a detour through translation into standard logical formalisms (cf. Kamp & Reyle 1990). Another promising linguistic-computational approach to aspect in natural languages is the dynamic calculus of aspectual change developed in Moens & Steedman 1988.

Next, a relatively theory-free look at the actual system of temporal expression in natural language will show how this differs from the temporal logics developed so far.

- The first major temporal construction are *tenses*, such as past, present or future. These position events in time. The same role is played by certain temporal *auxiliaries*, such as "have" or "will". Thus, there is a system for ordering events around the present moment, as well as among themselves (with iterated tenses like "would" or "will have").

• Another kind of positioning is performed by temporal *connectives*, such as "since", "until", "before", "after", "during", "while". Note that these can relate various categories of expression, including both sentences and noun phrases:

"after [Belinda broke up with him], George was not the same any more",

"after [the Clash], George was not the same any more".

• Specific temporal locations may be named by temporal *indexicals* or 'pronouns', such as "now", "then" , which can be used to refer deictically to temporal points (though variable across contexts). Grammatically, these fall into the broader class of temporal *adverbials*, which also include quite different forms, such as specific dates ("on the first of May"), as well as others yet to be mentioned.

• In addition to 'positioning', there is also temporal 'constitution': what are the special denotational properties of events described? This is the area of *aspect*, with such possibilities as 'terminative', 'inchoative' or 'iterative' aspect. Aspect can be triggered by lexical properties of certain verbs, but it can also be influenced by the earlier temporal operators (like the Progressive) and even by non-temporal linguistic constructions, such as putting in a direct object for a transitive verb, or applying a negation (see Verkuyl 1989). Note how various temporal adverbs supply aspectual information: for instance, concerning duration ("for an hour", "within a week"). Incidentally, linguistic theorizing about the semantics of aspect has led to a plea for what has been called 'natural language metaphysics' in Bach 1986, an enterprise which is quite close in spirit to the earlier-mentioned Common Sense Physics program in Artificial Intelligence.

• Finally, there exists an elaborate *quantificational* system over temporal entities, with expressions such as "always", "often", "sometimes", "once", "never", which shows striking analogies with the logical theory of Generalized Quantifiers (van Benthem 1986). In connection with the latter, it should be observed that temporal quantification over 'times' often refers to a more general notion, namely that of 'case' or 'occasion', rather than temporal points as such, witness Lewis 1975. And also, other distinctions occur that are known as well with quantification in general, such as the difference between 'cardinal' and 'ordinal' ways of counting. An example of the latter kind occurs in a sentence like: "He made a mistake with every second key-stroke". A thorough survey of this technical perspective may be found in de Swart 1991.

In our earlier logical systems, the cake was cut somewhat differently. For instance, many of the above distinctions are collapsed in the temporal predicate logic of Section II . The latter's one-place operators on propositions include tenses, aspectual operators, some temporal quantifiers, and its two-place operators include various temporal connectives.

Moreover, aspectual information will be available in certain special structural properties of point sets defined by formulas of the language, such as convexity, or other semantic forms of temporal closure.

Some essential linguistic phenomena still escape from this analysis. A prominent example is the *deictic* character of tenses, which may introduce specific reference points for further discourse. This has led some logicians to devise completely new temporal systems, e.g., in the tradition of 'Situation Semantics' (see Barwise & Perry 1983). Some deictic features of tenses and temporal quantification may actually be brought out with our standard tense-logical formalism too, under the dynamic mode of interpretation sketched in Section V . Then, e.g., the various linguistic analogies between temporal quantification and ordinary quantification over individual objects pointed out in Partee 1984 can be exploited to advantage (cf. Dekker 1990). All this is not restricted to point-based first-order formalisms. For instance, one can do a similar kind of deictic or dynamic analysis in the first-order interval languages of Section IV . For a sample treatment of tense in terms of an interval logic allowing deixis, see Fenstad et al. 1987.

Another phenomenon beyond the standard approach is the *non-first-order* character of temporal quantifiers like "mostly", "usually", "often" or "seldom". (These do remain, however, within the scope of the earlier-mentioned logical theory of Generalized Quantifiers, which is not essentially first-order.)

Our suggestion so far has been that analogies from natural language might become useful in the design of temporal *representations*. But current linguistic research may also provide interesting cues as to mechanisms of temporal *inference*, withness the various calculi of 'natural logic' close to linguistic forms in Kamp&Reyle 1990, Sánchez Valencia 1990, which might also be specialized to the more modest linguistic temporal 'subroutines' advocated in Guenthner 1989.

- <u>A Mathematical Perspective</u>

Some of the preceding observations suggest another, more *language-independent* view of temporal operators. For instance, generalized quantifiers are also mathematical objects in their own right, which do not need lexicalization in actual human languages for their existence. And a similar mathematical perspective can be taken toward temporal operators.

Consider any frame $(T, <)$. Temporal propositions correspond to subsets of T , and hence temporal operators may be seen as *unary operations* on the power set of T .

Not all a priori possibilities are plausible, however: certain 'temporal constraints' must be obeyed. But this can be done without specifying any formal language. For a start, it may be demanded that genuine temporal operators f be insensitive to 'imperceptible shifts' of the temporal order:

Automorphism Invariance

$\pi[f(A)] = f(\pi[A])$ for all $A \subseteq T$ and all $<$-automorphisms π of $(T, <)$.

This induces a certain uniformity. For instance, on the real number line $\mathbf{R}$, f-images of singleton propositions $\{t\}$ will arise through one uniform choice of a union of the three 'regions' $\{ t \}, \{ t' \mid t'<t \}, \{ t' \mid t<t' \}$.

Next, further constraints are possible too. For instance, a strong requirement of 'local computability' is expressed in the following notion of

Continuity

f commutes with arbitrary unions of all its arguments.

For such temporal operators, computation of a value $f(A)$ amounts to taking the union of all values at singletons $f(\{t\})$ with $t \in A$. Together, the two requirements so far are in fact characteristic for the basic Priorean operators (van Benthem 1986B):

Fact. On the reals, the automorphism-invariant continuous temporal operators are precisely the Priorean tenses F, P plus all disjunctions of these.

A natural invariant non-continuous tense is the earlier Progressive. Formation of interiors does not commute with arbitrary unions: witness the case of $\mathrm{int}((0, 1)) = (0, 1) \neq \bigcup\{\mathrm{int}(\{x\}) \mid 0<x<1\} = \emptyset$. Nevertheless, the latter is still 'weakly continuous' in the sense of computing its values on the basis of its values on all *convex subintervals* of A. Such operators remain at least *monotonic* with respect to set inclusion of their arguments. Thus, a hierarchy of possible semantic behaviour arises for unary temporal operators, that can be studied as such, perhaps asking for possible lexicalization of attractive candidates.

Moreover, a similar analysis is possible for other types of temporal operator. An example are two-place predicates R between temporal propositions, such as the above quantifiers, which are usually contextually restricted to some domain: "always *over* *period* A , B occurred". Another example are the earlier temporal connectives, such as "while", "during", "before". Here again, automorphism invariance makes sense:

$$R(A, B) \quad \text{iff} \quad R(\pi[A], \pi[B]) \qquad \text{for all } A, B \subseteq T,$$

and so do the earlier notions of continuity or monotonicity. And then, all automorphism-invariant continuous candidates on the reals can be finitely classified just like before.

Once again, the above non-first-order temporal quantifiers fall within this general scheme just as well as first-order ones. This suggests that temporal logic may also be developed from a more mathematical point of view as a theory of all possible temporal operators, disregarding the question whether these are definable in some particular logical language. So far, this perspective has remained a marginal one in the temporal literature.

VII Temporal Predicate Logic

VII.1 <u>Designing the System</u>

In the preceding Sections, temporal reasoning was studied on the basis of a relatively poor propositional formalism, as the main intention was to focus on temporal operators as such. Nevertheless, from a logical point of view, it is very natural to consider the addition of temporal operators to a richer predicate logic too. We shall start from some standard predicate-logical language here, and add two operators F and P as before, together with their defined universal duals G and H . Then, a number of new issues arises. In the syntax of the language, temporal operators and individual quantifiers start interacting, and one has to think about possible differences in meaning between such combinations as "everyone will always be foolish" and "always, everyone will be foolish" . Indeed, doubts have been voiced in the philosophical literature as to whether the new formalism admits of any coherent interpretation at all. Here is an example, adapted from Quine 1947, which, though not too serious in any deep sense, does emphasize the need for precision:

> "Mathematicians are always rational, but not always bipeds.
> Cyclists are not always rational, although they are always bipeds.
> Now, consider the mathematical cyclist Paul K. Zwier:
>> He is a mathematician, and therefore always rational,
>> but he is also a cyclist, and therefore, not always rational.
>
> And his legs show a contradictory behaviour similar to that of his mind ..."

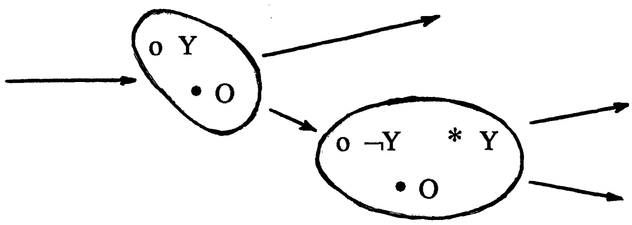
A systematic account of this puzzle, as well as the underlying language in general, presupposes various decisions as to the appropriate semantic structures for this language. In fact, this Chapter will be more discursive than the preceding ones, as the field abounds in options, among which no consensus has been reached yet.

We start with what is probably the most widespread approach.

Definition. Models for Temporal Predicate Logic.
Models are structures $\mathbb{M} = (T, <, \{D_t \mid t \in T\}, \{V_t \mid t \in T\})$ of temporal frames $(T, <)$ with a family of non-empty 'domains' of individuals D_t for each point in time (the individuals that 'exist' at t), as well as 'valuations' V_t interpreting all predicate and function symbols of the language at each point t in T .

This gives a pattern of ordinary predicate-logical structures ordered by temporal precedence, which may be viewed as instantaneous 'snap-shots' of the Universe, with possible repetitions of the same snap-shot occurring at different times:

o has aged
• has remained old
a youngster * arrived



Next, there is the matter of a suitable *truth definition.* This time, formulas ϕ will be interpreted at points t, as before, but now, also with an 'assignment' A from their (free) individual variables into the objects existing at t:

$$\mathbb{M}, t \models \phi\ [A]\ .$$

Here, all non-temporal clauses in the induction are as usual, with atoms referring to the facts as provided by V_t. In particular, one stipulates that individual quantifiers are to range over the domain at t:

$\mathbb{M}, t \models \exists x\, \phi\, [A]$ iff *there exists some* $d \in D_t$ *with* $\mathbb{M}, t \models \phi\ [A^x{}_d]$.

But, the case of the temporal operators calls for further decisions. The problem is that domains may change from one point in time to another, so that deciding whether, e.g., $\mathbb{M}, t \models G\, Rxy\, [A]$, may call for evaluation of Rxy with respect to A at other times than t : when $A(x), A(y)$ need not exist at all. Different authors have defended different options here. For instance, one may decide to evaluate all atomic statements at all times, calling atoms involving non-existent objects false (as advocated in Kripke 1963) or 'undefined' (as Hughes & Cresswell 1968 have done). We choose another road here:

Let ϕ have the free variables $x_1, ..., x_n$:
$\mathbb{M}, t \models F\phi\, [A]$ iff *there exists some* $t'>t$ *having all* $A(x_i)$ *in its domain* $D_{t'}$ $(1 \leq i \leq n)$ *with* $\mathbb{M}, t' \models \phi\, [A]$.
And likewise for the past tense $P\phi$.

Thus, the universal duals will only refer to future or past times when the objects involved are actually present. This is the sense in which the sentence "Napoleon always insisted on speed" only says that Napoleon was a devoted mover in his day.

This is not the only plausible policy. An elegant alternative would in fact be to have just one universal domain for all points in time, and enrich the language with an explicit *existence predicate* E, true at each point t of just the individuals in D_t. Then, evaluation would be completely straightforward, and the above readings of formulas like $F\,\phi(x)$, $G\,\phi(x)$ would rather be expressed by the restricted forms $F\,(Ex \wedge \phi(x))$, $G\,(Ex \rightarrow \phi(x))$.

With this semantics, one can look more systematically at combinations of temporal operators and quantifiers like those in Quine's example, noting analogies and differences between such patterns of expression as:

1 $\quad G\,\forall x(Mx \rightarrow Rx)$

2 $\quad \forall x\,G\,(Mx \rightarrow Rx)$

3 $\quad \forall x\,(Mx \rightarrow G\,Rx)$.

Here, formula 1 implies 2, but not conversely; while both are independent from 3 : which may be shown using pictures like the above. The first assertion is sometimes called 'de dicto' : it makes a general temporal claim about a closed statement, whereas the other two are 'de re' : they ascribe general temporal properties to individuals.

Actually, this is not yet the end of semantic options in this field. What has been made time-dependent so far are predicates and functions. Individuals may acquire or lose properties across different t in T (think of "being young"), and functions may come to produce different values (think of being "the president of the Soviet Union"). But the underlying individuals themselves have remained the same across points in time. In other versions of the semantics, however, individuals across different worlds can at best be 'counterparts' of each other (Lewis 1968). And in Hintikka 1969, Kripke 1963, individuals even become higher-order time-dependent entities, namely, *functions* from points in time to objects, rather like 'world-lines' assuming different denotations at different t in T. (Compare also the related use of 'individual concepts' in Montague 1974, where two such higher individuals can be the same for a while, and then fork out on their own.) The latter design for temporal predicate logic has some complexities (cf. Garson 1984), whence we shall not pursue it here. But related ideas will return for technical reasons at the end of this Section.

Finally, going back to the earlier Section IV, from an uncompromising philosophical point of view, one might even claim that individuals do not belong to the basic ontological furniture of the World at all. The primary data for humans are events, and individuals only arise as constructions out of these, when there is enough 'invariance' across a number of different experiences (cf. Barwise & Perry 1983, Seligman 1990). In that case again, our temporal semantics as it stands would be much too naive.

Thus, 'temporal predicate logic' is really a name for a diverse field, with many possible formal systems, whose relative merits may depend on considerations of philosophical intuition, mathematical elegance or computational utility.

VII.2 Exploring the Logic

- Axiomatics

The minimal calculus of deduction validated by the above models might be expected to look as follows:

- All valid principles from standard predicate logic
- All principles of the earlier minimal tense logic.

As it stands, however, this will not be the case.

Problem 1. Temporal Distribution.
On the above semantics, not all instances of the temporal distribution axiom are valid. For instances, there are easy counter-examples for

$$(G\,Px \wedge G\,(Px \rightarrow Qy)) \rightarrow G\,Qy,$$

due to the above stipulations about occurrence of assignment values for variables. (Think of a case where the objects $A(x)$, $A(y)$ never occur together in any domain D_t, so that $G\,(Px \rightarrow Qy)$ might be true for trivial reasons.) The standard remedy here has been to impose a further restriction on models, saying that no individuals get lost with the passage of time (that is, they remain 'present', though not necessarily 'alive' – much like Chinese ancestors in the family home):

$$\textit{for all}\ \ t, t' \in T,\ \ t < t' \rightarrow D_t \subseteq D_{t'} \qquad \textit{Domain Cumulation}.$$

This will also validate interchange principles for temporal operators and quantifiers like

$$\exists x\ G\ Px \rightarrow G\ \exists xPx \qquad \text{or} \qquad G\ \forall xPx \rightarrow \forall x\ G\ Px\ .$$

Note that these are also reasonable from a proof-theoretic point of view, witness the following derivation in the minimal axiomatic system:

$Px \rightarrow \exists xPx$	(ordinary predicate logic)
$G\ (Px \rightarrow \exists xPx)$	(Temporalization)
$G\ Px \rightarrow G\ \exists xPx$	(Distribution!)
$\exists x\ G\ Px \rightarrow G\ \exists xPx$	(ordinary predicate logic).

Problem 2. Individual Constants and Function Symbols.
A second problem is more insidious. In predicate logic, there is an evident principle of Existential Generalization, being

$$[t/x]\phi \rightarrow \exists x\phi \qquad \text{if } t \text{ is free for } x \text{ in } \phi\ .$$

But the presence of even just individual constants will spoil this for temporal predicate logic. For instance, the formula $G\ La \rightarrow \exists x\ G\ Lx$ is not universally valid, because the fact that "the loser will always lose" does not imply that there is any particular individual in our current sense which will always be the loser. This time, there is no easy remedy via constraints on domains: suitable *syntactic restrictions* are to be imposed on the principle of Existential Generalization, excluding its use for individual constants (or more generally, terms with function symbols) inside the scope of temporal operators.

With these provisos, at last, there is a general Completeness result for the modified axiomatic system (cf. Hughes & Cressswell 1968, Garson 1984):

Theorem. Provability in the modified minimal temporal predicate logic and universal validity in models satisfying domain cumulation are equivalent.

Next, further temporal predicate logics may be found on top of this. Many of these arise by merely taking any propositional tense logic L and then forming its minimal predicate-logical version 'LQ' by reading all its propositional axioms as general schemata for substitution, on top of the minimal apparatus given here. But other systems are sui generis, involving axioms without any propositional counterpart. A well-known example is the minimal logic plus the so-called *Barcan Axiom*

$$\forall x\, G\, Px \rightarrow G\, \forall x Px \ ,$$

which turns out to be complete with respect to the class of all models having one constant domain for all points in time.

Problem 3. Persistence of Identity and Distinctness.
Decisions taken here as to the nature of our individuals are reflected in the behaviour of *identity* in our language. On the standard object semantics, both of the following become universally valid:

$$\forall x \forall y\ (x{=}y \rightarrow G\, x{=}y) \qquad\qquad \forall x \forall y\ (\neg x{=}y \rightarrow G\, \neg x{=}y)\ .$$

This looks rather strong, but rejecting this tends to require a shift to the earlier higher-order function-like individuals, which seems a high price to pay. We shall not survey all possible ways-out of this predicament here.

- <u>Model Theory</u>

Given this perhaps irreducible conceptual diversity in setting up a proper semantics for temporal predicate logic, there is a special need for technical logical theory, in order to chart and understand the available options more systematically. In particular, some obvious extensions exist of the earlier *model-theoretic* themes. For instance, again, basic *invariances* may be defined for the language of temporal predicate logic: this time, via a combination of the temporal zigzags of Section II and predicate-logical Ehrenfeucht Games (cf. Doets 1987).

In the background of such notions, there is again a *translation* from temporal predicate logic into a more standard 'two-sorted' predicate logic, employing separate kinds of variable for temporal points and for individual objects. In general, k-place predicates of individuals become (k+1)-place predicates under the translation, having acquired one temporal variable. Moreover, in addition to 'precedence' $<$ between points, there is one new distinguished cross-sortal predicate of 'local existence':

Ext $\qquad$ x belongs to D_t .

For instance, the above Barcan Axiom will then read as follows:

$$\forall x \, G \, Px \rightarrow G \, \forall x Px :$$
$$\forall x \, (Ext_0 \rightarrow \forall t' \, ((t_0 < t' \wedge Ext') \rightarrow Pxt')) \rightarrow \forall t' \, (t_0 < t' \rightarrow \forall x \, (Ext' \rightarrow Pxt')) \, .$$

Moreover, as in Section II, a second-order version of the translation exists too, universally quantifying over all predicates except $<$ and E, so as to express properties of 'inhabited temporal frames'.

Example. Frame Correspondence.
On frames, the Barcan Axiom $\forall x \, G \, Px \rightarrow G \, \forall x Px$ corresponds to the first-order condition of 'domain inclusion':

$$\forall t' \, (t_0 < t' \rightarrow \forall x \, (Ext' \rightarrow Ext_0)) \, .$$

Its converse $G \, \forall x Px \rightarrow \forall x \, G \, Px$ defines the reverse inclusion, being the local version of the earlier 'domain cumulation'. An example of a non-first-order principle on frames is the interchange law $G \, \exists x Px \rightarrow \exists x \, G \, Px$ (van Benthem 1985).

The technical theory of temporal predicate logic has been investigated much more superficially than its propositional counterpart. (A laudable exception is Fine 1978.) Partly, this may have been due to a feeling that no new genuine temporal discoveries were awaiting discovery here. Partly also, the field seemed a mere pathological source of negative discoveries, such as failure of classical features like Interpolation. Perhaps, this reflects a certain 'instability' of our model theory so far, dissatisfaction with which has generated some promising recent developments in the area:

VII.3 Incompleteness and Functional Modelling

Recently, it has become clear that the above, apparently modest, framework for temporal predicate logic is seriously 'incomplete', showing a mismatch between reasonable systems of deduction and semantic validity: the former are usually too weak to produce all valid inferences. This discovery has produced some interesting new modellings as a positive side-effect.

- Incompleteness Results

Part of the trouble has to do with the earlier direction in axiomatic completeness theory 'from frames to logics'. For instance, there is already an old result, due independently to Per Lindström and to Dana Scott in the sixties, which says that frames with an effectively axiomatizable propositional temporal logic may change their behaviour drastically in the predicate-logical case:

Theorem. The full temporal predicate logic over the integers or the reals (with arbitrary domains attached at each point) is non-axiomatizable.

In fact, True Arithmetic can be effectively embedded into either logic. The reason is that the standard temporal order in the frame can be exploited to enforce standard interpretations for the basic arithmetical operations, via some suitable coding. This incompleteness phenomenon may be understood through the above frame translation. Essentially, the temporal predicate logic of a frame class is a fragment, no longer of the latter's *monadic* second-order logic as in Section II.7, but of its *polyadic* second-order logic, and as we have noted, the complexity of the latter system is generally much higher than that of the former.

There are problems too in the other direction for axiomatic completeness results, going 'from logics to frames'. To be sure, some positive results do exist here, concerning such natural modal predicate logics as S4Q (S5Q) defined as above, which are complete with respect to inhabited pre-orders (inhabited equivalence relations). Indeed, the following conjecture seems plausible (Hughes & Cresswell 1984):

> "Whenever a propositional tense logic L is complete with respect to a certain class of frames, its canonical predicate-logical version LQ as defined above will be complete with respect to some class of inhabited frames."

But, this statement turns out to be false (cf. Ono 1983). The resulting situation has been analyzed more generally in Shehtman & Skvortsov 1988, Ghilardi 1989. We state a striking result of Ghilardi's for modal logics, i.e., 'pure future' tense logics:

Theorem. Among the extensions of S4 , propositional logics L whose predicate companion LQ is frame complete must have either $L \supseteq S5$ or $L \subseteq S4.3$.

This excludes frame completeness for such natural modal predicate logics as S4.1Q or S4.3GrzQ. (S4.1 is complete with respect to *atomic* partial orders, S4.3Grz with respect to the class of all finite reflexive linear orders.) Van Benthem 1990A even claims that the possible 'zone of completeness' may be tightened to

' L=S4 or $S4.2 \subseteq L \subseteq S4.3$ or $S5 \subseteq L$ '.

Positive frame completeness results are known only for some of the boundaries here (Kripke, Ghilardi, Corsi and others):

Theorem. LQ is frame complete for L = S4, S4.2, S4.3, S5 and all its extensions.

These results are suggestive, though not yet definitive. For instance, it is not known to which extent the above picture changes when the predicate logics LQ are redefined so as to include the Barcan Axiom.

- Functional Models

In the analysis of these incompleteness phenomena, an interesting semantic issue arises. 'Incompleteness' means that there are many frame consequences of logics LQ which are not axiomatically derivable from them. So, the question becomes to design a more general semantics demonstrating this underivability. Here is one elegant proposal emerging from the above work.

Definition. Functional Models.

Functional frames are couples (D, F), where D is a family $\{ D_t \mid t \in T \}$ of domains, and F a family of maps between such domains. These become *functional models* upon addition of a valuation function V as before, interpreting all predicate and function symbols of the language at each $t \in T$. The *truth definition* then describes the notion of a formula being true at a point in a model under a certain assignment: $\mathbb{M}, t \models \phi\,[A]$. Its clauses are as above for atoms, as well as propositional connectives and quantifiers. But for the temporal operators, we now set:

$\mathbb{M}, t \models G\phi\,[A]$	iff	*for all maps* f *with domain* D_t *and range* $D_{t'}$: $\mathbb{M}, t' \models \phi\,[f \circ A]$,
$\mathbb{M}, t \models H\phi\,[A]$	iff	*for all maps* f *with range* D_t *and domain* $D_{t'}$, *and all assignments* A' *such that* $f \circ A' = A$: $\mathbb{M}, t' \models \phi\,[A']$.

Thus, there is a universal quantification here over further temporal points, as well as possibly different 'manifestations' of individuals. Note that the original models for temporal predicate logic represent the special case where all maps involved are identity functions.

Example. Additional Temporal Situations.

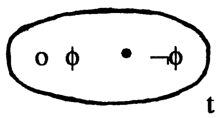
The difference with the earlier semantics shows already with one-point models $\{t\}$ having a family of maps from D_t to D_t. From a propositional point of view, these are just one-point reflexive frames, which would typically validate a formula like

$$\phi \rightarrow G\phi \ .$$

On the earlier predicate-logical semantics, one can only attach some individual domain D_t here: whence also the schematic formula

$$\phi\,(x_1, \ldots, x_n) \rightarrow G\,\phi\,(x_1, \ldots, x_n)$$

holds for all values of $x_1, \ldots, x_n$. But, with the addition of different maps from D_t to D_t here, the latter schema will no longer be valid. To see this, just consider the case of two individuals, one of them having a property ϕ and the other without it, in the presence of not only the identity map but also a map identifying them:



$f_1 = \{<o, o>, <\bullet, \bullet>\}$ $\qquad$ $f_2 = \{<o, \bullet>, <\bullet, \bullet>\}$

A simple inspection establishes that the new broader model class is still sound for our basic axiomatic calculi:

Proposition. All principles of the minimal temporal predicate logic are sound under interpretation in functional models.

What is still lacking so far are significant converse results in the area, namely completeness theorems for natural classes of functional frames.

Remark. Predecessors.

The above semantics is not entirely without precedent in the history of temporal logic. For instance, the view of 'individuals' emerging here seems rather analogous to the earlier-mentioned picture of 'world lines', for which various incompleteness phenomena had already been noted in the sixties by Kripke (cf. Garson 1984). Moreover, there has been a persistent folklore idea, even in the literature on propositional modal logic, that accessibility between worlds might also be induced by *mappings* from one to the other. 'Frames' will then be pairs (T, F) with F a set of functions on T so that a formula $G\phi$ may be called true at t if and only if ϕ is true at all worlds $f(t)$ (for $f \in F$). (See, for instance, Auffray 1989 or Anonymous 1989.) Then, well-known modal logics can be modelled by imposing natural mathematical restrictions on the set of mappings. Notably,

S4 demands that they be closed under *composition*

S5 demands that they be also closed under *inverses*.

Remark. Competitors.

There are also other interesting modelings for temporal predicate logic that have arisen in the study of incompleteness. A case in point are the 'Kripke bundles' of Shehtman & Skvortsov 1988. In the simplest formulation, these may be related to our original standard models by introducing a binary relation of 'similarity' on individuals, both across and inside worlds, and then explaining statements $F\phi$ of temporal possibility concerning certain individual objects by means of statements ϕ about suitable tuples of counterparts. This modelling, reminiscent of the 'counterpart theory' of Lewis 1968, may also be studied by ordinary semantic means, including frame correspondences (cf. Cepparello 1991).

Functional models are not mysterious entities: they may be investigated by the model-theoretic techniques from earlier Sections. In particular, 'frame correspondences' make sense again for various proposed temporal principles over them: seeing precisely what conditions are imposed on temporal patterns carrying families of mappings.

Example. Frame Correspondence for Basic Axioms.

Here are two well-known modal axioms. A functional frame validates the characteristic T axiom $\forall x\,(GAx \rightarrow Ax)$ under all valuations if and only if for each $t \in T$ and each $d \in D_t$, there exists some map f having domain and range both equal to D_t such that

$f(d) = d$. Thus, what is expressed is the existence of *local* identity maps. A similar phenomenon occurs with the schematic form of the S4 axiom $\forall x\ (GAx \rightarrow GGAx)$, which only requires 'local composition'. ('Global' versions of these structural conditions, such as the existence of identity maps or of composition functors, will arise only if we impose some uniformity condition, stating that 'local glueing' of mappings in the model leads to new maps still inside it.)

Two further examples concern earlier interchanges between temporality and quantification. The formula $G\ \forall x\ \phi \rightarrow \forall x\ G\ \phi$ is universally valid, and hence it defines no special condition on functional frames. But on the other hand, the Barcan Axiom again defines a form of 'reverse inclusion'. Frame validity of $\forall x\ G\ \phi \rightarrow G\ \forall x$ expresses that, at each point t, $\forall f : D_t \rightarrow D_{t'}\ \forall d \in D_{t'}\ \exists d' \in D_t\ \exists g : D_t \rightarrow D_{t'}\ g(d') = d$. ◆

Our next question concerns the relation between functional models and the original semantics of Section VII.1. The above one-point example showed that the propositional tense logic of a frame and its predicate-logical version, with individual domains added, need not have an obvious relation. Even so, a slightly more sophisticated reduction exists:

> Start from any functional model $\mathbb{M}$. Define a temporal frame $\mathbf{F}(\mathbb{M}) = (T, <)$ whose points are mappings in $\mathbb{M}$, ordered by the following relation:
>
> $f < g$ iff there exists some map h such that $g = h \circ f$.

Then, the following reduction is possible:

Fact. If $\mathbf{F}(\mathbb{M}) \models \phi$ for some propositional temporal formula ϕ,
then $\mathbb{M} \models \sigma(\phi)$ for every predicate-logical substitution instance $\sigma(\phi)$ of ϕ.

Proof. For convenience, here is the case of pure future formulas only. Suppose that $\mathbb{M}, t \not\models \sigma(\phi)\ [A]$ at some point t. Then define a valuation V on $\mathbf{F}(\mathbb{M})$ by setting

$$V(p) = \{ f \mid \mathbb{M}, t' \models \sigma(\pi)\ [f \circ A], \text{ where } f: D_t \rightarrow D_{t'} \}.$$

An easy induction then shows that, for all maps $f: D_t \rightarrow D_{t'}$:

$$((\mathbf{F}(\mathbb{M}), V), f) \models \psi \quad \text{iff} \quad \mathbb{M}, t' \models \sigma(\psi)\ [f \circ A].$$

And therefore, $\mathbf{F}(\mathbb{M}), id_t \not\models \phi$. ◆

Ghilardi 1989 also proves the converse of this observation.

We conclude with a more concrete illustration of the use of functional models.

Digression. Arguing With Functional Models.
As an illustration of the apparatus developed so far, here is a proof sketch for a slightly weaker version of Ghilardi's result mentioned above, whose main ideas may be of broader semantic interest.

Theorem. Among the extensions of S4, propositional tense logics L whose LQ is frame complete must have either $L \supseteq S5$ or $L \subseteq S4.3Grz$.

Proof. Let LQ be any frame complete modal predicate logic, with L extending S4. Now, consider the formula ϕ =

$$G\,\forall x\,(Ax \rightarrow GAx) \rightarrow F\,\forall x\,(FAx \rightarrow Ax)\,.$$

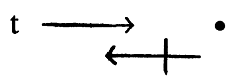
The reason for this particular choice will become clear in what follows.

Case 1. $LQ \nvdash \phi$.
Then, by frame completeness, there exists some LQ-frame $\mathbb{F}$ where ϕ fails. Now, starting from some point t where $G\,\forall x\,(Ax \rightarrow GAx) \wedge G\,\exists x(\neg Ax \wedge FAx)$ holds, such a frame will have an infinite strictly ascending chain of points. Consider the generated subframe $\mathbb{F}[t]$ with origin t: the logic L will still hold in it. Moreover, using the chain for 'measurement', the latter frame can be mapped by a zigzag morphism onto any finite linear order. Hence, by the p-Morphism Lemma, L is valid over all finite linear orders, and by the frame completeness of S4.3Grz for the latter class: $L \subseteq S4.3Grz$.

Case 2. $LQ \vdash \phi$.
In this case, we must have $L \supseteq S5$, because it may be shown that $L \vdash FGp \rightarrow p$. Thus, L being frame complete as a propositional temporal logic, it suffices to show the following assertion:

Claim. Every frame for L is symmetric.

To prove this assertion, suppose differently. Then some L-frame $\mathbb{F}$ must have a sub-situation of the following form:



Again, the logic L still holds in the generated subframe $\mathbb{F}[t]$, which latter can be mapped via a zigzag morphism onto the frame

$$\bullet \longrightarrow \bullet \qquad\qquad\qquad \mathbb{F}_2$$

So, L holds in the latter two-point frame as well.

Now comes a crucial idea. $\mathbb{F}_2$ is the frame representation $\mathbf{F}(\mathbb{M})$ of an earlier functional model $\mathbb{M}$ for temporal predicate logic:

points	$\{t\}$	
domains	$D_t = \{1, 2\}$	
functions	$\{f_1, f_2\}$	with f_1 the identity map on D_t
		and $f_2 = \{ <1, 1>, <2, 1> \}$.

By our general reduction, then, applied to the fact that $\mathbb{F}_2 \models L$, we have that :

$\mathbb{M} \models LQ$.

But, by the Soundness Theorem, this contradicts the earlier assumption that $LQ \vdash \phi$. For, $\mathbb{M}$ obviously falsifies the latter formula:

Set $V_t(A) = \{1\}$. We then have $G\,\forall x\,(Ax \to GAx)$ true at t , as no function leaves $V_t(A)$. But $F\,\forall x\,(FAx \to Ax)$ is false at t , because $\forall x\,(F\,Ax \to Ax)$ is refuted by the object 2 .

Note that the argument involves only *monadic* temporal predicate logic, having one-place predicates only. Incompleteness already strikes in the latter area.

The preceding result, and its line of argument raise many further questions. For instance, as for *completeness* by itself:

Would it be possible to *strengthen* the minimal temporal predicate logic in some principled way, so as to get a stronger canonical predicate version 'LQ' allowing for more transfer of frame completeness from the propositional case?

After all, the above negative result also excludes propositional logics L that are complete even with respect to single *finite* frames. And at least, the full temporal predicate logic of the latter kind of structure must always be effectively axiomatizable (and even *decidable*).

VIII Interactions: Temporality and Related Phenomena

What has become clear in many computational applications is that temporal logic can seldom function in isolation. It needs to be embedded in an environment of other notions, such as knowledge, action or communication, which all occur intertwined with it in the behaviour of intelligent systems (cf., e.g., Halpern & Moses 1985, Shoham 1988) . Technically, the most simple-minded approach here consists in adding a number of components into one big poly-modal logic, adding operators for modalities, actions, temporality, etcetera. In practice, however, many new questions arise precisely from the *interaction* between the various components. To conclude this Chapter, we merely identify some directions and issues here, whose more extensive investigation belongs to other parts of this Handbook.

We start with some technical observations. At first glance, systems of poly-modal logics seem an obvious sum of their parts. Properties of components may be 'pooled', and there are some interesting, but essentially straightforward phenomena of 'interaction'.

Example. Mixing Temporal Operators with Other Intensional Ones.
In temporal epistemic logic, one has possible interactions like

"If I will know that p is the case, then I know that p will be the case"

"If I know that p will be the case, then I will know that p is the case" .

On the relevant frames $\mathbb{F}$, having both a relation $<$ of temporal precedence and one of epistemic alternativeness R , the former principle expresses the following connection between the two:

$$\mathbb{F} \models FKp \rightarrow KFp \qquad \text{iff} \qquad \forall x\, \forall y\, (x<y \rightarrow \forall z\, (Rxz \rightarrow \exists u\, (z<u \wedge Ryu)))$$

This observation is in fact one instance of an obvious poly-modal version of the earlier 'Sahlqvist Theorem'. The second principle is not first-order definable, as follows from a result in van Benthem 1984 concerning the purely temporal formula $GFp \rightarrow FGp$.

But actually, poly-modal logic is a much less straightforward logical enterprise. For instance, there is a broad issue of what may be called *Transfer*:

Which known results for single-operator modal or temporal logics
generalize to poly-modal combinations?

An instance of this issue arose in the above example of frame correspondence. The Sahlqvist Theorem of Section II was originally a result concerning simple modal logic: which turns out to generalize directly to poly-modal logics. By contrast, van Benthem 1989D points at failure of transfer for an important completeness result. In modal logic, 'Bull's Theorem' says that all modal extensions of K4.3 (i.e., the complete logic of irreflexive linear frames) have the Finite Model Property, whence they are frame-complete. But, there exist frame-incomplete tense logics extending K4.3.

Also, there are interesting problems in putting together various components for a logic having desirable properties. For instance, it was shown only recently that

The obvious bimodal 'conservative sum' of two logics that are frame-complete is always a frame-complete logic again (Kracht & Wolter 1990, Goranko 1990).

This result may be extended to deal with other important semantic or axiomatic properties, such as Interpolation, Beth Definability or Finite Model Property. But there are some subtleties with complexity: notably, single modality S5 is NP-complete, but 'S5+S5' is PSPACE-complete (cf. Spaan 1991). Most of these questions are still open for poly-modal combinations satisfying additional 'interaction postulates' between the various modalities. (The case of tense logic itself is an example!) In such cases, at least *complexity* of poly-modal systems to which temporal operators are added may increase drastically, witness the tables in Halpern & Vardi 1986. Nevertheless, it seems fair to say that no satisfactory general understanding of Transfer in poly-modal logic has been achieved yet.

Finally, here are some areas of combination with particular importance. We merely list some useful references, without attempting any survey of their logical theory .

Time and Modality

Here, the setting is that of the Branching Time in Section III, namely that of branching possible histories. See Prior 1957 for an early study, van Eck 1981 for a system with many philosophical applications, Thomason 1985 for a general survey, Burgess 1984 for a number of technical results.

Time and Knowledge

Halpern & Vardi 1986 study combinations of temporal and epistemic logic (see also the survey in Halpern & Moses 1990), Shoham 1988 gives a more general, and quite influential account of changes in knowledge and ignorance over time tuned to the needs of Artificial Intelligence.

Time and Causality
Lewis 1973 is an early study of connections between causality and time-dependent conditionals. Also relevant is Gupta & Thomason 1981 on temporal conditionals. This type of interaction points back to the 'theories of events' mentioned in Section IV .

Time and Action
There is an obvious connection between action and the passage of time. Therefore, combinations of Temporal Logic and Dynamic Logic (Pratt 1976, Harel 1984) lie at hand. Indeed, the two paradigms have been regarded as competitors in some of the computational literature: should logic be about our actions which 'generate time' en route, or rather about the 'general stage' on which these take place? Galton 1987 discusses many of the important issues.

Many of these more general issues will also chime through in the various other contributions to the present Volume of this Handbook.

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