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THE CLOSED FRAGMENT OF DZHAPARIDZE'S POLYMODAL LOGIC AND THE LOGIC OF Σ_1-CONSERVATIVITY

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Abstract

Dzhaparidze's polymodal logic (referred in this paper as *GLP*) is an important joint provability logic. It corresponds to the case in which the powers of the theories grow so fast that every theory in the sequence proves everything that the previous theories prove and also proves each sentence unprovable in a previous theory to be unprovable in this theory. (Clearly, theories in such a sequence – except, possibly, the first – cannot be recursively enumerable). This logic was introduced by G.Dzhaparidze in [2], who also gave an axiomatization and a decision procedure for it. In [3] the author suggested a new approach to this logic and proved the fixed point property and the Craig interpolation property for *GLP*.

In this paper we investigate the *closed fragment* of *GLP*. As usual, there is a ordinal-indexed sequence of closed formulas (in the present case its length is ε_0) which plays the main role in our reasoning. We introduce all the standard notions connected with closed fragments (such as the *universal model*) and prove analogies of all the usual theorems. We also try to give a general approach to these standard notions, for example, to give a general definition of the *ordinal complexity* of an arbitrary modal logic. We also consider the arithmetical complexity of (arithmetical interpretation) of closed formulas.

Finally, we prove that the closed fragment of the provability logic for Σ_1-conservativity predicate is isomorphic to bimodal fragment of *GLP*. Thus, this closed fragment is decidable and its ordinal complexity is (exactly) ω^ω.

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Contents:

1. Introduction.

The notion of *ω-provability*, introduced by G.Boolos in [1] for theories in the language of arithmetic, is defined as follows: a sentence Q is ω-provable in the theory T if theory $T\cup\{\neg Q\}$ is ω-inconsistent, i.e. it simultaneously proves $\neg A(0)$, $\neg A(1)$, ..., $\neg A(\underline{n})$, ... and $\exists x\ A(x)$ for some formula $A(x)$. We obviously can work with this notion in arithmetic using a Σ_3-arithmetical formula $\mathrm{Pr}^{\omega}_{T}(\cdot)$ that formalizes the above definition, provided that we have some fixed provability predicate for theory T and it is Σ_1.

It is easy to check that this predicate satisfies all the usual conditions for the usual provability predicate, only we have to consider provable Σ_3-completeness instead of the usual provable Σ_1-completeness. Furthermore, if this predicate is Σ_3-sound, i.e. if every ω-provable Σ_3-sentence is true in the standard model, then the usual Solovay completeness proof works for ω-provability and the provability logic for predicate ω-provability (taken alone) is exactly the Gödel-Löb provability logic GL. This was also proved in [1].

In [2], G.Dzhaparidze solved the next natural problem – the joint provability logic for ordinary provability and ω-provability. Moreover, Dzhaparidze studied an infinite sequence of (iterated) ω-extensions of the given theory and found the joint provability logic for this sequence. So, this logic, which we call GLP and which is the main subject of this paper, is formulated in the modal language containing the infinite sequence of (unary) modal operators: [0], [1], [2], ..., [n], ...[1]. Sometimes we write □ instead of [0]. The logic is given by the following list of axioms:

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[1] We will be able to see soon, that it would be better to write [1], [2], ... But we have no choice.

$$[n](A\to B) \to ([n]A\to[n]B)$$
$$[n]A\to[n][n]A$$
$$[n]([n]A\to A)\to[n]A$$
$$[n]A\to[n+1]A$$
$$\langle n\rangle A\to[n+1]\langle n\rangle A$$

where the first three axioms are, of course, usual axioms of *GL* for $[n]$, and inference rules *modus ponens* ($\vdash A,\ \vdash A\to B \Rightarrow \vdash B$) and *[0]-necessitation* ($\vdash A \Rightarrow \vdash [0]A$). Of course, we can also apply $[n]$-necessitation (by the fourth axiom).

In [3] the author considered a more general arithmetical interpretation for *GLP*, based on an arbitrary sequence of arithmetical predicates which must satisfy several natural conditions. Maybe, however, a better way to deal with these matters is to speak about *theories*, i.e. about the sets of arithmetical formulas satisfying our predicates. The situation is not uncommon: we often reason in arithmetic about provability in arithmetic, although there are a lot of different (from the point of view of arithmetic) predicates representing it. So, we must always keep in mind that, when we say "theory", we mean some fixed provability predicate which has been chosen for every theory considered. Since we are going to deal with non-r.e. theories, we consider an important invariant associated with each theory, the *degree* of the theory, which is simply the arithmetical complexity of its provability predicate, i.e. the minimal number n such that this predicate is Σ_n.

It is now easy to introduce an arithmetical interpretation for *GLP*. We consider a sequence of theories $T_0,\ T_1,\dots,\ T_n,\dots$ such that $T_k \subseteq T_n$ (i.e. $\forall Q\ PA\vdash \mathrm{Pr}_{T_k}\ulcorner Q\urcorner \to \mathrm{Pr}_{T_n}\ulcorner Q\urcorner$, where *PA* is Peano Arithmetic; we may take any other theory as basic of it, which is sound and contains enough arithmetic to formalize our reasoning) and $deg(T_k)<deg(T_n)$ whenever $k<n$. This suffices for the arithmetical soundness of *GLP*. In order to get completeness, we have to require T_n to be $\Sigma_{deg(T_n)}$-sound. We call such sequences

increasing sequences.

The previous paragraph shows that we often will have to write something like Pr_{T_n} and $\Sigma_{deg(T_n)}$. Since this does not nice, we will write Pr_n and $\Sigma_{(n)}$ instead. Except in section 8, the reader can suppose that some increasing sequence has been fixed and we are considering it.

In this paper we consider the *closed fragment* of *GLP*, i.e. the modal formulas containing no propositional variables (up to *GLP*-provable equivalence). Let us consider an example here.

The well-known theorem of Kent asserts that for every n there is an arithmetical statement Q which is not *PA*-equivalent to any Σ_n-formula and such that $PA \vdash Q \rightarrow \mathrm{Pr}_{PA}\ulcorner Q \urcorner$. Proofs of this theorem usually use fixed points. But using the closed fragment of *GLP*, we can give examples of such formulas Q without using fixed points. Namely, it is enough to put $Q{:=}\mathrm{Pr}_0\ulcorner \mathrm{Pr}_1\ulcorner \bot \urcorner\urcorner \wedge \mathrm{Pr}_1\ulcorner \bot \urcorner$, where $T_0{=}PA$ and T_1 is chosen such that $deg(T_1)>n$, i.e. the arithmetical interpretation of the closed formula $\theta{:=}[1]\bot \wedge \Box[1]\bot$. Obviously, $PA \vdash Q \rightarrow \mathrm{Pr}_{PA}\ulcorner Q \urcorner$, which simply means $GLP \vdash \theta \rightarrow \Box\theta$. We claim that Q is not *PA*-equivalent to any $\Sigma_{(1)-1}(=\Sigma_{deg(T_1)-1})$-formula. Suppose not, then $\neg Q$ is *PA*-equivalent to some formula in $\Sigma_{(1)}$. By the provable $\Sigma_{(1)}$-completeness of T_1 we have $PA \vdash \neg Q \rightarrow \mathrm{Pr}_1\ulcorner \neg Q \urcorner$, or $GLP \vdash \neg\theta \rightarrow [1]\neg\theta$. We have:

$GLP \vdash \neg[1]\bot \vee \neg\Box[1]\bot \rightarrow [1](\neg[1]\bot \vee \neg\Box[1]\bot)$

(*) $GLP \vdash \neg[1]\bot \rightarrow [1](\neg[1]\bot \vee \neg\Box[1]\bot)$

$GLP \vdash [1](\neg[1]\bot \vee \neg\Box[1]\bot) \wedge \Box[1]\bot \rightarrow [1]\Box[1]\bot$

$\rightarrow [1](\neg[1]\bot)$

$\rightarrow [1]\bot$ (Löb's axiom)

$GLP \vdash \neg[1]\bot \wedge \Box[1]\bot \rightarrow [1]\bot$ (by (*))

$GLP \vdash \Box[1]\bot \rightarrow [1]\bot$

$GLP \vdash [1]\bot$ (Löb's rule)

which is obviously incorrect.[2]

It is wellknown that such sentence Q *cannot* be constructed by using the closed fragment of any other (known) provability logic. So, the closed fragment of *GLP* is not without its uses.

The plan of this paper is the following: in section 2 we give a brief summary of results about *GLP*. The reader can find all proofs in [3]. (As we noted above, the arithmetical completeness of *GLP* was originally proved in [2]). In fact, we will not use most of these results, except for the soundness results in theorems 2.6 and 2.11, which are quite routine.

In sections 3,4 we give a *syntactic* investigation of the closed fragment of *GLP*. So, to read these sections the reader does not have to know anything about *GLP* at all. Namely, in section 3 we define a closed formula $D(\alpha)$ for every ordinal $\alpha<\varepsilon_0$, and prove a kind of "monotonicity" theorem for this sequence. In section 4 we prove that every closed formulas has a *GLP*-equivalent in a special *normal* form. In particular, it will be shown that every closed formula is a *GLP*-equivalent of a Boolean combination of some formulas $D(\alpha)$.

In section 5 we introduce the *universal model* for the closed fragment of *GLP*, a useful tool in the investigation of this system.

Section 6 is a digression. We discuss some general concepts pertaining to the closed fragment of any modal system. In particular, we give a general definition of the *ordinal complexity* of a modal logic or of its closed fragment.

In section 7 we consider the problem of the arithmetical complexity of arithmetical interpretation of closed formulas. We also consider this problem from an "internal" point of view. In particular, we prove that if we add an additional predicate "Q is

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[2]Note that for this proof we do not need the arithmetical completeness of *GLP*. We can think of *GLP* as standing for the set of arithmetically valid modal formulas, the only thing we need is that certain fixed formulas (such as Lob's axiom) are arithmetically valid.

PA-equivalent to a Σ_1-formula" to our language, then (assuming $T_0=PA$) the closed fragment of the joint provability logic will not change, i.e. will be isomorphic to the closed fragment of pure *GLP* and thus decidable.

In section 8 we consider a very unexpected application of this technique, the *provability logic for* Σ_1-*conservativity*. This logic is still unknown, but we prove that its closed fragment is isomorphic to the closed fragment of *bimodal* fragment of *GLP* and hence is decidable.

And in conclusion, we mention one unsolved problem. Note that the proof of part 1)⇒4) of theorem 2.14, which is a basic theorem proved in [3], uses the arithmetical semantics of *GLP* and thus *cannot* be formalized in *PA*, but only in $PA' := PA \cup \{Pr\ulcorner Q \urcorner \to Q \mid Q$ is an arith. sentence$\}$. Note also that all reasoning in this paper about closed fragment of *GLP* use ε_0-induction and thus *cannot* be formalized in *PA*, but only in *PA*∪{the ε_0-induction schema}. Since this theory has the same theorems as *PA'*, a natural conjecture is that several modal properties of *GLP*, such as the Craig property, cannot be proved in *PA*. However, the author has no idea how to prove it.

2. Dzhaparidze's Polymodal Logic: Summary of Results.

We suppose the reader to understand that all our definitions and theorems concerning arithmetical semantics for *GLP* are "approximate", because they are formulated in terms of "theories" instead of arithmetical predicates with certain properties.

2.A. Modal language and arithmetical semantics

Definition 2.1. A *theory* T *of degree* n (we write $deg(T)=n$) is a set of arithmetical formulas with an associated arithmetical formula $\mathrm{Pr}(\cdot)$ (with one free variable) such that:

1. $\mathrm{Pr}(\cdot)\in\Sigma_n$

and for any arithmetical statements A,B

2. $PA\vdash \mathrm{Pr}\ulcorner A\to B\urcorner \to (\mathrm{Pr}\ulcorner A\urcorner \to \mathrm{Pr}\ulcorner B\urcorner)$
3. $A\in\Sigma_n \Rightarrow PA\vdash A\to \mathrm{Pr}\ulcorner A\urcorner$ (*provable* Σ_n*-completeness*)
4. $A\in\Sigma_n$, $\mathbb{N}\models \mathrm{Pr}\ulcorner A\urcorner \Rightarrow \mathbb{N}\models A$ (Σ_n*-soundness*)
5. $PA\vdash A \Rightarrow PA\vdash \mathrm{Pr}\ulcorner A\urcorner$.

and T is exactly set of arithmetical formulas Q such that $\mathrm{Pr}\ulcorner Q\urcorner$ holds in $\mathbb{N}$.

The theory T is *sound*, if property 4 holds for all n.

Definition 2.2. A sequence of theories T_0, $T_1,\ldots,$ $T_n,\ldots$ (finite or infinite) is an *increasing sequence*, if it satisfies the following conditions:

1. $deg(T_0)<deg(T_1)<\ldots<deg(T_n)<\ldots$
2. for any statement A and $n<k$ $PA\vdash \mathrm{Pr}_n\ulcorner A\urcorner \to \mathrm{Pr}_k\ulcorner A\urcorner$.

EXAMPLES:

(a) (see [1]) for any sound theory T the ω-extension of T is given by all theorems of T and all formulas of the form $\forall x\ Q(x)$, if $\forall n\in\omega\ T\vdash Q(\underline{n})$; we denote it T^{ω};

(b) for any number $n>0$ the theory T_n^{σ} is given by all axioms of PA and all *true* Σ_n-formulas[3].

CLAIM.

(a) for any sound theory T T^{ω} *is a theory and* $deg(T^{\omega})=deg(T)+2$;

(b) for any $n>0$ T_n^{σ} *is a sound theory and* $deg(T_n^{\sigma})=n$;

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[3]Or if you like all true Π_{n-1} formulas.

(c) for any sound theory T the (infinite) sequence T, T^{ω}, $(T^{\omega})^{\omega}$, ... *is increasing;*

(d) T^{σ}_1, T^{σ}_2, ... , T^{σ}_n, ... *is an (infinite) increasing sequence.*

Definition 2.3. Let $\mathcal{L}$ be the language consisting of propositional variables $p,q,\ldots$; Boolean connectives $\rightarrow,\bot$; modal operators $[i], i=0,1,\ldots$ .We use the standard abbreviations for $\wedge,\vee,\neg,\leftrightarrow$ and the following abbreviations:

$$\langle i\rangle A := \neg[i]\neg A$$

$\Box A := [0]A$	$\Diamond^{+}A := A\vee\Diamond A$
$\Diamond A := \langle 0\rangle A$	$[n]^{+}A := A\wedge[n]A$
$\Box^{+}A := A\wedge\Box A$	$\langle n\rangle^{+}A := A\vee\langle n\rangle A$

Consider an arbitrary sequence of theories T_0, T_1, ... T_n, ... (We do not specify the number of modal operators and the number of theories; we assume only that each theory corresponds to a modal operator). An *arithmetical interpretation* f is a mapping of $\mathcal{L}$-formulas to arithmetical sentences which commutes with Boolean connectives and translates $[n]$ by Pr_n, i.e. for every formula A

$$f([n]A) := \mathrm{Pr}_n\ulcorner f(A)\urcorner.^{4}$$

Definition 2.4. The logic *GLP* is the minimal set of $\mathcal{L}$-formulas containing the following axioms and closed under the following inference rules:

(in all axioms "for all $n\geq 0$" is understood)

0. All tautologies of propositional logic.
1. $[n](A\rightarrow B) \rightarrow ([n]A\rightarrow[n]B)$.
2. $[n]([n]A\rightarrow A) \rightarrow [n]A$.
3. $[n]A \rightarrow [n+1]A$.
4. $\langle n\rangle A \rightarrow [n+1]\langle n\rangle A$.

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[4] As we noted in section 1, we will write Pr_n instead of Pr_{T_n}.

Inference rules:

1. *Modus ponens.*

2. $\dfrac{A}{\Box A}$ (*[0]-necessitation*)

Some theorems of *GLP* are:

5. $[k]A \rightarrow [n]A.$
6. $[k]A \rightarrow [n][k]A$ $\Big\}\ k \le n$
7. $\langle k \rangle A \rightarrow [n]\langle k \rangle A$
8. $[n]([k]A \longrightarrow A)$ $\Big\}\ k < n$

Definition 2.5. The logic GLP^{ω} is the minimal set of $\mathcal{L}$-formulas closed under MP and containing the following axioms:

1. All theorems of *GLP*.
2. $[n]A \rightarrow A,\ n \ge 0.$

Theorem 2.6. *Arithmetical completeness of GLP.*
Let $T_0, T_1, \ldots, T_n, \ldots$ *be an increasing sequence. Then for any modal formula* A $GLP \vdash A$ *if and only if for every arithmetical interpretation* f $PA \vdash f(A)$.

Theorem 2.7. *Arithmetical completeness of* GLP^{ω} .
Let $T_0, T_1, \ldots, T_n, \ldots$ *be an increasing sequence of sound theories. Then for any modal formula* A $GLP^{\omega} \vdash A$ *if and only if for every arithmetical interpretation* f $\mathbb{N} \models f(A)$.

2.B. An auxiliary modal logic LN. Kripke semantics.

Definition 2.8. A *model* $\mathcal{K} = \langle K, R^0, R^1, \ldots, R^N, \Vdash \rangle$ consists of a nonempty set K (the *support* of $\mathcal{K}$), an *accessibility relation* R^i for the modal operator $[i]$ $(0 \le i \le N)$, and a *forcing relation* $\Vdash$, possessing the following properties:

1. for any i R^i is transitive, irreflexive and wellfounded;

2. $x \Vdash [i]A \iff \forall y(xR^i y \to y \Vdash A)$

If $x \Vdash A$ we say that x *forces* A or A is *true* in x; a formula A is *valid* in a model $\mathcal{K}$ ($\mathcal{K} \models A$) if it is true in each node of $\mathcal{K}$ ($\forall x \in K$ $x \Vdash A$).

Definition 2.9. The logic *LN* is the minimal set of $\mathcal{L}$-formulas containing the following axioms and closed under the following rules:

Axioms:

0. All tautologies of propositional logic.

$$\left.\begin{array}{ll} 1. & [n](A \to B) \to ([n]A \to [n]B) \\ 2. & [n]([n]A \to A) \to [n]A \end{array}\right\} 0 \le n \le N$$

3. $[k]A \to [n][k]A \qquad 0 \le k \le n \le N$

4. $\langle k \rangle A \to [n]\langle k \rangle A \qquad 0 \le k < n \le N$

Inference rules:

1. *Modus ponens.*
2. $\dfrac{A}{[n]A}$ (*[n]-necessitation*).

Definition 2.10. An *LN-model* $\mathcal{K}$ is a model $\langle K, R^0, R^1, \ldots, R^N, \Vdash \rangle$ such that:

$$\forall k,n:\ k<n \wedge xR^k y \wedge (xR^n z \vee zR^n x) \to zR^k y.$$

Theorem 2.11. *LN is complete with respect to LN-models, i.e. for any modal formula A $LN \vdash A$ if and only if A is valid in all LN-models. So, LN is decidable.*

2.C. Connections between LN, GLP and GLP$^{\omega}$.

In this subsection it will be convenient to consider the restricted variant of our language, with modal operators [0], [1], ..., [N] only, where N is arbitrary.

Definition 2.12. An *LN*-model $\mathcal{K}$ is *A-complete*, where A is a modal formula, if

$$\forall k\ \forall B: [k]B \subseteq A\ \forall n: k<n\leq N\ (\mathcal{K} \models [k]B \rightarrow [n]B).$$

Definition 2.13. For any modal formula A we define the modal formulas ΔA, $\Delta^{+}A$ and $M(A)$ as follows:

$$\Delta A := \bigwedge_{0\leq i_1<i_2<\dots<i_n\leq N} [i_1][i_2]\dots[i_n]A$$

$$\Delta^{+}A := A\wedge\Delta A$$

$$M(A) := \bigwedge_{[k]B\subseteq A,\, k<n\leq N} \Delta^{+}([k]B\rightarrow[n]B).$$

Theorem 2.14. *Consider an arbitrary increasing sequence of the theories $T_0(\cdot)$, $T_1(\cdot),\dots$ $T_N(\cdot)$. Then for any modal formula A the following statements are equivalent:*

1) $GLP\vdash A$.

2) For any arithmetical interpretation f $PA\vdash f(A)$.

3) for any A-complete LN-model $\mathcal{K}$ $\mathcal{K}\models A$.

4) $LN\vdash M(A)\rightarrow A$.

So, GLP is decidable.

Definition 2.15. For any modal formula A we put

$$H(A) := \bigwedge_{[n]B\subseteq A} ([n]B\rightarrow B)$$

Theorem 2.16. *Let $T_0,T_1,\dots,T_N$ be an increasing sequence of sound theories. Then for any modal formula A which does not contain $[N]$ the following statements are equivalent:*

1) $GLP^{\omega}\vdash A$.

2) $GLP\vdash H(A)\rightarrow A$.

3) $GLP\vdash [N]A$.

4) for all interpretation f $\mathbb{N}\models f(A)$.

So, GLP^{ω} is decidable.

2.D. Fixed point property and the Craig property.

Theorem 2.17. *Fixed point property for GLP.*
Let $A(p;q_1,q_2,\dots,q_n)$ be modalized in p (i.e. every occurrence of p in A lies in the scope of $[k]$ for some k). Then there exists a formula $F(q_1,q_2,\dots,q_n)$ (a "fixed point" of A) such that

$$GLP \vdash \Box(p \leftrightarrow A) \leftrightarrow \Box(p \leftrightarrow F)$$

$$GLP \vdash \Box^{+}(p \leftrightarrow A) \leftrightarrow \Box^{+}(p \leftrightarrow F)$$

•

Theorem 2.18. *The logics LN, GLP and GLP^{ω} possess the Craig interpolation property.*

3. The Fundamental Sequence of Closed Formulas.

Before reading this section, we ask the reader to read the beginning of section 7, up to corollary 7.4. In all formulas in sections 3,4,5 σ ranges over $\Sigma_{(0)}$-formulas, or if desired over $\Sigma^{LN}_{(0)}$-formulas.

We begin with certain necessary operations on ordinals (In this section *ordinal* denotes an ordinal below ε_0).

Definition 3.1. It is well-known that any ordinal $\alpha>0$ has the following normal form:

$$\alpha = \omega^{\lambda_1} + \omega^{\lambda_2} + \dots + \omega^{\lambda_n}, \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n.$$

We put ($\beta \geq 0$):

$$\alpha^{-} := \omega^{\lambda_1} + \omega^{\lambda_2} + \dots + \omega^{\lambda_{n-1}}$$

$$d(\alpha) := \lambda_n$$

$\alpha^{+\beta} := min\{\gamma>\alpha \,|\, d(\gamma)\geq\beta\}$
$\alpha^{-\beta} := (\alpha^{+\beta})^{-}$

Definition 3.2. It will be convenient also to use the following "linear" notations:

$\Omega(\lambda) := \omega^{\lambda}$

$B(\alpha,\beta) := \alpha^{+\beta}$

and their iterations:

$d^{0}(\alpha) = \Omega^{0}(\alpha) := \alpha$

$d^{n}(\alpha) := d(d(\ldots d(\alpha)\ldots)$ (n times)

$\Omega^{n}(\alpha) := \Omega(\Omega(\ldots\Omega(\alpha)\ldots)$ (n times)

$\omega_{n} := \Omega^{n}(1)$

$B(\alpha) := \alpha$

$B(\alpha_{0},\alpha_{1},\ldots\alpha_{n}) := B(\alpha_{0},B(\alpha_{1},\ldots B(\alpha_{n-1},\alpha_{n})\ldots)$

We summarize here several elementary properties of these operations:

Proposition 3.3.

a) $\alpha^{-}<\alpha$ *and* $d(\alpha^{-})\geq d(\alpha)$

b) $\alpha^{+\beta}>\alpha$ *and* $d(\alpha^{+\beta})=\beta$

c) $\alpha^{-\beta}\leq\alpha$ *and* $d(\alpha^{-\beta})\geq\beta$

d) if $\beta^{-}<\alpha<\beta$ *then* $\beta^{-}\leq\alpha^{-}$ *and* $d(\alpha)<d(\beta)$

e) $(\beta^{-})^{+d(\beta)}=\beta$

f) if $\gamma>\beta$, *then* $B(B(\alpha,\beta),\gamma)=B(\alpha,\gamma)$

We turn to closed formulas:

Definition 3.4. For any modal formula A the formula $\uparrow A$ is the result of raising each $[n]$ in A to $[n+1]$. We also will use the operator "$\uparrow$" for sets of formulas, models, etc. $\uparrow^{n}A$ denotes $\uparrow\uparrow\ldots\uparrow A$ (n times).

Definition 3.5. For each ordinal α we define the formula $D(\alpha)$

by induction on α:

$D(0) := \bot$

$D(\alpha) := \Box D(\alpha^-) \vee \uparrow D(d(\alpha))$.

It is easy to prove that if $\alpha^- = 0$ and $\alpha \neq 1$, then $GLP \vdash D(\alpha) \leftrightarrow \uparrow D(d(\alpha))$. Here are some examples of formulas $D(\alpha)$:

$D(0) = \bot$

$D(1) = [0]\bot$

$D(2) = [0][0]\bot$

$D(n) = [0]^n\bot$

$D(\omega) = [1]\bot$

$D(\omega+1) = [0][1]\bot$

$D(\omega+\mathrm{n}) = [0]^n[1]\bot$

$D(2\omega) = [0][1]\bot \vee [1]\bot$

$D(2\omega+n) = [0]^n([0][1]\bot \vee [1]\bot)$

$D(3\omega) = [0]([0][1]\bot \vee [1]\bot) \vee [1]\bot$

$D(\omega^2) = [1][1]\bot$

$D(2\omega^2) = [0][1][1]\bot \vee [1][1]\bot$

$D(\omega^n) = [1]^n\bot$

$D(\omega^\omega) = [2]\bot$

$D(\omega^{\omega+1}) = [1][2]\bot$

$D(\omega_n) = [n]\bot$

In fact, we will prove below that each closed formula is GLP-provable equivalent to a Boolean combination of $D(\alpha)$s. But the sequence $\{D_\alpha | \alpha < \varepsilon_0\}$ is not decreasing ($D(\omega) = [1]\bot$ does not imply $D(\omega+1) = \Box[1]\bot$). However, if we put

Definition 3.6.

$$H(\alpha) := \Box D(\alpha) \quad (= D(\alpha+1)),$$

then the sequence $\{H(\alpha) | \alpha < \varepsilon_0\}$ is decreasing. It follows that for any n the sequence $\{\uparrow^n H(\alpha) | \alpha < \varepsilon_0\}$ will also be decreasing . In the next section we will prove that each closed formula is GLP-equivalent to a Boolean combination of formulas of the form $\uparrow^n H(\alpha)$.

Definition 3.7. Let γ be an ordinal. For any formula A we put:

$\Delta_\gamma A := \Box(\neg\uparrow D(\gamma) \rightarrow A)$.

If Γ is a sequence of ordinals, we put

$\Delta_{\langle\rangle} A := A;$

$\Delta_{\Gamma*<\gamma>} := \Delta_\gamma \Delta_\Gamma A.$

We write $\Gamma = <\gamma_1, \ldots, \gamma_n> < \alpha$, if and only if $\gamma_1 < \alpha \wedge \gamma_2 < \alpha \wedge \ldots \wedge \gamma_n < \alpha$.

Thus using definitions 3.5 and 3.6 we can write

$$H(\omega^{\lambda_1} + \omega^{\lambda_2} + \ldots + \omega^{\lambda_n}) = \Delta_{<\lambda_1, \ldots, \lambda_n>} \Box\bot,$$

where as usual $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

Lemma 3.8. *Let $\beta^- \leq \alpha < \beta$. Then there is a sequence of ordinals $\Gamma < d(\beta)$ such that $H(\alpha) = \Delta_\Gamma H(\beta^-)$.*

Proof. Induction on α. The basis $\alpha = \beta^-$ is trivial (take $\Gamma = <>$).

Suppose that $\beta^- < \alpha < \beta$. By proposition 3.3.d), $\beta^- \leq \alpha^- < \alpha$ and $d(\alpha) < d(\beta)$. By the induction hypothesis for α^-, $H(\alpha^-) = \Delta_{\Gamma'} H(\beta^-)$ for some $\Gamma' < d(\beta)$, and therefore

$$H(\alpha) = \Box D(\alpha) = \Box(H(\alpha^-) \vee \uparrow D(d(\alpha))) = \Delta_{d(\alpha)} H(\alpha^-) = \Delta_{d(\alpha)} \Delta_{\Gamma'} H(\beta^-) = \\ = \Delta_\Gamma H(\beta^-),$$

where $\Gamma := \Gamma' * <d(\alpha)> < d(\beta)$. QED.

Theorem 3.9. *If $\alpha < \beta$, then $GLP \vdash \Box D(\alpha) \rightarrow D(\beta)$.*

Proof.

In this proof $\vdash A$ denotes $GLP \vdash A$.

The case $\alpha = 0$ is easy. Suppose that $\alpha > 0$.

Induction hypothesis:

for all $\alpha < \beta < \delta$, $\quad \vdash \Box D(\alpha) \rightarrow D(\beta)$

Our goal is to prove that

if $\alpha < \delta$, then $\vdash \Box D(\alpha) \rightarrow D(\delta)$

CLAIM. *If $\alpha < \beta < \delta$, then*

$$\vdash \Delta_\beta(\Delta_\alpha \sigma \rightarrow \sigma) \quad \text{or} \quad \vdash \neg\uparrow D(\beta) \rightarrow (\Delta_\alpha \sigma \rightarrow \sigma).$$

(The first formula is the box of the second).

Proof. We have:

$$\vdash \neg\sigma \wedge \Box(\neg{\uparrow}D(\alpha) \to \sigma) \to [1]\neg\sigma$$
$$\to [1](\neg{\uparrow}D(\alpha) \to \sigma)$$
$$\to [1]{\uparrow}D(\alpha)$$
$$\to {\uparrow}\Box D(\alpha)$$
$$\to {\uparrow}D(\beta),$$

because by the induction hypothesis, $\vdash \Box D(\alpha) \to D(\beta)$. Thus,

$$\vdash \neg{\uparrow}D(\beta) \to (\Box(\neg{\uparrow}D(\alpha) \to \sigma) \to \sigma).$$

QED.

COROLLARY. *If* $\Gamma<\beta<\delta$, *then*

$$\vdash \Delta_\beta(\Delta_\Gamma\sigma \to \sigma) \quad \text{or} \quad \vdash \neg{\uparrow}D(\beta) \to (\Delta_\Gamma\sigma \to \sigma).$$

Proof is trivial.

Now we can prove that $\vdash \Box D(\alpha) \to D(\delta)$. Since

$$D(\delta) = \Box D(\delta^-) \vee {\uparrow}D(d(\delta)),$$

we consider two cases:

Case 1. $\alpha<\delta^-$. Then by the induction hypothesis, $\vdash \Box D(\alpha) \to D(\delta^-)$, hence (by $\Box$-necessitation)

$$\vdash \Box D(\alpha) \to \Box\Box D(\alpha) \to \Box D(\delta^-) \to D(\delta).$$

Case 2. $\delta^- \leq \alpha < \delta$. By lemma 10.9, $H(\alpha) = \Delta_\Gamma H(\delta^-)$, where $\Gamma<d(\delta)$. Thus, it is necessary to prove that

$$\vdash \neg{\uparrow}D(d(\delta)) \to (\Delta_\Gamma H(\delta^-) \to H(\delta^-))$$

but this is a consequence of corollary above, because $\Gamma<d(\delta)<\delta$.

The theorem is thus proved.

4. Normal Forms of Closed Formulas. The Main Theorem.

Let GLP_0 be the logic LN + $\{\Box D(\alpha)\rightarrow D(\beta)\,|\,\alpha<\beta<\varepsilon_0\}$ + $\{\Box\bot\rightarrow[n]\bot\,|\,n\in\omega\}$ + a new inference rule ($\vdash A \Rightarrow \vdash\uparrow A$). As we proved above, $GLP_0\subseteq GLP$. In fact, GLP is conservative over GLP_0 with respect to closed formulas and we will prove this shortly; but it is essential for us to use in the following reasoning GLP_0 instead of GLP. So, *for the remainder of this section* $\vdash A$ *denotes* $GLP_0\vdash A$.

Corollary 4.1.

a) For any $n\in\omega$ *and* $\alpha\leq\beta$ $\vdash\uparrow^n H(\alpha)\rightarrow\uparrow^n H(\beta)$.

b) If $\alpha^-=0$ *and* $\alpha\neq 1$, *then* $\vdash D(\alpha)\leftrightarrow\uparrow D(d(\alpha))$.

Proof. Easy.

Lemma 4.2.

a) For any α,β

$$\vdash\neg\uparrow D(\beta) \rightarrow (H(\alpha) \leftrightarrow H(\alpha^{-\beta})).$$

Moreover, if $\beta>0$ *and* $\alpha^{-\beta}=0$, *then*

$$\vdash\neg\uparrow D(\beta) \rightarrow (H(\alpha) \leftrightarrow \bot).$$

b) For any α,β

$$\vdash\neg\uparrow H(\alpha) \rightarrow (H(\beta^{+\alpha}) \rightarrow H(\beta));$$

moreover, if $\beta=0$, *then*

$$\vdash\neg\uparrow H(\alpha) \rightarrow (H(\beta^{+\alpha}) \rightarrow \bot).$$

c) For any α,β

$$\vdash H(\alpha)\vee\uparrow D(\beta) \leftrightarrow D(\alpha^{+\beta});$$

moreover, for $\alpha=0$ *and* $\beta>0$

$$\vdash \uparrow D(\beta) \leftrightarrow D(\omega^{\beta}).$$

d) for any α *and* $n<m$

$$\vdash \uparrow^n H(0) \vee \uparrow^m H(\alpha) \leftrightarrow \uparrow^m H(\alpha).$$

Proof.

a) Evidently, $\vdash H(\alpha^{-\beta}) \to H(\alpha)$. Let $\gamma := \alpha^{+\beta}$. Then we have:

$$\vdash \Box D(\alpha) \to D(\gamma)$$

$$\to H(\gamma^{-}) \vee \uparrow D(d(\gamma))$$

(If $\beta>0$ and $\alpha^{-\beta}=\gamma^{-}=0$,

$$\to \uparrow D(d(\gamma)) \quad)$$

$$\vdash \neg\uparrow D(d(\gamma)) \to (H(\alpha) \to H(\gamma^{-}))$$

$$\vdash \neg\uparrow D(\beta) \to (H(\alpha) \to H(\alpha^{-\beta}))$$

(if $\beta>0$ and $\alpha^{-\beta}=0$,

$$\to \neg H(\alpha) \quad).$$ QED.

b) By the previous lemma,

$$\vdash \neg\uparrow H(\alpha) = \neg\uparrow D(\alpha+1) \to (H(\beta^{+\alpha}) \to H((\beta^{+\alpha})^{-(\alpha+1)}))$$

but $(\beta^{+\alpha})^{-(\alpha+1)} = ((\beta^{+\alpha})^{+(\alpha+1)})^{-} = (\beta^{+(\alpha+1)})^{-} = \beta^{-(\alpha+1)} \leq \beta$,

(here we have used proposition 3.3.f)), hence

$$\vdash H((\beta^{+\alpha})^{-(\alpha+1)}) \to H(\beta)$$

and we are done.

If $\beta=0$, we have: $(\beta^{+\alpha})^{-(\alpha+1)}=0$ and since $\alpha+1>0$, by the previous lemma,

$$\vdash \neg\uparrow H(\alpha) = \neg\uparrow D(\alpha+1) \to (H(\beta^{+\alpha}) \to \bot).$$

The proofs of the other claims are quite similar.

Definition 4.3. We say that a closed formula A is in *normal form*, if A is a conjunction (possibly, the empty conjunction) of formulas of the following form:

$$(H(\alpha_0) \to H(\beta_0)) \vee (\uparrow H(\alpha_1) \to \uparrow H(\beta_1)) \vee \dots \vee (\uparrow^{n} H(\alpha_n) \to \uparrow^{n} H(\beta_n)) \vee$$
$$\vee \uparrow^{n+1} \neg H(\alpha_{n+1}) \vee \dots \vee \uparrow^{m} \neg H(\alpha_m),$$

(here it is supposed that $m \geq 0$, but $n=-1,0,1,\dots$; we mean that if $n=-1$, we consider the form

$$\neg H(\alpha_0) \vee \uparrow \neg H(\alpha_1) \vee \dots \vee \uparrow^{m} \neg H(\alpha_m)),$$

where

$\alpha_i \in \varepsilon_0 \cup \{\infty\}$,

$H(\infty) := \top$,

$d(0) = d(\infty) := \infty$,

and

$\forall i \leq n \quad \alpha_i > \beta_i$

$$\forall i<n \quad d(\alpha_i),d(\beta_i) \geq B(\beta_{i+1},\beta_{i+2},\ldots,\beta_n+1)$$
(in particular,
$$\forall i<n \quad \alpha_i \geq B(\beta_i,\beta_{i+1},\ldots,\beta_n+1) \text{).}$$

For instance, the formula $\Box[1]\bot\to[1]\bot$ has the following normal form:
$$\vdash \Box[1]\bot\to[1]\bot \quad\leftrightarrow\quad (H(\omega)\to H(0)) \vee (\uparrow H(\infty)\to\uparrow H(0))$$

Theorem 4.4.

a) If A is a Boolean combination of the formulas of the form $\uparrow^n H(\alpha)$, then A has a GLP_0-equivalent in normal form.

b) If A has a normal form, then for any n the formula $[n]A$ is GLP_0-equivalent to a Boolean combination of the formulas of the form $\uparrow^n H(\alpha)$.

Proof.

a) Here is the algorithm for reducing any such formula A to the normal form:

Step 1. We write A in conjunctive normal form:
$$\vdash A \quad\leftrightarrow\quad \bigwedge\!\!\!\bigwedge \bigvee\!\!\!\bigvee \{\neg\uparrow^n H(\alpha),\ \uparrow^n H(\alpha)\}$$
and fix an arbitrary conjunct B:
$$B = \bigvee\!\!\!\bigvee \{\neg\uparrow^i H(\alpha_i)\,|\,i\in\Gamma\} \vee \bigvee\!\!\!\bigvee \{\uparrow^i H(\beta_i)\,|\,i\in\Delta\}.$$
(We have used here that two formulas $H(\alpha)$ and $H(\beta)$ are always comparable).
Put $m:=max(\Gamma\cup\Delta)$.

Step 2. Define $\alpha_i:=\infty$ for $i\in\Delta\setminus\Gamma$. Thus,
$$\vdash B \quad\leftrightarrow\quad \bigvee\!\!\!\bigvee \{\neg\uparrow^i H(\alpha_i)\,|\,i\leq m\} \vee \bigvee\!\!\!\bigvee \{\uparrow^i H(\beta_i)\,|\,i\in\Delta\}.$$
If $\Delta=\varnothing$, we are done.

Step 3. Put $n:=max(\Delta)$. By lemma 10.12.d) we can define for any $i\in\{0,1,\ldots,n\}\setminus\Delta$ $\beta_i:=0$. Thus,
$$\vdash B \quad\leftrightarrow\quad \bigvee\!\!\!\bigvee \{\neg\uparrow^i H(\alpha_i)\,|\,i\leq m\} \vee \bigvee\!\!\!\bigvee \{\uparrow^i H(\beta_i)\,|\,i\leq n\}.$$

Step 4. By lemma 10.12.c),
$$\vdash \uparrow^n H(\beta_n) \quad\leftrightarrow\quad \uparrow^n D(\beta_n+1),$$

$$\vdash \uparrow^{n-1}H(\beta_{n-1}) \vee \uparrow^{n}H(\beta_n) \leftrightarrow \uparrow^{n-1}D(\mathrm{B}(\beta_{n-1},\beta_n+1))$$
$$\vdash \uparrow^{n-2}H(\beta_{n-2}) \vee \uparrow^{n-1}H(\beta_{n-1}) \vee \uparrow^{n}H(\beta_n) \leftrightarrow \uparrow^{n-2}D(\mathrm{B}(\beta_{n-2},\beta_{n-1},\beta_n+1))$$

. . . .

for any $j<n$ (including $j=-1$)

$$\vdash \mathbb{W}\{\uparrow^{i}H(\beta_i) \mid j<i\le n\} \leftrightarrow \uparrow^{j+1}D(\mathrm{B}(\beta_{j+1},\beta_{j+2},\dots,\beta_{n-1},\beta_n+1)).$$

Hence, by lemma 4.2.a),

$$\vdash \neg \mathbb{W}\{\uparrow^{i}H(\beta_i) \mid j<i\le n\} \to (\uparrow^{j}H(\alpha_j) \leftrightarrow \uparrow^{j}H(\alpha'_j))$$
$$\to (\uparrow^{j}H(\beta_j) \leftrightarrow \uparrow^{j}H(\beta'_j))$$

where

$$d(\alpha'_j),d(\beta'_j) \ge \mathrm{B}(\beta_{j+1},\beta_{j+2},\dots,\beta_{n-1},\beta_n+1)$$

(we must put $\alpha'_j:=\infty$ whenever $\alpha_j=\infty$, and $\beta'_j:=0$ whenever $\beta_j=0$). Thus, we can replace in B the ordinals α_j,β_j by α'_j,β'_j. If we do so for all j from n-1 to 0, we can write B in the form:

$$\vdash B \leftrightarrow \mathbb{W}\{\neg\uparrow^{i}H(\alpha'_i) \mid i\le m\} \vee \mathbb{W}\{\uparrow^{i}H(\beta'_i) \mid i\le n\}.$$

where for any $j<n$

$$d(\alpha'_j),d(\beta'_j) \ge \mathrm{B}(\beta'_{j+1},\beta'_{j+2},\dots,\beta'_{n-1},\beta'_n+1)$$

(we put $\alpha'_n:=\alpha_n$, $\beta'_n:=\beta_n$).

In conclusion, if for some j $\alpha'_j\le\beta'_j$, by corollary 10.11 $\vdash B$. Therefore, we can assume $\alpha'_j>\beta'_j$ for any $j\le n$. Thus, we have reduced B to normal form.

b) Suppose the formula A has a normal form. Evidently, we can assume that A contains one conjunct only:

(*) $$A = \bigvee_{0\le i\le n} \uparrow^{i}H(\alpha_i) \to \uparrow^{i}H(\beta_i) \vee \bigvee_{n<i\le m} \neg\uparrow^{i}H(\alpha_i)$$

LEMMA.

$$\vdash \Box A \leftrightarrow H(\mathrm{B}(\beta_0,\beta_1,\dots,\beta_n+1)).$$

(if $n=-1$, $\vdash \Box A \leftrightarrow H(0)=\Box\bot$).

Proof. The case $n=-1$ is trivial. Suppose that $n\ge0$.

As we proved in the proof of claim a),

$$\vdash \bigvee_{0\le i\le n} \uparrow^{i}H(\beta_i) \leftrightarrow D(\mathrm{B}(\beta_0,\beta_1,\beta_2,\dots,\beta_{n-1},\beta_n+1)).$$

so the part "$\leftarrow$" is proved.

To prove the converse implication we will decrease each α_i as much as possible; namely, as we noted above

$$\alpha_i \geq B(\beta_i, \beta_{i+1}, \ldots, \beta_n+1),$$

hence, it is sufficient to consider the case

$$\alpha_i = B(\beta_i, \beta_{i+1}, \ldots, \beta_n+1),\ i\leq n; \quad \alpha_i=0,\ n<i\leq m.$$

But now by lemma 10.12.b),

$$(**) \qquad \vdash \neg\uparrow^{i+1}H(\alpha_{i+1}) \rightarrow (\uparrow^i H(\alpha_i) \rightarrow \uparrow^i H(\beta_i)),\quad 0\leq i\leq n$$
$$\rightarrow (\uparrow^i H(\alpha_i) \rightarrow \bot), \quad n<i<m$$

because for $i\leq n$

$$\alpha_i = \beta_i^{+(\alpha_{i+1})}.$$

Using (**) we can eliminate from A all disjuncts of the form $\neg\uparrow^i H(\alpha_i)$, except $H(\alpha_0)$. Thus we obtain

$$\vdash A \leftrightarrow \neg H(\alpha_0) \vee H(\beta_0) \vee \uparrow H(\beta_1) \vee \ldots \vee \uparrow^n H(\beta_n),$$

where

$$\alpha_0 = B(\beta_0, \beta_1, \ldots, \beta_n+1).$$

But we already noted above that

$$\vdash \bigvee_{0\leq i\leq n} \uparrow^i H(\beta_i) \leftrightarrow D(B(\beta_0, \beta_1, \ldots, \beta_n+1)) = D(\alpha_0);$$

thus,

$$\vdash A \leftrightarrow \neg H(\alpha_0) \vee D(\beta_0) = \Box D(\alpha_0) \rightarrow D(\alpha_0),$$

and by Löb's axiom

$$\vdash \Box A \leftrightarrow \Box D(\alpha_0) = H(B(\beta_0, \beta_1, \ldots, \beta_n+1)).$$

Using the lemma and corollary 7.4, one can see that if A has the normal form (*), then

$$\vdash [k]A \leftrightarrow \bigvee_{0\leq i\leq n,\ i<k} \uparrow^i H(\alpha_i) \rightarrow \uparrow^i H(\beta_i) \vee \bigvee_{n<i<k} \neg\uparrow^i H(\alpha_i) \vee \uparrow^k H(\chi),$$

where

$$\chi = B(\beta_k, \beta_{k+1}, \ldots, \beta_n+1),\ k\leq n; \quad \chi=0,\ k>n.$$

This completes the proof of the theorem.

Corollary 4.5.

a) Each closed formula is a GLP-equivalent to a Boolean combination of some formulas of the form $\uparrow^n H(\alpha)$.

b) Each closed formula is a GLP-equivalent to a Boolean *combination of some formulas of the form $D(\alpha)$*

Proof. $GLP \vdash \uparrow^n H(\alpha) \leftrightarrow D(\Omega^n(\alpha+1))$ by corollary 10.11.b).

c) if A has a normal form and A is not the empty conjunction, then $GLP \nvdash A$.

Proof. One should use lemma 10.14.1 and the simple fact that for any α the formula $H(\alpha)$ is *arithmetically false* (i.e. $GLP^{\omega} \nvdash H(\alpha)$ or $GLP^{\omega} \vdash \neg H(\alpha)$). We will give an alternative proof of this fact in the following section using the *universal model*.

d) For any closed formula A $GLP \vdash A \Rightarrow GLP_0 \vdash A$.

Proof. d) is a trivial consequence of c), because if $GLP_0 \nvdash A$, then by the previous theorem A has a GLP_0-equivalent in normal form which is not the empty conjunction. But we can prove this without arithmetical arguments.

Obviously, it is sufficient to prove that for any closed formula A $GLP_0 \vdash [k]A \rightarrow [k+1]A$. By the previous theorem, we can assume that A has the form (*). Here we consider the case $k<n$ only. By corollary 7.4, we must prove that

$$\vdash \uparrow^k H(B(\beta_k,\beta_{k+1},\ldots,\beta_n+1)) \rightarrow \uparrow^{k+1} H(B(\beta_{k+1},\ldots,\beta_n+1)) \vee \vee \uparrow^k (H(\alpha_k) \rightarrow H(\beta_k)),$$

or, if we put $\gamma := B(\beta_{k+1},\ldots,\beta_n+1)$, $\beta := \beta_k$ and $\alpha_k := \infty$ (this is sufficient),

$$\vdash H(\beta^{+\gamma}) \rightarrow H(\beta) \vee \uparrow H(\gamma).$$

but this holds by lemma 4.2.b). QED.

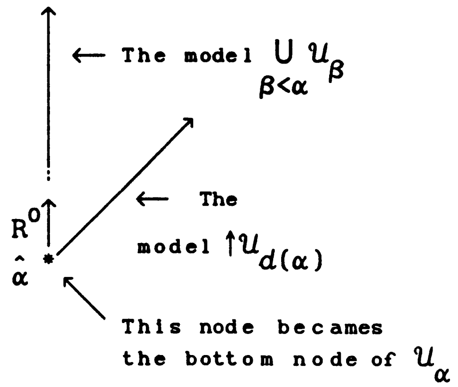
5. The Universal Model.

Definition 5.1. We define the *universal model* $\mathcal{U}$:

a) *Informal definition.* For any $\alpha<\varepsilon_0$ we define the model $\mathcal{U}_\alpha$:

$\mathcal{U}_0$ consists only of one node $\hat{0}$.

Suppose that $\{ \mathcal{U}_\beta \mid \beta<\alpha \}$ are already constructed. Here is a picture of $\mathcal{U}_\alpha$:



and we put $\mathcal{U} := \bigcup\{\mathcal{U}_\alpha \mid \alpha<\varepsilon_0\}$.

b) *Formal definition.*

$\mathcal{U} := \langle U, R^0, R^1, \ldots, R^n, \ldots\rangle$,

where

$$U := \{ \langle\alpha_0,\alpha_1,\ldots\alpha_n\rangle \mid \begin{array}{l} 1)\ \forall i<n \quad \alpha_i\neq 0 \wedge d(\alpha_i)\neq 0 \wedge \alpha_{i+1}\leq d(\alpha_i) \\ 2)\ \ d(\alpha_n)=0 \vee \alpha_n=0 \end{array} \};$$

$\langle\alpha_0,\alpha_1,\ldots,\alpha_n\rangle R^k \langle\beta_0,\beta_1,\ldots,\beta_m\rangle \ :\Longleftrightarrow$

1) $\forall i<k \quad \alpha_i=\beta_i$

2) $\alpha_k>\beta_k$

3) $\forall i\geq k, i<m \quad \beta_{i+1}=d(\beta_i)$.

$\hat{\alpha}$ denotes $\langle\alpha, d(\alpha), d^2(\alpha),\ldots,d^n(\alpha)\rangle$, where $n:=max\{i \mid d^i(\alpha)>0\}$.

($\hat{0} := \langle 0\rangle$)

$\mathcal{U}_\alpha := \langle U_\alpha, R^0, R^1, \ldots, R^n, \ldots\rangle$,

where

$U_\alpha := \{ \langle\alpha_0,\alpha_1,\ldots\alpha_n\rangle\in U \mid \alpha_0\leq\alpha \}$.

Thus, the universal model looks like a tree. The "trunk" of this tree will be called "the main axis" consists of nodes of the form $\hat{\alpha}$, $\alpha<\varepsilon_0$ and relations R^0; it has a lot of branches, each of them has the same structure (but uses R^1), etc.

Proposition 5.2.

a) $\mathcal{U}$ *is an LN-model.*

b) $\langle\alpha_0,\alpha_1,\ldots,\alpha_n,\ldots\rangle \vdash \uparrow^n A \iff \langle\alpha_n,\ldots\rangle \vdash A.$

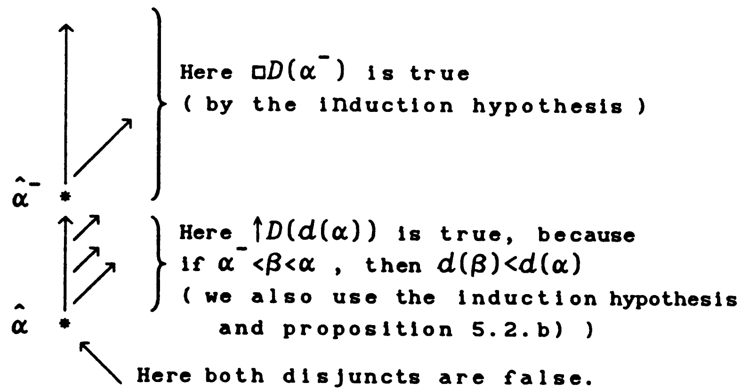
c) If $k<n$, $\langle\alpha_0,\alpha_1,\ldots\alpha_k\rangle \vdash \uparrow^n A \iff \hat{0} \vdash A.$

Proof left to the reader.

Proposition 5.3. *For any* $a\in\mathcal{U}_\alpha$ $\quad a \nVdash D(\alpha) \iff a=\hat{\alpha}.$

Proof. Induction on α. Observe:

$$D(\alpha) = \Box D(\alpha^-) \vee \uparrow D(d(\alpha))$$



Corollary 5.4.

a) $\langle\alpha_0,\ldots\rangle \vdash H(\alpha) \iff \alpha_0\le\alpha$

b) $\langle\alpha_0,\ldots,\alpha_n,\ldots\rangle \vdash \uparrow^n H(\alpha) \iff \alpha_n\le\alpha$

Proof. a) by the previous lemma; b) by proposition 5.2.b).

Theorem 5.5. *For any closed formula A* $\quad GLP\vdash A$ *if and only if* $\mathcal{U}\models A$ *(and also if and only if* $\forall\alpha\ \hat{\alpha}\vdash A$*).*

Proof. By corollary 9.15.b), $GLP \vdash A \Leftrightarrow GLP_0 \vdash A$.

Part 1. $GLP_0 \vdash A \Rightarrow \mathcal{U} \models A$.

Since $\mathcal{U}$ is an *LN*-model, we need to check three facts:

1) $\mathcal{U} \models \Box\bot \to [n]\bot$

2) $\mathcal{U} \models \Box D(\alpha) \to D(\beta)$ whenever $\alpha<\beta$

3) $\mathcal{U} \models A \Rightarrow \mathcal{U} \models \uparrow A$.

The first fact is trivial ($a \vdash \Box\bot \Leftrightarrow a=\hat{0}$), the second is a consequence of lemma 5.3 and the third is a consequence of proposition 5.2.b).

Part 2. $\mathcal{U} \models A \Rightarrow GLP_0 \vdash A$.

Suppose that $GLP_0 \nvdash A$. We can assume that A has the normal form; let B be a conjunct of A:

$$B = \bigvee_{0\le i\le n} \uparrow^i H(\alpha_i) \to \uparrow^i H(\beta_i) \vee \bigvee_{n<i\le m} \neg\uparrow^i H(\alpha_i).$$

If $n=-1$, $\hat{0} \nVdash B$. Assume that $n\ge 0$. Let $\alpha'_i := B(\beta_i, \beta_{i+1}, \ldots \beta_n + 1)$, where $i\le n$. Evidently, $\alpha_i \ge \alpha'_i$. Using corollary 11.4, one can prove that

$$\hat{\alpha}'_0 = \langle \alpha'_0, \alpha'_1, \ldots, \alpha'_n \rangle \nVdash B,$$

and thus $\hat{\alpha}'_0 \nVdash A$. QED.

In fact, we have proved that if A is in normal form and A is not the empty conjunction, then $\mathcal{U} \not\models A$, and therefore $GLP_0 \nvdash A$ and $GLP \nvdash A$. We also have proved that every non-provable (in *GLP* or GLP_0) closed formula is false in some node *on the main axle*. This allows us to give the following natural definition:

Definition 5.6. The *trace* of the closed formula A, denoted by $\mathrm{tr}(A)$, is the set of ordinals $\alpha < \varepsilon_0$ such that $\hat{\alpha} \nVdash A$.

As usual, trace "conversely commutes" with all Boolean connectives (i.e. $\mathrm{tr}(A \vee B) = \mathrm{tr}(A) \cap \mathrm{tr}(B)$, etc.); the trace of a formula is empty if and only if the formula is provable in *GLP* and so defines (closed) formula uniquely.

However, traces do not have to be clopen in the order topology. For example, $\mathrm{tr}([1]\bot) = \{\alpha > 0 \mid \alpha \text{ is limit}\}$ which is obviously closed, but not open. One consequence is that there is a sequence of formulas (for example, $\neg[1]\bot, \Box\bot, \Box^2\bot, \Box^3\bot, \ldots, \Box^n\bot, \ldots$) such that the intersection of all their traces is empty, but *GLP* does not prove any (finite) disjunction of them, which is obviously impossible for finite sequences. This non-compactness is our price for the restriction to the main axis.

However, we have several beautiful properties of traces instead (they will be obvious after section 7):

Proposition 5.7. *For any closed formula A,*

(a) $A \in \Sigma_0^{GLP}$ *if and only if* $tr(A)$ *is downward closed and clopen (in order topology);*

(b) $A \in \Pi_0^{GLP}$ *if and only if* $tr(A)$ *is upward closed and clopen;*

(c) $A \in \Sigma_1^{GLP}$ *if and only if* $tr(A)$ *is closed;*

(d) $A \in \Pi_1^{GLP}$ *if and only if* $tr(A)$ *is open;*

(e) $A \in \Delta_1^{GLP} = B_0^{GLP}$ *if and only if* $tr(A)$ *is clopen;*

For example,

$$tr(\Box^+[1]\bot) = tr(\Box[1]\bot) \cup tr([1]\bot) = \{\alpha \mid \alpha>\omega\} \cup \{\alpha>0 \mid \alpha \text{ is a limit}\} = \{\alpha \mid \alpha\geq\omega\},$$

this set is closed downward, but not open, it is only closed. Therefore, this formula is in Σ_1^{GLP} but not in Π_1^{GLP}.

6. A Digression: on the Ordinal Complexity of Modal Logics.

Definition 6.1. A modal formula is *closed*, if it contains no propositional variables, for instance: $\bot$, $\Box\bot$, $[1]\bot$, $[1]\Box\bot$, etc.

In the next section we are going to give a full description of the "closed fragment of *GLP*", i.e. a description of the closed modal formulas up to *GLP*-provable equivalence. However, before we undertake this investigation it is necessary to discuss some general concepts connected with provability logics and their closed fragments.

Let us begin by considering several examples.

The first is the usual Gödel-Löb provability logic *GL*, in the language with only one modal operator $\Box$. Assign to each natural number $n\in\omega$ the closed formula $C_n:=\Box^n\bot$. As G.Boolos proved in [4], each closed formula is *GL*-equivalent to a Boolean combination of the formulas C_n, $n\in\omega$. Furthermore, one can note that the sequence

$$C_1, C_2, \ldots, C_n, \ldots \quad , n\in\omega$$

is *decreasing*, i.e. $GL\vdash C_n\to C_m$ whenever $n\leq m$.

Next consider Carlson's bimodal provability logic CSM_1[5]. This logic formalizes provability in two r.e. extensions of *PA* such that one of them contains the other and is *reflexive* over it. Thus the modal language contains two modal operators $\Box=[0]$ and $[1]$ (Visser in [5] writes Δ and $\Box$ respectively) and the logic contains the schemas:

$$\Box A\to[1]A$$

and

$$[1](\Box A\to A).$$

Note also that $CSM_1\vdash[i]A\to[j][i]A$ for any $i,j\in\{0,1\}$.

Now for any ordinal α below ω^2, i.e. $\alpha=n+\omega.m$, we define the closed formula

$$C_\alpha := [1]^m\Box^n\bot.$$

(It is supposed that $\Box^0A=A$, etc.). In [5] it is proved that $\{C_\alpha|\alpha<\omega^2\}$ is a decreasing sequence and any closed formula is a Boolean combination of these formulas.

We can easily generalize this example by considering a sequence of r.e. extensions of *PA* (finite or infinite) $T_0, T_1, \ldots$

[5] There are many notations for this logic : $PRL_{ZF}, \ldots$. Ours is due to Visser [5].

such that T_n contains T_k and is reflexive over it whenever $k<n$. If there are N theories, $N\leq\omega$, we denote the joint provability logic $CSM_1^{(N)}$. Thus, $GL=CSM_1^{(1)}$ and $CSM_1=CSM_1^{(2)}$. If we define for $\alpha<\omega^N$

$$C_\alpha := [k]^{i_k}[k-1]^{i_{k-1}}\ldots[1]^{i_1}[0]^{i_0}\bot,$$

where $\alpha = i_0+\omega.i_1+\omega^2.i_2+\ldots+\omega^k.i_k$, the same properties as above will hold.

Thus, it is natural to speak about the *ordinal complexity* of these logics. But how to give a natural definition of this notion for an arbitrary provability logic (such that the ordinal complexity of $CSM_1^{(N)}$ is ω^N) ? One way is to use ordinal-indexed decreasing sequences with the corresponding properties. However, many questions arise, for example: why must any set of closed formulas possessing certain properties and linearly ordered by derivability in our logic be wellfounded ? Moreover, *GLP* might not have an ordinal complexity on this proposed definition. (How can we compare $[1]\bot$ and $\Box[1]\bot$?)

Now we can give our definitions: (L denotes any provability modal logic or its closed fragment; in the last case "formula" below means "closed formula")

Definition 6.2. We say that L is *wellfounded* if there is no infinite sequence of modal formulas $A_0,A_1,\ldots,A_n,\ldots$ such that for any i A_i is *L-consistent*, i.e. $L\nvdash\neg A_i$ and $L\vdash A_i\rightarrow\Diamond A_{i+1}$.

Definition 6.3. We define the partial order $\mathcal{P}=\langle P,\prec\rangle$ *associated* with L as follows:

where equivalence means "L-provable equivalence", P is the set of equivalence classes of all L-consistent modal formulas;

if $\mathfrak{a},\mathfrak{b}$ are equivalence classes and $A\in\mathfrak{a}$, $B\in\mathfrak{b}$, then

$$\mathfrak{a}\prec\mathfrak{b} :\Longleftrightarrow L\vdash B\rightarrow\Diamond A.$$

Trivially, L is well-founded if and only if $\mathcal{P}$ is.

Henceforth we will denote modal formulas and their equivalence classes by the same letters.

Definition 6.4. Let $\mathcal{P}$ be a wellfounded partial order. The *ordinal complexity of* $\mathcal{P}$ is the supremum of ordinals ξ such that there exists a $\mathcal{P}$-linear sequence of elements from P of length ξ:

$$\{ p_\alpha | \alpha<\xi \}, \ \forall\alpha,\beta<\xi \ (\alpha<\beta \Rightarrow p_\alpha \prec p_\beta).$$

Given a well-founded logic L we can now define the *ordinal complexity of* L as the ordinal complexity of the associated partial order.

The main question now is how to calculate ordinal complexities. We will use the following simple lemma (below ξ denotes an arbitrary limit ordinal):

Lemma 6.5. *Let $\mathcal{P}=\langle P,\prec\rangle$ be an arbitrary partial order. Suppose that there exists a sequence of elements from P $\{a_\alpha | \alpha<\xi\}$ ordered by ordinals $<\xi$ such that the following three conditions hold:*

1.(linearity) if $\alpha<\beta$, then $a_\alpha \prec a_\beta$;

2.(unboundedness) there is no $x\in P$ such that for all α $a_\alpha \prec x$;

3.(density) if $x\prec y$, then there exists α s.t. $a_\alpha \prec y$, but $a_\alpha \not\prec x$.

Then $\mathcal{P}$ is wellfounded and its ordinal complexity is ξ.

Proof. For any $x\in P$ we define $o(x)\in\xi$ as follows:

$$o(x) \ := \ min \ \{ \alpha<\xi \ | \ a_\alpha \not\prec x \}.$$

(This set is non-empty by unboundedness). Suppose that $x\prec y$. We claim that $o(x)<o(y)$. Indeed, linearity implies that $o(x)\le o(y)$ and if $o(x)=o(y)$, then $\{ \alpha<\xi \ | \ a_\alpha \not\prec x \} = \{ \alpha<\xi \ | \ a_\alpha \not\prec y \}$. The last equality is impossible by density. So we have proved that $o(x)<o(y)$.

But this property of o immediately implies that $\mathcal{P}$ is wellfounded and its ordinal complexity is $\le\xi$. On the other hand, by the definition, the ordinal complexity is $\ge\xi$. Thus, the lemma is proved.

Theorem 6.6. *As above, let L be an arbitrary provability logic. Suppose that there exists a sequence of L-unprovable formulas $\{D_\alpha | \alpha<\xi\}$ ordered by ordinals $<\xi$ such that the following three conditions hold:($H_\alpha:=\Box D_\alpha$; $A\vdash B$ means $L\vdash A\to B$):*

1.(linearity) if $\alpha<\beta$, then $L\vdash\Box D_\alpha\to D_\beta$;

2.(unboundedness) for any formula A, if for all $\alpha<\xi$ $H_\alpha\vdash A$, then $L\vdash A$;

3.(density) for any formula A and $\zeta<\xi$, if for all $\alpha<\zeta$ $H_\alpha\vdash A$, then $\Box^+D_\zeta\vdash A$. (For fixed ζ we will call it ζ-density; one can regard unboundedness as ξ-density).

Then L is wellfounded and its ordinal complexity is ξ.

Proof. Let $\mathcal{P}$ be the associated partial order. Define a_α for $\alpha<\xi$ to be the equivalence class of $\neg D_\alpha$. We claim that all conditions of lemma 6.5 hold. Indeed, linearity and unboundedness are trivial. In order to prove density suppose that $A\prec B$ (i.e. $L\vdash B\to\Diamond A$), but $a_\alpha\prec A$ (i.e. $L\vdash A\to\Diamond\neg D_\alpha$, or $H_\alpha\vdash\neg A$) whenever $a_\alpha\prec B$ (i.e. $L\vdash B\to\Diamond\neg D_\alpha$, or $H_\alpha\vdash\neg B$). Let β be minimal such that $a_\beta\not\prec A$. By density, $\Box^+D_\beta\vdash\neg A$, hence (using necessitation) $\Box\Box^+D_\beta=\Box D_\beta=H_\beta\vdash\Box\neg A$, hence we get $H_\beta\vdash\neg B$ and thus $a_\beta\prec B$ and $a_\beta\prec A$, contradicting the definition of β.

Now we can conclude that the ordinal complexity of $\mathcal{P}$ as well as L is ξ. QED.

In the case of *GL* we can define $D_n:=\Box^n\bot$. Now linearity and density are trivial, unboundedness is a consequence of the modal completeness of *GL* with respect to *finite* models. In the general case of $CSM_1^{(N)}$ we can define $D_\alpha:=C_\alpha$ and, using modal completeness, prove all necessary properties of these formulas (In fact, for $N<\omega$ unboundedness in the case of $CSM_1^{(N)}$ is *exactly* ω^N-density in the case of $CSM_1^{(\omega)}$).

Conclusion. *The ordinal complexity of (the closed fragment of)* $CSM^{(N)}_1$ *is exactly* ω^N.

Now we can consider our formulas $H(\alpha),D(\alpha)$ from the point of view of applying theorem 6.6. The first condition – linearity – is theorem 3.9. Unboundedness is obvious, because for any modal formula A which does not contain $[i]$ with $i\geq N$ we have

$$GLP\vdash A \iff \Box[N]\bot=H(\omega_N)\vdash A;$$

density for *closed* formulas is a simple consequence of the last theorem. Thus, we have proved the theorem:

Theorem 5. *The ordinal complexity of the closed fragment of GLP is* ε_0. *(or* ω_N *in the case of GLP with only N modal operators).*

Conjecture. *For any ordinal* $\zeta<\varepsilon_0$ *and for any modal formula A if for any* $\alpha<\zeta$ $GLP\vdash H(\alpha)\to A$, *then* $GLP\vdash H(\zeta)\wedge D(\zeta)\to A$.

If the conjecture holds, the ordinal complexity of GLP is ε_0.

It seems that the best way to prove this statement would be to introduce models for *GLP* which similar to the universal model. However, this seems rather difficult.

7. A Hierarchy of Closed Formulas.

Let us introduce classes of closed modal formulas Σ_n, Π_n, B_n, Σ^L_n, Π^L_n, Δ^L_n, B^L_n (where L is an arbitrary modal system) for every $n\geq0$ as follows:

Definition 7.1. Σ_0, Σ_1, ... , Σ_n, ... are the minimal sets of (closed) modal formulas such that for any (closed) modal formulas A, B and for any $n\geq 0$

1. $\top, \bot \in \Sigma_0$.
2. if A, $B \in \Sigma_n$, then $A\wedge B$, $A\vee B \in \Sigma_n$;
3. if $A \in \Sigma_n$, then A, $\neg A \in \Sigma_{n+1}$;
4. $[n]A \in \Sigma_n$.

Furthermore,

$\Pi_n = \{\neg A \mid A \in \Sigma_n\}$;

$B_n = \{A \mid A$ is a Boolean combination of Σ_n-formulas$\}$;

and for any modal system L

$\Sigma^L_n := \{ A \mid \exists A'\in\Sigma_n$ s.t. $L\vdash A\leftrightarrow A' \}$;

$\Pi^L_n := \{ A \mid \exists A'\in\Pi_n$ s.t. $L\vdash A\leftrightarrow A' \}$;

$\Delta^L_n := \Sigma^L_n \cap \Pi^L_n$

$B^L_n := \{ A \mid \exists A'\in B_n$ s.t. $L\vdash A\leftrightarrow A' \}$.

Lemma 7.2. *If $A \in \Sigma_n$, then $LN\vdash A\rightarrow[n]A$.*

Proof. By the definition, we can consider A as a positive Boolean combination of the formulas of the form $\neg[k]A$ for $k<n$ and $[k]A$ for $k\leq$n. All these formulas obviously satisfy the lemma. Now it is enough to note that the set of formulas A such that $LN\vdash A\rightarrow[n]A$ is closed under disjunctions and conjunctions.

Corollary 7.3. *If L=LN or GLP,*

(a) if $A \in \Sigma^L_n$, then $L\vdash(A\rightarrow[n]A) \wedge (\neg A\rightarrow[n+1]\neg A)$;

(b) if $A \in \Pi^L_n$, then $L\vdash(A\rightarrow[n+1]A) \wedge (\neg A\rightarrow[n]\neg A)$;

(c) if $A \in \Delta^L_n \supseteq B^L_{n-1} (n\geq 1)$, then $L\vdash(A\rightarrow[n]A) \wedge (\neg A\rightarrow[n]\neg A)$;

Our goal now is to prove the converse statement in each case (a)-(c) for L=GLP. But first, let us prove another useful corollary which we already used in section 3:

Corollary 7.4. *Assume, as above, L=LN or GLP. If $B \in \Delta^L_n$, then*

$$L \vdash [n](p \vee B) \leftrightarrow [n]p \vee B.$$

Proof. By corollary 7.3.(c),

$$L \vdash B \to [n]B \quad \text{and} \quad L \vdash \neg B \to [n]\neg B.$$

Hence, $L \vdash [n]p \vee B \to [n](p \vee B)$. On the other hand,

$$\begin{aligned} L \vdash \neg B \wedge [n](p \vee B) &\to [n]\neg B \\ &\to [n]p. \end{aligned}$$

QED.

In the sequel we will be interested only in the classes Σ_n^{GLP}, Π_n^{GLP}, Δ_n^{GLP}, B_n^{GLP}. In particular, we will prove that they are decidable and $\Delta_n^{GLP} = B_{n-1}^{GLP}$. So, we often will omit *GLP* and write Σ_n, Π_n, ...

Now we are going to introduce "the extended universal model" $\mathcal{V}$, which, intuitively, is $\mathcal{U}$ plus one "infinite" node, corresponding to the standard model of *PA*. We also will define its restrictions $\mathcal{V}_1$, $\mathcal{V}_2$, ..., $\mathcal{V}_n$ such that $\mathcal{V}_0$ will be exactly "the main axis" (of $\mathcal{V}$), $\mathcal{V}_1$ is $\mathcal{V}$ plus all its "immediate" branches, etc. Intuitively, $\mathcal{V}_n$ is a kind of "universal model" for B_{n-1} (or Δ_n).

Definition 7.5.

$$\mathcal{V} := \langle V, R^0, R^1, \dots, R^n, \dots \rangle,$$

where

$$V := \{ \langle \alpha_0, \alpha_1, \dots \alpha_n, \dots \rangle \mid \forall i \ \alpha_{i+1} \le d(\alpha_i) \}$$

and α_i range over ordinals less than *or equal* to ε_0.

We define $d(\varepsilon_0) := \varepsilon_0$.

$\langle \alpha_0, \alpha_1, \dots, \alpha_n, \dots \rangle R^k \langle \beta_0, \beta_1, \dots, \beta_m, \dots \rangle \quad :\Leftrightarrow$

1) $\forall i<k \quad \alpha_i = \beta_i$
2) $\alpha_k > \beta_k$
3) $\forall i \ge k \quad \beta_{i+1} = d(\beta_i)$ (i.e. $\langle \beta_0, \beta_1, \dots, \beta_m, \dots \rangle \in V_{k+1}$ – see below)

$\hat{\alpha}$ denotes $\langle \alpha, d(\alpha), d^2(\alpha), \dots, d^n(\alpha), \dots \rangle$.

$V_n := \{ \langle \alpha_0, \alpha_1, \dots \alpha_i, \dots \rangle \in V \mid \forall i \ge n \ \alpha_i = d(\alpha_{i-1}) \} \quad (n \ge 1).$

(We will not consider V_n as a submodel of V). Obviously $V = \bigcup_{n=1}^{\infty} V_n$.

Comparing definitions 5.1 and 7.4 one can note that if in section 5 we took our universal modal to consist only of *finite* sequence of ordinals, we now take it to contain only *infinite* sequences. Namely, all the old sequences in U are now terminated with an infinite sequence of zeros. We have also added some sequences of the form

$$\langle\varepsilon_0,\varepsilon_0,\ldots,\varepsilon_0, \text{[something below } \varepsilon_0\text{]}, 0, 0, \ldots\rangle$$

and only one essentially infinite sequence $\hat{\varepsilon}_0=\langle\varepsilon_0,\varepsilon_0,\ldots\rangle$. So, we will use proposition 5.2 and corollary 5.4 for V without any changes. We should also mention the following almost obvious fact (we will not use it):

Lemma 7.6. *For any closed formula* A $GLP^{\omega}\vdash A \Leftrightarrow \hat{\varepsilon}_0\vdash A$.

Proof is left to the reader.[6]

Now we are going to define a topological structure on every set V_n as well as on the whole of V. We give the following definition:

Definition 7.7. For every $n\geq1$, the topology τ_n on V_n is given by the following (clopen) base:

$\{ \alpha \mid \alpha\vdash A \}$, where A ranges over B_{n-1}.

Analogously, the topology τ on V is given by the following (clopen) base:

$\{ \alpha \mid \alpha\vdash A \}$, where A ranges over all closed formulas.[7]

---

[6] It is completely trivial using theorem 2.16. The reader may wish to consider how to prove it without using arithmetic.

[7] Note that V_n is *not* a subspace in V_{n+1} or in V.

We will need two facts about these topological spaces: 1) they are compact, and 2) the clopen sets listed above are exactly the clopen sets in these topological spaces. Proofs are quite similar for all cases, so, we consider the V_n only.

First of all, we give an equivalent definition of our topology:

Lemma 7.8. *The topological space V_n can be given by the following subbase:*

1. $\{ \langle\alpha_0,\alpha_1,\ldots,\alpha_i,\ldots,\alpha_{n-1},\ldots\rangle \in V \mid \alpha_i > \alpha \}$
2. $\{ \langle\alpha_0,\alpha_1,\ldots,\alpha_i,\ldots,\alpha_{n-1},\ldots\rangle \in V \mid \alpha_i \le \alpha \}$

(where $i<n$ in both cases).

Proof. In topological terminology, we have defined two topological spaces and have to prove that identity map is a homeomorphism.

Part 1. Fix an arbitrary formula $A \in B_{n-1}$. Our goal is to prove that for any $\mathfrak{a} \vdash A$ there exists a set $\mathfrak{A}$, containing $\mathfrak{a}$, open in sense of lemma 7.8 and such that $\forall \mathfrak{b} \in \mathfrak{A}\ \ \mathfrak{b} \vdash A$. Note that if A is a conjunction (disjunction) it is obviously enough to prove this fact for each conjunct (disjunct). Now the normal form theorem for closed formulas immediately implies that it is enough to consider the cases $A = \uparrow^i H(\alpha)$ and $A = \neg\uparrow^i H(\alpha)$, where $i \le n-1$. Now use corollary 5.4.(b).

Part 2. Here we shall prove that for every set mentioned in the statement of lemma 7.8 there exists a modal formula in B_{n-1} which is true in exactly this set. According to corollary 5.4.(b), it is enough to consider the formulas $\uparrow^i H(\alpha)$ and $\neg\uparrow^i H(\alpha)$.

Thus the lemma is proved. Note that this lemma immediately implies that V_n is Hausdorff.

Consider now the set of ordinals $\le\varepsilon_0$ with the natural (order) topology (the subbase consists of all sets of the form $\{\alpha|\alpha\le\beta\}$ and $\{\alpha|\alpha>\beta\}$) and denote this topological space by $\mathcal{E}$. It is well-known that $\mathcal{E}$ is compact. Every sequence in V_n is uniquely defined by its n first elements, so we can consider V_n as a subset of $\mathcal{E}^n:=\mathcal{E}\times\mathcal{E}\times\ldots\times\mathcal{E}$ (n times). Moreover, according to the definition of the Tychonoff product and lemma 7.8, V_n is a *subspace* of $\mathcal{E}^n$.

Lemma 7.9. *V_n is closed in $\mathcal{E}^n$.*

Proof. Inspection of definition 7.5 shows that it is enough to prove that the relation $d(\beta)<\alpha$ is *open*, i.e. that $\{\langle\alpha,\beta\rangle|d(\beta_0)<\alpha_0\}$ is open in $\mathcal{E}^2$. Indeed, suppose that $d(\beta_0)<\alpha_0$. Consider the set

$$O := O_1\cap O_2$$

where

$$O_1 := \{\langle\alpha,\beta\rangle|\alpha>d(\beta_0)\},$$
$$O_2 := \{\langle\alpha,\beta\rangle|\beta_0^-<\beta\le\beta_0\}.$$

Then O is open and $\langle\alpha_0,\beta_0\rangle\in O\subseteq\{\langle\alpha,\beta\rangle|d(\beta)<\alpha\}$. (We have used proposition 3.3.d))

Now we can easily prove the theorem:

Theorem 7.10.

(a) V (V_n) is Hausdorff and compact;

(b) set $A\in V$ (V_n) is clopen in V (V_n) if and only if there exists closed modal formula A (a formula A in B_{n-1}, respectively) such that $A = \{\ \alpha\in V\ (V_n)\ |\ \alpha\vdash A\ \}$.

Proof. (a) follows from lemma 7.9. (as we noted above, the proof for V does not differ from the proof for V_n). (b) is immediate consequence of the following simple topological lemma:

LEMMA. *Let $\mathcal{X} = \langle X,\tau\rangle$ be a compact Hausdorff topological space and let γ be a collection of clopen sets that is a base for $\mathcal{X}$. Then every clopen set is a finite union of sets in γ.*

PROOF. Let A be an arbitrary clopen set. It is well-known that every compact Hausdorff space is regular (T_3), so for every point x in A there is an open set O_x such that $x \in O_x \subseteq A$. (We have here used that $X \backslash A$ is closed) Since γ is a base, we can assume that $O_x \in \gamma$. Since A is compact (A is closed and $\mathfrak{X}$ is compact) and $A = \bigcup_{x \in A} O_x$, there exists a finite subset D of A such that $A = \bigcup_{x \in D} O_x$. This proves the lemma.

So, we have proved theorem 7.10. The following theorem is exactly the statement that V is compact in modal language:

Theorem 7.11. *Let Γ be an arbitrary GLP-consistent set of closed formulas. Then there exists $\alpha \in V$ such that $\alpha \vdash \Gamma$ ($:= \forall A \in \Gamma\ \alpha \vdash A$). If Γ is maximal GLP-consistent, then any such α is unique.*[8]

Proof. Let $\Gamma = \{A_0, A_1, \dots\}$; since V is the *universal* model, for each $n \in \omega$ the set $K_n := \{\alpha \in V \mid \alpha \vdash \bigwedge_{i \leq n} A_i\}$ is not empty. Of course, $K_0 \supseteq K_1 \supseteq K_2 \supseteq \dots$. Since each K_n is clopen and V is compact, for some α, $\alpha \in \bigcap_{i \in \omega} K_i$. Evidently, $\alpha \vdash \Gamma$.

The second part of our statement (which actually expresses that V is Hausdorff) is left to the reader.

Now we are ready to work with V and V_n. The following technical lemma is the main tool we use to investigate the hierarchy of closed formulas in *GLP*:

Lemma 7.12. *For every $n \geq 1$ and for every closed formula B the set $\{ \alpha \in V_n \mid \alpha \vdash [n]^+ B \}$ is open in V_n.*

---

[8] *α does not have to be on the main axis (V_1).* This is the explanation of some "strange" properties of traces. See the discussion at the end of section 3.

Proof. We may assume that B is in normal form (definition 4.3) and, moreover, that B consists of only one conjunct. Thus, we can write B in the form $B = A \vee \uparrow^{n}X$, where $A \in B_{n-1}$. By corollary 7.4,

$$\vdash [n]B \leftrightarrow A\vee [n]\uparrow X$$

$$\vdash [n]^{+}B \leftrightarrow A \vee [n]^{+}\uparrow X = A \vee \uparrow^{n}\Box^{+}X.$$

$$\{ \mathfrak{a} \in V_n \mid \mathfrak{a}\vdash [n]^{+}B \} = \{ \mathfrak{a} \in V_n \mid \mathfrak{a}\vdash A \} \cup \{ \mathfrak{a} \in V_n \mid \mathfrak{a}\vdash \uparrow^{n}\Box^{+}X \}$$

It is enough to prove that the second set is open, because the first is clopen. Suppose that

$$\mathfrak{a} = <\alpha_0,\alpha_1,\ldots,\alpha_{n-1},d(\alpha_{n-1}),d^2(\alpha_{n-1}),\ldots> \vdash \uparrow^{n}\Box^{+}X.$$

Consider any $\mathfrak{b}\in V_n$ of the form:

$$\mathfrak{b} = <\beta_0,\beta_1,\ldots,\beta_{n-1},d(\beta_{n-1}),d^2(\beta_{n-1}),\ldots>$$

where $\alpha^{-}_{n-1} < \beta_{n-1} \le \alpha_{n-1}$. It remains to prove that $\mathfrak{b}\vdash\uparrow^{n}\Box^{+}X$, because the set of possible $\mathfrak{b}$ is obviously open. By proposition 5.2.(b), $\mathfrak{a}\vdash\uparrow^{n}\Box^{+}X$ implies $\hat{d}(\alpha_{n-1})\vdash\Box^{+}X$ and $\mathfrak{b}\vdash\uparrow^{n}\Box^{+}X$ if and only if $\hat{d}(\beta_{n-1})\vdash\Box^{+}X$. But by proposition 3.3.d), $d(\beta_{n-1}) < d(\alpha_{n-1})$ and thus by the definition of our model $\hat{d}(\alpha_{n-1})R^{0}\hat{d}(\beta_{n-1})$ and we are done.

Now fix an arbitrary arithmetical interpretation f; f is based on the increasing sequence T_0, $T_1,\ldots$. Recall that $\Sigma_{(0)}:=\Sigma_{deg(T_0)}$, etc.

Theorem 7.13. *Let A be any closed modal formula.*

(a) The following conditions are equivalent ($n\ge1$):

1. $A \in B^{GLP}_{n-1}$.
2. $A \in \Delta^{GLP}_{n}$.
3. $f(A) \in B^{PA}_{(n-1)}$.
4. $f(A) \in \Delta^{PA}_{(n)}$.
5. $GLP\vdash (A \to [n]A) \wedge (\neg A \to [n]\neg A)$.

(b) The following conditions are equivalent ($n\ge0$):

1. $A \in \Sigma^{GLP}_{n}$.
2. $f(A) \in \Sigma^{PA}_{(n)}$.
3. $GLP\vdash (A \to [n]A) \wedge (\neg A \to [n+1]\neg A)$.

Proof.

(a). Implications $1\Rightarrow 3$, $1\Rightarrow 2$, $3\Rightarrow 4$, $2\Rightarrow 4$, $4\Rightarrow 5$ are trivial. So we only have to prove $5\Rightarrow 1$. Suppose that 5 holds for some formula A. Then $GLP\vdash (A\leftrightarrow [n]^{+}A)\ \wedge\ (\neg A\leftrightarrow [n]^{+}\neg A)$. Applying lemma 7.12 we get that $\{\ \alpha\in V_n\ \mid\ \alpha\vdash A\ \}$ is clopen in V_n. By theorem 7.10.(b), it coincides with $\{\ \alpha\in V_n\ \mid\ \alpha\vdash B\ \}$ for some $B\in \mathrm{B}_{n-1}$. In particular $\mathrm{tr}(A) = \mathrm{tr}(B)$. By an elementary property of traces, $GLP\vdash A\leftrightarrow B$ and we are done.

(b). As above, we only have to prove $3\Rightarrow 1$. Suppose 3 holds. From the previous paragraph we know that $A\in\mathrm{B}_n$. Thus (using the normal form theorem) we can suppose A has the form:

$$A = \bigwedge_{i=1}^{k} (\ B_i \vee \uparrow^n C_i \vee \neg\uparrow^n D_i\ ),\quad B_i\in\mathrm{B}_{n-1},\quad C_i, D_i\in\Sigma_0,\quad GLP\nvdash C_i\vee\neg D_i .$$

Let

$$A_1 = \bigwedge_{i=1}^{k} (\ B_i \vee \uparrow^n C_i\ ) \in \Sigma_n .$$

We claim that $GLP\vdash A\leftrightarrow A_1$. Suppose not. Obviously, $GLP\vdash A_1\rightarrow A$. So, there exists $\alpha<\varepsilon_0$ such that $\hat{\alpha}\vdash A\wedge\neg A_1$, which implies that for some i

$$\hat{\alpha}\vdash\ \neg B_i\ \wedge\ \neg\uparrow^n C_i\ \wedge\ \neg\uparrow^n D_i .$$

Or

$$\hat{d}^n(\alpha)\vdash\ \neg C_i\ \wedge\ \neg D_i .$$

Since $GLP\nvdash C_i\vee\neg D_i$, there exists β such that $\hat{\beta}\vdash\neg C_i\ \wedge\ D_i$. Because $D_i\in\Sigma_0$, this implies $\hat{d}^n(\alpha)R^0\hat{\beta}$, or $\beta<d^n(\alpha)$. Put

$$\mathfrak{b} := \langle\alpha, d(\alpha), d^2(\alpha), \ldots, d^{n-1}(\alpha), \beta, d(\beta), d^2(\beta), \ldots\rangle .$$

First, we have $\mathfrak{b}\vdash\ \neg\uparrow^n C_i\ \wedge\ \uparrow^n D_i$. Secondly, by the definition of our model, $\hat{\alpha}R^n\mathfrak{b}$, which implies $\mathfrak{b}\vdash\neg B_i$ (equivalently, $\hat{\alpha}\vdash\neg B_i$). Recall now that $GLP\vdash A\rightarrow [n]A$ and $\hat{\alpha}\vdash A$. This gives $\mathfrak{b}\vdash A$, but $\mathfrak{b}\vdash\neg B_i\wedge\neg\uparrow^n C_i\wedge\uparrow^n D_i$. Contradiction. Thus $GLP\vdash A\leftrightarrow A_1\in\Sigma_n$ and we are done.

Corollary 7.14.

(a) $\Sigma_n^{GLP} = \{\ A\ \mid\ GLP\vdash (A\ \rightarrow\ [n]A)\ \wedge\ (\neg A\ \rightarrow\ [n+1]\neg A)\ \}$;

(b) $\Pi_n^{GLP} = \{\ A\ \mid\ GLP\vdash (A\ \rightarrow\ [n+1]A)\ \wedge\ (\neg A\ \rightarrow\ [n]\neg A)\ \}$;

(c) $\mathrm{B}_n^{GLP} = \Delta_{n+1}^{GLP} = \{\ A\ \mid\ GLP\vdash (A\ \rightarrow\ [n+1]A)\ \wedge\ (\neg A\ \rightarrow\ [n+1]\neg A)\ \}$

The last corollary can be generalized in different ways. We will mention the generalization we shall need we will need in the next section. Define the Σ_0*-interpolation property* of two formulas A,B:

$$A \twoheadrightarrow B^{9} \;:\Leftrightarrow\; \exists\sigma\in\Sigma_0 \;\; GLP\vdash (A\to\sigma)\wedge(\sigma\to B).$$

Of course, $A\in\Sigma_0^{GLP} \Leftrightarrow A\twoheadrightarrow A$. We leave it to the reader, using the same technique as above, to prove the following lemma:

Lemma 7.15. $A\twoheadrightarrow B \;\Leftrightarrow\; GLP\vdash \langle 1\rangle^{+}A\to\Box^{+}B$

In fact, we will need the following simple corollary:

Corollary 7.16.

$$(\, B\wedge\Box(A\vee\langle 1\rangle^{+}B)\,)\twoheadrightarrow(\, A\vee\langle 1\rangle^{+}B\,).$$

Now we would like to consider our hierarchy *inside PA*, or as would be better, inside T_0 (we always can suppose that $T_0=PA$). The question is then the following: Let $\Sigma_{(n)}(\ulcorner Q\urcorner)$ be the *arithmetical formula (predicate)* representing the relation $Q\in\Sigma^{T_0}_{(n)}$, i.e Q is T_0-provably equivalent to some $\Sigma_{(n)}$-formula. We can regard $\Sigma_{(n)}$ as a new *(unary) modal operator. Can we prove* that

$$PA\vdash \Sigma_{(n)}(\ulcorner Q\urcorner) \leftrightarrow Pr_0\ulcorner Q\to Pr_n\ulcorner Q\urcorner\urcorner\wedge Pr_0\ulcorner\neg Q\to Pr_{n+1}\ulcorner\neg Q\urcorner\urcorner$$

(where Q is the arithmetical translation of a closed modal formula), or in the modal language,

$$\vdash \Sigma_n A \leftrightarrow \Box(A\to[n]A)\wedge\Box(\neg A\to[n+1]\neg A)\;?$$

(compare with corollary 7.14.(a)).

To prove this, we need the following lemma, a simple generalization of the statement 3⇒1 of theorem 7.13.(b):

---

[9] Perhaps, we should have written something like $A\twoheadrightarrow_0^{GLP}B$. But we will not use any other analogous notions.

Lemma 7.17. *For any closed formula A and $\alpha\in V$*

$$\alpha\vdash\Box(A\rightarrow[n]A)\wedge\Box(\neg A\rightarrow[n+1]\neg A) \implies \exists\sigma\in\Sigma_n\ \alpha\vdash\Box(A\leftrightarrow\sigma).$$

Now we can prove the theorem mentioned above:

Theorem 7.18. *Let A be a closed formula and f an arithmetical interpretation based on $T_0, T_1, \ldots$, $Q:=f(A)$. Then*

$$PA\vdash\ \Sigma_{(n)}(\ulcorner Q\urcorner) \leftrightarrow Pr_0\ulcorner Q\rightarrow Pr_n\ulcorner Q\urcorner\urcorner \wedge Pr_0\ulcorner\neg Q\rightarrow Pr_{n+1}\ulcorner\neg Q\urcorner\urcorner.$$

(where $\Sigma_{(n)}(\ulcorner Q\urcorner) := \exists\sigma\in\Sigma_{(n)} Pr_0\ulcorner Q\leftrightarrow\sigma\urcorner$).

Proof. The part "$\rightarrow$" holds for any arithmetical formula Q. So we will prove the converse statement.

Let $\mathcal{M}$ be a model of PA, $\mathcal{M}\vDash PA$. By the theorem 7.11, there exists a unique $\alpha\in V$ such that for any closed formula A

(*) $\qquad \mathcal{M}\vDash f(A) \iff \alpha\vdash A.$

Suppose that $\mathcal{M}\vDash \mathrm{Pr}_0\ulcorner f(A)\rightarrow \mathrm{Pr}_n\ulcorner f(A)\urcorner\urcorner \wedge \mathrm{Pr}_0\ulcorner\neg f(A)\rightarrow \mathrm{Pr}_{n+1}\ulcorner\neg f(A)\urcorner\urcorner$. By (*), $\alpha\vdash\Box(A\rightarrow[n]A)\wedge\Box(\neg A\rightarrow[n+1]\neg A)$. By lemma 7.17, there exists a formula $\sigma\in\Sigma_n$ (whence $f(\sigma)\in\Sigma_{(n)}$) such that $\alpha\vdash\Box(A\leftrightarrow\sigma)$ and by (*) $\mathcal{M}\vDash \mathrm{Pr}_0\ulcorner f(A)\leftrightarrow f(\sigma)\urcorner$. By the definition of $\Sigma_{(n)}$, $\mathcal{M}\vDash\Sigma_{(n)}(\ulcorner f(A)\urcorner)$. This completes the proof of the theorem.

Thus, we can express in our modal language great many additional modal operators. Actually, we have given proofs only for $\Sigma_{(n)}(\ulcorner Q\urcorner)$, but now the reader can believe that the same is true for the $\Sigma_{(0)}$-interpolation predicate, and many other analogous predicates. It is much more interesting and less trivial that, in some special cases, we can express the Σ_1-conservativity predicate. Our last section deals with this, perhaps the most beautiful, application of Dzhaparidze's logic.

8. The Logic of Σ_1-conservativity.

Let T_1, T_2 be arbitrary theories in the language of arithmetic, and Γ be a set of arithmetical formulas. We say that T_2 is *Γ-conservative over T_2, if every arithmetical statement in Γ which is provable in* T_2 *is also provable in T_1*. If Γ is decidable and the theories T_1, T_2 have natural provability predicates Pr_{T_1}, Pr_{T_2}, we can consider the arithmetical formula representing this relation:

$$Conserv_\Gamma(T_1, T_2) \ := \forall\gamma\in\Gamma \ (\ Pr_{T_2} \ulcorner\gamma\urcorner \ \rightarrow Pr_{T_1}\ulcorner\gamma\urcorner \ ).$$

Of course, we can use this predicate for the arithmetical interpretation of the modal language $\mathcal{L}(\Box,\rhd)$ (where $\rhd$ is a new binary modal operator); i.e. we translate $A\rhd B$ as $Conserv_\Gamma(PA+A, PA+B)$. As usual, the *provability logic for Γ-conservativity* (or the *logic for Γ-conservativity*) is the set of those modal formulas every arithmetical interpretation of which is provable in *PA*.

In [6] the logics of Σ_n-conservativity for n≥3 and the logics of Π_n-conservativity for $n\geq 2$ were found. Recall also that Π_1-conservativity coincides with relative interpretability, so the logic of Π_1-conservativity is also known. But the logics of Σ_1- and Σ_2-conservativity are still unsolved problems.

In this section we give an axiomatization and a decision procedure for the *closed fragment* of the logic of Σ_1-conservativity by reducing it to the closed fragment of the bimodal fragment of *GLP*. We also prove that the converse reduction is also possible.

Let $\mathcal{L}(\rhd;\Box,[1],[2],\ldots)$ be a modal language containing an infinite set of unary modal operators $\Box=[0],[1],[2],\ldots$ and a binary modal operator $\rhd$. We consider the following arithmetical interpretation of $\mathcal{L}$: $A\rhd B$ is translated as "*PA+B* is Σ_1-conservative over *PA+A*", $\Box A$ is translated as "*A* is provable in *PA*", $[1]A$ is translated as "$PA+\neg A$ is not Σ_1-sound", or, equivalently, "*A* is provable in *PA* plus all *true* Σ_2-formulas", and $[n]A$ for $n\geq 2$ are translated by an arbitrary increasing sequence of the strong provability predicates.

We will not introduce here any specific modal system for $\mathcal{L}$. So, we write $\vdash A$ if the (modal) formula A is arithmetically valid.

Our basic theorem is the following:

Theorem 8.1.

a) For any modal formula A,

$$\vdash [1]A \leftrightarrow \Box A \vee \neg(\neg A \rhd \neg A \wedge \neg\Box A).$$

b) If A,B are closed, then

$$\vdash A\rhd B \leftrightarrow \Diamond(A\wedge[1]^{+}\neg B) \rightarrow \langle 1\rangle B.$$

To prove this theorem we need to prove several Σ_1-conservativity principles.

Theorem 8.2. *The following list consists of arithmetically valid modal principles:*

- $A1.$ $A\rhd C \wedge B\rhd C \rightarrow A\vee B\rhd C$
- $A2.$ $A\rhd B \rightarrow (\Diamond A \rightarrow \Diamond B)$
- $A3.$ $A\rhd B \wedge B\rhd C \rightarrow A\rhd C$
- $A4.$ $\Box(A\rightarrow B) \rightarrow A\rhd B$
- $M^{*}.$ $A\rhd B \rightarrow A\wedge\Diamond C \rhd B\wedge\Diamond C$
- $A5^{*}.$ $\langle 1\rangle A \rhd A$
- $P^{*}.$ $\langle 1\rangle(A\rhd B) \rightarrow A\rhd B$
- $A2^{*}.$ $A\rhd B \rightarrow (\langle 1\rangle A \rightarrow \langle 1\rangle B)$
- $C.$ $\langle 1\rangle A \rightarrow \top\rhd A$
- $D.$ $A\rhd(B\wedge\Diamond A) \rightarrow (\Diamond A \rightarrow \langle 1\rangle B)$

Proof.

Axioms *A1*-*A4* together with the axioms of the pure provability logic *GL* form the *pure conservativity logic CL* (see [6]). The arithmetical soundness of the axioms $M^{*}, A5^{*}, P^{*}, A2^{*}, C$ is quite simple. So, we will prove the arithmetical soundness of principle D.

We will prove it in the language with quantifiers on Σ_1-formulas and with the Rosser orders $\prec, \preccurlyeq$, where

$$\exists x\delta_0(x) \prec \exists x\delta_1(x) \quad := \quad \exists x \; (\; \delta_0(x) \wedge \forall y{\le}x \; \neg\delta_1(y) \;),$$

$$\exists x\delta_0(x) \preccurlyeq \exists x\delta_1(x) \quad := \quad \exists x \; (\; \delta_0(x) \wedge \forall y{<}x \; \neg\delta_1(y) \;).$$

Here are four arithmetically valid principles of $\prec, \preccurlyeq$:

$R1.$ $A{\prec}B \;\rightarrow\; A{\preccurlyeq}B$

$R2.$ $A{\preccurlyeq}B \;\rightarrow\; A$

$R3.$ $A{\vee}B \;\rightarrow\; A{\preccurlyeq}B \vee B{\prec}A$

$R4.$ $A{\preccurlyeq}B \;\rightarrow\; \neg(B{\prec}A)$

Of course, we can write in our language:

$<1>A \;:=\; \forall\sigma \; (\; \Box(A\rightarrow\sigma) \;\rightarrow\; \sigma \;)$

$A{\rhd}B \;:=\; \forall\sigma \; (\; \Box(B\rightarrow\sigma) \;\rightarrow\; \Box(A\rightarrow\sigma) \;)$

Reason in PA. Suppose that

(1) $\quad A{\rhd}B{\wedge}\Diamond A$

and $\neg<1>B$. Then there is σ such that

(2) $\quad \Box(B\rightarrow\sigma)$

(3) $\quad \neg\sigma$

We define Σ_1-formulas ρ and S as the fixed points:

$\rho \;:=\; \Box(A\rightarrow \neg\rho){\preccurlyeq}\sigma$

$S \;:=\; \sigma{\prec}\Box(A\rightarrow \neg\rho)$

We have:

	$\rho \;\rightarrow\; \Box(A\rightarrow \neg\rho)$	$R2$
(4)	$\rho\rightarrow\Box\rho$	$\rho\in\Sigma_1$
	$\rho\rightarrow\Box\neg A$	two previous formulas
	$\Diamond A\rightarrow \neg\rho$	
	$\sigma\rightarrow\rho\vee S$	$R3$
	$\Box(B\wedge\Diamond A\rightarrow S)$	(2) and two previous formulas
	$\Box(A\rightarrow S)$	(1)
	$\Box(S\rightarrow \neg\rho)$	$R4$
(5)	$\Box(A\rightarrow \neg\rho)$	two previous formulas
	$\Box(A\rightarrow \neg\rho) \;\rightarrow\; \rho\vee S$	$R3$
	$S \;\rightarrow\; \sigma$	$R1,R2$

(6) ρ three previous formulas and (3)

$\Box\neg A$ (4),(5),(6)

QED.

Corollary 8.3. *The following modal principles and inference rules are arithmetically valid:*

B1. $A \rhd B \rightarrow (A\wedge\pi)\rhd(B\wedge\pi)$ by M^*, and A1–A4

B2. $\langle 1\rangle^+ A \rhd A$ by A4, $A5^*$

B3. if $\vdash A \rightarrow \pi$, then

$\vdash A\rhd B \leftrightarrow A\rhd B\wedge\pi$ "$\leftarrow$" by A3, A4

"$\rightarrow$" by B1

B4. if $\vdash B \rightarrow \Diamond A$, then

$\vdash A\rhd B \leftrightarrow (\Diamond A \rightarrow \langle 1\rangle B)$ "$\rightarrow$" by D

"$\leftarrow$"

$\Box\neg A \rightarrow A\rhd B$ by A4

$\langle 1\rangle B \rightarrow A\rhd B$ by C

B5

a). $\Box A \leftrightarrow (\neg A)\rhd\bot$ "$\leftarrow$" by A4

"$\rightarrow$" by A2

b). $[1]A \leftrightarrow \Box A \vee \neg(\neg A\rhd\neg A\wedge\neg\Box A)$ by B4.

Proof of theorem 8.1.

a) has been already proved (B5b)).

b) Let A,B be closed formulas. Evidently, to prove the theorem it is sufficient to consider the case when A,B do not contain $\rhd$. Thus, by corollary 7.16 there exists a closed formula $\sigma\in\Sigma_0$ such that

$$\vdash B\wedge\Box(\neg A\vee\langle 1\rangle^+ B) \rightarrow \sigma$$

$$\vdash \sigma \rightarrow \neg A\vee\langle 1\rangle^+ B$$

Or, if we put $\pi=\neg\sigma$,

(1) $\vdash A\wedge[1]^+\neg B \rightarrow \pi$

(2) $\vdash B\wedge\pi \rightarrow \Diamond(A\wedge[1]^+\neg B)$.

We have:

$$
\begin{aligned}
A \rhd B &\leftrightarrow (A \wedge [1]^{+}\neg B) \vee (A \wedge \langle 1\rangle^{+} B) \rhd B \\
&\leftrightarrow (A \wedge [1]^{+}\neg B \rhd B) \wedge (A \wedge \langle 1\rangle^{+} B \rhd B) && A1 \\
&\leftrightarrow A \wedge [1]^{+}\neg B \rhd B && B2 \\
&\leftrightarrow A \wedge [1]^{+}\neg B \rhd B \wedge \pi && B3 \text{ and } (1) \\
&\leftrightarrow (\ \Diamond(A \wedge [1]^{+}\neg B) \rightarrow \langle 1\rangle(B \wedge \pi)\ ) && B4 \text{ and } (2) \\
&\leftrightarrow (\ \Diamond(A \wedge [1]^{+}\neg B) \rightarrow \langle 1\rangle B \wedge \pi\ ) && \text{corollary 7.4.} \\
&\leftrightarrow (\ \Diamond(A \wedge [1]^{+}\neg B) \rightarrow \langle 1\rangle B\ )
\end{aligned}
$$

(since $\vdash \Diamond(A \wedge [1]^{+}\neg B) \rightarrow \Diamond\pi$ and $\vdash \Diamond\pi \rightarrow \pi$).

This concludes the proof of the theorem.

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Abstract

Dzhaparidze's polymodal logic (referred in this paper as *GLP*) is an important joint provability logic. It corresponds to the case in which the powers of the theories grow so fast that every theory in the sequence proves everything that the previous theories prove and also proves each sentence unprovable in a previous theory to be unprovable in this theory. (Clearly, theories in such a sequence – except, possibly, the first – cannot be recursively enumerable). This logic was introduced by G.Dzhaparidze in [2], who also gave an axiomatization and a decision procedure for it. In [3] the author suggested a new approach to this logic and proved the fixed point property and the Craig interpolation property for *GLP*.

In this paper we investigate the *closed fragment* of *GLP*. As usual, there is a ordinal-indexed sequence of closed formulas (in the present case its length is ε_0) which plays the main role in our reasoning. We introduce all the standard notions connected with closed fragments (such as the *universal model*) and prove analogies of all the usual theorems. We also try to give a general approach to these standard notions, for example, to give a general definition of the *ordinal complexity* of an arbitrary modal logic. We also consider the arithmetical complexity of (arithmetical interpretation) of closed formulas.

Finally, we prove that the closed fragment of the provability logic for Σ_1-conservativity predicate is isomorphic to bimodal fragment of *GLP*. Thus, this closed fragment is decidable and its ordinal complexity is (exactly) ω^ω.

THE CLOSED FRAGMENT OF DZHAPARIDZE'S POLYMODAL LOGIC AND THE LOGIC OF Σ_1-CONSERVATIVITY

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Contents:

increasing sequences.

The previous paragraph shows that we often will have to write something like Pr_{T_n} and $\Sigma_{deg(T_n)}$. Since this does not nice, we will write Pr_n and $\Sigma_{(n)}$ instead. Except in section 8, the reader can suppose that some increasing sequence has been fixed and we are considering it.

In this paper we consider the *closed fragment* of *GLP*, i.e. the modal formulas containing no propositional variables (up to *GLP*-provable equivalence). Let us consider an example here.

The well-known theorem of Kent asserts that for every n there is an arithmetical statement Q which is not *PA*-equivalent to any Σ_n-formula and such that $PA\vdash Q\rightarrow \mathrm{Pr}_{PA}\ulcorner Q\urcorner$. Proofs of this theorem usually use fixed points. But using the closed fragment of *GLP*, we can give examples of such formulas Q without using fixed points. Namely, it is enough to put $Q:=\mathrm{Pr}_0\ulcorner \mathrm{Pr}_1\ulcorner\bot\urcorner\urcorner\wedge \mathrm{Pr}_1\ulcorner\bot\urcorner$, where $T_0=PA$ and T_1 is chosen such that $deg(T_1)>n$, i.e. the arithmetical interpretation of the closed formula $\theta:=[1]\bot\wedge\Box[1]\bot$. Obviously, $PA\vdash Q\rightarrow \mathrm{Pr}_{PA}\ulcorner Q\urcorner$, which simply means $GLP\vdash\theta\rightarrow\Box\theta$. We claim that Q is not *PA*-equivalent to any $\Sigma_{(1)-1}(=\Sigma_{deg(T_1)-1})$-formula. Suppose not, then $\neg Q$ is *PA*-equivalent to some formula in $\Sigma_{(1)}$. By the provable $\Sigma_{(1)}$-completeness of T_1 we have $PA\vdash\neg Q\rightarrow \mathrm{Pr}_1\ulcorner\neg Q\urcorner$, or $GLP\vdash\neg\theta\rightarrow[1]\neg\theta$. We have:

$GLP\vdash\neg[1]\bot\vee\neg\Box[1]\bot\ \rightarrow\ [1](\neg[1]\bot\vee\neg\Box[1]\bot)$

(*) $GLP\vdash\neg[1]\bot\ \rightarrow\ [1](\neg[1]\bot\vee\neg\Box[1]\bot)$

$GLP\vdash[1](\neg[1]\bot\vee\neg\Box[1]\bot)\wedge\Box[1]\bot\ \rightarrow\ [1]\Box[1]\bot$

$\rightarrow[1](\neg[1]\bot)$

$\rightarrow[1]\bot$ (Löb's axiom)

$GLP\vdash\neg[1]\bot\wedge\Box[1]\bot\ \rightarrow\ [1]\bot$ (by (*))

$GLP\vdash\Box[1]\bot\ \rightarrow\ [1]\bot$

$GLP\vdash[1]\bot$ (Löb's rule)

which is obviously incorrect.[2]

It is wellknown that such sentence *Q cannot* be constructed by using the closed fragment of any other (known) provability logic. So, the closed fragment of *GLP* is not without its uses.

The plan of this paper is the following: in section **2** we give a brief summary of results about *GLP*. The reader can find all proofs in [3]. (As we noted above, the arithmetical completeness of *GLP* was originally proved in [2]). In fact, we will not use most of these results, except for the soundness results in theorems 2.6 and 2.11, which are quite routine.

In sections **3**,**4** we give a *syntactic* investigation of the closed fragment of *GLP*. So, to read these sections the reader does not have to know anything about *GLP* at all. Namely, in section **3** we define a closed formula $D(\alpha)$ for every ordinal $\alpha<\varepsilon_0$, and prove a kind of "monotonicity" theorem for this sequence. In section **4** we prove that every closed formulas has a *GLP*-equivalent in a special *normal* form. In particular, it will be shown that every closed formula is a *GLP*-equivalent of a Boolean combination of some formulas $D(\alpha)$.

In section **5** we introduce the *universal model* for the closed fragment of *GLP*, a useful tool in the investigation of this system.

Section **6** is a digression. We discuss some general concepts pertaining to the closed fragment of any modal system. In particular, we give a general definition of the *ordinal complexity* of a modal logic or of its closed fragment.

In section **7** we consider the problem of the arithmetical complexity of arithmetical interpretation of closed formulas. We also consider this problem from an "internal" point of view. In particular, we prove that if we add an additional predicate "*Q* is

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[2] Note that for this proof we do not need the arithmetical completeness of *GLP*. We can think of *GLP* as standing for the set of arithmetically valid modal formulas, the only thing we need is that certain fixed formulas (such as Lob's axiom) are arithmetically valid.

2.A. Modal language and arithmetical semantics

Definition 2.1. A *theory* T of *degree* n (we write $deg(T)=n$) is a set of arithmetical formulas with an associated arithmetical formula $Pr(\cdot)$ (with one free variable) such that:

1. $Pr(\cdot)\in\Sigma_n$

and for any arithmetical statements A,B

2. $PA\vdash Pr\ulcorner A\to B\urcorner \to (Pr\ulcorner A\urcorner \to Pr\ulcorner B\urcorner)$
3. $A\in\Sigma_n \Rightarrow PA\vdash A\to Pr\ulcorner A\urcorner$ (*provable* Σ_n*-completeness*)
4. $A\in\Sigma_n$, $\mathbb{N}\models Pr\ulcorner A\urcorner \Rightarrow \mathbb{N}\models A$ (Σ_n*-soundness*)
5. $PA\vdash A \Rightarrow PA\vdash Pr\ulcorner A\urcorner$.

and T is exactly set of arithmetical formulas Q such that $Pr\ulcorner Q\urcorner$ holds in $\mathbb{N}$.

The theory T is *sound*, if property 4 holds for all n.

Definition 2.2. A sequence of theories T_0, $T_1,\ldots,$ $T_n,\ldots$ (finite or infinite) is an *increasing sequence*, if it satisfies the following conditions:

1. $deg(T_0)<deg(T_1)<\ldots<deg(T_n)<\ldots$
2. for any statement A and $n<k$ $PA\vdash Pr_n\ulcorner A\urcorner \to Pr_k\ulcorner A\urcorner$.

EXAMPLES:

(a) (see [1]) for any sound theory T the ω-extension of T is given by all theorems of T and all formulas of the form $\forall x\ Q(x)$, if $\forall n\in\omega\ T\vdash Q(\underline{n})$; we denote it T^ω;

(b) for any number $n>0$ the theory T^σ_n is given by all axioms of PA and all *true* Σ_n-formulas[3].

CLAIM.

(a) for any sound theory T T^ω *is a theory and* $deg(T^\omega)=deg(T)+2$;

(b) for any $n>0$ T^σ_n *is a sound theory and* $deg(T^\sigma_n)=n$;

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[3] Or if you like all true Π_{n-1} formulas.

Inference rules:

1. *Modus ponens.*

2. $\frac{A}{\Box A}$ (*[0]-necessitation*)

Some theorems of *GLP* are:

$$\left.\begin{array}{l}5.\ [k]A \rightarrow [n]A. \\ 6.\ [k]A \rightarrow [n][k]A\end{array}\right\} k\le n$$

$$\left.\begin{array}{l}7.\ \langle k\rangle A \rightarrow [n]\langle k\rangle A \\ 8.\ [n]([k]A \rightarrow A)\end{array}\right\} k<n$$

Definition 2.5. The logic GLP^{ω} is the minimal set of $\mathcal{L}$-formulas closed under MP and containing the following axioms:

1. All theorems of *GLP*.
2. $[n]A \rightarrow A$, $n\ge 0$.

Theorem 2.6. *Arithmetical completeness of GLP.*
Let $T_0, T_1, \ldots, T_n, \ldots$ be an increasing sequence. Then for any modal formula A $GLP \vdash A$ if and only if for every arithmetical interpretation f $PA \vdash f(A)$.

Theorem 2.7. *Arithmetical completeness of GLP^{ω}.*
Let $T_0, T_1, \ldots, T_n, \ldots$ be an increasing sequence of sound theories. Then for any modal formula A $GLP^{\omega} \vdash A$ if and only if for every arithmetical interpretation f $\mathbb{N} \models f(A)$.

2.B. An auxiliary modal logic LN. Kripke semantics.

Definition 2.8. A *model* $\mathcal{K}=\langle K, R^0, R^1, \ldots, R^N, \Vdash\rangle$ consists of a nonempty set K (the *support* of $\mathcal{K}$), an *accessibility relation* R^i for the modal operator $[i]$ $(0\le i\le N)$, and a *forcing relation* $\Vdash$, possessing the following properties:

1. for any i R^i is transitive, irreflexive and wellfounded;

modal formula, if

$$\forall k\ \forall B\colon\ [k]B \subseteq A\ \forall n\colon\ k<n\leq N\ (\ \mathcal{K}\models\ [k]B\ \rightarrow\ [n]B\ ).$$

Definition 2.13. For any modal formula A we define the modal formulas ΔA, $\Delta^{+}A$ and $M(A)$ as follows:

$$\Delta A\ :=\ \bigwedge_{0\leq i_1<i_2<\ldots<i_n\leq N} [i_1][i_2]\ldots[i_n]A$$

$$\Delta^{+}A\ :=\ A\wedge\Delta A$$

$$M(A)\ :=\ \bigwedge_{[k]B\subseteq A,\ k<n\leq N} \Delta^{+}([k]B\rightarrow[n]B).$$

Theorem 2.14. *Consider an arbitrary increasing sequence of the theories $T_0(\cdot)$, $T_1(\cdot),\ldots$ $T_N(\cdot)$. Then for any modal formula A the following statements are equivalent:*

1) $GLP\vdash A$.

2) For any arithmetical interpretation f $PA\vdash f(A)$.

3) for any A-complete LN-model $\mathcal{K}$ $\mathcal{K}\models A$.

4) $LN\vdash M(A)\rightarrow A$.

So, GLP is decidable.

Definition 2.15. For any modal formula A we put

$$H(A)\ :=\ \bigwedge_{[n]B\subseteq A} ([n]B\rightarrow B)$$

Theorem 2.16. *Let $T_0, T_1,\ldots,T_N$ be an increasing sequence of sound theories. Then for any modal formula A which does not contain $[N]$ the following statements are equivalent:*

1) $GLP^{\omega}\vdash A$.

2) $GLP\vdash H(A)\rightarrow A$.

3) $GLP\vdash [N]A$.

4) for all interpretation f $\mathbb{N}\models f(A)$.

So, GLP^{ω} is decidable.

$\alpha^{+\beta} := min\{\gamma>\alpha \,|\, d(\gamma)\geq\beta\}$
$\alpha^{-\beta} := (\alpha^{+\beta})^{-}$

Definition 3.2. It will be convenient also to use the following "linear" notations:

$\Omega(\lambda) := \omega^{\lambda}$

$B(\alpha,\beta) := \alpha^{+\beta}$

and their iterations:

$d^{0}(\alpha) = \Omega^{0}(\alpha) := \alpha$

$d^{n}(\alpha) := d(d(\ldots d(\alpha)\ldots))$ (n times)

$\Omega^{n}(\alpha) := \Omega(\Omega(\ldots\Omega(\alpha)\ldots))$ (n times)

$\omega_{n} := \Omega^{n}(1)$

$B(\alpha) := \alpha$

$B(\alpha_{0},\alpha_{1},\ldots\alpha_{n}) := B(\alpha_{0},B(\alpha_{1},\ldots B(\alpha_{n-1},\alpha_{n})\ldots)$

We summarize here several elementary properties of these operations:

Proposition 3.3.

a) $\alpha^{-}<\alpha$ *and* $d(\alpha^{-})\geq d(\alpha)$

b) $\alpha^{+\beta}>\alpha$ *and* $d(\alpha^{+\beta})=\beta$

c) $\alpha^{-\beta}\leq\alpha$ *and* $d(\alpha^{-\beta})\geq\beta$

d) if $\beta^{-}<\alpha<\beta$ *then* $\beta^{-}\leq\alpha^{-}$ *and* $d(\alpha)<d(\beta)$

e) $(\beta^{-})^{+d(\beta)}=\beta$

f) if $\gamma>\beta$, *then* $B(B(\alpha,\beta),\gamma)=B(\alpha,\gamma)$

We turn to closed formulas:

Definition 3.4. For any modal formula A the formula $\uparrow A$ is the result of raising each $[n]$ in A to $[n+1]$. We also will use the operator "$\uparrow$" for sets of formulas, models, etc. $\uparrow^{n}A$ denotes $\uparrow\uparrow\ldots\uparrow A$ (n times).

Definition 3.5. For each ordinal α we define the formula $D(\alpha)$

$\Delta_{\Gamma*<\gamma>} := \Delta_\gamma \Delta_\Gamma A.$

We write $\Gamma=<\gamma_1,\ldots,\gamma_n><\alpha$, if and only if $\gamma_1<\alpha \wedge \gamma_2<\alpha \wedge \ldots \wedge \gamma_n<\alpha$.

Thus using definitions 3.5 and 3.6 we can write

$$H(\omega^{\lambda_1} + \omega^{\lambda_2} + \ldots + \omega^{\lambda_n}) = \Delta_{<\lambda_1,\ldots,\lambda_n>}\Box\bot,$$

where as usual $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

Lemma 3.8. *Let $\beta^- \leq \alpha < \beta$. Then there is a sequence of ordinals $\Gamma < d(\beta)$ such that $H(\alpha)=\Delta_\Gamma H(\beta^-)$.*

Proof. Induction on α. The basis $\alpha=\beta^-$ is trivial (take $\Gamma=<>$).

Suppose that $\beta^-<\alpha<\beta$. By proposition 3.3.d), $\beta^- \leq \alpha^- < \alpha$ and $d(\alpha)<d(\beta)$. By the induction hypothesis for α^-, $H(\alpha^-)=\Delta_{\Gamma'}H(\beta^-)$ for some $\Gamma'<d(\beta)$, and therefore

$$H(\alpha) = \Box D(\alpha) = \Box(H(\alpha^-)\vee\uparrow D(d(\alpha))) = \Delta_{d(\alpha)}H(\alpha^-) = \Delta_{d(\alpha)}\Delta_{\Gamma'}H(\beta^-) = $$
$$= \Delta_\Gamma H(\beta^-),$$

where $\Gamma:=\Gamma'*<d(\alpha)> < d(\beta)$. QED.

Theorem 3.9. *If $\alpha<\beta$, then $GLP \vdash \Box D(\alpha) \to D(\beta)$.*

Proof.

In this proof $\vdash A$ denotes $GLP \vdash A$.

The case $\alpha=0$ is easy. Suppose that $\alpha>0$.

Induction hypothesis:

for all $\alpha<\beta<\delta$, $\quad \vdash \Box D(\alpha) \to D(\beta)$

Our goal is to prove that

if $\alpha<\delta$, then $\vdash \Box D(\alpha) \to D(\delta)$

CLAIM. *If $\alpha<\beta<\delta$, then*

$$\vdash \Delta_\beta(\Delta_\alpha\sigma \to \sigma) \quad \text{or} \quad \vdash \neg\uparrow D(\beta) \to (\Delta_\alpha\sigma \to \sigma).$$

(The first formula is the box of the second).

4. Normal Forms of Closed Formulas. The Main Theorem.

Let GLP_0 be the logic LN + $\{\Box D(\alpha) \to D(\beta) \mid \alpha<\beta<\varepsilon_0\}$ + $\{\Box\bot \to [n]\bot \mid n\in\omega\}$ + a new inference rule ($\vdash A \Rightarrow \vdash \uparrow A$). As we proved above, $GLP_0 \subseteq GLP$. In fact, GLP is conservative over GLP_0 with respect to closed formulas and we will prove this shortly; but it is essential for us to use in the following reasoning GLP_0 instead of GLP. So, *for the remainder of this section* $\vdash A$ *denotes* $GLP_0 \vdash A$.

Corollary 4.1.

a) For any $n\in\omega$ *and* $\alpha\le\beta$ $\vdash \uparrow^n H(\alpha) \to \uparrow^n H(\beta)$.

b) If $\alpha^- = 0$ *and* $\alpha\ne 1$, *then* $\vdash D(\alpha) \leftrightarrow \uparrow D(d(\alpha))$.

Proof. Easy.

Lemma 4.2.

a) For any α, β

$$\vdash \neg\uparrow D(\beta) \to (H(\alpha) \leftrightarrow H(\alpha^{-\beta})).$$

Moreover, if $\beta>0$ *and* $\alpha^{-\beta}=0$, *then*

$$\vdash \neg\uparrow D(\beta) \to (H(\alpha) \leftrightarrow \bot).$$

b) For any α, β

$$\vdash \neg\uparrow H(\alpha) \to (H(\beta^{+\alpha}) \to H(\beta));$$

moreover, if $\beta=0$, *then*

$$\vdash \neg\uparrow H(\alpha) \to (H(\beta^{+\alpha}) \to \bot).$$

c) For any α, β

$$\vdash H(\alpha) \vee \uparrow D(\beta) \leftrightarrow D(\alpha^{+\beta});$$

moreover, for $\alpha=0$ *and* $\beta>0$

$$\vdash \uparrow D(\beta) \leftrightarrow D(\omega^{\beta}).$$

d) for any α *and* $n<m$

$$\vdash \uparrow^n H(0) \vee \uparrow^m H(\alpha) \leftrightarrow \uparrow^m H(\alpha).$$

Proof.

$$\vdash \uparrow^{n-1}H(\beta_{n-1}) \vee \uparrow^{n}H(\beta_n) \quad \leftrightarrow \quad \uparrow^{n-1}D(B(\beta_{n-1},\beta_n+1))$$
$$\vdash \uparrow^{n-2}H(\beta_{n-2}) \vee \uparrow^{n-1}H(\beta_{n-1}) \vee \uparrow^{n}H(\beta_n) \quad \leftrightarrow \quad \uparrow^{n-2}D(B(\beta_{n-2},\beta_{n-1},\beta_n+1))$$
. . . .

for any $j<n$ (including $j=-1$)

$$\vdash \mathbb{W}\,\{\uparrow^{i}H(\beta_i)\,|\,j<i\le n\} \quad \leftrightarrow \quad \uparrow^{j+1}D(B(\beta_{j+1},\beta_{j+2},\ldots,\beta_{n-1},\beta_n+1)).$$

Hence, by lemma 4.2.a),

$$\vdash \neg\,\mathbb{W}\,\{\uparrow^{i}H(\beta_i)\,|\,j<i\le n\} \quad \to \quad (\uparrow^{j}H(\alpha_j) \leftrightarrow \uparrow^{j}H(\alpha'_j))$$
$$\to \quad (\uparrow^{j}H(\beta_j) \leftrightarrow \uparrow^{j}H(\beta'_j))$$

where

$$d(\alpha'_j),d(\beta'_j) \ge B(\beta_{j+1},\beta_{j+2},\ldots,\beta_{n-1},\beta_n+1)$$

(we must put $\alpha'_j:=\infty$ whenever $\alpha_j=\infty$, and $\beta'_j:=0$ whenever $\beta_j=0$). Thus, we can replace in B the ordinals α_j,β_j by α'_j,β'_j. If we do so for all j from $n-1$ to 0, we can write B in the form:

$$\vdash B \quad \leftrightarrow \quad \mathbb{W}\,\{\neg\uparrow^{i}H(\alpha'_i)\,|\,i\le m\} \vee \mathbb{W}\,\{\uparrow^{i}H(\beta'_i)\,|\,i\le n\}.$$

where for any $j<n$

$$d(\alpha'_j),d(\beta'_j) \ge B(\beta'_{j+1},\beta'_{j+2},\ldots,\beta'_{n-1},\beta'_n+1)$$

(we put $\alpha'_n:=\alpha_n$, $\beta'_n:=\beta_n$).

In conclusion, if for some j $\alpha'_j\le\beta'_j$, by corollary 10.11 $\vdash B$. Therefore, we can assume $\alpha'_j>\beta'_j$ for any $j\le n$. Thus, we have reduced B to normal form.

b) Suppose the formula A has a normal form. Evidently, we can assume that A contains one conjunct only:

(*) $$A \;=\; \bigvee_{0\le i\le n} \uparrow^{i}H(\alpha_i)\to\uparrow^{i}H(\beta_i) \quad\vee\quad \bigvee_{n<i\le m} \neg\uparrow^{i}H(\alpha_i)$$

LEMMA.

$$\vdash \Box A \quad \leftrightarrow \quad H(B(\beta_0,\beta_1,\ldots,\beta_n+1)).$$

(if $n=-1$, $\vdash \Box A \;\leftrightarrow\; H(0)=\Box\bot$).

Proof. The case $n=-1$ is trivial. Suppose that $n\ge0$.

As we proved in the proof of claim a),

$$\vdash \bigvee_{0\le i\le n} \uparrow^{i}H(\beta_i) \quad \leftrightarrow \quad D(B(\beta_0,\beta_1,\beta_2,\ldots,\beta_{n-1},\beta_n+1)).$$

so the part "$\leftarrow$" is proved.

To prove the converse implication we will decrease each α_i as much as possible; namely, as we noted above

$$\alpha_i \geq B(\beta_i,\beta_{i+1},\ldots,\beta_n+1),$$

hence, it is sufficient to consider the case

$$\alpha_i = B(\beta_i,\beta_{i+1},\ldots,\beta_n+1),\ i\leq n; \qquad \alpha_i=0,\ n<i\leq m.$$

But now by lemma 10.12.b),

$$(**) \qquad \vdash \neg\uparrow^{i+1}H(\alpha_{i+1}) \to (\uparrow^{i}H(\alpha_i) \to \uparrow^{i}H(\beta_i)),\quad 0\leq i\leq n$$
$$\to (\uparrow^{i}H(\alpha_i) \to \bot), \quad n<i<m$$

because for $i\leq n$

$$\alpha_i = \beta_i^{+(\alpha_{i+1})}.$$

Using (**) we can eliminate from A all disjuncts of the form $\neg\uparrow^{i}H(\alpha_i)$, except $H(\alpha_0)$. Thus we obtain

$$\vdash A \leftrightarrow \neg H(\alpha_0) \vee H(\beta_0) \vee \uparrow H(\beta_1) \vee \ldots \vee \uparrow^{n}H(\beta_n),$$

where

$$\alpha_0 = B(\beta_0,\beta_1,\ldots,\beta_n+1).$$

But we already noted above that

$$\vdash \bigvee_{0\leq i\leq n} \uparrow^{i}H(\beta_i) \leftrightarrow D(B(\beta_0,\beta_1,\ldots,\beta_n+1)) = D(\alpha_0);$$

thus,

$$\vdash A \leftrightarrow \neg H(\alpha_0) \vee D(\beta_0) = \Box D(\alpha_0) \to D(\alpha_0),$$

and by Löb's axiom

$$\vdash \Box A \leftrightarrow \Box D(\alpha_0) = H(B(\beta_0,\beta_1,\ldots,\beta_n+1)).$$

Using the lemma and corollary 7.4, one can see that if A has the normal form (*), then

$$\vdash [k]A \leftrightarrow \bigvee_{0\leq i\leq n,\ i<k} \uparrow^{i}H(\alpha_i) \to \uparrow^{i}H(\beta_i) \vee \bigvee_{n<i<k} \neg\uparrow^{i}H(\alpha_i) \vee \uparrow^{k}H(\chi),$$

where

$$\chi = B(\beta_k,\beta_{k+1},\ldots,\beta_n+1),\ k\leq n; \qquad \chi=0,\ k>n.$$

This completes the proof of the theorem.

Corollary 4.5.

a) Each closed formula is a GLP-equivalent to a Boolean combination of some formulas of the form $\uparrow^{n}H(\alpha)$.

b) Each closed formula is a GLP-equivalent to a Boolean

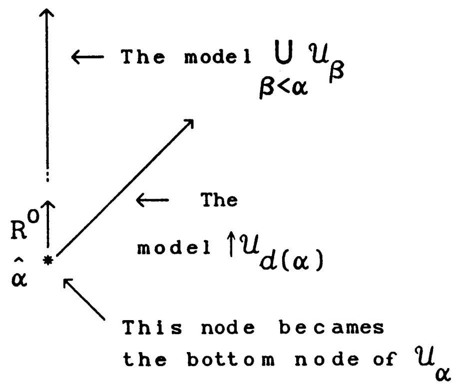
5. The Universal Model.

Definition 5.1. We define the *universal model* $\mathcal{U}$:

a) *Informal definition.* For any $\alpha<\varepsilon_0$ we define the model $\mathcal{U}_\alpha$:

$\mathcal{U}_0$ consists only of one node $\hat{0}$.

Suppose that $\{ \mathcal{U}_\beta \mid \beta<\alpha \}$ are already constructed. Here is a picture of $\mathcal{U}_\alpha$:



and we put $\mathcal{U} := \bigcup\{\mathcal{U}_\alpha \mid \alpha<\varepsilon_0\}$.

b) *Formal definition.*

$\mathcal{U} := \langle U, R^0, R^1, \dots, R^n, \dots \rangle$,

where

$U := \{ \langle\alpha_0,\alpha_1,\dots\alpha_n\rangle \mid$ 1) $\forall i<n \quad \alpha_i\neq 0 \wedge d(\alpha_i)\neq 0 \wedge \alpha_{i+1}\leq d(\alpha_i)$
2) $d(\alpha_n)=0 \vee \alpha_n=0 \quad \}$;

$\langle\alpha_0,\alpha_1,\dots,\alpha_n\rangle R^k \langle\beta_0,\beta_1,\dots,\beta_m\rangle \quad :\Longleftrightarrow$

1) $\forall i<k \quad \alpha_i=\beta_i$

2) $\alpha_k>\beta_k$

3) $\forall i\geq k, i<m \quad \beta_{i+1}=d(\beta_i)$.

$\hat{\alpha}$ denotes $\langle\alpha, d(\alpha), d^2(\alpha), \dots, d^n(\alpha)\rangle$, where $n:=max\{i \mid d^i(\alpha)>0\}$.

($\hat{0} := \langle 0\rangle$)

$\mathcal{U}_\alpha := \langle U_\alpha, R^0, R^1, \dots, R^n, \dots\rangle$,

where

$U_\alpha := \{ \langle\alpha_0,\alpha_1,\dots\alpha_n\rangle\in U \mid \alpha_0\leq\alpha \}$.

Thus, the universal model looks like a tree. The "trunk" of

Proof. By corollary 9.15.b), $GLP\vdash A \iff GLP_0\vdash A$.

Part 1. $GLP_0\vdash A \Rightarrow \mathcal{U}\models A$.

Since $\mathcal{U}$ is an *LN*-model, we need to check three facts:

1) $\mathcal{U}\models \Box\bot \to [n]\bot$

2) $\mathcal{U}\models \Box D(\alpha) \to D(\beta)$ whenever $\alpha<\beta$

3) $\mathcal{U}\models A \;\Rightarrow\; \mathcal{U}\models \uparrow A$.

The first fact is trivial ($a\vdash\Box\bot \iff a=\hat{0}$), the second is a consequence of lemma 5.3 and the third is a consequence of proposition 5.2.b).

Part 2. $\mathcal{U}\models A \Rightarrow GLP_0\vdash A$.

Suppose that $GLP_0\nvdash A$. We can assume that A has the normal form; let B be a conjunct of A:

$$B \;=\; \bigvee_{0\le i\le n} \uparrow^i H(\alpha_i) \to \uparrow^i H(\beta_i) \;\vee\; \bigvee_{n<i\le m} \neg\uparrow^i H(\alpha_i).$$

If $n=-1$, $\hat{0}\nVdash B$. Assume that $n\ge 0$. Let $\alpha'_i := B(\beta_i, \beta_{i+1}, \ldots \beta_n+1)$, where $i\le n$. Evidently, $\alpha_i \ge \alpha'_i$. Using corollary 11.4, one can prove that

$$\hat{\alpha}'_0 = \langle\alpha'_0, \alpha'_1, \ldots, \alpha'_n\rangle \nVdash B,$$

and thus $\hat{\alpha}'_0 \nVdash A$. QED.

In fact, we have proved that if A is in normal form and A is not the empty conjunction, then $\mathcal{U}\not\models A$, and therefore $GLP_0\nvdash A$ and $GLP\nvdash A$. We also have proved that every non-provable (in GLP or GLP_0) closed formula is false in some node *on the main axle*. This allows us to give the following natural definition:

Definition 5.6. The *trace* of the closed formula A, denoted by tr(A), is the set of ordinals $\alpha < \varepsilon_0$ such that $\hat{\alpha}\nVdash A$.

As usual, trace "conversely commutes" with all Boolean connectives (i.e. $\mathrm{tr}(A\vee B)=\mathrm{tr}(A)\cap\mathrm{tr}(B)$, etc.); the trace of a formula is empty if and only if the formula is provable in GLP and so defines (closed) formula uniquely.

However, traces do not have to be clopen in the order topology. For example, $\mathrm{tr}([1]\bot)=\{\alpha>0 \mid \alpha \text{ is limit}\}$ which is obviously the "closed fragment of *GLP*", i.e. a description of the closed modal formulas up to *GLP*-provable equivalence. However, before we undertake this investigation it is necessary to discuss some general concepts connected with provability logics and their closed fragments.

Let us begin by considering several examples.

The first is the usual Gödel-Löb provability logic *GL*, in the language with only one modal operator $\Box$. Assign to each natural number $n\in\omega$ the closed formula $C_n:=\Box^n\bot$. As G.Boolos proved in [4], each closed formula is *GL*-equivalent to a Boolean combination of the formulas C_n, $n\in\omega$. Furthermore, one can note that the sequence

$$C_1, C_2, \ldots, C_n, \ldots \quad , n\in\omega$$

is *decreasing*, i.e. $GL\vdash C_n\to C_m$ whenever $n\leq m$.

Next consider Carlson's bimodal provability logic CSM_1[5]. This logic formalizes provability in two r.e. extensions of *PA* such that one of them contains the other and is *reflexive* over it. Thus the modal language contains two modal operators $\Box=[0]$ and $[1]$ (Visser in [5] writes Δ and $\Box$ respectively) and the logic contains the schemas:

$$\Box A\to[1]A$$

and

$$[1](\Box A\to A).$$

Note also that $CSM_1\vdash[i]A\to[j][i]A$ for any $i,j\in\{0,1\}$.

Now for any ordinal α below ω^2, i.e. $\alpha=n+\omega.m$, we define the closed formula

$$C_\alpha := [1]^m\Box^n\bot.$$

(It is supposed that $\Box^0A=A$, etc.). In [5] it is proved that $\{C_\alpha|\alpha<\omega^2\}$ is a decreasing sequence and any closed formula is a Boolean combination of these formulas.

We can easily generalize this example by considering a sequence of r.e. extensions of *PA* (finite or infinite) $T_0, T_1, \ldots$

---

[5]There are many notations for this logic : $PRL_{ZF}, \ldots$. Ours is due to Visser [5].

Trivially, L is well-founded if and only if $\mathcal{P}$ is.

Henceforth we will denote modal formulas and their equivalence classes by the same letters.

Definition 6.4. Let $\mathcal{P}$ be a wellfounded partial order. The *ordinal complexity of* $\mathcal{P}$ is the supremum of ordinals ξ such that there exists a $\mathcal{P}$-linear sequence of elements from P of length ξ:

$$\{ p_\alpha | \alpha<\xi \},\ \forall\alpha,\beta<\xi\ (\ \alpha<\beta \Rightarrow p_\alpha \prec p_\beta\ ).$$

Given a well-founded logic L we can now define the *ordinal complexity of* L as the ordinal complexity of the associated partial order.

The main question now is how to calculate ordinal complexities. We will use the following simple lemma (below ξ denotes an arbitrary limit ordinal):

Lemma 6.5. *Let $\mathcal{P}=\langle P,\prec\rangle$ be an arbitrary partial order. Suppose that there exists a sequence of elements from P $\{a_\alpha|\alpha<\xi\}$ ordered by ordinals $<\xi$ such that the following three conditions hold:*

1.(linearity) if $\alpha<\beta$, then $a_\alpha \prec a_\beta$;

2.(unboundedness) there is no $x\in P$ such that for all α $a_\alpha \prec x$;

3.(density) if $x\prec y$, then there exists α s.t. $a_\alpha \prec y$, but $a_\alpha \not\prec x$.

Then $\mathcal{P}$ is wellfounded and its ordinal complexity is ξ.

Proof. For any $x\in P$ we define $a(x)\in\xi$ as follows:

$$a(x)\ :=\ \min\ \{\ \alpha<\xi\ |\ a_\alpha \not\prec x\ \}.$$

(This set is non-empty by unboundedness). Suppose that $x\prec y$. We claim that $a(x)<a(y)$. Indeed, linearity implies that $a(x)\le a(y)$ and if $a(x)=a(y)$, then $\{\ \alpha<\xi\ |\ a_\alpha \not\prec x\ \} = \{\ \alpha<\xi\ |\ a_\alpha \not\prec y\ \}$. The last equality is impossible by density. So we have proved that $a(x)<a(y)$.

But this property of a immediately implies that $\mathcal{P}$ is wellfounded and its ordinal complexity is $\le\xi$. On the other hand, by the definition, the ordinal complexity is $\ge\xi$. Thus, the lemma is

Conclusion. *The ordinal complexity of (the closed fragment of)* $CSM_1^{(N)}$ *is exactly* ω^N.

Now we can consider our formulas $H(\alpha), D(\alpha)$ from the point of view of applying theorem 6.6. The first condition – linearity – is theorem 3.9. Unboundedness is obvious, because for any modal formula A which does not contain $[i]$ with $i \geq N$ we have

$$GLP \vdash A \iff \Box[N]\bot = H(\omega_N) \vdash A;$$

density for *closed* formulas is a simple consequence of the last theorem. Thus, we have proved the theorem:

Theorem 5. *The ordinal complexity of the closed fragment of GLP is* ε_0. *(or* ω_N *in the case of GLP with only N modal operators).*

Conjecture. *For any ordinal* $\zeta < \varepsilon_0$ *and for any modal formula A if for any* $\alpha < \zeta$ $GLP \vdash H(\alpha) \to A$, *then* $GLP \vdash H(\zeta) \wedge D(\zeta) \to A$.

If the conjecture holds, the ordinal complexity of GLP is ε_0.

It seems that the best way to prove this statement would be to introduce models for *GLP* which similar to the universal model. However, this seems rather difficult.

7. A Hierarchy of Closed Formulas.

Let us introduce classes of closed modal formulas Σ_n, Π_n, B_n, Σ_n^L, Π_n^L, Δ_n^L, B_n^L (where L is an arbitrary modal system) for every $n \geq 0$ as follows:

Definition 7.1. Σ_0, Σ_1, ... , Σ_n, ... are the minimal sets of

$$L\vdash \ [n](p\vee B) \leftrightarrow [n]p\vee B.$$

Proof. By corollary 7.3.(c),

$$L\vdash \ B \rightarrow [n]B \quad \text{and} \quad L\vdash \ \neg B \rightarrow [n]\neg B.$$

Hence, $L\vdash [n]p\vee B \rightarrow [n](p\vee B)$. On the other hand,

$$L\vdash \ \neg B \wedge [n](p\vee B) \ \rightarrow \ [n]\neg B \ \rightarrow \ [n]p.$$

QED.

In the sequel we will be interested only in the classes Σ_n^{GLP}, Π_n^{GLP}, Δ_n^{GLP}, B_n^{GLP}. In particular, we will prove that they are decidable and $\Delta_n^{GLP}=B_{n-1}^{GLP}$. So, we often will omit *GLP* and write Σ_n, Π_n, ...

Now we are going to introduce "the extended universal model" $\mathcal{V}$, which, intuitively, is $\mathcal{U}$ plus one "infinite" node, corresponding to the standard model of *PA*. We also will define its restrictions $\mathcal{V}_1$, $\mathcal{V}_2$, ..., $\mathcal{V}_n$ such that $\mathcal{V}_0$ will be exactly "the main axis" (of $\mathcal{V}$), $\mathcal{V}_1$ is $\mathcal{V}$ plus all its "immediate" branches, etc. Intuitively, $\mathcal{V}_n$ is a kind of "universal model" for B_{n-1} (or Δ_n).

Definition 7.5.

$\mathcal{V} := \langle V, R^0, R^1, \ldots, R^n, \ldots\rangle$,

where

$V := \{ \langle\alpha_0,\alpha_1,\ldots\alpha_n,\ldots\rangle \mid \forall i \ \alpha_{i+1}\le d(\alpha_i) \}$

and α_i range over ordinals less than *or equal* to ε_0.

We define $d(\varepsilon_0):=\varepsilon_0$.

$\langle\alpha_0,\alpha_1,\ldots,\alpha_n,\ldots\rangle R^k \langle\beta_0,\beta_1,\ldots,\beta_m,\ldots\rangle \ :\Leftrightarrow$

1) $\forall i<k \quad \alpha_i=\beta_i$

2) $\alpha_k>\beta_k$

3) $\forall i\ge k \quad \beta_{i+1}=d(\beta_i)$ (i.e. $\langle\beta_0,\beta_1,\ldots,\beta_m,\ldots\rangle\in V_{k+1}$ – see below)

$\hat{\alpha}$ denotes $\langle\alpha, d(\alpha), d^2(\alpha),\ldots,d^n(\alpha),\ldots\rangle$.

$V_n := \{ \langle\alpha_0,\alpha_1,\ldots\alpha_i,\ldots\rangle \in V \mid \forall i\ge n \ \alpha_i=d(\alpha_{i-1}) \}$ $(n\ge 1)$.

We will need two facts about these topological spaces: 1) they are compact, and 2) the clopen sets listed above are exactly the clopen sets in these topological spaces. Proofs are quite similar for all cases, so, we consider the V_n only.

First of all, we give an equivalent definition of our topology:

Lemma 7.8. *The topological space V_n can be given by the following subbase:*

1. $\{ \langle\alpha_0,\alpha_1,\dots,\alpha_i,\dots,\alpha_{n-1},\dots\rangle \in V \mid \alpha_i > \alpha \}$
2. $\{ \langle\alpha_0,\alpha_1,\dots,\alpha_i,\dots,\alpha_{n-1},\dots\rangle \in V \mid \alpha_i \le \alpha \}$

(where $i<n$ in both cases).

Proof. In topological terminology, we have defined two topological spaces and have to prove that identity map is a homeomorphism.

Part 1. Fix an arbitrary formula $A \in B_{n-1}$. Our goal is to prove that for any $\mathfrak{a} \vdash A$ there exists a set $\mathfrak{A}$, containing $\mathfrak{a}$, open in sense of lemma 7.8 and such that $\forall \mathfrak{b} \in \mathfrak{A}\ \ \mathfrak{b} \vdash A$. Note that if A is a conjunction (disjunction) it is obviously enough to prove this fact for each conjunct (disjunct). Now the normal form theorem for closed formulas immediately implies that it is enough to consider the cases $A = \uparrow^i H(\alpha)$ and $A = \neg\uparrow^i H(\alpha)$, where $i \le n-1$. Now use corollary 5.4.(b).

Part 2. Here we shall prove that for every set mentioned in the statement of lemma 7.8 there exists a modal formula in B_{n-1} which is true in exactly this set. According to corollary 5.4.(b), it is enough to consider the formulas $\uparrow^i H(\alpha)$ and $\neg\uparrow^i H(\alpha)$.

Thus the lemma is proved. Note that this lemma immediately implies that V_n is Hausdorff.

Consider now the set of ordinals $\le\varepsilon_0$ with the natural (order)

PROOF. Let A be an arbitrary clopen set. It is well-known that every compact Hausdorff space is regular (T_3), so for every point x in A there is an open set O_x such that $x \in O_x \subseteq A$. (We have here used that $X \setminus A$ is closed) Since γ is a base, we can assume that $O_x \in \gamma$. Since A is compact (A is closed and $\mathcal{X}$ is compact) and $A = \bigcup_{x \in A} O_x$, there exists a finite subset D of A such that $A = \bigcup_{x \in D} O_x$. This proves the lemma.

So, we have proved theorem 7.10. The following theorem is exactly the statement that V is compact in modal language:

Theorem 7.11. *Let Γ be an arbitrary GLP-consistent set of closed formulas. Then there exists $\alpha \in V$ such that $\alpha \vdash \Gamma$ ($:= \forall A \in \Gamma$ $\alpha \vdash A$). If Γ is maximal GLP-consistent, then any such α is unique.*[8]

Proof. Let $\Gamma = \{A_0, A_1, \ldots\}$; since V is the *universal* model, for each $n \in \omega$ the set $K_n := \{\alpha \in V \mid \alpha \vdash \bigwedge_{i \leq n} A_i\}$ is not empty. Of course, $K_0 \supseteq K_1 \supseteq K_2 \supseteq \ldots$. Since each K_n is clopen and V is compact, for some α, $\alpha \in \bigcap_{i \in \omega} K_i$. Evidently, $\alpha \vdash \Gamma$.

The second part of our statement (which actually expresses that V is Hausdorff) is left to the reader.

Now we are ready to work with V and V_n. The following technical lemma is the main tool we use to investigate the hierarchy of closed formulas in *GLP*:

Lemma 7.12. *For every $n \geq 1$ and for every closed formula B the set $\{ \alpha \in V_n \mid \alpha \vdash [n]^+ B \}$ is open in V_n.*

---

[8] *α does not have to be on the main axis (V_1).* This is the explanation of some "strange" properties of traces. See the discussion at the end of section 3.

Proof.

(a). Implications 1⇒3, 1⇒2, 3⇒4, 2⇒4, 4⇒5 are trivial. So we only have to prove 5⇒1. Suppose that 5 holds for some formula A. Then $GLP \vdash (A \leftrightarrow [n]^{+}A) \wedge (\neg A \leftrightarrow [n]^{+}\neg A)$. Applying lemma 7.12 we get that $\{ \alpha \in V_n \mid \alpha \Vdash A \}$ is clopen in V_n. By theorem 7.10.(b), it coincides with $\{ \alpha \in V_n \mid \alpha \Vdash B \}$ for some $B \in \mathrm{B}_{n-1}$. In particular $\mathrm{tr}(A) = \mathrm{tr}(B)$. By an elementary property of traces, $GLP \vdash A \leftrightarrow B$ and we are done.

(b). As above, we only have to prove 3⇒1. Suppose 3 holds. From the previous paragraph we know that $A \in \mathrm{B}_n$. Thus (using the normal form theorem) we can suppose A has the form:

$$A = \bigwedge_{i=1}^{k} (B_i \vee \uparrow^n C_i \vee \neg\uparrow^n D_i), \quad B_i \in \mathrm{B}_{n-1}, \quad C_i, D_i \in \Sigma_0, \quad GLP \nvdash C_i \vee \neg D_i .$$

Let

$$A_1 = \bigwedge_{i=1}^{k} (B_i \vee \uparrow^n C_i) \in \Sigma_n .$$

We claim that $GLP \vdash A \leftrightarrow A_1$. Suppose not. Obviously, $GLP \vdash A_1 \rightarrow A$. So, there exists $\alpha < \varepsilon_0$ such that $\hat{\alpha} \Vdash A \wedge \neg A_1$, which implies that for some i

$$\hat{\alpha} \Vdash \neg B_i \wedge \neg\uparrow^n C_i \wedge \neg\uparrow^n D_i .$$

Or

$$\hat{d}^n(\alpha) \Vdash \neg C_i \wedge \neg D_i .$$

Since $GLP \nvdash C_i \vee \neg D_i$, there exists β such that $\hat{\beta} \Vdash \neg C_i \wedge D_i$. Because $D_i \in \Sigma_0$, this implies $\hat{d}^n(\alpha) R^0 \hat{\beta}$, or $\beta < d^n(\alpha)$. Put

$$\mathfrak{b} := \langle \alpha, d(\alpha), d^2(\alpha), \ldots, d^{n-1}(\alpha), \beta, d(\beta), d^2(\beta), \ldots \rangle .$$

First, we have $\mathfrak{b} \Vdash \neg\uparrow^n C_i \wedge \uparrow^n D_i$. Secondly, by the definition of our model, $\hat{\alpha} R^n \mathfrak{b}$, which implies $\mathfrak{b} \Vdash \neg B_i$ (equivalently, $\hat{\alpha} \Vdash \neg B_i$). Recall now that $GLP \vdash A \rightarrow [n]A$ and $\hat{\alpha} \Vdash A$. This gives $\mathfrak{b} \Vdash A$, but $\mathfrak{b} \Vdash \neg B_i \wedge \neg\uparrow^n C_i \wedge \uparrow^n D_i$. Contradiction. Thus $GLP \vdash A \leftrightarrow A_1 \in \Sigma_n$ and we are done.

Corollary 7.14.

(a) $\Sigma_n^{GLP} = \{ A \mid GLP \vdash (A \rightarrow [n]A) \wedge (\neg A \rightarrow [n+1]\neg A) \}$;

(b) $\Pi_n^{GLP} = \{ A \mid GLP \vdash (A \rightarrow [n+1]A) \wedge (\neg A \rightarrow [n]\neg A) \}$;

(c) $\mathrm{B}_n^{GLP} = \Delta_{n+1}^{GLP} = \{ A \mid GLP \vdash (A \rightarrow [n+1]A) \wedge (\neg A \rightarrow [n+1]\neg A) \}$

Lemma 7.17. *For any closed formula A and $\alpha\in V$*

$$\alpha\vdash\Box(A\to[n]A)\wedge\Box(\neg A\to[n+1]\neg A) \implies \exists\sigma\in\Sigma_n\ \alpha\vdash\Box(A\leftrightarrow\sigma).$$

Now we can prove the theorem mentioned above:

Theorem 7.18. *Let A be a closed formula and f an arithmetical interpretation based on $T_0,T_1,\dots$, $Q:=f(A)$. Then*

$$PA\vdash\ \Sigma_{(n)}(\ulcorner Q\urcorner)\ \leftrightarrow\ Pr_0\ulcorner Q\to Pr_n\ulcorner Q\urcorner\urcorner\wedge Pr_0\ulcorner\neg Q\to Pr_{n+1}\ulcorner\neg Q\urcorner\urcorner.$$

(where $\Sigma_{(n)}(\ulcorner Q\urcorner) := \exists\sigma\in\Sigma_{(n)}Pr_0\ulcorner Q\leftrightarrow\sigma\urcorner$).

Proof. The part "$\to$" holds for any arithmetical formula Q. So we will prove the converse statement.

Let $\mathcal{M}$ be a model of PA, $\mathcal{M}\models PA$. By the theorem 7.11, there exists a unique $\alpha\in V$ such that for any closed formula A

(*) $\qquad \mathcal{M}\models f(A) \iff \alpha\vdash A.$

Suppose that $\mathcal{M}\models Pr_0\ulcorner f(A)\to Pr_n\ulcorner f(A)\urcorner\urcorner\wedge Pr_0\ulcorner\neg f(A)\to Pr_{n+1}\ulcorner\neg f(A)\urcorner\urcorner$. By (*), $\alpha\vdash\Box(A\to[n]A)\wedge\Box(\neg A\to[n+1]\neg A)$. By lemma 7.17, there exists a formula $\sigma\in\Sigma_n$ (whence $f(\sigma)\in\Sigma_{(n)}$) such that $\alpha\vdash\Box(A\leftrightarrow\sigma)$ and by (*) $\mathcal{M}\models Pr_0\ulcorner f(A)\leftrightarrow f(\sigma)\urcorner$. By the definition of $\Sigma_{(n)}$, $\mathcal{M}\models\Sigma_{(n)}(\ulcorner f(A)\urcorner)$. This completes the proof of the theorem.

Thus, we can express in our modal language great many additional modal operators. Actually, we have given proofs only for $\Sigma_{(n)}(\ulcorner Q\urcorner)$, but now the reader can believe that the same is true for the $\Sigma_{(0)}$-interpolation predicate, and many other analogous predicates. It is much more interesting and less trivial that, in some special cases, we can express the Σ_1-conservativity predicate. Our last section deals with this, perhaps the most beautiful, application of Dzhaparidze's logic.

provable in PA plus all *true* Σ_2-formulas", and $[n]A$ for $n\geq 2$ are translated by an arbitrary increasing sequence of the strong provability predicates.

We will not introduce here any specific modal system for $\mathcal{L}$. So, we write $\vdash A$ if the (modal) formula A is arithmetically valid.

Our basic theorem is the following:

Theorem 8.1.

a) For any modal formula A,

$$\vdash [1]A \leftrightarrow \Box A \vee \neg(\neg A \rhd \neg A \wedge \neg\Box A).$$

b) If A,B are closed, then

$$\vdash A \rhd B \leftrightarrow \Diamond(A \wedge [1]^{+}\neg B) \rightarrow \langle 1 \rangle B.$$

To prove this theorem we need to prove several Σ_1-conservativity principles.

Theorem 8.2. *The following list consists of arithmetically valid modal principles:*

- $A1.$ $A \rhd C \wedge B \rhd C \rightarrow A \vee B \rhd C$
- $A2.$ $A \rhd B \rightarrow (\Diamond A \rightarrow \Diamond B)$
- $A3.$ $A \rhd B \wedge B \rhd C \rightarrow A \rhd C$
- $A4.$ $\Box(A \rightarrow B) \rightarrow A \rhd B$
- $M^*.$ $A \rhd B \rightarrow A \wedge \Diamond C \rhd B \wedge \Diamond C$
- $A5^*.$ $\langle 1 \rangle A \rhd A$
- $P^*.$ $\langle 1 \rangle (A \rhd B) \rightarrow A \rhd B$
- $A2^*.$ $A \rhd B \rightarrow (\langle 1 \rangle A \rightarrow \langle 1 \rangle B)$
- $C.$ $\langle 1 \rangle A \rightarrow \top \rhd A$
- $D.$ $A \rhd (B \wedge \Diamond A) \rightarrow (\Diamond A \rightarrow \langle 1 \rangle B)$

Proof.

Axioms $A1$-$A4$ together with the axioms of the pure provability logic GL form the *pure conservativity logic* CL (see [6]). The arithmetical soundness of the axioms $M^*, A5^*, P^*, A2^*, C$ is quite

(6) ρ three previous formulas and (3)

$\Box\neg A$ (4),(5),(6)

QED.

Corollary 8.3. *The following modal principles and inference rules are arithmetically valid:*

B1. $A \rhd B \to (A\wedge\pi)\rhd(B\wedge\pi)$ by M^*, and *A1-A4*

B2. $\langle 1\rangle^+ A \rhd A$ by *A4*, $A5^*$

B3. if $\vdash A \to \pi$, then

$\vdash A\rhd B \leftrightarrow A\rhd B\wedge\pi$ "$\leftarrow$" by *A3,A4*

"$\to$" by *B1*

B4. if $\vdash B \to \Diamond A$, then

$\vdash A\rhd B \leftrightarrow (\Diamond A \to \langle 1\rangle B)$ "$\to$" by *D*

"$\leftarrow$"

$\Box\neg A \to A\rhd B$ by *A4*

$\langle 1\rangle B \to A\rhd B$ by *C*

B5

a). $\Box A \leftrightarrow (\neg A)\rhd\bot$ "$\leftarrow$" by *A4*

"$\to$" by *A2*

b). $[1]A \leftrightarrow \Box A \vee \neg(\neg A\rhd\neg A\wedge\neg\Box A)$ by *B4*.

Proof of theorem 8.1.

a) has been already proved (*B*5b)).

b) Let A,B be closed formulas. Evidently, to prove the theorem it is sufficient to consider the case when A,B do not contain $\rhd$. Thus, by corollary 7.16 there exists a closed formula $\sigma\in\Sigma_0$ such that

$\vdash B\wedge\Box(\neg A\vee\langle 1\rangle^+B) \to \sigma$

$\vdash \sigma \to \neg A\vee\langle 1\rangle^+B$

Or, if we put $\pi=\neg\sigma$,

(1) $\vdash A\wedge[1]^+\neg B \to \pi$

(2) $\vdash B\wedge\pi \to \Diamond(A\wedge[1]^+\neg B)$.

We have:

R E F E R E N C E S

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