Reasoning About Space: The Modal Way

Marco Aiello[1,2] Johan van Benthem[1] and Guram Bezhanishvili[3]

[1] Institute for Logic, Language and Computation
University of Amsterdam
Plantage Muidergracht 24, 1018 TV Amsterdam
Phone: +31-20-525-5235
E-mail: {aiellom,johan}@science.uva.nl
[2] Intelligent Sensory and Information Systems
University of Amsterdam
Kruislaan 403, 1098 SJ Amsterdam
[3] Department of Mathematical Sciences
New Mexico State University
Las Cruces, NM 88003-0001, USA
E-mail: gbezhani@nmsu.edu

Abstract

We investigate the topological interpretation of modal logic in modern terms, using a new notion of bisimulation. Next, we look at modal logics with interesting topological content, presenting, amongst others, a new proof of McKinsey and Tarski's theorem on completeness of **S4** with respect to the real line, and a completeness proof for the logic of finite unions of convex sets of reals. We conclude with a broader picture of extended modal languages of space, for which the main logical questions are still wide open.

Keywords: spatial reasoning, modal logic, topological interpretation, topological bisimulations, chequered sets

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1 Introduction and purpose

The topological interpretation is one of the oldest semantics for modal languages. Reading the modal box as an interior operator, one can easily show that the modal logic **S4** is complete with respect to arbitrary topological spaces. But there are classical results with much more mathematical content, such as McKinsey and Tarski's beautiful theorem that **S4** is also the complete logic of the reals, and indeed of any metric separable space without isolated points. Even so, the topological interpretation has always remained something of a side-show in modal logic and intuitionistic logic, often tucked away in notes and appendices. The purpose of this paper is to take it one step further as a first stage in a program of independent interest, viz. the modal analysis of space – showing how one can get more generality, as well as some nice new questions. In particular, this paper contains (a) a modern analysis of the modal language as a topological formalism in terms of 'topo-bisimulation' (continuing [1]), (b) a number of connections between topological models and Kripke models, (c) a new proof of McKinsey and Tarski's Theorem (inspired by [17]), (d) an analysis of special topological logics on the reals, pointing toward a landscape of spatial logics above **S4**, and finally (e) an extension to richer modal languages of space, and their increased expressive power.

2 Modal language and topological semantics

2.1 Language and axioms

Let us first set the scene where we will operate. The basic language $\mathcal{L}$ of propositional modal logic is composed of

- a countable set of proposition letters,
- boolean connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$,
- modal operators $\Box$, $\Diamond$.

The standard axiomatization of our central logic **S4** is

$$\begin{array}{ll} \Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi) & \text{(K)} \\ \Box\varphi \rightarrow \Box\Box\varphi & \text{(4)} \\ \Box\varphi \rightarrow \varphi & \text{(T)} \end{array}$$

Modus Ponens and Necessitation are the only rules of inference:

$$\frac{\varphi \rightarrow \psi \qquad \varphi}{\psi}\quad \mathbf{MP} \qquad\qquad \frac{\varphi}{\Box\varphi}\quad \mathbf{N}$$

For a closer fit to topological reasoning, however, it is better to work with an equivalent axiomatization of **S4**:

$$
\begin{array}{ll}
\Box\top & \text{(N)} \\
(\Box\varphi \wedge \Box\psi) \leftrightarrow \Box(\varphi \wedge \psi) & \text{(R)} \\
\Box\varphi \rightarrow \varphi & \text{(T)} \\
\Box\varphi \rightarrow \Box\Box\varphi & \text{(4)}
\end{array}
$$

Modus Ponens and Monotonicity are the only rules of inference:

$$
\frac{\varphi \rightarrow \psi \qquad \varphi}{\psi} \quad \textbf{MP} \qquad\qquad \frac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi} \quad \textbf{M}
$$

As we shall see in a moment, these principles are valid in all topological spaces when we let formulas range over sets of points, with the Booleans as the obvious set operations, modal box as interior and dually, modal diamond as closure. On top of this base set, further modal axioms can be used to express more special topological properties. E.g., an additional 'axiom' $\Box\varphi \leftrightarrow \varphi$ would say that each set is open, meaning that the spaces satisfying it have the discrete topology.

2.2 Topological completeness

The first semantic completeness proof for **S4** did not use the by now dominant relational modal models, which go back to Kanger, Hintikka, and Kripke in the 1950s. It was actually an earlier *spatial* completeness argument of [16], in terms of the following notions. Recall that a topological space is a pair $\langle W, \tau \rangle$, where W is a non-empty set and τ a collection of subsets of W satisfying the following three conditions:

- $\emptyset, W \in \tau$,
- if $U, V \in \tau$, then $U \cap V \in \tau$,
- if $\{U_i\}_{i \in I} \in \tau$, then $\bigcup_{i \in I} U_i \in \tau$.

Let $Int(X)$ and $\overline{X}$ be the topological interior and closure operators of $\langle W, \tau \rangle$ respectively. It is well-known (cf. [13]) that these satisfy the following clauses for all $X, Y \subseteq W$:

$$
\begin{array}{ll}
Int(W) = W & \overline{\emptyset} = \emptyset \\
Int(X \cap Y) = Int(X) \cap Int(Y) & \overline{X \cup Y} = \overline{X} \cup \overline{Y} \\
Int(X) \subseteq X & X \subseteq \overline{X} \\
Int(X) \subseteq IntInt(X) & \overline{\overline{X}} \subseteq \overline{X},
\end{array}
$$

Moreover, there is a duality $Int(X) = W - \overline{W - X}$, and a topological space can also be defined in terms of an interior operator, or a closure operator satisfying the above four clauses.

McKinsey and Tarski defined a valuation ν of formulas of $\mathcal{L}$ into $\langle W, \tau \rangle$ by putting

- $\nu(P) \subseteq W$,
- $\nu(\neg\varphi) = W - \nu(\varphi)$,
- $\nu(\varphi \vee \psi) = \nu(\varphi) \cup \nu(\psi)$,
- $\nu(\varphi \wedge \psi) = \nu(\varphi) \cap \nu(\psi)$,
- $\nu(\varphi \rightarrow \psi) = (W - \nu(\varphi)) \cup \nu(\psi)$,
- $\nu(\Box\varphi) = Int(\nu(\varphi))$,
- $\nu(\Diamond\varphi) = \overline{\nu(\varphi)}$.

In definitions and arguments in this paper, we will often economize, and leave out the clauses for disjunction, implication and modal diamond, as these are automatic from the others. Now, call a triple $M = \langle W, \tau, \nu \rangle$ a *topological model*. A formula φ is said to be *true* in such a model M if $\nu(\varphi) = W$, and we say that φ is *topologically valid* if it is true in every topological model. Referring to the second axiomatization of **S4**, which highlights the interior operator, one easily sees its soundness:

$$\text{If } \mathbf{S4} \vdash \varphi, \text{ then } \varphi \text{ is topologically valid.}$$

McKinsey and Tarski's pioneering achievement [16] was a proof of completeness (for a new proof, see Section 4 below):

$$\text{If } \varphi \text{ is topologically valid, then } \mathbf{S4} \vdash \varphi.$$

Hence, for a modal logician, the topological semantics is adequate for **S4**, or – the other side of the same coin: for a topologist, **S4** axiomatizes the algebra of the interior operator completely.

2.3 The semantics amplified

In the rest of this paper, we will use the more modern format for a modal semantics. Given a topological model $M = \langle W, \tau, \nu \rangle$, we state what it means for a given formula φ to be *true in a point* w:

- $w \models P$ iff $w \in \nu(P)$,
- $w \models \neg\varphi$ iff not $w \models \varphi$,
- $w \models \varphi \wedge \psi$ iff $w \models \varphi$ and $w \models \psi$,
- $w \models \Box\varphi$ iff $(\exists U \in \tau)(w \in U$ and $(\forall v \in U)(v \models \varphi))$,

and hence also

- $w \models \Diamond\varphi$ iff $(\forall U \in \tau)(w \in U \Rightarrow (\exists v \in U)(v \models \varphi))$.

This semantics for the modal language looks different from the usual one, where models have a binary accessibility relation between points, and $w \models \Box\varphi$ if φ is true in all relational successors of w. Nevertheless, there are strong analogies, which we will sketch in Section 3. Indeed, all basic notions from relational Kripke models make sense for the topological semantics, too. Here are two typical examples that we shall need further on.

The topological semantics is *local* in that the truth value of a formula at a point w only depends on what happens inside the open neighborhoods of that point. More precisely, consider any topological model M with a point w inside, which lies in some open set U. Now define the obvious *restriction* of M to a topological model $M|U$ by taking U for the new universe, letting the open sets be all the old open sets inside U, and putting $\nu'(P) = \nu(P) \cap U$. It is easy to show by induction on formulas that

$$w \models \varphi \text{ in } M \text{ iff } w \models \varphi \text{ in } M|U.$$

Thus, e.g., to determine truth values for modal formulas at a point w on the real line, we only need to know how the model behaves in arbitrarily small open neighborhoods around w. Or conversely, we can change the model at a distance from a point w, without affecting the original truth values.

Our second illustration concerns the proper *semantic invariance* for our modal language. The connection between M and $M|U$ is a special case of a more general model relation investigated at length in [1], including versions in terms of Ehrenfeucht-Fraisse games.

Definition 2.1 (topological bisimulation) *Suppose two topological models $\langle X, \tau, \nu\rangle$, $\langle X', \tau', \nu'\rangle$ are given. A* topological bisimulation *is a non-empty relation $\sim \subseteq X \times X'$ such that if $x \sim x'$ then*

(base): *$x \in \nu(P)$ iff $x' \in \nu'(P)$ (for any variable P)*

(forth condition): *if $x \in U \in \tau$ then*
$(\exists U' \in \tau')(x' \in U' \;\&\; (\forall y' \in U')(\exists y \in U)(y \sim y'))$

(back condition): *if $x' \in U' \in \tau'$ then*
$(\exists U \in \tau)(x \in U \;\&\; (\forall y \in U)(\exists y' \in U')(y \sim y'))$.

As an example, the identity relation on U is a topo-bisimulation between the above models M and $M|U$. This also shows that the preceding definition does not require *totality*: some points need not have links at all. But much rougher 'contractions' and 'twists' are also possible. In general, topo-bisimulation is a coarse notion of similarity between topological spaces, much less fine-grained than homeomorphism or homotopy. But it is just right for the expressive power of the modal language:

Fact 2.2 *If $\sim$ is a topo-bisimulation between two models M, N such that $s \sim t$, then s, t satisfy the same modal formulas.*

The statement of true versions of converse results is a much more delicate matter (cf. [7] and [10]). Here we give just one simple illustration:

Fact 2.3 *If two worlds s and t satisfy the same modal formulas in two* finite *models M, N, then there exists a topo-bisimulation between these models which connects s with t.*

2.4 Extended modal languages

One striking feature of modern modal logic, which differs from earlier phases, is the use of languages with additional modal operators. This is an obvious move when thinking about modal languages for describing topological structure: one may want to express more than just the bare facts of interior and closure, while still sticking to the perspicuity of **S4** and its ilk. Here is one simple extension. One can add a *universal modality* $U\varphi$ expressing that φ holds in all worlds of the model, and a dual existential modality $E\varphi$ expressing that φ holds in at least one world. This allows us to express new topological properties, such as *connectedness*:

Recall that a topological space $\langle W, \tau \rangle$ is said to be *connected* if W can not be represented as the union of two disjoint open sets. As was shown independently in [19] and in [1], the 'defining' formula of connectedness is:

$$U(\Diamond P \to \Box P) \to (UP \vee U\neg P).$$

Note that connectedness is not definable in our basic language, as its failures are not invariant for topo-bisimulation. But one can easily *strengthen* topo-bisimulations to deal with this richer modal language, and one can also extend the logic **S4** to a complete system for it, adding amongst others the axioms of **S5** for U, E. We will return to such expressive extensions in Section 7.2. For more extensive information, cf. [4, 19, 1].

3 Topological spaces and Kripke models

The purpose of this section is a link-up with the better-known world of 'standard' semantics for modal logic. At the same time, this comparison increases our understanding of the 'topological content' of modal logic. What follows can be safely skipped by readers who already know, or do not care.

3.1 The basic connection

The standard Kripke semantics for **S4** is a particular case of its more general topological semantics. Recall that an **S4**-frame (henceforth 'frame', for short) is a couple $\langle W, R\rangle$, where W is a non-empty set and R a quasi-order (transitive and reflexive) on W. Call a set $X \subseteq W$ *upward closed* if $w \in X$ and wRv imply $v \in X$.

Fact 3.1 *Every frame $\langle W, R\rangle$ induces a topological space $\langle W, \tau_R\rangle$, where τ_R is the set of all upward closed subsets of $\langle W, R\rangle$.*

It is easy to check that τ_R is a topology on W, and that the closure and interior operators of $\langle W, \tau_R\rangle$ are respectively $R^{-1}(X)$ and $W - R^{-1}(W - X)$, where $R^{-1}(w) = \{v \in W : vRw\}$ and $R^{-1}(X) = \bigcup_{w\in X} R^{-1}(w)$, for $w \in W$, $X \subseteq W$. Indeed, τ_R is a rather special topology on W: for any family $\{X_i\}_{i\in I} \subseteq \tau_R$, we have $\bigcap_{i\in I} X_i \in \tau_R$. Such spaces are called *Alexandroff spaces*, in which every point has a least neighborhood. In frames, the least neighborhood of a point w is evidently $\{v \in W : wRv\}$, which is usually denoted by $R(w)$.

Conversely, every topological space $\langle W, \tau\rangle$ naturally induces a quasi-order R_τ defined by putting

$$wR_\tau v \text{ iff } w \in \overline{\{v\}} \text{ iff } w \in U \text{ implies } v \in U, \text{ for every } U \in \tau.$$

This is called the *specialization order* in the topological literature. Again it is easy to check that R_τ is transitive and reflexive, and that every open set of τ is R_τ-upward closed. Moreover, R_τ is anti-symmetric iff $\langle W, \tau\rangle$ satisfies the T_0 separation axiom (that is, any two different points are separated by an open set). Hence R_τ is a partial order iff $\langle W, \tau\rangle$ is a T_0-space.

Combining the two mappings, $R = R_{\tau_R}$, $\tau \subseteq \tau_{R_\tau}$, and $\tau = \tau_{R_\tau}$ iff $\langle W, \tau\rangle$ is an Alexandroff space. Indeed, $wR_{\tau_R}v$ iff $w \in \overline{\{v\}}$ iff $w \in R^{-1}(v)$ iff wRv. Also, as every open set of τ is R_τ-upward closed, $\tau \subseteq \tau_{R_\tau}$. Finally, $\tau = \tau_{R_\tau}$ iff every R_τ-upward closed set belongs to τ iff every point of W has a least neighborhood in $\langle W, \tau\rangle$ iff $\langle W, \tau\rangle$ is an Alexandroff space.

The upshot of all this is a one-to-one correspondence between quasi-ordered sets and Alexandroff spaces, and between partially ordered sets and Alexandroff T_0-spaces. Since every finite topological space is an Alexandroff space, this immediately gives a one-to-one correspondence between finite quasi-ordered sets and finite topological spaces, and finite partially ordered sets and finite T_0-spaces.

There is also a one-to-one correspondence between continuous maps and order preserving maps, as well as open maps and p-morphisms. Indeed, let two topological spaces $\langle W_1, \tau_1\rangle$ and $\langle W_2, \tau_2\rangle$ be given. Recall that a function $f : W_1 \to W_2$ is *continuous* if $f^{-1}(V) \in \tau_1$ for every $V \in \tau_2$. Moreover, f is *open* if it is continuous and $f(U) \in \tau_2$ for every $U \in \tau_1$. It is well-known that f is continuous iff $\overline{f^{-1}(X)} \subseteq f^{-1}(\overline{X})$, and that f is open iff $\overline{f^{-1}(X)} = f^{-1}(\overline{X})$, for every $X \subseteq W_2$.

Next, for two quasi-orders $\langle W_1, R_1\rangle$ and $\langle W_2, R_2\rangle$, $f : W_1 \to W_2$ is said to be *order preserving* if wR_1v implies $f(w)R_2f(v)$, for $w, v \in W_1$. f is a *p-morphism* if it is order preserving, and in addition $f(w)R_2v$ implies that there exists $u \in W_1$ such that wR_1u and $f(u) = v$, for $w \in W_1$ and $v \in W_2$. It is well-known that f is order preserving iff $R_1^{-1}f^{-1}(w) \subseteq f^{-1}R_2^{-1}(w)$, and that f is a p-morphism iff $R_1^{-1}f^{-1}(w) = f^{-1}R_2^{-1}(w)$, for every $w \in W_2$.

Putting this together, one easily sees that f is monotone iff f is continuous, and that f is p-morphism iff f is open.

As an easy consequence we obtain that the category **ATop** of Alexandroff spaces and continuous maps is isomorphic to the category **Qos** of quasi ordered sets and order preserving maps, and that the category $\mathbf{ATop}^+$ of Alexandroff spaces and open maps is isomorphic to the category $\mathbf{Qos}^+$ of quasi ordered sets and p-morphisms.

Similarly, the category $\mathbf{ATop}_{T_0}$ of Alexandroff T_0-spaces and continuous maps is isomorphic to the category **Pos** of partially ordered sets and order preserving maps, and the category $\mathbf{ATop}^+_{T_0}$ of Alexandroff T_0-spaces and open maps is isomorphic to the category $\mathbf{Pos}^+$ of partially ordered sets and p-morphisms.

In the finite case we get that the category Fin**Top** of finite topological spaces and continuous maps is isomorphic to the category Fin**Qos** of finite quasi ordered sets and order preserving maps, and that the category $\mathrm{Fin}\mathbf{Top}^+$ of finite topological spaces and open maps is isomorphic to the category $\mathrm{Fin}\mathbf{Qos}^+$ of finite quasi ordered sets and p-morphisms.

Similarly, the category $\mathrm{Fin}\mathbf{Top}_{T_0}$ of finite T_0-spaces and continuous maps is isomorphic to the category Fin**Pos** of finite partially ordered sets and order preserving maps, and the category $\mathrm{Fin}\mathbf{Top}^+_{T_0}$ of finite T_0-spaces and open maps is isomorphic to the category $\mathrm{Fin}\mathbf{Pos}^+$ of finite partially ordered sets and p-morphisms.

3.2 Analogies qua topics

The tight connection between modal frames and topological spaces explains the earlier-mentioned analogies in their semantic development, such as *locality* and *invariance for bisimulation.* It may be extended to include other basic modal topics, such as *correspondence theory* [6]. Likewise, the modern move toward extended modal languages makes equally good sense for the topological interpretation. Many natural topological notions need extra modal power for their definition: good examples are the basic *separation axioms.* We just saw that, among the quasi orders, partial orders correspond to topological spaces satisfying the T_0 separation axiom. But this difference does not show up in our basic modal language: **S4** is complete with respect to arbitrary partial orders. Defining separation axioms requires various expressive extensions of the modal base language.

Finally, in a more technical sense, there still seems a vast difference. The format of the topological interpretation looks more complex than the usual one which quantifies over accessible worlds only. For, it involves a *second-*

order quantification over sets of worlds, plus a first-order quantification over their members. But this difference is more apparent than real, because the quantification is over open sets only, and we may plausibly think of topological models as *two-sorted first-order models* with separate domains of 'points' and 'opens'. To bring this out more directly, one might also use an alternative 'bi-modal language' with *two* separate modalities: <U> ("for some open neighborhood of the current point"), $[x]$ ("for all points in the current open set"). (Cf. [15, 12, 1] for this decomposition.) On this approach, however, the base logic is no longer **S4**!

4 General Completeness

The preceding section shows that standard modal models are a particular case of a more general topological semantics. Hence, the known completeness of **S4** plus the topological soundness of its axioms immediately give us general topological completeness. Even so, we now give a direct model-theoretic proof of this result. It is closely related to the standard modal Henkin construction, but with some nice topological twists. (Compare [11] for the quite analogous case of modal 'neighborhood semantics'.)

4.1 The main argument

Soundness is immediate, and hence we move directly to completeness. Call a set Γ of formulas of $\mathcal{L}$ (**S4**–)*consistent* if for no finite set $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma$ we have that **S4** $\vdash \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)$. A consistent set of formulas Γ is called *maximally consistent* if there is no consistent set of formulas properly containing Γ. It is well-known that Γ is maximally consistent iff, for any formula φ of $\mathcal{L}$, either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$, but not both. Now we define a topological space out of maximally consistent sets of formulas.

Definition 4.1 (canonical topological space) *The* canonical topological space *is the pair $S^{\mathcal{L}} = \langle W^{\mathcal{L}}, \tau^{\mathcal{L}} \rangle$, where:*

- *$W^{\mathcal{L}}$ is the set of all maximally consistent sets Γ_{max};*
- *$\tau^{\mathcal{L}}$ is the set generated by arbitrary unions of the following* basic sets *$B^{\mathcal{L}} = \{\widehat{\Box\varphi} :\ \varphi$ is any formula $\}$, where $\widehat{\varphi} =_{def} \{\Gamma_{max} \in W^{\mathcal{L}} :\ \varphi \in \Gamma_{max}\}$. In other words, basic sets are the families of the form: $U_\varphi = \{\Gamma_{max} \in W^{\mathcal{L}} :\ \Box\varphi \in \Gamma_{max}\}$.*

Let us first check that $S^{\mathcal{L}}$ is indeed a topological space.

Lemma 4.2 *$B^{\mathcal{L}}$ forms a basis for the topology.*

Proof. We only need to show the following two properties:

- For any $U_\varphi, U_\psi \in B^{\mathcal{L}}$ and any $\Gamma_{max} \in U_\varphi \cap U_\psi$, there is $U_\chi \in B^{\mathcal{L}}$ such that $\Gamma_{max} \in U_\chi \subseteq U_\varphi \cap U_\psi$;
- For any $\Gamma_{max} \in W^{\mathcal{L}}$, there is $U_\varphi \in B^{\mathcal{L}}$ such that $\Gamma_{max} \in U_\varphi$.

Now, (N) implies that $\Box\top \in \Gamma_{max}$, for any Γ_{max}. Hence $W^{\mathcal{L}} = \widehat{\Box\top}$ and the second item is satisfied. As for the first item, thanks to (R), one can easily check that $\widehat{\Box(\varphi \wedge \psi)} = \widehat{\Box\varphi} \cap \widehat{\Box\psi}$. Hence $U_\varphi \cap U_\psi \in B^{\mathcal{L}}$, and so $B^{\mathcal{L}}$ is closed under finite intersections: whence the first item is satisfied. **q.e.d.**

Next we define the canonical topological model.

Definition 4.3 (Canonical topological model) *The* canonical topological model *is the pair $M^{\mathcal{L}} = \langle S^{\mathcal{L}}, \nu^{\mathcal{L}} \rangle$, where:*

- *$S^{\mathcal{L}}$ is the canonical topological space;*
- *$\nu^{\mathcal{L}}(P) = \{\Gamma_{max} \in X^{\mathcal{L}} : P \in \Gamma_{max}\}$.*

The valuation $\nu^{\mathcal{L}}$ equates truth of a proposition letter *at* a maximally consistent set with its membership *in* that set. We now show this harmony between the two viewpoints lifts to all formulas.

Lemma 4.4 (Truth lemma) For all modal formulas φ,

$$M^{\mathcal{L}}, w \models_{\mathcal{L}} \varphi \text{ iff } w \in \widehat{\varphi}.$$

Proof. Induction on the complexity of φ. The base case was just described. The case of the Booleans follows from the following well-known identities for maximally consistent sets:

- $\widehat{\neg\varphi} = W^{\mathcal{L}} - \widehat{\varphi}$;
- $\widehat{\varphi \wedge \psi} = \widehat{\varphi} \cap \widehat{\psi}$.

The interesting case is that of the modal operator $\Box$. We do the two relevant implications separately, starting with the easy one.

$\Leftarrow$ 'From membership to truth.' Suppose $w \in \widehat{\Box\varphi}$. By definition, $\widehat{\Box\varphi}$ is a basic set, hence open. Moreover, thanks to axiom (T), $\widehat{\Box\varphi} \subseteq \widehat{\varphi}$. Hence there exists an open neighborhood $U = \widehat{\Box\varphi}$ of w such that for any $v \in U$, $v \in \widehat{\varphi}$, and by the induction hypothesis, $M^{\mathcal{L}}, v \models_{\mathcal{L}} \varphi$. Thus $M^{\mathcal{L}}, w \models_{\mathcal{L}} \Box\varphi$.

$\Rightarrow$ 'From truth to membership.' Suppose $M^{\mathcal{L}}, w \models_{\mathcal{L}} \Box\varphi$. Then there exists a basic set $\widehat{\Box\psi} \in B^{\mathcal{L}}$ such that $w \in \widehat{\Box\psi}$ and for all $v \in \widehat{\Box\psi}$, $M^{\mathcal{L}}, v \models_{\mathcal{L}} \varphi$. By the induction hypothesis, $\forall v \in \widehat{\Box\psi}$, $v \in \widehat{\varphi}$: i.e., $\widehat{\Box\psi} \subseteq \widehat{\varphi}$. But this implies that the logic **S4** can prove the implication $\Box\psi \to \varphi$. (If not, then there would be some maximally consistent set containing both $\Box\psi$ and $\neg\varphi$.) But then we can prove the implication $\Box\Box\psi \to \Box\phi$, and hence, using the

S4 transitivity axiom, $\Box\psi \rightarrow \Box\phi$. It follows that $\widehat{\Box\psi} \subseteq \widehat{\Box\phi}$, and hence the world w belongs to $\widehat{\Box\phi}$. **q.e.d.**

Now we can clinch the proof of our main result.

Theorem 4.5 (Completeness) *For any set of formulas* Γ,

$$\textit{if} \quad \Gamma \models_{\mathcal{L}} \varphi \quad \textit{then} \quad \Gamma \vdash_{\mathbf{S4}} \varphi.$$

Proof. Suppose that $\Gamma \not\vdash_{\mathbf{S4}} \varphi$. Then $\Gamma \cup \{\neg\varphi\}$ is consistent, and by the Lindenbaum Lemma it can be extended to a maximally consistent set Γ_{max}. By the Truth Lemma, $M^{\mathcal{L}}, \Gamma_{max} \models_{\mathcal{L}} \neg\varphi$, whence $\Gamma_{max} \not\models_{\mathcal{L}} \varphi$, and we have constructed the required counter-model. **q.e.d.**

4.2 Topological comments

Let us now look at some topological aspects of this construction. In proving the box case of Truth Lemma, we did not use the standard modal argument, which crucially invokes the *distribution axiom* of the minimal modal logic. Normally, one shows that, if a formula $\Box\phi$ does not belong to a maximally consistent set Γ, then there exists some maximally consistent successor set of Γ containing $\neg\varphi$. This is not necessary in the topological version at this stage. We only needed the reflexivity and transitivity axioms, plus the Lindenbaum Lemma on maximally consistent extensions. The modal distribution axiom still plays a crucial role, but that was at the earlier stage of verifying that we had really defined a topology. This different way of 'cutting the cake' provides an additional proof-theoretic explanation why **S4** is the weakest axiom system complete for topological semantics. Moreover, the divergence with the 'standard' argument explodes the prejudice that one single 'well-known' interpretation for a language must be the only natural one.

Comparing our construction with the standard modal Henkin model for **S4** $\langle W^{\mathcal{L}}, R^{\mathcal{L}}, \models_{\mathcal{L}}\rangle$, the basic sets of our topology $S^{\mathcal{L}}$ are $R^{\mathcal{L}}$-upward closed. Hence every open of $S^{\mathcal{L}}$ is $R^{\mathcal{L}}$-upward closed, and $S^{\mathcal{L}}$ is weaker than the topology $\tau_{R^{\mathcal{L}}}$ corresponding to $R^{\mathcal{L}}$. In particular, our canonical topological space is *not* an Alexandroff space.

Here are some further topological aspects of the above construction. First, it is worthwhile to compare Stone's famous construction which uses the alternative basis $\{\widehat{\varphi} : \varphi \text{ any formula}\}$, yielding a space which we denote by $\langle W^{\mathcal{L}}, \tau^{S}\rangle$. It is well-known that $\langle W^{\mathcal{L}}, \tau^{S}\rangle$ is homeomorphic to the Cantor space – and so, up to homeomorphism, $\langle W^{\mathcal{L}}, \tau^{S}\rangle$ is compact, metric, 0-dimensional, and dense-in-itself. The basis of our topology, however, was the sub-family $\{\widehat{\Box\varphi} : \varphi \text{ any formula}\}$. Now every subtopology of one that is compact and dense-in-itself is also compact and dense-in-itself. Therefore, we get these same properties for our canonical topological space. But we can be more precise than this.

Fact 4.6 *The canonical topology is actually* intersection *of the Kripke and Stone topologies.*

In other words, $\tau^{\mathcal{L}} = \tau_{R^{\mathcal{L}}} \cap \tau^{\mathcal{S}}$. Indeed, since $\tau^{\mathcal{L}} \subseteq \tau_{R^{\mathcal{L}}}$ and $\tau^{\mathcal{L}} \subseteq \tau^{\mathcal{S}}$, obviously $\tau^{\mathcal{L}} \subseteq \tau_{R^{\mathcal{L}}} \cap \tau^{\mathcal{S}}$. Conversely, since every base set $\widehat{\varphi}$ of Stone's topology is $R^{\mathcal{L}}$-upward closed iff $\widehat{\varphi} = \widehat{\Box\psi}$ for some ψ, $\tau_{R^{\mathcal{L}}} \cap \tau^{\mathcal{S}} \subseteq \tau^{\mathcal{L}}$, and $\tau^{\mathcal{L}} = \tau_{R^{\mathcal{L}}} \cap \tau^{\mathcal{S}}$.

One can also connect modal formulas and topological properties more directly, by giving a direct proof of the fact that $S^{\mathcal{L}}$ is compact and dense-in-itself. The former fact goes just as for the Stone space, but we display it for the sake of illustration.

Lemma 4.7 $S^{\mathcal{L}}$ *is compact.*

Proof. Suppose otherwise. Then there is a family $\{\widehat{\Box\psi_i}\}_{i\in I} \subseteq B^{\mathcal{L}}$ such that $\bigcup_{i\in I} \widehat{\Box\psi_i} = W^{\mathcal{L}}$, and for no finite subfamily $\{\widehat{\Box\psi_{i_1}}, \ldots, \widehat{\Box\psi_{i_n}}\}$ we have $\widehat{\Box\psi_{i_1}} \cup \cdots \cup \widehat{\Box\psi_{i_n}} = W^{\mathcal{L}}$. Let $\Gamma = \{\neg\Box\psi_i\}_{i\in I}$.

Claim 4.8 Γ *is consistent.*

Proof. Suppose otherwise. Then there is a finite number of formulas $\neg\Box\psi_1, \ldots, \neg\Box\psi_n \in \Gamma$ such that $\mathbf{S4} \vdash \neg(\neg\Box\psi_1 \wedge \cdots \wedge \neg\Box\psi_n)$. Hence $\mathbf{S4} \vdash \Box\psi_1 \vee \cdots \vee \Box\psi_n$. But then $\widehat{\Box\psi_1} \cup \cdots \cup \widehat{\Box\psi_n} = W^{\mathcal{L}}$, which is a contradiction. **q.e.d.**

Since Γ is consistent, it can be extended to a maximally consistent set Γ_{max}. Obviously $\neg\Box\psi_i \in \Gamma_{max}$ for any $i \in I$. Hence $\Gamma_{max} \in \widehat{\neg\Box\psi_i}$ for any $i \in I$. Since $\widehat{\neg\Box\psi_i} = W^{\mathcal{L}} - \widehat{\Box\psi_i}$, $\Gamma_{max} \in W^{\mathcal{L}} - \widehat{\Box\psi_i}$ for any $i \in I$. Hence $\Gamma_{max} \in W^{\mathcal{L}} - \bigcup_{i\in I} \widehat{\Box\psi_i}$, which contradicts our assumption. Thus, $S^{\mathcal{L}}$ is compact. **q.e.d.**

Lemma 4.9 $S^{\mathcal{L}}$ *is dense-in-itself.*

Proof. Suppose there was an isolated point w in $S^{\mathcal{L}}$. Then there is a formula $\Box\varphi$ with $\widehat{\Box\varphi} = \{w\}$. This means $\Box\varphi \in w$ and for any ψ, $\psi \in w$ iff $\mathbf{S4} \vdash \Box\varphi \rightarrow \psi$, which is obviously a contradiction – since we are working in a language with infinitely many propositional letters. **q.e.d.**

Corollary 4.10 **S4** *is the logic of the class of all topological spaces which are compact and dense-in-itself.*

Still, the canonical topological space $S^{\mathcal{L}}$ is neither 0-dimensional nor metric (it is not even a T_0-space). So, $S^{\mathcal{L}}$ is not homeomorphic to the Cantor space. In the next section, we will show how to get completeness of **S4** with respect to the Cantor space by a different route.

4.3 Finite spaces suffice

We conclude with an observation that is important for later arguments. The whole construction in the completeness proof would also work if we restricted attention to the *finite* language consisting of the initial formula and all its subformulas. All definitions go through, and our arguments never needed to go beyond it. This means that we only get finitely many maximally consistent sets, and so non-provable formulas can be refuted on *finite models*, whose size is effectively computable from the formula itself. (Note however that the obtained finite model won't necessarily be dense-in-itself.)

Corollary 4.11 **S4** *has the* effective finite model property *w.r.t. the class of topological spaces.*

Incidentally, this also shows that validity in **S4** is *decidable*, but we forego such complexity issues in this paper.

The resulting models have some interesting topological extras. Consider any finite modal frame $\mathcal{F} = \langle W, R \rangle$. We define some auxiliary notions. For any $w \in W$, let $C(w) = \{v \in W : wRv \ \& \ vRw\}$. Call a set C a *cluster* if it is of the form $C(w)$ for some w: the *cluster generated by* w. $C(w)$ is *simple* if $C(w) = \{w\}$, and *proper* otherwise. $w \in W$ is called *minimal* if vRw implies wRv for any $v \in W$. A cluster C is *minimal* if there exists a minimal $w \in W$ such that $C = C(w)$. Next, call $\mathcal{F}$ *rooted* if there is $w \in W$ such that wRv for any $v \in W$: w is then a *root* of $\mathcal{F}$. This w need not be unique: any point from $C(w)$, the *initial cluster* of $\mathcal{F}$, will do.

Evidently, a finite Kripke frame $\mathcal{F}$ is rooted iff it has only one minimal cluster. Topologically, this property is related to the earlier notion of *connectedness*. We said in Section 2.4 that a topological space $\langle W, \tau \rangle$ is *connected* if its universe cannot be written as a union of two disjoint open sets. $\langle W, \tau \rangle$ is *well-connected* if $W = U \cup V$ implies $W = U$ or $W = V$, for any $U, V \in \tau$. Obviously well-connectedness is a stronger notion than connectedness. It corresponds to $\langle W, R_\tau \rangle$ being rooted. For this observe that, dually, well-connectedness can be stated as follows:

For any two closed subsets C and D of $\langle W, \tau \rangle$, $C \cap D = \emptyset$ implies either $C = \emptyset$ or $D = \emptyset$.

Lemma 4.12 *A finite Kripke frame is rooted iff the corresponding topological space is well-connected.*

Proof: Suppose $\langle W, R \rangle$ is a rooted Kripke frame with a root w, and $\langle W, \tau_R \rangle$ the corresponding topological space. Let X_1 and X_2 be closed sets of $\langle W, \tau_R \rangle$ such that $X_1 \cap X_2 = \emptyset$. By an easy dualization of the notions of Section 3.1, a set $X \subseteq W$ is topologically closed iff it is *downward closed* in the ordering, that is $u \in X$ and vRu imply $v \in X$, for any $u, v \in W$. Now if both X_1 and

X_2 are non-empty, then w belongs to both of them, which is a contradiction. Hence one of them should be empty, and $\langle W, \tau_R \rangle$ is well-connected.

Conversely, suppose $\langle W, R \rangle$ is not rooted. Then there are at least two different minimal clusters C_1 and C_2 in W. Since C_1 and C_2 are minimal clusters, they are downward closed, and hence closed in $\langle W, \tau_R \rangle$. Moreover, since they are different, $C_1 \cap C_2 = \emptyset$. Hence $\langle W, \tau_R \rangle$ is not well-connected. **q.e.d.**

This allows us to improve on Corollary 4.11.

Theorem 4.13 **S4** *is the logic of finite well-connected topological spaces.*

Proof: It suffices to observe the following. If a modal formula has a counter-example on a finite Kripke model, it fails in some point there. But then by standard 'Locality', it also fails in the submodel *generated* by that point and its relational successors, which is rooted – and hence transforms into a well-connected topological space. **q.e.d.**

Again, there is a downside to such an upgraded completeness result. What it also means is that the basic modal language cannot *define* such a nice topological property as well-connectedness. As we saw in Section 2.4, the definition of connectedness requires introduction of additional modalities. So does well-connectedness.

Finally, let us mention that for refuting non-theorems of **S4** it is enough to restrict ourselves to the class of those finite rooted models whose every cluster is proper. As we already mentioned in Section 3.1, having only simple clusters topologically corresponds to the T_0 separation axiom, which in finite case is equivalent to the T_D separation axiom (every point is obtained as intersection of an open and a closed sets). Consequently, having only proper clusters would topologically correspond to the fact that no point can be obtained as intersection of an open and a closed sets. Call spaces with this property *essentially non-T_D*. Then we can improve a little bit on Theorem 4.13:

Theorem 4.14 **S4** *is the logic of finite well-connected essentially non-T_D topological spaces.*

Proof: Suppose a modal formula φ has a counter-example on a finite rooted Kripke model $M = \langle W, R, \models \rangle$. Then replacing every cluster of W by an n-element cluster, where n is the maximum among the sizes of the clusters of W, we obtain a new frame $\langle W', R' \rangle$. Obviously $\langle W, R \rangle$ is a p-morphic image of $\langle W', R' \rangle$. This allows us to define $\models'$ on $\langle W', R' \rangle$ so that φ has also a counter-example on $M' = \langle W', R', \models' \rangle$. Now every cluster of W' is proper, hence $\langle W', R' \rangle$ transforms into a well-connected essentially non-T_D topological space. **q.e.d.**

5 Completeness on the reals

As early as 1944, McKinsey and Tarski proved the following beautiful result, which is an expansion of a completeness theorem by Tarski for intuitionistic propositional logic from 1938:

Theorem 5.1 (McKinsey-Tarski) **S4** *is the complete logic of any metric separable dense-in-itself space.*

Most importantly, this theorem implies completeness of **S4** with respect to the real line **R**. It also implies completeness of **S4** with respect to the Cantor space $\mathcal{C}$.

Our paper does not present any startling new results improving on this theorem. It rather takes a systematic look at its proof, and what it achieves. The original algebraic proof in [16] was very complex, the later more topological version in [18] is not much more accessible. Recently, Mints [17] replaced these by a much more perspicuous model-theoretic construction, extending earlier ideas of Beth and Kripke to get faster completeness of **S4** with respect to the Cantor space. We generalize its model-theoretic structure, using the topo-bisimulations of Section 2, and also provide a modification for completeness on the reals.

Our strategy in the following subsections starts from the standard modal completeness for **S4** involving counter-examples on finite rooted models, and then exhibits a topo-bisimulation resulting in "tree-like" topological model homeomorphic to the Cantor space $\mathcal{C}$. We then show how to extract completeness of **S4** with respect to the reals from the completeness of **S4** with respect to $\mathcal{C}$.

5.1 Cantorization

Our starting point is an arbitrary modal formula which is not provable in **S4**. We have already seen that such a non-theorem can be refuted on a finite rooted Kripke model. Now we will show how to transform the latter into a counterexample on the Cantor space $\mathcal{C}$. Our technique is *selective unraveling*, a refinement of the technique of *unraveling* in modal logic.

Suppose $M = \langle W, R, \models\rangle$ is a finite rooted model with a root w. Our goal is to select those infinite paths of M which are in a one-to-one correspondence with infinite paths of the full infinite binary tree T_2. In order to give an easier description of our construction, we assume that every cluster of W is proper. This can be done by Theorem 4.14. Now start with a root w, and announce (w) as a selective path. Then if $(w_1, \ldots, w_k)$ is already a selective path, introduce a *left* move by announcing $(w_1, \ldots, w_k, w_k)$ as a selective path; and introduce a *right* move by announcing $(w_1, \ldots, w_k, w_{k+1})$ as a selective path if $w_k R w_{k+1}$ and $w_k \neq w_{k+1}$. (Since we assumed that every cluster of W is proper, such w_{k+1} will exist for every w_k.)

To make this idea precise, we need some definitions. For $u, v \in W$, call v a *strong successor* of u if uRv and $u \neq v$. Write $SSuc(u)$ for the set of all strong successors of u. Since we assumed that every cluster of W is proper, $SSuc(u) \neq \emptyset$ for every $u \in W$. Suppose $v_1, \ldots, v_n$ is a complete enumeration of $SSuc(u)$ for every $u \in W$. Now define a *selective path of* W recursively:

1 (w) is a selective path;

2 If $(w_1, \ldots, w_k)$ is a selective path of length k, then $(w_1, \ldots, w_k, w_{k+1})$ is a selective path of length $k+1$, where $w_{k+1} = w_k$;

3 If $(w_1, \ldots, w_k)$ is a selective path of length k, then $(w_1, \ldots, w_k, w_{k+1})$ is a selective path of length $k+1$, where $w_{k+1} = v_i$ with $i \equiv k (mod\ n)$;[1]

4 That's all.

Denote by Σ the set of all infinite selective paths of W. For a finite selective path $(w_1, \ldots, w_k)$, let

$$B_{(w_1,\ldots,w_k)} = \{\sigma \in \Sigma : \ \sigma \text{ has an initial semgnet } (w_1, \ldots, w_k)\}.$$

Define topology τ_Σ on Σ by introducing

$$\mathcal{B}_\Sigma = \{B_{(w_1,\ldots,w_k)} : (w_1, \ldots, w_k) \text{ is a finite selective path of } W\}$$

as a basis.

To see that $\mathcal{B}_\Sigma$ is a basis, observe that $B_{(w)} = \Sigma$, and that

$$B_{(w_1,\ldots,w_k)} \cap B_{(v_1,\ldots,v_m)} = \begin{cases} B_{(w_1,\ldots,w_k)} & \text{if } (v_1, \ldots, v_m) \text{ is an initial segment} \\ & \text{of } (w_1, \ldots, w_k), \\ B_{(v_1,\ldots,v_m)} & \text{if } (w_1, \ldots, w_k) \text{ is an initial segment} \\ & \text{of } (v_1, \ldots, v_m), \\ \emptyset & \text{otherwise.} \end{cases}$$

In order to define $\models_\Sigma$ note that every infinite selective path σ of W either gets stable or keeps cycling. In other words, either $\sigma = (w_1, \ldots, w_k, w_k, \ldots)$, or $\sigma = (w_1, \ldots, w_n, w_{n+1}, \ldots)$, where w_i belongs to some cluster $C \subseteq W$ for $i > n$. In the former case we say that w_k stabilizes σ, and in the latter – that σ keeps cycling in C. Now define $\models_\Sigma$ on Σ by putting

$$\sigma \models_\Sigma P \text{ iff} \begin{cases} w_k \models P & \text{if } w_k \text{ stabilizes } \sigma, \\ \rho(C) \models P & \text{if } \sigma \text{ keeps cycling in } C \subseteq W, \text{ where } \rho(C) \text{ is some} \\ & \text{arbitrarily chosen representative of } C. \end{cases}$$

[1]In other words, w_{k+1} is the first strong successor of w_k in the complete enumeration of $SSuc(w_k)$ which has not appeared in any selective path of length k; if all strong successors of w_k have already appeared in one of selective paths of length k, then we start over again and put w_{k+1} to be the first strong successor of w_k in the complete enumeration of $SSuc(w_k)$.

All we need to show is that $\langle \Sigma, \tau_\Sigma \rangle$ is homeomorphic to the Cantor space, and that $M_\Sigma = \langle \Sigma, \tau_\Sigma, \models_\Sigma \rangle$ is topo-bisimilar to the initial M. In order to show the first claim, let us recall that the Cantor space is homeomorphic to the countable topological product of the two element set $\mathbf{2} = \{0, 1\}$ with the discrete topology. So, $\mathcal{C} \cong \mathbf{2}^\omega$ with the subbasic sets for the topology being $U = \prod_{i \in \omega} U_i$, where all but one U_i coincide with $\mathbf{2}$, or equivalently with the basic sets for the topology being $U = \prod_{i \in \omega} U_i$, where all but finitely many U_i coincide with $\mathbf{2}$.

To picture the Cantor space, one can think of the full infinite binary tree T_2: starting at the root, one associates 0 to every left-son of a node and 1 to every right-son. Then the points of the Cantor space are the infinite branches of T_2.

Proposition 5.2 *$\langle \Sigma, \tau_\Sigma \rangle$ is homeomorphic to $\mathcal{C}$.*

Proof. Suppose $\sigma = (w_1, w_2, w_3, \dots, w_k, \dots) \in \Sigma$, where $w_1 = w$ is a root of W. With each w_k ($k > 1$) associate 0 if $w_{k-1} = w_k$, and associate 1 if w_k is a strong successor of w_{k-1}. Denote an element of $\mathbf{2}$ associated with w_k by $g(w_k)$ and define $G : \Sigma \to \mathbf{2}^\omega$ by putting

$$G(w_1, w_2, w_3, \dots, w_k, \dots) = (g(w_2), g(w_3), \dots, g(w_k), \dots).$$

It should be clear from the definition that G is a bijection. In order to prove that it is a homeomorphism, we need to check that G is open. So, suppose $B_{(w_1,\dots,w_k)}$ is a basic open set of τ_Σ. Then

$$G(B_{(w_1,\dots,w_k)}) = \{g(w_2)\} \times \cdots \times \{g(w_k)\} \times \mathbf{2}^\omega$$

is a basic open of $\mathcal{C}$, G preserves basic opens, hence preserves opens. Conversely, suppose $U = \mathbf{2}^{k-1} \times \{c_k\} \times \mathbf{2}^\omega$, where $c_k = 0$ or 1, is a subbasic open of $\mathcal{C}$. Then

$$G^{-1}(U) = \bigcup_{g(w_k) = c_k} B_{(w_1,\dots,w_k)},$$

which obviously belongs to τ_Σ. Thus, G is open, hence a homeomorphism. **q.e.d.**

It is left to be shown that M_Σ is topo-bisimilar to M. Define $F : \Sigma \to W$ by putting

$$F(\sigma) = \begin{cases} w_k & \text{if } w_k \text{ stabilizes } \sigma, \\ \rho(C) & \text{if } \sigma \text{ keeps cycling in } C. \end{cases}$$

Obviously F is well-defined, and is actually surjective. (For any $w_k \in W$, $F(\sigma_0, w_k, w_k, \dots) = w_k$, where σ_0 is a (finite) selective path from w_1 to w_k.)

Proposition 5.3 *F is a total topo-bisimulation between $M_\Sigma = \langle \Sigma, \tau_\Sigma, \models_\Sigma \rangle$ and $M = \langle W, R, \models \rangle$.*

Proof. Recall from the previous section that with $\langle W, R \rangle$ is associated a finite topological space $\langle W, \tau_R \rangle$ (since $\langle W, R \rangle$ is rooted, $\langle W, \tau_R \rangle$ is actually well-connected). Let us check that $F : \langle \Sigma, \tau_\Sigma \rangle \to \langle W, \tau_R \rangle$ is open. Recall that $R(v)$, for $v \in W$, are basic opens of τ_R. So, in order to check that F is continuous, we need to show that the F inverse image of every $R(v)$ is open in τ_Σ. Observe that for any $v \in W$,

$$F^{-1}(R(v)) = \bigcup_{k \in \omega,\ vRw_k} B_{(w_1, \dots, w_k)},$$

which is an element of τ_Σ. Indeed, suppose $\sigma \in \bigcup_{k \in \omega,\ vRw_k} B_{(w_1, \dots, w_k)}$. Then σ belongs to one of $B_{(w_1, \dots, w_k)}$ with vRw_k. But then $w_k RF(\sigma)$, which together with vRw_k and transitivity of R imply that $vRF(\sigma)$. So, $F(\sigma) \in R(v)$, and $\sigma \in F^{-1}(R(v))$. Conversely, suppose $\sigma \in F^{-1}(R(v))$. Then $F(\sigma) \in R(v)$, and $vRF(\sigma)$. Now either w_k stabilizes σ, or σ keeps cycling in a cluster C. In the former case, $\sigma = (w_1, \dots, w_k, w_k, \dots)$, where $w_k = F(\sigma)$. Hence, $\sigma \in B_{(w_1, \dots, w_k)}$ with vRw_k. In the latter case, $\sigma = (w_1, \dots, w_n, w_{n+1}, \dots)$, where $w_i \in C$ for $i > n$, and $F(\sigma) = \rho(C)$. Hence, $\sigma \in B_{(w_1, \dots, w_n, w_{n+1})}$ with vRw_{n+1}. In either case,

$$F^{-1}(R(v)) \subseteq \bigcup_{k \in \omega,\ vRw_k} B_{(w_1, \dots, w_k)}$$

. Therefore, $F^{-1}(R(v)) = \bigcup_{k \in \omega,\ vRw_k} B_{(w_1, \dots, w_k)}$, and F is continuous.

In order to show that F preserves opens, consider any basic set $B_{(w_1, \dots, w_k)}$ of τ_Σ and show that $F(B_{(w_1, \dots, w_k)})$ is open in τ_R. For this we show that

$$F(B_{(w_1, \dots, w_k)}) = R(w_k).$$

Suppose $v \in F(B_{(w_1, \dots, w_k)})$. Then there exists $\sigma = (w_1, \dots, w_k, \dots) \in B_{(w_1, \dots, w_k)}$ such that $F(\sigma) = v$. Hence $w_k Rv$. Conversely, suppose $w_k Rv$. Consider a (finite) selective path σ_0 from w_1 to v containing $(w_1, \dots, w_k)$ as an initial segment. Then $\sigma = (\sigma_0, v, v, v, \dots) \in B_{(w_1, \dots, w_k)}$ and $F(\sigma) = v$. Hence $F(B_{(w_1, \dots, w_k)}) = R(w_k)$, which is a basic open of τ_R. So, F is open.

Moreover, as follows from the definition of $\models_\Sigma$,

$$\sigma \models_\Sigma P \text{ iff } F(\sigma) \models P.$$

Now since every continuous and open map satisfying this condition is a topo-bisimulation (see [1]), so is our F. **q.e.d.**

Theorem 5.4 **S4** *is complete with respect to the Cantor space.*

Proof. Suppose $\mathbf{S4} \not\vdash \varphi$. Then by Theorem 4.13 there is a finite rooted Kripke model M refuting φ. By Theorem 4.14 we can assume that every cluster of M is proper. By Propositions 5.2 and 5.3 there exists a valuation $\models_{\mathcal{C}}$ on the Cantor set $\mathcal{C}$ such that $\langle \mathcal{C}, \models_{\mathcal{C}} \rangle$ is topo-bisimilar to M. Hence, φ is refuted on $\mathcal{C}$. **q.e.d.**

5.2 Counterexamples on the reals

In the previous subsection we described how selective unraveling transforms counterexamples on a finite rooted Kripke model M into counterexamples on the Cantor space $\mathcal{C}$. In this subsection we show how to transfer counterexamples from M to $(0, 1)$. As a result, we obtain a new proof of completeness of **S4** with respect to the real line.

Our strategy is similar to that in Section 5.1: we start with a non-theorem of **S4** having a counterexample on a finite rooted Kripke model $M = \langle W, R, \models \rangle$ whose every cluster is proper. Then we construct the set Σ of all selective paths of W, and subtract a proper subset Λ of Σ, which is in a one-to-one correspondence with $(0, 1)$. After that we define a topology τ_Λ on Λ so that $\langle \Lambda, \tau_\Lambda \rangle$ is homeomorphic to $(0, 1)$ with its natural topology. Finally, we define a valuation $\models_\Lambda$ on Λ, and show that $\langle \Lambda, \tau_\Lambda, \models_\Lambda \rangle$ is topo-bisimilar to M. Note that since τ_Λ is pretty different from τ_Σ, the topo-bisimulation between $\langle \Lambda, \tau_\Lambda, \models_\Lambda \rangle$ and M is not simply the restriction of the topo-bisimulation between $\langle \Sigma, \tau_\Sigma, \models_\Sigma \rangle$ and M constructed in Section 5.1, but rather its appropriate modification.

Recall from Section 5.1 that in selective unraveling we had three different types of selective branches: going infinitely to the left, infinitely to the right, or infinitely zigzagging. Also recall that a selective branch σ is going infinitely to the left if $\sigma = (w_1, \ldots, w_k, w_k, \ldots)$; σ is going infinitely to the right if $\sigma = (w_1, \ldots, w_n, w_{n+1}, \ldots)$, where w_{k+1} is a strong successor of w_k for any $k \geq n$; and finally, σ is zigzagging if $\sigma = (w_1, \ldots, w_n, w_{n+1}, \ldots)$, where there are infinitely many $k \geq n$ with $w_{k+1} = w_k$, and there are also infinitely many $k \geq n$ with w_{k+1} being a strong successor of w_k.

In order to transfer counterexamples from M to $(0, 1)$, in the definition of selective unraveling we need to restrict ourselves only to those branches which are either going infinitely to the left or are infinitely zigzagging. In other words, we define a *real path* of W to be a selective path of W which is either going infinitely to the left or is infinitely zigzagging.

Denote by Π the set of all real infinite paths of W. So, Π is the subset of the set Σ of all selective infinite paths of W consisting of all selective paths going infinitely to the left or infinitely zigzagging. Hence, Π is in a one-to-one correspondence with the set of those infinite branches of the infinite binary tree T_2 which either have 0 from some node on or are infinitely zigzagging.

This correspondence sets desired connection between Π and $(0, 1)$. To see this recall the dyadic representation of a number from $[0, 1]$. Let $x \in [0, 1]$.

To construct an infinite branch $\alpha = (a_n)_{n\in\omega}$ of T_2 representing x observe that either $x \in [0, \frac{1}{2}]$ or $x \in [\frac{1}{2}, 1]$. In the former case put $a_1 = 0$ and in the latter case put $a_1 = 1$. Assume $x \in [0, \frac{1}{2}]$. Then either $x \in [0, \frac{1}{4}]$ or $x \in [\frac{1}{4}, \frac{1}{2}]$. Again in the former case put $a_2 = 0$ and in the latter case put $a_2 = 1$. Continuing this process, we will get an infinite branch $\alpha = (a_n)_{n\in\omega}$ of T_2 representing x.

Note that there are two ways for the dyadic representation of $\frac{1}{2}$: either as $(0, 1, 1, 1, \dots)$ or as $(1, 0, 0, 0, \dots)$. In general, there are two ways for the dyadic representation of any number $\frac{m}{2^n} \in [0, 1]$ ($m, n \in \omega$, $0 < m < 2^n$): either as $(a_1, \dots, a_k, 1, 0, 0, 0, \dots)$ or as $(a_1, \dots, a_k, 0, 1, 1, 1, \dots)$. Therefore, if we throw away all infinite branches of T_2 having 1 from some node on plus $(0, 0, 0, \dots)$, we obtain a one-to-one correspondence between $(0, 1)$ and the remaining infinite branches of T_2. Hence, there exists a one-to-one correspondence between $(0, 1)$ and $\Lambda = \Pi - \{(w, w, w, \dots)\}$.

Suppose $(w_1, \dots, w_{k-1}, w_k, w_k, \dots) \in \Lambda$ $(w_{k-1} \neq w_k)$ represents $\frac{m}{2^n} \in (0, 1)$. Also suppose

$$C_{(w_1,\dots,w_k)} = \{\lambda \in \Lambda : \text{ the initial segment of } \lambda \text{ is } (w_1, \dots, w_k)\}.$$

(Observe that $C_{(w_1,\dots,w_k)} = B_{(w_1,\dots,w_k)} \cap \Lambda$.)

In order to transfer topological structure of $(0, 1)$ to Λ observe that the family $\{(\frac{m}{2^n}, \frac{m+1}{2^n}) : \ m, n \in \omega,\ 0 < m + 1 < 2^n\}$ forms a basis for the topology on $(0, 1)$, and that the subset of Λ representing $(\frac{m}{2^n}, \frac{m+1}{2^n})$ is $D_{(w_1,\dots,w_k)} = C_{(w_1,\dots,w_k)} - \{(w_1, \dots, w_{k-1}, w_k, w_k, \dots)\}$.

Hence, if we define topology τ_Λ on Λ by introducing

$$\{D_{(w_1,\dots,w_k)} : (w_1, \dots, w_k) \text{ is a finite selective path of } \Lambda\}$$

as a basis, the following obvious fact holds:

Fact 5.5 (Λ, τ_Λ) *is homeomorphic to* $(0, 1)$.

Now we define $\models_\Lambda$ on Λ, and show that there exists a topo-bisimulation between $(\Lambda, \tau_\Lambda, \models_\Lambda)$ and M.

In order to define $\models_\Lambda$ observe that either $\lambda \in \Lambda$ gets stable or it keeps cycling. In other words, either $\lambda = (w_1, \dots, w_{k-1}, w_k, w_k, \dots)$, or $\lambda = (w_1, \dots, w_n, w_{n+1}, \dots)$, where w_i belongs to some cluster $C \subseteq W$, for $i > n$. In the former case we say that w_k stabilizes λ, and in the latter – that λ keeps cycling in C. Now define $\models_\Lambda$ on Λ by putting

$$\lambda \models_\Lambda P \text{ iff} \begin{cases} w_{k-1} \models P & \text{if } w_k \text{ stabilizes } \lambda, \\ \\ \rho(C) \models P & \text{if } \lambda \text{ keeps cycling in } C \subseteq W, \text{ where } \rho(C) \text{ is} \\ & \text{some arbitrarily chosen representative of } C. \end{cases}$$

Finally define a function $F : \Lambda \to W$ by putting

$$F(\lambda) = \begin{cases} w_{k-1} & \text{if } w_k \text{ stabilizes } \lambda, \\ \rho(C) & \text{if } \lambda \text{ keeps cycling in } C. \end{cases}$$

Proposition 5.6 *F is a total topo-bisimulation between $M_\Lambda = \langle \Lambda, \tau_\Lambda, \models_\Lambda \rangle$ and $M = \langle W, R, \models \rangle$.*

Proof. Obviously F is well-defined, and is actually surjective. (For any $w_k \in W$, $F(w_1, \ldots, w_k, w_{k+1}, w_{k+1}, \ldots) = w_k$, where $(w_1, \ldots, w_k)$ is a finite selective path from w_1 to w_k, and w_{k+1} is a strong successor w_k. Note that w_{k+1} exists, since every cluster of W is proper.) Let us check that $F : \langle \Lambda, \tau_\Lambda \rangle \to \langle W, \tau_R \rangle$ is open. Recall that $R(v)$, for $v \in W$, are basic opens of τ_R. So, in order to check that F is continuous, we need to show that the F inverse image of every $R(v)$ is open in τ_Λ. Observe that for any $v \in W$,

$$F^{-1}(R(v)) = \bigcup_{k \in \omega,\ vRw_k} D_{(w_1, \ldots, w_k)},$$

which is an element of τ_Λ. Indeed, suppose $\lambda \in \bigcup_{k \in \omega,\ vRw_k} D_{(w_1, \ldots, w_k)}$. Then λ belongs to one of $D_{(w_1, \ldots, w_k)}$ with vRw_k. Now $\lambda \in D_{(w_1, \ldots, w_k)}$ implies $w_k RF(\lambda)$, which together with vRw_k and transitivity of R yield $vRF(\lambda)$. Hence, $F(\lambda) \in R(v)$, and $\lambda \in F^{-1}(R(v))$. Conversely, suppose $\lambda \in F^{-1}(R(v))$. Then $F(\lambda) \in R(v)$, and $vRF(\lambda)$. Now either λ is going infinitely to the left or is infinitely zigzagging. In the former case, $\lambda = (w_1, \ldots, w_k, w_{k+1}, w_{k+1}, \ldots)$, where $w_k = F(\lambda)$. Hence, $\lambda \in D_{(w_1, \ldots, w_k)}$ with vRw_k. In the latter case, $\lambda = (w_1, \ldots, w_n, w_{n+1}, w_{n+2}, \ \ldots)$, where $F(\lambda) \in C(w_{n+1})$. Hence, $\lambda \in D_{(w_1, \ldots, w_n, w_{n+1})}$ with vRw_{n+1}. In either case, $\lambda \in \bigcup_{k \in \omega,\ vRw_k} D_{(w_1, \ldots, w_k)}$, and $F^{-1}(R(v)) = \bigcup_{k \in \omega,\ vRw_k} D_{(w_1, \ldots, w_k)}$. Hence, F is continuous.

In order to show that F preserves opens, consider any basic set $D_{(w_1, \ldots, w_k)}$ of τ_Λ and show that $F(D_{(w_1, \ldots, w_k)})$ is open in τ_R. For this we show that

$$F(D_{(w_1, \ldots, w_k)}) = R(w_k).$$

Suppose $v \in F(D_{(w_1, \ldots, w_k)})$. Then there exists $\lambda = (w_1, \ldots, w_k, \ldots) \in D_{(w_1, \ldots, w_k)}$ such that $F(\lambda) = v$. Now either λ is going infinitely to the left or is infinitely zigzagging. In the former case, $\lambda = (w_1, \ldots, w_k, \ldots, w_{k+l}, w_{k+l+1}, w_{k+l+1}, \ldots)$, where $w_{k+l} = v$. In the latter case, v is a representative of a cluster C where λ keeps cycling. In either case, $w_k Rv$. Hence, $v \in R(w_k)$. Conversely, suppose $v \in R(w_k)$. Then $w_k Rv$. Consider $\lambda = (w_1, \ldots, w_k, \ldots, v, u,\ u, \ldots)$, where $(w_1, \ldots, w_k, \ldots, v)$ is a finite selective path of W from w_1 to v containing $(w_1, \ldots, w_k)$ as an initial segment, and u is a strong successor of v. (u exists, since every cluster of W is proper.) Then $\lambda \in D_{(w_1, \ldots, w_k)}$ and $F(\lambda) = v$. Hence $F(D_{(w_1, \ldots, w_k)}) = R(w_k)$, which is a basic open of τ_R. So, F is open.

Moreover, as follows from the definition of $\models_\Lambda$,

$$\lambda \models_\Lambda P \text{ iff } F(\lambda) \models P.$$

Now since every continuous and open map satisfying this condition is a topo-bisimulation (see [1]), so is our F. **q.e.d.**

Corollary 5.7 **S4** *is complete with respect to* $(0,1)$.

Proof. Suppose **S4**$\not\vdash \varphi$. Then by Theorem 4.13 there is a finite rooted Kripke model M refuting φ. By Theorem 4.14 we can assume that every cluster of M is proper. By Proposition 5.6, M is topo-bisimilar to $M_\Lambda = \langle \Lambda, \tau_\Lambda, \models_\Lambda \rangle$. Hence, M_Λ is refuting φ. Now since $\langle \Lambda, \tau_\Lambda \rangle$ is homeomorphic to $(0,1)$, φ is refuted on $(0,1)$. **q.e.d.**

Theorem 5.8 **S4** *is complete with respect to the real line* **R**.

Proof. Suppose **S4**$\not\vdash \varphi$. Then by Corollary 5.7 there exists a valuation $\models_{(0,1)}$ on $(0,1)$ refuting φ. Now since $(0,1)$ is homeomorphic to $\mathbf{R}$, φ is refuted on $\mathbf{R}$. **q.e.d.**

This provides an alternative proof of McKinsey and Tarski's original proof. It should be noted that we can improve a little bit on their result. Indeed, McKinsey and Tarski proved that for any non-theorem φ of **S4** there exists a valuation ν on **R** falsifying φ.

Corollary 5.9 *There exists a single valuation* ν *on* **R** *falsifying all the non-theorems of* **S4**.

Proof. Enumerate all the non-theorems of **S4**. This can be done since the language of **S4** is countable. Let this enumeration be $\{\varphi_1, \varphi_2, \dots\}$. Since the interval $(n, n+1)$ is homeomorphic to **R**, from Theorem 5.8 it follows that there exists a valuation ν_n on $(n, n+1)$ such that $\langle (n, n+1), \nu_n \rangle$ falsifies φ_n. (Note that we need not know anything about the shape of $\nu_n(\varphi_n)$.) Now take $\bigcup_{n\in\omega}(n, n+1)$. For any propositional letter P let $\nu(P) = \bigcup_{n\in\omega} \nu_n(P)$ be the valuation of P on **R**. Note that each $\langle (n, n+1), \nu_n \rangle$ is an open submodel of $\langle \mathbf{R}, \nu \rangle$, where the 'identity embedding' is a topo-bisimulation. Hence, the truth values of modal formulas do not change moving from each $\langle (n, n+1), \nu_n \rangle$ to $\langle \mathbf{R}, \nu \rangle$. Therefore, φ_n is still falsified on the whole **R** for each n. Thus, we have constructed a single valuation ν on **R** falsifying all the non-theorems of **S4**. **q.e.d.**

This also shows that though very different from the standard canonical Kripke model of **S4**, **R** shares some of its universal properties.

5.3 Logical non-finiteness on the reals

Recall that two formulas φ and ψ are said to be **S4**-*equivalent* if $\mathbf{S4} \vdash \varphi \leftrightarrow \psi$. It is well known that there exist infinitely many formulas of one-variable which are not **S4**-equivalent. E.g., consider the following list of formulas:

$$\varphi_0 = P;$$

$$\varphi_n = \varphi_{n-1} \wedge \Diamond(\Diamond\varphi_{n-1} \wedge \neg\varphi_{n-1}).$$

We can easily construct a Kripke model on which all φ_n have different interpretations. Let $M = \langle \omega, R, \models \rangle$, where ω denotes the set of all natural numbers, nRm iff $m \leq n$, and $n \models P$ iff n is odd. Then one can readily check that φ_n is true at all odd points $> n$. Hence every φ_n has a different interpretation on M. It implies that the φ_n are not **S4**-equivalent. Now we will give a topological flavor to this result by showing that interpreting a propositional variable as a certain subset of $\mathbf{R}$ allows us to construct infinitely many **S4**-nonequivalent formulas of one variable. Corollary 5.9 already told us such a uniform choice must exist, but the proof does not construct $\nu(P)$ explicitly. The following argument does, and thereby also highlights the topological content of our modal completeness theorem.

We use $\Diamond$ and $\Box$ instead of the standard notations $\overline{()}$ and $Int()$ for the closure and interior operators of a topological space. This modal notation shows its basic use in topology because it allows us to write topological formulas in a much more perspicuous fashion.

To proceed further we need to recall the definition of Hausdorff's residue of a given set. Suppose a topological space $\langle W, \tau \rangle$ and $X \subseteq W$ are given. $\varrho(X) = X \cap \Diamond(\Diamond X - X)$ is called the *Hausdorff residue* of X. Let $\varrho^0(X) = X$, $\varrho^1(X) = \varrho(X)$ and $\varrho^{n+1}(X) = \varrho\varrho^n(X)$.

X is said to be of *rank* n, written $r(X) = n$, if n is the least natural number such that $\varrho^n(X) = \emptyset$. X is said to be of *finite rank* if there exists a natural n such that X is of rank n. X is said to be of *infinite rank* if it is not of finite rank.

$x \in X$ is said to be of *rank* n if $x \in \varrho^n(X)$, but $x \notin \varrho^{n+1}(X)$. $x \in X$ is said to be of *finite rank* if there exists a natural n such that x is of rank n. x is said to be of *infinite rank* if it is not of finite rank.

Obviously X is of rank n iff the rank of every element of X is strictly less than n, and there is at least one element of X of rank $n-1$; X is of finite rank iff there is a natural n such that the rank of every element of X is strictly less than n; and X is of infinite rank iff there is no finite bound on the ranks of elements of X.

It is obvious that if we interpret P as a subset X of $\mathbf{R}$, then φ_n will be interpreted as $\varrho^n(X)$. So, in order to show that different φ_n are **S4**-nonequivalent, it is sufficient to show that there is $X \subset \mathbf{R}$ such that $\varrho(X) \supset \varrho^2(X) \supset \cdots \supset \varrho^n(X) \supset \ldots$. Indeed, we have the following

Proposition 5.10 *There exists a subset X of $\mathbf{R}$ such that $\varrho(X) \supset \varrho^2(X) \supset \cdots \supset \varrho^n(X) \supset \ldots$.*

Proof: We will construct X inductively. Fix a natural number k.

Step 1: Consider a sequence $\{x_{i_1}\}_{i_1=1}^{\infty}$ from $(k-1,k)$ converging to $k-1$, and put

$$X_1 = \{k-1\} \cup \bigcup_{i_1=1}^{\infty} \{y_{i_2}^{i_1}\}_{i_2=1}^{\infty},$$

where $\{y_{i_2}^{i_1}\}_{i_2=1}^{\infty}$ is a sequence from (x_{i_1+1}, x_{i_1}) converging to x_{i_1+1}. Note that

$\Diamond X_1 = X_1 \cup \{x_{i_1}\}_{i_1=1}^{\infty}$,

$\Diamond X_1 - X_1 = \{x_{i_1}\}_{i_1=1}^{\infty}$,

$\Diamond(\Diamond X_1 - X_1) = \{k-1\} \cup \{x_{i_1}\}_{i_1=1}^{\infty}$, and

$\varrho(X_1) = \{k-1\}$.

So, $k-1$ is the only point of X_1 of rank 1, and $r(X_1) = 2$.

Step 2: Consider a sequence $\{x_{i_3}^{i_1,i_2}\}_{i_3=1}^{\infty}$ from $(y_{i_2+1}^{i_1}, y_{i_2}^{i_1})$ converging to $y_{i_2+1}^{i_1}$, and put

$$X_2 = \{k-1\} \cup \bigcup_{i_1=1}^{\infty} \{y_{i_2}^{i_1}\}_{i_2=1}^{\infty} \cup \bigcup_{i_1=1}^{\infty} \bigcup_{i_2=1}^{\infty} \bigcup_{i_3=1}^{\infty} \{y_{i_4}^{i_1,i_2,i_3}\}_{i_4=1}^{\infty},$$

where $\{y_{i_4}^{i_1,i_2,i_3}\}_{i_4=1}^{\infty}$ is a sequence from $(x_{i_3+1}^{i_1,i_2}, x_{i_3}^{i_1,i_2})$ converging to $x_{i_3+1}^{i_1,i_2}$. Note that $X_2 \supset X_1$, and

$\Diamond X_2 = X_2 \cup \{x_{i_1}\}_{i_1=1}^{\infty} \cup \bigcup_{i_1=1}^{\infty} \bigcup_{i_2=1}^{\infty} \{x_{i_3}^{i_1,i_2}\}_{i_3=1}^{\infty}$,

$\Diamond X_2 - X_2 = \{x_{i_1}\}_{i_1=1}^{\infty} \cup \bigcup_{i_1=1}^{\infty} \bigcup_{i_2=1}^{\infty} \{x_{i_3}^{i_1,i_2}\}_{i_3=1}^{\infty}$,

$\Diamond(\Diamond X_2 - X_2) = \{k-1\} \cup \bigcup_{i_1=1}^{\infty} \{y_{i_2}^{i_1}\}_{i_2=1}^{\infty} \cup \{x_{i_1}\}_{i_1=1}^{\infty} \cup \bigcup_{i_1=1}^{\infty} \bigcup_{i_2=1}^{\infty} \{x_{i_3}^{i_1,i_2}\}_{i_3=1}^{\infty}$,

$\varrho(X_2) = \{k-1\} \cup \bigcup_{i_1=1}^{\infty} \{y_{i_2}^{i_1}\}_{i_2=1}^{\infty}$, and

$\varrho^2(X_2) = \{k-1\}$.

So, the points of X_2 of rank 1 are $y_{i_2}^{i_1}$, for arbitrary i_1 and i_2, $k-1$ is the only point of X_2 of rank 2, and $r(X_2) = 3$.

Step n: For $n \geq 1$ consider a sequence $\{x_{i_{2n-1}}^{i_1,\ldots,i_{2n-2}}\}_{i_{2n-1}=1}^{\infty}$ from $(y_{i_{2n-2}+1}^{i_1,\ldots,i_{2n-3}}, y_{i_{2n-2}}^{i_1,\ldots,i_{2n-3}})$ converging to $y_{i_{2n-2}+1}^{i_1,\ldots,i_{2n-3}}$, and put

$$X_n = \{k-1\} \cup \bigcup_{i_1=1}^{\infty} \{y_{i_2}^{i_1}\}_{i_2=1}^{\infty} \cup \ldots \cup \bigcup_{i_1=1}^{\infty} \cdots \bigcup_{i_{2n-1}=1}^{\infty} \{y_{i_{2n}}^{i_1,\ldots,i_{2n-1}}\}_{i_{2n}=1}^{\infty},$$

where $\{y_{i_{2n}}^{i_1,\ldots,i_{2n-1}}\}_{i_{2n}=1}^{\infty}$ is a sequence from $(x_{i_{2n-1}+1}^{i_1,\ldots,i_{2n-2}}, x_{i_{2n-1}}^{i_1,\ldots,i_{2n-2}})$ converging to $x_{i_{2n-1}+1}^{i_1,\ldots,i_{2n-2}}$. Also let

$$A = \{x_{i_1}\}_{i_1=1}^{\infty} \cup \ldots \cup \bigcup_{i_1=1}^{\infty} \cdots \bigcup_{i_{2n-2}=1}^{\infty} \{x_{i_{2n-1}}^{i_1,\ldots,i_{2n-2}}\}_{i_{2n-1}=1}^{\infty}.$$

Then note that $X_n \supset X_{n-1} \supset \cdots \supset X_2 \supset X_1$, and

$$\Diamond X_n = X_n \cup A,$$

$$\Diamond X_n - X_n = A,$$

$$\Diamond(\Diamond X_n - X_n) = A \cup (X_n - [\textstyle\bigcup_{i_1=1}^{\infty} \cdots \bigcup_{i_{2n-1}=1}^{\infty} \{y_{i_{2n}}^{i_1,\ldots,i_{2n-1}}\}_{i_{2n}=1}^{\infty}]),$$

$$\varrho(X_n) = X_n - [\textstyle\bigcup_{i_1=1}^{\infty} \cdots \bigcup_{i_{2n-1}=1}^{\infty} \{y_{i_{2n}}^{i_1,\ldots,i_{2n-1}}\}_{i_{2n}=1}^{\infty}],$$

$$\varrho^2(X_n) = \rho(X_n) - [\textstyle\bigcup_{i_1=1}^{\infty} \cdots \bigcup_{i_{2n-3}=1}^{\infty} \{y_{i_{2n-2}}^{i_1,\ldots,i_{2n-3}}\}_{i_{2n-2}=1}^{\infty}],$$

$$\ldots$$

$$\varrho^n(X_n) = \{k-1\}.$$

So, the points of X_n of rank 1 are

$$X_n - [\bigcup_{i_1=1}^{\infty} \cdots \bigcup_{i_{2n-1}=1}^{\infty} \{y_{i_{2n}}^{i_1,\ldots,i_{2n-1}}\}_{i_{2n}=1}^{\infty}],$$

the points of X_n of rank 2 are

$$X_n - [\bigcup_{i_1=1}^{\infty} \cdots \bigcup_{i_{2n-3}=1}^{\infty} \{y_{i_{2n-2}}^{i_1,\ldots,i_{2n-3}}\}_{i_{2n-2}=1}^{\infty} \cup \bigcup_{i_1=1}^{\infty} \cdots \bigcup_{i_{2n-1}=1}^{\infty} \{y_{i_{2n}}^{i_1,\ldots,i_{2n-1}}\}_{i_{2n}=1}^{\infty}],$$

and so on; finally, $k-1$ is the only point of X_n of rank n, and $r(X_n) = n+1$.

Now let X_1 be constructed in $(0,1)$, X_2 in $(1,2)$, X_n in $(n-1,n)$, and so on. We put

$$X = \bigcup_{n=1}^{\infty} X_n.$$

Then it is obvious that $n-1 \in \varrho^n(X)$ and $n-1 \notin \varrho^{n+1}(X)$, for any natural n. So, $\varrho(X) \supset \varrho^2(X) \supset \cdots \supset \varrho^n(X) \supset \ldots$, and X contains points of every finite rank. **q.e.d.**

Remark 5.11 It is worth to be noted that the X constructed above does not contain elements of infinite rank. However, a little adjustment of the above construction will allow us to construct a subset of $\mathbf{R}$ with an element of infinite rank. Actually, it is possible to construct a subset of $\mathbf{R}$ containing elements of rank α, for any ordinal $\alpha < \aleph_1$.

Returning to our list of formulas, with P as the just constructed X, the interpretation of every φ_n in $\mathbf{R}$ will be different, in terms of some topologically significant phenomenon. In the next section we will show that if we restrict ourselves to only "good" subsets of $\mathbf{R}$, then the situation will be drastically changed.

6 Axiomatizing special kinds of regions

As we saw in the previous section, by interpreting propositional variables as certain subsets of the real line $\mathbf{R}$, we can refute every non-theorem of **S4** on $\mathbf{R}$. Certainly not all subsets of $\mathbf{R}$ are required for refuting the non-theorems of **S4**. In this section, we will analyze the complexity of the subsets of $\mathbf{R}$ required for refuting the non-theorems of **S4**. Similarly to Section 5.5, we prefer to use $\Diamond$ and $\Box$ to denote the closure and interior operators of a topological space. For consistency we also use $\wedge, \vee$ and $\neg$ to denote set-theoretical intersection, union and complement.

6.1 Serial sets on the real line

To start with, consider subsets of $\mathbf{R}$ with the simplest intuitive structure. Call $X \subseteq \mathbf{R}$ *convex* if all points lying in between any two points of X belong to X. In other words, X is convex if $x, y \in X$ and $x \leq y$ imply $[x, y] \subseteq X$. Every convex subset of $\mathbf{R}$ has one of the following forms:

$$\emptyset,\ (x,y),\ [x,y],\ [x,y),\ (x,y],\ (-\infty,x),\ (-\infty,x],\ (x,+\infty),\ [x,+\infty),\ \mathbf{R}.$$

Definition 6.1 *Call a subset of* $\mathbf{R}$ serial *if it is a finite union of convex subsets of* $\mathbf{R}$. *Denote the set of all serial subsets of* $\mathbf{R}$ *by* $\mathcal{S}(\mathbf{R})$. *So,*

$$\mathcal{S}(\mathbf{R}) = \{X \in \mathcal{P}(\mathbf{R}) :\ X \textit{ is a serial subset of } \mathbf{R}\}.$$

Obviously the X constructed in Proposition 5.10 is not serial, and actually this was absolutely crucial in showing that X had points of any finite rank. Indeed, we have the following

Lemma 6.2 $r(X) = 0$ *for any* $X \in \mathcal{S}(\mathbf{R})$.

Proof: First, $r(Y) = 0$ for any convex subset Y of $\mathbf{R}$. For, if Y is convex, then $\Diamond Y \wedge \neg Y$ consists of at most two points, $\Diamond(\Diamond Y \wedge \neg Y) = \Diamond Y \wedge \neg Y$, and $\varrho(Y) = Y \wedge \Diamond(\Diamond Y \wedge \neg Y) = Y \wedge (\Diamond Y \wedge \neg Y) = \emptyset$. Hence $r(Y) = 0$.

Now let X be a serial subset of $\mathbf{R}$. Then $X = \bigvee_{i=1}^{n} X_i$, where every X_i is a convex subset of $\mathbf{R}$, and actually we can assume that all X_i are disjoint. But then $\varrho(X) = \bigvee_{i=1}^{n} \varrho(X_i) = \emptyset$, and hence $r(X) = 0$. q.e.d.

It follows that if we interpret P as a serial subset of $\mathbf{R}$, then no two φ_n $(n \geq 1)$ from the previous section define sets equal to each other.

Call a valuation ν of our language $\mathcal{L}$ to subsets of $\mathbf{R}$ *serial* if $\nu(P) \in \mathcal{S}(\mathbf{R})$ for any propositional variable P. Since $\mathcal{S}(\mathbf{R})$ is closed with respect to $\neg, \wedge$ and $\Diamond$, we have that $\nu(\varphi) \in \mathcal{S}(\mathbf{R})$ for any serial valuation ν. Call a formula φ *$\mathcal{S}$-true* if it is true in $\mathbf{R}$ under a serial valuation. Call φ *$\mathcal{S}$-valid* if φ is $\mathcal{S}$-true for any serial valuation on $\mathbf{R}$. Let $L(\mathcal{S}) = \{\varphi : \varphi \text{ is } \mathcal{S}\text{-valid}\}$.

Fact 6.3 *$L(\mathcal{S})$ is a normal modal logic over* **S4**.

Obviously all φ_n $(n \geq 1)$ from the previous section are $L(\mathcal{S})$-equivalent. So, it is natural to expect that there are only finitely many formulas in one variable which are $L(\mathcal{S})$-nonequivalent, and indeed that $L(\mathcal{S})$ is a much stronger logic than **S4**.

As a first step in this direction, we show that the Grzegorczyk axiom

$$\mathbf{Grz} = \Box(\Box(P \to \Box P) \to P) \to P$$

belongs to $L(\mathcal{S})$.

Fact 6.4 **Grz** *is $\mathcal{S}$-valid.*

Proof. **Grz** is $\mathcal{S}$-valid iff $X \subseteq \Diamond(X \wedge \neg\Diamond(\Diamond X \wedge \neg X))$ for any $X \in \mathcal{S}(\mathbf{R})$. Suppose $X \in \mathcal{S}(\mathbf{R})$. Since $\Diamond X \wedge \neg X$ is finite, $\Diamond(\Diamond X \wedge \neg X) = \Diamond X \wedge \neg X$. Hence $\Diamond(X \wedge \neg\Diamond(\Diamond X \wedge \neg X)) = \Diamond(X \wedge \neg(\Diamond X \wedge \neg X)) = \Diamond(X \wedge (\neg\Diamond X \vee X)) = \Diamond X$, which clearly contains X. So, $X \subseteq \Diamond(X \wedge \neg\Diamond(\Diamond X \wedge \neg X))$. q.e.d.

As a next step, we show that the axioms

$\mathbf{BD}_2 = (\neg P \wedge \Diamond P) \to \Diamond\Box P$, and
$\mathbf{BW}_2 = \neg(P \wedge Q \ \wedge \ \Diamond(P \wedge \neg Q) \ \wedge \ \Diamond(\neg P \wedge Q) \ \wedge \ \Diamond(\neg P \wedge \neg Q))$,

bounding the depth and the width of a Kripke model to 2, are $\mathcal{S}$-valid.

Fact 6.5 $\mathbf{BD}_2$ *and* $\mathbf{BW}_2$ *are $\mathcal{S}$-valid.*

Proof: Note that $\mathbf{BD}_2$ is $\mathcal{S}$-valid iff $\Diamond X \wedge \neg X \subseteq \Diamond\Box X$ for any $X \in \mathcal{S}(\mathbf{R})$, and that $\mathbf{BW}_2$ is $\mathcal{S}$-valid iff $X \wedge Y \wedge \Diamond(X \wedge \neg Y) \wedge \Diamond(Y \wedge \neg X) \wedge \Diamond(\neg X \wedge \neg Y) = \emptyset$ for any $X, Y \in \mathcal{S}(\mathbf{R})$.

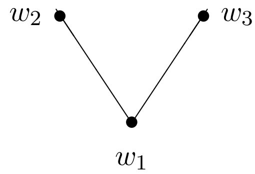
To show that $\Diamond X \wedge \neg X \subseteq \Diamond\Box X$ for any $X \in \mathcal{S}(\mathbf{R})$, suppose $x \in \Diamond X \wedge \neg X$. Then x is a limit point of X not belonging to X. Since X is serial, there is $y \in \mathbf{R}$ such that either $y < x$ and $(y, x) \subseteq X$, or $x < y$ and $(x, y) \subseteq X$. In both cases it is obvious that $x \in \Diamond\Box X$. So, $\Diamond X \wedge \neg X \subseteq \Diamond\Box X$.

To show that $X \wedge Y \wedge \Diamond(X \wedge \neg Y) \wedge \Diamond(Y \wedge \neg X) \wedge \Diamond(\neg X \wedge \neg Y) = \emptyset$ for any $X, Y \in \mathcal{S}(\mathbf{R})$, suppose $x \in X \wedge Y \wedge \Diamond(X \wedge \neg Y) \wedge \Diamond(Y \wedge \neg X)$. Then $x \notin \Box X$ and $x \notin \Box Y$. Hence there exist $y, z \in \mathbf{R}$ such that $y < x < z$ and $(y, z) \cap (\neg X \wedge \neg Y) = \emptyset$, which means that $x \notin \Diamond(\neg X \wedge \neg Y)$. So, $X \wedge Y \wedge \Diamond(X \wedge \neg Y) \wedge \Diamond(Y \wedge \neg X) \wedge \Diamond(\neg X \wedge \neg Y) = \emptyset$. **q.e.d.**

The following is an immediate consequence of our observations.

Corollary 6.6 $\mathbf{S4} \oplus \mathbf{Grz} \oplus \mathbf{BD}_2 \oplus \mathbf{BW}_2 \subseteq L(\mathcal{S})$.

In order to prove the converse, and hence complete our axiomatization of the logic of serial subsets of $\mathbf{R}$, observe that $\mathbf{S4} \oplus \mathbf{Grz} \oplus \mathbf{BD}_2 \oplus \mathbf{BW}_2$ is actually the complete modal logic of the following '2-fork' Kripke frame $\langle W, R \rangle$, where $W = \{w_1, w_2, w_3\}$ and $w_1 R w_1, w_2 R w_2, w_3 R w_3, w_1 R w_2, w_1 R w_3$:



Indeed, it is well known that **Grz** is valid on a Kripke frame iff it is a Noetherian partial order, that $\mathbf{BD}_2$ is valid on a partially ordered Kripke frame iff its depth is bounded by 2, and that $\mathbf{BW}_2$ is valid on a partially ordered Kripke frame of a depth ≤ 2 iff its width is bounded by 2. Now, denoting the logic of $\langle W, R \rangle$ by $L(\langle W, R \rangle)$, we have the following:

Theorem 6.7 $\mathbf{S4} \oplus \mathbf{Grz} \oplus \mathbf{BD}_2 \oplus \mathbf{BW}_2 = L(\langle W, R \rangle)$.

Proof: Denote $\mathbf{S4} \oplus \mathbf{Grz} \oplus \mathbf{BD}_2 \oplus \mathbf{BW}_2$ by L. It is obvious that $\langle W, R \rangle \models \mathbf{Grz}, \mathbf{BD}_2, \mathbf{BW}_2$. Hence $L \subseteq L(\langle W, R \rangle)$. Conversely, since **Grz** is a theorem of L, every L-frame is a Noetherian partial order. Since $\mathbf{BD}_2$ is a theorem of L, every L-frame is of the depth ≤ 2, hence L has the f.m.p., and thus is complete with respect to finite rooted partially ordered Kripke frames of depth ≤ 2. Since $\mathbf{BW}_2$ is a theorem of L, it is obvious that the width of finite rooted L-frames is also ≤ 2. But then it is routine to check that every such frame is a p-morphic image of $\langle W, R \rangle$. Hence $L(\langle W, R \rangle) \subseteq L$, and $L = L(\langle W, R \rangle)$. **q.e.d.**

As a final move, we show that $\langle W, \tau_R \rangle$ is an open and *serial* image of $\mathbf{R}$, meaning that there is an open map $f : \mathbf{R} \to W$ such that $f^{-1}(X) \in \mathcal{S}(\mathbf{R})$ for any subset X of W.

Recall that τ_R consists of the upward closed subsets of W, which obviously are $\emptyset$, $\{w_2\}$, $\{w_3\}$, $\{w_2, w_3\}$, and W. Fix any $x \in \mathbf{R}$ and define $f : \mathbf{R} \to W$ by putting

$$f(y) = \begin{cases} w_1 & \text{for } y = x, \\ w_2 & \text{for } y < x, \\ w_3 & \text{for } y > x. \end{cases}$$

Then it is routine to check that $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{w_2\}) = (-\infty, x)$, $f^{-1}(\{w_3\}) = (x, +\infty)$, $f^{-1}(\{w_2, w_3\}) = (-\infty, x) \cup (x, +\infty)$, and $f^{-1}(W) = \mathbf{R}$. So, f is continuous. Moreover, for any open subset U of $\mathbf{R}$, if $x \in U$, then $f(U) = W$; and if $x \notin U$, then $f(U) \subseteq \{w_2, w_3\}$, which is always open. Hence, f is open. Furthermore, from the definition of f it follows that the f-inverse image of any subset of W is a serial subset of $\mathbf{R}$. So, $\langle W, \tau_R\rangle$ is an open and serial image of $\mathbf{R}$.

As a trivial consequence of this observation, we obtain that for every valuation $\models$ on $\langle W, R\rangle$ there is a serial valuation $\models_S$ on $\mathbf{R}$ such that $\langle W, R, \models\rangle$ is topo-bisimilar to $\langle \mathbf{R}, \models_S\rangle$. Hence, every non-theorem of $L(\langle W, R\rangle)$ is a non-theorem of $L(S)$, and we have the following:

Corollary 6.8 $L(S) \subseteq L(\langle W, R\rangle)$.

Combining Corollaries 6.6 and 6.8 and Theorem 6.7 one obtains:

Theorem 6.9 $L(S) = L(\langle W, R\rangle) = \mathbf{S4} \oplus \mathbf{Grz} \oplus \mathbf{BD}_2 \oplus \mathbf{BW}_2$.

6.2 Formulas in one variable over the serial sets

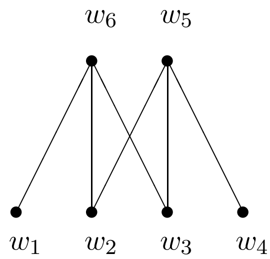
This section provides some more concrete information on 'serial sets'. As $L(\mathcal{S})$ is the logic of the finite '2-fork' frame, for every natural number $n \geq 0$, there are only finitely many $L(\mathcal{S})$-nonequivalent formulas built from the variables $P_1, \ldots, P_n$. In this subsection we show that there are exactly 64 $L(\mathcal{S})$-nonequivalent formulas in one variable, and describe them all.

Theorem 6.10 *Every formula in one variable is $L(\mathcal{S})$-equivalent to a disjunction of the following six formulas:*
$\Box P$,
$\Box\neg P$,
$P \wedge \Box\Diamond\neg P$,
$\neg P \wedge \Box\Diamond P$,
$P \wedge \Diamond\Box\neg P \wedge \Diamond\Box P$, *and*
$\neg P \wedge \Diamond\Box P \wedge \Diamond\Box\neg P$.

Hence, there are exactly 64 *$L(\mathcal{S})$-nonequivalent formulas in one variable.*

Proof. In line with our interest in tying up 'modal' and 'topological' ways of thinking, we will give two different proofs of this result. One proceeds by constructing the 1-universal Kripke model of $L(\mathcal{S})$, which is a standard technique in modal logic, the other is purely topological, using some obvious observations on serial subsets of $\mathbf{R}$.

First Proof. Since $L(\mathcal{S})$ is the logic of the '2-fork' frame, we can easily construct the 1-universal Kripke model $\langle W(1), \models_{(1)}\rangle$ of $L(\mathcal{S})$:



Here $w_n \models P$ iff n is even. Now one can readily check that each point of $W(1)$ corresponds to one of the six formulas in the condition of the theorem. Hence every formula in one variable is $L(\mathcal{S})$-equivalent to a disjunction of the above six formulas. Since there are exactly 2^6 different subsets of $W(1)$, we obtain that there are exactly 64 $L(\mathcal{S})$-nonequivalent formulas in one variable.

Second Proof. Observe that there exists a serial subset X of $\mathbf{R}$ such that $\Box X \neq \Box\neg X \neq X \wedge \Box\Diamond\neg X \neq \neg X \wedge \Box\Diamond X \neq X \wedge \Diamond\Box\neg X \wedge \Diamond\Box X \neq \neg X \wedge \Diamond\Box X \wedge \Diamond\Box\neg X$. For example, let $x < y < z < u$, and take $X = [x, y) \cup (y, z) \cup \{u\}$. Then one can readily check that

$\Box X = (x, y) \cup (y, z)$,
$\Box\neg X = (-\infty, x) \cup (z, u) \cup (u, +\infty)$,
$X \wedge \Box\Diamond\neg X = \{u\}$,
$\neg X \wedge \Box\Diamond X = \{y\}$,
$X \wedge \Diamond\Box\neg X \wedge \Diamond\Box X = \{x\}$, and
$\neg X \wedge \Diamond\Box X \wedge \Diamond\Box\neg X = \{z\}$.

Hence, we can always interpret P as a serial subset of $\mathbf{R}$ such that all the six formulas of the theorem correspond to different serial subsets of $\mathbf{R}$.

Now, let us prove that every subset of $\mathbf{R}$ obtained by repeatedly applying $\neg, \wedge, \Box$ to a serial set X is equal to a finite (including the empty) union of the following serial subsets:

$T_1 = \Box X,$
$T_2 = \Box\neg X,$
$T_3 = X \wedge \Box\Diamond\neg X,$
$T_4 = \neg X \wedge \Box\Diamond X,$
$T_5 = X \wedge \Diamond\Box\neg X \wedge \Diamond\Box X,$ and
$T_6 = \neg X \wedge \Diamond\Box X \wedge \Diamond\Box\neg X.$

For this first observe that $T_i \wedge T_j = \emptyset$ if $i \neq j$, and that $\bigvee_{i=1}^{6} T_i = \mathbf{R}$. So, these six serial subsets of $\mathbf{R}$ are mutually disjoint and jointly exhaustive. Next observe that $\neg T_i = T_j \vee T_k \vee T_l \vee T_m \vee T_n$, where $i, j, k, l, m, n \in \{1, 2, 3, 4, 5, 6\}$ are different from each other. Finally observe that $\Box T_1 = T_1$, $\Box T_2 = T_2$, and $\Box T_3 = \Box T_4 = \Box T_5 = \Box T_6 = \emptyset$.

Hence every subset of $\mathbf{R}$ obtained by repeatedly applying $\neg, \wedge, \Box$ to $\{T_1, \ldots, T_6\}$ is a finite (including the empty) union of $\{T_1, \ldots, T_6\}$.

Now suppose $Y \subseteq \mathbf{R}$ is obtained by repeatedly applying $\neg, \wedge, \Box$ to X. We prove by induction on the complexity of Y that Y is equal to a finite (including the empty) union of $\{T_1, \ldots, T_6\}$.

Base case. Since $X = T_1 \vee T_3 \vee T_5$ (and $\neg X = T_2 \vee T_4 \vee T_6$), the base case (that is when $Y = X$) is obvious.

Complement. Suppose $Y = \neg Z$ and $Z = T_{i_1} \vee \cdots \vee T_{i_k}$, where $i_1, \ldots, i_k \in \{1, \ldots, 6\}$. Then $Y = \neg(T_{i_1} \vee \cdots \vee T_{i_k}) = \neg T_{i_1} \wedge \cdots \wedge \neg T_{i_k}$. Since every $\neg T_{i_j}$ is equal to $\bigvee_{i_s \neq i_j} T_{i_s}$, using the distributivity law we obtain that $Y = \bigvee_{i_s, i_t \in \{1,\ldots,6\}} (T_{i_s} \wedge T_{i_t})$. Since for different i_s and i_t, $T_{i_s} \wedge T_{i_t} = \emptyset$, which is the empty union of T_is, we finally obtain that Y is a finite union of $\{T_1, \ldots, T_6\}$.

Intersection. Suppose $Y = Z_1 \wedge Z_2$, $Z_1 = T_{i_1} \vee \cdots \vee T_{i_k}$ and $Z_2 = T_{j_1} \vee \cdots \vee T_{j_m}$, where $i_1, \ldots, i_k, j_1, \ldots, j_m \in \{1, \ldots, 6\}$. Similarly to the above case, using the distributivity law we obtain that Y is a finite union of $\{T_1, \ldots, T_6\}$.

Interior. Suppose $Y = \Box Z$ and $Z = T_{i_1} \vee \cdots \vee T_{i_k}$, where $i_1, \ldots, i_k \in \{1, \ldots, 6\}$. Since T_is are mutually disjoint, $Y = \Box T_{i_1} \vee \cdots \vee \Box T_{i_k}$. Now since $\{T_1, \ldots, T_6\}$ is closed with respect to $\Box$, we obtain that Y is a finite union of $\{T_1, \ldots, T_6\}$.

Hence, every subset of $\mathbf{R}$ obtained by repeatedly applying $\neg, \wedge, \Box$ to a serial set X is equal to a finite (including the empty) union of $\{T_1, \ldots, T_6\}$. Since there are exactly 2^6 different subsets obtained as a union of $\{T_1, \ldots, T_6\}$, we obtain that there are exactly 64 different subsets of $\mathbf{R}$ obtained by repeatedly applying $\neg, \wedge, \Box$ to a serial set X. This directly implies that there are exactly 64 $L(\mathcal{S})$-nonequivalent formulas in one variable. **q.e.d.**

The same technique can also be used to prove the normal form theorem over $L(\mathcal{S})$ for every formula with more than one proposition variable.

6.3 Countable unions of convex sets on the real line

Let us now be a bit more systematic. By Theorem 5.8, **S4** is the complete logic of **R**, and hence sets of reals suffice as values $\nu(P)$ in refuting non-theorems. But how complex must these sets be? In first-order logic, e.g., we know that completeness requires atomic predicates over the integers which are at least Δ_2^0. With only simpler predicates in the arithmetic hierarchy, the logic gets richer. In a topological space like **R**, it seems reasonable to look at the Borel Hierarchy $\mathcal{G}$. How high up do we have to go for our **S4**-counterexamples? One could analyze our construction in Section 5.3 to have an upper bound. But here, we will state some more direct information.

Consider the set τ of all open subsets of **R**. Let $\mathcal{B}(\tau)$ denote the Boolean closure of τ. Since $\mathcal{B}(\tau)$ contains all closed subsets of **R**, it is obvious that $\mathcal{B}(\tau)$ is closed with respect to $\Diamond$. Obviously $\mathcal{S}(\mathbf{R})$ is properly contained in $\mathcal{B}(\tau)$. It is natural to ask whether the elements of $\mathcal{B}(\tau)$ are enough for refuting all the non-theorems of **S4**. The answer is negative: the modal logic is still richer.

Fact 6.11 *[9] The complete logic of* $\mathcal{B}(\tau)$ *is* **Grz**.

Hence, we need to seek something bigger than $\mathcal{B}(\tau)$. Let $\mathcal{C}^\infty(\mathbf{R})$ denote the set of countable unions of convex subsets of **R**. Since every open subset of **R** is a countable union of open intervals, it is obvious that $\tau \subseteq \mathcal{C}^\infty(\mathbf{R})$. Let $\mathcal{B}(\mathcal{C}^\infty(\mathbf{R}))$ denote the Boolean closure of $\mathcal{C}^\infty(\mathbf{R})$. Since $\tau \subseteq \mathcal{C}^\infty(\mathbf{R})$, we also have $\mathcal{B}(\tau) \subseteq \mathcal{B}(\mathcal{C}^\infty(\mathbf{R}))$. It follows that $\mathcal{B}(\mathcal{C}^\infty(\mathbf{R}))$ is also closed with respect to $\Diamond$. Moreover, $\mathcal{B}(\tau)$ is properly contained in $\mathcal{B}(\mathcal{C}^\infty(\mathbf{R}))$, since the set **Q** of rationals belongs to $\mathcal{B}(\mathcal{C}^\infty(\mathbf{R}))$ but does not belong to $\mathcal{B}(\tau)$.

Theorem 6.12 *[9]* **S4** *is the complete logic of* $\mathcal{B}(\mathcal{C}^\infty(\mathbf{R}))$.

So, the Boolean combinations of countable unions of convex subsets of **R** are all we need for refuting the non-theorems of **S4**. Since every countable union of convex subsets of **R** belongs to the Borel hierarchy $\mathcal{G}_2$ over the opens of **R**, very low level of the Borel hierarchy suffices for refuting the non-theorems of **S4**. So, $\mathcal{G}$ itself is more than sufficient for refuting the non-theorems of **S4**.

Summarizing, we constructed five Boolean algebras of subsets of **R** forming a chain under inclusion: $\mathcal{S}(\mathbf{R}) \subset \mathcal{B}(\tau) \subset \mathcal{B}(\mathcal{C}^\infty(\mathbf{R})) \subset \mathcal{G} \subset \mathcal{P}(\mathbf{R})$, where $\mathcal{S}(\mathbf{R})$ is the Boolean algebra of all serial subsets of **R**, $\mathcal{B}(\tau)$ the Boolean closure of the set of all open subsets of **R**, $\mathcal{B}(\mathcal{C}^\infty(\mathbf{R}))$ the Boolean closure of the set of all countable unions of convex subsets of **R**, $\mathcal{G}$ the Boolean algebra of all Borel subsets of **R**, and $\mathcal{P}(\mathbf{R})$ the power-set of **R**. All of these Boolean algebras are closed with respect to $\Diamond$. The modal logic of the last three algebras is **S4**, that of the second one is **Grz**, and the modal logic of the first is the logic of the '2-fork' Kripke frame.

6.4 Generalization to $\mathbf{R}^2$

In this final subsection, we shift aim in a different direction. We generalize our results on the serial subsets of $\mathbf{R}$ to the chequered subsets of $\mathbf{R}^2$, and indicate further generalizations to any Euclidean space $\mathbf{R}^n$.

A set $X \subseteq \mathbf{R}^2$ is *convex* if all points laying in between any two points of X belong to X. It is said to be *serial* if X is a finite union of convex subsets of $\mathbf{R}^2$. Denote the set of all serial subsets of $\mathbf{R}^2$ by $\mathcal{S}(\mathbf{R}^2)$.

Here is a real difference between $\mathbf{R}$ and $\mathbf{R}^2$. Unlike $\mathcal{S}(\mathbf{R})$, $\mathcal{S}(\mathbf{R}^2)$ is not closed with respect to complement. For instance, a full circle is obviously a convex subset of $\mathbf{R}^2$. However, its complement is not serial.

One natural way of overcoming this difficulty is to work with a smaller family of chequered subsets of $\mathbf{R}^2$, which also has a reasonable claim to being 'the two-dimensional generalization of the one-dimensional serial sets'.

A set $X \subseteq \mathbf{R}^2$ is a *rectangular convex* if $X = X_1 \times X_2$, where both X_1 and X_2 are convex subsets of $\mathbf{R}$ [5]. It is easy to see that every rectangular convex is a convex set in the usual sense, but not vice versa: a circle is not a rectangular convex.

A set $X \subseteq \mathbf{R}^2$ is said to be *chequered* if it is a finite union of rectangular convex subsets of $\mathbf{R}^2$. Denote the set of all chequered subsets of $\mathbf{R}^2$ by $\mathcal{CH}(\mathbf{R}^2)$. Obviously $\mathcal{CH}(\mathbf{R}^2) \subset \mathcal{S}(\mathbf{R}^2)$.

Note that unlike $\mathcal{S}(\mathbf{R}^2)$, $\mathcal{CH}(\mathbf{R}^2)$ does form a Boolean algebra. Moreover, $\Box X, \Diamond X \in \mathcal{CH}(\mathbf{R}^2)$ for any $X \in \mathcal{CH}(\mathbf{R}^2)$.

Fact 6.13 *$\mathcal{CH}(\mathbf{R}^2)$ forms a Boolean algebra closed with respect to $\Box$ and $\Diamond$.*

Proof. In order to show that $\mathcal{CH}(\mathbf{R}^2)$ forms a Boolean algebra it is sufficient to show that $\mathcal{CH}(\mathbf{R}^2)$ is closed with respect to $\neg$. For this observe that complement of a rectangular convex is union of at most four rectangular convexes, and that finite intersection of rectangular convexes is again a rectangular convex. Now suppose $A \in \mathcal{CH}(\mathbf{R}^2)$. Then there exist rectangular convexes $A_1, \ldots, A_n$ such that $A = \bigcup_{i=1}^n A_i$. But then $\neg A = \bigcap_{i=1}^n \neg A_i$, which is a chequered set by the above observation and the distributivity law.

Since $\mathcal{CH}(\mathbf{R}^2)$ forms a Boolean algebra, in order to show that $\mathcal{CH}(\mathbf{R}^2)$ is closed with respect to $\Box$ and $\Diamond$, it is sufficient to check that $\mathcal{CH}(\mathbf{R}^2)$ is closed with respect to $\Diamond$. For the latter observe that the closure of a rectangular convex is again a rectangular convex, and that the closure commutes with finite unions. Now suppose $A \in \mathcal{CH}(\mathbf{R}^2)$. Then there exist rectangular convexes $A_1, \ldots, A_n$ such that $A = \bigcup_{i=1}^n A_i$. But then $\Diamond A = \bigcup_{i=1}^n \Diamond A_i$, which is a chequered set by the above observation. **q.e.d.**

Hence, interpreting propositional variables as chequered subsets of $\mathbf{R}^2$, every formula of our language will be also interpreted as a chequered subset of $\mathbf{R}^2$.

This approach leads to a logic, which we will just sketch here. Call a valuation ν of $\mathcal{L}$ to subsets of $\mathbf{R}^2$ *chequered* if $\nu(P) \in \mathcal{CH}(\mathbf{R}^2)$ for any propositional variable P. Since $\mathcal{CH}(\mathbf{R}^2)$ is closed with respect to $\neg, \wedge$ and $\diamond$, we have that $\nu(\varphi) \in \mathcal{CH}(\mathbf{R}^2)$ for any chequered interpretation ν. Call a formula φ $\mathcal{CH}$*-true* if it is true in $\mathbf{R}^2$ under a chequered valuation. Call φ $\mathcal{CH}$*-valid* if φ is $\mathcal{CH}$-true for any chequered valuation on $\mathbf{R}^2$. Let $L(\mathcal{CH}) = \{\varphi : \varphi$ is $\mathcal{CH}$-valid$\}$.

Fact 6.14 *$L(\mathcal{CH})$ is a normal modal logic over* $\mathbf{S4}$.

Now we show that, similarly to $L(\mathcal{S})$, the Grzegorczyk axiom **Grz** is provable in $L(\mathcal{CH})$. For this it is sufficient to show that **Grz** is $\mathcal{CH}$-valid.

Fact 6.15 **Grz** *is $\mathcal{CH}$-valid.*

Proof. **Grz** is $\mathcal{CH}$-valid iff $X \subseteq \diamond(X \wedge \neg\diamond(\diamond X \wedge \neg X))$ for any $X \in \mathcal{CH}(\mathbf{R}^2)$. Suppose $X \in \mathcal{CH}(\mathbf{R}^2)$. Observe that, unlike $\mathcal{S}(\mathbf{R})$, $\diamond X \wedge \neg X$ is not finite. However, in this case the set $\diamond(\diamond X \wedge \neg X) - (\diamond X \wedge \neg X)$ is finite. Denote it by F. Then $\diamond(X \wedge \neg\diamond(\diamond X \wedge \neg X)) = \diamond(X \wedge \neg[(\diamond X \wedge \neg X) \vee F]) = \diamond(X \wedge (\neg\diamond X \vee X) \wedge \neg F) = \diamond(X - F)$. Now since F is finite, $\diamond(X - F) = \diamond X$. Therefore, $\diamond(X \wedge \neg\diamond(\diamond X \wedge \neg X)) = \diamond X$, which obviously contains X. So, $X \subseteq \diamond(X \wedge \neg\diamond(\diamond X \wedge \neg X))$. **q.e.d.**

Now we show that the axioms

$\mathbf{BD}_3 = \diamond(\Box P_3 \wedge \diamond(\Box P_2 \wedge \diamond\Box P_1 \wedge \neg P_1) \wedge \neg P_2) \rightarrow P_3$, and

$\mathbf{BW}_4 = \bigwedge_{i=0}^{4} \diamond P_i \rightarrow \bigvee_{0 \leq i \neq j \leq 4} \diamond(P_i \wedge \diamond P_j)$,

which bound the depth and the width of a Kripke model to 3 and 4, respectively, are also provable in $L(\mathcal{CH})$. For this we need to show that both $\mathbf{BD}_3$ and $\mathbf{BW}_4$ are $\mathcal{CH}$-valid.

Fact 6.16 *(1)* $\mathbf{BD}_3$ *is $\mathcal{CH}$-valid.*
(2) $\mathbf{BW}_4$ *is $\mathcal{CH}$-valid.*

Proof: (1) $\mathbf{BD}_3$ is $\mathcal{CH}$-valid iff $\diamond(\Box X_3 \wedge \diamond(\Box X_2 \wedge \diamond\Box X_1 \wedge \neg X_1) \wedge \neg X_2) \subseteq X_3$ for any $X_1, X_2, X_3 \in \mathcal{CH}(\mathbf{R}^2)$. Observe that $\diamond\Box X_1 \wedge \neg X_1$ is a subset of the frontier $Fr(X_1) = \diamond X_1 \wedge \neg\Box X_1$ of X_1. Hence, $\diamond(\Box X_3 \wedge \diamond(\Box X_2 \wedge \diamond\Box X_1 \wedge \neg X_1) \wedge \neg X_2) \subseteq \diamond(\Box X_3 \wedge \diamond(\Box X_2 \wedge Fr(X_1)) \wedge \neg X_2)$. Let $X_2^* = \Box X_2 \wedge Fr(X_1)$ and $X_3^* = \Box X_3 \wedge Fr(X_1)$. Also let $\neg^*, \diamond^*$ and $\Box^*$ denote the corresponding operations of a closed subspace $Fr(X_1)$ of $\mathbf{R}^2$. Then $\diamond(\Box X_3 \wedge \diamond(\Box X_2 \wedge Fr(X_1)) \wedge \neg X_2) = \diamond(\Box X_3 \wedge \diamond X_2^* \wedge \neg X_2) = \diamond(\Box X_3 \wedge \diamond^* X_2^* \wedge \neg X_2) \subseteq \diamond(\Box X_3 \wedge \diamond^* X_2^* \wedge \neg\Box X_2) = \diamond(\Box X_3 \wedge \diamond^* X_2^* \wedge \neg^* X_2^*) = \diamond(X_3^* \wedge \diamond^* X_2^* \wedge \neg^* X_2^*) = \diamond^*(X_3^* \wedge \diamond^* X_2^* \wedge \neg^* X_2^*)$. Since $Fr(X_1)$ is of dimension 1, $Fr(X_1)$ is homeomorphic to a closed serial subspace of $\mathbf{R}$. Since $\mathbf{BD}_2$ is $\mathcal{S}$-valid in $\mathbf{R}$, $\diamond^*(X \wedge \diamond^* Y \wedge \neg^* Y) \subseteq X$ for any open subsets

X, Y of $Fr(X_1)$. Hence, $\Diamond^*(X_3^* \wedge \Diamond^* X_2^* \wedge \neg^* X_2^*) \subseteq X_3^*$. Thus, $\Diamond(\Box X_3 \wedge \Diamond(\Box X_2 \wedge \Diamond\Box X_1 \wedge \neg X_1) \wedge \neg X_2) \subseteq X_3$, and $\mathbf{BD}_3$ is $\mathcal{CH}$-valid.

(2) $\mathbf{BW}_2$ is $\mathcal{CH}$-valid iff $\bigwedge_{i=0}^4 \Diamond X_i \subseteq \bigvee_{0\leq i\neq j\leq 4} \Diamond(X_i \wedge \Diamond X_j)$ for any $X_0, \ldots, X_4 \in \mathcal{CH}(\mathbf{R}^2)$. Suppose $x \in \bigwedge_{i=0}^4 \Diamond X_i$. Then x is a limit point of all X_i. Since there are five X_i, and every X_i belongs to $\mathcal{CH}(\mathbf{R}^2)$, there should exist X_i and X_j such that x is a limit point of $X_i \wedge X_j$. So, $x \in \bigvee_{0\leq i\neq j\leq 4} \Diamond(X_i \wedge \Diamond X_j)$. **q.e.d.**

As an immediate consequence we obtain that $L(\mathcal{CH}) \vdash \mathbf{Grz}, \mathbf{BD}_3, \mathbf{BW}_4$. Hence, like $L(\mathcal{S})$, $L(\mathcal{CH})$ is also a tabular logic. In a similar fashion, by induction on the dimension of $\mathbf{R}^n$, we can prove that the logic of chequered subsets of $\mathbf{R}^n$ is also tabular. In particular, it validates $\mathbf{BD}_{n+1}$ and $\mathbf{BW}_{2^n}$. Hence, we are capable of capturing the dimension of Euclidean spaces. For more details in this direction we refer to [3].

7 A general picture

7.1 The deductive landscape

The logics that we have studied in this paper fit into a more general environment. Typical for modal logic is its lattice of deductive systems such as **K, S4, S5** or **GL**. These form a large family describing different classes of relational frames, with often very different motivations (cf. the series of books “Advances in Modal Logic”, CSLI and FOLLi). Among the uncountably many modal logics, a small number are distinguished for one of two reasons. Logics like **S4** or **S5** were originally proposed as syntactic proof theories for notions of modality, and then turned out to be semantically complete with respect to natural frame classes, such as (for **S4**) transitive reflexive orders. Other modal logics, however, were discovered as the complete theories of important frames, such as the natural numbers with their standard ordering. What about a similar landscape of modal logics on the topological interpretation?

Some well-known modal logics extending **S4** indeed correspond to natural classes of topological spaces. E.g., it is easy to see that the ‘identity logic’ with axiom $\varphi \to \Box\varphi$ axiomatizes the complete logic of all *discrete spaces.* And it also defines them semantically through the usual notion of *frame correspondence* – which can be lifted to the topological semantics in a straightforward manner. But already **S5** corresponds to a less standard condition, viz. that every point has an open neighborhood all of whose points have x in all their open neighborhoods. (Alternatively, this says that every open set is closed.) Also, even rich topological spaces do not seem to validate very spectacular modal logics, witness the fact that $\mathbf{R}$ has just **S4** for its modal theory. We did find stronger logics with ‘general frames’ though, i.e., frames with a designated interior algebra of subsets, such as $\mathbf{R}$ with the serial sets.

The latter turned out to be a well-known modal 'frame logic', and we have not been able so far to find really new modal logics arising on the topological interpretation.

A related question is what becomes of the known general results on completeness and correspondence for modal logic in the topological setting. There appear to be some obstacles here. E.g., the substitution method for Sahlqvist correspondence (cf. [10]) has only a limited range. It does work for axioms like the above $\varphi \to \Box\varphi$, where it automatically generates the corresponding first-order condition

$$(\forall x)(\exists U \in \tau)(x \in U \;\&\; (\forall y \in U)(y = x)),$$

i.e., discreteness. Likewise, it works for the **S5** symmetry axiom $P \to \Box\Diamond P$, where it produces the above-mentioned

$$(\forall x)(\exists U \in \tau)(x \in U \;\&\; (\forall y \in U)(\forall V \in \tau)(y \in V \to x \in V)).$$

The method also works for antecedents of the form $\Box P$ – but things stop with antecedents like $\Diamond P$ or $\Box\Box P$. The reason is that, on the topological semantics, one modality $\Box$ expresses a *two-quantifier* combination

$$\exists U \in \tau \text{ such that } \forall x \in U,$$

so that syntactic complexity builds up more rapidly than in standard modal logic, where each modality is one quantifier over relational successors of the current world. General correspondence or completeness results for topological modal logics therefore seem harder to obtain — and we may need different syntactic notions for them (see [14] for recent results in that direction).

7.2 The expressive landscape

In any case, the basic modal language seems too poor to express many properties of topological interest. One earlier example was *connectedness*. This property cannot be modally defined. To see this, suppose there was a modal formula φ defining the connected topological spaces in the sense of frame correspondence. Now consider the non-connected discrete 2-element space with universe $\{1, 2\}$. The formula φ must fail here under some valuation ν – say at point 1. Now consider the one-point model $\{1\}$ copying 1's valuation. The link between just the worlds 1 in the two models is a topo-bisimulation, as is easy to see. But then, by modal invariance, φ would also fail in the connected one-point model: a contradiction.

As we have seen already, connectedness does have a definition in a modal language *extended* with a *universal modality* $U\varphi$ saying that φ *holds at all points of the topological space*:

$$U(\Diamond P \to \Box P) \to (UP \vee U\neg P).$$

This is one instance of a general trend in modal logic, toward moderate expressive extensions of the base language. The $\{\Box, U\}$ language is a natural candidate, as it can formulate 'global facts' about topological spaces such as inclusion of one region in another. Many of our earlier techniques apply such as frame correspondence, bisimulation and related model constructions. (Cf. ([1] for back-up to this section.) The general logic of this new language is known [4]: it is the system **S4+S5**, being **S4** for $\Box$, **S5** for U, plus the 'bridge axiom' $UP \to \Box P$. Moreover, according to [19], we have natural extension of the McKinsey and Tarski theorem: the $\{\Box, U\}$ modal theory of $\mathbf{R}$, and indeed of every Euclidean space $\mathbf{R}^n$, is exactly **S4+S5** plus the given connectedness axiom.

Thus, the concerns of this paper reproduce for richer modal languages, expressing more topological behavior. Most of the resulting questions seem completely open, as topological semantics does not seem to have had much of a follow-up in serious 'logic of space'.

Indeed, modal languages can also have much stronger topological modalities, such as the following 'Until' operator generalizing two well-known notions from temporal logic:

> *x has an open neighborhood all of whose interior points satisfy B while all its boundary points satisfy A.*

And even further extensions are needed to deal with modal separation axioms, such as a space being Hausdorff, which requires even stronger 'modalities' definable in the monadic second-order language over topological spaces. One can then see the art of the field in choosing 'good fragments' out of this total language, admitting of a good balance between expressive power and complexity.

Finally, the same modal methodology also extends to other similarity types. In particular, one can introduce geometrical structure. E.g., the affine geometry of betweenness suggest a 'convexity modality' CA:

> *x lies in between two points that satisfy A.*

This brings out differences between the spaces $\mathbf{R}^n$: as $\mathbf{R}$, but no higher-dimensional $\mathbf{R}^n$, satisfies the principle $CCA \leftrightarrow CA$. (A more extensive study of various modal languages for affine and metric geometry is made in [2] and [8].)

Thus, there is a lot of modal logic of space in between Tarski's work on topological structure and his work on the full first-order language of elementary geometry [20].

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References

[1] M. Aiello and J. van Benthem. Logical Patterns in Space. In D. Barker-Plummer, D. Beaver, J. van Benthem, and P. Scotto di Luzio, editors, *Logic Unleashed: Language, Diagrams and Computation*, Stanford, 2001. CSLI.

[2] M. Aiello and J. van Benthem. A Modal Walk Through Space. Manuscript, 2002.

[3] M. Aiello, J. van Benthem, and G. Bezhanishvili. Modal geometry, first steps, 2002. ILLC, University of Amsterdam. Manuscript.

[4] B. Bennett. Modal Logics for Qualitative Spatial Reasoning. *Bulletin of the IGPL*, 3:1 – 22, 1995.

[5] J. van Benthem. *The Logic of Time.* Reidel, Dordrecht, 1983.

[6] J. van Benthem. *Modal Logic and Classical Logic.* Bibliopolis, Napoli, 1985. The Humanities Press, Atlantic Heights (N.J.).

[7] J. van Benthem. *Exploring Logical Dynamics*, volume 156. CSLI Publications, Stanford — Cambridge University Press, 1996.

[8] J. van Benthem. Modal geometry. Manuscript, 2002.

[9] G. Bezhanishvili and M. Gehrke. A new proof of completeness of S4 with respect to the real line, 2001. Submitted.

[10] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic.* Cambridge University Press, 2001.

[11] B.F. Chellas. *Modal Logic: An Introduction.* Cambridge University Press, 1980.

[12] A. Dabrowski, A. Moss, and R. Parikh. Topological reasoning and the logic of knowledge. *Annals of Pure and Applied Logic*, 78:73–110, 1996.

[13] R. Engelking. *General Topology.* Heldermann Verlag, 1989.

[14] D. Gabelaia. Diamonds of space. Master's thesis, ILLC, University of Amsterdam, 2002.

[15] K. Georgatos. *Modal Logics for Topological Spaces*. PhD thesis, City University of New York, 1993.

[16] J. McKinsey and A. Tarski. The Algebra of Topology. *Annals of Mathematics*, 45:141–191, 1944.

[17] G. Mints. A Completeness Proof for Propositional S4 in Cantor space. In E. Orlowska, editor, *Logic at work : Essays dedicated to the memory of Helena Rasiowa*. Physica-Verlag, Heidelberg, 1998.

[18] H. Rasiowa and R. Sikorski. *The Mathematics of Metamatematics*. Panstwowe Wydawnictwo naukowe, 1963.

[19] V. Shehtman. "Everywhere" and "Here". *Journal of Applied Non-Classical Logics*, 9(2-3):369–379, 1999.

[20] A. Tarski. What is Elementary Geometry? In L. Henkin and P. Suppes and A. Tarski, editor, *The Axiomatic Method, with Special Reference to Geometry ad Physics*, pages 16–29. North-Holland, 1959.