Institute for Logic, Language and Computation

CANONICAL FORMULAS FOR K4

Part II: Cofinal Subframe Logics

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CANONICAL FORMULAS FOR K4

Part II: Cofinal Subframe Logics

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Canonical Formulas for K4. Part II: Cofinal Subframe Logics

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1 Introduction

This paper is a continuation of Zakharyaschev [1992], where the following basic results on modal logics with transitive frames were obtained:

- With every finite rooted transitive frame $\mathfrak{F}$ and every set $\mathfrak{D}$ of antichains (which were called *closed domains*) in $\mathfrak{F}$ we associated two formulas $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ and $\alpha(\mathfrak{F}, \mathfrak{D})$. We called them the *canonical* and *negation free canonical formulas*, respectively, and proved the Refutability Criterion characterizing the constitution of their refutation general frames in terms of subreduction (alias partial p-morphism), the cofinality condition and the closed domain condition.

- We proved also the Completeness Theorem for the canonical formulas providing us with an algorithm which, given a modal formula φ, returns canonical formulas $\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot)$, for $i = 1, \ldots, n$, such that

  $$\mathbf{K4} \oplus \varphi = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : i = 1, \ldots, n\};$$

  if φ is negation free then the algorithm instead of $\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot)$ can use the negation free canonical formulas $\alpha(\mathfrak{F}_i, \mathfrak{D}_i)$. Thus, every normal modal logic containing **K4** can be axiomatized by a set of canonical formulas.

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In this Part we apply the apparatus of the canonical formulas for establishing a number of results on the decidability, finite model property, elementarity and some other properties of modal logics within the field of **K4**.

Our attention will be focused on the class of logics which can be axiomatized by canonical formulas without closed domains, i.e. on the logics of the form

$$\mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \emptyset, \bot) : i \in I\}. \quad (1)$$

Adapting the terminology of Fine [1985], we call them the *cofinal subframe logics* and denote this class by $\mathcal{CSF}$. As was shown in Part I, almost all standard modal logics are in $\mathcal{CSF}$. The class $\mathcal{SF}$ of Fine's *subframe logics*, which can be represented in the form

$$\mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \emptyset) : i \in I\},$$

turns out to be a proper subclass of $\mathcal{CSF}$. In fact, this paper extends the results of Fine [1985] to the class of cofinal subframe logics. However our approaches are quite different in both their motivations and methods of obtaining results.

Fine introduces some special frame-based formulas - the subframe formulas - in such a way that they can axiomatize exactly those normal extensions of **K4** that are characterized by classes of Kripke frames which are closed under forming subframes and proves the finite model property of these logics using his powerful method of dropping points from the canonical model.

The canonical formulas of Part I, also frame-based ones, naturally arise in the course of analyzing the construction of general frames refuting an arbitrary given modal formula and so are able to axiomatize all extensions of **K4**. The classes of subframe and cofinal subframe logics appear then as relatively simple (but by no means trivial) classes which can serve as a good starting point in our attack on modal logics supported by the heavy artillery of the canonical formulas. The finite model property of logics in $\mathcal{CSF}$ turns out to be then just an easy consequence of the Refutability Criterion and Completeness Theorem.

This Part is organized in the following way.

Section 2 characterizes the canonical formulas which are provable in a given cofinal subframe logic. As a consequence we obtain that all finitely axiomatizable logics in $\mathcal{CSF}$ and even all the logics in the class which are recursively axiomatizable by canonical formulas are decidable.

In Section 3 we prove that every cofinal subframe logic has a unique representation of the form (1) with an independent set of canonical axioms. We show also that there are subframe logics with infinite independent sets of axioms, and so the cardinality of $\mathcal{SF}$ is that of continuum and there are undecidable recursively axiomatizable subframe logics.

The finite model property of all logics in $\mathcal{CSF}$ is proved in Section 4. Moreover, we obtain an (exponential) upper bound for the size of minimal frames separating $L \in \mathcal{CSF}$ from $\varphi \notin L$.

In Section 5 we give a purely frame-theoretic characterization of cofinal subframe logics: $L \in \mathcal{CSF}$ iff L is determined by a class of frames that is closed under forming cofinal subframes. Section 6 characterizes in a frame-theoretic way those logics in $\mathcal{CSF}$ that are elementary, canonical and compact. And Section 7 briefly considers the quasi-normal extensions of **K4** with normal and quasi-normal canonical axioms containing no closed domains. We characterize the canonical formulas belonging to such logics and prove that all finitely axiomatizable logics in this class are decidable, though not necessarily have the finite model property or are Kripke complete.

Now I am working on Part III which deals with the finite model property of logics outside of $\mathcal{CSF}$.

The results above can readily be transferred to the extensions of intuitionistic propositional logic **Int** (i.e. to the intermediate or superintuitionistic logics) which are axiomatizable by intuitionistic canonical formulas $\beta(\mathfrak{F}, \emptyset, \bot)$ or $\beta(\mathfrak{F}, \emptyset)$ without closed domains (see Zakharyaschev [1989, 1993]). Unlike the modal case, there is a purely syntactic characterization of subframe and cofinal subframe intermediate logics. For it was shown in Zakharyaschev [1983, 1989] that

- *an intermediate logic L is axiomatizable by implicative formulas iff it can be represented in the form*
$$L = \mathbf{Int} + \{\beta(\mathfrak{F}_i, \emptyset) : i \in I\}$$

and

- *L is axiomatized by disjunction free formulas iff it can be represented as*
$$L = \mathbf{Int} + \{\beta(\mathfrak{F}_i, \emptyset, \bot) : i \in I\}.$$

There are two ways of transferring those results to intermediate logics. The first one is just to translate the proofs into the intuitionistic language. Another one uses the fact that such properties of logics as the decidability, finite model property, etc., are preserved while passing from a modal logic M containing **S4** to its 'superintuitionistic fragment' ρM, which contains those intuitionistic formulas whose Gödel translations are in M, and the following Modal Companion Theorem proved in Zakharyaschev [1989]: *A normal logic M above* **S4** *is a modal companion of an intermediate logic*

$$L = \mathbf{Int} + \{\beta(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : i \in I\}$$

(i.e. $\rho M = L$) iff M can be represented in the form

$$M = \mathbf{S4} \oplus \{\alpha(\mathfrak{F}_i, \mathfrak{D}_i, \bot) : i \in I\} \oplus \{\alpha(\mathfrak{G}_j, \mathfrak{C}_j, \bot) : j \in J\}$$

where each $\mathfrak{G}_j$, for $j \in J$, contains at least one proper cluster.

I hope that the reader has Part I at hand and so shall freely use its terminology and notations.

2 The decidability

When proving such properties of logics as the decidability, finite model property or completeness, we may consider only the canonical formulas. Indeed, suppose a logic L and a formula φ are given. By the Completeness Theorem (of Part I), we can effectively construct canonical formulas $\alpha_1, \ldots, \alpha_n$ such that

$$\mathbf{K4} \oplus \varphi = \mathbf{K4} \oplus \alpha_1 \oplus \ldots \oplus \alpha_n.$$

Therefore, $\varphi \in L$ iff $\alpha_i \in L$ for every $i \in \{1, \ldots, n\}$, and so L is decidable iff there is an algorithm which is capable of deciding, given an arbitrary *canonical* formula α, whether or not $\alpha \in L$. It follows also from the equality above that, for every frame $\mathfrak{F}$, $\mathfrak{F} \not\models \varphi$ iff $\exists i \in \{1, \ldots, n\}$ $\mathfrak{F} \not\models \alpha_i$. Thus, L has the finite model property (or is Kripke complete) iff for every *canonical* formula $\alpha \notin L$ there is a finite (respectively, Kripke) frame for L refuting α.

The following lemma turns out to be very useful for establishing deducibility relations between canonical formulas.

Lemma 2.1 (Composition Lemma) *Suppose $\mathfrak{F}_i = \langle W_i, R_i, P_i \rangle$, for $i = 1, 2, 3$, are frames, f_1 is a (cofinal) subreduction of $\mathfrak{F}_1$ to $\mathfrak{F}_2$ and f_2 is a (cofinal) subreduction of $\mathfrak{F}_2$ to $\mathfrak{F}_3$. Then the composition $f_3 = f_2 f_1$ is a (cofinal) subreduction of $\mathfrak{F}_1$ to $\mathfrak{F}_3$.*

Proof. Since f_1 and f_2 are surjections, their composition is also a surjection. If $x, y \in \mathrm{dom} f_2 f_1$ and $x R_1 y$ then, by the condition (R1) in the definition of reduction (see Part I, Section 1), $f_1(x) R_2 f_1(y)$ and $f_2 f_1(x) R_3 f_2 f_1(y)$. If $f_2 f_1(x) R_3 z$ for some $x \in W_1$ and $z \in W_3$ then, by (R2), there are $v \in W_2$ and $y \in W_1$ such that $f_1(x) R_2 v$, $f_2(v) = z$ and $x R_1 y$, $f_1(y) = v$, i.e. $f_2 f_1(y) = z$. So $f_2 f_1$ satisfies both (R1) and (R2). If $X \in P_3$ then, by (R3), $f_2^{-1}(X) \in P_2$ and $f_1^{-1}(f_2^{-1}(X)) = (f_2 f_1)^{-1}(X) \in P_1$. Thus, f_3 satisfies (R1) - (R3), and so is a subreduction of $\mathfrak{F}_1$ to $\mathfrak{F}_3$.

Now suppose f_1 and f_2 are cofinal, $x \in W_1$ and $y R_1 x$ for some $y \in \mathrm{dom} f_2 f_1$. To prove that f_3 is cofinal, we must show that $x \in \mathrm{dom} f_3 \overline{\downarrow}$, i.e. either x is in $\mathrm{dom} f_3$ or sees a point in $\mathrm{dom} f_3$. Since f_1 is cofinal, either $x \in \mathrm{dom} f_1$ or $x R_1 z$ for some $z \in \mathrm{dom} f_1$. In the former case $f_1(y) R_2 f_1(x)$, and so, by the cofinality of f_2, either $f_1(x) \in \mathrm{dom} f_2$, i.e. $x \in \mathrm{dom} f_2 f_1$, or $f_1(x) R_2 v$ for some $v \in \mathrm{dom} f_2$, and then, by (R2), there is $u \in W_1$ such that $x R_1 u$ and $f_1(u) = v$, whence $u \in \mathrm{dom} f_2 f_1$. The latter case is considered analogously. $\dashv$

Theorem 2.2 (i) *Suppose $L = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \emptyset, \bot) : i \in I\}$. Then, for every canonical formula $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$ iff $\mathfrak{F} \not\models \alpha(\mathfrak{F}_i, \emptyset, \bot)$ for some $i \in I$, i.e. iff $\mathfrak{F}$ is cofinally subreducible to $\mathfrak{F}_i$ for some $i \in I$.*

(ii) *Suppose $L = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \emptyset) : i \in I\}$. Then, for every canonical formula $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$ iff $\alpha(\mathfrak{F}, \mathfrak{D}) \in L$ iff $\mathfrak{F} \not\models \alpha(\mathfrak{F}_i, \emptyset)$ for some $i \in I$, i.e. iff $\mathfrak{F}$ is subreducible to $\mathfrak{F}_i$ for some $i \in I$.*

Proof. (i) If $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$ then, for some $i \in I$, $\mathfrak{F} \not\models \alpha(\mathfrak{F}_i, \emptyset, \bot)$, since, by the Refutability Criterion, $\mathfrak{F} \not\models \alpha(\mathfrak{F}, \mathfrak{D}, \bot)$.

Now suppose that, for some $i \in I$, $\mathfrak{F} \not\models \alpha(\mathfrak{F}_i, \emptyset, \bot)$, i.e. there is a cofinal subreduction f of $\mathfrak{F}$ to $\mathfrak{F}_i$. Suppose also that $\mathfrak{G}$ is a general frame refuting $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$. Then there is a cofinal subreduction g of $\mathfrak{G}$ to $\mathfrak{F}$. By the Composition Lemma, fg is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}_i$, and so, by the Refutability Criterion, $\mathfrak{G} \not\models \alpha(\mathfrak{F}_i, \emptyset, \bot)$. Thus, $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ is valid in every general frame for L and hence $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$.

(ii) is proved analogously. ⊣

As an immediate consequence we obtain

Corollary 2.3 *Every finitely axiomatizable subframe or cofinal subframe logic is decidable.* ⊣

For the subframe logics this result was first proved by Fine [1985] and for the cofinal subframe logics above **S4** by Zakharyaschev [1984]. The intuitionistic analog of Corollary 2.3 is just McKay's [1968] Theorem on the decidability of all intermediate logics with a finite number of disjunction free additional axioms.

Corollary 2.4 (i) *L is a cofinal subframe logic iff, for every canonical formula $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, $\alpha(\mathfrak{F}, \emptyset, \bot) \in L$ whenever $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$.*

(ii) *L is a subframe logic iff, for every canonical formula $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$, $\alpha(\mathfrak{F}, \emptyset) \in L$ whenever $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$.* ⊣

Thus, dealing with subframe or cofinal subframe logics, we may consider only those canonical formulas that have no closed domains. We shall call the formulas of the form $\alpha(\mathfrak{F}, \emptyset)$ and $\alpha(\mathfrak{F}, \emptyset, \bot)$ the *subframe* and *cofinal subframe formulas*, respectively.

Corollary 2.5 (i) $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \emptyset, \bot) : i \in I\}$ *iff* $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in \mathbf{K4} \oplus \alpha(\mathfrak{F}_i, \emptyset, \bot)$ *for some* $i \in I$.

(ii) $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \emptyset) : i \in I\}$ *iff* $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in \mathbf{K4} \oplus \alpha(\mathfrak{F}_i, \emptyset)$ *for some* $i \in I$. ⊣

It is natural now to ask whether a subframe or cofinal subframe logic is decidable if it is recursively axiomatizable. In general, as we shall see in Section 3, the answer is negative. However, the answer turns out to be positive if the logic is recursively axiomatizable by *canonical* formulas.

Theorem 2.6 *Suppose $L \in \mathcal{SF}$ or $L \in \mathcal{CSF}$ and L is recursively axiomatizable by canonical formulas. Then L is decidable.*

Proof. Let L be a cofinal subframe logic. By Corollary 2.4, we may assume L to be recursively axiomatizable by some cofinal subframe formulas. According to Theorem 2.2, $\alpha(\mathfrak{G}, \mathfrak{D}, \bot) \in L$ iff there is a cofinal subreduct $\mathfrak{F}$ of $\mathfrak{G}$ such that $\alpha(\mathfrak{F}, \emptyset, \bot)$ is an axiom of L. So our decision algorithm may be as follows. Given a formula $\alpha(\mathfrak{G}, \mathfrak{D}, \bot)$, we construct all the cofinal subreducts $\mathfrak{F}_1, \dots, \mathfrak{F}_n$ of $\mathfrak{G}$ and then check whether at least one of the formulas $\alpha(\mathfrak{F}_1, \emptyset, \bot), \dots, \alpha(\mathfrak{F}_n, \emptyset, \bot)$ is an axiom of L. If the outcome of this check is positive then $\alpha(\mathfrak{G}, \mathfrak{D}, \bot) \in L$; otherwise $\alpha(\mathfrak{G}, \mathfrak{D}, \bot) \notin L$.

The case of a subframe L is considered in the same manner. ⊣

Corollary 2.7 *Every intermediate logic which is recursively axiomatizable by intuitionistic canonical formulas without closed domains is decidable.* ⊣

3 Independent axiomatization and cardinality

Suppose that Γ is a set of modal formulas and $L = \mathbf{K4} \oplus \Gamma$. The set of axioms Γ is called *independent* if, for no $\Delta \subset \Gamma$, $L = \mathbf{K4} \oplus \Delta$.

Theorem 3.1 *Every (cofinal) subframe logic L can be axiomatized by an independent set of (cofinal) subframe formulas, and such an axiomatization is unique.*

Proof. Suppose first that L is a subframe logic. Define on the set $\mathcal{FRF}$ of all finite rooted frames a relation $\leq$ by taking, for $\mathfrak{F}, \mathfrak{G} \in \mathcal{FRF}$,

$$\mathfrak{F} \leq \mathfrak{G} \text{ iff } \mathfrak{G} \text{ is subreducible to } \mathfrak{F}.$$

Using the Composition Lemma, it is not hard to show that $\leq$ is a partial order on $\mathcal{FRF}$. It is clear also that $\leq$ is well-founded, i.e. there is no infinite descending chain of distinct frames in $\mathcal{FRF}$.

Now suppose that $L = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \emptyset) : i \in I\}$ and $\{\mathfrak{F}_j : j \in J\}$ is the set of all minimal (with respect to $\leq$) frames in the set $\{\mathfrak{F}_i : i \in I\}$. Clearly, for every $i \in I$ there is $j \in J$ such that $\mathfrak{F}_j \leq \mathfrak{F}_i$. Then, by Theorem 2.2, $L = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_j, \emptyset) : j \in J\}$ and the subframe axioms $\{\alpha(\mathfrak{F}_j, \emptyset) : j \in J\}$ are independent. Furthermore, if there is some other independent axiomatization $L = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_k, \emptyset) : k \in K\}$ then, by Theorem 2.2, for each $j \in J$ there are $k \in K$ and $j' \in J$ such that $\mathfrak{F}_{j'} \leq \mathfrak{F}_k \leq \mathfrak{F}_j$. Then $j = j'$, from which it follows that $\mathfrak{F}_j$ and $\mathfrak{F}_k$ are isomorphic, and so $J \subseteq K$. Likewise $K \subseteq J$. Therefore $J = K$.

If L is a cofinal subframe logic then we define on $\mathcal{FRF}$ another well-founded partial order $\leq_c$:

$$\mathfrak{F} \leq_c \mathfrak{G} \text{ iff } \mathfrak{G} \text{ is cofinally subreducible to } \mathfrak{F}.$$

The rest of the proof remains the same as in the preceding case. ⊣

Unfortunately, this independent axiomatization result cannot be generalized to cover all logics above **K4**. For recently Alexander Chagrov and I have constructed normal modal logics containing **K4**, **S4**, **K4Grz** and an intermediate logic which have no independent axiomatizations.

Now we show that there are subframe and cofinal subframe logics with infinite independent sets of canonical axioms, or, which is equivalent, the partially ordered sets $\langle \mathcal{FRF}, \leq \rangle$ and $\langle \mathcal{FRF}, \leq_c \rangle$, defined in the proof of Theorem 3.1, contain infinite antichains. It will follow that the cardinality of the classes $\mathcal{SF}$ and $\mathcal{CSF}$ is that of continuum and that there are undecidable recursively axiomatizable logics in these classes. (It should be noted that these results were first formulated by Fine [1985], but his proof was incorrect. This problem was also discussed by Kracht [1990], who believed that all subframe logics are decidable. In Logic Notebook [1986] I mentioned the question on the cardinality of intermediate subframe logics (i.e. extensions of intuitionistic propositional logic with purely implicative axioms) as an open problem.)

Let $\mathfrak{F}_n = \langle W_n, R_n \rangle$, for $n = 3, 4, \ldots$, be the sequence of frames shown in Fig. 1.

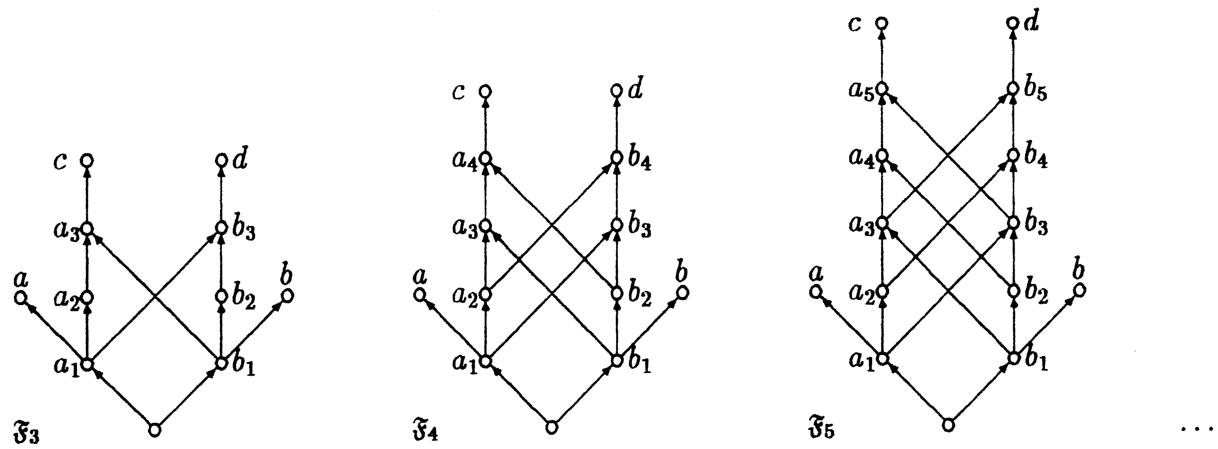


Figure 1:

Lemma 3.2 *For no $n \neq m$, $\mathfrak{F}_n$ is subreducible to $\mathfrak{F}_m$.*

Proof. Clearly $\mathfrak{F}_n$ is not subreducible to $\mathfrak{F}_m$ if $n < m$. So suppose that $n > m$ and f is a subreduction of $\mathfrak{F}_n$ to $\mathfrak{F}_m$. Since both a_1 and b_1 have three pairwise incomparable (with respect to R_m) successors in $\mathfrak{F}_m$, every point in $f^{-1}(a_1)$ and $f^{-1}(b_1)$ must also have at least three pairwise incomparable successors in $\mathfrak{F}_n$. Therefore, without loss of generality we may assume that $f^{-1}(a_1) = \{a_1\}$ and $f^{-1}(b_1) = \{b_1\}$. It should be clear also that $f^{-1}(a) = \{a\}$ and $f^{-1}(b) = \{b\}$. Since $a_1 R_m a_2$ and not $b_1 R_m a_2$, we must have $f^{-1}(a_2) = \{a_2\}$; symmetrically, $f^{-1}(b_2) = \{b_2\}$. And, by the same argument, for each i such that $1 \leq i \leq m$, $f^{-1}(a_i) = \{a_i\}$ and $f^{-1}(b_i) = \{b_i\}$. But then we come to a contradiction. For b_{m-1} does not see c in $\mathfrak{F}_m$, while in $\mathfrak{F}_n$ b_{m-1} sees all the points which are accessible from a_m except a_m itself, and so no point in $\mathfrak{F}_n$ can be mapped by f to c without violating (R1). ⊣

As a consequence of Lemma 3.2 and Theorem 2.2 one can readily prove the following

Theorem 3.3 (i) *There are subframe and cofinal subframe logics with infinite independent sets of canonical axioms.*

(ii) *The cardinality of both $\mathcal{SF}$ and $\mathcal{CSF}$ is that of continuum.*

(iii) *There are a continuum of undecidable logics in $\mathcal{SF}$ and $\mathcal{CSF}$, with infinitely many of them being recursively axiomatizable (but not by canonical formulas).* ⊣

Since all the frames $\mathfrak{F}_n$ defined above are partially ordered, using the results of Zakharyaschev [1989] we obtain also

Theorem 3.4 (i) *There are a continuum of intermediate logics with purely implicative additional axioms.*

(ii) *There are a continuum of undecidable intermediate logics with implicative additional axioms, and infinitely many of them are recursively axiomatizable.* ⊣

4 The finite model property

Another immediate consequence of Theorem 2.2 is the following

Theorem 4.1 *All subframe and cofinal subframe logics have the finite model property.*

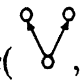
Proof. Suppose L is a subframe or cofinal subframe logic and $\alpha(\mathfrak{F},\mathfrak{D},\bot) \notin L$. Then, by Theorem 2.2, $\mathfrak{F}$ is a frame for L and, as we know, $\mathfrak{F} \not\models \alpha(\mathfrak{F},\mathfrak{D},\bot)$. ⊣

For the subframe logics this result was first obtained by Fine [1985]; for extensions of **S4** it was proved by Zakharyaschev [1984]. For intermediate logics Theorem 4.1 is equivalent (as if follows from Zakharyaschev [1983, 1989]) to McKay's [1968] Theorem, according to which all intermediate logics with disjunction free additional axioms have the finite model property.

Example 4.2 Using Theorem 4.1, we can give a simple proof of the well-known theorem first proved by Bull [1966] and Fine [1971]: *every extension of* **S4.3** *has the finite model property.* (Recall that all extensions of **S4.3** are normal.)

We know from Part I that

$$\mathbf{S4.3} = \mathbf{S4} \oplus \Box(\Box p \to q) \vee \Box(\Box q \to p)$$

$$= \mathbf{K4} \oplus \alpha(\bullet,\emptyset) \oplus \alpha($$  $$,\emptyset).$$

Now we show that every extension of **S4.3** is a cofinal subframe logic.

By Theorem 2.2, $\alpha(\mathfrak{F},\mathfrak{D},\bot) \in \mathbf{S4.3}$ iff $\mathfrak{F}$ contains either an irreflexive point or an antichain with at least two elements. Therefore, every extension of **S4.3** is axiomatized by some formulas $\alpha(\mathfrak{F},\mathfrak{D},\bot)$ such that $\mathfrak{F}$ is a finite chain of non-degenerate clusters. Each closed domain in such a formula consists of a single reflexive point. It remains to recall that, by Proposition 7 of Part I, each reflexive singleton is an open domain in every model, and so we may take $\mathfrak{D} = \emptyset$. ⊣

We can strengthen Theorem 4.1 by indicating an upper bound for the number of points in the minimal frame for $L \in \mathcal{CSF}$ (or $L \in \mathcal{SF}$) which refutes a given formula $\varphi \notin L$. Define $l(\varphi)$, the *length* of φ, as the number of subformulas in φ.

Theorem 4.3 (i) *Suppose L is a subframe logic and $\varphi \notin L$. Then there is a frame for L refuting φ and containing at most $2^{l(\varphi)}$ points.*

(ii) *Suppose L is a cofinal subframe logic and $\varphi \notin L$. Then there is a frame for L refuting φ and containing at most $2^{2l(\varphi)+1}$ points.*

Proof. (i) We consider first a subframe logic L. Let $\mathfrak{F} = \langle W, R\rangle$ be a finite frame for L refuting φ. We will extract a subframe $\mathfrak{F}'$ of $\mathfrak{F}$ which refutes φ and contains at most $2^{l(\varphi)}$ points; by the Composition Lemma and Refutability Criterion, $\mathfrak{F}'$ will also be a frame for L.

Fix a valuation $\mathfrak{V}$ under which φ is false at some point in $\mathfrak{F}$. By induction on the number of points in $\mathfrak{F}$ we can construct a finite tree of clusters $\mathfrak{S} = \langle V, S\rangle$ such that there is a reduction f of $\mathfrak{G}$ to $\mathfrak{F}$. Define a valuation $\mathfrak{U}$ on $\mathfrak{G}$ by taking $\mathfrak{U}(p,x) = \mathfrak{V}(p, f(x))$ for every variable p and every $x \in V$. By the P-morphism Theorem, $\mathfrak{U}(\chi, x) = \mathfrak{V}(\chi, f(x))$ for all formulas χ.

We say a point x in $\mathfrak{G}$ (or in $\mathfrak{F}$) *eliminates a formula* $\Box\psi$ if $x \not\models \psi$ and $y \models \psi$ for all $y \in x\uparrow - C(x)$, where $C(x)$ is the cluster generated by x. Let

$$\Gamma_x = \{\Box\psi \in \mathbf{Sub}\varphi : x \not\models \Box\psi \ \&\ x \text{ does not eliminate } \Box\psi\},$$

$$\Delta_x = \{\Box\psi \in \mathbf{Sub}\varphi : x \models \Box\psi\},$$
$$\Sigma_x = \{\psi \in \mathbf{Sub}\varphi : x \models \psi\}.$$

Suppose a point a eliminates $\Box\varphi$ in $\mathfrak{G}$ and V' is a *minimal* subset of V such that

(1) $a \in V'$ and
(2) if $x \in V'$ and $\Box\psi \in \Gamma_x$ then there is $y \in x\uparrow V'$ eliminating $\Box\psi$.

By induction on $|\Gamma_x|$, the cardinality of Γ_x, we show that for each $x \in V'$

$$|x\underline{\uparrow} V'| \leq 2^{|\Gamma_x|}.$$

If $|\Gamma_x| = 0$ then there is no point in $x\underline{\uparrow} V'$ except x, for otherwise we can remove all such points from V' and the remaining subset of V' will again satisfy (1) and (2), contrary to the requirement of minimality.

Suppose $|\Gamma_x| = n+1$. Consider first all the points $y_1, \dots, y_r$ in the set $C(x) \cap V' - \{x\}$. Each of them must eliminate some $\Box\psi \in \Gamma_x$, for otherwise it can be removed from V' without violating (1) and (2). Moreover, in total $y_1, \dots, y_r$ must eliminate at least r distinct formulas in Γ_x, and so the points in $C(x)\uparrow V' - C(x)$ eliminate only the remaining $\leq n+1-r$ formulas.

Let $z_1, \dots, z_s$ be an antichain in V' such that $\{z_1, \dots, z_s\}\underline{\uparrow} V' = C(x)\uparrow V' - C(x)$. Again, each z_i must eliminate some $\Box\psi \in \Gamma_x$ such that $z_j \models \Box\psi$ for all $j \neq i$. Therefore, $|\Gamma_{z_i}| \leq n+1-r-s$, with $r+s \geq 1$. So, by the induction hypothesis,

$$|z_i\underline{\uparrow} V'| \leq 2^{n+1-r-s}.$$

Thus, we have

$$|x\underline{\uparrow} V'| \leq 1 + r + s2^{n+1-r-s} \leq 2^{n+1},$$

and so

$$|V'| \leq 2^{|\Gamma_a|} \leq 2^{l(\varphi)}.$$

Consider the frame $\mathfrak{F}' = \langle W', R' \rangle$ where $W' = f(V')$ and R' is the restriction of R to W'. It is clear that $|W'| \leq 2^{l(\varphi)}$. Let $\mathfrak{V}'$ be the restriction of $\mathfrak{V}$ to $\mathfrak{F}'$. By induction on the construction of $\psi \in \mathbf{Sub}\varphi$ we can prove that, for every $x \in W'$, $\mathfrak{V}'(\psi, x) = \mathfrak{V}(\psi, x)$. The only non-trivial case is $\psi = \Box\chi$. If $\mathfrak{V}(\Box\chi, x) = T$ then $\mathfrak{V}(\chi, y) = \mathfrak{V}'(\chi, y) = T$ for all $y \in x\uparrow W'$, and so $\mathfrak{V}'(\Box\chi, x) = T$. If $\mathfrak{V}(\Box\chi, x) = F$ then $\mathfrak{U}(\Box\chi, z) = F$ for some $z \in f^{-1}(x) \cap V'$. By the definition of V', there is $v \in z\uparrow V'$ such that $\mathfrak{U}(\chi, v) = F$. Therefore, $xRf(v)$ and $\mathfrak{V}(\chi, f(v)) = \mathfrak{V}'(\chi, f(v)) = F$, whence $\mathfrak{V}'(\Box\chi, x) = F$.

Thus, we have constructed a subframe $\mathfrak{F}'$ of $\mathfrak{F}$ which refutes φ and has at most $2^{l(\varphi)}$ points.

(ii) Suppose now that L is a cofinal subframe logic and $\mathfrak{F} = \langle W, R\rangle$ is a frame for L refuting φ under a valuation $\mathfrak{V}$. As before, we construct a subframe $\mathfrak{F}' = \langle W', R'\rangle$ of $\mathfrak{F}$ such that $|W'| \leq 2^{l(\varphi)}$ and $\mathfrak{F}' \not\models \varphi$. But now $\mathfrak{F}'$ is not in general a frame for L, since it may be not a cofinal subreduct of $\mathfrak{F}$. So we add to W' all the the final clusters in $\mathfrak{F}$ that are accessible from W'; the resulting subframe $\mathfrak{F}_1 = \langle W_1, R_1\rangle$ is a cofinal subreduct of $\mathfrak{F}$ and obviously refutes φ under the valuation $\mathfrak{V}_1$ which is the restriction of $\mathfrak{V}$ to W_1. However, now $\mathfrak{F}_1$ may contain too many points. Our concluding step is to construct a reduct $\mathfrak{F}_2 = \langle W_2, R_2\rangle$ of $\mathfrak{F}_1$ which also refutes φ and contains at most $2^{2l(\varphi)+1}$ points.

Two final clusters $C(x)$ and $C(y)$ in $\mathfrak{F}_1$ are said to be *equivalent* (relative to φ) iff either they are both degenerate and $\Sigma_x = \Sigma_y$ or they are both non-degenerate and $\Delta_x = \Delta_y$. It is clear that, changing if necessary the valuation $\mathfrak{V}_1$ only on points from final clusters in $\mathfrak{F}_1$, we can achieve the situation when $\mathfrak{F}_1$ refutes φ under this valuation and, for every equivalent final clusters $C(x)$ and $C(y)$,

$$\forall u \in C(x) \exists v \in C(y)\ \Sigma_u = \Sigma_v.$$

Define on W_1 an equivalence relation $\equiv$ by taking $x \equiv y$ iff either $x = y$ or x and y belong to equivalent final clusters and $\Sigma_x = \Sigma_y$. By $[x]$ we denote the equivalence class containing x. Let $\mathfrak{F}_2 = \langle W_2, R_2\rangle$ be the frame with $W_2 = W_1/_\equiv$ and $[x]R_2[y]$ iff $[x] \subseteq [y]\!\downarrow$ in $\mathfrak{F}_1$. It is not hard to verify that the natural map $g(x) = [x]$ is a reduction of $\mathfrak{F}_1$ to $\mathfrak{F}_2$. Moreover, if $\mathfrak{V}_1(p,x) \neq \mathfrak{V}_1(p,y)$ for some $p \in \mathbf{Sub}\varphi$ then $g(x) \neq g(y)$, and so we may put $\mathfrak{V}_2(p, g(x)) = \mathfrak{V}_1(p,x)$ for all $x \in W_1$ and $p \in \mathbf{Sub}\varphi$. By the P-morphism Theorem, $\mathfrak{F}_2 \not\models \varphi$; by the Composition Lemma and Refutability Criterion, $\mathfrak{F}_2$ is a frame for L; and, finally, $|W_2| \leq 2^{2l(\varphi)+1}$. (Notice that if the frame $\mathfrak{F}$ is partially ordered then $|W_2| \leq 2^{l(\varphi)+1}$.) $\dashv$

Corollary 4.4 *Suppose L is an intermediate logic axiomatizable by disjunction free formulas and $\varphi \notin L$. Then there is a frame separating φ from L and containing at most $2^{l(\varphi)+1}$ points.* $\dashv$

It is worth noting that for some subframe and cofinal subframe logics the exponential upper bound for the complexity of refutation frames we have just obtained can be reduced to a polynomial or even linear one. For instance, Ono and Nakamura [1980] showed that **S4.3**, **S4.3Dum** and **S4.3Grz** have the linear finite model property, while Chagrov [1983] extended this result to all extensions of **S4.3** and established polynomial and linear upper bounds for some other known modal and intermediate logics. However, it is impossible to reduce essentially the upper bound of Theorem 4.3 for all subframe and cofinal subframe logics: in Zakharyaschev and Popov [1980] I proved the exponential lower bound for intuitionistic propositional logic **Int** and hence for **S4**; moreover, Chagrov [1983] showed that no intermediate logic L in the interval $\mathbf{Int} \subseteq L \subseteq \mathbf{Int} + \neg p \vee \neg\neg p$ and so no normal modal logic L in the interval $\mathbf{S4} \subseteq L \subseteq \mathbf{S4.2Grz}$ has the polynomial finite model property.

As a direct consequence of the proof of Theorem 4.3 we obtain the following

Corollary 4.5 (i) *Suppose* $L = \mathbf{K4} \oplus \varphi$ *is a subframe logic. Then*

$$L = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \emptyset) : i = 1, \ldots, n\}$$

for some $\mathfrak{F}_i$ *containing at most* $2^{l(\varphi)}$ *points.*

(ii) *Suppose* $L = \mathbf{K4} \oplus \varphi$ *is a cofinal subframe logic. Then*

$$L = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \emptyset, \bot) : i = 1, \ldots, n\}$$

for some $\mathfrak{F}_i$ *containing at most* $2^{2l(\varphi)+1}$ *points.* ⊣

An analogous corollary holds, of course, for $L = \mathbf{Int} + \varphi$ with (i) an implicative or (ii) disjunction free φ.

Note by the way that both the problems '$\mathbf{K4} \oplus \varphi \in \mathcal{CSF}$?' and '$\mathbf{K4} \oplus \varphi \in \mathcal{SF}$?' (and their intuitionistic counterparts) are algorithmically undecidable, as it follows from Chagrov and Zakharyaschev [1991].

5 Frame-theoretic characterization

Now we give a purely frame-theoretic characterization of the cofinal subframe logics; for the subframe ones the characterization was obtain by Fine [1985].

A general frame $\mathfrak{F} = \langle W, R, P\rangle$ is called a *subframe* of a general frame $\mathfrak{G} = \langle V, S, Q\rangle$ if $W \subseteq V$, R is the restriction of S to W and $P \subseteq Q$. In other terms, $\mathfrak{F}$ is (isomorphic to) a subframe of $\mathfrak{G}$ if $\mathfrak{G}$ is subreducible to $\mathfrak{F}$ by a map f which is a bijection from $\mathrm{dom} f \subseteq V$ onto W. $\mathfrak{F}$ is called a *cofinal subframe* of $\mathfrak{G}$ if $\mathfrak{F}$ is a subframe of $\mathfrak{G}$ and $W{\uparrow} \subseteq \overline{W{\downarrow}}$ in $\mathfrak{G}$[1]. Alternatively, $\mathfrak{F}$ is (isomorphic to) a cofinal subframe of $\mathfrak{G}$ if $\mathfrak{G}$ is cofinally subreducible to $\mathfrak{F}$ by a bijection f from $\mathrm{dom} f$ onto W. Finally, a class of frames $\mathcal{C}$ is said to be *closed under (cofinal) subframes* if every (cofinal) subframe of $\mathfrak{F}$ is in $\mathcal{C}$ whenever $\mathfrak{F} \in \mathcal{C}$.

Theorem 5.1 (i) *L is a cofinal subframe logic iff it is characterized by a class of frames that is closed under cofinal subframes.*

(ii) (Fine [1985]) *L is a subframe logic iff it is characterized by a class of frames that is closed under subframes.*

Proof. (i) Suppose L is a cofinal subframe logic. We show that the class of all (general or Kripke) frames for L is closed under cofinal subframes. Let $\mathfrak{G}$ be a frame for L and $\mathfrak{H}$ a cofinal subframe of $\mathfrak{G}$. Then $\mathfrak{H}$ is a frame for L, for otherwise $\mathfrak{H} \not\models \alpha(\mathfrak{F}, \emptyset, \bot)$ for some $\alpha(\mathfrak{F}, \emptyset, \bot) \in L$, and so, by the Composition Lemma and Refutability Criterion, $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \emptyset, \bot)$ which is a contradiction.

Now suppose that L is characterized by some class of frames $\mathcal{C}$ that is closed under cofinal subframes. We show that $L = L'$, where

$$L' = \mathbf{K4} \oplus \{\alpha(\mathfrak{F}, \emptyset, \bot) : \mathfrak{F} \not\models L\}.$$

[1]This definition is somewhat different from the conventional definition of cofinality, which requires that $V = \overline{W{\downarrow}}$. A cofinal subframe in our sense is a cofinal subframe of a generated subframe in the conventional terminology.

Indeed, if $\mathfrak{F}$ is a finite rooted frame and $\mathfrak{F} \not\models L$ then $\alpha(\mathfrak{F}, \emptyset, \bot) \in L$, for otherwise $\mathfrak{G} \not\models \alpha(\mathfrak{F}, \emptyset, \bot)$ for some $\mathfrak{G} \in \mathcal{C}$, and hence there is a cofinal subframe $\mathfrak{H}$ of $\mathfrak{G}$ which is reducible to $\mathfrak{F}$; but $\mathfrak{H} \in \mathcal{C}$, and so, by the P-morphism Theorem, $\mathfrak{F}$ is a frame for L, which is a contradiction. Thus, $L' \subseteq L$.

To prove the converse inclusion, suppose that $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$. Then $\mathfrak{F} \not\models L$, and hence $\alpha(\mathfrak{F}, \emptyset, \bot) \in L'$. Therefore, by Theorem 2.2, $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L'$.

(ii) is proved in exactly the same way. ⊣

Corollary 5.2 *If a logic L is characterized by a class of frames that is closed under cofinal subframes then L has the finite model property.* ⊣

Sometimes this Corollary makes it possible to prove the finite model property of a logic even without knowing the constitution of its frames. For illustration let us consider the following

Example 5.3 We are going to prove that if φ is a Boolean combination of formulas of the form Mp, where M is a modality (i.e. a sequence of $\Box$ and $\Diamond$), then $\mathbf{S4} \oplus \varphi$ has the finite model property. To this end we show that $\mathfrak{G} \not\models \varphi$ whenever $\mathfrak{F} \not\models \varphi$ for some cofinal subframe $\mathfrak{F}$ of $\mathfrak{G}$ and use Corollary 5.2. Without loss of generality we may assume that $\mathfrak{F}$ and $\mathfrak{G}$ are reflexive and have a common root. We may also assume that no modality M in φ has two adjacent $\Box$ or two adjacent $\Diamond$ (see Feys [1965]).

Suppose $\mathfrak{U}$ is a valuation on $\mathfrak{F} = \langle W, R, P \rangle$ under which φ is false at some point. Define a valuation $\mathfrak{V}$ on $\mathfrak{G} = \langle V, S, Q \rangle$ by taking

$$\mathfrak{V}(p,x) = \begin{cases} \mathfrak{U}(p,x) & \text{if } x \in W \\ F & \text{if } x \in V - W \ \&\ \exists y \in x{\uparrow}W\ \mathfrak{U}(p,x) = F \\ T & \text{otherwise.} \end{cases}$$

By induction on the length of M we show that, for every $x \in W$,

$$\mathfrak{V}(Mp, x) = \mathfrak{U}(Mp, x).$$

Suppose $M = \Box N$. If $\mathfrak{V}(\Box Np, x) = F$ then there is $y \in x{\uparrow}$ such that $\mathfrak{V}(Np, y) = F$. Suppose $N = \emptyset$. Then $\mathfrak{V}(p, z) = F$ for some $z \in y{\uparrow}W$, and so $\mathfrak{U}(Mp, x) = F$. Suppose $N = \Diamond K$. Since $\mathfrak{F}$ is a cofinal subframe of $\mathfrak{G}$, there is $z \in y{\uparrow}W$ and, clearly, $\mathfrak{V}(Np, z) = F$. Therefore, $\mathfrak{U}(Np, z) = F$, and so $\mathfrak{U}(Mp, x) = F$. If $\mathfrak{V}(\Box Np, x) = T$ then $\mathfrak{V}(Np, y) = T$ for all $y \in x{\uparrow}$; hence $\mathfrak{U}(Np, z) = T$ for all $z \in x{\uparrow}W$, and so $\mathfrak{U}(Mp, x) = T$.

The case $M = \Diamond N$ is considered analogously.

It follows immediately that, for all $x \in W$, $\mathfrak{V}(\varphi, x) = \mathfrak{U}(\varphi, x)$. Thus $\mathfrak{G}$ refutes φ under $\mathfrak{V}$, and so, by Corollary 5.2, $\mathbf{S4} \oplus \varphi$ has the finite model property.

It should be noted that we cannot replace $\mathbf{S4}$ in this proof with $\mathbf{K4}$. As we saw in Part 1 (Example 3), the Density Axiom $\Box\Box p \to \Box p$ is refuted in the frame  but valid in . ⊣

Corollary 5.4 $\mathcal{SF} \subset \mathcal{CSF}$.

Proof. The fact that $\mathcal{SF} \subseteq \mathcal{CSF}$ is an immediate consequence of Theorem 5.1. As was noted by Fine [1985, p.627], **S4.2** $\notin \mathcal{SF}$, but it follows from Example 5.3 that **S4.2** $=$ **S4** $\oplus \Diamond\Box p \to \Box\Diamond p$ is a cofinal subframe logic. The same, of course, is true for another well-known logic **S4.1** $=$ **S4** $\oplus \Box\Diamond p \to \Diamond\Box p$. ⊣

Corollary 5.5 *There are a continuum of cofinal subframe logics which are not subframe ones.*

Proof. There are a continuum of logics axiomatizable by the canonical formulas of the form $\alpha(\mathfrak{F}_i, \emptyset, \bot)$, where $\mathfrak{F}_i$ is the frame defined in Fig. 1. And none of them is a subframe logic, since the class of its frames is not closed under subframes. Indeed, add to $\mathfrak{F}_i$ a new point which is seen from all the points in $\mathfrak{F}_i$ and denote the result by $\mathfrak{G}_i$. Clearly, $\mathfrak{G}_i \models \alpha(\mathfrak{F}_j, \emptyset, \bot)$ for any j, but $\mathfrak{F}_i$, being a subframe of $\mathfrak{G}_i$, refutes $\alpha(\mathfrak{F}_i, \emptyset, \bot)$. ⊣

Corollary 5.6 *$\mathcal{CSF}$ is a complete sublattice of the lattice of all normal logics containing* **K4**. *$\mathcal{SF}$ is a complete sublattice of $\mathcal{CSF}$.*

Proof. Suppose $L_i \in \mathcal{CSF}$ for $i \in I$. Then, for each $i \in I$, there is a set Δ_i of cofinal subframe formulas such that $L_i = \mathbf{K4} \oplus \Delta_i$. So $\sum_{i\in I} L_i = \mathbf{K4} \oplus \bigcup_{i\in I} \Delta_i \in \mathcal{CSF}$.

As to the intersection $L = \bigcap_{i\in I} L_i$, it is clear that L is complete for the class $\bigcup_{i\in I}\{\mathfrak{F} : \mathfrak{F} \models L_i\}$ which is closed under cofinal subframes. Therefore, by Theorem 5.1, $L \in \mathcal{CSF}$.

The class $\mathcal{SF}$ is considered analogously. ⊣

The intuitionistic variant of Theorem 5.1 provides us with a nice frame-theoretic characterization of intermediate logics axiomatizable by implicative and disjunction free formulas.

Theorem 5.7 (i) *An intermediate logic L is axiomatized by purely implicative formulas iff it is characterized by a class of frames that is closed under subframes*[2].

(ii) *An intermediate logic L is axiomatizable by disjunction free formulas iff it is characterized by a class of frames that is closed under cofinal subframes.* ⊣

6 Elementarity, canonicity and compactness

First of all I remind the reader of the definitions of these terms. A modal logic L is called *elementary* if the class of all Kripke frames for L, treated as classical models for the first-order language with equality and binary predicate R, is elementary, i.e. there is a set Δ of first-order formulas in the language such that, for every Kripke frame $\mathfrak{F}$,

$$\mathfrak{F} \models L \text{ iff } \mathfrak{F} \models \Delta.$$

[2] An intuitionistic general frame $\mathfrak{F} = \langle W, R, P\rangle$ is a subframe of an intuitionistic general frame $\mathfrak{G} = \langle V, S, Q\rangle$ if $W \subseteq V$, R is the restriction of S to W and $V - (W - X){\downarrow} \in Q$ for every $X \in P$.

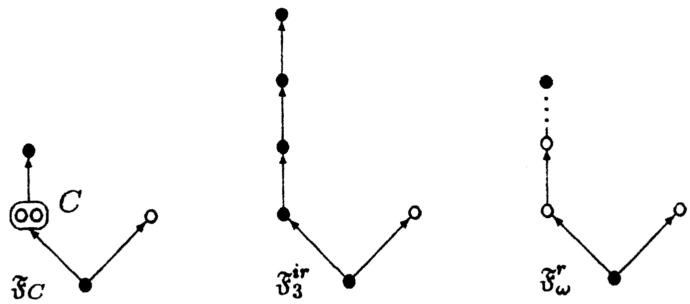


Figure 2:

(Here the former $\models$ is modal while the latter one is classical.) L is *canonical* if, for every descriptive frame $\mathfrak{F} = \langle W, R, P \rangle$,

$$\mathfrak{F} \models L \Rightarrow \langle W, R \rangle \models L.$$

Finally, L is *compact* (or *strongly complete*) if every L-consistent set of modal formulas has a model whose underlying Kripke frame validates L.

Fine [1985] gave a frame-theoretic characterization of these three properties for the subframe logics. He showed that a subframe logic L is elementary (and even universal) iff L is canonical iff L is compact iff the class of Kripke frames for L has the finite embedding property, i.e. $\mathfrak{F} \models L$ whenever every finite subframe of $\mathfrak{F}$ is a frame for L. Moreover, these properties turned out to be decidable for finitely axiomatizable subframe logics.

In this Section we obtain a generalization of Fine's characterization to the class of cofinal subframe logics. To formulate it we require two more definitions.

Let $\mathfrak{F}_C = \langle W_C, R_C \rangle$ be a frame containing a cluster C. For an ordinal ξ, $0 < \xi \leq \omega$, by $\mathfrak{F}_\xi^{ir} = \langle W_\xi, R_\xi^{ir} \rangle$ we denote the frame which is obtained from $\mathfrak{F}_C$ by replacing C with an ascending chain of ξ irreflexive points. More exactly, we put

$$W_\xi = (W - C) \cup \{i : 0 \leq i < \xi\}$$

and, for all $x, y \in W_\xi$,

$$\begin{array}{lll} xR_\xi^{ir}y & \text{iff} & xR_Cy \qquad\qquad\qquad\qquad\qquad \text{or} \\ & & \exists i,j < \xi(x = i \ \&\ y = j \ \&\ i < j) \ \ \text{or} \\ & & \exists i < \xi \exists z \in C(x = i \ \&\ zR_Cy) \qquad \text{or} \\ & & \exists i < \xi \exists z \in C(y = i \ \&\ xR_Cz). \end{array}$$

$\mathfrak{F}_\xi^r = \langle W_\xi, R_\xi^r \rangle$ is the result of replacing C in $\mathfrak{F}_C$ with an ascending chain containing ξ reflexive points, i.e.

$$R_\xi^r = R_\xi^{ir} \cup \{\langle i, i \rangle : 0 \leq i < \xi\}.$$

Fig. 2 illustrates this definition.

We say that a subreduction f of a frame $\mathfrak{G}$ to a finite frame $\mathfrak{F}$ is a *quasi-embedding* of $\mathfrak{F}$ into $\mathfrak{G}$ if $f^{-1}(x)$ is a singleton for every point x whose cluster $C(x)$ is not final in $\mathfrak{F}$. In

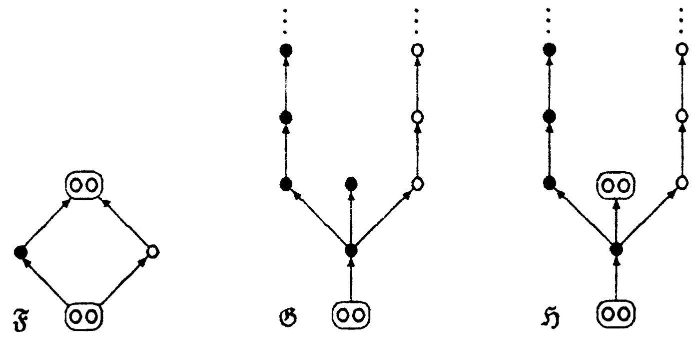


Figure 3:

such a case $\mathfrak{F}$ is called *quasi-embeddable* in $\mathfrak{G}$. For example, the frame $\mathfrak{F}$ in Fig. 3 is quasi-embeddable in $\mathfrak{G}$ and cofinally quasi-embeddable in $\mathfrak{H}$. Note also that the subreduction g of $\mathfrak{F}$ to $\mathfrak{F}_2$, constructed in the proof of Theorem 4.3, is a cofinal quasi-embedding of $\mathfrak{F}_2$ into $\mathfrak{F}$.

A logic L has the *finite cofinal quasi-embedding property* if a Kripke frame $\mathfrak{F}$ validates L whenever every finite frame which is cofinally quasi-embeddable in $\mathfrak{F}$ validates L.

We are now in a position to formulate and prove the main result of this Section.

Theorem 6.1 *The following conditions are equivalent for each cofinal subframe logic L:*

(1) *L is elementary;*

(2) *L is canonical;*

(3) *L is compact;*

(4) *for every finite rooted frame $\mathfrak{F}_C$ with a non-degenerate non-final cluster C*

$$(\forall \xi < \omega \ \mathfrak{F}_\xi^{ir} \models L) \Rightarrow \mathfrak{F}_C \models L$$

and

$$(\forall \xi < \omega \ \mathfrak{F}_\xi^{r} \models L) \Rightarrow \mathfrak{F}_C \models L;$$

(5) *L has the finite cofinal quasi-embedding property.*

Proof. (1) $\Rightarrow$ (2) follows from Theorem 4.1 above and Theorem 2.2 of Fine [1975]. (2) $\Rightarrow$ (3) is a direct consequence of the definition of the canonical models.

(3) $\Rightarrow$ (4). Suppose that $\mathfrak{F}_C = \langle W_C, R_C \rangle$ is a finite rooted frame with a non-degenerate non-final cluster C and $\forall \xi < \omega \ \mathfrak{F}_\xi^{ir} \models L$. We must prove that $\mathfrak{F}_C \models L$.

Let $\{a_i : i \in I\}$ be all the points in W_ω. With each a_i we associate a variable p_i different from p_j for any $j \neq i$ and construct from them the canonical formulas $\alpha(\mathfrak{F}_\xi^{ir}, \emptyset, \bot)$ for all ξ such that $0 < \xi < \omega$. Now take the set

$$\{\neg \alpha(\mathfrak{F}_\xi^{ir}, \emptyset, \bot) : 0 < \xi < \omega\}$$

and show that it is L-consistent.

Suppose otherwise. Then we shall have some $\xi < \omega$ for which

$$\alpha(\mathfrak{F}_1^{ir}, \emptyset, \bot) \vee \alpha(\mathfrak{F}_2^{ir}, \emptyset, \bot) \vee \ldots \vee \alpha(\mathfrak{F}_\xi^{ir}, \emptyset, \bot) \in L.$$

But on the other hand, the natural embedding of $\mathfrak{F}^{ir}_\zeta$ in $\mathfrak{F}^{ir}_\xi$, for $\zeta \leq \xi$, is cofinal (non-finality of C in $\mathfrak{F}_C$ is essential here), and so, according to the proof of Theorem 1 in Part I, there is a valuation $\mathfrak{V}$ on $\mathfrak{F}^{ir}_\xi$ such that all the formulas $\alpha(\mathfrak{F}^{ir}_\zeta, \emptyset, \bot)$, for $\zeta \leq \xi$, are false at the root of $\mathfrak{F}^{ir}_\xi$ under $\mathfrak{V}$, which is a contradiction, since $\mathfrak{F}^{ir}_\xi \models L$.

By the compactness of L, there is a model $\mathfrak{M} = \langle \mathfrak{G}, \mathfrak{V} \rangle$ on a Kripke frame $\mathfrak{G} = \langle V, S \rangle$ such that

(i) all $\alpha(\mathfrak{F}^{ir}_\xi, \emptyset, \bot)$, for $0 < \xi < \omega$, are simultaneously false at some point in $\mathfrak{M}$ and

(ii) $\mathfrak{G} \models L$.

Define a map f from V onto W_ω by taking

$$f(x) = \begin{cases} a_i & \text{if } x \not\models p_i \text{ and, for each } \xi < \omega, \text{ the} \\ & \text{premise of } \alpha(\mathfrak{F}^{ir}_\xi, \emptyset, \bot) \text{ is true at } x \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Using the proof of Theorem 1 in Part I, it is not hard to check that f is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}^{ir}_\omega$. On the other hand, we can easily construct a reduction g of $\mathfrak{F}^{ir}_\omega$ to $\mathfrak{F}_C$. Indeed, if $C = \{b_0, \dots, b_n\}$ then we may take

$$g(x) = \begin{cases} x & \text{if } x \in W_C - C \\ b_i & \text{if } x = m \text{ and } i = \mathrm{mod}_{n+1}(m). \end{cases}$$

By the Composition Lemma, there is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}_C$, and so $\mathfrak{F}_C \models L$, for otherwise $\mathfrak{G} \not\models L$, contrary to (ii).

The case with $\mathfrak{F}^{r}_\xi$ is considered in exactly the same way.

(4) $\Rightarrow$ (5). Suppose otherwise, i.e. there is a frame $\mathfrak{G}$ such that each finite frame which is cofinally quasi-embeddable in $\mathfrak{G}$ validates L but $\mathfrak{G} \not\models L$. Then there exists a cofinal subreduction f of $\mathfrak{G}$ to a finite rooted frame $\mathfrak{F} = \langle W, R \rangle$ such that $\mathfrak{F} \not\models L$. Starting with $\mathfrak{F}$ we construct by induction a finite rooted frame which is not a frame for L but is cofinally quasi-embeddable in $\mathfrak{G}$, contrary to our assumption. At the very beginning we mark by some signs all the non-final clusters in $\mathfrak{F}$ which means that all of them are to be analyzed in the sequel.

Suppose now that we have already constructed a finite rooted frame $\mathfrak{H} = \langle V, S \rangle$ and a cofinal subreduction g of $\mathfrak{G}$ to $\mathfrak{H}$ such that $\mathfrak{H} \not\models L$ and $g^{-1}(x)$ is a singleton for each x belonging to an unmarked non-final cluster in $\mathfrak{H}$. (At the first step $\mathfrak{H} = \mathfrak{F}$.)

Let $C = \{a_0, \dots, a_k\}$ be a marked cluster in $\mathfrak{H}$ whose all immediate predecessors $C_1, \dots, C_m$ ($m \geq 0$) are unmarked and let $b_1 \in C_1, \dots, b_m \in C_m$. By the induction hypothesis, $g^{-1}(b_i) = \{x_i\}$ for some $x_1, \dots, x_m$ in $\mathfrak{G}$. Choose a minimal number of disjoint sets $A_1, \dots, A_n$ of points in $\mathfrak{G}$ such that

- for each $i \in \{1, \dots, m\}$ there is $j \in \{1, \dots, n\}$ such that $A_j \subseteq x_i\uparrow$

and, for each $i \in \{1, \dots, n\}$, either

- $A_i = \{y_0, \dots, y_k\}$, $g(y_j) = a_j$, for $j = 0, \dots, k$, and A_i is a subset of a cluster in $\mathfrak{G}$

or

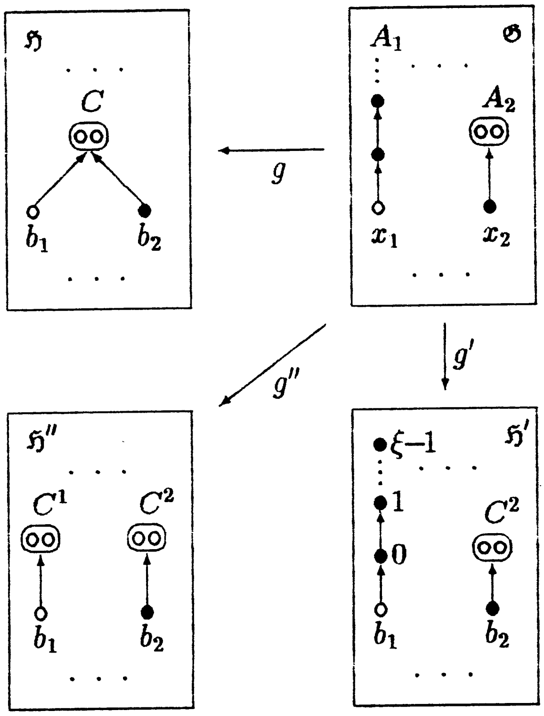


Figure 4:

- A_i is an infinite ascending chain $y_0, y_1, \dots$ all the points of which are either simultaneously irreflexive or simultaneously reflexive and $g(y_j) \in C$ for $j \geq 0$.

The existence of such $A_1, \dots, A_n$ follows from the fact that g is a subreduction of $\mathfrak{G}$ to $\mathfrak{H}$. (See Fig. 4.)

Our next action depends on the number of these $A_1, \dots, A_n$. Note by the way that $1 \leq n \leq m$.

Case 1. $n = 1$.

1.1. If $A_1 = \{y_0, \dots, y_k\}$, i.e. if A_1 is a part of a cluster in $\mathfrak{G}$, then we put $\mathfrak{H}' = \mathfrak{H}$, mark in $\mathfrak{H}'$ all the clusters that were marked in $\mathfrak{H}$ except C and define a partial map g' from $\mathfrak{G}$ onto $\mathfrak{H}'$ by taking

$$g'(x) = \begin{cases} g(x) & \text{if } x \in (\mathrm{dom} g - g^{-1}(C)) \cup A_1 \\ \text{undefined} & \text{otherwise.} \end{cases}$$

It is clear that $\mathfrak{H}' \not\models L$, g' is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{H}'$ and $g'^{-1}(x)$ is a singleton for each x belonging to an unmarked non-final cluster in $\mathfrak{H}'$. Note also that the number of marked clusters in $\mathfrak{H}'$ is less then that in $\mathfrak{H}$.

1.2. Suppose A_1 is an infinite ascending chain $y_0, y_1, \dots$ of irreflexive points. Then C is non-degenerate, and, since $\mathfrak{H} = \mathfrak{H}_C \not\models L$, there is, by (4), some $\xi < \omega$ such that $\mathfrak{H}_\xi^{ir} \not\models L$. (Recall that $\mathfrak{H}_\xi^{ir}$ is obtained from $\mathfrak{H}_C$ by replacing C with the ascending chain $0, \dots, \xi - 1$ of irreflexive points.) In this case we put $\mathfrak{H}' = \mathfrak{H}_\xi^{ir}$, mark in $\mathfrak{H}'$ all the clusters that were marked in $\mathfrak{H}$ (the new points $0, \ldots, \xi - 1$ are remained unmarked) and define a partial map g' from $\mathfrak{G}$ onto $\mathfrak{H}'$ by taking

$$g'(x) = \begin{cases} g(x) & \text{if } x \in \mathrm{dom} g - g^{-1}(C) \\ i & \text{if } x = y_i,\ 0 \leq i < \xi \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Again g' is a cofinal subreduction, $\mathfrak{H}' \not\models L$, $g'^{-1}(x)$ is a singleton for each x belonging to an unmarked non-final cluster in $\mathfrak{H}'$ and the number of marked clusters in $\mathfrak{H}'$ is less than that in $\mathfrak{H}$.

1.3. The case when A_1 is an ascending chain of reflexive points is considered in the same way as in 1.2, but using the second part of (4), i.e. $\mathfrak{H}^{r}_{\xi}$ instead of $\mathfrak{H}^{ir}_{\xi}$.

Case 2. Suppose now that $n > 1$. Then we first form a new frame $\mathfrak{H}'' = \langle V'', S'' \rangle$ by taking (see Fig. 4)

$$V'' = (V - C) \cup C^1 \cup \ldots \cup C^n,$$

where

$$C^i = \{a^i_0, \ldots, a^i_k\},\ i = 1, \ldots, n,$$

and, for all $x, y \in V''$,

$$\begin{array}{lll} xS''y & \text{iff} & x, y \in V - C \ \&\ xSy \qquad \text{or} \\ & & \exists i, j\ (x = a^i_j \ \&\ a_j S y) \qquad \text{or} \\ & & \exists i, j, l\ (y = a^i_j \ \&\ x \in \overline{b_l \downarrow} \ \&\ A_i \subseteq x_l \uparrow) \qquad \text{or} \\ & & \exists i, j, l\ (x = a^i_j \ \&\ y = a^i_l \ \&\ C \text{ is non-degenerate}). \end{array}$$

Mark in $\mathfrak{H}''$ all the clusters that were marked in $\mathfrak{H}$ and $C^1, \ldots, C^n$ as well. After that we define a map g'' from $\mathfrak{G}$ onto $\mathfrak{H}''$ by taking

$$g''(x) = \begin{cases} g(x) & \text{if } x \in \mathrm{dom} g - g^{-1}(C) \\ a^i_j & \text{if } x = y_l \in A_i \ \&\ \mathrm{mod}_{k+1}(l) = j \\ \text{undefined} & \text{otherwise.} \end{cases}$$

It is not difficult to see that g'' is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{H}''$. Moreover, $\mathfrak{H}'' \not\models L$, since $\mathfrak{H}''$ is reducible to $\mathfrak{H}$, and $g''^{-1}(x)$ contains only one point if $C(x)$ is an unmarked non-final cluster in $\mathfrak{H}''$. But the number of marked clusters in $\mathfrak{H}''$ has become greater than that in $\mathfrak{H}$. However, we need not worry. For we can now analyze the new clusters $C^1, \ldots, C^n$ which clearly satisfy the condition of Case 1, and so we shall eventually construct a frame $\mathfrak{H}'$ having all the desirable properties and less marked clusters than $\mathfrak{H}$. Fig. 4 will help the reader to complete the details.

(5) $\Rightarrow$ (1). Given a finite rooted frame $\mathfrak{F}$, one can construct a first-order formula $\Phi_{\mathfrak{F}}$ with the predicates $=$ and R and free variables corresponding to the points in $\mathfrak{F}$ such that a Kripke frame $\mathfrak{G}$ satisfies $\Phi_{\mathfrak{F}}$ iff $\mathfrak{F}$ is cofinally quasi-embeddable in $\mathfrak{G}$. (An example of

such $\Phi_{\mathfrak{F}}$ is presented below.) Then we shall have, for every Kripke frame $\mathfrak{G}$,

$\mathfrak{G} \not\models L$ iff (by (5))
there is a finite rooted frame $\mathfrak{F} \not\models L$ which
is cofinally quasi-embeddable in $\mathfrak{G}$ iff (by Lemma 6.2 below)
$\mathfrak{G} \models \exists\overline{x}\Phi_{\mathfrak{F}}$.

(Here $\exists\overline{x}\Phi_{\mathfrak{F}}$ is the existential closure of $\Phi_{\mathfrak{F}}$.) Therefore, we can take $\Delta = \{\neg\exists\overline{x}\Phi_{\mathfrak{F}} : \mathfrak{F} \not\models L\}$, and then the class of frames for L will coincide with the class of classical models for Δ. ⊣

We show now how one can construct $\Phi_{\mathfrak{F}}$ for a finite rooted frame $\mathfrak{F} = \langle W, R\rangle$. Let $a_0, \ldots, a_n$ be all the points in $\mathfrak{F}$. With each point a_i belonging to a non-final cluster in $\mathfrak{F}$ or to a final one having no predecessors in $\mathfrak{F}$ we associate the individual variable x_i. And if the final cluster $C(a_i)$ has immediate predecessors in $\mathfrak{F}$, say, $C(a_j), \ldots, C(a_k)$, then we associate with a_i the variables $x_i^j, \ldots, x_i^k$. The individual variables thus associated with points in $\mathfrak{F}$ will be denoted by x_i^s, where $0 \leq i \leq n$ and s is either blank or $0 \leq s \leq n$.

First we introduce two auxiliary formulas, namely

$$\Psi_k(x) = \exists y_1 \ldots \exists y_k (\bigwedge_{i \neq j} y_i \neq y_j \wedge R(x, y_1) \wedge R(y_1, y_2) \wedge \ldots \wedge R(y_{k-1}, y_k)),$$

which means 'x sees a chain of k distinct points', and

$$\Psi(x) = \neg\exists y R(x, y) \vee \exists z (R(x, z) \wedge \neg\exists y R(z, y)),$$

which means 'x is a final irreflexive point itself or sees such a point'.

Now we define $\Phi_{\mathfrak{F}}$ to be the conjunction of the following formulas under all admissible values of their parameters:

(0) $R(x_i, x_j^s)$: $a_i R a_j$, s is either blank or $s = i$ and the cluster $C(a_i)$ is not final in $\mathfrak{F}$;

(1) $\neg R(x_i^s, x_j^t)$: not $a_i R a_j$;

(2) $x_i^s \neq x_j^t$: $i \neq j$, $0 \leq i < j \leq n$;

(3) $\Psi_k(x_i^s)$: $C(a_i)$ is a final non-degenerate cluster in $\mathfrak{F}$ containing k points;

(4) $\neg\exists x \bigwedge_{a_i \in X} R(x_i^s, x)$: X is an antichain in $\mathfrak{F}$ such that $\widehat{X} = \emptyset$, where $\widehat{X} = \{y : X \subseteq y\!\downarrow\}$[3];

(5) $\forall x(\bigwedge_{a_i \in X} R(x_i^s, x) \to \Psi_k(x))$: X is an antichain in $\mathfrak{F}$ such that all final clusters in $\widehat{X}$ are non-degenerate and the smallest of them contains $k \geq 1$ points;

(6) $\forall x(\bigwedge_{a_i \in X} R(x_i^s, x) \to \Psi(x))$: X is an antichain in $\mathfrak{F}$ such that each final cluster in $\widehat{X}$ is degenerate;

[3]It should be noted that every formula of the form (4) contains only one conjunct $R(x_i^s, x)$ for each $a_i \in X$. If several variables are associated with a_i then there are several formulas of the form (4) corresponding to X. For example, if $X = \{a_i\}$ and x_i^j, x_i^k are the variables associated with a_i then we obtain two formulas, viz., $\neg\exists x R(x_i^j, x)$ and $\neg\exists x R(x_i^k, x)$. The same concerns the formulas of the form (5) - (7).

(7) $\forall x(\bigwedge_{a_i \in X} R(x_i^s, x) \to \Psi(x) \vee \Psi_k(x))$: X is an antichain in $\mathfrak{F}$ such that $\widehat{X}$ contains both degenerate and non-degenerate clusters and k is the number of points in the smallest non-degenerate one.

Lemma 6.2 *A Kripke frame $\mathfrak{G} = \langle V, S\rangle$ satisfies $\Phi_{\mathfrak{F}}$ iff $\mathfrak{F} = \langle W, R\rangle$ is cofinally quasi-embeddable in $\mathfrak{G}$.*

Proof. ($\Rightarrow$) Suppose $\mathfrak{G} \models \Phi_{\mathfrak{F}}[b_0^s, \dots, b_n^t]$ for some $b_0^s, \dots, b_n^t \in V$. By (2), $b_i^s \neq b_j^t$ if $i \neq j$, and so we can define a partial map f from V onto W by taking $f(b_i^s) = a_i$ for $i = 0, \dots, n$.

Now we extend f so that the resulting map g is a cofinal quasi-embedding of $\mathfrak{F}$ in $\mathfrak{G}$. First of all, if C is a final cluster in $\mathfrak{G}$, $b_i^s \in C$ and $C(a_i)$ is a non-degenerate final cluster in $\mathfrak{F}$ containing $k \geq 1$ points then, by (3), C contains at least k points and we evidently can extend f so that C is mapped onto $C(a_i)$. The same we do for all final clusters C containing some b_i^s. Let h be the resulting extension of f.

Denote by U the set $\mathrm{dom}h\uparrow - \mathrm{dom}h\overline{\downarrow}$. If $U = \emptyset$ then g, the extension of f we need, is just h. So suppose that $U \neq \emptyset$. Consider the set $\mathcal{A}$ of all antichains X in $\mathfrak{F}$ and define a quasi-order relation Q on $\mathcal{A}$ by taking, for every $X_1, X_2 \in \mathcal{A}$,

$$X_1 Q X_2 \text{ iff } X_1 \subseteq X_2\overline{\downarrow}.$$

We write $X_1 \equiv X_2$ if both $X_1 Q X_2$ and $X_2 Q X_1$ hold. Clearly, if $X = \{x_1, \dots, x_m\}$, $Y = \{y_1, \dots, y_k\}$ and $X \equiv Y$ then $m = k$ and $\{C(x_1), \dots, C(x_m)\} = \{C(y_1), \dots, C(y_k)\}$.

Let $X = \{a_i, \dots, a_j\} \in \mathcal{A}$ and let $\mathcal{B}_X$ be the set of all corresponding antichains in $\mathfrak{G}$ of the form $\{b_i^s, \dots, b_j^t\}$. Define the following three subsets of U:

$$U_X = \{y \in U : y \in \bigcup_{Y \in \mathcal{B}_X} \widehat{Y} \;\&\; \forall z \in y\underline{\uparrow}\; \forall Z \in \mathcal{A}(z \in \bigcup_{Y \in \mathcal{B}_Z} \widehat{Y} \Rightarrow ZQX)\},$$

$$FC_X = \{y \in U_X : C(y) \text{ is a final cluster in } \mathfrak{G}\},$$

$$\overline{FC}_X = \{y \in U_X : \forall z \in y\underline{\uparrow}\; z \notin FC_X\}.$$

It is clear that $U_X = U_X\underline{\uparrow}$, $FC_X = FC_X\underline{\uparrow}$ and $\overline{FC}_X = \overline{FC}_X\underline{\uparrow}$. Moreover, since $\mathfrak{F}$ is finite, $\bigcup_{X \in \mathcal{A}} U_X$ is a cover for U, while $FC_X \cup \overline{FC}_X$ is a cover for U_X, and so $\bigcup_{X \in \mathcal{A}} FC_X \cup \overline{FC}_X$ is a cover for U. Note also that $U_X \cap U_Y = \emptyset$ if $X \not\equiv Y$.

We are in a position now to define g. Its domain will be the set $\mathrm{dom}h \cup \bigcup_{X \in \mathcal{A}}(FC_X \cup \overline{FC}_X)$. First we put $g(x) = h(x)$ for all $x \in \mathrm{dom}h$. Then we consider $FC_X \cup \overline{FC}_X$ for $X \in \mathcal{A}$. Suppose that this set is not empty. Then $\widehat{X} \neq \emptyset$ in $\mathfrak{F}$, for otherwise (4) is not satisfied on $b_0^s, \dots, b_n^t$. The following three cases are possible.

Case 1. All the final clusters in $\widehat{X}$ are non-degenerate. Let $C = \{a_{i_1}, \dots, a_{i_k}\}$, for $k \geq 1$, be the smallest of them.

If $FC_X \neq \emptyset$ then, by (5), each cluster in FC_X contains at least k points. Define g on FC_X so that it maps each of its clusters on C.

If $\overline{FC}_X \neq \emptyset$ then (by transfinite induction) we can define g on $\overline{FC}_X$ so that $g(\overline{FC}_X) = C$ and for any $x \in \overline{FC}_X$ and $j \in \{1, \dots, k\}$ there is $y \in x\uparrow$ such that $g(y) = a_{i_j}$.

Case 2. All the final clusters in $\widehat{X}$ are degenerate. Then $\overline{FC_X} = \emptyset$, for otherwise (6) is not satisfied on $b_0^s, \dots, b_n^t$, while FC_X, for the same reason, consists of some final irreflexive points (i.e. dead ends) in $\mathfrak{G}$. Define g on FC_X so that it maps FC_X on some final point in $\widehat{X}$.

Case 3. If $\widehat{X}$ contains both degenerate and non-degenerate final clusters then we divide FC_X into two parts: one of them contains only irreflexive points while another only reflexive ones. (One of these sets or even both of them may be empty.) Define g on FC_X so that it maps the points from the first part onto some irreflexive final point in $\widehat{X}$ and the clusters from the second part on some minimal non-degenerate final cluster in $\widehat{X}$. $\overline{FC_X}$ is considered as in Case 1.

It is not hard to check now that g is a cofinal quasi-embedding of $\mathfrak{F}$ in $\mathfrak{G}$.

($\Leftarrow$) Suppose f is a cofinal quasi-embedding of $\mathfrak{F}$ in $\mathfrak{G}$. For each variable x_i in $\Phi_{\mathfrak{F}}$ associated with a point a_i we choose some $b_i \in f^{-1}(a_i)$ and after that for each x_i^j we choose some $b_i^j \in f^{-1}(a_i) \cap b_j\!\uparrow$. Using the definition of cofinal quasi-embedding one can readily show that our formula $\Phi_{\mathfrak{F}}$ is satisfied in $\mathfrak{G}$ on the chosen points. Indeed, take, for instance, the conjunct $R(x_i, x_j)$ corresponding to a_iRa_j. Since both a_i and a_j do not belong to final clusters in $\mathfrak{F}$, $f^{-1}(a_i) = \{b_i\}$, $f^{-1}(a_j) = \{b_j\}$ and, by the definition of reduction, b_iSb_j, whence $\mathfrak{G} \models R(x_i, x_j)[b_i, b_j]$. Now take $R(x_i, x_j^i)$ corresponding to a_iRa_j with a_j belonging to a final cluster in $\mathfrak{F}$. Then $\mathfrak{G} \models R(x_i, x_j^i)[b_i, b_j^i]$ by our choice of b_j^i. Thus all the conjuncts of the form (0) are satisfied in $\mathfrak{G}$ on the chosen points. The satisfiability of (1) - (7) can be proved in the same manner. ⊣

Remark. If we deal with a subframe logic L then to be equivalent to the elementarity, canonicity and compactness, the condition (4) in Theorem 6.1 must be satisfied for *all* (not only non-final) non-degenerate clusters C, while (5) becomes just the finite embedding property.

A close inspection of the definition of $\Phi_{\mathfrak{F}}$ shows that the class of Kripke frames for an elementary cofinal subframe logic L can be axiomatized by Π_4^0 sentences; for a subframe L we can do, as was shown by Fine [1985], only with Π_1^0, i.e. universal sentences. ⊣

Example 6.3 (a) $\mathbf{K4Grz} = \mathbf{K4} \oplus \alpha(\bullet, \emptyset) \oplus \alpha(\circledcirc\!\circledcirc, \emptyset)$ is neither elementary nor canonical nor compact, since every finite linearly ordered reflexive frame validates **K4Grz**, while the two point cluster is not a frame for it.

(b) $\mathbf{GL} = \mathbf{K4} \oplus \alpha(\circ, \emptyset)$ is not elementary, canonical and compact, for each finite linearly ordered irreflexive frame validates **GL**, while any non-degenerate cluster does not.

(c) $\mathbf{K4.1} = \mathbf{K4} \oplus \alpha(\bullet, \emptyset, \bot) \oplus \alpha(\circledcirc\!\circledcirc, \emptyset, \bot)$, on the contrary, is elementary, canonical and compact. Indeed, let $\mathfrak{F}_C$ be a finite frame with a non-final non-degenerate cluster C. Then $\mathfrak{F}_C \not\models \alpha(\bullet, \emptyset, \bot)$ iff $\mathfrak{F}_C$ has an irreflexive final point iff both $\mathfrak{F}_\xi^{ir} \not\models \alpha(\bullet, \emptyset, \bot)$ and $\mathfrak{F}_\xi^{r} \not\models \alpha(\bullet, \emptyset, \bot)$ hold for any finite ξ. Similarly, $\mathfrak{F}_C \not\models \alpha(\circledcirc\!\circledcirc, \emptyset, \bot)$ iff $\mathfrak{F}_C$ has a final cluster containing at least two points iff both $\mathfrak{F}_\xi^{ir} \not\models \alpha(\circledcirc\!\circledcirc, \emptyset, \bot)$ and $\mathfrak{F}_\xi^{r} \not\models \alpha(\circledcirc\!\circledcirc, \emptyset, \bot)$ hold for any finite ξ. ⊣

According to Chagrova [1991], the problem of determining, given a formula φ, whether

$\mathbf{K4} \oplus \varphi$ is elementary turns out to be algorithmically undecidable. But if we restrict the problem to those φ that axiomatize only cofinal subframe logics then it becomes decidable, as it is claimed by the following

Theorem 6.4 *There is an algorithm which, given a formula φ such that $\mathbf{K4} \oplus \varphi \in \mathcal{CSF}$, decides whether $\mathbf{K4} \oplus \varphi$ is elementary and, if it is, constructs a first-order equivalent of φ, i.e. such Φ (in R and $=$) that $\mathfrak{F} \models \varphi$ iff $\mathfrak{F} \models \Phi$ for every Kripke frame $\mathfrak{F}$.*

Proof. First we formulate the algorithm and then prove its correctness with respect to the given specification.

Construct two sets of rooted frames:

$$\mathcal{F}_1 = \{\mathfrak{F} : \mathfrak{F} \not\models \varphi \ \& \ |\mathfrak{F}| \leq 2^{2l(\varphi)+1}\}$$

and

$$\mathcal{F}_2 = \{\mathfrak{F} : \mathfrak{F} \not\models \varphi \ \& \ |\mathfrak{F}| \leq 2^{2l(\varphi)+2}\}.$$

Then, for each $\mathfrak{F}_C \in \mathcal{F}_1$ with a non-final non-degenerate cluster C, we check if $\mathfrak{F}^{ir}_\xi \in \mathcal{F}_2$ and $\mathfrak{F}^r_\zeta \in \mathcal{F}_2$ for some ξ and ζ. If the result of this check is positive for all $\mathfrak{F}_C \in \mathcal{F}_1$ then $L = \mathbf{K4} \oplus \varphi$ is elementary and $\Phi = \bigwedge_{\mathfrak{F} \in \mathcal{F}_1} \neg \exists \overline{x} \Phi_{\mathfrak{F}}$ is a first-order equivalent of φ. Otherwise L is not elementary.

Suppose now that the algorithm decides that L is elementary and show that L is really elementary. Let $\mathfrak{G}_C$ be a finite rooted frame with a non-final non-degenerate cluster C and $\mathfrak{G}_C \not\models \varphi$. By the proof of Theorem 4.3, there is a frame $\mathfrak{F} \in \mathcal{F}_1$ which is cofinally quasi-embeddable in $\mathfrak{G}_C$ via some subreduction f. If $C \cap \mathrm{dom} f = \emptyset$ then $\mathfrak{F}$ is cofinally quasi-embeddable in both $\mathfrak{G}^{ir}_\xi$ and $\mathfrak{G}^r_\xi$ for all ξ, and so $\mathfrak{G}^{ir}_\xi \not\models \varphi$ and $\mathfrak{G}^r_\xi \not\models \varphi$. Otherwise let $C' = f(C)$. Then, for $\mathfrak{F}_{C'} \in \mathcal{F}_1$, there are $\mathfrak{F}^{ir}_\xi \in \mathcal{F}_2$ and $\mathfrak{F}^r_\zeta \in \mathcal{F}_2$, from which it follows that $\mathfrak{G}^{ir}_\xi \not\models \varphi$ and $\mathfrak{G}^r_\zeta \not\models \varphi$. Thus, L satisfies (4) in Theorem 6.1 and so is elementary. The fact that Φ is a first-order equivalent of φ follows from Lemma 6.2 and the proofs of Theorems 4.3 and 6.1.

On the other hand, if our algorithm tells us that L is not elementary then there is $\mathfrak{F}_C \in \mathcal{F}_1$ such that either $\mathfrak{F}^{ir}_\xi \notin \mathcal{F}_2$ for all ξ or $\mathfrak{F}^r_\xi \notin \mathcal{F}_2$ for all ξ. Consider first the former alternative. If the algorithm was wrong then $\mathfrak{F}^{ir}_\xi \not\models \varphi$ for some finite ξ, and so, by the proof of Theorem 4.3, there exists a frame $\mathfrak{H} \in \mathcal{F}_1$ which is cofinally quasi-embeddable in $\mathfrak{F}^{ir}_\xi$ via some subreduction f. Without loss of generality we may assume that $\mathfrak{F}^{ir}_\zeta \models \varphi$ for all $\zeta < \xi$. This means that $\{0, \dots, \xi - 1\} \subseteq \mathrm{dom} f$. Moreover, it is clear that $f(i) \neq f(j)$ if $i \neq j$, for otherwise we can throw out all i such that there are $j > i$ with $f(i) = f(j)$, and $\mathfrak{H}$ will still be cofinally quasi-embeddable in the resulting frame, which is isomorphic to $\mathfrak{F}^{ir}_\zeta$ for some $\zeta < \xi$. But then $\xi \leq 2^{2l(\varphi)+1}$, and so $|\mathfrak{F}^{ir}_\xi| \leq 2^{2l(\varphi)+2}$, i.e. $\mathfrak{F}^{ir}_\xi \in \mathcal{F}_2$, which is a contradiction. The case with $\mathfrak{F}^r_\xi$ is considered in the same way. ⊣

It is worth also noting that as a consequence of Lemma 6.2 and the proof of Theorem 4.3 one can derive

Corollary 6.5 *Every cofinal subframe logic is elementary on the class of finite frames.* ⊣

To transfer Theorem 6.1 to the intermediate cofinal subframe logics we need to show that the canonicity is preserved while passing from a modal logic above **S4** to its superintuitionistic fragment. In fact we shall prove a somewhat more general preservation theorem. But first I remind the reader of some well-known notions and facts concerning modal companions of intermediate logics and the correspondence between general frames and algebras.

Let T be the *Gödel translation* prefixing $\Box$ to every subformula of intuitionistic formula. The *superintuitionistic fragment* of a normal modal logic M containing **S4** is the intermediate logic $\rho M = \{\varphi : T\varphi \in M\}$; M itself is called a *modal companion* of ρM. The set of all modal companions of an intermediate logic L forms the interval of logics between $\tau L = \mathbf{S4} \oplus \{T\varphi : \varphi \in L\}$ and $\sigma L = \tau L \oplus \mathbf{K4Grz}$, the *smallest* and *greatest modal companions* of L, respectively.

Given a modal or intuitionistic frame $\mathfrak{F} = \langle W, R, P\rangle$, by $\mathfrak{F}^+$ we denote the corresponding modal or pseudo-Boolean algebra of sets in the space W with the carrier P. If $\mathfrak{F}$ is a modal quasi-ordered frame then $\mathfrak{F}^+$ is a topological Boolean algebra. Conversely, given a modal or pseudo-Boolean algebra $\mathfrak{A}$, by $\mathfrak{A}_+$ we denote the Stone-Jónsson-Tarski representation of $\mathfrak{A}$, i.e. the general frame $\langle W_{\mathfrak{A}}, R_{\mathfrak{A}}, P_{\mathfrak{A}}\rangle$ where $W_{\mathfrak{A}}$ is the set of prime filters in $\mathfrak{A}$, $P_{\mathfrak{A}} = \{\{\nabla \in W_{\mathfrak{A}} : a \in \nabla\} : a \in \mathfrak{A}\}$ and $\nabla_1 R_{\mathfrak{A}} \nabla_2$ iff $\forall a \in \mathfrak{A}(\Box a \in \nabla_1 \Rightarrow a \in \nabla_2)$, if $\mathfrak{A}$ is modal, and $\nabla_1 R_{\mathfrak{A}} \nabla_2$ iff $\nabla_1 \subseteq \nabla_2$, if $\mathfrak{A}$ is pseudo-Boolean. A frame $\mathfrak{F}$ is *descriptive* if it is isomorphic to $(\mathfrak{F}^+)_+$.

For every quasi-ordered modal frame $\mathfrak{F} = \langle W, R, P\rangle$, we can construct the intuitionistic frame $\rho\mathfrak{F} = \langle \rho W, \rho R, \rho P\rangle$, called the *skeleton* of $\mathfrak{F}$, where $\rho W = \{C(x) : x \in W\}$, $C(x)\rho R C(y)$ iff xRy and $\rho P = \{\rho X : X \in P \;\&\; X = X\uparrow\}$. If $\mathfrak{F}$ is a frame for a modal logic M then $\rho\mathfrak{F}$ validates its superintuitionistic fragment ρM. As it follows from Maksimova [1975, Lemma 7], if $\mathfrak{F}$ is descriptive then the underlying Kripke frames of $\rho\mathfrak{F}$ and $((\rho\mathfrak{F})^+)_+$ are isomorphic.

Conversely, given an intuitionistic frame $\mathfrak{F} = \langle W, R, P\rangle$, by $\sigma\mathfrak{F} = \langle W, R, \sigma P\rangle$ we denote the modal frame in which σP is the Boolean closure of P in the space W. If $\mathfrak{F}$ is a frame for an intermediate logic L then, as was shown by Maksimova and Rybakov [1974], $\sigma\mathfrak{F} \models \sigma L$. For a descriptive $\mathfrak{F}$, by Maksimova's [1975] Lemma 8, we have that the underlying Kripke frames of $\sigma\mathfrak{F}$ and $((\sigma\mathfrak{F})^+)_+$ are isomorphic.

Theorem 6.6 (i) *If a normal modal logic $M \supseteq$ **S4** is canonical then the intermediate logic ρM is also canonical.*

(ii) *If an intermediate logic L is canonical then its smallest modal companion τL is also canonical.*

Proof. (i) Let $\mathfrak{F} = \langle W, R, P\rangle$ be a descriptive frame for ρM. Then $\sigma\mathfrak{F}$ and so $((\sigma\mathfrak{F})^+)_+$ are frames for $\sigma\rho M$, from which $((\sigma\mathfrak{F})^+)_+ \models M$. By Maksimova's Lemma, $\sigma\mathfrak{F}$ and $((\sigma\mathfrak{F})^+)_+$ are based on the same frame, namely $\langle W, R\rangle$, which, by the canonicity, validates M. Therefore, its skeleton, i.e. the same $\langle W, R\rangle$ considered as an intuitionistic frame, validates ρM.

(ii) Let $\mathfrak{F} = \langle W, R, P\rangle$ be a descriptive frame for τL. Then $\rho\mathfrak{F}$ and so $((\rho\mathfrak{F})^+)_+$ are frames for L. By Maksimova's Lemma $\langle \rho W, \rho R\rangle$ is the underlying Kripke frame of

$((\rho\mathfrak{F})^{+})_{+}$ and hence, by the canonicity, $\langle\rho W, \rho R\rangle \models L$. But then one can readily show by induction that $\langle W, R\rangle \models T\varphi$ for every $\varphi \in L$, which means that $\langle W, R\rangle \models \tau L$. ⊣

Remark. The canonicity is not in general preserved while passing from L to σL, witness the pair **Int** and σ**Int** = **K4Grz**. ⊣

Note by the way that using Theorem 6.6 we can transfer Fine's [1975] Theorem to intermediate logics.

Theorem 6.7 *If an intermediate logic L is elementary and Kripke complete then it is canonical.*

Proof. According to Chagrova [1990], τL is elementary and, as was shown by Zakharyaschev [1989, 1989a], it is Kripke complete. Therefore, by Fine's Theorem, τL is canonical, and so, by Theorem 6.6, $L = \rho\tau L$ is canonical as well. ⊣

As to the intermediate cofinal subframe logics, Theorem 6.1 degenerates into

Theorem 6.8 *All intermediate logics with disjunction free additional axioms are elementary, canonical and compact*[4].

Proof. As was shown in Zakharyaschev [1983, 1989], every intermediate logic L with disjunction free extra axioms can be axiomatized by intuitionistic canonical formulas $\beta(\mathfrak{F}, \emptyset, \bot)$ without closed domains, i.e. we have

$$L = \mathbf{Int} + \{\beta(\mathfrak{F}_i, \emptyset, \bot) : i \in I\}.$$

By the Modal Companion Theorem of Zakharyaschev [1989],

$$\tau L = \mathbf{K4} \oplus \alpha(\bullet, \emptyset) \oplus \{\alpha(\mathfrak{F}_i, \emptyset, \bot) : i \in I\}.$$

Since all $\mathfrak{F}_i$'s are partially ordered and by Theorem 6.1, τL is elementary and canonical. Then, according to Chagrova [1990], L is elementary and, by Theorem 6.6 (i), it is canonical and so compact. ⊣

The fact that all such intermediate logics are elementary was first proved by Chagrova [1986] and Rodenburg [1986] and their canonicity was established by Shimura [1992] who studied the canonical models of some cofinal subframe predicate intermediate logics and later by Zakharyaschev [1992a] with the help of Chagrova's and Rodenburg's result and the intuitionistic version of Fine's [1975] Theorem.

The last result in this section concerns the modal definability. We say a class $\mathcal{C}$ of Kripke frames is *modal* if there is a set Γ of modal formulas such that, for any frame $\mathfrak{F}$, $\mathfrak{F} \in \mathcal{C}$ iff $\mathfrak{F} \models \Gamma$. Fine [1985] gave a characterization of modal classes of transitive frames closed under subframes. Namely, such a class is modal iff it is closed under reducibility

[4] An intermediate logic L is called ***compact*** (or ***strongly complete***) if, for every pair (Γ, Δ) of sets of formulas such that $L \vdash \bigwedge_{\psi\in\Gamma'} \psi \to \bigvee_{\chi\in\Delta'} \chi$ for no finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$, all formulas in Γ are simultaneously true and all formulas in Δ are not true at some point in a model whose underlying Kripke frame validates L.

and has the finite subreduction property, i.e. $\mathfrak{F} \in \mathcal{C}$ whenever every finite rooted subreduct of $\mathfrak{F}$ is in $\mathcal{C}$.

Now we extend Fine's characterization to classes of transitive frames closed under cofinal subframes. Say that a class of frames $\mathcal{C}$ has the *finite cofinal subreduction* property if $\mathfrak{F} \in \mathcal{C}$ whenever every finite rooted cofinal subreduct of $\mathfrak{F}$ is in $\mathcal{C}$.

Theorem 6.9 *Suppose that a class $\mathcal{C}$ of transitive Kripke frames is closed under cofinal subframes. Then $\mathcal{C}$ is modal iff it is closed under reducibility and has the finite cofinal subreduction property.*

Proof. ($\Rightarrow$) Let L be the modal logic of the class $\mathcal{C}$. Since $\mathcal{C}$ is closed under cofinal subframes and by Theorem 5.1, $L \in \mathcal{CSF}$. Since $\mathcal{C}$ is modal, it coincides with the class of all Kripke frames for L. But then, by the P-morphism Theorem, $\mathcal{C}$ is closed under reducibility and, by the Refutability Criterion for the canonical formulas, it has the finite cofinal subreduction property.

($\Leftarrow$) Take $\Gamma = \{\alpha(\mathfrak{G}, \emptyset, \bot) : \mathfrak{G}$ is a finite rooted frame and $\mathfrak{G} \notin \mathcal{C}\}$ and show that $\mathcal{C}$ is the class of all Kripke frames for Γ. Suppose $\mathfrak{F} \models \Gamma$. Then every finite rooted cofinal subreduct of $\mathfrak{F}$ is in $\mathcal{C}$, for if $\mathfrak{G} \notin \mathcal{C}$ is such a subreduct then $\alpha(\mathfrak{G}, \emptyset, \bot) \in \Gamma$, and so, by the Refutability Criterion, $\mathfrak{F} \not\models \alpha(\mathfrak{G}, \emptyset, \bot)$, contrary to our assumption. Therefore, $\mathfrak{F} \in \mathcal{C}$.

Conversely, suppose $\mathfrak{F} \in \mathcal{C}$ but $\mathfrak{F} \not\models \alpha(\mathfrak{G}, \emptyset, \bot)$ for some $\mathfrak{G} \notin \mathcal{C}$. Then $\mathfrak{F}$ is cofinally subreducible to $\mathfrak{G}$, i.e. there is a cofinal subframe $\mathfrak{H}$ of $\mathfrak{F}$ which is reducible to $\mathfrak{G}$. Since $\mathcal{C}$ is closed under cofinal subframes, $\mathfrak{H} \in \mathcal{C}$, and so $\mathfrak{G} \in \mathcal{C}$, which is impossible. $\dashv$

7 Quasi-normal subframe and cofinal subframe logics

In the final section of this Part we briefly consider quasi-normal (i.e. not necessarily closed under necessitation $\varphi/\Box\varphi$) logics containing **K4** which can be axiomatized by normal and quasi-normal canonical formulas[5] without closed domains. Those quasi-normal logics that can be represented in the form

$$L = (\mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \emptyset) : i \in I\}) + \{\alpha(\mathfrak{F}_j, \emptyset) : j \in J\} + \{\alpha^-(\mathfrak{F}_k, \emptyset) : k \in K\} \qquad (2)$$

are called, as in the normal case, *(quasi-normal) subframe logics* and those of the form

$$L = (\mathbf{K4} \oplus \{\alpha(\mathfrak{F}_i, \emptyset, \bot) : i \in I\}) + \{\alpha(\mathfrak{F}_j, \emptyset, \bot) : j \in J\} + \{\alpha^-(\mathfrak{F}_k, \emptyset, \bot) : k \in K\} \qquad (3)$$

[5] In a few words I remind the reader of the difference between normal and quasi-normal canonical formulas; for details see Part I. Each quasi-normal canonical formula, denoted by $\alpha^-(\mathfrak{F}, \mathfrak{D}, \bot)$, is associated with a frame $\mathfrak{F}$ having irreflexive root u and a set $\mathfrak{D}$ of antichains in $\mathfrak{F}$. A general frame $\mathfrak{G}$ with actual world w refutes $\alpha^-(\mathfrak{F}, \mathfrak{D}, \bot)$ iff there is a cofinal quasi-subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (CD) for $\mathfrak{D}$ and the actual world condition (AW): $f(w) = u$. The only difference between subreduction and quasi-subreduction is that the latter can map any set of points to the irreflexive root of a frame. Finally, the difference between $\alpha^-(\mathfrak{F}, \mathfrak{D}, \bot)$ and $\alpha^-(\mathfrak{F}, \mathfrak{D})$, the quasi-normal negation free canonical formula, is the same as between $\alpha(\mathfrak{F}, \mathfrak{D}, \bot)$ and $\alpha(\mathfrak{F}, \mathfrak{D})$.

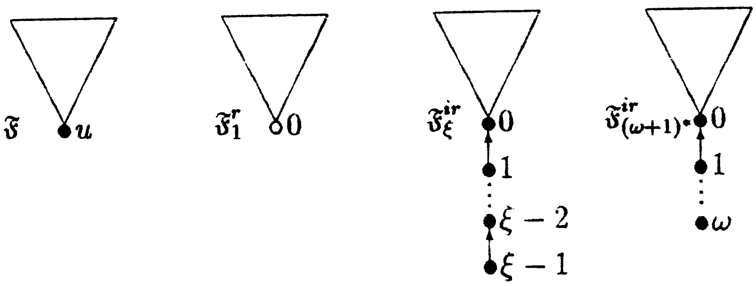


Figure 5:

are called (*quasi-normal*) *cofinal subframe logics*. The classes of quasi-normal subframe and cofinal subframe logics are denoted by $\mathcal{QSF}$ and $\mathcal{QCSF}$, respectively.

The example of Solovay's logic

$$\begin{aligned}\mathbf{S} &= (\mathbf{K4} \oplus \Box(\Box p \to p) \to \Box p) + \Box p \to p \\ &= (\mathbf{K4} \oplus \alpha(\circ, \emptyset)) + \alpha(\bullet, \emptyset),\end{aligned}$$

which clearly has no Kripke frames at all, shows that Theorem 4.1 cannot be extended to $\mathcal{QSF}$ and $\mathcal{QCSF}$. Yet we are going to prove that all finitely axiomatizable quasi-normal subframe and cofinal subframe logics are decidable.

We require the following notation. Given a frame $\mathfrak{F} = \langle W, R\rangle$ with irreflexive root u and an ordinal ξ, $0 < \xi < \omega$, by $\mathfrak{F}_\xi^{ir}$ and $\mathfrak{F}_\xi^r$ we denote the frames which are obtained from $\mathfrak{F}$ by replacing u with the descending chains $0, \dots, \xi - 1$ of irreflexive and reflexive points, respectively. And by $\mathfrak{F}_{(\omega+1)^\bullet}^{ir} = \left\langle W_{(\omega+1)^\bullet}, R_{(\omega+1)^\bullet}^{ir}, P_{(\omega+1)^\bullet}\right\rangle$ we denote the frame which is obtained from $\mathfrak{F}$ by replacing u with the infinite descending chain $0, 1, \dots$ of irreflexive points and then adding irreflexive root ω, with $P_{(\omega+1)^\bullet}$ containing all subsets of $W - \{u\}$, all finite subsets of natural numbers $\{0, 1, \dots\}$, all (finite) unions of these sets and all complements to them in the space $W_{(\omega+1)^\bullet}$ (see Fig. 5). Note that if $\omega \in X \in P_{(\omega+1)^\bullet}$ then X contains infinitely many natural numbers. Observe also that $\mathfrak{F}$ is a quasi-reduct of every frame of the form $\mathfrak{F}_\xi^{ir}$, $\mathfrak{F}_\xi^r$ or $\mathfrak{F}_{(\omega+1)^\bullet}^{ir}$ for $0 < \xi < \omega$.

The following theorem characterizes the canonical formulas belonging to logics in $\mathcal{QSF}$ and $\mathcal{QCSF}$. Its proof, as that of Theorem 2.2, heavily uses the Composition Lemma, which is obviously generalized to compositions of (cofinal) quasi-subreductions.

Theorem 7.1 *Suppose L is a subframe or cofinal subframe quasi-normal logic. Then*

(i) *for every finite frame $\mathfrak{F}$ with root u, $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in L$ iff $\langle \mathfrak{F}, u\rangle \not\models L$ and*

(ii) *for every finite frame $\mathfrak{F}$ with irreflexive root u, $\alpha^-(\mathfrak{F}, \mathfrak{D}, \bot) \in L$ iff $\langle \mathfrak{F}, u\rangle \not\models L$, $\langle \mathfrak{F}_1^r, 0\rangle \not\models L$ and $\left\langle \mathfrak{F}_{(\omega+1)^\bullet}^{ir}, \omega\right\rangle \not\models L$.*

Proof. (i) is proved similarly to Theorem 2.2. Details are left to the reader. (Do not forget that $\Box\alpha(\mathfrak{F}_i, \emptyset) \in L$, if L is of the form (1), and $\Box\alpha(\mathfrak{F}_i, \emptyset, \bot) \in L$, if L is of the form (2), for every $i \in I$.)

(ii) If $\alpha^-(\mathfrak{F},\mathfrak{D},\bot) \in L$ then none of the frames $\langle\mathfrak{F},u\rangle$, $\langle\mathfrak{F}_1^r,0\rangle$ and $\left\langle\mathfrak{F}^{ir}_{(\omega+1)^*},\omega\right\rangle$ validates L, since all of them are quasi-reducible to $\langle\mathfrak{F},u\rangle$ and so, by Theorem 3 of Part I, refute $\alpha^-(\mathfrak{F},\mathfrak{D},\bot)$.

To prove the converse suppose that a general frame $\mathfrak{G} = \langle V,S,Q\rangle$ with actual world w (which is the root of $\mathfrak{G}$) refutes $\alpha^-(\mathfrak{F},\mathfrak{D},\bot)$ and show that $\langle\mathfrak{G},w\rangle \not\models L$. By the Refutability Criterion for the quasi-normal canonical formulas, there is a cofinal quasi-subreduction f of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (AW), i.e. $f(w) = u$. Consider the set $U = f^{-1}(u) \in Q$. Without loss of generality we may obviously assume that $U = U\overline{\downarrow}$. There are three possible cases.

Case 1. The point w is irreflexive and $\{w\} \in Q$. Then the restriction of f to $\mathrm{dom}f - (U - \{w\})$ is a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}$ satisfying (AW), and so, by the Refutability Criterion and Composition Lemma, $\langle\mathfrak{G},w\rangle \not\models L$.

Case 2. There is a subset $X \subseteq U$ such that $w \in X \in Q$ and, for every $x \in X$ there exists $y \in x{\uparrow}X$. Then the restriction of f to $\mathrm{dom}f - (U - X)$ is clearly a cofinal subreduction of $\mathfrak{G}$ to $\mathfrak{F}_1^r$ satisfying (AW), and so again $\langle\mathfrak{G},w\rangle \not\models L$.

Case 3. If neither of the preceding cases holds then, for every $X \subseteq U$ such that $w \in X \in Q$, the set $D_X = X - X{\downarrow}$ of dead ends in X is a cover for X, i.e. $X \subseteq D_X\overline{\downarrow}$, and $w \in X - D_X \in Q$. Indeed, since Case 1 does not hold, $w \notin D_X$, for otherwise $\{w\} = D_X \in Q$. And if we assume that $X - D_X\overline{\downarrow} \neq \emptyset$ then $Y = (X - D_X\overline{\downarrow}){\downarrow} \subseteq U$, $w \in Y \in Q$ and $Y = Y{\downarrow}$, i.e. Case 2 holds, which is a contradiction.

Put

$$X_0 = D_U,\ X_1 = D_{U-X_0},\ \ldots,\ X_{n+1} = D_{U-(X_0\cup\ldots\cup X_n)},\ \ldots,$$

$$X_\omega = U - \bigcup_{\xi<\omega} X_\xi.$$

Each of these sets, save possibly X_ω, is an antichain of irreflexive points and belongs to Q. Besides, $X_\zeta \subset X_n{\downarrow} = \bigcup_{n<\xi\leq\omega} X_\xi$ for every $n < \zeta \leq \omega$. Therefore, the map g defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \in V - U \\ \xi & \text{if } x \in X_\xi,\ 0 \leq \xi \leq \omega \end{cases}$$

is a cofinal quasi-subreduction of $\mathfrak{G}$ to $\mathfrak{F}^{ir}_{(\omega+1)^*}$ satisfying (AW).

Suppose for definiteness that L is represented in the form (1). Since $\left\langle\mathfrak{F}^{ir}_{(\omega+1)^*},\omega\right\rangle$ does not validate L, it refutes at least one of L's axioms, and we again should consider three possible cases.

(a) $\mathfrak{F}^{ir}_{(\omega+1)^*} \not\models \alpha(\mathfrak{F}_i,\emptyset)$ for some $i \in I$, i.e. there is a subreduction h of $\mathfrak{F}^{ir}_{(\omega+1)^*}$ to $\mathfrak{F}_i$. Since $\{\omega\} \notin P_{(\omega+1)^*}$, either $\omega \notin \mathrm{dom}h$ or the root $h(\omega)$ of $\mathfrak{F}_i$ is reflexive. Then the composition hg is a subreduction of $\mathfrak{G}$ to $\mathfrak{F}_i$, from which $\mathfrak{G} \not\models \alpha(\mathfrak{F}_i,\emptyset)$ and so $\langle\mathfrak{G},w\rangle \not\models \Box\alpha(\mathfrak{F}_i,\emptyset)$, i.e. $\langle\mathfrak{G},w\rangle \not\models L$.

(b) $\left\langle\mathfrak{F}^{ir}_{(\omega+1)^*},\omega\right\rangle \not\models \alpha(\mathfrak{F}_j,\emptyset)$ for some $j \in J$, i.e. there is a subreduction h of $\mathfrak{F}^{ir}_{(\omega+1)^*}$ to $\mathfrak{F}_j$ satisfying (AW). Then, as we know, $h(\omega)$ is reflexive, and so hg is a subreduction of $\mathfrak{G}$ to $\mathfrak{F}_j$ satisfying (AW). Therefore, $\langle\mathfrak{G},w\rangle \not\models \alpha(\mathfrak{F}_j,\emptyset)$.

(c) $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega \right\rangle \not\models \alpha^-(\mathfrak{F}_k, \emptyset)$ for some $k \in K$, i.e. there is a quasi-subreduction h of $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ to $\mathfrak{F}_k$ satisfying (AW). But then hg is a quasi-subreduction of $\mathfrak{G}$ to $\mathfrak{F}_k$ satisfying (AW), whence $\langle \mathfrak{G}, w \rangle \not\models \alpha^-(\mathfrak{F}_k, \emptyset)$ and $\langle \mathfrak{G}, w \rangle \not\models L$.

Thus, every frame with actual world refuting $\alpha^-(\mathfrak{F}, \mathfrak{D}, \bot)$ is not a frame for L, which means that $\alpha^-(\mathfrak{F}, \mathfrak{D}, \bot) \in L$. ⊣

Corollary 7.2 *All subframe and cofinal subframe quasi-normal logics above* **S4** *have the finite model property.* ⊣

Example 7.3 As an illustration let us use Theorem 7.1 to characterize those normal and quasi-normal canonical formulas that belong to Solovay's logic **S**.

Clearly, either $\alpha(\circ, \emptyset)$ or $\alpha(\bullet, \emptyset)$ is refuted at the root of every rooted Kripke frame. So all normal canonical formulas are in **S**. Every quasi-normal formula $\alpha^-(\mathfrak{F}, \mathfrak{D}, \bot)$ associated with $\mathfrak{F}$ containing a reflexive point is also in **S**, since $\Box\alpha(\circ, \emptyset)$ is refuted at the roots of $\mathfrak{F}$, $\mathfrak{F}^r_1$ and $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$. But no quasi-normal formula $\alpha^-(\mathfrak{F}, \mathfrak{D}, \bot)$ built on irreflexive $\mathfrak{F}$ belongs to **S** because $\mathfrak{F}^{ir}_{(\omega+1)^\bullet} \models \alpha(\circ, \emptyset)$, for $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ contains neither infinite ascending chain nor reflexive point, and $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega \right\rangle \models \alpha(\bullet, \emptyset)$, for $\{\omega\} \notin P_{(\omega+1)^\bullet}$.

This characterization together with the Completeness Theorem for the canonical formulas (Theorem 4 of Part I) provide us with a new decision algorithm for **S**. ⊣

Example 7.4 $\alpha(\mathfrak{F}, \mathfrak{D}, \bot) \in \mathbf{S4} + \Box\Diamond p \to \Diamond\Box p = \mathbf{S4} + \alpha(\circledcirc\!\!\circledcirc, \emptyset, \bot)$ iff either $\mathfrak{F}$ contains an irreflexive point or $\mathfrak{F}$ is reflexive and all its final clusters are proper (cf. Segerberg [1971, p.177]). ⊣

Theorem 7.1 reduces the decision problem for a logic L in $\mathcal{QSF}$ or $\mathcal{QCSF}$ to the problem of verifying, given a finite frame $\mathfrak{F}$ with root u, whether or not the frames $\langle \mathfrak{F}, u \rangle$, $\langle \mathfrak{F}^r_1, 0 \rangle$ and $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega \right\rangle$ refute at least one axiom of L. The first two frames present no difficulty for a finitely axiomatizable L. And our aim now is to show that the condition $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega \right\rangle \not\models L$ can also be verified in finitely many steps.

Lemma 7.5 *Suppose L is a quasi-normal (cofinal) subframe logic represented in the form (1) (respectively, (2)) and $\mathfrak{F} = \langle W, R \rangle$ is a finite frame with irreflexive root u. Then $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega \right\rangle \not\models L$ iff one of the following conditions is satisfied:*

(i) *$\mathfrak{F}^{ir}_\xi$ is (cofinally) subreducible to $\mathfrak{F}_i$ for some $i \in I$ and some $\xi \leq |\mathfrak{F}_i|$;*

(ii) *for some $j \in J$, $\mathfrak{F}_j$ has reflexive root and $\mathfrak{F}$ is (cofinally) subreducible to $\mathfrak{F}_j$, with (AW) being satisfied;*

(iii) *$\mathfrak{F}^{ir}_\xi$ is (cofinally) quasi-subreducible to $\mathfrak{F}_k$ for some $k \in K$ and some $\xi \leq |\mathfrak{F}_k|$, with (AW) being satisfied.*

Proof. Let us suppose for definiteness that L is represented in the form (2); the form (1) is considered analogously.

($\Rightarrow$) If $\mathfrak{F}^{ir}_{(\omega+1)^\bullet} \not\models \alpha(\mathfrak{F}_i, \emptyset, \bot)$ for some $i \in I$ then there is a cofinal subreduction f of $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ to $\mathfrak{F}_i$. The map

$$g(x) = \begin{cases} f(x) & \text{if } x \text{ belongs to a final cluster in } f^{-1}(f(x)) \\ \text{undefined} & \text{otherwise} \end{cases}$$

is also a cofinal subreduction of $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ to $\mathfrak{F}_i$, with $g(\xi) \neq g(\zeta)$ for any distinct $\xi, \zeta \leq \omega$. Let $\mathfrak{F}'$ be the result of removing from $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ all those points $\xi \leq \omega$ that are not in dom$g$. Clearly, $\mathfrak{F}'$ is isomorphic to $\mathfrak{F}^{ir}_\xi$ for some $\xi \leq |\mathfrak{F}_i|$ and g is a cofinal subreduction of $\mathfrak{F}'$ to $\mathfrak{F}_i$.

If $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega \right\rangle \not\models \alpha(\mathfrak{F}_j, \emptyset, \bot)$ for some $j \in J$ then there is a cofinal subreduction f of $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ to $\mathfrak{F}_j$ satisfying (AW). Since $\{\omega\} \notin P_{(\omega+1)^\bullet}$, the root $v = f(\omega)$ of $\mathfrak{F}_j$ is reflexive, and so $f^{-1}(v)$ contains a reflexive point which belongs to $W - \{u\}$. But then the map

$$g(x) = \begin{cases} f(x) & \text{if } x \in W - \{u\} \\ v & \text{if } x = u \end{cases}$$

is a cofinal subreduction of $\mathfrak{F}$ to $\mathfrak{F}_j$ satisfying (AW).

Finally, if $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega \right\rangle \not\models \alpha^-(\mathfrak{F}_k, \emptyset, \bot)$ for some $k \in K$ then there is a cofinal quasi-subreduction f of $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ to $\mathfrak{F}_k$ satisfying (AW). Let v be the root of $\mathfrak{F}_k$. By the definition of $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$, every $X \in P_{(\omega+1)^\bullet}$ containing ω also contains some $\xi < \omega$. Let ζ be the minimal number such that $f(\zeta) = v$. Then the map

$$g(x) = \begin{cases} v & \text{if } x = \zeta \\ f(x) & \text{if } x \text{ belongs to a final cluster in } f^{-1}(f(x)) \\ \text{undefined} & \text{otherwise} \end{cases}$$

is a cofinal quasi-subreduction of $\mathfrak{F}^{ir}_{\zeta+1}$ to $\mathfrak{F}_k$ satisfying (AW). It remains, as we have already done before, to remove from $\mathfrak{F}^{ir}_{\zeta+1}$ all those points $\xi < \zeta$ that are not in dom$g$, thus obtaining a frame which is isomorphic to some $\mathfrak{F}^{ir}_\xi$, $\xi \leq |\mathfrak{F}_k|$, and cofinally quasi-subreducible by g to $\mathfrak{F}_k$ with $g(\xi - 1) = v$.

($\Leftarrow$) If the first condition holds then $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega \right\rangle$ refutes $\Box\alpha(\mathfrak{F}_i, \emptyset, \bot)$. The cofinal subreduction f of the second condition can be extended to the map

$$g(x) = \begin{cases} f(x) & \text{if } x \in W - \{u\} \\ v & \text{if } x = \xi \leq \omega \end{cases}$$

(v is the reflexive root of $\mathfrak{F}_j$) which is a cofinal subreduction of $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ to $\mathfrak{F}_j$ with $g(\omega) = v$, and hence $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega \right\rangle \not\models \alpha(\mathfrak{F}_j, \emptyset, \bot)$. And the third condition gives in the same way a cofinal quasi-subreduction of $\mathfrak{F}^{ir}_{(\omega+1)^\bullet}$ to $\mathfrak{F}_k$ satisfying (AW), from which $\left\langle \mathfrak{F}^{ir}_{(\omega+1)^\bullet}, \omega \right\rangle \not\models \alpha^-(\mathfrak{F}_k, \emptyset, \bot)$. $\dashv$

As a consequence of Theorem 7.1, Lemma 7.5 and the Completeness Theorem for the canonical formulas of Part I we immediately obtain

Theorem 7.6 *All finitely axiomatizable subframe and cofinal subframe quasi-normal logics are decidable.* ⊣

It is not hard also to give a frame-theoretic characterization of the classes $\mathcal{QSF}$ and $\mathcal{QCSF}$ similar to Theorem 5.1. Let us say that a frame $\mathfrak{F}$ with actual world u is a (*cofinal*) *subframe* of a frame $\mathfrak{G}$ with actual world w if $\mathfrak{F}$ is a (cofianl) subframe of $\mathfrak{G}$ and $u = w$.

Theorem 7.7 *L is a (cofinal) subframe quasi-normal logic iff L is characterized by a class of frames with actual worlds that is closed under (cofinal) subframes.*

Proof. Clearly, the class of all frames with actual worlds for a (cofinal) subframe L is closed under (cofinal) subframes.

Conversely, suppose L is characterized by a class $\mathcal{C}$ of frames with actual worlds that is closed under cofinal subframes. Then one can readily show that L is axiomatized by all the formulas $\alpha(\mathfrak{F}, \emptyset, \bot)$ such that $\langle \mathfrak{F}, u\rangle \not\models L$ and all the formulas $\alpha^{-}(\mathfrak{F}, \emptyset, \bot)$ such that the root u of $\mathfrak{F}$ is irreflexive, $\langle \mathfrak{F}, u\rangle \not\models L$, $\langle \mathfrak{F}_1^{ir}, 0\rangle \not\models L$ and $\left\langle \mathfrak{F}_{(\omega+1)^*}^{ir}, \omega\right\rangle \not\models L$. ⊣

References

R.A. BULL [1966], *That all normal extensions of* **S4.3** *have the finite model property*, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, vol.12 (1966), pp.341-344.

A. V. CHAGROV [1983], *On the polynomial finite model property of modal and intermediate logics*, In: Mathematical logic, mathematical linguistics and algorithm theory, pp.75-83, Kalinin State University, Kalinin. (Russian)

A.V. CHAGROV AND M.V. ZAKHARYASCHEV [1991], *The undecidability of the disjunction property of propositional logics and other related problems*, ITLI Prepublication Series, X-91-02, University of Amsterdam. To appear in the Journal of Symbolic Logic, 1993.

L.A. CHAGROVA [1986], *On the first-order definability of intuitionistic formulas with restrictions on occurrences of connectives*, in: Logical methods for constructing effective algorithms, pp.135-136, Kalinin State University, Kalinin. (Russian)

L.A. CHAGROVA [1990], *On the preservation of first-order properties under the embedding of intermediate logics in modal logics.* In: Proceedings of the Tenth USSR Conference for Mathematical Logic, Alma-Ata, 1990, p.163. (Russian)

L.A. CHAGROVA [1991], *An undecidable problem in correspondence theory*, the Journal of Symbolic Logic, vol.56 (1991), pp.1261-1272.

R. FEYS [1965], *Modal logics*, Louvain: E. Nauwelaerts, Paris, 1965.

K. FINE [1971], *The logics containing* **S4.3**, Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, vol.17 (1971), pp.371-376.

K. FINE [1975], *Some connections between elementary and modal logic*, in: S.Kanger (ed.), Proceedings of the Third Scandinavian Logic Symposium, pp.15-31, North-Holland, Amsterdam.

K. FINE [1985], *Logics containing* **K4**. *Part II*, the Journal of Symbolic Logic, vol.50 (1985), pp.619-651.

M. KRACHT [1990], *Internal definability and completeness in modal logic*, Dissertation, Institut für Mathematik II, Freie Universität Berlin, 1990.

LOGIC NOTEBOOK [1986], Novosibirsk. (Russian)

L.L. MAKSIMOVA [1975], *Pretabular extensions of Lewis' logic* **S4**, Algebra and Logic, vol.14 (1975), pp.28-55. (Russian)

L.L. MAKSIMOVA AND V.V. RYBAKOV [1974], *On the lattice of normal modal logics*, Algebra and Logic, vol.13 (1974), pp.188-216. (Russian)

C.G. MCKAY [1968], *The decidability of certain intermediate logics*, the Journal of Symbolic Logic, vol.33 (1968), pp.258-264.

H. ONO AND A. NAKAMURA [1980], *On the size of refutation Kripke models for some linear modal and tense logics*, Studia Logica, vol.39 (1980), pp.325-333.

P.H. RODENBURG [1986], *Intuitionistic correspondence theory*, Dissertation, University of Amsterdam.

K. SEGERBERG [1971], *An Essay in Classical Modal Logic*, Philosophical Studies, Uppsala.

T. SHIMURA [1992], *Kripke completeness of some intermediate predicate logics with the axiom of constant domain and a variant of canonical formulas*, to appear in Studia Logica.

M.V. ZAKHARYASCHEV [1983], *On intermediate logics*, Soviet Mathematics Doklady, vol.27 (1983), pp.274-277.

M.V. ZAKHARYASCHEV [1984], *Syntax and semantics of superintuitionistic and modal logics*, Dissertation, Moscow. (Russian)

M.V. ZAKHARYASCHEV [1988], *Syntax and semantics of modal logics containing* **S4**, Algebra and Logic, vol.27 (1988), pp.659-689. (Russian)

M.V. ZAKHARYASCHEV [1989], *Syntax and semantics of intermediate logics*, Algebra and Logic, vol.28 (1989), pp.402-429. (Russian)

M.V. ZAKHARYASCHEV [1989a], *Modal companions of intermediate logics: syntax, semantics and preservation theorems*, Mathematical Sbornik, vol.180 (1989), pp.1415-1427. (Russian)

M.V. ZAKHARYASCHEV [1992], *Canonical formulas for* **K4**. *Part I: Basic results*, the Journal of Symbolic Logic, vol.57 (1992), pp.1377-1402.

M.V. ZAKHARYASCHEV [1992a], *Intermediate logics with disjunction free axioms are canonical*, IGPL Newsletter, vol.1 (1992), no.4, pp.7-8.

M.V. ZAKHARYASCHEV [1993], *Canonical formulas for modal and superintuitionistic logics: a short outline*, to appear in: M. de Rijke (ed.), Modal logic and its neighbours '92.

M.V. ZAKHARYASCHEV AND S.V. POPOV [1980], *On the complexity of countermodels for intuitionistic calculus*, Preprint, Institute of Applied Mathematics, the USSR Academy of Sciences, no.45. (Russian)

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